Bifurcations with spherical symmetry

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ABSTRACT

Bifurcations from spherically symmetric states can occur in many physical and biological systems. These include the development of a spherical ball of cells into an asymmetrical state and the buckling of a sphere under pressure. They also occur in the evolution of reaction–diffusion systems on a spherical surface and in Rayleigh–Bénard convection in a spherical shell. Many of the behaviours of these systems can be explained by their underlying spherical symmetry alone. Using results from the area of mathematics known as equivariant bifurcation theory we can use group theoretical methods both to predict the symmetries of the solutions which are expected to result from bifurcations with symmetry and compute their stability. In this thesis both stationary and Hopf bifurcation with spherical symmetry are discussed.

Firstly, using group theoretical techniques, the symmetries of the periodic solutions which can be found at a Hopf bifurcation with spherical symmetry are computed. This computation has been carried out previously but contains some errors which are corrected here. For one particular representation of the group of symmetries of the sphere, $\mathbf{O}(\mathbf{3})$, the stability properties of the bifurcating branches of periodic solutions resulting from the Hopf bifurcation are analysed and a survey is carried out of other periodic and quasiperiodic solutions which can exist.

Secondly, symmetry considerations are used to investigate the existence and stability properties of symmetric spiral patterns on the surface of a sphere which result from stationary bifurcations. It is found that in the case of the Swift–Hohenberg equation spiral patterns with one or more arms can exist and be stable on spheres of certain radii. Although one-armed spirals in the Swift–Hohenberg equation are stationary solutions, it is shown that generically one-armed spirals on spheres must drift.

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CONTENTS

CHAPTER 1

INTRODUCTION

Bifurcations from spherical symmetry can be observed in many physical systems including the buckling of a sphere under pressure and Rayleigh–Bénard convection to name just two of the numerous examples. The bifurcation behaviour of such specific models can be studied directly, ignoring the symmetries, however some aspects of the analysis will be common to all models with the same underlying symmetry. This phenomenon that distinct but symmetrically related systems can exhibit remarkably similar behaviour is often called model independence [45]. Using the area of mathematics known as *equivariant bifurcation theory* one can study the model–independent behaviours of symmetric dynamical systems (those which depend on the symmetries alone) without any reference to the details of a particular model.

Equivariant bifurcation theory uses group theory to analyse bifurcations in dynamical systems with symmetry. Unlike other methods, equivariant bifurcation theory allows us to distinguish aspects of a problem which are a consequence of the underlying symmetries from those which are specific to the particular model. The advantage of equivariant bifurcation theory over other methods of bifurcation analysis in systems with symmetry is that we are able to study the behaviours of entire classes of systems with the same underlying symmetries in a more generic framework by using the symmetries alone. A great deal of information can be deduced in this way and explicit use of symmetry-based principles often simplifies the analysis. The symmetries determine the range of behaviours which it is possible for all symmetrically related systems to exhibit but the details of the particular model decides which of these behaviours the system chooses.

This thesis uses equivariant bifurcation theory to study the range of behaviours associated with a bifurcation from a spherically symmetric state. The wide range of physical systems where such a bifurcation can occur is explored in Section 1.2.1. The two main themes of this thesis are

(a) the time-periodic solutions which can be created as a result of a Hopf bifurcation from a spherically symmetric state and

(b) the symmetric spiral patterns which can exist on spheres as a result of a stationary bifurcation from spherical symmetry and subsequent secondary bifurcations.

The second topic is particularly interesting as it represents the first analytical study of symmetric spiral patterns on spheres. Such patterns have been found numerically and experimentally in a range of physical systems which is discussed in Section 1.2.2. Further details on the structure and results of this thesis are given in Section 1.3.

1.1 Introduction to dynamical systems with symmetry

Over the past thirty years equivariant bifurcation theory has developed into a powerful mathematical tool with applications including pattern formation, animal locomotion, speciation, fluid dynamics and magnetohydro dynamics. To allow us to describe current research in this area of mathematics, in particular to explain the results obtained in this thesis, we must first give brief definitions of some of the technical language which is used throughout the literature, and in this introduction. A more thorough introduction to the general theory of bifurcations with symmetry can be found in Chapter 2 of this thesis.

When we study bifurcations of dynamical systems with symmetry we consider systems of ordinary differential equations

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(\mathbf{x}, \lambda), \tag{1.1.1}$$

where $\mathbf{x} \in V$ for some finite dimensional vector space V, $\lambda \in \mathbb{R}$ is a bifurcation parameter and f is a smooth nonlinear mapping. We specify the symmetries of (1.1.1) in terms of a group Γ where

$$\gamma f(\mathbf{x}, \lambda) = f(\gamma\mathbf{x}, \lambda) \quad \text{for all} \quad \mathbf{x} \in V, \quad \lambda \in \mathbb{R} \quad \text{and} \quad \gamma \in \Gamma. \tag{1.1.2}$$

We say that the vector field f is equivariant with respect to the action (or representation) of Γ on V. A consequence of $\gamma \in \Gamma$ being a symmetry of (1.1.1) is that if $\mathbf{x}(t)$ is a solution then so is $\gamma\mathbf{x}(t)$. An element $\sigma \in \Gamma$ is a symmetry of a stationary solution, $\mathbf{x}$, of (1.1.1) if $\sigma\mathbf{x} = \mathbf{x}$. The set of such symmetries form a subgroup $\Sigma_{\mathbf{x}} \subset \Gamma$ called the *isotropy subgroup* of $\mathbf{x}$. Similarly the isotropy subgroup of a periodic solution $\mathbf{x}(t)$ of (1.1.1) is the subgroup of elements $(\gamma, \theta) \in \Gamma \times S^1$ for which $\gamma\mathbf{x}(t + \theta) = \mathbf{x}(t)$ for all t.

We assume that (1.1.1) has a trivial solution $\mathbf{x} = \mathbf{0}$ with isotropy subgroup Γ which undergoes a bifurcation at $\lambda = 0$. We call this a bifurcation with Γ symmetry. If this is a stationary bifurcation then (under certain hypotheses) a result called the equivariant branching lemma [25] guarantees that branches of stationary solutions with certain isotropy subgroups bifurcate. These isotropy subgroups fix a one-dimensional subspace of V and are called *axial* isotropy subgroups. If the bifurcation at $\lambda = 0$ is a Hopf bifurcation then any bifurcating branches of solutions are periodic and have the symmetries of isotropy subgroups of $\Gamma \times S^1$. There is an analogue of the equivariant branching lemma called the equivariant Hopf theorem which (again, under certain hypotheses) guarantees the existence of periodic solution branches with isotropy subgroups which fix a two-dimensional subspace of $V \oplus V$. Such isotropy subgroups are called $\mathbb{C}$-axial.

Which subgroups of Γ ($\Gamma \times S^1$) are axial ($\mathbb{C}$-axial) isotropy subgroups depends on the action (or representation) of Γ on the vector space V. Thus the representation is as important as the symmetry group Γ in determining the symmetries of the solutions of (1.1.1) which can exist. This will be reflected throughout this thesis where we consider the symmetries of solutions which can occur in different representations of certain groups.

For a given representation of Γ, the general form of the equivariant vector field $f(\mathbf{x}, \lambda)$ can be computed using the fact that it must satisfy (1.1.2). This vector field can be used to determine whether solutions branch supercritically or subcritically and their stability. In addition, the vector field can be used to determine when it is possible for solutions with non-axial symmetry to exist.

The general theory of bifurcations with symmetry, which has been very briefly outlined above, has been used in a wide range of applications. Here we give just a few of the large number of examples of situations where equivariant bifurcation theory has been used to deduce information about systems with certain symmetries.

- Stationary bifurcations with S_N symmetry (where S_N is the symmetric group on N symbols) can be used to describe speciation models [26, 36].
- Stationary bifurcations leading to the spontaneous formation of regular patterns with symmetries determined by geometry can be studied for many different examples including the elementary example of a bifurcation in a box considered by Hoyle [52].
- Spatially periodic patterns on planar lattices result from stationary bifurcations with $H_{\mathcal{L}} \dot{+} \mathbf{T}^2$ symmetry where $H_{\mathcal{L}}$ is the group of rotations and reflections of the lattice $\mathcal{L}$. The equivariant branching lemma leads to a series of planforms which are guaranteed to exist bifurcating from a trivial homogeneous state [33]. These bifurcations can be used to explain patterns seen in certain kinds of reaction–diffusion systems, convection, geometric hallucination patterns in the visual cortex [12, 37], stripes and spots on animal skins and nematic liquid crystals [21].
- Hopf bifurcations can also occur on lattices leading to patterns which are periodic in space and time. Examples studied include the case of a square lattice by Silber and Knobloch [77] and cubic lattices by Callahan [17].
- Hopf bifurcation has been used to describe four identical nonlinear oscillators coupled with the symmetry of a square. In addition to the periodic solutions with maximal symmetry, Swift [79] found that there can be a branch of quasiperiodic solutions with two frequencies bifurcating from the origin.
- For small Reynolds numbers, an ABC flow is a stable solution of the Navier–Stokes equations with a particular forcing. Hopf bifurcation with the rotational symmetries of a cube can be used to study the instability of ABC flow for increasing Reynolds numbers in the case $A = B = C = 1$ [6].
- Hopf bifurcation with S_N symmetry can be used to describe the behaviour of N all-to-all coupled nonlinear oscillators [32].

- Abreu and Dias [1] have studied Hopf bifurcation in reaction–diffusion equations defined on the hemisphere with Neumann boundary conditions on the equator. They obtain periodic solutions for the hemisphere problem by extending to the sphere and finding solutions with spherical spatial symmetries containing reflection across the equator.

In addition to these examples, there has been much research concerning bifurcations with spherical symmetry where the group of symmetries, Γ, contains the orthogonal group $\mathbf{O}(\mathbf{3})$. Recall that in this thesis we will be considering the Hopf bifurcation with spherical symmetry and the existence of symmetric spiral patterns on spheres resulting from stationary bifurcations. Research concerning particular systems which undergo bifurcations from spherically symmetric states provides the motivation for the work in this thesis, while previous studies of such bifurcations using equivariant bifurcation theory form the basis on which the current work builds. In Section 1.2 we review the research which is of particular relevance for the work presented in this thesis.

1.2 Previous studies of pattern formation on a sphere

In this section we give an overview of the previous research concerning pattern formation on a sphere which is relevant for the work presented in this thesis. This includes motivating examples of systems which can undergo a bifurcation from a spherically symmetric state, and also systems which have been found to exhibit spiral wave behaviours, particularly in spherical domains. We also review results which have been obtained by studying stationary and Hopf bifurcations with $\mathbf{O}(\mathbf{3})$ symmetry using the techniques of equivariant bifurcation theory.

1.2.1 Bifurcations with $\mathbf{O}(\mathbf{3})$ symmetry

There are many physical and biological examples of systems where a spherically symmetric state undergoes a bifurcation to a state with less symmetry. For instance, bifurcations to stationary patterns occur in Rayleigh–Bénard convection in a spherical shell [13, 14, 76]. If the fluid within the spherical shell is subjected to a magnetic field (and is electrically conducting) or concentration gradient in addition to the temperature gradient then it is possible for a Hopf bifurcation to oscillating solutions to occur. Examples of such oscillating solutions in convection have been found by Cross and Hohenberg [29], Knobloch [56] and Knobloch and Proctor [58]. Convection within a spherical shell has applications including continental drift driven by convection currents in the Earth's mantle and also convection within the Sun where the strong magnetic field has an influence on the convective motion.

Another physical example of a bifurcation from a spherically symmetric state is the buckling of a sphere or spherical shell under uniform external pressure. This has applications including the evolution of a gas bubble in a liquid [55, 70].

Both stationary and Hopf bifurcations can occur in reaction–diffusion systems on a sphere, as discussed by Turing [80]. Stationary patterns resulting from reaction–diffusion systems on a sphere are considered by Varea et al. [82] and Callahan [18] while a specific example of a reaction–diffusion system which undergoes a Hopf bifurcation is discussed in [34, 35, 89]. Hopf bifurcations can also occur in excitable reaction–diffusion systems which will be discussed in the context of spiral waves later in this introductory chapter.

Biological examples of bifurcations from states with spherical symmetry include a spherical ball of cells developing into an asymmetric shape. Such a ball of cells could be an embryo as in [80] or a solid tumour as in [15].

Stationary bifurcations with spherical symmetry have been widely studied using the methods of equivariant bifurcation theory. Recall that the main result in the study of symmetric stationary bifurcations is the equivariant branching lemma which guarantees the existence of branches of stationary solutions to (1.1.1) with the symmetries of the axial isotropy subgroups of the group Γ in the representation of interest. There may also be solutions with the symmetries of the other isotropy subgroups in this representation but there is no result which guarantees their existence. In the case where the group Γ is $\mathbf{O}(\mathbf{3})$ the irreducible representations are on the spaces V_ℓ of spherical harmonics of degree ℓ. The representations and subgroups of $\mathbf{O}(\mathbf{3})$ will be introduced in Chapter 3.

The problem of computing all isotropy subgroups of $\mathbf{O}(\mathbf{3})$ in every irreducible representation has been tackled by a number of people. Michel [68] first listed the isotropy subgroups for the representations on the spaces V_ℓ for even values of ℓ using a result called the chain criterion (see Theorem 2.4.2). Ihrig and Golubitsky [53] noticed that this criterion was incorrect for computing isotropy subgroups of $\mathbf{O}(\mathbf{3})$ due to its continuous symmetries and suggested a more appropriate version. They used this new criterion to compute all isotropy subgroups of $\mathbf{O}(\mathbf{3})$ for the representations on V_ℓ for *every* value of ℓ. However, Linehan and Stedman [63] noticed that this improved chain criterion still gave incorrect solutions in some cases. They gave a result which they called the 'massive chain criterion' which allowed them to correctly list the isotropy subgroups of $\mathbf{O}(\mathbf{3})$ in every irreducible representation on V_ℓ. We will make use of the massive chain criterion (Theorem 3.4.1) in Chapter 6.

Having used the equivariant branching lemma to prove the existence of branches of solutions to (1.1.1) with certain symmetries, it is possible compute conditions for these solutions to be stable. Chossat et al. [23] considered the stability of the solution branches with axial symmetry in the representations of $\mathbf{O}(\mathbf{3})$ on V_ℓ for $\ell = 3$, 4 and 5. In addition they discussed the existence of other solutions with submaximal isotropy (i.e. symmetry $\Sigma \subsetneq \mathbf{O}(\mathbf{3})$ where Σ fixes a subspace of V_ℓ of dimension larger than one and $\Sigma \subset \Delta$ where Δ is a larger isotropy subgroup of $\mathbf{O}(\mathbf{3})$).

Other studies of stationary bifurcations with $\mathbf{O}(\mathbf{3})$ symmetry include that of Matthews [66] who discusses the transcritical bifurcations from spherical symmetry that occur when the representation of $\mathbf{O}(\mathbf{3})$ is on V_ℓ for ℓ even. Results on the existence and stability of solution branches are given for the even values of ℓ up to $\ell = 18$ including all solutions in subspaces of dimension 3 or lower. A criterion for the existence of solutions with dihedral symmetry in two-dimensional spaces is given and it is shown that when ℓ is large, although none of the bifurcating branches of stationary solutions are stable, there is a preferred solution with only one positive eigenvalue and this is never the axisymmetric solution.

In applications it is not always the case that the spherical symmetry is perfect. For example, although the Earth can be modelled as a sphere with $\mathbf{O}(\mathbf{3})$ symmetry, the rotation of the Earth and the fact that it is not a perfect sphere break the $\mathbf{O}(\mathbf{3})$ symmetry. This can be reflected in models by adding small inhomogeneities to the equivariant vector field as in [19] or by adding small terms which are equivariant with respect only to a subgroup $\Delta \subset \mathbf{O}(\mathbf{3})$ to weakly break the $\mathbf{O}(\mathbf{3})$ symmetry to Δ symmetry as in [61]. In the latter case it is found that equilibrium solutions which exist when the system has perfect spherical symmetry can persist when the symmetry is broken and in addition heteroclinic cycles can occur.

The dynamics in a system which contains a heteroclinic cycle are complex. Equilibria are connected by trajectories and a state on such a trajectory will appear to cycle among the fixed-points in turn. Heteroclinic cycles can be structurally stable in systems with symmetry [39]. Another situation in which it has been found that heteroclinic cycles can exist in systems with $\mathbf{O}(\mathbf{3})$ symmetry is when there is a mode interaction where the representation of $\mathbf{O}(\mathbf{3})$ is on the direct sum of vector spaces $V_\ell \oplus V_{\ell+1}$. Armbruster and Chossat [5] found heteroclinic cycles in the interaction between the $\ell = 1$ and $\ell = 2$ spherical harmonics. In an extension of this work by Chossat and Guyard [22] it was found that in most cases heteroclinic cycles can be found in $(\ell, \ell+1)$ mode interactions. Additionally, heteroclinic cycles have been found when the $\mathbf{O}(\mathbf{3})$ symmetry is broken to $\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c$ symmetry in the representation of $\mathbf{O}(\mathbf{3})$ on $V_1 \oplus V_2$ [24].

Hopf bifurcations with spherical symmetry have also been studied previously. In the original work of Golubitsky and Stewart [43] on Hopf bifurcations with symmetry the example of the Hopf bifurcation with spherical symmetry was considered. The authors listed the $\mathbb{C}$-axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ in the representations on $V_\ell \oplus V_\ell$. (Recall that periodic solutions of (1.1.1) with these symmetries are guaranteed to exist by the equivariant Hopf theorem.) One small error in this list was corrected by Golubitsky et al. [46]; however, a small number of errors remain. It may be possible for solutions of (1.1.1) other than those guaranteed by the equivariant Hopf theorem to exist under certain conditions. These solutions would have the symmetries of isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ with fixed-point subspaces of dimension larger than two. These isotropy subgroups are yet to be computed for each representation.

By computing the general form of the equivariant vector field f, it is possible to compute the stability of the periodic solution branches predicted by the equivariant Hopf theorem. For the specific example of the Hopf bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry where $\mathbf{O}(\mathbf{3})$ acts naturally on $V_2 \oplus V_2$, Iooss and Rossi [54] use analytical methods to find five different types of bifurcating periodic solutions. They compute the stability of these solution branches and show that a family of quasiperiodic solutions can bifurcate directly from the trivial solution together with the periodic solutions.

Subsequently, Haaf et al. [51] showed that these results could be found more efficiently by realising V_2 as the set of traceless symmetric 3×3 matrices. They too describe the stability of the five periodic solution branches and discuss the restriction of the dynamics to higher dimensional invariant subspaces of $V_2 \oplus V_2$ and the various possible degeneracies which can occur in the stability conditions. Both Iooss and Rossi [54] and Haaf et al. [51] found that the stability of two of the five axial solution branches in this representation depends on the coefficients of quintic order terms in the equivariant vector field. There have been no studies of the dynamics which can occur near Hopf bifurcations with $\mathbf{O}(\mathbf{3})$ symmetry for representations on $V_\ell \oplus V_\ell$ for values of ℓ larger than two.

1.2.2 Spiral Waves

In addition to the periodic patterns which can occur as a result of Hopf bifurcations with spherical symmetry we will also study in this thesis another type of pattern which can occur in spherical geometry— spiral patterns.

Spiral patterns or spiral waves arise in various chemical and biological systems as well as in numerical simulations of reaction–diffusion systems. For instance, spiral waves have been observed in the Belousov–Zhabotinsky chemical reaction [88] and in the oxidation of carbon monoxide on the surface of a platinum catalyst [71] as well as in Rayleigh–Bénard convection and excitable systems such as those described by the FitzHugh–Nagumo model. It is thought that spiral waves and their three-dimensional analogues, scroll waves, appear in heart muscle during cardiac arrhythmias (see for example [10, 49, 72, 85]). In addition, there is speculation that spiral waves may be involved in epileptic seizures where a spiral wave manifests as the local synchronization of large groups of neurons [69].

Spiral waves in planar domains have been widely studied and observed in numerical simulations and experiments (see [7–9, 30] for example). In planar domains, spiral waves rotate rigidly about a centre where the front of the wave has a tip. Far from the rotation centre the spiral wave is well approximated by an Archimedean spiral. This rigid rotation is a relative equilibrium since in a frame rotating at the same speed as the spiral the tip position is fixed. In addition, spiral waves can meander (the tip traces out a flower pattern with either inward or outward petals depending on parameter values) or drift (the tip drifts off along a line drawing little loops as it goes). These motions are two-frequency quasiperiodic. Barkley [8] realised that these spiral wave dynamics could be explained by the Euclidean symmetry $SE(2)$ of the plane. His ideas were extended by Sandstede et al. [73, 74] and Wulff [86]. In particular, the Euclidean symmetry can be used to study the transition (via Hopf bifurcation) from rigidly rotating to meandering planar spirals [74]. Multiarmed spirals can also occur in planar domains [83] and have been observed in the Belousov–Zhabotinsky reaction [41]. For an overview of results on spiral waves in the plane see Boily [11] or Hoyle [52].

Spiral waves can also occur in spherical geometry. In contrast to the large volume of work on planar spirals, there has been relatively little research concerning spiral patterns on spheres. Spiral waves on the surface of a sphere must have two tips and so the dynamics of such patterns are expected to be qualitatively different from the planar case. In this thesis we will investigate another difference between planar and spherical spirals; while one-armed planar spirals have trivial isotropy (i.e. no symmetries) we will show that one-armed spherical spirals which have symmetry in the equator can exist generically.

Various spiral patterns on spheres have been found to exist. Grindrod and Gomatam [50] showed that a rotating spiral wave on a sphere which is symmetric in the equator can exist and be stable [48] under the condition that the tips are fixed at the north and south poles. Indeed such solutions have been found experimentally [64] and in numerical simulations of excitable reaction–diffusion systems on a sphere [3, 16, 47, 87, 91]; however, spiral waves which are asymmetric with respect to the equator have also been observed [67].

Meandering spiral waves have been found in simulations of systems with inhomogeneous excitability [31, 87] on the sphere. The transition from rotating spiral waves to meandering spiral waves on a sphere has been studied using the group of rotations of a sphere, $\mathbf{SO}(\mathbf{3})$, by Chan [20] and Comanici [27, 28]. They independently studied the Hopf bifurcation of a rotating spiral relative equilibrium with trivial isotropy which leads to the meandering of the spiral wave.

In addition to rotating spirals, stationary spiral patterns have also been observed in spherical geometries. For example, numerical simulations of Rayleigh–Bénard convection in a thin spherical shell have been found to give a stable stationary spiral roll covering the whole surface of the sphere without any defects [62, 90]. A similar stable stationary spiral pattern has also been found in numerical simulations of a variation of the Swift–Hohenberg equation by Matthews [65]. With certain parameter values the single armed spiral was found but in addition double spirals and 'tennis ball' patterns were observed. As yet there has been no analytical study of the existence properties of spiral patterns with symmetries on the sphere.

1.3 Structure of this thesis

There are two main themes to the work in this thesis – the study of the Hopf bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry and also the investigation into the existence of symmetric spiral patterns on spheres resulting from stationary bifurcations. Throughout this thesis we will use definitions and results from the general theory of bifurcations with symmetry, an overview of which is given in Chapter 2. We will also require the results concerning the representations and subgroups of the group $\mathbf{O}(\mathbf{3})$ which are reviewed in Chapter 3.

We begin the work on the Hopf bifurcation with spherical symmetry in Chapter 4. We repeat the computations of Golubitsky et al. [46] regarding the enumeration of the $\mathbb{C}$-axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ in order to correct the errors which remain in the list given in [46]. As a result of these computations we are able to present a corrected list of the $\mathbb{C}$-axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ and in addition we compute the isotropy subgroups $\Sigma \subset \mathbf{O}(\mathbf{3}) \times S^1$ which fix a four-dimensional subspace of $V_\ell \oplus V_\ell$. If these subgroups are maximal (i.e. there is no isotropy subgroup Δ satisfying $\Sigma \subsetneq \Delta \subsetneq \mathbf{O}(\mathbf{3})$) then a result of Fiedler [38] (see Theorem 2.5.3) guarantees the existence of periodic solutions with these isotropy subgroups bifurcating from the Hopf bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry in addition to the periodic solutions with $\mathbb{C}$-axial symmetry. If $\Sigma \subset \mathbf{O}(\mathbf{3}) \times S^1$ is a submaximal isotropy subgroup which fixes a subspace of dimension greater than two then it is possible, depending on the values of coefficients in the equivariant vector field f, for solutions with Σ symmetry to exist. A result of van Gils and Golubitsky [40] says that when the vector field f in the restriction to the fixed-point subspace of Σ decomposes into phase and amplitude equations then we expect the submaximal solution with Σ symmetry to be quasiperiodic when it exists. In this thesis we use these results to consider the submaximal solutions which can exist in the representation on $V_3 \oplus V_3$.

In Chapter 5 of this thesis we consider the Hopf bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry where $\mathbf{O}(\mathbf{3})$ acts naturally on $V_3 \oplus V_3$. Here we find that there are six $\mathbb{C}$-axial subgroups. We use the general form of the equivariant vector field to compute the stability conditions for each of the six bifurcating periodic solution branches with $\mathbb{C}$-axial symmetry. In this representation we find that the cubic order truncation of the general equivariant vector field is sufficient to determine the stability of all six solution branches. In this chapter we also investigate solutions in the equivariant vector field for the representation on $V_3 \oplus V_3$ which have symmetry Σ, where Σ is an isotropy subgroup which fixes a four-dimensional subspace of $V_3 \oplus V_3$. These subgroups are all submaximal. We find that it is possible for both periodic and quasiperiodic submaximal solutions to exist.

We then move on to consider symmetric spiral patterns on spheres. The most symmetric spiral patterns on the sphere have symmetries which are a subgroup of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$. Systems which have this symmetry have an $\mathbf{x} \to -\mathbf{x}$ symmetry in addition to the spherical symmetry. In Chapter 6 we consider the solutions which can exist as a result of a stationary bifurcation with this symmetry for both irreducible and reducible representations of $\mathbf{O}(\mathbf{3})$. Many of these solutions exist only for certain values of the coefficients in the equivariant vector field. We show how these coefficients can be computed for the example of the Swift–Hohenberg equation [78]. We consider reducible representations of $\mathbf{O}(\mathbf{3})$ in this chapter due to the fact that the spiral solutions, for which we are interested in deducing the existence properties, can only result from mode interactions where the representation of $\mathbf{O}(\mathbf{3})$ is a reducible representation on $V_\ell \oplus V_{\ell+1}$ for some value of ℓ.

In Chapter 7 we begin by investigating the existence properties of spirals which have symmetries which are a subgroup of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$. In the reducible representations on $V_2 \oplus V_3$ and $V_3 \oplus V_4$ we show that such spiral patterns can exist as stationary solution branches resulting from a stationary bifurcation with $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry and subsequent secondary bifurcations.

For the example of the Swift–Hohenberg equation with no quadratic terms we consider when the stationary spirals can exist and be stable and how they bifurcate from other solutions in the representations of $\mathbf{O}(\mathbf{3})$ on $V_2 \oplus V_3$ and $V_3 \oplus V_4$. Finally we show that if the symmetry is broken from $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ to $\mathbf{O}(\mathbf{3})$ then generically multiarmed spiral patterns persist as stationary solutions with slightly broken symmetry. One-armed spirals (when they persist) are generically forced by symmetry to drift.

CHAPTER 2

BACKGROUND

GENERAL THEORY OF BIFURCATION WITH SYMMETRY

2.1 Introduction

In this chapter we provide, without proofs, background results required for this thesis. This constitutes an overview of the area of mathematics known as equivariant bifurcation theory. A much more detailed account, including proofs, can be found in [46, Chapters XII, XIII and XVI].

Equivariant bifurcation theory can be thought of as the study of systems of ordinary differential equations with symmetry where the symmetries of a system of ODEs are specified in terms of a group Γ. Let V be a finite dimensional vector space and let

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, \lambda) \tag{2.1.1}$$

be a system of ODEs where $\mathbf{x} \in V$, $\lambda \in \mathbb{R}$ is a bifurcation parameter and $f : V \times \mathbb{R} \to V$ is a smooth, nonlinear map. We say that a transformation γ is a symmetry of (2.1.1) if

$$f(\gamma \cdot \mathbf{x}, \lambda) = \gamma \cdot f(\mathbf{x}, \lambda) \qquad \forall \mathbf{x} \in V. \tag{2.1.2}$$

Here, $\cdot$ represents some 'action' of γ on the vector space V, which must be defined. A consequence of (2.1.2) is that if $\mathbf{x}(t)$ is a solution to (2.1.1) then so is $\gamma \cdot \mathbf{x}(t)$. We assume that the set of transformations γ that are symmetries of (2.1.1) form a group Γ. Throughout this thesis the type of groups, Γ, we will be dealing with are compact Lie groups. These groups and the way they act on vector spaces are defined in Section 2.2.

Using the symmetry group, Γ, of the system (2.1.1) alone we are able to compute the generic form of the nonlinear map f. Such a mapping which commutes with the action of Γ on V is said to be Γ equivariant. In Section 2.3 we present the required results for the computation of this mapping.

The system (2.1.1) is said to describe a *steady-state bifurcation problem with* Γ *symmetry* if it has a fixed-point $\mathbf{x}_0 = \mathbf{0}$ such that $f(\mathbf{0}, \lambda) = \mathbf{0}$ for all values of λ and the Jacobian $(\mathrm{d}f)|_{(\mathbf{0},\lambda)}$ has a real eigenvalue passing through zero at a bifurcation point $\lambda = \lambda_c$. At this bifurcation a number of branches of steady-state solutions are created which have less symmetry than Γ. In Section 2.4 we will introduce the *equivariant branching lemma* which proves the existence of branches of steady-state solutions with the symmetries of certain subgroups of Γ.

If $\mathbf{x}_0$ is a fixed-point solution of (2.1.1) such that $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}$ has purely imaginary eigenvalues at a bifurcation point $\lambda = \lambda_c$ then the solution undergoes a Hopf bifurcation with Γ symmetry. Under certain hypotheses the *equivariant Hopf theorem*, which we shall introduce in Section 2.5, proves the existence of branches of periodic solutions emanating from the bifurcation point with the symmetries of certain subgroups of $\Gamma \times S^1$.

In this chapter we also consider how symmetries can be used to compute the stability properties of the solution branches created at bifurcations with symmetry.

2.2 Group Theory

Throughout this thesis we will be using results which apply for certain actions of Lie groups. We now review the properties of these groups and their representations which we will require for this thesis. Further details on all ideas introduced in this section can be found in [46, Chapter XII].

2.2.1 Lie groups and their representations

Definition 2.2.1. A *Lie group* is a differentiable manifold, where the group operation is an analytic map and the inversion operation which gives the inverse of a group element is also analytic. A Lie group is a way of describing a continuous symmetry since group elements can be varied continuously. A Lie group is *compact* if its manifold is compact. This is equivalent to the parameters of the Lie group varying over a closed interval. Every compact Lie group is isomorphic to a closed subgroup of $\mathbf{GL}(n)$, the group of all invertible $n \times n$ matrices over $\mathbb{R}$.

The compact Lie groups which we will encounter in this thesis are the orthogonal group $\mathbf{O}(3)$, consisting of all 3×3 matrices A satisfying

$$AA^T = I_3,$$

and its subgroups. Here, I_3 is the 3×3 identity matrix. The group $\mathbf{O}(\mathbf{3})$ will be studied in some detail in Chapter 3.

Let Γ be a compact Lie group and V a finite dimensional vector space. We say that Γ *acts linearly on* V if there is a continuous mapping (called the action) $\Gamma \times V \to V$ sending $(\gamma, v) \to \gamma \cdot v$ such that

(a) For each $\gamma \in \Gamma$ the mapping $\rho_\gamma : V \to V$ defined by $\rho_\gamma(v) = \gamma \cdot v$ is linear.

(b) If $\gamma_1, \gamma_2 \in \Gamma$ then $\gamma_1 \cdot (\gamma_2 \cdot v) = (\gamma_1\gamma_2) \cdot v$ for all $v \in V$.

Definition 2.2.2. The mapping $\rho : \Gamma \to \mathbf{GL}(V)$ which sends γ to ρ_γ is a *representation* of Γ on V. Here $\mathbf{GL}(V)$ is the group of all invertible linear transformations $V \to V$.

If V is n dimensional, then the representation ρ is n dimensional and consists of invertible $n \times n$ matrices ρ_γ for $\gamma \in \Gamma$. A group can have many different representations of various dimensions. Every representation has $\rho_I = I_n$, where I_n is the $n \times n$ identity element.

Within the collection of all representations of a group Γ there are two types which we are interested in for the purposes of studying systems of ODEs with symmetry. These are the irreducible and absolutely irreducible representations.

Definition 2.2.3. A subspace $W \subset V$ is Γ-invariant under the representation ρ of the group Γ if

$$\rho(\gamma)w \in W, \quad \forall\gamma \in \Gamma, \quad \forall w \in W.$$

A representation of Γ is said to be *irreducible* if the only Γ-invariant subspaces are $\{\mathbf{0}\}$ and V.

Let Γ be a compact Lie group acting linearly on V. We say that a map $f : V \to V$ is Γ-equivariant or commutes with Γ if

$$f(\gamma \cdot v) = \gamma \cdot f(v) \qquad \forall\gamma \in \Gamma, \quad \forall v \in V. \tag{2.2.1}$$

Definition 2.2.4. A representation of Γ is said to be *absolutely irreducible* if the only linear mappings that commute with the action of Γ on V are scalar multiples of the identity.

It can be shown that every absolutely irreducible representation is irreducible.

Remark 2.2.5. For representations over $\mathbb{C}$, there is no distinction between irreducibility and absolute irreducibility, but real representations can be irreducible without being absolutely irreducible.

2.2.2 Isotypic decomposition and linear commuting maps

The study of a representation of a compact Lie group is often simplified by observing that it decomposes into a direct sum of simpler, irreducible representations.

Definition 2.2.6. A subspace $W \subset V$ is said to be Γ-*irreducible* if W is Γ-invariant and the action of Γ on W is irreducible.

Under the action of a compact Lie group Γ a vector space V can be decomposed into the sum of a finite number, m, of Γ-irreducible subspaces V_i, giving

$$V = V_1 \oplus \cdots \oplus V_m.$$

See [46, Chapter XII, Corollary 2.2]. This decomposition is not in general unique. Some of the V_i may be isomorphic to each other. Subspaces V_i and V_j are Γ-isomorphic to each other if there is a linear isomorphism $\theta : V_i \to V_j$ which commutes with the action of Γ. To get a unique decomposition we use the following theorem.

Theorem 2.2.7. *Let Γ be a compact Lie group acting on V.*

(a) Up to isomorphism there are a finite number of distinct Γ-irreducible subspaces of V. Call these $U_1, \dots, U_k$.

(b) Define W_k to be the sum of all Γ-irreducible subspaces W of V such that W is isomorphic to U_j. Then

$$V = W_1 \oplus \cdots \oplus W_k \tag{2.2.2}$$

is the unique isotypic decomposition of V for the action of Γ.

Proof. See [46, Chapter XII, Theorem 2.5]. □

The subspaces W_j are called the *isotypic components* of V for the action of Γ.

The following results about linear maps which commute with nonirreducible (or reducible) representations will be useful in Section 2.4.

Lemma 2.2.8. *Let Γ be a compact Lie group acting on V, let $A : V \to V$ ba a linear mapping that commutes with Γ and let $W \subset V$ be a Γ-irreducible subspace. Then $A(W)$ is Γ-invariant and either $A(W) = \{\mathbf{0}\}$ or the representations of Γ on W and $A(W)$ are isomorphic.*

Proof. See [46, Chapter XII, Lemma 3.4]. □

This lemma together with Theorem 2.2.7 implies the following result.

Theorem 2.2.9. *Let Γ be a compact Lie group acting on V. Decompose V into isotypic components*

$$V = W_1 \oplus \cdots \oplus W_k.$$

Let $A : V \to V$ be a linear mapping commuting with Γ. Then $A(W_j) \subset W_j$ for $j = 1, \dots, k$.

Proof. See [46, Chapter XII, Theorem 3.5]. □

2.3 Classification of equivariant mappings

Suppose that the mapping $f : V \to V$ commutes with the action of a Lie group Γ on V, as in (2.2.1). Such a mapping is said to be Γ–equivariant. The action of Γ on V imposes restrictions on the possible form of f enabling us to compute its general form. In this section we present all results required to show that when we compute the generic form of the nonlinear mapping f we need only consider polynomial maps. These results are technical and we do not give any proofs here.

In Section 2.3.1 we give results which describe the smooth nonlinear mappings which are equivariant with respect to Γ. In Section 2.3.2 we show how to compute the number of Γ–equivariant mappings of a chosen degree using character methods.

2.3.1 Technical results

Invariant functions

Let Γ be a Lie group acting on a vector space V. A real-valued function $g : V \to \mathbb{R}$ is Γ-*invariant* if

$$g(\gamma\mathbf{x}) = g(\mathbf{x}) \qquad \forall\mathbf{x} \in V, \qquad \forall\gamma \in \Gamma. \tag{2.3.1}$$

An *invariant polynomial* is a real-valued polynomial satisfying (2.3.1). Let $\mathcal{P}_\Gamma$ denote the ring of Γ-invariant polynomials and $\mathcal{E}_\Gamma$, the ring of Γ-invariant functions.

Definition 2.3.1. Let $\mathcal{U} = \{u_1, \ldots, u_s\}$ be a collection of Γ-invariant polynomials. Then $\mathcal{U}$ forms a *Hilbert basis* for $\mathcal{P}_\Gamma$ is for every $g \in \mathcal{P}_\Gamma$ there exists a polynomial $p : \mathbb{R}^s \to \mathbb{R}$ such that

$$g(\mathbf{x}) = p(u_1(\mathbf{x}), \ldots, u_s(\mathbf{x})).$$

The Hilbert-Weyl Theorem states that when Γ is a compact Lie group, there is always a finite Hilbert basis, $\mathcal{U}$, for $\mathcal{P}_\Gamma$ (see [46, Chapter XII, Theorem 4.2]). This result can be extended to any Γ-invariant function in $\mathcal{E}_\Gamma$ by the following result:

Theorem 2.3.2 (Schwarz). *Let Γ be a compact Lie group acting on a vector space V. Let $u_1, \ldots, u_s$ be a Hilbert basis for $\mathcal{P}_\Gamma$ and let $g \in \mathcal{E}_\Gamma$. Then there exists a smooth function $h : \mathbb{R}^s \to \mathbb{R}$ such that*

$$g(\mathbf{x}) = h(u_1(\mathbf{x}), \ldots, u_s(\mathbf{x})).$$

Proof. See [46, Chapter XII, Theorem 4.3]. □

This result reduces the study of Γ-invariant functions to the study of Γ-invariant polynomials. In particular, we need only find a Hilbert basis for $\mathcal{P}_\Gamma$ to have characterised the Γ-invariant functions.

Equivariant mappings

The results in this section use the following lemma.

Lemma 2.3.3. *Let $g : V \to \mathbb{R}$ be a Γ-invariant function and let $f : V \to V$ be a Γ-equivariant mapping. Then the product $gf : V \to V$ is Γ-equivariant.*

Proof. See [46, Chapter XII, Lemma 5.1]. □

Let $\overrightarrow{\mathcal{P}}_\Gamma$ and $\overrightarrow{\mathcal{E}}_\Gamma$ denote the set of Γ-equivariant polynomials and smooth functions respectively. We say that the equivariant polynomials $f_1, \ldots, f_r$ *generate* $\overrightarrow{\mathcal{P}}_\Gamma$ *over* $\mathcal{P}_\Gamma$ if every Γ-equivariant polynomial f may be written $f = g_1 f_1 + \cdots + g_r f_r$ for invariant polynomials $g_1, \ldots, g_r$. The definition for generating equivariants of $\overrightarrow{\mathcal{E}}_\Gamma$ over $\mathcal{E}_\Gamma$ is analogous.

The Hilbert-Weyl Theorem generalises to equivariant polynomial mappings: When Γ is a compact Lie group there exists a finite set of Γ-equivariant polynomials $f_1, \ldots, f_r$ that generate $\overrightarrow{\mathcal{P}}_\Gamma$. See [46, Chapter XII, Theorem 5.2]. We can now give the equivariant version of Theorem 2.3.2:

Theorem 2.3.4 (Poénaru). *Let Γ be a compact Lie group acting on a vector space V and let $f_1, \ldots, f_r$ generate $\overrightarrow{\mathcal{P}}_\Gamma$ over $\mathcal{P}_\Gamma$, then $f_1, \ldots, f_r$ generate $\overrightarrow{\mathcal{E}}_\Gamma$ over $\mathcal{E}_\Gamma$.*

Proof. See [46, Chapter XII, Theorem 5.3]. □

This theorem reduces a search for generating equivariants to a search for generating polynomial equivariants.

The results of this section allow us to determine the general form of a Γ-equivariant map. We wish to use the form of this general Γ-equivariant map to compute the stability of solutions to (2.1.1). To do this we will need to use the Taylor expansion of this map.

It is possible (although less elegant) to compute this Taylor expansion directly. To compute the Taylor expansion to order k we consider a map $V \to V$ containing all possible homogeneous polynomial terms of degree i for every $i \leq k$. We then use the fact that the map f must satisfy (2.1.2) for all $\gamma \in \Gamma$ for the action of Γ on V to discover that some terms are not permitted and others occur in certain ratios in the Taylor expansion. Note that it is sufficient to impose that (2.1.2) hold for the actions of a set of generators of the group Γ. We must bear in mind that choice of representation of Γ will affect the outcome of this computation. We will use this method throughout this thesis to compute Taylor expansions of general forms of Γ-equivariant mappings near bifurcation points.

At any order k the Taylor expansion of f is linear combination of a number, $E(k)$, of linearly independent Γ-equivariant maps of order k. When computing the k^{th} order terms in the Taylor expansion of f it is useful to know this number $E(k)$ so that we know that all possible equivariants have been found. In Section 2.3.2 we will show how to compute $E(k)$ using character methods.

2.3.2 Character formula for the number of Γ-equivariant maps

In this section we state results required to compute the number of Γ-equivariant maps of degree k for a given representation of Γ on a finite dimensional vector space V.

Let $\mathcal{P}_\Gamma^k$ denote the vector space of all homogeneous polynomials of degree k which are invariant under the action of Γ on V and let $\overrightarrow{\mathcal{P}}_\Gamma^k$ denote the vector space of all homogeneous polynomial maps of degree k which are equivariant under the action of Γ on V. Then

$$\begin{aligned} \dim \mathcal{P}_\Gamma^k &= \#\text{ linearly independent polynomial } \Gamma\text{-invariants of degree } k = I(k) \\ \dim \overrightarrow{\mathcal{P}}_\Gamma^k &= \#\text{ linearly independent polynomial } \Gamma\text{-equivariant maps of degree } k = E(k). \end{aligned}$$

Suppose that $V = \mathbb{C}^n$ and ρ_γ is the $n \times n$ matrix which describes the action of $\gamma \in \Gamma$ on V. Then we define the character of the element $\gamma \in \Gamma$ for the representation ρ of Γ on V to be the function $\chi : \Gamma \to \mathbb{C}$ given by

$$\chi(\gamma) = \text{trace}(\rho_\gamma) = \sum_{i=1}^{n} (\rho_\gamma)_{ii}, \qquad \forall \gamma \in \Gamma.$$

The action of Γ on V induces a natural action on $\mathcal{P}_{\Gamma}^{k}$ and the corresponding character is denoted by $\chi_{(k)}$. The following theorem of Sattinger [75] tells us how to compute $I(k)$ and $E(k)$ using characters.

Theorem 2.3.5. *Let Γ be a compact Lie group acting linearly on a vector space V with corresponding character χ. Then*

$$\begin{aligned} I(k) = \dim \mathcal{P}_{\Gamma}^{k} &= \int_{\Gamma} \chi_{(k)}(\gamma)\, \mathrm{d}\mu_{\Gamma}(\gamma) && (2.3.2)\\ E(k) = \dim \overrightarrow{\mathcal{P}}_{\Gamma}^{k} &= \int_{\Gamma} \chi_{(k)}(\gamma)\, \chi(\gamma)\, \mathrm{d}\mu_{\Gamma}(\gamma) && (2.3.3) \end{aligned}$$

where $\mathrm{d}\mu_{\Gamma}(\gamma)$ is the normalised invariant (Haar) measure of Γ.

The calculations of $I(k)$ and $E(k)$ can be simplified by noting the fact that elements which are conjugate in Γ have the same character for a given representation.

In order to use Theorem 2.3.5 we need to calculate the character $\chi_{(k)}$. We use the recursive formula

$$k\chi_{(k)}(\gamma) = \sum_{i=0}^{k-1} \chi(\gamma^{k-i})\chi_{(i)}(\gamma) \tag{2.3.4}$$

with $\chi_{(0)}(\gamma) = 1$. A proof of this formula can be found in Antoneli et al. [4]. Using this formula one can compute

$$\begin{aligned} \chi_{(1)}(\gamma) &= \chi(\gamma) && (2.3.5)\\ 2\chi_{(2)}(\gamma) &= \chi(\gamma^2) + (\chi(\gamma))^2 && (2.3.6)\\ 6\chi_{(3)}(\gamma) &= 2\chi(\gamma^3) + 3\chi(\gamma)\chi(\gamma^2) + (\chi(\gamma))^3 . && (2.3.7) \end{aligned}$$

We will use these formulae and Theorem 2.3.5 to compute $E(3)$ for reducible representations of the group $\mathbf{O(3)}$ in Chapter 6.

2.4 Steady-state bifurcation with symmetry

Throughout this section let V be a finite dimensional vector space. Consider the system of ODEs

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(\mathbf{x}, \lambda), \tag{2.4.1}$$

where $\mathbf{x} \in V$, $\lambda \in \mathbb{R}$ is a bifurcation parameter and $f : V \times \mathbb{R} \to V$ is a smooth, nonlinear map which satisfies

$$f(\gamma\mathbf{x}, \lambda) = \gamma f(\mathbf{x}, \lambda) \qquad \forall \mathbf{x} \in V, \qquad \forall \gamma \in \Gamma, \tag{2.4.2}$$

for a compact Lie group Γ. In other words f commutes with the action of Γ on V.

In this section we introduce a result known as the equivariant branching lemma. This lemma makes predictions about the symmetry of solutions at steady-state bifurcations, based on the symmetry of the bifurcation problem. It was proved by Vanderbauwhede [81] and Cicogna [25].

In Section 2.4.1 we give the required definitions before stating the equivariant branching lemma in Section 2.4.2. Results related to computing the stability of the bifurcating solution branches are given in Section 2.4.4.

2.4.1 Group orbits, isotropy subgroups and fixed-point subspaces

Further details on the results in this section can be found in [46, Chapter XIII].

Group orbits and isotropy subgroups

Let $\mathbf{x}_0 \in V$ be a steady-state solution of (2.4.1) for some value of λ so that

$$f(\mathbf{x}_0, \lambda) = \mathbf{0}.$$

Since f commutes with the action of Γ, $\gamma\mathbf{x}_0$ is also a steady-state solution for all $\gamma \in \Gamma$. Thus steady-state solutions of (2.4.1) occur in group orbits where the *group orbit* of $\mathbf{x}_0$ is defined to be the set

$$\Gamma\mathbf{x}_0 = \{\gamma\mathbf{x}_0 : \gamma \in \Gamma\}.$$

The symmetry of a fixed-point $\mathbf{x}_0 \in V$ is the set of all $\gamma \in \Gamma$ that leaves $\mathbf{x}_0$ invariant. This set is a subgroup of Γ called the *isotropy subgroup* of $\mathbf{x}_0$ and is denoted

$$\Sigma_{\mathbf{x}_0} = \{\gamma \in \Gamma \ : \ \gamma\mathbf{x}_0 = \mathbf{x}_0\}.$$

It can be shown that points on the same group orbit have conjugate isotropy subgroups (see [46, Chapter XIII, Lemma 1.1]). We consider conjugate steady-states to represent the same steady-state and solutions are classified in terms of their isotropy subgroup.

We say that $\Sigma \subset \Gamma$ is an isotropy subgroup if it fixes some vector $\mathbf{x} \in V$ and contains all the group elements that fix $\mathbf{x}$. Whether a given subgroup of Γ is an isotropy subgroup will depend upon the action of Γ on V. In other words different subgroups of Γ will be isotropy subgroups in different representations of Γ.

Given a representation of Γ on a vector space V it is possible to compute all conjugacy classes of isotropy subgroups of Γ. Let Σ and Δ be two conjugacy classes of isotropy subgroups of Γ. Then we can define a partial order, $\leq$, by $\Sigma \leq \Delta$ if and only if there are subgroups $\Sigma_j \in \Sigma$ and $\Delta_k \in \Delta$ such that $\Sigma_j \subset \Delta_k$. This partial ordering allows us to construct the *lattice of isotropy subgroups* for this representation of Γ.

Group orbits and isotropy subgroups satisfy the following proposition.

Proposition 2.4.1. *Let Γ be a compact Lie group acting on V. Then for any $\mathbf{x} \in V$*

(a) *If Γ is finite, then $|\Gamma| = |\Sigma_{\mathbf{x}}||\Gamma\mathbf{x}|$*

(b) $\dim \Gamma = \dim \Sigma_{\mathbf{x}} + \dim \Gamma\mathbf{x}$.

Proof. See [46, Chapter XIII, Proposition 1.2]. □

Fixed-point subspaces

The *fixed-point subspace* of a subgroup $\Sigma \subset \Gamma$ is defined as

$$\mathrm{Fix}(\Sigma) = \{\mathbf{x} \in V \;:\; \sigma\mathbf{x} = \mathbf{x},\;\; \forall\, \sigma \in \Sigma\}.$$

Fixed-point subspaces are flow invariant under Γ-equivariant mappings since if $\sigma \in \Sigma$,

$$f(\mathbf{x}, \lambda) = f(\sigma\mathbf{x}, \lambda) = \sigma f(\mathbf{x}, \lambda), \qquad \forall\, \mathbf{x} \in \mathrm{Fix}(\Sigma).$$

Hence we have that $f(\mathrm{Fix}(\Sigma), \lambda) \subset \mathrm{Fix}(\Sigma)$ i.e. a trajectory which starts in $\mathrm{Fix}(\Sigma)$ remains in $\mathrm{Fix}(\Sigma)$ for all time. This means that if we are looking for a solution to (2.4.1) with a certain isotropy subgroup $\Sigma_{\mathbf{x}_0}$ we can restrict f to $\mathrm{Fix}(\Sigma_{\mathbf{x}_0})$ and solve the equations there.

If $\Sigma_{\mathbf{x}_0}$ is the isotropy subgroup of the point $\mathbf{x}_0$ then the largest group to leave $\mathrm{Fix}(\Sigma_{\mathbf{x}_0})$ invariant is the *normaliser* of $\Sigma_{\mathbf{x}_0}$ in Γ defined by

$$N(\Sigma_{\mathbf{x}_0}) = \{\gamma \in \Gamma \;:\; \gamma^{-1}\Sigma_{\mathbf{x}_0}\gamma = \Sigma_{\mathbf{x}_0}\}.$$

The group that we expect to govern the bifurcation in $\mathrm{Fix}(\Sigma_{\mathbf{x}_0})$ is $N(\Sigma_{\mathbf{x}_0})/\Sigma_{\mathbf{x}_0}$, where we factor out $\Sigma_{\mathbf{x}_0}$ because it acts trivially on $\mathrm{Fix}(\Sigma_{\mathbf{x}_0})$. Thus the restriction of (2.4.1) to $\mathrm{Fix}(\Sigma_{\mathbf{x}_0})$ results in equations which have $N(\Sigma_{\mathbf{x}_0})/\Sigma_{\mathbf{x}_0}$ symmetry. Moreover, the size of the quotient group, $|N(\Sigma_{\mathbf{x}_0})/\Sigma_{\mathbf{x}_0}|$, gives the number of solutions within $\mathrm{Fix}(\Sigma_{\mathbf{x}_0})$ which are equivalent i.e. have the same isotropy.

Suppose that Γ acts absolutely irreducibly on V. Then by definition $\mathrm{Fix}(\Gamma) = \{\mathbf{0}\}$ or V. If Γ acts non-trivially then $\mathrm{Fix}(\Gamma) = \{\mathbf{0}\}$ and due to the flow-invariance of $\mathrm{Fix}(\Gamma)$ there is a solution $f(\mathbf{0}, \lambda) = \mathbf{0}$ of (2.4.1) for all λ.

An isotropy subgroup $\Sigma \subset \Gamma$ is said to be *maximal* if there does not exist an isotropy subgroup $\Delta \subset \Gamma$ satisfying $\Sigma \subsetneq \Delta \subsetneq \Gamma$. If an isotropy subgroup $\Sigma \subset \Gamma$ has $\dim \mathrm{Fix}(\Sigma) = 1$ then we say that Σ is an *axial* isotropy subgroup. It can be shown that axial isotropy subgroups must be maximal.

Determining isotropy subgroups of Γ

To decide which subgroups of a finite group Γ are isotropy subgroups for a given representation we can use the following result.

Lemma 2.4.2 (Chain criterion). *Suppose that Γ is group of finite order. A subgroup $\Sigma \subset \Gamma$ is an isotropy subgroup if and only if* $\dim \mathrm{Fix}(\Sigma) > 0$ *and* $\dim \mathrm{Fix}(\Delta) < \dim \mathrm{Fix}(\Sigma)$ *for all* $\Delta \supset \Sigma$.

Proof. See, for example, [68]. □

Remark 2.4.3. When Γ contains continuous symmetries Lemma 2.4.2 provides only a necessary condition for a subgroup $\Sigma \subset \Gamma$ to be an isotropy subgroup. To compute isotropy subgroups in the case where $\Gamma = \mathbf{O(3)}$ we use an different version of Lemma 2.4.2 called the 'massive chain criterion' [63]. This is discussed in Section 3.4.

To compute the isotropy subgroups of a group Γ for a given representation using Lemma 2.4.2 we first need to find the dimension of the fixed-point subspaces of the subgroups $\Sigma \subset \Gamma$. To compute the dimension of a fixed-point subspace we use the following theorem.

Theorem 2.4.4 (**Trace formula**). *Let Γ be a compact Lie group acting on V and let $\Sigma \subset \Gamma$ be a Lie subgroup. Then*

$$\dim \mathrm{Fix}(\Sigma) = \int_{\Sigma} \chi(\sigma)$$

where the integral is with respect to the normalised Haar measure on Σ and $\chi(\sigma) = trace(\rho_\sigma)$ is the character of $\sigma \in \Sigma$ and ρ_σ is the matrix of the element $\sigma \in \Sigma$ in the representation ρ on V. If the group Γ is finite then we have

$$\dim \mathrm{Fix}(\Sigma) = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} \chi(\sigma).$$

Proof. See [46, Chapter XIII, Theorem 2.3]. □

2.4.2 The equivariant branching lemma

Suppose that $\mathbf{x}_0$ is a fixed-point of (2.4.1) and at some parameter value $\lambda = \lambda_c$ there is a steady-state bifurcation. This means that the Jacobian at this point, $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda_c)}$, has one or more zero eigenvalues. From now on we will assume that the system has been reduced to the centre manifold so that the Jacobian vanishes at the bifurcation point. That is $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda_c)} = 0$. We also assume that the origins of $\mathbf{x}$ and λ have been chosen such that the fixed-point is at $\mathbf{x}_0 = \mathbf{0}$ and the bifurcation point is at $\lambda = 0$. Then the existence of the fixed-point at the bifurcation point requires that $f(\mathbf{0}, \lambda) = \mathbf{0}$. Note that this is automatically satisfied if Γ acts absolutely irreducibly on V.

We now make the following definition.

Definition 2.4.5. Let Γ be a compact Lie group acting on a vector space V. A *steady-state bifurcation problem with* Γ *symmetry* is a Γ-equivariant mapping $f : V \times \mathbb{R} \to V$ satisfying $f(\mathbf{0}, 0) = \mathbf{0}$ and $(\mathrm{d}f)|_{(\mathbf{0},0)} = 0$.

Recall that the Γ-equivariant mapping, f, satisfies (2.4.2). By differentiating the equivariance condition (2.4.2) using the chain rule we have

$$\gamma(\mathrm{d}f)|_{(\mathbf{x},\lambda)} = (\mathrm{d}f)|_{(\gamma\mathbf{x},\lambda)}\gamma, \qquad \forall \mathbf{x} \in V, \qquad \forall \gamma \in \Gamma. \tag{2.4.3}$$

Applying this at the fixed point $\mathbf{x} = \mathbf{0}$ we see that $(\mathrm{d}f)|_{(\mathbf{0},\lambda)}$ commutes with the action of Γ. If Γ acts absolutely irreducibly on V then the only linear mappings which commute with the action of Γ are scalar multiples of the identity so

$$(\mathrm{d}f)|_{(\mathbf{0},\lambda)} = c(\lambda)I$$

with $c(0) = 0$ since by definition $(\mathrm{d}f)|_{(\mathbf{0},0)} = 0$. This excludes the possibility of a Hopf bifurcation, where we would have a pair of purely imaginary eigenvalues at the bifurcation point.

(Hopf bifurcations will be considered in Section 2.5 and are associated with nonabsolutely irreducible complex representations.) We can assume generically that

$$c'(0) \neq 0. \tag{2.4.4}$$

We are now able to state one version of the equivariant branching lemma.

Theorem 2.4.6 (Equivariant branching lemma). *Let Γ be a Lie group acting absolutely irreducibly on V and let f be a Γ-equivariant bifurcation problem satisfying (2.4.4). Let Σ be an axial isotropy subgroup of Γ. Then there exists a unique smooth solution branch to $f = 0$ such that the isotropy subgroup of each solution is Σ.*

It is possible to prove a more general version of this theorem.

Theorem 2.4.7 (Generalised equivariant branching lemma). *Let Γ be a Lie group acting on V. Assume*

(a) $\mathrm{Fix}(\Gamma) = \{\mathbf{0}\}$

(b) $\Sigma \subset \Gamma$ *is an axial isotropy subgroup*

(c) $f : V \times \mathbb{R} \to V$ *is a Γ-equivariant bifurcation problem satisfying*

$$(\mathrm{d}f_\lambda)|_{(\mathbf{0},0)}\mathbf{v} \neq \mathbf{0} \tag{2.4.5}$$

for some nonzero $\mathbf{v} \in \mathrm{Fix}(\Sigma)$.

Then there exists a unique branch of solutions to $f(\mathbf{x}, \lambda) = 0$ emanating from $(\mathbf{0}, 0)$ where the symmetry of the solution is Σ.

Here $(\mathrm{d}f_\lambda)$ is defined by

$$(\mathrm{d}f_\lambda)_{ij} = \frac{\partial(\mathrm{d}f)_{ij}}{\partial\lambda}.$$

Proof. See [46, Chapter XIII, Theorem 3.5] □

Theorem 2.4.6 follows from Theorem 2.4.7 since nontrivial absolutely irreducible actions satisfy $\mathrm{Fix}(\Gamma) = \{\mathbf{0}\}$ and when Γ acts absolutely irreducibly $(\mathrm{d}f_\lambda)|_{(\mathbf{0},0)}\mathbf{v} = c'(0)\mathbf{v}$. So conditions (2.4.4) and (2.4.5) are equivalent. There are certain advantages to each version of the equivariant branching lemma. The generalised version does not require the action of Γ to be absolutely irreducible – it holds even for reducible actions as long as $\mathrm{Fix}(\Gamma) = \{\mathbf{0}\}$. However, Theorem 2.4.7 has condition (2.4.5) which must be checked for each axial isotropy subgroup whereas condition (2.4.4) holds simultaneously for all subgroups $\Sigma \subset \Gamma$.

The equivariant branching lemma guarantees that if (2.4.1) satisfies the relevant conditions then at a steady-state bifurcation there will be a branch of solutions with Σ symmetry if Σ is an axial isotropy subgroup of Γ. There may also be solution branches that bifurcate from the origin with isotropy subgroup Σ such that $\dim \mathrm{Fix}(\Sigma) > 1$ but the equivariant branching lemma says nothing about them. In general they will have to be found directly from (2.4.1). All branches which bifurcate at the origin are known as *primary* branches.

Remark 2.4.8. In the conditions of the equivariant branching lemma it is assumed that $c'(0) \neq 0$. This means that the trivial equilibrium $\mathbf{x} = \mathbf{0}$ undergoes an exchange of stability (for λ near 0). We say that a primary bifurcating branch of solutions is *subcritical* if the branch occurs for values of λ where the trivial solution is stable and *supercritical* otherwise.

2.4.3 Bifurcations from group orbits of equilibria

It is possible for primary branches of group orbits of equilibria to undergo a secondary symmetry breaking bifurcation for example in a mode interaction problem where the representation of the group Γ is reducible. We now consider the solutions which can result from a secondary bifurcation from a group orbit of fixed-points. All results in this section can be found in Golubitsky and Stewart [45]. We consider bifurcations from equilibria with less than full Γ symmetry.

Let $f : V \times \mathbb{R} \to V$ be a Γ equivariant vector field where Γ is a compact Lie group. Let $\mathbf{x}_0$ be a fixed-point of f and $\Gamma\mathbf{x}_0$ the group orbit through $\mathbf{x}_0$. Assume that $\dim \Gamma\mathbf{x}_0 \geq 1$ and denote by $\Sigma_{\mathbf{x}_0}$ the isotropy subgroup of $\mathbf{x}_0$. By Proposition 2.4.1 the group orbit $\Gamma\mathbf{x}_0 \subset V$ is a smooth manifold of dimension $\dim \Gamma - \dim \Sigma_{\mathbf{x}_0}$. This means that it is possible for a group orbit to be flow invariant rather than just consisting of equilibria. In that case the group orbit is called a *relative equilibrium*.

The following theorem shows that flows on relative equilibria generically fill out k-dimensional tori where the number k is uniquely determined by the isotropy subgroup $\Sigma_{\mathbf{x}_0}$. In other words, solutions that are relative equilibria are quasiperiodic with k frequencies.

Theorem 2.4.9. *Let $\Gamma\mathbf{x}_0$ be a relative equilibrium and let $\Sigma_{\mathbf{x}_0}$ be the isotropy subgroup of $\mathbf{x}_0$. Then relative equilibria are quasiperiodic motions with k frequencies where generically*

$$k = \operatorname{rank}(N(\Sigma_{\mathbf{x}_0})/\Sigma_{\mathbf{x}_0}).$$

Here $N(\Sigma_{\mathbf{x}_0})$ is the normaliser of $\Sigma_{\mathbf{x}_0}$ in Γ and the rank of a Lie group is the maximal dimension of any Torus group

$$\mathbf{T}^m = \overbrace{S^1 \times \cdots \times S^1}^{m}$$

contained in that group.

This means that it is possible for steady-state bifurcations from group orbits of equilibria to lead to relative equilibria rather than just new equilibria.

2.4.4 Stability of solution branches

In this section we consider the stability of branches of fixed-point solutions of (2.4.1).

We say that a fixed-point, $\mathbf{x}_0$, of a system of ODEs is *asymptotically stable* if every trajectory $\mathbf{x}(t)$ which begins near $\mathbf{x}_0$ stays near $\mathbf{x}_0$ for all $t > 0$ and also $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_0$. The fixed-point is *neutrally stable* if the trajectory stays near $\mathbf{x}_0$ for all $t > 0$.

Linear stability is a condition for asymptotic stability which says that if all the eigenvalues of the Jacobian evaluated at the fixed-point $\mathbf{x}_0$ have negative real part then $\mathbf{x}_0$ is linearly stable. If it has no eigenvalues with the real part equal to zero then $\mathbf{x}_0$ is a hyperbolic fixed-point. The Hartman–Grobman Theorem says that for hyperbolic fixed-points $\mathbf{x}_0$, if $\mathbf{x}_0$ is linearly stable then $\mathbf{x}_0$ is asymptotically stable.

When, as in (2.4.1), the system of ODEs commutes with the action of a Lie group Γ the following issues arise:

(a) If the isotropy subgroup $\Sigma_{\mathbf{x}_0}$ of a fixed-point $\mathbf{x}_0$ has $\dim \Sigma_{\mathbf{x}_0} < \dim \Gamma$ then neither linear stability nor asymptotic stability is possible. The Jacobian, $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}$, is forced to have zero eigenvalues. We must introduce the concepts of linear orbital stability and orbital stability.

(b) The explicit computation of the Jacobian, $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}$ is aided by knowledge of the representation of the isotropy subgroup $\Sigma_{\mathbf{x}_0}$.

Orbital stability

Let Γ be a Lie group acting on V and let f be a Γ-equivariant map as in (2.4.1). Let $\mathbf{x}_0$ be a fixed-point of (2.4.1) with isotropy subgroup Σ. Using Proposition 2.4.1 we can see that if $\dim \Sigma < \dim \Gamma$ then $\dim \Gamma \mathbf{x}_0 > 0$. This means that there are steady states of (2.4.1) arbitrarily close to $\mathbf{x}_0$. Trajectories which start at these fixed-points remain there for all time and so do not tend to $\mathbf{x}_0$. This means that $\mathbf{x}_0$ cannot be asymptotically stable. We must define a new type of stability.

We say that the fixed-point $\mathbf{x}_0$ is *orbitally stable* if $\mathbf{x}_0$ is neutrally stable and if whenever $\mathbf{x}(t)$ is a trajectory beginning near $\mathbf{x}_0$, then $\lim_{t\to\infty} \mathbf{x}(t)$ exists and lies in $\Gamma\mathbf{x}_0$.

We can also show that if $\dim \Sigma < \dim \Gamma$ then the fixed-point cannot be linearly stable: Since $\dim \Gamma \mathbf{x}_0 > 0$ the orbit $\Gamma\mathbf{x}_0$ contains a smooth curve $y(s) = \gamma(s)\mathbf{x}_0$ with $\gamma(s)$ a smooth curve in Γ and $\gamma(0) = 1$. Since $\mathbf{x}_0$ is a stationary point of (2.4.1) then $\gamma(s)\mathbf{x}_0$ is also for all s so we have $f(y(s), \lambda) = 0$. Differentiating this with respect to s and evaluating at $s = 0$ gives

$$\frac{\mathrm{d}}{\mathrm{d}s} f(y(s), \lambda)\bigg|_{s=0} = (\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)} \left(\frac{\mathrm{d}\gamma}{\mathrm{d}s}\bigg|_{s=0} \mathbf{x}_0 \right) = \mathbf{0}, \tag{2.4.6}$$

and so $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}$ has a zero eigenvalue with eigenvector $\frac{\mathrm{d}\gamma}{\mathrm{d}s}\big|_{s=0} \mathbf{x}_0$, which is tangent to the group orbit $\Gamma\mathbf{x}_0$. This means that the fixed point $\mathbf{x}_0$ has a zero growth rate eigenvalue corresponding to perturbations along the group orbit. We can note that (2.4.6) provides a method for computing the zero eigenvectors of $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}$.

There is a linear criterion for orbital stability: Let $\mathbf{x}_0$ be an equilibrium of (2.4.1) where f commutes with the action of Γ. The fixed-point $\mathbf{x}_0$ is *linearly orbitally stable* if the eigenvalues of $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}$, other than those forced to be zero by symmetry, have negative real part.

It can be shown that if a fixed-point $\mathbf{x}_0$ is linearly orbitally stable then it is orbitally stable.

Symmetry restrictions on the Jacobian

Let $\mathbf{x}_0$ be a fixed-point of (2.4.1). It is possible to use the action of the isotropy subgroup Σ of $\mathbf{x}_0$ on V to block diagonalise the Jacobian $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}$ and thereby simplify the explicit computation of the eigenvalues. By computing the eigenvalues of $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}$ we can determine whether or not the fixed-point $\mathbf{x}_0$ is orbitally stable.

We have already seen in (2.4.3) that the Jacobian satisfies the commutativity condition

$$\gamma(\mathrm{d}f)|_{(\mathbf{x},\lambda)} = (\mathrm{d}f)|_{(\gamma\mathbf{x},\lambda)}\gamma.$$

Since $\Sigma \subset \Gamma$ is the isotropy subgroup of $\mathbf{x}_0$, by definition, $\sigma\mathbf{x}_0 = \mathbf{x}_0$ for every $\sigma \in \Sigma$ and so

$$\sigma(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)} = (\mathrm{d}f)|_{(\sigma\mathbf{x}_0,\lambda)}\sigma = (\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}\sigma.$$

Thus $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}$ is a linear map which commutes with the isotropy subgroup Σ. We can decompose V into isotypic components for the action of Σ:

$$V = W_1 \oplus \cdots \oplus W_k$$

as in Theorem 2.2.7. Then by Theorem 2.2.9

$$(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}(W_j) \subset W_j.$$

If Σ acts absolutely irreducibly on an isotypic component W_j then the restriction of $(\mathrm{d}f)|_{(\mathbf{x}_0,\lambda)}$ to W_j is a scalar multiple of the identity. The subspace $\mathrm{Fix}(\Sigma)$ is always an isotypic component since it is the sum of all subspaces of V on which Σ acts trivially.

2.5 Hopf bifurcation with symmetry

In this section we consider the case where a system of ODEs with Γ symmetry undergoes a Hopf bifurcation. There is an analogue of the equivariant branching lemma which tells us that periodic branches of solutions with certain symmetries will be created at a Hopf bifurcation with Γ symmetry.

2.5.1 Existence of periodic solutions

Consider the system of ODEs given by

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(\mathbf{x}, \lambda). \tag{2.5.1}$$

Assume that there is an equilibrium solution $\mathbf{x} = \mathbf{0}$ for all values of λ. This system undergoes a standard Hopf bifurcation (i.e. without symmetry) at $\lambda = 0$ if $(\mathrm{d}f)|_{(\mathbf{0},0)}$ has a single pair of complex conjugate eigenvalues which cross the imaginary axis (with non-zero speed) at $\lambda = 0$. The standard Hopf theorem implies that a branch of periodic solutions is created but it uses the hypothesis that the imaginary eigenvalues are simple. When system (2.5.1) has Γ symmetry we cannot use the standard Hopf theorem directly since there are expected to be multiple pairs of complex conjugate eigenvalues crossing the imaginary axis at a Hopf bifurcation with symmetry.

Assume now that in (2.5.1), $\mathbf{x} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ is a bifurcation parameter and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth mapping which commutes with the action of a compact Lie group Γ on $\mathbb{R}^n$ as in (2.1.2). Further assume that $f(\mathbf{0}, \lambda) = \mathbf{0}$ so there is a trivial Γ-invariant equilibrium solution, $\mathbf{x} = \mathbf{0}$.

For Hopf bifurcation of this solution to occur at $\lambda = 0$ we require that $(\mathrm{d}f)|_{(\mathbf{0},0)}$ have purely imaginary eigenvalues. We assume that (2.5.1) is already reduced to the imaginary eigenspace. Note that since eigenvalues occur in complex conjugate pairs the number of purely imaginary eigenvalues at the bifurcation point must be even, so we must have $\mathbf{x} \in \mathbb{R}^{2p}$ where $n = 2p$. Sometimes it is useful to make the identification $\mathbb{R}^{2p} \cong \mathbb{C}^p$.

It turns out that if $(\mathrm{d}f)|_{(\mathbf{0},0)}$ is to have purely imaginary eigenvalues then the imaginary eigenspace, $\mathbb{R}^n$, must be Γ-simple. This means that either $\mathbb{R}^n = V \oplus V$ where V is an absolutely irreducible representation of Γ, or Γ acts irreducibly but not absolutely irreducibly on $\mathbb{R}^n$. In either case, in suitable coordinates and rescaling time if necessary, at the bifurcation point the Jacobian generically takes the form

$$(\mathrm{d}f)|_{(\mathbf{0},0)} = J \equiv \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix} \tag{2.5.2}$$

and the eigenvalues of $(\mathrm{d}f)|_{(\mathbf{0},\lambda)}$ are

$$\mu_{\pm} = \sigma(\lambda) \pm \mathrm{i}\omega(\lambda) \tag{2.5.3}$$

each of multiplicity p, where σ and ω are smooth functions of λ satisfying $\sigma(0) = 0$ and $\omega(0) = 1$. This implies that the eigenvalues at the bifurcation point are $\pm\mathrm{i}$. See [46, Chapter XVI, Section 1].

Near a Hopf bifurcation we expect to see branches of periodic solutions. Let $\mathbf{x}(t)$ be a periodic solution of (2.5.1) with period 2π. A symmetry of $\mathbf{x}(t)$ is an element $(\gamma, \theta) \in \Gamma \times S^1$ such that

$$(\gamma, \theta) \cdot \mathbf{x}(t) \equiv \gamma \mathbf{x}(t + \theta) = \mathbf{x}(t), \qquad \forall t.$$

Here S^1 is the circle group of phase shifts acting on the space of 2π periodic functions. We say that (γ, θ) is a *spatiotemporal symmetry*. Notice that if $\theta = 0$ then the symmetry is purely spatial. We can write the isotropy subgroup of $\mathbf{x}(t)$ as

$$\Sigma_{\mathbf{x}(t)} = \{(\gamma, \theta) \in \Gamma \times S^1 \ : \ \gamma \mathbf{x}(t + \theta) = \mathbf{x}(t)\} \subset \Gamma \times S^1.$$

Remark 2.5.1. In Chapters 4 and 5 of this thesis we will consider representations of $\mathbf{O}(\mathbf{3})$ on a space of the type $V \oplus V$ where $\mathbf{O}(\mathbf{3})$ acts absolutely irreducibly on V. In this case, by [46, Chapter XVI, Remark 3.3(d)], we can take a basis for V as a real vector space and consider $V \oplus V$ to be the vector space over $\mathbb{C}$ with this basis. Elements of $\Gamma = \mathbf{O}(\mathbf{3})$ act on $V \oplus V$ by the same matrices as for V and $\theta \in S^1$ acts as scalar multiplication by $\mathrm{e}^{\mathrm{i}\theta}$.

We now state the equivariant Hopf theorem which is the analogue of the equivariant branching lemma.

Theorem 2.5.2 (Equivariant Hopf theorem). *Consider the system of ODEs given by (2.5.1) where $\mathbf{x} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth mapping which commutes with the action of a compact Lie group Γ on $\mathbb{R}^n$. Suppose that Γ acts Γ-simply on $\mathbb{R}^n$ so that we can assume that (2.5.2) and (2.5.3) hold. Assume also that*

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\right|_{\lambda=0} \neq 0. \tag{2.5.4}$$

Then if $\Sigma \subset \Gamma \times S^1$ is an isotropy subgroup satisfying

$$\dim \operatorname{Fix}(\Sigma) = 2, \tag{2.5.5}$$

there exists a unique branch of periodic solutions to (2.5.1) with period near 2π bifurcating from the origin having Σ as their group of symmetries.

Proof. See [46, Chapter XVI, Theorem 4.1]. □

Condition (2.5.4) is the hypothesis from the standard Hopf theorem that the eigenvalues of $(\mathrm{d}f)|_{(\mathbf{0},\lambda)}$ cross the imaginary axis with non-zero speed. We say that an isotropy subgroup $\Sigma \subset \Gamma \times S^1$ is $\mathbb{C}$-axial if is satisfies condition (2.5.5). The equivariant Hopf theorem guarantees that if system (2.5.1) satisfies the relevant conditions then at a Hopf bifurcation with Γ symmetry a branch of periodic solutions with Σ symmetry is created if Σ is an isotropy subgroup of $\Gamma \times S^1$ with $\dim \operatorname{Fix}(\Sigma) = 2$.

It is possible for condition (2.5.5) in the equivariant Hopf theorem to be weakened to the subgroup Σ being a maximal isotropy subgroup of $\Gamma \times S^1$:

Theorem 2.5.3 (Fiedler [38]). *Assume that system* (2.5.1) *satisfies the conditions* (2.5.2) *and* (2.5.4) *stated above and suppose that Σ is a maximal isotropy subgroup of $\Gamma \times S^1$. Then there exist small amplitude periodic solutions to (2.5.1) with period near 2π, having Σ as their group of symmetries.*

In addition to the branches of solutions to (2.5.1) guaranteed to exist by the equivariant Hopf theorem, Theorem 2.5.3 guarantees the existence of branches of solutions with symmetry Σ where Σ has $\dim \operatorname{Fix}(\Sigma) > 2$ but Σ is maximal.

In Section 2.5.3 we will discuss how to compute the isotropy subgroups of $\Gamma \times S^1$. Before that, we will consider the stability of periodic solutions to (2.5.1).

2.5.2 Stability of periodic solutions

In this section we will consider how to compute the stability of periodic solutions to (2.5.1).

Suppose that $\mathbf{x}(t)$ is a periodic solution of (2.5.1) with period $\frac{2\pi}{1+\tau}$ for a period-scaling parameter τ near 0. A Liapunov-Schmidt reduction of (2.5.1) gives a reduced equation $g(\mathbf{x}, \lambda, \tau)$, the zeros of which are in one-to-one correspondence with the periodic solutions $\mathbf{x}(t)$ of (2.5.1).

To compute the stability of these periodic solutions we first assume that the map f is in (exact) Birkhoff normal form. That is f commutes with $\Gamma \times S^1$ at all orders. It is only possible to find a suitable change of coordinates to put f in Birkhoff normal form up to a given order k. There is no change of coordinates that puts f into Birkhoff normal form to all orders but we will deal with this issue later.

Stability in Birkhoff normal form

The following theorem gives us the form of the reduced equation g when f is in Birkhoff normal form.

Theorem 2.5.4. *Suppose that the vector field f in (2.5.1) is in Birkhoff normal form. Then it is possible to perform a Liapunov-Schmidt reduction on (2.5.1) such that the reduced equation g has the form*

$$g(\mathbf{x}, \lambda, \tau) = f(\mathbf{x}, \lambda) - (1+\tau)J\mathbf{x} \tag{2.5.6}$$

where τ is the period-scaling parameter.

Proof. See [46, Chapter XVI, Theorem 10.1]. □

Remark 2.5.5. When the representation of Γ is as in Remark 2.5.1 we identify J with i and so the reduced equation is

$$g(\mathbf{x}, \lambda, \tau) = f(\mathbf{x}, \lambda) - (1+\tau)\mathrm{i}\mathbf{x} \tag{2.5.7}$$

Let $\mathbf{x}(t)$ be a periodic solution of (2.5.1) with isotropy subgroup $\Sigma \subset \Gamma \times S^1$ which corresponds to a solution, $(\mathbf{x}_0, \lambda_0, \tau_0)$, to $g = 0$. There is a one-to-one correspondence between the Floquet multipliers of the periodic solution $\mathbf{x}(t)$ and the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{x}_0,\lambda_0,\tau_0)}$. A multiplier lies inside the unit circle if and only if the corresponding eigenvalue of $(\mathrm{d}g)|_{(\mathbf{x}_0,\lambda_0,\tau_0)}$ has negative real part (see [46, Chapter XVI, Proposition 6.4]). This is reflected in the following result.

Corollary 2.5.6. *Suppose that the vector field f in (2.5.1) is in Birkhoff normal form and that g is the mapping obtained by using the Liapunov-Schmidt procedure. Let $(\mathbf{x}_0, \lambda_0, \tau_0)$ be a solution to $g = 0$ and let $\mathbf{x}(t)$ be the corresponding periodic solution of (2.5.1). Then $\mathbf{x}(t)$ is orbitally asymptotically stable if the $n - d_\Sigma$ eigenvalues of $(\mathrm{d}g)|_{(\mathbf{x}_0,\lambda_0,\tau_0)}$ which are not forced to be zero by the group action have negative real parts. Here we define*

$$d_\Sigma = \dim \Gamma + 1 - \dim \Sigma.$$

Proof. See [46, Chapter XVI, Corollary 10.2]. □

Remark 2.5.7. When $\dim \mathrm{Fix}(\Sigma) = 2$, the assumption that f is in Birkhoff normal form implies that we can apply the standard Hopf theorem to (2.5.1) restricted to $\mathrm{Fix}(\Sigma) \times \mathbb{R}$. In this case exchange of stability occurs at the bifurcation point so that if the steady-state solution $\mathbf{x} = \mathbf{0}$ is stable subcritically, then a subcritical branch of periodic solutions with isotropy subgroup Σ is unstable. Supercritical branches may be either stable or unstable depending on the signs of the real parts of the eigenvalues on the complement of $\mathrm{Fix}(\Sigma)$.

