The Effect of Physical Manipulation on Children's Numerical Strategies – Evaluating the Potential for Tangible Technology

by

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Abstract

Despite their prevalence in early years' education, there seems to be a lack of agreement over how or indeed whether physical objects support children's learning. Understanding the role of physically manipulating representations has gained impetus with the increasing potential to integrate digital technology into physical objects: tangible technology. This thesis aimed to evaluate the potential for tangible technologies to support numerical development by examining young children's (4-8 years) use of physical objects in a numerical task. This task required them to find all the different ways in which a number (e.g., 7) can be decomposed (e.g., into 2 & 5).

Seven carefully designed studies compared children's numerical strategies using physical objects (cubes) with other materials (paper/virtual representations) or no materials. The studies showed that physical objects not only helped children identify solutions through simple physical actions, but fostered strategies that related solutions such as swapping groups of cubes or moving just one cube to get a new solution. This led to predictions about how a computer might influence strategies by constraining children's actions to moving just one object at a time using the mouse. These predictions were confirmed, and a further study showed how using materials that changed colour according to the number grouped could support strategies by drawing children's attention to numerical changes.

The research showed that, to help children identify ways to break down a number efficiently, it was more effective to constrain their actions using a graphical, rather than tangible, interface. However, when multiple (physical) objects could be manipulated, children were able to constrain their own actions and used a wider range of strategies. Although moving multiple objects can be facilitated through interfaces such as tabletop computers, this research indicated certain cognitive benefits of physically manipulating representations for children's numerical development that may inform tangible designs.

Publications arising from the thesis

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Table of Contents

List of Figures

List of Tables

Extended Summary

Physical learning materials, or 'manipulatives' (e.g., cubes or tiles), are a common learning resource in early years' education across several cultures. However, despite their historical and widespread use, it remains unclear how, or even if, they support children's learning. Establishing the role of manipulatives in education has therefore been identified as an important goal for educational research. More recently, the development of our understanding in this area has gained increased impetus through the potential to augment physical materials with technology ('tangible technologies') to support learning. Understanding the potential of tangible technologies is not only key to designing novel and effective pedagogical materials, but can also help identify the value, or limitations, of other forms of interaction with technology, such as a mouse or tabletop interface.

The aim of this doctoral research was to evaluate the potential of tangible technologies for learning by examining the role of physically manipulating representations within a specific learning activity. This activity centred on a key concept in young children's numerical development – *additive composition*, which refers to an understanding of how numbers are composed of smaller numbers (e.g., 7 is composed of 3 and 4). Understanding how numbers can be decomposed in different ways is key to the use of more flexible calculation strategies and has been proposed as a foundation to understanding how multidigit numbers are composed of different multiples of ten (e.g., that 17 can be decomposed into 10 and 7 (Nunes & Bryant, 1996). Indeed, developing an understanding of additive composition has been described as a key step in numerical development for children (Fuson, 1992). The main question in this research was therefore: does physically manipulating digital representations present any unique benefits for supporting children's understanding of additive composition?

In order to address the main research question, children's interactions with physical representations (small plastic cubes) were compared with other forms of representation (e.g., pictorial/virtual materials) as a means of helping young children (aged 4-8 years) explore the concept of additive composition. The task used to explore this concept throughout the research studies was adapted from Jones et al (1996), and required children to identify all the ways of partitioning a single digit number (e.g., 7) into combinations of two parts (e.g., 2 & 5, 6 & 1) using a story context. This activity was hence called the partitioning task. In order to answer the main research question, a total of seven studies were carried out. These addressed four subsidiary questions that were identified from a review of the research literature. Findings relating to these subsidiary questions and the main research question are summarised as follows:

1. Do physical objects support children's strategies for partitioning numbers?

The first, exploratory, study compared the use of physical materials (cubes) with pictorial materials (squares) and a control condition (no materials) in helping children (aged 4-8 years) solve two numerical tasks: an addition task and a partitioning task. Measures were taken of the number of correct scores and whether the representation was used for each problem. No significant differences were found between the numbers of correct scores for the different conditions. Although children used cubes more than squares in both tasks, this did not lead to more correct solutions. This lack of advantage seemed attributable to the demands of having to count out objects. Indeed, children often used more efficient counting strategies (e.g., counting-on) when they chose not to use materials. This study thereby highlighted the need to take account of the initial demands of counting out the total amount when starting the partitioning task.

The second study addressed this issue by examining whether children identified more partitioning solutions using physical materials than no materials when given the correct number of counted out cubes. As expected, children identified more correct solutions using cubes than with no external representation. However, arguably of greater interest was the effect of physical materials on children's strategies. By developing a coding system, solutions were categorised according to their relationship with the previous solution. A *compensation* solution was coded when the solution was one different from the previous solution (e.g., 2 & 6 following 1 & 7) and a *commutative* solution when the solution was the reverse of the previous one (e.g., 2 & 6 following 6 & 2). It was found that when children used physical objects they not only identified more correct solutions but identified a significantly greater proportion of solutions that were related (those coded as *compensation* and *commutative*). This is significant because relating solutions reflects important numerical relationships. Another interesting finding from this study was how children were more likely to begin with a solution that partitioned a number into two equal groups when using materials than no materials.

2. What are the advantages/limitations of physically manipulating representations for children's partitioning strategies?

Study 3 examined what properties of physical objects influenced children's partitioning strategies by comparing performance in four conditions in a 2x2 between subjects design: where children were provided with either physical or pictorial materials, and were provided with either a record or no record of their previous solutions. It was found that providing children with a record of the representational configurations they had created for previous solutions did not affect their strategies even though this record showed which solutions had been identified (and hence which solutions remained). In contrast, physically manipulating representations seemed to be important: children identified significantly more correct partitioning solutions using physical than pictorial materials. Furthermore, children in the Physical condition identified significantly more solutions that were related – i.e. more *compensation* and *commutative* solutions.

Study 4 was designed to examine how using physical materials influenced children's partitioning strategies using video observations. Children solved problems first with no materials, and then in counterbalanced conditions using physical and pictorial materials. Supporting the previous studies' findings, children identified more correct solutions using physical materials than in the other two conditions. Video records of children's actions showed how physical objects allowed them to create new spatial partitioning configurations with ease and then enumerate (most of) these as valid solutions. This study also examined the role of different properties of the physical materials. Children touched objects both to help count and to keep track of the position of objects (freeing up demands of visual attention). Objects were sometimes stacked vertically or moved relative to the child's position. However, it was not clear how much advantage this provided over the pictorial materials, especially as the amounts being counted were small (hence posing limited computational demands). More important seemed to be the type of actions children made with the materials when relating consecutive solutions. *Commutative* solutions overwhelmingly reflected children interchanging groups of objects. This action involved moving multiple objects using both hands, sometimes picking up groups or simply pushing them. In contrast, *compensation* solutions involved more constrained manipulation, where children would move just a single object with one hand.

3. What is the effect of constraining manipulation on children's partitioning strategies?

Having identified a relationship between the manipulative properties of representations and partitioning strategies, it was predicted that constraining the number of objects that could be manipulated at one time would significantly affect the strategies used to identify solutions. This prediction addressed the main research question by looking at whether physically manipulating representations leads to differences in strategies in comparison to other forms of interaction. This study examined the effect of constraining actions so that only one object could be manipulated at a time, as this action reflected one of the key strategies identified in the previous study (*compensation*). As predicted, children identified significantly less *commutative* solutions when their actions were constrained than not constrained. However, although children identified more *compensation* solutions, the difference was not significant (possibly because children in the constraints condition tended to move objects one at a time as quickly as possible using both hands and often needed reminding of the constraining rule).

Study 6 was therefore carried out to examine the effect of constraining actions using a graphical user interface where children could only move one object (on-screen square) at a time more slowly using a mouse. Although there were no differences in the number of solutions identified in each condition, as predicted, there were significant differences in the strategies used. Children identified significantly less *commutative* solutions and significantly more *compensation* solutions when their actions were constrained using the mouse than when manipulating physical materials. This study thereby highlighted how different forms of interaction can impact on the strategies children use to partition numbers. As these strategies reflect different aspects of additive composition, it is possible that different forms of interface will affect the ideas that develop.

4. Can children's partitioning strategies be supported by augmenting the representation's perceptual information?

The previous study showed how children could be encouraged to identify *compensation* solutions when they could move only one object at a time. However, they would often create a new configuration without identifying it as a new solution. The final study therefore examined the effect of a digital perceptual effect on children's partitioning strategies. With this effect, on-screen squares would change colour according to the number grouped together, and changes of groupings would thereby be visually emphasised. As predicted, it was found that this prompt led children to identify significantly more incremental changes to the representation – i.e. significantly more *compensation* solutions.

Does physically manipulating digital representations present any unique benefits for supporting children's understanding of additive composition?

The findings from the seven studies were then used to address the main research question. The studies highlighted how the ability to spatially manipulate representations not only helped children identify more ways to partition a number, but also helped them relate solutions to each other better than with representations they could not spatially manipulate (paper), or with no materials. However, it is possible to spatially manipulate representations on a computer using a mouse, and it was shown in this research how constraining children's actions using this form of interface can actually help children to identify unitary changes to the representation (solutions differing by just one in each part). In this partitioning problem, the verbal identification of such incremental changes reflects an efficient strategy – *compensation*.

Hence, when exploring additive composition in this particular partitioning problem, constraining children's actions using a graphical (rather than tangible) interface seemed to encourage more efficient strategies. However, it was shown that constraining their actions almost eliminated their use of another strategy – the *commutative* strategy. It was thereby concluded that physically manipulating representations led to a wider range of strategies. Furthermore, when children identified *compensation* solutions using physical objects, they had to constrain their own actions. Indeed, many children were observed to change strategy, realising the advantage of moving only one object at a time. It might be argued that this use of more varied strategies (and potential for children to constrain their own actions) is pedagogically advantageous, providing the opportunity for children to reflect upon their actions and then select the most appropriate strategy. Further research might investigate this possibility through more developmental interventions.

Although it is possible to design ways of allowing children to select and manipulate multiple objects using a mouse, this research has demonstrated how physically manipulating representations allows children to move single or multiple objects with ease. Yet, other interfaces such as multi-point touch surface table computers also present accessible ways to move multiple objects and moreover, such interfaces can present designers with greater flexibility over when (and when not) to constrain actions to foster certain numerical strategies. Further research however would be needed to explore how certain representational differences may lead to differences in children's strategies. It is possible that certain physical affordances observed in this research, such as the ability to touch objects (to help count), or to gather multiple objects, or even the ability to lift objects over one another as seen in many *commutative* strategies, indicate that a tangible interface would be advantageous.

The augmented materials in the final study demonstrated the potential to create dynamic perceptual effects to help children identify numerical changes to the representation. The particular effect used - where colour corresponded to quantity - meant that certain numerical relations were highlighted, such as whether parts were the same or when parts were reordered (i.e. *commutative*). Since actions such as splitting groups in half and swapping over groups were more prevalent in the Physical condition in this research, this raises an interesting question of whether augmenting physical objects with such perceptual effects might support children's ability to reflect on certain numerical changes. This possibility suggests that physically manipulating digital representation may present unique benefits for supporting children's understanding of additive composition.

Chapter 1

Review of the Literature

1.1 Introduction

1.1.1 Importance of mathematical development

Mathematics is an essential life skill (Burr, 2008) – from organising budgets to checking change when shopping and has important economic implications (NCTM, 2000, p.5). Considering the importance of being 'mathematically able' it is of some concern that, despite substantial investment, many children still have difficulties with some or most aspects of arithmetic (Dowker, 2009). As children's mathematical abilities at a young age have a close bearing on their later success (Burr, 2008), understanding and addressing young children's difficulties is a significant goal, and highlights the importance of research in this area.

One challenge for research is to try to understand the role of mathematical tools in supporting mathematical activity. As Sutherland (2007) states:

> *"From the perspective of mathematics education, it is important to analyse what a particular tool privileges or potentially enables a person to do and the potential purpose of each tool for learning and doing mathematics"* (p. 6)

Some tools, such as manipulatives, are materials that have been specifically designed or chosen to support children's learning. Manipulatives are physical materials such as tiles and cubes intended to represent more abstract concepts such as numbers, and are used widely across early years' educational settings to help children learn certain ideas. Nevertheless, despite their history and widespread use, it remains unclear how or even if they support children's learning (see McNeil & Jarvin, 2007). Consequently, establishing the role of physical materials in learning has been identified as an important goal for educational research (Ginsburg & Golbeck, 2004).

The fast evolving capacity of digital technology has played a significant role in the development of mathematical tools and this is clearly observable in the development of manipulatives. Graphical objects resembling physical manipulatives can now be presented and interacted with on computers using a keyboard or mouse, giving rise to the term 'virtual manipulatives' (Moyer, Bolyard, & Spikell, 2002). The benefits of these virtual representations have been well described (e.g., Clements, 1999) and have led to a fast growing generation of online tools. This novel mode of interaction raises important new questions – for example (in particular) what is the impact of this form of interaction on children's learning?

Research into the effect of the interface on children's interactions has been carried out over the last thirty years, and has arguably gained significance in light of emerging forms of interaction such as tabletop computers and tangible technologies. Tangible technologies, or more simply 'Tangibles', are digitally augmented physical learning materials. Although their use in education can be traced back to the first physical embodiments of Logo (floor robots such as Roamer[1] for example) that have built upon the seminal work of Seymour Papert (see section 1.3.2.3), the mouse and keyboard remain the pervasive form of interaction. However, more recently, the increasing capacity to integrate more sophisticated technology seamlessly into smaller devices has generated a wealth of novel possibilities for developing effective learning materials.

The ability to combine the possible benefits of digital technology and physical manipulation to support young children's learning has generated significant research interest as well as novel designs (M. Resnick et al., 1998). Nevertheless, evaluating the potential of this form of interface returns us to an important question, namely: what is the impact of this form of manipulation on children's learning?

1.1.2 Summary

It is possible that Tangibles offer the potential to help children learn key numerical ideas. In order to evaluate this potential however, it is necessary to develop our understanding of what advantages this form of interaction brings. By identifying specific advantages and limitations of physical manipulation in learning, it may be possible to inform the design of effective new learning materials.

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[1] (Roamer is a floor robot where children are able to input directional instructions through keys on the robot (or via a computer).

1.1.3 Aim of thesis

The aim of this thesis is to evaluate the potential of Tangibles to support young children's[2] numerical development. This will be achieved by first reviewing the literature in the following areas:

- Children's numerical development
- Physical learning materials
- Digitally augmented manipulatives (including Tangibles and virtual representations manipulated using a graphical interface)

From this review, more specific research questions will be identified. These research questions will then be summarised in the final section of the review.

1.2 Children's Numerical Development

This section aims to review the literature on young children's numerical development in order to identify what key concept might be supported through the design of more effective learning materials.

2 Children aged 4-8 years who are in their first few years of schooling.

1.2.1 Preschool ability

1.2.1.1 Innate numerical ability

In 1954, Tobias Dantzig wrote a book entitled "*Number, the language of science*" which suggested that people were born with a faculty that the author referred to as 'number sense' (Dantzig, 1954). The book was written whilst the Piagetian thought dominated, which was quite conservative with regard to young children's numerical abilities. It took nearly twenty years for Dantzig's insight to be confirmed.

In a pioneering experiment, Starky and Cooper (1980) showed that 4-6 month year olds were more likely to attend to visual arrays (dots) that had changed in numerosity using a 'habituation-dishabituation' paradigm: they looked for longer at a novel stimulus (different number of objects). Clearly, with each change of numerosity, there were other factors that could vary such as the area or darkness of the array, therefore Starkey and Cooper tried to control for these by changing the arrangement of dots in each trial. Nevertheless, Mix, Huttenlocher & Levine (2002) have emphasised the difficulty in ruling out other perceptual clues such as shape, size or density rather than quantity in children's judgements and suggested that children may be responding to continuous rather than discrete quantity. Wynn, Bloom, & Chiang (2002) have attempted to address this possibility by using a group of moving dots, controlling for factors such as area and contour, thereby showing that infants do indeed respond to numerosity.

Wynn's research has also demonstrated that infants are able to compute basic arithmetic consequences of adding and subtracting (e.g., 1+1, 2-1) again using the 'habituation-dishabituation' paradigm. Although her assertions have been questioned (e.g., Cohen & Marks, 2002), the balance of evidence does suggest that infants are able to represent the numerosity of sets and carry out mental manipulations of these representations.

Infants do have an upper limit for their numerical concept: up to around 4 objects (Starkey & Cooper, 1980), which is most likely to reflect their ability to identify the numerosity of an array at a glance without counting. This perceptual ability is shared with adults and has been called *subitising* (Mandler & Shebo, 1982), which allows children and adults to enumerate small numbers (up to around 5) without having to count.

Research continues to develop our understanding of innate mechanisms that may provide the foundations of later ability, such as a possible internal number line that enables children and adults to approximate the addition and subtraction of larger numbers of perceptual objects (Gilmore, McCarthy, & Spelke, 2007). However, in order to succeed mathematically, it is necessary to understand how to manipulate and communicate mathematical symbols, starting with number words and progressing to more complex operations.

1.2.1.2 Pre-School Experience

Children's numerical ability is founded on their early experiences (Baroody, Eiland, & Thompson, 2009). Jordan, Kaplan, Ramineni, & Locuniak (2009), for example, have demonstrated a strong and significant relationship between children's kindergarten (aged 5.5 years) number competence and their mathematical achievement three years later. The start of school does not however mark the beginning of children's numerical development, because children bring to school a range of skills and understanding gained from prior informal experiences (Canobi, 2007) such as the ability to add or subtract (Martin Hughes, 1981). Indeed, in a study of over 1,400 children in Australia, Clarke, Clarke, & Cheesman (1996) found that much of what had traditionally formed the maths curriculum for the first year of school was already understood by many children on arrival at school. When children enter school, they already therefore have some knowledge of the number system and possibly some basic operations. However, their limited understanding of numbers will still make certain maths problems inaccessible. To identify these difficulties, it is important to examine the development of children's understanding of number words, and how this relates to their ability to apply this understanding to more complex problems.

1.2.2 Children's developing understanding of number words

When children first say the number words, they do so without understanding exactly what these words mean. Indeed, the words may just be part of an inseparable linguistic sequence, such as part of a nursery rhyme. Eventually, children not only learn the symbolic significance of these words, but also how they are related within a specific culturally determined decade system. Fuson (1992) identified specific stages to this development. These will be outlined before looking at one of the most difficult numerical concepts children have to learn – multidigit understanding.

Fuson identified five key levels of development: String, Unbreakable list, Breakable chain, Numerable chain and Bidirectional chain.

- *String*

Children initially learn the number words, possibly through songs or counting activities, and may be able to recite them, albeit not actually being able to distinguish individual number words within the linguistic 'string'.

- *Unbreakable list*

In the next stage, children learn to identify the number words, allowing them to take part in counting activities which involve reciting these words in the correct order, using each word to correspond to each item counted. This one-to-one correspondence between object and count words is not however immediately clear; it is a skill that is developed.

Although children may become proficient in counting, this does not mean that they understand the significance of each count word. For this, children need to realise that the last word counted represents the whole set, i.e. 'three' is not just the last word counted but represents three objects. The notion that number words refer to a set is referred to as the *cardinal principle*, leading researchers to talk about children's 'understanding of cardinality'.

In Fuson's Unbreakable stage, children make a key developmental step – they are able to enumerate quantities, through counting, or possibly subitising if the set is small, and understand that the number words can represent quantities. Children are consequently able to approach questions asking '*how many*?' However, their calculation strategies are limited, mainly because they do not yet understand that number words can be 'broken'. In other words, given two amounts to add, children will want to combine the amounts and 'count-all', starting at the first object. Consequently, addition is still dependent on objects, or *perceptual items* as Fuson refers. In fact, young children can often find the actual question '*how many?*' difficult to understand unless there is a concrete referent (Hughes, 1986).

- *Breakable chain level*

Children's first step away from their dependence on perceptual items in counting is integral to Fuson's Breakable chain level. This level refers to how children can 'break' the sequence of numbers by using a number word to represent a quantity within an addition (or subtraction) sum. Instead of counting from one, children can begin 'counting on' from the number word of the first addend (number to be added). The transition from count-all to count-on is considered to reflect a key conceptual step forward and various attempts have been made to evaluate interventions supporting this graduation. Secada, Fuson, & Hall (1983), for example, analysed this transition and identified three sub-skills: a) counting up from an arbitrary point, b) shifting from the cardinal to the counting meaning of the first addend and c) beginning the count of the second addend with the next counting word. The authors demonstrated the success of measuring these three sub-skills on predicting counting-on behaviour and furthermore demonstrated the success of interventions supporting these skills. In order to assess children's ability to count-on, the authors examined children's strategies for adding two amounts when dots representing the first addend were visible and then hidden as illustrated in Figure 1.1, thereby emphasising how counting-on marks children's first steps away from depending on perceptual units to add amounts.

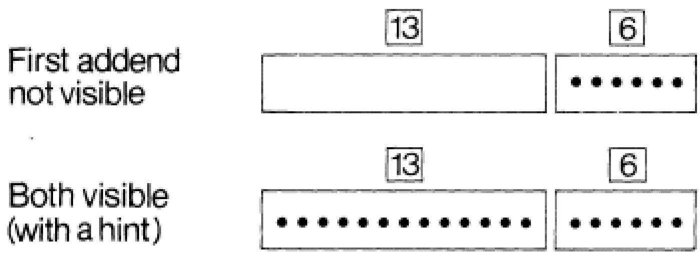

Figure 1.1: Materials used to assess children's ability to count-on (Secada et al., 1983)

- *Numerable chain level*

In the Numerable chain level, both addends are described as embedded within the sum. Consequently, when counting on the second addend, children are not dependent on perceptual items to know when to stop counting. Instead, they use the number word itself as a means to count to the result. For example given the sum 5 + 4, children count-on from 5 (6, 7, 8, 9), and stop counting when they know they have counted out the second addend. This example illustrates how children require a method of knowing how many they have counted-on in the absence of perceptual clues. Three methods have been proposed (Steffe, von Glaserfield, Richards, & Cobb, 1983) for how this can be managed: a) by keeping track of the auditory pattern of the words counted-on; b) by using known finger patterns and matching each addend word to a finger extended as the word is said; and c) by double counting (alternating between the amount counted-on and the total). The skills needed for counting-on are relatively demanding for children, yet are highly significant in that they mark the point at which children become independent of external materials to carry out basic addition and subtraction problems. This progression from a reliance on physical objects has important theoretical implications for this thesis; and it is worth examining in greater detail how children are able to achieve this.

When counting-on, children have the dual task of keeping track of the amount counted-on and the total. This places considerable cognitive demands on children and could explain the almost universal strategy of children using fingers to help them without instruction to do so. Fingers help because they provide perceptual structures that children can enumerate without counting (Fuson, 1992). This is particularly the case for smaller numbers where, by raising fingers when counting-on, children can identify when to stop counting. However, even with fingers, counting-on is quite a demanding task, especially if the amount to count-on is large. In order to manage larger numbers, children therefore need a means to simplify calculations. This can be achieved through more flexible strategies that can facilitate calculations requiring a greater understanding of numbers than is reflected in Fuson's Bidirectional chain level.

- *Bidirectional chain*

In Fuson's final level, the Bidirectional chain level, the whole number sequence becomes a series of embedded cardinal amounts, where each word is part of a series but is also a separable cardinal amount. Understanding how each number is composed of smaller cardinal amounts enables children to transform calculations to take advantage of the number facts that they have begun to learn. For example, the sum 7 + 8 can be decomposed to 7 + 7 + 1. As a result, knowledge of doubles allows children to transform the problem to 14 + 1 which places fewer demands on counting. It is also possible for children to use the decade structure to simplify calculations in a similar way. For example the sum 8 + 9 might be broken down into 8 + 2 + 7. Children can consequently draw upon possible number knowledge of both the number bonds to ten, and their understanding of how ten plus units corresponds to teen numbers.

1.2.2.1 Base ten understanding

The developmental levels described above refer to understanding the structure of single digit numbers. A key challenge and great difficulty for children is in developing an understanding of multidigit numbers (Baroody, 1990; Fuson, 1990; Varelas & Becker, 1997) - how a number such as 16 or 47 is composed of two parts – tens and units.

Significantly, this symbolic system, which uses place value according to a base ten grouping, is a culturally defined system.

According to Fuson (1990), multidigit understanding is difficult because it requires children to understand not only how numbers can be partitioned according to the decade structure, but also how these values interrelate. Resnick (1983b) uses the term 'Unique partitioning' to describe the more basic ability to partition multidigits into tens and units and 'Multiple partitioning' to describe the ability to partition multidigits in non-standard ways that demonstrate the deeper understanding required for competence with multidigit calculations (e.g., a number such as 34 is not just composed of 3 tens and 4 units (unique partitioning) but can also be decomposed into 2 tens and 14 units). Such understanding is challenging; indeed, Resnick described the introduction to the decimal system as the most difficult (and important) instructional task in mathematics in the early years (1983b, p.126).

According to Nunes and Bryant (1996), the reason that children's understanding of the decimal structure does not develop until a later age is likely to be that they do not understand one or both of the two mathematical principles that underlie its structure. These are a) that units can be of different sizes – for example, tens and units, and b) that any positive integer can be decomposed into two or more others that precede it in the ordinal list of numbers.

This understanding of how numbers can be decomposed into smaller numbers is reflected in Fuson's Bidirectional chain level and is also encompassed in other developmental models such as, for example, Saxton and Cakir (2006) who identify children's ability to partition single digit numbers as a significant predictor of base ten understanding or Jones et al. (1996) who describe the ability to partition single digit numbers in different ways as a key prerequisite for place value understanding.

1.2.3 Summary

Children are born with certain perceptual mechanisms that allow them to make non-symbolic quantitative judgements. Although these may support later abilities, children need to learn the number words and, importantly, the structural relationship between them (e.g., how the number 7 can be broken down into 3 and 4). This understanding of how numbers can be partitioned into smaller numbers is referred to as *additive composition* (see following section) and allows children to decompose and recompose addition and subtraction problems, thereby providing them with more flexible and efficient calculation strategies. It is also possible that understanding additive composition provides a foundation for understanding how multidigits are composed.

According to Martins-Mourao & Cowan (1998), additive composition is thought to form a conceptual base for the development of children's elemental arithmetic and their understanding of the decade numeration system. This concept will therefore be examined in more detail.

1.2.4 Additive composition

1.2.4.1 Defining Additive composition

Piaget (1965) coined the term 'additive composition' to refer to the way in which a whole relates to its parts, and involves understanding how the whole is the sum of the parts. Piaget examined part-whole understanding as an ability to simultaneously process a subordinate and basic level concept (e.g., a set of wooden beads consisting of brown and white beads). Piaget described three levels of children's understanding. At the first level, children are unable to process both levels simultaneously – they are unable to hold in mind the relationship of a bead between its inclusion in the basic level set (white/brown) and the superordinate level (wooden). At the second level, where children are aged around 6/7, they begin to discover this relationship through experience, perhaps through trial and error. In the third state, when children are aged around 7-8 years, this "*discovery is spontaneous and immediate*" (Piaget, 1952; p.176) children are able to reason simultaneously about the whole and the parts.

Piaget then compared children's understanding of the additive composition of classes with their understanding of number, where *"a whole remains constant irrespective of the various additive compositions of its parts, e.g., 4 + 4 = 1 + 7 = 2 + 6 = 3 + 5"* (p.183). Piaget argued that a similar pattern of development was apparent, where at the first stage the two sets are not seen as equivalent (i.e. children cannot distinguish changes to the parts from changes to the whole), at the third stage they are seen as equivalent, whilst in between these two stages, children demonstrate intermediary reactions.

Piaget summarised children's understanding in terms of the equation A + A' = B, where A is one part, A' is the other part (members not belonging to A), and B is the whole. This part-whole relationship describes the schema that plays a role in several models of children's development of number understanding (Putnam, Debettencourt, & Leinhardt, 1990; L. B. Resnick, 1983b; Riley & Greeno, 1988), where $P_A + P_B = W$ (reflected in Figure 1.2). Irwin (1996) chose to use numbering instead of lettering for the parts, and is the convention used in this thesis as it helps refer to the order in which parts are presented. Therefore: $P_1 + P_2 = W$.

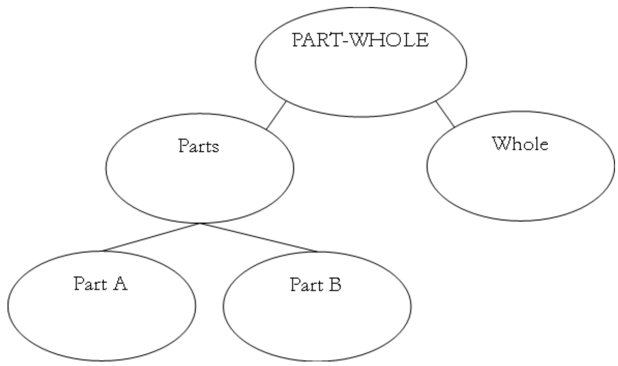

Figure 1.2: Part-Whole Schema (adapted from L. B. Resnick, 1983b)

Piaget explored this concept of additive composition with a question in which participants were asked if a child who had four sweets to eat in the morning and four sweets to eat in the evening would have had the same number in total as a child who had been given one sweet in the morning and seven sweets in the evening. Concrete materials were provided for the task. Piaget concluded that only children of 7 years and older could answer this question successfully. Presented symbolically, these children needed to know that:

$$P_1 + P_2 = (P_1 - 3) + (P_2 + 3), \text{ where } P_1 = P_2 = 4$$

- *Compensation*

The above statement reflects one of two key aspects of Resnick's (1983a) description of the part-whole schema: *compensation*: - that if one part is increased by the same amount as the other part is decreased, then the whole will remain the same. Therefore:

$$\text{If } P1 + P2 = W \text{ then } (P1 + x) + (P2 - x) = W$$

In this statement, the same amount taken from one part is added to the other. If the amount taken from one part is equal in value but a different token[3] the following statement applies:

$$\text{If } P_1 + P_2 = W \text{ then } (P_1 + m) + (P_2 - n) = W, \text{ if } m = n$$

- *Covariation*

The other part-whole key aspect reflects an understanding of how an increase or decrease in one part will lead to an equal increase or decrease in the total as long as the other part remains unchanged. This is referred to as the *covariation* principle:

$$\text{If } P_1 + P_2 = W \text{ then } (P_1 + x) + P_2 = W + x$$

or:

$$\text{If } P_1 + P_2 = W \text{ then } (P_1 - x) + P_2 = W - x$$

A concept of additive composition refers to an understanding of the relationship between parts and wholes. For example, Resnick describes additive composition as the principle that numbers are composed of other numbers, and that any number can be decomposed into parts. Farrington-Flint, Canobi, Wood, & Faulkner (2007) use a similar description:

[3] Token here refers to an individual instance of the same symbol – e.g., different instances of the amount 3.

"Additive composition is the principle that larger numbers are made up of smaller amounts, that is, most natural numbers are composed by addition and therefore can be additively decomposed in various ways" (p.228)

Although Baroody (2004a) does not actually use the term 'additive composition', he concludes that one of the 'big ideas' children must acquire in their numerical development is that:

"a quantity (a whole) can consist of parts and can be broken down (decomposed) into them, and the parts can be combined (composed) to form the whole" (p.199)

Clearly, it is possible to decompose or partition a number into many parts, although the simplest, and most frequently referred to, is breaking a number into two parts. Indeed, Nunes refers to additive composition as how "*any number can be expressed as the sum of two other numbers (or decomposed into two other numbers*)". Similarly, in her Bidirectional stage, Fuson makes reference to learning the ways in which a number can be partitioned into combinations of two numbers (using five as an example). The next section looks at how this concept may develop.

1.2.4.2 Emergence of Additive Composition understanding

- *'Protoquantitative' concept of additive composition*

According to Resnick (1992a), children possess a part-whole understanding from an early age, stemming from their interactions with physical objects. Therefore, even before children have reliably learnt to quantify objects, they know that a whole quantity can be cut into two or more parts, that the parts can then be recombined to make the whole and that the order in which parts are combined does not matter in reconstituting the original amount. Then, as children learn to quantify amounts, this 'protoquantitative' knowledge supports the development of a quantitative understanding of part-whole relations. Resnick describes certain activities, such as combining and comparing sets of objects, as providing the basis of this understanding but does also emphasise the importance of the context in which these occur, which encourages mathematical statements to be made about these actions as indicated in Figure 1.3 (adapted from Resnick, Biull and Lesgold (1992).

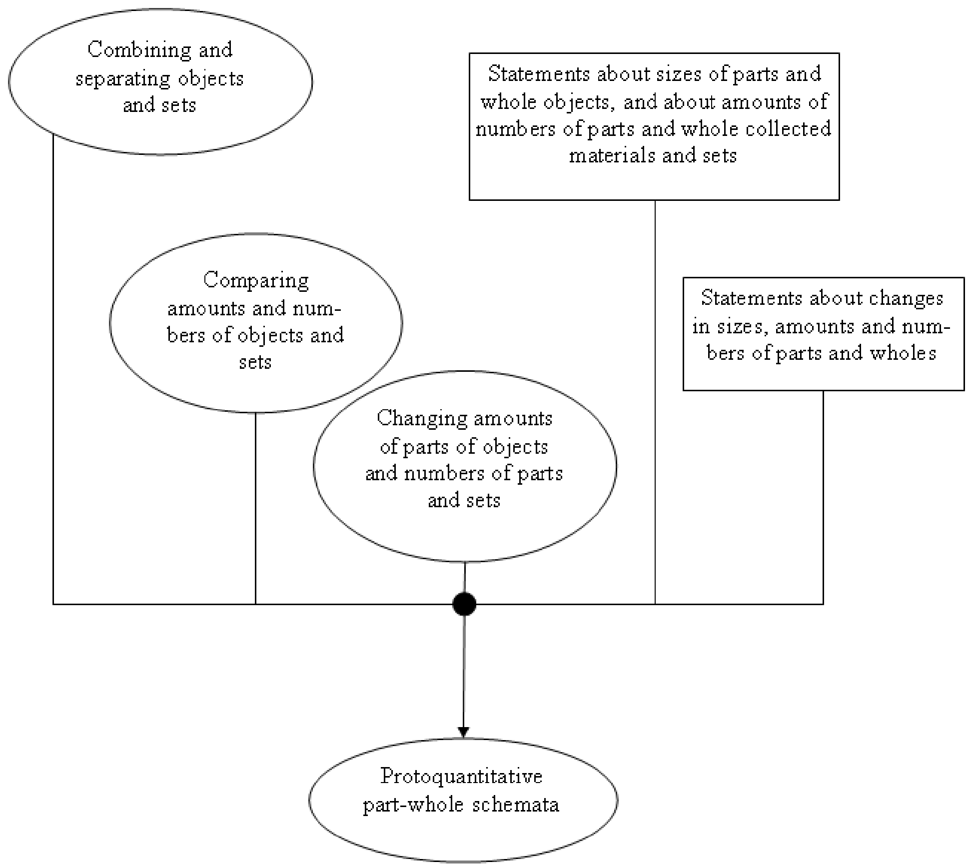

Figure 1.3: Development of the protoquantitative part-whole schema (adapted from L. B. Resnick et al., 1992)

The proposal that children's initial conceptions of additive composition will develop from their experiences with objects also reflects Nunes and Bryant's (1996) suggestion that adding and subtracting with concrete referents can impart knowledge about the decomposition of numbers that is crucial for the development of numeracy concepts and acquisition of later mathematical skills.

- *Additive composition from procedural experience*

Rather than developing from any protoquantitative understanding, an alternative view is that children's understanding of how numbers can be broken down is developed simply through procedural experience. According to this view, it is through repeated calculations that children come to realise certain patterns that reflect the relationship between quantities. This view consequently reflects a different order between conceptual and procedural understanding which Bisanz, Sherman, Rasmussen, & Ho (2005) describe as "*application before evaluation"*. Accordingly, it is argued that children are able to activate a procedure without having any former knowledge of the underlying principles or concepts that make the procedure valid. For example, a child may realise that the best way to calculate 2 + 8 is to begin by counting on from the larger amount without realising the quantitative relationship – that parts can be combined in any order without change to the total (i.e. commutativity - see Baroody, Wilkins, & Tiilikainen, 2003).

Applying this line of argument to additive composition, children may develop their understanding through repeated experiences of adding different amounts and 'seeing' how this often results in the same total. This approach seems to be suggested in Fuson's developmental model of number previously described in section 1.2.2.

According to this model, the concept of additive composition seems to reflect the Bidirectional stage where "*any given small number can be broken down into all of its possible addend pairs*". The Bidirectional stage follows the Numerable chain stage where Fuson describes children's ability to use procedures such as count-on, tending thereby to suggest that children's understanding of the decomposition of number succeeds rather than underlies their developing notion of additive composition.

- *Additive composition - Iterative relationship between concept and procedure*

It appears therefore that there exist two distinct accounts of how children develop their initial understanding of additive composition. On closer inspection however, the task of distinguishing the two accounts becomes less simple. This is because one interpretation - the 'procedures first' view (e.g., Baroody, Lai, & Mix, 2006) - describes how children's experiences with small collections may support their understanding of the additive composition of quantities by allowing them to quickly enumerate the parts and whole through subitising. This argument is difficult to test, because the process of subitising is perceptual and therefore provides little means of identifying whether children are in fact developing their understanding from this form of quantification procedure rather than from a non-quantitative schema.

Baroody (2004b) describes and illustrates a hypothetical trajectory of some key number and arithmetical skills in which children's understanding of part-whole relationships (including composition and decomposition) is developed from their existing cardinal concepts of number and verbal number recognition. Significantly, Baroody highlights how the relationship between children's part-whole understanding and other developing concepts of number (ordinal concepts, and addition and subtraction concepts) is iterative. In other words, children develop an initial understanding of additive composition from quantifying small amounts, and this understanding then develops from mathematical experiences with larger numbers.

1.2.4.3 Summary

A possible way to describe children's developing understanding of additive composition is as a balance between the two viewpoints previously described. On the one hand, children may first develop an understanding of how amounts can be decomposed and recomposed on a perceptual level. Experience with small collections may provide a foundation to interpret this concept quantitatively and may even help children to interpret certain mathematical situations. On the other hand, it is through experience with numerical calculations that children are able to integrate this schema with a more developed understanding of how numbers can be broken down and recomposed.

In other words, the development of additive composition seems to reflect an iterative relationship between children's informal understanding of how collections can be broken down, and their more formal mathematical experiences from part-whole problems.

1.2.4.4 Assessing the emergence of additive composition

The difficulties of assessing children's initial understanding of additive composition reflect the difficulties of defining what constitutes such an understanding. Two key studies by Sophian & McCorgray (1994), and Irwin (1996), have examined the age at which children first demonstrate an understanding of part-whole relations.

Sophian & McCorgray (1994) conducted two studies on the development of 4 to 6 year olds' understanding of part-whole relations. The first study looked at the children's ability to appreciate the structure of arithmetic problems – where their answers to addition and subtraction problems showed their awareness that P1 + P2 = W (e.g., that P1 could not be greater than W). Problems were presented using objects that were then covered to prevent counting strategies. It was shown that 5 and 6 year olds showed a sensitivity to the part-whole structure, whereas 4 year olds tended not to. The second study examined young children's appreciation of part-whole relations further, using a class inclusion task. As in the first study, it was the older children (5-6 years), and not the 4-year-olds, who were able to perform reliably in this problem. In this way, although the first study showed that only the oldest children were successful in quantifying parts (rather than just providing an appropriate estimate) both studies demonstrated how children as young as 5 had developed a basic appreciation of part-whole relations.

Irwin (1996) also examined the development of children's quantitative part-whole knowledge. In her study, children aged 4 to 7 years were given a range of problems requiring them to predict the effect of changes to one of more parts of uncounted quantities, counted quantities and numerical equations. The results were used to show how children as young as 4 were able to predict the effect of changes to one or more parts of an uncounted whole but were less competent in predicting changes to counted quantities. This age is lower than that found by Sophian, although Irwin's study does contain several methodological problems identified by Baroody (2004b). Baroody highlighted how children may not have needed a part-whole schema to solve some problems, relying instead on counting strategies. In other words, because the amounts given were small (e.g., 4) children may simply have calculated the resultant changes to parts and not needed to apply any part-whole schema.

Irwin's study also showed how it was only the older, 7 year old, children who were able to apply the part-whole schema to a purely numerical context: of derived equations. Children were asked to identify three doubles that they knew (e.g., 2 + 2 = 4) and were then asked to calculate the answer to a related equation by adding or taking away from one of the parts. This calculation is clearly more cognitively demanding, and it is perhaps not surprising that none of the 4 or 5 year olds, and only 25% of the 6 year olds, were able to do it successfully. Unfortunately, the paper does not make clear what the assessment criteria were, nor how one could be certain that children were using part-whole relations to solve the problems rather than more simple addition strategies.

By showing that children are sensitive to part-whole relations earlier than the age at which they can apply this knowledge to symbolic problems, both Irwin and Sophian's studies seem to support Resnick's descriptions of a protoquantitative concept – emerging around 4 to 5 years - although performance depends greatly on how the problem is presented. However, it is not until children are at around school age, 5-6 years, that they develop the ability to apply this part-whole understanding within a numerical context, and perhaps another year older before they develop strategies in which to apply this part-whole schema to calculate numerical changes.

Sophian and Irwin's research focused on the bridge between children's pre-quantitative and quantitative understanding of additive composition. When children are in the early years of school, they face increasingly difficult numerical problems to solve. This section looks at the development of additive composition by looking at different tasks where this concept may play a key role.

1.2.4.5 Assessing a developing understanding of additive composition

- *Conservation tasks*

Several tests of children's part-whole understanding seem to reflect Piaget's (1965) conservation task. For example, Fischer (1990) used the following test to assess children's part-whole knowledge. Children were presented with two sets of cubes that shared the same total but were composed of different parts of two types of coloured sets. They were required to identify that these quantities were the same. Then, in a similar task, they were asked if a total had changed when objects were turned over to reveal a different colour. Unfortunately, these part-whole tests were administered with other number concepts tests, and data on performance for this task were not provided.

Saxton & Cakir (2006) also examined children's part-whole knowledge by devising a 'partitioning' task which was used to predict base ten knowledge (as well as with tests of grouping and counting-on). Children were aged between 78 and 86 months and partitioning was assessed using two tasks. In the first, children had to enumerate a collection of cubes; the collection was then divided into two groups and children were asked again to say how many were in the whole collection. In the second 'mirror' task, children first counted the total of two groups of cubes, and were then asked the total again when the two groups were combined. Knowledge of partitioning was attributed to children who did not hesitate or attempt to recount objects.

It was found that performance on Saxton and Cakir's two partitioning tasks was strongly correlated ($r=0.84$, $p<0.001$) and 54.6% children met the criterion for possessing knowledge of partitioning (3 out 4 correct). As the average age of children was nearly 7 years, this seems quite a low score considering that children were required only to know that a partitioning of the whole did not change its quantity. It is possible however that both Saxton & Cakir's and Fischer's tasks are susceptible to the same criticisms made of

Piaget's conversation task, namely linguistic demands (McGarrigle & Donaldson, 1974) and double questioning (children may think they should change their answer when asked the same question by an adult) (Samuel & Bryant, 1984).

In these conservations tasks described, children are being asked to recognise that perceptual changes to parts do not change the whole. However, aside from the double question issue, it is not clear how much this approach can really assess children's understanding of how quantitative changes to parts affect the whole. Indeed, although these problems have been used to assess children's part-whole or partitioning understanding, only three tasks have been explicitly related to additive composition (Cowan, 2003): *missing part questions*, the *shop task*, and *decomposition* problems.

- *Missing part questions*

According to Resnick (1983b), children's ability to interpret certain word problems in terms of a part-whole schema is good evidence that they have informally understood additive composition. One type of problem that requires such understanding is an 'unknown start' problem. In this question, children have to identify an initial part when told the other part and the resultant total. For example, *"Paul had some marbles; Charles gave him five more. He now he has eight marbles. How many did he have to start with?"* It is argued that children need a part-whole schema in order to select a suitable strategy for enumerating this initial part. The difficulty children have with this type of question was demonstrated by Riley and Greeno (1988), who showed that it was not until children were around U.S Grade 2 level (7 years old) that they were confident with this type of problem.

- *Shop task*

The shop task was developed by Nunes (reported in Nunes & Bryant, 1996) to examine children's understanding of the composition of the decade structure. In this task, children are asked to give the examiner a specific amount of money which can only be achieved by using single unit coins and a higher value coin (e.g., give 7p when the child has 5 1p coins and a 5p coin).

Nunes (in Nunes & Bryant, 1996. p.53) described a study examining the relationship between children's ability to solve the shop task and their use of addition strategies. It was found that those children who solved the shop task tended to be able to count-on in simple addition problems. This led Nunes to postulate that children's understanding of the numeration system (namely the ability to count-on from the cardinal value of one addend) is necessary, but not by itself sufficient, for understanding additive composition. This argument seems to reflect Fuson's developmental levels where the Breakable chain level (children can break a number, leading to strategies such as counting-on) precedes the Bidirectional level (children understand how numbers are embedded within others). However, it could be argued that the shop task is still a relatively simple test of additive composition because children are simply required to identify how a whole can be partitioned rather than reason about how a whole can be partitioned in different ways.

- *Decomposition task*

In describing the Bidirectional chain level, Fuson refers to how, when given the addition problem 7 + 6, children might decompose 7 into 6 and 1, and then use their knowledge of doubles to conclude that the answer is one more than 12. In order to use this strategy, children need to decompose and recompose numbers, and it has therefore been argued that the use of this decomposition strategy reflects knowledge of additive composition (Canobi, Reeve, & Pattison, 2003; Cowan, 2003; Steinberg, 1985).

Children use a wealth of strategies for additive problems, such as guessing, counting-all, counting-on, retrieval and decomposition. The decomposition strategy has been identified as cognitively efficient and numerically more developed (Baroody et al., 2006; Canobi, Reeve, & Pattison, 1998), but it is much less frequently used and generally applied only by older children. For example, Siegler (1987) examined the use of different strategies for addition problems and found that decomposition was used on only 11% of problems by US Grade 2 children (7 years) compared to only 2% problems by Kindergarteners (5 years).

It seems from this that decomposition strategies are only used by children at an age when they have already developed an understanding of additive composition. Nevertheless, it has been argued that some children may appreciate the concept of decomposition but lack the procedural skills to apply it spontaneously (Putnam et al., 1990). To address this, Putnam looked at children's ability to identify decomposition as a valid strategy when used by a puppet (children are more comfortable identifying that a puppet, rather than an adult, is wrong or confused). He found that out of 22 normally achieving third graders (8-9 years), about half were able to give adequate explanations of decomposition strategies.

Providing verbal explanations for whether a strategy is valid or not is arguably quite difficult. Canobi at al (2003) therefore looked at children's ability to identify decomposition strategies using a different set up. In their study, children were asked whether a puppet was able to identify the solution to an addition question without counting, by using the answer to the previous addition question. Their study focused on children's (5 – 8 years) ability to identify problems that were decompositions of the previous (e.g., to identify that '4 + 2 + 3' can be solved by referring to the previous question of '6 + 3') and problems that were reordering of the previous ('3 + 6' following '6 + 3'). Problems were also presented to each child in three counterbalanced conditions: with physical counters, with numbers, and with abstract symbols. Children were asked to judge whether the puppet could use the previous problem (which required children to identify that the problem was decomposition/reordering of the previous) and then justify their answer. It was found that children were better able to notice that addends had been reordered than decomposed, and that decomposition of addends was noticed more when presented with objects than with symbols. This shift from being able to reason with concrete materials before more abstract concepts seems to echo Resnick's (1992a) description of a protoquantitative to quantitative development in children's understanding of additive composition.

1.2.4.5 Summary

Children's understanding of additive composition of numbers seems to develop between the approximate ages of 5 to 7 years. Unfortunately, differences in forms of assessment make it difficult to be more precise. Three key tasks have been postulated as engaging children's knowledge of additive composition: use of the decomposition strategy in addition problems, missing start addition problems, and the shop task. As Cowan notes, no study seems to have attempted to compare children's performances on all three of these tasks, although one study, by Martins-Mourao & Cowan (1998) examined children's ability on the shop task and the missing part problem. The study looked at 152 children aged between 4 and 7 years, finding that children were more likely to succeed in the shop task than in the missing part problems. However, these tasks still only require children to apply a part-whole schema to identify single rather than multiple solutions. As Nunes & Bryant (1996) state, it would be interesting to discover the age at which children actually know that a number such as 6 can be partitioned into different combinations such as 4 and 2, or 1 and 5. Several tasks described below have explored this ability.

1.2.4.6 Partitioning tasks

Additive composition refers to an understanding of how a number is made up of smaller numbers. Decomposition (partitioning a number into different parts) is therefore central to this concept, and the term 'partitioning task' will be used in this thesis to refer to tasks requiring children to decompose a number into different combinations – typically of two parts. Three tasks have been identified that might be considered partitioning tasks.

- ***Jones et al (1996) partitioning task***

Jones et al describe their partitioning task in the context of a framework for the assessment and intervention of multidigit number sense. It is argued that partitioning is a key aspect of the multidigit understanding identified by Resnick (1992a), and the authors propose five levels of understanding. The first of these is a pre-place value level of understanding. The activity for this pre-place value level of partitioning requires children to identify all the ways that the numbers 5, 8 and 10 can be decomposed into combinations of two numbers (e.g., 5 into 2 & 3). The problem is presented using a story context and concrete materials:

> *"The man in the yellow hat shook 2 bags. 'I had 10 candies and put some in one bag and the rest in the other", he told George. How many could be in each bag?'"* (p. 316)

Unfortunately, the authors do not provide a detailed account of what prompts were provided to children, or what criteria were used for their assessment of understanding. Nevertheless, the task raises interesting questions concerning its relation to additive composition that will be returned to later. More than this, rather than just an assessment task, Jones et al's partitioning problem is also a learning activity where children can explore multiple solutions to a single question.

- ***Fischer's (1990) part-whole activity***

Fischer describes an activity which asks children to separate a set of five objects into two parts and enumerate the parts. For this activity, lessons were designed to foster comparisons across the children's configurations in order to promote understanding of the various combinations of subsets that compose a whole set. An interesting point to note in this activity was the use of external configurations to help children infer numerical relationships from their solutions. This use of a record of solutions is described more recently by Clements (2009). Clements describes how children can be encouraged to see patterns in different combinations by listing them in order (6 & 0, 5 & 1, 4 & 2 etc.). By listing the solutions in order, it is likely that children can see how one part goes up and the other down in consecutive solutions, yet it is not exactly clear how easily they will understand *why* this is so.

- ***Baroody's (2006) double decomposition game***

Baroody describes a game where children have some cars to move from one side of a board to the other. The board is separated into hexagons and children are required to move the cars according to the number on a card they select at random. In one version called '*double additive decomposition'*, children select a card, partition its number mentally into two parts and then move two cars according to each part:

> *"In double additive decomposition, a child draws a number card such as 5 and can decompose it into parts any way she or he wishes (e.g., moving one car five spaces and the other none or moving one car three spaces and the other two"* (p.28)

Although children have a concrete referent in which they can act out the decomposition, children cannot simply partition the objects to identify a solution. Instead, they must identify a solution mentally and then use the cars to externalise that solution.

1.2.4.7 Summary

This final section has briefly reviewed three tasks that share a common goal – to encourage children to identify different ways a number can be partitioned. However, one key difference between the tasks is the form of representation used to support children. With Jones' and Fisher's task, children are given concrete materials to identify solutions. With Baroody's task, they are not. In Fisher's task (and the reference to Clements), children are encouraged to compare records of solutions. It might be argued that differences between these tasks reflect differences in arguments for the development of additive composition. If additive composition develops from children's protoquantitative understanding, it might beneficial for children to use objects to help map this knowledge to a quantitative understanding. Alternatively, if children develop additive composition by noticing patterns from repeated numerical calculations, it may be beneficial to provide them with a means to record and compare their solutions. In reality this distinction may not be so clear – physical objects might be used to facilitate calculations; numerical records might be used to encourage children to map their physical understanding to symbols. Nevertheless, the tasks described, along with the arguments put forward for the development of additive composition, do seem to indicate different roles for the types of learning materials that may be most supportive.

1.2.5 Numerical development - Summary

This section has reviewed children's numerical development in their early years of schooling and identified a concept – *additive composition* - that plays a key role in their numerical understanding. Additive composition refers to an understanding of the way numbers are composed, and has consequently been related to various numerical abilities from calculation strategies to multidigit understanding. The concept seems to develop during children's first few school years, although differences in assessment tasks make it difficult to be more exact. What is also not clear is the extent to which children's numerical understanding builds upon their understanding of how physical collections can be composed and decomposed in different ways, or whether children need to discover these relations through calculation experience. These two possibilities seem to have important implications for the type of materials children are given to learn about how numbers can be broken down.

Whether physical materials support children in developing numerical concepts such as additive composition is a key question, and one that has attracted much research. The next section aims to review the literature concerning the role of physical materials in learning mathematics in order to help identify when, or if, such mathematical tools can support the development of additive composition.

1.3 Physical Learning Materials

The previous section described activities where children explored the different ways in which numbers could be broken down. Certain activities suggested a role for physical objects in helping children identify different combinations. In contrast, other activities did not involve physical objects, thereby making it unclear whether physical objects are the most effective representation for learning, or whether they are even necessary. This section intends to review the literature on the role of physical materials in supporting children's numerical development and then draw together these arguments to evaluate both the advantages and limitations of physical materials for supporting children in tasks such as the partitioning tasks described.

1.3.1 Manipulatives – physical learning materials

Manipulatives are physical materials used in education to support children's learning, particularly in mathematics. Manipulatives can vary in many ways, including their shape, size, colour and the quantity used. They may be more or less 'concrete'. 'Concrete' is defined here as pertaining to everyday objects (for a more detailed discussion of the origin and meaning of the term see Clements (1999)). A more concrete manipulative might be exemplified by a lollypop stick or toy cookie, whilst a less concrete object might be a simple plastic cube – one that is most familiar within an educational context.

Manipulatives are intended to present mathematical ideas, in this case, as representations of number (s). The term *representation* will be used in this thesis to refer to both internal and external manifestations of number, although, as emphasised by Cobb, Yackel, & Wood (1992), it is acknowledged that the representational meaning of mathematical tools such as manipulatives is socially constructed rather than being a property inherent in materials. In other words, the representational meaning of simple physical objects is generated through the particular social context - between the teacher and students for example.

Lesh, Post, & Behr (1987) describe five forms of external representation: manipulative models, static pictures, written symbols, spoken language and real scripts. According to Lesh et al, a key goal is to support children's ability to reason between these modes of representation. Pape & Tchoshanov (2001) describe children's learning in terms of an interplay between internal and external representations within a social context. As illustrated in their diagram (Figure 1.4), they list five forms of external representation: written number words, written numerals, pictorial materials, verbal number words and manipulatives models.

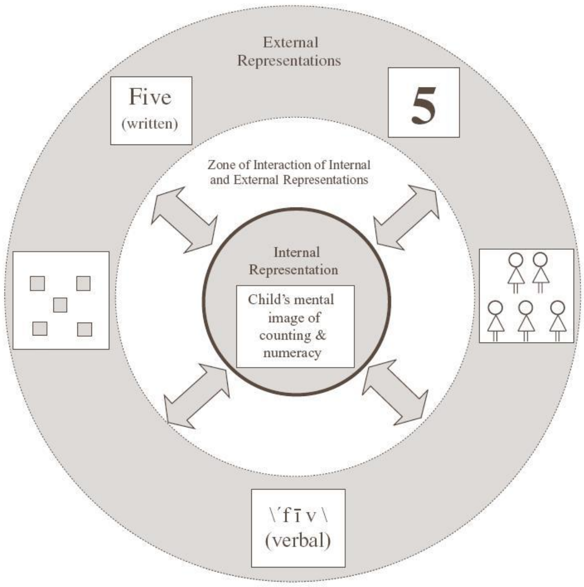

Figure 1.4: The relationship between internal and external representations in developing children's understanding of the concept of numeracy (Pape & Tchoshanov, 2001)

Comparing the modes of external representation proposed by Lesh et al and by Pape & Tchoshanov, it is possible to identify one key distinction: representations that have a one-to-one correspondence with the number of items they represent, and those that use a symbolic reference to the cardinal value of the set. For example, the written numeral "5", or the spoken word "*five*" are cardinal terms, whilst distinct images or physical cubes have one-to-one correspondence with the quantity of items in the set. This distinction is certainly problematic – certain manipulatives may have different symbolic values (e.g., Dienes' cubes) – but it does suggest one apparent appeal of physical objects in that they seem to provide a means of communicating certain quantitative relations, such as how amounts can be added together or partitioned in different ways. However, it is not clear how easily children are able to interpret such numerical relationships from these materials. It is also not clear why physically manipulating a representation of quantity would be advantageous. In order to address these questions, it is important to examine the history of manipulatives and the different arguments surrounding their use.

1.3.2 History of manipulatives

1.3.2.1 Frobel and Montessori

Two of the first key proponents and designers of manipulatives, Montessori (1912) and Froebel (1826), both advocated the importance of playful discovery in learning. Interestingly, Froebel (cited in Theissen, 2005) actually proposed a specific system for the use of materials in learning mathematics. This system built upon the materials he presented: 'Froebel's gifts', of which the third gift is most relevant to the level of mathematics discussed here. Froebel's third gift consisted of a set of eight one inch wooden cubes, presented together to form a 2 inch cube. Froebel (cited in Theissen, 2005, p.16)

> *"The principle cube appears separated by the mentioned division in this play into eight equal cubes. The child thus distinguishes here as a given fact and without any words (purely as the perception of an object), a whole and a part, for each component cube is part of the principle cube"*
>
> Froebel (cited in Theissen, 2005, p.16)

As well as using accompanying rhymes to foster mathematical language, Froebel described types of activities with the gifts, which he separated into three stages. The first consisted of constructing real life structures with the materials (e.g., a building or chair). In the second stage, children were encouraged to create systematic arrangements, or patterns. In the third stage, activity was intended to be more formally mathematical (i.e. applying to number problems).

Although written more than a century ago, it might be argued that some of the most prevalent manipulatives used in early learning, such as rods and cubes, are not too dissimilar to those proposed by Froebel. What we do have, however, is a stronger theoretical framework for the role of such objects in helping children to construct their understanding of the world.

1.3.2.2 Piaget and Bruner

The importance of a child actively exploring the environment and 'discovering' new ideas received theoretical support in the work of Piaget in the 1960's. Although Piaget's (1965) work was more a theory of the development of knowledge than a theory of instruction, it did highlight a role for concrete materials in helping younger children develop and articulate their understanding of the world, whilst also indicating that developmental progress was reflected in gradual independence from these materials. This view of a concrete to abstract progression of knowledge, and the possible implications for instruction, received more support from the work of Bruner (1966). According to Bruner, the instructor should try to encourage students to discover principles by themselves. The task of the instructor is to translate information to be learned into a format appropriate to the learner's current state of understanding. Bruner described children's understanding in terms of three levels of representation: *enactive*, *iconic* and *symbolic*. Although Bruner did not specifically relate these modes of representation to the stages of development proposed by Piaget, he did suggest a sequential graduation through these representational forms – with children progressing from working with hands-on physical materials, to reasoning with iconic and ultimately symbolic representations.

1.3.2.3 Papert

Papert, who worked under Piaget at the University of Geneva, also proposed an educational theory greatly influenced by Piaget's constructivism. Papert argued that the most effective way in which children are able to develop their internal models of the outside world is to externalise these models: through construction, hence the term *Constructionism* given to Papert's theoretical approach.

Constructionism has clear parallels to Constructivism in its developmental, child-centred emphasis that views children as the builders of their own cognitive world. One area of greatest difference, however, concerns their approaches to the development of intelligence. Whilst Piaget's emphasis was on the construction of internal stability, Papert was more interested in the dynamics of change (Ackermann, 2001) which had implications for the role of the teacher. Papert viewed Constructionism in direct contrast to *Instructionism* (Papert & Harel, 1991) and was highly critical of current approaches to the teaching of subjects such as mathematics that required children to absorb numerous abstract rules – arguing that this approach led to a negative attitude to the subject, which in his book, 'Mindstorms' (Papert, 1980), he referred to as '*Mathaphobia'*. Instead, Papert has argued strongly for the need for children to be able to freely engage in constructing and sharing public entities, be they physical models or articulated theories.

Papert's emphasis on the power of learning tools to allow children to externalise their thinking led him to his, arguably visionary, belief in the future role of the computer in helping children learn. Papert's work was epitomised by the development of *Logo* – a simple computer language in which children must externalise rules to guide an on-screen object in order to construct geometric shapes. The success of Logo is evidenced by its prevalence today in educational settings in many countries; however, it is not without its critics. Whilst Papert described Logo as a means for children to develop various cognitive skills from problem solving to planning and reasoning, there is mixed evidence for whether the programming skills developed do generalise to such higher order thinking (Yelland, 1995). Pea (1983), for example, reports on three studies examining the learning benefits of Logo. In the first, particular difficulties children had with Logo are discussed (e.g., debugging, using variables). In the second, the depth of children's understanding is questioned, highlighting instances where children's programs often "*displayed production without comprehension*". In the third study, it was shown that children using Logo over a school year did not outperform their non-programming peers in measures of planning ability.

As well as raising doubts over some of the claims surrounding Logo, Pea draws attention to the discovery-learning pedagogy advocated by Papert. Arguably, with

Papert's emphasis on the learner, it is not always clear what role is intended for the teacher. This issue is discussed by Sutherland (1993) who emphasises the need for educators to become more explicit in their underlying theories that influence their teaching. Indeed, differences in teacher involvement may help explain differences in findings from Logo research. In trying to resolve discussions over the teacher's role, Sutherland relates the work of Vygotsky (1962) who focuses on the role of social interaction in children's development and Wood, Bruner and Ross (1976) who introduced the term 'scaffolding' to describe the actions of the teacher in reducing some of the cognitive demands of the task in more complex problems.

Although it remains unclear to what extent certain tools such as Logo do foster children's cognitive and social development, Papert's work has remained highly influential, particularly in the development of Microworlds to support mathematics (e.g., Geraniou, Mavrikis, Hoyles, & Noss, 2008; Hoyles, Noss, & Adamson, 2002) Importantly for this thesis, Papert's theoretical arguments are highly applicable to the use of physical objects. By constructing external models using the materials, children are provided with a way to externalise, communicate and reflect upon their understanding. Indeed, Papert's work has been applied to digitally augmented physical objects through the creation of floor robots and, more recently, programmable *Lego* (called *Mindstorms* in honour of his book).

1.3.2.4 Dienes

The work of Piaget, Bruner and Papert has clearly shaped theoretical and instructional approaches to learning, and the physical embodiments of Papert's Logo highlight their relationship to current developments in tangible designs. Nevertheless, with a focus on the use of physical materials to explore numerical relationships, it is also important to consider the pioneering work of the mathematical theorist and practitioner: Zoltan Dienes.

Arguably, one of the greatest legacies of Dienes' (1964) work relates to the structured materials he developed for supporting children's concepts of place value: the base ten version of his Multibase Arithmetic Blocks (hence referred to as Dienes' blocks). The appeal of these materials is demonstrated through their educational use and focus in research literature, although empirical evidence for their effectiveness is less clear (Fuson & Briars, 1990; L. B. Resnick & Omanson, 1987; P. Thompson, 1995). Dienes focused his work on mathematics, as he believed this differed from other domains in the nature of the structural relationships between concepts (such as the relationship between groups in different numerical bases). He argued that learning mathematics consisted of apprehending such relationships and applying the resulting concepts to real world situations. Dienes drew upon Piaget's assertion that learning is an active process and proposed three stages of learning instruction. These stages were described according to the type of 'games' (engaging activities) that might be played and reflected Piaget's stages of knowledge development. Within these games, Dienes proposed that children should be presented with materials that varied perceptually but were all consistent in their structural correspondence to the concept to be learnt. The first stage, 'Preliminary games', describes a form of undirected activity where children start actively exploring materials and making observations about their properties without necessarily understanding their significance. The second stage, 'Structured games', describes how activities become more directed and purposeful (such as addition or subtraction activities with materials). During this stage, Dienes describes a move toward more abstract reasoning by using a variety of materials that vary perceptually, but are used in identical tasks. In the final stage, 'Practice games', children begin to record activities using symbolic notation, which Dienes believed was only a small step away from working without materials. It is interesting to note the similarities of Dienes activities with those proposed by Frobel some 100 years earlier.

1.3.2.5 Summary

The work of Piaget, Bruner and Papert, amongst others, has helped provide a theoretical foundation for children's cognitive development and identified a possible role for physical objects in exploring and articulating ideas when children lack the ability to do so more abstractly. Dienes' work is of great significance in that it describes not only types of activity but the types of material that may be used to support certain mathematical concepts. These types of material (e.g., unit cubes, Dienes' blocks) are still in common use today, thereby raising the question: what evidence do we have that they work?

1.3.3 Empirical evidence for the effectiveness of manipulatives

Several investigators have examined the use of Dienes' base ten blocks in supporting children's understanding of multidigits and multidigit calculations. Fuson & Briars (1990), for example, reported success in the use of the blocks for improving children's multidigit calculations although their study did not include a control group and many of the younger children needed frequent reminders that they should draw upon their experiences with the blocks to help solve the multidigit problems. Resnick and Omanson (1987) also examined the use of base blocks to support children's multidigit algorithm procedures. They found that children's capabilities for doing arithmetic with the blocks were not reflected in a comparable ability for following written arithmetic procedures.

Other studies have also highlighted children's difficulties in mapping between base ten structures and symbolic notation (I. Thompson, 2000). Children, especially younger children, seem to have difficulty in mapping procedures using manipulatives with symbolic procedures; although this may reflect the more specific difficulties of mapping to base ten and place value notation, both of which are cultural conventions. Indeed, Baroody (1990) suggests that it is necessary to introduce a specific sequence of different manipulative and recording procedures (see Figure 1.5) that can graduate children's understanding. As Figure 1.5 illustrates, Baroody describes how this graduation might be achieved through both concrete and pictorial materials.

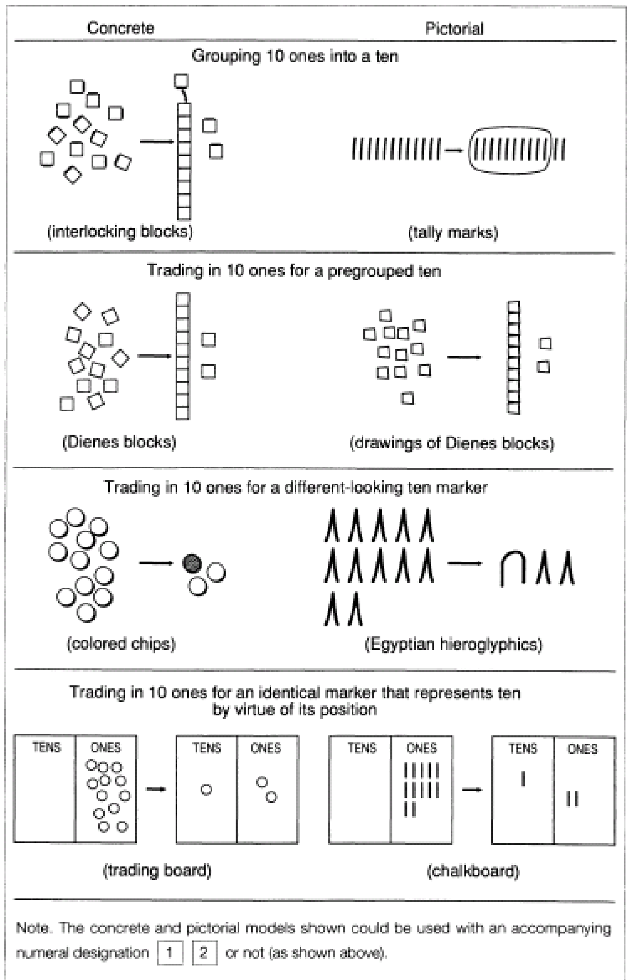

Figure 1.5: Graduation in representations to support place value understanding (Baroody, 1990)

Despite many other studies looking at the potential for manipulatives to support learning, there still remains no clear consensus about their effectiveness. Some studies have reported benefits (Canobi, 2005; Martin & Schwartz, 2005; Suydam & Higgins, 1977), whilst others have reported no differences, or even negative effects, from the use of physical materials (Ball, 1992; Fennema, 1972; L. B. Resnick & Omanson, 1987; Uttal, Scudder, & DeLoache, 1997). Sowell (1989) combined the findings of 60 studies in a meta-analysis to evaluate the effectiveness of manipulative instruction (compared to pictorial or abstract) and found a positive effect of long term manipulative use. It is noted however that this form of instruction tended to be favoured by teachers who had had specific teaching instruction, and that this may have led to a more general positive effect.

1.3.3.1 Summary

Although there have been many class-based studies examining whether manipulatives 'work', conflicting findings leave this question unresolved. The discrepancies between findings highlight a key difficulty in trying to establish the effectiveness of manipulatives, namely: materials are used to teach a wide range of concepts, to children who differ significantly in their abilities, by teachers using a variety of approaches. Manipulatives may be implemented in many different ways based on a range of factors: these include how structured the activity is (e.g., the different levels described by Dienes), how much time the children have to use the materials, how much effort is put into mapping the materials to abstract reasoning and differences in the types of manipulatives available

(Mix, in press). In short, whether manipulatives 'work' or not in each study may be influenced by any of the multiple factors determining the learning context, rather than by the properties of the physical materials per se. Manipulatives in no way guarantee success (Baroody, 1989) and rather than asking 'if' manipulatives work, the question should perhaps focus on *when* manipulatives work, or do not, and importantly, *why*. Understanding the mechanisms through which physical materials may support children's learning can help to evaluate the potential of these materials, and the circumstances under which this potential can be realised.

1.3.4 Mechanisms

There have been many reasons put forward on how manipulatives might support learning. Halford & Boulton-Lewis (1992), for example, list seven: as a memory aide, verifying truth, increasing flexibility, facilitating retrieval, mediating transfer, indirectly facilitating abstraction, and generating predictions of unknown facts. Reasons such as these are quite high level however, and do not explain why manipulatives may confer these benefits. This section therefore attempts to describe some of the possible mechanisms in which manipulatives might help children develop certain numerical concepts such as additive composition. These will be organised around four themes proposed by Mix (in press) for the possible benefits of concrete materials: *conceptual metaphor*, *offloading intelligence*, *focusing attention* and *generating actions*.

1.3.4.1 Conceptual metaphors

Manipulatives might provide children with perceptual experiences that they can draw upon when trying to reason abstractly. For example, Hughes (1986) describes how young children were not able to solve an addition problem when it was presented symbolically, but could when the same question was presented with concrete referents. Significantly, they also understood the problem when it referred just to hypothetical objects, thereby suggesting that concrete experience can provide a reference point with which to reason in the absence of materials. Various other studies have shown how young children are able to reason about certain concepts such as *commutativity* (Canobi, Reeve, & Pattison, 2002); *inverse relations* (Canobi, 2005); *equivalence* (Sherman & Bisanz, 2009) and *additive composition* (Sophian & McCorgray, 1994), and are able to do so with physical objects before they can with symbols. It is possible therefore that children are able to draw upon this experience in order to rationalise about these concepts at a later stage without these materials. In other words, manipulatives might provide a conceptual metaphor for subsequent symbolic reasoning.

It can be questioned why children need manipulatives when they have experience with everyday objects. One reason might be that manipulatives provide an external prompt that encourages them to draw upon this previous experience when using more formal, symbolic mathematical language in the classroom. Indeed, McNeil & Jarvin (2007) propose that one of the key advantages of manipulatives is they allow children to draw upon real life experiences. Moreover, in contrast to 'real life' objects, manipulatives can be designed to emphasise the features that are most relevant for the mathematical concepts being discussed in the classroom – in the way that Dienes designed the Multiple Arithmetic Cubes (MAB) to reflect the structure of the base system.

- *Embodied Cognition*

By describing how children can draw upon their concrete experience to reason abstractly, the above arguments imply a separation between concrete and symbolic reasoning, where the first is dependent on physical experiences, and the latter more abstract and hence independent of any perceptual experiences. However, this separation between concrete and abstract reasoning has been criticised by proponents of Embodied Cognition (e.g., Lackoff & Núñez, 2000). Embodied Cognition is a theoretical viewpoint that argues that adults' thinking should not be considered as abstract but rather as grounded in prior perceptual experiences, and that the tight coupling between experiences and cognition should not be separated. This is saying that children's concrete experiences do not simply serve as a reference for more abstract thinking, but rather become embodied in higher order thinking. This view is not without its critics (e.g., Clark, 1999; Mahon & Caramazza, 2008), but the growing evidence of the role of visual-spatial imagery (Hatano, Shimizu, & Amaiwa, 1987; Hegarty & Kozhevnikov, 1999), and motor activation (Wilson, 2001) when individuals are solving problems 'abstractly', does support these arguments. It has been shown, with particular relevance to manipulatives, that abacus masters have expansive digit spans (retention of numerical digit strings) and arithmetic abilities because they have internalised a mental model that can simulate the structure of an abacus (Hatano & Osawa, 1983). However, although this research shows the potential to internalise a particular external structure to support recall and calculation, it remains unclear whether experiences with manipulatives can be internalised to support the formation of numerical concepts.

One explanation for how perceptual experiences can become embodied in concepts has been presented by Lackoff and Nunez (2000). The authors describe a process of *conflation*: the simultaneous activation of distinct areas of the brain that are concerned with different aspects of experience, resulting in relevant neural links. One example provided is children's concept of numbers – if children walk up some stairs whilst simultaneously counting them, the conflation of these experiences could develop the concept of numbers as points on a line. From this, it might be argued that children's concepts of decomposition (how numbers are composed) may be developed by the simultaneous experience of partitioning objects whilst verbally identifying numerical decomposition solutions. Unfortunately, there is limited neurological evidence at present that supports the process of conflation in concept formation.

- *Analogical Reasoning*

Rather than actually becoming embodied in developing concepts, it is possible to consider how actions with manipulatives may provide a metaphor for children to reason about concepts symbolically. As Gentner (1983, p.162) states: "*Many (perhaps most) metaphors are predominantly relational comparisons, and are thus essentially analogies*."

According to Gentner, the ability to use the source domain as an analogy for the target domain is determined mainly by the structural relationships between the two. In the case of decomposition, the structural mapping between the source domain (concrete materials) and the target domain (numerical symbols) is strong (Halford & Boulton-Lewis, 1992) as it is possible to map the way numbers can be decomposed to the way physical materials can be partitioned in different ways. Children's familiarity with the structural relationships between physical objects thereby provides a base from which to reason analogically about the relationships between numerical symbols. If a collection of objects can be partitioned into two groups, so might the number 6. However, this mapping requires children to appreciate the structural relationship in both the base and source domains. Children might know that objects can be partitioned in different ways, but they may not focus on the quantitative aspect of partitioning – focusing instead on other properties such as how overall length may change with different configurations (as highlighted in Piaget's class inclusion tasks). With Dienes' blocks, children may not focus on the intended properties of the materials – namely that a collection of ten objects can be grouped to together to create one collection (Baroody, 1990; Varelas & Becker, 1997). Halford & Boulton-Lewis (1992) described such problems as processing loads (the cognitive demands of having to process objects and their intended representational meanings simultaneously) required by certain types of concrete material.

- *Difficulties with linking representations*

The difficulties that children may have in processing the relevant features of both concrete materials and numerical symbols is highlighted in the work of Uttal et al (1997). Uttal et al conducted research focused on young children's ability to use a scale model of a room to orientate themselves in order to search for hidden objects. It was found that children had significant difficulties in interpreting the scale model as a symbolic representation as well as a play object itself. Extending these arguments to the use of manipulatives, it was argued that children may still have the same problem of 'dual representation': processing the manipulatives as symbolic representations as well as objects of interest in themselves. The authors do not go as far as to denounce the use of manipulatives but suggest that care should be taken to help children process them as symbols - through explicit instruction and the use of simple materials with fewer extraneous features that have not been used in non-mathematical contexts.

Even if children do address the relevant properties of the materials, they may have difficulty in mapping their structural relationship to symbols because they have yet to develop this understanding. This paradox is summarised by the experiences of Holt (1982, p.138-139).

> *"Bill [a colleague] and I were excited about [Cuisenaire] rods because we could see strong connections between the world of rods and the world of numbers. We therefore assumed that*

children, looking at the rods and doing things with them, could see how the world of numbers and numerical operations worked. The trouble with this theory was that Bill and I already knew how the world of numbers worked. We could say, "Oh, the rods behave just the way the numbers do." But if we had not known how number behaved, would looking at the rods have helped us to find out?"

The difficulty that children may have in interpreting certain physical representations highlights a key criticism of manipulatives – namely that the value of manipulatives cannot be considered in isolation from the context in which they are used (Cobb et al., 1992). As Ball (1992) states: "*understanding does not travel through the fingertips and up the arm*". In other words, it is problematic to think that mathematical meaning is transparent within manipulatives (Moyer, 2001); it is more the activity with the manipulatives, and the context of this activity, through which transparency emerges (Meira, 1998). Reflecting a more socio-cultural perspective therefore, it is important to consider the role of manipulatives as a mediating tool within a particular context (Vygotsky, 1978). As Dienes (1964, p.55) himself states: "*one cannot over-emphasise that it is not the material itself which creates the true mathematical learning-situation*".

1.3.4.2 Manipulatives focus attention

There are many factors in the learning context, some of which may potentially distract children from focusing on the intended learning activity. Manipulatives might therefore help by focusing attention on numerical ideas. For example, if a teacher is discussing an addition problem with a child using a story context, the manipulative may help focus the child's attention on the key information being given – the amounts referred to in the story, rather than other aspects. Indeed, the role of representations in helping constrain inferences has been discussed by Scaife & Roger (1996).

The previous example highlights the way in which the materials may support joint attention – important in the social construction of knowledge (Tomasello, 1995). In this context, this could help both teacher and child attend to the same representational properties while communicating mathematically. However, the reservations raised by Uttal et al are relevant – namely, how clear is it that children's attention is focused upon the numerical properties of manipulatives? In this regard, it may be possible to focus children's attention on numerical ideas by applying Uttal's recommendations: use materials that minimize extraneous features (i.e. 'non-relevant' properties such as colour or shape). This proposition is supported by other research indicating that students are more likely to be successful in extracting information when the materials used are more 'abstract' and less concrete (Kaminski, Sloutsky, & Heckler, 2006; Sloutsky, Kaminski, & Heckler, 2005a). However, it should be noted that these studies were carried out with undergraduate students who might have been expected to have been more able to reason with more abstract representations.

An example of a more basic manipulative design is the MAB presented by Dienes. These blocks differed only in shape and size in order to represent the decade structure of numbers. It is worth noting that Dienes actually advocated using a variety of materials to support numerical concepts. These materials included a range of extraneous features such as different colours/shapes, but maintained the same structural relationships (e.g., groupings of ten).

The effect of different types of physical materials on children's number concepts was examined by Chao, Stigler, & Woodward (2000), who designed a series of nine numerical games for a total of 157 kindergarten aged children in three schools over a five week maths program. Classes were allocated to one of two groups: a structured material condition (using the same generic materials), and a variety material condition (using a variety of materials sharing a similar structure). It was found that the two kinds of materials had varying effects on learning in certain tasks (recognising numerical patterns) but no effect in others (e.g., numerical inferences, number sequencing, base ten). Further research is clearly needed to examine the effects of different types of manipulatives although it is again important to consider the activity in which the materials are used.

1.3.4.3 Manipulatives generate actions

Using manipulatives generates physical actions, typically using both hands. These actions tend to be more expansive and expressive than those generated from interacting with other representations such as paper or computer based materials. Indeed, the actions generated through the use of manipulatives are often put forward as a key advantage of this representational medium (Gravemeijer, 1991; McNeil & Jarvin, 2007). Unfortunately, despite the popularity of terms such as 'kinaesthetic learning' (e.g., Begel, Garcia, & Wolfman, 2004), there is limited evidence that physical actions support learning, particularly in 'more abstract' subjects such as maths. Indeed, in describing the benefits of computer based materials for maths education, Kaput (1992) specifically notes that there is no evidence of physical actions having any benefit on learning in this domain.

A central theme around the importance of physical actions in learning abstract concepts relates to how easily certain ideas can be labelled as 'abstract' in the sense of being disconnected from perceptual experience. Returning to the embodied cognition arguments, where thinking is described as being grounded in perceptual and sensory experiences, it is possible to think of the relationship between certain mathematical terms such as adding, and the actions generated when carrying out this operation with objects.

Indeed, many numerical terms, such as *adding*, *taking away* and *partitioning,* apply to actions that can be carried out with objects. The difficulty lies in understanding the role of such actions in learning related concepts, and how this might be demonstrated empirically.

If physical actions are embodied in certain numerical concepts, then it might be expected that evidence of certain motor activation would be found when individuals engage with these concepts. Arguably, support for this comes from a study showing muscle activation in young adults' fingers when they were asked to make judgements about the parity (odd/even) of visually presented Arabic numbers (Sato, Cattaneo, Rizzolatti, & Gallese, 2007). It was found that when judging small numbers ($n<5$), muscle activity was produced in the right hand despite participants claiming not to use any finger strategies. In a different study asking participants to make parity judgements, it was found that participants were more likely to make precision grip responses when presented with small numbers and power grips for larger numbers (Moretto & di Pellegrino, 2008). As precision grip is associated with physically grabbing small objects and power grips for larger, this study argued for a shared processing of symbolic and physical information in the coordination of actions.

- *Gestures*

Further support for the embodiment of physical actions in numerical concepts has come from studies looking at gesture use. Edwards (2005), for example, investigated young adults' concepts of fractions and recorded the gestures used when describing fraction concepts. Numerous gestures were observed, many of which were judged to originate from previous experiences using manipulatives – gestures such as partitioning groups of objects.

The last ten years have seen a great number of studies that examine the role of gestures in understanding and learning (e.g., Abrahamson, 2004; Broaders, Cook, Mitchell, & Goldin-Meadow, 2007; Goldin-Meadow, 2000; Sabena, 2004). This research has emphasised the role of gesture in supporting thinking independently of the listener. For example, Iverson & GoldinMeadow (1997) showed how congenitally blind children used gestures when communicating with other blind children. By supporting thinking, gesturing may help free up valuable cognitive resources. Indeed, one study showed that children were more able to hold a word list in memory when asked to gesture while explaining a maths task (Goldin-Meadow, Nusbaum, Kelly, & Wagner, 2001). The potential of gesturing to support cognition was also indicated in a study by Cook (2007) showing that children who were required to gesture while learning a new mathematical concept were more likely to retain the knowledge.

If gestures are linked to individuals' thinking, they may also provide a way to communicate ideas. Indeed, it has been suggested that they provide teachers with an effective means of assessing children's understanding (Goldin-Meadow, 2000; Herbert & Pierce, 2007; Kelly, Singer, Hicks, & Goldin-Meadow, 2002). Furthermore, the use of gesture by the teacher has also been found to support children's understanding (Flevares & Perry, 2001; Valenzeno, Alibali, & Klatzky, 2003), and it has even been shown that children can replicate teachers' gestures to help them change their mind about their pre-existing ideas (Cook & Goldin-Meadow, 2006).

Clearly, the link between gesture use and manipulatives is still not clear. Certain gestures may be used with objects that could just as easily be enacted with other representations. A good example might be the use of pointing to support counting (Alibali & DiRusso, 1999; Carlson, Avraamides, Cary, & Strasberg, 2007). Alibali & DiRusso (1999) demonstrated how young children would use pointing gestures towards objects to help offload the cognitive demands of keeping track of items counted, and to help correspondence between objects and number words. Such gestures could also be enacted toward pictorial representations. Indeed, the authors argue that touching objects is simply an extension of pointing (albeit that the tactile feedback may also support visual processes). Therefore, despite the increasing evidence for some form of motoric encoding in cognition, further research is required to understand the extent to which the benefit of using gestures is fostered by manipulatives.

1.3.4.4 Manipulatives help offload intelligence

Mix (in press) also identifies the potential for manipulatives to support children in 'offloading intelligence'. Offloading intelligence in this context seems to reflect an external cognition perspective where intelligence, or cognitive activity, is seen as an interaction between internal (mental) and external representations (Rogers, 2004). Rogers and Scaife (1998) characterize this relationship in terms of '*Computational offloading'* where different representations require a different amount of effort to solve problems with equivalent information.

Manipulatives may help children learn through 'offloading cognition' by reducing the cognitive effort to solve numerical problems. However, in order to evaluate this possibility it is necessary to examine what information manipulatives provide, as well as how interaction with this information can support learning.

According to McNeil & Jarvin (2007), one of the key advantages of manipulatives is that they provide an additional channel of information which might be regarded as predominately visual or tactile. Sensory input provides children with information on certain properties of the manipulatives, such as their size, shape, colour, and any markings (e.g., numerical values inscribed on coins). This may have symbolic value – for example, Cuisenaire rods use length and colour to represent different numerical values, while Dienes' blocks use size (and shape) to signify the decade structure. Information about the properties of the materials presents certain affordances: perceptual information that facilitates certain physical actions (Gibson, 1977; see Hartson, 2003). For example, the size of cubes will determine how many can be grabbed by children in one hand and moved simultaneously. The shape of objects will determine how they will rest of a flat surface or possibly adjoin with other materials. Indeed, physical knowledge – knowledge about the rule governing physical materials (e.g., they will collide when moved against other materials) – is developed at a young age - around six months (Spelke, 1990). In addition, this knowledge may underlie more domain specific numerical skills (Carey & Spelke, 1994).

It was previously discussed (section 1.2.11) how infants' numerical ability is strongly related to their capacity to enumerate small collections – by subitising. The spatial arrangement of manipulatives may support children by activating such mechanisms. Importantly, with respect to manipulatives, this information does not need to be visual – children can subitise a collection through touch (Riggs et al., 2006).

Spatial properties provide other information to support cognition. Larkin and Simon (1987) describe how the spatial relationships between (diagrammatic) objects can encode information about their relatedness. Similarly therefore, manipulatives may also help offload cognition by allowing information to be spatially organised. According to the Gestalt principles of visual perception (see Rock, 1993), items that are closer together are more likely to be associated; therefore it is possible to partition a collection of objects into parts by moving objects into spatial groups. Again this spatial information can be processed through touch – *proprioception* allows an individual to know the position of objects relative to the body.

- *Offloading intelligence and problem solving*

Zhang and Norman (1994) describe the way in which information from an external representation helps to define the structure of a task. They demonstrate, using the Tower of Hanoi as an example, how the problem structure, and hence individuals' cognitive activity, can be shaped by changes to the external representation. According to Neth and Muller (2008), it is possible to describe the way cognitive activity adapts to the environment in two ways:

> *"On one hand, the cognitive system adapts itself to the structure of its environment to transcend its inherent limitations (e.g. of attention and memory). On the other hand, cognitive systems exhibit a pervasive tendency to adapt and structure their environments in service of their goals"* (p. 993)

Manipulatives can not only help children offload cognition by providing visual and tactile information, but they can also allow children to adapt this information (spatially) to support cognitive activity. Kirsch & Maglio (1994) distinguish two types of actions: *pragmatic* and *epistemic*. Pragmatic actions are defined as those that adapt the representation intentionally toward a goal state. In contrast, epistemic actions are those that adapt the representation in order to provide the user with information that can support the activity. This distinction is described with reference to a game of *Tetris* (a computer game that involves using a mouse to rotate falling objects so that they fit together). Pragmatic actions are those that rotate objects to a desired angle to 'fit'; epistemic actions are those that help the user explore different angles of rotation in order to identify the most appropriate – distinguishable from pragmatic actions in that they may involve initial actions that rotate objects away from the most efficient orientation. Although the terminology used by Kirsch has been criticised (H. Neth & Muller, 2008), it does highlight the role of manipulating information in order to support problem solving. Other studies have provided further support of this, for example: rearranging scrabble tiles to help identify words (Maglio, Matlock, Raphaely, Chernicky, & Kirsh, 1999), or arranging coins to support addition (H Neth & Payne, 2001).

- *Offloading intelligence and learning*

If, as it is argued, manipulatives provide ways in which children can offload some of the cognitive task demands, it is important to question how this will ultimately lead to learning. One possibility is that externalising information may reduce the demands on working memory, thereby freeing up valuable cognitive resources to encode information to memory (Sweller & Chandler, 1994). Manipulatives may therefore confer an advantage over other materials such as paper by presenting information in parallel via the motor system as well as visually. Indeed, it has been argued (Wilson, 2001) that sensory and motoric encoding should be considered as a separate system (in addition to the auditory loop and visual-spatial sketchpad described in standard models of working memory (Baddeley & Hitch, 1974)).

The potential to reduce cognitive processing demands may be significant considering that working memory plays a key role in various numerical procedures such as addition (Adams & Hitch, 1997; Hecht, 2002; Passolunghi, Vercelloni, & Schadee, 2007). Using manipulatives may therefore allow children to develop ideas by supporting working memory during problem solving. It may even be possible that the use of materials supports the development of internal structures that may in turn support working memory at a later stage. Lee, Lu & Ko (2007), for example, demonstrated that abacus training was able to improve adults and children's ability to retain visual-spatial information (measured through span tasks, e.g., forward digit span) supporting the proposal that domain-specific training enhances the efficiency of storing and assessing task-relevant information (Ericsson & Kintsch, 1995).

Evidence that manipulatives can support learning by reducing cognitive task demands remains unclear, and it possible that such support (if indeed manipulatives do support working memory) is detrimental. In order to reduce certain procedural demands, children are able to use more efficient strategies – counting-on instead of counting-all for example. Therefore, by reducing certain procedural demands, external representations may have the result of reducing the motivation to develop such efficient strategies. Indeed, in the study by Secada et al (1983) (see section 1.22), the authors needed to cover up the first addend (i.e. remove access to the external representation) in order to motivate children to use the more efficient count-on procedure. As Muldoon, Lewis, & Towse (2005) have shown, providing objects for numerical problems may sometimes encourage children to just count them rather than try to infer numerical relationships.

The possibly detrimental effect of reducing problem solving demands by providing an 'easy to use' external representation can be compared to research with adults that demonstrates how increasing the costs of interacting with external information (e.g., a time delay) can increase the use of both planning (O'Hara & Payne, 1998; Van Nimwegen, Van Oostendorp, Burgos, & Koper, 2006) and of memory based strategies (Gray & Fu, 2004) . For example, O'Hara & Payne (1998) demonstrated how individuals made less moves to solve a particular puzzle (8-puzzle[4]) when implementation costs (inputting instructions) were introduced for each move. Gray and Fu (2004)

[4] The aim of the 8-puzzle is to arrange a 3x3 matrix of tiles by moving them one at a time into the empty space until the desired arrangement is reached.

demonstrated that individuals were more likely to use memory based strategies when the costs of externalising information were increased, preferring to rely on 'imperfect knowledge in the head' rather than 'perfect knowledge in the world'.

The research described above raises the possibility that facilitating problem solving by providing an external representation may have the unintended effect of reducing the use of more efficient mental strategies. Using the partitioning tasks previously described as an example, allowing children to identify ways to partition a number by simply moving physical objects and then counting them may reduce their motivation to develop more efficient mental strategies that infer relationships between solutions. However, there are two key reasons why the literature on problem solving may not generalise well to learning tasks for children. These concern differences in the structure of the problems and the abilities of the problem solvers.

Problem structure

Problem solving research has generally focused on 'well defined' problems, where the initial and goal states and legal moves are known. Indeed, many of the problems used, such as the Tower of Hanoi (TOH) (Zhang & Norman, 1994), Slide Jump puzzle (O'Hara & Payne, 1999) and Ball and Boxes (Van Nimwegen, Van Oostendorp, & Schijf, 2004) permit as few as 2-5 possible operations on different states and have only one correct solution. The known solution state and relatively constrained problem space may therefore allow the user to consider possible moves and choose the one most appropriate before acting on the representation. Evaluating possible actions before performing them may consequently result in greater efficiency.

In contrast to this, in less well defined problems, possible states and operations are less clear. There may be multiple solutions and multiple pathways (possible transformations between different states). Thinking about the range of actions will therefore be more cognitively demanding and may consequently increase the advantage of supporting cognitive operations with perceptual ones (i.e. use external representations).

Problem solver

Individuals often fail to plan despite the potential gain in terms of problem solving efficiency. Children in particular find planning difficult due to cognitive demands and motivational reasons (Ellis & Siegler, 1997). Indeed, despite attempts to make the task more accessible (such as by displaying the end state and using a cover story) children between 4 and 6 have been shown to have quite limited ability to plan in problems based on a 3 and even 2 disk version of the Tower of Hanoi (Klahr & Robinson, 1981). It is not clear, however, whether such poor performance reflects more motivational factors given the decontextualised and abstract nature of tasks such as Tower of Hanoi. Indeed, children's planning abilities might be more positively exposed through computer games where it has been shown that careful designs are able to increase motivation for learning (Habgood, Ainsworth, & Benford, 2005).

Individuals' ability to plan will also be influenced by their domain understanding, as this will determine knowledge of what states, actions or solution(s) are possible in a task. It is possible, therefore, that when children lack the understanding to plan, the most efficient way to progress is to act - thereby changing the external representation and generating information to help inform planning in the task. This suggestion is reflected in Martin and Schwartz's (2005) theory of Physically Distributed Learning (PDL).

According to this theory, in problems where the user only has "*incipient*" (emerging) knowledge, actions on the environment can lead to reinterpretation of the problem, and thereby lead to learning. Support for this theory has come from examining children's use of manipulatives in numerical problems and will therefore by looked at more closely in the next section.

- *Physically Distributed Learning (PDL)*

In their paper describing PDL, Martin and Schwartz (2005) present a simple framework for how individuals learn with physical objects using two dimensions: the stability or adaptability of the environment (in this case physical objects), and the stability or adaptability of one's ideas (Figure 1.6). Quadrant 1 of this framework refers to the way in which learning is possible just from the structure of the environment, as might be the case with the tens and units pieces supporting base ten understanding. The second quadrant is referred to as *off-loading* – where an individual uses a representation simply to externalise existing ideas. Quadrant 3, *repurposing*, describes the same processing of offloading, but in this case by actively manipulating the external representation – comparable to Kirsch's pragmatic and epistemic actions. Quadrant 4, PDL, also describes manipulation of the external representation ('environment') but in this case leading to the development of qualitatively new ideas. In other words, physical manipulation allows an individual to reinterpret the external representation, and this re-interpretation reflects the development of new schemata.

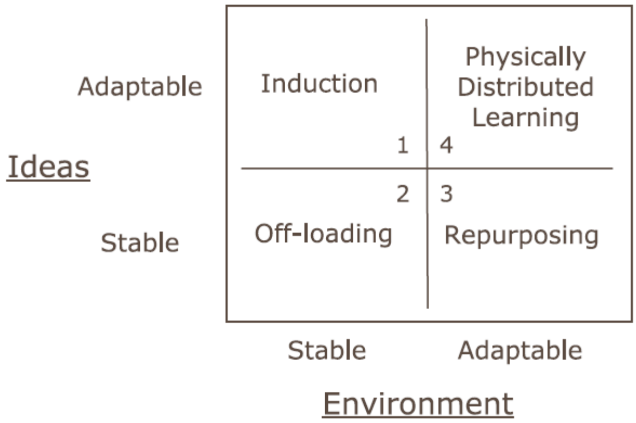

Figure 1.6: Physical actions and learning (Martin & Schwartz, 2005)

To test this theory, Martin and Schwartz compared children's (11-12 year old) learning of fraction concepts using two materials: one that could be physically manipulated (tiles and pie pieces), with one that had the same structure but could not be physically manipulated (squares on paper). Children solved fraction operator problems (such as one third of 12) using both materials in counterbalanced conditions. Each solution received an interpretation score that reflected the child's verbal answer and an adaptation score that reflected the child's physical arrangement of the pieces. It was found that physical materials conferred an advantage for both the number of adaptations (manipulations) and interpretations (correct answers). Although the two scores were not significantly related, the finding that children made more changes and provided more correct answers with physical objects was used to support PDL. This theory has also received support in a study comparing younger children's (4-5 years old) use of physical and pictorial materials in addition and geometry tasks (Martin, Lukong, & Reaves, 2007), although only verbal scores were measured in this instance.

In many ways, PDL can be compared to other display based theories (e.g., D. Kirsh & Maglio, 1994; Larkin, 1989; Zhang, 1997) where users act on and interpret information within an external system to support their cognitive activity. Indeed, with its emphasis on externalising and reconstructing ideas through external models, the theory appears to reflect the key arguments described by Papert's constructivism. It does seem to distinguish itself, however, by focusing on a particular property of manipulatives, namely the ability to spatially adapt the representation through physical actions, as well as a certain outcome: the development of qualitatively new ideas. The theory is therefore particularly apt for examining the potential of manipulatives to develop young children's numerical concepts such as decomposition. PDL does raise certain questions that should first be considered.

One question concerns the process by which children are able to 'reinterpret the representation'. In particular, it is not made clear what role the context plays in structuring interpretations. Considering the highly structured nature of Martin and Schwartz's fraction study (i.e. in a school classroom with an adult asking numerical questions), it seems likely that contextual factors do play a key role in guiding children in their interpretations, and 're-interpretations', of the representation – in this case as verbal numerical solutions.

It is also not clear what is meant by 'physically' adapting the representation. Annotating paper requires physical actions, so it might be assumed that physical action infers spatial manipulation. In this case, is it important to question whether such spatial manipulation needs to be made through direct physical contact with the representation or can be achieved more indirectly through a graphical interface.

Finally, it is not clear what concepts might be supported through PDL. Although the authors describe how PDL is most effective when children have 'incipient knowledge' of a certain concept, it is not clear whether this applies to all concepts. It is possible that PDL may be highly effective for some concepts, but not for others. A key challenge would therefore be to understand the mechanisms of the theory sufficiently to predict when physically manipulating representations may or may not be effective.

1.3.4.5 Summary

This section has examined the possible mechanisms by which using manipulatives might support numerical development. These possible mechanisms were discussed under four headings adapted from Mix *(in press): conceptual metaphors, focus attention, generate actions* and *offload intelligence.* Broadly speaking these mechanisms might be described as those where actions with objects become directly integrated into numerical concepts or provide a reference for numerical ideas, and those that describe how manipulatives may support children‘s problem solving: focusing their attention on relevant information and freeing up valuable cognitive resources to encode this information to memory.

The review also elucidated possible limitations of objects, most importantly that their numerical significance is only granted by the context in which they are used. Children need support in interpreting their physical interactions. It is also important to acknowledge the demands placed on children in having to simultaneously process physical materials as objects in themselves, as well as representations in a mathematical domain. This is a key limitation raised by Kaput (1992; 1993) about physical materials – they provide no means of mapping physical changes to symbolic changes. Kaput identifies a further key criticism: that manipulatives are constrained to the 'eternal present' (i.e. that changes to the representation necessarily remove evidence of the previous states). Kaput provides an example of how this prevents the materials from being able to simultaneously display both process and result in numerical operations. Unlike the written notation 2 + 3 = 5, which shows both the process (2 add 3) and the result (is 5) simultaneously, physical materials necessarily show this transformation sequentially.

What does seem clear from the analysis is that the potential role of manipulatives depends on the concept being learnt. Different numerical activities present different demands and have different relationships to children's prior knowledge. Concepts will also vary in how they can be represented through physical actions. Consequently, in order to predict the potential of physically manipulating representations to support a certain concept, it is necessary to consider the numerical activity and the ways in which different properties of the physical representation may support or hinder learning.

1.3.5 Manipulatives to support learning in a partitioning task

Section one of the literature review identified additive composition as a key numerical concept for children. This concept reflects an understanding of how numbers can be decomposed and recomposed into smaller numbers. The analysis identified several learning tasks that require children to identify different ways in which a number can be composed (Baroody, 2006; Clements, 2009; Fischer, 1990; Jones, Thornton, & Putt, 1994). One key factor that seemed to differ between tasks concerned the materials used to support children – whether, for example, physical objects were provided.

It is possible to consider many of the possible learning mechanisms identified for manipulatives with respect to the partitioning learning task. For example, the activity may generate physical actions and visual-spatial experiences that become embodied in children's concepts of decomposition. Physical objects may also provide conceptual metaphors for numerical decomposition by helping children map their prior knowledge of how physical collections can be partitioned in different ways. This possibility would support Resnick's (1992a) notion of a protoquantitative concept of additive composition, although it is possible that physical objects may also be advantageous in helping children calculate numerical solutions, encouraging them thereby to explore patterns between solutions. Physical objects may support cognition by allowing children to use visual and tactile stimuli to offload the task of enumerating solutions. For example, objects can provide an external representation of the total amount to be partitioned and allow children to enumerate parts using perceptual processes such as subitising. It is also possible to consider PDL in relation to a partitioning task. If children have only incipient ideas about how numbers can be decomposed, physical objects may allow them to manipulate the representation and interpret changes to develop new ideas in this domain.

Despite all the advantages promulgated, some consideration should be given to the way in which the use of manipulatives might hinder the learning task. If children are able to act on the representation with ease, they may be less inclined to plan their strategies. More than this, by facilitating the process of identifying solutions and enumerating parts, children may be less motivated to develop efficient strategies that relate solutions. Importantly, actions on physical objects will remove any record of previous solutions. Therefore, unlike other materials such as paper, physical materials provide no means for children to examine, compare and reflect on the relationship between different solutions.

1.3.6 Summary

It can be seen that understanding the role of physical materials in the partitioning task draws together many of the arguments concerning the role of manipulatives in numerical development generally. Consequently, in order to evaluate the potential of physically manipulating representations for supporting children's numerical development, it is possible to identify two key research questions:

- *Do physical objects support children's strategies for partitioning numbers?*
- *What are the advantages/limitations of physically manipulating representations for children's partitioning strategies?*

As well as providing educators with important information, understanding the learning mechanisms of manipulatives is important in understanding the potential role for technology and novel learning materials. Technology has presented the means to build on the advantages as well address the limitations of different representations, and indeed, with respect to manipulatives, there has been a proliferation of computer representations – virtual manipulatives – to support numerical development. With this has also come the recent advance of an increasing ability to integrate technology into physical materials – tangible technologies. In order to evaluate the potential of tangible technology to support numerical development, it is not only necessary to identify the advantages and limitations of using physical objects, but also to understand how different learning mechanisms are affected by interaction with other interfaces (e.g., mouse, tabletop computers, etc). The next section will examine the literature on digitally augmented manipulatives in order to identify key research questions for evaluating the potential of tangible technologies in this domain.

1.4 Digital manipulatives

1.4.1 Introduction

The increasing use of technology in schools is testament to confidence in the potential for 'the digital' to support learning. This confidence has extended to the development of computer based representations: 'virtual manipulatives', with the hope that they might combine the advantages brought by the use of manipulatives with those of technology. Notwithstanding this, it should be noted that the previous section identified various possible learning mechanisms for physical manipulatives that might not extend to the use of computer materials controlled through a standard mouse/keyboard interface, particularly those concerned with the role of actions.

Identifying the benefits, if any, of physical manipulation is important, especially in order to understand the potential of emerging technologies that offer novel ways for interacting with digital technology. This can be seen in the development of tangible (or hand held) technologies, where the ability to integrate smaller and more sophisticated technology into physical materials has generated novel opportunities for supporting children's learning.

1.4.2 Tangible technologies

The traditional and most common (certainly in schools) form of interacting with digital technology is through a computer, where objects, both textual and graphical, can be manipulated on screen using a keyboard and/or mouse. This set up, with its clear distinction between input and output, has consequently been referred to as a graphical user interface (GUI). Tangible technologies ('Tangibles') attempt to transform this input-output distinction by presenting novel ways to interact with digital technology that blend the physical and digital worlds together (Ullmer & Ishii, 2000). There exist various frameworks to distinguish types of Tangibles (Fishkin, 2004; Hornecker & Buur, 2006; Koleva, Benford, Ng, & Rodden, 2003) but common to these designs is the emphasis on touch and physicality in both input and output. Tangibles therefore present exciting ways to design novel relationships between children's interactions and digital technology. This form of technology has consequently generated substantial interest in the possibilities for designing effective learning materials.

1.4.3 Tangible technology for learning

In a seminal paper, Mitch Resnick and colleagues (1998) at MIT describe a new generation of computationally enhanced manipulative materials called 'digital manipulatives'. These materials are described as those which embed computational capabilities inside traditional children's toys – such as cubes, beads, and balls (M. Resnick, 1998). According to Resnick, digital manipulatives could enable more difficult concepts to be explored through physical manipulation. Coming from the same research laboratory, it is perhaps not surprising that this belief in the potential of technology to make difficult concepts more engaging and accessible echoes the vision described by Papert almost 20 years previously. What has changed is the wealth of new possibilities generated by advances in technology.

Resnick does not identify the particular learning mechanisms in which digital manipulatives may lead to learning but places them in a historical context – as extensions of manipulatives designed by pioneers such as Montessori and Froebel, whose designs have withstood the test of time in educational contexts. Zuckerman (2005), also at MIT, used the different approaches of these early works to categorise two different types of digital manipulatives: *Montessori inspired Manipulatives (MiMs)* – that focus on more abstract concepts, and *Froebel inspired Manipulatives (FiMs)* – that focus on more real life processes. These labels are used to taxonomise both traditional and digitally augmented manipulative designs.

The validity of Zuckerman's definitions can be queried, particular with reference to Froebel who actually created quite generic materials and activities designed to foster symbolic understanding. Nevertheless, the distinction does help define the focus of this thesis: on the potential of MiMs, or rather tangible technologies focusing on children's understanding of more abstract concepts. Zuckerman describes how Tangibles research has tended to focus more on FiMs, and hence aims to address this imbalance by presenting two research projects reflecting MiMs: *SystemBlocks* and *Flowblocks*. The latter of these is described briefly below.

- *Flowblocks – an example of Tangibles for learning*

Flowblocks (Figure 1.7) were designed to model concepts related to counting, probability, looping and branching. The system consists of blocks which sequentially display a light, giving the appearance of the light 'flowing' through the cubes. Different blocks allow children to explore the dynamic system, for example, by speeding up and slowing down the flow. There is a counter cube which displays the number of times light has flowed through.

Figure 1.7: FlowBlocks (Zuckerman et al., 2005)

The authors report success with *FlowBlocks* in terms of children's engagement. Unfortunately, no information is provided on the important question of what the children actually learnt. Answering this question with empirical evidence is difficult, requiring more time and attention to other effects on learning such as the amount of adult support provided. Nor is it clear why *FlowBlocks* might support certain concepts such as counting better than a computer based representation - it is not clear what unique advantages are offered through physical interaction.

1.4.4 Learning Benefits of digital manipulatives

In a paper entitled "*Do Tangibles support learning?",* Marshall (2007), summarises the different approaches that have been taken to identify the benefits of this form of technology. These approaches are summarised in Figure 1.8 below.