Stability in truncated Birkhoff normal form

It is possible to use Corollary 2.5.6 to determine the asymptotic stability of some periodic solutions of (2.5.1) even when f is not in Birkhoff normal form.

By a suitable change of coordinates, up to any given order k the Γ-equivariant vector field f can be made to commute with S^1 also. Thus to order k the Taylor expansion of f can be assumed to commute with $\Gamma \times S^1$. We call this the k^{th} order truncated Birkhoff normal form of (2.5.1). The dynamics of the truncated Birkhoff normal form are related to, but not identical with the local dynamics of the system (2.5.1) around the equilibrium point $\mathbf{x} = \mathbf{0}$. In truncating the Taylor series, we are ignoring terms of higher order which do not commute necessarily with S^1 and that can change the dynamics and also possibly the stability of those periodic solutions which, by the equivariant Hopf theorem, exist even for the nontruncated system.

Assume that

$$f(\mathbf{x}, \lambda) = \tilde{f}(\mathbf{x}, \lambda) + o(\|\mathbf{x}\|^k)$$

where $\tilde{f}$ commutes with $\Gamma \times S^1$ but the perturbation $o(\|\mathbf{x}\|^k)$ commutes only with Γ. Here, as usual, $h(x) = o(\|\mathbf{x}\|^k)$ means that $h(x)/\|\mathbf{x}\|^k \to 0$ as $\|\mathbf{x}\| \to 0$. It is possible to show that, provided k is large enough, Corollary 2.5.6 remains true for the reduced function $\tilde{g}$ corresponding to the truncation $\tilde{f}$.

Definition 2.5.8. Suppose that $\dim \mathrm{Fix}(\Sigma) = 2$. Then Σ has *p-determined stability* if all eigenvalues of

$$(\mathrm{d}\tilde{g})|_{(\mathbf{x}_0, \lambda_0, \tau_0)} = (\mathrm{d}\tilde{f})|_{(\mathbf{x}_0, \lambda_0)} - (1 + \tau_0)J,$$

other than those forced to be zero by Σ, have the form

$$\mu_j = \alpha_j a^{m_j} + o(a^{m_j}),$$

where $\mathbf{x}(t)$ is a branch of periodic solutions to $\dot{\mathbf{x}} = \tilde{f}(\mathbf{x}, \lambda)$ with symmetry Σ, $a = \|\mathbf{x}(t)\|$ and α_j is a $\mathbb{C}$-valued function of the Taylor coefficients of terms of degree $\leq p$ in $\tilde{f}$.

We say that $\tilde{f}$ is *nondegenerate for* Σ if all α_j have non-zero real parts. This allows us to state the following theorem.

Theorem 2.5.9. *Suppose that the hypotheses of Theorem 2.5.2 hold, and that the isotropy subgroup $\Sigma \subset \Gamma \times S^1$ has p-determined stability. Let $k \geq p$ and assume that $\tilde{f}$ is nondegenerate for Σ. Then for λ sufficiently near 0, the stabilities of a periodic solution of (2.5.1) with isotropy subgroup Σ are given by the same expressions in the coefficients of f as those that determine the stability of a solution of the truncated Birkhoff normal form*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \tilde{f}(\mathbf{x}, \lambda),$$

with isotropy subgroup Σ.

Proof. See [46, Chapter XVI, Theorem 11.2]. □

Remark 2.5.10. By Theorem 2.5.9, the result given in Remark 2.5.7 holds even when f is not in Birkhoff normal form.

This means that we can use the k^{th} order Taylor series of f which commutes with $\Gamma \times S^1$ to compute the stability of a periodic solution with isotropy subgroup Σ whose existence is guaranteed by the equivariant Hopf theorem, as long as $k \geq p$ when Σ has p-determined stability. Theorem 2.5.9 completes the results required for a stability analysis of the $\mathbb{C}$-axial periodic solutions of (2.5.1).

2.5.3 Isotropy subgroups of $\Gamma \times \mathbf{S}^1$

In order to apply the equivariant Hopf theorem we need to consider which subgroups $\Sigma \subset \Gamma \times S^1$ can be isotropy subgroups and which of these subgroups have two-dimensional fixed point subspaces. We begin by discussing how to compute the isotropy subgroups of $\Gamma \times S^1$.

Computing isotropy subgroups of $\Gamma \times S^1$

In this section we outline the method of Golubitsky and Stewart [43] and Golubitsky et al. [46, Chapter XVI, Section 7] for computing the isotropy subgroups of $\Gamma \times S^1$. An alternative method for computing the isotropy subgroups $\Sigma \subset \Gamma \times S^1$ with $\dim \mathrm{Fix}(\Sigma) = 2$ is given by Golubitsky and Stewart in [44]. Although this alternative method requires less computation, the reasons for some of the steps in the procedure are less intuitive then the method of [46, Chapter XVI, Section 7] summarised in this section. In this thesis we will use the method outlined below.

Definition 2.5.11. Suppose that $H \subset \Gamma$ is a subgroup and $\theta : H \to S^1$ is a group homomorphism. We call

$$H^\theta = \{(h, \theta(h)) \in \Gamma \times S^1 : h \in H\}$$

a *twisted subgroup* of $\Gamma \times S^1$. We call the homomorphism $\theta : H \to S^1$ the *twist* of H.

All isotropy subgroups of $\Gamma \times S^1$ are twisted subgroups, see [46, Chapter XVI, Proposition 7.2].

We intuitively think of elements of Γ as spatial symmetries and elements of S^1 as temporal symmetries, acting on periodic solutions by a phase shift. Thus an element $(h, \theta(h)) \in \Gamma \times S^1$ is a spatial symmetry if $\theta(h) = 0$ and a combined spatiotemporal symmetry if $\theta(h) \neq 0$.

For a given twisted subgroup $H^\theta \subset \Gamma \times S^1$, the spatial symmetries form a subgroup $K = \ker \theta$. Since K is the kernel of a homomorphism θ it is a normal subgroup of H. Furthermore the quotient group H/K is isomorphic to a closed subgroup of S^1, namely $\mathrm{Im}(\theta)$. The only closed subgroups of S^1 are $\mathbb{1}$, $\mathbb{Z}_n$ $(n \geq 2)$ and S^1.

We wish to compute the conjugacy classes of isotropy subgroups of $\Gamma \times S^1$. To do this we need to know when two twisted subgroups are conjugate in $\Gamma \times S^1$. The following lemma provides two sufficient conditions.

Lemma 2.5.12.

(a) *Let H^θ and L^ϕ be conjugate twisted subgroups in $\Gamma \times S^1$. Then H and L are conjugate subgroups of Γ.*

(b) *Let H^θ and H^ϕ be conjugate twisted subgroups in $\Gamma \times S^1$. Then there exists $\gamma \in N_\Gamma(H)$ such that $\ker \phi = \gamma \ker \theta \gamma^{-1}$. Here $N_\Gamma(H) = \{\gamma \in \Gamma : \gamma H \gamma^{-1} = H\}$ is the normaliser of H in Γ.*

Proof. See [46, Chapter XVI, Lemma 7.3]. □

This lemma allows us to determine the conjugacy class of the pair (H, K) in $\Gamma \times S^1$ but this does not determine H^θ uniquely since (H, K) does not uniquely determine θ. If θ is a homomorphism

$H \to S^1$ with $\ker\theta = K$ then all other such homomorphisms are of the form $\alpha \circ \theta$ where α is an automorphism of $\mathrm{Im}(\theta)$. The twisted groups H^θ and $H^{\alpha\circ\theta}$ are conjugate in $\Gamma \times S^1$ if α is induced by conjugation by elements in $N_\Gamma(H)$.

In summary, the conjugacy classes of twisted subgroups of $\Gamma \times S^1$ can be found as follows:

1. Find the conjugacy classes of closed subgroups of Γ. For each conjugacy class choose a representative H.
2. Find all closed normal subgroups $K \subset H$ such that H/K is isomorphic to $\mathbb{1}$, $\mathbb{Z}_n$ or S^1.
3. Choose one representative of each conjugacy class of K's under the action of $N_\Gamma(H)/H$. This gives a list of all pairs (H, K).
4. Find the possible homomorphisms θ for each pair by listing the automorphisms of H/K, not including those that are induced by conjugation by elements $\gamma \in N_\Gamma(H)$.

This procedure gives a complete list of the conjugacy classes of twisted subgroups of $\Gamma \times S^1$. Two simplifications are often useful:

(i) For twist types $\mathbb{1}$ and $\mathbb{Z}_2$, there are no such automorphisms.

(ii) If there exists an element κ which acts by conjugation to invert each element of H, then for twist types $\mathbb{Z}_3$ and S^1 the only non-trivial automorphism of H/K is inversion, but this is induced by conjugation by κ and hence can be eliminated.

Dimensions of fixed-point subspaces

To determine which of the twisted subgroups H^θ, computed by the method above, are isotropy subgroups for a particular action of the group $\Gamma \times S^1$ we use the chain criterion, Lemma 2.4.2. To use this we need to know how to compute $\dim \mathrm{Fix}(H^\theta)$.

Using the trace formula, Theorem 2.4.4, an argument given in Golubitsky et al. [46, Chapter XVI, Section 8] shows that for twisted subgroups H^θ,

$$\dim \mathrm{Fix}(H^\theta) = \int_{H^\theta} \mathrm{trace}(h, \theta(h)) = \int_H 2\cos(\theta(h))\mathrm{trace}(h). \qquad (2.5.8)$$

This can be used to prove the following proposition.

Proposition 2.5.13.

(a) If $\theta(H) = \mathbb{1}$ *then* $\dim \mathrm{Fix}(H^\theta) = 2\dim \mathrm{Fix}(H)$.

(b) If $\theta(H) = \mathbb{Z}_2$ *then* $\dim \mathrm{Fix}(H^\theta) = 2(\dim \mathrm{Fix}(K) - \dim \mathrm{Fix}(H))$.

(c) If $\theta(H) = \mathbb{Z}_3$ *then* $\dim \mathrm{Fix}(H^\theta) = \dim \mathrm{Fix}(K) - \dim \mathrm{Fix}(H)$.

(d) If $\theta(H) = \mathbb{Z}_4$ *then* $\dim \mathrm{Fix}(H^\theta) = \dim \mathrm{Fix}(K) - \dim \mathrm{Fix}(L)$ *where* L *is the unique subgroup such that* $K \subset L \subset H$ *and* $|H : L| = 2$.

(e) *If* $\theta(H) = \mathbb{Z}_6$ *then* $\dim \mathrm{Fix}(H^\theta) = \dim \mathrm{Fix}(H) + \dim \mathrm{Fix}(K) - \dim \mathrm{Fix}(L) - \dim \mathrm{Fix}(M)$ *where* L, M *are the unique subgroups between* K *and* H *such that* $|H : L| = 3$ *and* $|H : M| = 2$.

Proof. See [46, Chapter XVI, Section 8]. □

The method of proof of Proposition 2.5.13 works only for twist types $\mathbb{Z}_k$ when $k = 1, 2, 3, 4$ or 6. For other values of k and twist type S^1 we have to use (2.5.8) directly. Provided we know how to compute $\dim \mathrm{Fix}(H)$ for subgroups $H \subset \Gamma$ (which will depend on the representation of Γ), we are now able to compute the $\mathbb{C}$-axial isotropy subgroups of $\Gamma \times S^1$ with two-dimensional fixed-point subspaces. This enables us to use the equivariant Hopf theorem, Theorem 2.5.2. We can also use this method and the chain criterion (Lemma 2.4.2) to compute the isotropy subgroups of $\Gamma \times S^1$ with higher dimensional fixed-point subspaces.

This concludes the background results on the general theory of bifurcations with symmetry which will be required for this thesis.

CHAPTER 3

BACKGROUND

THE GROUP $\mathbf{O(3)}$

3.1 Introduction

In this thesis we will be studying bifurcations from states with spherical symmetry. The symmetry group of the sphere is $\mathbf{O(3)}$. In this chapter we will give information about this group which will be required throughout this thesis. In Section 3.2 we define the group $\mathbf{O(3)}$ and introduce its representations. In Section 3.3 we consider the subgroups of $\mathbf{O(3)}$, their containment relations and the dimension of the fixed-point subspace of each subgroup in each representation of $\mathbf{O(3)}$. Further details on the results in Sections 3.2 and 3.3 of this chapter can be found in [46, Chapter XIII]. In Section 3.4 we will outline the method used for determining, in each representation, which subgroups of $\mathbf{O(3)}$ are isotropy subgroups.

3.2 The group $\mathbf{O(3)}$ and its representations

The orthogonal group $\mathbf{O(3)}$ consists of all 3×3 matrices A satisfying $A^{-1} = A^T$. These matrices have $\det(A) = \pm 1$ and represent the rotations and reflections of a sphere. Algebraically

$$\mathbf{O(3)} = \mathbf{SO(3)} \times \mathbb{Z}_2^c,$$

where $\mathbf{SO(3)}$ is the group of all rotations of the sphere, i.e. $A \in \mathbf{O(3)}$ with $\det(A) = 1$, and $\mathbb{Z}_2^c = \{I, -I\}$, where I is the identity element and $-I$ is inversion in the centre of the sphere. If a point on the surface of the sphere is given in spherical polar coordinates by (θ, ϕ) then the action of the element $-I$ on this point is

$$(\theta, \phi) \to (\pi - \theta, \pi + \phi) \qquad \text{where} \qquad 0 \le \theta \le \pi \quad \text{and} \quad 0 \le \phi < 2\pi.$$

For each irreducible representation of $\mathbf{SO}(3)$ there are two irreducible representations of $\mathbf{O}(3)$, where the element $-I$ either acts as plus or minus the identity, giving rise to the plus and minus representations of $\mathbf{O}(3)$ respectively. The group $\mathbf{SO}(3)$ has precisely one irreducible representation in each odd dimension $2\ell+1$ for $\ell \geq 0$, denoted by V_ℓ, where V_ℓ is the space of spherical harmonics of degree ℓ.

The natural representation of $\mathbf{O}(3)$ on V_ℓ is defined to be the plus representation, where $-I$ acts as the identity, if ℓ is even and the minus representation, where $-I$ acts as minus the identity, if ℓ is odd.

3.2.1 Spherical harmonics

Let (θ, ϕ) denote a point on the surface of a sphere of constant radius R where $\theta \in [0, \pi]$ is the angle measuring the distance to the north pole (the z-axis) and $\phi \in [0, 2\pi]$ is the azimuthal angle.

The spherical harmonics of degree ℓ, $Y_\ell^m(\theta, \phi)$, are the eigenfunctions of the angular part of the spherical Laplacian operator:

$$\nabla^2 \mathbf{U}(R, \theta, \phi) = \frac{1}{R^2}\left[\frac{\partial}{\partial R} R^2 \frac{\partial}{\partial R} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\mathbf{U},$$

with eigenvalue $-\frac{\ell(\ell+1)}{R^2}$. The functions are given by

$$Y_\ell^m(\theta, \phi) = (-1)^m \left(\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right)^{1/2} \mathcal{P}_\ell^m(\cos\theta) \mathrm{e}^{\mathrm{i}m\phi} \tag{3.2.1}$$

for $-\ell \leq m \leq \ell$, where

$$\mathcal{P}_\ell^m(x) = \frac{(1-x^2)^{m/2}}{2^\ell\ \ell!}\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell$$

is the associated Legendre function. The spherical harmonics satisfy

$$Y_\ell^{-m}(\theta, \phi) = (-1)^m \overline{Y_\ell^m}(\theta, \phi), \tag{3.2.2}$$

where the bar denotes complex conjugate. They also satisfy the orthogonality condition

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta, \phi)\ \overline{Y_{\ell'}^{m'}}(\theta, \phi)\ \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi = \delta_{\ell,\ell'}\ \delta_{m,m'}. \tag{3.2.3}$$

In some sections of this thesis we will explicitly use the spherical harmonics of degrees $\ell = 2$, 3, and 4. These are given in appendix A. We now consider the matrices for the action of $\mathbf{O}(3)$ on the spherical harmonics of degree ℓ.

3.2.2 Matrices for the natural action of $\mathbf{O}(3)$ on V_ℓ

In this section we consider how the elements of the group $\mathbf{O}(3)$ act on the functions $Y_\ell^m(\theta, \phi) \in V_\ell$ for the natural action on V_ℓ. We show how to compute the $(2\ell+1) \times (2\ell+1)$ matrices which generate the natural action of $\mathbf{O}(3)$ on V_ℓ. These will be used in later chapters of this thesis to compute the general form of equivariant vector fields.

Let (θ, ϕ) denote a point on the surface of the sphere as in Section 3.2.1. We will consider the actions of the following set of generators of $\mathbf{O(3)}$ on the spherical harmonics of degree ℓ:

- An infinitesimal rotation, ϕ', in the ϕ direction taking $(\theta, \phi) \to (\theta, \phi + \phi')$.
- An infinitesimal rotation, θ', in the θ direction taking $(\theta, \phi) \to (\theta + \theta', \phi)$.
- The inversion element $-I$ which takes $(\theta, \phi) \to (\pi - \theta, \pi + \phi)$.

Using (3.2.1) we can compute that

$$Y_\ell^m(\pi - \theta, \pi + \phi) = (-1)^\ell Y_\ell^m(\theta, \phi)$$

so when ℓ is even the element $-I$ acts as the identity on all spherical harmonics of degree ℓ and when ℓ is odd $-I$ acts as multiplication by -1.

Similarly we can use (3.2.1) to show that

$$Y_\ell^m(\theta, \phi + \phi') = \mathrm{e}^{\mathrm{i}m\phi'} Y_\ell^m(\theta, \phi) \tag{3.2.4}$$

and that in the limit $\theta' \to 0$,

$$\begin{aligned} Y_\ell^m(\theta + \theta', 0) \quad = \quad & -\frac{1}{2}\sqrt{(\ell + m)(\ell - m + 1)}\, \theta'\, Y_\ell^{m-1}(\theta, 0) + Y_\ell^m(\theta, 0) \\ & \qquad + \frac{1}{2}\sqrt{(\ell - m)(\ell + m + 1)}\, \theta'\, Y_\ell^{m+1}(\theta, 0). \end{aligned} \tag{3.2.5}$$

Since it is not obvious that (3.2.5) holds, a proof is given in appendix B.

Suppose that w is the physical variable in a pattern-forming system and that it can be written as

$$w(\theta, \phi) = \sum_{m=-\ell}^{\ell} A_m Y_\ell^m(\theta, \phi) = (A_{-\ell},\ A_{-\ell+1},\ \cdots\ ,\ A_\ell) \begin{pmatrix} Y_\ell^{-\ell}(\theta, \phi) \\ Y_\ell^{-\ell+1}(\theta, \phi) \\ \vdots \\ Y_\ell^{\ell}(\theta, \phi) \end{pmatrix} = \mathbf{A}\, \mathbf{Y}_\ell(\theta, \phi)^{\mathbf{T}}.$$

i.e. as a linear combination of the spherical harmonics of degree ℓ where '$\cdot^{\mathbf{T}}$' indicates the transpose.

We want to find the $(2\ell + 1) \times (2\ell + 1)$ matrices $M_{\phi'}$ and $M_{\theta'}$ for the actions of the infinitesimal rotations ϕ' and θ' on the vector of amplitudes $\mathbf{A} = (A_{-\ell},\ A_{-\ell+1},\ \cdots\ ,\ A_\ell)$.

For the infinitesimal rotation $\phi' \in \mathbf{O(3)}$,

$$\phi' \cdot w(\theta, \phi) = w(\theta, \phi + \phi') = \sum_{m=-\ell}^{\ell} A_m Y_\ell^m(\theta, \phi + \phi') = \sum_{m=-\ell}^{\ell} A_m \mathrm{e}^{\mathrm{i}m\phi'}\, Y_\ell^m(\theta, \phi).$$

Hence $\phi' : A_m \to \mathrm{e}^{\mathrm{i}m\phi'} A_m$ and therefore the matrix which multiplies the column vector of amplitudes $\mathbf{A}^{\mathbf{T}}$ on the left to execute the transformation $\phi \to \phi + \phi'$ is

$$M_{\phi'} = \operatorname{diag}\left(\mathrm{e}^{-\mathrm{i}\ell\phi'}, \mathrm{e}^{\mathrm{i}(-\ell+1)\phi'}, \ldots, \mathrm{e}^{\mathrm{i}(\ell-1)\phi'}, \mathrm{e}^{\mathrm{i}\ell\phi'}\right). \tag{3.2.6}$$

For the infinitesimal rotation θ', by (3.2.5) we have

$$\begin{aligned}\theta' \cdot w(\theta,0) &= w(\theta+\theta',0) = \sum_{m=-\ell}^{\ell} A_m Y_\ell^m(\theta+\theta',0) \\ &= \mathbf{A}\,\mathbf{M}\,\mathbf{Y}_\ell(\theta,0)^{\mathbf{T}} \\ &= \mathbf{Y}_\ell(\theta,0)\,\mathbf{M}^{\mathbf{T}}\,\mathbf{A}^{\mathbf{T}}\end{aligned}$$

where $\mathbf{M}$ is the $(2\ell+1)\times(2\ell+1)$ matrix with m^{th} row

$$\mathbf{v}_m = \left(0,\ \ldots,0,-\frac{1}{2}\sqrt{(\ell+m)(\ell-m+1)}\theta',\ 1,\ \frac{1}{2}\sqrt{(\ell-m)(\ell+m+1)}\,\theta',\ 0,\ \ldots,0\right)$$

for $m=-\ell,\ldots,\ell$ where the entry 1 lies in the m^{th} column. Therefore the matrix which multiplies the column vector of amplitudes $\mathbf{A}^{\mathbf{T}}$ on the left to execute the transformation $\theta \to \theta+\theta'$ is

$$M_{\theta'} = \mathbf{M}^{\mathbf{T}} = \left[\mathbf{v}^{\mathbf{T}}_{-\ell} \mid \mathbf{v}^{\mathbf{T}}_{-\ell+1} \mid \cdots \mid \mathbf{v}^{\mathbf{T}}_{\ell}\right], \tag{3.2.7}$$

i.e. the matrix with columns $\mathbf{v}^{\mathbf{T}}_m$.

Finally, since

$$-I\cdot w(\theta,\phi) = w(\pi-\theta,\pi+\phi) = \sum_{m=-\ell}^{\ell} A_m Y_\ell^m(\pi-\theta,\pi+\phi) = \sum_{m=-\ell}^{\ell} A_m(-1)^\ell\, Y_\ell^m(\theta,\phi)$$

the element $-I$ acts on the column vector $\mathbf{A}^{\mathbf{T}}$ by scalar multiplication by $(-1)^\ell$ or equivalently by multiplication by the matrix $M_{-I}=(-1)^\ell I_{2\ell+1}$ where $I_{2\ell+1}$ is the $(2\ell+1)\times(2\ell+1)$ identity matrix.

Remark 3.2.1. Suppose that $w(\theta,\phi,t)$ is a solution to some pattern forming system and it can be written as a time-dependent linear combination of spherical harmonics of degree ℓ:

$$w(\theta,\phi,t) = \sum_{m=-\ell}^{\ell} A_m(t) Y_\ell^m(\theta,\phi)$$

where $A_m(t)\in\mathbb{C}$. Since w is real the amplitudes $A_m(t)$ must satisfy

$$A_{-m}(t) = (-1)^m \overline{A}_m(t) \qquad \forall t.$$

Hence $A_0(t)\in\mathbb{R}$ and in general the dimension of the vector of amplitudes $\mathbf{A}=(A_{-\ell},A_{-\ell+1},\ldots,A_\ell)$ (and hence the representation) is $2\ell+1$.

3.3 Subgroups of $\mathbf{O}(\mathbf{3})$

In this section we consider the subgroups of $\mathbf{O}(\mathbf{3})$ and their containment relations. This information will be required when computing isotropy subgroups of groups containing $\mathbf{O}(\mathbf{3})$ which we will do in Chapters 4 and 6 of this thesis. In Section 3.3.2 we state without proof the theorems of Golubitsky et al. [46] which tell us the dimension of the fixed-point subspace of each subgroup in the representations of $\mathbf{O}(\mathbf{3})$ on V_ℓ, the space of spherical harmonics of degree ℓ.

The subgroups of $\mathbf{O}(\mathbf{3})$ fall into three classes:

I Subgroups of $\mathbf{SO}(3)$,

II Subgroups containing the inversion element $-I$,

III Subgroups not in $\mathbf{SO}(3)$ and not containing $-I$.

In this section we will consider each of these classes of subgroups in turn.

Class I subgroups

The group $\mathbf{SO}(3)$ is the group of rotations of a sphere. It can be generated by rotations in the x-, y- and z-axes. The subgroup consisting of rotations in the z-axis and a rotation through π in the x-axis is isomorphic to the group of symmetries of the circle, $\mathbf{O}(2)$, where the reflection in $\mathbf{O}(2)$ is realised by the rotation in the x-axis. Removing this rotation we are left with only the rotations in the z-axis and the subgroup $\mathbf{SO}(2)$.

The subgroup generated by rotation through $2\pi/n$ in the z-axis and rotation through π in the x-axis is isomorphic to $\mathbf{D}_n$. By removing the rotation in the x-axis we are left with $\mathbb{Z}_n$.

In addition there are the *exceptional* subgroups, $\mathbb{T}, \mathbb{O}$ and $\mathbb{I}$, the groups of rotations of a tetrahedron, octahedron and icosahedron respectively. They are finite and of orders 12, 24 and 60 respectively. Finally there is the trivial subgroup $\mathbb{1}$.

Class II subgroups

The subgroups of $\mathbf{O}(3)$ of class II all have the form $\Sigma \times \mathbb{Z}_2^c$ where Σ is a subgroup of $\mathbf{SO}(3)$.

Class III subgroups

Each class III subgroup, $H \subset \mathbf{O}(3)$, is isomorphic to a subgroup, $\pi(H)$, of $\mathbf{SO}(3)$, though Σ is never conjugate to that subgroup. Every class III subgroup is uniquely determined by the subgroups $\pi(H)$ and $H \cap \mathbf{SO}(3)$ of $\mathbf{SO}(3)$. In [46, Chapter XIII, Section 9] it is shown that the subgroup $H \cap \mathbf{SO}(3)$ has index 2 in $\pi(H)$ and that all class III subgroups of $\mathbf{O}(3)$ are conjugate to one of the subgroups H given in Table 3.1.

H	$\pi(H)$	$H \cap \mathbf{SO}(3)$
$\mathbf{O}(2)^-$	$\mathbf{O}(2)$	$\mathbf{SO}(2)$
$\mathbb{O}^-$	$\mathbb{O}$	$\mathbb{T}$
$\mathbf{D}_{2m}^d$	$\mathbf{D}_{2m}$	$\mathbf{D}_m$
$\mathbf{D}_m^z$	$\mathbf{D}_m$	$\mathbb{Z}_m$
$\mathbb{Z}_{2m}^-$	$\mathbb{Z}_{2m}$	$\mathbb{Z}_m$

Table 3.1: The class III subgroups of $\mathbf{O}(3)$

The subgroup $\mathbf{O}(2)^-$ can be generated by rotations in the z-axis and reflection in the xz-plane. The group generated by a rotation through $2\pi/m$ in the z-axis and reflection in the xz-plane is isomorphic to $\mathbf{D}_m^z$. The subgroup $\mathbb{O}^-$ is the group of rotations and reflections of a tetrahedron.

The subgroup $\mathbf{D}^d_{2m}$ can be generated by an element $-R^z_{\pi/m}$, which is a rotation through π/m in the z-axis combined with inversion in the origin, and a rotation through π in the x-axis. Finally, by removing the rotation in the x-axis from $\mathbf{D}^d_{2m}$ we are left with $\mathbb{Z}^-_{2m}$.

3.3.1 Containment relations

In this section we describe the containment relations between the conjugacy classes of subgroups of $\mathbf{O(3)}$.

Class I subgroups

It is clear that

(a) $\mathbb{Z}_n < \mathbf{D}_n < \mathbf{O(2)}$,

(b) $\mathbb{Z}_n < \mathbb{Z}_m$ and $\mathbf{D}_n < \mathbf{D}_m$ if n divides m,

(c) $\mathbb{Z}_2 < \mathbf{D}_n$ $(n \geq 2)$ due to the rotation through π symmetry of $\mathbf{D}_n$,

(d) $\mathbb{Z}_n < \mathbf{SO(2)} < \mathbf{O(2)}$ $(n \geq 2)$.

The containment relations for the exceptional subgroups of $\mathbf{SO(3)}$ are shown in Figure 3.1.

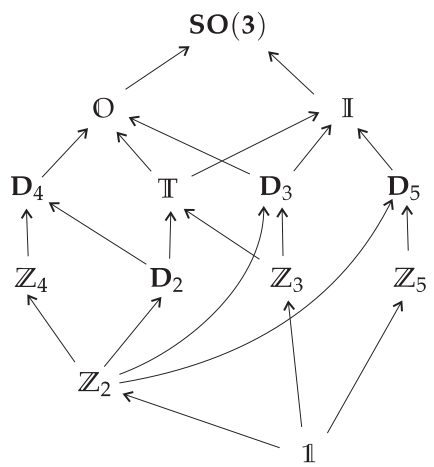


Figure 3.1: The containment relations for the exceptional subgroups of $\mathbf{SO(3)}$. Arrows indicate the direction of containment.

Class II subgroups

The subgroups of a class II subgroup $\Sigma \times \mathbb{Z}^c_2$ where Σ is a subgroup of $\mathbf{SO(3)}$ are:

(a) Subgroups of Σ,

(b) Subgroups of the form $K \times \mathbb{Z}^c_2$ where K is a subgroup of Σ,

(c) The class III subgroups of $\mathbf{O(3)}$ which are isomorphic to a subgroup of Σ.

Class III subgroups

Proposition 3.3.1. *The containments between conjugacy classes of subgroups of class III groups are as follows:*

(a) $\mathbb{O}^-$ *contains* $\mathbf{D}_4^d$, $\mathbb{Z}_4^-$, $\mathbf{D}_3^z$, $\mathbf{D}_2^z$, $\mathbb{Z}_2^-$ *and all subgroups of* $\mathbb{T}$.

(b) $\mathbf{O(2)}^-$ *contains* $\mathbf{D}_m^z$ $(m \geq 2)$, $\mathbb{Z}_2^-$ *and subgroups of* $\mathbf{SO(2)}$.

(c) $\mathbb{Z}_{2m}^-$ *contains* $\mathbb{Z}_{2k}^-$ *where k divides m and $2k$ does not divide m, and subgroups of* $\mathbb{Z}_m$.

(d) $\mathbf{D}_{2m}^d$ *contains* $\mathbf{D}_{2k}^d$ *and* $\mathbb{Z}_{2k}^-$ *when k divides m and $2k$ does not divide m,* $\mathbf{D}_k^z$ *when k divides m,* $\mathbf{D}_2^z$, $\mathbb{Z}_2^-$ *and all subgroups of* $\mathbf{D}_m$.

(e) $\mathbf{D}_m^z$ *contains* $\mathbf{D}_k^z$ *where k divides m,* $\mathbb{Z}_2^-$*, and subgroups of* $\mathbb{Z}_m$.

Proof. See [46, Chapter XIII, Section 9]. □

Remark 3.3.2. Note that if H is a class III subgroup of $\mathbf{O(3)}$ then subgroups of $H \cap \mathbf{SO(3)}$ are contained in H and all other subgroups of H are of class III.

3.3.2 Dimensions of fixed-point subspaces

In order to determine the isotropy subgroups of any group containing $\mathbf{O(3)}$ we will need to know the dimensions of the fixed-point subspaces of the subgroups of $\mathbf{O(3)}$ in the representations on the spherical harmonics of degree ℓ for both the plus and minus representations. These are given by the following two theorems.

Remark 3.3.3. For the plus representation of $\mathbf{O(3)}$, $-I$ acts trivially and therefore $\mathrm{Fix}(\Sigma \times \mathbb{Z}_2^c) = \mathrm{Fix}(\Sigma)$ for subgroups $\Sigma \subset \mathbf{SO(3)}$. In the minus representation where $-I$ acts as minus the identity, $-I$ fixes only the origin and hence $\mathrm{Fix}(\Sigma \times \mathbb{Z}_2^c) = 0$ for all subgroups $\Sigma \subset \mathbf{SO(3)}$. Hence we only need formulae for the dimensions of the fixed-point subspaces of the class I and class III subgroups of $\mathbf{O(3)}$.

Theorem 3.3.4. *Let* $\mathbf{SO(3)}$ *act irreducibly on the space* V_ℓ *of spherical harmonics of degree ℓ. The dimensions of the fixed-point subspaces of closed subgroups are:*

(a) $d(\mathbb{Z}_m) = 2\,[\ell/m] + 1 \qquad (m \geq 1)$

(b) $d(\mathbf{D}_m) = \begin{cases} [\ell/m] & (\ell \text{ odd}) \\ [\ell/m] + 1 & (\ell \text{ even}) \end{cases}$

(c) $d(\mathbb{T}) = 2\,[\ell/3] + [\ell/2] - \ell + 1$

(d) $d(\mathbb{O}) = [\ell/4] + [\ell/3] + [\ell/2] - \ell + 1$

(e) $d(\mathbb{I}) = [\ell/5] + [\ell/3] + [\ell/2] - \ell + 1$

(f) $d(\mathbf{O(2)}) = \begin{cases} 0 & (\ell \text{ odd}) \\ 1 & (\ell \text{ even}) \end{cases}$

(g) $d(\mathbf{SO}(\mathbf{2})) = 1$,

where $d(\Sigma) = \dim \mathrm{Fix}(\Sigma)$ *and* $[x]$ *is the greatest integer less than or equal to* x.

Proof. See [46, Chapter XIII, Section 8]. □

From parts (c), (d) and (e) of Theorem 3.3.4 we can observe that we have the results in Table 3.2. It can also be seen that

$$d(\mathbb{T})(\ell+6) = d(\mathbb{T})(\ell) + 1$$
$$d(\mathbb{O})(\ell+12) = d(\mathbb{O})(\ell) + 1$$
$$d(\mathbb{I})(\ell+30) = d(\mathbb{I})(\ell) + 1.$$

ℓ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$d(\mathbb{I})$	0	0	0	0	0	1	0	0	0	1	0	1	0	0	1
$d(\mathbb{O})$	0	0	0	1	0	1	0	1	1	1	0	2			
$d(\mathbb{T})$	0	0	1	1	0	2	1	1	2	2	1	3			
ℓ	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$d(\mathbb{I})$	1	0	1	0	1	1	1	0	1	1	1	1	1	0	2

Table 3.2: Dimensions of the fixed-point subspaces of $\mathbb{I}$, $\mathbb{O}$ and $\mathbb{T}$ for $\ell = 1, \ldots 30$.

Theorem 3.3.5. *Let* $\mathbf{O}(\mathbf{3})$ *act irreducibly on* V_ℓ *with* $-I$ *acting as minus the identity. Then the dimensions of the fixed-point subspaces for class III subgroups are*

(a) $d(\mathbb{Z}_{2m}^-) = 2\,[(\ell+m)/2m]$

(b) $d(\mathbf{D}_m^z) = \begin{cases} [\ell/m] & (\ell \text{ even}) \\ [\ell/m]+1 & (\ell \text{ odd}) \end{cases}$

(c) $d(\mathbf{D}_{2m}^d) = [(\ell+m)/2m]$

(d) $d(\mathbb{O}^-) = [\ell/3] - [\ell/4]$

(e) $d(\mathbf{O}(\mathbf{2})^-) = \begin{cases} 0 & (\ell \text{ even}) \\ 1 & (\ell \text{ odd}), \end{cases}$

where $d(\Sigma) = \dim \mathrm{Fix}(\Sigma)$ *and* $[x]$ *is the greatest integer less than or equal to* x.

Proof. See [46, Chapter XIII, Section 9]. □

Using part (d) of Theorem 3.3.5 we can observe that we have the results in Table 3.3. It can also be seen that

$$d(\mathbb{O}^-)(\ell+12) = d(\mathbb{O}^-)(\ell) + 1.$$

Remark 3.3.6. Theorems 3.3.4 and 3.3.5 can be proved using the trace formula, Theorem 2.4.4.

ℓ	1	2	3	4	5	6	7	8	9	10	11	12
$d(\mathbb{O}^-)(\ell)$	0	0	1	0	0	1	1	0	1	1	1	1

Table 3.3: Dimensions of Fixed-Point subspaces for $\mathbb{O}^-$

3.4 Determining isotropy subgroups of $\mathbf{O(3)}$

Recall from Section 2.4.1 that for finite groups Γ, Lemma 2.4.2, the standard chain criterion, provides a necessary and sufficient condition for a subgroup $\Sigma \subset \Gamma$ to be an isotropy subgroup. However, when Γ contains continuous symmetries, as for $\Gamma = \mathbf{O(3)}$, Lemma 2.4.2 provides only a necessary condition for Σ to be an isotropy subgroup. When $\Gamma = \mathbf{O(3)}$, a necessary and sufficient condition for a subgroup $\Sigma \subset \mathbf{O(3)}$ to be an isotropy subgroup in the representation on V_ℓ is provided by a result of Linehan and Stedman [63] called the 'massive chain criterion'.

In this section we state the massive chain criterion and also illustrate why it is required by considering an example where the standard chain criterion fails to correctly identify the isotropy subgroups of $\mathbf{O(3)}$. Throughout this thesis we will use different notation from Linehan and Stedman [63] since the authors use notation which makes analogies with areas of physics. This is also the reason for the seemingly strange name of the criterion.

Theorem 3.4.1 (Massive chain criterion). *The subgroup $\Sigma \subset \mathbf{O(3)}$ is an isotropy subgroup in the representation on V_ℓ if and only if for each strictly larger and adjacent group Δ (so that $\Sigma \subset \Delta \subset \cdots \subset \mathbf{O(3)}$)*

$$\dim \mathrm{Fix}(\Delta) - r(\Delta) < \dim \mathrm{Fix}(\Sigma) - r(\Sigma)$$

where

$$r(\Sigma) = \min\{\dim V_\ell - 1,\ q(\Sigma)\} \tag{3.4.1}$$

and

$$q(\Sigma) = \dim N_{\mathbf{O(3)}}(\Sigma) - \dim \Sigma.$$

Proof. See Linehan and Stedman [63]. □

This differs from the standard chain criterion, Lemma 2.4.2, by the quantity $r(\Sigma)$. The subspace of V_ℓ which is invariant under Σ is $\mathrm{Fix}(\Sigma)$. We can partition $\mathrm{Fix}(\Sigma)$ into two sets. In the first set, $V_\ell^m(\Sigma)$, we place one copy of each basis pattern. We then place any duplicates into the second set of basis functions, $V_\ell^0(\Sigma)$. Functions in $V_\ell^0(\Sigma)$ can be transformed by an element in $\mathbf{O(3)}$ to a function in $V_\ell^m(\Sigma)$. Then $r(\Sigma) = \dim V_\ell^0(\Sigma)$, so $r(\Sigma)$ is a measure of the extent to which the members of the subset $\mathrm{Fix}(\Sigma)$ are equivalent under transformations in $\mathbf{O(3)}$. If $\dim V_\ell > 3$ then $r(\Sigma) = 0, 1$ or 3.

An example

Example 3.4.2. Suppose that $\mathbf{O(3)}$ acts on V_3 with the natural representation. In this case the subgroup $\mathbb{Z}_2 \subset \mathbf{O(3)}$ has $\dim \mathrm{Fix}(\mathbb{Z}_2) = 3$ and $\mathbf{D}_2^z$ is a larger and adjacent subgroup – it lies immediately above $\mathbb{Z}_2$ in the lattice of subgroups of $\mathbf{O(3)}$. Since $\dim \mathrm{Fix}(\mathbf{D}_2^z) = 2$, using the standard chain criterion (Lemma 2.4.2) there is no reason to rule out $\mathbb{Z}_2$ from being an isotropy subgroup of $\mathbf{O(3)}$ for this representation. Indeed, by checking all other subgroups larger and adjacent to $\mathbb{Z}_2$ we conclude, using the standard chain criterion, that $\mathbb{Z}_2$ is an isotropy subgroup. This conclusion is however incorrect and we now show that any solution in $\mathrm{Fix}(\mathbb{Z}_2)$ in fact has $\mathbf{D}_2^z$ symmetry, with respect to a particular choice of symmetry axes.

One copy of $\mathbb{Z}_2$ is generated by a rotation through π in the z-axis. For the natural action of $\mathbf{O(3)}$ on V_3 this gives

$$\mathrm{Fix}(\mathbb{Z}_2) = \{Y_3^{-2}(\theta,\phi),\ Y_3^0(\theta,\phi),\ Y_3^2(\theta,\phi)\}$$

or equivalently

$$\mathrm{Fix}(\mathbb{Z}_2) = \{Y_3^0(\theta,\phi),\ \mathrm{Re}(Y_3^2(\theta,\phi)),\ \mathrm{Im}(Y_3^2(\theta,\phi))\}.$$

Hence, any solution $w(\theta,\phi)$ with $\mathbb{Z}_2$ symmetry can be written as a linear combination

$$w(\theta,\phi) = a\, Y_3^0(\theta,\phi) + b\,\mathrm{Re}(Y_3^2(\theta,\phi)) + c\,\mathrm{Im}(Y_3^2(\theta,\phi)),$$

where $a,\ b,\ c \in \mathbb{R}$. If this solution is rotated through any angle in the z-axis then it still lies in the same subspace $\mathrm{Fix}(\mathbb{Z}_2)$ but the values of the coefficients a, b and c will change. Hence there are infinitely many solutions with $\mathbb{Z}_2$ symmetry in $\mathrm{Fix}(\mathbb{Z}_2)$. This is due to the fact that the normaliser $N_{\mathbf{O(3)}}(\mathbb{Z}_2) = \mathbf{O(2)} \times \mathbb{Z}_2^c$ is infinite. Without loss of generality, there is some choice of rotation which makes $c = 0$. This is equivalent to noticing that $\mathrm{Im}(Y_3^2)$ is just a rotation of $\mathrm{Re}(Y_3^2)$ through $\pi/4$ in the z-axis and hence that

$$V_3^m(\mathbb{Z}_2) = \{Y_3^0,\ \mathrm{Re}(Y_3^2)\} \qquad V_3^0(\mathbb{Z}_2) = \{\mathrm{Im}(Y_3^2)\}.$$

This means that $r(\mathbb{Z}_2) = 1$ and since we can compute $r(\mathbf{D}_2^z) = 0$ we find that $\mathbb{Z}_2$ is not an isotropy subgroup in this representation by the massive chain criterion. For the choice of symmetry axes which make $c = 0$ the solution $w(\theta,\phi)$ lies in $\mathrm{Fix}(\mathbf{D}_2^z)$ and thus has $\mathbf{D}_2^z$ symmetry.

Note that we have explicitly shown that $r(\mathbb{Z}_2) = 1$ by considering the basis functions. Using the formula (3.4.1) we can simply compute that

$$\begin{aligned} r(\mathbb{Z}_2) &= \min\{\dim V_\ell - 1,\ \dim N_{\mathbf{O(3)}}(\mathbb{Z}_2) - \dim \mathbb{Z}_2\} \\ &= \min\{6,\ \dim(\mathbf{O(2)} \times \mathbb{Z}_2^c) - \dim \mathbb{Z}_2\} = 1. \end{aligned}$$

We have now summarised all the information about the group $\mathbf{O(3)}$ which will be required throughout this thesis in order to compute isotropy subgroups of groups containing $\mathbf{O(3)}$. Together with the results in Chapter 2 we have now given all background results for this thesis.

CHAPTER 4

HOPF BIFURCATION ON A SPHERE

ISOTROPY SUBGROUPS AND EQUIVARIANT MAPPINGS

4.1 Introduction

In this chapter we investigate the symmetries of branches of periodic solutions which are created at a Hopf bifurcation with $\mathbf{O}(3)$ symmetry. At such a bifurcation, pairs of complex conjugate eigenvalues of the trivial solution with spherical symmetry cross the imaginary axis. We require that at this point the Jacobian has purely imaginary eigenvalues. In Chapter 2 we saw that this is the case when the representation of $\mathbf{O}(3)$ is on $V_\ell \oplus V_\ell$, where $\mathbf{O}(3)$ acts absolutely irreducibly on V_ℓ, the space of spherical harmonics of degree ℓ. A vector $\mathbf{x} \in V_\ell \oplus V_\ell$ can be written as

$$\mathbf{x} = \sum_{m=-\ell}^{\ell} z_m Y_\ell^m(\theta, \phi) + \bar{z}_m \overline{Y}_\ell^m(\theta, \phi).$$

The action of $\mathbf{O}(3)$ on $\mathbf{x} \in V_\ell \oplus V_\ell$ is determined by its action on

$$\mathbf{z} = \left(z_{-\ell},\, z_{-(\ell-1)},\, \ldots,\, z_\ell\right)^{\mathbf{T}} \in \mathbb{C}^{2\ell+1}.$$

Elements in $\mathbf{O}(3)$ act on $\mathbf{z}$ via the same matrices as for the action of $\mathbf{O}(3)$ on V_ℓ i.e. those given in Section 3.2.2. In addition, an element $\psi \in S^1$ acts on $\mathbf{z}$ as scalar multiplication by $e^{i\psi}$.

We consider the system of ODEs

$$\frac{d\mathbf{z}}{dt} = f(\mathbf{z}, \lambda), \tag{4.1.1}$$

where $\mathbf{z} \in \mathbb{C}^{2\ell+1}$, $\lambda \in \mathbb{R}$ is a bifurcation parameter and $f : \mathbb{C}^{2\ell+1} \times \mathbb{R} \to \mathbb{C}^{2\ell+1}$ is a smooth mapping which commutes with the action of the compact Lie group $\mathbf{O}(3)$ on $V_\ell \oplus V_\ell$. By the notion of Birkhoff normal form, f can also be assumed to commute with the action of S^1 to some order k. This equivariant vector field is a function of the complex amplitudes z_m and $\bar{z}_m$ for $-\ell \leq m \leq \ell$. In Section 4.4 we consider how to compute the general form of such mappings.

Since the action of $\mathbf{O}(\mathbf{3})$ on $V_\ell \oplus V_\ell$ is $\mathbf{O}(\mathbf{3})$-simple we can assume that (4.1.1) satisfies all conditions of the equivariant Hopf theorem (Theorem 2.5.2). Thus if $\Sigma \subset \mathbf{O}(\mathbf{3}) \times S^1$ is a $\mathbb{C}$-axial isotropy subgroup (i.e. it fixes a two dimensional subspace of $V_\ell \oplus V_\ell$) then there exists a branch of periodic solutions to (4.1.1) with period near 2π bifurcating from the origin with Σ as their group of symmetries. Furthermore, by Theorem 2.5.3, branches of solutions with the symmetries of other maximal isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ are also guaranteed.

To determine the symmetries of the branches of periodic solutions of (4.1.1) which are guaranteed to exist, it remains to compute the maximal isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$. The $\mathbb{C}$-axial isotropy subgroups were first listed by Golubitsky and Stewart [43]. One error in this list was corrected by Golubitsky et al. [46, Chapter XVIII, Section 5], however a small number of other errors remain. In this chapter we repeat the computations of Golubitsky and Stewart [43] in order to correct these errors.

Recall from Section 2.5.3 that all isotropy subgroups of $\Gamma \times S^1$ are twisted subgroups. In Section 4.2 we compute the conjugacy classes of twisted subgroups H^θ of $\mathbf{O}(\mathbf{3}) \times S^1$ for which it is possible that H^θ could be an isotropy subgroup of $\mathbf{O}(\mathbf{3}) \times S^1$ for some representation on $V_\ell \oplus V_\ell$. In Section 4.3.2 we decide which of these twisted subgroups are $\mathbb{C}$-axial isotropy subgroups in the representation on $V_\ell \oplus V_\ell$ for every value of ℓ. We correct the errors in Table 5.1 of [46, Chapter XVIII, Section 5] and present an amended list of the $\mathbb{C}$-axial subgroups, giving reasons why the changes are required.

In Section 4.3.3 we consider the isotropy subgroups, $\Sigma \subset \mathbf{O}(\mathbf{3}) \times S^1$ which have four-dimensional fixed-point subspaces. If Σ is maximal then by Theorem 2.5.3 a branch of periodic solutions with symmetry Σ is guaranteed to bifurcate from the origin. If Σ is submaximal (i.e. contained in a $\mathbb{C}$-axial subgroup) then it is possible that solutions to (4.1.1) with Σ symmetry may exist depending on the values of the coefficients in the Taylor expansion of the equivariant vector field f.

4.2 Twisted subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$

In this section we follow the method of Golubitsky et al. [46, Chapter XVI, Section 7] (which we summarised in Section 2.5.3 of this thesis) to compute the conjugacy classes of twisted subgroups H^θ of $\Gamma \times S^1$ in the case where Γ is the orthogonal group $\mathbf{O}(\mathbf{3})$.

Step 1 of this method is to find the conjugacy classes of subgroups of $\mathbf{O}(\mathbf{3})$ and to choose a representative H. These subgroups were given in Section 3.3 of this thesis.

For step 2 we must find all closed normal subgroups $K \subset H$ such that H/K is isomorphic to $\mathbb{1}$, $\mathbb{Z}_n$ or S^1 for each subgroup $H \subset \mathbf{O}(\mathbf{3})$. Then step 3 says that we must choose one representative of each conjugacy class of K's under the action of $N_\Gamma(H)/H$. This gives a list of conjugacy classes of all pairs (H, K). The point of these computations is to produce a list of pairs (H, K)

which, together with a homomorphism $\theta : H \to S^1$, give the conjugacy classes of twisted subgroups of $\mathbf{O(3)} \times S^1$ which can be isotropy subgroups for the representation on $V_\ell \oplus V_\ell$ where V_ℓ is the space of spherical harmonics of degree ℓ. The computation of this list can be greatly simplified by noticing that certain pairs (H, K) cannot give a twisted subgroup H^θ which is an isotropy subgroup of $\mathbf{O(3)} \times S^1$ for any homomorphism θ.

Remark 4.2.1. Recall from Remark 2.5.1 that the element $\psi \in S^1$ acts as multiplication by $\mathrm{e}^{\mathrm{i}\psi}$. For any value of ℓ, in the plus representation of $\mathbf{O(3)} \times S^1$ on $V_\ell \oplus V_\ell$, the element $-I \in \mathbf{O(3)}$ acts as the identity and therefore the element $(-I, 0) \in \mathbf{O(3)} \times S^1$ must lie in every isotropy subgroup. This means that H and K must both be class II subgroups of $\mathbf{O(3)}$ since $-I \in H$ and $-I \in \ker\theta = K$.

In the minus representation, $-I$ acts as minus the identity the time shift by $\psi = \pi$ acts as multiplication by -1. Hence $(-I, \pi) \in \mathbf{O(3)} \times S^1$ acts as the identity and must therefore be contained in every isotropy subgroup. This means that H must be a class II subgroup and K must be either a class I or class III subgroup of $\mathbf{O(3)}$ since $-I \in H$ and $-I \notin \ker\theta = K$.

We now compute the conjugacy classes of pairs (H, K) for which H is a class II subgroup of $\mathbf{O(3)}$ and K is normal in H with quotient group H/K isomorphic to $\mathbb{1}$, $\mathbb{Z}_n$ or S^1. Note that any such K must contain the commutator subgroup

$$H' = \langle\, h^{-1}k^{-1}hk \;:\; h, k \in H \,\rangle$$

since this is the smallest normal subgroup of H such that the quotient H/H' is abelian. Here $\langle\cdot\rangle$ indicates 'group generated by'. We will call pairs (H, K) which satisfy the conditions above permitted pairs.

Before we list the conjugacy classes of permitted pairs (H, K) we recall some facts about normal subgroups. A subgroup K is normal in H if $hKh^{-1} = K$ for all $h \in H$. This is equivalent to $hkh^{-1} \in K$ for all $h \in H$ and $k \in K$. We write this as $K \lhd H$. We can note that

(a) the subgroups H and $\mathbb{1}$ are always normal in H (so (H, H) is always a permitted pair)

(b) any subgroup of an abelian group is normal.

Lemma 4.2.2. *If H is any group and K is a subgroup with $|H : K| = 2$ then K is a normal subgroup of H and $H/K \cong \mathbb{Z}_2$.*

Proof. See, for example, [2]. □

A consequence of this lemma is that if $H = J \times \mathbb{Z}_2^c$ for some $J \subset \mathbf{SO(3)}$ then $J \lhd H$ with $H/J \cong \mathbb{Z}_2$ and so $(J \times \mathbb{Z}_2^c, J)$ is always a permitted pair.

Proposition 4.2.3. *The conjugacy classes of pairs (H, K) which can give a twisted subgroup of $\mathbf{O(3)} \times S^1$ which is an isotropy subgroup are as given in Table 4.1.*

Proof. By Remark 4.2.1 H must be a class II subgroup of $\mathbf{O(3)}$ in order for the twisted subgroup, H^θ, given by the pair (H, K) and homomorphism θ, to be an isotropy subgroup of $\mathbf{O(3)} \times S^1$ in any representation. We will consider each class II subgroup, $J \times \mathbb{Z}_2^c$, in turn.

J	K	H/K	J	K	H/K
$\mathbf{SO}(\mathbf{3})$	$\mathbf{O}(\mathbf{3})$	$\mathbb{1}$	$\mathbf{D}_{2m}$	$\mathbf{D}_m \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$
	$\mathbf{SO}(\mathbf{3})$	$\mathbb{Z}_2$		$\mathbf{D}_{2m}^d$	$\mathbb{Z}_2$
$\mathbf{O}(\mathbf{2})$	$\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c$	$\mathbb{1}$	$\mathbb{Z}_{md}$	$\mathbb{Z}_m \times \mathbb{Z}_2^c$	$\mathbb{Z}_d$
	$\mathbf{O}(\mathbf{2})$	$\mathbb{Z}_2$	$\mathbb{Z}_{(2d-1)m}$	$\mathbb{Z}_m$	$\mathbb{Z}_{2(2d-1)}$
	$\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$\mathbb{Z}_{2md}$	$\mathbb{Z}_{2m}^-$	$\mathbb{Z}_{2d}$
	$\mathbf{O}(\mathbf{2})^-$	$\mathbb{Z}_2$	$\mathbb{T}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{1}$
$\mathbf{SO}(\mathbf{2})$	$\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c$	$\mathbb{1}$		$\mathbb{T}$	$\mathbb{Z}_2$
	$\mathbf{SO}(\mathbf{2})$	$\mathbb{Z}_2$		$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$
	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	S^1		$\mathbf{D}_2$	$\mathbb{Z}_6$
	$\mathbb{Z}_{2n}^-$	S^1	$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{1}$
$\mathbf{D}_n$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbb{1}$		$\mathbb{O}$	$\mathbb{Z}_2$
	$\mathbf{D}_n$	$\mathbb{Z}_2$		$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$
	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$		$\mathbb{O}^-$	$\mathbb{Z}_2$
	$\mathbf{D}_n^z$	$\mathbb{Z}_2$	$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$\mathbb{1}$
				$\mathbb{I}$	$\mathbb{Z}_2$

Table 4.1: The normal subgroups of class II subgroups of $\mathbf{O}(\mathbf{3})$ which have quotient subgroups isomorphic to a subgroup of S^1. Here $H = J \times \mathbf{Z}_2^c$. These pairs (H, K) can give twisted subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ which could be isotropy subgroups.

$J = \mathbf{SO}(3)$**:** By computing the commutator subgroup we find that when $H = \mathbf{O}(\mathbf{3})$ we have $H' = \mathbf{SO}(\mathbf{3})$ and $H/H' \cong \mathbb{Z}_2$. Thus for (H, K) to be a permitted pair K must contain $\mathbf{SO}(\mathbf{3})$. This leaves only the pairs $(\mathbf{O}(\mathbf{3}), \mathbf{O}(\mathbf{3}))$ and $(\mathbf{O}(\mathbf{3}), \mathbf{SO}(\mathbf{3}))$.

$J = \mathbf{O}(2)$**:** When $H = \mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c$ we have $H' = \mathbf{SO}(\mathbf{2})$ and $H/H' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus for (H, K) to be a permitted pair K must contain $\mathbf{SO}(\mathbf{2})$. Notice that since $\mathbb{Z}_2 \times \mathbb{Z}_2$ is abelian but not cyclic $K = \mathbf{SO}(\mathbf{2})$ does not give a permitted pair. This leaves only the pairs $(\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c, \mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c)$ and $(\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2^c, \mathbf{O}(\mathbf{2}))$, $(\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2^c, \mathbf{O}(\mathbf{2})^-)$ and $(\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2^c, \mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c)$ which are permitted by Lemma 4.2.2.

$J = \mathbf{SO}(2)$**:** Since $\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c$ is abelian all of its subgroups are normal. The subgroups K which give a permitted pair (H, K) are $\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c$, $\mathbf{SO}(\mathbf{2})$, $\mathbb{Z}_{2n}^-$ and $\mathbb{Z}_n \times \mathbb{Z}_2^c$. The pairs with $K = \mathbb{Z}_n$ are not permitted since in this case $H/K \cong S^1 \times \mathbb{Z}_2$.

$J = \mathbf{D}_n$ **for** n **odd:** When $H = \mathbf{D}_n \times \mathbb{Z}_2^c$ and n is odd we have $H' = \mathbb{Z}_n$ and $H/H' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus for (H, K) to be a permitted pair K must contain $\mathbb{Z}_n$. Notice that since $\mathbb{Z}_2 \times \mathbb{Z}_2$ is abelian but not cyclic $K = \mathbb{Z}_n$ does not give a permitted pair. This leaves only the pairs with $K = \mathbf{D}_n \times \mathbb{Z}_2^c$, $\mathbf{D}_n^z$, $\mathbf{D}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_2^c$. All of these subgroups K are normal in $H = \mathbf{D}_n \times \mathbb{Z}_2^c$ and give permitted pairs (H, K).

$J = \mathbf{D}_n$ **for** n **even:** Let $n = 2m$. When $H = \mathbf{D}_{2m} \times \mathbb{Z}_2^c$ we have $H' = \mathbb{Z}_m$ and $H/H' \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus for (H, K) to be a permitted pair K must contain $\mathbb{Z}_m$. Notice that since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is abelian but not cyclic $K = \mathbb{Z}_m$ does not give a permitted pair. All of the subgroup types K which give permitted pairs when n is odd also give permitted pairs when n is even. In addition to these pairs we have that $K = \mathbf{D}_{2m}^d$ and $\mathbf{D}_m \times \mathbb{Z}_2^c$ give permitted pairs by Lemma 4.2.2. The remaining normal subgroups of $H = \mathbf{D}_{2m} \times \mathbb{Z}_2^c$ which contain $\mathbb{Z}_m$ are $K = \mathbf{D}_m^z$, $\mathbf{D}_m$, $\mathbb{Z}_m \times \mathbb{Z}_2^c$ and $\mathbb{Z}_{2m}^-$. However, all of these subgroups give $H/K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and as such, do not give permitted pairs.

$J = \mathbb{Z}_n$**:** Since $\mathbb{Z}_n \times \mathbb{Z}_2^c$ is abelian all of its subgroups are normal. Suppose that m divides n so that $n = md$ for some $d \in \mathbb{N}$. Then $K = \mathbb{Z}_m$ is a subgroup of $H = \mathbb{Z}_n \times \mathbb{Z}_2^c$ with $H/K \cong \mathbb{Z}_d \times \mathbb{Z}_2$. This is a subgroup of S^1 only when d is odd. The subgroup $K = \mathbb{Z}_m \times \mathbb{Z}_2^c$ has quotient group $H/K \cong \mathbb{Z}_d$ and hence the pair (H, K) is permitted for all values of d. Suppose now that $2m$ divides n so that $n = 2md$. Then $K = \mathbb{Z}_{2m}^-$ has $H/K \cong \mathbb{Z}_{2d}$ and so the pair (H, K) is permitted for all values of d.

$J = \mathbb{T}$**:** When $H = \mathbb{T} \times \mathbb{Z}_2^c$ we have $H' = \mathbf{D}_2$ and $H/H' \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$. Thus, for (H, K) to be a permitted pair, K must contain $\mathbf{D}_2$. The subgroups containing $\mathbf{D}_2$ are $\mathbb{T} \times \mathbb{Z}_2^c$, $\mathbb{T}$, $\mathbf{D}_2 \times \mathbb{Z}_2^c$ and $\mathbf{D}_2$. All of these subgroups are normal in H and give permitted pairs (H, K).

$J = \mathbb{O}$**:** When $H = \mathbb{O} \times \mathbb{Z}_2^c$ we have $H' = \mathbb{T}$ and $H/H' \cong \mathbb{Z}_2$. Thus, for (H, K) to be a permitted pair, K must contain $\mathbb{T}$. The subgroups containing $\mathbb{T}$ are $\mathbb{O} \times \mathbb{Z}_2^c$, $\mathbb{O}$, $\mathbb{T} \times \mathbb{Z}_2^c$ and $\mathbb{T}$. All of these subgroups are normal in H and give permitted pairs (H, K).

$J = \mathbb{I}$**:** When $H = \mathbb{I} \times \mathbb{Z}_2^c$ we have $H' = \mathbb{I}$ and $H/H' \cong \mathbb{Z}_2$. The only subgroups K which give a permitted pair are then $\mathbb{I} \times \mathbb{Z}_2^c$ and $\mathbb{I}$.

These arguments justify all entries in Table 4.1. □

In order to give a list of conjugacy classes of twisted subgroups of $\mathbf{O(3)} \times S^1$ it remains only to carry out the fourth and final step in the procedure given in Section 2.5.3. This says that for every pair (H, K) in Table 4.1 we must determine the possible homomorphisms $\theta : H \to H/K$. To do this we must find all automorphisms of H/K which are not induced by conjugation by elements in the normaliser of H.

Homomorphisms

For each pair (H, K) in Table 4.1 we must find all of the homomorphisms $\theta : H \to H/K$ to determine the conjugacy classes of twisted subgroups of $\mathbf{O(3)} \times S^1$. Recall that if θ is a homomorphism $H \to S^1$ with $\ker \theta = K$ then all other such homomorphisms are of the form $\alpha \circ \theta$ where α is an automorphism of $\mathrm{Im}(\theta) = H/K$. The twisted groups H^θ and $H^{\alpha\circ\theta}$ are conjugate in $\mathbf{O(3)} \times S^1$ if α is induced by conjugation by elements in $N_{\mathbf{O(3)}}(H)$. Thus for each pair (H, K) in Table 4.1 we must find all automorphisms of H/K which are not induced by conjugation by elements in the normaliser of H.

When H/K is $\mathbb{1}$ there are no such automorphisms and the only homomorphism $\theta : H \to H/K$ is given by

$$\theta(h) = 0 \qquad \forall\, h \in H.$$

When H/K is $\mathbb{Z}_2$ there are again no such automorphisms and the homomorphism θ is given by

$$\theta(h) = \begin{cases} 0, & h \in K \\ \pi, & h \in H - K \end{cases}$$

We now consider the remaining pairs (H, K) in turn.

Pairs $(\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c, K)$: Consider the pairs $(\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c, K)$ in Table 4.1 where $\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c / K = S^1$. There are no non-trivial automorphisms $\alpha : S^1 \to S^1$ which are not induced by an element in $N_{\mathbf{O}(\mathbf{3})}(\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c) = \mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c$. Hence, up to conjugacy, there is only one possible homomorphism $\theta : \mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c \to S^1$ for each pair $(\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c, K)$ which is as follows:

- When $K = \mathbb{Z}_n \times \mathbb{Z}_2^c$ the pair $(\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c, K)$ can only give an isotropy subgroup in the plus representation of $\mathbf{O}(\mathbf{3})$ by Remark 4.2.1. In this representation $-I$ acts as the identity and $-I \in K = \ker \theta$. Thus the homomorphism θ is given by

$$\theta(\psi) = \theta(-\psi) = n\psi \qquad \forall \psi \in \mathbf{SO}(\mathbf{2}). \tag{4.2.1}$$

- When $K = \mathbb{Z}_{2n}^-$ the pair $(\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c, K)$ can only give an isotropy subgroup in the minus representation of $\mathbf{O}(\mathbf{3})$ by Remark 4.2.1. The homomorphism θ is given by

$$\theta(\psi) = \begin{cases} n\psi, & \psi \in \mathbf{SO}(\mathbf{2}) \\ n\psi + \pi, & \psi \in \mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c - \mathbf{SO}(\mathbf{2}). \end{cases} \tag{4.2.2}$$

Pairs $(\mathbb{T} \times \mathbb{Z}_2^c, K)$: Consider the pairs $(\mathbb{T} \times \mathbb{Z}_2^c, K)$ in Table 4.1 where $K = \mathbf{D}_2 \times \mathbb{Z}_2^c$ or $\mathbf{D}_2$ and $\mathbb{T} \times \mathbb{Z}_2^c / K = \mathbb{Z}_3$ or $\mathbb{Z}_6$ respectively. In both cases there are no non-trivial automorphisms which are not induced by an element in $N_{\mathbf{O}(\mathbf{3})}(\mathbb{T} \times \mathbb{Z}_2^c) = \mathbb{O} \times \mathbb{Z}_2^c$. In each case there is (up to conjugacy) just one homomorphism, θ, which is as follows:

- When $K = \mathbf{D}_2 \times \mathbb{Z}_2^c$ the homomorphism, θ, is given by

$$\theta(h) = \frac{2\pi k}{3} \qquad \forall\, h \in r^k \mathbf{D}_2 \ \text{ for } k = 0, 1, 2$$

where $r = R_{2\pi/3}$ is a rotation through $2\pi/3$ and so is a generator of $\mathbb{Z}_3$. By Remark 4.2.1 the pair (H, K) can only give an isotropy subgroup in the plus representation of $\mathbf{O}(\mathbf{3})$ so $\theta(-h) = \theta(h)$ in this case.

- When $K = \mathbf{D}_2$ the homomorphism, θ, is given by

$$\theta(h) = \frac{2\pi k}{6} \qquad \forall\, h \in r^k \mathbf{D}_2 \ \text{ for } k = 0, \ldots, 5$$

where $r = -R_{2\pi/3}$ is a rotation through $2\pi/3$ combined with inversion in the origin and so is a generator of $\mathbb{Z}_6$.

Pairs $(\mathbb{Z}_n \times \mathbb{Z}_2^c, K)$: For the pairs $(\mathbb{Z}_n \times \mathbb{Z}_2^c, K)$ in Table 4.1 the number of non-trivial automorphisms $\alpha : H/K \to H/K$ which are not induced by conjugation by elements in $N_{\mathbf{O}(\mathbf{3})}(\mathbb{Z}_n \times \mathbb{Z}_2^c) = \mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c$ depends on the size of H/K. We will consider each case in turn.

- Consider the pair $(\mathbb{Z}_{md} \times \mathbb{Z}_2^c, \mathbb{Z}_m \times \mathbb{Z}_2^c)$ where $H/K = \mathbb{Z}_d$. One possible homomorphism $\theta : \mathbb{Z}_{md} \times \mathbb{Z}_2^c \to \mathbb{Z}_d$ is given by

$$\theta(r^k) = \theta(-r^k) = 2\pi k/d \qquad \text{for } k = 0, \ldots, md-1 \tag{4.2.3}$$

where $r = R_{2\pi/md}$ is a rotation through $2\pi/md$ and is the generator of $\mathbb{Z}_{md}$. All other homomorphisms are given by $\alpha \circ \theta$ where $\alpha : \mathbb{Z}_d \to \mathbb{Z}_d$ is an automorphism. Suppose that $\mathbb{Z}_d = \langle \omega \rangle$ where $\omega = 2\pi/d$. When $d = 1$ or 2 there are no non-trivial automorphisms. When $d \geq 3$ the automorphisms α_j of $\mathbb{Z}_d$ are given by

$$\alpha_j(\omega) = j\omega \qquad \text{for } j = 1, \ldots, d-1.$$

Notice that if $j = 1$ then we have the trivial automorphism.

Let $\kappa \in \mathbf{O(2)}$ be the order two element such that

$$\kappa R_\phi \kappa^{-1} = R_{2\phi} \qquad \text{for rotations } R_\phi \in \mathbf{SO(2)}.$$

Then for elements $r^k \in \mathbb{Z}_{md}$ where $r = R_{2\pi/md}$,

$$\theta(\kappa r^k \kappa^{-1}) = \theta(r^{2k}) = 4\pi k/d \qquad \text{for } k = 0, \ldots, md-1$$

and

$$(\alpha_j \circ \theta)(r^k) = \alpha_j(2\pi k/d) = \alpha_j(k\omega) = jk\omega = 2\pi jk/d.$$

Hence, when $j = 2$ the homomorphism ψ_j given by $\alpha_j \circ \theta$ is induced by conjugatation by $\kappa \in \mathbf{O(2)}$.

This means that the homomorphisms from $\mathbb{Z}_{md} \times \mathbb{Z}_2^c$ to $\mathbb{Z}_d$ are given by θ and $\psi_j = \alpha_j \circ \theta$ for $j = 3, \ldots, d-1$ where

$$\psi_j(r^k) \;=\; \psi_j(-r^k) \;=\; \alpha_j(\theta(r^k)) \;=\; \alpha_j(2\pi k/d) \;=\; 2\pi kj/d. \tag{4.2.4}$$

- Consider the pair $(\mathbb{Z}_{mb} \times \mathbb{Z}_2^c, \mathbb{Z}_m)$ where $H/K = \mathbb{Z}_{2b}$ and $b = 2d-1$ is odd. One possible homomorphism $\theta : \mathbb{Z}_{mb} \times \mathbb{Z}_2^c \to \mathbb{Z}_{2b}$ is given by

$$\begin{aligned} \theta(r^k) &= 2\pi k/b && \text{for } k = 0, \ldots, mb-1 \\ \theta(-r^k) &= 2\pi k/b + \pi && \text{for } k = 0, \ldots, mb-1. \end{aligned} \tag{4.2.5}$$

where $r = R_{2\pi/mb}$ is the generator of $\mathbb{Z}_{mb}$. All other homomorphisms are given by $\alpha \circ \theta$ where $\alpha : \mathbb{Z}_{2b} \to \mathbb{Z}_{2b}$ is a non-trivial automorphism. When $b = 1$ there are no such automorphisms. Notice that since $(-I, \pi) \in H^{(\alpha\circ\theta)}$ for all α, the automorphisms must satisfy $\alpha(\pi) = \pi$. Suppose that $\mathbb{Z}_{2b} = \langle \omega \rangle$ where $\omega = \pi/b$. When $b \geq 3$ the automorphisms α_j of $\mathbb{Z}_{2b}$ are given by

$$\alpha_j(\omega) = j\omega \qquad \text{for } j = 1, \ldots, 2b-1 \text{ and } j \text{ odd}.$$

We can only have j odd since we need $\alpha_j(\pi) = \pi$. Notice that if $j = 1$ then we have the trivial automorphism and if $j = b$ then α_b is not an automorphism (it is not bijective).

For $\kappa \in \mathbf{O(2)}$ and $r^k \in \mathbb{Z}_{mb}$ where $r = R_{2\pi/mb}$,

$$\begin{aligned} \theta(\kappa r^k \kappa^{-1}) &= \theta(r^{2k}) \;=\; 4\pi k/b && \text{for } k = 0, \ldots, mb-1 \\ \theta(\kappa(-r^k)\kappa^{-1}) &= \theta(-r^{2k}) \;=\; 4\pi k/b + \pi && \text{for } k = 0, \ldots, mb-1 \end{aligned} \tag{4.2.6}$$

and

$$\begin{aligned}(\alpha_j \circ \theta)(r^k) &= \alpha_j(2\pi k/b) = 2\pi jk/b \\ (\alpha_j \circ \theta)(-r^k) &= \alpha_j(2\pi k/b + \pi) = 2\pi jk/b + j\pi = 2\pi jk/b + \pi \end{aligned} \tag{4.2.7}$$

since j is odd. Hence, when $j = 2 + b$ the homomorphism ψ_j given by $\alpha_j \circ \theta$ is induced by conjugatation by $\kappa \in \mathbf{O(2)}$.

This means that the homomorphisms from $\mathbb{Z}_{mb} \times \mathbb{Z}_2^c$ to $\mathbb{Z}_{2b}$ are given by θ and $\psi_j = \alpha_j \circ \theta$ for $j = 3, 5, \ldots, b-2, b+4, \ldots 2b-1$ where

$$\begin{aligned}\psi_j(r^k) &= \alpha_j(\theta(r^k)) = \alpha_j(2\pi k/b) = 2\pi kj/b \\ \psi_j(-r^k) &= \alpha_j(\theta(-r^k)) = \alpha_j(2\pi k/b + \pi) = 2\pi kj/b + \pi. \end{aligned} \tag{4.2.8}$$

Notice that the homomorphism given by $j = b + s$ is the same as a homomorphism given by $j = s$ so we can take $j = 3, 4, \ldots, b-1$ in (4.2.8).

- Finally, consider the pair $(\mathbb{Z}_{2md} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-)$ where $H/K = \mathbb{Z}_{2d}$. One possible homomorphism $\theta : \mathbb{Z}_{2md} \times \mathbb{Z}_2^c \to \mathbb{Z}_{2d}$ is given by

$$\begin{aligned}\theta(r^k) &= \pi k/d \qquad \text{for } k = 0, \ldots, 2md-1 \\ \theta(-r^k) &= \pi k/d + \pi \qquad \text{for } k = 0, \ldots, 2md-1. \end{aligned} \tag{4.2.9}$$

where $r = R_{\pi/md}$ is the generator of $\mathbb{Z}_{2md}$. All other homomorphisms are given by $\alpha \circ \theta$ where $\alpha : \mathbb{Z}_{2d} \to \mathbb{Z}_{2d}$ is a non-trivial automorphism. When $d = 1$ there are no such automorphisms. Notice that since $(-I, \pi) \in H^{(\alpha\circ\theta)}$ for all α, the automorphisms must satisfy $\alpha(\pi) = \pi$. Suppose that $\mathbb{Z}_{2d} = \langle \omega \rangle$ where $\omega = \pi/d$. When $d \geq 2$ the automorphisms α_j of $\mathbb{Z}_d$ are given by

$$\alpha_j(\omega) = j\omega \qquad \text{for } j = 1, \ldots, 2d-1 \text{ and } j \text{ odd}.$$

We can only have j odd since we need $\alpha_j(\pi) = \pi$. Notice that if $j = 1$ then we have the trivial automorphism and if $j = d$ where d is odd then α_d is not an automorphism (it is not bijective).

For $\kappa \in \mathbf{O(2)}$ and $r^k \in \mathbb{Z}_{2md}$ where $r = R_{\pi/md}$,

$$\begin{aligned}\theta(\kappa r^k \kappa^{-1}) &= \theta(r^{2k}) = 2\pi k/d \qquad \text{for } k = 0, \ldots, 2md-1 \\ \theta(\kappa(-r^k)\kappa^{-1}) &= \theta(-r^{2k}) = 2\pi k/d + \pi \qquad \text{for } k = 0, \ldots, md-1 \end{aligned} \tag{4.2.10}$$

and

$$\begin{aligned}(\alpha_j \circ \theta)(r^k) &= \alpha_j(\pi k/d) = \pi jk/d \\ (\alpha_j \circ \theta)(-r^k) &= \alpha_j(\pi k/d + \pi) = \pi jk/d + j\pi = \pi jk/d + \pi \end{aligned} \tag{4.2.11}$$

since j is odd. Hence there is no homomorphism $\psi_j = \alpha_j \circ \theta$ induced by conjugatation by $\kappa \in \mathbf{O(2)}$. The homomorphisms from $\mathbb{Z}_{2md} \times \mathbb{Z}_2^c$ to $\mathbb{Z}_{2d}$ are given by θ and $\psi_j = \alpha_j \circ \theta$ for $j = 3, 5, \ldots, 2d-1$ where

$$\begin{aligned}\psi_j(r^k) &= \alpha_j(\theta(r^k)) = \alpha_j(\pi k/d) = \pi kj/d \\ \psi_j(-r^k) &= \alpha_j(\theta(-r^k)) = \alpha_j(\pi k/d + \pi) = \pi kj/d + \pi. \end{aligned} \tag{4.2.12}$$

Table 4.1 together with the homomorphisms given above completes the list of conjugacy classes of twisted subgroups of $\mathbf{O(3)} \times S^1$. The next step is to identify those twisted subgroups which are isotropy subgroups for some representation on $V_\ell \oplus V_\ell$. This is the subject of the next section.

4.3 Isotropy subgroups of $\mathbf{O(3)} \times S^1$

In Section 4.2 we computed the conjugacy classes of twisted subgroups, H^θ, of $\mathbf{O(3)} \times S^1$. We found that the pairs of subgroups (H, K) given in Table 4.1, with the exception of the pairs with $H = \mathbb{Z}_n \times \mathbb{Z}_2^c$, uniquely identify these conjugacy classes. For each pair where $H \neq \mathbb{Z}_n \times \mathbb{Z}_2^c$ there is only one possible homomorphism $\theta : H \to H/K$.

In this section we will identify which of these twisted subgroups are isotropy subgroups with two or four-dimensional fixed-point subspace in each representation of $\mathbf{O(3)} \times S^1$ on $V_\ell \oplus V_\ell$. To do this we will use the chain criterion (Lemma 2.4.2) and the formulae for computing dimensions of fixed point subspaces of twisted subgroups given in Section 2.5.3. We first compute the dimensions of the fixed-point subspaces of each of the twisted subgroups, H^θ, of $\mathbf{O(3)} \times S^1$ for each representation on $V_\ell \oplus V_\ell$.

4.3.1 Dimensions of fixed-point subspaces of twisted subgroups

In this section we compute the dimension of the fixed-point subspace of each conjugacy class of twisted subgroups, H^θ, of $\mathbf{O(3)} \times S^1$. The dimension of the fixed-point subspace of H^θ for the representation on $V_\ell \oplus V_\ell$ is a formula in terms of ℓ. We only give formulae for $\dim \mathrm{Fix}(H^\theta)$ in the representations where it is possible for H^θ to be an isotropy subgroup of $\mathbf{O(3)} \times S^1$. Recall from Remark 4.2.1 that twisted subgroups $H^\theta \subset \mathbf{O(3)} \times S^1$ given by pairs (H, K) can only be isotropy subgroups in the plus representation if K is a class II subgroup of $\mathbf{O(3)}$ and in the minus representation K must be either class I or class III.

Proposition 4.3.1. *The dimension of the fixed-point subspace of each conjugacy class of twisted subgroups H^θ, is as given in Table 4.2 for the representation in which it is possible for H^θ to be an isotropy subgroup of $\mathbf{O(3)} \times S^1$. In the cases where $H = \mathbb{Z}_n \times \mathbb{Z}_2^c$ the dimension of the fixed-point subspace of H^{ψ_j} depends on the value of j where the homomorphisms ψ_j are as given in (4.2.4), (4.2.8) or (4.2.12) depending on the subgroup K.*

Proof. We will use Proposition 2.5.13, as well as Theorems 3.3.4 and 3.3.5. Note that in the plus representation $-I$ acts as the identity and so for subgroups $J \subset \mathbf{SO(3)}$, $\dim \mathrm{Fix}(J \times \mathbb{Z}_2^c) = \dim \mathrm{Fix}(J)$. In the minus representation $-I$ acts as minus the identity and so fixes only the origin and hence $\dim \mathrm{Fix}(J \times \mathbb{Z}_2^c) = 0$.