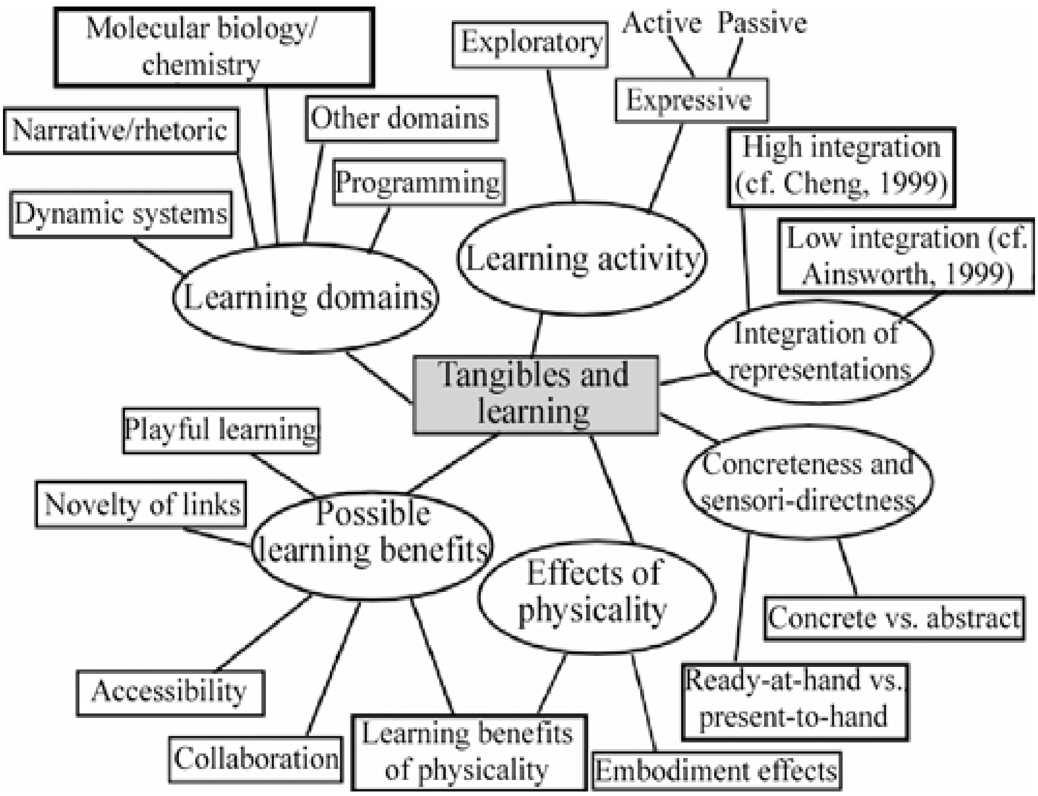

Figure 1.8: Analytical framework for Tangibles for learning (Marshall, 2007)

Marshall identifies many lines of research not focused upon in this thesis, such as the potential for Tangibles to support collaboration (Africano et al., 2004; Price, Rogers, Scaife, Stanton, & Neale, 2003; Stanton, Bayon, Abnett, Cobb, & O'Malley, 2002) and accessibility and enjoyment (Price et al., 2003; Xie, Antle, & Motamedi, 2008). However, one dimension identified which is particularly relevant is the '*effects of physicality*'. Unfortunately, considering the broad coverage within a relatively short paper, Marshall is only able to indicate research in this area, such as the possible role of embodiment. Nevertheless, the conclusions reached are similar to those made in the debates surrounding manipulatives:

"Thus, despite the common view that the physical materials used in tangible interfaces are particularly suitable for learning tasks, there is only limited evidence to support this claim. This

suggests that intuitions about the benefits of physical manipulation should be abandoned. Instead, empirical research is required to investigate in which (if any) domains and situations physical manipulation will be of benefit to the learner." (p.168)

A more thorough analysis of the possible learning benefits of Tangibles was presented by O'Malley & Stanton-Fraser (2004). The review examines both the theoretical and empirical arguments surrounding physical manipulation and learning, and links this to frameworks as well as case studies of Tangibles. The main focus of the analysis is centred on children's ability to map between physical representations and the domain they are intended to represent. This focus reflects how, in contrast to analogue materials, digital manipulatives (such as computer representations) present a separation between input and output. The authors go on to describe at least three levels to such interactive learning environments (p. 23):

1) Representation of the learning domain

2) Representation of the learning activity

3) Representation embodied in the tools themselves

These three levels help describe some of the issues surrounding manipulatives and the role of technology.

1.4.4.1 Traditional Manipulatives

With analogue manipulatives like plastic cubes, the learning representation is also the tool – input is also output. This direct relationship reflects many of the arguments put forward in the previous section for the advantages of manipulatives, such as embodiment and tactile feedback. However, it was also suggested that these materials have no implicit link to more formal symbolic mathematics: this relationship needs to be created through the activities presented by the teacher.

1.4.4.2 Virtual manipulatives

According to Moyer (2002), a virtual manipulative is defined as "*an interactive web-based visual representation of a dynamic object the presents opportunities for constructing mathematical knowledge*". In this thesis, virtual manipulatives will be defined as on-screen objects that can be manipulated using a graphical user interface, but which are not necessarily accessed through the internet (i.e. not necessarily web-based).

Kaput (1992) identifies several key advantages of virtual representations that address limitations of physical materials: the potential to link representations, provide feedback and provide a trace of past actions. These advantages are echoed by others such as Moyer et al (2002) who add more pragmatic factors to the list such as: adaptability, availability, ease of setting up and clearing away, and ability to print. Moyer et al also highlight how the materials may overcome the stigma that is sometimes associated with the use of concrete materials for younger, less able children.

Clements (1999) makes the case that the emphasis on the use of physical materials results from a desire to make learning concrete, but argues that the benefits of concreteness are not simply due to physicality so much as to how well the materials connect ideas to the real world. Using this definition, Clements describes ways in which computer based representations can achieve this more effectively, referring to various advantages such as how the materials can be designed to help externalise mathematical ideas and processes, thus helping to reinforce the link between concrete and symbolic representations.

Despite the purported benefits of virtual manipulatives, their advantages have yet to receive much empirical support. However, the aim here is not so much to evaluate the different learning opportunities presented as to focus on certain aspects that help examine representational differences with physical manipulatives. Three key aspects are discussed below: how the materials can help map to symbolic representations; provide a record of representational change; and the different forms of manipulation.

- *Mapping to symbolic representations*

One key argument in support of virtual manipulatives is that, unlike analogue materials, they can provide a means to link learning representation to symbolic representations (Clements, 1999; Kaput, 1992; P. W. Thompson, 1992). Materials can be designed to create a dynamic link between the learning representation and more symbolic notation. For example, in an activity provided by the National Library of Virtual Manipulatives (NLVM, 2007), children are able to manipulate virtual Dienes' blocks to try to match a written number (Figure 1.9).

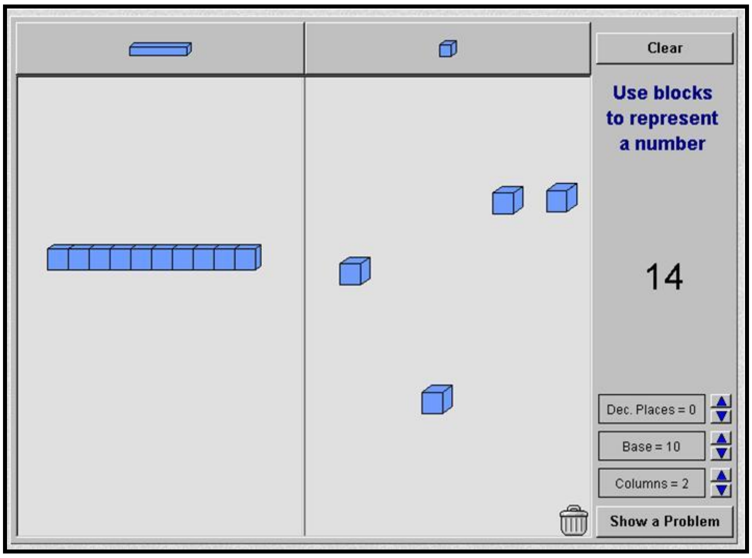

Figure 1.9: Tens and units activity with Virtual Manipulatives (NLVM, 2007)

With virtual manipulatives, it is also possible to manipulate symbols and explore the resultant changes on the learning representation (P. W. Thompson, 1992). Unfortunately, it is not clear whether such a transparent link between representations is desirable. As argued by O'Malley (1992), such transparency does not require the learner to reflect upon their actions. Recognising this, designers may opt to incorporate some degree of opacity to foster more reflection on the mapping between levels of representation. A key challenge might therefore be to design technologies in a way to augment representations that draw children's attention to important numerical concepts without making the link so explicit as to limit reflection.

Virtual manipulatives present various ways in which the representation of the learning activity can be augmented. Objects can be designed to emulate familiar physical materials such as cubes or rods but can also be extended - allowing them, for example, to change colour or make sounds. The materials are not constrained by physical laws – they can be designed to change shape and size, or be made to appear and disappear instantaneously. On the other hand, certain physical aspects are more difficult to emulate – for example, creating the illusion of three dimensional structures and movement. However, it is still not clear what design of learning representation is most effective for building children's understanding of number; including certain features may only serve to distract (McNeil & Jarvin, 2007; Uttal et al., 1997).

- *Record of Representational change*

Kaput (1992) identified a further limitation of physical materials that can be addressed by virtual manipulatives. Unlike physical materials, it is possible to keep a log of actions with virtual materials so that a record can be presented of the changes made to a representation, thereby facilitating a review of these changes. According to Kaput, this ability is important in maths where changes in representational state reflect key numerical processes, such as how two objects have been combined in different ways to create a whole. This particular feature may be highly relevant for activities such as the partitioning task where comparing a list of different solutions may be beneficial (Clements, 2009). It is unclear, however, how easily young children can identify and then reflect on such symbolic relationships.

- *Manipulation*

With virtual manipulatives, representations on screen are typically mouse controlled. There is therefore a physical separation between the tool and the learning activity representation. In this set up manipulation is indirect, and the designer needs to decide what physical actions with the mouse relate to actions on screen. For some actions, such as moving on-screen objects, this mapping is quite simple. Indeed, Donker & Reitsma

(2007) showed that even 4 year old children were proficient at 'drag and drop' actions (although less so than 5 year olds, which indicates possible difficulties for younger children and children with physical disabilities). However, other actions, such as attaching or detaching objects or moving groups of objects, may require less obvious mouse actions, or combinations of mouse and keyboard actions (such as pressing the shift key to select multiple objects). The actions can obviously be learnt, but this does raise questions about how seamless they will be to young children. Importantly, the indirect relationship between actions with the tool (mouse) and the learning representation may limit many of the benefits of physical actions (e.g., embodiment, tactile feedback). If these learning mechanisms are important, the potential advantages of virtual manipulatives may be greatly limited by the form of interaction. Certain limitations of mouse-controlled virtual manipulatives may be addressed by emerging interfaces such as tabletops where multiple virtual representations can be manipulated through touch, but it is possible that there are still limitations presented by the inability to physically interact with representations. Digital manipulative may address this limitation by providing a tangible interface.

1.4.4.3 Digital manipulatives (including Tangibles)

Digital manipulatives allow designers to create a tight coupling between physical actions and the learning representation. Indeed, the learning representation may actually be embodied in the object being manipulated[5]. It is thereby possible to augment the learning representation with digital technology, and to explore the effects of so doing through direct physical manipulation. Nevertheless, considering the comparative ease and availability of virtual manipulatives, it is important to consider what benefits are provided by such physical interaction.

Several studies have attempted to identify the possible benefits of physical manipulation by comparing performance between physical and virtual representations. Of these, relatively few have attempted to limit confounding variables, and those that have tend to report no significant differences (Klahr, Triona, & Williams, 2007; Triona & Klahr, 2003; Zacharia & Constantinou, 2008). Unfortunately, this indicates a key difficulty in comparing physical and virtual representations: by controlling variables to examine effects, it is easy to 'design out' many of the advantages of either medium. This point is exemplified in a study by Triona & Klahr (2003), who compared the effects of using virtual and physical materials (springs) on children's ability to design experiments. No differences were found between the physical and virtual materials although, in order to balance conditions, the authors noted that they focused on the length and width of the spring rather than the weight of an attached object as "*the effect of the mass of the weight used on the springs is not visually discernable*" (p. 159). In other words, to balance conditions, possible unique advantages of either medium may have been eliminated.

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[5] Various taxonomies have been created to describe the range of couplings between interface and digital representation in Tangibles. These will not be discussed here but the reader is directed toward work such as Fishkin (2004) , Koleva et al (2003), and Price (2008).

It is possible that studies comparing physical and virtual representations have not been designed specifically to detect certain benefits of physical manipulation as identified in the previous section. Some mechanisms, such as the embodiment of motor actions, may be hard to detect. Others, such as the use of tactile information to reduce cognitive demands, may be easier. Clearly, the extent to which these processes play a role, and consequently the potential benefit of a physical interface, will depend on the nature of the task.

1.4.5 Physical versus Graphical interfaces and the partitioning task

The previous section summarised some of the possible advantages and limitations of physical manipulation in a task requiring children to identify different ways to partition numbers. It is possible that digital manipulatives offer the potential to build upon these mechanisms, although, as discussed with the *FlowBlocks* as an example, it is important to question why physical manipulation offers advantages over and above manipulation through a graphical user interface. It has been argued that computers are also able to offer concrete experiences (Clements, 1999). Nevertheless, there were some mechanisms identified that pertain to physical manipulation, such the role of tactile information in supporting the cognitive system, or the generation of motor actions in developing embodied concepts.

It is not clear how some arguments extend to manipulating objects using a graphical user interface. Martin & Schwartz (2005), for example, describe how physical manipulation can lead to new ideas yet Martin (2007) has more recently applied the theory of Physically Distributed Learning to describe the benefits of virtual manipulatives. This suggests that it is the result of manipulation rather than the form of interaction that is important. Clearly, different forms of interaction will affect how easily certain representational changes can be made. Young children are able to manipulate physical objects with ease, moving individual or multiple objects with one or both hands, and moving multiple objects in this way may well allow changes in representational states that foster certain ideas. Although it is possible to design ways for such actions to be possible using a virtual interface, this may have bearing on how easy they are to enact, and hence how likely they are to be used. Indeed, it has been shown that the strategies individual employ can be highly sensitive to even the smallest implementation costs of an interface (Gray & Boehm-Davis, 2000).

If children's strategies are affected by the ease with which objects can be manipulated, it is possible that some of the constraints presented by an interface have an effect on the strategies that children employ, and hence on the ideas that they develop. Nevertheless, this does not suggest that physical manipulation would be preferable – it is possible that graphical user interfaces offer the potential to allow certain actions not possible through physical manipulation or constrain manipulation in ways that foster certain advantageous strategies.

1.4.6 Summary

Digital manipulatives provide exciting new ways to interact with digital technology. Evaluating the benefits of this technology to support children's numerical development is however hindered by a lack of understanding of the unique benefits of physical manipulation. Although various arguments have been put forward, it is often unclear why certain benefits may not extend to manipulating objects using a graphical user interface. Virtual objects are still manipulated using physical actions, albeit that these actions are mediated through the mouse or keyboard. This more indirect form of interaction makes it easier for the designer to constrain what actions can be made on the representation. This presents a third key research question:

- *What is the effect of constraining manipulation on children's partitioning strategies?*

Virtual manipulatives have gained popularity in research and educational practice, reflecting the advantages offered by digital technology. One key advantage is that designs can include links to more formal symbolic mathematics, thereby addressing a key limitation of physical materials. However, it is not clear how representations should be augmented to encourage children to reflect on different mappings.

With physical manipulatives, it is the activity designed by the teacher that provides the context in which children explore numerical relationships. The materials tend to be simple to help focus attention and limit distracting features, designed to focus attention on mathematical structures, such as using colour and size to represent different quantities (e.g., Cuisenaire rods), or different shape and size materials to represent the base structure (e.g., Dienes' blocks). Digital technology provides ways to introduce new structural representations, such as Dienes' blocks that can be broken down (NLVM, 2007), or objects whose colour can be changed to emphasise part-whole relationships (NNS, 1997) as shown in Figure 1.10.

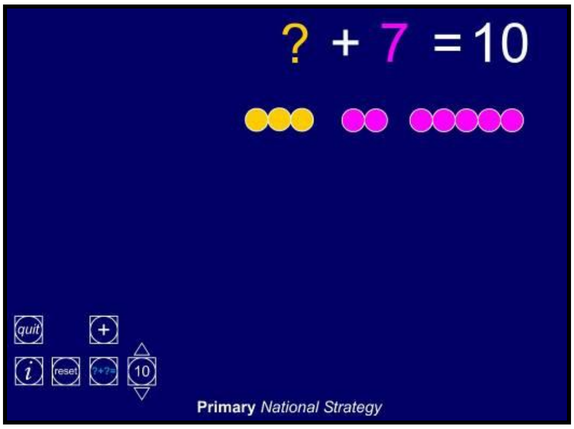

Figure 1.10: National Numeracy Strategy Interactive teaching program (NNS, 1997)

It is expected that these perceptual effects will influence the manner in which children interact with materials. Tangible technologies provide an opportunity to integrate such effects into physical materials, although it is not exactly clear how certain perceptual features influence the way children then use the materials. Nevertheless, understanding how certain perceptual effects shape children's strategies can guide us on how best to develop materials to support certain numerical ideas. A final key research question is therefore:

- *Can children's partitioning strategies be supported by augmenting the representation's perceptual information?*

1.5. Literature Review Summary

This thesis aims to help evaluate the potential of Tangibles to support children numerical development. In order to identify more specific research questions, the literature review has examined three key areas: children's numerical development, physical representations, and digitally augmented manipulatives. The review of children's numerical development identified a key numerical concept – additive composition and possible 'partitioning' tasks to support this concept. The second section reviewed the literature on the role of physical learning materials – manipulatives, and identified various arguments for how these materials might support concepts directly or indirectly by facilitating problem solving. However, it is not clear which, if any, of these mechanisms play a role in a partitioning task. Importantly, if manipulating representations is supportive, it is not clear whether interaction needs to be physical or whether it could be achieved through a graphical user interface. As well as comparing the relative benefits/limitations of a tangible compared with a graphical interface, the final section highlighted the potential for using digital effects to influence children's partitioning strategies in order to help them explore numerical relationships. By examining the result of certain effects on children's interactions, it may be possible to predict how these might be integrated into tangible designs to support children's understanding of additive composition.

Consequently, in the literature review four key research questions were identified:

1. Do physical objects support children's strategies for partitioning numbers?

2. What are the advantages/limitations of physically manipulating representations for children's partitioning strategies?

3. What is the effect of constraining manipulation on children's partitioning strategies?

4. Can children's partitioning strategies be supported by augmenting the representation's perceptual information?

This thesis will report studies that examined these questions in order to address the main research question: *does physically manipulating digital representations present any unique* *benefits for supporting children's understanding of additive composition?* In doing so, the thesis will contribute to our understanding of the potential for tangible technologies to support young children's numerical development.

Chapter 2

The Role of Physical Representations for solving Addition and Partitioning Problems - Study 1

2.1 Introduction

As was discussed in the literature review, it remains unclear if and when manipulatives support young children's learning. A key difficulty has been that arguments for and against the materials have tended to be quite high level, focusing on how effectively materials can represent and communicate abstract concepts to young children (see Gravemeijer, 1991; Halford, 1992; Mix, in press; Williams & Kamii, 1986). Although such questions are important, they are limited in helping to identify particular affordances of physical materials that would imply they are better than other materials, such as paper, or providing any means to predict what numerical tasks or ideas are best supported by this form of representation.

Identifying when physical materials will confer an advantage is certainly challenging and needs to take account of many variables such as the teacher, the task and domain, the materials used and individual children's abilities and experiences. The literature review emphasised the need to consider these different contextual factors when attempting to evaluate the potential of learning materials such as manipulatives. Nevertheless, different materials have different properties which affect children's interactions. Examining how these properties influence children's strategies can not only inform educators on when they might use the materials but also help with the design of novel materials.

The literature review examined the different properties of physical materials and the mechanisms in which these might support learning. One key physical affordance that was identified was the ability to spatially manipulate one or many objects with simple actions using one or both hands. Physical materials also provide tactile information that may help process information such as keeping track of the last object counted. However, it was also highlighted in the literature review how the roles of different properties would depend upon the task in which they were used. One task that was described was a partitioning task – where children were asked to identify all the ways a number could be partitioned into different combinations of two parts.

In order to identify the procedural and conceptual demands involved in carrying out particular numerical tasks, it is important to first consider how they relate to children's numerical development.

2.1.1 Numerical development and numerical problems

The literature review discussed children's early numerical development in relation to Fuson's model which describes four developmental levels: *Unbreakable list*, *Breakable Chain*, *Numerable chain* and *Bidirectional Chain*. These levels describe children's developing understanding of key concepts (such as one to one correspondence, cardinality and the decomposition of number) and are fundamental to the development of more efficient strategies for calculating part-whole number problems such as addition or subtraction, where numerical problems can be interpreted and possibly mentally adapted to reduce the computational costs of calculating a solution.

Fuson described these stages in relation to children's developing addition and subtraction strategies. One key step is when children understand that the last word counted represents the totality of the set – the cardinal concept. Although children may already be familiar with counting as an activity, this understanding allows them to recognise that questions asking *'how many?'* require a numerical answer that refers to the total of a set. Children can thereby start to answer simple problems such as *"how many is 4 add 4?"* However, in order to identify the solution, children require all the objects (or 'perceptual items') present which they can then count (hence the 'count-all' strategy). An important part of counting is maintaining one to one correspondence between objects and the count words, and keeping track of the last object counted. Tagging gestures support this activity (Alibali & DiRusso, 1999), and tactile feedback may help offload the need to visually keep track of objects (Carlson et al., 2007). It may also be helpful to move objects to create spatial information showing which objects have been counted. Indeed, this may explain why Martin, Lukong and Reaves (2007) found that young children identified more correct solutions in addition problems using physical materials than pictorial ones.

With an understanding of cardinality, children develop a more efficient strategy for adding amounts – they are able to count-on from one addend. This is more efficient because it only requires children to count the objects of one addend. Although this may still be demanding if the second part to be added is large, children learn that they can count-on from the largest addend – whether it is the first or second. In other words, given the problem 2 + 9, children learn they can count-on from 9. This strategy embodies the concept of *commutativity* – that two amounts can be added in any order with no change in the total. However, as with other concepts (including additive composition) it is not clear whether children's understanding precedes or follows use of the strategy (Baroody et al., 2003). If children do have a 'protoquantitative' understanding of commutativity, it is possible that using objects in a numerical task may help them draw upon this and apply a more efficient strategy (count-on from the largest) to add amounts.

At Fuson's Breakable chain level, children are able to count-on but still require perceptual items for one addend. In the next stage (the Numerable chain level), they learn to enumerate this second part without perceptual items. This procedure is cognitively demanding, requiring children to 'double count' (keep track of the total and the amount counted-on simultaneously). Interestingly, Fuson does refer to the use of fingers as one method to support this process (Steffe, von Glaserfield, Richards, & Cobb, 1983), suggesting that this level describes the ability to enumerate without the need for external *materials*. Finally, in the Bidirectional chain level, Fuson describes the ability to decompose numerical problems to facilitate counting strategies. By decomposing the parts of an addition problem, children are able to take advantage of certain number facts that they have learnt: most commonly doubles and problems around the decade structure (e.g., 10 + 5 = 15)(Carpenter et al., 1999).

2.1.2 Partitioning problem

Fuson's model helps provide a structure for children's developing ability to solve addition problems. However, it may also be also possible to use this model to reflect on children's developing ability to solve a partitioning problem. The structure of a partitioning problem was described in general terms in the literature review: requiring children to identify the different ways in which a number can be broken down into different combinations. However, as the problem has been described more in terms of a learning activity than an assessment task, it is not clear what criteria are used to evaluate children's understanding. Jones, Thorton & Putt (1994) do, however, describe in simple terms the performance of several children:

> *"When he was told the number of candies in one bag, Bill was able to tell the missing part in the other bag by counting back. He claimed there were only four partitions for 10. Nathan and Jeanie successfully gave four of the partitions for 10 and Tom and Shannon Figured out 9 of the 11 pairs mentally"* (p. 134)

In a later paper, Jones et al (1996) also describe:

> *"The level 1 partitioning further underscores the difficulty Sally had in thinking in terms of composite units or, for that matter, any kind of group greater than "one". She could only generate one partition of 10 candies – 9 in one bag and 1 in the other"* (p.321)

From these descriptions it is possible to deduce the following points concerning the task:

- It was presented in a story context.
- The interviewer played an active role (for example, by telling children one part).
- Counting strategies are one way to identify one part when given another.
- The number of solutions was deemed to reflect ability – with identifying more than one solution being considered significant.
- The total number of solutions is one more than the amount to partition (11 pairs for partitioning 10).

- With respect to the previous point, 'none and all' must be regarded as a valid solution.

- If Tom and Shannon solved the problem 'mentally', it seems that other children may have used external representations.

2.1.3 Developing ability to solve the partitioning problem

Using the above observations, as well as Fuson's model of developing numerical competence, it may be possible to distinguish several progressive stages of ability in solving the partitioning task:

1) Identify a single solution

To make sense of a question asking how to break a number in different ways, children need to know that a number *can* be broken. This understanding is reflected in Fuson's Breakable chain level. Without it, children may find a question asking them how many ways a number such as '7' can be partitioned quite difficult, although they may understand the question if it is presented using physical materials. Children know that a collection of objects can be separated into two groups and may therefore reason that a specific collection, e.g., 7 objects, can be partitioned and each part enumerated. Indeed Canobi, Reeve & Pattison (2003) demonstrated how children were more likely to notice how addends could be decomposed when problems were presented using objects than with symbols. However, as demonstrated by Hughes (1981), although children may have difficulty with symbolically presented problems, they may not actually need physical materials but simply a reference to concrete objects to solve problems. Presenting the partitioning problem in a story context may therefore be sufficient to help children recognise how an amount can be partitioned into two parts.

If children recognise that a number can be broken, they need to identify a strategy for enumerating each part. Jones et al (1996) describe how a child was able to count back to identify one part. However, this was possible because the interviewer provided the first part. Therefore, without support, a key demand for children is to identify this first part. For this, children need to identify that a part must be any positive number less than (or equal to) the whole.

The fact that Jones et al refer to the last pair of children "*figuring out pairs... mentally*" suggests that the other children used materials to support them. Indeed, the authors describe earlier in the paper how manipulatives were given. Consequently, children could use these materials to reduce the calculation demands – hence reflecting the use of perceptual items in Fuson's Breakable chain level. Perceptual items may support children's counting by helping them maintain correspondence between the count word and object, and to keep track of the last object counted. Once children have counted out the total amount to partition, they can then use this to identify a first part simply by counting a selection of these objects. They can then enumerate the other part by counting the remaining objects. If children physically partition objects into two spatial groups, this may help by a) providing perceptual clues as to which objects need to be included in which group when counting and b) creating smaller collections of objects that can be enumerated by subitising. Considering the small amounts used by Jones et al (5, 8 and 10), this may be highly relevant as most parts will be less than five.

2) Identify multiple solutions (but less than half)

Central to the partitioning problem is the fact that there is more than one solution and children need to recognise that a number can be decomposed in more than one way. This may be unfamiliar to children as many numerical problems only have one solution, but the idea is central to the concept of additive composition and seems to reflect Fuson's Bidirectional level. Fuson even refers to children '*knowing each number as all the combinations'*.

Following Resnick's (1992a) arguments, if children are familiar with the way that collections of objects can be decomposed in different ways (protoquantitative understanding), this may help them recognise how numbers can also be decomposed in different ways. Indeed, it is possible to apply Martin & Schwartz's (2005) theory of Physically Distributed Learning to this problem. If children have incipient knowledge of additive composition (or protoquantitative understanding), physically manipulating objects may help them to develop numerical ideas (quantitative understanding). This may occur because children's understanding of the problem may be sufficient to constrain their actions to partitioning objects into two groups. Children are then able to count objects in each group to identify a correct solution. Then, through simply physical actions afforded by the materials, children can create different configurations that they can enumerate.

By acting on the representation physically, children are hence able to identify multiple partitioning solutions. This process may help them map their protoquantitative and quantitative understanding of decomposition or, alternatively, may help them develop numerical understanding of decomposition simply through the experience of identifying repeated numerical combinations (the 'application before evaluation' process described by Bisanz, Sherman, Rasmussen & Ho (2005)). As argued in the literature review, both accounts might be possible: where children's understanding is developed through an iterative process of both building on former knowledge and gaining experience by identifying numerical solutions.

3) Identify multiple solutions (more than half)

In order to solve the partitioning problem, children need to identify multiple solutions, but they also need some idea of the range of solutions possible – the problem space. Of course, children can just continue to identify different combinations independently of one another, but the greater the number of solutions that are identified, the greater the chance that children will repeat a solution if they have no means to track what solutions have been given. Without a strategy, this would certainly require substantial memorising.

Certain representations, such as paper, may help children by providing a record of previous solutions. If annotations reflect previous configurations, children can use this to determine what solutions have been given as well as an indication of what solutions remain. Physical materials do not provide such a record – they are confined to the '*eternal present*' (Kaput, 1993). However, physical materials may still help children by providing a visual (and tactile) representation: children then have an additional source of information to recall past solutions (in addition to remembering verbal solutions given), although this may still be cognitively demanding. More likely perhaps is that physical objects help children by fostering the use of efficient strategies for keeping track of solutions. The external representation may help children recognise a simple strategy of progressing through different configurations such as moving one object at a time from one group to another.

In order to identify more than half the solutions, children will need to identify '*commutative*' solutions - those that have the same parts in different orders. This may be difficult for children working mentally because these solutions may sound highly similar – they have the same numerical parts, although reordered. Physical objects may help children recognise the difference, as it may be clearer to see that objects in a different order present a different configuration.

4) Identifying all solutions

Again it is possible for children to identify all solutions simply through repeatedly identifying solutions independently of each other. Without a strategy this may however be quite laborious, nor is it clear how easily children will recognise solutions with zero in one part and the whole in the other. Contextual clues may help, for example, by presenting the validity of choosing to put all biscuits in one bag and none in the other; although this may seem quite unpragmatic (why have the other bag?).

It is possible that certain strategies help children recognise that 'all and none' is a possible solution. For example, by moving one object at a time from one group to another, children will eventually reach this configuration. It seems however that without previous experience of identifying such a solution children would probably need support to recognise its validity.

2.1.4 Summary

This section has described Fuson's model of numerical development and how it relates to the development of children's strategies in problems such as addition. This model was then adapted to consider how children's numerical development might influence their ability to solve the partitioning problem, and described four possible levels of increasing ability.

It was highlighted in this section how children's success may be significantly supported by the use of external materials. More specifically, it was discussed how certain properties of physical objects may support strategies: for example, by helping children create spatial configurations through simple actions or the use of tactile information to offload counting demands. Alternatively, there may be certain limitations: they might not be as easy to manipulate as fingers and, unlike material such as paper, provide no trace of previous solutions.

2.1.5 Aim of Study 1

The aim of this initial exploratory study was to investigate the role of physical materials in young children's numerical development by examining the use of the materials in two types of problems: addition and partitioning. The addition questions were designed to vary in their computational demands, so that they would become increasingly more difficult for children who lacked more flexible strategies. The partitioning problem was intended to reveal children's understanding of how numbers can be decomposed.

In order to help identify the unique advantages of physical materials, children's performances using physical materials were compared to a pictorial and control condition. The Pictorial condition was created in order to identify whether any differences were attributable to the particular characteristics of physical representations such as their dynamic or tactile nature. In the control condition, children were given no materials but were able to use their fingers[6]. In this context, fingers can be described as a unique external representation – one where children are able to manipulate two groups (hands) of five units (fingers).

2.1.6 Study predictions

In the study, children were asked to use the representation in each condition if it helped, and were expected to use this when lacking adequate mental strategies. It was predicted that children in the physical materials condition would identify more solutions than children in other conditions in both the addition and partitioning problems. In the addition problems, physical objects would allow children to break the problem into two stages: count out one or both addends; and then count out the solution. Tactile feedback and spatial manipulation would facilitate these processes. Although children can also manipulate their fingers, they would be limited by only having ten fingers to count with. In the partitioning problems, the physical objects would support children by allowing them to count out the whole amount and then use this to easily identify multiple solutions by partitioning this amount into two spatially distinct groups in different ways.

[6] It was decided, following a pilot, that it would be difficult to prevent children from using their fingers in the absence of other external support.

2.2 Method

2.2.1 Design

A mixed design was used with Condition (Physical, Paper, No Materials) as the between subjects factor, and Problem type (Addition, Partitioning) as the within subjects factor. The primary dependent measures were Response (correct/incorrect), and Representation Use – whether the representation was or was not used for each problem. Observational notes were also made of the strategy that children used in the addition problems.

2.2.2 Participants

Seventy-five children took part in this study. Children were from Year 1 and Year 2 (US equivalent: Kindergarten, Grade 1) of a local infant school in Nottingham where the number of children having free school meals is comparable to the national average (a measure of Socio Economic Status). All children had English as their first language and no special needs were reported. Because class sizes are limited to 30, these two year groups were split across three classes with one year one class (lower ability), a mixed Year 1/2 class (higher ability Year 1, lower ability Year 2) and a higher ability Year 2 class. Classes in the UK are not typically mixed, although, this is not uncommon practice.

An initial session used the British Ability Scales (second edition (Elliot, 1983)) in order to create a numerical score for each child in which to create three equal groups. The scale requires completion of mathematics problems presented in various formats, but of numerical content only – no reading is required. Children's scores from this test were put in rank order following which children were systematically allocated to one of the three conditions (Physical, Paper, No Materials). Data from two children were not used as these children were not able to complete the tasks in the main session. The final sample was therefore 73 (39 girls and 34 boys, range 68 to 92 months; M=88; SD=6.7 months). Between groups analysis of variance was carried out to ensure that numerical ability between groups was balanced. As expected, this revealed no significant differences between conditions for numerical score ($F(2,72)=0.018$, $p=ns$).

2.2.3 Materials and Procedure

Sessions took place in a familiar room adjoining one of the classrooms. The interviewer[7] had spent a day in class with the children previously. Sessions lasted between ten and twenty minutes and were conducted individually.

All children were presented with 12 addition problems followed by three partitioning problems. This fixed order was chosen so that children could begin with familiar addition problems before progressing to the unfamiliar partitioning problems. Before starting the problems, the interviewer put the materials in front of children in the Physical or Pictorial conditions. The physical representation consisted of 20 randomly placed black *Unifix* cubes (2cm plastic cubes that can be adjoined linearly - Figure 2.1a). The pictorial representations consisted of 20 grey squares randomly located on a sheet of laminated paper (Figure 2.1b). A marker pen and board rubber was also provided.

7 The interviewer for all studies reported in this thesis was the Doctoral candidate and a qualified infant teacher

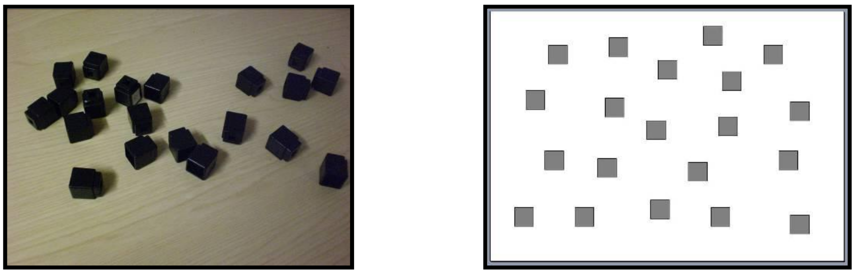

Figure 2.1: a) Physical and b) Pictorial materials used in Study 1

2.2.3.1 Addition problems

The addition problems were presented using laminated cards. Each card had a number problem written in the format 'a + b =__' using black size 72 Ariel font. Before reading out the sum, the interviewer would present the materials and then ask children to use *"the cubes/the squares/your fingers if they helped."* The questions were presented in the same order and consisted of four blocks. The blocks varied according to two factors: *Total*: total less than 10/total more than 10, and *Addend order*: biggest addend first/biggest addend second. The first six questions had a larger first addend with a total less than 10 for the first three questions and more than 10 for the second. The second six questions had a smaller first addend with a total less than 10 for the first three questions and more than 10 for the second. This fixed order of questions was intended to represent questions of increasing difficulty. The questions and their order are shown in Appendix A.

Children were given ten seconds to answer each question. If there was no answer, the interviewer would ask the child if they he/she were still thinking. Children in the Physical and Pictorial condition were told not to use their fingers. If any children did start using fingers in the Physical or Pictorial condition, the interviewer would remind them not to for now, also reminding them that they could use the cubes/paper if it helped. The problem would end if the child gave an answer, said they were not thinking any more, or on the third wait of 10 seconds. For each problem, the interviewer would record the solution children gave, whether the representation was used, and the strategy used if so. The coding of the strategy is described in the results section.

2.2.3.2 Partitioning problems

The partitioning problems were all characterised in the form of the same vignette, accompanied by an illustration (see Figure 2.2). Children were 'introduced' to a character called Mary, and told that she was going shopping. Children were given an initial demonstration question to ensure understanding. They were shown a picture of three bananas and asked *"can you find all the ways that Mary can put the bananas in the bags?"* There were four acceptable solutions: 0 & 3, 1 & 2, 2 & 1, and 3 & 0. If children independently gave two or more solutions then the interviewer would move to the three assessed partitioning problems: *"well done, see there are different ways Mary can put the fruit in the two bags"*. If children did not identify any solutions, the interviewer would support the child's understanding by pointing to one image of the bananas and then to one bag saying: *"so, one way is there could be one banana is this bag and …"* Children would then identify the second part, with the interviewer pointing to the image of the other two bags if necessary. The interviewer would then ask children *"can you find another way Mary could put the bananas in the bags?"* Again, if children were not able, the interviewer would provide prompts as before for the solution 2 & 1. After children had identified at least two solutions, the interviewer would move onto the three assessed partitioning problems.

Figure 2.2: Supporting illustration

The children were given three partitioning problems; requiring them to partition 5, then 8, and then 10. These were the amounts used by Jones et al (1996) from which this partitioning problem was adapted. For each problem, an image of a new character was presented along with supporting images of objects to partition which were removed after asking the question. Children were introduced to the character and problem context (e.g., character packing shirts to go on holiday with two suitcases) and then asked the partitioning question: "*How many can be in each bag/suitcase*?" Children were then reminded about using the materials for that condition: "*remember to use the cubes/squares/your fingers if they help*".

Children were given a prompt if they did not provide a solution: *"there are 5/8/10 bananas/shirts/ties, try to tell me how many can be in each suitcase/bag."* For pauses after children had identified a first solution, the interviewer would prompt: *"is that all the ways or can you think of any more ways?"* The session ended after two prompts had been given, or if the child said he/she had finished. If a child used non specific words such as *'some'* or *'the rest'* when identifying solutions, the interviewer would prompt by asking *"so how many is 'some'/'the rest'?"*

The interviewer recorded the verbal solutions children gave and whether the representation was used on that problem. Much positive praise was given throughout.

2.3 Results

2.3.1 Addition problems

Children all solved 12 addition problems each. These were made of three problems in each of four blocks. For each problem, there were two dependent variables: Response (correct/incorrect) and Representation Use (used/not used). Children consequently received a score of 0-12 for each of these measures.

In the Physical condition, use of the representation was defined by children pointing to or moving cubes. In the Pictorial condition, Representation Use was judged by children pointing to or marking squares. In the No Materials condition, Representation Use was coded if children showed signs of deliberately extending fingers on either hand. The addition problems were also coded according to the strategy used: *count all, count-on, recall,* and *other*, where *other* was used to refer to strategies not falling into the first three listed (see section 2.3.1.4 for coding description).

2.3.1.1 Correct Scores and Representation Use

The distribution of group data was tested (Kolmogorov-Smirnov) and revealed no significant departures from normality for Addition scores on any of the conditions: Physical ($D(24)=0.17$, $p=ns$); Pictorial ($D(25)=0.12$, $p=ns$); and No Materials ($D(24)=0.17$, $p=ns$). Further tests on Representation Use revealed no significant departure from normality for Physical ($D(24)=0.15$, $p=ns$) and No Materials ($D(24)=0.14$, $p=ns$) conditions. Although tests revealed that the Representation Use for Pictorial condition was non-normal ($D(25)=0.24$, $p<0.05$)- likely attributable to low representation use, multivariate analysis of variance was carried out with Condition (*Physical/Pictorial/No Materials*) as the independent factor and Addition score and Representation Use as dependent variables. Age and BAS numerical score were entered as covariates. Analysis revealed no significant main effect for Condition (Physical/Pictorial/No Materials) for correct scores ($F(2,68)=0.32$, $p=ns$). Age was not significantly related to correct score ($F(1,68)=0.18$, $p=ns$), but BAS number score was ($F(1,68)=46.07$, $p<0.001$).

For Representation Use (see Table 2.2), analysis revealed a significant main effect for Condition ($F(2,68)=3.65$, $p<0.05$), Post hoc Bonferoni tests (at $p<0.05$) were conducted to explore these effects further. Pairwise comparison revealed a significant difference between the Physical and Pictorial conditions. Neither Age ($F(1,68)=0.46$, $p=ns$) nor BAS number score ($F(1,68)=0.87$, $p=ns$) were significantly related to Representation Use.

Differences between Correct scores and Representation Use for the different subset addition problems were then analysed using Repeated Measures analysis of variance with problem type as a within subject factor and Condition as a between. As expected, there were significant differences in children's addition scores for different subsets ($F(3,210)=37.22$, $p<0.001$). Post hoc Bonferoni tests (at $p<0.05$) were conducted to explore these effects further. Pairwise comparisons revealed significant differences between each problem type. The order of difficulty is illustrated in the means for each problem shown in Table 2.1. There was no significant interaction effects found between Problem type and Condition. Repeated measures analysis of variance was also conducted on Representation Use but revealed no differences between problem types ($F(3,210)=1.99$, $p=ns$).

Table 2.1: Group mean (standard deviation) for Correct scores (out of 3) for each block of Addition problems

	Addition Scores by problem type				
Condition	1st addend largest. Sum <10	1st addend largest. Sum >10	2nd addend largest. Sum <10	2nd addend largest. Sum >10	Total score
Fingers	2.50 (0.83)	1.67 (1.24)	2.17 (1.01)	1.17 (1.24)	7.37 (3.42)
Paper	2.72 (0.74)	1.72 (1.06)	1.96 (1.10)	1.08 (1.29)	8.24 (2.99)
Physical	2.58 (0.78)	1.71 (1.20)	2.12 (1.15)	1.67 (1.13)	7.42 (3.66)
Total	2.60 (0.78)	1.70 (1.15)	2.08 (1.08)	1.30 (1.123)	7.68 (3.34)

Table 2.2: Group mean (standard deviation) for Representation Use (out of 3) for each Problem Type in each block of Addition problems

	Representation Use by problem type				
Condition	1st addend largest. Sum <10	1st addend largest. Sum >10	2nd addend largest. Sum <10	2nd addend largest. Sum >10	Total use
Fingers	1.38 (1.28)	1.33 (1.27)	1.29 (1.30)	1.29 (1.23)	5.29 (3.84)
Paper	0.56 (0.96)	0.84 (1.07)	0.68 (1.03)	1.04 (1.34)	3.12 (3.67)
Physical	1.29 (1.33)	1.58 (1.32)	1.42 (1.32)	1.83 (1.37)	6.12 (4.50)
Total	1.07 (1.24)	1.25 (1.25)	1.12 (1.25)	1.38 (1.34)	4.82 (4.16)

2.3.1.2 Relationship between Addition Score and Representation Use

Analyses were carried out to examine whether children in each condition identified a greater proportion of addition problems as correct when they used the representation than when they did not use the representation. For each child, a proportional score was calculated for Representation Use (Correct with representation / Correct and Incorrect with representation) and for No Representation Use (Correct without representation / Correct and Incorrect without representation). As some children used or did not use the representation throughout the 12 problems, this generated some missing data in each condition, however, this was generally low (< 6), apart from in the Paper condition where 9 children did not use the representation[8]. Median and interquartile ranges for proportional scores are illustrated in Table 2.3.

Proportional scores in each condition were then examined using non-parametric within subjects tests (Wilcoxon) and revealed a significant difference in the No Materials condition ($Z=-3.01$, $p<0.005$) but not in the Physical ($Z=-1.26$. p=ns) or Pictorial ($Z=-0.58$. p=ns). In other words, children identified a significantly higher proportion of correct scores when using their fingers than when not.

Table 2.3: Medians (IQR) for the proportion of solutions correct when Representation Used or not Used

	Representation Used	Representation Not Used
No Materials (fingers)	0.80 (0.64, 1.0) n=20	0.54 (0.06, 0.89) n=22
Pictorial	0.71 (0.34, 1.0) n=16	0.71 (0.30, 1.00) n=24
Physical	0.75 (0.53, 0.95) n=21	0.68 (0.43, 0.91) n=18

2.3.1.4 Addition Strategies

Identifying children's strategies was relatively simple when children were using representations as it was possible to see which addends of the problem children had externalised and counted. Consequently, a coding scheme was created to code strategies

[8] As differences were not significant, this discrepancy in sample sizes was not analysed further

in the problems when children used the representation. The coding scheme simplified counting behaviours into four categories, intended to reflect addition strategies (see Fuson, 1992b). The coding scheme is shown in Table 2.5. *Count-all* describes children who counted out both addends of the problem. *Count-on* refers to when children begin counting-on one addend from another. As this is computationally demanding when the second addend is large, children develop flexible strategies for manipulating the problem – counting-on the first addend from the second or decomposing and recomposing the problem (e.g., counting-on 4 from 10 when given the problem 9 + 5). Although children are still counting-on, these more advanced strategies were coded separately as *more developed counting-on*. The most efficient strategy of recall was not part of the coding system as children would not use the representation for this strategy. Any use of the representation that did not fall into the above categories was coded as *other* and typically reflected where children were confused and identified an unrelated solution.

Table 2.4: Coding scheme for Addition strategies

Strategy	Behaviour
Count-all	Children count out both addends and then count all
Count-on the second addend	Children count out the second addend and then count on from the first
More developed counting-on	Children count out the smaller first addend and count on from larger second addend or children count out amount that shows decomposition of problem using decade structure (e.g., counting on 4 from 10 for 9 + 5)
Other	Children use the representation unsuccessfully and with no clear strategy from above

The coding scheme used is a relatively crude measure of strategies used by children. It does, however, illustrate differences between conditions for strategy use. For the More developed counting-on strategy, whilst 11 children used this strategy in the No Materials condition, only one children used this in the Physical and Pictorial conditions. Similarly, whilst 13 children used the Count-on strategy in the No Materials condition, no child used this in the Physical condition. Interestingly, 5 children used this strategy in the Pictorial condition. The Count-all strategy was used by many more children, 16 in the No Materials, 12 in the Pictorial and 19 in the Physical condition. Consequently, a non-parametric between subjects analysis (Kruskal-Wallis) was carried and revealed a significant difference in the number solutions children used this strategy between conditions ($\chi^2(2)=9.75$, $p<0.01$). Mann-Whitney showed that children identified significantly more Count-all solutions in the Physical condition (Mdn=5) than both the No Materials (Mdn=1) ($U=179.50$, $Z=-2.27$, $p<0.05$) and Pictorial condition (Mdn=0) ($U=160.50$, $Z=-2.87$, $p<0.005$).

2.3.2 Partitioning Problems

2.3.2.1 Coding

- *Correct Scores*

Children solved three partitioning problems: requiring them to partition 5, 8 and 10 respectively. Consequently, the total number of partitions was different for each problem. Rather than convert scores to percentages that would generate misleading differences (e.g., children identifying one solution would receive a different score in each problem), it was decided that it was more appropriate to apply a crude coding system that would allow comparisons between problems. The coding system converted scores to an ordinal scale of 0 to 3. The levels of this coding system were designed to reflect the stages of ability identified in the section 2.1.3.

In this coding system, children received a score of zero if they identified no solutions and 1 if they identified a single solution. Two levels were assigned for children identifying multiple solutions. Children scored 2 if they identified less than half the solutions and 3 if they identified more than half. This distinction was not so much reflective of conceptual competency as of the efficiency of children's strategies for identifying the majority of solutions. Identifying the majority of solutions suggests that children are more aware of the problem space, and importantly means that children have identified at least one set of solutions that are commutative (e.g., 2 & 5 and 5 & 2) although these may not necessarily be identified in succession.

Therefore, with a score of 0-3 for each problem, children were able to receive a total score between 0-9 for the three partitioning problems.

- *Use of Representation*

Although children were able to identify multiple solutions for each partitioning problem, it was difficult to identify if and how the representation was used for each solution. Moreover, children may have manipulated the representation for an initial solution, and simply then used the representation as a visual support to help identify further solutions. It was therefore decided to simply code each problem according to whether the representation was used at least once or not. Consequently, children were given a total score of 0-3 for Representation Use for all three partitioning problems.

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2.3.2.2 Correct Scores and Representation Use

The distribution of group data was tested (Kolmogorov-Smirnov) and revealed no significant departures from normality for Partitioning scores on any of the conditions: Physical ($D(24)=0.05$, $p=ns$); Pictorial ($D(25)=0.14$, $p=ns$); and No Materials ($D(24)=0.06$, $p=ns$). In contrast, further tests revealed that the data for Representation Use in each condition was significantly non-normal: Physical ($D(24)=0.38$, $p<0.01$); Pictorial ($D(25)=0.51$, $p<0.01$); and No Materials ($D(24)=0.49$, $p<0.01$). Although Representation Use for partitioning problems did not meet assumptions of normality, parametric analyses are reported as non-parametric tests revealed differences in the same direction and effect size.

Multivariate analysis of variance was carried out on Correct scores and Representation Use as dependent variables, Condition as an independent factor and Age and BAS Number score as covariates. Analysis revealed no significant main effect for Conditions for Partitioning scores ($F(2,68)=1.85$, p=ns). Age was not significantly related to Addition score ($F(1,68)=0.35$, p=ns), but BAS number score was ($F(1,68)=35.85$, $p<0.001$).

For Representation Use, analysis revealed a significant main effect for Condition ($F(2,68)=3.46$, $p<0.05$). Post hoc Bonferoni tests (at $p<0.05$) were conducted to explore further these effects. Pairwise comparison revealed a significant difference between the between Physical and No Materials conditions and Physical and Pictorial conditions[10]. Neither Age ($F(1,68)=1.26$, p=ns) nor BAS number score ($F(1,68)=0.75$, p=ns) were significantly related to Representation Use.

Differences between Correct scores and Representation Use for the three different partitioning problems (partitioning 5, 8 and 10) were then analysed using a mixed design Analysis of Variance with Partitioning problem (5, 8, 10) as a within subjects factor and Condition as between. Significant differences were found between the scores on the three problems (Partitioning 5 (M=1.92, SD=0.89); Partitioning 8 (M=1.42, SD=1.08); Partitioning 10 (M=1.53, SD=1.09), but there were no significant interaction effects found between the three problems and Condition. Repeated measures analysis of variance was also conducted on Representation Use which revealed no differences between problem types ($F(3,210)=2.37$, p=ns). Considering the small use of

[10] It should be highlighted that this significant difference is attributable to the almost lack of use of representations in the Paper and No Materials (fingers) conditions compared with 9 children using materials in the Physical condition.

representations in each condition, no further analysis was carried out on the relationship between Correct scores and Representation Use for Partitioning problems.

2.4 Discussion

It was predicted in this study that children who had access to physical representations would solve more addition and partitioning problems than children who had access to paper or their simply their fingers. This was not found. There were no significant differences between the three conditions for the number of correct solutions on both types of problems. In itself, this finding cannot lead one to conclude that physical representations were not supportive; they simply offered no advantage over pictorial materials or fingers. However, the data generated in this study does provide a window onto how the materials were used and whether children would have been more successful had they been encouraged to use the materials more.

2.4.1 Addition problems

It was found that an addition problem was more likely to be correct if children used their fingers in the No Materials condition than when they did not. There was no such effect found in the two material conditions. However, this relationship does not show causation. There are at least three possible interpretations: that using fingers helped children identify more solutions correctly, that children who were more able to answer a solution correctly were more likely to use their fingers or that another factor affected both these variables leading to a relationship between them. The first explanation was predicted by this study: that using external representations would help children identify more solutions correctly.

However, if this were so we would expect that children might do better than when they did not use representations as much. Instead, it was found that children in the Pictorial condition were less likely to use the materials, were not that successful when they did use them, and yet identified the same number of correct addition solutions as children in the Physical and No Materials conditions. Indeed, more solutions in the Pictorial condition were solved with no materials than in other conditions. This finding suggests that children's use of materials was attributable to their understanding and motivation to use the materials rather than because they were needed to solve the problem. Children used physical materials because they were more motivated to do so, not because they were more helpful. Indeed, this explanation helps explain why physical materials were just as likely to be used on simple addition problem as the more difficult ones and why Representation Use was not related to numerical ability. Contrary to predictions, children did not use the materials to help when they lacked adequate mental strategies. Unfortunately, it was difficult to identify from the measures taken in this study why some children chose to use the materials whilst others did not.

Children were less likely to use the pictorial than physical materials. There are several possibilities for this. Children may simply have been confused by how they were meant to use the materials since, unlike fingers and cubes, children are able to annotate paper in many ways. This was indeed demonstrated. Several children for example chose to write out the problem using numerals or re-represent the problem as shown in Figures 2.3a and 2.3b. Figure 2.3a also highlights how the paper allows children to annotate numerical symbols as well as provide a record of solutions. One child did this as a way to remember the problem asked. Another reason children may have been more reluctant to use the paper was because annotating paper is more time consuming and demanding in terms of fine motor control than manipulating objects.