For the pairs (H, K) in Table 4.1 with $H/K = \mathbb{1}$, by Proposition 2.5.13(a), since these pairs can only give isotropy subgroups in the plus representation,

$$\dim \mathrm{Fix}(H^\theta) = 2 \dim \mathrm{Fix}(H) = 2 \dim \mathrm{Fix}(J)$$

J	K	$\dim \mathrm{Fix}(H^\theta)$ plus representation	$\dim \mathrm{Fix}(H^\theta)$ minus representation
$\mathbf{SO(3)}$	$\mathbf{O(3)}$	0	–
$\mathbf{SO(3)}$	$\mathbf{SO(3)}$	–	0
$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\begin{cases} 2, & \ell \text{ even} \\ 0, & \ell \text{ odd} \end{cases}$	–
$\mathbf{O(2)}$	$\mathbf{O(2)}$	–	$\begin{cases} 2, & \ell \text{ even} \\ 0, & \ell \text{ odd} \end{cases}$
$\mathbf{O(2)}$	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	$\begin{cases} 0, & \ell \text{ even} \\ 2, & \ell \text{ odd} \end{cases}$	–
$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	–	$\begin{cases} 0, & \ell \text{ even} \\ 2, & \ell \text{ odd} \end{cases}$
$\mathbf{SO(2)}$	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	2	–
$\mathbf{SO(2)}$	$\mathbf{SO(2)}$	–	2
$\mathbf{SO(2)}$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\begin{cases} 2 & \text{when } n = 1, 2, \ldots, \ell \\ 0 & \text{otherwise} \end{cases}$	–
$\mathbf{SO(2)}$	$\mathbb{Z}_{2n}^-$	–	$\begin{cases} 2 & \text{when } n = 1, 2, \ldots, \ell \\ 0 & \text{otherwise} \end{cases}$
$\mathbf{D}_n$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\begin{cases} 2[\ell/n] + 2, & \ell \text{ even} \\ 2[\ell/n], & \ell \text{ odd} \end{cases}$	–
$\mathbf{D}_n$	$\mathbf{D}_n$	–	$\begin{cases} 2[\ell/n] + 2, & \ell \text{ even} \\ 2[\ell/n], & \ell \text{ odd} \end{cases}$
$\mathbf{D}_n$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\begin{cases} 2[\ell/n], & \ell \text{ even} \\ 2[\ell/n] + 2, & \ell \text{ odd} \end{cases}$	–
$\mathbf{D}_n$	$\mathbf{D}_n^z$	–	$\begin{cases} 2[\ell/n], & \ell \text{ even} \\ 2[\ell/n] + 2, & \ell \text{ odd} \end{cases}$
$\mathbf{D}_{2m}$	$\mathbf{D}_m \times \mathbb{Z}_2^c$	$2[(\ell+m)/2m]$	–
$\mathbf{D}_{2m}$	$\mathbf{D}_{2m}^d$	–	$2[(\ell+m)/2m]$
$\mathbb{Z}_{md}$	$\mathbb{Z}_m \times \mathbb{Z}_2^c$	$2\lvert P_{d,j,m}(\ell)\rvert$	–
$\mathbb{Z}_{(2d-1)m}$	$\mathbb{Z}_m$	–	$2\lvert P_{(2d-1),j,m}(\ell)\rvert$
$\mathbb{Z}_{2md}$	$\mathbb{Z}_{2m}^-$	–	$2\lvert P_{2d,j,m}(\ell)\rvert$
$\mathbb{T}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$4[\ell/3] + 2[\ell/2] - 2\ell + 2$	–
$\mathbb{T}$	$\mathbb{T}$	–	$4[\ell/3] + 2[\ell/2] - 2\ell + 2$
$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\begin{cases} \ell - 2[\ell/3], & \ell \text{ even} \\ \ell - 2[\ell/3] - 1, & \ell \text{ odd} \end{cases}$	–
$\mathbb{T}$	$\mathbf{D}_2$	–	$\begin{cases} \ell - 2[\ell/3], & \ell \text{ even} \\ \ell - 2[\ell/3] - 1, & \ell \text{ odd} \end{cases}$
$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$2([\ell/4] + [\ell/3] + [\ell/2] - \ell + 1)$	–
$\mathbb{O}$	$\mathbb{O}$	–	$2([\ell/4] + [\ell/3] + [\ell/2] - \ell + 1)$
$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$2([\ell/3] - [\ell/4])$	–
$\mathbb{O}$	$\mathbb{O}^-$	–	$2([\ell/3] - [\ell/4])$
$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$2([\ell/5] + [\ell/3] + [\ell/2] - \ell + 1)$	–
$\mathbb{I}$	$\mathbb{I}$	–	$2([\ell/5] + [\ell/3] + [\ell/2] - \ell + 1)$

Table 4.2: The dimensions of the fixed-point subspaces of the twisted subgroups H^θ of $\mathbf{O(3)} \times S^1$ in the representation on $V_\ell \oplus V_\ell$ where H^θ can be an isotropy subgroup. Here the set $P_{d,j,m}(\ell)$ is defined by $P_{d,j,m}(\ell) = \left\{ p \;:\; -\ell \le p \le \ell \quad \text{and} \quad \frac{mj+p}{md} \in \mathbb{Z} \right\}$ and the value of j determines the homomorphism ψ_j which is given by (4.2.4), (4.2.8) or (4.2.12) for the different subgroups K.

where $H = J \times \mathbb{Z}_2^c$. The formula in terms of ℓ for $\dim \mathrm{Fix}(J)$ is given by Theorem 3.3.4.

Consider next the pairs (H, K) in Table 4.1 with $H/K = \mathbb{Z}_2$, where $K = \Sigma \times \mathbb{Z}_2^c$ is a class II subgroup of $\mathbf{O(3)}$. By Proposition 2.5.13(b), and since these pairs can only give isotropy subgroups in the plus representation,

$$\dim \mathrm{Fix}(H^\theta) = 2\,(\dim \mathrm{Fix}(K) - \dim \mathrm{Fix}(H)) = 2\,(\dim \mathrm{Fix}(\Sigma) - \dim \mathrm{Fix}(J))$$

where $H = J \times \mathbb{Z}_2^c$. The formulae in terms of ℓ for $\dim \mathrm{Fix}(\Sigma)$ and $\dim \mathrm{Fix}(J)$ are given by Theorem 3.3.4. In many cases it is possible to simplify the expression for $\dim \mathrm{Fix}(\Sigma) - \dim \mathrm{Fix}(J)$ to find the formula given in Table 4.2.

Now consider the pairs (H, K) in Table 4.1 with $H/K = \mathbb{Z}_2$, where K is a class I or III subgroup of $\mathbf{O(3)}$. By Proposition 2.5.13(b), and since these pairs can only give isotropy subgroups in the minus representation,

$$\dim \mathrm{Fix}(H^\theta) = 2\,(\dim \mathrm{Fix}(K) - \dim \mathrm{Fix}(H)) = 2 \dim \mathrm{Fix}(K)$$

where $H = J \times \mathbb{Z}_2^c$. The formula in terms of ℓ for $\dim \mathrm{Fix}(K)$ is given by Theorem 3.3.4 (for class I K) or Theorem 3.3.5 (for class III K).

We consider the remaining pairs (H, K) in turn.

$(\mathbf{SO(2)} \times \mathbb{Z}_2^c, \mathbb{Z}_n \times \mathbb{Z}_2^c)$**:** Recall from Section 4.2 that for this twisted subgroup the twist homomorphism, $\theta : \mathbf{SO(2)} \times \mathbb{Z}_2^c \to S^1$, is given by

$$\theta(\psi) = \theta(-\psi) = n\psi \qquad \text{for} \qquad \psi \in \mathbf{SO(2)}$$

where $-\psi = -I \cdot \psi$. For each $\psi \in \mathbf{SO(2)}$ in the representation on V_ℓ, the trace of ψ is given by

$$\mathrm{Trace}(\psi) = \chi(\psi) = \sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\psi}.$$

In our representation on $V_\ell \oplus V_\ell$, by Remark 2.5.1, $\theta(\psi)$ acts as scalar multiplication by $\mathrm{e}^{\mathrm{i}\theta(\psi)}$ and hence

$$\mathrm{Trace}(\psi, \theta(\psi)) = \mathrm{e}^{\mathrm{i}n\psi} \sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\psi}.$$

Then by the trace formula we have

$$\begin{aligned}
\dim \mathrm{Fix}(H^\theta) &= \int_{\mathbf{SO(2)} \times \mathbb{Z}_2^c} \mathrm{Trace}(h, \theta(h)) \;=\; 2 \int_{\mathbf{SO(2)}} \mathrm{Trace}(\psi, n\psi) \\
&= \frac{1}{\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}n\psi} \sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\psi} d\psi \\
&= \begin{cases} 2 & \text{if } n = 1, 2, \ldots, \ell \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$(\mathbf{SO(2)} \times \mathbb{Z}_2^c, \mathbb{Z}_{2n}^-)$**:** Recall from Section 4.2 that for this twisted subgroup the twist homomorphism, $\theta : \mathbf{SO(2)} \times \mathbb{Z}_2^c \to S^1$, is given by

$$\theta(\psi) = n\psi \qquad \theta(-\psi) = n\psi + \pi$$

where $\psi \in \mathbf{SO(2)}$ and $-\psi = -I \cdot \psi$. For each element $\pm\psi \in \mathbf{SO(2)} \times \mathbb{Z}_2^c$ we then have

$$\mathrm{Trace}(\psi, n\psi) = \mathrm{e}^{\mathrm{i}n\psi} \sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\psi} \qquad \text{and} \qquad \mathrm{Trace}(-\psi, n\psi + \pi) = -\mathrm{e}^{\mathrm{i}(n\psi+\pi)} \sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\psi}.$$

Then by the trace formula we have

$$\begin{aligned}
\dim \mathrm{Fix}(H^\theta) &= \int_{\mathbf{SO(2)}\times\mathbb{Z}_2^c} \mathrm{Trace}(h, \theta(h)) \\
&= \int_{\mathbf{SO(2)}} \mathrm{Trace}(\psi, n\psi) + \int_{\mathbf{SO(2)}} \mathrm{Trace}(-\psi, n\psi + \pi) \\
&= 2\int_{\mathbf{SO(2)}} \mathrm{Trace}(\psi, n\psi) \\
&= \begin{cases} 2 & \text{if } n = 1, 2, \ldots, \ell \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

$(\mathbb{Z}_{md} \times \mathbb{Z}_2^c, \mathbb{Z}_m \times \mathbb{Z}_2^c)$: For these pairs the twisted subgroups are given by H^{ψ_j} where the homomorphisms $\psi_j : \mathbb{Z}_{md} \times \mathbb{Z}_2^c \to \mathbb{Z}_d$ are given by (4.2.4) for $j = 1$ and $j = 3, 4, \ldots, d-1$. Thus for each element $r^k \in \mathbb{Z}_{md}$, where $r = R_{2\pi/md}$ is a rotation through $2\pi/md$,

$$\mathrm{Trace}(r^k, \psi_j(r^k)) = \mathrm{e}^{2\pi\mathrm{i}jk/d} \sum_{p=-\ell}^{\ell} \mathrm{e}^{2\pi\mathrm{i}pk/md},$$

and hence

$$\begin{aligned}
\dim \mathrm{Fix}(H^{\psi_j}) &= \int_{\mathbb{Z}_{md}\times\mathbb{Z}_2^c} \mathrm{Trace}(h, \psi_j(h)) \;=\; 2\int_{\mathbb{Z}_{md}} \mathrm{Trace}(h, \psi_j(h)) \\
&= \frac{2}{md} \sum_{k=0}^{md-1} \mathrm{Trace}(r^k, \psi_j(r^k)) \\
&= \frac{2}{md} \sum_{p=-\ell}^{\ell} \sum_{k=0}^{md-1} \left[\mathrm{e}^{2\pi\mathrm{i}(mj+p)/md}\right]^k.
\end{aligned}$$

Since

$$\sum_{k=0}^{md-1} \left[\mathrm{e}^{2\pi\mathrm{i}(mj+p)/md}\right]^k = \begin{cases} md & \text{when } \frac{mj+p}{md} \in \mathbb{Z} \\ 0 & \text{otherwise,} \end{cases}$$

we find that if we define the set

$$P_{d,j,m}(\ell) = \left\{ p \in \mathbb{Z} \;:\; -\ell \le p \le \ell \quad \text{and} \quad \frac{mj+p}{md} \in \mathbb{Z} \right\} \tag{4.3.1}$$

then

$$\dim \mathrm{Fix}(H^\theta) = 2|P_{d,j,m}(\ell)|.$$

Notice that for $p \in P_{d,j,m}(\ell)$,

$$p = m(dq - j) \qquad \text{for some } q \in \mathbb{Z}$$

This means that

$$\begin{aligned}
P_{d,j,m}(\ell) &= \{q \in \mathbb{Z} \;:\; -\ell \le m(dq - j) \le \ell\} \\
&= \left\{ q \in \mathbb{Z} \;:\; \frac{mj-\ell}{md} \le q \le \frac{mj+\ell}{md} \right\}.
\end{aligned} \tag{4.3.2}$$

$(\mathbb{Z}_{(2d-1)m} \times \mathbb{Z}_2^c, \mathbb{Z}_m)$**:** Let $b = 2d-1$. For these pairs the twisted subgroups are given by H^{ψ_j} where the homomorphisms $\psi_j : \mathbb{Z}_{mb} \times \mathbb{Z}_2^c \to \mathbb{Z}_{2b}$ are given by (4.2.8) for $j = 1$ and $j = 3, 4, \ldots, b-1$ and $b = 2d-1$. Thus for each element $r^k \in \mathbb{Z}_{mb}$, where $r = R_{2\pi/mb}$ is a rotation through $2\pi/mb$,

$$\mathrm{Trace}(r^k, \psi_j(r^k)) = \mathrm{Trace}(-r^k, \psi_j(-r^k)) = \mathrm{e}^{2\pi \mathrm{i} jk/b} \sum_{p=-\ell}^{\ell} \mathrm{e}^{2\pi \mathrm{i} pk/mb}$$

and hence, as in the case above,

$$\begin{aligned} \dim \mathrm{Fix}(H^{\psi_j}) &= 2 \int_{\mathbb{Z}_{mb}} \mathrm{Trace}(h, \psi_j(h)) \\ &= 2|P_{b,j,m}(\ell)| \;=\; 2|P_{(2d-1),j,m}(\ell)|, \end{aligned}$$

where the set $P_{b,j,m}(\ell)$ is as in (4.3.1) and (4.3.2).

$(\mathbb{Z}_{2md} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-)$**:** For these pairs the twisted subgroups are given by H^{ψ_j} where the homomorphisms $\psi_j : \mathbb{Z}_{2md} \times \mathbb{Z}_2^c \to \mathbb{Z}_{2d}$ are given by (4.2.12) for the odd values $j = 1, 3, 5, \ldots, 2d-1$. Thus for each element $r^k \in \mathbb{Z}_{2md}$, where $r = R_{\pi/md}$ is a rotation through π/md,

$$\mathrm{Trace}(r^k, \psi_j(r^k)) = \mathrm{Trace}(-r^k, \psi_j(-r^k)) = \mathrm{e}^{\pi \mathrm{i} jk/d} \sum_{p=-\ell}^{\ell} \mathrm{e}^{\pi \mathrm{i} pk/md}$$

and hence

$$\begin{aligned} \dim \mathrm{Fix}(H^{\psi_j}) &= \int_{\mathbb{Z}_{2md} \times \mathbb{Z}_2^c} \mathrm{Trace}(h, \psi_j(h)) \;=\; 2 \int_{\mathbb{Z}_{2md}} \mathrm{Trace}(h, \psi_j(h)) \\ &= \frac{2}{2md} \sum_{k=0}^{2md-1} \mathrm{Trace}(r^k, \psi_j(r^k)) \\ &= \frac{1}{md} \sum_{p=-\ell}^{\ell} \sum_{k=0}^{2md-1} \left[\mathrm{e}^{\pi \mathrm{i}(mj+p)/md}\right]^k. \end{aligned}$$

Since

$$\sum_{k=0}^{2md-1} \left[\mathrm{e}^{\pi \mathrm{i}(mj+p)/md}\right]^k = \begin{cases} 2md & \text{when } \frac{mj+p}{md} \in 2\mathbb{Z} \\ 0 & \text{otherwise,} \end{cases}$$

we find that

$$\begin{aligned} \dim \mathrm{Fix}(H^{\theta}) &= 2\left|\left\{p \;:\; -\ell \le p \le \ell \quad \text{and} \quad \frac{mj+p}{md} \;\text{ is an even integer}\right\}\right| \\ &= 2|P_{2d,j,m}(\ell)|, \end{aligned}$$

where the set $P_{d,j,m}(\ell)$ is as in (4.3.1) and (4.3.2).

$(\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c)$**:** By Proposition 2.5.13(c) and Theorem 3.3.4(b) and (c),

$$\begin{aligned} \dim \mathrm{Fix}(H^{\theta}) &= \dim \mathrm{Fix}(\mathbf{D}_2) - \dim \mathrm{Fix}(\mathbb{T}) \\ &= \begin{cases} \ell - 2\,[\ell/3], & \ell \text{ even} \\ \ell - 2\,[\ell/3] - 1, & \ell \text{ odd} \end{cases} \end{aligned}$$

in the plus representation.

$(\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2)$: By Proposition 2.5.13(e) and Theorem 3.3.4(b) and (c),

$$\begin{aligned}\dim \mathrm{Fix}(H^\theta) &= \dim \mathrm{Fix}(\mathbb{T} \times \mathbb{Z}_2^c) + \dim \mathrm{Fix}(\mathbf{D}_2) - \dim \mathrm{Fix}(\mathbf{D}_2 \times \mathbb{Z}_2^c) - \dim \mathrm{Fix}(\mathbb{T}) \\ &= \dim \mathrm{Fix}(\mathbf{D}_2) - \dim \mathrm{Fix}(\mathbb{T}) \\ &= \begin{cases} \ell - 2\,[\ell/3]\,, & \ell \text{ even} \\ \ell - 2\,[\ell/3] - 1, & \ell \text{ odd} \end{cases}\end{aligned}$$

in the minus representation. □

4.3.2 $\mathbb{C}$-axial isotropy subgroups

In Section 4.3.1 we computed formulae for the dimensions of the fixed-point subspaces of the twisted subgroups, H^θ, which may be isotropy subgroups for a representation of $\mathbf{O}(\mathbf{3}) \times S^1$ on $V_\ell \oplus V_\ell$. We now wish to determine for which values of ℓ each twisted subgroup, H^θ, is a $\mathbb{C}$-axial subgroup of $\mathbf{O}(\mathbf{3}) \times S^1$. That is, the values of ℓ for which H^θ is an isotropy subgroup with $\dim \mathrm{Fix}(H^\theta) = 2$. We first list in Table 4.3 the values of ℓ where each twisted subgroup has $\dim \mathrm{Fix}(H^\theta) = 2$ before determining, using the chain criterion (Lemma 2.4.2), when each of these twisted subgroups are $\mathbb{C}$-axial isotropy subgroups.

Proposition 4.3.2. *The values of ℓ for which the twisted subgroups $H^\theta \subset \mathbf{O}(\mathbf{3}) \times S^1$ have two-dimensional fixed-point subspaces in the representation on $V_\ell \oplus V_\ell$ are as given in Table 4.3.*

Proof. For the twisted subgroups, H^θ, with $H = J \times \mathbb{Z}_2^c$ where $J = \mathbf{O}(\mathbf{2})$ or $\mathbf{SO}(\mathbf{2})$, the entries in Table 4.3 follow directly from Table 4.2. Similarly, when J and K are both exceptional subgroups of $\mathbf{O}(\mathbf{3})$ then the entries in Table 4.3 follow directly from Table 4.2 when combined with Tables 3.2 and 3.3.

For the pairs $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c)$ and $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2)$ we can see that

$$\begin{aligned}\ell - 2\,[\ell/3] = 2 \quad \text{when } \ell \text{ is even} \quad &\Rightarrow \quad [\ell/3] = \ell/2 - 1 \quad \text{when } \ell \text{ is even} \\ &\Rightarrow \quad 0 < \ell \leq 6 \quad \text{and } \ell \text{ is even} \\ \ell - 2\,[\ell/3] - 1 = 2 \quad \text{when } \ell \text{ is odd} \quad &\Rightarrow \quad [\ell/3] = (\ell - 3)/2 \quad \text{when } \ell \text{ is odd} \\ &\Rightarrow \quad 3 < \ell \leq 9 \quad \text{and } \ell \text{ is odd.}\end{aligned}$$

Hence, for the twisted subgroups H^θ given by these pairs, the values of ℓ for which $\dim \mathrm{Fix}(H^\theta) = 2$ are $2, 4, 5, 6, 7$ and 9.

The entries in Table 4.3 for the twisted subgroups, H^θ, where H is a dihedral subgroup of $\mathbf{O}(\mathbf{3})$ follow from the fact that

$$\begin{aligned}[][\ell/n] = 1 &\quad \text{when } n \leq \ell < 2n \\ [\ell/n] = 0 &\quad \text{when } 0 \leq \ell < n \\ [(\ell + m)/2m] = 1 &\quad \text{when } m \leq \ell < 3m\end{aligned}$$

Finally we consider the range of values of ℓ for which $|P_{d,j,m}(\ell)| = 1$ where $P_{d,j,m}(\ell)$ is the set defined by (4.3.1). By (4.3.2), $|P_{d,j,m}(\ell)| = 1$ when there is only one integer q which satisfies

$$\frac{j}{d} - \frac{\ell}{md} \leq q \leq \frac{j}{d} + \frac{\ell}{md}. \tag{4.3.3}$$

J	K	Values of ℓ plus representation	Values of ℓ minus representation
$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	Even ℓ	–
$\mathbf{O(2)}$	$\mathbf{O(2)}$	–	Even ℓ
$\mathbf{O(2)}$	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	Odd ℓ	–
$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	–	Odd ℓ
$\mathbf{SO(2)}$	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	All ℓ	–
$\mathbf{SO(2)}$	$\mathbf{SO(2)}$	–	All ℓ
$\mathbf{SO(2)}$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	All ℓ for $n = 1, 2, \ldots, \ell$	–
$\mathbf{SO(2)}$	$\mathbb{Z}_{2n}^-$	–	All ℓ for $n = 1, 2, \ldots, \ell$
$\mathbf{D}_n$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\begin{cases} 0 \le \ell < n, & \ell \text{ even} \\ n \le \ell < 2n, & \ell \text{ odd} \end{cases}$	–
$\mathbf{D}_n$	$\mathbf{D}_n$	–	$\begin{cases} 0 \le \ell < n, & \ell \text{ even} \\ n \le \ell < 2n, & \ell \text{ odd} \end{cases}$
$\mathbf{D}_n$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\begin{cases} n \le \ell < 2n, & \ell \text{ even} \\ 0 \le \ell < n, & \ell \text{ odd} \end{cases}$	–
$\mathbf{D}_n$	$\mathbf{D}_n^z$	–	$\begin{cases} n \le \ell < 2n, & \ell \text{ even} \\ 0 \le \ell < n, & \ell \text{ odd} \end{cases}$
$\mathbf{D}_{2m}$	$\mathbf{D}_m \times \mathbb{Z}_2^c$	$m \le \ell < 3m$	–
$\mathbf{D}_{2m}$	$\mathbf{D}_{2m}^d$	–	$m \le \ell < 3m$
$\mathbb{Z}_{md}$[*]	$\mathbb{Z}_m \times \mathbb{Z}_2^c$	$\begin{cases} 0 \le \ell < m, & d = 1 \\ m \le \ell < m(d-1), & d \ge 3,\ j = 1 \\ m(d-j) \le \ell < mj, & d > 3,\ j \ge 3 \end{cases}$	–
$\mathbb{Z}_{(2d-1)m}$[*]	$\mathbb{Z}_m$	–	$\begin{cases} 0 \le \ell < m, & d = 1,\ j = 1 \\ m \le \ell < 2m, & d = 2,\ j = 1 \\ m \le \ell < m(2d-2), & d \ge 3, j = 1 \\ m(2d-1-j) \le \ell < mj, & d \ge 3, j \ge 3 \end{cases}$
$\mathbb{Z}_{2md}$[*]	$\mathbb{Z}_{2m}^-$	–	$\begin{cases} m \le \ell < m(2d-1), & d \ge 2,\ j = 1 \\ m(2d-j) \le \ell < mj, & d \ge 2,\ j \ge 3 \end{cases}$
$\mathbb{T}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$3, 4, 7, 8, 11$	–
$\mathbb{T}$	$\mathbb{T}$	–	$3, 4, 7, 8, 11$
$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$2, 4, 5, 6, 7, 9$	–
$\mathbb{T}$	$\mathbf{D}_2$	–	$2, 4, 5, 6, 7, 9$
$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$4, 6, 8, 9, 10, 13, 14, 15, 17, 19, 23$	–
$\mathbb{O}$	$\mathbb{O}$	–	$4, 6, 8, 9, 10, 13, 14, 15, 17, 19, 23$
$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$3, 6, 7, 9$–$14, 16, 17, 20$	–
$\mathbb{O}$	$\mathbb{O}^-$	–	$3, 6, 7, 9$–$14, 16, 17, 20$
$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$6, 10, 12, 15, 16, 18, 20, 21, 22, 24$–$28$ 31–$35, 37$–$39, 41, 43, 44, 47, 49, 53, 59$	–
$\mathbb{I}$	$\mathbb{I}$	–	$6, 10, 12, 15, 16, 18, 20, 21, 22, 24$–$28$ 31–$35, 37$–$39, 41, 43, 44, 47, 49, 53, 59$

Table 4.3: The values of ℓ for which the twisted subgroups $H^\theta \subset \mathbf{O(3)} \times S^1$ given by the pairs (H, K) have two-dimensional fixed-point subspaces in the representation on $V_\ell \oplus V_\ell$ where H^θ can be an isotropy subgroup. Here $H = J \times \mathbb{Z}_2^c$. [*] The homomorphism is $\psi_j : H \to H/K$ which is given by (4.2.4), (4.2.8) or (4.2.12) depending on K.

When $d = 1$ we must have $j = 1$ and this integer, q must be 1 so

$$0 < \frac{m-\ell}{m} \leq 1 \leq \frac{m+\ell}{m} < 2 \qquad \Rightarrow \qquad 0 \leq \ell < m.$$

When $d = 2$ we must also have $j = 1$. The nearest integers to $1/d = 1/2$ are 0 and 1 which are equal distances away and hence we cannot have $|P_{2,1,m}(\ell)| = 1$ for any values of ℓ.

When $d \geq 3$ and $j = 1$ the nearest integer to j/d is $q = 0$ and so

$$-1 < \frac{m-\ell}{md} \leq 0 \leq \frac{m+\ell}{md} < 1 \qquad \Rightarrow \qquad m \leq \ell < m(d-1).$$

Finally, when $d > 3$ and $j \geq 3$ the nearest integer to j/d is $q = 1$ and so

$$0 < \frac{mj-\ell}{md} \leq 1 \leq \frac{mj+\ell}{md} < 2 \qquad \Rightarrow \qquad m(d-j) \leq \ell < mj. \qquad \square$$

Theorem 4.3.3. *The $\mathbb{C}$-axial subgroups of $\mathbf{O(3)} \times S^1$ in the representations on $V_\ell \oplus V_\ell$ are as given in Table 4.4.*

J	K	$\theta(H)$	Plus representation	Minus representation
$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbb{1}$	Even ℓ	
$\mathbf{O(2)}$	$\mathbf{O(2)}$	$\mathbb{Z}_2$		Even ℓ
$\mathbf{O(2)}$	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	Odd ℓ	
$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$		Odd ℓ
$\mathbf{SO(2)}$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	S^1	All ℓ for $n = 1, 2, \ldots, \ell$	
$\mathbf{SO(2)}$	$\mathbb{Z}_{2n}^-$	S^1		All ℓ for $n = 1, 2, \ldots, \ell$
$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$\mathbb{1}$	6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 47, 49, 53, 59	
$\mathbb{I}$	$\mathbb{I}$	$\mathbb{Z}_2$		6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 47, 49, 53, 59
$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{1}$	4, 6, 8, 9, 10, 13, 14, 15, 17, 19, 23	
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Z}_2$		4, 6, 8, 9, 10, 13, 14, 15, 17, 19, 23
$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	3, 6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 20	
$\mathbb{O}$	$\mathbb{O}^-$	$\mathbb{Z}_2$		3, 6, 7, 9, 10,11, 12, 13, 14, 16, 17, 20
$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$	2, 4, 5, 6, 7, 9	
$\mathbb{T}$	$\mathbf{D}_2$	$\mathbb{Z}_6$		2, 4, 5, 6, 7, 9
$\mathbf{D}_{2m}$	$\mathbf{D}_m \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$m \leq \ell < 3m, \quad (m \geq 3)$	
$\mathbf{D}_{2m}$	$\mathbf{D}_{2m}^d$	$\mathbb{Z}_2$		$m \leq \ell < 3m, \quad (m \geq 3)$
$\mathbf{D}_4$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	2, 4, 5	
$\mathbf{D}_4$	$\mathbf{D}_4^d$	$\mathbb{Z}_2$		2, 4, 5

Table 4.4: The $\mathbb{C}$-axial subgroups of $\mathbf{O(3)} \times S^1$ for the representations $V_\ell \oplus V_\ell$. The last two columns give the values of ℓ for which the subgroups are isotropy subgroups. Here $H = J \times \mathbb{Z}_2^c$.

Proof. We consider each row in Table 4.3 and determine for which values of ℓ the twisted subgroup H^θ given by the pair (H, K) is an isotropy subgroup by using the chain criterion (Lemma 2.4.2).

If a twisted subgroup, H^θ is maximal (i.e. not contained in any other twisted subgroup G^θ in Table 4.3) then it is a $\mathbb{C}$-axial isotropy subgroup for all values of ℓ where $\dim \mathrm{Fix}(H^\theta) = 2$ as given in Table 4.3.

Remark 4.3.4. Notice that a pair (H, K) gives a twisted subgroup, H^θ that contains the twisted subgroup, L^ψ, given by the pair (L, M) only if $L \subset H$, $M \subset K$ and the quotient groups satisfy $L/M \subset H/K$. Also note that this is a necessary but not sufficient condition for $L^\psi \subset H^\theta$. We will sometimes use the notation $(L, M) \subset (H, K)$ to mean that $L^\psi \subset H^\theta$.

Since the pairs $(H, K) =$

$$\begin{array}{ccc}
(\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{O(2)} \times \mathbb{Z}_2^c), & (\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{O(2)}), & (\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{SO(2)} \times \mathbb{Z}_2^c), \\
(\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{O(2)}^-), & (\mathbf{SO(2)} \times \mathbb{Z}_2^c, \mathbb{Z}_n \times \mathbb{Z}_2^c), & (\mathbf{SO(2)} \times \mathbb{Z}_2^c, \mathbb{Z}_{2n}^-), \\
(\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c), & (\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I}), & (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c), \\
(\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}), & (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c), & (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}^-), \\
(\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c), & \text{and} & (\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2)
\end{array}$$

define maximal twisted subgroups, H^θ, they give $\mathbb{C}$-axial isotropy subgroups for all values of ℓ given in Table 4.3.

The twisted subgroup given by the pair $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_m \times \mathbb{Z}_2^c)$ is contained in that given by the pair $(\mathbf{D}_{2md} \times \mathbb{Z}_2^c, \mathbf{D}_{md} \times \mathbb{Z}_2^c)$ for any odd value of d. However, for any odd value of d the values of ℓ for which each of these twisted subgroups have two-dimensional fixed-point subspaces do not overlap. Hence by the chain criterion, when $m \geq 3$, $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_m \times \mathbb{Z}_2^c)$ gives a $\mathbb{C}$-axial subgroup when $m \leq \ell < 3m$. Similarly when $m \geq 3$ the pair $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_{2m}^d)$ also defines a $\mathbb{C}$-axial subgroup when $m \leq \ell < 3m$. When $m = 2$, from Table 4.3 we can see that both of the pairs $(\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c)$ and $(\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_4^d)$ define twisted subgroups with two-dimensional fixed-point subspaces when $\ell = 2, 3, 4$ and 5. However

$$(\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c) \subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c) \quad \text{and} \quad (\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_4^d) \subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}^-).$$

Hence, by the chain rule, neither pair defines a $\mathbb{C}$-axial isotropy subgroup when $\ell = 3$.

This accounts for all of the entries in Table 4.4. It remains to explain why the rest of the pairs (H, K) in Table 4.3 do not define $\mathbb{C}$-axial isotropy subgroups.

By the chain criterion the twisted subgroup, H^θ, defined by $(\mathbf{SO(2)} \times \mathbb{Z}_2^c, \mathbf{SO(2)} \times \mathbb{Z}_2^c)$ cannot be an isotropy subgroup since it is contained in the twisted subgroup, L^ψ defined by $(\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{O(2)} \times \mathbb{Z}_2^c)$ and for all values of ℓ where $\dim \mathrm{Fix}(H^\theta) = 2$, $\dim \mathrm{Fix}(L^\psi) = 2$ also. Similarly H^θ defined by $(\mathbf{SO(2)} \times \mathbb{Z}_2^c, \mathbf{SO(2)})$ cannot be an isotropy subgroup because it does not satisfy the chain criterion with L^ψ defined by $(\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{O(2)})$ for any value of ℓ.

Since $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c) \subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c)$, by the chain criterion the twisted subgroup H^θ defined by $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c)$ cannot be a $\mathbb{C}$-axial isotropy subgroup when $\ell = 4$ or 8 even though $\dim \mathrm{Fix}(H^\theta) = 2$ for these values. Also $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c) \subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c)$, and so by the chain criterion H^θ cannot be a $\mathbb{C}$-axial isotropy subgroup when $\ell = 3, 7$ or 11 either. This leaves no values of ℓ for which $\dim \mathrm{Fix}(H^\theta) = 2$ so we conclude that H^θ is never a $\mathbb{C}$-axial isotropy subgroup. We come to the same conclusion for the twisted subgroup L^{ψ} defined by $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbb{T})$ since

$$(\mathbb{T} \times \mathbb{Z}_2^c, \mathbb{T}) \subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}) \quad \text{and} \quad (\mathbb{T} \times \mathbb{Z}_2^c, \mathbb{T}) \subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}^-).$$

The pairs $(\mathbb{Z}_{md} \times \mathbb{Z}_2^c, \mathbb{Z}_m \times \mathbb{Z}_2^c)$, $(\mathbb{Z}_{(2d-1)m} \times \mathbb{Z}_2^c, \mathbb{Z}_m)$ and $(\mathbb{Z}_{2md} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-)$ cannot define $\mathbb{C}$-axial isotropy subgroups for any value of ℓ for any homomorphism $\psi_j : H \to H/K$. For all ℓ, the twisted subgroups defined by these pairs do not satisfy the chain criterion when compared with the twisted subgroup defined by the pairs $(\mathbf{SO(2)} \times \mathbb{Z}_2^c, \mathbb{Z}_m \times \mathbb{Z}_2^c)$, $(\mathbf{SO(2)} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-)$ and $(\mathbf{SO(2)} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-)$ respectively.

Notice that

$$(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c) \quad \subset \quad (\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{O(2)} \times \mathbb{Z}_2^c) \tag{4.3.4}$$

$$(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c) \quad \subset \quad (\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c). \tag{4.3.5}$$

By (4.3.4) and the chain criterion the twisted subgroup, H^θ, defined by the pair $(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c)$ cannot be a $\mathbb{C}$-axial isotropy subgroup for any even values of ℓ. By (4.3.5) and the chain criterion, H^θ, is also not a $\mathbb{C}$-axial isotropy subgroup for the remaining odd values of ℓ. Similarly $(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n)$ does not give a $\mathbb{C}$-axial isotropy subgroup for any value of ℓ due to the fact that the pair is contained in both $(\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{O(2)})$ and $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n}^d)$.

In addition

$$(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbb{Z}_n \times \mathbb{Z}_2^c) \quad \subset \quad (\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{SO(2)} \times \mathbb{Z}_2^c) \tag{4.3.6}$$

$$(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbb{Z}_n \times \mathbb{Z}_2^c) \quad \subset \quad (\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c). \tag{4.3.7}$$

By (4.3.6) and the chain criterion the twisted subgroup, H^θ, defined by the pair $(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbb{Z}_n \times \mathbb{Z}_2^c)$ cannot be a $\mathbb{C}$-axial isotropy subgroup for odd values of ℓ. By (4.3.7) and the chain criterion, H^θ, is also not a $\mathbb{C}$-axial isotropy subgroup for the remaining even values of ℓ. Similarly $(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n^z)$ does not give a $\mathbb{C}$-axial isotropy subgroup for any value of ℓ due to the fact that the pair is contained in both $(\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{O(2)}^-)$ and $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n}^d)$.

Having now considered all rows in Table 4.3, this completes the proof. $\square$

Differences from previously published results

As mentioned in Section 4.1, the $\mathbb{C}$-axial isotropy subgroups of $\mathbf{O(3)} \times S^1$ were first listed by Golubitsky and Stewart [43]. One error in this list was corrected in Golubitsky et al. [46, Chapter XVIII, Section 5], however a small number of other errors remained. In this Chapter so far, we have repeated the computations of Golubitsky and Stewart [43] and Golubitsky et al. [46, Chapter XVIII, Section 5] in order to correct these errors. Table 4.4 represents our corrected list of the $\mathbb{C}$-axial isotropy subgroups of $\mathbf{O(3)} \times S^1$. In this section we outline how and why our results differ from those of [43] and [46].

The differences between our results in Table 4.4 and the results of Golubitsky et al. given in Table 5.1 of [46, Chapter XVIII] are as follows:

1. We have found it necessary to include the value $\ell = 15$ in the lists of values where the pairs $(H, K) = (\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c)$ and $(\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I})$ give twisted subgroups which are $\mathbb{C}$-axial subgroups in the plus and minus representations respectively. This is because the twisted subgroups given by both of these pairs are maximal and in the given representations the fixed-point subspace of the twisted subgroups is two-dimensional.

2. In Table 5.1 of [46, Chapter XVIII] the final row states that in the plus representation the pair $(H, K) = (\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c)$ defines a twisted subgroup which is a $\mathbb{C}$-axial subgroup when $\ell/2 < n \leq \ell$. [1] We have found in our computations that the range of values for which this twisted subgroup is a $\mathbb{C}$-axial isotropy subgroup is $\ell/3 < n \leq \ell$ when $n \geq 3$ and $\ell = 2, 4$ and 5 when $n = 2$.

   In all previous enumerations of the $\mathbb{C}$-axial subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$, [43, 44, 46], it is assumed that the twisted subgroup, L^ψ given by $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c)$ is contained in H^θ given by $(\mathbf{D}_{4n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n} \times \mathbb{Z}_2^c)$. This results in the reduction in the range of values for which L^ψ is a $\mathbb{C}$-axial subgroup which is reported in [43, 44, 46]. However, the containment relation $L^\psi \subset H^\theta$ does not hold. For example, the element $(R_{\pi/n}, \pi)$, where $R_{\pi/n} \in \mathbf{O}(\mathbf{3})$ is a rotation through an angle π/n and $\pi \in S^1$ is the non-identity element in $\mathbb{Z}_2$, is contained in the smaller group, L^ψ, but not in any copy of the larger group, H^θ. When we choose a copy of the group $L = \mathbf{D}_{2n} \times \mathbb{Z}_2^c$ to make the twisted subgroup L^ψ we choose the two axes of rotation required. There is only one copy of the group $H = \mathbf{D}_{4n} \times \mathbb{Z}_2^c$ which contains L but in this case $L = \ker\theta$ and hence $L^\psi \not\subseteq H^\theta$. This is an example of the insufficiency of the condition given in Remark 4.3.4 to determine all containment relations between twisted subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$.

3. In Table 5.1 of [46, Chapter XVIII] the penultimate row states that in the minus representation the pair $(H, K) = (\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n)$ gives a twisted subgroup, H^θ which is a $\mathbb{C}$-axial subgroup when $\ell/2 < n \leq \ell$. However, as we noted in the proof of Theorem 4.3.3, H^θ is not a $\mathbb{C}$-axial subgroup by the chain criterion since it is contained in the twisted subgroup, L^ψ given by the pair $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n}^d)$.

   We have shown that L^ψ is a $\mathbb{C}$-axial subgroup for the values given in Table 4.4. This twisted subgroup does not appear in the list of $\mathbb{C}$-axial subgroups given in [43] nor Table 5.1 of [46, Chapter XVIII] since it is assumed that L^ψ is contained in the twisted subgroup given by $(\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c, \mathbf{O}(\mathbf{2})^-)$ for all values of n. This assumption is false due to the fact that $\mathbf{D}_{2n}^d$ is not contained in $\mathbf{O}(\mathbf{2})^-$ by Proposition 3.3.1.

$\mathbb{C}$-axial subgroups in the natural representation

Using Table 4.4 we can find the $\mathbb{C}$-axial isotropy subgroups for the natural representations of $\mathbf{O}(\mathbf{3}) \times S^1$ on $V_\ell \oplus V_\ell$. Recall that the natural representation is the plus representation for even values of ℓ and the minus representation for odd values of ℓ. Table 4.5 gives the $\mathbb{C}$-axial isotropy

[1] By comparing Table 14.1 of [43] and Table 5.1 of [46, Chapter XVIII] and the subsequent remarks it is clear that there is a misprint in footnote [2] to Table 5.1 [46, Chapter XVIII] and it should say that the class II subgroup is $\mathbf{D}_{n/2} \times \mathbb{Z}_2^c$ and n is even.

subgroups in the natural representation for $\ell = 1, \ldots, 6$. This table is the equivalent to Table 5.2 of [46, Chapter XVIII], taking into consideration the errors in Table 5.1 of [46, Chapter XVIII] which we have corrected. We find that when $\ell = 3$, there are fewer branches of periodic solutions guaranteed to exist by the equivariant Hopf theorem than previously thought and when $\ell = 4, 5$ and 6 there are more solution branches. In Chapter 5 we will study the $\mathbb{C}$-axial periodic solutions in the natural representation on $V_3 \oplus V_3$ in detail.

ℓ	J	K	$\theta(H)$	Number of branches given by equivariant Hopf theorem
1	$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$	2
	$\mathbf{SO(2)}$	$\mathbb{Z}_{2n}^-$	S^1 $[n = 1]$	
2	$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbb{1}$	5
	$\mathbf{SO(2)}$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	S^1 $[n = 1, 2]$	
	$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$	
	$\mathbf{D}_4$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	
3	$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$	6
	$\mathbf{SO(2)}$	$\mathbb{Z}_{2n}^-$	S^1 $[1 \le n \le 3]$	
	$\mathbb{O}$	$\mathbb{O}^-$	$\mathbb{Z}_2$	
	$\mathbf{D}_6$	$\mathbf{D}_6^d$	$\mathbb{Z}_2$	
4	$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbb{1}$	10
	$\mathbf{SO(2)}$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	S^1 $[1 \le n \le 4]$	
	$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{1}$	
	$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$	
	$\mathbf{D}_{2n}$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$ $[2 \le n \le 4]$	
5	$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$	11
	$\mathbf{SO(2)}$	$\mathbb{Z}_{2n}^-$	S^1 $[1 \le n \le 5]$	
	$\mathbb{T}$	$\mathbf{D}_2$	$\mathbb{Z}_6$	
	$\mathbf{D}_{2n}$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$ $[2 \le n \le 5]$	
6	$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbb{1}$	15
	$\mathbf{SO(2)}$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	S^1 $[1 \le n \le 6]$	
	$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$\mathbb{1}$	
	$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{1}$	
	$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	
	$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$	
	$\mathbf{D}_{2n}$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$ $[3 \le n \le 6]$	

Table 4.5: The $\mathbb{C}$-axial subgroups of $\mathbf{O(3)} \times S^1$ for the natural representations on $V_\ell \oplus V_\ell$ for $\ell = 1, \ldots, 6$. Here $H = J \times \mathbb{Z}_2^c$.

4.3.3 Isotropy subgroups with four-dimensional fixed-point subspaces

In this section we compute the isotropy subgroups, $\Sigma \subset \mathbf{O(3)} \times S^1$, which have four-dimensional fixed-point subspaces for all representations on $V_\ell \oplus V_\ell$. If Σ is maximal then by Theorem 2.5.3 a branch of periodic solutions with Σ symmetry is guaranteed to bifurcate from the origin. If Σ is submaximal (i.e. contained in a $\mathbb{C}$-axial subgroup) then it is possible that solutions to (4.1.1) with Σ symmetry may exist depending on the values of the coefficients in the Taylor expansion of the equivariant vector field f. We call such a solution (if it exists) a submaximal solution.

In this section we compute in which representations, $V_\ell \oplus V_\ell$, the twisted subgroups in Table 4.2 are isotropy subgroups of $\mathbf{O(3)} \times S^1$ with four-dimensional fixed-point subspaces. To do this we first list in Table 4.6 the values of ℓ where each twisted subgroup has $\dim \mathrm{Fix}(H^\theta) = 4$. We then determine, using the chain criterion (Lemma 2.4.2), when the twisted subgroups are isotropy subgroups with four-dimensional fixed-point subspaces.

Proposition 4.3.5. *The values of ℓ for which the twisted subgroups $H^\theta \subset \mathbf{O(3)} \times S^1$ in Table 4.2 have four-dimensional fixed-point subspaces in the representation on $V_\ell \oplus V_\ell$ are as given in Table 4.6.*

Proof. The twisted subgroups, H^θ, with $H = J \times \mathbb{Z}_2^c$ where $J = \mathbf{O(2)}$ or $\mathbf{SO(2)}$ do not appear in Table 4.6 since they never have a four-dimensional fixed-point subspace. When J and K are both exceptional subgroups of $\mathbf{O(3)}$ then the entries in Table 4.6 follow directly from Table 4.2 when combined with Tables 3.2 and 3.3.

For the pairs $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c)$ and $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2)$ we can see that

$$\begin{aligned}
\ell - 2\,[\ell/3] = 4 \quad \text{when } \ell \text{ is even} \quad &\Rightarrow \quad [\ell/3] = \ell/2 - 2 \quad \text{when } \ell \text{ is even} \\
&\Rightarrow \quad 6 < \ell \leq 12 \quad \text{and } \ell \text{ is even} \\
\ell - 2\,[\ell/3] - 1 = 2 \quad \text{when } \ell \text{ is odd} \quad &\Rightarrow \quad [\ell/3] = (\ell-5)/2 \quad \text{when } \ell \text{ is odd} \\
&\Rightarrow \quad 9 < \ell \leq 15 \quad \text{and } \ell \text{ is odd.}
\end{aligned}$$

Hence the values of ℓ for which $\dim \mathrm{Fix}(H^\theta) = 2$ are $8, 10, 11, 12, 13$ and 15.

The entries in Table 4.3 for the twisted subgroups, H^θ, where H is a dihedral subgroup of $\mathbf{O(3)}$ follow from the fact that

$$\begin{aligned}
[\ell/n] &= 1 \qquad \text{when } n \leq \ell < 2n \\
[\ell/n] &= 2 \qquad \text{when } 2n \leq \ell < 3n \\
[(\ell+m)/2m] &= 2 \qquad \text{when } 3m \leq \ell < 5m
\end{aligned}$$

Finally we consider the range of values of ℓ for which $|P_{d,j,m}(\ell)| = 2$ where $P_{d,j,m}(\ell)$ is the set defined by (4.3.1). By (4.3.2), $|P_{d,j,m}(\ell)| = 2$ when there are two integers q which satisfy (4.3.3).

When $d = 1$ we must have $j = 1$. This means that one of the integers must be 1 but the next nearest integers to 1 are 0 and 2 which are equal distances away and hence $|P_{1,1,m}(\ell)| \neq 2$ for any values of ℓ.

When $d = 2$ we must also have $j = 1$. The nearest integers to $1/d$ are 0 and 1 and so we have

$$-1 < \frac{m-\ell}{2m} \leq 0 < 1 \leq \frac{m+\ell}{2m} < 2 \qquad \Rightarrow \qquad m \leq \ell < 3m.$$

When $d \geq 3$, $|P_{d,j,m}(\ell)| = 2$ when

$$-1 < \frac{mj-\ell}{md} \leq 0 < 1 \leq \frac{mj+\ell}{md} < 2$$

which implies that

$$\max\{mj, m(d-j)\} \leq \ell < \min\{m(d+j), m(2d-j)\}.$$

The results in Table 4.6 follow directly from these computations. $\square$

J	K	Values of ℓ plus representation	Values of ℓ minus representation
$\mathbf{D}_n$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\begin{cases} n \le \ell < 2n, & \ell \text{ even} \\ 2n \le \ell < 3n, & \ell \text{ odd} \end{cases}$	–
$\mathbf{D}_n$	$\mathbf{D}_n$	–	$\begin{cases} n \le \ell < 2n, & \ell \text{ even} \\ 2n \le \ell < 3n, & \ell \text{ odd} \end{cases}$
$\mathbf{D}_n$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\begin{cases} 2n \le \ell < 3n, & \ell \text{ even} \\ n \le \ell < 2n, & \ell \text{ odd} \end{cases}$	–
$\mathbf{D}_n$	$\mathbf{D}_n^z$	–	$\begin{cases} 2n \le \ell < 3n, & \ell \text{ even} \\ n \le \ell < 2n, & \ell \text{ odd} \end{cases}$
$\mathbf{D}_{2m}$	$\mathbf{D}_m \times \mathbb{Z}_2^c$	$3m \le \ell < 5m$	–
$\mathbf{D}_{2m}$	$\mathbf{D}_{2m}^d$	–	$3m \le \ell < 5m$
$\mathbb{Z}_{md}{}^{[*]}$	$\mathbb{Z}_m \times \mathbb{Z}_2^c$	$\begin{cases} m \le \ell < 3m & d = 2,\ j = 1 \\ \text{[A]} & d \ge 3 \end{cases}$	–
$\mathbb{Z}_{(2d-1)m}{}^{[*]}$	$\mathbb{Z}_m$	–	$\begin{cases} 2m \le \ell < 4m & d = 2,\ j = 1 \\ \text{[B]} & d \ge 3 \end{cases}$
$\mathbb{Z}_{2md}{}^{[*]}$	$\mathbb{Z}_{2m}^-$	–	$\begin{cases} m \le \ell < 3m & d = 1,\ j = 1 \\ \text{[C]} & d \ge 2 \end{cases}$
$\mathbb{T}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$6, 9, 10, 13, 14, 17$	–
$\mathbb{T}$	$\mathbb{T}$	–	$6, 9, 10, 13, 14, 17$
$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$8, 10, 11, 12, 13, 15$	–
$\mathbb{T}$	$\mathbf{D}_2$	–	$8, 10, 11, 12, 13, 15$
$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$12, 16, 18, 20–22, 25–27, 29, 31, 35$	–
$\mathbb{O}$	$\mathbb{O}$	–	$12, 16, 18, 20–22, 25–27, 29, 31, 35$
$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$15, 18, 19, 21–26, 28, 29, 32$	–
$\mathbb{O}$	$\mathbb{O}^-$	–	$15, 18, 19, 21–26, 28, 29, 32$
$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$30, 36, 40, 42, 45, 46, 48, 50, 51, 52, 54–58$ $61–65, 67–69, 71, 73, 74, 77, 79, 83, 89$	–
$\mathbb{I}$	$\mathbb{I}$	–	$30, 36, 40, 42, 45, 46, 48, 50, 51, 52, 54–58$ $61–65, 67–69, 71, 73, 74, 77, 79, 83, 89$

Table 4.6: The values of ℓ for which the twisted subgroups $H^\theta \subset \mathbf{O}(\mathbf{3}) \times S^1$ given by the pairs (H, K) have four-dimensional fixed-point subspaces in the representation on $V_\ell \oplus V_\ell$ where H^θ can be an isotropy subgroup. Here $H = J \times \mathbb{Z}_2^c$.
[*] The homomorphism is $\psi_j : H \to H/K$ which is given by (4.2.4), (4.2.8) or (4.2.12) depending on K.
[A]: $\max\{mj, m(d-j)\} \le \ell < \min\{m(d+j), m(2d-j)\}$
[B]: $\max\{mj, m(2d-1-j)\} \le \ell < \min\{m(2d-1+j), m(4d-2-j)\}$
[C]: $\max\{mj, m(2d-j)\} \le \ell < \min\{m(2d+j), m(4d-j)\}$.

Theorem 4.3.6. *If $H^\theta \subset \mathbf{O(3)} \times S^1$ is a twisted subgroup with four-dimensional fixed-point subspace in the representation on $V_\ell \oplus V_\ell$ then it is an isotropy subgroup in that representation. In other words, Table 4.6 is a list of the isotropy subgroups of $\mathbf{O(3)} \times S^1$ with four-dimensional fixed-point subspaces in the representation on $V_\ell \oplus V_\ell$.*

Proof. We consider each twisted subgroup H^θ in Table 4.6 and show that for all values of ℓ where $\dim \mathrm{Fix}(H^\theta) = 4$, the twisted subgroup is an isotropy subgroup. We do this using the chain criterion (Lemma 2.4.2).

If a twisted subgroup, H^θ is maximal (not contained in any other subgroup of $\mathbf{O(3)} \times S^1$) then it is an isotropy subgroup with four-dimensional fixed point subspace for all values of ℓ where $\dim \mathrm{Fix}(H^\theta) = 4$. The pairs $(H, K) =$

$$\begin{array}{ccc} (\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c), & (\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I}), & (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c), \\ (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}), & (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c), & (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}^-), \\ (\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c), & \text{and} & (\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2) \end{array}$$

give maximal twisted subgroups and are therefore maximal isotropy subgroups with four-dimensional fixed point subspace for all values of ℓ given in Table 4.6. Hence, by Theorem 2.5.3, a branch of periodic solutions of (4.1.1) with these symmetries is guaranteed to exist for the representation on $V_\ell \oplus V_\ell$ for the values of ℓ in Table 4.6.

The twisted subgroup given by the pair $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_m \times \mathbb{Z}_2^c)$ is contained in that given by the pair $(\mathbf{D}_{2md} \times \mathbb{Z}_2^c, \mathbf{D}_{md} \times \mathbb{Z}_2^c)$ for any odd value of d. However, for any odd value of d the values of ℓ for which each of these twisted subgroups have four-dimensional fixed-point subspaces do not overlap. Hence by the chain criterion, when $m \geq 3$, $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_m \times \mathbb{Z}_2^c)$ gives an isotropy subgroup with four-dimensional fixed-point subspace for all values of ℓ in Table 4.6. We can also observe that $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_m \times \mathbb{Z}_2^c)$ never gives a maximal isotropy subgroup since for all values of ℓ for which it has a four-dimensional fixed-point subspace, $(\mathbf{D}_{6m} \times \mathbb{Z}_2^c, \mathbf{D}_{3m} \times \mathbb{Z}_2^c)$ is a $\mathbb{C}$-axial subgroup which contains $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_m \times \mathbb{Z}_2^c)$.

Similarly when $m \geq 3$ the pair $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_{2m}^d)$ also defines a submaximal isotropy subgroup with four-dimensional fixed-point subspace in the minus representation when $3m \leq \ell < 5m$.

When $m = 2$, from Table 4.3 we can see that both of the pairs $(\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c)$ and $(\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_4^d)$ define twisted subgroups with four-dimensional fixed-point subspaces when $\ell = 6, 7, 8$ and 9. Although

$$(\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c) \subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c) \quad \text{and} \quad (\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_4^d) \subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}^-),$$

the larger groups have fixed-point subspaces of dimension less than four when $\ell = 6, 7, 8$ and 9. Hence, by the chain criterion, both pairs define a submaximal isotropy subgroup when $\ell = 6, 7, 8$ and 9.

Now consider the twisted subgroup, H^θ given by the pair $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c)$. Although it is contained in the twisted subgroups given by the pairs $(\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c)$, $(\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c)$ and $(\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c)$; for the values of ℓ for which $\dim \mathrm{Fix}(H^\theta) = 4$, these subgroups have a smaller fixed-point subspace and so H^θ is a submaximal isotropy subgroup by the chain criterion. Similarly the pair $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbb{T})$ gives a submaximal isotropy subgroup with four-dimensional fixed-point subspace for all values of ℓ given in Table 4.6.

For general n the twisted subgroup H^θ given by the pair $(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c)$ is contained in the twisted subgroups given by the pairs $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c)$ and $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n} \times \mathbb{Z}_2^c)$. However for the values of ℓ where $\dim \mathrm{Fix}(H^\theta) = 4$, these pairs always have a fixed-point subspace of dimension less than 4 and so by the chain criterion H^θ is an isotropy subgroup with four-dimensional fixed-point subspace for the values of ℓ given in Table 4.6. It can never be a maximal isotropy subgroup since it is contained in the $\mathbb{C}$-axial subgroup given by the pair $(\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{O(2)} \times \mathbb{Z}_2^c)$. We now consider the fact that for $n = 2, 3, 4$ and 5, H^θ may be contained in a twisted subgroup L^ψ where $L = M \times \mathbb{Z}_2^c$ and M is an exceptional subgroup of $\mathbf{O(3)}$:

(a) $(\mathbf{D}_5 \times \mathbb{Z}_2^c, \mathbf{D}_5 \times \mathbb{Z}_2^c) \subset (\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c)$ but when $\dim \mathrm{Fix}(\mathbf{D}_5 \times \mathbb{Z}_2^c, \mathbf{D}_5 \times \mathbb{Z}_2^c) = 4$, $\dim \mathrm{Fix}(\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c) < 4$.

(b) $(\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_4 \times \mathbb{Z}_2^c) \subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c)$ but when $\dim \mathrm{Fix}(\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_4 \times \mathbb{Z}_2^c) = 4$, $\dim \mathrm{Fix}(\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c) < 4$.

(c) $(\mathbf{D}_3 \times \mathbb{Z}_2^c, \mathbf{D}_3 \times \mathbb{Z}_2^c) \subset (\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c)$ and$(\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c)$ but when $\dim \mathrm{Fix}(\mathbf{D}_3 \times \mathbb{Z}_2^c, \mathbf{D}_3 \times \mathbb{Z}_2^c) = 4$, $\dim \mathrm{Fix}(\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c) < 4$ and $\dim \mathrm{Fix}(\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c) < 4$.

(d) $(\mathbf{D}_2 \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c) \subset (\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c)$, $(\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c)$, $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c)$ and $(\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c)$ but when $\dim \mathrm{Fix}(\mathbf{D}_2 \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c) = 4$, these larger groups have fixed-point subspaces of dimension less than 4.

Hence for all values of n, by the chain criterion, $(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c)$ gives a submaximal isotropy subgroup with four-dimensional fixed-point subspace for the values of ℓ in Table 4.6.

Similar arguments show that the twisted subgroups given by the pairs

$$(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n) \qquad (\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbb{Z}_n \times \mathbb{Z}_2^c) \qquad \text{and} \qquad (\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n^z)$$

are also submaximal isotropy subgroups with four-dimensional fixed-point subspace for the values of ℓ given in Table 4.6.

Finally we consider the twisted subgroups, H^{ψ_j}, where H is a cyclic subgroup of $\mathbf{O(3)}$.

(a) Let H^{ψ_j} be the twisted subgroup given by the pair $(\mathbb{Z}_{md} \times \mathbb{Z}_2^c, \mathbb{Z}_m \times \mathbb{Z}_2^c)$ with homomorphism ψ_j defined by (4.2.4). When $d > 3$ this subgroup is not contained in any other twisted subgroup with four-dimensional fixed-point subspace. It is contained in the $\mathbb{C}$-axial subgroup given by the pair $(\mathbf{SO(2)} \times \mathbb{Z}, \mathbb{Z}_m \times \mathbb{Z}_2^c)$ and hence is a submaximal isotropy subgroup with $\dim \mathrm{Fix}(H^{\psi_j}) = 4$ for all values of ℓ given in Table 4.6 and all homomorphisms ψ_j. When $d = 3$ and $m = 1$, j must be equal to 1 and $\dim \mathrm{Fix}(H^{\psi_1}) = 4$ when $\ell = 2$ or 3. So even though $(\mathbb{Z}_3 \times \mathbb{Z}_2^c, \mathbb{Z}_2^c) \subset (\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c)$, by the chain criterion, H^{ψ_1} is still a submaximal isotropy subgroup when $\ell = 2$ and 3 since $\dim \mathrm{Fix}(\mathbb{T} \times$

$\mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c) \leq 2$ for these values of ℓ. Similarly when $d = 2$, $j = 1$ and

$$\begin{aligned}(\mathbb{Z}_{2m} \times \mathbb{Z}_2^c, \mathbb{Z}_m \times \mathbb{Z}_2^c) &\subset (\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_m \times \mathbb{Z}_2^c)\\ (\mathbb{Z}_4 \times \mathbb{Z}_2^c, \mathbb{Z}_2 \times \mathbb{Z}_2^c) &\subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c)\end{aligned}$$

but the larger group always has a fixed-point subspace of dimension less than four for all values of ℓ where the smaller group has a fixed-point subspace of dimension four and hence by the chain criterion, H^{ψ_1} is a submaximal isotropy subgroup for all values of ℓ given in Table 4.6.

(b) Let H^{ψ_j} be the twisted subgroup given by the pair $(\mathbb{Z}_{(2d-1)m} \times \mathbb{Z}_2^c, \mathbb{Z}_m)$ with homomorphism ψ_j defined by (4.2.8). This never has four-dimensional fixed-point subspace when $d = 1$. For all other values of d

$$(\mathbb{Z}_{(2d-1)m} \times \mathbb{Z}_2^c, \mathbb{Z}_m) \subset (\mathbb{Z}_{2(2d-1)m} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-)$$

for any odd value of $j = 1, 3, \ldots, 2d-3$. However, the larger group always has a fixed-point subspace of dimension less than four for all values of ℓ where the smaller group has a fixed-point subspace of dimension four. Also when $d = 2$ and $m = 1$, $j = 1$ and

$$(\mathbb{Z}_3 \times \mathbb{Z}_2^c, \mathbb{1}) \subset (\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2)$$

however again the larger group always has a fixed-point subspace of dimension less than four for all values of ℓ where the smaller group has a fixed-point subspace of dimension four and hence by the chain criterion, H^{ψ_j} is a submaximal isotropy subgroup for all values of ℓ given in Table 4.6.

(c) Let H^{ψ_j} be the twisted subgroup given by the pair $(\mathbb{Z}_{2md} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-)$ with homomorphism ψ_j defined by (4.2.12). When $d = 1$

$$\begin{aligned}(\mathbb{Z}_{2m} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-) &\subset (\mathbb{Z}_{6m} \times \mathbb{Z}_2^c, \mathbb{Z}_{6m}^-)\\ (\mathbb{Z}_{2m} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-) &\subset (\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_{2m}^d)\\ (\mathbb{Z}_4 \times \mathbb{Z}_2^c, \mathbb{Z}_4^-) &\subset (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}^-)\\ (\mathbb{Z}_2 \times \mathbb{Z}_2^c, \mathbb{Z}_2^-) &\subset (\mathbf{D}_2 \times \mathbb{Z}_2^c, \mathbf{D}_2^z) \quad \text{and other larger groups}\end{aligned}$$

but the larger group always has a fixed-point subspace of dimension less than four for all values of ℓ where the smaller group has a fixed-point subspace of dimension four and hence by the chain criterion, H^{θ} is a submaximal isotropy subgroup for all values of ℓ given in Table 4.6.

This completes the proof. □

Conclusions

We have now computed the isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ with two and four dimensional fixed-point subspaces in the representations on $V_\ell \oplus V_\ell$ for all values of ℓ. The equivariant Hopf theorem guarantees the existence of branches of periodic solutions to (4.1.1) with the symmetries of the $\mathbb{C}$-axial subgroups at a Hopf bifurcation with $\mathbf{O}(\mathbf{3}) \times S^1$ symmetry. To determine whether these solution branches bifurcate supercritically or subcritically and whether the solutions can be stable we must compute, to high enough order, the Taylor expansion of the general form of the $\mathbf{O}(\mathbf{3}) \times S^1$ vector field for a specific value of ℓ. We also need this Taylor expansion to determine if it is possible for solutions with Σ symmetry to exist where Σ is a submaximal isotropy subgroup of $\mathbf{O}(\mathbf{3}) \times S^1$ with four-dimensional fixed-point subspace. One method of computing this Taylor expansion is discussed in the next section.

4.4 $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant mappings

In this section we discuss in general how to compute the Taylor expansion of a $\mathbf{O}(\mathbf{3})$ equivariant mapping $f(\mathbf{z})$ where $\mathbf{z} \in \mathbb{C}^{2\ell+1}$ is the vector which describes the amplitudes of the spherical harmonics of degree ℓ in the space $V_\ell \oplus V_\ell$ as in Section 4.1. Recall from Chapter 2 that by the notion of Birkhoff normal form the Taylor expansion of f to order k can be assumed to commute with $\mathbf{O}(\mathbf{3}) \times S^1$. Thus to compute the k^{th} order truncated Birkhoff normal form of f for the natural representation of $\mathbf{O}(\mathbf{3})$ on $V_\ell \oplus V_\ell$ it is equivalent to compute the k^{th} order Taylor expansion of a mapping which is equivariant with respect to $\mathbf{O}(\mathbf{3}) \times S^1$.

Throughout this section we will assume that $\mathbf{O}(\mathbf{3})$ acts on $\mathbf{z} \in \mathbb{C}^{2\ell+1}$ by the natural action of $\mathbf{O}(\mathbf{3})$ on V_ℓ. This means that elements of $\mathbf{O}(\mathbf{3})$ act on $\mathbf{z}$ by the same matrices as for V_ℓ i.e. those given in Section 3.2.2, and the phase shifts $\psi \in S^1$ act as scalar multiplication by $e^{i\psi}$.

Given a specific value of ℓ it is possible to compute the Taylor expansion of a $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant mapping $f(\mathbf{z})$ for the representation on $V_\ell \oplus V_\ell$ to any given order. However, it is not possible to carry out this computation for general ℓ. In this section we will discuss the method of computation for the representation of $\mathbf{O}(\mathbf{3}) \times S^1$ on $V_\ell \oplus V_\ell$ for a given value of ℓ and prove that two certain cubic maps are $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant for all values of ℓ. In Chapter 5 we will carry out the computation of the Taylor expansion of the $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant mapping, f, to cubic order for the action of $\mathbf{O}(\mathbf{3}) \times S^1$ on $V_3 \oplus V_3$.

4.4.1 Equivariant mappings on $\mathbb{C}^{2\ell+1}$

In this section we describe a method for computing the Taylor expansion of a $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant mapping $f(\mathbf{z})$ for the natural representation on $V_\ell \oplus V_\ell$ to any given order.

Recall that the mapping $f : \mathbb{C}^{2\ell+1} \to \mathbb{C}^{2\ell+1}$ is $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant if

$$f(\gamma \cdot \mathbf{z}) = \gamma \cdot f(\mathbf{z}) \qquad \forall \gamma \in \mathbf{O}(\mathbf{3}) \times S^1. \tag{4.4.1}$$

This mapping is in exact Birkhoff normal form. Here '$\cdot$' represents an action of $\mathbf{O}(\mathbf{3}) \times S^1$ on the vector

$$\mathbf{z} = (z_{-\ell}, z_{-\ell+1}, \ldots, z_\ell)^{\mathbf{T}} \in \mathbb{C}^{2\ell+1},$$

where '$\mathbf{T}$' denotes the transpose. The action, '$\cdot$', of $\gamma \in \mathbf{O}(\mathbf{3}) \times S^1$ on $\mathbf{z}$ is given by multiplication on the left by a $(2\ell+1)\times(2\ell+1)$ matrix, M_γ. Hence the mapping f must satisfy

$$f(M_\gamma \mathbf{z}) = M_\gamma f(\mathbf{z}) \qquad \forall \gamma \in \mathbf{O}(\mathbf{3}) \times S^1. \tag{4.4.2}$$

Let F_k denote the Taylor expansion of f to order k. In order to compute F_k we impose that (4.4.2) hold for the mapping $F_k = (F_{k,-\ell}, F_{k,-\ell+1}, \ldots, F_{k,\ell})^{\mathbf{T}}$ where each component $F_{k,r}$, $r = -\ell, \ldots, \ell$ is a linear combination of all possible terms in $z_{-\ell}, \ldots, z_\ell$ and their complex conjugates of order less than or equal to k. This will force the coefficients of some terms to be zero and others to occur in certain ratios. Thus F_k is a linear combination of a number of basis $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant mappings of order less than or equal to k.

We can note that it is sufficient to impose that F_k satisfies (4.4.2) for a set of generators of $\mathbf{O}(\mathbf{3}) \times S^1$. In Section 3.2.2 we computed the matrices, $M_{\phi'}$ and $M_{\theta'}$, for the action of the generators ϕ' and θ' which are infinitesimal rotations in the z- and y- axes respectively. Recall that

$$\begin{aligned} M_{\phi'} &= \operatorname{diag}\left(\mathrm{e}^{-\mathrm{i}\ell\phi'}, \mathrm{e}^{-\mathrm{i}(\ell-1)\phi'}, \ldots, \mathrm{e}^{\mathrm{i}(\ell-1)\phi'}, \mathrm{e}^{\mathrm{i}\ell\phi'}\right) && (4.4.3)\\ M_{\theta'} &= \left[\mathbf{v}^{\mathbf{T}}_{-\ell} \mid \mathbf{v}^{\mathbf{T}}_{-\ell+1} \mid \cdots \mid \mathbf{v}^{\mathbf{T}}_{\ell}\right] && (4.4.4)\end{aligned}$$

i.e. the matrix with columns $\mathbf{v}^{\mathbf{T}}_m$ where

$$\mathbf{v}_m = \left(0, \ldots, 0, -\frac{1}{2}\sqrt{(\ell+m)(\ell-m+1)}\theta', 1, \frac{1}{2}\sqrt{(\ell-m)(\ell+m+1)}\,\theta', 0, \ldots, 0\right)$$

for $m = -\ell, \ldots, \ell$ with the entry 1 lying in the m^{th} column.

In the natural representation on V_ℓ, the element $-I$ acts as the identity when ℓ is even and minus the identity when ℓ is odd. This means that $-I$ acts on $\mathbf{z} \in \mathbb{C}^{2\ell+1}$ by scalar multiplication by $(-1)^\ell$ or equivalently by multiplication by the matrix $M_{-I} = (-1)^\ell I_{2\ell+1}$ where $I_{2\ell+1}$ is the $(2\ell+1)\times(2\ell+1)$ identity matrix. Recall that in our representation the phase shifts $\psi \in S^1$ act as scalar multiplication by $\mathrm{e}^{\mathrm{i}\psi}$ or equivalently by multiplication by the matrix $M_\psi = \mathrm{e}^{\mathrm{i}\psi} I_{2\ell+1}$. Hence the $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant vector field f satisfies

$$f(\mathrm{e}^{\mathrm{i}\psi}\mathbf{z}) = \mathrm{e}^{\mathrm{i}\psi} f(\mathbf{z}) \qquad \forall \psi \in S^1.$$

This means that in the Taylor expansion F_k, each component $F_{k,r}$ contains only terms of odd order which are of the form

$$z_{j_1} z_{j_2} \cdots z_{j_p} \bar{z}_{j_{(p+1)}} \bar{z}_{j_{(p+2)}} \cdots \bar{z}_{j_{(2p-1)}} \qquad j_q \in -\ell, \ldots, \ell \qquad \text{for} \quad q = 1, \ldots, 2p-1. \tag{4.4.5}$$

Moreover, equivariance with respect to rotations ϕ' implies that the only terms in $F_{k,r}$ are those of the form (4.4.5) where $j_1 + j_2 + \cdots + j_p - j_{(p+1)} - \cdots - j_{(2p-1)} = r$.

We have seen that for any value of ℓ, the Taylor expansion to order k, F_k, of the $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant vector field, f, for the representation of $\mathbf{O}(\mathbf{3})$ on V_ℓ is a linear combination of a number of basis $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant mappings of order less than or equal to k. The basis mappings must be of odd order and contain only terms of the form given above.

We now show that two particular mappings are cubic $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant mappings for all values of ℓ.

4.4.2 Two cubic $\mathbf{O(3)} \times S^1$ equivariant mappings

Proposition 4.4.1. *Let $\mathbf{z} = (z_{-\ell}, z_{-\ell+1}, \ldots, z_\ell)^{\mathbf{T}} \in \mathbb{C}^{2\ell+1}$ be the vector which describes the amplitudes of the spherical harmonics of degree ℓ. The mappings*

$$\mathbf{P}_1(\mathbf{z}) = \mathbf{z}|\mathbf{z}|^2 \qquad \text{and} \qquad \mathbf{P}_2(\mathbf{z}) = \hat{\mathbf{z}}\left(z_0^2 + 2\sum_{m=1}^{\ell}(-1)^m z_m z_{-m}\right)$$

where

$$|\mathbf{z}|^2 = \sum_{m=-\ell}^{\ell}|z_m|^2 \qquad \text{and} \qquad \hat{\mathbf{z}} = \left((-1)^\ell \bar{z}_\ell,\ (-1)^{\ell-1}\bar{z}_{\ell-1},\ \ldots,\ (-1)^{-\ell}\bar{z}_{-\ell}\right)^{\mathbf{T}}$$

are cubic equivariant maps for all natural representations of $\mathbf{O(3)} \times S^1$.

Proof. We must show that for $j = 1$ and 2,

$$\mathbf{P}_j(\gamma \cdot \mathbf{z}) = \gamma \cdot \mathbf{P}_j(\mathbf{z}) \qquad \forall \gamma \in \mathbf{O(3)} \times S^1.$$

It is sufficient to show that this condition holds for the set of generators of $\mathbf{O(3)} \times S^1$ discussed in Section 3.2.2. Since these mappings are both cubic and each component contains only terms of the form given in (4.4.5) they are both equivariant with respect to the actions of the inversion element $-I$ and the phase shifts $\psi \in S^1$. It remains to show the equivariance of $\mathbf{P}_1$ and $\mathbf{P}_2$ with respect to the infinitesimal rotations ϕ' and θ'. The matrices $M_{\phi'}$ and $M_{\theta'}$ for the actions of these two infinitesimal rotations are given by (4.4.3) and (4.4.4). These matrices act by multiplication of $\mathbf{z}$ on the left.

Let $\mathbf{v}_k$ denote the entry in the k^{th} row of the column vector $\mathbf{v}$ where $k = -\ell, \ldots, \ell$. Then

$$\left[\mathbf{P}_1(M_{\phi'}\mathbf{z})\right]_k = \mathrm{e}^{\mathrm{i}k\phi'} z_k \sum_{m=-\ell}^{\ell} |\mathrm{e}^{\mathrm{i}m\phi'} z_m|^2 = \mathrm{e}^{\mathrm{i}k\phi'}\,[\mathbf{P}_1(\mathbf{z})]_k = \left[M_{\phi'}\mathbf{P}_1(\mathbf{z})\right]_k \qquad \forall k$$

$$\begin{aligned}
\left[\mathbf{P}_2(M_{\phi'}\mathbf{z})\right]_k &= (-1)^{-k}\mathrm{e}^{\mathrm{i}k\phi'}\bar{z}_{-k}\left(z_0^2 + 2\sum_{m=1}^{\ell}(-1)^m \mathrm{e}^{\mathrm{i}m\phi'} z_m \mathrm{e}^{-\mathrm{i}m\phi'} z_{-m}\right) \\
&= \mathrm{e}^{\mathrm{i}k\phi'}\,[\mathbf{P}_2(\mathbf{z})]_k = \left[M_{\phi'}\mathbf{P}_2(\mathbf{z})\right]_k \qquad \forall k.
\end{aligned}$$

We also have that

$$[M_{\theta'}\mathbf{z}]_k = \frac{1}{2}\sqrt{(\ell - k + 1)(\ell + k)}\theta' z_{k-1} + z_k - \frac{1}{2}\sqrt{(\ell + k + 1)(\ell - k)}\theta' z_{k+1}$$

and subsequently in the limit $\theta' \to 0$

$$\begin{aligned}
[\mathbf{P}_1(M_{\theta'}\mathbf{z})]_k &= [M_{\theta'}\mathbf{z}]_k \left\{ \sum_{m=-\ell}^{\ell} \left| \frac{1}{2}\sqrt{(\ell - m + 1)(\ell + m)}\theta' z_{m-1} + z_m - \right.\right. \\
&\qquad\qquad \left.\left. \frac{1}{2}\sqrt{(\ell + m + 1)(\ell - m)}\theta' z_{m+1} \right|^2 \right\} \\
&= [M_{\theta'} \cdot \mathbf{z}]_k \left\{ \sum_{m=-\ell}^{\ell} z_m \bar{z}_m + \frac{1}{2}\sqrt{(\ell - m + 1)(\ell + m)}\theta'(z_{m-1}\bar{z}_m + \bar{z}_{m-1}z_m) \right. \\
&\qquad\qquad \left. -\frac{1}{2}\sqrt{(\ell + m + 1)(\ell - m)}\theta'(z_{m+1}\bar{z}_m + \bar{z}_{m+1}z_m) \right\} \\
&= [M_{\theta'} \cdot \mathbf{z}]_k \sum_{m=-\ell}^{\ell} |z_m|^2 \qquad (4.4.6)
\end{aligned}$$

since the other terms in the sum cancel out by telescoping. We also compute that

$$[M_{\theta'}\mathbf{P}_1(\mathbf{z})]_k \quad = \quad [M_{\theta'}\mathbf{z}]_k \sum_{m=-\ell}^{\ell} |z_m|^2 \; = \; [\mathbf{P}_1(M_{\theta'}\mathbf{z})]_k \qquad \forall k.$$

and hence the map $\mathbf{P}_1$ is equivariant with respect to the action of the infinitesimal rotation θ'.

Similarly

$$[\mathbf{P}_2(M_{\theta'}\mathbf{z})]_k \quad = \quad (-1)^k \left[\overline{M_{\theta'}\mathbf{z}}\right]_{-k} \left\{ \frac{1}{2}\left([M_{\theta'}\mathbf{z}]_0\right)^2 + \sum_{m=1}^{\ell} (-1)^m \left[M_{\theta'}\mathbf{z}\right]_m \left[M_{\theta'}\mathbf{z}\right]_{-m} \right\}$$

and

$$[M_{\theta'}\mathbf{P}_2(\mathbf{z})]_k = (-1)^k \left[\overline{M_{\theta'}\mathbf{z}}\right]_{-k} \left\{ \frac{1}{2} z_0^2 + \sum_{m=1}^{\ell} (-1)^m z_m z_{-m} \right\}.$$

But since θ' is infinitesimal in the limit $\theta' \to 0$

$$\begin{aligned} \frac{1}{2}\left([M_{\theta'} \cdot \mathbf{z}]_0\right)^2 + \sum_{m=1}^{\ell} (-1)^m \left[M_{\theta'}\mathbf{z}\right]_m \left[M_{\theta'}\mathbf{z}\right]_{-m} = \frac{1}{2}\left(z_0^2 + \sqrt{\ell(\ell+1)}\theta'(z_{-1}z_0 - z_1 z_0) \right) \\ + \sum_{m=1}^{\ell} (-1)^m \left\{ z_m z_{-m} + \frac{1}{2}\sqrt{(\ell - m + 1)(\ell + m)}\theta'(z_{m-1}z_{-m} - z_{-m+1}z_m) \right. \\ \left. + \frac{1}{2}\sqrt{(\ell + m + 1)(\ell - m)}\theta'(z_{-m-1}z_m - z_{m+1}z_{-m}) \right\}. \end{aligned}$$

Telescoping this sum leaves us with

$$\frac{1}{2}\left([M_{\theta'}\mathbf{z}]_0\right)^2 + \sum_{m=1}^{\ell} (-1)^m \left[M_{\theta'}\mathbf{z}\right]_m \left[M_{\theta'}\mathbf{z}\right]_{-m} = \frac{1}{2} z_0^2 + \sum_{m=1}^{\ell} (-1)^m z_m z_{-m}$$

and hence

$$[\mathbf{P}_2(M_{\theta'}\mathbf{z})]_k = [M_{\theta'}\mathbf{P}_2(\mathbf{z})]_k \qquad \forall k.$$

The map $\mathbf{P}_2$ is equivariant with respect to the action of the infinitesimal rotation θ'.

We have shown that $\mathbf{P}_1$ and $\mathbf{P}_2$ are both equivariant with respect to a set of generators of $\mathbf{O(3)} \times S^1$ for the representation on V_ℓ for any value of ℓ and hence they are $\mathbf{O(3)} \times S^1$ equivariant mappings for all representations. $\square$

CHAPTER 5

HOPF BIFURCATION ON A SPHERE

THE NATURAL REPRESENTATION ON $V_3 \oplus V_3$

5.1 Introduction

In this chapter we investigate the dynamics which can occur near a Hopf bifurcation with spherical symmetry for a specific representation of the group $\mathbf{O(3)}$. Throughout this chapter we assume that $\mathbf{O(3)} \times S^1$ acts on the $\mathbf{O(3)}$-simple space $V_3 \oplus V_3$ where $\mathbf{O(3)}$ acts absolutely irreducibly on the spherical harmonics of degree three, V_3, by the natural representation. The natural representation on V_3 is the minus representation where $-I \in \mathbf{O(3)}$ acts as minus the identity. Recall that the action of $\mathbf{O(3)}$ on $V_3 \oplus V_3$ is given by the action of $\mathbf{O(3)}$ on a vector

$$\mathbf{z}(t) = (z_{-3},\, z_{-2},\, z_{-1},\, z_0,\, z_1,\, z_2,\, z_3)^{\mathbf{T}} \in \mathbb{C}^7,$$

which describes the amplitudes of the spherical harmonics of degree three as in Section 4.1.

Consider the system of ODEs given by

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = f(\mathbf{z}, \lambda), \tag{5.1.1}$$

where $\lambda \in \mathbb{R}$ is a bifurcation parameter, $\mathbf{z} \in \mathbb{C}^7$ and $f : \mathbb{C}^7 \times \mathbb{R} \to \mathbb{C}^7$ is a smooth mapping which commutes with the action of $\mathbf{O(3)}$ on $V_3 \oplus V_3$. Using the notion of Birkhoff normal form, we can assume that for any positive integer, k, the Taylor expansion of f to order k, which we denote by F_k, is equivariant with respect to the action of $\mathbf{O(3)} \times S^1$ on $V_3 \oplus V_3$. In Section 5.2 we use the method given in Section 4.4 to compute F_3, the Taylor expansion of f to order 3 which commutes with the action of $\mathbf{O(3)} \times S^1$ on $V_3 \oplus V_3$.

Recall that under certain hypotheses, the equivariant Hopf theorem, Theorem 2.5.2, guarantees that if $\Sigma \subset \mathbf{O(3)} \times S^1$ is an isotropy subgroup satisfying $\dim \mathrm{Fix}(\Sigma) = 2$ in the representation on $V_3 \oplus V_3$ then there is a unique branch of periodic solutions to (5.1.1) with period near 2π bifurcating from the Hopf bifurcation point at the origin with Σ as their group of symmetries.

In Chapter 4 we computed the $\mathbb{C}$-axial isotropy subgroups of $\mathbf{O(3)} \times S^1$ for all representations of $\mathbf{O(3)} \times S^1$ on $V_\ell \oplus V_\ell$ as well as the isotropy subgroups which fix a four-dimensional subspace. In Section 5.3 we will consider all of the isotropy subgroups of $\mathbf{O(3)} \times S^1$ for the representation on $V_3 \oplus V_3$ and give images of the periodic solution branches with the symmetries of the $\mathbb{C}$-axial subgroups, whose existence is guaranteed by the equivariant Hopf theorem. In Section 5.4 we investigate the stability of these solution branches using the vector field computed in Section 5.2.

Depending on the values of coefficients in the $\mathbf{O(3)} \times S^1$ equivariant vector field f it is possible for (5.1.1) to admit solutions with symmetry Σ where Σ is an isotropy subgroup with $\dim \mathrm{Fix}(\Sigma) > 2$. In Section 5.5 we will investigate the conditions under which solutions to (5.1.1) with symmetry Σ where $\dim \mathrm{Fix}(\Sigma) = 4$ can exist. It turns out that in this case all isotropy subgroups, Σ, with four-dimensional fixed-point subspaces are submaximal isotropy subgroups – that is, they are contained within a $\mathbb{C}$-axial subgroup. We refer to a solution with submaximal symmetry as a submaximal solution and solutions with $\mathbb{C}$-axial symmetry are refered to as maximal solutions (since all the maximal isotropy subgroups are $\mathbb{C}$-axial in this representation).

5.2 The $\mathbf{O(3)} \times S^1$ equivariant vector field

In this section we compute the Taylor expansion of the general $\mathbf{O(3)}$ equivariant vector field, f, for the action of $\mathbf{O(3)}$ on $V_3 \oplus V_3$. Due to the notion of Birkhoff normal form we can assume that the Taylor expansion also commutes with the action of S^1 to any given order k. Here we compute the Taylor expansion of f to cubic order. We denote this Taylor expansion by F_3. It must satisfy

$$F_3(M_\gamma \mathbf{z}) = M_\gamma F_3(\mathbf{z}) \qquad \forall \gamma \in \mathbf{O(3)} \times S^1 \tag{5.2.1}$$

where

$$\mathbf{z}(t) = (z_{-3},\, z_{-2},\, z_{-1},\, z_0,\, z_1,\, z_2,\, z_3)^{\mathbf{T}} \in \mathbb{C}^7,$$

is the vector which describes the amplitudes of the spherical harmonics of degree 3 and the group of 7×7 matrices M_γ which give the action of $\mathbf{O(3)}$ on $\mathbf{z}$ is generated by the matrices $M_{\phi'}$, $M_{\theta'}$, M_{-I} and M_ψ which are defined for general values of ℓ in Section 3.2.2. We find that the generating matrices for the action of $\mathbf{O(3)}$ on $\mathbf{z} \in \mathbb{C}^7$ are

$$M_{\phi'} = \mathrm{diag}\left(\mathrm{e}^{-3\mathrm{i}\phi'}, \mathrm{e}^{-2\mathrm{i}\phi'}, \mathrm{e}^{-\mathrm{i}\phi'}, 1, \mathrm{e}^{\mathrm{i}\phi'}, \mathrm{e}^{2\mathrm{i}\phi'}, \mathrm{e}^{3\mathrm{i}\phi'}\right) \tag{5.2.2}$$

$$M_{-I} = -I_7 \tag{5.2.3}$$

$$M_\psi = \mathrm{e}^{\mathrm{i}\psi} I_7 \tag{5.2.4}$$

$$M_{\theta'} = \begin{bmatrix} 1 & -\sqrt{\frac{3}{2}}\theta' & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}}\theta' & 1 & -\sqrt{\frac{5}{2}}\theta' & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}}\theta' & 1 & -\sqrt{3}\theta' & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3}\theta' & 1 & -\sqrt{3}\theta' & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3}\theta' & 1 & -\sqrt{\frac{5}{2}}\theta' & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}}\theta' & 1 & -\sqrt{\frac{3}{2}}\theta' \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}}\theta' & 1 \end{bmatrix} \tag{5.2.5}$$

where diag $(\ldots)$ indicates a diagonal matrix with elements as listed. By imposing that F_3 satisfy (5.2.1) for this set of generating matrices, we find that the general form of a cubic vector field which commutes with the action of $\mathbf{O(3)} \times S^1$ as above is

$$F_3(\mathbf{z}, \lambda) = \mu \mathbf{z} + A\ \mathbf{z}|\mathbf{z}|^2 + B\ \mathbf{P}(\mathbf{z})\hat{\mathbf{z}} + C\ \mathbf{Q}(\mathbf{z}) + D\ \mathbf{R}(\mathbf{z}) \tag{5.2.6}$$

where $\mu,\ A,\ B,\ C,\ D$ are smooth complex-valued functions of λ and

$$\begin{aligned} |\mathbf{z}|^2 &= |z_{-3}|^2 + |z_{-2}|^2 + |z_{-1}|^2 + |z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 \\ \mathbf{P}(\mathbf{z}) &= z_0^2 - 2z_{-1}z_1 + 2z_{-2}z_2 - 2z_{-3}z_3 \\ \hat{\mathbf{z}} &= (-\bar{z}_3, \bar{z}_2, -\bar{z}_1, \bar{z}_0, -\bar{z}_{-1}, \bar{z}_{-2}, -\bar{z}_{-3})^{\mathbf{T}} \\ \mathbf{Q}(\mathbf{z}) &= (\mathbf{Q}_{-3},\ \mathbf{Q}_{-2},\ \mathbf{Q}_{-1},\ \mathbf{Q}_0,\ \mathbf{Q}_1,\ \mathbf{Q}_2,\ \mathbf{Q}_3)^{\mathbf{T}} \\ \mathbf{R}(\mathbf{z}) &= (\mathbf{R}_{-3},\ \mathbf{R}_{-2},\ \mathbf{R}_{-1},\ \mathbf{R}_0,\ \mathbf{R}_1,\ \mathbf{R}_2,\ \mathbf{R}_3)^{\mathbf{T}}, \end{aligned}$$

where $\mathbf{Q}_m(\tilde{\mathbf{z}}) = \mathbf{Q}_{-m}(\mathbf{z})$, $\mathbf{R}_m(\tilde{\mathbf{z}}) = \mathbf{R}_{-m}(\mathbf{z})$, with $\tilde{\mathbf{z}} = (z_3,\ z_2,\ z_1,\ z_0,\ z_{-1},\ z_{-2},\ z_{-3})^{\mathbf{T}}$ and

$$\begin{aligned} \mathbf{Q}_{-3}(\mathbf{z}) &= 5z_{-3}(\,5|z_{-3}|^2 + 5|z_{-2}|^2 - |z_{-1}|^2 - 4|z_0|^2 - 5|z_1|^2 - 5|z_2|^2 - 8|z_3|^2) \\ &\quad + 5\bar{z}_3\left(2z_0^2 - 3z_1z_{-1} + 3z_2z_{-2}\right) + \sqrt{15}\left(2z_{-1}^2\bar{z}_1 + 5z_{-2}^2\bar{z}_{-1}\right) \\ &\quad + 5\sqrt{2}\,(z_0z_{-1}\bar{z}_2 + z_0z_{-2}\bar{z}_1 + 3z_{-2}z_{-1}\bar{z}_0) \\ \mathbf{Q}_{-2}(\mathbf{z}) &= 5z_{-2}(\,5|z_{-3}|^2 + 3|z_{-1}|^2 - 3|z_1|^2 - 8|z_2|^2 - 5|z_3|^2) + 4\sqrt{30}z_{-1}z_0\bar{z}_1 \\ &\quad + 5\bar{z}_2\,(5z_1z_{-1} + 3z_3z_{-3}) + 10\sqrt{15}z_{-1}z_{-3}\bar{z}_{-2} + 3\sqrt{30}z_{-1}^2\bar{z}_0 \\ &\quad + 5\sqrt{2}\,(z_1z_{-3}\bar{z}_0 + z_0z_1\bar{z}_3 + 3z_0z_{-3}\bar{z}_{-1}) \\ \mathbf{Q}_{-1}(\mathbf{z}) &= z_{-1}\left(-5|z_{-3}|^2 + 15|z_{-2}|^2 - 3|z_{-1}|^2 + 12|z_0|^2 - 16|z_1|^2 - 15|z_2|^2 - 25|z_3|^2\right) \\ &\quad + \bar{z}_1\left(24z_0^2 + 25z_2z_{-2} - 15z_3z_{-3}\right) + \sqrt{15}\left(4z_1z_{-3}\bar{z}_{-1} + 2z_1^2\bar{z}_3 + 5z_{-2}^2\bar{z}_{-3}\right) \\ &\quad + 5\sqrt{2}\,(z_2z_{-3}\bar{z}_0 + z_0z_2\bar{z}_3 + 3z_0z_{-3}\bar{z}_{-2}) \\ &\quad + 2\sqrt{30}\,(3z_{-2}z_0\bar{z}_{-1} + 2z_{-2}z_1\bar{z}_0 + 2z_1z_0\bar{z}_2) \\ \mathbf{Q}_0(\mathbf{z}) &= z_0\left(-20|z_{-3}|^2 + 12|z_{-1}|^2 - 12|z_0|^2 + 12|z_1|^2 - 20|z_3|^2\right) \\ &\quad + 4\bar{z}_0\,(12z_1z_{-1} + 5z_3z_{-3}) + 15\sqrt{2}\,(z_1z_2\bar{z}_3 + z_{-2}z_{-1}\bar{z}_{-3}) \\ &\quad + 5\sqrt{2}\,(z_1z_{-3}\bar{z}_{-2} + z_2z_{-3}\bar{z}_{-1} + z_3z_{-2}\bar{z}_1 + z_3z_{-1}\bar{z}_2) \\ &\quad + \sqrt{30}\left(4z_{-2}z_1\bar{z}_{-1} + 4z_2z_{-1}\bar{z}_1 + 3z_1^2\bar{z}_2 + 3z_{-1}^2\bar{z}_{-2}\right) \end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{-3}(\mathbf{z}) &= 3z_{-3}(\,3|z_{-3}|^2+3|z_{-2}|^2+|z_{-1}|^2-|z_1|^2-2|z_2|^2-3|z_3|^2)+3z_{-2}z_2\bar{z}_3\\
&\quad+3\sqrt{2}\,(z_0z_{-2}\bar{z}_1+z_{-2}z_{-1}\bar{z}_0)+\sqrt{15}\left(z_{-2}^2\bar{z}_{-1}+z_1z_{-2}\bar{z}_2\right)\\
\mathbf{R}_{-2}(\mathbf{z}) &= z_{-2}\left(9|z_{-3}|^2+4|z_{-2}|^2+7|z_{-1}|^2-2|z_1|^2-4|z_2|^2-6|z_3|^2\right)+3z_{-3}z_3\bar{z}_2\\
&\quad+3\sqrt{2}\,(z_0z_{-3}\bar{z}_1+z_1z_{-3}\bar{z}_0)+5z_{-1}z_1\bar{z}_2+\sqrt{30}\left(z_{-1}^2\bar{z}_0+z_{-1}z_0\bar{z}_1\right)\\
&\quad+\sqrt{15}\,(z_{-3}z_2\bar{z}_1+z_2z_{-1}\bar{z}_3+2z_{-1}z_{-3}\bar{z}_{-2})\\
\mathbf{R}_{-1}(\mathbf{z}) &= z_{-1}\left(3|z_{-3}|^2+7|z_{-2}|^2+|z_{-1}|^2+6|z_0|^2-|z_1|^2-2|z_2|^2-3|z_3|^2\right)+6z_0^2\bar{z}_1\\
&\quad+3\sqrt{2}\,(z_0z_{-3}\bar{z}_{-2}+z_0z_2\bar{z}_3)+\sqrt{30}\,(2z_{-2}z_0\bar{z}_{-1}+z_{-2}z_1\bar{z}_0+z_0z_1\bar{z}_2)\\
&\quad+\sqrt{15}\left(z_{-2}^2\bar{z}_{-3}+z_{-2}z_3\bar{z}_2\right)+5z_{-2}z_2\bar{z}_1\\
\mathbf{R}_0(\mathbf{z}) &= 6z_0\left(|z_{-1}|^2+|z_1|^2\right)+3\sqrt{2}\,(z_{-3}z_1\bar{z}_{-2}+z_1z_2\bar{z}_3+z_{-2}z_{-1}\bar{z}_{-3}+z_{-1}z_3\bar{z}_2)\\
&\quad+12z_1z_{-1}\bar{z}_0+\sqrt{30}\left(z_{-2}z_1\bar{z}_{-1}+z_1^2\bar{z}_2+z_{-1}^2\bar{z}_{-2}+z_{-1}z_2\bar{z}_1\right)
\end{aligned}$$

Remark 5.2.1. We have found four cubic $\mathbf{O(3)}\times S^1$ equivariant maps for this representation. This is in agreement with the results of the computations of the number of equivariants by Antoneli et al. [4].

In Section 5.3 we will use the Taylor expansion, F_3, to determine conditions on the coefficients A, B, C and D for the maximal solutions to (5.1.1) to be stable. It will also be used in Section 5.5 to discover when branches of submaximal solutions can exist.

5.3 Isotropy subgroups and maximal solution branches

From Table 4.5 we can see that in the natural representation on $V_3 \oplus V_3$, $\mathbf{O(3)}\times S^1$ has six $\mathbb{C}$-axial isotropy subgroups. Thus, by the equivariant Hopf theorem, (5.1.1) is guaranteed to have a branch of periodic solutions with each of these symmetry groups. In this section we compute one possible form of the fixed-point subspace of each of the six $\mathbb{C}$-axial isotropy subgroups and use this information to discover what the solutions with these symmetries look like. We also find the isotropy subgroups, Σ with fixed-point subspaces of dimension larger than two by using Theorem 4.3.6 when $\dim \mathrm{Fix}(\Sigma) = 4$ and the chain criterion when $\dim \mathrm{Fix}(\Sigma) > 4$.

5.3.1 Isotropy subgroups and their fixed-point subspaces

We now compute all isotropy subgroups of $\mathbf{O(3)}\times S^1$ in the representation on $V_3 \oplus V_3$ and one possible form of their fixed-point subspaces.

Proposition 5.3.1. *The isotropy subgroups of $\mathbf{O(3)}\times S^1$ for the representation on $V_3 \oplus V_3$ and one possible form of their fixed-point subspaces are as given in Table 5.1. Also listed is $N(\Sigma)/\Sigma$ for each isotropy subgroup Σ where*

$$N(\Sigma) = \{\gamma \in \mathbf{O(3)}\times S^1 : \gamma\Sigma\gamma^{-1} = \Sigma\}$$

is the normaliser of Σ *in* $\mathbf{O(3)} \times S^1$.

Σ	J	K	$\theta(H)$	$\mathrm{Fix}(\Sigma)$	$\dim \mathrm{Fix}(\Sigma)$	$N(\Sigma)/\Sigma$
$\widetilde{\mathbf{O(2)}}$	$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$	$\{(0,0,0,w_1,0,0,0)\}$	2	S^1
$\widetilde{\mathbf{SO(2)}}_1$	$\mathbf{SO(2)}$	$\mathbb{Z}_2^-$	S^1	$\{(0,0,w_1,0,0,0,0)\}$	2	S^1
$\widetilde{\mathbf{SO(2)}}_2$	$\mathbf{SO(2)}$	$\mathbb{Z}_4^-$	S^1	$\{(0,w_1,0,0,0,0,0)\}$	2	S^1
$\widetilde{\mathbf{SO(2)}}_3$	$\mathbf{SO(2)}$	$\mathbb{Z}_6^-$	S^1	$\{(w_1,0,0,0,0,0,0)\}$	2	S^1
$\widetilde{\mathbb{O}}$	$\mathbb{O}$	$\mathbb{O}^-$	$\mathbb{Z}_2$	$\{(0,w_1,0,0,0,-w_1,0)\}$	2	S^1
$\widetilde{\mathbf{D}_6}$	$\mathbf{D}_6$	$\mathbf{D}_6^d$	$\mathbb{Z}_2$	$\{(w_1,0,0,0,0,0,-w_1)\}$	2	S^1
$\widetilde{\mathbb{Z}}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_6^-$	$\mathbb{Z}_2$	$\{(w_1,0,0,0,0,0,w_2)\}$	4	$\mathbf{O(2)} \times S^1$
$\widetilde{\mathbb{Z}}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4^-$	$\mathbb{Z}_2$	$\{(0,w_1,0,0,0,w_2,0)\}$	4	$\mathbf{O(2)} \times S^1$
$\widetilde{\mathbf{D}_3}$	$\mathbf{D}_3$	$\mathbf{D}_3^z$	$\mathbb{Z}_2$	$\{(w_1,0,0,w_2,0,0,-w_1)\}$	4	$\mathbf{D}_2 \times S^1$
$\widetilde{\mathbf{D}_2}$	$\mathbf{D}_2$	$\mathbf{D}_2^z$	$\mathbb{Z}_2$	$\{(0,w_1,0,w_2,0,w_1,0)\}$	4	$\mathbf{D}_2 \times S^1$
$\widetilde{\mathbb{Z}}_3^1$	$\mathbb{Z}_3$	$\mathbb{1}$	$\mathbb{Z}_6$	$\{(0,0,w_1,0,0,w_2,0)\}$	4	$\mathbf{SO(2)} \times S^1$
$\widetilde{\mathbb{Z}}_5$	$\mathbb{Z}_5$	$\mathbb{1}$	$\mathbb{Z}_{10}$[*]	$\{(w_1,0,0,0,0,w_2,0)\}$	4	$\mathbf{SO(2)} \times S^1$
$\widetilde{\mathbb{Z}}_3^2$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\{(w_1,0,0,w_2,0,0,w_3)\}$	6	$\mathbf{O(2)} \times S^1$
$\widetilde{\mathbb{Z}}_2^1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\{(0,w_1,0,w_2,0,w_3,0)\}$	6	$\mathbf{O(2)} \times S^1$
$\widetilde{\mathbb{Z}}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2$	$\{(w_1,0,w_2,0,w_3,0,w_4)\}$	8	$\mathbf{O(2)} \times S^1$
$\widetilde{\mathbb{1}}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{Z}_2$	$\mathbb{C}^7$	14	$\mathbf{O(3)} \times S^1$

Table 5.1: The isotropy subgroups Σ of $\mathbf{O(3)} \times S^1$ for the representation on $V_3 \oplus V_3$. Here $H = J \times \mathbb{Z}_2^c$ and all vectors a column vectors. [*] The homomorphism $\theta : H \to H/K$ is given by (4.2.8) with $b = 5$ and $j = 3$.

Figure 5.1 shows the partial ordering of the conjugacy classes of isotropy subgroups for this representation.

Remark 5.3.2. We use the notation of Golubitsky et al. [46], whereby $\Sigma = \widetilde{J}$ is an isotropy subgroup with $H = J \times \mathbb{Z}_2^c$ and $\theta(H)$ a nontrivial subgroup of S^1. A subscript or superscript is added in cases of ambiguity. Notice also that since H is a class II subgroup of $\mathbf{O(3)}$ and K is a class I or III subgroup for all isotropy subgroups in this representation, $\theta(H) \neq \mathbb{1}$.

Remark 5.3.3. The final column in Table 5.1 lists the group $N(\Sigma)/\Sigma$ for each isotropy subgroup Σ, where $N(\Sigma)$ is the normaliser of Σ in $\mathbf{O(3)} \times S^1$ which leaves $\mathrm{Fix}(\Sigma)$ invariant. Recall from Section 2.4.1 that since the vector field f in (5.1.1) is equivariant with respect to the action of $\mathbf{O(3)} \times S^1$ to any order k, in the restriction to $\mathrm{Fix}(\Sigma)$, f restricts to a $N(\Sigma)/\Sigma$ equivariant system to order k. Considering the restriction of f to $\mathrm{Fix}(\Sigma)$ enables us to deduce information about the possible existence and bifurcations of periodic and quasiperiodic solutions with submaximal symmetry, i.e. the symmetries of isotropy subgroups with $\dim \mathrm{Fix}(\Sigma) > 2$. In Section 5.5 we investigate possible submaximal solutions in $\mathrm{Fix}(\Sigma)$ for the isotropy subgroups Σ in Table 5.1 with $\dim \mathrm{Fix}(\Sigma) = 4$.

Proof of Proposition 5.3.1 The isotropy subgroups of $\mathbf{O(3)} \times S^1$ with fixed-point subspaces of dimensions 2 and 4 in the natural representation on $V_3 \oplus V_3$ are given by Theorems 4.3.3 and 4.3.6 respectively. Using Table 4.2 we find that when $\ell = 3$ the only twisted subgroups, $H^\theta \subset \mathbf{O(3)} \times S^1$, with fixed-point subspaces of dimension greater than 4 are those listed in

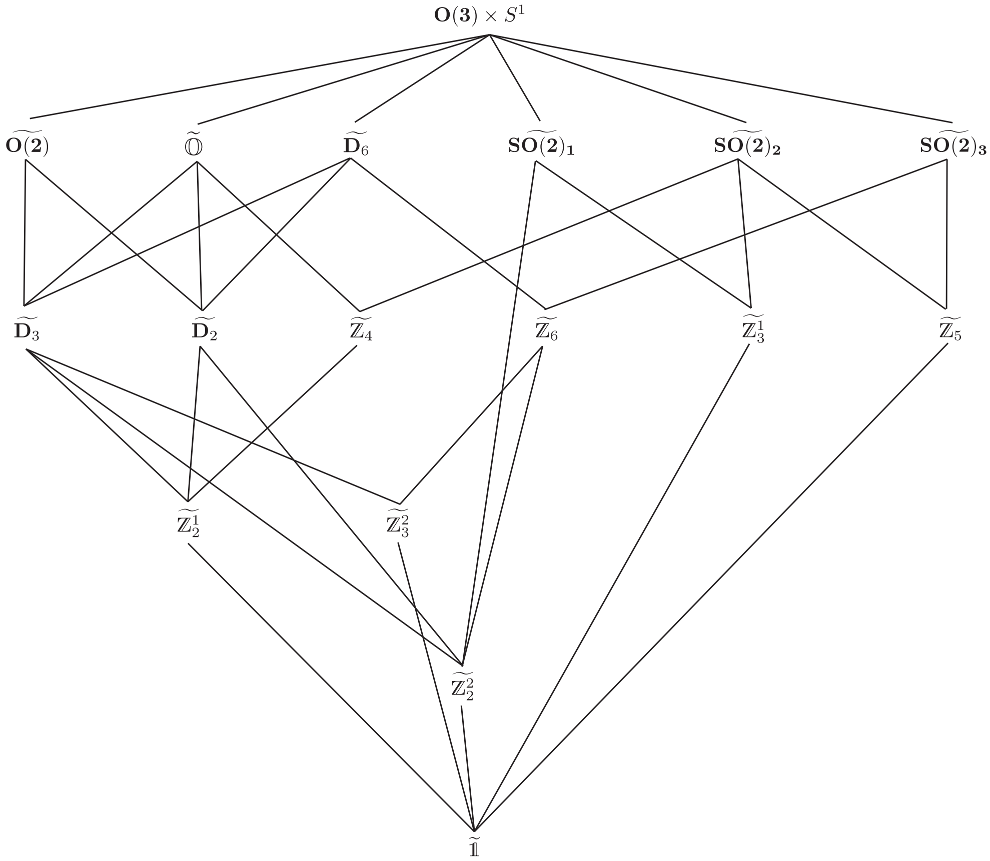


Figure 5.1: The partial ordering of conjugacy classes of isotropy subgroups of $\mathbf{O(3)} \times S^1$ in the representation on $V_3 \oplus V_3$.

Table 5.1. Using the chain criterion it can be seen that each of these twisted subgroups is an isotropy subgroup.

It remains to show that the form of the fixed-point subspace for each isotropy subgroup in Table 5.1 is correct for one set of generators of the subgroup. In Table 5.2 we list the set of generators for each isotropy subgroup which give the fixed-point subspaces in Table 5.1.

The action of each of the generators, $\sigma \in \Sigma$ on $\mathbf{z}$ is given by multiplication by the matrix M_σ where

- $(R^z_\alpha, 0)$ is a rotation through an angle α in the z-axis with
$$M_{(R^z_\alpha,0)} = \operatorname{diag}\left(\mathrm{e}^{-3\alpha\mathrm{i}}, \mathrm{e}^{-2\alpha\mathrm{i}}, \mathrm{e}^{-\alpha\mathrm{i}}, 1, \mathrm{e}^{\alpha\mathrm{i}}, \mathrm{e}^{2\alpha\mathrm{i}}, \mathrm{e}^{3\alpha\mathrm{i}}\right)$$
where diag $(\ldots)$ indicates a diagonal matrix with elements as listed.
- $(0, \alpha)$ is a phase shift by α with $M_{(0,\alpha)} = \mathrm{e}^{\alpha\mathrm{i}} I_7$.
- $(\kappa_{xz}, 0)$ is reflection in the xz-plane which sends $y \to -y$. The matrix for its action on

Σ	Generators	Fix(Σ)
$\widetilde{\mathbf{O}(\mathbf{2})}$	$(R^z_\alpha, 0)$, $(\kappa_{xz}, 0)$, $(-I, \pi)$	$\{(0,0,0,w_1,0,0,0)\}$
$\widetilde{\mathbf{SO}(\mathbf{2})}_1$	(R^z_α, α), $(-I, \pi)$	$\{(0,0,w_1,0,0,0,0)\}$
$\widetilde{\mathbf{SO}(\mathbf{2})}_2$	$(R^z_\alpha, 2\alpha)$, $(-I, \pi)$	$\{(0,w_1,0,0,0,0,0)\}$
$\widetilde{\mathbf{SO}(\mathbf{2})}_3$	$(R^z_\alpha, 3\alpha)$, $(-I, \pi)$	$\{(w_1,0,0,0,0,0,0)\}$
$\widetilde{\mathbb{O}}$	$(-R^z_{\pi/2}, 0)$, $(\mathcal{R}_{2\pi/3}, 0)$, $(\kappa_{x=y}, 0)$, $(-I, \pi)$	$\{(0,w_1,0,0,0,-w_1,0)\}$
$\widetilde{\mathbf{D}}_6$	$(-R^z_{\pi/3}, 0)$, $(R^x_\pi, 0)$, $(-I, \pi)$	$\{(w_1,0,0,0,0,0,-w_1)\}$
$\widetilde{\mathbb{Z}}_6$	$(-R^z_{\pi/3}, 0)$, $(-I, \pi)$	$\{(w_1,0,0,0,0,0,w_2)\}$
$\widetilde{\mathbb{Z}}_4$	$(-R^z_{\pi/2}, 0)$, $(-I, \pi)$	$\{(0,w_1,0,0,0,w_2,0)\}$
$\widetilde{\mathbf{D}}_3$	$(R^z_{2\pi/3}, 0)$, $(\kappa_{xz}, 0)$, $(-I, \pi)$	$\{(w_1,0,0,w_2,0,0,-w_1)\}$
$\widetilde{\mathbf{D}}_2$	$(R^z_\pi, 0)$, $(\kappa_{xz}, 0)$, $(-I, \pi)$	$\{(0,w_1,0,w_2,0,w_1,0)\}$
$\widetilde{\mathbb{Z}}^1_3$	$(R^z_{2\pi/3}, 2\pi/3)$, $(-I, \pi)$	$\{(0,0,w_1,0,0,w_2,0)\}$
$\widetilde{\mathbb{Z}}_5$	$(R^z_{2\pi/5}, 6\pi/5)$, $(-I, \pi)$	$\{(w_1,0,0,0,0,w_2,0)\}$
$\widetilde{\mathbb{Z}}^2_3$	$(R^z_{2\pi/3}, 0)$, $(-I, \pi)$	$\{(w_1,0,0,w_2,0,0,w_3)\}$
$\widetilde{\mathbb{Z}}^1_2$	$(R^z_\pi, 0)$, $(-I, \pi)$	$\{(0,w_1,0,w_2,0,w_3,0)\}$
$\widetilde{\mathbb{Z}}^2_2$	$(-R^z_\pi, 0)$, $(-I, \pi)$	$\{(w_1,0,w_2,0,w_3,0,w_4)\}$
$\widetilde{\mathbb{1}}$	$(-I, \pi)$	$\mathbb{C}^7$

Table 5.2: One set of generators for each isotropy subgroup of $\mathbf{O}(\mathbf{3}) \times S^1$ in the representation on $V_3 \oplus V_3$ and the corresponding fixed-point subspace.

$\mathbf{z} \in \mathbb{C}^7$ is

$$M_{(\kappa_{xz},0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $(R^x_\pi, 0)$ is rotation through π in the x-axis which sends $y \to -y$ and $z \to -z$. The matrix for its action on $\mathbf{z} \in \mathbb{C}^7$ is

$$M_{(R^x_\pi,0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $(\kappa_{x=y}, 0)$ is a reflection in the plane where $x = y$ which sends $x \leftrightarrow y$. The matrix for its action on $\mathbf{z} \in \mathbb{C}^7$ is

$$M_{(\kappa_{x=y},0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- $(\mathcal{R}_{2\pi/3}, 0)$ is a rotation through $2\pi/3$ in the line $x = y = z$ which sends $x \to y \to z \to x$. We do not require the matrix for the action of this transformation on $\mathbf{z} \in \mathbb{C}^7$ since the other generators of $\widetilde{\mathbb{O}}$ are sufficient to compute the form of $\mathrm{Fix}(\widetilde{\mathbb{O}})$ in this representation.

To see that these generators give the fixed-point subspaces shown, observe that for the action of each generator, $\sigma \in \Sigma$, on $\mathbf{w} \in \mathrm{Fix}(\Sigma)$ we have $M_\sigma \mathbf{w} = \mathbf{w}$.

5.3.2 Maximal solution branches

For each of the $\mathbb{C}$-axial isotropy subgroups, Σ, in Table 5.1 it is possible to give images of the branch of periodic solutions to (5.1.1) with symmetry Σ. Solution branches with these symmetries are guaranteed to exist by the equivariant Hopf theorem.

Suppose that $w(\theta, \phi, t)$ is a time-dependent function on a sphere which can be written as a linear combination of the spherical harmonics of degree 3. Then

$$\begin{aligned} w(\theta,\phi,t) &= \sum_{m=-3}^{3} z_m(t) Y_3^m(\theta,\phi) + c.c. \\ &= \sum_{m=-3}^{3} \left(z_m(t) + (-1)^m \overline{z}_{-m}(t)\right) Y_3^m(\theta,\phi) \end{aligned}$$

by (3.2.2). Notice that $w(\theta, \phi, t)$ is real. If $w(\theta, \phi, t)$ has symmetry Σ then

$$\sigma \mathbf{z} = \mathbf{z}, \qquad \forall \sigma \in \Sigma \qquad \Rightarrow \qquad \mathbf{z} \in \mathrm{Fix}(\Sigma)$$

where

$$\mathbf{z}(t) = (z_{-3},\ z_{-2},\ z_{-1},\ z_0,\ z_1,\ z_2,\ z_3)^{\mathbf{T}} \in \mathbb{C}^7.$$

Assume that $\mathbf{z}(t)$ is a periodic solution of (5.1.1) which has $\mathbb{C}$-axial symmetry. Using the forms of the fixed-point subspaces of each of the $\mathbb{C}$-axial isotropy subgroups given in Table 5.1 we can compute the form of the periodic solution pattern $w(\theta, \phi, t)$ corresponding to each $\mathbb{C}$-axial isotropy subgroup. In each case the vector $\mathbf{z}(t) \in \mathrm{Fix}(\Sigma)$ depends only on $w_1(t) \in \mathbb{C}$. We can assume that $w_1(t) = R\mathrm{e}^{\mathrm{i}\omega t}$ where $R \in \mathbb{R}$ is constant and $\omega = 2\pi/T$, so $w(\theta, \phi, t)$ has period T.

(a) If $\mathbf{z}(t) \in \mathrm{Fix}(\widetilde{\mathbf{O(2)}})$ then

$$\begin{aligned} w(\theta,\phi,t) &= (w_1(t) + \overline{w}_1(t))\, Y_3^0(\theta,\phi) \ = \ 2\,\mathrm{Re}(w_1(t))\, Y_3^0(\theta,\phi) \\ &= \frac{R}{2}\sqrt{\frac{7}{\pi}} \left(5\cos^3\theta - 3\cos\theta\right) \cos(\omega t). \end{aligned}$$

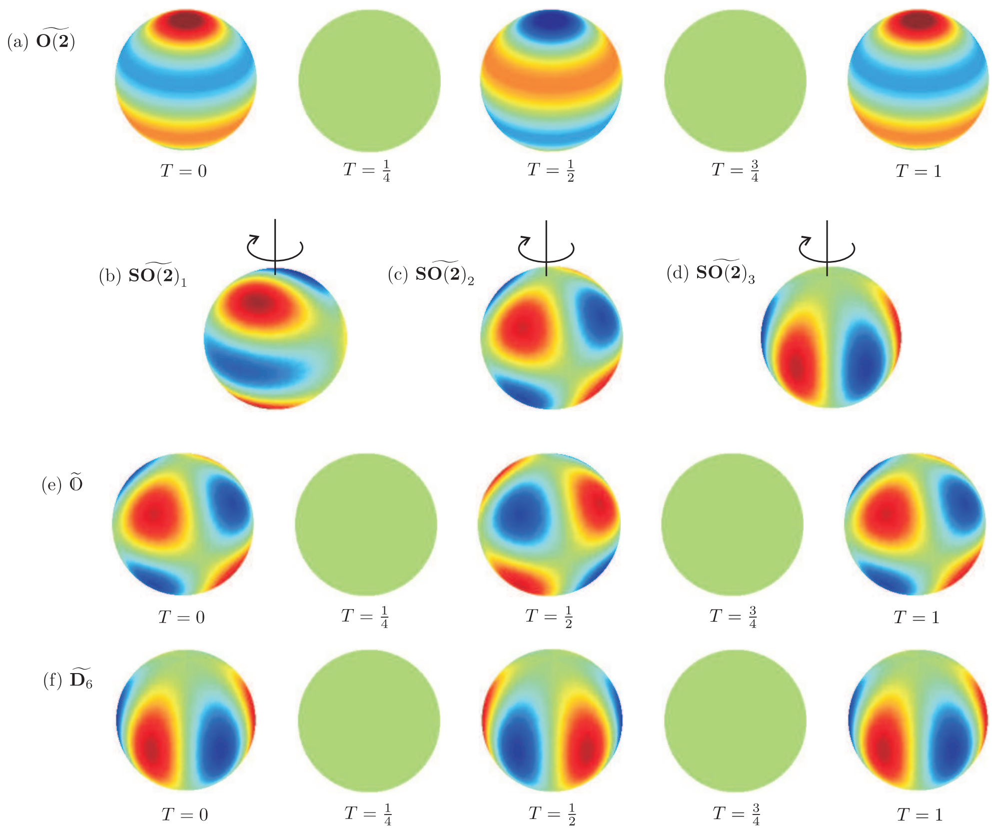


Figure 5.2: The six periodic solution branches with $\mathbb{C}$-axial symmetry. (a), (e) and (f) illustrate the evolution of the three standing waves over one period and (b), (c) and (d) illustrate the travelling wave solutions showing the axis and direction of rotation. Red areas show where the solution is positive and blue areas where the solution is negative.

This is a standing wave solution and it is drawn in Figure 5.2(a) for values of t over one period.

(b) If $\mathbf{z}(t) \in \mathrm{Fix}(\widetilde{\mathbf{SO(2)}_1})$ then

$$\begin{aligned} w(\theta,\phi,t) &= w_1(t)Y_3^{-1}(\theta,\phi) - \overline{w}_1(t)Y_3^1(\theta,\phi) \;=\; 2\,\mathrm{Re}\left(w_1(t)Y_3^{-1}(\theta,\phi)\right) \\ &= \frac{R}{4}\sqrt{\frac{21}{\pi}}\sin\theta\left(5\cos^2\theta - 1\right)\cos(\omega t - \phi). \end{aligned}$$

This is a travelling wave solution and it is drawn in Figure 5.2(b) where the direction the pattern travels around the sphere is shown.

(c) If $\mathbf{z}(t) \in \mathrm{Fix}(\widetilde{\mathbf{SO(2)}_2})$ then

$$\begin{aligned} w(\theta,\phi,t) &= w_1(t)Y_3^{-2}(\theta,\phi) + \overline{w}_1(t)Y_3^2(\theta,\phi) \;=\; 2\,\mathrm{Re}\left(w_1(t)Y_3^{-2}(\theta,\phi)\right) \\ &= \frac{R}{2}\sqrt{\frac{105}{\pi}}\sin^2\theta\cos\theta\cos(\omega t - 2\phi). \end{aligned}$$

This is a travelling wave solution and it is drawn in Figure 5.2(c) where the direction the pattern travels around the sphere is shown.

(d) If $\mathbf{z}(t) \in \mathrm{Fix}(\widetilde{\mathbf{SO(2)}}_3)$ then

$$\begin{aligned} w(\theta,\phi,t) &= w_1(t)Y_3^{-3}(\theta,\phi) - \overline{w}_1(t)Y_3^3(\theta,\phi) \;=\; 2\,\mathrm{Re}\left(w_1(t)Y_3^{-3}(\theta,\phi)\right) \\ &= \frac{R}{4}\sqrt{\frac{35}{\pi}}\sin^3\theta\cos(\omega t - 3\phi). \end{aligned}$$

This is a travelling wave solution and it is drawn in Figure 5.2(d) where the direction the pattern travels around the sphere is shown.

(e) If $\mathbf{z}(t) \in \mathrm{Fix}(\widetilde{\mathbb{O}})$ then

$$\begin{aligned} w(\theta,\phi,t) &= \left(w_1(t) - \overline{w}_1(t)\right)Y_3^{-2}(\theta,\phi) + \left(-w_1(t) + \overline{w}_1(t)\right)Y_3^2(\theta,\phi) \\ &= 2\mathrm{i}\;\mathrm{Im}\left(w_1(t)\right)\left(Y_3^{-2}(\theta,\phi) - Y_3^2(\theta,\phi)\right) \\ &= R\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta\sin(2\phi)\sin(\omega t). \end{aligned}$$

This is a standing wave solution and it is drawn in Figure 5.2(e) for values of t over one period.

(f) If $\mathbf{z}(t) \in \mathrm{Fix}(\widetilde{\mathbf{D}}_6)$ then

$$\begin{aligned} w(\theta,\phi,t) &= \left(w_1(t) + \overline{w}_1(t)\right)Y_3^{-3}(\theta,\phi) - \left(w_1(t) + \overline{w}_1(t)\right)Y_3^3(\theta,\phi) \\ &= 2\,\mathrm{Re}(w_1(t))\;\left(Y_3^{-3}(\theta,\phi) - Y_3^3(\theta,\phi)\right) \\ &= \frac{R}{2}\sqrt{\frac{35}{\pi}}\sin^3\theta\cos(3\phi)\cos(\omega t). \end{aligned}$$

This is a standing wave solution and it is drawn in Figure 5.2(f) for values of t over one period.

In the next section we will compute when these solution branches can be stable.

5.4 Stability of maximal solution branches

Recall that in Chapter 4 we found that the equivariant Hopf theorem guarantees that (5.1.1) has six branches of periodic solutions with the symmetries of the $\mathbb{C}$-axial subgroups of $\mathbf{O(3)} \times S^1$ given in Table 5.1. Images of these solutions were given in Section 5.3. In Section 5.2 we computed F_3, the Taylor expansion to cubic order of a vector field which is equivariant with respect to the action of $\mathbf{O(3)} \times S^1$ on $V_3 \oplus V_3$. This Taylor expansion is given by (5.2.6). In this section we will use F_3 and the isotypic decomposition of V_3 for the action of each of the $\mathbb{C}$-axial subgroups to determine the branching direction and stability of each of the six periodic solutions.

Stability of periodic solutions with $\mathbb{C}$-axial symmetry

In order to meet the conditions of the equivariant Hopf theorem we assume that $\mu(\lambda) \in \mathbb{C}$ in (5.2.6) satisfies $\mu(0) = \mathrm{i}$ and $\mathrm{Re}(\mu'(0)) \neq 0$. We will assume that

$$\mathrm{Re}(\mu(\lambda)) = \lambda + \text{ higher order terms in } \lambda,$$

so the trivial solution $\mathbf{z} = 0$ is stable for $\lambda < 0$ and unstable for $\lambda > 0$. This means that for a branch of solutions bifurcating from the trivial solution at $\lambda = 0$ to be stable it must bifurcate supercritically. To determine the dependence of the direction of branching on the coefficients A, B, C and D in (5.2.6), for each periodic solution we compute the branching equation by restricting (5.2.6) to $\mathrm{Fix}(\Sigma)$ for each of the corresponding $\mathbb{C}$-axial subgroups Σ. We do this in Section 5.4.1.

Suppose that $\mathbf{z}(t)$ is a periodic solution of (5.1.1) with $\mathbb{C}$-axial symmetry group Σ which has p-determined stability. By Theorem 2.5.9, the stability of the this periodic solution near $\mathbf{0}$ can be determined by the expressions for the stability of the periodic solution of

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = F_k(\mathbf{z}, \lambda) \tag{5.4.1}$$

with the same symmetry Σ, where the k^{th} order Taylor expansion, F_k, commutes with $\mathbf{O}(\mathbf{3}) \times S^1$ and $k \geq p$. We will see that each of the $\mathbb{C}$-axial subgroups in Table 5.1 has 3-determined stability so we can use F_3 to determine the stability of all of the maximal periodic solution branches.

The equivariant Hopf theorem states that the bifurcating branches of periodic solutions, $\mathbf{z}(t)$, of (5.1.1) have period near 2π. Suppose then that they have period $\frac{2\pi}{1+\tau}$ so that τ is the period perturbing parameter. Then by Theorem 2.5.4, and the fact that the symmetry group, Σ, of $\mathbf{z}(t)$ has 3-determined stability, the periodic solutions of

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = F_3(\mathbf{z}, \lambda) \tag{5.4.2}$$

with period $\frac{2\pi}{1+\tau}$ are in one–to–one correspondence with the zeroes of

$$g(\mathbf{z}, \lambda, \tau) = F_3(\mathbf{z}, \lambda) - (1+\tau)\mathrm{i}\mathbf{z}. \tag{5.4.3}$$

Moreover, the expressions in terms of the coefficients A, B, C, and D in F_3 which determine the stability of the periodic solutions of (5.4.2) determine also the stability of the periodic solution of (5.1.1) with the same symmetry. If $\mathbf{z}(t)$ is a periodic solution of (5.4.2) with $\lambda = \lambda_0$ and $\tau = \tau_0$ then the corresponding solution to (5.4.3) is $(\mathbf{z}_0, \lambda_0, \tau_0)$. By Corollary 2.5.6, since the Floquet multipliers of $\mathbf{z}(t)$ correspond to the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$, the periodic solution $\mathbf{z}(t)$ is stable if the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ which are not forced to be zero by symmetry have negative real part.

Computing eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$

The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ are expressions in terms of the coefficients A, B, C, and D in F_3. Our task is to compute these eigenvalues for each of the periodic solutions with $\mathbb{C}$-axial symmetry. Throughout this section we will use the subscripts r and i on the coefficients A, B, C, and D to denote the real and imaginary parts respectively.

For our representation of $\mathbf{O}(\mathbf{3}) \times S^1$ on $V_3 \oplus V_3$, the Jacobian $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for each periodic solution is a 14×14 matrix. It is most convenient for us to choose as basis functions for $\mathbb{C}^7$ the coordinate functions

$$z_{-3},\ \bar{z}_{-3},\ z_{-2},\ \bar{z}_{-2},\ z_{-1},\ \bar{z}_{-1},\ z_0,\ \bar{z}_0,\ z_1,\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ z_3,\ \bar{z}_3. \tag{5.4.4}$$

With this basis the Jacobian is given by

$$(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \begin{pmatrix} M_{(-3,-3)} & M_{(-3,-2)} & M_{(-3,-1)} & M_{(-3,0)} & M_{(-3,1)} & M_{(-3,2)} & M_{(-3,3)} \\ M_{(-2,-3)} & M_{(-2,-2)} & M_{(-2,-1)} & M_{(-2,0)} & M_{(-2,1)} & M_{(-2,2)} & M_{(-2,3)} \\ M_{(-1,-3)} & M_{(-1,-2)} & M_{(-1,-1)} & M_{(-1,0)} & M_{(-1,1)} & M_{(-1,2)} & M_{(-1,3)} \\ M_{(0,-3)} & M_{(0,-2)} & M_{(0,-1)} & M_{(0,0)} & M_{(0,1)} & M_{(0,2)} & M_{(0,3)} \\ M_{(1,-3)} & M_{(1,-2)} & M_{(1,-1)} & M_{(1,0)} & M_{(1,1)} & M_{(1,2)} & M_{(1,3)} \\ M_{(2,-3)} & M_{(2,-2)} & M_{(2,-1)} & M_{(2,0)} & M_{(2,1)} & M_{(2,2)} & M_{(2,3)} \\ M_{(3,-3)} & M_{(3,-2)} & M_{(3,-1)} & M_{(3,0)} & M_{(3,1)} & M_{(3,2)} & M_{(3,3)} \end{pmatrix} \tag{5.4.5}$$

where

$$M_{(i,j)} = \begin{pmatrix} m_{(i,j)} & m'_{(i,j)} \\ \overline{m'_{(i,j)}} & \overline{m_{(i,j)}} \end{pmatrix}$$

and

$$m_{(i,j)} = \frac{\partial g_i}{\partial z_j} \qquad m'_{(i,j)} = \frac{\partial g_i}{\partial \bar{z}_j}$$

where the derivatives are evaluated at $(\mathbf{z}_0, \lambda_0, \tau_0)$. Suppose that

$$\mathbf{a} = (a_{-3},\ a_{-2},\ a_{-1},\ a_0,\ a_1,\ a_2,\ a_3)^{\mathbf{T}} \in \mathbb{C}^7$$

is an eigenvector of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$. With respect to the basis (5.4.4) this vector is

$$(a_{-3},\ \bar{a}_{-3},\ a_{-2},\ \bar{a}_{-2},\ a_{-1},\ \bar{a}_{-1},\ a_0,\ \bar{a}_0,\ a_1,\ \bar{a}_1,\ a_2,\ \bar{a}_2,\ a_3\ \bar{a}_3,\ ).$$

We can simplify the computation of the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in two ways.

1. We can use the isotypic decomposition of V_3 for the action of the corresponding $\mathbb{C}$-axial subgroup Σ. This allows us to block diagonalise the matrix.
2. We can compute the zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$. Knowing which isotypic components the zero eigenvalues lie in will help us to compute the other eigenvalues in those components.

Zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$

For a solution $(\mathbf{z}_0, \lambda_0, \tau_0)$ with symmetry Σ, the number of distinct zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ is given by

$$d_\Sigma = \dim(\mathbf{O(3)}) + 1 - \dim(\Sigma) = 4 - \dim(\Sigma).$$

Since $\dim(\mathbf{O(3)} \times S^1) = 4$, as discussed in Section 2.4.4, there are four smooth curves

$$y_j(s) = \gamma_j(s)\mathbf{z}_0, \qquad j = 1, 2, 3, 4$$

in the group orbit $(\mathbf{O(3)} \times S^1)\mathbf{z}_0$ where $\gamma_j(s)$ is a smooth curve in $\mathbf{O(3)} \times S^1$ with $\gamma(0) = 1$. For each of these curves,

$$\left.\frac{\mathrm{d}\gamma_j}{\mathrm{d}s}\right|_{s=0} \mathbf{z}_0$$

is an eigenvector of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ with zero eigenvalue. Of these vectors, only $d_\Sigma = 4 - \dim(\Sigma)$ are distinct.

We now discuss the four smooth curves in $\mathbf{O(3)} \times S^1$ which give rise to these zero eigenvectors. The three smooth curves in $\mathbf{O(3)}$ can be thought of rotations in the z-,y- and x-axes respectively.

Infinitesimal rotations in the z-axis are the given by the matrix $M_{\phi'}$ as in (3.2.6). Let the curve given by rotating a point about the z-axis be $y_1(s)$ then

$$\gamma_1(s) = \mathrm{diag}\left(\mathrm{e}^{-3s\mathrm{i}},\ \mathrm{e}^{-2s\mathrm{i}},\ \mathrm{e}^{-s\mathrm{i}},\ 1,\ \mathrm{e}^{s\mathrm{i}},\ \mathrm{e}^{2s\mathrm{i}},\ \mathrm{e}^{3s\mathrm{i}}\right)$$

and so

$$\left.\frac{\mathrm{d}\gamma_1}{\mathrm{d}s}\right|_{s=0} = \mathrm{diag}\left(-3\mathrm{i},\ -2\mathrm{i},\ -\mathrm{i},\ 0,\ \mathrm{i},\ 2\mathrm{i},\ 3\mathrm{i}\right). \tag{5.4.6}$$

Similarly, infinitesimal rotations in the y-axis are the given by the matrix $M_{\theta'}$ as in (3.2.7). Let the curve given by rotating a point $(\theta, 0)$ an infinitesimal amount about the y-axis be $y_2(s)$ then

$$\gamma_2(s) = \begin{bmatrix} 1 & -\sqrt{\frac{3}{2}}s & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}}s & 1 & -\sqrt{\frac{5}{2}}s & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}}s & 1 & -\sqrt{3}s & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3}s & 1 & -\sqrt{3}s & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3}s & 1 & -\sqrt{\frac{5}{2}}s & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}}s & 1 & -\sqrt{\frac{3}{2}}s \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}}s & 1 \end{bmatrix}$$

and so

$$\left.\frac{\mathrm{d}\gamma_2}{\mathrm{d}s}\right|_{s=0} = \begin{bmatrix} 0 & -\sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & -\sqrt{\frac{5}{2}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}} & 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & -\sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} & 0 & -\sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 \end{bmatrix} \tag{5.4.7}$$

Finally, the curve given by rotating a point $(\theta, \pi/2)$ about the x-axis, $y_3(s)$ can be found by rotating the curve $y_2(s)$ through an angle $\pi/2$ in the z-axis. We find that

$$\gamma_3(s) = \begin{bmatrix} 1 & \sqrt{\frac{3}{2}}s & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}}s & 1 & \sqrt{\frac{5}{2}}s & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}}s & 1 & \sqrt{3}s & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3}s & 1 & \sqrt{3}s & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3}s & 1 & \sqrt{\frac{5}{2}}s & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}}s & 1 & \sqrt{\frac{3}{2}}s \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}}s & 1 \end{bmatrix}$$

and so

$$\left.\frac{\mathrm{d}\gamma_3}{\mathrm{d}s}\right|_{s=0} = \begin{bmatrix} 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{5}{2}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}} & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} & 0 & \sqrt{\frac{3}{2}} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 \end{bmatrix} \tag{5.4.8}$$

The smooth curve in S^1 is given by $\gamma_4(s) = \mathrm{e}^{\mathrm{i}s} I_7$ where I_7 is the 7×7 identity matrix. Hence

$$\left.\frac{\mathrm{d}\gamma_4}{\mathrm{d}s}\right|_{s=0} = \mathrm{i}I_7. \tag{5.4.9}$$

Zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ can then be found for the solution $(\mathbf{z}_0, \lambda_0, \tau_0)$ with Σ symmetry by multiplying (5.4.6)–(5.4.9) by a vector $\mathbf{z}_0 \in \mathrm{Fix}(\Sigma)$.

5.4.1 Branching equations

For each of the branches of periodic solutions $\mathbf{z}(t)$ of (5.4.2) we now compute the equation which determines whether the branch bifurcates subcritically or supercritically.

Suppose that $(\mathbf{z}_0, \lambda, \tau)$ satisfies $g(\mathbf{z}_0, \lambda, \tau) = 0$ and $\mathbf{z}_0 \in \mathrm{Fix}(\Sigma)$ for some $\mathbb{C}$-axial subgroup Σ. Then in the restriction of (5.4.3) to $\mathrm{Fix}(\Sigma)$,

$$0 = F_3(\mathbf{z}_0, \lambda) - (1+\tau)\mathrm{i}\mathbf{z}_0.$$

Since w_1 is the only non-zero entry in the vector $\mathbf{z}_0 \in \mathrm{Fix}(\Sigma)$, this implies that

$$0 = \mu(\lambda)w_1 + h_\Sigma(A,B,C,D)w_1|w_1|^2 - (1+\tau)\mathrm{i}w_1 \tag{5.4.10}$$

where $h_\Sigma(A,B,C,D)$ is a (real) linear combination of the coefficients in (5.2.6) which is different for each $\mathbb{C}$-axial group Σ. If we define $\nu(\lambda) = \mu(\lambda) - (1+\tau)\mathrm{i}$ then dividing (5.4.10) by w_1 we have the branching equation for the solution with Σ symmetry:

$$0 = \nu(\lambda) + h_\Sigma(A,B,C,D)|w_1|^2. \tag{5.4.11}$$

We can compute that the branching equations for each the solutions with $\mathbb{C}$-axial symmetry are as given in Table 5.3.

Notice that since $\mathrm{Re}(\mu(\lambda)) = \lambda$ to linear order in λ, we have $\mathrm{Re}(\nu(\lambda)) = \lambda$ to linear order also. Taking the real part of the branching equation (5.4.11) we have

$$\lambda = -\mathrm{Re}\ (h_\Sigma(A,B,C,D))|w_1|^2. \tag{5.4.12}$$

Recall that in order for the branch of solutions to be stable the branch must bifurcate supercritically. The branch of periodic solutions with symmetry Σ bifurcates supercritically when $\lambda > 0$. This occurs when $\mathrm{Re}\ (h_\Sigma(A,B,C,D)) < 0$.

Σ	Branching equation	Real part of branching equation
$\widetilde{\mathbf{O}(\mathbf{2})}$	$0 = \nu(\lambda) + (A + B - 12C)\|w_1\|^2$	$\lambda = -(A_r + B_r - 12C_r)\|w_1\|^2$
$\widetilde{\mathbf{SO}(\mathbf{2})}_1$	$0 = \nu(\lambda) + (A - 3C + D)\|w_1\|^2$	$\lambda = -(A_r - 3C_r + D_r)\|w_1\|^2$
$\widetilde{\mathbf{SO}(\mathbf{2})}_2$	$0 = \nu(\lambda) + (A + 4D)\|w_1\|^2$	$\lambda = -(A_r + 4D_r)\|w_1\|^2$
$\widetilde{\mathbf{SO}(\mathbf{2})}_3$	$0 = \nu(\lambda) + (A + 25C + 9D)\|w_1\|^2$	$\lambda = -(A_r + 25C_r + 9D_r)\|w_1\|^2$
$\widetilde{\mathbb{O}}$	$0 = \nu(\lambda) + (2A + 2B - 40C)\|w_1\|^2$	$\lambda = -(2A_r + 2B_r - 40C_r)\|w_1\|^2$
$\widetilde{\mathbf{D}_6}$	$0 = \nu(\lambda) + (2A + 2B - 15C)\|w_1\|^2$	$\lambda = -(2A_r + 2B_r - 15C_r)\|w_1\|^2$

Table 5.3: Branching equations for each of the six bifurcating branches of periodic solutions. For the branch to bifurcate supercritically we require that $\lambda > 0$.

5.4.2 Eigenvalues of the solution branches

We now compute the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for each of the branches of periodic solutions with $\mathbb{C}$-axial symmetry group, Σ, in turn using the isotypic decomposition of V_3 for the action of Σ and the cubic order truncation of the $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant vector field given by (5.2.6).

The $\widetilde{\mathbf{O}(\mathbf{2})}$ symmetric branch

We will compute the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for the periodic solution with $\widetilde{\mathbf{O}(\mathbf{2})}$ symmetry. Since

$$d_{\widetilde{\mathbf{O}(\mathbf{2})}} = \dim(\mathbf{O}(\mathbf{3})) + 1 - \dim(\widetilde{\mathbf{O}(\mathbf{2})}) = 3$$

we expect to find that $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ has three zero eigenvalues.

The isotypic decomposition of V_3 for the action of $\widetilde{\mathbf{O}(\mathbf{2})}$: The subspace

$$W_0 = \{(0,0,0,u_1,0,0,0)\} = \mathrm{Fix}(\widetilde{\mathbf{O}(\mathbf{2})})$$

is an isotypic component since it is the subspace on which $\widetilde{\mathbf{O}(\mathbf{2})}$ acts trivially. This corresponds to the trivial representation of $\widetilde{\mathbf{O}(\mathbf{2})}$. The action of $\widetilde{\mathbf{O}(\mathbf{2})}$ for the other irreducible representations is given by

$$\begin{aligned} \phi \cdot (z_{-j},\, z_j) &= \left(\mathrm{e}^{-\mathrm{i}j\phi} z_{-j},\, \mathrm{e}^{\mathrm{i}j\phi} z_j\right) \qquad \phi \in \mathbf{SO}(\mathbf{2}) \\ \kappa_{xz} \cdot (z_{-j},\, z_j) &= (-1)^j \left(z_j,\, z_{-j}\right) \end{aligned}$$

for $j \geq 1$. The representation on V_3 is a sum of the trivial representation on W_0 and the representations above for $j = 1, 2$ and 3 on the subspaces W_1, W_2 and W_3 respectively where

$$\begin{aligned} W_1 &= \{(0,0,u_1,0,u_2,0,0)\} \\ W_2 &= \{(0,u_1,0,0,0,u_2,0)\} \\ W_3 &= \{(u_1,0,0,0,0,0,u_2)\}. \end{aligned}$$

Thus the isotypic decomposition of V_3 for the action of $\widetilde{\mathbf{O}(\mathbf{2})}$ is

$$V_3 = W_0 \oplus W_1 \oplus W_2 \oplus W_3.$$

Zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$**:** The three zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ are found by multiplying (5.4.7)–(5.4.9) by a vector $(0,0,0,w_1,0,0,0) \in \mathrm{Fix}(\widetilde{\mathbf{O(2)}})$. This gives

$$\begin{aligned}
\mathbf{a}_2(w_1) &= (0,0,-w_1,0,w_1,0,0)^{\mathbf{T}} \in W_1 \\
\mathbf{a}_3(w_1) &= (0,0,w_1,0,w_1,0,0)^{\mathbf{T}} \in W_1 \\
\mathbf{a}_4(w_1) &= (0,0,0,\mathrm{i}w_1,0,0,0)^{\mathbf{T}} \in W_0
\end{aligned}$$

for the curves γ_2, γ_3 and γ_4 respectively.

Block diagonalised form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$**:** Using the isotypic decomposition of V_3 above we can block diagonalise $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ by reordering the basis functions of V_3 and thus the basis coordinate functions as

$$z_0,\ \bar{z}_0,\ z_{-1},\ \bar{z}_{-1},\ z_1,\ \bar{z}_1,\ z_{-2},\ \bar{z}_{-2},\ z_2,\ \bar{z}_2,\ z_{-3},\ \bar{z}_{-3},\ z_3\ \bar{z}_3. \tag{5.4.13}$$

This gives the block diagonal form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ as

$$(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \begin{pmatrix}
M_{(0,0)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & M_{(-1,-1)} & M_{(-1,1)} & 0 & 0 & 0 & 0 \\
0 & M_{(1,-1)} & M_{(1,1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{(-2,-2)} & M_{(-2,2)} & 0 & 0 \\
0 & 0 & 0 & M_{(2,-2)} & M_{(2,2)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & M_{(-3,-3)} & M_{(-3,3)} \\
0 & 0 & 0 & 0 & 0 & M_{(3,-3)} & M_{(3,3)}
\end{pmatrix}.$$

Eigenvalues in W_0**:** The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_0 are given by the eigenvalues of

$$M_{(0,0)} = \begin{pmatrix} m_{(0,0)} & m'_{(0,0)} \\ \overline{m'_{(0,0)}} & \overline{m_{(0,0)}} \end{pmatrix}.$$

Since W_0 contains the zero eigenvector $\mathbf{a}_4$, $M_{(0,0)}$ has a zero eigenvalue and the other eigenvalue is given by $2\mathrm{Re}(m_{(0,0)})$ where

$$m_{(0,0)} = \left.\frac{\partial g_0}{\partial z_0}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (2A+2B-24C)\,|w_1|^2 = (A+B-12C)\,|w_1|^2$$

using the branching equation. Thus the eigenvalues in W_0 are

$$\xi_0^+ = (2A_r + 2B_r - 24C_r)\,|w_1|^2 = -2\lambda \qquad \text{and} \qquad \xi_0^- = 0.$$

Eigenvalues in W_j**,** $j = 1,2,3$ **:** The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_j are given by the eigenvalues of

$$\mathbf{M}_j = \begin{pmatrix} M_{(-j,-j)} & M_{(-j,j)} \\ M_{(j,-j)} & M_{(j,j)} \end{pmatrix}.$$

Since $g_j(\mathbf{z}) = g_{-j}(\tilde{\mathbf{z}})$ where $\tilde{\mathbf{z}} = (z_3, z_2, z_1, z_0, z_{-1}, z_{-2}, z_{-3})$ it follows that for this solution $M_{(-j,-j)} = M_{(j,j)}$ and $M_{(-j,j)} = M_{(j,-j)}$. Also, since g_{-j} cannot contain any terms in $\bar{z}_{-j}z_0^2$ or $z_j|z_0|^2$, we must have that

$$m'_{(-j,-j)} = \left.\frac{\partial g_{-j}}{\partial \bar{z}_{-j}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0 \qquad \text{and} \qquad m_{(-j,j)} = \left.\frac{\partial g_{-j}}{\partial z_j}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0$$

and hence

$$\mathbf{M}_j = \begin{pmatrix} m_{(-j,-j)} & 0 & 0 & m'_{(-j,j)} \\ 0 & \overline{m_{(-j,-j)}} & \overline{m'_{(-j,j)}} & 0 \\ 0 & m'_{(-j,j)} & m_{(-j,-j)} & 0 \\ \overline{m'_{(-j,j)}} & 0 & 0 & \overline{m_{(-j,-j)}} \end{pmatrix}.$$

We can see that the eigenvalues of $\mathbf{M}_j$ are double and given by the eigenvalues of

$$E_j = \begin{pmatrix} m_{(-j,-j)} & m'_{(-j,j)} \\ \overline{m'_{(-j,j)}} & \overline{m_{(-j,-j)}} \end{pmatrix}.$$

When $j = 1$, since W_1 contains the zero eigenvectors $\mathbf{a}_2$ and $\mathbf{a}_3$, we know that one pair of double eigenvalues of $\mathbf{M}_1$ are zero and the other is given by $2\mathrm{Re}(m_{(-1,-1)})$ where

$$m_{(-1,-1)} \;=\; \left.\frac{\partial g_{-1}}{\partial z_{-1}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A + 12C + 6D)\,|w_1|^2 \;=\; (-B + 24C + 6\,D)\,|w_1|^2.$$

Hence the eigenvalues in W_1 are

$$\xi_1^+ = (-2B_r + 48C_r + 12D_r)\,|w_1|^2 \qquad \text{and} \qquad \xi_1^- = 0.$$

When $j = 2$ or 3 the eigenvalues are the roots of

$$\xi^2 - 2\,\mathrm{Re}(m_{(-j,-j)})\xi + |m_{(-j,-j)}|^2 - |m'_{(-j,j)}|^2 = 0$$

where

$$\begin{aligned} m_{(-2,-2)} &= \left.\frac{\partial g_{-2}}{\partial z_{-2}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + A|w_1|^2 \;=\; (-B + 12C)\,|w_1|^2 \\ m'_{(-2,2)} &= \left.\frac{\partial g_{-2}}{\partial \bar{z}_2}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = Bw_1^2 \\ m_{(-3,-3)} &= \left.\frac{\partial g_{-3}}{\partial z_{-3}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A - 20C)\,|w_1|^2 \;=\; (-B - 8C)\,|w_1|^2 \\ m'_{(-3,3)} &= \left.\frac{\partial g_{-3}}{\partial \bar{z}_3}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = (-B + 10C)\,w_1^2. \end{aligned}$$

Thus the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_2 are

$$\xi_2^{\pm} = \left[-B_r + 12C_r \pm \sqrt{B_r^2 + 24B_iC_i - 144C_i^2}\right]|w_1|^2,$$

and the eigenvalues in W_3 are

$$\xi_3^{\pm} = \left[-B_r - 8C_r \pm \sqrt{B_r^2 - 20B_rC_r + 100C_r^2 - 36B_iC_i + 36C_i^2}\right]|w_1|^2.$$

Note that we have found three zero eigenvalues as expected and hence if the values of A, B, C and D are such that the other eigenvalues are non-zero then $\widetilde{\mathbf{O}(\mathbf{2})}$ has 3-determined stability.

The $\widetilde{\mathbf{SO}(2)}_1$ symmetric branch

We will next compute the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for the periodic solution with $\widetilde{\mathbf{SO}(2)}_1$ symmetry. Since

$$d_{\widetilde{\mathbf{SO}(2)}_1} = \dim(\mathbf{O}(3)) + 1 - \dim(\widetilde{\mathbf{SO}(2)}_1) = 3$$

we expect to find that $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ has three zero eigenvalues.

The isotypic decomposition of V_3 for the action of $\widetilde{\mathbf{SO}(2)}_1$: The subspace

$$W_0 = \{(0,0,u_1,0,0,0,0)\} = \mathrm{Fix}(\widetilde{\mathbf{SO}(2)}_1)$$

is an isotypic component since it is the subspace on which $\widetilde{\mathbf{SO}(2)}_1$ acts trivially. This corresponds to the trivial representation of $\widetilde{\mathbf{SO}(2)}_1$. The action of $\widetilde{\mathbf{SO}(2)}_1$ for the other irreducible representations is given by

$$(R_\alpha,\alpha)\cdot(z_j) \quad = \quad \left(\mathrm{e}^{\mathrm{i}(j+1)\alpha}z_j\right), \qquad (R_\alpha,\alpha)\in\widetilde{\mathbf{SO}(2)}_1$$

for $j \neq -1$. The representation on V_3 is a sum of the trivial representation on W_0 and the representations above for $j = -3$, -2, 0, 1, 2 and 3. However the representations for $j = -3$ and $j = 1$ are $\widetilde{\mathbf{SO}(2)}_1$-isomorphic, as are the pair of representations given by $j = -2$ and $j = 0$. Hence the isotypic decomposition for the action of $\widetilde{\mathbf{SO}(2)}_1$ on V_3 is

$$V_3 = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus W_4,$$

where

$$\begin{aligned}
W_1 &= \{(0,0,0,0,0,u_1,0)\}\\
W_2 &= \{(0,0,0,0,0,0,u_1)\}\\
W_3 &= \{(0,u_1,0,u_2,0,0,0)\}\\
W_4 &= \{(u_1,0,0,0,u_2,0,0)\}.
\end{aligned}$$

Zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$: The three zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ are found by multiplying (5.4.7)–(5.4.9) by the vector $(0,0,w_1,0,0,0,0) \in \mathrm{Fix}(\widetilde{\mathbf{SO}(2)}_1)$. They are

$$\begin{aligned}
\mathbf{a}_2(w_1) &= \left(0,-\sqrt{5}w_1,0,\sqrt{6}w_1,0,0,0\right)^{\mathbf{T}} \in W_3\\
\mathbf{a}_3(w_1) &= \left(0,\sqrt{5}w_1,0,\sqrt{6}w_1,0,0,0\right)^{\mathbf{T}} \in W_3\\
\mathbf{a}_4(w_1) &= (0,0,\mathrm{i}w_1,0,0,0,0)^{\mathbf{T}} \in W_0
\end{aligned}$$

for the curves γ_2, γ_3 and γ_4 respectively.

Block diagonalised form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$: Using the isotypic decomposition of V_3 above we can block diagonalise $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ by reordering the basis functions of V_3 and thus the basis coordinate functions as

$$z_{-1},\ \bar{z}_{-1},\ z_2,\ \bar{z}_2,\ z_3\ \bar{z}_3,\ z_{-2},\ \bar{z}_{-2},\ z_0,\ \bar{z}_0,\ z_{-3},\ \bar{z}_{-3},\ z_1,\ \bar{z}_1. \tag{5.4.14}$$

This gives the block diagonal form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ as

$$(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \begin{pmatrix} M_{(-1,-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_{(2,2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{(3,3)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{(-2,-2)} & M_{(-2,0)} & 0 & 0 \\ 0 & 0 & 0 & M_{(0,-2)} & M_{(0,0)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{(-3,-3)} & M_{(-3,1)} \\ 0 & 0 & 0 & 0 & 0 & M_{(1,-3)} & M_{(1,1)} \end{pmatrix}.$$

Eigenvalues in W_0: The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_0 are given by the eigenvalues of $M_{(-1,-1)}$. Since W_0 contains the zero eigenvector $\mathbf{a}_4$, $M_{(-1,-1)}$ has a zero eigenvalue and the other eigenvalue is given by $2\mathrm{Re}(m_{(-1,-1)})$ where

$$m_{(-1,-1)} \;=\; \left.\frac{\partial g_{-1}}{\partial z_{-1}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} \;=\; \nu(\lambda) + (2A - 6C + 2D)\,|w_1|^2 \;=\; (A - 3C + D)\,|w_1|^2$$

using the branching equation. Thus the eigenvalues in W_0 are

$$\xi_0^+ = (2A_r - 6C_r + 2D_r)\,|w_1|^2 = -2\lambda \qquad \text{and} \qquad \xi_0^- = 0.$$

Eigenvalues in W_1 and W_2: The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_1 and W_2 are given by the eigenvalues of $M_{(2,2)}$ and $M_{(3,3)}$ respectively. Notice that since g_j cannot contain any terms in $\bar{z}_j z_{-1}^2$ for $j = 2$ or 3,

$$m'_{(2,2)} = \left.\frac{\partial g_2}{\partial \bar{z}_2}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0 \qquad \text{and} \qquad m'_{(3,3)} = \left.\frac{\partial g_3}{\partial \bar{z}_3}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0$$

and hence the eigenvalues are given by $m_{(j,j)}$ and $\overline{m_{(j,j)}}$ for $j = 2$ or 3. We compute that

$$\begin{aligned} m_{(2,2)} \;&=\; \left.\frac{\partial g_2}{\partial z_2}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A - 15C - 2D)\,|w_1|^2 \;=\; (-12C - 3D)\,|w_1|^2 \\ m_{(3,3)} \;&=\; \left.\frac{\partial g_3}{\partial z_3}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A - 25C - 3D)\,|w_1|^2 \;=\; (-22C - 4D)\,|w_1|^2 \end{aligned}$$

and hence the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_1 are

$$\xi_1 = (-12C - 3D)\,|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_1 = \left(-12\overline{C} - 3\overline{D}\right)|w_1|^2$$

and in W_2 the eigenvalues are

$$\xi_2 = (-22C - 4D)\,|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_2 = \left(-22\overline{C} - 4\overline{D}\right)|w_1|^2.$$

Eigenvalues in W_3 The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_3 are given by the eigenvalues of

$$(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}|W_3 = \begin{pmatrix} M_{(-2,-2)} & M_{(-2,0)} \\ M_{(0,-2)} & M_{(0,0)} \end{pmatrix}.$$

Notice that since g_{-2} cannot contain any terms in $z_{-1}^2\bar{z}_{-2}$ or $z_0|z_{-1}|^2$, and g_0 cannot contain any terms in $z_{-1}^2\bar{z_0}$ or $z_{-2}|z_{-1}|^2$,

$$m'_{(-2,-2)} = \left.\frac{\partial g_{-2}}{\partial \bar{z}_{-2}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0, \qquad m_{(-2,0)} = \left.\frac{\partial g_{-2}}{\partial z_0}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0,$$

$$m_{(0,-2)} = \left.\frac{\partial g_0}{\partial z_{-2}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0 \quad \text{and} \quad m'_{(0,0)} = \left.\frac{\partial g_0}{\partial \bar{z}_0}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0.$$

We are then left with

$$(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}|W_3 = \begin{pmatrix} m_{(-2,-2)} & 0 & 0 & m'_{(-2,0)} \\ 0 & \overline{m_{(-2,-2)}} & \overline{m'_{(-2,0)}} & 0 \\ 0 & m'_{(0,-2)} & m_{(0,0)} & 0 \\ \overline{m'_{(0,-2)}} & 0 & 0 & \overline{m_{(0,0)}} \end{pmatrix}.$$

Thus the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_3 are given by the eigenvalues of

$$E_3 = \begin{pmatrix} m_{(-2,-2)} & m'_{(-2,0)} \\ \overline{m'_{(0,-2)}} & \overline{m_{(0,0)}} \end{pmatrix}$$

and their complex conjugates. Since W_3 contains the zero eigenvectors $\mathbf{a}_2$ and $\mathbf{a}_3$, the eigenvalues of E_3 are zero and $\mathrm{Trace}(E_3) = m_{(-2,-2)} + \overline{m_{(0,0)}}$ where

$$\begin{aligned} m_{(-2,-2)} &= \left.\frac{\partial g_{-2}}{\partial z_{-2}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A + 15C + 7D)\,|w_1|^2 \;=\; (18C + 6D)\,|w_1|^2 \\ m_{(0,0)} &= \left.\frac{\partial g_0}{\partial z_0}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A + 12C + 6D)\,|w_1|^2 \;=\; (15C + 5D)\,|w_1|^2. \end{aligned}$$

Hence the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_3 are $\xi_3^- = 0$ twice,

$$\xi_3^+ = (33C_r + 11D_r + 3\mathrm{i}C_i + \mathrm{i}D_i)\,|w_1|^2, \qquad \text{and} \qquad \bar{\xi}_3^+ = (33C_r + 11D_r - 3\mathrm{i}C_i - \mathrm{i}D_i)\,|w_1|^2.$$

Eigenvalues in W_4: The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_4 are given by the eigenvalues of

$$(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}|W_4 = \begin{pmatrix} M_{(-3,-3)} & M_{(-3,1)} \\ M_{(1,-3)} & M_{(1,1)} \end{pmatrix}.$$

Similar arguments to those above show that these eigenvalues are the eigenvalues of

$$E_4 = \begin{pmatrix} m_{(-3,-3)} & m'_{(-3,1)} \\ \overline{m'_{(1,-3)}} & \overline{m_{(1,1)}} \end{pmatrix}$$

and their complex conjugates. We compute that

$$\begin{aligned} m_{(-3,-3)} &= \left.\frac{\partial g_{-3}}{\partial z_{-3}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A - 5C + 3D)\,|w_1|^2 \;=\; (-2C + 2D)\,|w_1|^2 \\ m'_{(-3,1)} &= \left.\frac{\partial g_{-3}}{\partial \bar{z}_1}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 2\sqrt{15}C\, w_1^2 \\ m'_{(1,-3)} &= \left.\frac{\partial g_1}{\partial \bar{z}_{-3}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 2\sqrt{15}C\, w_1^2 \\ m_{(1,1)} &= \left.\frac{\partial g_1}{\partial z_1}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A + 2B - 16C - D)\,|w_1|^2 \;=\; (2B - 13C - 2D)\,|w_1|^2. \end{aligned}$$

Hence the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_4 are the roots of

$$\xi^2 - \left(m_{(-3,-3)} + \overline{m_{(1,1)}}\right)\xi + m_{(-3,-3)}\overline{m_{(1,1)}} - m'_{(-3,1)}m'_{(1,-3)} = 0$$

and their complex conjugates. They are

$$\xi_4^{\pm} = \left(-C + D + \overline{B} - \frac{13}{2}\overline{C} - \overline{D} \pm \sqrt{\delta}\right)|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_4^{\pm}$$

where

$$\delta = \left(-C + D + \overline{B} - \frac{13}{2}\overline{C} - \overline{D}\right)^2 + (2C - 2D)\left(2\overline{B} - 13\overline{C} - 2\overline{D}\right) + 60|C|^2.$$

Since we have found three zero eigenvalues as expected, if the values of A, B, C and D are such that the other eigenvalues are non-zero then $\widetilde{\mathbf{SO}(\mathbf{2})}_1$ has 3-determined stability.

The $\widetilde{\mathbf{SO}(\mathbf{2})}_2$ symmetric branch

We now compute the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for the periodic solution with $\widetilde{\mathbf{SO}(\mathbf{2})}_2$ symmetry. Again, since

$$d_{\widetilde{\mathbf{SO}(\mathbf{2})}_2} = \dim(\mathbf{O}(\mathbf{3})) + 1 - \dim(\widetilde{\mathbf{SO}(\mathbf{2})}_2) = 3$$

we expect to find that $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ has three zero eigenvalues.

The isotypic decomposition of V_3 for the action of $\widetilde{\mathbf{SO}(\mathbf{2})}_2$: The subspace

$$W_0 = \{(0, u_1, 0, 0, 0, 0, 0)\} = \mathrm{Fix}(\widetilde{\mathbf{SO}(\mathbf{2})}_2)$$

is an isotypic component since it is the subspace on which $\widetilde{\mathbf{SO}(\mathbf{2})}_2$ acts trivially. This corresponds to the trivial representation of $\widetilde{\mathbf{SO}(\mathbf{2})}_2$. The action of $\widetilde{\mathbf{SO}(\mathbf{2})}_2$ for the other irreducible representations is given by

$$(R_\alpha, 2\alpha)\cdot(z_j) \;=\; \left(\mathrm{e}^{\mathrm{i}(j+2)\alpha} z_j\right), \qquad (R_\alpha, 2\alpha) \in \widetilde{\mathbf{SO}(\mathbf{2})}_2$$

for $j \neq -2$. The representation on V_3 is a sum of the trivial representation on W_0 and the representations above for $j = -3$, -1, 0, 1, 2 and 3. However the representations for $j = -3$ and $j = -1$ are $\widetilde{\mathbf{SO}(\mathbf{2})}_2$-isomorphic. Hence the isotypic decomposition for the action of $\widetilde{\mathbf{SO}(\mathbf{2})}_2$ on V_3 is

$$V_3 = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5,$$

where

$$\begin{aligned}
W_1 &= \{(0, 0, 0, u_1, 0, 0, 0)\} \\
W_2 &= \{(0, 0, 0, 0, u_1, 0, 0)\} \\
W_3 &= \{(0, 0, 0, 0, 0, u_1, 0)\} \\
W_4 &= \{(0, 0, 0, 0, 0, 0, u_1)\} \\
W_5 &= \{(u_1, 0, u_2, 0, 0, 0, 0)\}.
\end{aligned}$$

Zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$**:** The three zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ are found by multiplying (5.4.7)–(5.4.9) by the vector $(0,w_1,0,0,0,0,0) \in \mathrm{Fix}(\widetilde{\mathbf{SO(2)}}_2)$. They are

$$\begin{aligned}
\mathbf{a}_2(w_1) &= \left(-\sqrt{3}w_1, 0, \sqrt{5}w_1, 0,0,0,0\right)^{\mathbf{T}} \in W_5 \\
\mathbf{a}_3(w_1) &= \left(\sqrt{3}w_1, 0, \sqrt{5}w_1, 0,0,0,0\right)^{\mathbf{T}} \in W_3 \\
\mathbf{a}_4(w_1) &= (0, \mathrm{i}w_1, 0,0,0,0,0)^{\mathbf{T}} \in W_0
\end{aligned}$$

for the curves γ_2, γ_3 and γ_4 respectively.