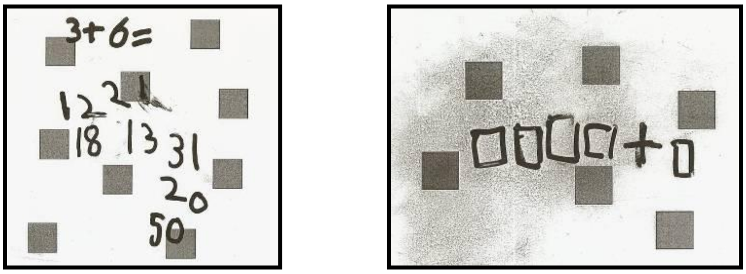

Figure 2.3: a) Annotating numerals and b) Re-representing the problem

When using pictorial materials, most children did so as intended, marking off squares to support counting, and did not seem to have difficulty in doing so. However, observations of children's annotations showed that a key difficulty for children was keeping track of which objects had been counted. In the Physical condition, children tended to manipulate objects to help keep track of counting by creating a linear configuration (which provides children with a means to track which objects by tagging objects one by one in a single direction), or by moving objects when they had been counted. In contrast, although children could mark squares to show they had been counted, the random arrangement meant that children often had difficulty in keeping track of which objects they had already counted and which still remained. Indeed, children often created a line to help keep track as shown in Figure 2.4. Children's fingers are linearly arranged, clearly thereby allowing them to keep track of counting by extending fingers one by one in a single direction. If being able to create a linear configuration does present an advantage for counting, this may be a possible factor negatively affecting children's performance using paper in studies by Martin & Schwartz (2005) where children were presented with random configurations.

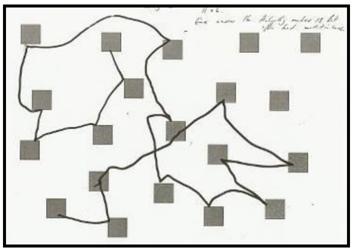

Figure 2.4: Drawing a line through squares to monitor those counted

Generally, children did not use the representations as much as predicted, especially for the more difficult problems. A key reason may be the ordering of problems: by presenting the easiest problem first (which nearly all children identified correctly), the study was designed in a way that the representations were not as necessary at the start. It is possible that by presenting more difficult problems initially, children may have begun by using the representations and this initial use might have prompted greater use for later problems.

Many children did use the objects for more difficult problems, although the findings suggest that this does not confer an advantage. Examination of the strategies children used with the materials helps explain this finding. Coding of the strategies was relatively crude, yet highlighted how children would use a count-all strategy when using the physical materials. The fact that children in the No Materials (and Pictorial) condition tended to use more developed count-on strategies suggests that the physical materials fostered the less developed count-all strategy even though children were able to use more developed strategies. Indeed, although strategy use without materials was not measured, it is likely that children not using materials used more developed strategies as the count-all strategy would be extremely demanding to carry out mentally.

The count-all strategy is quite a time consuming procedure. For example, for the problem 7 + 12, children must count out each of these addends and then recount the total. Consequently, children need to count objects 38 times (7 then 12 then 19). This is time consuming and prone to count errors. An important question then is why children did not use the materials more efficiently (e.g., to support counting-on).

A key reason for children using the objects to count-all may have been because this was their experience of using the materials. In contrast, they had had more experience using their fingers for more advanced counting-on strategies. Indeed, the teachers of the school believed that the children did not have any experience in using manipulatives to count-on. Instead, children had used the number line; which may explain why several children used the squares in the Pictorial condition to count-on. Another reason that children may have counted-all with the objects was because they were able to do so. In a study by Muldoon, Lewis and Towse (2005), it was shown that children will tend to count objects if they are there rather than infer numerical relationships. Indeed, it is telling that in an intervention by Secada, Fuson and Hall (1983), when children were presented with two sets of dots and asked how many, they covered up the first set in order to assess if children could count-on. Although it is possible to count-all using fingers, children are limited by not being able to simultaneously represent addends with a sum greater than 10 which they can then count. This constraint may possibly foster the count-on strategy. They may also have had more experience counting-on using fingers.

With respect to addition problems therefore, the findings from this paper would suggest that providing children with physical materials does not confer an advantage in simple addition problems. Although less able children may be more likely to use the materials, use does not lead to more accurate solutions. Indeed, the findings suggest that there is a danger that use of materials actually reduces the likelihood that children will use more advanced count-on strategies. What cannot be deduced from this study is whether using objects for addition problems helps children's develop their numerical understanding that may later help them use more advanced strategies: for example, by helping them understanding how numbers can be decomposed and recomposed.

2.4.2 Partitioning problems

In the addition problems, children had to count out the representations for each problem. Therefore, the demands of using the representation for each problem were arguably relatively high, especially if children had an efficient mental strategy available. In contrast, in the partitioning problem, children only needed to count out the initial amount once and could then use this to help identify multiple solutions. However, use of materials was even less than for the addition problems. One key reason may be ordering effects: partitioning problems always followed addition problems and children's use might have been affected by experiences with the addition problems. However, it is possible to identify other reasons why materials tended not to be used or failed to provide an advantage when they were used.

One reason why children may have chosen not to use the materials is because the initial partition amount (5) was small and was relatively easy to partition mentally. In contrast, the initial demand of counting out five objects and then using these to count an answer may be comparatively greater. This may also have been the case for partitioning 10: where children had prior experience in class with number combinations to 10, recall of at least one combination would be easier than starting to count out 10 objects. Indeed, children identified more solutions partitioning 10 than they did for 8. However, although children showed they understood the problems (only six children did not identify a solution), only 20 out of the 73 children got more than half the solutions when partitioning 5 (and less for larger numbers). It was predicted that physical objects would help children identify multiple solutions and possibly foster the use of an efficient strategy for keeping track of the different solutions. However, children may not have known that the physical objects could provide this support, or may have had relatively poor planning skills (see Ellis & Siegler, 1997) with which to identify how an initial investment of counting out objects to partition would reduce the demands of identifying each solution. Of the children who did use materials to count out the initial amount; several seemed to forget the task demands (they simply counted out the partitioning amount) or treated the problem differently – as a question about finding half the total. Unfortunately, as only 9 out of 24 children used the materials at least once in the Physical condition and no video data was captured, it is difficult to draw any firm conclusions about how the materials were used. Nevertheless, it would be interesting to see if, and how, children used the materials if the initial amount was provided. Indeed, in Martin and Schwartz's (2005) study where children used objects and paper for fraction problems, the initial amount was provided. Furthermore, as the partitioning problem in this study involved identifying multiple solutions from the initial amount; it would be relatively pragmatic to provide this amount in a classroom context. Indeed, some materials such as the bead string in Figure 2.5 are designed for children to explore how to partition a given amount (typically ten).

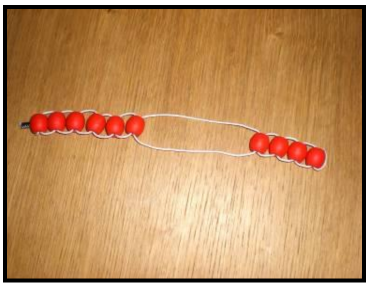

Figure 2.5: Bead string with 10 objects

It is also possible that the problems could have been presented differently in order to facilitate the use of materials. Indeed, in a typical classroom context, children might usually be given at least a demonstration by the class teacher before using materials in a task. In order to assess children's tendency to choose materials to support problem solving, no demonstration was given; it is hence possible that this actually discouraged the use of external representations in all conditions.

2.4.3 Summary

In this study, it was predicted that children in the Physical condition would identify more correct addition and partitioning solutions because the representation would provide a means to support counting when children lacked adequate mental strategies and would provide a means to identify solutions and control the problem space in the partitioning problems. These predictions were not supported. However, it is possible to use the findings to draw several conclusions about the role of physical materials in numerical tasks. Firstly, when evaluating effectiveness, it is important to consider relative performance with fingers as a representation. Fingers provide a linear representation that can be manipulated. Although we only have ten fingers, more advanced strategies can reduce count amounts to smaller amounts, and it is possible that this constraint may foster such strategies instead of needing to provide multiple physical objects. A second conclusion is that it cannot be assumed that less able children will be more inclined to use materials. Indeed, it seems that children may be more likely to externalise if they know how to solve the problem. Less able children may therefore need explicit prompts and demonstration in using the materials. This point raises a key issue: namely that children's use and success of using materials will greatly depend on the context in which they are presented. In this study, children were assessed in a relatively unfamiliar context and it is unclear how previous experiences affected their interpretations of why certain materials were presented in the way they were. This may be particularly important for the pictorial materials that are not only less familiar but present a range of ways they can be annotated. Clearer instructions on how to use the materials may be important to evaluate this form of representation, although it does highlight how this medium allows children to construct their own representations – an important means to externalise thinking (Cox, 1999). A further conclusion is that children may be inclined to count all objects for quantities in a question using physical materials. This may be helpful when starting to learn to add; supporting the count-all strategy (helping to explain the advantage found by Martin, et al (2007) for physical objects over paper in addition problem with younger children), but less productive when wishing to encourage the development of more efficient counting procedures.

An important final conclusion from this study is that when considering the demands of using physical objects in numerical problems, it is important to consider the demands of initially counting out amounts. Although counting out objects may ultimately be cognitively beneficial, children may lack sufficient understanding and planning ability to make this initial investment in time and effort. For example, in the partitioning problem, it would be interesting to examine differences between representations if children were presented with the initial amount to partition. Presenting materials to children in this way may greatly influence how they are subsequently used by the children.

Chapter 3

The effect of physical representations on children's partitioning strategies - Study 2

3.1 Introduction

Study 1 examined whether physical materials would help children solve two types of numerical problem: addition and partitioning. The children were not given instructions in how to use the materials but simply provided with the materials and asked to use them 'if they helped'. Contrary to predictions, it was found that the children did not use the materials to support them when they were unable to solve the problems mentally. Moreover, not only was there no relationship found between ability and Representation Use, but it appeared that physical materials often fostered the use of less developed strategies.

Study 1 also helped identify a key reason why physical objects may not have conferred the predicted advantage. The initial demands of counting out addends in the addition problem or the initial amount to partition in the partitioning problem were relatively high compared to attempting the problem mentally. Indeed, an individual's preference to solve problems mentally when a more accurate solution might be obtained using external materials was discussed by Gray and Fu (2004). In this paper, the authors examined how individuals' strategies rely on memory or action with an interface when the costs of using these are manipulated. It was found that constraints in accessing information from the external interface led to individuals' reliance on memory strategies even when the absolute difference in perceptual-motor versus memory retrieval effort was small, and even when relying on memory led to a higher error rate and lower performance.

The advantages of physical materials were predicted to be greater in the partitioning problems for the following reason: having made the initial investment of counting the whole, children could manipulate the representation easily to create parts that they could then count to identify a solution. Furthermore, it would be possible to use this external representation to help identify all the solutions more systematically. Unfortunately, the physical materials were not used on about three quarters of problems. One reason for this may have been children's lack of experience in using the materials for this type of problem. Indeed, several children who did use the materials seemed to confuse the task demands once they had counted out the initial amount to partition – e.g., by simply identifying this initial amount verbally as 'the' answer. The initial requirement to count out objects may therefore not only have deterred their use for many children but also compromised the potential for the materials to support children when they were used.

Study 1 was in many ways a preliminary study examining how children would use materials to support problem solving in two different types of numerical problem. The study highlighted how the demands of counting out the initial amount in the partitioning task presented a greater limitation than was predicted. Study 2 therefore was intended to re-examine the role of physical materials in this task when the initial task demands were reduced. By providing children with a demonstration of how they might use the materials and providing them with the initial amount to partition, it was intended to re-evaluate the potential of physical objects to support young children's partitioning strategies. However, it is not clear exactly *how* the materials would influence children's strategies. In order to predict the effect of physical materials on children's partitioning strategies, it is necessary to examine in more detail the demands of the partitioning task and how these are changed with the introduction of physical representations.

3.1.1 Partitioning task demands

In the partitioning problem, children are given a specific problem within a story context in which the aim is to identify all the different combinations of two parts (P1 and P2) for a given whole (W). For each valid solution, these parts combine to make the whole: P1 + P2 = W. As P1 or P2 can equal zero there are a total of W + 1 solutions. For example, when partitioning the amount 3 into two partitions, there are four solutions (3 & 0, 2 & 1, 1 & 2, 0 & 3). The children's task is therefore to identify valid numerical values for P1 and P2, to then identify more solutions ensuring that the value of P1 and P2 are different each time (keeping track of what solutions have been given), and to continue so that all possible values of P1 and P2 have been identified (keeping track of solutions left to identify). There are at least five identifiable strategies for how a child might identify solutions mentally:

1) Identify P1 such that P1 ≤ W. Then identify P2 through approximation
2) Identify P1 such that P1 ≤ W. Then calculate P2 by counting on down from W or up to W
3) Recall P1 and P2 of previous solution and reverse such that P1=P2 and P2=P1 (*commutative*)

4) Recall P1 and P2 of previous solution and change values by one (P1+/-1, P2 +/-1) maintaining P1 + P2 = W (*compensation*)

5) Recall solution from declarative memory

Each of these strategies listed above can be evaluated with respect to their costs and benefits in time and effort. For example, for strategy 2, it may be relatively easy to identify one part as being one less than the whole (e.g., 7 when partitioning 8) then count up 1 to get the second part. However, this strategy becomes more difficult when the count amount is larger and, moreover, a method is needed to keep track of the solutions already given. Strategy 3 (*commutative)* is relatively undemanding procedurally, requiring children to simply hold the numerical values of the previous solution in memory. This strategy does however require an understanding that reversal of the parts does not affect the whole (commutativity) and is limited because a different strategy is needed to identify other pairs of solutions. Strategy 5 (*recall*) is quick and relatively undemanding once combinations have been committed to declarative memory. However, it is unlikely that young children have been exposed to sufficient problems to have achieved this, and even less likely they are able to recall solutions in such a way that they are able to ensure all solutions are given.

Arguably, it is strategy 4 (*compensation*) that provides the most efficient way to identify solutions whilst keeping track of the problem space. By starting at one 'extreme' (all in one part and none in the other, and working incrementally by identifying parts that are one different from the previous), it is possible to identify successive solutions whilst monitoring what solutions have been given and what solutions are left to give. However, this strategy does present certain procedural and conceptual demands. On a procedural level, children must remember the previous solution, and then mentally subtract one from one part and add one to the other part. On a conceptual level, children need to understand that numbers can be decomposed in this way. This understanding reflects the concept of *additive composition*, in which, more specifically, Irwin (1996) actually refers to one important principle of additive composition: *compensation* – an understanding that taking from one part and adding this to the other leaves the whole unchanged.

Strategies 3 and 4 are of particular interest in this study because both require children to relate one solution to the previous. The relationships for each strategy can be defined as follows:

- *Strategy 3 (commutative)*

  If P1 + P2 = W then P2 + P1 = W

  E.g., If '6 & 1' is a solution then '1 & 6' is also a solution

- *Strategy 4 (compensation)*

  If P1 + P2 = W then (P1 + x) + (P2 – x) = W

  E.g., If '5 & 4' is a solution then '6 & 3' and '4 & 5' are also solutions[11]

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[11] For this example, x=1. In other words, each part is only one less or one more than the previous. Henceforth, when referring to the '*compensation* strategy', x=1. This is because a) this amount is considered small enough to be mentally computed by children and b) there is no logical reason to apply a strategy where x>1.

These strategies reflect important quantitative relations between parts and the whole which play an important role in children's development of number concepts (Nunes et al., 2007), although it is not clear how much understanding is required by children in order to apply these strategies in the partitioning problem. Nevertheless, research highlighting the iterative relationship between children's conceptual and procedural knowledge (e.g., Rittle-Johnson, Siegler, & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001) suggests that developing the use of these strategies may itself help develop children's conceptual knowledge in this area.

3.1.2 The effect of physical material on partitioning task demands

When solving partitioning problems mentally, children initially have to remember the amount to partition (W). If children are given a physical instantiation of this amount, these demands can be externalised, and when they want to calculate a solution, such as in strategy 2, they are able to count the external representations of each part. To help clarify to which part an object belongs, children may choose to partition the objects physically, thereby creating two spatial collections – as shown in Figure 3.1. Indeed, the initial demands of identifying the first part (such that $P1 \leq W$) can be offloaded; simply by grouping objects physically, children can identify P1 by enumerating one part (e.g., by counting/subitising). Moreover, in order to calculate the second part, children now need only to enumerate the other group of objects.

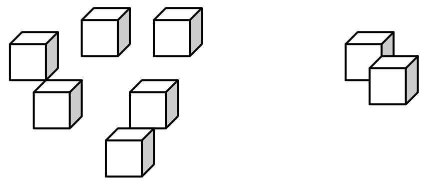

Figure 3.1: Spatial partitioning of 8 objects

The demands of strategy 1 and 2 can be supported using physical objects: children are able to identify a solution simply by physically acting on the objects, creating two spatial groups, and then enumerating each of these groups. Although children may find it easier to recall previous spatial configurations than numerical values, it is important to highlight how the physical materials do not provide any way of keeping record of which solutions have and have not been given previously. Therefore, similarly to the arguments made previously, the problem with strategies 1 and 2 is that they do not provide a means to monitor the problem space. Indeed, children could simply keep rearranging the cubes and identifying solutions without knowing if these had or had not been identified previously.

It is also possible that physical objects might support the use of strategies 3 and 4. Several authors have described how children may hold an understanding of logical relations in the physical prior to numerical domain (Herscovics, 1996; Kamii, Lewis, & Kirkland, 2001; Piaget, 1965; L. B. Resnick, 1992a) and it is possible that this is reflected by children's ability to use the physical representation to relate solutions. For example, children may recognise that changing the order of two parts is another way to present the whole, even if they have yet to develop an understanding of what impact this will have on the numerical total (cf. understanding of conservation – see Chapter 1). They may also recognise that moving only one object from one pile to another is not only a way to create a unique configuration, but is a sustainable strategy that can be repeated to identify further solutions. In other words, children may carry out strategies physically before having the conceptual or procedural knowledge of how to apply them numerically. Once children have enacted the strategy, they can they enumerate the resultant solution.

The process of acting and then interpreting the representation is described by Martin and Schwartz (2005) in their theory of Physically Distributed Learning. According to Martin and Schwartz, physical changes can help individuals reinterpret the environment, leading to learning. Applying this theory to this task, it is possible that physical objects foster the use of strategies that relate successive solutions and, by doing so, develop children's understanding in this domain. Indeed, it is possible that children's experience of identifying solutions using these strategies with physical objects leads them to apply these strategies at a later stage in the absence of physical support. Conversely though, it might also be argued that using physical objects will actually decrease the likelihood that children use strategies that relate solutions. This is because the computational demands of counting out unrelated solutions (strategy 2) are greatly reduced. In other words, being able to easily create new groupings and count the resultant configuration will decrease the likelihood that children will develop strategies to help identify new solutions and monitor the problem space.

Two possibilities have been put forward about the effect of physical materials on children's strategies in the partitioning task; with important differences for how the materials might support learning. By examining what solutions children identify when using objects, it may be possible to deduce what strategies they are using, and hence determine which of these two possibilities is more accurate.

3.1.3 Study aims and predictions

The aim of Study 2 was to determine the effect of physical objects on children's partitioning strategies by comparing children's performance using cubes with no materials. It was predicted that by providing the initial amount of cubes, the procedural task demands would be reduced and children would identify more correct solutions with cubes than without. However, the study also aimed to compare the strategies used, which was achieved by developing a means to code the solutions given. By comparing the type of solutions given using cubes or using no materials, this study tested whether physically manipulating representations encouraged or discouraged the use of strategies that relate successive solutions (*compensation* and *commutative* strategies).

3.2 Method

3.2.1 Design

A within subjects design was used with Condition (Physical/No Materials) as the within subjects independent variable. The primary dependent variable was the number of correct solutions. These solutions were then coded according to a coding scheme developed in this study, thereby creating a further dependent measure: frequency of solutions in particular strategy categories.

3.2.2 Participants

Thirty two children took part in this study (17 girls and 15 boys, range 68 to 82 months; M=74.2; SD=3.86 months). Children were from two Year 1 groups in a local school in

Nottingham, and their parents had signed and returned a consent form asking if they would like their child to take part in the study (56% response). This school was chosen from previous research with the university but children had not participated in a previous study in this research. The school is a larger than average primary school, with 345 pupils and situated in a suburb of Nottingham that is recognised as having a high social, educational and economic level. This is reflected in the very small proportion (2%) of children that receive free school meals.

In this study, all but one child had English as their first language and one child was reported as having special needs. These children were competent in the task and were included in the analysis. The school requested that no personal information including individual ability levels be taken.

3.2.3 Materials and Procedure

Sessions took place individually on a table in the corridor outside the class. They were held during lessons when noise levels in this area were acceptably low, and lasted between five and ten minutes. Children were presented with two partitioning problems: partitioning 6 followed by partitioning 7, always in this order. The order of condition (Physical/No Materials) was counterbalanced, changing for each child in turn. The order of children reflected an alphabetic class list, which made it easier for the class teacher to know who was next and was deemed sufficiently randomised for this within subjects design.

As children had only briefly been introduced to the interviewer by the teacher in class, the interviewer spent up to a minute at the start of the session chatting informally to help each child relax. The interviewer would then explain that the purpose of the research was to find out what children find easy and difficult about number questions.

Children were then presented with a partitioning problem which was characterised in the form of a vignette, accompanied by an illustration (see Figure 3.2). Children were introduced to a character called Jon, and told that he had bought some bananas. Children were told that Jon likes to come home and keep the bananas in the two bowls, and that Jon was confused because there were "*so many ways to put the bananas in the bowls*". The interviewer explained that the aim was to try to help Jon by telling him all the different ways he could keep his bananas in the two bowls.

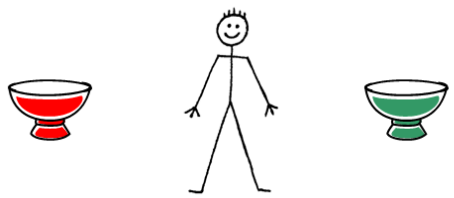

Figure 3.2: Image used in Study 2 to support problem understanding

3.2.4.1 Example problem

Before each of the two partitioning problems in each condition, the interviewer presented an example to help children understand the task demands and what constituted a valid solution. The interviewer would explain: *"One day, Jon bought 3 bananas* [interviewer shows image of 3 bananas]. *Watch how I use [my head/these cubes] to help me find all the ways the three bananas could be in the two bowls."* In the Physical condition the interviewer placed three *Unifix* (2cm plastic cubes) on the table. In the No Materials condition, the interviewer pointed to his head (the teacher of the class had explained how this prompt was used when children were being asked to solve problems mentally).

The interviewer would then identify the four ways to partition three in the following order: 3 & 0, 1 & 2, 2 & 1, and 0 & 3. In the Physical condition, the interviewer would partition the cubes before identifying the solution. Partitioning cubes involved moving the cubes into left and right groups in front of the interviewer. In the No Materials condition, the interviewer would simply point to the corresponding bowls when saying the verbal solutions. In the demonstration, the interviewer would explain that there could be *"three in the red bowl and none in the green", "one in the red bowls and two in the green", "two in the red bowl and ..."* On this third solution, the interviewer would purposefully pause and look at the child to prompt the child to say the solution (two in the green). If the child did not answer, the interviewer would use the image of the bananas and repeat *"two in the red bowl and …"* All children were able to complete this, as well as the final solution which again the interviewer prompted *"and none in the red bowl and …."* (three in the green). The prompts for children to complete the solution were to ensure understanding and for children to practise giving numerical answers for each part.

3.2.4.2 Partitioning problem

After the demonstration problem, the interviewer removed the picture of three bananas but kept the picture of the stick figure and the two bowls. The children were then told that on another day Jon went shopping and bought 6/7 bananas. The order of total amount to partition was the same for all children: 6 followed by 7. Similarly to the example, in the Physical condition, children were presented with the correct total number of cubes to partition, which were placed in a line in front of the child. Children were then asked to use the cubes (use their heads in the No Materials condition) to tell the interviewer all the ways Jon could put the 6/7 bananas in the two bowls. The children were reminded that, for each way, they were to say how many there were altogether in each bowl so that the interviewer could write down their answers. After solving the first partitioning problem, the interviewer would present the example and partitioning problem in the other condition. Condition order was counterbalanced between children.

3.2.4.3 Prompts during problem solving

For all problems, if children did not respond after 10 seconds they were prompted by the interviewer: *"can you think of any ways that Jon can put the 6/7 bananas in the two bags?"* For pauses after children had identified the first solution, the interviewer would prompt saying *"is that all the ways or can you think of any more ways?"* The session would end after two prompts had been given or if the child indicated that he/she had finished. If a child used non specific words such as 'some' or 'the rest' when identifying solutions, the interviewer would prompt by asking *"so how many is 'some'/'the rest'?*

The interviewer wrote down all solutions given by the children so that they could see that their answers were being recorded (and that they were therefore important to the task) although they could not see what was actually being written down. Children would generally say or point to the bowl to which they were referring (e.g., *"three in that one")* but if it was not clear the interviewer would prompt *"three in which bowl?"* The interviewer recorded the left bag as referring to the first part and right as the second.

For several children it was necessary to remind them of the need to identify partitions verbally in the Physical condition by stating the total amount, not to just show the configuration. Although it might be argued that this provided an unfair prompt for this condition, the prompt was a) only required for three children and b) only provided once; if the child created a configuration and then looked to the interviewer, this was taken to mean that the child had created a solution[12].

3.3 Results

3.3.1 Correct solutions

Solutions were initially coded as correct or incorrect. Correct solutions were then further coded as being unique or repeated (see Figure 3.3). A repeated solution was any solution that had been given previously (in the same order of parts). Each child received a score for the number of unique correct solutions identified in each condition. Henceforth, unique correct solutions will simply be referred to as *correct* solutions and repeated correct solutions will be referred to as *repeated* solutions. If a score was incorrect, it did not matter whether it was repeated or not. The distribution of group data was tested (Kolmogorov-Smirnov) and revealed significant departures from normality for scores on the first problem, partitioning 6 ($D(32)=0.17$, $DF=32$, $p<0.05$) although not the second ($D(32)=0.13$, p=ns). A Wilcoxon test was therefore carried out and showed there were no significant differences for correct solutions between the first (Mdn=5) and second problems (Mdn=5) ($Z=-0.70$, p=ns).

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[12] Although not expanded upon in this thesis, it is an interesting point to note that in this situation, the representation provides children with a means to communicate answers to the adult interviewer.

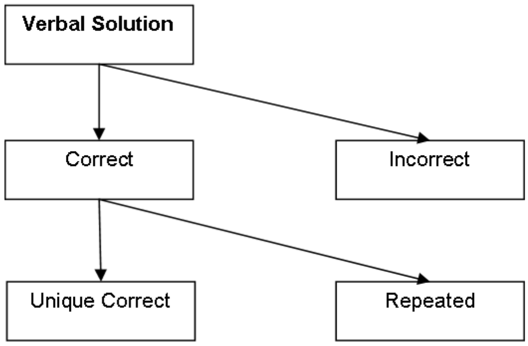

Figure 3.3: Coding of correct solutions

The distribution of group data was tested (Kolmogorov-Smirnov) and revealed no significant departures from normality for scores in the Physical condition (Kolmogorov-Smirnov: $D(32)=0.161$, $p<0.05$). A Wilcoxon test revealed that children identified significantly more correct solutions in the Physical condition (Mdn=6) than the No Materials condition (Mdn=4) ($Z=-4.50$, $p<0.0005$)[13] (see Table 3.1). In addition, the effect size was found to be fairly large ($d=1.09$, $r=0.48$) using Cohen's d for paired samples (Cohen, 1988). Children gave more Incorrect solutions in the No Materials (18 children) than Physical condition (4 children).

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13 In Study 1, scores were coded to compare condition as the total number of solutions possible for partitioning 5, 8 and 10 were substantially different. In this and subsequent studies, the partitioning amounts were more comparable (e.g., 7 and 8), and it was found that analyses using uncoded data revealed differences in the same direction and magnitude. Therefore, the analyses reported henceforth just examined the absolute number of correct solutions.

3.3.2 Strategy

In order to examine differences in the possible strategies used between conditions, a coding scheme was first developed for correct solutions.

3.3.2.1 Coding Scheme

Two key strategies for partitioning were previously identified: *commutative* and *compensation*. A *commutative* strategy was defined as when the parts for a solution are identified by reversing the order of parts of the previous solution. A *compensation* strategy was defined as when a solution is derived by adding one from one part and taking one from the other from the previous solution. It is thereby possible to examine each solution children give (after the first solution) in terms of its relationship to the previous solution and use this to infer strategy. For example, the solution '1 & 6' after '6 & 1' might arguably reflect a *commutative* strategy. Similarly, the solution '2 & 5' after '1 & 6' might reflect a *compensation* strategy.

Clearly, this form of coding allows both type 1 and 2 errors: a solution identified using a strategy might not be coded because children did not actually verbalise the initial solution. Equally a solution might be coded although it only followed the previous by chance. However, as these errors should be equally as likely to occur in each condition, it should be possible to compare conditions to examine any significant differences.

The coding scheme was devised to identify *compensation* or *commutative* strategies. However, for partitioning odd numbers, such as 7, there is a sequence of solutions that falls under both coding descriptions: 3 & 4 following 4 & 3. It is possible in the Physical condition to infer how this solution might have been identified by looking at how objects were moved, although this is less easy in the No Materials condition. It was therefore decided to code all solutions of this type in the same way, and to code this sequence of solutions as *compensation* since observations in the Physical condition suggested that children were more inclined to identify this pattern of solutions by moving one from the previous solution.

It is important to note that a solution that is coded as neither *compensation* nor *commutative* does not mean that children were not relating successive solutions. Indeed a couple of children seemed to apply a combination of *commutative* and *compensation* at the same time (e.g., swapping over and moving one object: e.g., '1 & 6' following '7 & 0'). However, these were less clear and not frequent, and any solution after the first that was not codable as *compensation* or *commutative* was coded as *other*. The coding flow diagram is presented in Figure 3.4.

A final point concerning the coding scheme: although the coding scheme only applied to unique correct solutions (i.e. not repeated), a solution could be coded according to strategy even if the previous one was repeated. However, in line with the coding definitions, a solution would be coded as *other* if the previous one was incorrect.

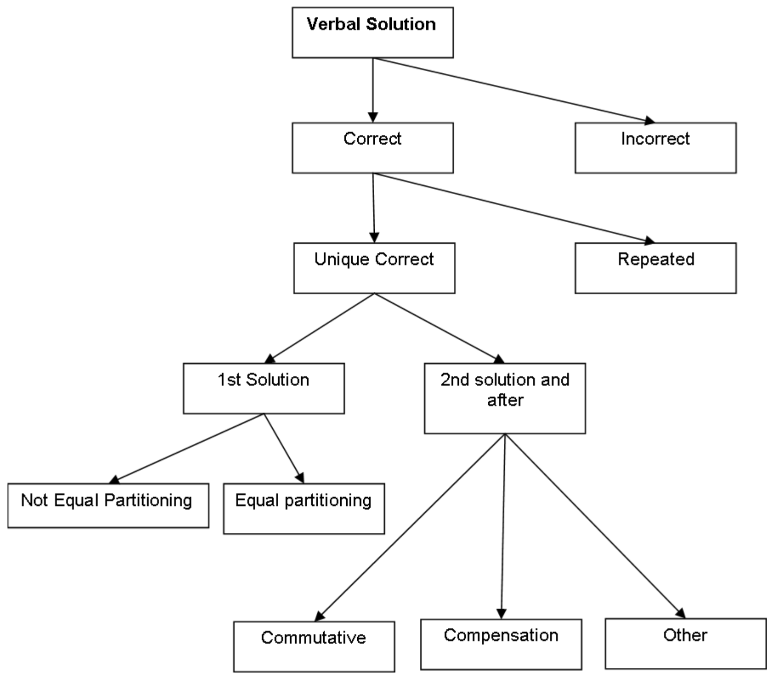

Figure 3.4: Coding of Strategies

3.3.2.2 Differences in strategy use between conditions

Using the coding scheme, it was possible to give each child a score in each condition for the number of *compensation*, *commutative* and *other* solutions given. The maximum number of *commutative* solutions possible for partitioning 6 and 7 was three. The maximum number of *compensation* and other solutions for partitioning 6 was 6, and for partitioning 7 was 7. The median and interquartile scores are shown in Table 3.1. Whilst 19 children identified at least one *commutative* solution in the Physical condition, less than half (10) did so when solving the partitioning problems without materials. Similarly, whilst most children (28) identified at least one *compensation* solution in the Physical condition, only 14 did so in the No Materials condition. Wilcoxon tests[14] showed that children identified significantly more *commutative* solutions ($Z=-2.25$, $p<0.05$) and significantly more *compensation* ($Z=-3.69$, $p<0.01$) solutions in the Physical condition than the No Materials condition (see Figure 3.5). There were no significant differences between conditions for the number of *other* solutions ($Z=-0.39$, p=ns).

Table 3.1: Medians (IQR) for strategy solutions in the Physical and No Materials conditions

	Commutative	Compensation	Other
Physical	1 (0, 2)	1.5 (1, 2.75)	2 (1, 2)
No Materials	0 (0, 1)	0 (0, 1)	1 (0.25, 2)

Although these tests revealed a significantly greater number of *commutative* and *compensation* solutions in the Physical condition, it might be argued that this can be explained by children in the Physical condition simply identified more correct solutions overall (although the difference in *other* solutions was not significant). Indeed, Spearman Rank order correlations revealed significant positive relationships between *compensation* solutions and overall solutions in the No Materials ($\rho=0.465$, $p<0.01$) and Physical

[14] Considering the median scores of zero in the No Materials condition, another way to approach analysis would have been to categorise scores according to whether children identified at least one solution or not, and then carry out paired sampled tests on the binomial distributions. However, Wilcoxon tests will be reported in this thesis as a) this acknowledges the interval data for the majority of children in one of the within subjects conditions and b) significance levels for differences between conditions were unchanged.

conditions (ρ=0.506, $p<0.005$), and similarly, significant positive relationships between *commutative* solutions and overall solutions in the No Materials (ρ=0.606, $p<0.001$) and Physical (ρ=0.471, $p<0.01$) conditions. However, whilst the correlation between *other* solutions and overall solutions was large in the No Materials condition (ρ=0.718, $p<0.001$), this was not significant in the Physical condition (ρ=0.231, p=ns).

Therefore, analysis on the relationship between the number of coded solutions identified and overall solutions supports the prior analysis on differences between conditions for the strategies used. Children identified more solutions overall in the Physical condition and this is reflected in a greater number of *compensation* and *commutative* solutions but not in *other* solutions. In contrast, the greater number of solutions identified in the No Materials condition seemed to reflect a greater number of all strategy solutions. This helps explain why a significantly greater number of *compensation* and *commutative* but not *other* solutions were found in the Physical condition compared to the No Materials condition. The total numbers of solutions identified by children in each condition are illustrated in Figure 3.5.

Further comparisons of strategies between conditions in this thesis focused only on scores for strategy solutions unless analysis accounting for overall solutions affected interpretations of these findings.

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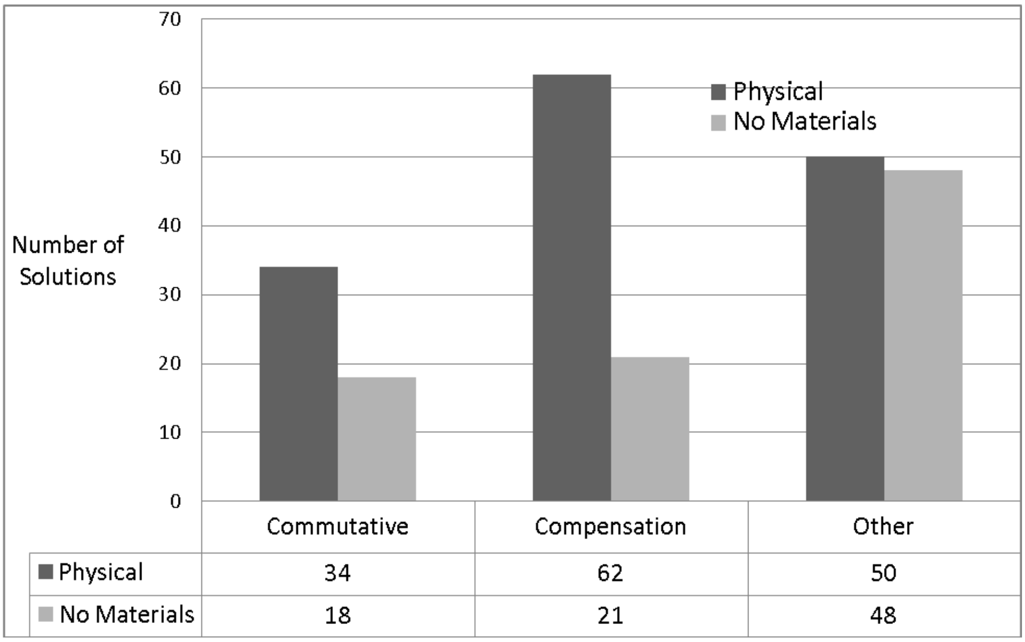


Figure 3.5: Total number of coded strategies identified by children in each condition

3.3.2.3 Initial Solution

The strategies analysed above were for solutions given after the first. However, it was interesting to see differences in the pattern of first solutions given. For many children, the first solutions given for partitioning 6 were 3 & 3. This is half of 6. For partitioning 7, many children identified a solution that was as close to halving as possible: 3 & 4 or 4 & 3. These 'halving' solutions will hence be referred to as 'Equal partitioning'. By coding these solutions, it was possible to examine differences in the number of Equal partitioning solutions between the two conditions. A signed ranked test was carried out to test differences between binomial data for each condition and found significantly more Equal Partitioning solutions in the Physical condition (+ve=18, -ve=4, ties=10, p=0.004).

3.4 Discussion

This study examined the effect of physical representations on children's partitioning strategies. The main prediction was that children would identify more correct solutions using physical materials than without. This prediction was confirmed: children identified significantly more correct solutions with the cubes, and the effect was quite strong.

It is difficult to draw comparisons with Study 1 as the findings from Study 1 led to substantial methodological changes in this study. Importantly, the cubes were used by children on all problems in the Physical condition compared to only 37.5% of problems in the Physical condition in Study 1. Clearly, this could be attributed to the wording differences: children in Study 1 were asked to use the cubes *'if they helped'*; children in this study were simply asked to use the cubes. Children were also provided with a demonstration of how to use the cubes to identify solutions, although care was taken to ensure that the same solutions and order were provided in both conditions. Study 1 helped to identify a key demand in using physical objects to solve this problem: the need to count out the initial amount. This demand was removed in this study by presenting the initial amount. Therefore, by comparing the results of Study 1 and Study 2, it might be possible to draw the tentative conclusion that physical objects can support partitioning, albeit only when children are clear about how to use the materials and have pre-counted materials to reduce this initial task demand.

Study 1 showed that children may use different strategies when using physical objects and these strategies may be less developed than those used without materials. Therefore, an important aim of this current study was to examine what effect physical objects had, if any, on children's partitioning strategies. More precisely, the study examined whether physical objects increased or decreased the use of strategies that related solutions: *commutative* and *compensation* strategies. Two possibilities were discussed for how physical objects might influence children's strategies. On the one hand, physical objects might *reduce* the use of *commutative* and *compensation* solutions because the computational demands of creating and enumerating parts for unrelated solutions are substantially lower than having to calculate unrelated solutions mentally. The motivation to relate solutions when solving problems mentally might therefore be greater in order to reduce computational demands. On the other hand, physical objects might *increase* the incidence of *commutative* and *compensation* strategies by providing an external representation that can be spatially manipulated to create 'related configurations'; i.e. swapping over parts to create a symmetrically opposite configuration and thereby identify a *commutative* solution.

The findings from the study clearly supported the second possibility: physical objects increased the use of *compensation* and *commutative* strategies. Furthermore, analysis demonstrated that the differences in strategies between conditions was not solely attributable to differences in the total number of solutions identified; no differences were found in the number of unrelated (*other*) solutions between conditions and, unlike in the No Materials condition, the number of *other* solutions was not significantly related to the number of overall solutions in the Physical condition .The finding that children identified more *compensation* solutions in the Physical condition has important implications. In contrast with Study 1, where physical materials often resulted in the use of less developed strategies, in this study, physical materials increased the use of a more developed strategy. It has been argued that increasing the use of this more developed *compensation* strategy is beneficial for learning, as the strategy is not only procedurally important in helping children manipulate difficult part-whole problems (e.g., $9 + 4 = 10 + 3$), but reflects one of the numerical 'big ideas' – i.e. that numbers can be broken down in different ways (Baroody et al., 2006). As children identified more solutions using physical materials, the findings of this study offer support for Martin and Schwartz's (2005) theory of Physically

Distributed Learning which describes how actions on the representation can lead to new interpretations – and thereby learning. Indeed, Nunes and Bryant (1996) have described how children who have developed strategies for calculation without materials may still develop ideas about how numbers can be decomposed and recomposed through their experiences with physical materials.

Unfortunately, by only comparing differences in partitioning strategies when using physical materials or no materials, this study is limited in identifying what particular representational affordances of the physical materials were beneficial. Whilst children did physically manipulate the cubes - generating different spatial configuration to identify different solutions, this may not have been necessary. It is possible that children were simply supported by the external representation of quantity in the question. If so, children might have benefited as much from a different representational medium such as a paper.

Although children in this study identified more correct ways to partition a number when they used physical objects, many solutions still seemed to be unrelated to previous solutions, as indicated by the number of solutions coded as *other*. Unlike related solutions, unrelated solutions do not emphasise the relationship between parts and whole. For example, although the solutions 2 & 6 and 4 & 4 still embody the concept of *compensation* (that taking from one part and adding to the other leaves the whole unchanged), it is arguably more difficult for children to see this relationship or apply it as a strategy in comparison to solutions that only differ by one, such as 2 & 6 and 3 & 5.

Physical materials therefore fostered but did not guarantee the use of more efficient strategies. It is not clear whether further use would have led to improvements, but the findings in this study do highlight how the materials seemed to encourage an initial strategy (partitioning objects equally) that does not seem the most efficient way to begin this problem. By partitioning objects equally in the first question (6 into 3 & 3), children could not identify a *commutative* solution, whilst applying a *compensation* strategy from this point would only identify half the solutions. Children would then need to find a way to identify the other half. It is arguably more efficient and more reflective of an expert strategy to begin at one 'extreme' (e.g., 6 & 0) and identifying solutions incrementally to reach the other extreme (0 & 6). It is not clear why children tended toward an initial Equal partitioning solution. Possibly it was identified through recall as children had experiences in working with halves and doubles, but this would predict no differences between conditions as children could recall as easily without materials. It seems therefore that the materials did foster this initial solution. One explanation for this is that children had prior experience in activities in halving groups of objects, another is that the perceptual properties encouraged such a solution – perhaps because it maintains a form of visual symmetry. Nevertheless, although the reasons for this initial strategy are unclear, it is possible that by encouraging children to start differently, for example with an 'extreme' solution of all in one part and none in the other, children would be more successful in this problem.

3.4.1 Summary

This study has shown that children can identify more partitioning solutions using physical materials than with no materials. Furthermore, physical materials can increase the likelihood of using more developed strategies that relate solutions to each other. The materials may have supported children in different ways: by providing a visual and tactile means to enumerate parts from a representation of the whole and/or allowing them to recreate new valid groupings through simple physical actions. Whilst some of these properties are shared with other representations, others are unique to physical representations.

The ability to create new spatial groups with simple physical actions using both hands is an affordance of physical materials, and may indeed have helped children explore the problem space and identify new configurations with minimal demands on the motor system. However, the trade-off from spatially manipulating the physical representation is that no record is left of previous configurations. In other words, the representation provides no means of identifying what solutions have been given previously – and without a record of previous solutions, children have limited means to determine what solutions they have yet to identify. Indeed, despite their relative success in identifying solutions when using materials compared to no materials, children still failed to identify all the solutions. In fact, only three out of 32 children managed to do so using materials. It is possible, that with a record of what solutions they had identified, children would be more apt at deducing what solutions remained.

In contrast to physical materials, using pictorial representations (pen and paper) can provide a record of previous changes to the representation. Children can annotate a solution and thereby create a record of the solution, which they can then use to monitor what solutions have been given and what solutions remain. However, although pictorial materials can provide a visual representation of the whole in the same way as physical materials, it is not possible to transform the spatial position of objects (although marks can be made to indicate such transformations). A question is thereby raised concerning the importance of spatial manipulation for solving partitioning problems. According to Martin and Schwartz (2005), it is physical adaptations (arrangements) that help children to identify new ideas, and indeed children identified new solutions in this problem by creating new spatial groups for objects. However, it could be argued that annotating pictorial materials also provides a visual clue for groupings. Although it may be more demanding in terms of time and fine motor control for young children to annotate paper than to simply move objects, research has shown how increasing the costs of manipulating the external representation can actual lead to more efficient problem solving by encouraging reflection and planning (O'Hara & Payne, 1998).

The current study has shown that physical materials can help children identify more correct partitioning solutions and encourage the use of efficient strategies. However, there is a 'trade-off' in using the materials between the ability to create spatial groups with ease and the ability to create a trace of previous actions.

Chapter 4

Examining the trade-off between spatial manipulation and representational trace for solving partitioning problems - Study 3

4.1 Introduction

It was shown in Study 2 that physical materials helped children to identify correct ways of partitioning a number into two parts. This conclusion may seem unsurprising: by providing an external representation of the whole, the cubes helped offload the demands of calculating each partitioning solution. However, examination of the strategies used, such as identifying incremental solutions or swapping over parts to identify a new solution, suggested that the representational properties of the materials fostered children's strategies for identifying successive solutions. Indeed, it was found that children identified significantly more solutions that were related to the previous solution with the cubes than without. Unfortunately, it was not clear from Study 2 whether other external representations (such as paper) would offer the same benefits.

Chapter one described various properties of physical materials that distinguish them from other representations, for example, they provide tactile information. They can also be manipulated into different spatial configurations; although in so doing they remove all trace of previous representational states. This trade-off helps draw comparisons of physical representations with another representation medium: pictorial representations. Pictorial representations cannot be spatially manipulated; however, as a consequence, changes to the representation (made through annotation) leave a record of previous activity. This record of previous changes to the representation will be referred to as 'representational trace'.

The aim of the current study is to understand some of the advantages and limitations of physical representations by examining this trade off between spatial manipulation and representational trace in solving the partitioning problems. Before predicting differences, these two features are examined in further detail.

4.1.1 Spatial manipulation

Physical objects can be physically manipulated into different spatial configurations. Objects can be placed closer together or further apart. Reflecting Gestalt principles of proximity (see Rock, 1993), objects that are close together tend to be perceived as belonging to the same group under certain circumstances. As discussed in Chapter 2, spatially distinct groups may also supporting counting by providing children with a visual cue of when to stop counting as well as the faster enumeration process of *subitising* groups of objects smaller than about five (see Mandler & Shebo, 1982). Consequently, manipulating cubes may help children to create different numerical groups (with relative ease) and then enumerate these groups.

4.1.1.1 Actions versus Planning

The physical and cognitive demands of manipulating physical objects are quite low. Children understand the physical laws of materials (such as how objects move as connected wholes) from as young as six months (Spelke, 1990). Young children are also generally able to manipulate multiple objects using both hands with relative ease (depending on the size and shape of the objects). In contrast, working with other representations such as paper may be more difficult. Although children of the age focused upon in this research are generally competent at using a pencil to make basic annotations, the fine motor and attentional demands are greater than those required in moving objects. However, it is not quite clear how this 'implementation cost' may affect problem solving. It is possible that, as found with adults, increased costs may foster more 'planful' behaviour (O'Hara & Payne, 1999; Van Nimwegen et al., 2006). In other words, because it is more difficult to annotate paper than move cubes, children may think more about their actions before carrying them out.

Compared to adults, however, children have greater motivational and cognitive difficulties with planning (Ellis & Siegler, 1997). Furthermore, whilst adults may find planning easier in the well-structured puzzle problems used in many studies, children may have much greater difficulty in learning tasks when they have only incipient knowledge. It may therefore be easier for children to act on the representation to support cognition rather than plan before acting. Indeed, the benefits of manipulating representations to support cognition in problem solving have been described in various studies (Anzai & Simon, 1979; D. Kirsh & Maglio, 1994).

- *Physically Distributed Learning (PDL)*

Martin and Schwartz (2005) describe how acting on representations can support ideas, although their work makes a distinction between physical actions that help offload cognition and physical actions that lead to conceptual learning. They make the argument that individuals with incipient knowledge in a domain can physically manipulate the environment, perceptually interpret these changes, and hence develop new ideas in the domain. Martin and Schwartz support their theory by demonstrating how individuals are able to solve more operator fraction problems correctly using physical representations that can be spatially manipulated than static pictorial representations that can only be annotated. Applying PDL to the partitioning task therefore, it might be argued that physically manipulating the cubes helps develop children's ideas about how numbers are decomposed.

The methodological approach used by Martin and Schwartz of comparing a material that could be manipulated spatially (physical) with a static material (pictorial) was integrated into the design of Study 1. The predictions were not supported: physical objects were not found to confer an advantage. However, as previously suggested, the lack of any significant difference between the conditions may have been attributable to the lack of any demonstration by the experimenter and the lack of an initial counted out amount to partition (as was provided by Martin & Schwartz in their own studies). Indeed, Study 2 showed a clear advantage of physical materials over no materials when these limitations were addressed, and demonstrated that physical objects not only helped children identify more ways of partitioning a number, but fostered the use of strategies that related one solution to the previous one. The use of such strategies is important in the partitioning task because they provide a way of identifying unique solutions. In particular, the *compensation* strategy provides a way of identifying incremental (differing by one) solutions. However, children often failed to identify many solutions or identified solutions that were not related to one another. This possibly highlights a key limitation of physical objects that was previously discussed – they do not provide a record of previous solutions with which to identify what solutions have been given and what solutions remain.

4.1.1.2 Trace of interim solutions

Pictorial materials may support children in the partitioning task because they provide a record of interim solutions. Annotations can be used to identify what solutions have been given and what solutions remain. It might be argued that multiple sets of physical materials could also provide the means to create a record - children could create a configuration and move to the next set, leaving a record of the previous solution. However, this is much more difficult to achieve practically. Spatially manipulating objects not only requires more workspace but groups of objects can easily be moved unintentionally or be confused with one another.

Pictorial materials may therefore address a key limitation of physical materials: they provide an easier means to create a trace of previous activity and hence a means of identifying previous solutions. However, the value of this representational characteristic is not clear. Not only may it be difficult for children to identify previous solutions from their annotations, but they may simply lack a developed understanding of the value of these previous annotations. Such reflective activity may be difficult for children without explicit prompts or even instruction. Importantly, by using an efficient strategy for identifying unique solutions (i.e. *compensation*) children do not actually need a record of solutions.

4.1.1.3 Summary and study aims

Physical objects have unique representational properties that may be beneficial or limiting depending on the problem. Physical objects allow children to manipulate the representation with ease, which may help them to act on and interpret the representation to develop new ideas. This may explain the increased use of the *commutative* and *compensation* solutions in Study 2. However, it is possible that using pictorial materials will increase the costs of manipulating the representation that will encourage children to plan and hence use more efficient strategies. Furthermore, pictorial representations provide a record of previous solutions. This record may help children identify what solutions they have given and what solutions remain. Physical representations do not provide this benefit, although it is not clear whether children possess sufficient understanding to be able to recognise the value of using these records of past actions.

The aim of Study 3 was to evaluate the representational properties of physical materials by examining the role of spatial manipulation and representational trace on children's strategies to solve the partitioning problem used in the previous studies. This was addressed by conducting a 2x2 controlled design study, manipulating these two representational characteristics (Physical/Pictorial and, Trace/No Trace). It was predicted that children in the Physical and Trace conditions would identify significantly more correct partitioning solutions.

4.2 Method

4.2.1 Design

A 2x2 between subjects design was used with Material (Physical/Paper) and Trace (Trace/No Trace) as the two independent variables, resulting in four independent groups: *Physical Trace, Physical No Trace, Pictorial Trace* and *Pictorial No Trace* (see Table 4.1). The primary dependent measure was the verbal solutions provided by children, which were then coded according to strategy using the coding scheme developed and defined in Study 2.

Table 4.1: Four conditions in the study design

	Trace of solutions provided	
Representation	No Trace	Trace
Physical	Physical No Trace	Pictorial No Trace
Pictorial	Physical Trace	Pictorial Trace

4.2.2 Participants

One hundred children took part in this study (54 girls and 46 boys; range 53 months to 87 months; M=70.79 months; SD=9.98 months). Children were from Reception, Year 1 and Year 2 classes at a local primary school in the Nottingham area. The School had recently amalgamated an Infant and Junior school, and had yet to receive a formal inspection. Based on the reports from the pre-existing schools, the percentage of children receiving free school meals is slightly above the national average (a measure of Social Economic Status). The classes were mixed, those taking part in the study being: one Reception class, two Reception/Year 1 mixed classes (former Infant school building) and three mixed Year 2/Year 3 classes (former Junior school building). Only children in Year 2 from the Year 2/3 mixed classes took part. There were 2 children with English as a second language and 1 with Special needs. The teacher judged that these children would not have particular difficulties with the problems so they were included in the study and analysis.

Children were randomly allocated to one of the four conditions by assigning a number to each child using a random number generator in Excel, applying a different (sequential) range of numbers to each condition, and then allocating each child to the condition corresponding to his/her number.

4.2.3 Materials and Procedure

Sessions took place in two locations. For children in the former Infant School, sessions took place in the lower school library area. Although there were sometimes other individual work sessions occurring at the same time, noise levels and distractions were low. Children in the former Primary school were tested in a small meeting room off one of the corridors. Although the door was left open, the occasional noises of other children passing in the corridor did not seem to cause distraction.

Sessions lasted between ten and fifteen minutes. The interviewer began by welcoming the child and thanking him/her for coming. It was then explained in general terms that the aim of the study was to find out what helps children learn about numbers, and asked the child if he/she like to help by '*having a go at a few questions about numbers'*. Every child seemed keen to do so, and the interviewer then explained the story problem.

It was decided to present the children in this study with a different story context from that given to the children in Study 1. The problem structure was isomorphic but used cows in fields rather than fruit in bags for two main reasons. Firstly, because some children were younger, it was felt that a clear visual image of the two partitioning areas would support children's understanding. Secondly, it was expected that this problem was less hypothetical: cows can change fields over time, whereas a person is not likely to change objects in two bags (or reflect on the change). Importantly, it is also less logical for cows to be equally partitioned between two fields than fruit in bags.

The interviewer recounted the story problem about a farmer, his two fields and the cows he kept in his fields. The laminated image was placed 50cm away from the children (to prevent children placing objects actually on the image) and showed a fence separating the two fields with a gate in the middle that had been left open. The interviewer then explained the problem: the farmer kept cows in the fields but, because the gate was open, the cows kept wandering from one field to the other. The interviewer used a laminated image of three cows to help children visualise the cows moving between fields. A single image of the three cows was used rather than three separate images in order not to provide a prompt of spatially partitioning objects.

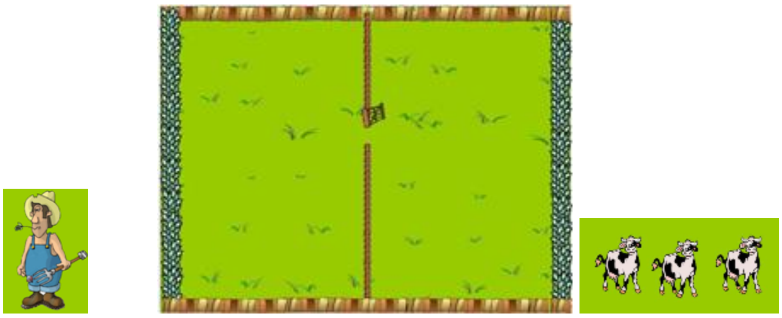


Figure 4.1: Laminated objects used to support understanding

The interviewer then told the child that because the cows keep moving from one field to the other, the farmer is confused: he doesn't know how many cows are in each field. The interviewer then explained what was required: to help the farmer by telling him *"all the different ways the cows can be in the two fields"*, and then told the children to watch an example showing them what he meant. The materials used in the demonstration and problem are described below as they differed according to which condition the child was in.

4.2.3.1 Materials in each Condition

- ***Physical No Trace***

Children in this group were presented with a line of counted out red cubes (2cm^3 wooden cubes) in front of them for each problem (see Figure 4.2a)

- *Physical Trace*

Similarly to the Physical No Trace condition, children in this group were presented with a line in front of them of counted out red cubes ($2cm^3$ wooden cubes) for each problem. However, whenever children verbally identified a solution, the interviewer quickly recreated the configuration of the cubes children had made on the right hand side of their workspace using black wooden cubes (as illustrated in Figure 4.2b). It was decided that the interviewer, not child, would create this record in order not to interrupt children's use of the physical representation. For the same reason, the interviewer used a separate set of cubes (not the cubes children were using) in order not to disrupt how children might identify one solution from the previous. The interviewer would start at the top of this space and create successive configurations under each other so that a total of 13 configurations would fit in this space (therefore balancing the 13 rows provided in the Pictorial Trace condition).

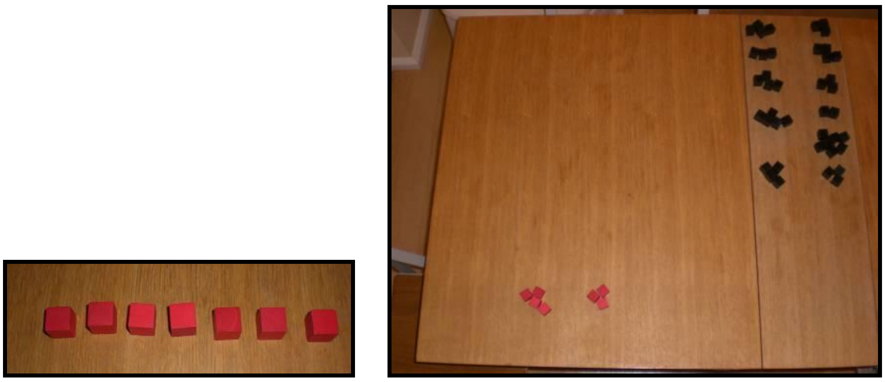


Figure 4.2: a) Physical materials as presented in both Physical conditions and b) Example of trace solutions created in the Physical Trace condition

- *Pictorial No Trace*

Children in this group were provided with a sheet of paper with rows of squares (equal to the partitioning amount). The squares were $2cm^2$ white with a black border separated by a 1.5cm gap (see Figure 4.3). Each sheet of paper was about 6cm (three times the height of the squares) by 30cm (the width of A3 paper). However, after each verbal solution, the interviewer removed this piece of paper, turned it over and placed it to the right hand side of the children's workspace. He then gave the children an identical sheet of paper with a row of the number of squares.

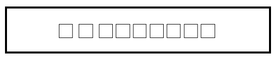


Figure 4.3: Pictorial materials used in conditions

- *Pictorial Trace*

Children in this group were provided with an A3 (Portrait) sheet of paper with 13 aligned rows of the number of squares to partition[15] (Figure 4.4). The squares were identical to the *Pictorial No Trace* condition, and were aligned in order to facilitate comparison between solutions.

[15] In all conditions, it was decided to set a maximum number of solutions for the children. As the maximum number of correct solutions was 10 it was decided to stop children after 13 solutions (where children would have given at least four incorrect or repeated solutions)

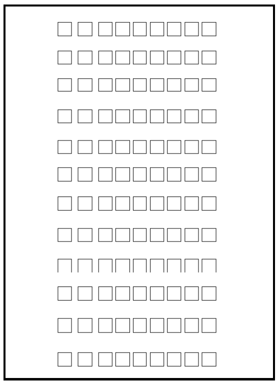


Figure 4.4: Pictorial materials used in Pictorial Trace condition (13 rows)

4.2.3.2 Example question

In the example question, the interviewer drew children's attention to the number of cows on the laminated image of three cows, and explained that the aim was to '*find all the different ways the 3 cows could be in the two fields'*. The interviewer then placed the cubes (three cubes in both Physical conditions) and the paper (sheet with one row of three squares in the *Pictorial No Trace* and sheet with 13 rows of three squares in the Pictorial Trace) in front of the children, and asked them to watch how these *three cubes/three squares* could be used to help find the different ways. In all four conditions, the interviewer then used the materials to create the four solutions; 3 & 0, 1 & 2, 2 & 1 and 0 & 3; always in this order. In the two cubes conditions, the interviewer moved the cubes into 2 spatial groups. In the Pictorial condition, the interviewer drew a circle round the number of squares using a pencil.

In the Physical No Trace condition, the interviewer simply created the configuration, verbally identified the solution from the configuration, and then continued to create and identify all solutions (similarly to Study 2). In the Physical Trace condition, after the first solution (3 & 0), the interviewer recreated the configuration at the top right side of the workspace using the black cubes whilst telling the children: *"I am going to use these cubes so I have this [pointing to this new row of squares] to remember my answer".* The interviewer then proceeded to identify solutions in the same way as for the Physical No Trace condition, although recreating each of the four configurations below the previous on the right hand side of the workspace. In the Pictorial No Trace condition, the interviewer circled the squares, verbally identified the solution, and moved the paper to the right turning it upside down. He then repeated this for the remaining solutions using new sheets with 3 squares on. In the Pictorial Trace condition, after the first solution, the interviewer said "*I am going to use the next row of squares so I have this [pointing to the first solution] to remember my answer".* Although only three rows were used, there were 13 rows in the Pictorial Trace example (see Figure 4.3 for example of annotation of first three rows).

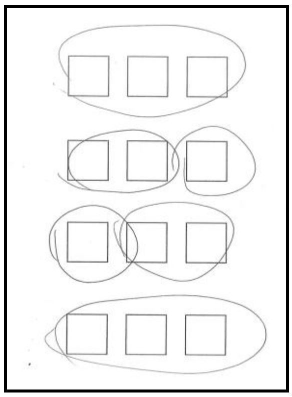


Figure 4.5: Example of interviewer's annotation for demonstration with three objects in Physical Trace condition (only four of 13 rows shown)

At the end of the example, the interviewer would say *"see, there are lots of ways the cows can be in the fields".* In the Physical Trace and Pictorial Trace conditions, the interviewer would point clearly to the rows of four configurations in order to emphasise how these provided a record of all the solutions given.

4.2.3.3 Partitioning problem

Following the demonstration problem, the interviewer explained to the children that the farmer bought some more cows and now had 7 altogether. The order of partition amount was the same in all conditions: 7 for the first problem, 8 for the second and 9 for the last. The interviewer then placed the correct amount of cubes/sheet with the correct amount of squares, in front of children and asked them to '*use the 7 cubes/squares to show the farmer all the ways the 7 cows could be in the 2 fields'*. The interviewer asked the children to '*remember, for each answer, to say how many were in this field (pointing to left field) and how many in this field [pointing in right field]*'. He then said *'keep going and let me know when you think you have found all the different ways.'*

The interviewer recorded solutions and gave prompts as in Study 2. In addition, children were encouraged to use the representations at all times: *"remember to use the cubes/squares*". Although there was inevitably a slight delay in the Physical Trace condition while the interviewer recreated the configuration, the time was kept to a minimum (about 3-5 seconds) as a) the interviewer knew the number of cubes being partitioned and b) the re-created configuration did not have to be an exact replica. Following the final problem (partitioning 9), the interviewer thanked the child and gave them a sticker.

4.3 Results

4.3.1 Correct solutions

All children were presented with three partitioning questions: partitioning 7, 8 and 9. The solutions children gave were coded according to being *Correct* (and unique), *Repeated* (correct but not unique) and *Incorrect*. The maximum number of correct solutions for each problem was is 8, 9, and 10 respectively. All children therefore received a score between 0 and 27 for the number of correct solutions identified. The distribution of group data was tested (Kolmogorov-Smirnov) and revealed no significant departures from normality for scores on any of the conditions: Physical No Trace ($D(25)=0.12$, p=ns); Physical Trace ($D(25)=0.12$, p=ns); Pictorial No Trace ($D(25)=0.14$, p=ns); and Pictorial Trace ($D(25)=0.17$, p=ns). Analysis of Variance was therefore carried out with Material (Physical/Pictorial) and Trace (Trace/No Trace) as between subjects variables.

Analysis revealed a significant main effect for Condition ($F(3,96)=4.29$, $p<0.01$) but failed to reveal a main effect for Trace ($F(1,96)=0.64$, p=ns). There were also no significant interaction effects ($F(1,96)=0.05$, p=ns). The means for each condition and factor are shown in Figure 4.6.

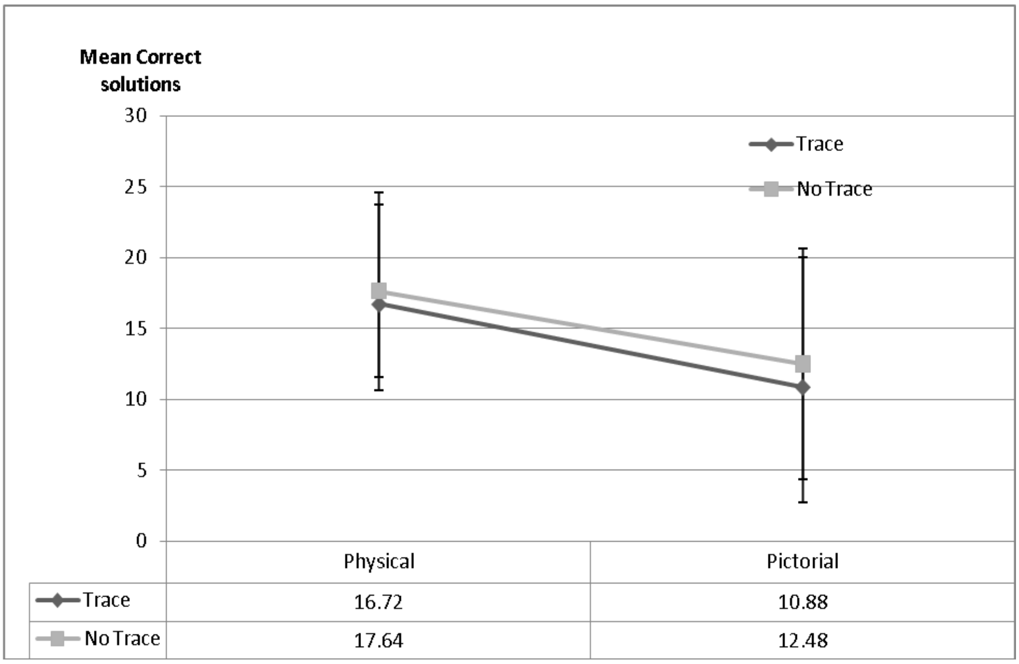


Figure 4.6: Mean Correct Solutions in the four conditions (Physical/Pictorial – Trace/No Trace)

A Freidman test showed that there were no significant differences in the total number of solutions identified between the three partitioning problems (χ^2=0.88, DF=2 p=ns).

- *Incorrect solutions*

Since the number of incorrect scores was independent of the number of correct scores, separate analyses were carried out. The distribution of group data was tested (Kolmogorov-Smirnov) and revealed significant departures from normality for incorrect scores on all conditions. Mann-Whitney tests were therefore carried out to examine main effects. These showed significantly more incorrect solutions in the Pictorial Conditions than in the Physical Conditions (U=737, Z=-3.73, $p<0.0005$) but no difference between the Trace and No Trace Conditions (U=995, Z=-1.86, p=ns).

- *Repeated solutions*

A Kruskal-Wallis test revealed a significant difference between conditions for the number of repeated solutions identified ($\chi^2(3)=15.91$, $p<0.005$). Mann-Whitney tests revealed that this finding reflected that children in the Physical No Trace condition identified significantly more repeated solutions than the other three conditions.

4.3.2 Difference between age groups

Children from three age groups took part in the study: Reception, Year 1 and Year 2. A Kruskal-Wallis test revealed significant differences for Correct scores between the three year groups. As expected, separated Mann-Whitney tests showed that Year 1 (Mdn=15) scored significantly higher than Reception (Mdn=3) ($U=138.0$, $Z=-4.12$, $p<0.001$), while Year 2 (Mdn=21) scored significantly higher than Year 1 ($U=471.5$, $Z=-3.30$, $p<0.01$).

Man-Whitney tests were then carried out to examine the main effects of Materials and Trace in each of the three age groups. Reflecting analysis carried out on all children, no significant differences were found between Trace and No Trace in any of the age groups. In contrast, it was found that children identified more correct solutions using physical than pictorial materials in all three age groups: Reception ($U=10.5$, $Z=-2.68$, $p<0.01$), Year 1($U=166.0$, $Z=-2.35$, $p<0.05$), Year 2 ($U=91.5$, $Z=-2.04$, $p<0.05$). Median scores for each condition for the three age groups are shown in Table 4.1.

Table 4.2: Median (IQR) correct scores for Physical and Pictorial conditions for children in Reception, Year 1 and Year 2

	Physical	Pictorial
Reception	9 (3, 13.5) n=8	0 (0, 4) n=10
Year 1	16.5 (14.25, 22) n=24	14 (3, 17) n=23
Year 2	22.5 (18, 24.25) n=18	18 (12, 22) n=22

4.3.3 Strategy

Using the Coding scheme developed in Study 2, children's solutions were coded according to the number of related solutions. Related solutions comprised of *commutative* (reverse of the previous solution) and *compensation* (one different from the previous solution). Mann-Whitney tests revealed no differences in the number of Related solutions identified between the Trace (Mdn=5) and No Trace (Mdn=6) conditions, but showed that children in the Physical conditions (Mdn=8) identified significantly more Related solutions than those in the Pictorial conditions (Mdn=4) ($U=810.0$, $Z=-3.06$, $p<0.005$). Breaking this down by strategy, there were significantly more *compensation* ($U=937.5$, $Z=-2.18$, $p<0.05$) solutions identified in the *Physical* conditions than *Pictorial*. There was a more significant difference in commutative solutions ($U=722.00$, $z=-3.98$, $p<0.0005$), however, there was arguably insufficient variation in commutative scores in the Pictorial conditions for comparing scores by rank. Whilst 32 out of 50 children identified at least 1 commutative solution in the Physical condition, only 14 out of 50 did in the Pictorial conditions and half of these only identified 1 commutative solution. Therefore differences were reanalysed by coding each child according to using the commutative strategy or not. As this was a between subjects design, a $\chi2$ analysis was conducted and found a significant difference between the Physical and Pictorial condition in the number of children who identified at least one commutative solution ($\chi2=13.04$, df=1, $p<0.0005$)

As well as related solutions, it was found that children in the Physical condition also identified significantly more *other* solutions than children in the Pictorial conditions ($U=941.5$, $Z=-2.14$, $p<0.05$). Median scores for strategies in the Physical and Pictorial conditions are shown in Table 4.2.

Table 4.3: Median (IQR) scores for coded strategies in the Physical and Pictorial conditions

	Compensation	Commutative	Other
Physical	6 (1.75, 9)	1 (0, 3)	7 (2, 9)
Pictorial	3 (0, 6.25)	0 (0, 1)	4 (0, 8)

4.3.4. Equal partitioning

There were no significant differences in the number of equally partitioned first solutions between the groups or main conditions.

4.4 Discussion

The aim of this study was to develop an understanding of the representational benefits and limitations of physical materials by comparing children's partitioning strategies using physical with pictorial representations, and also by examining the possible advantages of representation trace. It was predicted that spatial manipulation of the physical materials would help children to identify more solutions as would providing a record of each representational state created to identify a solution. The findings were clear: children using physical materials identified significantly more correct solutions than children using pictorial materials, thus supporting the predictions put forward by Martin and Schwartz (2005) in their theory of Physically Distributed Learning. However, it was also found in this study that, contrary to predictions, providing children with a trace of their previous solutions did not lead to more correct solutions.

4.4.1. Role of Trace

It is possible that children simply forgot, or didn't understand, that they had a trace of their solutions, especially in the Physical conditions that involved the unusual aspect of the interviewer replicating solutions with other cubes. However, not only were explicit references made in the demonstration problem, but informal enquiries made after formal questioning had been completed suggested that children *did* understand that these conditions presented a trace of previous solutions. Two reasons might therefore be put forward for why they did not use this to help the problem solving: (a) they were unable to identify the value of the representational trace or (b) they judged the demands of using this record as too high.

Identifying the value of the trace, unprompted, requires children to identify the need in the task to identify multiple but a finite number of solutions and recognise how progress can be monitored by using the trace. This may have been beyond most children who may have perceived the representation as a means to simply identify a new solution. Furthermore, children who had a more developed understanding of the problem may have realised how they could keep track of previous solutions by using a specific strategy for identifying solutions such as *compensation*.

Alternatively, children may have understood the value of the trace but chosen not to use it due to the procedural demands involved. In order to use the record, children needed a quick means of recognising which solutions they had identified previously. If this was done numerically, children would need to count the parts of these previous solutions whilst trying to track the numerical values they had not yet identified. Alternatively, children could have used visual clues to identify which solutions had not been given. However, whilst this may be relatively easy with clear and ordered solutions, it is quite challenging when previous configurations have been made that are difficult to compare. Unfortunately, if configurations *were* ordered, this would likely be because children were using a specific strategy (e.g., *compensation),* and would therefore remove the need for children to use the record.

One finding that is not easily explained is the significantly greater number of repeated solutions in the P*hysical No Trace* condition. This finding actually implies that children did use the trace in the *Physical Trace* condition, and by doing so were able to identify and avoid repeated solutions. However, an alternative explanation is that in the *Physical Trace* condition children were simply slowed down by having to wait for the interviewer to recreate the solution and, by slowing down, were more likely to think about the solutions they had identified previously. This point highlights a possible indirect effect of providing a trace in the Physical condition. However, if children were slowed down and did think more about their previous solutions, it is interesting to speculate why this did not also foster more efficient strategies.

The findings in this study thereby demonstrate that, in this partitioning problem, children do not use a trace of their solutions without explicit instruction. It is possible that children could be encouraged to use a record of their solutions with more prompts or if the record was easier to interpret, such as a symbolic record of solutions, but there is no strong reason to believe this would encourage children to relate solutions, especially as a symbolic record would remove any visual-spatial clues as to how one solution may relate to the previous one.

4.4.2. Role of Physical manipulation

The findings of this study suggest that the advantages of physical materials over no material in Study 2 were not simply attributable to an external representation of units. In this study, children were able to identify more correct solutions when they were able to physically manipulate the representation. There are several possible reasons why physical materials provided this advantage.

Firstly, it seemed that the materials were more accessible for children with little domain knowledge. Whilst six children in Reception and three in Year 1 failed to identify any solutions at all in the Pictorial condition, no child failed to do so in the Physical condition. This may be because physical objects limit how the representation can be acted upon. Therefore, with little understanding, children in this condition may have been more likely to use the materials appropriately by creating two spatial groups, than to annotate the pictorial materials appropriately by creating lines to separate two groups.

The physical materials may also have supported the procedure of identifying two amounts accurately. By creating two spatially separate groups, not only are the boundaries of each group clearer but children's enumeration may be supported through subitising. Tactile feedback and moving objects may also have supported counting. Indeed, there were significantly more incorrect solutions in the Pictorial condition. However, although there were more incorrect solutions in the Pictorial condition, the reason for there being less correct solutions was because far fewer verbal solutions were provided.

It is predicted in PDL that children will identify more solutions using physical materials because they allow children to act on the interpretation with ease, creating spatial configurations that can be interpreted to support new ideas. This study did not record the number of changes to the representations children made, but informal observations indicated that children created more configurations than they identified – suggesting that the representation did help children to explore the range of solutions possible, and allowed children to identify new solutions by first acting on the representation and then enumerating (interpreting) the resulting solution.

4.4.3 Motivation

Another possible reason that children in the Pictorial condition identified fewer solutions may be motivational. The advantages of physical materials may not be so much that children become aware of the greater number of solutions possible, but rather that the lower demands of manipulating the representations (and/or familiarity) motivate them to continue. Physical materials are more easily and quickly manipulated than pictorial materials as well as providing both tactile and visual stimuli. Significantly, unlike pictorial materials, manipulation does not leave a trace, so that children may be less concerned about 'going wrong'. Even in the *Physical Trace* condition, a trace was only made *after* children had identified a solution.

It is not clear how PDL accounts for possible motivational effects. Motivation might encourage children to adapt the materials more, thereby leading them to develop more ideas, but it is difficult to isolate motivation as a factor. Nevertheless, there was reason to believe that the advantage of the materials in this study was not purely motivational. Firstly, there were no clear signs of loss of motivation in either condition (e.g., loss of visual concentration). Secondly, sessions were relatively short (around 12 minutes on average), especially for the older children where the advantage of physical materials was still clear. Finally, if children were losing motivation, a fall in performance over the three problems might have been expected, yet there were no such differences in either condition. Therefore, although it is not possible to rule out motivation as a key factor in differences between conditions, it is unlikely to be the only factor.

4.4.4 Strategies

Differences in the ways children identified successive solutions in each condition provide a window onto how the representations may have influenced problem solving. Using the coding scheme developed in Study 2, it was possible to compare and contrast strategies that children may have used. Similarly to Study 2, a large number of solutions were related, seemingly derived from the previous solution, with older children identifying more related solutions. As this suggests developmental progress, it is possible to argue that because children identified more related solutions using physical objects, this representation fosters the use of more developed strategies for partitioning. However, children in the Physical condition also identified a greater amount of *other* solutions.

Therefore, it seems that the greater number of related solutions reflects a general effect of identifying more solutions overall using physical objects.

The possibility that the larger number of *related* solutions in the Physical conditions may simply reflect the larger number of solutions found overall does not itself negate the benefits of this representation: a greater number of solutions identified this way means that children will have more experience of such strategies and hence a possibly greater chance of developing related numerical ideas. However, it does suggest that the manipulative or perceptual properties that may have fostered certain strategies in Study 2 are not unique to physical objects. In other words, it may simply be the external, linear representation of objects that helps children identify related solutionsIt should be noted, however, that difference in the number of commutative solutions identified between conditions appeared more substantial. This raises the possibility that the manipulative or perceptual properties of physical objects do foster this strategy. Indeed, with cubes, it is easy to change a configuration (e.g., 2 and 5) into a unique but symmetrical configuration of (i.e.. 5 and 2) through simple actions: grabbing a group with each hand and then swapping over hands. Unfortunately, without video data it is difficult to conclude that such actions were indeed responsible for the greater use of this strategy.

4.4.5 Summary

This study has shown that children are able to identify more ways to partition a number when using physical than when using pictorial representations. This supports PDL and suggests that the implementation costs of having to annotate paper does not result in more planful behaviour, as has been found with adult studies (O'Hara & Payne, 1999). This may not be surprising considering the evidence suggesting that children find planning cognitively challenging and unappealing (Ellis & Siegler, 1997).

The problem used in this study required children to keep track of which solutions they had given and which solutions still remained to be identified. The prediction that children would utilise a trace of their solutions to meet these demands was not supported. This suggests that the cognitive demands of interpreting these previous solutions to inform strategies were too high, although it is not clear how children would respond to more explicit instruction before starting the problem solving.

Although the trace may have been of limited value to children, the value of representational trace for the teacher should not be ignored. Having a trace of children's actions allows the teacher to see the child's progress without constant supervision. Whilst children's use of pictorial materials provides this trace, physical materials do not. Developing our understanding of the value of a trace of solutions is important, not only in evaluating this representational characteristic but also in evaluating novel digital technologies that can provide a trace of solutions even when objects are manipulated spatially. Indeed, the ability for computer based manipulatives to provide a trace of actions is referred to as a key advantage that these materials have over physical objects (Clements, 1999; Kaput, 1993).

This study also examined the effect of using the different representations on children' strategies, and showed that children in the Physical conditions identified more solutions that related to the previous one. Although this could simply be a reflection of a greater number of solutions having been found overall, it appearedthat one particular strategy (*commutativity)* was more likely to be used in the Physical condition. More focused video analysis of children solving partitioning problems may help to explain which representational characteristics of the materials encourage the use of this strategy. Similarly, there are other questions raised in the studies conducted so far that might be addressed using more qualitative analysis. These include questions about the relationship between children's actions and the solutions identified, and how certain unique properties of physical materials such as sensorimotoric information may play a role. It might also be possible to identify which behaviours are related to children's use of the most efficient strategy for solving this problem (*compensation*) and possibly thereby begin to suggest ways in which the materials themselves can be adapted to foster such a strategy.

Chapter 5

The role of physical actions in solving partitioning problems - Study 4

5.1 Introduction

It was shown in Study 2 that children were able to identify more correct partitioning solutions using physical objects than with no materials. Study 3 then focused on physical and pictorial representations to examine the representational trade-off between manipulating objects spatially and leaving a record of previous solutions, and demonstrated that the conclusions of Study 2 were not simply attributable to children having an external representation of the whole. It was found that physical manipulation of objects did help children's problem solving, whereas having a record of previous solutions made no significant difference.

Although Studies 2 and 3 both showed that manipulating physical cubes supported partitioning, the studies were limited in providing an account of *how* children's actions with the material allowed them to identify more verbal solutions. Physical materials have many unique representational qualities, both in terms of perceptual properties (e.g., spatial configuration/tactile feedback) and how these properties can be manipulated. Identifying the role of more specific representational characteristics can not only develop our understanding of the advantages and limitations of physical materials but also help suggest ways in which we might design more effective materials to support learning.

5.1.1 Physical properties

In the literature review, it was discussed how physical materials have certain representational qualities that may support cognitive activity. For example, objects provide tactile information about properties such as the position and quantity being touched. The spatial configuration of objects may also help children by emphasising the group to which objects belong and allowing children to quickly enumerate objects though perceptual mechanisms (subitising). Different sources of information potentially help children process greater amounts of information in parallel (e.g., one can hold in a form of tactile memory the information that four objects are in the left hand whilst counting out two or more objects with the right hand under visual control of movement). The potential for different multimodal forms of encoding in memory, or even of offloading memory demands onto external representations, may thus provide added advantages of using physical materials for certain problems.

It was also discussed in the literature review how a key benefit of physical representations may reflect the benefits of manipulation. When children are partitioning with physical objects, they are able to move objects with ease using both hands into different groups that can then be enumerated as parts. This notion that children can act on and then interpret the representation is central to the Theory of Physically Distributed Learning (PDL) (Martin & Schwartz, 2005). According to PDL, individuals are able to adapt the environment to help 'adapt ideas'. In this context, the term *adapt* is synonymous with *change*.

In their paper, Martin and Schwartz examine the relationship between changes to the representation and interpretations by developing a means to quantify each of these measures. Interpretations are defined in terms of the verbal solutions children give. Changes to the representation are defined in terms of the number of 'adaptations'. The term 'adaptation' is initially defined as a physical arrangement of pieces; however, because a measure is also presented of the number of adaptations using pictorial materials; it seems that the term is used more generally to refer to how items, physical or pictorial, have been grouped perceptually. In the case of physical objects, groups may be spatial, while with pictorial materials, groups may be those objects encircled by annotation. Using this definition, changes to the way objects have been grouped lead to new adaptations, so that it is possible to talk about the 'number of adaptations' generated using representations.

The finding in Study 3 that children identified more correct solutions using physical objects than pictorial seemed to support PDL by suggesting that children were able to manipulate the representation – i.e. generate more adaptations – to develop new ideas. The study also suggested that representational properties affected how children adapted materials leading to different strategies for identifying solutions. However, from the measures taken in the study, it was not possible to identify *how* children interacted with the representations – for example, whether they did adapt the physical representation more than the pictorial representation. It was also not possible to examine how physical properties affected children's actions and consequently strategies.

5.1.2 Study aim and hypothesis

The aim of Study 4 was to address the limitations of Studies 2 and 3 by examining in closer detail the role of physical representations in the partitioning problem. As in the previous two studies, comparisons were made with no materials and with pictorial representations. Again, it was predicted that children would identify more correct solutions and more related solutions using physical objects. Study 4 was designed to carry out a more in depth qualitative analysis with a smaller sample of children, with video data captured to examine the relationship between children's actions using the representations and the verbal solutions they provided. In particular, it was possible to evaluate the role of physical manipulation in the task by analysing the number and types of adaptations made to the representations. It was predicted that children would adapt the physical representation more than the pictorial representation and would identify more correct solutions.

The study also examined the use of strategies by coding solutions in the same way as for Studies 2 and 3. However, observational analysis aimed to explain the relationship between certain actions with the representations and the strategies used. Finally, the study also examined other interactions with the representations (e.g., pointing, touching) to identify the role of certain representational properties in supporting problem solving.

5.2 Method

5.2.1 Design

The study used a within subject design with Representation (No Materials/Paper/Physical) as the independent variable. All children solved three partitioning problems: one with No Materials first (the baseline condition), then the two others with order of condition for Paper and Physical counterbalanced. The dependent measure was the number of correct (and unique – i.e. not repeated) partitions for each problem.

5.2.2 Participants

Participants were children invited to the University of Nottingham for the day as part of a 'summer scientist week'. This event was advertised around several local schools in the Nottingham area, describing how children could act as 'scientists' by taking part in different projects. This opportunistic sampling resulted in 12 children from different social economic backgrounds and schooling (6 girls and 6 boys, range: 62 months to 87 months; M=73 months; SD=7.0 months). No further details (e.g., English language, special needs) were requested.

5.2.3 Materials and Procedure

Sessions took place in a large room where five other studies were taking place. Each study area was partitioned and noise levels were generally low. Children were all accompanied by their parents who were asked to sit slightly behind their children to avoid unintended prompts. The interviewer, who was unfamiliar to the children, spent a few minutes conversing with each child to put him/her at ease before sessions began.

The story context was that used for Study 1. This is because the context of the fields in the previous study was considered to provide too strong a prompt for how to partition objects into two groups (this study aimed to effect of representation on grouping). The problems were all characterised in the form of the same vignette, accompanied by an illustration (Figure 5.1). Children were 'introduced' to a character called Mary, and told that she was going shopping. They were then shown a picture of three bananas and asked if they could *"say all the ways that Mary could put the bananas in the bags"*. The interviewer helped children identify solutions, and after allowing them a short time to explore the problem presented them with all the solutions in the following fixed order: 0 & 3, 1 & 2, 2 & 1 and, 3 & 0.

The picture of three bananas was then removed and not replaced, while the picture of the character and bags remained on the table.

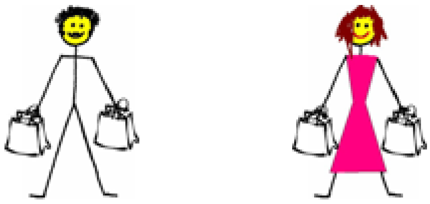


Figure 5.1: a) & b): Supporting images provided

The children were then given the partitioning problem requiring them to partition the amount 7 with no materials, as follows: *"The next day, Mary buys seven bananas. She puts some in one bag and some in the other bag. Can you tell me all the ways she can put the seven bananas into the two bags?"*

For all problems, if children did not respond after 10 seconds they were prompted by the interviewer: *"can you think of any ways that Mary can put the seven bananas in the two bags?"* For pauses after children had identified the first solution, the interviewer would prompt saying *"is that all the ways, or can you think of any more ways?"* The session would end after two prompts had been given or if the child said they had finished. If a child used non specific words such as 'some' or 'the rest' when identifying solutions, the interviewer would prompt by asking *"so how many is 'some'/'the rest'?"*

The No Materials condition was followed by the Physical or Paper condition, the order of which was counterbalanced between children. In each condition, a demonstration was given using the materials before sessions began. Children were again presented with a picture of three objects to be bought and told all the ways these could be partitioned between the two bags. However, in this demonstration the interviewer asked the children to watch how the cubes/paper could be used to help find all the ways. The cubes were blue *Unifix* cubes (Figure 5.2a). The Paper representation consisted of 2cm dark grey squares aligned horizontally and separated by a 1.5cm gap across a sheet of A4 paper (landscape)(Figure 5.2b). A pencil and eraser were also provided; although it was decided to provide only one piece of paper per problem to balance conditions (children were only given one set of cubes)[16].

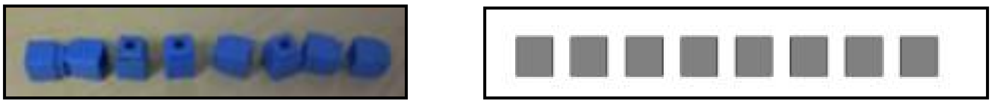


Figure 5.2: a) Pictorial and b) Physical materials used

Each child partitioned 8 followed by 9 in counterbalanced conditions. The interviewer used the materials to model all the answers by moving the cubes into different groups, or by drawing circles around squares in the Pictorial condition to make groups. Again the possibilities were identified in the same order; 0 & 3, 1 & 2, 2 & 1 and,

[16] This also allowed this study to examine whether annotating the same representation would help children in the Pictorial condition to relate consecutive solutions – not found in the previous study where children used different sheets of paper.

3 & 0. In the Pictorial condition the interviewer briefly showed how the eraser could be used to remove any unwanted lines if this helped.

All sessions were videoed and the interviewer also wrote children's solutions on paper so that they could see that their answers were being recorded (and that they were therefore important to the task), although they could not see what was actually being written. Children would generally point to the bag to which they were referring (e.g., *"three in that one")* but if it was not clear the interviewer would prompt *"three in which bag?"* The interviewer recorded the left bag as referring to the first part and the right as the second.

For several children it was necessary to remind them of the need to verbally identify partitions by stating the cardinal amounts, and not just show the configuration. As identifying the numerical solution from the physical state was integral to the study, it was important to apply the same rules for prompting children to verbalise solutions across the different conditions. Children were prompted by saying *"remember to tell me how many in each bag when you have a new answer"* if they created a new configuration but made no signs of adapting (moving, annotating) the materials for more than several seconds. Although it is possible that this criterion resulted in prompting children to enumerate when they did not realise they had identified a solution, the same prompts were provided across both conditions and were deemed necessary to avoid children creating many solutions physically without identifying any verbally due to forgetting or misinterpreting the task demands.

5.3 Results

5.3.1 Correct solutions

Children's solutions were scored according to the number of unique correct partitioning solutions they gave. The maximum number possible was different for each question, with a maximum of 8, 9 and 10 for partitioning 7, 8 and 9 respectively. Because the size and direction of effect sizes did not differ when scores were coded (from 0-3 as they were in Study 1), the analyses reported were carried out on absolute scores. Nevertheless, despite the Physical and Pictorial conditions being counterbalanced, they always followed the No Materials condition, therefore order effects cannot be ruled out.

The distribution of group data was tested (Kolmogorov-Smirnov) and revealed significant departures from normality; non-parametric analyses were therefore carried out. A non-parametric Friedman test was used to examine differences in correct scores between conditions and revealed significant differences ($\chi^2(2)=9.90$, $p<0.01$). Wilcoxon tests were therefore used to examine differences between conditions and revealed significant differences between No Materials (Mdn=1) and Physical conditions (Mdn=4) ($Z=-2.35$, $p<0.05$) but not between No Materials and Paper conditions (Mdn=1) ($Z=-0.28$, p=ns). Children identified significantly more correct solutions in the Physical than Paper condition ($Z=-2.68$, $p<0.01$).

5.3.2 Adaptations

A key aim of this study was to examine children's use of representations to help identify verbal solutions. Consequently, it was important to identify a way to measure and compare children's use of physical and pictorial representations. It was decided to use the term '*adapt'* to describe changes to the representation and the term '*adaptation'* as a unit in which to quantify the number of changes made.

The term 'adaptation' is taken from Martin and Schwartz (2005) who used the term to refer to when children arranged objects into equally partitioned groups. Unfortunately, no further description was provided to help define this behaviour. In this study, the aim was to examine children's partitioning behaviour with the representation, and the term 'adaptation' was therefore defined more expansively as *any configuration that resulted from a change in the number of objects grouped together.* What constituted as grouped together in each condition is described presently. It was also decided to define groupings according to a left to right ordering (i.e. swapping objects 2 & 5, and 5 & 2, would constitute two adaptations). It was thereby also possible to describe each adaptation by the number of groups and objects in each group.

In the Physical condition, a grouping was defined as objects placed in close proximity to each other relative to another group of objects[17]. For example, in Figure 5.3, this adaptation would be described as 4 & 4 & 1. In the Pictorial condition a grouping was defined as objects that were separated by annotation between objects. For example, in Figure 5.3b, this adaptation would be coded as 1 & 8. Because of the scope for subjectivity in these descriptions, a secondary coder was employed to quantify the number of adaptations made by the 12 children in the Physical and Pictorial conditions (this could not be done in the No Materials condition). Inter-rater agreement was

[17] As actual measurements would reflect the space children used, coding necessarily involved an element of coders' interpretation of when children had placed objects together as part of the same group.

calculated using an Intra-Class reliability coefficient as data was at least interval. The model used assumed the same raters rated all cases and each rating score came from the same rater. The coefficient for physical and pictorial materials was 0.978 which shows that inter-rater reliability was high.

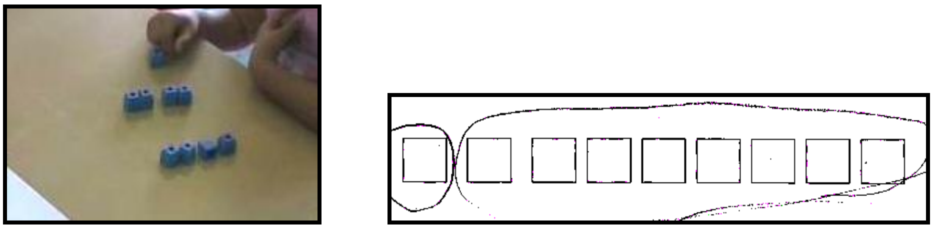


Figure 5.3: a) & b): Examples of physical and pictorial configurations – coded as 4 & 4 & 1.and 1 & 8 respectively

In total, 119 adaptations were coded in the Physical condition and 55 in the Pictorial condition. A Kolmogorov-Smirnov test of normality revealed that the data met assumptions of normality (Physical: $D(12)=0.15$, p=ns); Pictorial: $D(12)=0.22$) p=ns), so a paired sample t-test was carried out to examine differences in the number of adaptations in each condition. This showed that children created significantly more adaptations in the Physical condition ($M=9.92$, $SD=7.78$) than Pictorial condition ($M=4.58$, $SD=4.96$) ($t=3.26$, $p<0.01$).

5.3.3 Relationship between Adaptations and Correct scores

In order to examine the relationship between the number of adaptations and the number of verbally identified correct scores, a Spearman correlation was carried on these two measures in the Physical and Pictorial conditions. This revealed a significant correlation in the Physical condition ($\rho=0.74$, $p<0.01$), but not in the Pictorial condition ($\rho=0.12$, p=ns).

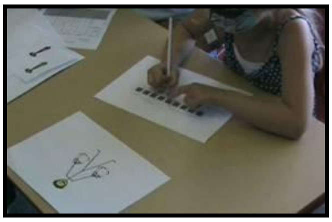
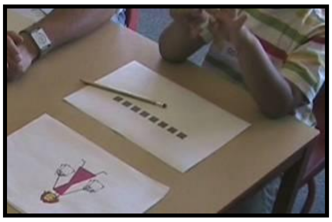
A significant correlation between the number of solutions identified verbally and the number of adaptations does not necessary imply causation. It might be expected, however, that if adapting the representation led to new correct solutions, there would be more adaptations than correct solutions, as children may not verbally identify some adaptations. Indeed, a Wilcoxon test (correct scores were non-normal) revealed that children identified significantly more adaptations than correct solutions in the Physical condition ($Z=-2.95$, $p<0.05$); but that the difference was not significant in the Pictorial condition ($Z=-1.55$, p=ns).

As shown in Table 5.1 there were more adaptations than verbal solutions in the Pictorial condition, although this difference does highlight a difficulty with the coding scheme for adaptations. A couple of children in both the Physical and Pictorial condition began partitioning by verbalising a strategy of placing objects one at a time into two groups *"one in this bag, one in this bag"*. In the Physical condition, these children moved objects one at a time into two new groups. In the Pictorial condition, they annotated around each square one at a time. In both conditions each action was coded as a new adaptation but not a verbal solution. This behaviour accounts for the greater number of adaptations than verbal solutions in the Pictorial condition: there were no other instances where children annotated (adaptation) without identifying a verbal solution. In contrast, there were many instances where children moved physical objects during problem solving without identifying a new solution.

Table 5.1: Median (IQR) scores for Correct solutions and Adaptations in the Physical and Pictorial conditions

	Correct Solutions	Adaptations
Physical	4 (1.25,6)	8 (4.25,13.5)
Pictorial	1 (1,2.75)	4 (1,6)

Another way to examine whether children identified a solution prior to or following adaptation of the representation is to examine counting behaviour. Counting could be identified when a child enumerated the number of objects verbally; so that by examining counting behaviour it was possible to identify whether children had identified a verbal solution before or after adapting the materials for each solution. Unfortunately, the small amounts involved in questions meant that instances of observable counting behaviour were generally few. Whilst no child was observed counting physical objects before moving them, there were 7 observations (3 children) where children counted the pictorial squares before annotating them (e.g., Figure 5.5a). One child was even observed counting out the initial amount on his fingers before annotating (Figure 5.5b).

Figures 5.5: a) & b): Children counting prior to adaptation in the Pictorial condition

5.3.4 Strategies

5.3.4.1 Equal partitioning

The majority of children's initial strategy was to partition objects into two equal groups. In order to compare the use of equal partitioning in each condition, initial solutions were categorised into: Equal partitioning[18]; Correct but not Equal partitioning; and Incorrect. Table 5.1 shows the distribution of these three categories for the No Materials, Physical, and Pictorial conditions. The table illustrates how the majority of children identified an Equal Partitioning in both the Physical (8) and Pictorial conditions (7) but not in the No Materials conditions (0).

[18] As stated in Study 2, for odd numbers, the two solutions that are nearest to equal partitioning were also coded as Equal partitioning (i.e. 3 & 4 and 4 & 3 for partitioning 7; 4 & 5 and 5 & 4 for partitioning 9).

Table 5.2: Frequency of first solutions in each condition coded as Equal partitioning, Correct but not Equal partitioning and Incorrect

Condition	Equal partitioning	Correct Not Equal Partitioning	Incorrect first solution
No Materials	0	7	5
Physical	8	2	2
Pictorial	7	3	2

5.3.4.2 Relating solutions

- *Verbal solutions*

Children's verbal solutions were coded in the same way as for Studies 2 and 3. As shown in Table 5.2, children identified 15 *compensation* solutions in the Physical condition, 8 with No Materials, and 5 with Paper. Using a Freidman within subjects test, these differences were found not to be significant ($\chi^2(2)=3.36$, p=ns). The number of *commutative* solutions in respective conditions was 4, 2 and 0 – which were too small to detect any differences ($\chi^2(2)=5.00$, p=ns).

- *Adaptations*

As well as measuring the quantity of adaptations, the numerical grouping of each adaptation was recorded – e.g., 4 & 4 & 1 or 4 & 5 for partitioning 9 objects. It was thereby possible to apply the scheme used to code verbal solutions to children's adaptations (i.e. changes to the representations). Accordingly, a *compensation* adaptation was coded if successive groupings of objects into two groups differed by one (e.g., with groups of 3 and 5 cubes, move one object to create groups of 2 and 6). *Commutative* adaptations were coded if the order of the parts was reversed (e.g., annotating around groups of 3 and 5 squares after annotating around 5 and 3). The frequency of coded strategies for verbal solutions and adaptations for each condition are shown in Table 5.2.

Table 5.2 illustrates that in the Pictorial condition the number of coded strategies for verbal solutions and adaptations was the same. In contrast, in the Physical condition, the number of *compensation* adaptations (28) was greater than the number of *compensation* solutions identified verbally (15) A Wilcoxon within subjects test revealed the difference to be significant ($Z=-1.98$, $p<0.05$). In other words, children often moved one object from one group to another, but did not verbally identify this as a new solution.

Table 5.3: Frequency of Adaptations and Verbal solutions coded as Commutative or Compensation in each condition

	Verbal solution		Adaptation	
	Compensation	Commutative	Compensation	Commutative
No Materials	8	2	No external representation	
Physical	15	4	28	4
Pictorial	5	0	5	0

5.3.4.3 Abstracting Strategies

With respect to the difference between *compensation* adaptations and *compensation* verbal solutions in the Physical condition, it may be important to highlight one child's behaviour. Having partitioned the cubes and counted out this solution (pointing to cubes whilst counting), the child moved one cube at a time from one group to another creating the following adaptations: 5 & 3, 6 & 2, 7 & 1 and 8 & 0. However, this child's corresponding verbal solutions were: 5 & 3, 4 & 2, 3 & 1. She realised her mistake on the final solution.

It appeared from this that the child was applying a mental algorithm for *compensation* but making the error of taking from both parts. It might be argued that this behaviour thereby indicates that this child did not need the physical representation to identify the *compensation* strategy. However, as this study (as well as Studies 2 and 3) has shown how the physical representations increased the use of the *compensation* strategy, an alternative suggestion is that the representation prompted the strategy and that, rather than count each group, children sometimes chose to apply the calculation mentally. Indeed, observations showed another child beginning to identify *compensation* solutions by counting groups and then continuing to move objects one at a time, but looking away from the representation while identifying solutions verbally. When apparently challenged by the mental calculation, the child returned to counting objects.

5.3.5 Qualitative Analysis of actions observed

A further aim of this study was to use observations to examine children's interactions with the physical objects when problem solving. The most salient actions were to move the cubes into separate spatial groups. Children moved individual or multiple cubes at a time: a couple of children often counted objects in twos, moving them in pairs, while others did not just move cubes between two groups but would gather all the objects together in front of them after each solution and then place them into groups further away one by one. Children would often place the cubes just in front or on top of the laminated image (Figure 5.6a). One child placed a couple of cubes further away from the working area with the result that these cubes were left out in subsequent solutions. Although the cubes could be joined together, only one child actually did so. This child joined the cubes in order to stack them vertically (Figure 5.6b).

Figures 5.6: a) Actions with the cubes and b) Using the laminate objects and stacking cubes

Children would sometimes move cubes by picking them up and placing them or pushing them with their finger or side of hand (e.g., Figure 5.7a) (often using both hands). Children would also touch cubes, or make a touching gesture near a cube when counting (e.g., Figure 5.7b) (counting behaviour being identified by number words spoken aloud or mimed). They would often hold or touch a single or group of cubes when looking at another group, possibly as a prompt to remember what to count next (e.g., Figure 5.7c).

Figures 5.7: a) moving group of cubes b) pointing gesture to count c) holding cube to remember what to place next

5.3.5.1 Actions and strategies

Only four *commutative* solutions were coded in the Physical condition. However, the same behaviour was observed in three of these: swapping over groups of objects using both hands (Figure 5.8a). The other *commutative* solution was identified by moving the whole group of objects from right to left (Figure 5.8b). 15 *compensation* solutions were identified. Almost all of these reflected children moving one object at a time (*compensation* adaptation - Figure 5.8c).

Figures 5.8: a) Swapping groups (commutativity), b) Moving the whole group from right to left (commutativity) and c) Moving one object (compensation)

5.4 Discussion

Although the study sample was small for this study, comparisons between conditions revealed similar findings to Studies 2 and 3: children identified significantly more partitioning solutions in the Physical than Paper and No Materials (baseline) conditions. This advantage seemed attributable to how children were able to identify a new solution through simple physical actions. This finding consequently provides further support for PDL that predicts that physical actions on representations will generate more interpretations.

5.4.1 Adaptations and ideas

Further support for PDL came from analysis of children's adaptations of the representations. Children adapted the physical representation significantly more in the Physical than Pictorial condition, and these adaptations were significantly correlated with verbal solutions. In contrast, in the Pictorial conditions, there was no significant difference between adaptations and verbal solutions. It was also shown how children often moved objects one at a time from one group to another in the Physical condition and then verbally identified many of these changes as verbal *compensation* solutions. This may help explain the greater use of the *compensation* strategy found in Study 2. Finally, although it was observed how several children counted out a solution before annotating paper, observations of counting always followed adaptations in the Physical condition. In other words, there seemed to be tentative evidence that, in contrast to the physical representation, children were using the pictorial representation to record rather than generate ideas.

Observations of children's actions highlighted how physical materials fostered adaptations and new solutions. Tactile feedback may have supported children's visual attention in these actions as they were able to move objects quickly and easily with both hands. In contrast, adapting the pictorial representation was more procedurally demanding, requiring fine motor control and constant visual attention to make pencil annotations. Similar conclusions to Study 3 might therefore be drawn – that simple physical actions on the representation fostered ideas (number of partitioning solutions and related solutions) and that the greater cost of using pictorial materials did not lead to more planned behaviour.

5.4.2 Problem solving and the problem context

When describing the influence of the representations on children's ideas, it is clearly important to recognise the role of the context. In this way, differences between conditions reflect an interaction between representational properties and the problem context. For example, the problem described a character and the ways of arranging fruit between two bags. For the younger children, this prompted an activity of placing objects one by one into different piles. When this was carried out with cubes the children would be end up with two groups of objects, but when carried with squares they would end up with a series of encircled squares. In other words, unlike squares, partitioning cubes one by one resulted in two clear groups. It was also observed that several children placed objects on top of the laminate image in the Physical condition – an action that may have supported problem solving by emphasising the need to create two groups.

The disparity between the number of adaptations and number of verbal solutions in the Physical condition highlighted the task demands of verbally identifying new configurations. In this regard, it is important to emphasise the role of the interviewer.

Many factors, including the example problem and problem question as well as certain verbal prompts to quantify certain solutions ("*how many*?") or simply the act of recording solutions, were all factors that encouraged children to verbally identify numerical solutions. Therefore, the problem context was central to constraining how the objects should be manipulated and how children were meant to interpret their actions. In other words, it is problematic to think that mathematical meaning is transparent within manipulatives (Moyer, 2001). It is more the activity with the manipulatives, and the context of this activity, in which transparency emerges (Meira, 1998).

5.4.3 Representational properties

This study also helped identify the effect of certain representational properties on children's interactions. For example, one child manipulated cubes by stacking them vertically which, interestingly, may have supported problem solving by facilitating comparison between groups of objects (using height from the table). Another child dropped the cubes in order to create a random arrangement to begin problem solving. There were numerous observations of children using the tactile properties of the cubes although again it is not clear how much this actually helped offload task demands. For example, although children touched cubes to help them count (by keeping track of the objects counted), they were also able to touch the squares. Children also used tactile feedback to keep track of the position of cubes when looking at other objects. Although this seemed to be done to help remind them that a certain cube had still to be moved (children would move this object next), it is again not clear how much this behaviour supported cognition. It is possible that the use of tactile information provides subtle benefits in helping children offload some of the cognitive demands of the activity, but this task is not sensitive enough to confirm this advantage. Although a similar study using larger amounts of objects (i.e. more procedurally demanding) may reveal this representational benefit, this would change the nature of the task by asking children to decompose multidigit numbers.

The size and shape of the objects may also have influenced children's actions. Children seemed to be able to hold about 4-5 objects in one hand. Indeed, several children dropped cubes as a result of trying to grab more. It was also possible to attach the cubes, although this was only done by the one child who stacked objects vertically. Interestingly, the shape of the objects did seem to play a role – a couple of children spent a small amount of time moving the cubes so that they all had the attached part facing upwards and were roughly aligned on one side.

5.4.4 Representational properties and strategy use

Although the numbers were too small to detect significant differences, the pattern of strategies between conditions reflected the findings from Studies 2 and 3: namely that children identified more related solutions using cubes. This study helped identify how the representational properties of the physical materials may have fostered these strategies. It was shown that a common action was to move one object at a time from one group to the next. This action is a systematic way of changing the grouping of objects incrementally and presents a way to identify solutions by either counting the cubes after each move or applying this 'incremental change analogy' to calculate solutions using a mental algorithm (add to one part, take from the other). On the other hand, children often moved more than one object at a time. Therefore, an interesting question is whether encouraging children to *only* move one object at a time would increase the number of incremental solutions - in other words, a greater number of *compensation* solutions.