Block diagonalised form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$**:** Using the isotypic decomposition of V_3 above we can block diagonalise $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ by reordering the basis functions of V_3 and thus the basis coordinate functions as

$$z_{-2},\ \bar{z}_{-2},\ z_0,\ \bar{z}_0,\ z_1\ \bar{z}_1,\ z_2,\ \bar{z}_2,\ z_3,\ \bar{z}_3,\ z_{-3},\ \bar{z}_{-3},\ z_{-1},\ \bar{z}_{-1}. \tag{5.4.15}$$

This gives the block diagonal form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ as

$$(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \begin{pmatrix}
M_{(-2,-2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & M_{(0,0)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_{(1,1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{(2,2)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_{(3,3)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & M_{(-3,-3)} & M_{(-3,-1)} \\
0 & 0 & 0 & 0 & 0 & M_{(-1,-3)} & M_{(-1,-1)}
\end{pmatrix}.$$

Eigenvalues in W_0**:** The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_0 are given by the eigenvalues of $M_{(-2,-2)}$. Since W_0 contains the zero eigenvector $\mathbf{a}_4$, $M_{(-2,-2)}$ has a zero eigenvalue and the other eigenvalue is given by $2\mathrm{Re}(m_{(-2,-2)})$ where

$$m_{(-2,-2)} = \left.\frac{\partial g_{-2}}{\partial z_{-2}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (2A+8D)\,|w_1|^2 = (A+4D)\,|w_1|^2$$

using the branching equation. Thus the eigenvalues in W_0 are

$$\xi_0^+ = (2A_r + 8D_r)\,|w_1|^2 = -2\lambda \qquad \text{and} \qquad \xi_0^- = 0.$$

Eigenvalues in W_j **for** $j = 1,2,3$ **and** 4**:** The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_j for $j = 1,2,3$ and 4 are given by the eigenvalues of $M_{(j-1,j-1)}$. Notice that since g_{j-1} cannot contain any terms in $\bar{z}_{j-1}z_{-2}^2$ for $j = 1,2,3$ or 4,

$$m'_{(j-1,j-1)} = \left.\frac{\partial g_{j-1}}{\partial \bar{z}_{j-1}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0$$

and hence the eigenvalues are given by $m_{(j-1,j-1)}$ and $\overline{m_{(j-1,j-1)}}$ for $j = 1, 2, 3$ and 4. We compute that

$$
\begin{aligned}
m_{(0,0)} &= \left.\frac{\partial g_0}{\partial z_0}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + A|w_1|^2 \;=\; -4D|w_1|^2 \\
m_{(1,1)} &= \left.\frac{\partial g_1}{\partial z_1}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A - 15C - 2D)\,|w_1|^2 \;=\; (-15C - 6D)\,|w_1|^2 \\
m_{(2,2)} &= \left.\frac{\partial g_2}{\partial z_2}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A + 2B - 40C - 4D)\,|w_1|^2 \;=\; (2B - 40C - 8D)\,|w_1|^2 \\
m_{(3,3)} &= \left.\frac{\partial g_3}{\partial z_3}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A - 25C - 6D)\,|w_1|^2 \;=\; (-25C - 10D)\,|w_1|^2
\end{aligned}
$$

and hence the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_1 are

$$\xi_1 = -4D|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_1 = -4\overline{D}|w_1|^2,$$

in W_2,

$$\xi_2 = (-15C - 6D)\,|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_2 = \left(-15\overline{C} - 6\overline{D}\right)|w_1|^2,$$

in W_3,

$$\xi_3 = (2B - 40C - 8D)\,|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_3 = \left(2\overline{B} - 40\overline{C} - 8\overline{D}\right)|w_1|^2$$

and in W_4 the eigenvalues are

$$\xi_4 = (-25C - 10D)\,|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_4 = \left(-25\overline{C} - 10\overline{D}\right)|w_1|^2.$$

Eigenvalues in W_5**:** The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_5 are given by the eigenvalues of

$$(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}|W_5 = \begin{pmatrix} M_{(-3,-3)} & M_{(-3,-1)} \\ M_{(-1,-3)} & M_{(-1,-1)} \end{pmatrix}.$$

It can be shown that these eigenvalues are the eigenvalues of

$$E_5 = \begin{pmatrix} m_{(-3,-3)} & m'_{(-3,-1)} \\ \overline{m'_{(-1,-3)}} & \overline{m_{(-1,-1)}} \end{pmatrix}$$

and their complex conjugates. Since W_5 contains the zero eigenvectors $\mathbf{a}_2$ and $\mathbf{a}_3$ the eigenvalues of E_5 are zero and $\mathrm{Trace}(E_5) = m_{(-3,-3)} + \overline{m_{(-1,-1)}}$ where

$$
\begin{aligned}
m_{(-3,-3)} &= \left.\frac{\partial g_{-3}}{\partial z_{-3}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A + 25C + 9D)\,|w_1|^2 \;=\; (25C + 5D)\,|w_1|^2 \\
m_{(-1,-1)} &= \left.\frac{\partial g_{-1}}{\partial z_{-1}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A + 15C + 7D)\,|w_1|^2 \;=\; (15C + 3D)\,|w_1|^2.
\end{aligned}
$$

Hence the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_5 are $\xi_5^- = 0$ twice,

$$\xi_5^+ = (40C_r + 8D_r + 10\mathrm{i}C_i + 2\mathrm{i}D_i)\,|w_1|^2, \qquad \text{and} \qquad \bar{\xi}_5^+ = (40C_r + 8D_r - 10\mathrm{i}C_i - 2\mathrm{i}D_i)\,|w_1|^2.$$

Since we have found three zero eigenvalues as expected, if the values of A, B, C and D are such that the other eigenvalues are non-zero then $\widetilde{\mathbf{SO}(\mathbf{2})}_2$ has 3-determined stability.

The $\widetilde{\mathbf{SO(2)}}_3$ symmetric branch

In this section we compute the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for the periodic solution with $\widetilde{\mathbf{SO(2)}}_3$ symmetry. As in all previous cases, since

$$d_{\widetilde{\mathbf{SO(2)}}_3} = \dim(\mathbf{O(3)}) + 1 - \dim(\widetilde{\mathbf{SO(2)}}_3) = 3$$

we expect to find that $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ has three zero eigenvalues.

The isotypic decomposition of V_3 for the action of $\widetilde{\mathbf{SO(2)}}_3$: The subspace

$$W_0 = \{(u_1, 0, 0, 0, 0, 0, 0)\} = \mathrm{Fix}(\widetilde{\mathbf{SO(2)}}_3)$$

is an isotypic component since it is the subspace on which $\widetilde{\mathbf{SO(2)}}_3$ acts trivially. This corresponds to the trivial representation of $\widetilde{\mathbf{SO(2)}}_3$. The action of $\widetilde{\mathbf{SO(2)}}_3$ for the other irreducible representations is given by

$$(R_\alpha, 3\alpha) \cdot (z_j) \quad = \quad \left(\mathrm{e}^{\mathrm{i}(j+3)\alpha} z_j\right), \qquad (R_\alpha, 3\alpha) \in \widetilde{\mathbf{SO(2)}}_3$$

for $j \neq -3$. The representation on V_3 is a sum of the trivial representation on W_0 and the representations above for $j = -2$, -1, 0, 1, 2 and 3. Since none of these representations are $\widetilde{\mathbf{SO(2)}}_3$-isomorphic, the isotypic decomposition for the action of $\widetilde{\mathbf{SO(2)}}_3$ on V_3 is

$$V_3 = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 \oplus W_6,$$

where

$$\begin{aligned}
W_1 &= \{(0, u_1, 0, 0, 0, 0, 0)\} \\
W_2 &= \{(0, 0, u_1, 0, 0, 0, 0)\} \\
W_3 &= \{(0, 0, 0, u_1, 0, 0, 0)\} \\
W_4 &= \{(0, 0, 0, 0, u_1, 0, 0)\} \\
W_5 &= \{(0, 0, 0, 0, 0, u_1, 0)\} \\
W_6 &= \{(0, 0, 0, 0, 0, 0, u_1)\}.
\end{aligned}$$

Zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$: The three zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ are found by multiplying (5.4.7)–(5.4.9) by the vector $(w_1, 0, 0, 0, 0, 0, 0) \in \mathrm{Fix}(\widetilde{\mathbf{SO(2)}}_3)$. They are

$$\begin{aligned}
\mathbf{a}_2(w_1) &= (0, w_1, 0, 0, 0, 0, 0)^{\mathbf{T}} \in W_1 \\
\mathbf{a}_3(w_1) &= (0, w_1, 0, 0, 0, 0, 0)^{\mathbf{T}} \in W_1 \\
\mathbf{a}_4(w_1) &= (\mathrm{i}w_1, 0, 0, 0, 0, 0, 0)^{\mathbf{T}} \in W_0
\end{aligned}$$

for the curves γ_2, γ_3 and γ_4 respectively.

Block diagonalised form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$: Using the isotypic decomposition of V_3 above we can see that $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ must already be diagonal in the basis given by (5.4.4). Thus the block diagonal form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ is

$$(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \begin{pmatrix} M_{(-3,-3)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & M_{(-2,-2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{(-1,-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{(0,0)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{(1,1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{(2,2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{(3,3)} \end{pmatrix}.$$

Eigenvalues in W_0: The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_0 are given by the eigenvalues of $M_{(-3,-3)}$. Since W_0 contains the zero eigenvector $\mathbf{a}_4$, $M_{(-3,-3)}$ has a zero eigenvalue and the other eigenvalue is given by $2\mathrm{Re}(m_{(-3,-3)})$ where

$$m_{(-3,-3)} = \left.\frac{\partial g_{-3}}{\partial z_{-3}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (2A + 50C + 18D)\,|w_1|^2 = (A + 25C + 9D)\,|w_1|^2$$

using the branching equation. Thus the eigenvalues in W_0 are

$$\xi_0^+ = (2A_r + 50C_r + 18D_r)\,|w_1|^2 = -2\lambda \qquad \text{and} \qquad \xi_0^- = 0.$$

Eigenvalues in W_j for $j = 1, \ldots, 6$: The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_j for $j = 1, \ldots, 6$ are given by the eigenvalues of $M_{(k,k)}$ for $k = -2, \ldots, 3$. Notice that since g_k cannot contain any terms in $\bar{z}_k z_{-3}^2$ for $k = -2, \ldots, 3$,

$$m'_{(k,k)} = \left.\frac{\partial g_k}{\partial \bar{z}_k}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0$$

and hence the eigenvalues are given by $m_{(k,k)}$ and $\overline{m_{(k,k)}}$ for $k = -2, \ldots, 3$. Since the zero eigenvectors $\mathbf{a}_2$ and $\mathbf{a}_3$ lie in W_1 we must have $m_{(-2,-2)} = 0$ and $\xi_1 = 0$. We compute that

$$\begin{aligned}
m_{(-1,-1)} &= \left.\frac{\partial g_{-1}}{\partial z_{-1}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A - 5C + 3D)\,|w_1|^2 = (-30C - 6D)\,|w_1|^2 \\
m_{(0,0)} &= \left.\frac{\partial g_0}{\partial z_0}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A - 20C)\,|w_1|^2 = (-45C - 9D)\,|w_1|^2 \\
m_{(1,1)} &= \left.\frac{\partial g_1}{\partial z_1}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A - 25C - 3D)\,|w_1|^2 = (-50C - 12D)\,|w_1|^2 \\
m_{(2,2)} &= \left.\frac{\partial g_2}{\partial z_2}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A - 25C - 6D)\,|w_1|^2 = (-50C - 15D)\,|w_1|^2 \\
m_{(3,3)} &= \left.\frac{\partial g_3}{\partial z_3}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (A + 2B - 40C - 9D)\,|w_1|^2 = (2B - 65C - 18D)\,|w_1|^2
\end{aligned}$$

and hence the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_2 are

$$\xi_2 = (-30C - 6D)\,|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_2 = \left(-30\overline{C} - 6\overline{D}\right)|w_1|^2,$$

in W_3,

$$\xi_3 = (-45C - 9D)\,|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_3 = \left(-45\overline{C} - 9\overline{D}\right)|w_1|^2,$$

in W_4,

$$\xi_4 = (-50C - 12D)\,|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_4 = \left(-50\overline{C} - 12\overline{D}\right)|w_1|^2,$$

in W_5,

$$\xi_5 = (-50C - 15D)\,|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_5 = \left(-50\overline{C} - 15\overline{D}\right)|w_1|^2$$

and in W_6 the eigenvalues are

$$\xi_6 = (2B - 65C - 18D)\,|w_1|^2 \qquad \text{and} \qquad \bar{\xi}_6 = \left(2\overline{B} - 65\overline{C} - 18\overline{D}\right)|w_1|^2.$$

Since we have found three zero eigenvalues as expected, if the values of A, B, C and D are such that the other eigenvalues are non-zero then $\widetilde{\mathbf{SO}(\mathbf{2})}_3$ has 3-determined stability.

The $\widetilde{\mathbb{O}}$ symmetric branch

We now compute the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for the periodic solution with $\widetilde{\mathbb{O}}$ symmetry. Since

$$d_{\widetilde{\mathbb{O}}} = \dim(\mathbf{O}(\mathbf{3})) + 1 - \dim(\widetilde{\mathbb{O}}) = 4$$

we expect to find that $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ has four zero eigenvalues.

The isotypic decomposition of V_3 for the action of $\widetilde{\mathbb{O}}$: The subspace

$$W_0 = \{(0, u_1, 0, 0, 0, -u_1, 0)\} = \mathrm{Fix}(\widetilde{\mathbb{O}})$$

is an isotypic component since it is the subspace on which $\widetilde{\mathbb{O}}$ acts trivially. This corresponds to the trivial representation of $\widetilde{\mathbb{O}}$. Our representation of $\widetilde{\mathbb{O}}$ on V_3 is a sum of irreducible representations of $\widetilde{\mathbb{O}}$. The irreducible representations of $\widetilde{\mathbb{O}}$ are given by the character table, Table 5.4 where

$$[I] \quad \left[(\kappa_{x=y})\right] \quad \left[(\mathcal{R}_{2\pi/3})\right] \quad \left[(R^z_\pi)\right] \quad \left[(-R^z_{\pi/2})\right]$$

are the five conjugacy classes of elements in $\widetilde{\mathbb{O}}$.

Representation	$[I]$	$\left[(\kappa_{x=y})\right]$	$[(\mathcal{R}_{2\pi/3})]$	$[(R^z_\pi)]$	$\left[(-R^z_{\pi/2})\right]$
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	-1	0	-1	1
χ_5	3	1	0	-1	-1

Table 5.4: Character table for the group $\widetilde{\mathbb{O}}$

In our representation on V_3

$$\chi\,[I] = 7 \quad \chi\left[(\kappa_{x=y})\right] = 1 \quad \chi\left[(\mathcal{R}_{2\pi/3})\right] = 1 \quad \chi\left[(R^z_\pi)\right] = -1 \quad \chi\left[(-R^z_{\pi/2})\right] = 1$$

so we can see that

$$\chi = \chi_1 + \chi_4 + \chi_5.$$

Since none of these representations are $\widetilde{\mathbb{O}}$-isomorphic, the subspaces which are invariant under each of the representations are the isotypic components. The computation of the form of these subspaces is greatly simplified if, instead of using the spherical harmonics of degree 3 as our basis functions, we use the set of basis functions given by

$$\begin{aligned}
B_0 &= xyz = \mathrm{i}\sqrt{\frac{2\pi}{105}}\left(Y_3^{-2} - Y_3^2\right) \\
B_1 &= x(z^2 - y^2) = \sqrt{\frac{\pi}{21}}\left(Y_3^{-1} - Y_3^1\right) + \sqrt{\frac{\pi}{35}}\left(Y_3^{-3} - Y_3^3\right) \\
B_2 &= y(x^2 - z^2) = -\mathrm{i}\sqrt{\frac{\pi}{21}}\left(Y_3^{-1} + Y_3^1\right) + \mathrm{i}\sqrt{\frac{\pi}{35}}\left(Y_3^{-3} + Y_3^3\right) \\
B_3 &= z(y^2 - x^2) = -2\sqrt{\frac{2\pi}{105}}\left(Y_3^{-2} + Y_3^2\right) \\
B_4 &= x\left(x^2 - \frac{3}{2}(y^2 + z^2)\right) = \frac{5}{2}\sqrt{\frac{\pi}{35}}\left(Y_3^{-3} - Y_3^3\right) - \frac{3}{2}\sqrt{\frac{\pi}{21}}\left(Y_3^{-1} - Y_3^1\right) \\
B_5 &= y\left(y^2 - \frac{3}{2}(z^2 + x^2)\right) = -\frac{5\mathrm{i}}{2}\sqrt{\frac{\pi}{35}}\left(Y_3^{-3} + Y_3^3\right) - \frac{3\mathrm{i}}{2}\sqrt{\frac{\pi}{21}}\left(Y_3^{-1} + Y_3^1\right) \\
B_6 &= z\left(z^2 - \frac{3}{2}(x^2 + y^2)\right) = 2\sqrt{\frac{\pi}{7}}Y_3^0
\end{aligned}$$

Let us denote the vector of amplitudes for this basis by $\mathbf{b} = (b_0, b_1, b_2, b_3, b_4, b_5, b_6)$. Since all elements of $\widetilde{\mathbb{O}}$ act trivially on B_0 the space which is invariant under the representation χ_1 is

$$W_0 = \{(b, 0, 0, 0, 0, 0, 0)\} = \{(0, u_1, 0, 0, 0, -u_1, 0)\} = \mathrm{Fix}(\widetilde{\mathbb{O}}).$$

Using the actions of the generating elements of $\widetilde{\mathbb{O}}$ on the point (x, y, z) on the surface of the sphere as given in Table 5.2 we can see that

$$\begin{aligned}
I \cdot (B_1, B_2, B_3) &= (B_1, B_2, B_3) \quad \Rightarrow \quad \chi(I) = 3 \\
(\kappa_{x=y}) \cdot (B_1, B_2, B_3) &= (-B_2, -B_1, -B_3) \quad \Rightarrow \quad \chi(\kappa_{x=y}) = -1 \\
(\mathcal{R}_{2\pi/3}) \cdot (B_1, B_2, B_3) &= (B_2, B_3, B_1) \quad \Rightarrow \quad \chi(\mathcal{R}_{2\pi/3}) = 0 \\
(R_\pi^z) \cdot (B_1, B_2, B_3) &= (-B_1, -B_2, B_3) \quad \Rightarrow \quad \chi(R_\pi^z) = -1 \\
(-R_{\pi/2}^z) \cdot (B_1, B_2, B_3) &= (-B_2, B_1, B_3) \quad \Rightarrow \quad \chi(-R_{\pi/2}^z) = 1
\end{aligned}$$

This means that the three-dimensional subspace

$$W_1 = \{(0, b_1, b_2, b_3, 0, 0, 0)\} = \{(\sqrt{3}u_1, u_3, \sqrt{5}u_2, 0, -\sqrt{5}u_1, u_3, -\sqrt{3}u_2)\}$$

is invariant under the action of the representation χ_4. Similarly

$$\begin{aligned}
I \cdot (B_4, B_5, B_6) &= (B_4, B_5, B_6) \quad \Rightarrow \quad \chi(I) = 3 \\
(\kappa_{x=y}) \cdot (B_4, B_5, B_6) &= (B_5, B_4, B_6) \quad \Rightarrow \quad \chi(\kappa_{x=y}) = 1 \\
(\mathcal{R}_{2\pi/3}) \cdot (B_4, B_5, B_6) &= (B_4, B_6, B_4) \quad \Rightarrow \quad \chi(\mathcal{R}_{2\pi/3}) = 0 \\
(R_\pi^z) \cdot (B_4, B_5, B_6) &= (-B_4, -B_5, B_6) \quad \Rightarrow \quad \chi(R_\pi^z) = -1 \\
(-R_{\pi/2}^z) \cdot (B_4, B_5, B_6) &= (B_5, -B_4, -B_6) \quad \Rightarrow \quad \chi(-R_{\pi/2}^z) = -1
\end{aligned}$$

so the three-dimensional subspace

$$W_2 = \{(0, 0, 0, 0, b_4, b_5, b_6)\} = \{(\sqrt{5}u_1, 0, \sqrt{3}u_2, u_3, \sqrt{3}u_1, 0, \sqrt{5}u_2)\}$$

is invariant under the action of the representation χ_5. Therefore the isotypic decomposition of V_3 with respect to the action of $\widetilde{\mathbb{O}}$ is

$$V_3 = W_0 \oplus W_1 \oplus W_2.$$

Zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$: The four zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ are found by multiplying (5.4.6)–(5.4.9) by the vector $(0, w_1, 0, 0, 0, -w_1, 0) \in \mathrm{Fix}(\widetilde{\mathbb{O}})$. They are

$$\begin{aligned}
\mathbf{a}_1(w_1) &= (0, iw_1, 0, 0, 0, iw_1, 0)^{\mathbf{T}} \in W_1 \\
\mathbf{a}_2(w_1) &= \left(-\sqrt{3}w_1, 0, \sqrt{5}w_1, 0, \sqrt{5}w_1, 0, -\sqrt{3}w_1\right)^{\mathbf{T}} \in W_1 \\
\mathbf{a}_3(w_1) &= \left(\sqrt{3}w_1, 0, \sqrt{5}w_1, 0, -\sqrt{5}w_1, 0, -\sqrt{3}w_1\right)^{\mathbf{T}} \in W_1 \\
\mathbf{a}_4(w_1) &= (0, \mathrm{i}w_1, 0, 0, 0, -\mathrm{i}w_1, 0)^{\mathbf{T}} \in W_0
\end{aligned}$$

for the curves γ_1, γ_2, γ_3 and γ_4 respectively.

Block diagonalised form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$: Using the isotypic decomposition of V_3 above we can see that to diagonalise $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ we must use the basis functions

$$v_1 = z_{-2} - z_2, \qquad v_2 = \sqrt{3}z_{-3} - \sqrt{5}z_1, \qquad v_3 = \sqrt{5}z_1 - \sqrt{3}z_3, \qquad v_4 = z_{-2} + z_2$$

$$v_5 = \sqrt{5}z_{-3} + \sqrt{3}z_1, \qquad v_6 = \sqrt{3}z_{-1} + \sqrt{5}z_3, \qquad v_7 = z_0$$

and their complex conjugates. Under this change of basis $g(\mathbf{z}, \lambda, \tau)$ becomes $h(\mathbf{v}, \lambda, \tau)$ where $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ and

$$(\mathrm{d}h)|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \begin{pmatrix}
N_{(1,1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & N_{(2,2)} & N_{(2,3)} & N_{(2,4)} & 0 & 0 & 0 \\
0 & N_{(3,2)} & N_{(3,3)} & N_{(3,4)} & 0 & 0 & 0 \\
0 & N_{(4,2)} & N_{(4,3)} & N_{(4,4)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & N_{(5,5)} & N_{(5,6)} & N_{(5,7)} \\
0 & 0 & 0 & 0 & N_{(6,5)} & N_{(6,6)} & N_{(6,7)} \\
0 & 0 & 0 & 0 & N_{(7,5)} & N_{(7,6)} & N_{(7,7)}
\end{pmatrix}$$

where

$$N_{(i,j)} = \begin{pmatrix} n_{(i,j)} & n'_{(i,j)} \\ \overline{n'_{(i,j)}} & \overline{n_{(i,j)}} \end{pmatrix} \qquad \text{and} \qquad n_{(i,j)} = \left.\frac{\partial h_i}{\partial v_j}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)}, \quad n'_{(i,j)} = \left.\frac{\partial h_i}{\partial \bar{v}_j}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)}.$$

Eigenvalues in W_0: The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_0 are given by the eigenvalues of $N_{(1,1)}$. Since W_0 contains the zero eigenvector $\mathbf{a}_4$, one eigenvalue is zero and the other is $2\mathrm{Re}(n_{(1,1)})$ where

$$n_{(1,1)} = \frac{\partial h_1}{\partial v_1} = \frac{\partial g_{-2}}{\partial v_1} - \frac{\partial g_2}{\partial v_1} = \frac{\partial g_{-2}}{\partial z_{-2}}\frac{\partial z_{-2}}{\partial v_1} + \frac{\partial g_{-2}}{\partial z_2}\frac{\partial z_2}{\partial v_1} - \frac{\partial g_2}{\partial z_{-2}}\frac{\partial z_{-2}}{\partial v_1} - \frac{\partial g_2}{\partial z_2}\frac{\partial z_2}{\partial v_1}$$

where all derivatives are evaluated at $(\mathbf{z}_0, \lambda_0, \tau_0)$. Since

$$\begin{array}{l} z_{-2} = \frac{1}{2}(v_1 + v_4) \\ z_2 = \frac{1}{2}(-v_1 + v_4) \end{array} \qquad \text{and} \qquad \frac{\partial g_{-2}}{\partial z_{-2}} = \frac{\partial g_2}{\partial z_2}, \qquad \frac{\partial g_{-2}}{\partial z_2} = \frac{\partial g_2}{\partial z_{-2}},$$

when the derivatives are evaluated at $(\mathbf{z}_0, \lambda_0, \tau_0)$, we find that

$$\begin{aligned} n_{(1,1)} &= \left.\frac{\partial g_{-2}}{\partial z_{-2}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} - \left.\frac{\partial g_{-2}}{\partial z_2}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} \\ &= \nu(\lambda) + (3A + 2B - 40C + 4D)\,|w_1|^2 - (-A - 2B + 40C + 4D)\,|w_1|^2 \\ &= (2A + 2B - 40C)\,|w_1|^2. \end{aligned}$$

Hence the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_0 are

$$\xi_0^+ = (4A_r + 4B_r - 80C_r)\,|w_1|^2 \qquad \text{and} \qquad \xi_0^- = 0.$$

Eigenvalues in W_1: The two distinct eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_1 have multiplicity three. Since W_1 contains the three zero eigenvectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$, one of the two distinct eigenvalues is zero. Since all of the entries in $N_{(2,4)}$, $N_{(3,4)}$, $N_{(4,2)}$ and $N_{(4,3)}$ are zero, due to the fact that they vanish upon evaluation at $(\mathbf{z}_0, \lambda_0, \tau_0)$ the two distinct eigenvalues in W_1 are given by the eigenvalues of $N_{(4,4)}$. Since one of these eigenvalues is zero, the other is $2\mathrm{Re}(n_{(4,4)})$ where

$$\begin{aligned} n_{(4,4)} &= \frac{\partial h_4}{\partial v_4} = \frac{\partial g_{-2}}{\partial v_4} + \frac{\partial g_2}{\partial v_4} = \frac{\partial g_{-2}}{\partial z_{-2}}\frac{\partial z_{-2}}{\partial v_4} + \frac{\partial g_{-2}}{\partial z_2}\frac{\partial z_2}{\partial v_4} - \frac{\partial g_2}{\partial z_{-2}}\frac{\partial z_{-2}}{\partial v_4} - \frac{\partial g_2}{\partial z_2}\frac{\partial z_2}{\partial v_4} \\ &= \left.\frac{\partial g_{-2}}{\partial z_{-2}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} + \left.\frac{\partial g_{-2}}{\partial z_2}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} \\ &= \nu(\lambda) + (3A + 2B - 40C + 4D)\,|w_1|^2 + (-A - 2B + 40C + 4D)\,|w_1|^2 \\ &= (-2B + 40C + 8D)\,|w_1|^2. \end{aligned}$$

Hence the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_1 are

$$\xi_1^+ = (-4B_r + 80C_r + 16D_r)\,|w_1|^2 \qquad \text{and} \qquad \xi_1^- = 0,$$

each of multiplicity 3.

Eigenvalues in W_2: The two distinct eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_2 also have multiplicity three. Since all of the entries in $N_{(5,7)}$, $N_{(6,7)}$, $N_{(7,5)}$ and $N_{(7,6)}$ are zero, due to the fact that they vanish upon evaluation at $(\mathbf{z}_0, \lambda_0, \tau_0)$ the two distinct eigenvalues in W_2 are given by the eigenvalues of $N_{(7,7)} = M_{(0,0)}$ since $v_7 = z_0$. Since

$$\begin{aligned} m_{(0,0)} &= \left.\frac{\partial g_0}{\partial z_0}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + 2A|w_1|^2 = (-2B + 40C)\,|w_1|^2 \\ m'_{(0,0)} &= \left.\frac{\partial g_0}{\partial \bar{z}_0}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = -2Bw_1^2, \end{aligned}$$

the eigenvalues are the roots of

$$\xi^2 - (-4B_r + 80C_r)\,|w_1|^2\xi + \left(|2B - 40C|^2 - |2B|^2\right)|w_1|^4 = 0.$$

Thus the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_2 are

$$\xi_2^{\pm} = \left[-2B_r + 40C_r \pm 2\sqrt{B_r^2 + 40B_iC_i - 400C_i^2}\right] |w_1|^2$$

each of multiplicity 3. Since we have found four zero eigenvalues as expected, if the values of A, B, C and D are such that the other eigenvalues are non-zero then $\widetilde{\mathbb{O}}$ has 3-determined stability.

The $\widetilde{\mathbf{D}_6}$ symmetric branch

Finally, we will compute the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for the periodic solution with $\widetilde{\mathbf{D}_6}$ symmetry. Since

$$d_{\widetilde{\mathbf{D}_6}} = \dim(\mathbf{O}(\mathbf{3})) + 1 - \dim(\widetilde{\mathbf{D}_6}) = 4$$

we expect to find that $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ has four zero eigenvalues.

The isotypic decomposition of V_3 for the action of $\widetilde{\mathbf{D}_6}$: The subspace

$$W_0 = \{(u_1, 0, 0, 0, 0, 0, -u_1)\} = \mathrm{Fix}(\widetilde{\mathbf{D}_6})$$

is an isotypic component since it is the subspace on which $\widetilde{\mathbf{D}_6}$ acts trivially. This corresponds to the trivial representation of $\widetilde{\mathbf{D}_6}$. Our representation of $\widetilde{\mathbf{D}_6}$ on V_3 is a sum of irreducible representations of $\widetilde{\mathbf{D}_6}$. The irreducible representations of $\widetilde{\mathbf{D}_6}$ are given by the character table, Table 5.5 where

$$[I] \quad [(-R_{\pi/3})] \quad [(R_{2\pi/3})] \quad [(-R_{\pi})] \quad [(\kappa)] \quad [(-R_{\pi/3}\kappa)]$$

are the six conjugacy classes of elements in $\widetilde{\mathbf{D}_6}$. Here R_α is a rotation through an angle α in some axis and κ is a rotation through π in an orthogonal axis. The element $(-I, \pi) \in \widetilde{\mathbf{D}_6}$ acts as the identity and is a member of $[I]$.

Representation	$[I]$	$[(-R_{\pi/3})]$	$[(R_{2\pi/3})]$	$[(-R_{\pi})]$	$[(\kappa)]$	$[(-R_{\pi/3}\kappa)]$
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	1	-1	1	-1
χ_4	1	-1	1	-1	-1	1
χ_5	2	1	-1	-2	0	0
χ_6	2	-1	-1	2	0	0

Table 5.5: Character table for the group $\widetilde{\mathbf{D}_6}$.

In our representation on V_3,

$$\chi[I] = 7 \quad \chi[(-R_{\pi/3})] = 1 \quad \chi[(R_{2\pi/3})] = 1 \quad \chi[(-R_{\pi})] = 1 \quad \chi[(\kappa)] = -1 \quad \chi[(-R_{\pi/3}\kappa)] = 1$$

so we can see that

$$\chi = \chi_1 + \chi_2 + \chi_4 + \chi_5 + \chi_6.$$

Since none of these representations are $\widetilde{\mathbf{D}_6}$-isomorphic, the subspaces which are invariant under each of the representations are the isotypic components. The subspaces which are invariant under χ_1, χ_2, χ_4, χ_5 and χ_6 are respectively

$$\begin{aligned}
W_0 &= \{(u_1, 0, 0, 0, 0, 0, -u_1)\} \;=\; \mathrm{Fix}(\widetilde{\mathbf{D}_6}) \\
W_1 &= \{(u_1, 0, 0, 0, 0, 0, u_1)\} \\
W_2 &= \{(0, 0, 0, u_1, 0, 0, 0)\} \\
W_3 &= \{(0, u_1, 0, 0, 0, u_2, 0)\} \\
W_4 &= \{(0, 0, u_1, 0, u_2, 0, 0)\}.
\end{aligned}$$

Hence the isotypic decomposition of V_3 with respect to the action of $\widetilde{\mathbf{D}_6}$ is given by

$$V_3 = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus W_4.$$

Zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$**:** The four zero eigenvectors of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ are found by multiplying (5.4.6)–(5.4.9) by the vector $(w_1, 0, 0, 0, 0, 0, -w_1) \in \mathrm{Fix}(\widetilde{\mathbf{D}_6})$. They are

$$\begin{aligned}
\mathbf{a}_1(w_1) &= (\mathrm{i}w_1, 0, 0, 0, 0, 0, \mathrm{i}w_1)^{\mathbf{T}} \in W_1 \\
\mathbf{a}_2(w_1) &= \left(0, \sqrt{3}w_1, 0, 0, 0, \sqrt{3}w_1, 0\right)^{\mathbf{T}} \in W_3 \\
\mathbf{a}_3(w_1) &= \left(0, \sqrt{3}w_1, 0, 0, 0, -\sqrt{3}w_1, 0\right)^{\mathbf{T}} \in W_3 \\
\mathbf{a}_4(w_1) &= (-\mathrm{i}w_1, 0, 0, 0, 0, 0, \mathrm{i}w_1)^{\mathbf{T}} \in W_0
\end{aligned}$$

for the curves γ_1, γ_2, γ_3 and γ_4 respectively.

Block diagonalised form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$**:** Using the isotypic decomposition of V_3 above we can see that to diagonalise $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ we must use the basis functions

$$v_1 = z_{-3} - z_3, \quad v_2 = z_{-3} + z_3, \quad z_0, \quad z_{-2}, \quad z_2, \quad z_{-1}, \quad z_1$$

and their complex conjugates.

Due to the fact that $g_j(\mathbf{z}) = g_{-j}(\tilde{\mathbf{z}})$ where $\tilde{\mathbf{z}} = (z_3, z_2, z_1, z_0, z_{-1}, z_{-2}, z_{-3})$, it follows that for this solution, $M_{(-j,-j)} = M_{(j,j)}$ and $M_{(-j,j)} = M_{(j,-j)}$. This results in the following block diagonal form of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$:

$$(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \begin{pmatrix}
M_{(-3,-3)} - M_{(-3,3)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & M_{(-3,-3)} + M_{(-3,3)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_{(0,0)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{(-2,-2)} & M_{(-2,2)} & 0 & 0 \\
0 & 0 & 0 & M_{(-2,2)} & M_{(-2,-2)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & M_{(-1,-1)} & M_{(-1,1)} \\
0 & 0 & 0 & 0 & 0 & M_{(-1,1)} & M_{(-1,-1)}
\end{pmatrix}.$$

Eigenvalues in W_0**:** The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_0 are given by the eigenvalues of $M_{(-3,-3)} - M_{(-3,3)}$. Since W_0 contains the zero eigenvector $\mathbf{a}_4$, $M_{(-3,-3)} - M_{(-3,3)}$ has a zero eigenvalue and the other eigenvalue is given by $2\mathrm{Re}(m_{(-3,-3)} - m_{(-3,3)})$ where

$$\begin{aligned} m_{(-3,-3)} &= \left.\frac{\partial g_{-3}}{\partial z_{-3}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (3A+2B+10C+9D)\,|w_1|^2 = (A+25C+9D)\,|w_1|^2 \\ m_{(-3,3)} &= \left.\frac{\partial g_{-3}}{\partial z_3}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = (-A-2B+40C+9D)\,|w_1|^2 \end{aligned}$$

using the branching equation. Thus the eigenvalues in W_0 are

$$\xi_0^+ = (4A_r + 4B_r - 30C_r)\,|w_1|^2 = -2\lambda \qquad \text{and} \qquad \xi_0^- = 0.$$

Eigenvalues in W_1: The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_1 are given by the eigenvalues of $M_{(-3,-3)} + M_{(-3,3)}$. Since W_1 contains the zero eigenvector $\mathbf{a}_1$, $M_{(-3,-3)} + M_{(-3,3)}$ has a zero eigenvalue and the other eigenvalue is given by $2\mathrm{Re}(m_{(-3,-3)} + m_{(-3,3)})$. Thus the eigenvalues in W_1 are

$$\xi_1^+ = (-4B_r + 130C_r + 36D_r)\,|w_1|^2 \qquad \text{and} \qquad \xi_1^- = 0.$$

Eigenvalues in W_2: The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_2 are given by the eigenvalues of $M_{(0,0)}$. Since

$$\begin{aligned} m_{(0,0)} &= \left.\frac{\partial g_0}{\partial z_0}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = \nu(\lambda) + (2A-40C)\,|w_1|^2 = (-2B-25C)\,|w_1|^2 \\ m'_{(0,0)} &= \left.\frac{\partial g_0}{\partial \bar{z}_0}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = (2B-20C)\,w_1^2 \end{aligned}$$

the eigenvalues are the roots of

$$\xi^2 - (-4B_r - 50C_r)\,|w_1|^2\xi + \left(|2B+25C|^2 - |2B-20C|^2\right)|w_1|^4 = 0.$$

Thus the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_2 are

$$\xi_2^{\pm} = \left[-2B_r - 25C_r \pm \sqrt{4(B_r - 10C_r)^2 - 45C_i(4B_i + 5C_i)}\right]|w_1|^2.$$

Eigenvalues in W_3: The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_3 are given by the eigenvalues of

$$\begin{pmatrix} M_{(-2,-2)} & M_{(-2,2)} \\ M_{(-2,2)} & M_{(-2,-2)} \end{pmatrix} = \begin{pmatrix} m_{(-2,-2)} & 0 & 0 & m'_{(-2,2)} \\ 0 & \overline{m_{(-2,-2)}} & \overline{m'_{(-2,2)}} & 0 \\ 0 & m'_{(-2,2)} & m_{(-2,-2)} & 0 \\ \overline{m'_{(-2,2)}} & 0 & 0 & \overline{m_{(-2,-2)}} \end{pmatrix}$$

due to the fact that g_{-2} cannot contain any terms which do not disappear upon differentiating with respect to $\bar{z}_{-2}$ or z_2 and evaluating at $\mathbf{z}_0$ and as such

$$m'_{(-2,-2)} = \left.\frac{\partial g_{-2}}{\partial \bar{z}_{-2}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0 \qquad \text{and} \qquad m_{(-2,2)} = \left.\frac{\partial g_{-2}}{\partial z_2}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} = 0.$$

Hence the eigenvalues are double and given by the eigenvalues of

$$E_3 = \begin{pmatrix} m_{(-2,-2)} & m'_{(-2,2)} \\ \overline{m'_{(-2,2)}} & \overline{m_{(-2,-2)}} \end{pmatrix}.$$

Since the zero eigenvectors $\mathbf{a}_2$ and $\mathbf{a}_3$ are contained in W_3 the eigenvalues of E_3 are zero and $2\mathrm{Re}(m_{(-2,-2)})$ where

$$m_{(-2,-2)} \;=\; \left.\frac{\partial g_{-2}}{\partial z_{-2}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} \;=\; \nu(\lambda) + (2A+3D)\,|w_1|^2 \;=\; (-2B+15C+3D)\,|w_1|^2.$$

Hence the double eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_3 are

$$\xi_2^+ = (-4B_r + 30C_r + 6D_r)\,|w_1|^2 \qquad \text{and} \qquad \xi_2^- = 0.$$

Eigenvalues in W_4: By an argument similar to that above, the eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_4 are double and equal to the eigenvalues of

$$E_4 = \begin{pmatrix} m_{(-1,-1)} & m'_{(-1,1)} \\ \overline{m'_{(-1,1)}} & \overline{m_{(-1,-1)}} \end{pmatrix}.$$

These are given by the roots of

$$\xi^2 - 2\mathrm{Re}\left(m_{(-1,-1)}\right)\xi + |m_{(-1,-1)}|^2 - |m'_{(-1,1)}|^2 = 0$$

where

$$\begin{aligned} m_{(-1,-1)} \;&=\; \left.\frac{\partial g_{-1}}{\partial z_{-1}}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} \;=\; \nu(\lambda) + (2A-30C)\,|w_1|^2 \;=\; (-2B-15C)\,|w_1|^2 \\ m'_{(-1,1)} \;&=\; \left.\frac{\partial g_{-1}}{\partial \bar{z}_1}\right|_{(\mathbf{z}_0,\lambda_0,\tau_0)} \;=\; (-2B+15C)\,w_1^2. \end{aligned}$$

Thus the double eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ in W_4 are

$$\xi_4^{\pm} = \left[-2B_r - 15C_r \pm \sqrt{(2B_r-15C_r)^2 - 120B_iC_i}\right]|w_1|^2.$$

Since we have found four zero eigenvalues as expected, if the values of A, B, C and D are such that the other eigenvalues are non-zero then $\widetilde{\mathbf{D}_6}$ has 3-determined stability.

Summary

The eigenvalues of each of the $\mathbb{C}$-axial periodic solutions in each of the isotypic components for that solution are listed in Table 5.6. The isotypic components for the actions of each of the $\mathbb{C}$-axial subgroups are summarised in Table 5.7.

Table 5.6: The eigenvalues of $(\mathrm{d}g)|_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for each $\mathbb{C}$-axial branch of periodic solutions by isotypic component.

Isotropy subgroup	Isotypic component	Eigenvalues	Multiplicity
$\widetilde{\mathbf{O(2)}}$	W_0	$(2A_r+2B_r-24C_r)\lvert w_1\rvert^2=-2\lambda$	1
		0	1
	W_1	$(-2B_r+48C_r+12D_r)\lvert w_1\rvert^2$	2
		0	2
	W_2	$\left[-B_r+12C_r+\sqrt{B_r^2+24B_iC_i-144C_i^2}\right]\lvert w_1\rvert^2$	2
		$\left[-B_r+12C_r-\sqrt{B_r^2+24B_iC_i-144C_i^2}\right]\lvert w_1\rvert^2$	2
	W_3	$\left[-B_r-8C_r+\sqrt{B_r^2-20B_rC_r+100C_r^2-36B_iC_i+36C_i^2}\right]\lvert w_1\rvert^2$	2
		$\left[-B_r-8C_r-\sqrt{B_r^2-20B_rC_r+100C_r^2-36B_iC_i+36C_i^2}\right]\lvert w_1\rvert^2$	2
$\widetilde{\mathbf{SO(2)}}_1$	W_0	$(2A_r-6C_r+2D_r)\lvert w_1\rvert^2=-2\lambda$	1
		0	1
	W_1	$\xi=(-12C-3D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_2	$\xi=(-22C-4D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_3	$\xi=(33C_r+11D_r+3\mathrm{i}C_i+\mathrm{i}D_i)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
		0	2
	W_4	$\xi_\pm=\left(-C+D+\overline{B}-\frac{13}{2}\overline{C}-\overline{D}\pm\sqrt{\delta}\right)\lvert w_1\rvert^2$ and $\overline{\xi}_\pm$	1 of each
		where $\delta=\left(-C+D+\overline{B}-\frac{13}{2}\overline{C}-\overline{D}\right)^2$	
		$+(2C-2D)\left(2\overline{B}-13\overline{C}-2\overline{D}\right)+60\lvert C\rvert^2$	
$\widetilde{\mathbf{SO(2)}}_2$	W_0	$(2A_r+8D_r)\lvert w_1\rvert^2=-2\lambda$	1
		0	1
	W_1	$\xi=-4D\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_2	$\xi=(-15C-6D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_3	$\xi=(2B-40C-8D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_4	$\xi=(-25C-10D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_5	$\xi=(40C_r+8D_r+10\mathrm{i}C_i+2\mathrm{i}D_i)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
		0	2
$\widetilde{\mathbf{SO(2)}}_3$	W_0	$(2A_r+50C_r+18D_r)\lvert w_1\rvert^2=-2\lambda$	1
		0	1
	W_1	0	2
	W_2	$\xi=(-30C-6D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_3	$\xi=(-45C-9D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_4	$\xi=(-50C-12D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_5	$\xi=(-50C-15D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_6	$\xi=(2B-65C-18D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
$\widetilde{\mathrm{O}}$	W_0	$(4A_r+4B_r-80C_r)\lvert w_1\rvert^2=-2\lambda$	1
		0	1
	W_1	$(-4B_r+80C_r+16D_r)\lvert w_1\rvert^2$	3
		0	3
	W_2	$\left[-2B_r+40C_r+2\sqrt{B_r^2+40B_iC_i-400C_i^2}\right]\lvert w_1\rvert^2$	3
		$\left[-2B_r+40C_r-2\sqrt{B_r^2+40B_iC_i-400C_i^2}\right]\lvert w_1\rvert^2$	3

Continued on next page

Table 5.6 : – continued from previous page

Isotropy subgroup	Isotypic component	Eigenvalues	Multiplicity
$\widetilde{\mathbf{D}_6}$	W_0	$(4A_r + 4B_r - 30C_r)\lvert w_1\rvert^2 = -2\lambda$	1
		0	1
	W_1	$(-4B_r + 130C_r + 36D_r)\lvert w_1\rvert^2$	1
		0	1
	W_2	$\left[-2B_r - 25C_r + \sqrt{4(B_r - 10C_r)^2 - 45C_i(4B_i + 5C_i)}\right]\lvert w_1\rvert^2$	1
		$\left[-2B_r - 25C_r - \sqrt{4(B_r - 10C_r)^2 - 45C_i(4B_i + 5C_i)}\right]\lvert w_1\rvert^2$	1
	W_3	$(-4B_r + 30C_r + 6D_r)\lvert w_1\rvert^2$	2
		0	2
	W_4	$\left[-2B_r - 15C_r + \sqrt{(2B_r - 15C_r)^2 - 120B_iC_i}\right]\lvert w_1\rvert^2$	2
		$\left[-2B_r - 15C_r - \sqrt{(2B_r - 15C_r)^2 - 120B_iC_i}\right]\lvert w_1\rvert^2$	2

Σ	Isotypic components
$\widetilde{\mathbf{O(2)}}$	$W_0 = \{(0,0,0,u_1,0,0,0)\} = \mathrm{Fix}(\widetilde{\mathbf{O(2)}})$
	$W_1 = \{(0,0,u_1,0,u_2,0,0)\}$
	$W_2 = \{(0,u_1,0,0,0,u_2,0)\}$
	$W_3 = \{(u_1,0,0,0,0,0,u_2)\}$
$\widetilde{\mathbf{SO(2)}}_1$	$W_0 = \{(0,0,u_1,0,0,0,0)\} = \mathrm{Fix}(\widetilde{\mathbf{SO(2)}}_1)$
	$W_1 = \{(0,0,0,0,0,u_1,0)\}$
	$W_2 = \{(0,0,0,0,0,0,u_1)\}$
	$W_3 = \{(0,u_1,0,u_2,0,0,0)\}$
	$W_4 = \{(u_1,0,0,0,u_2,0,0)\}$
$\widetilde{\mathbf{SO(2)}}_2$	$W_0 = \{(0,u_1,0,0,0,0,0)\} = \mathrm{Fix}(\widetilde{\mathbf{SO(2)}}_2)$
	$W_1 = \{(0,0,0,u_1,0,0,0)\}$
	$W_2 = \{(0,0,0,0,u_1,0,0)\}$
	$W_3 = \{(0,0,0,0,0,u_1,0)\}$
	$W_4 = \{(0,0,0,0,0,0,u_1)\}$
	$W_5 = \{(u_1,0,u_2,0,0,0,0)\}$
$\widetilde{\mathbf{SO(2)}}_3$	$W_0 = \{(u_1,0,0,0,0,0,0)\} = \mathrm{Fix}(\widetilde{\mathbf{SO(2)}}_3)$
	$W_1 = \{(0,u_1,0,0,0,0,0)\}$
	$W_2 = \{(0,0,u_1,0,0,0,0)\}$
	$W_3 = \{(0,0,0,u_1,0,0,0)\}$
	$W_4 = \{(0,0,0,0,u_1,0,0)\}$
	$W_5 = \{(0,0,0,0,0,u_1,0)\}$
	$W_6 = \{(0,0,0,0,0,0,u_1)\}$
$\widetilde{\mathbb{O}}$	$W_0 = \{(0,u_1,0,0,0,-u_1,0)\} = \mathrm{Fix}(\widetilde{\mathbb{O}})$
	$W_1 = \{(\sqrt{3}u_1, u_3, \sqrt{5}u_2, 0, -\sqrt{5}u_1, u_3, -\sqrt{3}u_2)\}$
	$W_2 = \{(\sqrt{5}u_1, 0, \sqrt{3}u_2, u_3, \sqrt{3}u_1, 0, \sqrt{5}u_2)\}$
$\widetilde{\mathbf{D}_6}$	$W_0 = \{(u_1,0,0,0,0,0,-u_1)\} = \mathrm{Fix}(\widetilde{\mathbf{D}_6})$
	$W_1 = \{(u_1,0,0,0,0,0,u_1)\}$
	$W_2 = \{(0,0,0,u_1,0,0,0)\}$
	$W_3 = \{(0,u_1,0,0,0,u_2,0)\}$
	$W_4 = \{(0,0,u_1,0,u_2,0,0)\}$

Table 5.7: Isotypic components for the actions of the $\mathbb{C}$-axial subgroups Σ on V_3.

5.4.3 Conditions for stability of the solution branches

We now state a theorem which lists conditions in terms of the coefficients A, B, C and D for each of the individual solution branches to be stable.

Theorem 5.4.1. *For each $\mathbb{C}$-axial subgroup, Σ, listed in Table 5.1 let $\Delta_0, \ldots, \Delta_k$ be the functions of the coefficients A, B, C and D given in Table 5.8. Then*

(i) For each Σ, the corresponding branch of periodic solutions to (5.1.1) is supercritical if $\Delta_0 < 0$ and subcritical if $\Delta_0 > 0$.

(ii) For each Σ, the corresponding branch of periodic solutions to (5.1.1) is stable near $\lambda = 0$ if $\Delta_j < 0$ for all j. If $\Delta_j > 0$ for some $j = 0, \ldots, k$ then the branch of periodic solutions is unstable.

Proof. The conditions in Table 5.8 are those which must be satisfied for each of the branches of solutions to have eigenvalues with negative real part. These eigenvalues were found using F_3, the cubic order truncation of the Taylor expansion of the general $\mathbf{O(3)} \times S^1$ equivariant vector field. Since each of the six $\mathbb{C}$-axial isotropy subgroups Σ have 3-determined stability, these conditions are sufficient to determine the stability of the periodic solutions of (5.1.1) with axial symmetry by Theorem 2.5.9. □

5.4.4 Remarks and Examples

From our analysis of the stability of the six branches of periodic solutions to (5.1.1) with maximal symmetry we make the following observations.

1. We have found that for each of the six $\mathbb{C}$-axial isotropy subgroups of $\mathbf{O(3)} \times S^1$ in the natural representation on $V_3 \oplus V_3$, the periodic solution with this symmetry (whose existence is guaranteed by the equivariant Hopf theorem) has 3-determined stability. That is, the cubic order truncation of the Taylor series of the general $\mathbf{O(3)} \times S^1$ equivariant mapping is sufficient to determine the stability of each of the six solution branches. The number of zero eigenvalues for each solution branch is the number forced to be zero by symmetry. This is in contrast with the representation on $V_2 \oplus V_2$ where Iooss and Rossi [54] and Haaf et al. [51] found that some $\mathbb{C}$-axial isotropy subgroups had 5-determined stability, so the conditions for stability of the corresponding branches of periodic solutions included coefficients of order 5 terms in the Taylor expansion of the general equivariant mapping on $V_2 \oplus V_2$.

2. For each of the six solution branches it is possible to find a set of values for A, B, C and D such that the solution is stable:

   Example 5.4.2. Suppose that

   $$A_r = -20 \qquad B_r = \frac{5}{2} \qquad B_i = 5 \qquad C_r = -\frac{1}{6} \qquad C_i = \frac{3}{2} \qquad D_r = \frac{3}{5}$$

Σ	Δ_0	$\Delta_1, \ldots, \Delta_k$
$\widetilde{\mathbf{O(2)}}$	$A_r + B_r - 12C_r$	$-B_r + 24C_r + 6D_r$
		$-B_r + 12C_r$
		$\mathrm{Re}(B\overline{C}) - 6\|C\|^2$
		$-B_r - 8C_r$
		$\|C\|^2 - \mathrm{Re}(B\overline{C})$
$\widetilde{\mathbf{SO(2)}}_1$	$A_r - 3C_r + D_r$	$2B_r - 15C_r$
		$-4C_r - D_r$
		$-11C_r - 2D_r$
		$3C_r + D_r$
		$\left\|\mathrm{Re}(\sqrt{\delta})\right\| - \left\|B_r - \frac{15}{2}C_r\right\|^{(*)}$
$\widetilde{\mathbf{SO(2)}}_2$	$A_r + 4D_r$	$-D_r$
		$-5C_r - 2D_r$
		$B_r - 20C_r - 4D_r$
		$5C_r + D_r$
$\widetilde{\mathbf{SO(2)}}_3$	$A_r + 25C_r + 9D_r$	$-5C_r - D_r$
		$-25C_r - 6D_r$
		$2B_r - 65C_r - 18D_r$
		$-10C_r - 3D_r$
$\widetilde{\mathbb{O}}$	$A_r + B_r - 20C_r$	$-B_r + 20C_r + 4D_r$
		$-B_r + 20C_r$
		$\mathrm{Re}(B\overline{C}) - 10\|C\|^2$
$\widetilde{\mathbf{D}_6}$	$2A_r + 2B_r - 15C_r$	$-2B_r - 25C_r$
		$-5\|C\|^2 - 4\mathrm{Re}(B\overline{C})$
		$-2B_r - 15C_r$
		$-\mathrm{Re}(B\overline{C})$
		$-2B_r + 15C_r + 3D_r$
		$-2B_r + 65C_r + 18D_r$

Table 5.8: Stability conditions for the six branches of periodic solutions. If $\Delta_j < 0$ for all j then the branch of periodic solutions is stable near $\lambda = 0$.
(*): $\delta = \left(-C + D + \overline{B} - \frac{13}{2}\overline{C} - \overline{D}\right)^2 + (2C - 2D)\left(2\overline{B} - 13\overline{C} - 2\overline{D}\right) + 60|C|^2$.

and A_i and D_i take any values. Then we can see that the three standing wave solutions (with symmetries $\widetilde{\mathbf{O(2)}}$, $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbf{D}_6}$) are all stable and the travelling wave solutions (with symmetries $\widetilde{\mathbf{SO(2)}}_k$, $k = 1, 2, 3$) are all unstable. The bifurcation diagram near the bifurcation point $\lambda = 0$ is then as in Figure 5.3.

Example 5.4.3. Suppose that

$$A_r = 10 \qquad B_r = 1 \qquad B_i = -20 \qquad C_r = 3 \qquad C_i = 3 \qquad D_r = -10 \qquad D_i = -5$$

and A_i takes any value. Then we find that the solution with $\widetilde{\mathbf{SO(2)}}_1$ symmetry is stable. The bifurcation diagram near the bifurcation point $\lambda = 0$ is then as in Figure 5.4. The stability of the branch of solutions with $\widetilde{\mathbf{SO(2)}}_3$ symmetry is not determined since, with these parameter values, this solution has zero eigenvalues in addition to those forced to

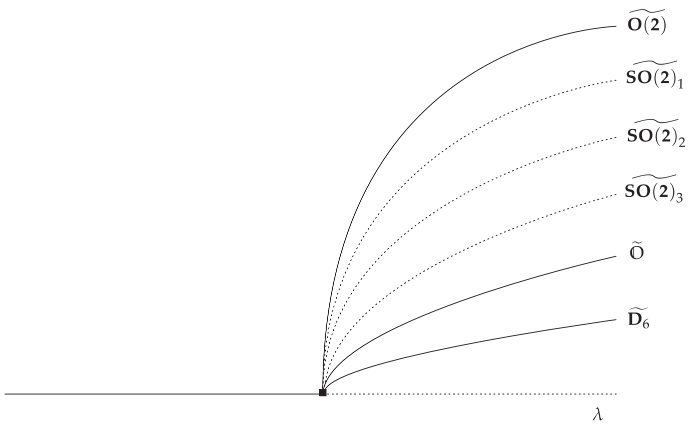


Figure 5.3: Bifurcation diagram for the values of the parameters A, B, C and D as in Example 5.4.2. Stable solutions are denoted by solid lines and unstable solutions by dashed lines.

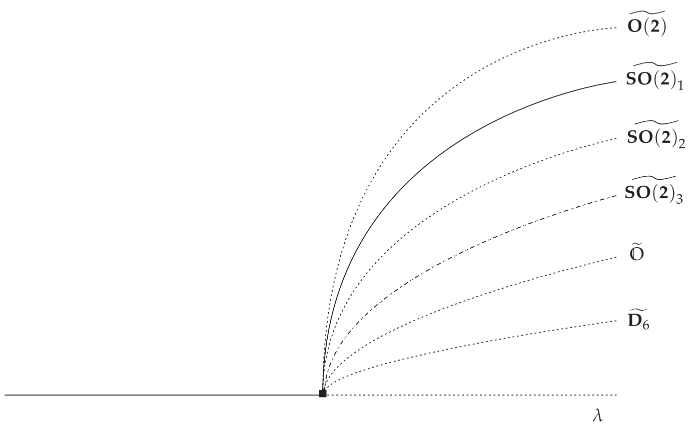


Figure 5.4: Bifurcation diagram for the values of the parameters A, B, C and D as in Example 5.4.3. Stable solutions are denoted by solid lines and unstable solutions by dashed lines. The dot-dashed line indicates that the stability of the solution with $\widetilde{\mathbf{SO(2)}}_3$ symmetry is not determined at cubic order for these parameter values.

be zero by symmetry.

Example 5.4.4. Suppose that

$$A_r = -2 \qquad B_r = -1 \qquad C_r = -\frac{1}{10} \qquad D_r = \frac{1}{3}$$

and A_i, B_i, C_i and D_i take any values. Then we find that the solution with $\widetilde{\mathbf{SO(2)}}_2$ symmetry is stable. The bifurcation diagram near the bifurcation point $\lambda = 0$ is then as in Figure 5.5.

Example 5.4.5. Suppose that

$$A_r = -40 \qquad B_r = 10 \qquad C_r = 3 \qquad D_r = -5$$

and A_i, B_i, C_i and D_i take any values. Then we find that the solution with $\widetilde{\mathbf{SO(2)}}_3$ symmetry is stable. The bifurcation diagram near the bifurcation point $\lambda = 0$ is then as in Figure 5.6.

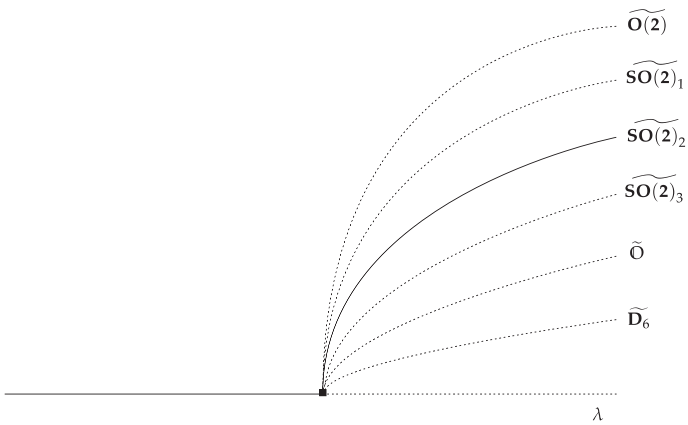


Figure 5.5: Bifurcation diagram for the values of the parameters A, B, C and D as in Example 5.4.4. Stable solutions are denoted by solid lines and unstable solutions by dashed lines.

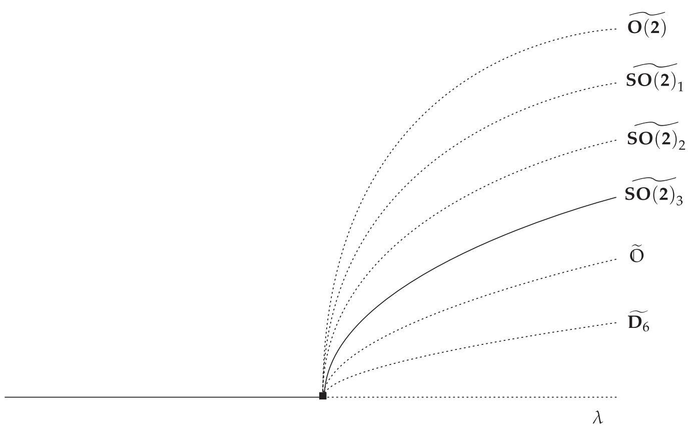


Figure 5.6: Bifurcation diagram for the values of the parameters A, B, C and D as in Example 5.4.5. Stable solutions are denoted by solid lines and unstable solutions by dashed lines.

3. It is possible for all six branches of periodic solutions to bifurcate supercritically and be unstable for some values of the parameters A, B, C and D. This could mean that a submaximal solution is stable, there is a heteroclinic orbit or the behaviour of the system with these parameter values is chaotic.

   Example 5.4.6. Suppose that

   $$A_r = -30 \qquad B_r = 50 \qquad B_i = 50 \qquad C_r = 3 \qquad C_i = -150 \qquad D_r = -13$$

   and A_i and D_i take any values. Then we find that all six maximal branches of periodic solutions bifurcate supercritically yet are unstable. The bifurcation diagram near the bifurcation point $\lambda = 0$ is then as in Figure 5.7.

4. There are a number of pairs of solution branches which are never simultaneously stable. Using Table 5.8 these can be seen to be

   (a) $\widetilde{\mathbf{SO(2)}}_1$ and $\widetilde{\mathbf{SO(2)}}_2$

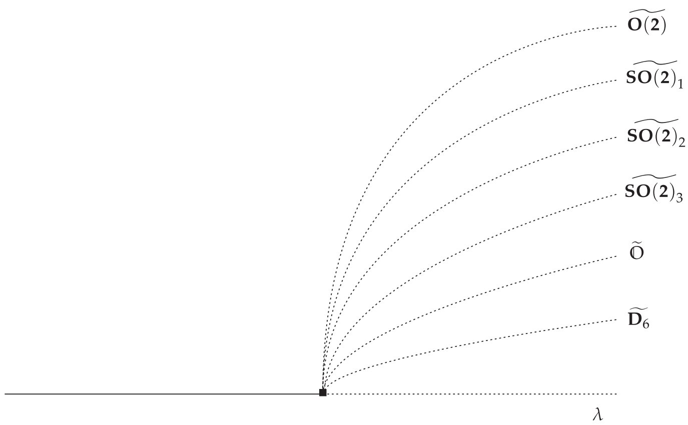


Figure 5.7: Bifurcation diagram for the values of the parameters A, B, C and D as in Example 5.4.6. Stable solutions are denoted by solid lines and unstable solutions by dashed lines.

(b) $\widetilde{\mathbf{SO(2)}}_2$ and $\widetilde{\mathbf{SO(2)}}_3$

(c) $\widetilde{\mathbf{SO(2)}}_2$ and $\widetilde{\mathbb{O}}$

(d) $\widetilde{\mathbf{SO(2)}}_3$ and $\widetilde{\mathbf{D}_6}$.

This concludes our analysis of the stability of the solutions to (5.1.1) with maximal $\mathbb{C}$-axial symmetry. We now move on to consider the existence and stability of solutions with submaximal symmetry, in particular solutions with symmetry Σ where Σ is an isotropy subgroup with $\dim \mathrm{Fix}(\Sigma) = 4$.

5.5 Submaximal solution branches

In Section 5.4 we considered only the solutions of (5.1.1) which have $\mathbb{C}$-axial symmetry. These solutions are guaranteed to exist for all values of the coefficients A, B, C and D in the cubic order truncation of the Taylor series of f, the general $\mathbf{O(3)} \times S^1$ equivariant vector field. However, these are not the only solutions of (5.1.1). In this section we will find conditions on the values of the coefficients A, B, C and D which allow the existence of solutions with symmetry groups Σ where Σ is an isotropy subgroup of $\mathbf{O(3)} \times S^1$ in the representation on $V_3 \oplus V_3$ with a four-dimensional fixed-point subspace. These subgroups are listed in Table 5.1.

Remark 5.5.1. In the natural representation of $\mathbf{O(3)} \times S^1$ on $V_3 \oplus V_3$ all of the isotropy subgroups, Σ, with four-dimensional fixed-point subspaces lie inside $\mathbb{C}$-axial subgroups. Hence the subgroups Σ are submaximal and we refer to solutions with symmetry Σ as submaximal solutions.

Suppose that
$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = f(\mathbf{z}, \lambda),$$
where $f : \mathbb{C}^7 \times \mathbb{R} \to \mathbb{C}^7$ is equivariant with respect to $\mathbf{O(3)} \times S^1$ to all orders i.e. f is the exact Birkhoff normal form, not just a truncated Taylor series. Then recall from Remark 5.3.3 that

f restricts to a $N(\Sigma)/\Sigma$ equivariant system on $\mathrm{Fix}(\Sigma)$ for some action of $N(\Sigma)/\Sigma$. For each isotropy subgroup Σ the group $N(\Sigma)/\Sigma$ is given in Table 5.1. For the isotropy subgroups with $\dim \mathrm{Fix}(\Sigma) = 4$ the action of $N(\Sigma)/\Sigma$ on $\mathrm{Fix}(\Sigma)$ is given in Table 5.9.

Σ	$N(\Sigma)/\Sigma$	Action	
$\widetilde{\mathbb{Z}}_6$	$\mathbf{O(2)} \times S^1$	$\phi(w_1, w_2) = (e^{-3i\phi} w_1, e^{3i\phi} w_2)$	$\phi \in \mathbf{SO(2)}$
		$\kappa(w_1, w_2) = (-w_2, -w_1)$	$\kappa \in \mathbf{O(2)}$
$\widetilde{\mathbb{Z}}_4$	$\mathbf{O(2)} \times S^1$	$\phi(w_1, w_2) = (e^{-2i\phi} w_1, e^{2i\phi} w_2)$	$\phi \in \mathbf{SO(2)}$
		$\kappa(w_1, w_2) = (-w_2, -w_1)$	$\kappa \in \mathbf{O(2)}$
$\widetilde{\mathbf{D}}_3$	$\mathbf{D}_2 \times S^1$	$R^z_\pi(w_1, w_2) = (-w_1, w_2)$	$R^z_\pi \in \mathbf{D}_2$
		$R^y_\pi(w_1, w_2) = (w_1, w_2)$	$R^y_\pi \in \mathbf{D}_2$
$\widetilde{\mathbf{D}}_2$	$\mathbf{D}_2 \times S^1$	$R^z_\pi(w_1, w_2) = (w_1, w_2)$	$R^z_\pi \in \mathbf{D}_2$
		$R^y_\pi(w_1, w_2) = (-w_1, w_2)$	$R^y_\pi \in \mathbf{D}_2$
$\widetilde{\mathbb{Z}}^1_3$	$\mathbf{SO(2)} \times S^1$	$\phi(w_1, w_2) = (e^{-i\phi} w_1, e^{2i\phi} w_2)$	$\phi \in \mathbf{SO(2)}$
$\widetilde{\mathbb{Z}}_5$	$\mathbf{SO(2)} \times S^1$	$\phi(w_1, w_2) = (e^{-3i\phi} w_1, e^{2i\phi} w_2)$	$\phi \in \mathbf{SO(2)}$

Table 5.9: The action of $N(\Sigma)/\Sigma$ on $\mathrm{Fix}(\Sigma)$ for isotropy subgroups Σ with $\dim \mathrm{Fix}(\Sigma) = 4$. For each Σ, $\psi \in S^1$ acts as $\psi(w_1, w_2) = e^{i\psi}(w_1, w_2)$.

We now consider the restriction of f to $\mathrm{Fix}(\Sigma)$ for each isotropy subgroup Σ given in Table 5.9. To cubic order the restriction of f to $\mathrm{Fix}(\Sigma)$ is equal to the restriction of F_3 to $\mathrm{Fix}(\Sigma)$ where F_3 is as in (5.2.6). We will look for changes in stability of the maximal solution branches within $\mathrm{Fix}(\Sigma)$ and identify additional periodic and quasiperiodic solutions to (5.1.1) which lie in these subspaces.

5.5.1 Solutions in $\mathrm{Fix}(\widetilde{\mathbb{Z}}_6)$ and $\mathrm{Fix}(\widetilde{\mathbb{Z}}_4)$

We first look for submaximal solutions with symmetry $\Sigma = \widetilde{\mathbb{Z}}_6$ or $\widetilde{\mathbb{Z}}_4$. For both of these isotropy subgroups $N(\Sigma)/\Sigma = \mathbf{O(2)} \times S^1$. Using the actions of $\mathbf{O(2)} \times S^1$ given in Table 5.9 for these two isotropy subgroups we compute that to cubic order the Taylor expansion of the general mapping which commutes with these actions is in both cases of the form

$$\begin{aligned} \dot{w}_1 &= \mu w_1 + \alpha w_1 |w_1|^2 + \beta w_1 |w_2|^2 \\ \dot{w}_2 &= \mu w_2 + \alpha w_2 |w_2|^2 + \beta w_2 |w_1|^2. \end{aligned} \tag{5.5.1}$$

In the restriction of F_3 to $\mathrm{Fix}(\widetilde{\mathbb{Z}}_6)$,

$$\alpha = A + 25C + 9D \qquad \text{and} \qquad \beta = A + 2B - 40C - 9D$$

and in the restriction to $\mathrm{Fix}(\widetilde{\mathbb{Z}}_4)$,

$$\alpha = A + 4D \qquad \text{and} \qquad \beta = A + 2B - 40C - 4D.$$

The system of equations (5.5.1) has previously been studied in the context of a Hopf bifurcation with $\mathbf{O(2)}$ symmetry. Provided that none of the coefficients of the cubic terms in the Birkhoff normal form of this bifurcation vanish (i.e. there are no degeneracies, so there are two cubic equivariant mappings) then there are only two types of solutions which bifurcate from a Hopf bifurcation with $\mathbf{O}(\mathbf{2})$ symmetry: standing waves and rotating or travelling waves.

In Fix($\widetilde{\mathbb{Z}}_6$) the standing waves are solutions with $\widetilde{\mathbf{D}_6}$ symmetry and the rotating waves are solutions with $\widetilde{\mathbf{SO}(\mathbf{2})}_3$ symmetry. In Fix($\widetilde{\mathbb{Z}}_4$) the standing waves are solutions with $\widetilde{\mathbb{O}}$ symmetry and the rotating waves are solutions with $\widetilde{\mathbf{SO}(\mathbf{2})}_2$ symmetry. Since generically these are the only solutions which bifurcate, in the Hopf bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry there are no solutions with $\widetilde{\mathbb{Z}}_6$ or $\widetilde{\mathbb{Z}}_4$ symmetry.

Only if we allow degeneracies is it possible for solutions with $\widetilde{\mathbb{Z}}_6$ or $\widetilde{\mathbb{Z}}_4$ symmetry to exist. For example if $\alpha_r = \beta_r$ then a solution with submaximal symmetry can exist: The standing wave solution in Fix(Σ) is given by $w_1 = w_2$ and the eigenvalues of this solution are

$$2\mathrm{Re}(\alpha+\beta)|w_1|^2, \quad 2\mathrm{Re}(\alpha-\beta)|w_1|^2, \quad \text{and} \quad 0 \text{ twice.}$$

The travelling wave solution in Fix(Σ) is given by $w_2 = 0$ and the eigenvalues of this solution are

$$2\mathrm{Re}(\alpha)|w_1|^2, \quad (\beta-\alpha)|w_1|^2, \quad (\bar{\beta}-\bar{\alpha})|w_1|^2, \quad \text{and} \quad 0.$$

Suppose that $\alpha_r < 0$ and $\alpha_r + \beta_r < 0$ so that both solutions bifurcate supercritically. If $\alpha_r - \beta_r < 0$ then the standing wave is stable in Fix(Σ) and if $\alpha_r - \beta_r > 0$ the travelling wave solution is stable. When $\alpha_r = \beta_r$ the stability of neither solution is determined by the cubic order truncation and a solution with submaximal symmetry exists.

For a classification of the possible degeneracies in the $\mathbf{O}(\mathbf{2})$ Hopf bifurcation see [42].

5.5.2 Solutions in Fix($\widetilde{\mathbf{D}_3}$) and Fix($\widetilde{\mathbf{D}_2}$)

Next, we look for submaximal solutions with symmetry groups $\Sigma = \widetilde{\mathbf{D}_3}$ and $\widetilde{\mathbf{D}_2}$ where $N(\Sigma)/\Sigma = \mathbf{D}_2 \times S^1$. If we compute to cubic order the equations which are equivariant with respect to the actions of $\mathbf{D}_2 \times S^1$ on Fix($\widetilde{\mathbf{D}_3}$) and Fix($\widetilde{\mathbf{D}_2}$) described in Table 5.9 then we find

$$\begin{aligned}
\dot{w}_1 &= \mu_1 w_1 + \alpha_1 w_1|w_1|^2 + \beta_1 w_1|w_2|^2 + \gamma_1 w_2^2\overline{w}_1 \\
\dot{w}_2 &= \mu_2 w_2 + \alpha_2 w_2|w_2|^2 + \beta_2 w_2|w_1|^2 + \gamma_2 w_1^2\overline{w}_2.
\end{aligned} \tag{5.5.2}$$

In the case where $\mu_1 = \mu_2$, these equations also occur in the context of a Hopf bifurcation on a rotating rhombic lattice in the restriction to certain four-dimensional subspaces. See, for example, [57, 59]. In the restriction of (5.2.6) to Fix($\widetilde{\mathbf{D}_3}$) or Fix($\widetilde{\mathbf{D}_2}$) we have $\mu_1 = \mu_2 = \mu$, $\beta_2 = 2\beta_1 = 2\beta$, $\gamma_2 = 2\gamma_1 = 2\gamma$ and in Fix($\widetilde{\mathbf{D}_3}$),

$$\begin{aligned}
\alpha_1 &= 2A + 2B - 15C \\
\alpha_2 &= A + B - 12C = \frac{1}{5}(2\alpha_1 + \beta + \gamma) \\
\beta &= A - 20C \\
\gamma &= B - 10C
\end{aligned} \tag{5.5.3}$$

and in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$,

$$\begin{aligned}\alpha_1 &= 2A + 2B - 40C \\ \alpha_2 &= A + B - 12C = \frac{1}{10}(3\alpha_1 + 4\beta + 4\gamma) \\ \beta &= A \\ \gamma &= B.\end{aligned} \tag{5.5.4}$$

In either case there are three branches of standing wave solutions, with symmetries $\widetilde{\mathbf{O}(\mathbf{2})}$, $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbf{D}_6}$, which bifurcate from the Hopf bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry. Depending on the values of the coefficients α_1, β and γ in $\mathrm{Fix}(\Sigma)$ for $\Sigma = \widetilde{\mathbf{D}_3}$ or $\widetilde{\mathbf{D}_2}$ it is possible to find solutions to (5.5.2) with Σ symmetry.

Here we will consider the equations in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$, where the values of the coefficients are given by (5.5.4). Since the equations in $\mathrm{Fix}(\widetilde{\mathbf{D}_3})$ have the same form, a similar analysis yields similar results.

In $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ the three standing wave solutions are $w_1 = 0$, with $\widetilde{\mathbf{O}(\mathbf{2})}$ symmetry, $w_2 = 0$, with $\widetilde{\mathbb{O}}$ symmetry and $w_1 = \sqrt{\frac{3}{10}}w_2$, with $\widetilde{\mathbf{D}_6}$ symmetry. This is because alternative forms of $\mathrm{Fix}(\widetilde{\mathbf{D}_6})$ and $\mathrm{Fix}(\widetilde{\mathbb{O}})$ to those given in Table 5.1 which lie inside $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ are

$$\mathrm{Fix}(\widetilde{\mathbf{D}_6}) = \left\{ \left(0, \sqrt{\frac{3}{10}}w_2, 0, w_2, 0, \sqrt{\frac{3}{10}}w_2, 0 \right) \right\} \qquad \mathrm{Fix}(\widetilde{\mathbb{O}}) = \{(0, w_1, 0, 0, 0, w_1, 0)\}. \tag{5.5.5}$$

This alternative form of $\mathrm{Fix}(\widetilde{\mathbb{O}})$ is found using the set of generators $(-R^z_{\pi/2}, 0)$, $(\mathcal{R}_{2\pi/3}, 0)$, $(\kappa_{xz}, 0)$, $(-I, \pi)$ with the actions as given in Section 5.3. To see that (5.5.5) gives a form of $\mathrm{Fix}(\widetilde{\mathbf{D}_6})$ we show that it just a rotation of the form of the fixed-point subspace given in Table 5.1. In this subspace

$$w(\theta, \phi, t) = 2\,\mathrm{Re}(w_1(t)) \left(Y_3^{-3}(\theta, \phi) - Y_3^3(\theta, \phi) \right) = \frac{R\cos(\omega t)}{2}\sqrt{\frac{35}{\pi}} \left(x^3 - 3xy^2 \right).$$

Suppose that we apply the transformation R^y_π which rotates this combination through π in the y-axis sending $x \to z$ and $z \to -x$. Then

$$\begin{aligned}R^y_\pi \cdot w(\theta, \phi, t) &= \frac{R\cos(\omega t)}{2}\sqrt{\frac{35}{\pi}} \left(z^3 - 3zy^2 \right) \\ &= 2\sqrt{\frac{3}{8}}R\cos(\omega t) \left(Y_3^{-2}(\theta, \phi) + Y_3^2(\theta, \phi) \right) + 2\sqrt{\frac{5}{4}}R\cos(\omega t) Y_3^0(\theta, \phi) \\ &= 2\sqrt{\frac{5}{4}}R\cos(\omega t) \left[\sqrt{\frac{3}{10}} \left(Y_3^{-2}(\theta, \phi) + Y_3^2(\theta, \phi) \right) + Y_3^0(\theta, \phi) \right].\end{aligned}$$

Hence the form of $\mathrm{Fix}(\widetilde{\mathbf{D}_6})$ given in (5.5.5) is just a rotated version of the form given in Table 5.1.

We now consider the points where the stability of these standing wave solution branches within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ change. For the periodic solution with $\widetilde{\mathbf{O}(\mathbf{2})}$ symmetry, $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ is contained in the direct sum of the isotypic components W_0 and W_2. Thus the eigenvalues of this solution within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ are

$$\xi_0^- = 0, \quad \xi_1^+ = (2A_r + 2B_r - 24C_r)\,|w_2|^2,$$

$$\xi_2^{\pm} = \left[-B_r + 12C_r \pm \sqrt{B_r^2 + 24B_iC_i - 144C_i^2} \right] |w_2|^2.$$

We see that this branch of solutions undergoes a stationary bifurcation if $\mathrm{Re}(B\overline{C}) = 6|C|^2$. It is also possible for this solution branch to undergo a Hopf bifurcation at $-B_r + 12C_r = 0$ if $6|C|^2 - \mathrm{Re}(B\overline{C}) > 0$ there.

The eigenvalues of the $\widetilde{\mathbb{O}}$ symmetric branch in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ are

$$\xi_0^- = 0, \quad \xi_1^+ = (4A_r + 4B_r - 80C_r)\,|w_1|^2,$$

$$\xi_2^{\pm} = \left[-2B_r + 40C_r \pm 2\sqrt{B_r^2 + 40B_iC_i - 400C_i^2} \right] |w_1|^2.$$

Hence this branch of solutions undergoes a stationary bifurcation if $\mathrm{Re}(B\overline{C}) = 10|C|^2$. It can also undergo a Hopf bifurcation at $-B_r + 20C_r = 0$ if $10|C|^2 - \mathrm{Re}(B\overline{C}) > 0$ there.

Finally, the periodic solution with $\widetilde{\mathbf{D}_6}$ symmetry has eigenvalues

$$\xi_0^- = 0, \quad \xi_1^+ = (4A_r + 4B_r - 30C_r)\,|w|^2,$$

$$\xi_4^{\pm} = \left[-2B_r - 15C_r \pm \sqrt{(2B_r - 15C_r)^2 - 120B_iC_i} \right] |w|^2$$

in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ where w is some combination of w_1 and w_2. This solution undergoes a stationary bifurcation when $\mathrm{Re}(B\overline{C}) = 0$. It also has a zero eigenvalue at $-2B_r - 15C_r = 0$ which represents a Hopf bifurcation if $\mathrm{Re}(B\overline{C}) > 0$ there.

The bifurcations of these solution branches allow for the possibility of the existence of periodic and quasiperiodic solutions with $\widetilde{\mathbf{D}_2}$ symmetry. Using the numerical continuation package AUTO, it is possible to demonstrate the existence of these branches of periodic and quasiperiodic solutions with $\widetilde{\mathbf{D}_2}$ symmetry for some particular values of the coefficients A, B and C.

Remark 5.5.2. The numerical branch continuation package AUTO requires that the input equations are real. This means that instead of finding periodic solutions to (5.5.2) with coefficients given by (5.5.4), we set $w_1 = Re^{i\phi}$ and $w_2 = Se^{i\psi}$ where R, S, ϕ and ψ are real functions of time and find fixed points of the resulting set of real differential equations. Separating the real and imaginary parts and letting $\alpha = \alpha_1$ and $\theta = 2\phi - 2\psi$, (5.5.2) with coefficients given by (5.5.4) becomes

$$\dot{R} = \lambda R + \alpha_r R^3 + \beta_r R S^2 + R S^2 \left(\gamma_r \cos(\theta) + \gamma_i \sin(\theta) \right) \tag{5.5.6}$$

$$\dot{S} = \lambda S + \frac{1}{10}\left(3\alpha_r + 4\beta_r + 4\gamma_r\right) S^3 + 2\beta_r S R^2 + 2SR^2 \left(\gamma_r \cos(\theta) - \gamma_i \sin(\theta) \right) \tag{5.5.7}$$

$$\begin{aligned} \dot{\theta} = {} & 2R^2 \left(\alpha_i - 2\beta_i - 2\gamma_i \cos(\theta) - 2\gamma_r \sin(\theta) \right) \\ & + 2S^2 \left(\gamma_i \cos(\theta) - \gamma_r \sin(\theta) - \frac{3}{10}\alpha_i + \frac{3}{5}\beta_i - \frac{2}{5}\gamma_i \right). \end{aligned} \tag{5.5.8}$$

The solution of these equations with $R = 0$ has $\widetilde{\mathbf{O(2)}}$ symmetry, the solution with $S = 0$ has $\widetilde{\mathbb{O}}$ symmetry and the solution with $R = \sqrt{\frac{3}{10}}S$ and $\theta = 0$ has $\widetilde{\mathbf{D}_6}$ symmetry. Making this change of coordinates to a system of 3 real differential equations introduces complications. If $R = 0$ or $S = 0$ then the phase difference θ is not defined. This is due to the fact that the values of the frequencies ϕ and ψ depend on a parameter $\mathrm{Im}(\mu) = \omega$ which does not appear in the system

(5.5.6) – (5.5.8). A consequence of this is that for certain values of the coefficients α, β and γ, when $R = 0$ or $S = 0$ it may be that there is no value of θ for which $\dot{\theta} = 0$ and hence AUTO will not find a branch of stationary solutions to (5.5.6) – (5.5.8). The phase difference θ keeps changing as one of the periodic solutions with $\widetilde{\mathbf{O}(\mathbf{2})}$ or $\widetilde{\mathbb{O}}$ symmetry is approached. This occurs when the eigenvalues $\xi_2^{\pm}$ for the solutions with $\widetilde{\mathbf{O}(\mathbf{2})}$ or $\widetilde{\mathbb{O}}$ symmetry are complex. In this case trajectories spiral towards (away from) a stable (unstable) periodic orbit as shown in Figure 5.8. In the case where a value of θ which gives $\dot{\theta} = 0$ can be found the eigenvalues are real and the direction in which trajectories approach (move away from) a stable (unstable) periodic orbit is defined as in Figure 5.9.

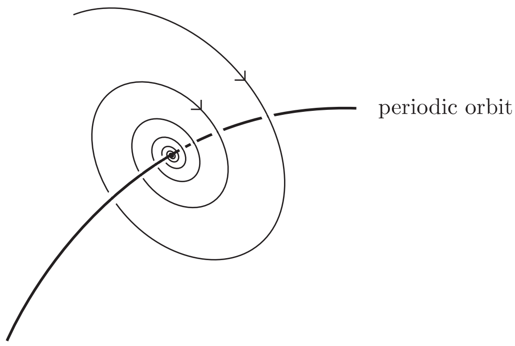


Figure 5.8: When $\dot{\theta} \neq 0$ for any value of $\theta \in \mathbb{R}$ trajectories spiral towards a stable periodic orbit where $R = 0$ or $S = 0$.

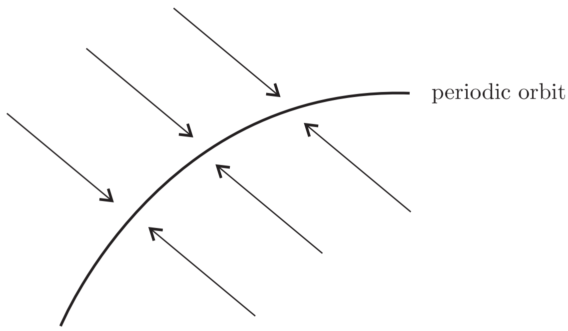


Figure 5.9: When $\dot{\theta} = 0$ trajectories approach the stable periodic orbit where $R = 0$ or $S = 0$ in a defined direction given by the eigenvector corresponding to the eigenvalue ξ_2.

Despite this complication, we wish to study the dynamics of the system near the points where $R = 0$ or $S = 0$. We do this using AUTO for a particular set of values of the coefficients A, B and C.

Example 5.5.3. Suppose that when $\lambda = 1$

$$A = -3 + \mathrm{i}, \qquad B = 1 + 3\mathrm{i}, \qquad C = C_r + \frac{3}{40}\mathrm{i}$$

and we vary the value of C_r. Then

$$\alpha = \alpha_1 = 2A + 2B - 40C = \alpha_r + 5\mathrm{i}, \qquad \beta = A = -3 + \mathrm{i}, \qquad \gamma = B = 1 + 3\mathrm{i}, \tag{5.5.9}$$

where $\alpha_r = -4 - 40C_r$. For these values

1. The $\widetilde{\mathbf{O(2)}}$ symmetric branch of solutions bifurcates supercritically when $\alpha_r < \frac{8}{3}$ and undergoes a stationary bifurcation at $\alpha_r = \frac{1}{3}\left(\sqrt{559} - 22\right)$.

2. The $\widetilde{\mathbb{O}}$ symmetric branch of solutions bifurcates supercritically when $\alpha_r < 0$ and undergoes a stationary bifurcation at $\alpha_r = \sqrt{31} - 6$.

3. The $\widetilde{\mathbf{D}_6}$ symmetric branch of solutions bifurcates supercritically when $\alpha_r < \frac{20}{3}$ and undergoes a stationary bifurcation at $\alpha_r = 5$ and a Hopf bifurcation at $\alpha_r = \frac{4}{3}$.

Using AUTO we find that there is a branch of periodic solutions connecting the $\widetilde{\mathbf{O(2)}}$ and $\widetilde{\mathbf{D}_6}$ symmetric branches and a branch of periodic solutions connecting the $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbf{D}_6}$ symmetric branches. These bifurcate at the stationary bifurcations and have $\widetilde{\mathbf{D}_2}$ symmetry. Neither of these solutions is stable. In addition there is a branch of stable quasiperiodic solutions which bifurcates from the solution with $\widetilde{\mathbf{D}_6}$ symmetry at the Hopf bifurcation. This solution branch also has $\widetilde{\mathbf{D}_2}$ symmetry. These branches of solutions can be seen in Figure 5.10.

As $\alpha_r \to \alpha_c \approx 2.17806$ the quasiperiodic solution spends an increasing amount of time near the unstable branch of solutions connecting the $\widetilde{\mathbf{O(2)}}$ and $\widetilde{\mathbf{D}_6}$ symmetric branches. This can be seen in Figure 5.11. At $\alpha_r = \alpha_c$ the system undergoes a global bifurcation to a homoclinic orbit.

Types of stationary bifurcations in Example 5.5.3

With the values of the coefficients as in (5.5.9), the system of real three differential equations for which AUTO finds the branches of fixed-points becomes

$$\dot{R} = R + \alpha_r R^3 - 3RS^2 + RS^2\left(\cos(\theta) + 3\sin(\theta)\right) \tag{5.5.10}$$

$$\dot{S} = S + \frac{1}{10}\left(3\alpha_r - 8\right)S^3 - 6SR^2 + 2SR^2\left(\cos(\theta) - 3\sin(\theta)\right) \tag{5.5.11}$$

$$\dot{\theta} = 2R^2\left(3 - 6\cos(\theta) - 2\sin(\theta)\right) + 2S^2\left(3\cos(\theta) - \sin(\theta) - \frac{21}{10}\right). \tag{5.5.12}$$

We can use these equations to determine the nature of the stationary bifurcations of the solution branches.

Bifurcation of $\widetilde{\mathbf{O(2)}}$ symmetric solution This branch of solutions has a stationary bifurcation at

$$\alpha_r = \alpha_0 = \frac{\sqrt{559} - 22}{3} \approx 0.5477269\ldots$$

At this value of α_r we have the stationary solution

$$R_0 = 0, \quad S_0 = \sqrt{\frac{10}{30 - \sqrt{559}}}, \quad \theta_0 = 2\arctan\left(\frac{-10 + \sqrt{559}}{51}\right).$$

Suppose we expand

$$\begin{aligned}
\alpha_r &= \alpha_0 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \ldots \\
R &= R_0 + \epsilon R_1 + \epsilon^2 R_2 + \ldots \\
S &= S_0 + \epsilon S_1 + \epsilon^2 S_2 + \ldots \\
\theta &= \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + \ldots
\end{aligned}$$

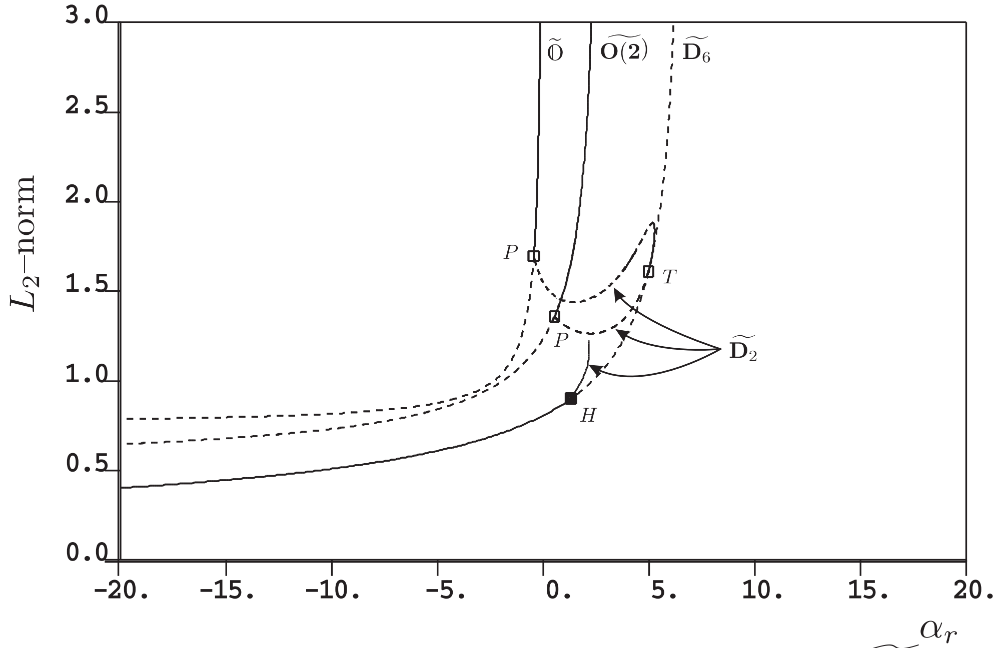


Figure 5.10: AUTO generated diagram of the three standing wave solutions with $\widetilde{\mathbf{O(2)}}$, $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbf{D}_6}$ symmetry in Fix($\widetilde{\mathbf{D}_2}$). The diagram shows the bifurcations of these solution branches and the bifurcating branches of solutions with $\widetilde{\mathbf{D}_2}$ symmetry. P denotes a pitchfork bifurcation, T a transcritical bifurcation and H a Hopf bifurcation. Stable solutions are denoted by solid lines and unstable solutions by dashed lines. The unstable solutions with $\widetilde{\mathbf{D}_2}$ symmetry are periodic and the stable solution with $\widetilde{\mathbf{D}_2}$ symmetry is quasiperiodic.

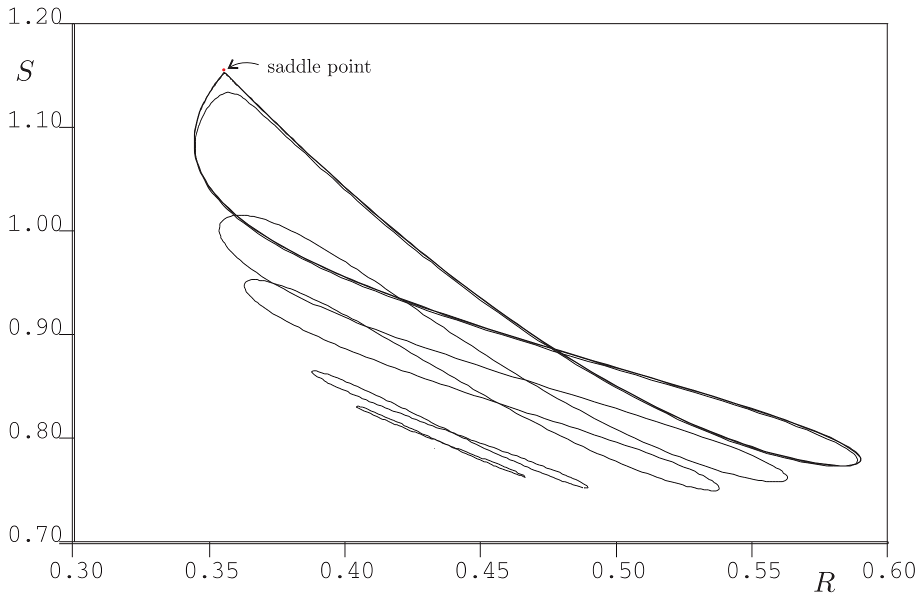


Figure 5.11: AUTO generated diagram of the periodic solution branch in the RS plane for different values of α_r. As α_r approaches 2.17806 the periodic solution approaches the saddle point on the branch connecting the $R = 0$ solution branch to the $\theta = 0$ branch which is marked.

Then

$$\begin{aligned}\sin(\theta) &= \sin(\theta_0) + \epsilon\theta_1\cos(\theta_0) + \epsilon^2\left(\theta_2\cos(\theta_0) - \frac{1}{2}\theta_1^2\sin(\theta_0)\right) + \ldots \\ \cos(\theta) &= \cos(\theta_0) - \epsilon\theta_1\sin(\theta_0) - \epsilon^2\left(\theta_2\sin(\theta_0) + \frac{1}{2}\theta_1^2\cos(\theta_0)\right) + \ldots\end{aligned}$$

Substituting into the right hand sides of equations (5.5.10) – (5.5.12) and setting equal to zero we find that at $O(\epsilon^1)$

$$\mathbf{M}\cdot\mathbf{R}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} R_1 \\ S_1 \\ \theta_1 \end{pmatrix} = \alpha_1\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$$

where $a \approx -7.43868\ldots$ and $b \approx -0.59191\ldots$. This has solutions

$$\begin{pmatrix} R_1 \\ S_1 \\ \theta_1 \end{pmatrix} = K\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_1\begin{pmatrix} 0 \\ -b/2 \\ 0 \end{pmatrix}$$

where $K \in \mathbb{R}$. At $O(\epsilon^2)$ we find

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} R_2 \\ S_2 \\ \theta_2 \end{pmatrix} = \alpha_2\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} cK\alpha_1 \\ d\alpha_1^2 + eK^2 \\ fK^2 \end{pmatrix}$$

where c, d, e and $f \in \mathbb{R}$. Multiplying on the left by $\mathbf{l} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, the left zero eigenvector of $\mathbf{M}$, we find that we must have $cK\alpha_1 = 0$. If $K = 0$ then we find that $R_1 = R_2 = \ldots = R_n = 0$ for any n so we never switch onto a different branch of solutions. Hence we must choose $\alpha_1 = 0$ and so we have

$$\begin{pmatrix} R_2 \\ S_2 \\ \theta_2 \end{pmatrix} = K'\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2\begin{pmatrix} 0 \\ -b/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{e}{2}K^2 \\ \frac{f}{a}K^2 \end{pmatrix}$$

where $K' \in \mathbb{R}$. At $O(\epsilon^3)$ we find

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & a \end{pmatrix}\begin{pmatrix} R_3 \\ S_3 \\ \theta_3 \end{pmatrix} = \alpha_3\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} + \begin{pmatrix} cK\alpha_2 - gK^3 \\ hKK' \\ jKK' \end{pmatrix}$$

where g, h and $j \in \mathbb{R}$. Again multiplying on the left by the left zero eigenvector $\mathbf{l}$ we find that K must satisfy

$$cK\alpha_2 - gK^3 = 0$$

so $K = 0$ or $K = \pm\sqrt{\frac{c\alpha_2}{g}}$. Since we find that c, and g are both positive this means that the bifurcation at $\alpha_r = \frac{\sqrt{559}-22}{3}$ is a subcritical pitchfork. If $K = 0$ in the expansion above then we remain on the solution branch with $\widetilde{\mathbf{O(2)}}$ symmetry and if $K = \pm\sqrt{\frac{c\alpha_2}{g}}$ then we switch onto one of the bifurcating branches. These bifurcating branches are submaximal solutions with $\widetilde{\mathbf{D}}_2$ symmetry.

Bifurcation of $\widetilde{\mathbb{O}}$ symmetric solution A similar analysis to that above shows that the bifurcation of this branch of solutions at

$$\alpha_r = \sqrt{31} - 6 \approx -0.4322356\ldots$$

is also a subcritical pitchfork.

Stationary bifurcation of $\widetilde{\mathbf{D}_6}$ symmetric solution This branch of solutions undergoes a stationary bifurcation at $\alpha_r = \alpha_0 = 5$. At this value of α_r we have the stationary solution

$$R_0 = \sqrt{\frac{3}{5}}, \qquad S_0 = \sqrt{2}, \qquad \theta_0 = 0.$$

Expanding as for the $\widetilde{\mathbf{O(2)}}$ symmetric case and substituting into the right hand sides of equations (5.5.10) – (5.5.12) and setting equal to zero we find that at $O(\epsilon^1)$

$$\mathbf{M}\cdot\mathbf{R}_1 = \begin{pmatrix} 6 & -\frac{4}{5}\sqrt{30} & \frac{6}{5}\sqrt{15} \\ -\frac{8}{5}\sqrt{30} & \frac{14}{5} & -\frac{18}{5}\sqrt{2} \\ -\frac{12}{5}\sqrt{15} & \frac{18}{5}\sqrt{2} & -\frac{32}{5} \end{pmatrix} \begin{pmatrix} R_1 \\ S_1 \\ \theta_1 \end{pmatrix} = \alpha_1 \begin{pmatrix} -\frac{3}{25}\sqrt{15} \\ -\frac{3}{5}\sqrt{2} \\ 0 \end{pmatrix} \tag{5.5.13}$$

The matrix $\mathbf{M}$ above has eigenvalues -2, 0 and $\frac{22}{5}$. The right and left zero eigenvectors of $\mathbf{M}$ are

$$\mathbf{r} = \begin{pmatrix} 1 \\ -\frac{1}{5}\sqrt{30} \\ -\frac{3}{5}\sqrt{15} \end{pmatrix} \qquad \text{and} \qquad \mathbf{l} = \begin{pmatrix} 1 & -\frac{1}{10}\sqrt{30} & \frac{3}{10}\sqrt{15} \end{pmatrix}$$

respectively. Hence equation (5.5.13) has solutions

$$\begin{pmatrix} R_1 \\ S_1 \\ \theta_1 \end{pmatrix} = K \begin{pmatrix} 1 \\ -\frac{1}{5}\sqrt{30} \\ -\frac{3}{5}\sqrt{15} \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ \frac{12}{25}\sqrt{2} \\ \frac{27}{50} \end{pmatrix}$$

where $K \in \mathbb{R}$. At $O(\epsilon^2)$ we find

$$\begin{aligned} \begin{pmatrix} 6 & -\frac{4}{5}\sqrt{30} & \frac{6}{5}\sqrt{15} \\ -\frac{8}{5}\sqrt{30} & \frac{14}{5} & -\frac{18}{5}\sqrt{2} \\ -\frac{12}{5}\sqrt{15} & \frac{18}{5}\sqrt{2} & -\frac{32}{5} \end{pmatrix} \begin{pmatrix} R_2 \\ S_2 \\ \theta_2 \end{pmatrix} &= \alpha_2 \begin{pmatrix} -\frac{3}{25}\sqrt{15} \\ -\frac{3}{5}\sqrt{2} \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} -\frac{94}{25}\sqrt{15}K^2 - \frac{4743}{12500}\sqrt{15}\alpha_1^2 + \frac{1101}{125}K\alpha_1 \\ -\frac{76}{5}\sqrt{2}K^2 - \frac{1809}{2500}\sqrt{2}\alpha_1^2 + \frac{177}{125}\sqrt{30}K\alpha_1 \\ \frac{84}{5}K^2 + \frac{243}{125}\alpha_1^2 - \frac{396}{125}\sqrt{15}K\alpha_1 \end{pmatrix} \end{aligned} \tag{5.5.14}$$

Multiplying on the left by the left zero eigenvector $\mathbf{l}$ we find that K must satisfy

$$\frac{108}{25}\sqrt{15}K^2 - \frac{1212}{125}K\alpha_1 + \frac{1089}{3125}\sqrt{15}\alpha_1^2 = 0$$

Hence the two possible values of K are

$$K_0 = \frac{3}{50}\sqrt{15}\alpha_1 \qquad \text{and} \qquad K_1 = \frac{121}{1350}\sqrt{15}\alpha_1$$

With $K = K_0$ we have to $O(\epsilon^1)$

$$\begin{pmatrix} R \\ S \\ \theta \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{3}{5}} \\ \sqrt{2} \\ 0 \end{pmatrix} + \epsilon\alpha_1 \begin{pmatrix} \frac{3}{50}\sqrt{15} \\ \frac{3}{10}\sqrt{2} \\ 0 \end{pmatrix}$$

so we stay on the original branch of solutions and with $K = K_1$ we have

$$\begin{pmatrix} R \\ S \\ \theta \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{3}{5}} \\ \sqrt{2} \\ 0 \end{pmatrix} + \epsilon\alpha_1 \begin{pmatrix} \frac{121}{1350}\sqrt{15} \\ \frac{19}{90}\sqrt{2} \\ -\frac{4}{15} \end{pmatrix}$$

so we switch to a different branch of solutions where the value of θ is not always zero. Hence the bifurcation at $\alpha_r = 5$ is transcritical.

Example 5.5.4. Suppose that when $\lambda = 1$

$$A = -1 + 2\mathrm{i}, \qquad B = -1 + 2\mathrm{i}, \qquad C = C_r + \frac{1}{4}\mathrm{i}$$

and we vary the value of C_r. Then

$$\alpha = \alpha_1 = 2A + 2B - 40C = \alpha_r - 2\mathrm{i}, \qquad \beta = A = -1 + 2\mathrm{i}, \qquad \gamma = B = -1 + 2\mathrm{i}, \tag{5.5.15}$$

where $\alpha_r = -4 - 40C_r$. For these values

1. The $\widetilde{\mathbf{O(2)}}$ symmetric branch of solutions bifurcates supercritically when $\alpha_r < \frac{8}{3}$ and undergoes a stationary bifurcation at $\alpha_r = -\frac{22}{3}$. This bifurcation can be found to be a subcritical pitchfork.

2. The $\widetilde{\mathrm{O}}$ symmetric branch of solutions bifurcates supercritically when $\alpha_r < 0$ and undergoes no bifurcations. For these parameter values the eigenvalues $\zeta_2^{\pm}$ are complex and so by Remark 5.5.2 AUTO will not find this branch of solutions as there is not a corresponding branch of stationary solutions to (5.5.6)–(5.5.8).

3. The $\widetilde{\mathbf{D}_6}$ symmetric branch of solutions bifurcates supercritically when $\alpha_r < \frac{20}{3}$ and undergoes a stationary bifurcation at $\alpha_r = -24$, which can be found to be transcritical, and a Hopf bifurcation at $\alpha_r = -\frac{28}{3}$.

Using AUTO we find that there is a branch of unstable periodic solutions with $\widetilde{\mathbf{D}_2}$ symmetry connecting the branches of solutions with $\widetilde{\mathbf{O(2)}}$ and $\widetilde{\mathbf{D}_6}$ symmetry. At the transcritical bifurcation on the $\widetilde{\mathbf{D}_6}$ symmetric branch, a branch of stable periodic solutions with $\widetilde{\mathbf{D}_2}$ is created. This solution goes through two saddle node bifurcations but remains stable for large negative values of α_r. In addition there is an unstable branch of quasiperiodic solutions bifurcating from the Hopf bifurcation point on the $\widetilde{\mathbf{D}_6}$ symmetric branch. As $\alpha_r \to \alpha_c \approx -10.1$ the quasiperiodic solution spends an increasing amount of time near the unstable branch of solutions connecting the $\widetilde{\mathbf{O(2)}}$ and $\widetilde{\mathbf{D}_6}$ symmetric branches. This can be seen in Figure 5.12. At $\alpha_r = \alpha_c$ the system undergoes a global bifurcation to a homoclinic orbit.

All of these solution branches can be seen in Figure 5.13.

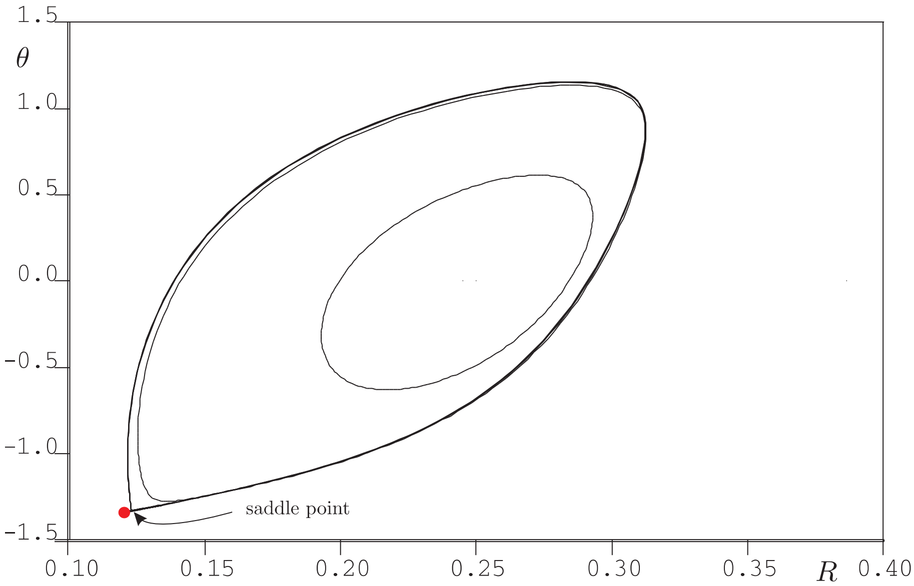


Figure 5.12: AUTO generated diagram of the periodic solution branch in the $R\theta$ plane for different values of α_r. As α_r approaches -10.1 the periodic solution approaches the saddle point on the branch connecting the $\widetilde{\mathbf{O(2)}}$ symmetric solution branch to the $\widetilde{\mathbf{D}_6}$ symmetric branch which is marked.

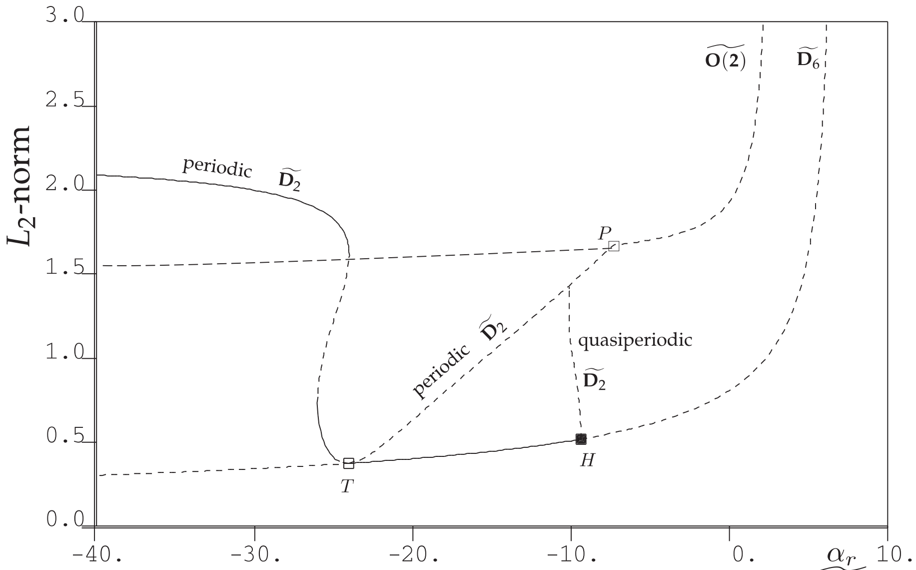


Figure 5.13: AUTO generated diagram of the maximal solution branches with $\widetilde{\mathbf{O(2)}}$ and $\widetilde{\mathbf{D}_6}$ symmetry showing the connector branches, the periodic solution and also another branch of submaximal solutions with $\widetilde{\mathbf{D}_2}$ symmetry. H denotes a Hopf bifurcation, T a transcritical bifurcation and P a pitchfork bifurcation. Stable solutions are denoted by solid lines and unstable solutions by dashed lines.

5.5.3 Solutions in Fix($\widetilde{\mathbb{Z}}_3^1$) and Fix($\widetilde{\mathbb{Z}}_5$)

Finally, we look for submaximal solutions with symmetry groups $\Sigma = \widetilde{\mathbb{Z}}_3^1$ and $\widetilde{\mathbb{Z}}_5$ where $N(\Sigma)/\Sigma = \mathbf{SO(2)} \times S^1$. Computing to cubic order the $\mathbf{SO(2)} \times S^1$ equivariant vector field for the actions of $\mathbf{SO(2)} \times S^1$ described in Table 5.9 we find that in both cases the mapping is of the form

$$\begin{aligned} \dot{w}_1 &= \mu_1 w_1 + a w_1 |w_1|^2 + b w_1 |w_2|^2 \\ \dot{w}_2 &= \mu_2 w_2 + c w_2 |w_2|^2 + d w_2 |w_1|^2. \end{aligned}$$

By rescaling $w_1 \to \sqrt{\frac{d}{b}} w_1$ these equations are of the form

$$\begin{aligned} \dot{w}_1 &= \mu_1 w_1 + \alpha_1 w_1 |w_1|^2 + \beta w_1 |w_2|^2 \\ \dot{w}_2 &= \mu_2 w_2 + \alpha_2 w_2 |w_2|^2 + \beta w_2 |w_1|^2. \end{aligned} \tag{5.5.16}$$

These equations describe the interaction of two Hopf bifurcations with $\mathbf{SO(2)}$ symmetry at $1:1$ resonance, but with different $\mathbf{SO(2)}$ actions. In the case where $\mu_1 = \mu_2$, similarly to (5.5.2), these equations also occur in the context of a Hopf bifurcation on a rotating rhombic lattice in the restriction to certain four-dimensional subspaces [57, 59].

In the restriction of (5.2.6) to Fix($\widetilde{\mathbb{Z}}_3^1$) or Fix($\widetilde{\mathbb{Z}}_5$) we have $\mu_1 = \mu_2 = \mu$ and in Fix($\widetilde{\mathbb{Z}}_3^1$),

$$\begin{aligned} \alpha_1 &= A + 4D \\ \alpha_2 &= A - 3C + D \\ \beta &= A - 15C - 2D \end{aligned} \tag{5.5.17}$$

and in Fix($\widetilde{\mathbb{Z}}_5$),

$$\begin{aligned} \alpha_1 &= A + 25C + 9D \\ \alpha_2 &= A + 4D \\ \beta &= A - 25C - 6D. \end{aligned} \tag{5.5.18}$$

In either case there are two 'pure mode' travelling wave solutions (the maximal solution branches) and branches of 'mixed mode' solutions (submaximal solutions) which exist for some values of the coefficients α_1, α_2 and β. The pure mode solutions correspond to $w_1 = 0$ with eigenvalues

$$0, \qquad 2(\alpha_2)_r |w_2|^2{}_{(\star)}, \qquad (\beta - \alpha_2)\,|w_2|^2, \qquad (\overline{\beta} - \overline{\alpha}_2)\,|w_2|^2$$

and $w_2 = 0$ with eigenvalues

$$0, \qquad 2(\alpha_1)_r |w_1|^2{}_{(\star)}, \qquad (\beta - \alpha_1)\,|w_1|^2, \qquad (\overline{\beta} - \overline{\alpha}_1)\,|w_1|^2.$$

Subscript r denotes the real part. The starred eigenvalues $(\star)$ are required to be negative for the branch to bifurcate supercritically from the Hopf bifurcation with $\mathbf{O(3)}$ symmetry. The maximal solution branches undergo Hopf bifurcations when $\beta_r = (\alpha_2)_r$ and $\beta_r = (\alpha_1)_r$ respectively. At these bifurcations it is possible for a quasiperiodic branch of mixed mode solutions with submaximal symmetry to be created.

Remark 5.5.5. In $\mathrm{Fix}(\widetilde{\mathbb{Z}}_3^1)$ the travelling wave solution with $w_1 = 0$ has $\widetilde{\mathbf{SO}(\mathbf{2})}_1$ symmetry and when $w_2 = 0$ the corresponding solution has $\widetilde{\mathbf{SO}(\mathbf{2})}_2$ symmetry. The branch of mixed mode solutions (where it exists) has $\widetilde{\mathbb{Z}}_3^1$ symmetry.

Similarly in $\mathrm{Fix}(\widetilde{\mathbb{Z}}_5)$ the travelling wave solution with $w_1 = 0$ has $\widetilde{\mathbf{SO}(\mathbf{2})}_2$ symmetry and when $w_2 = 0$ the corresponding solution has $\widetilde{\mathbf{SO}(\mathbf{2})}_3$ symmetry. The quasiperiodic branch of mixed mode solutions (where it exists) has $\widetilde{\mathbb{Z}}_5$ symmetry.

It is possible for the quasiperiodic submaximal solutions to be stable within $\mathrm{Fix}(\Sigma)$ for $\Sigma = \widetilde{\mathbb{Z}}_3^1$ or $\widetilde{\mathbb{Z}}_5$. For example, suppose that the pure mode solutions bifurcate supercritically at the Hopf bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry. Then $(\alpha_1)_r < 0$ and $(\alpha_2)_r < 0$. Suppose further that $(\alpha_2)_r < (\alpha_1)_r$. By letting $w_1 = Re^{\mathrm{i}\phi}$ and $w_2 = Se^{\mathrm{i}\psi}$ and separating the phase and amplitude equations we find that the 'mixed mode' solution is given by

$$R^2 = \frac{\lambda((\alpha_2)_r - \beta_r)}{\beta_r^2 - (\alpha_1)_r(\alpha_2)_r} \qquad\qquad S^2 = \frac{\lambda((\alpha_1)_r - \beta_r)}{\beta_r^2 - (\alpha_1)_r(\alpha_2)_r}$$

and exists when $R^2 > 0$ and $S^2 > 0$. This occurs if

1. $(\alpha_1)_r < \beta_r$ and $\beta_r^2 < (\alpha_1)_r(\alpha_2)_r$ or
2. $\beta_r < (\alpha_2)_r$ and $\beta_r^2 > (\alpha_1)_r(\alpha_2)_r$.

The real parts of the eigenvalues of the mixed mode solutions are the roots of

$$\xi^2 - 2R^2\left(\frac{(\alpha_1)_r\left((\alpha_2)_r - \beta_r\right) + (\alpha_2)_r\left((\alpha_1)_r - \beta_r\right)}{(\alpha_2)_r - \beta_r}\right)\xi + 4R^2S^2\left((\alpha_1)_r(\alpha_2)_r - \beta_r^2\right) = 0.$$

Thus when

$$(\alpha_1)_r < \beta_r \qquad \text{and} \qquad \beta_r^2 < (\alpha_1)_r(\alpha_2)_r$$

the quasiperiodic mixed mode solution exists and is stable within $\mathrm{Fix}(\Sigma)$. A bifurcation diagram varying the value of β_r is given in Figure 5.14.

5.5.4 Conclusions

By studying the restriction of F_3 given by (5.2.6) to four-dimensional invariant subspaces we have been able to find periodic and quasiperiodic solutions to (5.1.1) with submaximal symmetry. Although there are no such solutions in the four dimensional spaces $\mathrm{Fix}(\widetilde{\mathbb{Z}_4})$ and $\mathrm{Fix}(\widetilde{\mathbb{Z}_6})$, we found that periodic and quasiperiodic solutions with the symmetries of each of the other isotropy subgroups $\Sigma \subset \mathbf{O}(\mathbf{3}) \times S^1$ with four dimensional fixed-point subspaces can exist for some values of the coefficients A, B, C and D in F_3. Moreover, it is possible for these solutions to be stable within $\mathrm{Fix}(\Sigma)$.

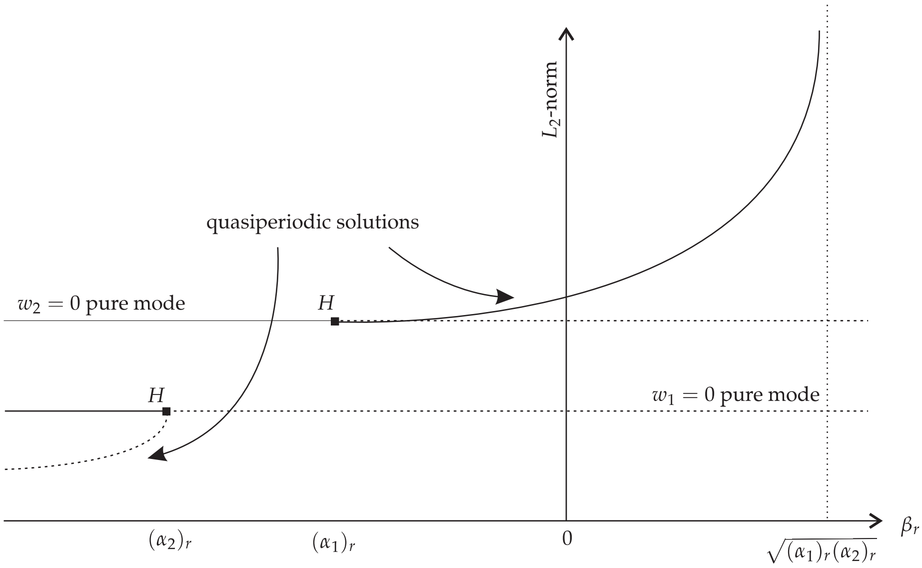


Figure 5.14: Bifurcation diagram showing a situation where quasiperiodic submaximal solution branches with symmetry $\Sigma = \widetilde{\mathbb{Z}}_3^1$ or $\widetilde{\mathbb{Z}}_5$ exist and one such branch is stable. The pure mode solutions are periodic maximal solutions with $\mathbb{C}$-axial symmetry. Here H indicates a Hopf bifurcation. Stable solutions are denoted by solid lines and unstable solutions by dashed lines. The stability is computed in the restricted space $\mathrm{Fix}(\Sigma)$.

CHAPTER 6

STATIONARY BIFURCATION WITH $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ SYMMETRY

6.1 Introduction

Some dynamical systems, including pattern forming systems such as Boussinesq Rayleigh–Bénard convection, are invariant under a change in sign of the physical variable w. For example the Swift–Hohenberg equation [78]

$$\frac{\partial w}{\partial t} = \mu w - (1 + \nabla^2)^2 w - w^3 \tag{6.1.1}$$

is invariant under the transformation $w \to -w$. Hence if w is a solution then $-w$ is also a solution. Suppose we study the dynamics of such a system on a sphere. Then the geometry forces the solutions to be invariant under the group $\mathbf{O}(\mathbf{3})$ and the $w \to -w$ symmetry forces the solutions to be invariant under the group

$$\mathbb{Z}_2 = \{1, -1\}.$$

Hence the overall symmetry of the system is $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$. Suppose that the trivial solution $w = 0$ undergoes a stationary bifurcation as the parameter μ is varied. After reducing equation (6.1.1) to the centre manifold we have the following system of ODEs

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(\mathbf{x}, \lambda), \tag{6.1.2}$$

where $\mathbf{x} \in V$ is the position in phase space and λ is a bifurcation parameter. The vector field, f, is equivariant with respect to some action of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ on V where $-1 \in \mathbb{Z}_2$ acts on the vector $\mathbf{x}$ by $-1 \cdot \mathbf{x} = -\mathbf{x}$. Equivariance with respect to this symmetry implies that

$$f(-\mathbf{x}, \lambda) = -f(\mathbf{x}, \lambda)$$

i.e. the vector field f is odd in $\mathbf{x}$ so a Taylor expansion of this vector field will not contain any terms of even order. Since f is odd,

$$f(\mathbf{0}, \lambda) = -f(\mathbf{0}, \lambda) \quad \Rightarrow \quad f(\mathbf{0}, \lambda) = \mathbf{0},$$

and hence (6.1.2) has a trivial solution $\mathbf{x} = \mathbf{0}$ for all values of λ. We assume that this solution undergoes a stationary bifurcation when $\lambda = 0$ where stationary solutions with certain symmetries are created. The symmetries of the fixed-point solutions of equation (6.1.2) will be isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for some representation of the group. In the case of the Swift–Hohenberg equation (6.1.1) the radius, R, of the sphere dictates the relevant representation of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$. For some values of R this representation will be an irreducible representation of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ on V_ℓ, the space of spherical harmonics of degree ℓ, for a particular value of ℓ, but for other values of R the solutions will have symmetries that are isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for a reducible representation on $V_\ell \oplus V_{\ell+1}$ for some value of ℓ. That is, they result from an interaction between the ℓ and $\ell+1$ modes. In Chapter 7 we will study some particular solution patterns which can only result from mode interactions.

In this chapter we study the isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for several natural representations of the group $\mathbf{O}(\mathbf{3})$. In Section 6.2.1 we will consider the isotropy subgroups for the irreducible representations on V_ℓ. These isotropy subgroups are related to the isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ which we considered in Chapter 4. In Section 6.2.2 we will investigate the isotropy subgroups for reducible representations on $V_\ell \oplus V_{\ell+1}$ where there is an interaction between the ℓ and $\ell+1$ modes. We refer to such a representation as the $(\ell, \ell+1)$ mode interaction. Although all isotropy subgroup of $\mathbf{O}(\mathbf{3})$ in the representations on $V_1 \oplus V_2$ and $V_2 \oplus V_3$ have previously been computed (see [5, 22, 24]), there has been no study of the isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for mode interaction problems. In Section 6.2.2 we will consider the specific example of the (2,3) mode interaction.

In Section 6.3 we discuss the relationship between $\mathbf{O}(\mathbf{3})$ and $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector fields for irreducible representations of the groups. We also consider $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector fields for reducible representations and compute explicitly to cubic order the Taylor expansion of a general $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field for the representations on $V_2 \oplus V_3$ and $V_3 \oplus V_4$. These vector fields will be used in Chapter 7 for the investigation of spiral patterns on spheres.

Finally in Section 6.4 we show how to compute the values of the coefficients in the equivariant vector fields for the specific example of the Swift–Hohenberg equation (6.1.1).

6.2 Isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$

In this section we first discuss how to compute the isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ in the irreducible representations of $\mathbf{O}(\mathbf{3})$ on V_ℓ, the space of spherical harmonics of degree ℓ. We then consider the subgroups which can be isotropy subgroups in reducible representations on $V_\ell \oplus V_{\ell+1}$, the space of spherical harmonics of degrees ℓ and $\ell+1$.

Recall that all solutions of (6.1.2) have as their group of symmetries an isotropy subgroup of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for a given representation. The equivariant branching lemma, Theorem 2.4.6, guarantees the existence of solutions with the symmetries of the axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ at the stationary bifurcation of the trivial solution $\mathbf{x} = \mathbf{0}$. It may be possible for (6.1.2) to have solutions with the symmetries of other isotropy subgroups depending on values of the coefficients in the Taylor expansion of f.

For any representation of $\mathbf{O(3)} \times \mathbb{Z}_2$, the isotropy subgroups are twisted subgroups H^θ where H is a subgroup of $\mathbf{O(3)}$ and $\theta : H \to \mathbb{Z}_2$ is a group homomorphism. These twisted subgroups are a subset of the twisted subgroups of $\mathbf{O(3)} \times S^1$ containing only the subgroups, H^θ with twist types $\mathbb{Z}_2$ or $\mathbb{1}$. These twisted subgroups are uniquely determined by pairs of subgroups (H, K) where H is a subgroup of $\mathbf{O(3)}$ and K is a normal subgroup of H such that $H/K \cong \mathbb{Z}_2$ or $\mathbb{1}$. Since there are no automorphisms of H/K in either case, the group homomorphism $\theta : H \to H/K$ is always given by

$$\theta(h) = \begin{cases} 1 & \text{if } h \in K \\ -1 & \text{if } h \in H - K. \end{cases} \tag{6.2.1}$$

A complete list of the twisted subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ contains the pairs (H, H) for all subgroups $H \subset \mathbf{O(3)}$ and all pairs (H, K) where $|H : K| = 2$ (by Lemma 4.2.2, K is then normal in H). All such pairs can be found in Table 6.7.

For different representations of the group $\mathbf{O(3)}$, different twisted subgroups will be isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$. To determine which twisted subgroups are isotropy subgroups for a particular representation on the space V (where $V = V_\ell$ or $V_\ell \oplus V_{\ell+1}$ for any value of ℓ) we use the massive chain criterion (Theorem 3.4.1). This says that the twisted subgroup $H^\theta \subset \mathbf{O(3)} \times \mathbb{Z}_2$ is an isotropy subgroup in the representation on V if and only if for each strictly larger and adjacent group Δ (so that $H^\theta \subset \Delta \subset \cdots \subset \mathbf{O(3)} \times \mathbb{Z}_2$)

$$\dim \mathrm{Fix}(\Delta) - r(\Delta) < \dim \mathrm{Fix}(H^\theta) - r(H^\theta)$$

where

$$r(H^\theta) = \min\{\dim V - 1,\ q(H^\theta)\} \qquad \text{and} \qquad q(H^\theta) = \dim N_{\mathbf{O(3)} \times \mathbb{Z}_2}(H^\theta) - \dim H^\theta.$$

Remark 6.2.1. Although the statement of Theorem 3.4.1 relates to subgroups $\Sigma \subset \mathbf{O(3)}$ in irreducible representations, the theorem remains valid for twisted subgroups $H^\theta \subset \mathbf{O(3)} \times \mathbb{Z}_2$ and reducible representations.

To use Theorem 3.4.1 we must compute the values of $\dim \mathrm{Fix}(H^\theta)$ and $r(H^\theta)$ for each twisted subgroup H^θ. For pairs (H, H) the twisted subgroup is not really twisted at all and

$$\dim \mathrm{Fix}(H, H) = \dim \mathrm{Fix}(H).$$

For the pairs twisted subgroups given by pairs (H, K) where $|H : K| = 2$ we use the trace formula (Theorem 2.4.4) to compute $\dim \mathrm{Fix}(H^\theta)$. Let $\int \mathrm{d}\mu_H$ denote the normalised Haar integral on H. Then since the index $|H : K| = 2$ we have that $\int \mathrm{d}\mu_H = \frac{1}{2} \int \mathrm{d}\mu_K$. This means that

$$\dim \mathrm{Fix}(K) = \int_K \chi(h) \mathrm{d}\mu_K = 2 \int_K \chi(h) \mathrm{d}\mu_H$$

We also have that

$$\dim \mathrm{Fix}(H, K) = \int_K \chi(h) \mathrm{d}\mu_H - \int_{H-K} \chi(h) \mathrm{d}\mu_H,$$

since $\chi((\sigma, -1)) = -\chi(\sigma)$, and

$$\dim \operatorname{Fix}(H) = \int_K \chi(h) \mathrm{d}\mu_H + \int_{H-K} \chi(h) \mathrm{d}\mu_H.$$

Hence we have

$$\dim \operatorname{Fix}(H, K) = \dim \operatorname{Fix}(K) - \dim \operatorname{Fix}(H).$$

We can use this to compute the twisted subgroups which can be isotropy subgroups in irreducible and reducible representations of $\mathbf{O(3)} \times \mathbb{Z}_2$.

Remarks on notation

From now on in this thesis if the subgroup $H^\theta = (H,\ H)$ is an isotropy subgroup in a particular representation of $\mathbf{O(3)} \times \mathbb{Z}_2$ then we will denote H^θ simply by H.

If the twisted subgroup $H^\theta = (H,\ K)$ is an isotropy subgroup then usually we will denote H^θ by $\widetilde{(H)}_K$ where the tilde over the symbol denotes that the isotropy subgroup has twist type $\mathbb{Z}_2$. We use this notation because in some representations there may be two or more isotropy subgroups with the same subgroup H but different subgroups K.

Notice that a pair (H, K) will only be given a label $\widetilde{(H)}_K$ if it is an isotropy subgroup in the representation being discussed.

For example, if in a particular representation the twisted subgroups given by pairs $(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_4 \times \mathbb{Z}_2^c)$, $(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c)$ and $(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_4^d)$ are isotropy subgroups then we denote

$$\begin{aligned} &(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_4 \times \mathbb{Z}_2^c) \quad \text{by} \quad \mathbf{D}_4 \times \mathbb{Z}_2^c, \\ &(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c) \quad \text{by} \quad (\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c})_{\mathbf{D}_2 \times \mathbb{Z}_2^c} \\ \text{and} \quad &(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_4^d) \quad \text{by} \quad (\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c})_{\mathbf{D}_4^d}. \end{aligned}$$

When there is only one isotropy subgroup with twist type $\mathbb{Z}_2$ for the subgroup H then we will drop the subscript K since there will be no confusion as to which isotropy subgroup we mean. For example, if in another representation only the twisted subgroups given by pairs $(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_4 \times \mathbb{Z}_2^c)$ and $(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c)$ are isotropy subgroups then we denote

$$\begin{aligned} &(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_4 \times \mathbb{Z}_2^c) \quad \text{by} \quad \mathbf{D}_4 \times \mathbb{Z}_2^c \\ \text{and} \quad &(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c) \quad \text{by} \quad \widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}. \end{aligned}$$

6.2.1 Isotropy subgroups in irreducible representations

In this section we compute the twisted subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ which can be isotropy subgroups in an irreducible representation on V_ℓ. For any value of ℓ, in the plus representation of $\mathbf{O(3)}$ the element $-I \in \mathbf{O(3)}$ acts as the identity and therefore the element $(-I, 1) \in \mathbf{O(3)} \times \mathbb{Z}_2$ must lie in every isotropy subgroup. This means that H and K must both be class II subgroups of $\mathbf{O(3)}$ since $-I \in H$ and $-I \in \ker \theta = K$.

In the minus representation, $-I$ acts as minus the identity and hence $(-I, -1) \in \mathbf{O(3)} \times \mathbb{Z}_2$ acts as the identity and must therefore be contained in every isotropy subgroup. This means that H must be a class II subgroup and K must be either a class I or class III subgroup of $\mathbf{O(3)}$ since $-I \in H$ and $-I \notin \ker\theta = K$.

Thus the twisted subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ which can be isotropy subgroups in an irreducible representation on V_ℓ are those listed in Table 6.1. Also given is the formula in terms of ℓ for the dimension of their fixed-point subspaces for the representation on V_ℓ and the value of $q(H^\theta)$ (which is required in order to use the massive chain criterion, Theorem 3.4.1).

Remark 6.2.2. Notice that Table 6.1 is a restriction of the list of twisted subgroups of $\mathbf{O(3)} \times S^1$ given in Table 4.2 to those with twist types $\mathbb{Z}_2$ or $\mathbb{1}$. Notice also that in Table 4.2 the formulae for $\dim \mathrm{Fix}(H^\theta)$ are twice those in Table 6.1. This is due to the fact that in Table 4.2 the representation of $\mathbf{O(3)}$ is the $\mathbf{O(3)}$-simple representation on the direct sum of two copies of V_ℓ.

The axial isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ which fix a one-dimensional subspace in the representation on V_ℓ are then as in Table 6.2. This is a restriction of Table 4.4 to those entries with twist types $\mathbb{Z}_2$ or $\mathbb{1}$.

Similarly the isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ which fix a two-dimensional subspace in the representation on V_ℓ are as in Table 6.3 which is almost a restriction of Table 4.6 to those entries with twist types $\mathbb{Z}_2$ or $\mathbb{1}$. In addition to making this restriction we must also remove the twisted subgroups H^θ given by the pairs $(\mathbb{Z}_{2m} \times \mathbb{Z}_2^c, \mathbb{Z}_m \times \mathbb{Z}_2^c)$ and $(\mathbb{Z}_{2m} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-)$. These pairs both give twisted subgroups with $q(H^\theta) = 1$. For every value of ℓ where they have a two-dimensional fixed-point subspace, the twisted subgroups given by the pairs $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_m \times \mathbb{Z}_2^c)$ and $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_{2m}^d)$ have one-dimensional fixed-point subspaces. Hence $(\mathbb{Z}_{2m} \times \mathbb{Z}_2^c, \mathbb{Z}_m \times \mathbb{Z}_2^c)$ and $(\mathbb{Z}_{2m} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m}^-)$ do not satisfy the massive chain criterion when compared with $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_m \times \mathbb{Z}_2^c)$ and $(\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_{2m}^d)$ respectively.

Remark 6.2.3. In the natural representation when ℓ is odd all isotropy subgroups have twist type $\mathbb{Z}_2$. Also, since H is a class II subgroup of $\mathbf{O(3)}$, $-I \in H$. This means that all isotropy subgroups in these representations have $\dim \mathrm{Fix}(H^\theta) = \dim \mathrm{Fix}(K)$. Hence, if K is an isotropy subgroup of $\mathbf{O(3)}$ for some odd value of ℓ then the twisted subgroup, H^θ, given by (H, K) where $H = K \cup (-I, -1)K$ is an isotropy subgroup of $\mathbf{O(3)} \times \mathbb{Z}_2$ for the same value of ℓ. Consequently the work of [23, 53, 63] in finding the isotropy subgroups of $\mathbf{O(3)}$ has already determined the isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ for natural representations on V_ℓ where ℓ is odd.

Remark 6.2.4. In the natural representation on V_ℓ for even values of ℓ the isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ with twist type $\mathbb{1}$ are precisely the isotropy subgroups of $\mathbf{O(3)}$ for the same value of ℓ and as such have already been determined in [23, 53, 63]. However, in addition, $\mathbf{O(3)} \times \mathbb{Z}_2$ has isotropy subgroups with twist type $\mathbb{Z}_2$.

Using the massive chain criterion, Theorem 3.4.1, it is possible to determine all isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in the representation on V_ℓ for a given value of ℓ. We will consider here the examples of the natural representations on V_ℓ for $\ell = 2, 3$ and 4.

J	K	$\dim \mathrm{Fix}(H^\theta)$ plus representation	$\dim \mathrm{Fix}(H^\theta)$ minus representation	$q(H^\theta)$
$\mathbf{SO(3)}$	$\mathbf{O(3)}$	0	–	0
$\mathbf{SO(3)}$	$\mathbf{SO(3)}$	–	0	0
$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\begin{cases} 1, & \ell \text{ even} \\ 0, & \ell \text{ odd} \end{cases}$	–	0
$\mathbf{O(2)}$	$\mathbf{O(2)}$	–	$\begin{cases} 1, & \ell \text{ even} \\ 0, & \ell \text{ odd} \end{cases}$	0
$\mathbf{O(2)}$	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	$\begin{cases} 0, & \ell \text{ even} \\ 1, & \ell \text{ odd} \end{cases}$	–	0
$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	–	$\begin{cases} 0, & \ell \text{ even} \\ 1, & \ell \text{ odd} \end{cases}$	0
$\mathbf{SO(2)}$	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	1	–	1
$\mathbf{SO(2)}$	$\mathbf{SO(2)}$	–	1	1
$\mathbf{D}_n$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\begin{cases} [\ell/n]+1, & \ell \text{ even} \\ [\ell/n], & \ell \text{ odd} \end{cases}$	–	0
$\mathbf{D}_n$	$\mathbf{D}_n$	–	$\begin{cases} [\ell/n]+1, & \ell \text{ even} \\ [\ell/n], & \ell \text{ odd} \end{cases}$	0
$\mathbf{D}_n$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\begin{cases} [\ell/n], & \ell \text{ even} \\ [\ell/n]+1, & \ell \text{ odd} \end{cases}$	–	0
$\mathbf{D}_n$	$\mathbf{D}_n^z$	–	$\begin{cases} [\ell/n], & \ell \text{ even} \\ [\ell/n]+1, & \ell \text{ odd} \end{cases}$	0
$\mathbf{D}_{2m}$	$\mathbf{D}_m \times \mathbb{Z}_2^c$	$[(\ell+m)/2m]$	–	0
$\mathbf{D}_{2m}$	$\mathbf{D}_{2m}^d$	–	$[(\ell+m)/2m]$	0
$\mathbb{Z}_m$	$\mathbb{Z}_m \times \mathbb{Z}_2^c$	$2[\ell/m]+1$	–	$\begin{cases} 3, & m=1 \\ 1, & m \geq 2 \end{cases}$
$\mathbb{Z}_{2m}$	$\mathbb{Z}_m \times \mathbb{Z}_2^c$	$2[(\ell+m)/2m]$	–	1
$\mathbb{Z}_m$	$\mathbb{Z}_m$	–	$2[\ell/m]+1$	$\begin{cases} 3, & m=1 \\ 1, & m \geq 2 \end{cases}$
$\mathbb{Z}_{2m}$	$\mathbb{Z}_{2m}^-$	–	$2[(\ell+m)/2m]$	1
$\mathbb{T}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$2[\ell/3]+[\ell/2]-\ell+1$	–	0
$\mathbb{T}$	$\mathbb{T}$	–	$2[\ell/3]+[\ell/2]-\ell+1$	0
$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$[\ell/4]+[\ell/3]+[\ell/2]-\ell+1$	–	0
$\mathbb{O}$	$\mathbb{O}$	–	$[\ell/4]+[\ell/3]+[\ell/2]-\ell+1$	0
$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$[\ell/3]-[\ell/4]$	–	0
$\mathbb{O}$	$\mathbb{O}^-$	–	$[\ell/3]-[\ell/4]$	0
$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$[\ell/5]+[\ell/3]+[\ell/2]-\ell+1$	–	0
$\mathbb{I}$	$\mathbb{I}$	–	$[\ell/5]+[\ell/3]+[\ell/2]-\ell+1$	0

Table 6.1: The twisted subgroups H^θ of $\mathbf{O(3)} \times \mathbb{Z}_2$ and the dimensions of their fixed-point subspaces in the representations on V_ℓ where H^θ can be an isotropy subgroup. Here $H = J \times \mathbb{Z}_2^c$

J	K	$\theta(H)$	Plus representation	Minus representation
$\mathbf{O}(\mathbf{2})$	$\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c$	$\mathbb{1}$	Even ℓ	
$\mathbf{O}(\mathbf{2})$	$\mathbf{O}(\mathbf{2})$	$\mathbb{Z}_2$		Even ℓ
$\mathbf{O}(\mathbf{2})$	$\mathbf{SO}(\mathbf{2}) \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	Odd ℓ	
$\mathbf{O}(\mathbf{2})$	$\mathbf{O}(\mathbf{2})^-$	$\mathbb{Z}_2$		Odd ℓ
$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$\mathbb{1}$	6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 47, 49, 53, 59	
$\mathbb{I}$	$\mathbb{I}$	$\mathbb{Z}_2$		6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 47, 49, 53, 59
$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{1}$	4, 6, 8, 9, 10, 13, 14, 15, 17, 19, 23	
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Z}_2$		4, 6, 8, 9, 10, 13, 14, 15, 17, 19, 23
$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	3, 6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 20	
$\mathbb{O}$	$\mathbb{O}^-$	$\mathbb{Z}_2$		3, 6, 7, 9, 10,11, 12, 13, 14, 16, 17, 20
$\mathbf{D}_{2m}$	$\mathbf{D}_m \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$m \leq \ell < 3m, \quad (m \geq 3)$	
$\mathbf{D}_{2m}$	$\mathbf{D}_{2m}^d$	$\mathbb{Z}_2$		$m \leq \ell < 3m, \quad (m \geq 3)$
$\mathbf{D}_4$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	2, 4, 5	
$\mathbf{D}_4$	$\mathbf{D}_4^d$	$\mathbb{Z}_2$		2, 4, 5

Table 6.2: The axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for the representations V_ℓ. The last two columns give the values of ℓ for which the subgroups are isotropy subgroups. Here $H = J \times \mathbb{Z}_2^c$.

J	K	$\theta(H)$	plus representation	minus representation
$\mathbf{D}_n$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbb{1}$	$\begin{cases} n \leq \ell < 2n, & \ell \text{ even} \\ 2n \leq \ell < 3n, & \ell \text{ odd} \end{cases}$	–
$\mathbf{D}_n$	$\mathbf{D}_n$	$\mathbb{Z}_2$	–	$\begin{cases} n \leq \ell < 2n, & \ell \text{ even} \\ 2n \leq \ell < 3n, & \ell \text{ odd} \end{cases}$
$\mathbf{D}_n$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$\begin{cases} 2n \leq \ell < 3n, & \ell \text{ even} \\ n \leq \ell < 2n, & \ell \text{ odd} \end{cases}$	–
$\mathbf{D}_n$	$\mathbf{D}_n^z$	$\mathbb{Z}_2$	–	$\begin{cases} 2n \leq \ell < 3n, & \ell \text{ even} \\ n \leq \ell < 2n, & \ell \text{ odd} \end{cases}$
$\mathbf{D}_{2m}$	$\mathbf{D}_m \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$3m \leq \ell < 5m$	–
$\mathbf{D}_{2m}$	$\mathbf{D}_{2m}^d$	$\mathbb{Z}_2$	–	$3m \leq \ell < 5m$
$\mathbb{T}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{1}$	6, 9, 10, 13, 14, 17	–
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{Z}_2$	–	6, 9, 10, 13, 14, 17
$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{1}$	12, 16, 18, 20–22, 25–27, 29, 31, 35	–
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Z}_2$	–	12, 16, 18, 20–22, 25–27, 29, 31, 35
$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	15, 18, 19, 21–26, 28, 29, 32	–
$\mathbb{O}$	$\mathbb{O}^-$	$\mathbb{Z}_2$	–	15, 18, 19, 21–26, 28, 29, 32
$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$\mathbb{1}$	30, 36, 40, 42, 45, 46, 48, 50, 51, 52, 54–58 61–65, 67–69, 71, 73, 74, 77, 79, 83, 89	–
$\mathbb{I}$	$\mathbb{I}$	$\mathbb{Z}_2$	–	30, 36, 40, 42, 45, 46, 48, 50, 51, 52, 54–58 61–65, 67–69, 71, 73, 74, 77, 79, 83, 89

Table 6.3: The isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ with two-dimensional fixed-point subspaces for the representations V_ℓ. The last two columns give the values of ℓ for which the subgroups are isotropy subgroups. Here $H = J \times \mathbb{Z}_2^c$.

Example 6.2.5 (The natural representation on V_2). Using Table 6.2 we can see that in the natural representation on V_2 the axial isotropy subgroups are

$$\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c = (\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c,\ \mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c) \qquad \text{and} \qquad \widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c} = (\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c).$$

Similarly, using Table 6.3 the only isotropy subgroup with two-dimensional fixed-point subspace is

$$\mathbf{D}_2 \times \mathbb{Z}_2^c = (\mathbf{D}_2 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c)$$

and this subgroup has $r(\mathbf{D}_2 \times \mathbb{Z}_2^c) = 0$. Using Table 6.1 we find that the twisted subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ which fix a subspace of dimension greater than two in the natural representation on V_2 are the two given in Table 6.4. Neither of these twisted subgroups are isotropy subgroups by the massive chain criterion. This means that any combination of spherical harmonics of degree $\ell = 2$ has $\mathbf{D}_2 \times \mathbb{Z}_2^c$ symmetry about some axes.

H	K	H/K	$\dim \mathrm{Fix}(H^\theta)$	$r(H^\theta)$
$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{1}$	3	1
$\mathbb{Z}_2^c$	$\mathbb{Z}_2^c$	$\mathbb{1}$	5	3

Table 6.4: The twisted subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ which have a fixed-point subspace of dimension greater than 2 when $\ell = 2$.

The lattice of isotropy subgroups is as in Figure 6.1.

$$\begin{tikzcd}
 & \mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c \arrow[dr, dash] & & \widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c} \arrow[dl, dash] \\
 & \mathbf{D}_2 \times \mathbb{Z}_2^c &
\end{tikzcd}$$

Figure 6.1: Lattice of isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for the natural representation on V_2.

Example 6.2.6 (The natural representation on V_3). Using Table 6.2 we can see that when $\ell = 3$ there are three axial isotropy subgroups

$$\begin{aligned}
\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c} &= (\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c, \mathbf{O}(\mathbf{2})^-) \\
\widetilde{\mathbb{O} \times \mathbb{Z}_2^c} &= (\mathbb{O} \times \mathbb{Z}_2^c,\ \mathbb{O}^-) \\
\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c} &= (\mathbf{D}_6 \times \mathbb{Z}_2^c,\ \mathbf{D}_6^d)
\end{aligned}$$

and using Table 6.3 there are two isotropy subgroups with two-dimensional fixed-point subspaces:

$$\widetilde{\mathbf{D}_3 \times \mathbb{Z}_2^c} = (\mathbf{D}_3 \times \mathbb{Z}_2^c, \mathbf{D}_3^z) \qquad \text{and} \qquad \widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c} = (\mathbf{D}_2 \times \mathbb{Z}_2^c, \mathbf{D}_2^z).$$

Using Table 6.1 we find that the twisted subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ which fix a subspace of dimension greater than two in the natural representation on V_3 are the four given in Table 6.5. We use the massive chain criterion (Theorem 3.4.1) to determine which of these twisted subgroups are isotropy subgroups.

The lattice of isotropy subgroups when $\ell = 3$ is then as in Figure 6.2.

H	K	H/K	$\dim \mathrm{Fix}(H^\theta)$	$r(H^\theta)$	Isotropy subgroup?
$\mathbb{Z}_3 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	3	1	No
$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	3	1	No
$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2$	4	1	Yes, $\widetilde{\mathbb{Z}_2 \times \mathbb{Z}_2^c}$
$\mathbb{Z}_2^c$	$\mathbb{1}$	$\mathbb{Z}_2$	7	3	Yes, $\widetilde{\mathbb{Z}_2^c}$

Table 6.5: The twisted subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ which have a fixed-point subspace of dimension greater than 2 when $\ell = 3$. The dimension of the fixed-point subspace is shown. The right-hand column indicates whether or not each twisted subgroup is an isotropy subgroup. If it is an isotropy subgroup then its label is given.

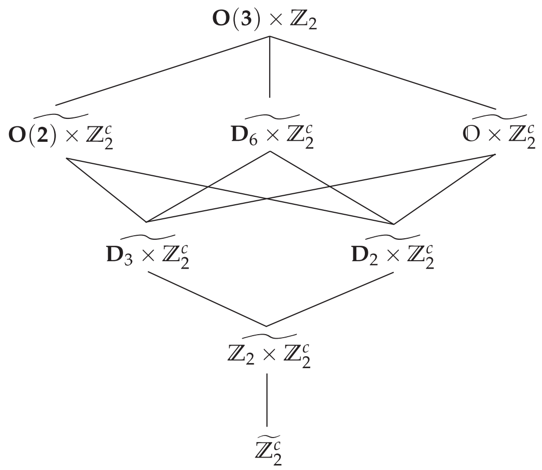


Figure 6.2: Lattice of isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for the natural representation on V_3.

Example 6.2.7 (The natural representation on V_4). Using Table 6.2 we can see that when $\ell = 4$ there are five axial isotropy subgroups

$$\begin{aligned}
\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c} &= (\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c,\ \mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c) \\
\widetilde{\mathbb{O} \times \mathbb{Z}_2^c} &= (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c) \\
\widetilde{\mathbf{D}_8 \times \mathbb{Z}_2^c} &= (\mathbf{D}_8 \times \mathbb{Z}_2^c, \mathbf{D}_4 \times \mathbb{Z}_2^c) \\
\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c} &= (\mathbf{D}_6 \times \mathbb{Z}_2^c, \mathbf{D}_3 \times \mathbb{Z}_2^c) \\
\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c} &= (\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c)
\end{aligned}$$

and using Table 6.3 there are three isotropy subgroups with two-dimensional fixed-point subspaces:

$$\begin{aligned}
\mathbf{D}_4 \times \mathbb{Z}_2^c &= (\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_4 \times \mathbb{Z}_2^c) \\
\mathbf{D}_3 \times \mathbb{Z}_2^c &= (\mathbf{D}_3 \times \mathbb{Z}_2^c,\ \mathbf{D}_3 \times \mathbb{Z}_2^c) \\
\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c} &= (\mathbf{D}_2 \times \mathbb{Z}_2^c,\ \mathbb{Z}_2 \times \mathbb{Z}_2^c).
\end{aligned}$$

Using Table 6.1 we find that the twisted subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ which fix a subspace of dimension greater than two in the natural representation on V_4 are the six given in Table 6.6. We use the massive chain criterion (Theorem 3.4.1) to determine which of these twisted subgroups are isotropy subgroups.

The lattice of isotropy subgroups is as in Figure 6.3.

H	K	H/K	$\dim \mathrm{Fix}(H^\theta)$	$r(H^\theta)$	Isotropy subgroup?
$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{1}$	3	0	Yes, $\mathbf{D}_2 \times \mathbb{Z}_2^c$
$\mathbb{Z}_4 \times \mathbb{Z}_2^c$	$\mathbb{Z}_4 \times \mathbb{Z}_2^c$	$\mathbb{1}$	3	1	No
$\mathbb{Z}_3 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3 \times \mathbb{Z}_2^c$	$\mathbb{1}$	3	1	No
$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2^c$	$\mathbb{Z}_2$	4	1	Yes, $\widetilde{\mathbb{Z}_2 \times \mathbb{Z}_2^c}$
$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{1}$	5	1	Yes, $\mathbb{Z}_2 \times \mathbb{Z}_2^c$
$\mathbb{Z}_2^c$	$\mathbb{Z}_2^c$	$\mathbb{1}$	9	3	Yes, $\mathbb{Z}_2^c$

Table 6.6: The twisted subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ which have a fixed-point subspace of dimension greater than 2 when $\ell = 4$. The dimension of the fixed-point subspace is shown. The right-hand column indicates whether or not each twisted subgroup is an isotropy subgroup. If it is an isotropy subgroup then its label is given.

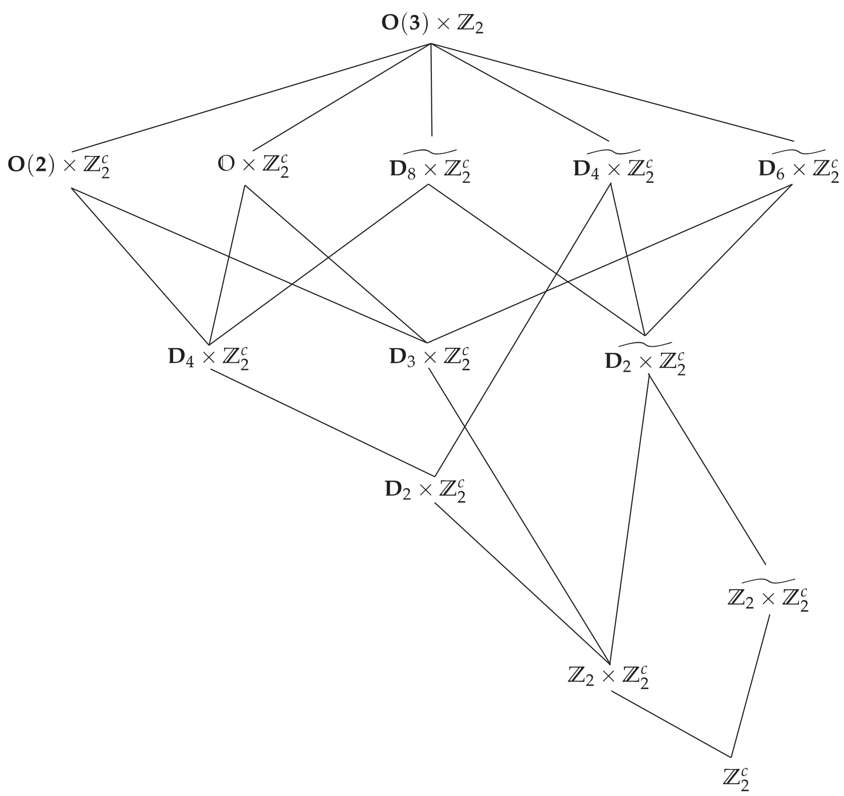


Figure 6.3: Lattice of isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ for the natural representation on V_4.

6.2.2 Isotropy subgroups in reducible representations

We now compute the twisted subgroups of $\mathbf{O}(3) \times \mathbb{Z}_2$ which can be isotropy subgroups in the reducible representations on $V_\ell \oplus V_{\ell+1}$. This will be useful in Chapter 7 where we will study specific patterns on spheres which can only occur in mode interactions.

Here we assume that $\mathbf{O}(3)$ acts on the V_ℓ and $V_{\ell+1}$ components via the natural actions on these spaces. Let

$$\mathbf{z} = (\mathbf{x}; \mathbf{y}) = \Big(x_{-\ell}, x_{-(\ell-1)}, \ldots, x_\ell \, ; \, y_{-(\ell+1)}, y_{-\ell}, \ldots, y_{\ell+1} \Big)$$

be the vector of amplitudes of the spherical harmonics of degrees ℓ and $\ell+1$ where

$$x_{-m} = (-1)^m \overline{x}_m \qquad \text{and} \qquad y_{-m} = (-1)^m \overline{y}_m.$$

The isotropy subgroups of $\mathbf{O}(3) \times \mathbb{Z}_2$ for the natural reducible representation on $V_\ell \oplus V_{\ell+1}$ fall into three categories

1. Isotropy subgroups which contain the element $(-I, (-1)^\ell) \in \mathbf{O}(3) \times \mathbb{Z}_2$. This element acts as the identity on V_ℓ but fixes only the origin in $V_{\ell+1}$. These isotropy subgroups are the isotropy subgroups of $\mathbf{O}(3) \times \mathbb{Z}_2$ in the irreducible representation on V_ℓ and have a fixed-point subspace containing only amplitudes x_m.

2. Isotropy subgroups which contain the element $(-I, (-1)^{\ell+1}) \in \mathbf{O}(3) \times \mathbb{Z}_2$. This element acts as the identity on $V_{\ell+1}$ but fixes only the origin in V_ℓ. These isotropy subgroups are the isotropy subgroups of $\mathbf{O}(3) \times \mathbb{Z}_2$ in the irreducible representation on $V_{\ell+1}$ and have a fixed-point subspace containing only amplitudes y_m.

3. Isotropy subgroups containing neither $(-I, 1)$ nor $(-I, -1)$. These isotropy subgroups are twisted subgroups H^θ where H is a class I or III subgroup of $\mathbf{O}(3)$. Subgroups of this type are never isotropy subgroups in irreducible representations. They have fixed-point subspaces containing amplitudes x_m and y_m. Since the action of $\mathbf{O}(3) \times \mathbb{Z}_2$ is irreducible on V_ℓ and $V_{\ell+1}$, every symmetry maps $V_\ell \to V_\ell$ and $V_{\ell+1} \to V_{\ell+1}$. This means that there cannot be a one-dimensional fixed-point subspace containing elements in V_ℓ and $V_{\ell+1}$. Hence, although these subgroups can be isotropy subgroups, they cannot be axial isotropy subgroups because they always fix a subspace of dimension two or larger.

This means that it is possible for twisted subgroups H^θ with H any class of subgroup of $\mathbf{O}(3)$ to be isotropy subgroups of $\mathbf{O}(3) \times \mathbb{Z}_2$ in reducible representations. There are many more isotropy subgroups in reducible representations on $V_\ell \oplus V_{\ell+1}$ than in irreducible representations since all of the isotropy subgroups in the representations on V_ℓ and $V_{\ell+1}$ are isotropy subgroups in the representation on $V_\ell \oplus V_{\ell+1}$ and there are additional isotropy subgroups H^θ with H a class I or III subgroup of $\mathbf{O}(3)$.

All twisted subgroups of $\mathbf{O}(3) \times \mathbb{Z}_2$ are listed in Table 6.7 along with formulae for the dimension of their fixed-point subspaces in the representation on $V_\ell \oplus V_{\ell+1}$ and the value of $q(H^\theta) = \dim N_{\mathbf{O}(3)}(H^\theta) - \dim(H^\theta)$. We wish to determine which of these subgroups are isotropy subgroups in a given reducible representation.

Remark 6.2.8. The formulae for the dimensions of the fixed-point subspaces in Table 6.7 are computed as follows. The dimension of the fixed-point subspace of a twisted subgroup H^θ in the representation on $V_\ell \oplus V_{\ell+1}$ is just the sum of the dimensions of its fixed-point subspaces in the representations on V_ℓ and $V_{\ell+1}$. In other words

$$\dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(H^\theta) = \dim \mathrm{Fix}_{V_\ell}(H^\theta) + \dim \mathrm{Fix}_{V_{\ell+1}}(H^\theta). \tag{6.2.2}$$

However, the values of $\dim \mathrm{Fix}_{V_\ell}(H^\theta)$ depend on the classes of the subgroups H and K which define the twisted subgroup H^θ and whether ℓ is even or odd. Recall that

$$\dim \mathrm{Fix}_{V_\ell}(H^\theta) = \begin{cases} \dim \mathrm{Fix}_{V_\ell}(H) & \text{when } H/K = \mathbb{1} \\ \dim \mathrm{Fix}_{V_\ell}(K) - \dim \mathrm{Fix}_{V_\ell}(H) & \text{when } H/K = \mathbb{Z}_2. \end{cases} \tag{6.2.3}$$

1. When H is a class I subgroup of $\mathbf{O}(\mathbf{3})$ then K must also be a class I subgroup. Since H and K are both subgroups of $\mathbf{SO}(\mathbf{3})$, formulae for $\dim \mathrm{Fix}_{V_\ell}(H)$ and $\dim \mathrm{Fix}_{V_\ell}(K)$ are given by Theorem 3.3.4.