This study also helped explain the greater use of the *commutative* strategy identified with cubes than without in Study 2 and with paper in Study 3. It was observed that children would identify *commutative* solutions when they interchanged groups of objects: either by pushing groups across the table or grabbing cubes with both hands and swapping them over. This reflects a key affordance of the physical representation – *the ability to move multiple objects with ease*. It is possible to envisage ways to facilitate this action. If objects were slightly smaller, for example, it may be easier to move larger amounts. Alternatively, if cubes were larger or more awkward, children may find it more difficult to move multiple objects. It might be expected that this would hinder this strategy.

5.4.5 Summary

In conclusion, this study has shown how the manipulative properties of the representation may affect children's strategies for identifying verbal partitioning solutions. It would be expected therefore that changing these properties would lead to changes in strategy. For example, if children were asked to attach cubes, it would not only be easier to move groups of cubes (thereby possibly increasing the *commutative* strategy) but would hinder the ease of moving individual cubes that would have to be unattached first (thereby possibly reducing the *compensation* strategy). Alternatively, it might be expected that requiring children to move only one object at a time would foster the use of *compensation* strategy whilst hindering the number of *commutative* solutions.

Chapter 6

The Effect of Constraining Actions on Children's Partitioning Strategies – Study 5

6.1 Introduction

The findings from Study 4 supported those of the previous two studies, showing that physical materials can provide an advantage over no materials or pictorial materials for helping children identify ways to partition a number into different combinations. The study also helped to explain possible mechanisms: physical materials were manipulated frequently and with ease, and included two key actions – moving all objects simultaneously and moving single objects incrementally. These two actions led children to identify related solutions: i.e. *compensation* and *commutative* solutions.

6.1.1 Changes to the representation

In order to discuss the role of certain actions in solving the partitioning problem, it may be useful to represent diagrammatically the different possible representational states within the partitioning problem. A 'representational state' is defined here as a unique numerical grouping for decomposing a number. The number of combinations is quite large if also considering the order of parts – i.e. 3 & 6 is different from 6 & 3. For example, there would be 9 ways to partition 4: 4 & 0, 0 & 4, 3 &1, 1 & 3, 2 & 2, 2 & 1 & 1, 1 & 2 & 1, 1 & 1 & 2, 1 & 1& 1 & 1. However, in this partitioning problem, the problem is presented in way to constrain the number of different parts to two. The story, the example problem, the laminate image and even certain prompts by the interviewer all help encourage children to partition objects into two parts. Consequently, the number of combinations is much less (equal to n + 1). For example, there would be 5 ways to partition 4: 4 & 0, 0 & 4, 3 & 1, 1 & 3, 2 & 2.

When using physical materials, groupings are represented spatially. Therefore, to illustrate possible representational states, spatial configuration will be presented diagrammatically. Figure 6.1, for example, illustrates the 7 possible representational states for partitioning 6. These states are presented symmetrically to highlight *commutative* states.

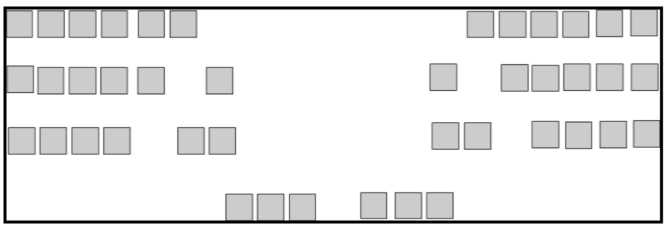


Figure 6.1: Diagrammatic representation of the 7 configurations for partitioning 6 into 2 parts

In the previous study, the term 'adaptation' was used to define a different numerical grouping – a change in representational state. However, the term was used to describe the resultant state rather than the process of changing one state to another. This *process* of changing one adaptation to another will be referred to as 'transformation'. Because children are able to move one or many objects from one group to another, there are various transformations possible. In fact, when partitioning n objects, there are n possible transformations from each state. For example, if a child is partitioning 6 objects and has made the adaptation 5 & 1, there are 6 possible transformations possible (still assuming two groups), as illustrated in Figure 6.2.

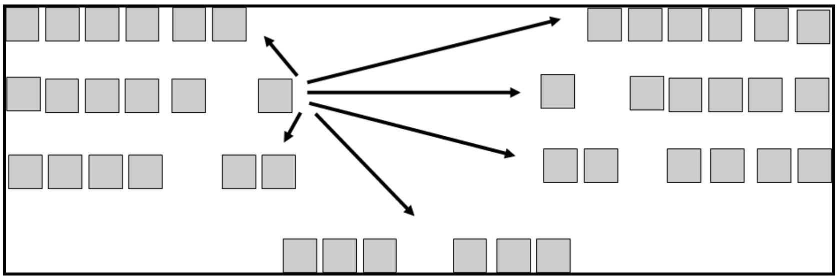


Figure 6.2: Diagrammatic representation of transformation from one state when partitioning 6

Using this form of diagram, it is now possible to illustrate the transformations reflecting the two key strategies: *compensation* and *commutative*. In Study 4, a '*compensation* adaptation' was defined as an adaptation resulting from moving one object. Therefore, from 5 & 1, moving one object to create 6 & 0 or 4 & 2 would be coded as *compensation* adaptations. These two transformations are illustrated in Figure 6.3a. Alternatively, if the new adaptation reflected a reversing of the parts of the previous, this was coded as a *commutative* adaptation. For example, from 5 & 1, swapping over objects or simply moving 4 objects to create the adaptation 1 & 5 would be coded as a *commutative* adaptation. This single transformation is shown in Figure 6.3b.

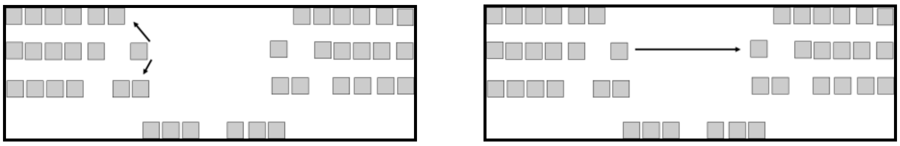


Figure 6.3: Diagrammatic representation of a) Compensation and b) Commutative adaptations from the configuration 5 and 1

In Study 4 it was found that, of 119 adaptations coded in the Physical condition, 28 were *compensation*. Furthermore, it was observed that all 15 *compensation* solutions identified verbally reflected *compensation* adaptations. The findings in Studies 2, 3 and 4 that children identified significantly more *compensation* solutions using physical materials than no materials or pictorial materials suggests that this action with physical materials is important in helping children identify solutions using the *compensation* strategy. This presents the possibility, raised at the end of the previous chapter, that encouraging children to move one object at a time (increasing *compensation* adaptations) would lead to an increase in the number of solutions that differed by one (increase in *compensation* solutions). This makes sense: if children move only one object at a time; they only need to recognise that each change can be enumerated verbally in order to identify a *compensation* solution.

It was also observed in Study 4 that 4 of the 119 adaptations were *commutative* adaptations, all of which were enumerated - resulting in 4 *commutative* solutions. Similarly, the findings in Studies 2, 3 and 4 that children identified more *commutative* solutions using physical materials than no materials or pictorial materials suggests that this action with physical materials is important in helping children identify solutions using the *commutative* strategy. It is possible therefore that encouraging children to move one object at a time would actually hinder this strategy. By having to move objects incrementally, the costs of creating a reverse configuration would be greater. For example, as illustrated in Figure 6.4, creating the configuration 1 & 5 from 5 & 1 would involve four transformations rather than just one. It would therefore be expected that encouraging children to move only one object at a time would reduce the number of *commutative* solutions.

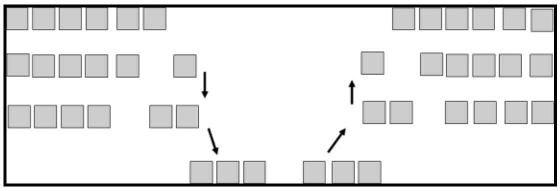


Figure 6.4: Four transformations needed to create the configuration 1 and 5 from 5 and 1 when only moving one object at a time

6.1.2 Summary and predictions

In the theory of Physically Distributed Learning (2005), it is proposed that children's physical actions on a representation can help them to develop new ideas. This theory was supported in the previous studies which showed that children's actions with physical objects not only helped them identify more ways to partition a number, but also helped them to relate consecutive solutions. Two strategies were identified for how children related solutions: *compensation* and *commutative*, and it was shown how these strategies reflected different actions: moving individual cubes or moving groups of cubes. This study examined the potential to change the strategies children used by manipulating the type of actions that could be made on the representation. With reference to PDL, the study looked at the potential to influence children's ideas by influencing the type of physical actions that could be made on the external representation.

In order to encourage children to move one object at a time, children's actions were constrained externally (i.e. through verbal instructions and a demonstration). The effect of constraining actions was examined by comparing children's strategies using physical materials in two conditions: one where they could move as many objects as they wished, and another where they could move only one object at a time. It was predicted that when children's actions were constrained to moving one object at a time, they would identify significantly more *compensation* solutions and significantly less *commutative* solutions than when they were able to move as many objects at a time as they wished.

6.2 Method

6.2.1 Design

The study used a within subjects design with Manipulation (*Constraints/No Constraints*) as the independent variable. Children used physical objects in both conditions, the presentation order of which was counterbalanced. All children solved one partitioning question in each condition, the primary dependent measure being their verbal solutions which were scored as being either correct (and unique) or not. The number of *compensation* and *commutative* solutions were then coded from the correct scores.

6.2.2 Participants

58 children took part in the study (28 girls and 30 boys, ranging from 54 months to 94 months; M=74 months; SD=13 months). Similarly to Study 4, the sample was taken from children who had been invited to the University of Nottingham for the day as part

of a 'summer scientist week' (not the same children as for Study 4). This event was advertised around several local schools in the Nottingham area, describing how children could act as 'scientists' by taking part in different projects. This opportunistic sampling resulted in a range of children being selected from different social economic backgrounds and schooling.

6.2.3 Materials and Procedure

Sessions took place in a large room where six other studies were taking place (noise levels however were generally low). Children were interviewed individually although almost all were accompanied by a parent or guardian who was asked to sit slightly behind their child to avoid unintended prompts. The interviewer, who was unfamiliar to the children, spent a couple of minutes conversing with each child before presenting the demonstration problem.

The interviewer explained the story problem to each child. It was decided to use the farmer story in Study 3 again in this study as the laminate image of the two fields seemed to provide a strong perceptual clue for children to partition into two groups. Unlike Study 3, children used physical objects in all conditions and were actively encouraged to place objects on the image when partitioning (to further encourage children to move objects between two groups only).

The interviewer used the materials to recount the story problem: a farmer owned two fields that were separated by a fence, but there was also a gate between the fields and this had been left open. The interviewer explained the problem: "*because cows kept wandering through the open gate, the farmer was confused; he didn't know how many cows could be in each field*". The interviewer explained that the task was to help the farmer by finding all the different ways in which his cows could be in the two fields. Children were then provided with an example in each condition before solving the partitioning problem.

6.2.3.1 Example

In the example question, the interviewer showed children a picture of three cows and explained the aim to *'find all the different ways in which the three cows can be in the two fields'*. The interviewer then asked children to watch how three cubes could be used to help find the different ways. No further attempt was made to make the representational link between the cows and cubes explicit. The interviewer then manipulated the cubes to present the children with the following partitioning solutions: 3 & 0, 1 & 2, 2 & 1 and 0 & 3, with the first part of the solutions referring to cows in the left field. This order was always used and was intended to show all possible partitioning solutions without prompting any specific strategy. The cubes always started just in front of the laminated picture of the fields and were then moved onto the image of the first field with the first solution of 3 & 0.

The interviewer manipulated the cubes differently according to the condition. In the *Constraints* condition, the interviewer moved only one cube at a time. In the *No Constraints* condition, the interviewer moved as many objects as were needed to create the solution configuration. The interviewer only moved one cube when showing the solution 2 & 1 following 1 & 2 in both conditions; he did not swap over the cubes in the *No Constraints* condition.

Following the demonstration problem, it was explained to the children that the farmer then bought some more cows. The order of conditions was counterbalanced across children but the order of partition amounts was kept the same: 7 for the first problem and 8 for the second. It was decided to start with 7, similarly to Studies 3 and 4, to avoid any unnecessary prompting of partition into two equal groups as a first strategy. The interviewer counted out the appropriate number of cubes in front of the image of the fields and then asked the children to '*use the 7/8 cubes to find all the ways in which the 7/8 cows can be in the 2 fields'*. According to condition, the interviewer would then say how the cubes could be manipulated: *'for this question you can only move one cube at a time/move as many cubes as you like at a time'*.

The interviewer recorded solutions and gave prompts as in the previous study. However, if any children in the *Constraints* condition moved more than one object simultaneously, the interviewer would ask them to replace the objects, reminding them that in this question they could only move one at a time.

6.3 Results

6.3.1 Solutions

The distribution of group data was tested (Kolmogorov-Smirnov). As this revealed significant departures from normality, non-parametric analyses were then carried out in which Wilcoxon tests revealed a significant difference between the first problem (Mdn=6) and the second (Mdn=6) ($Z=-2.14$, $p<0.05$). This suggested potential learning effects, although when the data were re-coded to scores of 0-3 (using the coding from Study 1) the difference was not significant ($Z=-1.60$, p=ns). It is possible therefore that the difference was attributable to there being an additional solution when partitioning 8. As conditions were counterbalanced for order, further analysis was carried out on absolute scores. Wilcoxon tests revealed no difference between the total number of scores identified in the *No Constraints* condition (Mdn=6) and the *Constraints* condition (Mdn=6)

(Z=-1.52, p=ns). However, the distribution of scores did suggest possible ceiling effects (as illustrated in Figure 6.5).

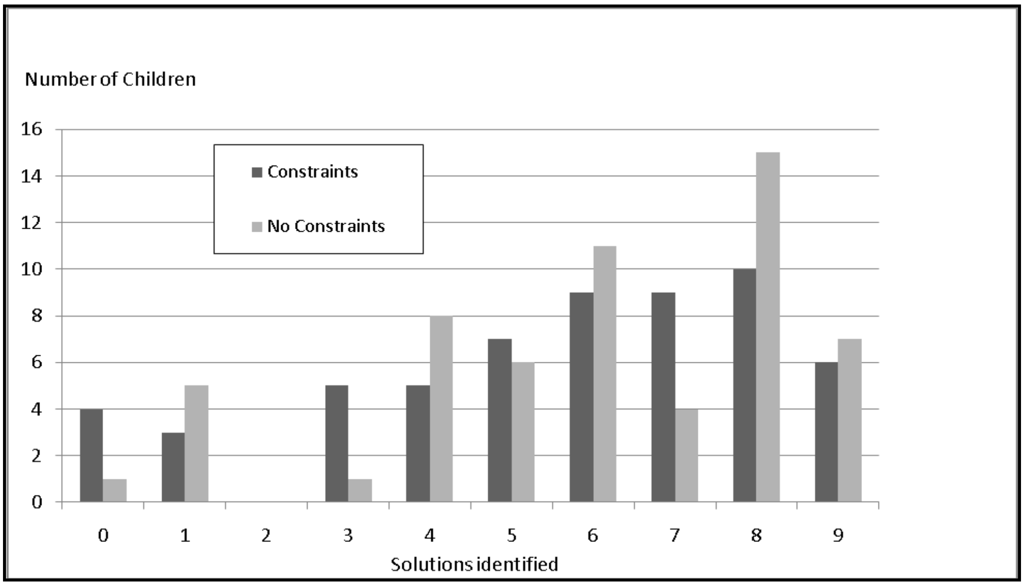


Figure 6.5: Number of children identifying number of solutions in Constraints and No Constraints conditions

6.3.2 Strategies

6.3.2.1 Coding

The correct solutions following the first solution were coded used the coding scheme developed in Study 2: into *compensation*, *commutative* or *other*. A *commutative* solution was scored if a solution was the reverse of the previous solution (e.g., 2 & 5 following 5 & 2). A *compensation* solution was scored if a solution differed by one from the previous (e.g., 2 & 5 following 1 & 6). This means that the total number of *compensation* solutions possible is 7 for partitioning 7, and 8 for partitioning 8 (first solution not included). There were 4 possible *commutative* solutions possible for partitioning 8 and 3 for partitioning 7[19].

6.3.2.2 Differences in total number of commutative and compensation solutions

All but 7 children identified at least one solution coded as *compensation* or *commutative*. Wilcoxon tests revealed that, as predicted, there were more *compensation* solutions in the *Constraints* condition (Mdn=2) than in the *No Constraints* condition (M=1.5), although the difference was not significant (Z=-1.30, p=ns). In contrast, and in line with predictions, there were more *commutative* solutions in the *No Constraints* condition (Mdn=1) than in the *Constraints* condition (Mdn=0) (Z=-3.29, p<0.005). As there were no significant differences in the total number of solutions given between conditions, no further analysis on proportional scores were carried out.

6.3.2.4 Equal partitioning

The first partitioning amount was odd (7). However, by coding as equal the two solutions closest to an equal partitioning (3 & 4, 4 & 3), it was possible to analyse the first solution given in both partitioning problems as being Equal partitioning or not. The majority of first solutions were coded as Equal partitioning: 72.5% of correct first solutions for partitioning 7, and 63.0% for partitioning 8. A Wilcoxon test showed this difference was

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19 The solution 4 & 3 after 3 & 4 (and vice-versa) was coded as compensation as discussed in Chapter 2

not significant (Z=-1.15, p=ns), although it did reveal that children identified significantly more Equal partitioning solutions in the *Constraints* condition than the *No Constraints*. (Z=-2.07, p<0.05).

6.3.3 Qualitative analysis

6.3.3.1 Children's actions

Video observations supported explanations for predicted differences in the number of *commutative* solutions found in each condition. In the *No Constraints* condition, children verbally identified nearly all *commutative* solutions when swapping over all objects simultaneously; either by picking up or sweeping groups with their hands (Figure 6.6).

Figure 6.6: Moving all objects simultaneously in the No Constraints condition

Observations also helped explain why there were no significant differences between conditions for *compensation* solutions although this was predicted. Although children only moved objects one at a time, they would often do so with great haste and many needed prompting to only move one object when they initially went to grasp several. Furthermore, children would often use both hands, moving objects individually but in quick succession (Figure 6.7a & b). Consequently, in the *Constraints* condition, successive changes to the representations followed quickly, so that children had very little time to see and possibly reflect on incremental adaptations. This lack of visual access to new representational states may also have been exacerbated by the fact that children's hands would often block their sight of several cubes.

Figure 6.7: a & b) Moving individual objects quickly in succession in Constraints condition and c) Moving two objects at a time in No Constraints condition

6.3.3.2 Effect of laminate image

Video observations also indicated why children's partitioning scores seemed higher than in previous studies (although the different samples make comparison difficult). As was predicted, the use of the laminate image helped partitioning into two groups - where most children would simply move objects from one field to another. It was interesting to observe, however, that many children (n=11) continued to remove objects from the laminate board after they had given a solution even though this was not demonstrated in the example. Several children actually changed strategy, beginning by moving objects off the laminate image to start with, and then simply moving objects from one field to the other later. It was expected that this behaviour would be less common when children could only move one object at a time; however, this was not found.

The way children moved objects onto the board may also help explain why more first solutions were equally partitioned in the *Constraints* condition. In this condition, children would move objects one by one in alternate fields (6.8a). In contrast, in the *No Constraints* condition, children tended to grab multiple objects to place on the board (Figure 6.8b). Although this usually resulted in an equal distribution, it was clearly not as effective as a 'one for one, one for the other' partitioning strategy fostered by the *Constraints* condition.

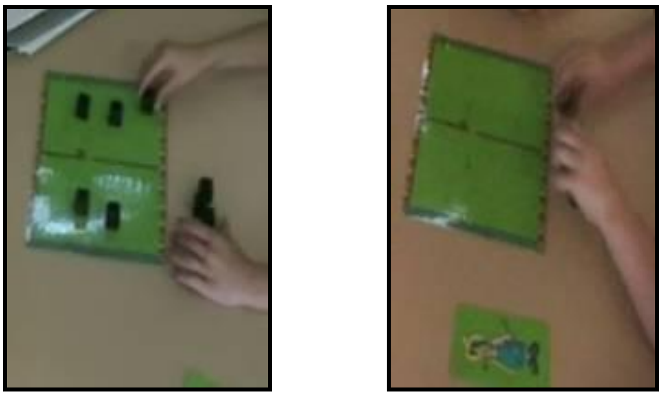

Figure 6.8: Moving objects for first solution a) one by one in Constraints b) as groups in No Constraints

6.4 Discussion

The aim of the current study was to examine the effect of constraining actions on children's partitioning strategies. The previous studies have shown the prevalence of two key strategies: *compensation* and *commutative*, and Study 4 then demonstrated how these actions reflected two mechanisms for adapting the representation: moving objects incrementally and moving all objects simultaneously. It was thereby predicted in this study that constraining children's actions would lead to an increase in *compensation* solutions – as children would be exposed to incremental changes in representation – and to a decrease in *commutative* strategies – as the costs of changing over quantities would now be greater.

The findings from this study showed that constraining actions did have a significant effect on strategies. As predicted, children identified a significantly lower number of *commutative* solutions when asked to move only one object at a time than when they could move as many as they wanted. There was also a greater number of *compensation* solutions in the Constraints condition although this difference was not significant.

Video observations provided further qualitative support for how constraining actions affected strategies. Most *commutative* solutions were identified by swapping over amounts – as observed in Study 4. This action was not possible in the *Constraints* condition where children would have had to move multiple objects in succession in order to identify such solutions. Preventing children from moving multiple objects simultaneously therefore significantly reduced a key strategy for identifying related solutions.

6.4.1 Limitations to the effect of constraining actions

Together, children in the *Constraints* condition identified 125 *compensation* solutions (about 28% of the total possible). However, children in this condition also identified many solutions not coded as *compensation*: 36 *commutative* and 109 *other*. In order to have identified these solutions, children needed to have 'passed through' as least two configurations that could have been identified and consequently coded as *compensation*. Video observations provide some indication of why children did not identify these intermediate states as valid solutions. Children in the *Constraints* condition often adapted the representation at great speed, moving objects one at a time but in quick succession using both hands. Furthermore, the positions of children's hands in both conditions may have prevented them from seeing all of the cubes. This may have hindered children's ability to reflect on the unique groupings created after moving individual cubes in the *Constraints* condition.

Another possible reason why children may not have identified incremental changes in the *Constraints* condition is that they had an additional cognitive demand of remembering to only move one object. It is difficult to assess how much effort this required, but the fact that the interviewer needed to prompt several children demonstrated difficulties in suppressing a tendency to grab multiple objects. A further possible reason why children did not identify as many *compensation* solutions as predicted in the *Constraints* condition may be due to the demonstration problem. In order to allow fair comparisons, the interviewer created the same configurations using three objects in both conditions. Whilst in the *No Constraints* condition this meant creating and identifying four configurations, in the *Constraints* condition the interviewer would create the same four configurations but make eight adaptations as changes were incremental. In this demonstration, therefore, the interviewer passed through incremental states without identifying them as partitioning solutions, and this may have increased the chances that children did likewise.

6.4.2 Effect of laminate image

It was intended in this study to encourage children to move objects between two groups by using the laminate image of the fields. Unlike the previous studies, the laminate image was intentionally placed in front of the children, and the demonstration involved moving objects on top of it. This seemed to have the desired effect: in contrast to previous studies where children often created more than two groups (as demonstrated in Study 4), nearly all children in this study moved objects onto the laminate image and continued by moving objects between two groups. Although it is difficult to compare performance between studies as different children took part, the children in this study, who were similar in age to those in the previous studies, identified a higher proportion of correct solutions; indeed there may have been ceiling effects. The laminated image therefore seemed to foster correct solutions by encouraging children to move objects between only two groups. This highlights the need to consider the effect of other external materials on children's interactions with different representations.

6.4.3 Initial configuration

It was decided not to start the session with objects already on the images in order to avoid influencing children's strategies by presenting an initial solution. It would alternatively have been possible to start with objects on the laminate image in a prearranged grouping. Indeed, starting with all objects in one field is arguably the most efficient way to identify all solutions using a *compensation* strategy. However, it is not clear whether children would identify the value of this initial configuration. Indeed, in a second study reported by Martin and Schwartz's (2005) it was shown that children began moving objects even when they were presented in a configuration reflecting the solution.

An unexpected finding in this study was the difference between the *Constraints* and *No Constraints* conditions on children's tendency to partition objects equally in their first solution. When children partitioned objects in the *Constraints* condition, they were significantly more likely to partition the objects into two equal groups. Video observations showed how moving objects one at a time encouraged many children to adopt a 'one for one, one for the other' sharing strategy. As this strategy is important in young children's developing understanding of one to one correspondence (Nunes & Bryant, 1996), it is an interesting possibility that the manipulative properties of objects may influence this strategy.

6.4.4 Summary

The extent to which children's understanding of additive composition will develop from experiences in the partitioning task is not known. However, it argued that, in accordance with PDL, children's actions with physical objects may lead to new ideas in this domain by increasing the use of related strategies. The current study examined the effect on strategies of manipulating the actions that children could make on the representation and found that, in line with predictions, constraining the number of cubes that could be moved at one time significantly affected strategies.

By suggesting that changing the type of physical actions that are possible may change the type of ideas developed, this study intends to extend the arguments of PDL. More specifically, constraining children's actions to moving one object at a time may reduce children's tendency to use strategies that reflect moving multiple objects (such as the *commutative* strategy in the partitioning task). The findings also suggest that constraining actions may encourage children's strategies that reflect incremental changes to the representation (such as the *compensation* strategy in this task).

Although children identified more *compensation* solutions in the *Constraints* condition, the difference was not significant as was predicted. A number of potential reasons for this were identified, many of which are attributable to the constraints being external: children were required to remember how they should manipulate objects. Even though children knew the interviewer was watching their actions, it was still necessary to provide occasional prompts. Such one to one attention would be rather impractical in a classroom context, and it may be possible instead to provide a representation that only allowed children to move one object at a time. For example, if the activity was carried out in a larger area (e.g., school playing field) with much larger cubes, children may only be physically able to move one object at a time. Alternatively, with certain physical designs, such as a bead string (illustrated in the discussion in Study 1) it is difficult to move more than one object at a time (due to friction).

Another way to externalise manipulation constraints is to use a graphical user interface. As the designer chooses what actions are possible, it is easy to control how and how many objects can be manipulated. Using a graphical user interface in this way, it is possible to constrain children's actions so they can only move one object at a time. It is also likely that this form of interface would affect the speed at which children could move objects. Although children as young as four are able to use actions such as drag and drop using a mouse (Donker & Reitsma, 2007), the fine motor control required means that the movement of objects will be slower than the actions observed with physical objects in this study. It is possible that using a different interface such as touch screen would facilitate actions, although the slower manipulation involved in using a mouse may actually be beneficial for this partitioning task. It has been argued that a key reason why children did not identify many valid intermediate representational states was because they moved objects too quickly and, furthermore, that their hands may have hindered their ability to see all the cubes. Consequently, it is possible that manipulating objects indirectly through a mouse would lead children to identify more incremental changes because manipulation would be slower and children's hands would not get in the way of seeing the representation.

Chapter 7

The Effect of Constraining Actions using a Graphical Interface on Children's Partitioning Strategies- Study 6

7.1 Introduction

Studies in this thesis have shown that physically manipulating representations can help children identify multiple ways in which to partition numbers and, furthermore, that use of materials may foster two key strategies allowing children to relate consecutive solutions: *compensation* (where objects are moved incrementally from one group to another) and *commutative* (swapping over whole groups of objects) of objects. From this it was predicted in the last study that constraining children's actions so that only one object could be moved at a time would raise the prevalence of *compensation* and reduce the prevalence of *commutative* solutions. Indeed, children did identify significantly fewer *commutative* solutions when their actions were constrained, but although they identified more *compensation* solutions, the increase was not found to be significant. It was argued that this finding might be explained by the quick adaptations that children made with physical cubes - thereby minimizing the amount of time children could see changes to the representation, and the cognitive demands of having to remember the instructions for manipulating objects. This led to the suggestion that it may be possible to help children identify incremental changes to the representation by a) increasing the amount of time children could see changes and b) externalising manipulation constraints to the external representation (thereby reducing the cognitive demands of having to remember to move only one object at a time).

It is possible to use particular physical materials to influence children's actions. By using large objects, for example, children may only be able to move one at a time and consequently take longer to make changes. However, it is also possible that it would then be more difficult for children to see all the objects at the same time, thereby hindering their ability to identify different configurations. Another way in which it is possible to constrain the actions children make on a representation is to use a graphical user interface. It was discussed in the literature review that computer or 'virtual' representations provide a means to design what actions are permitted. This would make it possible to constrain both the number of objects that could be manipulated simultaneously, as well as the time taken for each manipulation.

The literature review also described other potential benefits of virtual manipulatives, such as a means of providing children with dynamically linked representations. With the potential benefits attributed to virtual manipulatives, it is important to consider how this form of interaction may influence children's actions and, importantly whether anything might be lost in terms of the perceptual and manipulative properties of physical manipulatives. According to Kaput (1992), there is limited evidence that physical representations present any unique advantages for problem solving in mathematics. Indeed, this assertion has been supported by various studies attempting to compare the use of physical and virtual representations in learning activities (e.g., Klahr et al., 2007; Triona & Klahr, 2003; Zacharia & Constantinou, 2008). Furthermore, the fact that Martin (2007) has applied PDL to virtual manipulatives does suggest that actions on the representations do not need to be made through direct physical interaction.

Previous studies, therefore, suggest that the benefits of physically manipulating representations can be extended to virtual manipulatives. Another possibility is that the design of these studies has not been able to detect important differences. Indeed, in studies comparing physical and virtual manipulatives, it is often unclear, what differences are expected. Even if there are cognitive differences resulting from different forms of interface, the task may not be sufficiently demanding to detect these differences. For example, although it was shown in Chapter 5 that children often touched objects to help offload the cognitive demands of keeping track of their position, children would also sometimes just point to objects, suggesting that any cognitive benefits of this tactile information for the partitioning task are small (or even negligible).

As well as touching objects when counting, Study 4 demonstrated other instances where certain affordances of physical objects could be identified in problem solving. Such instances included stacking objects, touching objects to remember to move them next, and moving the position of objects in relation to the body. It was unclear, however, how significantly these instances affected children's strategies, and therefore unclear what impact there would be from the use of an interface that did not make such physical affordances available. Study 4 did however reveal one property that seemed to significantly affect problem solving strategies – the number of objects moved at a time.

It was shown in Studies 4 and 5 that the two key strategies for identifying related solutions reflected two types of actions: moving objects one at a time and moving multiple objects simultaneously. This finding has important implications for the use of different forms of interface that may affect how single or multiple objects can be manipulated. For example, with a standard mouse controlled computer, there are various design options for selecting and manipulating objects. 'Drag and drop' is a method in which even young children are able to select and move single objects (Donker & Reitsma, 2007). There are also different ways to allow multiple objects to be selected and moved (e.g., 'lassoing'). A key feature of a graphical user interface is that it is possible to design what actions are possible. Such an interface thereby has the potential to constrain manipulation on representations.

It was shown in Study 5 that constraining the number of objects that could be moved at a time significantly influenced strategies – children identified significantly less *commutative* solutions. Contrary to predictions, however, constraining manipulation did not lead to a significantly greater number of *compensation* solutions. It was argued that this may be attributable to the speed at which children moved individual cubes, and the fact that children were required to remember the constraint rule. Using a graphical user interface to constrain manipulation may therefore overcome these limitations. By manipulating representations using a mouse, it is not only possible to externalize rules of manipulation (which children do not then need to remember), but likely that it would take children more time to move objects using a method such as drag and drop than by moving objects physically. It might be predicted therefore that manipulating representations through a graphical interface, where objects can only be manipulated singly, would lead to significantly different strategies than manipulating representations physically. It may be expected that the additional demands of having to move several objects one at a time would lead to children identifying significantly fewer *commutative* solutions using virtual representations. In contrast however, children may identify significantly more *compensation* solutions when constraints on manipulation limit them to making only incremental changes to the representation, and also, significantly, when the demands of moving objects with the mouse mean that they would be seeing each representational state for a longer time.

7.1.3 Study aims and predictions

This study aimed to examine the effect of constraining manipulation on children's partitioning strategies. By using a graphical interface to constrain manipulation to moving one object at a time, it was predicted that children would identify more *compensation* and less *commutative* solutions than when using physical materials (when manipulation is unconstrained).

*7.*2 Method

7.2.1 Design

The study used a mixed design with Representation (Physical/Virtual) as the within subjects variable and Age group (Reception, Year 1, Year 2) as a between subjects variable. All children solved two partitioning questions, one in each condition. The order of conditions was counterbalanced across participants. Verbal solutions were scored as being correct (and unique) or not. The number of *compensation* and *commutative* strategies were then coded from correct solutions as described previously.

7.2.2 Participants

Sixty-five children took part in this study (36 girls and 29 boys, range 57 months to 92 months; M=73.1 months; SD=10 months). In order to compare any developmental differences between the use of representations, participants were children from

Reception, Year 1 and Year 2. Children attended a local infant school in Nottingham whose parents had returned a consent form allowing video data to be captured (54% positive response rate). The percentage of children receiving free school meals is within the national average (a measure of Social Economic Status) and the proportion of pupils with learning difficulties is slightly lower than the national average. Because class sizes are limited to 30, these three year groups were actually split across five classes with two Reception classes, one Year 1 class (lower ability), a mixed Year 1/2 class (higher ability Year 1, lower ability Year 2) and a higher ability Year 2 class.

7.2.3 Materials and Procedure

Sessions took place in a room adjoining one of the classrooms. Children were interviewed individually, and were reasonably familiar with the interviewer from previous observational work in the school. The structure of the partitioning problem was kept the same as the partitioning problem used throughout this research; however, the story vignette from Study 2 was used again for this study. This is because, unlike the farmer and two fields scenario used in the previous study, the scenario of a man and two bowls in Study 2 seemed to be less constrictive, i.e. children often created more than two groups. This was considered important in investigating possible representational differences for creating groups in this study.

In the task, the interviewer explained how a man had bought some bananas and was thinking of all the different ways to keep the bananas in his two bowls. A laminate picture was presented showing the character between two bowls (coloured red and green – Figure 7.1). The picture was on folded laminate paper placed on the left hand side of children's workspace for both the Physical and Virtual conditions (children were not allowed to place cubes on the image in the Physical condition).

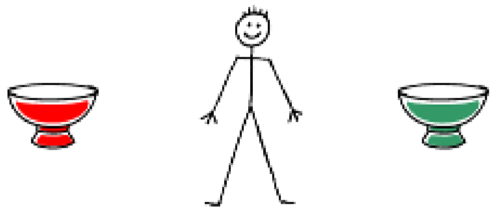

Figure 7.1: Laminated image of character and bowls

Similarly to the previous studies, the interviewer explained the task in each condition by first demonstrating the partitioning of three objects, showing children the different ways the character could partition three bananas using three cubes in the Physical condition and three squares in the Virtual condition. The same order of partitions was used; 3 & 0, 1 & 2, 2 & 1 and 0 & 3. The objects in the Virtual condition were dark grey squares with a thick black border (to help distinguish overlapping squares). These squares were aligned horizontally in the centre and covered about half of the screen width. The physical cubes were presented in a left to right line in front of the children. The virtual materials squares could be manipulated individually by drag and drop (left mouse button held down to drag, released to drop). There was no way of moving objects as a group. These materials were created in Macromedia Flash, exported as Shockwave Flash files and opened in Adobe Flash player; full screen size.

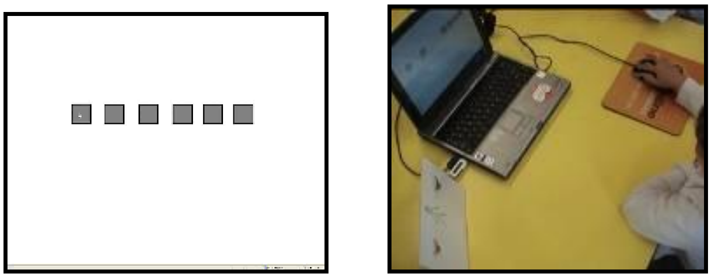

Figure 7.2: Screenshot and set up of Virtual condition

After the demonstration, the interviewer explained that the man bought six bananas the next week and was thinking about all the different ways he could keep them in his two bowls. The decision to use six bananas was a) to make the task easier for the younger children used in the study and b) to provide an interesting comparison with previous studies by starting with an even number. The interviewer then asked children to "*use the cubes/squares to find all the different ways the 6 bananas can be in the two bowls*". After the first problem, the interviewer provided an example (using 3 again) in the other condition before the final task of partitioning 7.

The prompts provided were the same as for the previous study except for one key difference. It was decided in this study to reduce the prompts for children to identify more than one solution. If children paused for ten seconds, rather than ask *"is that all the ways or can you think of more ways"*, the interviewer simply asked *"are you still thinking?"* It was agreed with the teachers of the school that this question would help establish whether the children were still thinking about another solution without providing a strong prompt for them to identify more solutions.

7.3 Results

7.3.1 Correct Solutions

The distribution of group data was tested (Kolmogorov-Smirnov). This revealed significant departures from normality, and non-parametric analyses were therefore carried out. Figure 7.3 helps illustrate why the data were non-normal despite the relatively large data set. Many children in the first two year groups (11 in Reception, 9 in Year 1) identified just one solution in both problems, whereas no children in Year 2 identified just one solution in either condition.

Wilcoxon tests were carried out on the number of correct solutions and revealed no significant differences between the Physical (Mdn=4) or Virtual (Mdn=4) conditions ($Z=-0.11$, $p=ns$), similarly, no differences were found when these groups were broken down by age group. There were also no differences between the number of correct solutions identified for the first and second partitioning problems ($Z=-0.74$, $p=ns$).

Kruskal-Wallis tests revealed Year group effects for correct solutions in both the Physical ($\chi^2(2)=21.72$, $p<0.001$) and Virtual ($\chi^2(2)=21.43$, $p<0.001$) conditions. Mann-Whitney tests showed that although Year 1 children identified more correct solutions than Reception children, the difference was not significant for either the Physical ($U=229$, $Z=-1.72$, $p=ns$) or Virtual ($U=239$, $Z=-1.50$, $p=ns$) conditions. In contrast, children in Year 2 identified significantly more correct solutions than Year 1 in both the Physical ($U=70$, $Z=-3.34$, $p<0.001$) and Virtual ($U=60.5$, $Z=-3.61$, $p<0.0005$) conditions. Clearly, this large difference is attributable largely to the fact that all Year 2 children identified more than one correct solution in each condition.

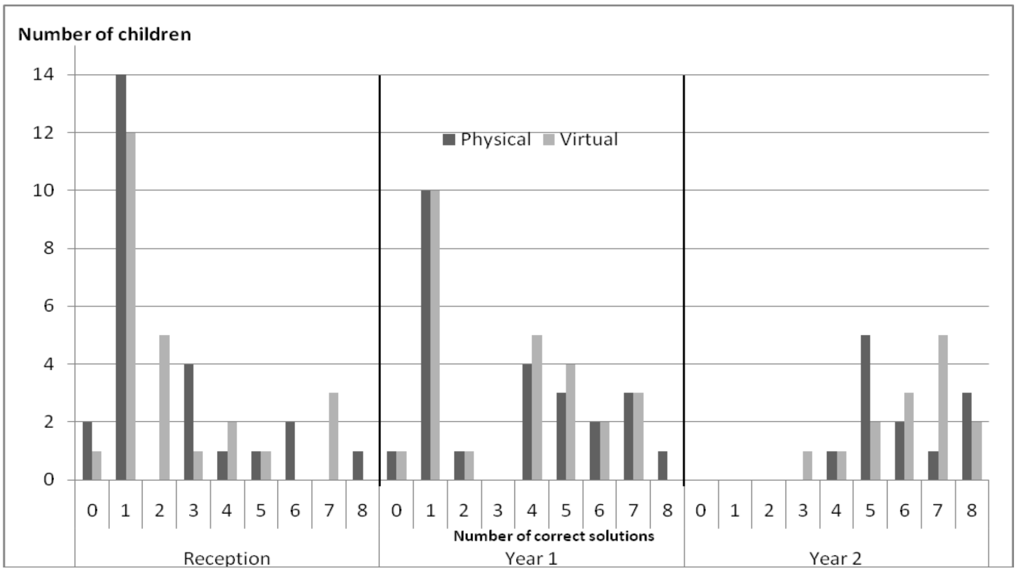

Figure 7.3: Frequency of children for partitioning scores in the Physical and Virtual conditions in each age group

7.3.4 Strategy

The correct solutions (after the first solution) were coded according to strategy: *commutativity*, *compensation* and *other*. As the number of solutions identified by children in Year 1 and Year 2 was generally low, there was limited data to compare strategy use in these two age groups. Table 7.1 illustrates the number of children identifying at least one strategy solution using physical or virtual materials. Although this data is in the direction of the study hypotheses (more children identifying *compensation* solutions in the Virtual condition, more children identifying *commutative* solutions in the Physical) these difference were not significant. In contrast, the large number of strategy solutions identified by children in Year 2 allowed comparisons between conditions for strategy use (Wilcoxon). In line with predictions, there were significantly more *compensation* solutions ($Z=-2.14$, $p<0.05$) in the Virtual condition ($Mdn=4$) than Physical ($Mdn=2$) and significantly more *commutative* solutions ($Z=-2.00$, $p<0.05$) in the Physical condition (Mdn=2) than Virtual (Mdn=0).

Table 7.1: Number of Reception and Year 1 children identifying at least one strategy solution in the Physical and Virtual conditions

		Compensation	Commutative	Other
Reception (n=25)	Physical	4	2	9
	Virtual	8	1	7
Year 1 (n=25)	Physical	10	6	10
	Virtual	15	2	10

7.4.3.3 Equal partitioning

As indicated previously, a large proportion (82%) of first solutions for partitioning 6 were equally partitioned (3 & 3) in the first solution. This proportion fell to 51% for partitioning 7. This might be expected as 6 is an even number and 7 is an odd number (although the coding scheme means that both 3 & 4 and 4 & 3 were coded as Equal partitioning for 7). However, it is interesting to note that the fall in proportion of Equal partitioning solutions was not the same for each age group. Wilcoxon tests revealed that the number of Equal partitioning solutions between the two tasks was not different for the Reception children ($Z=-1.34$, p=ns). However, there were significantly fewer Equal partitioning solutions in the second task (partitioning 7) than the first (partitioning 6) for both the Year 1 ($Z=-2.67$ $p<0.01$) and Year 2 children ($Z=-2.33$, $p<0.05$). There were no significant differences for any age group in the number of Equal partitioning solutions identified between the Physical or Virtual conditions.

7.4.4 Qualitative analysis: Comparing use of physical and virtual representations

7.4.4.1Visuo-Spatial characteristics

Observational analysis of the video data showed that, as might be expected, children solved the partitioning problems by moving objects into two spatially separate groups using both physical and virtual materials. Objects could be manipulated in three dimensions; although, with the exception of two children who created towers with the cubes (e.g., Figure 7.5a), physical objects were manipulated on the horizontal surface of the table. Although some children used more space (e.g., Figure 7.5b), the limited number of objects meant that children did not generally require the extra work space afforded in the Physical condition. Indeed, apart from the occasional object moved slightly off screen (e.g., Figure 7.6a), the limited screen size did not seem to present problems. There were also no clear difficulties presented by the two dimensional nature of the squares, although one child did enumerate the objects incorrectly when one square was 'hidden' behind another (see Figure 7.6b).

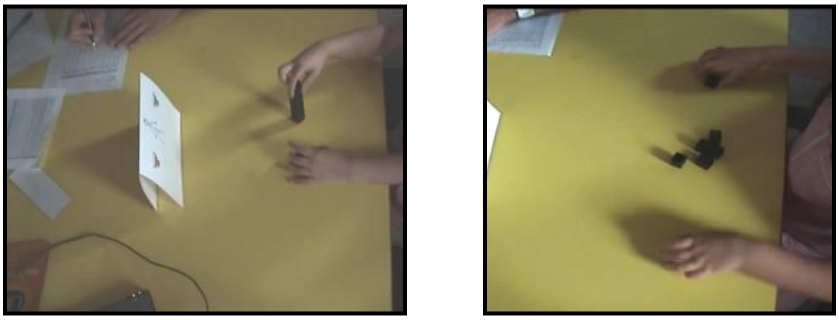

Figure 7.4: a) Stacking cubes b) use of wider space in Physical condition

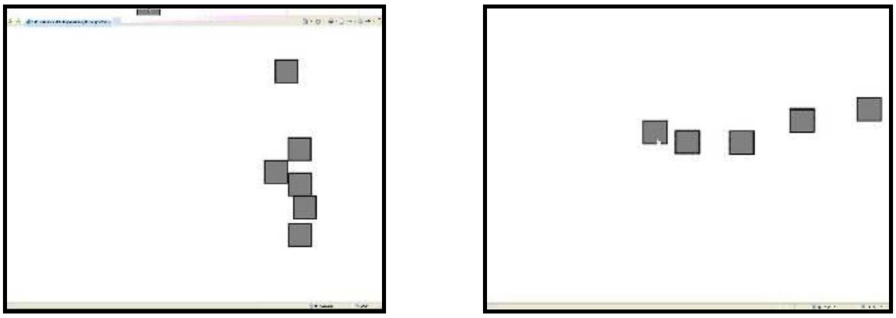

Figure 7.5: Squares a) moved off screen and b) hidden behind other in Virtual condition

There did seem to be one clear visual advantage of the mouse interface in the Virtual condition – children always had a clear view of objects manipulated on screen. In contrast, when manipulating physical objects, children's hands and arms would often block their line of sight (e.g., Figures 7.7). Although this was not a problem when children wanted to count objects (they could just move their hands away), children's hands did obscure the representation much of the time and they would often therefore move objects from one group to another without actually seeing the resulting configuration.

Figure 7.6: Hand obscuring view of configurations in the Physical condition

7.4.4.2 Tactile or haptic characteristics

Children often touched the cubes or made a touching gesture to support counting (Figures 7.8a). In the Virtual condition, children used the mouse pointer in a similar fashion, hovering over each object when counting. However, this action with the mouse did seem to place greater demands on fine motor control skills and several children preferred to point to objects directly on the screen when counting (Figure 7.8b). If touching objects helped children offload the demands of keeping track of them, it might be expected that a larger number of partitioning errors would be found in the Virtual condition. However, this was not the case, although it is possible that the small number of objects used in this study minimized any benefits of such tactile information.

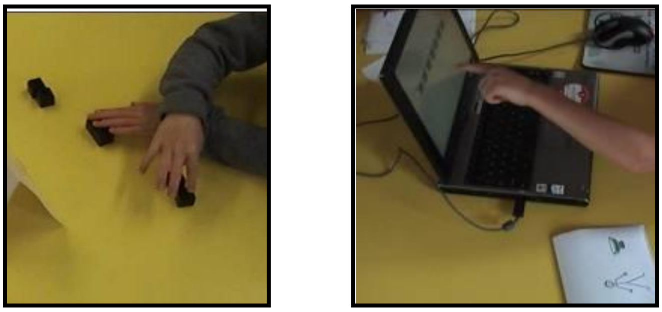

Figure 7.7: a) Children touching objects and b) The screen to support enumeration

7.4.4.3 Manipulative characteristics

In the Physical condition, children manipulated cubes in several ways, including picking them up individually, and sliding them along the table using the side of the finger or hand (e.g., Figure 7.9). Children would often use both hands, notably when partitioning amounts into two equal parts at the start. When identifying *commutative* solutions, children would usually manipulate both groups of objects simultaneously, as described in the Study 4. Several children also moved and counted cubes in twos. It was interesting to observe that most children seemed to make continual contact with the cubes; often just fumbling with cubes when not actually making new adaptations.

Figure 7.8: Moving multiple objects using both hands in Physical condition

In contrast, manipulation in the Virtual condition was constrained to making clear and distinct changes moving one object at a time using the mouse. All children were able to do this, although some of the younger children had difficulties keeping the mouse button depressed when dragging objects, or needed help replacing the mouse if it reached the edge of the mouse mat. Furthermore, because children could only move one object at a time in the Virtual condition, there was no gathering up of objects after a solution (as was observed in the Physical condition). Interestingly, although no child used the physical cubes to create any pattern (beyond simple groupings), several children did begin to create patterns while problem solving in the Virtual condition (Figure 7.10).

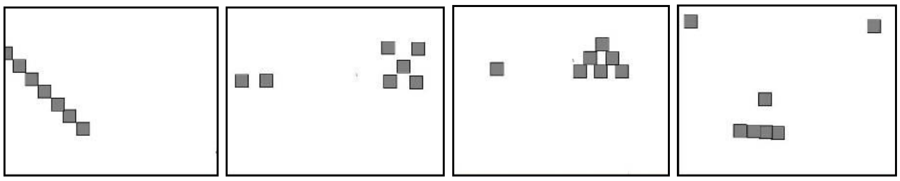

Figure 7.10: Patterns created by different children in the Virtual condition

7.4 Discussion

This study examined the effect of constraining children's actions on objects so that representational changes were incremental and slower. This was achieved by comparing children's partitioning strategies using physical cubes with those using virtual squares manipulated with a mouse. It was predicted that constraining actions in the Virtual condition would result in fewer *commutative* solutions and more *compensation* solutions. In the two younger age groups, more children did identify *compensation* solutions in the Virtual condition and more *commutative* in the Physical. Unfortunatley, the data was too limited to detect any significant differences. In contrast, analysis on the strategies of children in the older age group supported predictions. Children identified significantly more *compensation* solutions using virtual materials and significantly more *commutative* solutions using physical. . These findings have important implications for the theory of Physically Distributed Learning (Martin & Schwartz, 2005) by suggesting that changing what type of actions can be made on the representation may lead to different strategies and hence new ideas in this domain.

7.4.1 Multiple solutions

A key reason for lack of differences found between conditions for younger children seemed to be that many children only identified one correct solution. Although identifying a solution shows that nearly all children had a grasp of the task demands (i.e. to partition a whole into two parts), the lack of more than one solution raises the possibility that children did not fully understand the task demands to identify multiple solutions. It is possible that this finding differs from previous studies (where children did generally identify multiple solutions) as a result of the change in prompt from "*is that all the ways or are there any more ways?"* to "*are you still thinking?"* However, children were provided with several clues to identify multiple solutions – emphasis was placed on the initial explanation on identifying the different *ways* to partition, and was then demonstrated by partitioning three in multiple ways in the example. Furthermore *"are you still thinking?"* was still considered a prompt for children to continue. It also possible that children's concept of number is such that the notion that a whole number has multiple ways of being decomposed is difficult. Indeed, many children were younger than the age that additive composition is reported to fully develop (Bryant & Nunes, 1996). Another interpretation for this finding is that children are simply not used to providing multiple solutions to one question. Indeed, this reason was offered by several teachers when asked about the pattern of solutions. If this is the case, it might be regarded as a methodological limitation; although an alternative view is that exposure to multiple solution problems plays a role in numerical development. Indeed, interventions aiming to encourage children to identify multiple solutions have often been effective (e.g., Ainsworth, Wood, & O'Malley, 1998). It would be interesting to test whether a more general intervention to encourage multiple solutions would transfer to this task.

7.4.2 Equal partitioning

The finding that even the older children began by partitioning equally (even though they then went on to identify multiple solutions) suggests that there is a strong initial propensity to divide the materials into two equal groups. This may reflect the affordance of the materials: that dividing the objects equally maintains visual symmetry. Indeed, in Study 4 children did not tend to partition equally in the initial *No Materials* baseline condition (although children partitioned an odd number in this condition and there was quite a small sample size). However, another strong possibility is that children's propensity to partition objects reflects previous experience of partitioning objects – both in more informal sharing activities and in more formal mathematical tasks.

Closer analysis of the children's initial adaptations highlighted the way in which many children began the problem in both the Physical and Virtual conditions by partitioning the presented objects one by one into two different groups. This meant that a configuration of '1 & 5' or '1 & 6' was often coded as the first adaptation. It would be interesting to investigate how easy it might be to draw children's attention to this configuration as a potential solution. Doing so may even affect further strategies by prompting children to identify incremental changes. Indeed, this may in turn foster multiple solutions since when children in this study gave only one solution it was always an Equal partitioning one.

It was also found in this study that all but the youngest age group significantly reduced the number of Equal partitioning solutions in the second task. Although there may be more complex interactional effects (older children may have known that seven could not be partitioned equally), it is possible that children's experiences in the first task may have helped them recognise that starting by partitioning equally was not the most efficient strategy. Therefore, it is possible that introducing prompts for children to identify a solution that is not equal partitioning initially may help them develop ideas about how to solve this problem. Consequently, younger children may benefit most from such a prompt.

7.4.3 Adaptations and interpretations

The Virtual condition changed the way children could manipulate objects. Changes were constrained to increments of one object at a time, preventing children from moving multiple objects as they did when identifying many *commutative* solutions in the Physical condition. As might be expected, changes were slower using a mouse, possibly explaining why the impact on strategies was greater than when children were simply asked to move objects one by one in Study 5. Furthermore, children may have benefited from the full visual access to changes in groupings of objects afforded by mouse interaction in the Virtual condition.

Although children were able to move objects more quickly in the Physical condition, observations suggested that children would often pause – fumbling with objects before making the next change. In contrast, actions with virtual objects were more discreet – manipulation was only in order to change the numerical groupings. Despite this, however, children did not identify more partitioning solutions in the Virtual condition. This may be because children were simply creating a specific solution they had in mind – i.e. following out a plan. Alternatively, children may have been exploring changes to the representation and interpreting their actions to inform their ideas for partitioning solutions. This latter explanation reflects the arguments of the theory of PDL: that actions lead to new ideas. If this is the case, it is again possible that prompts might be provided to encourage children to recognise and identify the valid solutions created by each incremental change to groups of objects. These prompts might consist of influencing children's actions on the representation – internally or externally constraining manipulation in order to create a delay between representational changes. Alternatively the prompts might be perceptual; drawing children's attention to the validity of the new solutions after incremental changes.