2. If H and K are both class II subgroups of $\mathbf{O}(\mathbf{3})$ then H^θ contains the element $(-I, 1)$ and hence $\dim \mathrm{Fix}_{V_\ell}(H^\theta) = 0$ when ℓ is odd. Therefore

$$\dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(H^\theta) = \begin{cases} \dim \mathrm{Fix}_{V_\ell}(H^\theta) & \text{when } \ell \text{ even} \\ \dim \mathrm{Fix}_{V_{\ell+1}}(H^\theta) & \text{when } \ell \text{ odd.} \end{cases}$$

   The formulae for $\dim \mathrm{Fix}_{V_\ell}(H^\theta)$ can then be computed using (6.2.3) and Theorem 3.3.4.

3. If H is a class II subgroup and K is a class I or III subgroup of $\mathbf{O}(\mathbf{3})$ then H^θ contains the element $(-I, -1)$ and hence $\dim \mathrm{Fix}_{V_\ell}(H^\theta) = 0$ when ℓ is even. Therefore

$$\dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(H^\theta) = \begin{cases} \dim \mathrm{Fix}_{V_{\ell+1}}(H^\theta) & \text{when } \ell \text{ even} \\ \dim \mathrm{Fix}_{V_\ell}(H^\theta) & \text{when } \ell \text{ odd.} \end{cases}$$

   The formulae for $\dim \mathrm{Fix}_{V_\ell}(H^\theta)$ can then be computed using (6.2.3) and Theorems 3.3.4 and 3.3.5.

4. If H is a class III subgroup of $\mathbf{O}(\mathbf{3})$ then for odd values of ℓ where $-I$ acts as the identity, $\dim \mathrm{Fix}_{V_\ell}(H)$ is as given by Theorem 3.3.5. However, for even values of ℓ where $-I$ acts as the identity the group H acts on V_ℓ in exactly the same way as the group $\pi(H)$ which is the subgroup of $\mathbf{SO}(\mathbf{3})$ which is isomorphic to H. Thus for even values of ℓ, $\dim \mathrm{Fix}_{V_\ell}(H) = \dim \mathrm{Fix}_{V_\ell}(\pi(H))$. Using (6.2.2) and (6.2.3) we then find that when $K = H$

$$\dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(H^\theta) \quad = \quad \begin{cases} \dim \mathrm{Fix}_{V_\ell}(H) + \dim \mathrm{Fix}_{V_{\ell+1}}(\pi(H)) & \text{when } \ell \text{ odd} \\ \dim \mathrm{Fix}_{V_\ell}(\pi(H)) + \dim \mathrm{Fix}_{V_{\ell+1}}(H) & \text{when } \ell \text{ even.} \end{cases}$$

   When $H/K = \mathbb{Z}_2$,

$$\dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(H^\theta) = \dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(K) - \dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(H)$$

   where $\dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(H)$ is as above and $\dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(K)$ depends on the class of K.

Table 6.7: Twisted subgroups H^θ of $\mathbf{O(3)} \times \mathbb{Z}_2$ and formulae for $\dim \mathrm{Fix}(H^\theta)$ for the reducible representations on $V_\ell \oplus V_{\ell+1}$.
(†): $q(H^\theta) = 3$ when $n = 1$.

	H	K	H/K	$\dim \mathrm{Fix}(H^\theta)$ for ℓ odd	$\dim \mathrm{Fix}(H^\theta)$ for ℓ even	$q(H^\theta)$
	$\mathbf{SO(3)}$	$\mathbf{SO(3)}$	$\mathbb{1}$	0	0	0
	$\mathbf{O(2)}$	$\mathbf{O(2)}$	$\mathbb{1}$	1	1	0
	$\mathbf{O(2)}$	$\mathbf{SO(2)}$	$\mathbb{Z}_2$	1	1	0
	$\mathbf{SO(2)}$	$\mathbf{SO(2)}$	$\mathbb{1}$	2	2	0
H class I	$\mathbf{D}_n$	$\mathbf{D}_n$	$\mathbb{1}$	$\left[\frac{\ell}{n}\right] + \left[\frac{\ell+1}{n}\right] + 1$	$\left[\frac{\ell}{n}\right] + \left[\frac{\ell+1}{n}\right] + 1$	0
	$\mathbf{D}_{2n}$	$\mathbf{D}_n$	$\mathbb{Z}_2$	$\left[\frac{\ell+n}{2n}\right] + \left[\frac{\ell+1+n}{2n}\right]$	$\left[\frac{\ell+n}{2n}\right] + \left[\frac{\ell+1+n}{2n}\right]$	0
	$\mathbf{D}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_2$	$\left[\frac{\ell}{n}\right] + \left[\frac{\ell+1}{n}\right] + 1$	$\left[\frac{\ell}{n}\right] + \left[\frac{\ell+1}{n}\right] + 1$	0
	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{1}$	$2\left[\frac{\ell}{n}\right] + 2\left[\frac{\ell+1}{n}\right] + 2$	$2\left[\frac{\ell}{n}\right] + 2\left[\frac{\ell+1}{n}\right] + 2$	1(†)
	$\mathbb{Z}_{2n}$	$\mathbb{Z}_n$	$\mathbb{Z}_2$	$2\left[\frac{\ell+n}{2n}\right] + 2\left[\frac{\ell+1+n}{2n}\right]$	$2\left[\frac{\ell+n}{2n}\right] + 2\left[\frac{\ell+1+n}{2n}\right]$	1
	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{1}$	$2\left[\frac{\ell}{3}\right] + 2\left[\frac{\ell+1}{3}\right] - \ell + 1$	$2\left[\frac{\ell}{3}\right] + 2\left[\frac{\ell+1}{3}\right] - \ell + 1$	0
	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{1}$	$\left[\frac{\ell}{4}\right] + \left[\frac{\ell+1}{4}\right] + \left[\frac{\ell}{3}\right] + \left[\frac{\ell+1}{3}\right] - \ell + 1$	$\left[\frac{\ell}{4}\right] + \left[\frac{\ell+1}{4}\right] + \left[\frac{\ell}{3}\right] + \left[\frac{\ell+1}{3}\right] - \ell + 1$	0
	$\mathbb{O}$	$\mathbb{T}$	$\mathbb{Z}_2$	$\left[\frac{\ell}{3}\right] + \left[\frac{\ell+1}{3}\right] - \left[\frac{\ell+1}{4}\right] - \left[\frac{\ell}{4}\right]$	$\left[\frac{\ell}{3}\right] + \left[\frac{\ell+1}{3}\right] - \left[\frac{\ell+1}{4}\right] - \left[\frac{\ell}{4}\right]$	0
	$\mathbb{I}$	$\mathbb{I}$	$\mathbb{1}$	$\left[\frac{\ell}{5}\right] + \left[\frac{\ell+1}{5}\right] + \left[\frac{\ell}{3}\right] + \left[\frac{\ell+1}{3}\right] - \ell + 1$	$\left[\frac{\ell}{5}\right] + \left[\frac{\ell+1}{5}\right] + \left[\frac{\ell}{3}\right] + \left[\frac{\ell+1}{3}\right] - \ell + 1$	0
	$\mathbf{O(3)}$	$\mathbf{O(3)}$	$\mathbb{1}$	0	0	0
	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbb{1}$	1	1	0
	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	0	0	0
	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	$\mathbb{1}$	1	1	0
	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbb{1}$	$\left[\frac{\ell+1}{n}\right] + 1$	$\left[\frac{\ell}{n}\right] + 1$	0
	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$\left[\frac{\ell+1}{n}\right]$	$\left[\frac{\ell}{n}\right]$	0
H class II	$\mathbf{D}_{2n} \times \mathbb{Z}_2^c$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$\left[\frac{\ell+1+n}{2n}\right]$	$\left[\frac{\ell+n}{2n}\right]$	0
K class II	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\mathbb{1}$	$2\left[\frac{\ell+1}{n}\right] + 1$	$2\left[\frac{\ell}{n}\right] + 1$	1(†)
	$\mathbb{Z}_{2n} \times \mathbb{Z}_2^c$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$2\left[\frac{\ell+1+n}{2n}\right]$	$2\left[\frac{\ell+n}{2n}\right]$	1
	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{1}$	$2\left[\frac{\ell+1}{3}\right] + \left[\frac{\ell+1}{2}\right] - \ell$	$2\left[\frac{\ell}{3}\right] + \left[\frac{\ell}{2}\right] - \ell + 1$	0
	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{1}$	$\left[\frac{\ell+1}{4}\right] + \left[\frac{\ell+1}{3}\right] + \left[\frac{\ell+1}{2}\right] - \ell$	$\left[\frac{\ell}{4}\right] + \left[\frac{\ell}{3}\right] + \left[\frac{\ell}{2}\right] - \ell + 1$	0
	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$\left[\frac{\ell+1}{4}\right] - \left[\frac{\ell+1}{3}\right]$	$\left[\frac{\ell}{4}\right] - \left[\frac{\ell}{3}\right]$	0
	$\mathbb{I} \times \mathbb{Z}_2^c$	$\mathbb{I} \times \mathbb{Z}_2^c$	$\mathbb{1}$	$\left[\frac{\ell+1}{5}\right] + \left[\frac{\ell+1}{3}\right] + \left[\frac{\ell+1}{2}\right] - \ell$	$\left[\frac{\ell}{5}\right] + \left[\frac{\ell}{3}\right] + \left[\frac{\ell}{2}\right] - \ell + 1$	0

Continued on next page

Table 6.7 – continued from previous page

	H	K	H/K	$\dim \mathrm{Fix}(H^\theta)$ for ℓ odd	$\dim \mathrm{Fix}(H^\theta)$ for ℓ even	$q(H^\theta)$
	$\mathbf{O(3)}$	$\mathbf{SO(3)}$	$\mathbb{Z}_2$	0	0	0
	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbf{O(2)}$	$\mathbb{Z}_2$	0	0	0
	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$	1	1	0
	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	$\mathbf{SO(2)}$	$\mathbb{Z}_2$	1	1	0
H class II	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbf{D}_n$	$\mathbb{Z}_2$	$\left[\frac{\ell}{n}\right]$	$\left[\frac{\ell+1}{n}\right]$	0
K class	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbf{D}_n^z$	$\mathbb{Z}_2$	$\left[\frac{\ell}{n}\right]+1$	$\left[\frac{\ell+1}{n}\right]+1$	0
I or III	$\mathbf{D}_{2n} \times \mathbb{Z}_2^c$	$\mathbf{D}_{2n}^d$	$\mathbb{Z}_2$	$\left[\frac{\ell+n}{2n}\right]$	$\left[\frac{\ell+n+1}{2n}\right]$	0
	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_n$	$\mathbb{Z}_2$	$2\left[\frac{\ell}{n}\right]+1$	$2\left[\frac{\ell+1}{n}\right]+1$	$1(\dagger)$
	$\mathbb{Z}_{2n} \times \mathbb{Z}_2^c$	$\mathbb{Z}_{2n}^-$	$\mathbb{Z}_2$	$2\left[\frac{\ell+n}{2n}\right]$	$2\left[\frac{\ell+1+n}{2n}\right]$	1
	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{T}$	$\mathbb{Z}_2$	$2\left[\frac{\ell}{3}\right]+\left[\frac{\ell}{2}\right]-\ell+1$	$2\left[\frac{\ell+1}{3}\right]+\left[\frac{\ell+1}{2}\right]-\ell$	0
	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{O}$	$\mathbb{Z}_2$	$\left[\frac{\ell}{4}\right]+\left[\frac{\ell}{3}\right]+\left[\frac{\ell}{2}\right]-\ell+1$	$\left[\frac{\ell+1}{4}\right]+\left[\frac{\ell+1}{3}\right]+\left[\frac{\ell+1}{2}\right]-\ell$	0
	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{O}^-$	$\mathbb{Z}_2$	$\left[\frac{\ell}{4}\right]-\left[\frac{\ell}{3}\right]$	$\left[\frac{\ell+1}{4}\right]-\left[\frac{\ell+1}{3}\right]$	0
	$\mathbb{I} \times \mathbb{Z}_2^c$	$\mathbb{I}$	$\mathbb{Z}_2$	$\left[\frac{\ell}{5}\right]+\left[\frac{\ell}{3}\right]+\left[\frac{\ell}{2}\right]-\ell+1$	$\left[\frac{\ell+1}{5}\right]+\left[\frac{\ell+1}{3}\right]+\left[\frac{\ell+1}{2}\right]-\ell$	0
	$\mathbf{O(2)}^-$	$\mathbf{O(2)}^-$	$\mathbb{1}$	2	2	0
	$\mathbf{O(2)}^-$	$\mathbf{SO(2)}$	$\mathbb{Z}_2$	0	0	0
	$\mathbb{O}^-$	$\mathbb{O}^-$	$\mathbb{1}$	$\left[\frac{\ell}{3}\right]-\left[\frac{\ell}{4}\right]+\left[\frac{\ell+1}{4}\right]+\left[\frac{\ell+1}{3}\right]+\left[\frac{\ell+1}{2}\right]-\ell$	$\left[\frac{\ell+1}{3}\right]-\left[\frac{\ell+1}{4}\right]+\left[\frac{\ell}{4}\right]+\left[\frac{\ell}{3}\right]+\left[\frac{\ell}{2}\right]-\ell+1$	0
	$\mathbb{O}^-$	$\mathbb{T}$	$\mathbb{Z}_2$	$\left[\frac{\ell}{4}\right]+\left[\frac{\ell}{3}\right]-\left[\frac{\ell+1}{4}\right]+\left[\frac{\ell+1}{3}\right]-\left[\frac{\ell+1}{2}\right]+1$	$\left[\frac{\ell+1}{4}\right]+\left[\frac{\ell+1}{3}\right]-\left[\frac{\ell}{4}\right]+\left[\frac{\ell}{3}\right]-\left[\frac{\ell}{2}\right]$	0
	$\mathbb{Z}_{2n}^-$	$\mathbb{Z}_{2n}^-$	$\mathbb{1}$	$2\left[\frac{\ell+n}{2n}\right]+2\left[\frac{\ell+1}{2n}\right]+1$	$2\left[\frac{\ell+1+n}{2n}\right]+2\left[\frac{\ell}{2n}\right]+1$	1
	$\mathbb{Z}_{2n}^-$	$\mathbb{Z}_n$	$\mathbb{Z}_2$	$2\left[\frac{\ell}{2n}\right]+2\left[\frac{\ell+1+n}{2n}\right]+1$	$2\left[\frac{\ell+n}{2n}\right]+2\left[\frac{\ell+1}{2n}\right]+1$	1
H class III	$\mathbf{D}_{2n}^d$	$\mathbf{D}_{2n}^d$	$\mathbb{1}$	$\left[\frac{\ell+n}{2n}\right]+\left[\frac{\ell+1}{2n}\right]+1$	$\left[\frac{\ell+1+n}{2n}\right]+\left[\frac{\ell}{2n}\right]+1$	0
	$\mathbf{D}_{2n}^d$	$\mathbf{D}_n^z$	$\mathbb{Z}_2$	$\left[\frac{\ell}{2n}\right]+\left[\frac{\ell+1+n}{2n}\right]+1$	$\left[\frac{\ell+n}{2n}\right]+\left[\frac{\ell+1}{2n}\right]+1$	0
	$\mathbf{D}_{2n}^d$	$\mathbf{D}_n$	$\mathbb{Z}_2$	$\left[\frac{\ell}{2n}\right]+\left[\frac{\ell+1+n}{2n}\right]$	$\left[\frac{\ell+n}{2n}\right]+\left[\frac{\ell+1}{2n}\right]$	0
	$\mathbf{D}_{2n}^d$	$\mathbb{Z}_{2n}^-$	$\mathbb{Z}_2$	$\left[\frac{\ell+n}{2n}\right]+\left[\frac{\ell+1}{2n}\right]$	$\left[\frac{\ell+n+1}{2n}\right]+\left[\frac{\ell}{2n}\right]$	0
	$\mathbf{D}_n^z$	$\mathbf{D}_n^z$	$\mathbb{1}$	$\left[\frac{\ell}{n}\right]+\left[\frac{\ell+1}{n}\right]+2$	$\left[\frac{\ell}{n}\right]+\left[\frac{\ell+1}{n}\right]+2$	0
	$\mathbf{D}_n^z$	$\mathbb{Z}_n$	$\mathbb{Z}_2$	$\left[\frac{\ell}{n}\right]+\left[\frac{\ell+1}{n}\right]$	$\left[\frac{\ell}{n}\right]+\left[\frac{\ell+1}{n}\right]$	0
	$\mathbf{D}_{2n}^z$	$\mathbf{D}_n^z$	$\mathbb{Z}_2$	$\left[\frac{\ell+n}{2n}\right]+\left[\frac{\ell+1+n}{2n}\right]$	$\left[\frac{\ell+n}{2n}\right]+\left[\frac{\ell+1+n}{2n}\right]$	0
	$\mathbf{D}_2^z$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2$	$\frac{3\ell}{2}-\frac{1}{2}-\left[\frac{\ell}{2}\right]$	$\frac{3\ell}{2}+1-\left[\frac{\ell+1}{2}\right]$	0

The axial isotropy subgroups for the representation of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ on $V_\ell \oplus V_{\ell+1}$ are the axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for the representations on V_ℓ and $V_{\ell+1}$ which can be found from Table 6.2.

There are two ways to determine all of the isotropy subgroups for a given reducible representation of a group Γ. One way is to use the massive chain criterion just as we have for the irreducible representations of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ in examples 6.2.5–6.2.7 above for the representations on V_2, V_3 and V_4. The second way is to use the following proposition of Chossat and Guyard [22].

Proposition 6.2.9. *Let ρ_1 and ρ_2 be two irreducible representation of Γ acting respectively on the vector spaces W_1 and W_2 and let $W = W_1 \oplus W_2$. Then Σ is an isotropy subgroup for the action $\rho = \rho_1 + \rho_2$ of Γ on W, if and only if there exist isotropy subgroups Σ_1 and Σ_2 for the representations ρ_1 and ρ_2 respectively such that*

$$\Sigma = \Sigma_1 \cap \Sigma_2.$$

Proof. Let (w_1, w_2) be any element of $W_1 \oplus W_2$ and let Σ be its isotropy subgroup. Clearly Σ must fix both elements w_1 and w_2 so $\Sigma \subset \Sigma_{w_1} \cap \Sigma_{w_2}$ with Σ_{w_1} and Σ_{w_2} the isotropy subgroups of w_1 and w_2 respectively. Conversely, any element in $\Sigma_{w_1} \cap \Sigma_{w_2}$ fixes w_1 as well as w_2 and hence is in Σ. $\square$

To use Proposition 6.2.9 we must consider all isotropy subgroups of Γ not just the conjugacy classes. All possible orientations of the isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ on V_ℓ and $V_{\ell+1}$ must be considered. Since this complicates this method of computation we will use the massive chain criterion method to determine the isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for representations on $V_\ell \oplus V_{\ell+1}$ for the examples in this thesis. We now consider the example where the representation is on $V_2 \oplus V_3$.

Example 6.2.10 (The natural representation on $V_2 \oplus V_3$). For the reducible representation of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ on $V_2 \oplus V_3$ the axial isotropy subgroups are the axial isotropy subgroups for the irreducible representations of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ on V_2 and V_3. These isotropy subgroups were found in Examples 6.2.5 and 6.2.6 and are given in Table 6.8 along with one possible form of the fixed-point subspace of each group. These fixed-point subspaces are given in the form

$$\{(x_{-2},\, x_{-1},\, x_0,\, x_1,\, x_2\,;\, y_{-3},\, y_{-2},\, y_{-1},\, y_0,\, y_1,\, y_2,\, y_3)\}$$

where x_j is the amplitude of $Y_2^j(\theta, \phi)$ and y_j is the amplitude of $Y_3^j(\theta, \phi)$. Recall that these amplitudes satisfy $x_{-j} = (-1)^j \overline{x}_j$ and $y_{-j} = (-1)^j \overline{y}_j$.

Using Table 6.7 we find that the twisted subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ with a fixed-point subspace of dimension greater than 1 when $\ell = 2$ are as listed in Table 6.9. Using the massive chain criterion we can determine which of these twisted subgroups are isotropy subgroups. The penultimate column in Table 6.9 shows whether or not each twisted subgroup is an isotropy subgroup. If it is an isotropy subgroup, a label for the subgroup is also given in this column. The final column gives one possible form of the fixed-point subspace if the twisted subgroup is an isotropy subgroup. If the twisted subgroup H^θ is not an isotropy subgroup then this column

Isotropy subgroup	H	K	H/K	Fixed-point subspace
$\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c$	$\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c$	$\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c$	$\mathbb{1}$	$\{(0,0,a,0,0;0,0,0,0,0,0,0)\}$
$\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$	$\mathbf{D}_4 \times \mathbb{Z}_2^c$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$\{(a,0,0,0,a;0,0,0,0,0,0,0)\}$
$\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c}$	$\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c$	$\mathbf{O}(\mathbf{2})^-$	$\mathbb{Z}_2$	$\{(0,0,0,0,0;0,0,0,a,0,0,0)\}$
$\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$	$\mathbf{D}_6 \times \mathbb{Z}_2^c$	$\mathbf{D}_6^d$	$\mathbb{Z}_2$	$\{(0,0,0,0,0;\mathrm{i}a,0,0,0,0,0,\mathrm{i}a)\}$
$\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{O}^-$	$\mathbb{Z}_2$	$\{(0,0,0,0,0;0,a,0,0,0,a,0)\}$

Table 6.8: Axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ for the $(2,3)$ mode interaction and one possible form of their fixed-point subspaces. In all fixed-point subspaces, $a \in \mathbb{R}$.

gives an example of a twisted subgroup G^θ for which the massive chain criterion (Theorem 3.4.1) fails.

Figure 6.4 shows images of patterns with the symmetries of some of the isotropy subgroups. Patterns with symmetry groups containing the isotropy subgroup $\widetilde{\mathbf{D}_2}$ can be seen in Figure 7.4. We can construct the lattice of isotropy subgroups for the $(2,3)$ mode interaction as in Figure 6.5 by using the dimensions of the fixed-point subspaces of the isotropy subgroups given in Table 6.9. We also must consider the containment relations between the twisted subgroups. In some cases it may be clear from the form of the fixed-point subspace given in Table 6.9 that one twisted subgroup lies inside another, however in other cases the fixed-point subspaces may need to be rotated in order to see the containment.

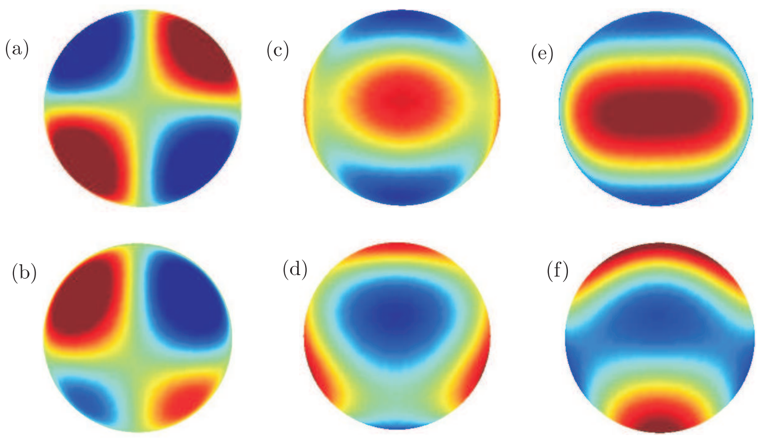


Figure 6.4: Images of patterns with symmetries of some isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ in the representation on $V_2 \oplus V_3$ as given in Table 6.9. Images (a) and (b) have $\widetilde{\mathbf{D}_4^z}$ symmetry, (c) and (d) have $\widetilde{\mathbf{D}_4^d}$ symmetry and (e) and (f) have $\mathbf{D}_4^d$ symmetry all viewed from the top and side respectively. See Table 6.9 for definitions of these symmetry groups.

H	K	$\dim \mathrm{Fix}(H^\theta)$	$r(H^\theta)$	Isotropy subgroup?	Fixed-point subspace/ Example of larger group G^θ
$\mathbf{SO}(\mathbf{2})$	$\mathbf{SO}(\mathbf{2})$	2	0	No	$(\mathbf{O}(\mathbf{2})^-,\ \mathbf{O}(\mathbf{2})^-)$
$\mathbf{D}_3$	$\mathbf{D}_3$	2	0	No	$(\mathbf{D}_6^d,\ \mathbf{D}_6^d)$
$\mathbf{D}_4$	$\mathbf{D}_2$	2	0	Yes, $\widetilde{\mathbf{D}_4}$	$\{(a,0,0,0,a;0,ib,0,0,0,-ib,0)\}$
$\mathbf{D}_3$	$\mathbb{Z}_3$	2	0	No	$(\mathbf{D}_3 \times \mathbb{Z}_2^c,\ \mathbf{D}_3^z)$
$\mathbb{Z}_n$	$\mathbb{Z}_n$	2 when $n \geq 4$	1	No	$(\mathbf{O}(\mathbf{2})^-,\ \mathbf{O}(\mathbf{2})^-)$
$\mathbb{Z}_6$	$\mathbb{Z}_3$	2	1	No	$(\mathbf{D}_6 \times \mathbb{Z}_2^c,\ \mathbf{D}_6^d)$
$\mathbf{D}_2$	$\mathbf{D}_2$	3	0	Yes, $\mathbf{D}_2$	$\{(a,0,b,0,a;0,ic,0,0,0,-ic,0)\}$
$\mathbf{D}_2$	$\mathbb{Z}_2$	3	0	Yes, $\widetilde{\mathbf{D}_2}$	$\{(ia,0,0,0,-ia;0,b,0,c,0,b,0)\}$
$\mathbb{Z}_3$	$\mathbb{Z}_3$	4	1	No	$(\mathbf{D}_3^z,\ \mathbf{D}_3^z)$
$\mathbb{Z}_4$	$\mathbb{Z}_2$	4	1	Yes, $\widetilde{\mathbb{Z}_4}$	$\{(A,0,0,0,\overline{A};0,B,0,0,0,\overline{B},0)\}$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	6	1	Yes, $\mathbb{Z}_2$	$\{(A,0,c,0,\overline{A};0,B,0,d,0,\overline{B},0)\}$
$\mathbb{Z}_2$	$\mathbb{1}$	6	1	Yes, $\widetilde{\mathbb{Z}_2}$	$\{(0,A,0,-\overline{A},0;B,0,C,0,-\overline{C},0,-\overline{B})\}$
$\mathbb{1}$	$\mathbb{1}$	12	3	Yes, $\mathbb{1}$	$V_2 \oplus V_3$
$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	2	0	Yes, $\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\{(a,0,b,0,a;0,0,0,0,0,0,0)\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	2	1	No	$(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c)$
$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2^c$	2	1	No	$(\mathbf{D}_4 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c)$
$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	3	1	No	$(\mathbf{D}_2 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c)$
$\mathbb{Z}_2^c$	$\mathbb{Z}_2^c$	5	3	No	$(\mathbf{D}_2 \times \mathbb{Z}_2^c,\ \mathbf{D}_2 \times \mathbb{Z}_2^c)$
$\mathbf{D}_3 \times \mathbb{Z}_2^c$	$\mathbf{D}_3^z$	2	0	Yes, $\widetilde{\mathbf{D}_3 \times \mathbb{Z}_2^c}$	$\{(0,0,0,0,0;a,0,0,b,0,0,-a)\}$
$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbf{D}_2^z$	2	0	Yes, $\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c}$	$\{(0,0,0,0,0;0,a,0,b,0,a,0)\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2^c$	$\mathbb{Z}_6^-$	2	1	No	$(\mathbf{D}_6 \times \mathbb{Z}_2^c,\ \mathbf{D}_6^d)$
$\mathbb{Z}_4 \times \mathbb{Z}_2^c$	$\mathbb{Z}_4^-$	2	1	No	$(\mathbb{O} \times \mathbb{Z}_2^c,\ \mathbb{O}^-)$
$\mathbb{Z}_3 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$	3	1	No	$(\mathbf{D}_3 \times \mathbb{Z}_2^c,\ \mathbf{D}_3^z)$
$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	3	1	No	$(\mathbf{D}_2 \times \mathbb{Z}_2^c,\ \mathbf{D}_2^z)$
$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2^-$	4	1	Yes, $\widetilde{\mathbb{Z}_2 \times \mathbb{Z}_2^c}$	$\{(0,0,0,0,0;a,b,c,d,-c,b,-a)\}$
$\mathbb{Z}_2^c$	$\mathbb{1}$	7	3	Yes, $\widetilde{\mathbb{Z}_2^c}$	V_3
$\mathbf{O}(\mathbf{2})^-$	$\mathbf{O}(\mathbf{2})^-$	2	0	Yes, $\mathbf{O}(\mathbf{2})^-$	$\{(0,0,a,0,0;0,0,0,b,0,0,0)\}$
$\mathbf{D}_6^d$	$\mathbf{D}_6^d$	2	0	Yes, $\mathbf{D}_6^d$	$\{(0,0,a,0,0;ib,0,0,0,0,0,ib)\}$
$\mathbf{D}_4^d$	$\mathbf{D}_4^d$	2	0	Yes, $\mathbf{D}_4^d$	$\{(0,0,a,0,0;0,ib,0,0,0,-ib,0)\}$
$\mathbf{D}_4^d$	$\mathbf{D}_2^z$	2	0	Yes, $\widetilde{\mathbf{D}_4^d}$	$\{(a,0,0,0,a;0,0,0,b,0,0,0)\}$
$\mathbf{D}_4^z$	$\mathbf{D}_4^z$	2	0	No	$(\mathbf{O}(\mathbf{2})^-,\ \mathbf{O}(\mathbf{2})^-)$
$\mathbf{D}_2^z$	$\mathbb{Z}_2$	2	0	No	$(\mathbf{D}_4^z,\ \mathbf{D}_2^z)$
$\mathbf{D}_4^z$	$\mathbf{D}_2^z$	2	0	Yes, $\widetilde{\mathbf{D}_4^z}$	$\{(a,0,0,0,a;0,b,0,0,0,b,0)\}$
$\mathbb{Z}_6^-$	$\mathbb{Z}_6^-$	3	1	No	$(\mathbf{D}_6^d,\ \mathbf{D}_6^d)$
$\mathbb{Z}_4^-$	$\mathbb{Z}_4^-$	3	1	No	$(\mathbf{D}_4^d,\ \mathbf{D}_4^d)$
$\mathbb{Z}_4^-$	$\mathbb{Z}_2$	3	1	No	$(\mathbf{D}_4^d,\ \mathbf{D}_2^z)$
$\mathbf{D}_3^z$	$\mathbf{D}_3^z$	3	0	Yes, $\mathbf{D}_3^z$	$\{(0,0,a,0,0;b,0,0,c,0,0,-b)\}$
$\mathbf{D}_2^z$	$\mathbb{Z}_2^-$	3	0	Yes, $\widetilde{\mathbf{D}_2^z}$	$\{(ia,0,0,0,-ia;b,0,c,0,-c,0,-b)\}$
$\mathbf{D}_2^z$	$\mathbf{D}_2^z$	4	0	Yes, $\mathbf{D}_2^z$	$\{(a,0,b,0,a;0,c,0,d,0,c,0)\}$
$\mathbb{Z}_2^-$	$\mathbb{1}$	5	1	Yes, $\widetilde{\mathbb{Z}_2^-}$	$\{(0,A,0,-\overline{A},0;0,B,0,c,0,\overline{B},0)\}$
$\mathbb{Z}_2^-$	$\mathbb{Z}_2^-$	7	1	Yes, $\mathbb{Z}_2^-$	$\{(a,b,c,-b,a;d,e,f,g,-f,e,-d)\}$

Table 6.9: The twisted subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ with a fixed-point subspace of dimension greater than 1 in the representation on $V_2 \oplus V_3$. In the fixed-point subspaces, lower-case letters represent real values and upper-case, complex values.

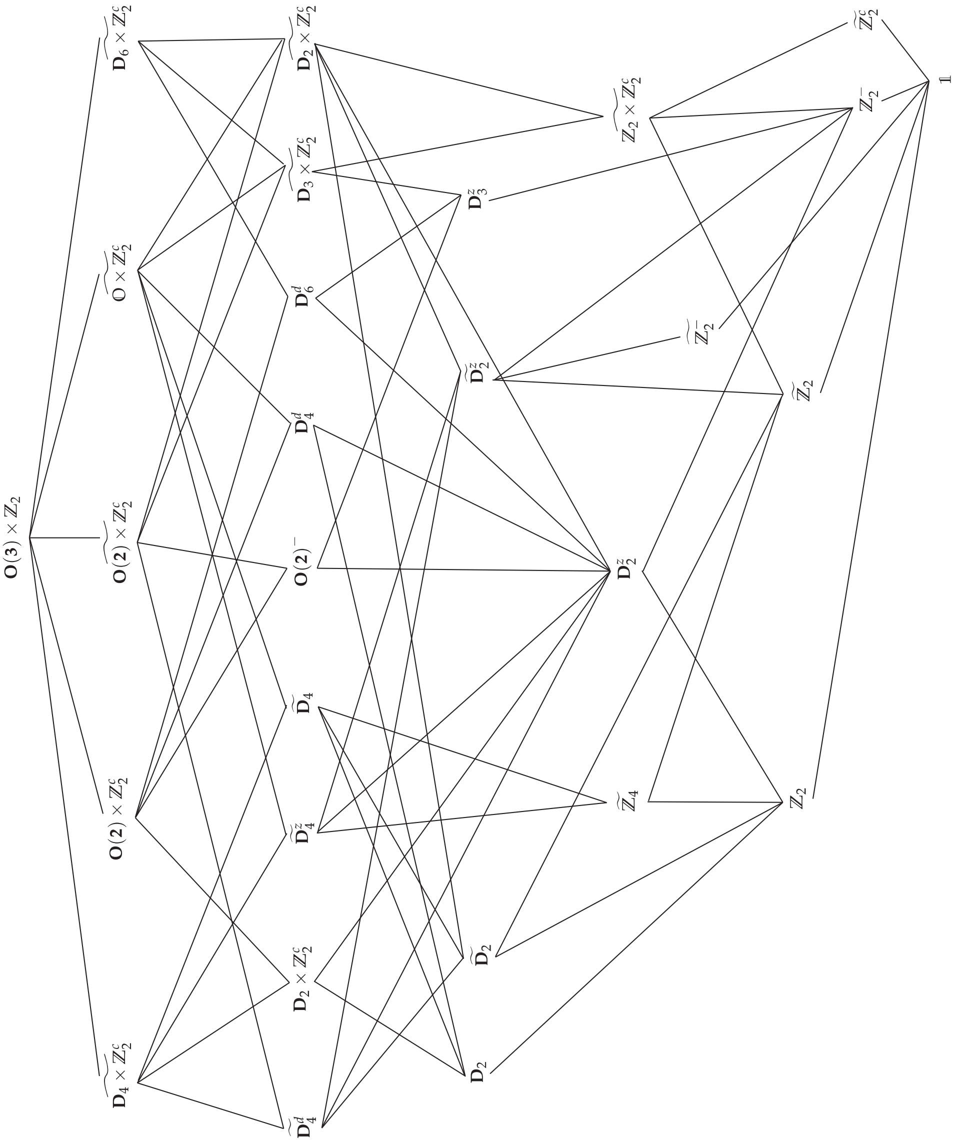


Figure 6.5: Lattice of isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ for the $(2, 3)$ mode interaction.

6.3 $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector fields

In this section we first consider the relationship between $\mathbf{O(3)} \times \mathbb{Z}_2$ and $\mathbf{O(3)}$ equivariant vector fields for natural irreducible representations on V_ℓ. Differences between these vector fields (and hence different dynamics) occur for even values of ℓ. We consider the example where $\ell = 2$ in some detail. We then move on to discuss the mappings which are equivariant with respect to reducible actions of $\mathbf{O(3)} \times \mathbb{Z}_2$ on $V_\ell \oplus V_{\ell+1}$. For general ℓ we compute the number of cubic $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant mappings using character methods. Finally we will compute to cubic order the Taylor expansion of the general $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field for the natural representations on $V_2 \oplus V_3$ and $V_3 \oplus V_4$. Throughout this section we will consider only the natural representation of $\mathbf{O(3)} \times \mathbb{Z}_2$ where $-I \in \mathbf{O(3)}$ acts as $(-1)^\ell$ on the spherical harmonics of degree ℓ and $-1 \in \mathbb{Z}_2$ acts as -1 on all spherical harmonics.

6.3.1 Irreducible representations

Consider the system of ODEs

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(\mathbf{x}, \lambda), \tag{6.3.1}$$

where $\lambda \in \mathbb{R}$ is a bifurcation parameter, $\mathbf{x}$ is the vector of amplitudes of the spherical harmonics $Y_\ell^m(\theta, \phi)$ and f is a smooth mapping which is equivariant with respect to the natural representation of $\mathbf{O(3)} \times \mathbb{Z}_2$ on V_ℓ.

Hence f satisfies

$$f(M_\gamma \cdot \mathbf{x}, \lambda) = M_\gamma \cdot f(\mathbf{x}, \lambda)$$

for the matrices M_γ which generate the action of $\mathbf{O(3)}$ on V_ℓ. These matrices are given in Section 3.2.2. Recall in particular that $M_{-I} = (-1)^\ell I_{2\ell+1}$ where $I_{2\ell+1}$ is the $(2\ell+1) \times (2\ell+1)$ identity matrix. In addition, f is equivariant with respect to the action of $-1 \in \mathbb{Z}_2$ on V_ℓ. The matrix for this action is $M_{-1} = -I_{2\ell+1}$.

If ℓ is odd then $M_{-I} = M_{-1}$ and hence an $\mathbf{O(3)}$ equivariant mapping on V_ℓ when ℓ is odd is equivariant with respect to $\mathbf{O(3)} \times \mathbb{Z}_2$. The $\mathbf{O(3)}$ equivariant vector fields for $\ell = 3$ and 5 have been computed to cubic order by Chossat et al. [23].

If ℓ is even then imposing that an $\mathbf{O(3)}$ equivariant vector field also commutes with $-1 \in \mathbb{Z}_2$ removes all of the terms of even order from the $\mathbf{O(3)}$ equivariant vector field. Hence in the representation on V_ℓ for ℓ even the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field is the same as the $\mathbf{O(3)}$ equivariant vector field with the even order terms removed. The cubic order terms in such a vector field for $\ell = 4$ and 6 have been computed by Callahan [18].

When ℓ is even the dynamics in an $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field will be different from those in an $\mathbf{O(3)}$ equivariant vector field and have not previously been studied. We consider now the example where the representation is the natural representation on V_2.

6.3.2 Example: The representation on V_2

In this section we consider the solutions which can occur in the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field for the representation on V_2. We compute the stability of the axial solution branches and look for submaximal solutions.

We can compute using the method in Section 6.3.1 that to cubic order for the representation of $\mathbf{O(3)} \times \mathbb{Z}_2$ on V_2 the equivariant vector field f is given by

$$\dot{\mathbf{x}} = f(\mathbf{x}, \lambda) = \mu \mathbf{x} + \alpha \mathbf{x}|\mathbf{x}|^2, \tag{6.3.2}$$

where

$$\mathbf{x} = (x_{-2}, x_{-1}, x_0, x_1, x_2)^{\mathbf{T}}$$

is the vector of amplitudes of the spherical harmonics of degree two. Here $x_{-m} = (-1)^m \overline{x}_m$ and

$$|\mathbf{x}|^2 = \sum_{m=-2}^{2} |x_m|^2.$$

The coefficients α and μ are smooth, real functions of λ. We will assume that $\mu = \lambda +$ higher order terms in λ so that a stationary bifurcation occurs at $\lambda = 0$ and the trivial solution is stable for $\lambda < 0$.

Recall from Example 6.2.5 that in this representation there are two axial isotropy subgroups, $\mathbf{O(2)} \times \mathbb{Z}_2^c$ and $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$. The only other isotropy subgroup in this representation is $\mathbf{D}_2 \times \mathbb{Z}_2^c$ and every combination of spherical harmonics of degree two has $\mathbf{D}_2 \times \mathbb{Z}_2^c$ symmetry about some axes. (Any combination of spherical harmonics $Y_2^m(\theta, \phi)$ automatically has $\mathbb{Z}_2^c$ symmetry. Since $\mathrm{Re}(Y_2^j(\theta, \phi))$ and $\mathrm{Im}(Y_2^j(\theta, \phi))$ are equivalent patterns under a rotation for $j = 1, 2$, $r(\mathbb{Z}_2^c) = 3$ and we have three degrees of freedom with the choice of our rotation axes. We can choose these in such a way that the combination has $\mathbf{D}_2 \times \mathbb{Z}_2^c$ symmetry about these axes.) The group of symmetries of any solution to $\dot{\mathbf{x}} = f(\mathbf{x}, \lambda)$ must contain $\mathbf{D}_2 \times \mathbb{Z}_2^c$ and all solutions lie in $\mathrm{Fix}(\mathbf{D}_2 \times \mathbb{Z}_2^c)$. The fixed-point subspaces of the isotropy subgroups are

$$\begin{aligned}
\mathrm{Fix}(\mathbf{O(2)} \times \mathbb{Z}_2^c) &= \{(0, 0, a, 0, 0)\} \\
\mathrm{Fix}(\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}) &= \{(b, 0, 0, 0, b)\} \\
\mathrm{Fix}(\mathbf{D}_2 \times \mathbb{Z}_2^c) &= \{(b, 0, a, 0, b)\}
\end{aligned}$$

where $a, b \in \mathbb{R}$. By the equivariant branching lemma (Theorem 2.4.6) stationary solutions with $\mathbf{O(2)} \times \mathbb{Z}_2^c$ and $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetries are guaranteed to exist. By (2.4.6) these solutions must have 2 and 3 zero eigenvectors respectively, none of which lie in $\mathrm{Fix}(\mathbf{D}_2 \times \mathbb{Z}_2^c)$. However, using the cubic order truncation (6.3.2) of the equivariant vector field, f, we find that both solution branches have four zero eigenvalues. This means that there are degeneracies at cubic order and we must include the quintic order terms in f to determine the stability of the axial solution branches. Recall that a $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field contains no quartic terms. To quintic order the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field is

$$\dot{\mathbf{x}} = f(\mathbf{x}, \lambda) = \lambda \mathbf{x} + \alpha \mathbf{x}|\mathbf{x}|^2 + \beta \mathbf{x} \left(\sum_{m=-2}^{2} |x_m|^2 \right)^2 + \gamma\, \mathbf{S}(\mathbf{x}) \tag{6.3.3}$$

where

$$\begin{aligned}
S_{-2}(\mathbf{x}) &= x_{-2}\left(10x_0^4 - 36x_{-1}x_0^2x_1 + 24x_{-2}^2x_2^2 - 48x_2x_1x_{-1}x_{-2} + 15x_{-1}^2x_1^2\right) - 9x_2x_{-1}^4 \\
&\quad +\sqrt{6}x_0\left(3x_1x_{-1}^3 + 12x_2x_{-1}^2x_{-2} - x_0^2x_{-1}^2\right) \\
S_{-1}(\mathbf{x}) &= x_{-1}\left(4x_0^4 - 18x_{-1}x_1x_0^2 + 36x_2x_{-2}x_0^2 + 24x_{-2}^2x_2^2 + 24x_{-1}^2x_1^2 - 30x_2x_1x_{-1}x_{-2}\right) \\
&\quad +18x_1^3x_{-2}^2 + \sqrt{6}x_0\left(2x_0^2x_1x_{-2} - 12x_2x_1x_{-1}^2 - 3x_2x_{-1}^3 - 9x_1^2x_{-1}x_{-2}\right) \\
S_0(\mathbf{x}) &= x_0\left(4x_0^4 - 16x_{-1}x_1x_0^2 + 40x_2x_{-2}x_0^2 + 18x_{-1}^2x_1^2 - 72x_2x_1x_{-1}x_{-2}\right) \\
&\quad +3\sqrt{6}\left(2x_2x_{-2} + x_1x_{-1} - x_0^2\right)\left(x_1^2x_{-2} + x_{-1}^2x_2\right)
\end{aligned}$$

and $S_{-m} = (-1)^m\overline{S}_m$. We wish to determine conditions on $\alpha, \beta, \gamma \in \mathbb{R}$ for the axial solution branches to be stable and also whether it is possible for solutions with $\mathbf{D}_2 \times \mathbb{Z}_2^c$ symmetry to exist. All eigenvalues of each axial solution which lie in the complement of $\mathrm{Fix}(\mathbf{D}_2 \times \mathbb{Z}_2^c)$ are either zero or equal to an eigenvalue in $\mathrm{Fix}(\mathbf{D}_2 \times \mathbb{Z}_2^c)$. This means that we can use the restriction of (6.3.3) to determine the stability of the axial solution branches in addition to looking for submaximal solutions in this subspace. The restriction of (6.3.3) to $\mathrm{Fix}(\mathbf{D}_2 \times \mathbb{Z}_2^c)$ is

$$\dot{a} = \lambda a + \alpha a(2b^2 + a^2) + (\beta + 4\gamma)\, a^5 + 4\,(\beta + 10\gamma)\, a^3b^2 + 4\beta ab^4 \tag{6.3.4}$$
$$\dot{b} = \lambda b + \alpha b(2b^2 + a^2) + 4\,(\beta + 6\gamma)\, b^5 + (\beta + 10\gamma)\, a^4b + 4\beta a^2b^3. \tag{6.3.5}$$

These equations have residual symmetries $N(\mathbf{D}_2 \times \mathbb{Z}_2^c)/\mathbf{D}_2 \times \mathbb{Z}_2^c = \mathbf{D}_6$. The stationary points of these equations are

1. The trivial solution where $a = b = 0$.

2. The solution where $b = 0$ and a satisfies $\lambda + \alpha a^2 + (\beta + 4\gamma)a^4 = 0$. These solutions are of the form $(a_{0,\pm}, 0)$ where

$$a_{0,\pm}^2 = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\lambda(\beta + 4\gamma)}}{2(\beta + 4\gamma)}.$$

   These solutions have $\mathbf{O(2)} \times \mathbb{Z}_2^c$ symmetry and exist only when $a_{0,\pm}^2$ is real and positive. See Figure 6.6 for a picture of this solution.

3. The solution where $a = 0$ and b satisfies $\lambda + 2\alpha b^2 + 4(\beta + 6\gamma)b^4 = 0$. These solutions are of the form $(0, b_{0,\pm})$ where

$$b_{0,\pm}^2 = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\lambda(\beta + 6\gamma)}}{4(\beta + 6\gamma)}.$$

   These solutions have $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry and exist only when $b_{0,\pm}^2$ is real and positive. See Figure 6.6 for a picture of this solution.

4. Solutions where both $a \neq 0$ and $b \neq 0$. Then a and b satisfy

$$\lambda + \alpha(2b^2 + a^2) + (\beta + 4\gamma)\, a^4 + 4\,(\beta + 10\gamma)\, a^2b^2 + 4\beta b^4 = 0 \tag{6.3.6}$$
$$\lambda + \alpha(2b^2 + a^2) + 4\,(\beta + 6\gamma)\, b^4 + (\beta + 10\gamma)\, a^4 + 4\beta a^2b^2 = 0. \tag{6.3.7}$$

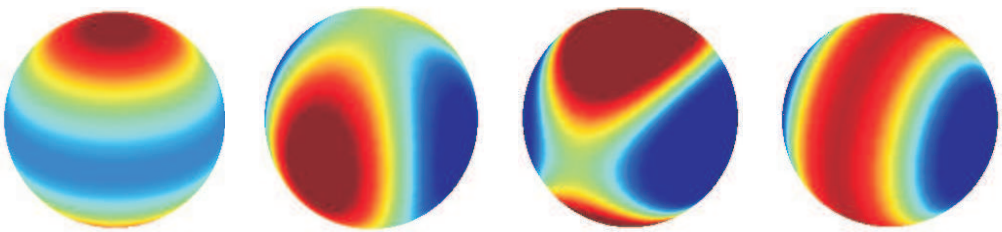
By subtracting (6.3.6) from (6.3.7) we see that

$$3a^4 - 20a^2b^2 + 12b^4 = (3a^2 - 2b^2)(a^2 - 6b^2) = 0$$

and hence there are solutions where

(a) $a^2 = 6b^2$. Using (6.3.6) we see that b must satisfy $\lambda + 8\alpha b^2 + 64(\beta + 6\gamma)b^4 = 0$ and hence $b^2 = \frac{1}{4}b_{0,\pm}^2$. This solution has the same existence properties as solution 3 above and hence it has $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry. This solution is a rotation of solution 3 as can be seen in Figure 6.6.

(b) $3a^2 = 2b^2$. Using (6.3.6) we see that b must satisfy $9\lambda + 24\alpha b^2 + 64(\beta 4\gamma)b^4 = 0$ and hence $b^2 = \frac{3}{8}a_{0,\pm}^2$. This solution has the same existence properties as solution 2 above and hence it has $\mathbf{O(2)} \times \mathbb{Z}_2^c$ symmetry as can be seen in Figure 6.6.



Solution 2 with $\mathbf{O(2)} \times \mathbb{Z}_2^c$ symmetry	Solution 3 with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry	Solution 4a with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry	Solution 4b with $\mathbf{O(2)} \times \mathbb{Z}_2^c$ symmetry

Figure 6.6: Images of solutions to (6.3.4)–(6.3.5). All solutions have axial symmetry i.e. $\mathbf{O(2)} \times \mathbb{Z}_2^c$ or $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry.

There are no further solutions and hence there are no submaximal solutions with $\mathbf{D}_2 \times \mathbb{Z}_2^c$ symmetry. We now investigate the stability of the maximal solution branches, solutions 2 and 3. Solutions 4(a) and 4(b) have the same existence and stability properties as solutions 3 and 2 respectively. We will assume that $\lambda > 0$ so that the trivial solution is unstable and we will find the branches of solutions which bifurcate supercritically at $\lambda = 0$.

Solutions with $\mathbf{O(2)} \times \mathbb{Z}_2^c$ symmetry exist when $a_{0,\pm} \in \mathbb{R}$. For this to occur we require that $\alpha^2 - 4\lambda(\beta + 4\gamma) > 0$ and $a_{0,\pm}^2 > 0$.

- When $\beta + 4\gamma < 0$, $a_{0,-}^2 > 0$ for all values of α but $a_{0,+}^2 < 0$.
- When $0 < 4\lambda(\beta + 4\gamma) < \alpha^2$, $a_{0,\pm}^2 > 0$ when $\alpha < 0$.

These solutions have eigenvalues

$$\xi_1 = 6\gamma a_{0,\pm}^4 \qquad \text{and} \qquad \xi_2 = -2\lambda + 2(\beta + 4\gamma)a_{0,\pm}^4 \ \text{(double in } V_2\text{)}$$

and hence the solutions with $\mathbf{O(2)} \times \mathbb{Z}_2^c$ symmetry are stable when $\gamma < 0$ and $\lambda > (\beta + 4\gamma)a_{0,\pm}^4$.

Solutions with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry exist when $b_{0,\pm} \in \mathbb{R}$. For this to occur we require that $\alpha^2 - 4\lambda(\beta + 6\gamma) > 0$ and $a_{0,\pm}^2 > 0$.

- When $\beta + 6\gamma < 0$, $b_{0,-}^2 > 0$ for all values of α but $b_{0,+}^2 < 0$.

- When $0 < 4\lambda(\beta + 6\gamma) < \alpha^2$, $b_{0,\pm}^2 > 0$ when $\alpha < 0$.

These solutions have eigenvalues

$$\xi_1 = -2\lambda + 8(\beta + 6\gamma)b_{0,\pm}^4 \qquad \text{and} \qquad \xi_2 = -24\gamma b_{0,\pm}^4$$

and hence the solutions with $\mathbf{O(2)} \times \mathbb{Z}_2^c$ symmetry are stable when $\gamma > 0$ and $\lambda > 4(\beta + 6\gamma)b_{0,\pm}^4$.

Hence we can see that the two axial solutions can never be simultaneously stable. There are many different possible phase portraits depending on the values of λ, α, β and γ. The maximum number of solutions occurs when $\alpha^2 - 4\lambda(\beta + 4\gamma) > 0$, $\alpha^2 - 4\lambda(\beta + 6\gamma) > 0$ with $(\beta + 4\gamma) > 0$, $(\beta + 6\gamma) > 0$ and $\alpha < 0$. Then solutions

$$(0,0),\ (\pm a_{0,\pm}, 0),\ (0, \pm b_{0,\pm}),\ \left(\pm\sqrt{\frac{3}{2}}b_{0\pm}, \pm\sqrt{\frac{1}{4}}b_{0\pm}\right),\ \left(\pm\sqrt{\frac{1}{4}}a_{0\pm}, \pm\sqrt{\frac{3}{8}}a_{0\pm}\right)$$

exist–a total of 24 non-trivial solutions to (6.3.6)–(6.3.7) but only two distinct symmetry types. For the example coefficient values $\lambda = 1$, $\alpha = -5$, $\beta = -2$ and $\gamma = 1$ the phase portrait is as in Figure 6.7.

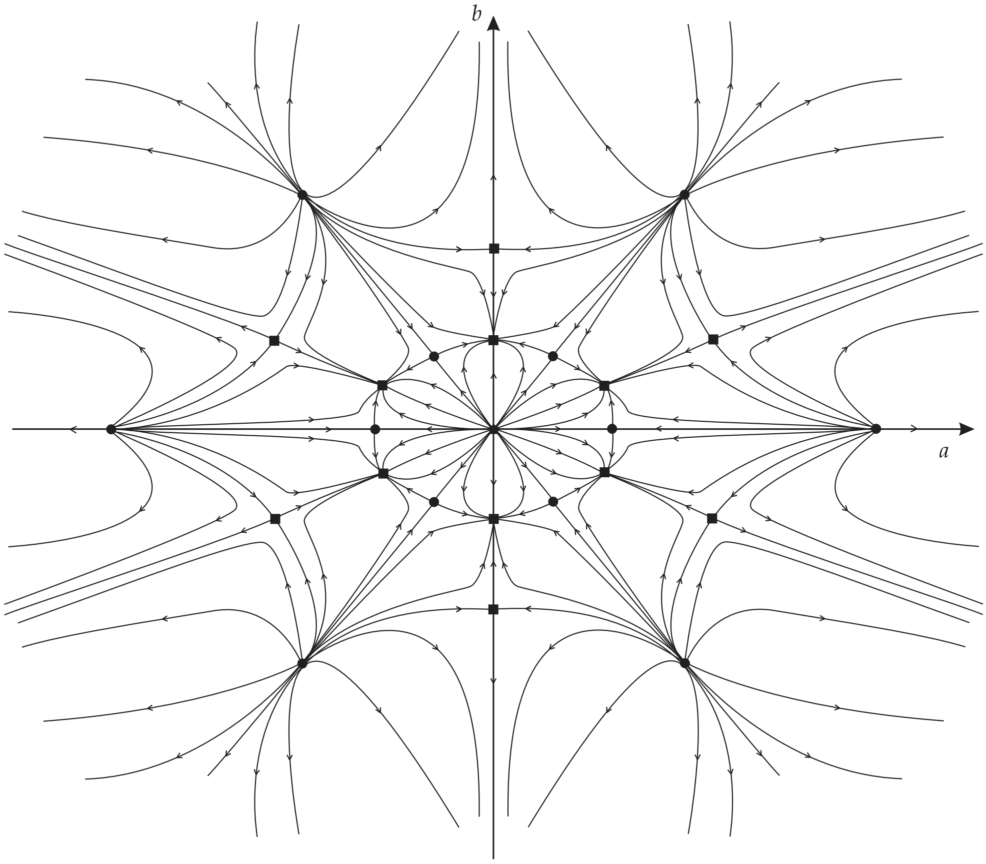


Figure 6.7: Phase portrait showing the solutions to (6.3.6)–(6.3.7) when $\lambda = 1$, $\alpha = -5$, $\beta = -2$ and $\gamma = 1$ and their stability. Solutions with $\mathbf{O(2)} \times \mathbb{Z}_2$ are represented by dots and solutions with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2}$ by squares. This phase portrait has hexagonal symmetry due to the residual $\mathbf{D}_6$ symmetry of the equations in $\mathrm{Fix}(\mathbf{D}_2 \times \mathbb{Z}_2^c)$.

6.3.3 Example: The representation on V_3

We now consider the stability of solutions with axial symmetry which occur in the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field for the representation on V_3. Recall that in the representation on V_3 the $\mathbf{O(3)}$ equivariant vector field contains no cubic terms so is already equivariant with respect to $\mathbf{O(3)} \times \mathbb{Z}_2$. Hence the results in this section are the same as those found by Chossat et al. [23]. We can compute using the method of Section 6.3.1 that to cubic order for the representation of $\mathbf{O(3)} \times \mathbb{Z}_2$ on V_3 the equivariant vector field f is given by

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(\mathbf{x}, \lambda) = \mu\mathbf{x} + \alpha\mathbf{x}|\mathbf{x}|^2 + \beta\mathbf{R}(\mathbf{x}) \tag{6.3.8}$$

where $\mu, \alpha, \beta \in \mathbb{R}$ are smooth functions of λ,

$$|\mathbf{x}|^2 = \sum_{m=-3}^{3} |x_m|^2 = x_0^2 - 2x_{-1}x_1 + 2x_{-2}x_2 - 2x_{-3}x_3$$

and $\mathbf{R}(\mathbf{x}) = (R_{-3},\ R_{-2},\ R_{-1},\ R_0,\ R_1,\ R_2,\ R_3)$ where $R_{-k} = (-1)^k\overline{R_k}$ and

$$\begin{aligned}
R_{-3}(\mathbf{x}) &= 15x_{-3}\left(4x_{-1}x_1 - 3x_0^2\right) - 10\sqrt{15}x_{-2}^2x_1 - 4\sqrt{15}x_{-1}^3 + 30\sqrt{2}x_{-2}x_{-1}x_0 \\
R_{-2}(\mathbf{x}) &= 5x_{-2}\left(3x_0^2 + 4x_{-1}x_1 - 10x_{-2}x_2\right) - 20\sqrt{15}x_{-3}x_{-1}x_2 - 30\sqrt{2}x_{-3}x_0x_1 \\
&\quad -2\sqrt{30}x_{-1}^2x_0 \\
R_{-1}(\mathbf{x}) &= x_{-1}\left(60x_{-3}x_3 - 20x_{-2}x_2 + 8x_{-1}x_1 - 9x_0^2\right) + 30\sqrt{2}x_{-3}x_0x_2 \\
&\quad -12\sqrt{15}x_1^2x_{-3} - 10\sqrt{15}x_{-2}^2x_3 + 4\sqrt{30}x_{-2}x_0x_1 \\
R_0(\mathbf{x}) &= 3x_0\left(30x_{-3}x_3 + 10x_{-2}x_2 + 6x_{-1}x_1 - 3x_0^2\right) - 2\sqrt{30}\left(x_{-2}x_1^2 + x_{-1}^2x_2\right) \\
&\quad -30\sqrt{2}\left(x_{-3}x_1x_2 + x_3x_{-2}x_{-1}\right).
\end{aligned}$$

As in Section 6.3.2, we will assume that $\mu = \lambda +$ higher order terms in λ so that a stationary bifurcation occurs at $\lambda = 0$ and the trivial solution is stable for $\lambda < 0$. By the equivariant branching lemma (Theorem 2.4.6), (6.3.8) has branches of stationary solutions with the symmetries of the axial isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in this representation.

Recall from Example 6.2.6 that in this representation there are three axial isotropy subgroups, $\widetilde{\mathbf{O(2)} \times \mathbb{Z}_2^c}$, $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ and $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$. The fixed-point subspaces of these axial isotropy subgroups are

$$\begin{aligned}
\mathrm{Fix}(\widetilde{\mathbf{O(2)} \times \mathbb{Z}_2^c}) &= \{(0,0,0,a,0,0,0)\} \\
\mathrm{Fix}(\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}) &= \{(0,a,0,0,0,a,0)\} \\
\mathrm{Fix}(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}) &= \{(a,0,0,0,0,0,-a)\}
\end{aligned}$$

where $a, b \in \mathbb{R}$. By restricting (6.3.8) to each of these subspaces we find that the branching equations for each of these isotropy subgroups are

$$\widetilde{\mathbf{O(2)} \times \mathbb{Z}_2^c} \quad : \quad 0 = \lambda + (\alpha - 9\beta)\, a^2 \tag{6.3.9}$$

$$\widetilde{\mathbb{O} \times \mathbb{Z}_2^c} \quad : \quad 0 = \lambda + (2\alpha - 50\beta)\, a^2 \tag{6.3.10}$$

$$\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c} \quad : \quad 0 = \lambda + 2\alpha a^2. \tag{6.3.11}$$

By (2.4.6) these three solution branches must have 2, 3 and 3 zero eigenvectors respectively. We compute that the branch of solutions with $\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c}$ symmetry has eigenvalues

$$-36\beta a^2 \ (2) \qquad 24\beta a^2 \ (2) \qquad 0 \ (2) \qquad -2\lambda = 2(\alpha - 9\beta)a^2 \ (1)$$

where the number in brackets indicates the multiplicity of each eigenvalue. This solution branch can never be stable.

The branch of solutions with $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ symmetry has eigenvalues

$$80\beta a^2 \ (3) \qquad 0 \ (3) \qquad -2\lambda = 2(2\alpha - 50\beta)a^2 \ (1)$$

and hence is stable when it bifurcates supercritically (when $2\alpha - 50\beta < 0$) and $\beta < 0$. Finally the branch of solutions with $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$ symmetry has eigenvalues

$$-60\beta a^2 \ (2) \qquad -90\beta a^2 \ (1) \qquad 0 \ (3) \qquad -2\lambda = 4\alpha a^2 \ (1)$$

and hence this solution branch is stable when $\alpha < 0$ and $\beta > 0$. The solution branches with symmetries $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ and $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$ cannot be simultaneously stable.

Notice that for this representation on V_3, the stability of each of the axial solution branches is determined by the cubic order truncation of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field in contrast to the representation on V_2 studied in Section 6.3.2 where the quintic order expansion was required to determine the stability of the axial solution branches.

In Section 6.4 we will compute the values of α and β in (6.3.8) for the specific example of the Swift–Hohenberg equation (6.1.1) in order to determine which of the axial solution branches are stable for this example.

6.3.4 Reducible representations

Consider the system of ODEs

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = f(\mathbf{z}, \lambda), \tag{6.3.12}$$

where $\lambda \in \mathbb{R}$ is a bifurcation parameter,

$$\mathbf{z} = (\mathbf{x}\,;\,\mathbf{y})^{\mathbf{T}} = \left(x_{-\ell}, x_{-(\ell-1)}, \ldots, x_\ell\,;\, y_{-(\ell+1)}, y_{-\ell}, \ldots, y_{(\ell+1)}\right)^{\mathbf{T}}$$

is the vector of amplitudes of the spherical harmonics $Y_\ell^m(\theta, \phi)$ and $Y_{\ell+1}^n(\theta, \phi)$ where

$$x_{-m} = (-1)^m \overline{x}_m \qquad \text{and} \qquad y_{-n} = (-1)^n \overline{y}_n$$

and f is a smooth mapping which is equivariant with respect to the natural representation of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ on $V_\ell \oplus V_{\ell+1}$. Hence f satisfies

$$f(M_\gamma^{(\ell,\ell+1)} \cdot \mathbf{z}, \lambda) = M_\gamma^{(\ell,\ell+1)} \cdot f(\mathbf{z}, \lambda)$$

for the matrices $M_\gamma^{(\ell,\ell+1)}$ which generate the action of $\mathbf{O}(\mathbf{3})$ on $V_\ell \oplus V_{\ell+1}$. The matrices $M_\gamma^{(\ell,\ell+1)}$ are given by

$$M_\gamma^{(\ell,\ell+1)} = \begin{bmatrix} M_\gamma^\ell & \mathbf{0}_{\ell,\ell+1} \\ \mathbf{0}_{\ell+1,\ell} & M_\gamma^{\ell+1} \end{bmatrix} \tag{6.3.13}$$

where M^{ℓ}_{γ} is the matrix for the action of γ on V_ℓ and $\mathbf{0}_{j,k}$ is the $(2j+1) \times (2k+1)$ zero matrix. The set of generating matrices M^{ℓ}_{γ} are given in Section 3.2.2.

We make the following observations about mappings which are equivariant with respect to the action of $\mathbf{O(3)} \times \mathbb{Z}_2$ on $V_\ell \oplus V_{\ell+1}$:

1. Equivariance with respect to the element $-1 \in \mathbb{Z}_2$ implies that
$$f(-\mathbf{z}, \lambda) = -f(\mathbf{z}, \lambda),$$
i.e. f is odd in $\mathbf{z}$ and hence contains only terms of odd order.

2. Suppose that
$$f(\mathbf{z}) = (g(\mathbf{z}); h(\mathbf{z}))^{\mathbf{T}} = \left(g_{-\ell},\, g_{-(\ell-1)},\, \ldots,\, g_\ell\,;\, h_{-(\ell+1)},\, h_{-\ell},\, \ldots,\, h_{(\ell+1)}\right)^{\mathbf{T}}. \quad (6.3.14)$$
Since the actions of $\mathbf{O(3)} \times \mathbb{Z}_2$ on V_ℓ and $V_{\ell+1}$ are absolutely irreducible, there are two linear equivariant maps $\mu_x(\lambda)(\mathbf{x}; 0)$ and $\mu_y(\lambda)(0; \mathbf{y})$. This means that the solution branches which are guaranteed to exist by the equivariant branching lemma with the symmetries of an axial isotropy subgroup for the irreducible representation on V_ℓ need not bifurcate at the same value of λ as those for the irreducible representation on $V_{\ell+1}$. Moreover, all of the linearly independent equivariant mappings comprising f are of the form $(P(\mathbf{z}); 0)$ or $(0; Q(\mathbf{z}))$.

3. Imposing that f is equivariant with respect to the action of $-I \in \mathbf{O(3)}$ on $V_\ell \oplus V_{\ell+1}$ we find that all cubic order terms in $g(\mathbf{z})$ are of the form
$$x_i x_j x_k \qquad \text{for } \; i, j, k \in -\ell, \ldots, \ell$$
or
$$y_i y_j x_k \qquad \text{for } \; i, j, \in -(\ell+1), \ldots, (\ell+1) \quad \text{and} \quad k \in -\ell, \ldots, \ell.$$
Similarly all cubic order terms in $h(\mathbf{z})$ are of the form
$$y_i y_j y_k \qquad \text{for } \; i, j, k \in -(\ell+1), \ldots, (\ell+1)$$
or
$$x_i x_j y_k \qquad \text{for } \; i, j, \in -\ell, \ldots, \ell \quad \text{and} \quad k \in -(\ell+1), \ldots, (\ell+1).$$

4. Imposing that f is also equivariant with respect to the action of the infinitesimal rotation $\phi' \in \mathbf{O(3)}$ on $V_\ell \oplus V_{\ell+1}$ we find that all cubic order terms in $g_m(\mathbf{z})$ are of the form
$$x_i x_j x_k \quad \text{where} \quad i+j+k = m \quad \text{for} \quad i, j, k \in -\ell, \ldots, \ell$$
or
$$y_i y_j x_k \quad \text{where} \quad i+j+k = m \quad \text{for} \quad i, j \in -(\ell+1), \ldots, (\ell+1) \quad \text{and} \quad k \in -\ell, \ldots, \ell.$$
Similarly all cubic order terms in $h_m(\mathbf{z})$ are of the form
$$y_i y_j y_k \quad \text{where} \quad i+j+k = m \quad \text{for} \quad i, j, k \in -(\ell+1), \ldots, (\ell+1)$$
or
$$x_i x_j y_k \quad \text{where} \quad i+j+k = m \quad \text{for} \quad i, j \in -\ell, \ldots, \ell \quad \text{and} \quad k \in -(\ell+1), \ldots, (\ell+1).$$

Given a small value of ℓ it is possible to compute the form of the equivariant vector field $f(\mathbf{z}, \lambda)$ to cubic order. We find that the cubic equivariant maps containing terms in $x_i x_j x_k$ are the cubic equivariants for the representation of $\mathbf{O(3)}$ on V_ℓ and the cubic equivariant maps containing terms in $y_i y_j y_k$ are the cubic equivariants for the representation of $\mathbf{O(3)}$ on $V_{\ell+1}$.

In order to know when we have found all cubic $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant maps it is possible to compute the number of such maps using character methods.

The number of cubic $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant maps in the representation on $V_\ell \oplus V_{\ell+1}$

In this section we will follow the method of Antoneli et al. [4] using characters to compute the number of cubic $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant maps in reducible representations on $V_\ell \oplus V_{\ell+1}$. This method is outlined in Section 2.3.2.

We can note that equivariance with respect to the action of the element $-1 \in \mathbb{Z}_2$ does not place any restrictions on cubic maps so we need only consider equivariance with respect to the rotations and inversion symmetries of $\mathbf{O(3)}$. Since all rotations through an angle θ in $\mathbf{SO(3)}$ are conjugate we have that in the representation on $V_\ell \oplus V_{\ell+1}$ the character of such a rotation R_θ is given by

$$\begin{aligned}
\chi(R_\theta) &= \sum_{m=-\ell}^{\ell} \mathrm{e}^{im\theta} + \sum_{m=-\ell-1}^{\ell+1} \mathrm{e}^{im\theta} \\
&= 2\sum_{m=-\ell}^{\ell} \mathrm{e}^{im\theta} + \mathrm{e}^{-\mathrm{i}\theta(\ell+1)} + \mathrm{e}^{\mathrm{i}\theta(\ell+1)} \\
&= 2\left(\frac{\cos(\ell\theta) - \cos((\ell+1)\theta)}{1-\cos(\theta)} + \cos((\ell+1)\theta)\right) \\
&= 2\left(\frac{\cos(\ell\theta) - \cos(\theta)\cos((\ell+1)\theta)}{1-\cos(\theta)}\right).
\end{aligned}$$

The Haar integral of a class function f on $\mathbf{SO(3)}$ is

$$\frac{1}{\pi}\int_0^\pi f(R_\theta)(1-\cos(\theta))\mathrm{d}\theta.$$

(See Wigner [84].) The conjugacy classes of elements of $\mathbf{O(3)}$ are also parameterised by θ, however there are two classes for each θ. One class is represented by the rotation R_θ and the other is represented by $-R_\theta$. In this case the Haar integral of a class function on $\mathbf{O(3)}$ is

$$\frac{1}{2\pi}\int_0^\pi \left[f(R_\theta) + f(-R_\theta)\right](1-\cos(\theta))\mathrm{d}\theta.$$

Thus we also need to compute $\chi(-R_\theta)$. Using the action of $-I$ on $V_\ell \oplus V_{\ell+1}$ we find that

$$\begin{aligned}
\chi(-R_\theta) &= (-1)^\ell \sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\theta} + (-1)^{\ell+1}\sum_{m=-\ell-1}^{\ell+1} \mathrm{e}^{\mathrm{i}m\theta} \\
&= (-1)^{\ell+1}\left(\mathrm{e}^{-\mathrm{i}\theta(\ell+1)} + \mathrm{e}^{\mathrm{i}\theta(\ell+1)}\right) \\
&= (-1)^{\ell+1}\, 2\cos((\ell+1)\theta)
\end{aligned}$$

Using (2.3.3) we see that the number of cubic $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariants for the representation on $V_\ell \oplus V_{\ell+1}$ is given by

$$\begin{aligned} E(3) &= \frac{1}{12\pi}\int_0^\pi (1-\cos(\theta))\Big[\chi(R_\theta)^4 + 3\chi(R_\theta)^2\chi(R_{2\theta}) + 2\chi(R_{3\theta})\chi(R_\theta) \\ &\qquad + \chi(-R_\theta)^4 + 3\chi(-R_\theta)^2\chi(R_{2\theta}) + 2\chi(-R_{3\theta})\chi(-R_\theta)\Big]\,\mathrm{d}\theta \\ &= \frac{1}{12\pi}\left(I_1 + I_2 + 3I_3 + 3I_4 + 2I_5 + 2I_6\right), \end{aligned}$$

where

$$I_1 = \int_0^\pi (1-\cos(\theta))\chi(R_\theta)^4 \mathrm{d}\theta = 2\pi(16\ell+7) \tag{6.3.15}$$

$$I_2 = \int_0^\pi (1-\cos(\theta))\chi(-R_\theta)^4 \mathrm{d}\theta = 6\pi \tag{6.3.16}$$

$$I_3 = \int_0^\pi (1-\cos(\theta))\chi(R_\theta)^2\chi(R_{2\theta}) \mathrm{d}\theta = 2\pi \tag{6.3.17}$$

$$I_4 = \int_0^\pi (1-\cos(\theta))\chi(-R_\theta)^2\chi(R_{2\theta}) \mathrm{d}\theta = 6\pi \tag{6.3.18}$$

$$I_5 = \int_0^\pi (1-\cos(\theta))\chi(R_\theta)\chi(R_{3\theta}) \mathrm{d}\theta = \begin{cases} 2\pi & \text{if } \ell = 0 \bmod 3 \\ -2\pi & \text{if } \ell = 1 \bmod 3 \\ 0 & \text{if } \ell = 2 \bmod 3. \end{cases} \tag{6.3.19}$$

$$I_6 = \int_0^\pi (1-\cos(\theta))\chi(-R_\theta)\chi(-R_{3\theta}) \mathrm{d}\theta = 0. \tag{6.3.20}$$

The details of the computations of these integrals can be found in Appendix B. Thus

$$\begin{aligned} E(3) &= \frac{1}{12\pi}\left(2\pi(16\ell+7) + 6\pi + 6\pi + 18\pi + \begin{cases} 4\pi & \text{if } \ell = 0 \bmod 3 \\ -4\pi & \text{if } \ell = 1 \bmod 3 \\ 0 & \text{if } \ell = 2 \bmod 3 \end{cases}\right) \\ &= \begin{cases} (8\ell+12)/3 & \text{if } \ell = 0 \bmod 3 \\ (8\ell+10)/3 & \text{if } \ell = 1 \bmod 3 \\ (8\ell+11)/3 & \text{if } \ell = 2 \bmod 3. \end{cases} \end{aligned} \tag{6.3.21}$$

6.3.5 Example: The representation on $V_2 \oplus V_3$

We now compute to cubic order the general form of a mapping which is equivariant with respect to the action of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ on $V_2 \oplus V_3$. This vector field will be required in Section 7.2.1. Using (6.3.21) we see that in the Taylor expansion of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field for the representation on $V_2 \oplus V_3$ there are 9 cubic equivariant maps. Using equivariance with respect to the matrices $M_\gamma^{(2,3)}$ defined by (6.3.13) for $\gamma = \phi', \theta', -I$ and -1 we find that

$$f(\mathbf{z}, \lambda) = (g(\mathbf{z}, \lambda); h(\mathbf{z}, \lambda))$$

where

$$g(\mathbf{z}, \lambda) = \mu_x \mathbf{x} + \alpha_1 \mathbf{x}|\mathbf{x}|^2 + \beta_1 \mathbf{x}|\mathbf{y}|^2 + \gamma_1 \mathbf{P}(\mathbf{x}, \mathbf{y}) + \gamma_2 \mathbf{Q}(\mathbf{x}, \mathbf{y}) \tag{6.3.22}$$

$$h(\mathbf{z}, \lambda) = \mu_y \mathbf{y} + \alpha_2 \mathbf{y}|\mathbf{x}|^2 + \beta_2 \mathbf{y}|\mathbf{y}|^2 + \delta_1 \mathbf{R}(\mathbf{y}) + \delta_2 \mathbf{S}(\mathbf{x}, \mathbf{y}) + \delta_3 \mathbf{T}(\mathbf{x}, \mathbf{y}). \tag{6.3.23}$$

Here $\mu_1, \mu_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ and $\delta_3 \in \mathbb{R}$ are smooth functions of λ,

$$\begin{aligned}
|\mathbf{x}|^2 &= |x_{-2}|^2 + |x_{-1}|^2 + |x_0|^2 + |x_1|^2 + |x_2|^2 \\
&= x_0^2 - 2x_1x_{-1} + 2x_2x_{-2} \\
|\mathbf{y}|^2 &= |y_{-3}|^2 + |y_{-2}|^2 + |y_{-1}|^2 + |y_0|^2 + |y_1|^2 + |y_2|^2 + |y_3|^2 \\
&= y_0^2 - 2y_1y_{-1} + 2y_2y_{-2} - 2y_3y_{-3}
\end{aligned}$$

and

- $\mathbf{P}(\mathbf{x},\mathbf{y}) = (P_{-2},\ P_{-1},\ P_0,\ P_1,\ P_2)$ where $P_{-k} = (-1)^k\overline{P}_k$ and

$$\begin{aligned}
P_{-2}(\mathbf{x},\mathbf{y}) &= 18y_{-3}y_3x_{-2} - 2y_{-2}y_2x_{-2} - 10y_{-2}^2x_2 + 4\sqrt{15}y_{-3}y_{-1}x_2 - 5\sqrt{6}y_{-3}y_2x_{-1} \\
&\quad + \sqrt{10}\,(y_{-2}y_1x_{-1} + 4y_{-3}y_1x_0 + 2y_{-2}y_{-1}x_1) - 2\sqrt{5}y_0\,(y_{-2}x_0 + 3y_{-3}x_1) \\
P_{-1}(\mathbf{x},\mathbf{y}) &= 10y_{-1}^2x_1 + 5y_{-3}y_2x_0 - 3y_{-3}y_3x_{-1} - 18y_{-2}y_2x_{-1} + 5y_{-1}y_1x_{-1} \\
&\quad -2\sqrt{30}y_{-2}y_0x_1 - 5\sqrt{2}y_{-1}y_0x_0 + 6\sqrt{5}y_{-3}y_0x_2 + 5\sqrt{6}y_3y_{-2}x_{-2} \\
&\quad -\sqrt{10}y_{-1}\,(y_2x_{-2} + 2y_{-2}x_2) + \sqrt{15}y_1\,(3y_{-2}x_0 - 2y_{-3}x_1) \\
P_0(\mathbf{x},\mathbf{y}) &= 4\sqrt{10}\,(y_{-1}y_3x_{-2} + y_{-3}y_1x_2) - 3\sqrt{15}\,(y_{-2}y_1x_1 + y_{-1}y_2x_{-1}) \\
&\quad -2\sqrt{5}y_0\,(y_{-2}x_2 + y_2x_{-2}) + 5\sqrt{2}y_0\,(y_1x_{-1} + y_{-1}x_1) \\
&\quad -5\,(y_{-3}y_2x_1 + y_{-2}y_3x_{-1}) + 2x_0\left(y_{-3}y_3 + 4y_{-2}y_2 + 5y_{-1}y_1 - 5y_0^2\right)
\end{aligned}$$

- $\mathbf{Q}(\mathbf{x},\mathbf{y}) = (Q_{-2},\ Q_{-1},\ Q_0,\ Q_1,\ Q_2)$ where $Q_{-k} = (-1)^k\overline{Q}_k$ and

$$\begin{aligned}
Q_{-2}(\mathbf{x},\mathbf{y}) &= x_{-2}\left(y_0^2 - 6y_{-2}y_2 + 16y_{-3}y_3\right) + 3\sqrt{10}y_{-2}y_1x_{-1} + 2\sqrt{10}y_{-3}y_1x_0 \\
&\quad -4\sqrt{5}y_{-2}y_0x_0 - 2\sqrt{3}y_{-1}y_0x_{-1} + 2\sqrt{6}y_{-1}^2x_0 - 5\sqrt{6}y_{-3}y_2x_{-1} \\
Q_{-1}(\mathbf{x},\mathbf{y}) &= x_{-1}\left(y_{-3}y_3 - 6y_{-2}y_2 + 9y_{-1}y_1 - 5y_0^2\right) + 6y_{-1}^2 - 5y_{-3}y_2x_0 \\
&\quad -3\sqrt{10}y_{-1}y_2x_{-2} + 2\sqrt{3}y_1y_0x_{-2} - 2\sqrt{30}y_{-2}y_0x_1 - \sqrt{2}y_0y_{-1}x_0 \\
&\quad + 5\sqrt{6}y_3y_{-2}x_{-2} + \sqrt{15}y_1\,(y_{-2}x_0 + 2y_{-3}x_1) \\
Q_0(\mathbf{x},\mathbf{y}) &= x_0\left(-7y_0^2 + 12y_{-1}y_1 - 6y_{-2}y_2 - 4y_{-3}y_3\right) + 5y_{-3}y_2x_1 + 5y_{-2}y_3x_{-1} \\
&\quad -\sqrt{15}\,(y_1y_{-2}x_1 + y_{-1}y_2x_{-1}) + 2\sqrt{10}\,(y_{-3}y_1x_2 + y{-}1y_3x_{-2}) \\
&\quad - 4\sqrt{5}y_0\,(y_{-2}x_2 + y_2x_{-2}) + \sqrt{2}y_0\,(y_1x_{-1} + y_{-1}x_1) \\
&\quad + 2\sqrt{6}\left(y_1^2x_{-2} + y_{-1}^2x_2\right)
\end{aligned}$$

- $\mathbf{R}(\mathbf{y}) = (R_{-3},\ R_{-2},\ R_{-1},\ R_0,\ R_1,\ R_2,\ R_3)$ where $R_{-k} = (-1)^k\overline{R_k}$ and

$$\begin{aligned}
R_{-3}(\mathbf{y}) &= 15y_{-3}\left(4y_{-1}y_1 - 3y_0^2\right) - 10\sqrt{15}y_{-2}^2y_1 - 4\sqrt{15}y_{-1}^3 + 30\sqrt{2}y_{-2}y_{-1}y_0 \\
R_{-2}(\mathbf{y}) &= 5y_{-2}\left(3y_0^2 + 4y_{-1}y_1 - 10y_{-2}y_2\right) - 20\sqrt{15}y_{-3}y_{-1}y_2 - 30\sqrt{2}y_{-3}y_0y_1 \\
&\quad -2\sqrt{30}y_{-1}^2y_0 \\
R_{-1}(\mathbf{y}) &= y_{-1}\left(60y_{-3}y_3 - 20y_{-2}y_2 + 8y_{-1}y_1 - 9y_0^2\right) + 30\sqrt{2}y_{-3}y_0y_2 \\
&\quad -12\sqrt{15}y_1^2y_{-3} - 10\sqrt{15}y_{-2}^2y_3 + 4\sqrt{30}y_{-2}y_0y_1 \\
R_0(\mathbf{y}) &= 3y_0\left(30y_{-3}y_3 + 10y_{-2}y_2 + 6y_{-1}y_1 - 3y_0^2\right) - 2\sqrt{30}\left(y_{-2}y_1^2 + y_{-1}^2y_2\right) \\
&\quad -30\sqrt{2}\,(y_{-3}y_1y_2 + y_3y_{-2}y_{-1})
\end{aligned}$$

- $\mathbf{S}(\mathbf{x},\mathbf{y}) = (S_{-3},\ S_{-2},\ S_{-1},\ S_0,\ S_1,\ S_2,\ S_3)$ where $S_{-k} = (-1)^k\overline{S_k}$ and

$$\begin{aligned}
S_{-3}(\mathbf{x},\mathbf{y}) &= 15y_{-3}\left(2x_{-1}x_1 - x_0^2\right) - 10\sqrt{6}x_{-2}x_1y_{-2} + 2\sqrt{15}x_{-2}^2y_1 - 6\sqrt{5}x_{-2}x_{-1}y_0 \\
&\quad + 6\sqrt{10}x_0x_{-2}y_{-1} \\
S_{-2}(\mathbf{x},\mathbf{y}) &= -5y_{-2}\left(3x_0^2 - 2x_{-1}x_1 + 4x_{-2}x_2\right) - 2\sqrt{10}x_{-2}\left(2x_1y_{-1} + x_{-1}y_1\right) \\
&\quad + 4\sqrt{15}x_{-1}x_0y_{-1} - 2\sqrt{30}x_{-1}^2y_0 + 10x_{-2}^2y_2 + 10\sqrt{6}x_{-1}x_2y_{-3} \\
&\quad + 6\sqrt{5}x_{-2}x_0y_0 \\
S_{-1}(\mathbf{x},\mathbf{y}) &= -y_{-1}\left(3x_0^2 - 14x_{-1}x_1 + 28x_{-2}x_2\right) + 2\sqrt{10}x_{-1}\left(2x_2y_{-2} + x_{-2}y_2\right) \\
&\quad + 6\sqrt{10}x_0x_2y_{-3} + 2\sqrt{15}x_{-2}^2y_3 - 4\sqrt{15}x_0x_1y_{-2} - 2\sqrt{3}x_{-2}x_1y_0 \\
&\quad + 4\sqrt{6}x_{-2}x_0y_1 - 16x_{-1}^2y_1 + 6\sqrt{2}x_0x_{-1}y_0 \\
S_0(\mathbf{x},\mathbf{y}) &= 3y_0\left(x_0^2 + 6x_{-1}x_1 - 10x_{-2}x_2\right) + 2\sqrt{3}\left(x_{-2}x_1y_1 + x_2x_{-1}y_{-1}\right) \\
&\quad -6\sqrt{2}x_0\left(x_1y_{-1} + x_{-1}y_1\right) - 2\sqrt{30}\left(x_{-1}^2y_2 + x_1^2y_{-2}\right) \\
&\quad + 6\sqrt{5}x_0\left(x_2y_{-2} + x_{-2}y_2\right) + 6\sqrt{5}\left(x_1x_2y_{-3} + x_{-1}x_{-2}y_3\right)
\end{aligned}$$

- $\mathbf{T}(\mathbf{x},\mathbf{y}) = (T_{-3},\ T_{-2},\ T_{-1},\ T_0,\ T_1,\ T_2,\ T_3)$ where $T_{-k} = (-1)^k\overline{T_k}$ and

$$\begin{aligned}
T_{-3}(\mathbf{x},\mathbf{y}) &= 5y_{-3}\left(2x_0^2 - 3x_{-1}x_1\right) + 5\sqrt{6}x_{-2}x_1y_{-2} - 5x_{-1}x_0y_{-2} - 2\sqrt{10}x_0x_{-2}y_{-1} \\
&\quad +\sqrt{15}x_{-1}^2y_{-1} \\
T_{-2}(\mathbf{x},\mathbf{y}) &= 5y_{-2}\left(x_0^2 - 2x_{-1}x_1 + 2x_{-2}x_2\right) - 5\sqrt{6}x_{-1}x_2y_{-3} + 5x_0x_1y_{-3} \\
&\quad -\sqrt{15}x_1x_{-2}y_{-1} + \sqrt{30}x_{-1}^2y_0 - 4\sqrt{5}x_{-2}x_0y_0 + 3\sqrt{10}x_{-2}x_1y_{-1} \\
T_{-1}(\mathbf{x},\mathbf{y}) &= y_{-1}\left(2x_0^2 - 7x_{-1}x_1 + 16x_{-2}x_2\right) - 2\sqrt{10}x_0x_2y_{-3} + \sqrt{15}x_1^2y_{-3} \\
&\quad + 6x_{-1}^2y_1 + 2\sqrt{3}x_{-2}x_1y_0 - 4\sqrt{6}x_{-2}x_0y_1 + \sqrt{15}x_0x_1y_{-2} \\
&\quad -\sqrt{2}x_0x_{-1}y_0 - 3\sqrt{10}x_{-1}x_2y_{-2} \\
T_0(\mathbf{x},\mathbf{y}) &= y_0\left(x_0^2 - 6x_1x_{-1} + 18x_2x_{-2}\right) - 4\sqrt{5}x_0\left(x_2y_{-2} + x_{-2}y_2\right) \\
&\quad +\sqrt{30}\left(x_1^2y_{-2} + x_{-1}^2y_2\right) - 2\sqrt{3}\left(x_{-2}x_1y_1 + x_{-1}x_2y_{-1}\right) \\
&\quad +\sqrt{2}x_0\left(x_1y_{-1} + x_{-1}y_1\right)
\end{aligned}$$

Remark 6.3.1. The mapping $\mathbf{x}|\mathbf{x}|^2$ is the cubic equivariant for the representation of $\mathbf{O(3)}$ on V_2 and $\mathbf{y}|\mathbf{y}|^2$ and $\mathbf{R}(\mathbf{y})$ are the cubic equivariants for the representation of $\mathbf{O(3)}$ on V_3 as found in [23] and Section 6.3.3.