7.4.4 Potential role for technology to support partitioning

Clearly, a key source of support for understanding and solving the partitioning problem effectively can be provided by an adult or perhaps even a more able peer. For example, the simply verbal cue of '*can you think of any other ways?*' seemed to prompt children to identify more solutions in the previous studies. Unfortunately, such support, which also needs to be careful not to simply tell children what to do, is impractical in a classroom context. Instead, it may be possible to augment representations in order to provide prompts for children to develop their understanding of key concepts integral to the problem. These might address concepts that relate to identified difficulties throughout the studies, such as understanding that: there is more than one solution, that equal partitioning is simply one of many solutions, that identifying incremental solutions can help enumerate parts and keep track of the problem space, and that there is a limited but wide range of possible solutions. In order to support these task demands, it may be possible to use digital technology to augment the manipulative and/or perceptual features of the representation.

7.4.5.1 Varying manipulative properties

This and the previous study have examined the effect of constraining the manipulative properties of the representation and showed that, as predicted, this significantly affects children's strategies. It was argued that the virtual representation may have supported children by increasing the time they could see incremental changes of groupings. If so, longer exposure, by increasing the time to manipulate objects, may help them further.

This suggestion seems to reflect work showing how increasing implementation costs can foster planning (e.g., O'Hara & Payne, 1999). However, planning may be difficult for young children (Ellis & Siegler, 1997), especially when they only have incipient understanding. It was also shown in Studies 2 and 4 that the implementation costs of using paper did not improve children's planning. An alternative to increasing implementation costs might be to simply introduce a delay *after* each adaptation in order to foster children's interpretations of the representation state. In other words, if actions lead to ideas, delaying time between actions may encourage interpretation and development of new ideas. It is possible however, that introducing such delays between adaptations may frustrate children, especially as this behavior would probably not be expected.

7.4.5.2 Varying perceptual properties

It might also be possible to change the perceptual features of the representation to prompt certain strategies. The objects used throughout these studies have intentionally been perceptually invariant: the same size, colour and shape. However, it was discussed how the perceptual property of symmetry may still have influenced strategies, even if this was not the most efficient cue. It may be interesting to consider therefore how varying other perceptual cues may affect problem solving strategies. For example, including objects of different size, colour and shape may encourage children to group objects differently. It was argued previously that encouraging children to identify initial incremental changes might support problem solving. Perceptual prompts might therefore be used to encourage children to begin partitioning in ways other than an equal partitioning, or highlight how an incremental change in the way objects have been grouped is itself a unique numerical solution. Perceptual prompts might also help children to explore the whole range of different configurations.

7.4.6 Efficiency and innovation

Knowing what prompts to give children to solve the problem is difficult: too little prompting may lead children not to explore and develop ideas about the problem (as was the case with the many children who only identified a single solution). On the other hand, too much prompting may simply teach procedure at the cost of developing more conceptual understanding – an identified problem that can arise when teaching with manipulatives (Ball, 1992; P. Thompson, 1994). In this study for example, the Virtual condition constrained children's actions to move only one object at a time. This constraint consequently encouraged a more efficient strategy (*compensation*). However, in the Physical condition, although children could move multiple objects with ease, they still often moved one object at a time. Some children even began by moving multiple objects and then constrained their own actions to moving one at a time. There seems therefore to be trade off – promoting a more efficient strategy through constraining actions and promoting discovery by allowing children to constrain their own actions. This trade off is discussed by Schwartz, Bransford and Sears (2005) as a balance between *efficiency* and *innovation*. It is argued that the latter is important for transferring learning to new contexts. In other words, allowing children to identify their own best way of manipulating objects may have developed understanding that is best measured through transfer tasks than through measures of efficiency in this particular problem. Unfortunately, assessing transfer was beyond the remit of this thesis.

7.4.7 Summary

This study has shown that constraining children's actions on representations can significantly influence the strategies for identifying solutions in a partitioning problem. Constraining children's actions so that only one object could be moved at a time led children to identify more solutions that differed by one. Understanding and applying the concept that taking one from one part and adding it to another is important in numerical development, and it is possible, therefore, that using virtual representations such as those used in this study would best support the development of this concept. However, the gains accrued from greater exposure to this strategy in the Virtual condition must be considered in light of possible benefits from children constraining their own actions in the Physical condition.

Observations of children using the two representations suggested that although children took advantage of certain visual and tactile properties of the physical materials, such as touching or stacking them, these properties provided no great advantage over the virtual materials. It is possible, however, that these physical attributes confer a greater advantage in tasks in a different domain (one exploring three dimensional shapes, for example) or in tasks which present greater procedural demands such as partitioning larger amounts.

In order to investigate the role of manipulation, it was intended to match the physical and virtual materials for perceptual features such as size, colour and shape. As a result this study does not compare the relative value of physical and virtual materials since this reduces some of the key benefits of digital materials where properties can be designed to support learning. One possibility might be to encourage children to interpret configurations by introducing a delay after each manipulation. Alternatively, the possibility was raised of integrating specific perceptual features that could influence children's strategies. Such features might help children to recognise that there are multiple solutions, or to identify possible solutions from incremental changes made to the representation.

Chapter 8

The effect of augmenting representations with perceptual prompts on children's partitioning strategies- Study 7

8.1 Introduction

The previous study showed that constraining children's actions on a numerical representation could lead to differences in the strategies used for identifying ways to partition a number. As predicted, children were more likely to identify solutions where the parts differed by one (*compensation* solution) when their actions were constrained to moving one object at a time using a graphical interface. It has been argued that increasing the use of this strategy has important implications for learning in this domain, as it not only reflects an important procedure used to facilitate calculation in various part-whole problems but also emphasises an important numerical concept – that taking something from one part and adding it to the other leaves the whole unchanged.

It was shown in Study 5 that simply *asking* children to move one object at a time was not sufficient to increase the use of the *compensation* strategy significantly. Although the verbal instructions in the study meant children moved objects one by one, it is likely that the rapid way they actually moved the cubes (often using both hands) gave them insufficient time to look at each separate resulting change to the representation, and consequently therefore insufficient time to identify it as a new and valid partitioning solution. In contrast to this, when using the computer in Study 6, children's actions were constrained by the interface, and manipulation was slowed down by using the mouse. As predicted, this increased the likelihood that children would identify incremental representational states – demonstrated by the significantly greater number of *compensation* solutions. Despite this constraint however, children still identified many solutions that were not *compensation* (generally '*other*' solutions). In other words, children often generated a potential new solution by moving one object, but did not identify this solution verbally.

Identifying solutions that differ incrementally by one (*compensation*) is an efficient way of identifying different partitioning solutions. Importantly, children are able to quantify each part in relation to the previous solution – a strategy that can be applied in the absence of objects. It was shown in Chapter 7 that this strategy reflected more successful problem solving and was used significantly more by children with greater numerical ability. An important question therefore is how this strategy might be encouraged – how might children be prompted to identify incremental changes to the representation? Clearly, this could be achieved through verbal prompts. The interviewer could explicitly ask children to pause and reflect on the novel configuration each time an object was moved from one group to another. However, this form of prompting would be quite demanding and arguably impractical in a classroom context, in addition, it may be necessary to help children understand why it is beneficial to identify each change. Another approach would be to augment the graphical representation used in the previous study to provide prompts for numerical changes to the representation. If designed well, such prompts could not only foster an effective strategy without adult support but also help children understand why it is advantageous to identify each new numerical configuration.

8.1.1 Perceptual prompt for representational changes

There are many ways in which digital technology might be applied to prompt children to identify incremental solutions. Designing the most effective way needs consideration of what kind of effect to use, and when it should occur. The digital effect might assume a variety of forms, which, with a computer, might typically be visual or auditory. A visual stimulus might be beneficial for various reasons. It is, for example, less confusing to present multiple effects simultaneously, whilst effects can also be continuous if required (i.e. they can remain on screen for children to attend to). For this design, the aim of the effect is to provide a simple yet salient prompt for a change in the representation.

Colour is a feature that is simple to process and distinguish. One possible design approach would therefore be to use a change in colour to emphasise numerical changes to the representation. In other words, when children create a new configuration (i.e. move an object from one group to another), the objects themselves could change colour to emphasise the change in numerical grouping. This effect might therefore prompt children to recognise that a novel partitioning solution had been created.

A key challenge, however, is know how to define a 'new numerical configuration' so that effects can be presented appropriately. In the previous study, numerical groupings seemed to reflect the relative distance of virtual objects to each other. Although this form of spatial grouping could be programmed (for example, by using an algorithm where the groups were identified by the relative distances of on-screen objects), there are potential difficulties in deciding when objects should be defined as belonging to a certain group. One solution, therefore, is to generate a clear and discrete rule. For example, objects linked together are considered grouped; objects not linked are considered as not grouped. Although objects could be linked in various ways, observations from the previous studies highlighted how children often placed objects linearly (horizontally) using both graphical and physical representations (see Figure 8.1). This seemed to facilitate enumeration by helping children keep track of the count amount.

Figure 8.1: Placing objects in a line to support enumeration

8.1.1 Design of materials to support partitioning strategy

Based on the design requirements discussed above, the graphical representation from the previous study was developed in two key ways. Firstly, by placing objects next to one another, or overlapping, it was possible to link them. This was intended to provide clear identification for a prompt when objects were considered grouped together. The second change was to provide a perceptual prompt to highlight numerical changes. The graphical objects were designed so that a change in the number of objects grouped together would result in a change in colour. It was decided that, as the change in colour reflected a change in quantity, it was appropriate for all objects grouped together to change colour (the cardinal principle – that each object is part of the set). It was also decided to use a different colour for each quantity represented. Clearly this would mean a potentially infinite number of colours, although for the purposes of this study, children would not be presented with more than ten objects at a time.

The use of colour to reflect quantity is not unique in mathematical materials. Indeed, Cuisenaire rods (a common mathematical manipulative), uses colour to reflect quantity (Figure 8.2). In this design, number is represented by length (1 unit=1 cm) and colour. It was decided to use the same 'colour to quantity' mapping of Cuisenaire rods as this helped to communicate the resource with the class teacher (who was able to use the resource after the study if wanted).

Figure 8.2: Cuisenaire rods

There are examples of virtual resources which use Cuisenaire rods for numerical tasks (e.g., http://www.arcytech.org/java/integers/). What was unique about the representation created for this study, however, was that it was possible to decompose groupings, which would then change colour accordingly. The quantity to colour mapping used for the design is shown in Table 8.1 Figure 8.3 illustrates how the change in grouping resulting from moving an object from one group to another results in a change in colour (5 & 2 changed to 4 & 3).

Table 8.1: Colour to Quantity mapping used in the Study 7 virtual materials

Quantity	Colour	Image
1	White	
2	Red	
3	Light Green	
4	Purple	
5	Yellow	
6	Dark Green	
7	Dark Grey	
8	Brown	
9	Dark Blue	
10	Orange	

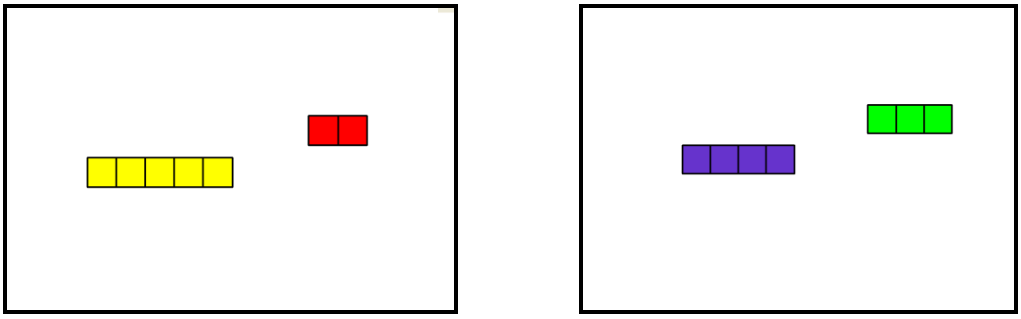

Figure 8.3: Screen shots of virtual squares in Colour Prompt condition

8.1.2 Summary and predictions

Study 6 showed that constraining manipulation to just one object at a time using a graphical interface increased the number of *compensation* solutions children identified. It was also shown that many intermediate representational states were not identified as potentially unique and valid solutions. This study examined the potential to help children identify changes to the representation by augmenting the representation with perceptual prompts. In order to examine whether such a prompt did help children identify changes, this study compared two representations: virtual squares *without* a perceptual prompt (white squares that did not change colour), and virtual squares *with* a perceptual prompt. It was predicted that more incremental changes to the representation (*compensation* solutions) would be identified using the virtual objects with the colour prompt than without the colour prompt. A pilot study was first carried out to ensure that children were able to manipulate the objects appropriately.

8.2 Pilot study

8.2.1 Method

8.2.1.1 Design

A within subjects design was used with Prompt (*Colour Prompt/No Prompt*) as the independent variable. The order of condition was counterbalanced. The primary dependent measure was the verbal solutions provided which were then coded for strategy (*compensation/commutative/other*) using the scheme developed in Study 2.

8.2.1.2 Participants

20 children took part in this pilot study (9 girls and 11 boys, range 69 to 91 months; M=76.1months; SD=6.19 months). Children were randomly selected from two year groups: Year 1 and Year 2, at a local infant and primary school in the Nottingham area. The percentage of children receiving free school meals is average and the proportion of pupils with learning difficulties is slightly lower than average. Because class sizes are limited to 30, these two year groups were actually split across three classes: a Year 1 class (lower ability), a mixed Year 1/2 class (higher ability Year 1, lower ability Year 2) and a higher ability Year 2 class.

8.2.1.3 Materials

- *Virtual squares*

The squares used in both conditions were identical in size and shape to the squares used in Study 6 (i.e. 1.5cm^2). However, whilst the squares in the No Prompt condition were always white, the squares in the Colour Prompt condition were designed to change colour according to the number of squares grouped together. In order to provide a clear and discrete means of identifying when squares were grouped, they were programmed to 'snap to join' (i.e. if an object was released in a position touching or overlapping another object, it would automatically move so it was joined horizontally). Accordingly, objects joined together were considered as grouped, objects not joined: as not grouped. Squares could only be joined horizontally in a line. If an attempt was made to attach the squares vertically, the joining square would jump to one end of the horizontal line of squares (thus avoiding the need to allow all objects to relocate whenever there was 'insertion' into an existing group).

The representation in the No Prompt condition could be manipulated identically to the Colour Prompt condition, the only difference being that squares would not change colour (they remained white). In the Colour Prompt condition, squares would change colour according to the number of squares attached using the mapping shown in Figure 8.4. In this mapping, only individual objects were white, as shown in Figure 8.5 which contrasts the same numerical groupings in the two conditions.

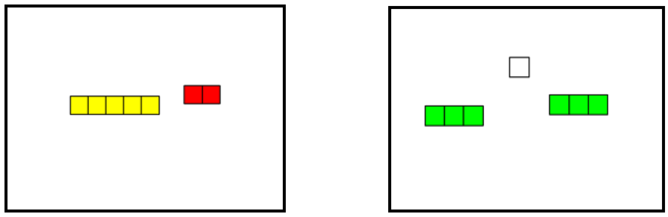

Figure 8.4: Screen shots of virtual squares in Colour Prompt condition

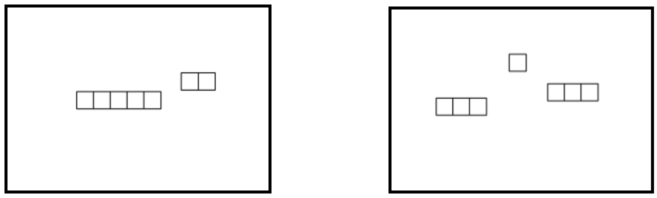

Figure 8.5: Screen shots of virtual squares in No Prompt condition

- *Other materials*

It was decided to use the same story context as for Studies 3 and 5 – a farmer trying to find all the ways a number of cows can be in two fields – as this context seemed to help communicate the need to create only two groups. The materials consisted of images of the farmer, the cows and the fields. The field image was placed on the keyboard throughout the tasks (the keyboard was not needed as input was through a mouse) while the keyboard itself was placed directly in front of the screen (thereby helping prompt children to partition objects into two groups). Although it was possible to provide an on screen image of the fields, it was decided not to in case this hindered how easily children could see the colour prompts of the objects in the Colour Prompt condition.

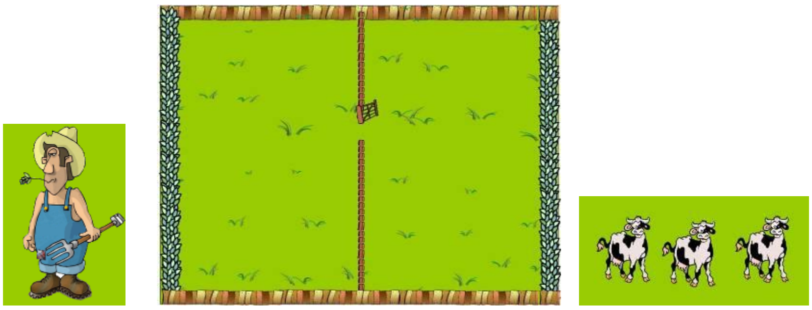

Figure 8.6: Materials used for partitioning story context

8.2.1.4 Procedure

- ***Example partitioning question***

Similarly to the previous studies, the interviewer preceded the task in each condition by demonstrating partitioning of three objects. The interviewer first presented the story context using the laminated images, and then asked children to watch whilst he used '*these three squares*' to find all the ways that 3 cows can be in the two fields. The order of condition was counterbalanced between children. The squares that were presented reflected the condition and were initially presented in the centre of the screen, attached in a horizontal line. Whilst these squares were white in the No Prompt condition, they were light green in the Colour Prompt condition (corresponding to three – see Table 8.1).

Similarly to previous studies, the interviewer demonstrated moving the squares into groups on the left or right side of the screen reflecting the following solutions in this order: 3 & 0, 1 & 2, 2 & 1 and 0 & 3. This quantity change was also illustrated in a change of colour as two grouped objects were both coloured red and an individual object would be white. The interviewer did *not* make reference to the colour or change of colour in the Colour Prompt condition.

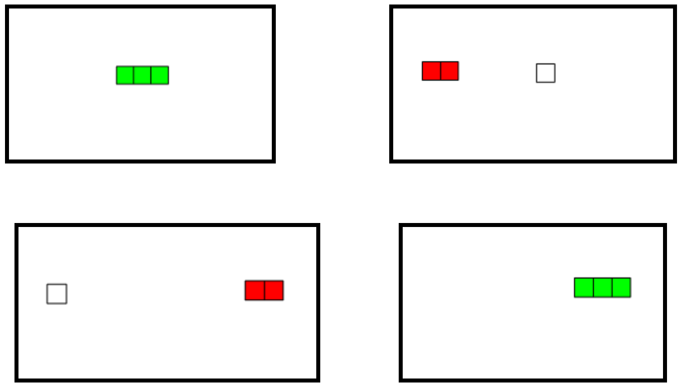

Figure 8.7: Screen shots of partitions in the Colour Prompt condition example

- *Partitioning problem*

Following the demonstration problem, the interviewer explained to the children that the farmer then bought some more cows; he now had 7 cows. The order of partition amount was the same in all conditions; 7 for the first problem and 8 for the second. The interviewer started the program with the appropriate number of squares for that condition. Similarly to the example, squares were presented attached in the centre of the screen. Therefore, whilst squares were white in the No Prompt condition, in the Colour Prompt condition they were initially dark green when partitioning 7 and dark grey when partitioning 8.

The instructions and prompts were similar to previous studies; children were asked to use the squares to find all the ways that 7/8 cows could be in the two fields. As for previous studies (other than Study 6), children were given prompts to identify more solutions if they stopped after the first solution: "*is that all the ways or can you think of any more ways*?" The prompt was given to encourage children to identify multiple solutions, thereby creating more solutions in which to compare strategies between conditions.

After the first problem, the interviewer returned to the example question with the three squares for the other condition before presenting the final problem requiring children to partition 8 using the squares for that condition. Structuring the study in this way (demonstration; then problem for the first; then second condition) ensured that conditions were counterbalanced.

8.2.3 Results

8.2.3.1 Correct Scores

Kolmogorov-Smirnov tests showed that the correct scores in both conditions were significantly non-normal; non-parametric analysis was therefore carried out. A Wilcoxon repeated measures test revealed no significant differences between correct scores in the Colour Prompt (Mdn=6) and No Prompt conditions (Mdn=5.5) (Z=-0.51, p=ns) or between partitioning 7 and 8 (Z=-1.07, p=ns).

8.2.3.2 Strategies

Solutions were coded according to strategy (*other* for all solutions not coded as *compensation* or *commutative*). Wilcoxon tests revealed no differences between Colour Prompt and No Prompt conditions for *compensation* (Z=-0.60, p=ns) or *other* (Z=-0.09, p=ns) strategies. As expected considering the use of the graphical user interface, children identified very few *commutative* solutions (5 in the Colour prompt condition and 4 in the No Colour promt). Median and IQR scores are shown in Table 8.2.

Table 8.2: Median (IQR) scores for strategies in Colour Prompt and No Prompt conditions

	Colour Prompt	No Prompt
Compensation	2 (1,3)	2 (0,3)
Commutative	0 (0,0.75)	0 (0,0)
Other	1.5 (1,3.75)	2 (1,3)

8.2.3.3 Qualitative observation

Observations of children's actions highlighted a key design issue in this pilot – when children manipulated squares in the Colour Prompt condition, they often did not attach squares: they moved squares close together but not joined as shown in Figure 8.8. As squares in this condition only changed colour when attached, this clearly compromised the independent variable differentiating conditions. Several children actually separated all objects in the Colour Prompt condition such that all squares were white, as they would be in the No Prompt condition.

There were two key observations about the reasons objects were not attached. Firstly, children had difficulties in attaching objects. The squares were programmed to attach when touching, but it appeared that children were expecting objects to attach simply when they were close (whereas they actually had to be touching). Consequently, children would tend to move squares gradually closer to one another and often stop before they were attached. Although children in this study, as in Study 6, had no difficulty in moving objects using the mouse, attaching objects did seem to demand additional fine motor control as children had to move squares to a more specific position.

This problem seemed compounded by the fact that children did not generally realise that they could attach objects by simply dropping them onto another object rather than having to closely align the sides.

It was also clear that children often moved objects into new groups without trying to attach them. Figures 8.8a and 8.8b show examples where children have attached objects or have placed them close in the expectation of their attaching. The screenshots also show where children have moved objects into groups without attaching them. Figure 8.8c shows a situation where no objects have been attached, and highlights a key problem identified in this pilot. When children did not attach objects, the difference between the conditions was eliminated – cubes would not change colour when regrouped in the Colour Prompt condition.

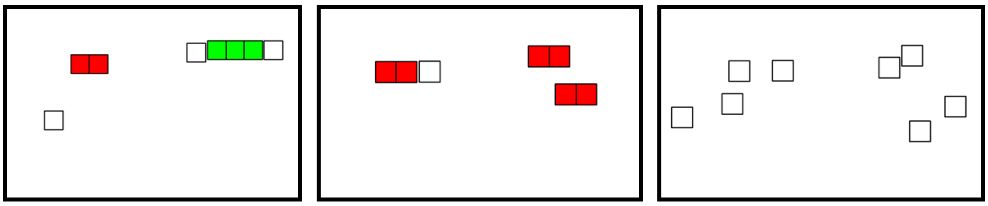

Figure 8.8: a), b), & c): Squares not attached in the Colour Prompt condition

There was also an indication that children were slightly confused by the change of representations from one condition to the next. Although a demonstration was provided before each condition, there were several occasions when children verbalised their expectation of a colour change for objects in the No Prompt condition following the Colour Prompt condition. This may be attributable to the fact that children were not given different instructions when they changed conditions that helped explain how the representation differed.

Another key problem was that children often appeared to be exploring the representation rather than using the objects to solve the problem. This may be understandable as children were not given any time in the study to 'play' with the representation in order to familiarise themselves with the way objects could be attached and the range of colours that could be generated. It is possible that providing children with time to explore the representation before using it to solve the problem may reduce this potential distraction.

8.2.4 Discussion

This pilot study aimed to identify any methodological issues in the proposed study examining the effect of perceptual prompts to support partitioning strategies. Twenty children took part in the study, which was considered a large enough number to provide an indication of any main effects. Although children did identify more *compensation* solutions in the Colour Prompt condition, the difference was small and non-significant. However, several issues were raised that might explain why the predicted differences between conditions were not found.

The key problem highlighted in this pilot was that children often grouped objects by moving them close together but not attaching them. Unlike the previous studies, the design of the materials in the Colour Prompt condition required children to attach squares. It would be possible to adapt the materials to address this issue: for example programming squares to change colour when within a certain proximity. However, this could introduce new problems – the technology might define objects as grouped when children had not intended to group them (and vice versa). Squares were designed to change colour when attached as this provided a discrete definition of grouping, yet children were not provided with any explanation of this digital behaviour, nor had they any opportunity to accustom themselves to this behaviour before using the objects in the partitioning problem. In addition, children were also required to accept that in the other condition, objects similarly attached did not change colour. It is entirely possible that the demonstration problem was insufficient for children to become familiar with the materials so that, as a consequence, not only were objects used in a way not intended (i.e. not attached to a group) but important cognitive resources may have been used up in trying to understand the behaviour of these novel materials.

- *8.2.4.1 Study design changes*

It was decided to make several methodological changes in the light of the findings from the pilot.

- Children would be given a chance to familiarise themselves with the materials before problem solving.
- The study would be a between subjects design so that children would only be required to familiarise themselves with one type of material.
- Children would be given explicit instruction in how to join the squares.
- If children identified a group verbally (in their solution) and objects were not all attached, they would be reminded to attach objects in the same group.
- Children would be given a tablet computer to manipulate objects with a pen as a small test showed that this would be easier than manipulating objects using the mouse (the need to attach squares requires greater motor control than simply moving objects).

8.3 Main Study

8.3.1. Aims and predictions

The study addressed the methodological issues highlighted in the pilot study in order to examine whether a perceptual clue (change of colour) could prompt children to identify numerical changes in the representation. As the interface constrained children's actions so that only one object could be moved at a time, it was expected that prompting children to identify changes to the representation would lead to a greater number of *compensation* solutions (partition solutions that differ by one in each part). Consequently, it was predicted that children in the Colour Prompt condition would identify a greater number of *compensation* solutions than children in the No Prompt condition.

8.3.2. Method

8.3.2.1 Design

A between subjects design was used with Prompt (*Colour Prompt*/*No Prompt*) as the independent variable. The primary dependent measure was the verbal solutions provided which were then coded for strategy (*compensation*/*commutative*/*other*) using the scheme developed in Study 2.

8.3.2.2 Participants

Thirty eight children took part in this study (20 girls and 18 boys, range 69 to 93 months; M=80.84; SD=6.55 months). These children (who had not taken part in the pilot) were selected from those in the same classes. The selection was made from Year 1 and Year 2, split across three classes: a Year 1 class (lower ability), a mixed Year 1/2 class (higher ability Year 1, lower ability Year 2) and a higher ability Year 2 class. The selected children were randomly allocated (using Excel Random number generator) to one of the two conditions: Colour Prompt or No Prompt. There were no significant age differences between conditions (U=160.5, Z=-0.59, p=ns).

8.3.2.3 Materials and procedure

- *Initial familiarisation with materials*

The same virtual materials were used in this study as the pilot. However, in order to address the possible issue of fine motor control skills being needed to manipulate objects, it was decided to present the task on a tablet computer (15 inch HP Compaq) after a small pilot test established that children were able to move and attach objects with greater ease using the pen interface on the tablet than a mouse.

Before explaining the task, the interviewer quickly showed children the tablet computer pen drawing a line in a paint program and asked children if they had used a tablet computer before: no children said they had. The interviewer explained to the children that they were going to be asked to solve some problems using squares then opened a file with ten squares arranged linearly in the centre of the screen. In line with the colour-quantity relationship shown in Figure 8.8, these ten attached squares appeared as orange in the Colour Prompt condition (but white in the No Prompt condition). The children were then encouraged to move the squares around using the Tablet pen. After 30 seconds the interviewer stopped the child and explained exactly how the squares could be attached. The interviewer demonstrated how the squares needed to be touching in order to join; however, an easy way to join the squares is to 'drop' them when they were overlapping. The children were then given a further 30 seconds to continue exploring the squares and practise joining them.

- *Demonstration problem*

As in the pilot study, after the initial presentation of the materials, the interviewer explained the problem. Similarly to the pilot, the laminate image was placed on the keyboard to support problem solving (Figure 8.9). The interviewer then showed the children the example question with the three squares: three white squares in the No Prompt condition and three green squares (changing to red and white for 2 & 1 respectively) in the Colour Prompt condition. The interviewer drew attention to the attaching of squares: *"see how I join the squares together if they are in the same group"*. Similar to previous studies, the interview proceeded to demonstrate solutions in the following order: 3 & 0, 1 & 2, 2 & 1 and 0 & 3.

- *Partitioning tasks*

Following the demonstration, children in each condition were given the problems requiring them to partition 6 and 7 respectively. It was decided to use 6 and 7 because it was discovered that the teacher in one class (the youngest group) had recently given a numeracy lesson looking at number pairs to 10 and had used 8 to demonstrate how to break a number down into pairs. Although this demonstration had been short, it is possible that children might have remembered the solutions given, thereby creating an external influence for partitioning this amount. [21]

As this was a between subjects design, children used the same materials in both partitioning problems. It was not necessary therefore to provide a further example problem between the two partitioning problems. Prompting was the same as in the pilot and previous studies, although, if children in either condition moved objects close together without actually joining them, the interviewer said *"remember to join the squares if they are in the same group".*

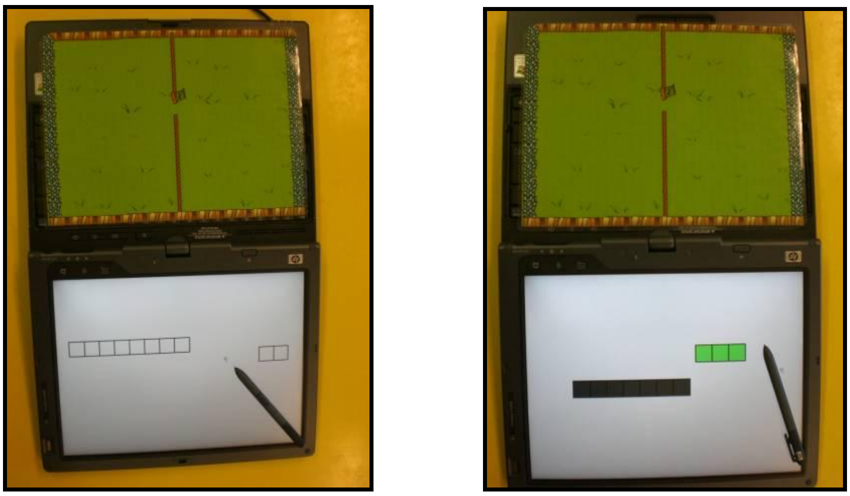

Figure 8.9: Examples of setup of Tablet computers in No Prompt and Colour Prompt condition

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21 It is possible that this teacher demonstration also prompted a certain strategy – especially as solutions were given in a way that reflected the *compensation* strategy (8 & 0, 7 & 1 etc). However, this possibility is not discussed further as there were an equal number of children from this teacher's class in each condition.

8.3.3 Results

8.3.3.1 Correct Scores

Since Kolmogorov-Smirnov tests revealed that the correct scores in both conditions were not normally distributed; non-parametric analyses were carried out. A Mann-Whitney test revealed no significant differences between correct scores in the Colour Prompt (Mdn=12) and No Prompt conditions (Mdn=12) ($U=161$, $Z=-0.57$, $p=ns$). A Wilcoxon test also revealed no differences between partitioning 6 and 7 ($Z=-1.07$, $p=ns$). Kruskal-Wallis tests revealed no differences between the three class groups for partitioning 6 ($\chi^2(2)=3.55$, $p=ns$) or 7 ($\chi^2(2)=2.11$, $p=ns$).

8.3.3.2 Strategies

Mann-Whitney tests revealed no differences between Colour Prompt (Mdn=1) and No Prompt (Mdn=0) conditions for *commutative* solutions ($U=134$, $Z=-1.46$, $p=ns$). However, in line with predictions, there were significantly more *compensation* solutions found in the Colour Prompt (Mdn=6) than No Prompt (Mdn=3) condition ($U=109$, $Z=-2.10$, $p<0.05$), and significantly less *other* solutions in the Colour prompt condition (Mdn=2) than No Prompt condition (Mdn=4)($U=94$, $Z=-2.55$, $p<0.05$).

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Table 8.3: Median (IQR) scores for strategies in the Colour Prompt and No Prompt conditions

	Colour Prompt (n=19)	No Prompt (n=19)
Compensation	6 (3,9)	3 (2,5)
Commutative	1 (0,2)	0 (0,1)
Other	2 (1,4)	4 (3,5)

8.3.3.3 Equal partitioning

In the Colour Prompt condition, children identified an Equal partitioning solution (3 & 3 for partitioning 6, and 3 & 4 or 4 & 3 for partitioning 7) on 23 out of 38 problems (60.5% with 1 incorrect). In the No Prompt condition, children identified 22 out of 38 (57.9% with 2 first solutions incorrect). As expected therefore, there were no significant differences between conditions for partitioning 6 ($Z=0.60$, p=ns) or partitioning 7 ($Z=0.71$, p=ns). There were also no differences found in the number of equal partitioning solutions identified when partitioning 6 or 7 ($Z=1.15$, p=ns).

8.3.4 Discussion

Study 7 examined the effect of perceptual prompts on children's partitioning strategies. Materials were designed to draw children's attention to changes in quantity by a change of colour according to the number of objects attached. As the graphical interface constrained actions to allow only one object to be moved at a time, it was predicted that the perceptual prompts would encourage children to identify more solutions that differed by one – i.e. more *compensation* solutions. This prediction was supported: children did identify a greater number of *compensation* solutions in the Colour Prompt than No Prompt condition. This difference was not attributable to children identifying more solutions in this condition so much as to the type of strategies used to identify correct solutions that differed between conditions. Whilst children in the Colour Prompt condition identified more *compensation* solutions, children in the No Prompt condition identified more *other* solutions. Interestingly, children in the Colour Prompt condition also identified more *commutative* solutions, although the numbers were too small to detect any significant effects.

Observations of children's manipulations of squares indicated that the issues raised in the pilot had been addressed by the changes made in this study. The problem of attaching objects seemed to have been eliminated, firstly because children were able to manipulate objects with greater ease using the tablet computer, and secondly because they not only had a chance to familiarise themselves with the representation before the problem solving started, but were also given explicit instruction in how to attach objects. Consequently, very few prompts were needed for children to attach objects during sessions.

8.3.4.1 Colour Prompt and strategy

The conditions in this study were designed so that the only differences between representations were the colour of the squares. As children successfully attached objects when grouping them, a significant difference in strategies between the conditions can be attributed to this perceptual clue. It was predicted that this prompt would help children identify discrete incremental changes in the representation that could be identified as new solutions. This prediction was indeed supported – children identified more *compensation* solutions in the Colour Prompt condition. Figure 8.10 illustrates how each new grouping was emphasised by colour changes in this condition.

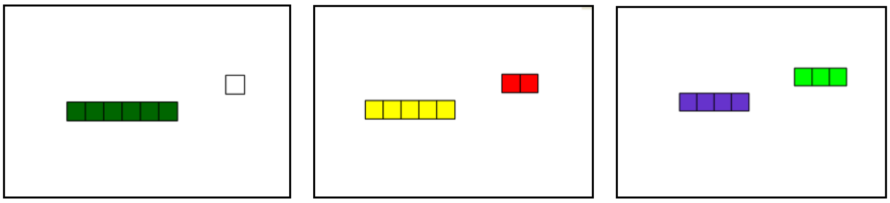

Figure 8.10: Change in colour as a perceptual clue for consecutive solutions in Colour Prompt condition

It seems therefore that the colour prompt did draw children's attention to numerical changes in the representation. It might be argued that this effect was partly attributable to motivation – the colour representation was more engaging and hence increased children's general levels of attention to the problem. However, if this was the case, it might be expected to have led to a greater number of correct solutions – this was not found. Another possibility is that children learnt the colour to quantity mapping of the representation in the Colour prompt condition and that this helped them identify solutions more easily (removing the need to calculate new parts). However, children did not have previous experience in using these materials and it is unlikely that they managed to learn the mapping in the duration of the session.

Although the changing colour of squares may have helped children identify incremental changes in the representation, this prompt was clearly insufficient for children to identify all solutions in this way. The proportion of solutions coded as *compensation* in the Colour Prompt condition was high (62.2%) but this still meant that 37.8% of solutions identified were not *compensation*. As squares could only be moved one by one, this meant that children in this condition would have seen the changes in colour of the new groups but not have identified these as potential new solutions. This is clearly demonstrated in children's initial solutions. In the Colour Prompt condition, 60.5% of first solutions were equal partitioning, and this did not significantly differ in the No Prompt condition. In order to identify a fair share solution initially, children had to move at least three objects, and each of these changes would have been emphasised by a change in colour in the Colour Prompt condition. The colour prompt, therefore, was insufficient to draw most children's attention to the intermediate solutions generated when partitioning objects equally at the start of problem solving. However, as illustrated in Figure 8.11 below, apart from the first change (creating groups of 1 and 5), children's tendency to move objects one by one into two different groups meant that many intermediate representational states did not consist of two groups (i.e. they did not reflect valid partitioning solutions). It is possible therefore that encouraging children to create only two groups (by locating the objects on a virtual image of two fields for example) may have helped draw children's attention to the valid intermediate representational states.

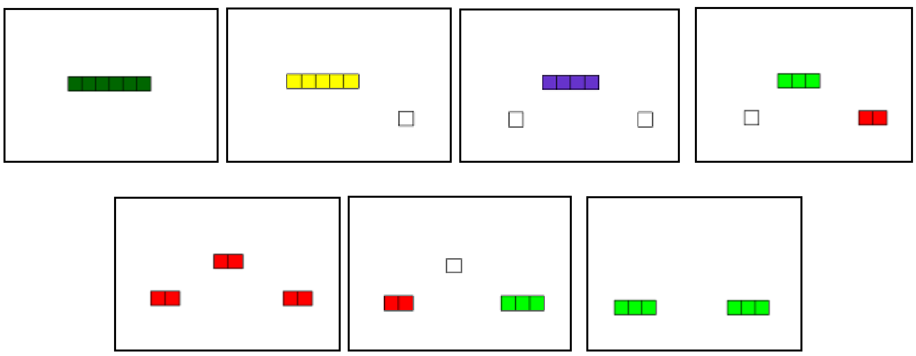

Figure 8.11: Example of adaptations made to identify initial Equal partitioning solutions in Colour Prompt condition

8.3.4.2 Colour Prompt and learning

In this study, the representation in the Colour Prompt condition increased the use of an efficient *compensation* strategy. Importantly, this is an effective strategy for distinguishing one solution from the previous one, and can hence be used in the absence of materials. Indeed, Study 2 showed instances of children moving objects but applying this strategy mentally (not looking at the objects to count out the solution). Unfortunately, this study did not include a transfer task to examine whether the increased use of the *compensation* strategy in the Colour Prompt condition would transfer to problem solving without materials.

It has been argued that in order to facilitate transfer to different contexts, more generic manipulative materials with less specific perceptual features should be used (Sloutsky, Kaminski, & Heckler, 2005b; Uttal et al., 1997). Therefore it is possible that the salient perceptual features of the colour representation could actually impede transfer. However, although the colour representation does include more perceptual features, these are not irrelevant features – they provide a visual representation of quantity and, importantly, a perceptual prompt for numerical change. It is possible, therefore, that these features help draw children's attention to the more abstract principle that a new partitioning solution can be generated by simply taking one from one group and adding to the other. If so, the perceptual prompt may facilitate transfer of this strategy to use in the absence of materials.

The possibility that the colour prompt leads to successful transfer is supported by an interesting study by Frydman and Bryant (1988) investigating the development of a concept of division in young children. Their studies centred on dividing sweets between two people and it was shown that young children were able to partition individual items between two groups with ease. However, when the context was changed so that some of the sweets were wrapped in pairs and the children were told that one of the individuals preferred to receive sweets in groups of two, children had great difficulty partitioning an equal amount. Instead, they tended to treat the group of two sweets as a single item. The authors then examined children's partitioning behaviour when given colour cues. Individual sweets now consisted of two colours, blue and yellow, and the groups of two sweets consisted of one of each of these colours. This colour cue significantly helped children partition correctly, giving an equal quantity of sweets to the two individuals even though one received sweets in groups of two. What was arguably most interesting in this study, however, was the finding that when the colour cue was removed, improved performance remained. In other words, the colour cue not only helped children to problem solve, but also helped them abstract strategies in the absence of the prompt. Clearly, the problem in Frydman and Bryant's study is different from the partitioning problem in this present study, but it does nevertheless raise the possibility that the changes in partitioning strategies in the Colour Prompt condition may transfer to problem solving in the absence of such prompts.

8.3.4.3 Summary

It was found in this study that providing children with a salient perceptual prompt for changes in quantity increased the use of a *compensation* strategy. The representation was manipulated on a graphical interface which constrained children's actions to moving one object at a time and this representational feature provided an additional visual stimulus for changes in quantity. However, the study raised questions over how well this representational feature may support or possibly hinder learning in this domain. Whilst it has been argued that this augmented representation may help children by encouraging the use of a strategy, it is possible that the inclusion of a more specific representational feature (colour) actually limits children's ability to transfer any learning to novel situations.

It is also not clear what impact this feature might have on different numerical concepts. It has been argued in previous chapters that the partitioning task encapsulates other important concepts such as commutativity, and it was therefore interesting to see that children identified more *commutative* solutions in the Colour Prompt condition albeit that the number of solutions was too small to detect any possible significant differences. As shown in Study 5, the limited number of *commutative* solutions is likely to be attributable to the constraining actions of the interface. Therefore, if an interface was used that did allow multiple objects to be moved simultaneously, then it might be possible to examine whether the colour perceptual prompt could significantly foster the use of the *commutative* strategy. It is possible (although clearly in need of empirical support) that the colour prompt fosters the use of the *commutative* strategy by emphasising the symmetrical nature of *commutative* solutions as shown in Figure 8.12.

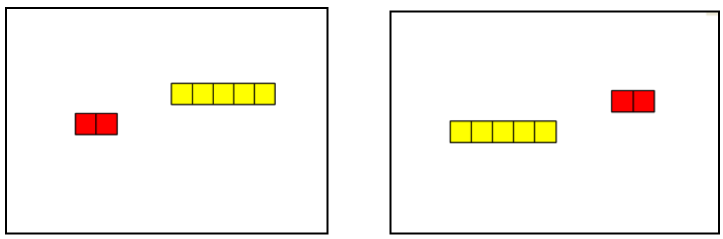

Figure 8.12: Commutative representational states with the Colour Prompt representation

Because the colour prompts used in the representation mapped to the number of objects in each group, it is possible to identify other numerical concepts that were illustrated through this perceptual cue. For example, the inability to create two equal groups when their total is an odd number was highlighted by the inability to create two groups of the same colour (when partitioning 7). This perceptual cue also highlighted how numbers could or could not be partitioned into larger groups of equal size: for example, the screenshot in Figure 8.14 illustrates how 6, but not 7, could be partitioned into 3 equal groups. Furthermore, the principle of inversion was possibly represented in the way some children added one object to a group and then took away another (i.e.: a + b - c = a, if b = c) as illustrated in Figure 8.15.

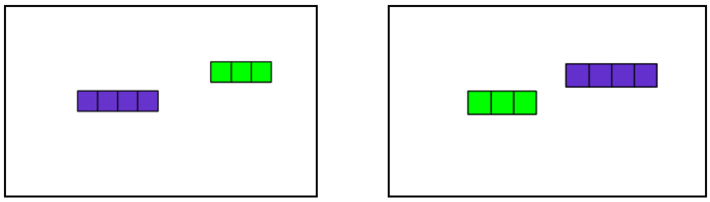

Figure 8.13: Representations for 4 & 3 and 3 & 4 in Colour Prompt condition, highlighting commutative parts

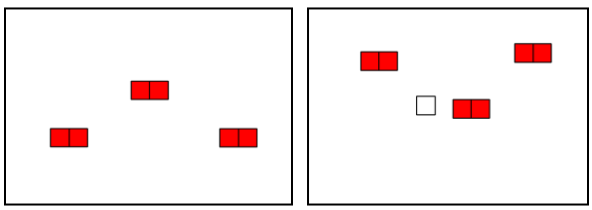

Figure 8.14: Partitioning 6 and 7 in larger groups

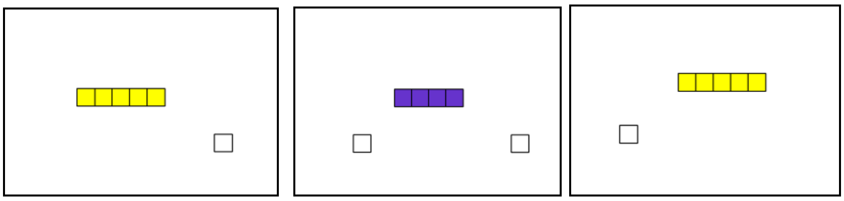

Figure 8.15: Inversion

It is important to stress that just because the colour prompts may seem to embody certain numerical relationships, it is unknown how these prompts may or may not affect children's understanding. Indeed, as discussed Chapter 1, it has been argued how it the transparency of certain representations is only clear once the concepts they are meant to represent are understood (Holt, 1982).

Researchers (e.g., Ball, 1992; Sutherland, 2007) have emphasised the role of the teacher and the learning activity to help foster certain mathematical ideas. What is not clear, and arguably worthy of further research, is whether the representation generated in this study presents a potential tool to explore and/or communicate certain numerical relationships. This final study has demonstrated the potential to influence children's strategies, and hence possibly their ideas, by using digital technology to draw attention to certain representational changes.

Chapter 9

Final Discussion

9.1 Summary of Thesis aims

The aim of this thesis has been to evaluate the potential for tangible technologies to support children's understanding of the numerical concept of additive composition. In order to design tangible technologies that are effective, it is important to first identify the possible advantages, as well as limitations, of using physical representations in this domain. Additive composition is a key part of children's numerical development (L. B. Resnick, 1983b) and it has been suggested that physical materials may support children's understanding of this concept (Nunes & Bryant, 1996). However, it is not yet clear how, or even if, actions with physical materials, as opposed to other materials (or even no materials) might lead to learning. This challenge was addressed in this thesis by designing a task that required children to decompose single digit numbers into combinations of two parts, and then conducting a series of studies examining the role of physical representations in the task. The findings of each study were discussed at the end of their respective chapters, whilst this discussion chapter draws everything together in order to look at the main research question, and examine what implications the findings have for related research and practice.

9.2 Structure of the discussion

This discussion begins with a summary of the findings, first addressing each sub-question and then the main research question. These will then be evaluated in the light of the implications they have for the design of effective tangible technologies in this domain. The partitioning problem will then be examined in more detail, identifying the role of other contextual factors and their effect in drawing out the key arguments relevant to the main question. The limitations of the research will then be discussed before considering the wider implications of the findings and possibilities for further research.

9.3 Summary of findings

9.3.1 Do physical objects support children's strategies for partitioning numbers?

Study 1 was an exploratory study into the use of physical objects in an addition and partitioning task. It was shown that when children were given an uncounted number of objects and simply asked to use them if they helped, physical objects did not assist them any more than using paper or even no materials (simply their fingers). Although the children used physical materials more than paper or fingers in the partitioning task, this did not seem to confer any advantage. Study 1 thereby highlighted the need to take account of the way materials are presented, and the effort that children must make to first count out the initial partitioning amount before they can identify any partitioning solutions.

Study 2 addressed the issues arising in Study 1, examining whether children found more partitioning solutions using physical objects than no materials if they were first given the initial number to be partitioned. As expected, this help proved effective – the use of physical objects clearly did help children identify more correct partitioning solutions. Arguably of greater interest was the effect that the use of physical materials had on children's strategies for identifying solutions. Using a coding system, solutions were categorised according to their relationship to the previous solution. A *compensation* solution was coded when the solution was one different from the previous one, and a *commutative* solution when the solution was the reverse of the previous one. Other solutions were coded as *other*. From this, it was found that when children used physical objects they identified significantly more solutions that were coded as *compensation* and *commutative* than when they did not use materials. Another finding from this study related to children's first solutions: when physical materials were used it was found that a significantly higher proportion of children's first solutions were Equal partitioning - i.e. an equal division of the objects, or (in the case of an odd number) as close to equal as possible.

9.3.1.1 Summary

It was found that the use of physical objects did help develop children's strategies for partitioning numbers when the initial demand of counting out the amount to partition was carried out by the interviewer. Not only did children identify more ways to partition a number using objects, but they were also more likely to then relate consecutive solutions. Relating solutions is an efficient approach to this problem and, importantly, leads to strategies that can be carried out in the absence of materials. Moreover, these strategies embody important quantitative relationships.

9.3.2. What are the advantages/limitations of physically manipulating representations for children's partitioning strategies?

Study 3 sought to discover what properties of physical objects supported or limited children's partitioning strategies. This was done by comparing children's performances under four different conditions in a 2x2 design: using physical/pictorial materials and providing a record/no record of previous states. It was found that providing children with a record of previous representational states did not affect their strategies. Children did not use this record even though it was demonstrated that this could show previous solutions, and despite its potential value as a means of finding solutions that had not yet been identified. In contrast to this, there was a significant effect from using physical materials: children identified significantly more correct partitioning solutions using physical than pictorial materials. Furthermore, using physical materials, children identified significantly more solutions that were related (i.e. more *compensation* and *commutative* solutions) although, it is possible that the greater number of *compensation* solutions may simply reflect the greater number solutions identified overall. A further finding from this study was that children's propensity to start by partitioning objects into two equal parts did not differ between the Physical and Pictorial conditions.

Study 4 examined children's strategies in greater detail. Children solved problems first using no materials, and then using physical and pictorial materials (in counterbalanced conditions). In line with the previous studies' findings, children identified more correct solutions using physical materials than under the other two conditions. Video records of children's actions highlighted the way in which they were able to create new spatial partitioning configurations easily when they were using physical materials, and that they were then able to identify most of these as valid solutions. In contrast to this, children created significantly less groupings when using paper. The way in which children gave a verbal solution whenever they annotated paper (and sometimes before annotating) suggested that they may have used this form of representation to record rather than generate ideas. There were also possible signs that children were abstracting strategies in the Physical condition: they would begin by moving single objects and counting each part and then continue moving objects but calculate each part mentally (looking away from the representation). This is speculative but has important implications as possible evidence for children using concrete materials to help them develop abstract strategies.

Study 4 also examined the potential role of different properties of the physical materials. Children touched objects to help them count as well as to keep track of their position. Objects were sometimes stacked vertically or moved relative to the child's position, although it was not clear how much advantage this provided over pictorial materials, especially as the numbers being counted were small (hence posing limited computational demands). More important seemed to be the types of action that children made with the materials when relating consecutive solutions. *Commutative* strategies involved children swapping over groups of objects. This action involved moving multiple objects using both hands, sometimes picking up groups or simply pushing them. In contrast, *compensation* solutions involved more constrained manipulation, where children would move a single object with one hand.

9.3.2.2 Summary

These studies suggest that the key advantage of physical materials for this problem lies in the way they allow children to create new spatial configurations with simple actions, and then enumerate these to identify more correct solutions. In so doing the studies provide support for the theory of Physically Distributed Learning (Martin & Schwartz, 2005) that describes how such actions can lead to new interpretations. However, rather than simply moving objects randomly, the physical representation seemed to foster strategies for relating solutions through specific actions: moving all objects in groups (*commutative*) or moving individual objects one by one (*compensation*). A *compensation* strategy is arguably a more efficient strategy for progressing incrementally through the problem space, although, unfortunately, it was not clear from the studies what properties of the materials might encourage particular actions. One possibility is their visuo-spatial properties: while an Equal partitioning solution creates symmetrical groups, a *commutative* solution creates a symmetrically opposite configuration.

9.3.3 What is the effect of constraining physical manipulation on children's partitioning strategies?

Study 5 examined the effect on children's strategies when their actions were constrained by allowing them to move only one object at a time. The outcome, as predicted, was to find that they then identified significantly fewer *commutative* solutions. However, although they identified more *compensation* solutions, the difference was not significant (albeit most probably because children in the constraints condition tended to move objects quickly using both hands and often needed reminding of the constraining rule).

In Study 6, the effect of constraining actions was examined using a graphical interface. As predicted, children again identified significantly fewer *commutative* solutions, but were now found to identify significantly more *compensation* solutions than when they were manipulating physical materials. The significant increase in *compensation* solutions resulting from constraining manipulation using a graphical interface (as opposed to through instructions in Study 5) was attributed to children having longer to see each numerical change. Their actions were not only slowed but their hands did not block their view of the representation. Children also did not have to remember the need to move one object at a time as the constraints were built into the system.

Video analysis in Study 6 seemed to support the previous suggestion that although children touched objects to support cognition, this affordance did not seem to play an important role: no clear disadvantages (such as count errors) were identifiable in the graphical condition. An interesting finding in Study 6 was the number of representational states that were *not* identified in either the physical or graphical condition. Of particular interest was the number of incremental changes in the Physical condition that were not identified verbally. In other words, children often moved physical objects one by one in quick succession but did not actually identify the ensuing intermediate states as solutions.

9.3.3.1 Summary

These studies showed that constraining children's actions can significantly affect their strategies for identifying partitioning solutions. By requiring children to move objects one by one using a graphical interface, it is possible to encourage them towards using a *compensation* strategy, which is the most efficient for solving the problem. Nevertheless, there were still many solutions in this study that children did not identify, albeit that this may partly be explained by a tendency, in both the Physical and Virtual conditions, not to identify many of the incremental changes to the representations.

9.3.4 Can children's partitioning strategies be supported by augmenting the representation's perceptual information?

The final study was Study 7, which examined the effect of a digital perceptual prompt on children's partitioning strategies. The effect used was for objects to change colour according to the number grouped together so that changes of groupings would result in a perceptual prompt of colour change. As expected, it was found that children were significantly more likely to identify changes to the representation when manipulating squares with this prompt than without it. In other words, children identified significantly more *compensation* solutions with this augmented representation.

9.3.4.1 Summary

The final study in this thesis demonstrated that it is possible to influence children's strategies by augmenting the representation with visual prompts. A simple colour prompt was enough to significantly increase the number of *compensation* solutions identified. The digital effect was examined using the virtual representation, but this type of perceptual prompt could theoretically be integrated into physical objects – and hence articulates a possible tangible design to support children's numerical development.

9.3.5 Does physically manipulating digital representations present any unique benefits for supporting children's understanding of additive composition?

The aim of this section is to draw together the findings in order to answer the main research question; and in doing so help evaluate the potential for tangible technologies to support children's numerical development.

By comparing children's scores and strategies for solving the partitioning problem, it was found that physically manipulating representations supported children's ability to decompose a single digit number into composite pairs. However, this advantage did not seem unique to physical representations: children were able to identify as many correct solutions using virtual manipulative manipulated with a mouse. Moreover, using a graphical user interface increased the use of an efficient strategy for solving the partitioning problem (*compensation*) – a strategy that allows children to move incrementally through the range of solutions, identifying one solution from the previous. Therefore, the studies in this research did not identify any clear advantage for physically manipulating representations to support children's understanding of additive composition. However, the research did show significant differences in the strategies used when physically manipulating representations and possible cognitive benefits from touching objects. Furthermore, by demonstrating the potential to use digital effects to draw children's attention to numerical changes, the final study raised interesting questions around the possible unique benefits of an augmented physical representation.

Therefore, this final section will examine in more detail some of the themes to emerge from this research that are relevant in evaluating the potential for Tangibles in this domain. Although it is not the aim of this discussion to advocate specific design ideas, examples such as a tangible version of the materials used in Study 7 may be drawn upon to illustrate certain arguments.

9.3.5.1 Record of solution

One key limitation of physical objects that has been identified is that no trace is provided of previous activity (Kaput, 1993). However, it was shown in Study 3 that, without scaffolding or providing any explicit instruction on how to use such a trace, children will not make use of it in this task. Furthermore, it is not clear how much the use of a trace would actually encourage the development of efficient strategies that work through the problem systematically.

Although Tangibles might address this limitation: by providing a means of recording actions or specific representational states, this research does not show that this will necessarily help children. It might however be noted that such a trace may actually be more useful for the teacher, not only as a formal record to help subsequent assessment, but also as a means of encouraging general class discussion about strategies.

9.3.5.2 Spatial manipulation

Additive composition involves an understanding of how a number can be broken down into smaller numbers, and it has been argued that in a task requiring children to break a number into different parts, it is beneficial for them to identify as many partitioning combinations as possible. These studies have shown that children identify more partitioning combinations when they are able to spatially manipulate the representation. Spatial configurations may help important cognitive tasks such as enumerating by subitising small groups, or keeping track of objects when counting by creating a linear configuration. Spatially manipulating objects also seems to allow children to act (physically) and then interpret the representation (numerically) as described in PDL. Importantly, when children manipulate objects, simple transformations can embody key numerical concepts (such as swapping over groups, which creates a symmetrical opposite configuration and embodies the concept of commutativity). Moreover, moving objects from one group to another introduces children to the important concept that the quantity of objects in parts can change without any objects being added or taken away from the collection as a whole – the central tenet of additive composition.

Tangible technologies therefore have the potential to allow children to explore the concept of additive composition by transforming the spatial configuration of the representation. Although this may seem obvious, some Tangibles do not provide this opportunity. For example, the *Teaching Table* (Khandelwal & Mazalek, 2007) has been designed to support children by providing a means of manipulating numerals on tiles (Figure 9.1) and giving feedback on answers to numerical questions. However, by using numerical symbols, children will not be exposed to spatial configurations as discussed. Similarly, when David Merrill presented *Siftables* at the *Technology, Entertainment and Development* conference in 2009 (see Merrill, Kalanithi, & Maes, 2007 for design description), it was demonstrated how tiles could be manipulated to explore numerical equations (Figure 9.1). Admittedly, the authors of these designs do not express any specific purpose of helping children explore quantitative relations, yet they do raise questions about how easily children can explore such relations with designs that require them to spatially manipulate numerical symbols.

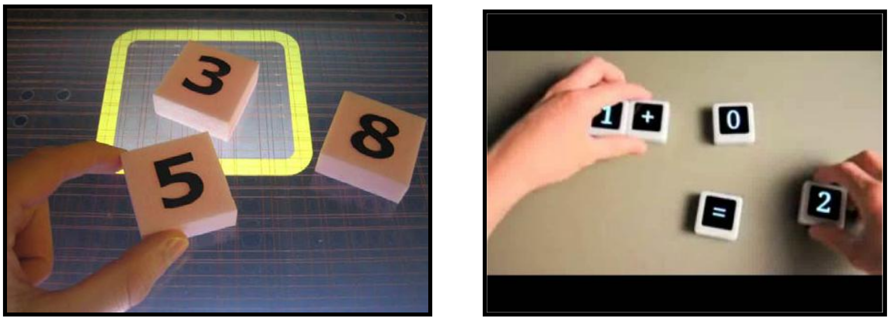

Figure 9.1: Manipulating symbols: Teaching table (Khandelwal & Mazalek, 2007) and Siftables (Merrill et al., 2007)

9.3.5.3 Tangible v Graphical Interface

It is possible to spatially manipulate objects using a graphical interface as well as Tangibles. Computer representations have an advantage over Tangibles, in that they are relatively easy to create using a range of different digital effects, as demonstrated in the design of the representations in Study 7. The relatively low cost and ease of creating virtual manipulatives helps explain their growing number in schools (e.g., NLVM, 2007). It is important therefore to ask what added value, if any, is offered by tangible interfaces to help children develop their understanding of additive composition.

9.3.5.4 Tactile feedback

Tangible designs can provide children with tactile information. The possible role of this affordance was highlighted in the research by numerous observations of children collecting multiple objects with ease, touching objects when counting, or placing fingers on objects as a 'marker' when looking at other objects. However, there was no evidence that these actions significantly influenced children's strategies in the task. It is possible that this simply reflected the task demands: as amounts were small, any cognitive benefits gained from touching objects may have been too small to detect. Unfortunately, without conducting further tasks that are procedurally more demanding, it is not possible to conclude that the tactile feedback afforded by physical objects offers any advantage.

9.3.5.5 Controlling manipulation

A key focus in the studies has been on the strategies children used to identify solutions. The pattern of solutions identified provided an indicator of these strategies and a coding scheme was devised to quantify solutions identified using different strategies. Two key strategies were identified: identifying a solution by reversing the parts of the previous solution (*commutative*) and identifying a solution by adding and taking away from each part of the previous solution (*compensation*). These strategies embody important part-whole relationships that are central to additive composition, and a key finding from this research was that activity with physical materials seemed to foster these strategies. However, it was found that constraining actions using a graphical interface significantly affected strategies - increasing the use of the efficient *compensation* strategy. This finding generated an important theme from this research: that *with a graphical user interface it is relatively easy for the designer to influence children's numerical problem solving strategies by controlling how many objects can be manipulated at a time and by slowing down the speed at which objects can be manipulated.*

It may be possible to use digital effects to influence the way physical objects are manipulated - for example, introducing delays in perceptual prompts might encourage children to slow down manipulation. However, it is arguably more difficult to control manipulation with tangible interfaces because manipulation is highly dependent on physical properties such as size and shape, and these are not as easily changed[22]. Although it may be possible to design a way that tangible technology can address this limitation, (e.g., if materials were attached using some form of digitally controlled mechanism such as electromagnetism it might be possible to control how many and how easily objects could be separated), this would probably be more expensive and difficult to achieve than simply programming virtual objects.

It might be argued that, unlike manipulating objects physically, a device such as mouse presents a barrier to certain forms of manipulation (such as moving multiple objects with ease). Indeed, tablet computers were used in the final study as children had experienced difficulty in attaching objects on screen using the mouse. However, graphical interfaces are evolving. Multi-touch surfaces already make it possible for multiple objects to be moved with simple hand gestures, although further research would be needed to establish how easy or seamless manipulation would then be for children compared with moving physical objects. Possible limitations were identified from observations in this research: for example in the way children used tactile feedback to select and move multiple objects, and in the way objects were often moved over one another.

[22] The potential to transform the structure and behaviour of tangible digital materials was discussed by Hiroshi Ishii on behalf of his group's *Radical Atoms* project at the CHI 2009 panel on April 9th 2009 in Boston (Ishii, 2009).

9.3.6 Summary

The ability to spatially adapt objects may help children to act on and interpret representations, thereby providing a means of developing their ideas about how numbers can be decomposed into smaller numbers. However, the form of interface may significantly affect the strategies children use, and hence their ideas about numbers. It was found, for example, that a graphical interface fostered the use of a more efficient strategy in this research by constraining children's actions. The immediate findings from the studies therefore suggest that a graphical, rather than tangible, interface may be more effective in supporting children in this particular problem. In other words, in response to the main research question, the findings from the studies did not identify any clear benefits for physically manipulating representation for supporting children's understanding of additive composition. However, in order to evaluate more fully the potential for Tangibles to support children's understanding of additive composition, it is important to re-examine the task used in this research and the context in which it was presented in light of the findings.

9.4 The partitioning task

The aim of the partitioning task was to create a context in which children could solve a problem using different representations so that differences in their strategies could be attributed to unique representational properties. However, in order to create these conditions, it is important to acknowledge that the research created a unique context in which the representational medium was simply one factor. Significantly, as highlighted by Nilholm and Säljö (1996, p.342), interpretations of the findings do make certain assumptions in terms of how activity in this context might generalise to others as well as whether certain activity is as reflective of cognitive factors as predicted.

It is therefore the aim in this section to examine differences in children's strategies in the partitioning task in order to try to understand how their interpretation of the problem context may have affected their problem solving behaviour and their use of the different materials presented. This critical examination of the partitioning task is not intended to undermine the findings – rather, it is intended to provide another viewpoint from which to evaluate the potential for Tangibles to support children's understanding of additive composition in an educational setting. In order to break down the partitioning task, differences in the types of children's responses will be examined.

9.4.1 No solution – single solution

It was previously argued that children would need at least an initial understanding of additive composition to identify a single partitioning solution. A story context was presented in an attempt to ensure that any lack of answers did not stem simply from a misunderstanding of what was being asked. In this respect, the problem context seemed successful – nearly all children in the research identified at least one correct solution. Unfortunately, the studies provide limited information as to what age children have insufficient understanding to make sense of the task. Whilst Studies 2 and 4 examined children's problem solving without materials, the children in Study 2 were older (Year 1) and there were only a few young children in the small Study 4 (in which the No Materials condition was always presented first). It is not therefore possible to identify from this research the age at which children have difficulties in identifying even a single solution and, importantly, whether materials can help them.

The supportive role of physical materials demonstrated in these studies suggests that they would help children who could not solve the partitioning problem without any external materials. In comparison to other representations such as paper, physical materials are more limited in what changes can be made - and that is to change their spatial configuration. With sufficient prompts, as given in this study, children may understand that the requirement is to partition objects into two groups. They may then need prompts in enumerating these groups – as was done in the research. By allowing a problem to be tackled in two stages however (creating a configuration, and then enumerating that configuration) the use of physical materials may provide a means for younger children to identify a solution that they cannot identify without materials.

It is possible that virtual representations are equally supportive for younger children. Study 6 showed that young children are quite able to understand how to partition virtual objects into groups. Furthermore, whilst it might be argued that physical materials are more accessible for younger children (e.g., requiring a lower degree of fine motor control), it might also be possible that their prior experience with physical materials such as a cubes in a non numerical context is distracting. Indeed the difficulties children experience with the dual representation of objects has been raised by Uttal et al (1997). In contrast, young children may be less distracted and more focused on the task when asked to partition novel digital materials in the context of numerical problem solving.

9.4.2 Single - multiple solutions

What seemed to separate the younger and older children in this research was that the youngest age group (4-5 years) tended to give just a single solution. As all children had been given a demonstration with multiple solutions, and had been given clear instructions to "*find all the different ways",* it might be argued that this marks a developmental step – an awareness that numbers can be partitioned in more than one way. However, there is a danger that this difference in ability reflects other factors such as children's experiences in problems of this type, or confusion of the task demands. Indeed, it was shown how the simple prompt "*is that all the ways or can you think of any more ways?"* had a clear effect on encouraging multiple solutions. Nevertheless, it is still interesting to note that older children identified multiple solutions unprompted. An important question therefore is whether particular representations might be able to foster this behaviour.

It was not possible from the studies to examine whether external representations help children identify multiple solutions more than no materials. The children in Study 2 who did not use materials were all able to identify more than a single solution. It is possible, however, that children's understanding is supported by physical materials - their prior knowledge of how physical objects can be grouped in multiple ways may help them understand how numbers can be partitioned in multiple ways (as argued by L. B. Resnick, 1992b). Nevertheless, findings from this research showed that younger children were *not* more likely to identify multiple solutions using physical materials than pictorial or virtual. The greatest factor affecting whether children identified multiple answers was not the type of material so much as prompts by the interviewer, such as *"is that all the ways, or can you think of any more ways?"*

9.4.3 First solution

In all the studies conducted, it was clear that children had a tendency to begin by identifying a solution that partitioned the whole into two equal groups or as close to this as possible. This was true for children who identified multiple solutions as well as for those who identified only a single solution, suggesting therefore that this was not simply due to misunderstanding the question. This finding is made more interesting by the fact that this initial strategy had not been demonstrated by the interviewer, and also that it is arguably not the most efficient strategy for the problem.

It appeared that the external representation fostered the use of this strategy. Children identified significantly more Equal partitioning solutions in Studies 2 and 4 using materials than no materials, although there were no differences between physical, pictorial and virtual material in the other studies. In Study 5 there were signs that constraining children's actions increased the number of Equal partitioning solutions (based on observations that children would place objects one by one in different groups). On the other hand Study 6 did not support this – children did not identify more Equal partitioning solutions using the graphical interface (albeit that this might have been partly explained by the small quantities used).

The initial Equal partitioning strategy is not perhaps surprising - it is certainly the most logical way to partition objects in a context such as placing fruit in two bags, although the change to cows in fields did not seem to make any difference. It is possible therefore that partitioning equally marks an initial step in children's understanding of how numbers are partitioned. Interestingly, in Study 6, the older children actually demonstrated a decrease in this initial strategy in the second problem – possible learning effects.

This studies conducted did not attempt to address the ways in which digital augmentation of a representation might encourage children to *begin* partitioning differently. It was suggested, however, that the colour prompt in Study 7 might affect strategies. It was anticipated that a visual prompt would encourage children to identify each initial incremental change to the representation but this was not found to be the case. Children's motivation to partition objects equally was highlighted in many observations of their taking time to place the last 'odd' object. Interestingly, the colour representation in Study 7 visually emphasised when groups were or were not equal (Figure 9.2), raising the possibility that these materials could be used to help children explore what amounts can and cannot be partitioned into two equal amounts (an important learning point for younger children).

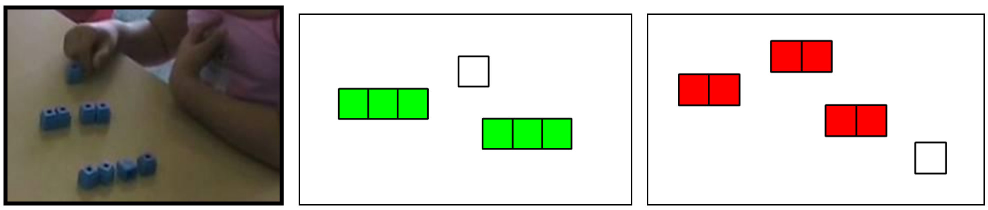

Figure 9.2: Partitioning equally: a) using cubes b) & c) emphasised with colour prompts

9.4.4 Relating solutions

9.4.4.1 Compensation

When identifying multiple solutions, older children were more likely to relate consecutive solutions. The *compensation* strategy has been described as the most efficient, and it was found that constraining manipulation using the graphical interface increased the use of this strategy. The concept of *compensation* is summarised as follows: if P1 + P2 = W, then (P1 + a) + (P2 - a) = W (Irwin, 1996). For the *compensation* strategy coded in this research, a = 1, which allows children to calculate one solution from the previous. However, although children may not be able to quantify the change in amounts as easily, the concept of *compensation* is still embodied in changes of groups of more than one. In fact, it might be argued that certain changes can make this concept more salient: for example if children separate the whole into two parts and then recompose the whole before creating a new grouping, they will have more visual information about the relationship between different parts and the unchanged whole. This raises an important theoretical argument in this research: *although constraining manipulation may foster a particular strategy for identifying new parts that relate to the previous, more unconstrained action may still help children explore the relationship between different parts and the whole*. Clearly, this assertion is speculative, although it does reflect Nunes and Bryant (1996) suggestion that decomposing and recomposing physical objects may help develop children's understanding of additive composition. Arguably, this numerical relationship could be further emphasised by the use of visual prompts: children could see how they can create different parts (i.e. different colours) when partitioning the whole, but always the same whole (i.e. same colour) when objects are recomposed as illustrated in Figure 9.3. Again this is speculation, and further research would be needed to investigate what impact, if any, such visual information would have.

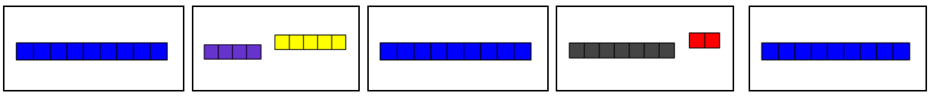

Figure 9.3: Moving multiple objects allows the whole to be decomposed and recomposed in single actions (9 into 4 & 5, and then into 7 & 2)

9.4.4.2 Commutativity

The most efficient procedure for solving the partitioning task was the *compensation* strategy, although the *commutative* strategy also required children to identify one solution from the previous. In Study 6, when 75 children used the graphical interface, only 7 *commutative* solutions were identified (compared with 28 in the Physical condition); this might perhaps have been expected in the light of Study 4, which showed that this strategy reflected the way in which children moved multiple objects with both hands. The *commutative* strategy embodies the important concept of how parts can be reversed without change to the whole and the research raises the possibility that this concept could be supported by the unconstrained manipulation allowed in the Physical condition. Although this strategy may be limited in this problem (it would need to be used alongside another strategy to identify all solutions), it does highlight how the unconstrained actions in the Physical condition led to greater variation in the ways children related one solution to another. Importantly, although this strategy may not be the most efficient in this problem, it is possible that this will support ideas for other problems. Indeed, the teacher may see it as a learning opportunity to discuss this aspect of numbers: that they can be added in any order.

Tangible designs may build upon this affordance. It may even be possible to use digital effects to highlight the concept – for example, through perceptual effects. Although not significant, there were more *commutative* solutions in the prompts condition in Study 7 even though manipulation was constrained. It is possible that when moving objects one by one, children's attention was drawn to a commutative configuration that had the same colours, reversed (e.g., Figure 9.4). By using the knowledge that the same colour meant the same quantity, children may be able to reflect on how the quantities had reversed without the need to count each part. It would be interesting to know whether this perceptual effect would have encouraged the use of the *commutative* strategy if the representation had allowed manipulation of multiple objects.

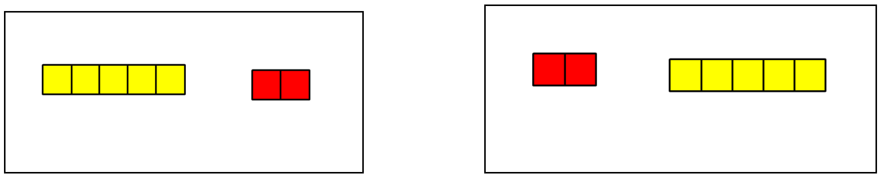

Figure 9.4: Colour prompt highlighting commutative solutions

9.4.5 Actions – Gestures

In Chapter 1, it was suggested that one key benefit of using physical objects is that they allow actions with objects that might become embodied in the concepts being learnt. Although, the research did not focus on gesture, it is important to be aware of the way in which their use can have an effect on children's actions with different interfaces.

Constraining actions in the graphical condition meant that objects were manipulated by small indirect actions using the mouse with one hand. In contrast, in the Physical condition, children were able to make unconstrained actions using both hands. By accommodating a role for sensorimotoric encoding in working memory (Wilson, 2001), it is possible that these physical actions become encoded in children's developing ideas. In other words, children's concepts of additive composition may become embodied in their actions with objects in the partitioning task. Such encoding may then be observable through gestures used when expressing thinking at a later stage. Roth (2002), for example, demonstrated how students developed certain iconic gestures that reflected actions with physical materials when learning about electrostatics. Edwards (2005) has also shown how young adults' concepts of fractions appeared to embody previous physical actions with physical objects (e.g., splitting objects into two groups).

Clearly, as well as helping individuals externalise their thinking, gestures play an important role in helping communication, and have been shown to help teachers communicate ideas (Valenzeno et al., 2003) as well as provide a way to assess children's understanding (Kelly et al., 2002). It is important therefore to consider whether certain gestures relating to actions with objects might help communication between teacher and child. In this research, care was taken not to explain the problem using gestures in case this led to an unfair advantage – with children using the gestures to infer how to move physical objects, or feeling encouraged towards particular strategies. When the interviewer demonstrated the change of 1 & 2 to 2 & 1 in the example problem, only one object was moved. Had the two groups just been swapped over, the interviewer would effectively have been modelling a *commutative* strategy. Seeing this gesture may have significantly affected children's later performance.

The importance of gestures might be greater in the classroom context – a teacher may wish to communicate ideas to children from a distance without objects to hand, or monitor children's actions from various locations in the classroom. This section has therefore sought to highlight the way in which both the children's and interviewer's actions with physical objects in this task may have important implications for how the materials used can support learning (particularly in a classroom context). In this light, it is important to note how the physical materials (hence Tangibles) may offer a key advantage over other representations such as paper and virtual, where actions are less pronounced.

9.4.6 Summary

It was previously mentioned that, despite physical objects helping children identify partitioning solutions and even fostering the use of strategies that relate solutions, constraining actions using a graphical interface encouraged the use of a more efficient *compensation* strategy. This section attempted to describe children's developing strategies in more detail within the particular context of the research task, and in the process provide a more thorough evaluation of the potential for Tangibles to support children's understanding of additive composition.

One point raised concerned the importance of considering the particular context in which the representations were compared in the studies. As Ball (1992, p.47) emphasises: "*understanding does not travel through the fingers tips and up the arm*". In the partitioning task, the interviewer provided numerous prompts to ensure children understood the nature of the problem and to remind them to provide numerical solutions. Prompts were even given in some studies for children to identify multiple solutions. Clearly, the teacher could provide similar prompts when initially presenting a task in class, although it might be difficult to ensure having every child's full attention in a classroom situation. Effective Tangibles may however be used as a means of giving prompts – i.e. as a neutral way of encouraging children to enumerate solutions or of providing feedback.

A key way in which Tangibles might help is by drawing attention to certain numerical changes by using perceptual prompts. This was demonstrated by the digital colour prompts used in Study 7, although this effect was simply designed to draw children's attention to representational changes. It is however possible that providing a consistent effect for particular numbers, in this case colour, has unforeseen detrimental effects such as removing the need for children to develop calculation strategies.

This section has highlighted the way in which physical objects provide a means for children to quickly explore different relationships between parts and wholes, and how digital augmentation may help draw attention to certain key numerical relations. One key theme to emerge is that constraining manipulation to incremental changes may have important implications for developing ideas. With respect to PDL, if actions lead to ideas, it might be anticipated that changes to actions may result in changes to the nature of ideas developed. Incremental changes may encourage children to enumerate changes when parts change by only one. This may explain how children identified incremental solutions when their actions were constrained in the graphical condition. In contrast, by moving greater numbers of objects simultaneously, children may benefit from exploring certain relationships between parts and whole, such as how parts can be swapped without changes to the whole, or how the whole can be decomposed and recomposed in different ways. By moving multiple objects children may be less likely to enumerate changes, but have instead the opportunity to notice important numerical relations between parts and whole.

The suggestion that that moving multiple objects may support understanding of different part-whole relationships clearly needs empirical support. However, the studies reported have presented findings demonstrating that moving multiple objects does foster an alternative strategy embodying a key numerical principle (commutativity). Different tasks may be designed that allow children to explore different part-whole concepts by moving multiple objects.

9.4.6.1 Efficiency v Innovation

The previous section highlights a key pedagogical issue. The graphical interface increased the use of the *compensation* strategy by requiring children to move objects incrementally. In contrast, if children identified a *compensation* solution in the Physical condition, they constrained their own actions. This means that the physical objects allowed children to discover for themselves the benefits of constraining their actions. This was clearly demonstrated by one child in Study 4, who began by moving multiple objects but then demonstrated a clear change of strategy by moving objects one by one. It was also shown by children changing strategy from moving objects off the laminate object to moving objects from field to field in Study 5.

This trade off between constraining actions to foster an efficient strategy or allowing more unconstrained action to foster more exploratory behaviour can be compared to the work of Schwartz, Bransford and Sears (2005) who describe a trade off between efficiency and innovation. It is argued that the benefits of fostering innovation are best revealed in tests of transfer as the learner has had an opportunity to practise identifying what is and what is not efficient to solve a certain problem. It may be possible therefore, that the Physical condition supported innovation by allowing children a greater range of actions from which to decide which were the most efficient. In this case, the learning benefits may be better revealed through transfer tasks, although further developmental research would need to test this argument.

9.5 Limitations of this research

Many of the limitations of this research have been raised during this discussion. These are briefly summarised under the headings of design limitations and theoretical limitations.

9.5.1 Design limitations

This research has focused mainly on children aged around 5 to 8, and demonstrated children's developing ability from identifying a single solution to identifying all solutions using efficient strategies. Unfortunately, although there were several children who did not identify a single solution, and some children who identified all solutions in the most efficient way, there were not enough data to analyse the impact of different representations at these stages. Consequently, the findings are limited in the extent to which they can establish the ages at which there is greatest potential for different materials to support young children's incipient understanding or 'expert' behaviour.

Study 1 showed that it did not really help to just give children materials without a good understanding of how they could adapt them. Following Study 1 therefore, it was decided to give children a small actual demonstration before the problem solving began. A demonstration problem was designed using three objects to help children's understanding. The demonstration was short however, and although children were provided with a story context to support their understanding, the study design did require them to begin problem solving with a novel problem that was different from the one they had been learning in class. This point is particularly relevant to Study 3 which examined children's use of 'representational trace', since although children had been provided with the demonstration, no explicit instruction had been given on how the trace could be used beneficially. It is possible that providing this prior instruction might have significantly affected the use and hence advantage of this representational property.

In addition to the short demonstration, most studies in the research adopted a within subjects design in which children had only a single problem to solve with a certain representation. This did not give children an opportunity to familiarise themselves with the materials beforehand, and it is possible that this might have had the effect of minimising any differences between representational effects in the studies undertaken. Although no improvements in performance were found in Study 3 where children used the same representation on three consecutive problems, it might be argued that this still offered only limited opportunity for children's problem solving to develop according to each representation.

It is also possible that the demonstration problem itself influenced children's strategies, thereby disguising the effect of the materials. In the demonstration, four solutions were shown: (with the fixed order 3 & 0, 1 & 2, 2 & 1 and 0 & 3). This provided three codable solutions (2 *other* solutions and 1 *compensation*[23] solution), and it is possible that demonstrating these solutions in a different order (e.g., 3 & 0, 2 & 1, 1 & 2 and 0 & 3) might have affected the strategies subsequently used. This might particularly have been the case if the demonstration had been accompanied by clear gestures such as swapping over objects, as it has been shown that a teacher's gestures can help children's understanding (Valenzeno et al., 2003). However, it is interesting to note that although each demonstration began with a particular solution (all in one part and none in the other), this did not seem to detract from children's tendency to begin by Equal partitioning.

The demonstration was provided as a prompt to highlight that this problem required multiple solutions and which types of solution were considered valid (i.e. that *commutative* solutions were unique, and that 'none' was a valid amount in one part). This prompt remained constant throughout the studies, although other prompts were given that did change between studies. For example, it was decided to verbally prompt children by asking *"is that all the ways or are there any more ways?"* if children paused for ten seconds. This prompt was different in the sixth study: *"are you still thinking?"* Although the same prompts were used for all conditions within each study, the differences make

[23] It was decided to code such changes as *compensation* although it also is a reverse of parts (*commutative*)

comparisons between studies difficult and, significantly, highlight the strength of other factors in influencing children's partitioning behaviour.

9.5.2 Theoretical arguments

9.5.2.1 Context

The importance of considering the context in which representations were examined in the studies was previously discussed. Although the research was carried out in school and centred on a curriculum relevant task, there are clear differences between the research and an everyday classroom context (e.g., one to one attention, absence of peers, etc.). Whilst the aim of this thesis was to examine differences between representations rather than how well children's performance might generalise to the classroom, it is important to consider interactional effects – how the particular research context may have benefited children's performance with one type of representation.

It is likely that children's prior experiences with materials influenced the way they used them in the sessions. Whilst children had used the cubes in class, they had not seen or used the specific pictorial or virtual materials. This may have created an unfair advantage for physical materials, although it is equally possible that their prior use acted as a deterrent. For example, it was shown in Study 1 that children used less efficient procedures with objects when adding, whilst several even made comments suggesting that they saw the need to use objects at all as a sign of poor numerical ability: *"I don't need cubes any more"*. Prior activity with materials may also have affected their strategies - a possible explanation put forward for children's inclination to partition objects equally.

It is also important to consider how the nature of the task may have minimized the possible advantages of the other representations. After Study 1 it was decided to give children the total number of objects at outset. The interviewer would therefore count out and present the materials to each child in each problem. In contrast, in the Virtual condition, it was possible to create a program showing the total initial number. Although there are considerations about how easily children can access and open certain files, this does highlight an advantage of these materials in this task. A similar argument could be made for pictorial materials, where teachers are able to provide the total by simply photocopying and handing out sheets.

Another affordance of paper discussed in Study 3 was that it provides a trace of solutions. Although this did not seem to influence problem solving, it did provide a record of work that the teacher could use to assess progress. In the task, the interviewer recorded solutions; this not only provided a record but also made it clear to children that they were required to provide a numerical solution. In the classroom, the teacher is able to communicate the task and task demands, but realistically children will be given the task to solve amongst peers with far less adult supervision. A pictorial representation thereby provides a simple way for children to record their own solutions. Tangible designs therefore need to take account of these (and other) aspects that may play a greater role when activities take place in the classroom (or home).

9.5.2.2 Problem solving and learning

A key assumption made in these studies is that differences in the ways children solve problems with materials will help develop their ideas and strategies to solve problems without them. Support for this argument came from observations that children seemed to be abstracting strategies – manipulating representations but applying calculations without counting the materials. However, the research did not carry out pre- and post-tests to examine learning effects, so that it is not clear what gains may or may not have been made.

Some of the points made in this discussion suggest that the benefits of physical materials may have been more clearly demonstrated by extending the design of the studies to examine learning effects. It was suggested that tactile information may have helped reduce the cognitive demands in the task thereby freeing up more working memory for children to learn. It is also possible that motoric coding could have helped children recall strategies at a later stage (e.g., children may be able to recall a simple gesture of moving objects one at a time using one finger). Importantly, though, the process of constraining their own actions and identifying which actions are most efficient may support children's learning in a way that could be demonstrated through transfer tasks (Schwartz et al., 2005). Clearly, these proposals are speculative; they are mainly intended to acknowledge that the benefits of physical materials may not have been best evaluated through short one-off problem sessions.

9.5.3 Summary

Although this research has contributed to our understanding of the potential for Tangibles to support children's learning, it is important to take into account some of the limitations when using the findings to generalise on the evaluation of Tangibles in a different context. This research has highlighted how children's actions with physical objects may foster different strategies that relate solutions when exploring how numbers are composed. However, it is important to examine how these strategies develop over time and how they affect children's capabilities in the absence of materials. Importantly, in evaluating the potential for Tangibles to support children's understanding of additive composition it is necessary to consider how the materials will be implemented in an educational context that introduces a variety of different contextual factors.

9.6 Implications

This research has brought together work from three main research areas: children's numerical development, external representations and digital manipulatives. This section of the discussion looks at the contribution that this research has for these different areas.

9.6.1 Numerical development

One implication of this research concerns the tendency children displayed toward partitioning into two equal groups. It is possible that this tendency marks the emergence of children's understanding of how numbers are made up of smaller numbers, and may influence later learning. This may have implications for teaching – it may, for example, be possible to frame difficult part-whole questions around equal partitioning.

Another question highlighted in the research was whether younger children's tendency to identify just a single solution reflected conceptual development, prior experience, or possibly an interaction of both. It is likely that children's prior maths experience may reinforce the notion of there being just a single solution, particularly as '*how many?*' questions tend to require just one solution. It would be interesting to examine the impact of interventions encouraging children to identify multiple solutions on their concepts of how numbers can be broken down in different ways (additive composition); indeed this has proven effective with older children (Ainsworth et al., 1998).

9.6.1.1 Assessment of additive composition

One key contribution this research may have for other research into children's numerical development is the task used throughout the studies. The task is adapted from Jones et al (1996) assessing children's skills for multidigit understanding, although the research has presented an opportunity to quantify not only the number of correct solutions but also the key strategies used. Although the task focuses on the composition of numbers, it has not previously been discussed within the literature on additive composition. Previously two main tasks have been described as assessing children's understanding of additive composition: Nunes' shop task (in Nunes & Bryant, 1996) and the missing addend addition problem. It is likely that the current task is comparatively demanding as children are effectively being asked to identify repeated combinations of parts from a given whole. Children's understanding of additive composition has also been related to more difficult tasks such as recomposing an addition problem: e.g., recomposing 9 + 4 to 10 + 3, or 8 + 7 to 7 + 7 + 1 (Canobi et al., 2002). It would be interesting to compare children's performances in the partitioning task used in this research with scores on other tasks related to the concept of additive composition. It may even be found that the partitioning task provides a valuable assessment tool by being able to measure different stages of children's development from their initial ability to identify just a single solution through to a full awareness of being able to successfully identify all solutions using an efficient procedure.

9.6.2 External representations

9.6.2.1 Constructionism

Chapter 1 highlighted the work of Seymour Papert as highly relevant to the theoretical approach adopted in this thesis. According to Papert, children's learning can be supported by constructing public entities: allowing them to externalise, share and reflect on their own thinking. In the partitioning tasks, children were able to externalise their thinking using cubes; however, their constructions were greatly constrained by the task demands presented by the interviewer. Nevertheless, it has been argued that children's actions were less constrained using physical objects than virtual materials. In this way, children were able to reflect on their actions on the representation and change their strategies accordingly. Unfortunately, in the same way that the studies did not assess how this supported learning or transfer to other tasks, it is not possible to determine whether externalising thinking using the cubes helped children develop any ideas beyond the specific task (e.g., planning skills) as suggested by Papert (1980).

This discussion has also emphasised the role of the interviewer in scaffolding children's understanding and behaviour in the task. In doing so, the research re-iterates the conclusions of Sutherland (1993) that is it important to consider the interactions between the learner *and* teacher within a Constructionist paradigm.

9.6.2.2 Physically Distributed Learning

When children manipulated objects in this research, they created new spatial configurations. The research showed how they would create these new configurations and then interpret some (but not all) of them as new solutions. This finding - that actions on the representation supported problem solving, echoes previous literature describing the interactive nature of actions and cognition in problem solving (Anzai & Simon, 1979; D Kirsh, 1995; H. Neth & Muller, 2008). However, the task presented was one in which the children had incipient understanding and the advantages of using physical materials indicated possible benefits for learning in this domain. By showing how children's actions with physical materials may help them develop new ideas, this research provides support for PDL. Indeed, the findings support predictions made by Martin and Schwartz (2005) that children will make more adaptations and identify more solutions using physical materials than pictorial.

Rather than just show that children identified more solutions using physical materials, this research has demonstrated how these materials may actually encourage the use of strategies that relate consecutive solutions. These strategies can be applied in the absence of materials, and observations suggested that some children were indeed beginning to apply these strategies mentally whilst manipulating the materials. There were also several children who demonstrated important changes in strategy when using materials despite the limited opportunity for learning in the short sessions. For example, several children constrained their own actions, identifying incremental solutions by changing from moving multiple objects each time to moving them one by one. The research thereby provides strong support for the potential for actions on the representation to lead to new ideas.

If actions can lead to new ideas, as proposed by PDL, it might be argued that changing the actions that can be made on a representation might influence the nature of the ideas developed. The design of this research might indeed be interpreted as testing and supporting this suggestion. By changing the way in which children could manipulate objects, it was shown that the strategies they used were significantly different. These findings highlight the potential differences in learning that might occur when acting on virtual rather than physical representations – a difference not discussed in Martin's (2007) more recent work which has applied the arguments of PDL to the use of virtual manipulatives.

Whilst it was shown in this research that physical objects could help children partition, the findings from the first study showed that manipulatives may not always be beneficial; indeed, children used less developed strategies for addition problems when using materials than when using their fingers. Although Martin and Schwartz do propose that PDL may only occur when children have incipient domain knowledge, it is still not clear when the advantages of manipulating physical objects can be predicted. This research provides important points for predicting when PDL may occur.

Firstly, it is important that the initial demands of using the materials are not too high. In this task it was important to provide children with the total number of objects to partition. In Martin and Schwartz's (2005) reported studies, children were also given the initial number to partition, and it might be expected that the benefits would have been more limited had this not been the case. Secondly, physical materials have properties that may help offload task demands, such as lining up objects to help keep track when counting as demonstrated in Study 1. This may explain why Martin et al (2007) found that children were able to solve more addition problems using physical materials than pictorial materials - the children were younger than those in Study 1 and might well not have been able to use more developed strategies such as count on. A third aspect for predicting when PDL may occur reflects a consideration of what visual (or tactile) properties may encourage children to create configurations that can be interpreted as solutions. This research provides one possibility: visual symmetry. As discussed, there was a strong tendency for children to begin by partitioning objects into equal parts, and it is possible that this tendency to partition into symmetrical groups also affected the way children partitioned objects in the fraction tasks in Martin's studies.

Mathematics has a strong relationship with geometry. It may well therefore be possible to identify other activities where manipulating objects might help children create configurations that can be interpreted numerically to solve a particular problem. Examples might include arranging rods of different lengths, or sorting certain manipulatives into odds and evens.

The way objects are manipulated will largely depend on the context of the numerical activity; but the argument being made is in this section is that, when considering the circumstances in which actions may lead to ideas, it is important to consider what properties of materials may foster certain actions. Different properties may foster different actions which may lead to the development of different ideas.

9.6.2.3 Tactile information

Tactile information was identified as a key affordance of physical representations and frequent observations were made throughout the studies of children touching objects to support cognition; from tagging objects to count or touching objects to remember to move them next. However, it was not clear how much this supported problem solving, and indeed children did not seem disadvantaged in the Virtual condition where objects could not be touched. Interestingly, in the Virtual condition, children would often point the cursor to objects when counting. This indicates that the some of the cognitive benefits of tagging an object with fingers can be extended to tagging an onscreen object using the mouse pointer. However, it is probable that manipulating the cursor accurately using a mouse requires a greater degree of fine motor control than can be expected at a young age; importantly, it also requires greater visual attention as, unlike using fingers, no tactile feedback is provided for the position of cursor. The small numbers of objects to be counted in the numerical task may have rendered the benefits of tactile feedback negligible, but it is possible that they would become significant in a more demanding task (counting larger arrays for example).

9.6.2.4 Embodied cognition

Physical objects can be manipulated in space which can generate actions that can be emulated as gestures. For example, various actions observed in this research such as taking away, adding, partitioning or swapping over groups of objects can all be enacted without materials through gestures. This may have implications, not only for communicating these actions, but also for how the concepts refelcted in these actions could become encoded in memory. Therefore this task presents a possible platform in which to examine the role of embodiment in young children's developing concepts. If, for example, children were observed to develop gestures (similar to the actions observed in the studies) to communicate or support thinking, this would provide strong evidence for the embodiment literature and highlight the importance of physical actions in learning.

9.6.3 Manipulative debate

An important goal for education is developing our understanding of when manipulatives support learning (Ginsburg & Golbeck, 2004). The previous sections have attempted to expand on PDL by describing how particular representational properties may affect the actions taken, and how this in turn may lead to differences in the ideas developed. Key to all of this, however, is how children are able to interpret their actions with the representations numerically.

9.6.3.1 Linking representations

A key criticism of manipulatives is their dual representation: they represent both numbers and objects themselves (see Uttal et al., 1997). Indeed, in a relatively recent summary of the 'manipulatives debate' (McNeil & Jarvin, 2007), the main conclusion for educators was the importance of using materials that do not distract children from interpreting the objects as representations of number: for example, not to use objects from unrelated activities. The authors highlighted the importance of bridging the gap between the potential for physical manipulatives to tap into children's intuitive knowledge and the formal language of mathematics they need to develop:

> *"Assuming manipulative do, indeed, foster children's understanding, the use of manipulatives may simply result in a greater divide between intuitive and formal knowledge. Thus one of the primary goals of teachers should be to develop lessons that help students make connections between intuitive and formal knowledge"* (p. 315)

In the study sessions of this research, children did use physical materials according to the demands of the numerical problem. However, prompts were often needed by the interviewer for children to enumerate their solutions. Children were usually then able to continue independently; although it is not clear whether younger children who identified only one solution would have benefited from more support. Unfortunately, there is rarely enough time in an actual classroom to verbally encourage children to enumerate solutions. Instead, the teacher needs to provide a means for children to interpret representations numerically – often achieved using other materials such as paper. Indeed, there was an opportunity during the period of this research to observe a class at a school in Australia, and it was interesting to note that the teacher there used physical materials (seeds) to explore ways to partition ten and a piece of paper for children to record their solutions.

One key advantage of pictorial materials is that they provide a means of ensuring that children quantify solutions – and a record for the teacher to examine their solutions.

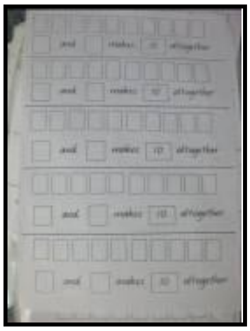

Figure 9.5: Manipulating objects and recording on paper

In summary, although children may use manipulatives according to the numerical context of a problem, it is important to consider how these actions are interpreted numerically. In designing effective Tangibles, it is important to consider what will help children to interpret their actions without compromising any of the advantages of physical interaction. The virtual design used in Study 7 raised the possibility of using colour as a bridging metaphor to encourage children to reflect on their actions and interpret the representation numerically. Clearly, this proposal would need empirical support.

9.6.3.2 Implications for Teachers

A key limitation identified by this research related to differences between the study context and the classroom. Finding a way for children to interpret their actions numerically when the teacher cannot attend is difficult. Teachers may understandably be keen for children to learn efficient procedures (particularly if these are described in the curriculum and will help with certain tests), and they may need to focus children's attention on learning the most efficient ways of identifying different partitioning solutions. This suggestion is supported by a variety of teaching resources available that structure activities such as those shown in Figure 9.6.

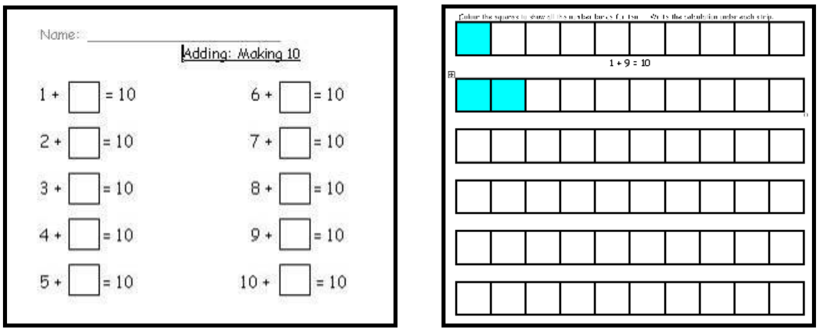

Figure 9.6: Teaching resources used to support children's learning of number combinations

A key theme raised in this thesis concerns whether children will benefit from finding out for themselves how to identify solutions in the most efficient way (as opposed to being told). This theme is iterated by Thompson (1994) who argues that the key question that should drive the use of manipulatives is not what we want children to do, but what we want them to understand. Indeed, an important finding of this research for teachers is that providing children with manipulatives may not just allow them to explore different strategies for identifying ways to partition numbers, but may actually help them discover ways to relate solutions. Although children may not identify all solutions, or indeed may use a variety of strategies, the task used in this research does offer an opportunity for class discussion. Whilst further research would be needed to investigate the extent to which children's learning is helped by the use of materials, just providing them with the experience of relating solutions may help them learn ways to partition numbers such as ten – which in itself is a key curriculum objective. Indeed, Baroody (2006) discusses the importance of helping children understand how parts relate in order to develop mastery of number combinations. This research has also shown that, if the goal is to foster the use of an efficient procedure for identifying different number combinations, then constraining actions using a graphical interface may be most effective.

9.6.4 Tangibles to support numerical development

In discussing the potential for Tangibles to support children's understanding of additive composition, many points were raised that apply to wider arguments about the use of Tangibles to support children's numerical ability. In particular, it was discussed how the representation used in the final study exemplified the way in which physical representations might be augmented to draw children's attention to certain numerical relationships (such as how objects can be partitioned equally or added in any order).

The digital effects used in the material in Study 7 were designed simply to highlight changes in quantity. However, by using colours to represent unique numbers the materials also highlighted how Tangibles might be designed to embody certain numerical concepts that are not possible with analogue materials. Physical Cuisenaire rods, for example, use colour and length to represent different numbers, but consequently each rod cannot be broken down into smaller rods. In contrast, the *Unifix* cubes used in the study can be broken down but cannot change perceptual features such as colour to represent their numerosity. Arguably, the design in Study 7 exemplifies how a tangible design may be able to capture both these concepts – augmented cubes that use colour to represent number that can also be broken down (changing colour automatically).

9.7 Future research

During this discussion, many references have been made to the need for further research. These are summarised in this final section, and, reflecting the different areas brought together in this research, are discussed under three headings: numerical development, external representations and digital manipulatives.

9.7.1 Numerical development

Further research might examine the effects of using physical materials for young children who cannot identify a partitioning solution without materials. It would be interesting to see if materials help develop children's incipient understanding of a problem, as well as to examine what role materials have in children's tendency to divide objects equally, especially if they have not had school experience in a related activity. Furthermore, as young children have less developed fine motor control skills and less experience with graphical interfaces, further research might also examine whether the benefits of physical manipulation are more effective at younger ages.

Since this research compared children's approaches to different representation problems in short problem solving sessions, it was not possible to examine changes over time. Further research may establish whether children are able to develop their strategies using different materials, and possibly throw more light on whether they can abstract strategies, a possibility raised in Chapter 5 where video observations showed children manipulating objects physically at the same time as they were calculating parts mentally.

This would offer a chance to examine whether the use of physical materials leads to greater post intervention gains without materials than other representations.

One of the arguments made in this discussion is that, since physical manipulation has helped children identify what is and is not relevant when solving a problem, the advantages of using physical materials may be measured through transfer tasks (Schwartz et al., 2005) better than through tests of procedural efficiency. Further research could examine whether any gains in using manipulatives in the partitioning task transfer to gains in other assessment tasks for additive composition such as Nunes' shopping task (Nunes & Bryant, 1996) or missing part addition problems. This could also examine the relationship between children's performance on these tasks and their performance on the partitioning tasks developed in this research, increasing further our understanding of the development of children's concepts in this domain.

9.7.2 External representations

This research has examined the affordances of physical materials and the roles that certain properties of a specific manipulative play in a specific task. The discussion has described how these affordances might affect problem solving in other tasks. As well as examining these predictions, further research might also examine the role of specific physical properties on problem solving. For example, it might be predicted that larger or differently shaped objects might influence children's actions, whilst comparing tiles with cubes might help determine the importance of certain actions such as moving objects over each other. Research could also use interfaces such as tabletop computers to examine whether tactile feedback (as opposed to just moving objects by hand) can support cognition in more demanding problems. In short, there are many ways in which further research can build on the findings of this present research in trying to unpick the complex interaction between representational properties and children's problem solving strategies, all of which helps to improve the design of novel materials.

9.7.2.1 Gestures

One affordance of the physical materials used in this study is that their manipulation led to actions that could be emulated in the absence of the materials. It is possible therefore that the materials themselves foster gestures that support communication or embodied cognition. The value of gestures in mathematics has received growing research interest (e.g., Abrahamson & Howison, 2008; Cook, 2007), and has indicated a role for physical objects in developing such gestures (Edwards, 2005). However, further research is certainly needed to establish such a causal link, and the current task may provide a means to examine such mechanisms. Studies might be made to examine the use of gestures in this partitioning task and whether the introduction of manipulatives results in a greater use of gestures than other materials such as virtual representations.

9.7.2.2 Physically Distributed Learning

PDL describes how children can act on representations, and then interpret their actions to develop new ideas. It was suggested in this discussion that in the absence of other plans children may tend towards creating geometric patterns. These can often be interpreted numerically, and the suggestion therefore offers scope for future research to test predictions about which concepts may be supported through this process. The present research has also raised the possibility that changing the nature of interaction with the representation will influence strategies. Further research might investigate this possibility, and in particular examine whether the ability to move single or multiple objects affects strategies in other domains such as fractions. Any differences found may have important implications, especially when there is considerable scope for variation in the way that virtual materials can be designed to allow manipulation of multiple objects – e.g., the instructions for moving multiple objects in Taylor Martins' online virtual materials require children to press the shift key to allow squares to stick to each other when moving (Martin, 2004 - see figure 10.1).

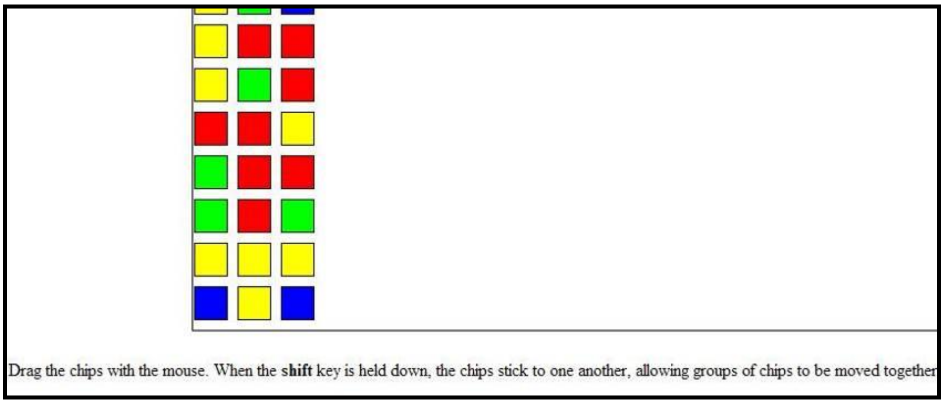

Figure 9.7: Screenshot of virtual materials (Martin, n.d.)

The effects of different actions can also be examined with the physical manipulatives used in this research. The *Unfix* cubes used across studies can be joined linearly, but doing so changes the relative cost of moving individual or groups of objects. Further research could usefully examine whether asking children to join cubes significantly affects the strategies they use.

9.7.3 Interface design

Graphical interfaces allow actions on representations to be controlled by the designer. They thereby offer a way of testing some of the hypotheses put forward in this discussion – for example a tabletop computer can provide a platform in which to tease apart certain arguments such as the importance of tactile feedback when manipulating objects on a touch screen. Importantly, tabletop interfaces can provide ways to control how objects can be manipulated in groups or individually – research in this area might help by identifying the most effective way to allow young children to manipulate multiple objects. This would also provide valuable information on whether there are limitations to how easily objects can be manipulated using a graphical, as opposed to tangible, interface.

The materials used in Study 7 can be adapted to allow children to manipulate multiple objects, and further research might usefully test some of the possibilities described: namely, how colour prompts may draw children's attention to some important numerical principles – e.g., how adding and taking away the same amount leaves a quantity unchanged, or how a collection of objects can and cannot be decomposed into equal groups.

9.7.4 Tangibles in school context

The possibilities for further research described above should help develop our understanding of the potential for tangible designs to support children's numerical development. However, it has been emphasised throughout this chapter how the effects of materials will largely depend on the context in which they are used. If Tangibles are to be designed to support learning in an educational context, it is important to extend research to this setting. It is possible, for example, that a tangible design such as that discussed in Chapter 8 might provide a basis for class discussion on numerical relations. On the other hand, it may be found that such digitally augmented materials simply distract children as they focus on the technology rather than the numerical principles they are intended to represent.

9.7.5 Summary

This thesis has focused on examining the potential for Tangibles to support children's understanding of additive composition – a key concept in children's numerical development. Clearly, the focused nature of the research means there remains a significant gap between the findings and knowledge of the effectiveness of novel materials in educational settings. However, the research has produced significant findings that shed light on representational properties that influence children's strategies for exploring how numbers can be decomposed into smaller numbers. This section has sought to identify areas where further research can build on these findings to examine how problem solving in the task may generalise to other tasks and, importantly, lead to learning. This could also test some of the predictions made in this research in terms of how actions on representations may support some concepts and not others, and importantly, how influencing actions can in turn influence strategies, and ultimately therefore the ideas developed. Finally, research might examine the potential for augmenting materials to foster certain interpretations. The materials described in the final chapters of this thesis provide an example of the type of resource that might be used to compare and contrast differences between interfaces and help identify the extent to which the use of Tangibles offers unique benefits in this domain.

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Appendix A

Addition and Partitioning problem order used in Study 1

Addition problems

3+2

5+3

7+2

9+3

11+6

13+7

1+4

3+6

2+8

4+11

5+9

7+12

Open partitioning problems

Partition 5

Partition 8

Partition 10