We could now use this equivariant vector field to determine the direction of branching and the stability of the axial solution branches i.e. those with symmetries as in Table 6.8, for given values of the coefficients $\mu_1,\ \mu_2,\ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \gamma_1,\ \gamma_2,\ \delta_1,\ \delta_2$ and $\delta_3 \in \mathbb{R}$. We will use this equivariant vector field in Chapter 7 when studying spiral patterns on spheres.

6.3.6 Example: The representation on $V_3 \oplus V_4$

We can also compute to cubic order the general form of a mapping which is equivariant with respect to the action of $\mathbf{O(3)} \times \mathbb{Z}_2$ on $V_3 \oplus V_4$. This vector field will be required in Section

7.2.2. Using (6.3.21) we see that in the Taylor expansion of the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field for the representation on $V_3 \oplus V_4$ there are 12 cubic equivariant maps. Using equivariance with respect to the matrices $M_\gamma^{(3,4)}$ defined by (6.3.13) for $\gamma = \phi'$, θ', $-I$ and -1 we find that $f(\mathbf{z}, \lambda) = (g(\mathbf{z}, \lambda); h(\mathbf{z}, \lambda))$ where

$$g(\mathbf{z}, \lambda) = \mu_x \mathbf{x} + \alpha_1 \mathbf{x}|\mathbf{x}|^2 + \beta_1 \mathbf{x}|\mathbf{y}|^2 + \gamma_1 \mathbf{P}(\mathbf{x}) + \gamma_2 \mathbf{Q}(\mathbf{x}, \mathbf{y}) + \gamma_3 \mathbf{R}(\mathbf{x}, \mathbf{y}) + \gamma_4 \mathbf{S}(\mathbf{x}, \mathbf{y}) \quad (6.3.24)$$

$$h(\mathbf{z}, \lambda) = \mu_y \mathbf{y} + \alpha_2 \mathbf{y}|\mathbf{x}|^2 + \beta_2 \mathbf{y}|\mathbf{y}|^2 + \delta_1 \mathbf{T}(\mathbf{y}) + \delta_2 \mathbf{U}(\mathbf{x}, \mathbf{y}) + \delta_3 \mathbf{V}(\mathbf{x}, \mathbf{y}) + \delta_4 \mathbf{W}(\mathbf{x}, \mathbf{y}) \quad (6.3.25)$$

Here μ_1, μ_2, α_1, α_2, β_1, β_2, γ_1, γ_2, γ_3, γ_4 δ_1, δ_2, δ_3 and $\delta_4 \in \mathbb{R}$ are smooth functions of λ.

Instead of writing explicitly the cubic equivariant maps $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$, Tables 6.10–6.18 give the coefficients of each term in each component of the equivariant vector field. For example, Table 6.10 lists all of the terms which occur in the component g_{-3} of f and their coefficients in terms of α_1, β_1, γ_1, γ_2, γ_3 and γ_4. Using Table 6.10 we can see that the coefficient of $x_1 y_{-3} y_{-1}$ in g_{-3} is $-20\sqrt{105}\gamma_2 + 3\sqrt{105}\gamma_3$ which means that the term $x_1 y_{-3} y_{-1}$ does not occur in $\mathbf{x}|\mathbf{x}|^2$, $\mathbf{x}|\mathbf{y}|^2$, $\mathbf{P}(\mathbf{x})$ or $\mathbf{S}(\mathbf{x}, \mathbf{y})$ and has coefficient $-20\sqrt{105}$ in $\mathbf{Q}(\mathbf{x}, \mathbf{y})$ and $3\sqrt{105}$ in $\mathbf{R}(\mathbf{x}, \mathbf{y})$.

Since $x_{-m} = (-1)^m \overline{x}_m$ and $y_{-m} = (-1)^m \overline{y}_m$ we must have $g_{-m} = (-1)^m \overline{g}_m$ and $h_{-m} = (-1)^m \overline{h}_m$ so we only give the form of the components g_{-3}, g_{-2}, g_{-1}, g_0, h_{-4}, h_{-3}, h_{-2}, h_{-1} and h_0.

Remark 6.3.2. The mappings $\mathbf{x}|\mathbf{x}|^2$ and $\mathbf{P}(\mathbf{x})$ are the cubic equivariants for the representation of $\mathbf{O(3)}$ on V_3 as found in [23] and Section 6.3.3 and $\mathbf{y}|\mathbf{y}|^2$ and $\mathbf{T}(\mathbf{y})$ are the cubic equivariants for the representation of $\mathbf{O(3)}$ on V_4 as found in [18].

Term	α_1	β_1	γ_1	γ_2	γ_3	γ_4
$x_{-3}^2x_3$	-2	0	0	0	0	0
$x_{-3}x_{-2}x_2$	2	0	0	0	0	0
$x_{-3}x_{-1}x_1$	-2	0	1	0	0	0
$x_{-3}x_0^2$	1	0	-45	0	0	0
$x_{-2}^2x_1$	0	0	$-10\sqrt{15}$	0	0	0
$x_{-2}x_{-1}x_0$	0	0	$30\sqrt{2}$	0	0	0
x_{-1}^3	0	0	$-4\sqrt{15}$	0	0	0
$x_{-3}y_{-4}y_4$	0	2	0	0	0	0
$x_{-3}y_{-3}y_3$	0	-2	0	315	-42	105
$x_{-3}y_{-2}y_3$	0	2	0	-390	54	-180
$x_{-3}y_{-1}y_1$	0	-2	0	375	-51	225
$x_{-3}y_0^2$	0	1	0	-180	24	-120
$x_{-2}y_{-4}y_3$	0	0	0	$-210\sqrt{3}$	$28\sqrt{3}$	$-70\sqrt{3}$
$x_{-2}y_{-3}y_2$	0	0	0	$25\sqrt{21}$	$-4\sqrt{21}$	$25\sqrt{21}$
$x_{-2}y_{-2}y_1$	0	0	0	$15\sqrt{3}$	$-3\sqrt{3}$	$-45\sqrt{3}$
$x_{-2}y_{-1}y_0$	0	0	0	$-5\sqrt{30}$	$\sqrt{30}$	$5\sqrt{30}$
$x_{-1}y_{-4}y_2$	0	0	0	$32\sqrt{105}$	$-4\sqrt{105}$	$4\sqrt{105}$
$x_{-1}y_{-3}y_1$	0	0	0	$-18\sqrt{105}$	$3\sqrt{105}$	$6\sqrt{105}$
$x_{-1}y_{-2}y_0$	0	0	0	0	0	$30\sqrt{6}$
$x_{-1}y_{-1}^2$	0	0	0	$10\sqrt{15}$	$-2\sqrt{15}$	$10\sqrt{15}$
$x_0y_{-4}y_1$	0	0	0	$-30\sqrt{70}$	$3\sqrt{70}$	0
$x_0y_{-3}y_0$	0	0	0	$90\sqrt{7}$	$-15\sqrt{7}$	0
$x_0y_{-2}y_{-1}$	0	0	0	$-30\sqrt{10}$	$6\sqrt{10}$	0
$x_1y_{-4}y_0$	0	0	0	$20\sqrt{42}$	0	0
$x_1y_{-3}y_{-1}$	0	0	0	$-20\sqrt{105}$	$3\sqrt{105}$	0
$x_1y_{-2}^2$	0	0	0	$30\sqrt{15}$	$-6\sqrt{15}$	0
$x_2y_{-4}y_{-1}$	0	0	0	0	$-6\sqrt{21}$	0
$x_2y_{-3}y_{-2}$	0	0	0	0	$3\sqrt{21}$	0
$x_3y_{-4}y_{-2}$	0	0	0	0	$12\sqrt{7}$	0
$x_3y_{-3}^2$	0	0	0	0	-21	0

Table 6.10: The form of the component g_{-3} of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant mapping f in the representation on $V_3 \oplus V_4$. For example, the coefficient of $x_1y_{-3}y_{-1}$ in g_{-3} is $-20\sqrt{105}\gamma_2 + 3\sqrt{105}\gamma_3$.

Term	α_1	β_1	γ_1	γ_2	γ_3	γ_4
$x_{-3}x_{-2}x_3$	-2	0	0	0	0	0
$x_{-3}x_{-1}x_2$	0	0	$20\sqrt{15}$	0	0	0
$x_{-3}x_0x_1$	0	0	$-30\sqrt{2}$	0	0	0
$x_{-2}^2x_2$	2	0	-50	0	0	0
$x_{-2}x_{-1}x_1$	-2	0	20	0	0	0
$x_{-2}x_0^2$	1	0	15	0	0	0
$x_{-1}^2x_0$	0	0	$-2\sqrt{30}$	0	0	0
$x_{-3}y_{-3}y_4$	0	0	0	$210\sqrt{3}$	$-28\sqrt{3}$	$70\sqrt{3}$
$x_{-3}y_{-2}y_3$	0	0	0	$-25\sqrt{21}$	$4\sqrt{21}$	$-25\sqrt{21}$
$x_{-3}y_{-1}y_2$	0	0	0	$-15\sqrt{3}$	$3\sqrt{3}$	$45\sqrt{3}$
$x_{-3}y_0y_1$	0	0	0	$5\sqrt{30}$	$-\sqrt{30}$	$-5\sqrt{30}$
$x_{-2}y_{-4}y_4$	0	2	0	-420	56	-140
$x_{-2}y_{-3}y_3$	0	-2	0	70	-14	140
$x_{-2}y_{-2}y_2$	0	2	0	-170	17	-140
$x_{-2}y_{-1}y_1$	0	-2	0	370	-50	140
$x_{-2}y_0^2$	0	1	0	-230	34	-70
$x_{-1}y_{-4}y_3$	0	0	0	$14\sqrt{5}$	0	$-42\sqrt{5}$
$x_{-1}y_{-3}y_2$	0	0	0	$35\sqrt{35}$	$-3\sqrt{35}$	$15\sqrt{35}$
$x_{-1}y_{-2}y_1$	0	0	0	$-111\sqrt{5}$	$18\sqrt{5}$	$-27\sqrt{5}$
$x_{-1}y_{-1}y_0$	0	0	0	$75\sqrt{2}$	$-15\sqrt{2}$	$10\sqrt{2}$
$x_0y_{-4}y_2$	0	0	0	$2\sqrt{210}$	$-\sqrt{210}$	$4\sqrt{210}$
$x_0y_{-3}y_1$	0	0	0	$8\sqrt{210}$	0	$-6\sqrt{210}$
$x_0y_{-2}y_0$	0	0	0	$110\sqrt{3}$	$-15\sqrt{3}$	$60\sqrt{3}$
$x_0y_{-1}^2$	0	0	0	$-20\sqrt{30}$	$4\sqrt{30}$	$-10\sqrt{30}$
$x_1y_{-4}y_1$	0	0	0	$-20\sqrt{35}$	$6\sqrt{35}$	0
$x_1y_{-3}y_0$	0	0	0	$30\sqrt{14}$	0	0
$x_1y_{-2}y_{-1}$	0	0	0	$-20\sqrt{5}$	$-3\sqrt{5}$	0
$x_2y_{-4}y_0$	0	0	0	$20\sqrt{70}$	$-6\sqrt{70}$	0
$x_2y_{-3}y_{-1}$	0	0	0	$-100\sqrt{7}$	$12\sqrt{7}$	0
$x_2y_{-2}^2$	0	0	0	150	-9	0
$x_3y_{-4}y_{-1}$	0	0	0	0	$6\sqrt{21}$	0
$x_3y_{-3}y_{-2}$	0	0	0	0	$-3\sqrt{21}$	0

Table 6.11: The form of the component g_{-2} of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant mapping f in the representation on $V_3 \oplus V_4$. For example, the coefficient of $x_1y_{-4}y_1$ in g_{-2} is $-20\sqrt{35}\gamma_2 + 6\sqrt{35}\gamma_3$.

Term	α_1	β_1	γ_1	γ_2	γ_3	γ_4
$x_{-3}x_{-1}x_3$	-2	0	60	0	0	0
$x_{-3}x_0x_2$	0	0	$30\sqrt{2}$	0	0	0
$x_{-3}x_1^2$	0	0	$-12\sqrt{15}$	0	0	0
$x_{-2}^2x_3$	0	0	$-10\sqrt{15}$	0	0	0
$x_{-2}x_{-1}x_2$	2	0	-20	0	0	0
$x_{-2}x_0x_1$	0	0	$4\sqrt{30}$	0	0	0
$x_{-1}^2x_1$	-2	0	8	0	0	0
$x_{-1}x_0^2$	1	0	-9	0	0	0
$x_{-3}y_{-2}y_4$	0	0	0	$32\sqrt{105}$	$-4\sqrt{105}$	$4\sqrt{105}$
$x_{-3}y_{-1}y_3$	0	0	0	$-18\sqrt{105}$	$3\sqrt{105}$	$-6\sqrt{105}$
$x_{-3}y_0y_2$	0	0	0	0	0	$30\sqrt{6}$
$x_{-3}y_1^2$	0	0	0	$10\sqrt{15}$	$-2\sqrt{15}$	$-10\sqrt{15}$
$x_{-2}y_{-3}y_4$	0	0	0	$-14\sqrt{5}$	0	$42\sqrt{5}$
$x_{-2}y_{-2}y_3$	0	0	0	$-35\sqrt{35}$	$3\sqrt{35}$	$-15\sqrt{35}$
$x_{-2}y_{-1}y_2$	0	0	0	$111\sqrt{5}$	$-18\sqrt{5}$	$27\sqrt{5}$
$x_{-2}y_0y_1$	0	0	0	$-75\sqrt{2}$	$15\sqrt{2}$	$-15\sqrt{2}$
$x_{-1}y_{-4}y_4$	0	2	0	-392	56	-224
$x_{-1}y_{-3}y_3$	0	-2	0	343	-35	161
$x_{-1}y_{-2}y_2$	0	2	0	-258	50	-116
$x_{-1}y_{-1}y_1$	0	-2	0	187	-26	89
$x_{-1}y_0^2$	0	1	0	-80	4	-40
$x_0y_{-4}y_3$	0	0	0	$28\sqrt{6}$	$-7\sqrt{6}$	$-14\sqrt{6}$
$x_0y_{-3}y_2$	0	0	0	$15\sqrt{42}$	$-5\sqrt{42}$	$5\sqrt{42}$
$x_0y_{-2}y_1$	0	0	0	$-57\sqrt{6}$	$3\sqrt{6}$	$-9\sqrt{6}$
$x_0y_{-1}y_0$	0	0	0	$16\sqrt{15}$	$\sqrt{15}$	$2\sqrt{15}$
$x_1y_{-4}y_2$	0	0	0	$-48\sqrt{7}$	$4\sqrt{7}$	$24\sqrt{7}$
$x_1y_{-3}y_1$	0	0	0	$-28\sqrt{7}$	$4\sqrt{7}$	$-36\sqrt{7}$
$x_1y_{-2}y_0$	0	0	0	$88\sqrt{10}$	$-12\sqrt{10}$	$36\sqrt{10}$
$x_1y_{-1}^2$	0	0	0	-180	15	-60
$x_2y_{-4}y_1$	0	0	0	$20\sqrt{35}$	$-6\sqrt{35}$	0
$x_2y_{-3}y_0$	0	0	0	$-30\sqrt{14}$	0	0
$x_2y_{-2}y_{-1}$	0	0	0	$20\sqrt{5}$	$3\sqrt{5}$	0
$x_3y_{-4}y_0$	0	0	0	$20\sqrt{42}$	0	0
$x_3y_{-3}y_{-1}$	0	0	0	$-20\sqrt{105}$	$3\sqrt{105}$	0
$x_3y_{-2}^2$	0	0	0	$30\sqrt{15}$	$-6\sqrt{15}$	0

Table 6.12: The form of the component g_{-1} of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant mapping f in the representation on $V_3 \oplus V_4$. For example, the coefficient of $x_1y_{-4}y_2$ in g_{-1} is $-48\sqrt{7}\gamma_2 + 4\sqrt{7}\gamma_3 + 24\sqrt{7}\gamma_4$.

Term	α_1	β_1	γ_1	γ_2	γ_3	γ_4
$x_{-3}x_0x_3$	-2	0	90	0	0	0
$x_{-3}x_1x_2$	0	0	$-30\sqrt{2}$	0	0	0
$x_{-2}x_{-1}x_3$	0	0	$-30\sqrt{2}$	0	0	0
$x_{-2}x_0x_2$	2	0	30	0	0	0
$x_{-2}x_1^2$	0	0	$-2\sqrt{30}$	0	0	0
$x_{-1}x_0x_1$	-2	0	18	0	0	0
$x_{-1}^2x_2$	0	0	$-2\sqrt{30}$	0	0	0
x_0^3	1	0	-9	0	0	0
$x_{-3}y_{-1}y_4$	0	0	0	$30\sqrt{70}$	$-3\sqrt{70}$	0
$x_{-3}y_0y_3$	0	0	0	$-90\sqrt{7}$	$15\sqrt{7}$	0
$x_{-3}y_1y_2$	0	0	0	$30\sqrt{10}$	$-6\sqrt{10}$	0
$x_{-2}y_{-2}y_4$	0	0	0	$2\sqrt{210}$	$-\sqrt{210}$	$4\sqrt{210}$
$x_{-2}y_{-1}y_3$	0	0	0	$-8\sqrt{210}$	0	$-6\sqrt{210}$
$x_{-2}y_0y_2$	0	0	0	$110\sqrt{3}$	$-15\sqrt{3}$	$60\sqrt{3}$
$x_{-2}y_1^2$	0	0	0	$-20\sqrt{30}$	$4\sqrt{30}$	$-10\sqrt{30}$
$x_{-1}y_{-3}y_4$	0	0	0	$-28\sqrt{6}$	$7\sqrt{6}$	$14\sqrt{6}$
$x_{-1}y_{-2}y_3$	0	0	0	$-15\sqrt{42}$	$5\sqrt{42}$	$-5\sqrt{42}$
$x_{-1}y_{-1}y_2$	0	0	0	$57\sqrt{6}$	$-3\sqrt{6}$	$9\sqrt{6}$
$x_{-1}y_0y_1$	0	0	0	$-16\sqrt{15}$	$-\sqrt{15}$	$-2\sqrt{15}$
$x_0y_{-4}y_4$	0	2	0	-336	42	-252
$x_0y_{-3}y_3$	0	-2	0	504	-84	168
$x_0y_{-2}y_2$	0	2	0	-324	24	-108
$x_0y_{-1}y_1$	0	-2	0	96	-12	72
$x_0y_0^2$	0	1	0	0	9	-30
$x_1y_{-4}y_3$	0	0	0	$-28\sqrt{6}$	$7\sqrt{6}$	$14\sqrt{6}$
$x_1y_{-3}y_2$	0	0	0	$-15\sqrt{42}$	$5\sqrt{42}$	$-5\sqrt{42}$
$x_1y_{-2}y_1$	0	0	0	$57\sqrt{6}$	$-3\sqrt{6}$	$9\sqrt{6}$
$x_1y_{-1}y_0$	0	0	0	$-16\sqrt{15}$	$-\sqrt{15}$	$-2\sqrt{15}$
$x_2y_{-4}y_2$	0	0	0	$2\sqrt{210}$	$-\sqrt{210}$	$4\sqrt{210}$
$x_2y_{-3}y_1$	0	0	0	$-8\sqrt{210}$	0	$-6\sqrt{210}$
$x_2y_{-2}y_0$	0	0	0	$110\sqrt{3}$	$-15\sqrt{3}$	$60\sqrt{3}$
$x_2y_{-1}^2$	0	0	0	$-20\sqrt{30}$	$4\sqrt{30}$	$-10\sqrt{30}$
$x_3y_{-4}y_1$	0	0	0	$30\sqrt{70}$	$-3\sqrt{70}$	0
$x_3y_{-3}y_0$	0	0	0	$-90\sqrt{7}$	$15\sqrt{7}$	0
$x_3y_{-2}y_{-1}$	0	0	0	$30\sqrt{10}$	$-6\sqrt{10}$	0

Table 6.13: The form of the component g_0 of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant mapping f in the representation on $V_3 \oplus V_4$. For example, the coefficient of $x_1y_{-4}y_3$ in g_0 is $-28\sqrt{6}\gamma_2 + 7\sqrt{6}\gamma_3 + 14\sqrt{6}\gamma_4$.

Term	α_2	β_2	δ_1	δ_2	δ_3	δ_4
$y_{-4}^2y_4$	2	0	0	0	0	0
$y_{-4}y_{-3}y_3$	-2	0	0	0	0	0
$y_{-4}y_{-2}y_2$	2	0	-280	0	0	0
$y_{-4}y_{-1}y_1$	-2	0	420	0	0	0
$y_{-4}y_0^2$	1	0	-224	0	0	0
$y_{-3}^2y_2$	0	0	$70\sqrt{7}$	0	0	0
$y_{-3}y_{-2}y_1$	0	0	-210	0	0	0
$y_{-3}y_{-1}y_0$	0	0	$14\sqrt{10}$	0	0	0
$y_{-2}^2y_0$	0	0	$12\sqrt{70}$	0	0	0
$y_{-2}y_{-1}^2$	0	0	$-20\sqrt{7}$	0	0	0
$y_{-4}x_{-3}x_3$	0	-2	0	-140	0	0
$y_{-4}x_{-2}x_2$	0	2	0	0	0	0
$y_{-4}x_{-1}x_1$	0	-2	0	0	0	-28
$y_{-4}x_0^2$	0	1	0	7	7	21
$y_{-3}x_{-3}x_2$	0	0	0	$70\sqrt{3}$	0	0
$y_{-3}x_{-2}x_1$	0	0	0	0	0	$14\sqrt{5}$
$y_{-3}x_{-1}x_0$	0	0	0	$-7\sqrt{6}$	$-7\sqrt{6}$	$-7\sqrt{6}$
$y_{-2}x_{-3}x_1$	0	0	0	$-10\sqrt{105}$	0	$-2\sqrt{105}$
$y_{-2}x_{-2}x_0$	0	0	0	$\sqrt{210}$	$\sqrt{210}$	$-\sqrt{210}$
$y_{-2}x_{-1}^2$	0	0	0	$6\sqrt{7}$	$6\sqrt{7}$	$6\sqrt{7}$
$y_{-1}x_{-3}x_0$	0	0	0	$9\sqrt{70}$	$-\sqrt{70}$	$3\sqrt{70}$
$y_{-1}x_{-2}x_{-1}$	0	0	0	$-6\sqrt{35}$	$-6\sqrt{35}$	$-2\sqrt{35}$
$y_0x_{-3}x_{-1}$	0	0	0	$-6\sqrt{42}$	$4\sqrt{42}$	$-2\sqrt{42}$
$y_0x_{-2}^2$	0	0	0	$3\sqrt{70}$	$3\sqrt{70}$	$\sqrt{70}$
$y_1x_{-3}x_{-2}$	0	0	0	0	$-10\sqrt{21}$	0
$y_2x_{-3}^2$	0	0	0	0	$10\sqrt{7}$	0

Table 6.14: The form of the component h_{-4} of the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant mapping f in the representation on $V_3 \oplus V_4$. For example, the coefficient of $y_1x_{-3}x_{-2}$ in h_{-4} is $-10\sqrt{21}\delta_3$.

Term	α_2	β_2	δ_1	δ_2	δ_3	δ_4
$y_{-4}y_{-3}y_4$	2	0	0	0	0	0
$y_{-4}y_{-2}y_3$	0	0	$-140\sqrt{7}$	0	0	0
$y_{-4}y_{-1}y_2$	0	0	210	0	0	0
$y_{-4}y_0y_1$	0	0	$-14\sqrt{10}$	0	0	0
$y_{-3}^2y_3$	-2	0	245	0	0	0
$y_{-3}y_{-2}y_2$	2	0	-105	0	0	0
$y_{-3}y_{-1}y_1$	-2	0	175	0	0	0
$y_{-3}y_0^2$	1	0	-154	0	0	0
$y_{-2}^2y_1$	0	0	$-45\sqrt{7}$	0	0	0
$y_{-2}y_{-1}y_0$	0	0	$23\sqrt{70}$	0	0	0
y_{-1}^3	0	0	$-30\sqrt{7}$	0	0	0
$y_{-4}x_{-2}x_3$	0	0	0	$-70\sqrt{3}$	0	0
$y_{-4}x_{-1}x_2$	0	0	0	0	0	$-14\sqrt{5}$
$y_{-4}x_0x_1$	0	0	0	$7\sqrt{6}$	$7\sqrt{6}$	$7\sqrt{6}$
$y_{-3}x_{-3}x_3$	0	-2	0	-35	0	0
$y_{-3}x_{-2}x_2$	0	2	0	105	0	35
$y_{-3}x_{-1}x_1$	0	-2	0	-21	-21	-14
$y_{-3}x_0^2$	0	1	0	-14	-14	0
$y_{-2}x_{-3}x_2$	0	0	0	$-10\sqrt{21}$	0	$-5\sqrt{21}$
$y_{-2}x_{-2}x_1$	0	0	0	$-12\sqrt{35}$	$3\sqrt{35}$	$\sqrt{35}$
$y_{-2}x_{-1}x_0$	0	0	0	$5\sqrt{42}$	$5\sqrt{42}$	0
$y_{-1}x_{-3}x_1$	0	0	0	$-6\sqrt{105}$	$-\sqrt{105}$	0
$y_{-1}x_{-2}x_0$	0	0	0	$3\sqrt{210}$	$-2\sqrt{210}$	$-\sqrt{210}$
$y_{-1}x_{-1}^2$	0	0	0	$-6\sqrt{7}$	$-6\sqrt{7}$	$4\sqrt{7}$
$y_0x_{-3}x_0$	0	0	0	$27\sqrt{7}$	$7\sqrt{7}$	$9\sqrt{7}$
$y_0x_{-2}x_{-1}$	0	0	0	$-9\sqrt{14}$	$6\sqrt{14}$	$-3\sqrt{14}$
$y_1x_{-3}x_{-1}$	0	0	0	$-6\sqrt{105}$	$-\sqrt{105}$	$-2\sqrt{105}$
$y_1x_{-2}^2$	0	0	0	$15\sqrt{7}$	0	$5\sqrt{7}$
$y_2x_{-3}x_{-2}$	0	0	0	0	$-5\sqrt{21}$	0
$y_3x_{-3}^2$	0	0	0	0	35	0

Table 6.15: The form of the component h_{-3} of the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant map f in the representation on $V_3 \oplus V_4$. For example, the coefficient of $y_1x_{-3}x_{-1}$ in h_{-3} is $-6\sqrt{105}\delta_2 - \sqrt{105}\delta_3 - 2\sqrt{105}\delta_4$.

Term	α_2	β_2	δ_1	δ_2	δ_3	δ_4
$y_{-4}y_{-2}y_4$	2	0	-280	0	0	0
$y_{-4}y_{-1}y_3$	0	0	-210	0	0	0
$y_{-4}y_0y_2$	0	0	$24\sqrt{70}$	0	0	0
$y_{-4}y_1^2$	0	0	$-20\sqrt{7}$	0	0	0
$y_{-3}^2y_4$	0	0	$70\sqrt{7}$	0	0	0
$y_{-3}y_{-2}y_3$	-2	0	105	0	0	0
$y_{-3}y_{-1}y_2$	0	0	$90\sqrt{7}$	0	0	0
$y_{-3}y_0y_1$	0	0	$-23\sqrt{70}$	0	0	0
$y_{-2}^2y_2$	2	0	-240	0	0	0
$y_{-2}y_{-1}y_1$	-2	0	135	0	0	0
$y_{-2}y_0^2$	1	0	76	0	0	0
$y_{-1}^2y_0$	0	0	$-21\sqrt{10}$	0	0	0
$y_{-4}x_{-1}x_3$	0	0	0	$-10\sqrt{105}$	0	$-2\sqrt{105}$
$y_{-4}x_0x_2$	0	0	0	$\sqrt{210}$	$\sqrt{210}$	$-\sqrt{210}$
$y_{-4}x_1^2$	0	0	0	$6\sqrt{7}$	$6\sqrt{7}$	$6\sqrt{7}$
$y_{-3}x_{-2}x_3$	0	0	0	$-10\sqrt{21}$	0	$5\sqrt{21}$
$y_{-3}x_{-1}x_2$	0	0	0	$12\sqrt{35}$	$-3\sqrt{35}$	$-\sqrt{35}$
$y_{-3}x_0x_1$	0	0	0	$-5\sqrt{42}$	$-5\sqrt{42}$	0
$y_{-2}x_{-3}x_3$	0	-2	0	-5	0	-15
$y_{-2}x_{-2}x_2$	0	2	0	75	15	25
$y_{-2}x_{-1}x_1$	0	-2	0	-51	24	-9
$y_{-2}x_0^2$	0	1	0	16	16	0
$y_{-1}x_{-3}x_2$	0	0	0	0	$-5\sqrt{3}$	$-15\sqrt{3}$
$y_{-1}x_{-2}x_1$	0	0	0	$-36\sqrt{5}$	$-6\sqrt{5}$	$-3\sqrt{5}$
$y_{-1}x_{-1}x_0$	0	0	0	$18\sqrt{6}$	$-7\sqrt{6}$	$3\sqrt{6}$
$y_0x_{-3}x_1$	0	0	0	$-3\sqrt{6}$	$2\sqrt{6}$	$9\sqrt{6}$
$y_0x_{-2}x_0$	0	0	0	$39\sqrt{3}$	$-\sqrt{3}$	$-7\sqrt{3}$
$y_0x_{-1}^2$	0	0	0	$-15\sqrt{10}$	0	$\sqrt{10}$
$y_1x_{-3}x_0$	0	0	0	$9\sqrt{10}$	$4\sqrt{10}$	$3\sqrt{10}$
$y_1x_{-2}x_{-1}$	0	0	0	$-6\sqrt{5}$	$9\sqrt{5}$	$-2\sqrt{5}$
$y_2x_{-3}x_{-1}$	0	0	0	$-18\sqrt{15}$	$-8\sqrt{15}$	$-6\sqrt{15}$
$y_2x_{-2}^2$	0	0	0	45	-15	15
$y_3x_{-3}x_{-2}$	0	0	0	0	$5\sqrt{21}$	0
$y_4x_{-3}^2$	0	0	0	0	$10\sqrt{7}$	0

Table 6.16: The form of the component h_{-2} of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant mapping f in the representation on $V_3 \oplus V_4$. For example, the coefficient of $y_1x_{-3}x_0$ in h_{-2} is $9\sqrt{10}\delta_2 + 4\sqrt{10}\delta_3 + 3\sqrt{10}\delta_4$.

Term	α_2	β_2	δ_1	δ_2	δ_3	δ_4
$y_{-4}y_{-1}y_4$	2	0	-420	0	0	0
$y_{-4}y_0y_3$	0	0	$-14\sqrt{10}$	0	0	0
$y_{-4}y_1y_2$	0	0	$40\sqrt{7}$	0	0	0
$y_{-3}y_{-2}y_4$	0	0	210	0	0	0
$y_{-3}y_{-1}y_3$	-2	0	175	0	0	0
$y_{-3}y_0y_2$	0	0	$23\sqrt{70}$	0	0	0
$y_{-3}y_1^2$	0	0	$-90\sqrt{7}$	0	0	0
$y_{-2}^2y_3$	0	0	$-45\sqrt{7}$	0	0	0
$y_{-2}y_{-1}y_2$	2	0	-135	0	0	0
$y_{-2}y_0y_1$	0	0	$42\sqrt{10}$	0	0	0
$y_{-1}^2y_1$	-2	0	65	0	0	0
$y_{-1}y_0^2$	1	0	-64	0	0	0
$y_{-4}x_0x_3$	0	0	0	$-9\sqrt{70}$	$\sqrt{70}$	$-3\sqrt{70}$
$y_{-4}x_1x_2$	0	0	0	$6\sqrt{35}$	$6\sqrt{35}$	$2\sqrt{35}$
$y_{-3}x_{-1}x_3$	0	0	0	$-6\sqrt{105}$	$-\sqrt{105}$	0
$y_{-3}x_0x_2$	0	0	0	$3\sqrt{210}$	$-2\sqrt{210}$	$-\sqrt{210}$
$y_{-3}x_1^2$	0	0	0	$-6\sqrt{7}$	$-6\sqrt{7}$	$4\sqrt{7}$
$y_{-2}x_{-2}x_3$	0	0	0	0	$5\sqrt{3}$	$15\sqrt{3}$
$y_{-2}x_{-1}x_2$	0	0	0	$36\sqrt{5}$	$6\sqrt{5}$	$3\sqrt{5}$
$y_{-2}x_0x_1$	0	0	0	$-18\sqrt{6}$	$7\sqrt{6}$	$-3\sqrt{6}$
$y_{-1}x_{-3}x_3$	0	-2	0	-5	-5	-30
$y_{-1}x_{-2}x_2$	0	2	0	15	0	5
$y_{-1}x_{-1}x_1$	0	-2	0	-75	0	-8
$y_{-1}x_0^2$	0	1	0	52	2	6
$y_0x_{-3}x_2$	0	0	0	$-\sqrt{30}$	$-\sqrt{30}$	$-2\sqrt{30}$
$y_0x_{-2}x_1$	0	0	0	$-24\sqrt{2}$	$-9\sqrt{2}$	$-3\sqrt{2}$
$y_0x_{-1}x_0$	0	0	0	$5\sqrt{15}$	$-5\sqrt{15}$	$\sqrt{15}$
$y_1x_{-3}x_1$	0	0	0	$4\sqrt{15}$	$4\sqrt{15}$	$8\sqrt{15}$
$y_1x_{-2}x_0$	0	0	0	$14\sqrt{30}$	$4\sqrt{30}$	$-2\sqrt{30}$
$y_1x_{-1}^2$	0	0	0	-60	15	0
$y_2x_{-3}x_0$	0	0	0	$-9\sqrt{10}$	$-4\sqrt{10}$	$-3\sqrt{10}$
$y_2x_{-2}x_{-1}$	0	0	0	$6\sqrt{5}$	$-9\sqrt{5}$	$2\sqrt{5}$
$y_3x_{-3}x_{-1}$	0	0	0	$-6\sqrt{105}$	$-\sqrt{105}$	$-2\sqrt{105}$
$B_3x_{-2}^2$	0	0	0	$15\sqrt{7}$	0	$5\sqrt{7}$
$y_4x_{-3}x_{-2}$	0	0	0	0	$10\sqrt{21}$	0

Table 6.17: The form of the component h_{-1} of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant mapping f in the representation on $V_3 \oplus V_4$. For example, the coefficient of $y_1x_{-3}x_1$ in h_{-1} is $4\sqrt{15}\delta_2 + 4\sqrt{15}\delta_3 + 8\sqrt{15}\delta_4$.

Term	α_2	β_2	δ_1	δ_2	δ_3	δ_4
$y_{-4}y_0y_4$	2	0	-448	0	0	0
$y_{-4}y_1y_3$	0	0	$14\sqrt{10}$	0	0	0
$y_{-4}y_2^2$	0	0	$12\sqrt{70}$	0	0	0
$y_{-3}y_{-1}y_4$	0	0	$14\sqrt{10}$	0	0	0
$y_{-3}y_0y_3$	-2	0	308	0	0	0
$y_{-3}y_1y_2$	0	0	$-23\sqrt{70}$	0	0	0
$y_{-2}^2y_4$	0	0	$12\sqrt{70}$	0	0	0
$y_{-2}y_{-1}y_3$	0	0	$-23\sqrt{70}$	0	0	0
$y_{-2}y_0y_2$	2	0	152	0	0	0
$y_{-2}y_1^2$	0	0	$-21\sqrt{10}$	0	0	0
$y_{-1}^2y_2$	0	0	$-21\sqrt{10}$	0	0	0
$y_{-1}y_0y_1$	-2	0	128	0	0	0
y_0^3	1	0	-64	0	0	0
$y_{-4}x_1x_3$	0	0	0	$-6\sqrt{42}$	$4\sqrt{42}$	$-2\sqrt{42}$
$y_{-4}x_2^2$	0	0	0	$3\sqrt{70}$	$3\sqrt{70}$	$\sqrt{70}$
$y_{-3}x_0x_3$	0	0	0	$-27\sqrt{7}$	$-7\sqrt{7}$	$-9\sqrt{7}$
$y_{-3}x_1x_2$	0	0	0	$9\sqrt{14}$	$-6\sqrt{14}$	$3\sqrt{14}$
$y_{-2}x_{-1}x_3$	0	0	0	$-3\sqrt{6}$	$2\sqrt{6}$	$9\sqrt{6}$
$y_{-2}x_0x_2$	0	0	0	$39\sqrt{3}$	$-\sqrt{3}$	$-7\sqrt{3}$
$y_{-2}x_1^2$	0	0	0	$-15\sqrt{10}$	0	$\sqrt{10}$
$y_{-1}x_{-2}x_3$	0	0	0	$\sqrt{30}$	$\sqrt{30}$	$2\sqrt{30}$
$y_{-1}x_{-1}x_2$	0	0	0	$24\sqrt{2}$	$9\sqrt{2}$	$3\sqrt{2}$
$y_{-1}x_0x_1$	0	0	0	$-5\sqrt{15}$	$5\sqrt{15}$	$-\sqrt{15}$
$y_0x_{-3}x_3$	0	-2	0	-8	-8	36
$y_0x_{-2}x_2$	0	2	0	-12	-12	-4
$y_0x_{-1}x_1$	0	-2	0	-84	-24	-8
$y_0x_0^2$	0	1	0	67	-13	9
$y_1x_{-3}x_{-2}$	0	0	0	$\sqrt{30}$	$\sqrt{30}$	$2\sqrt{30}$
$y_1x_{-2}x_{-1}$	0	0	0	$24\sqrt{2}$	$9\sqrt{2}$	$3\sqrt{2}$
$y_1x_{-1}x_0$	0	0	0	$-5\sqrt{15}$	$5\sqrt{15}$	$-\sqrt{15}$
$y_2x_{-3}x_1$	0	0	0	$-3\sqrt{6}$	$2\sqrt{6}$	$9\sqrt{6}$
$y_2x_{-2}x_0$	0	0	0	$39\sqrt{3}$	$-\sqrt{3}$	$-7\sqrt{3}$
$y_2x_{-1}^2$	0	0	0	$-15\sqrt{10}$	0	$\sqrt{10}$
$y_3x_{-3}x_0$	0	0	0	$-27\sqrt{7}$	$-7\sqrt{7}$	$-9\sqrt{7}$
$y_3x_{-2}x_{-1}$	0	0	0	$9\sqrt{14}$	$-6\sqrt{14}$	$3\sqrt{14}$
$y_4x_{-3}x_{-1}$	0	0	0	$-6\sqrt{42}$	$4\sqrt{42}$	$-2\sqrt{42}$
$y_4x_{-2}^2$	0	0	0	$3\sqrt{70}$	$3\sqrt{70}$	$\sqrt{70}$

Table 6.18: The form of the component h_0 of the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant mapping f in the representation on $V_3 \oplus V_4$. For example, the coefficient of $y_1x_{-3}x_{-2}$ in h_0 is $\sqrt{30}\delta_2 + \sqrt{30}\delta_3 + 2\sqrt{30}\delta_4$.

6.4 Coefficients for the Swift–Hohenberg equation

The motivation for the work in this chapter on bifurcations with $\mathbf{O(3)} \times \mathbb{Z}_2$ symmetry was the fact that dynamical systems on a sphere which are invariant under a change in sign of the physical variable have $\mathbf{O(3)} \times \mathbb{Z}_2$ as their group of symmetries. One such system is the Swift–Hohenberg equation [78],

$$\frac{\partial w}{\partial t} = \mu w - (1 + \nabla^2)^2 w - w^3. \tag{6.4.1}$$

In this section we will discuss how to compute the values of coefficients in $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector fields for this specific example system. We will compute the values explicitly in some of the representations for which we computed the general form of the equivariant vector field in Section 6.3. This will allow us to determine which solutions of (6.4.1) are stable in these cases.

Since we are considering the Swift–Hohenberg equation on a spherical domain, we assume that the solutions can be written as a linear combination of spherical harmonics i.e.

$$w(\theta,\phi,t) = \sum_{\ell\geq 0}\sum_{m=-\ell}^{\ell} x_{\ell,m}(t)Y_\ell^m(\theta,\phi)$$

where $x_{\ell,m} = (-1)^m \overline{x}_{\ell,m}$ since $w(\theta,\phi,t) \in \mathbb{R}$. Recall that spherical harmonics are eigenfunctions of the angular part of the spherical Laplacian operator

$$\nabla^2 \mathbf{U}(R,\theta,\phi) = \frac{1}{R^2}\left[\frac{\partial}{\partial R}R^2\frac{\partial}{\partial R} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\mathbf{U}$$

with

$$\nabla^2 Y_\ell^m(\theta,\phi) = -\frac{\ell(\ell+1)}{R^2}Y_\ell^m(\theta,\phi)$$

where R is the radius of the sphere which we assume is constant.

Equation (6.4.1) has an equilibrium solution $w = 0$. Linearising about this solution by letting $w = \epsilon w_1$ we find that

$$\frac{\partial w_1}{\partial t} = \mu w_1 - (1+\nabla^2)^2 w_1. \tag{6.4.2}$$

In Section 6.4.1 we will consider the case where the representation of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ is irreducible so that the solution of the linear problem (6.4.2) can be written as a sum of spherical harmonics of a single degree ℓ. We then carry out the computation of the coefficients in the equivariant vector field for the specific case where $\ell = 3$. In Section 6.4.2 we consider the case where the representation of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ is the reducible representation on $V_\ell \oplus V_{\ell+1}$ where w_1 is assumed to be a linear combination of spherical harmonics of degrees ℓ and $\ell+1$. We carry out the computations of the coefficients of the cubic terms in the equivariant vector field for the specific examples where $\ell = 2$ and $\ell = 3$.

6.4.1 Irreducible representations of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$

In this section we consider how to compute coefficients in an $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field for the specific example of the Swift–Hohenberg equation (6.4.1) when the representation of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ is irreducible. In this case the solution of the linearised Swift–Hohenberg equation (6.4.2) can be written as a sum of spherical harmonics of a single degree ℓ.

If we assume that

$$w_1 = \sum_{m=-\ell}^{\ell} x_m(t)Y_\ell^m(\theta,\phi) \tag{6.4.3}$$

for some value of ℓ then substituting into (6.4.2) we find that

$$\frac{\partial w_1}{\partial t} = \mu w_1 - (1-\ell(\ell+1)/R^2)^2 w_1,$$

which has general solution

$$w_1(t) = \exp\left[(\mu - (1-\ell(\ell+1)/R^2)^2)t\right] w_1(0).$$

Hence the critical value of μ where the modes of degree ℓ have zero growth rate occurs at

$$\mu_c = (1-\ell(\ell+1)/R^2)^2$$

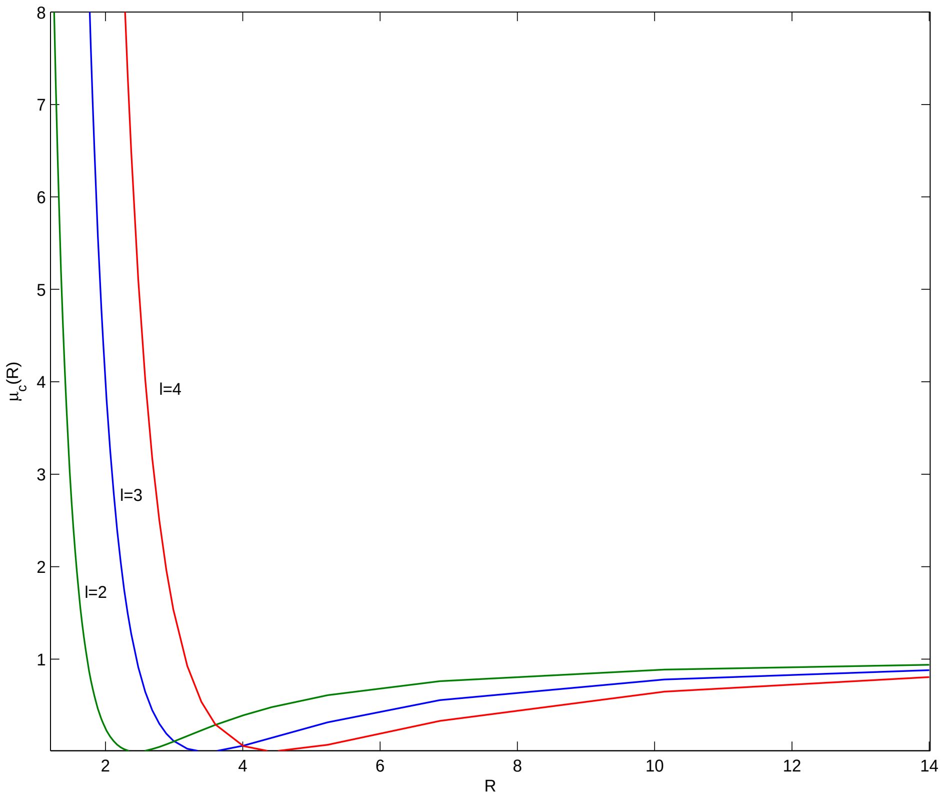


Figure 6.8: A plot of the function $\mu_c(R) = (1 - \ell(\ell+1)/R^2)^2$ for $\ell = 2, 3$ and 4. The minimum value of $\mu_c = 0$ for $\ell = 2$ occurs at $R = \sqrt{6}$, for $\ell = 3$ the minimum occurs at $R = \sqrt{12}$ and for $\ell = 4$ the minimum occurs at $R = \sqrt{20}$.

and the value of R which minimises μ_c for a given value of ℓ is

$$R_c = \sqrt{\ell(\ell+1)}.$$

Figure 6.8 show plots of μ_c as a function of R for some specific values of ℓ.

We now consider the full equation (6.4.1) with the nonlinear term w^3. Let

$$\mu = \mu_c + \epsilon^2 \mu_2 = \epsilon^2 \mu_2 \tag{6.4.4}$$
$$T = \epsilon^2 t \tag{6.4.5}$$
$$w = \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3. \tag{6.4.6}$$

Substituting (6.4.4) – (6.4.6) into (6.4.1) we find that to cubic order in epsilon

$$\epsilon^3 \frac{\partial w_1}{\partial T} = \epsilon^3 \mu_2 w_1 - (1+\nabla^2)^2(\epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3) - \epsilon^3 w_1^3.$$

At order ϵ we recover the linearised stability problem as before which is satisfied for w_1 as in (6.4.3). If we let $w_2 = 0$ then the equation at order ϵ^2 is also satisfied. At order ϵ^3 we then have

$$\frac{\partial w_1}{\partial T} = \mu_2 w_1 - (1+\nabla^2)^2 w_3 - w_1^3 \tag{6.4.7}$$

If we multiply (6.4.7) by $\overline{Y_\ell^m}$ and integrate over the sphere we find

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\frac{\partial w_1}{\partial T}\overline{Y_\ell^m}\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi &= \int_0^{2\pi}\int_0^{\pi}\mu_2 w_1\overline{Y_\ell^m}\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi \\ &\quad -\int_0^{2\pi}\int_0^{\pi}(1+\nabla^2)^2 w_3\overline{Y_\ell^m}\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi \\ &\quad -\int_0^{2\pi}\int_0^{\pi} w_1^3\overline{Y_\ell^m}\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi \\ &= I_1 - I_2 - I_3. \end{aligned} \tag{6.4.8}$$

We can then see that by using the form of w_1 as in equation (6.4.3) and the orthogonality of the spherical harmonics (3.2.3) the left hand side becomes $\dot{x}_m$ where the dot denotes $\frac{\mathrm{d}}{\mathrm{d}T}$ and the first integral on the right hand side, I_1, becomes $\mu_2 x_m$. By integrating by parts we find that

$$I_2 = \int_0^{2\pi}\int_0^{\pi}(1+\nabla^2)^2 w_3\overline{Y_\ell^m}\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi = -\int_0^{2\pi}\int_0^{\pi} w_3(1+\nabla^2)^2\overline{Y_\ell^m}\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi = 0$$

since $(1+\nabla^2)^2\overline{Y_\ell^m} = (-1)^m(1+\nabla^2)^2 Y_\ell^m = 0$ for any m. This means that (6.4.8) becomes

$$\dot{x}_m = \mu_2 x_m - \int_0^{2\pi}\int_0^{\pi}\left(\sum_{n=-\ell}^{\ell} x_n Y_\ell^n\right)^3\overline{Y_\ell^m}\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi. \tag{6.4.9}$$

By evaluating all terms in this integral we can find the exact form of the component $\dot{x}_m$ in the $\mathbf{O(3)}\times\mathbb{Z}_2$ equivariant vector field for the Swift–Hohenberg equation for the representation on V_ℓ. Using the orthogonality of the spherical harmonics we can see that

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m Y_\ell^n Y_\ell^p\overline{Y_\ell^q}\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi = 0 \qquad \text{unless} \qquad m+n+p=q. \tag{6.4.10}$$

This reduces the number of integrals we need to evaluate. Alternatively, if we already have the general form of the equivariant vector field found using symmetries we need only compute as many integrals as there are coefficients in the vector field.

We will now consider the case where the representation is on V_3.

Example: The representation on V_3

In Section 6.3.3 we computed that the general form of the $\mathbf{O(3)}\times\mathbb{Z}_2$ equivariant vector field for the representation on V_3 is given by (6.3.8). We now compute the values of α and β for the specific example of the Swift–Hohenberg equation. This will allow us to determine which of the three axial solution branches guaranteed to exist by the equivariant branching lemma is stable for the Swift–Hohenberg equation on a sphere of radius near $R_c = \sqrt{12}$.

When $\ell = 3$ we can see from equation (6.4.9) that for $-3 \le m \le 3$ we have

$$\dot{x}_m = \mu_2 x_m - \int_0^{2\pi}\int_0^{\pi}\left(\sum_{n=-3}^{3} x_n Y_3^n\right)^3\overline{Y_3^m}\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi. \tag{6.4.11}$$

Comparing this and the form of the equivariant vector field (6.3.8) we can see immediately that $\mu_2 = \lambda$. Since the coefficient of $x_{-3}x_1x_2$ in the $\dot{x}_0$ component of (6.3.8) is $-30\sqrt{2}\beta$ and the term

$x_{-3}x_1x_2$ occurs 6 times in the expression $\left(\sum_{n=-3}^{3} x_n\right)^3$ in (6.4.11) with $m = 0$ we can see that

$$\begin{aligned}
-30\sqrt{2}\beta &= -6\int_0^{2\pi}\int_0^{\pi} Y_3^{-3}Y_3^1Y_3^2\overline{Y_3^0}\sin\theta\, d\theta d\phi \\
&= \frac{2205\sqrt{2}}{512\pi}\int_0^{\pi}\Big(-25\cos^{12}\theta + 95\cos^{10}\theta - 138\cos^8\theta + 94\cos^6\theta \\
&\qquad\qquad -29\cos^4\theta + 3\cos^2\theta\Big)\sin\theta d\theta \\
&= \frac{21\sqrt{2}}{286\pi}
\end{aligned}$$

Similarly we see that the coefficient of x_0^3 in the $\dot{x}_0$ component of (6.3.8) is $\alpha - 9\beta$ and the term x_0^3 appears only once in the expression $\left(\sum_{n=-3}^{3} x_n\right)^3$ in (6.4.11) with $m = 0$ so

$$\begin{aligned}
\alpha - 9\beta &= -\int_0^{2\pi}\int_0^{\pi}(Y_3^0)^4\sin\theta\, d\theta d\phi \\
&= -\frac{49}{128\pi}\int_0^{\pi}\Big(625\cos^{12}\theta - 1500\cos^{10}\theta + 1350\cos^8\theta \\
&\qquad\qquad -540\cos^6\theta + 81\cos^4\theta\Big)\sin\theta d\theta \\
&= -\frac{1687}{2860\pi}
\end{aligned}$$

Hence we find that

$$\beta = -\frac{7}{2860\pi} \qquad \text{and} \qquad \alpha = -\frac{175}{286\pi} \tag{6.4.12}$$

are the values of the coefficients in the equivariant vector field (6.3.8) for the Swift–Hohenberg equation. Using the results of Section 6.3.3, since

$$\beta = -\frac{7}{2860\pi} < 0 \qquad \text{and} \qquad 2\alpha - 50\beta = -\frac{315}{286\pi} < 0$$

we can see that the solution branch with $\widetilde{\mathbb{O}\times\mathbb{Z}_2^c}$ symmetry is stable for the Swift–Hohenberg equation on a sphere of radius near $R_c = \sqrt{12}$.

6.4.2 Reducible representations of $\mathbf{O(3)} \times \mathbb{Z}_2$

In this section we consider how to compute coefficients in an $\mathbf{O}(3) \times \mathbb{Z}_2$ equivariant vector field for the specific example of the Swift–Hohenberg equation (6.4.1) when the representation of $\mathbf{O}(3) \times \mathbb{Z}_2$ is reducible. In this case the solution of the linearised Swift–Hohenberg equation (6.4.2) can be written as a sum of spherical harmonics of degrees ℓ and $\ell + 1$.

If we assume that $w_1 = w_1^{(1)} + w_1^{(2)}$ where

$$w_1^{(1)} = \sum_{m=-\ell}^{\ell} x_m(t)Y_\ell^m(\theta,\phi) \qquad \text{and} \qquad w_1^{(2)} = \sum_{n=-\ell-1}^{\ell+1} y_n(t)Y_{\ell+1}^n(\theta,\phi)$$

for some value of ℓ, then substituting into the linearised Swift–Hohenberg equation (6.4.2) we find that

$$\frac{\partial w_1}{\partial t} = \mu w_1 - (1 - \ell(\ell+1)/R^2)^2 w_1^{(1)} - (1 - (\ell+1)(\ell+2)/R^2)^2 w_1^{(2)}. \tag{6.4.13}$$

Notice that

$$(1-\ell(\ell+1)/R^2)^2 \;=\; (1-(\ell+1)(\ell+2)/R^2)^2 \;=\; \frac{1}{(\ell+1)^2} \quad \text{when} \quad R=\ell+1$$

and hence substituting $R=\ell+1$ into (6.4.13) we have

$$\frac{\partial w_1}{\partial t} = \mu w_1 - \frac{1}{(\ell+1)^2} w_1$$

which has general solution

$$w_1(t) = \exp\left[\left(\mu - \frac{1}{(\ell+1)^2}\right) t\right] w_1(0).$$

Hence $R_c=\ell+1$ and the critical value of μ where the ℓ and $\ell+1$ modes both have zero growth rate occurs at $\mu_c = 1/(\ell+1)^2$. Notice that this is the point in Figure 6.8 where the ℓ and $\ell+1$ lines cross.

We now consider the full equation (6.4.1) with the nonlinear term w^3. Let

$$\begin{aligned} \mu &= \mu_c + \epsilon^2\mu_2 && (6.4.14)\\ R &= R_c + \epsilon^2 R_2 && (6.4.15)\\ T &= \epsilon^2 t && (6.4.16)\\ w &= \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 && (6.4.17) \end{aligned}$$

where $R_c = \ell+1$ and $\mu_c = 1/(\ell+1)^2$. We find that the linear differential operator $L = \mu - (1+\nabla^2)^2$ acts as

$$\begin{aligned} L &= \mu - \left(1 - \frac{\ell(\ell+1)}{R^2}\right)^2 \\ &\sim \mu_c + \epsilon^2\mu_2 - \left(1 - \frac{\ell(\ell+1)}{R_c^2} + \frac{2\ell(\ell+1)}{R_c^3}R_2\epsilon^2\right)^2 \\ &\sim \mu_c - \left(1 - \frac{\ell(\ell+1)}{R_c^2}\right)^2 + \epsilon^2\left[\mu_2 - 2\left(1 - \frac{\ell(\ell+1)}{R_c^2}\right)\left(\frac{2\ell(\ell+1)}{R_c^3}R_2\right)\right] \\ &= L_0 + \epsilon^2 L_2 && (6.4.18) \end{aligned}$$

on the spherical harmonics of degree ℓ. On the spherical harmonics of degree ℓ we find that

$$L_0 Y_\ell^m = 0 \qquad \text{and} \qquad L_2 Y_\ell^m = \left(\mu_2 - \frac{4\ell}{(\ell+1)^3}R_2\right) Y_\ell^m$$

and on the spherical harmonics of degree $\ell+1$ we find that

$$L_0 Y_{\ell+1}^m = 0 \qquad \text{and} \qquad L_2 Y_{\ell+1}^m = \left(\mu_2 + \frac{4(\ell+2)}{(\ell+1)^3}R_2\right) Y_{\ell+1}^m.$$

Substituting (6.4.14) – (6.4.17) into (6.4.1) we find that to cubic order in ϵ

$$\epsilon^3\frac{\partial w_1}{\partial T} = \epsilon L_0 w_1 + \epsilon^2 L_0 w_2 + \epsilon^3(L_0 w_3 + L_2 w_1 - w_1^3).$$

At order ϵ we recover the linearised stability problem which is satisfied since

$$w_1 = \sum_{m=-\ell}^{\ell} x_m(t) Y_\ell^m(\theta,\phi) + \sum_{n=-\ell-1}^{\ell+1} y_n(t) Y_{\ell+1}^n(\theta,\phi). \qquad (6.4.19)$$

At order ϵ^2 we have $0 = L_0 w_2$. Let $w_2 = 0$, then we find that at order ϵ^3 we have

$$\frac{\partial w_1}{\partial T} = L_0 w_3 + L_2 w_1 - w_1^3 \tag{6.4.20}$$

If we multiply equation (6.4.20) by $\overline{Y}_\ell^p$ and integrate over the sphere we find

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi} \frac{\partial w_1}{\partial T}\overline{Y}_\ell^p \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi &= \int_0^{2\pi}\int_0^{\pi} L_2 w_1 \overline{Y}_\ell^p \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi \\
&\quad - \int_0^{2\pi}\int_0^{\pi} L_0 w_3 \overline{Y}_\ell^p \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi \\
&\quad\quad - \int_0^{2\pi}\int_0^{\pi} w_1^3 \overline{Y}_\ell^p \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi \\
&= I_1 - I_2 - I_3.
\end{aligned} \tag{6.4.21}$$

We can then see that by using the form of w_1 as in equation (6.4.19) and orthogonality of the spherical harmonics (3.2.3) the left hand side becomes $\dot{x}_p$ where the dot denotes $\frac{\mathrm{d}}{\mathrm{d}T}$ and the first integral on the right hand side, I_1, becomes

$$\left(\mu_2 - \frac{4\ell}{(\ell+1)^3}R_2\right) x_p.$$

By integrating by parts we find that

$$I_2 = \int_0^{2\pi}\int_0^{\pi} L_0 w_3 \overline{Y}_\ell^p \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = -\int_0^{2\pi}\int_0^{\pi} w_3 L_0 \overline{Y}_\ell^p \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = 0.$$

This means that equation (6.4.21) becomes

$$\dot{x}_p = \left(\mu_2 - \frac{4\ell}{(\ell+1)^3}R_2\right) x_p - \int_0^{2\pi}\int_0^{\pi} \left(\sum_{m=-\ell}^{\ell} x_m Y_\ell^m + \sum_{n=-\ell-1}^{\ell+1} y_n Y_{\ell+1}^n\right)^3 \overline{Y}_\ell^p \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi. \tag{6.4.22}$$

Similarly if we multiply equation (6.4.20) by $\overline{Y}_{\ell+1}^p$ and integrate over the sphere we find

$$\dot{y}_p = \left(\mu_2 + \frac{4(\ell+2)}{(\ell+1)^3}R_2\right) y_p - \int_0^{2\pi}\int_0^{\pi} \left(\sum_{m=-\ell}^{\ell} x_m Y_\ell^m + \sum_{n=-\ell-1}^{\ell+1} y_n Y_{\ell+1}^n\right)^3 \overline{Y}_{\ell+1}^p \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi. \tag{6.4.23}$$

By evaluating all terms in these integrals we can find the exact form of the components $\dot{x}_p$ and $\dot{y}_p$ in the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field for the Swift–Hohenberg equation for the representation on $V_\ell \oplus V_{\ell+1}$. Using orthogonality of the spherical harmonics we can see that

$$\int_0^{2\pi}\int_0^{\pi} Y_{\ell_1}^m Y_{\ell_2}^n Y_{\ell_3}^p \overline{Y}_{\ell_4}^q \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = 0 \qquad \text{unless} \qquad m+n+p = q. \tag{6.4.24}$$

This reduces the number of integrals we need to evaluate. Alternatively, if we already have the general form of the equivariant vector field found using symmetries we need only compute as many integrals as there are coefficients in the vector field.

We will now evaluate the coefficients in the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field for the Swift–Hohenberg equation when the representation is on $V_2 \oplus V_3$ or $V_3 \oplus V_4$.

Example: The coefficients for the representation on $V_2 \oplus V_3$

In Section 6.3.5 we computed that the general form of the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field for the representation on $V_2 \oplus V_3$ is given by (6.3.22)–(6.3.23). We now compute the values of the coefficients in this vector field for the specific example of the Swift–Hohenberg equation.

When $\ell = 2$ we can see from equations (6.4.22) and (6.4.23) that

$$\dot{x}_p = \left(\mu_2 - \frac{8}{27}R_2\right) x_p - \int_0^{2\pi}\int_0^{\pi}\left(\sum_{m=-2}^{2} x_m Y_2^m + \sum_{n=-3}^{3} y_n Y_3^n\right)^3 \overline{Y}_2^p \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi \qquad (6.4.25)$$

$$\dot{y}_p = \left(\mu_2 + \frac{16}{27}R_2\right) y_p - \int_0^{2\pi}\int_0^{\pi}\left(\sum_{m=-2}^{2} x_m Y_2^m + \sum_{n=-3}^{3} y_n Y_3^n\right)^3 \overline{Y}_3^p \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi. \qquad (6.4.26)$$

Comparing these equations and the general form of the equivariant vector field for this representation (6.3.22)–(6.3.23) we can see immediately that $\mu_x = \mu_2 - \frac{8}{27}R_2$ and $\mu_y = \mu_2 + \frac{16}{27}R_2$. By comparing the cubic terms in the general form of the equivariant vector field with equations (6.4.25) and (6.4.26) we can compute the values of the coefficients α_1, α_2, β_1, β_2, γ_1, γ_2, δ_1, δ_2 and δ_3 as follows:

- The coefficient of x_0^3 in $\dot{x}_0$ of the general equivariant vector field is α_1. The term x_0^3 occurs only once in the cubed sum of equation (6.4.25) so we have
$$\alpha_1 = -\int_0^{2\pi}\int_0^{\pi} (Y_2^0)^4 \sin\theta \, \mathrm{d}\theta\mathrm{d}\phi = -\frac{15}{28\pi}.$$
- The coefficient of $x_1 y_{-2} y_{-1}$ in $\dot{x}_{-2}$ of the general equivariant vector field is $2\sqrt{10}\gamma_1$. The term $x_1 y_{-2} y_{-1}$ occurs 6 times in the cubed sum of equation (6.4.25) so we have
$$\gamma_1 = -\frac{3\sqrt{10}}{10}\int_0^{2\pi}\int_0^{\pi} Y_2^1 Y_3^{-2} Y_3^{-1} \overline{Y}_2^{-2} \sin\theta \, \mathrm{d}\theta\mathrm{d}\phi = \frac{3}{44\pi}.$$
- The coefficient of $x_{-1} y_{-1} y_0$ in $\dot{x}_{-2}$ of the general equivariant vector field is $-2\sqrt{3}\gamma_2$. The term $x_{-1} y_{-1} y_0$ occurs 6 times in the cubed sum of equation (6.4.25) so we have
$$\gamma_2 = \sqrt{3}\int_0^{2\pi}\int_0^{\pi} Y_2^{-1} Y_3^{-1} Y_3^0 \overline{Y}_2^{-2} \sin\theta \, \mathrm{d}\theta\mathrm{d}\phi = \frac{1}{44\pi}.$$
- The coefficient of $x_{-2} y_0^2$ in $\dot{x}_{-2}$ of the general equivariant vector field is $\beta_1 + \gamma_2$. The term $x_{-2} y_0^2$ occurs 3 times in the cubed sum of equation (6.4.25) so we have
$$\beta_1 = -3\int_0^{2\pi}\int_0^{\pi} Y_2^{-2} (Y_3^0)^2 \overline{Y}_2^{-2} \sin\theta \, \mathrm{d}\theta\mathrm{d}\phi - \gamma_2 = -\frac{23}{44\pi} - \frac{1}{44\pi} = -\frac{6}{11\pi}.$$
- The coefficient of $y_{-3} y_1 y_2$ in $\dot{y}_0$ of the general equivariant vector field is $-30\sqrt{2}\delta_1$. The term $y_{-3} y_1 y_2$ occurs 6 times in the cubed sum of equation (6.4.26) so we have
$$\delta_1 = \frac{\sqrt{2}}{10}\int_0^{2\pi}\int_0^{\pi} Y_3^{-3} Y_3^1 Y_3^2 Y_3^0 \sin\theta \, \mathrm{d}\theta\mathrm{d}\phi = -\frac{7}{2860\pi}.$$
- The coefficient of y_0^3 in $\dot{y}_0$ of the general equivariant vector field is $\beta_2 - 9\delta_1$. The term y_0^3 occurs only once in the cubed sum of equation (6.4.26) so we have
$$\beta_2 = -\int_0^{2\pi}\int_0^{\pi} (Y_3^0)^4 \sin\theta \, \mathrm{d}\theta\mathrm{d}\phi + 9\delta_1 = -\frac{1687}{2860\pi} - \frac{63}{2860\pi} = -\frac{175}{286\pi}.$$

- The coefficient of $y_{-3}x_{-2}x_2$ in $\dot{y}_{-3}$ of the general equivariant vector field is $2\alpha_2$. The term $y_{-3}x_{-2}x_2$ occurs 6 times in the cubed sum of equation (6.4.26) so we have

$$\alpha_2 = -3\int_0^{2\pi}\int_0^{\pi} Y_3^{-3}Y_2^{-2}Y_2^2\overline{Y}_3^{-3}\sin\theta\, d\theta d\phi = -\frac{25}{22\pi}.$$

- The coefficient of $y_1x_{-2}^2$ in $\dot{y}_{-3}$ of the general equivariant vector field is $2\sqrt{15}\delta_2$. The term $y_1x_{-2}^2$ occurs 3 times in the cubed sum of equation (6.4.26) so we have

$$\delta_2 = -\frac{\sqrt{15}}{10}\int_0^{2\pi}\int_0^{\pi} Y_3^1(Y_2^{-2})^2\overline{Y}_3^{-3}\sin\theta\, d\theta d\phi = -\frac{3}{44\pi}.$$

- The coefficient of $y_{-2}x_{-1}x_0$ in $\dot{y}_{-3}$ of the general equivariant vector field is $-5\delta_3$. The term $y_{-2}x_{-1}x_0$ occurs 6 times in the cubed sum of equation (6.4.26) so we have

$$\delta_3 = \frac{6}{5}\int_0^{2\pi}\int_0^{\pi} Y_3^{-2}Y_2^{-1}Y_2^0\overline{Y}_3^{-3}\sin\theta\, d\theta d\phi = -\frac{1}{22\pi}.$$

It is now possible to use these values of the coefficients in the general form of the equivariant vector field (6.3.22)–(6.3.23) to study the stability and bifurcations of the various branches of solutions of the Swift–Hohenberg equation. We will do this in Chapter 7.

Example: The coefficients for the representation on $V_3 \oplus V_4$

In Section 6.3.6 we computed that the general form of the $\mathbf{O}(3)\times\mathbb{Z}_2$ equivariant vector field for the representation on $V_3 \oplus V_4$ is given by (6.3.24)–(6.3.25). We now compute the values of the coefficients in this vector field for the specific example of the Swift–Hohenberg equation.

When $\ell = 3$ we can see from (6.4.22) and (6.4.23) that

$$\dot{x}_p = \left(\mu_2 - \frac{3}{16}R_2\right)x_p - \int_0^{2\pi}\int_0^{\pi}\left(\sum_{m=-3}^{3} x_m Y_3^m + \sum_{n=-4}^{4} y_n Y_4^n\right)^3 \overline{Y}_3^p \sin\theta\, d\theta d\phi \qquad (6.4.27)$$

$$\dot{y}_p = \left(\mu_2 + \frac{5}{16}R_2\right)y_p - \int_0^{2\pi}\int_0^{\pi}\left(\sum_{m=-3}^{3} x_m Y_3^m + \sum_{n=-4}^{4} y_n Y_4^n\right)^3 \overline{Y}_4^p \sin\theta\, d\theta d\phi. \qquad (6.4.28)$$

Comparing these equations and the equivariant vector field for this representation (6.3.24)–(6.3.25) we can see immediately that $\mu_x = \mu_2 - \frac{3}{16}R_2$ and $\mu_y = \mu_2 + \frac{5}{16}R_2$. By comparing the cubic terms in the general form of the equivariant vector field with equations (6.4.27) and (6.4.28) we can compute the values of the coefficients α_1, α_2, β_1, β_2, γ_1, γ_2, γ_3, γ_4, δ_1, δ_2, δ_3 and δ_4 as follows:

- The coefficient of $x_{-3}x_{-2}x_2$ in $\dot{x}_{-3}$ of the general equivariant vector field is $2\alpha_1$. The term $x_{-3}x_{-2}x_2$ occurs 6 times in the cubed sum of equation (6.4.27) so we have

$$\alpha_1 = -3\int_0^{2\pi}\int_0^{\pi} Y_3^{-3}Y_3^{-2}Y_3^2\overline{Y}_3^{-3}\sin\theta\, d\theta d\phi = -\frac{175}{286\pi}.$$

- The coefficient of $x_{-2}x_{-1}x_0$ in $\dot{x}_{-3}$ of the general equivariant vector field is $30\sqrt{2}\gamma_1$. The term $x_{-2}x_{-1}x_0$ occurs 6 times in the cubed sum of equation (6.4.27) so we have

$$\gamma_1 = -\frac{6}{30\sqrt{2}}\int_0^{2\pi}\int_0^{\pi} Y_3^{-2}Y_3^{-1}Y_3^0\overline{Y}_3^{-3}\sin\theta\, d\theta d\phi = -\frac{7}{2860\pi}.$$

- The coefficient of $x_{-3}y_{-4}y_4$ in $\dot{A}x_{-3}$ of the general equivariant vector field is $2\beta_1$. The term $x_{-3}y_{-4}y_4$ occurs 6 times in the cubed sum of equation (6.4.27) so we have

$$\beta_1 \quad = \quad -3\int_0^{2\pi}\int_0^{\pi} Y_3^{-3}Y_4^{-4}Y_4^{4}\overline{Y_3^{-3}}\sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = -\frac{735}{572\pi}.$$

- The coefficient of $x_1y_{-4}y_0$ in $\dot{x}_{-3}$ of the general equivariant vector field is $20\sqrt{42}\gamma_2$. The term $x_1y_{-4}y_0$ occurs 6 times in the cubed sum of equation (6.4.27) so we have

$$\gamma_2 \quad = \quad -\frac{6}{20\sqrt{42}}\int_0^{2\pi}\int_0^{\pi} Y_3^{1}Y_4^{-4}Y_4^{0}\overline{Y_3^{-3}}\sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = -\frac{21}{2860\pi}.$$

- The coefficient of $x_2y_{-3}y_{-2}$ in $\dot{x}_{-3}$ of the general equivariant vector field is $3\sqrt{21}\gamma_3$. The term $x_2y_{-3}y_{-2}$ occurs 6 times in the cubed sum of equation (6.4.27) so we have

$$\gamma_3 \quad = \quad -\frac{6}{3\sqrt{21}}\int_0^{2\pi}\int_0^{\pi} Y_3^{3}Y_4^{-3}Y_4^{-2}\overline{Y_3^{-3}}\sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = -\frac{5}{143\pi}.$$

- The coefficient of $x_{-1}y_{-2}y_0$ in $\dot{x}_{-3}$ of the general equivariant vector field is $30\sqrt{6}\gamma_4$. The term $x_{-1}y_{-2}y_0$ occurs 6 times in the cubed sum of equation (6.4.27) so we have

$$\gamma_4 \quad = \quad -\frac{6}{30\sqrt{6}}\int_0^{2\pi}\int_0^{\pi} Y_3^{-1}Y_4^{-2}Y_4^{0}\overline{Y_3^{-3}}\sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = -\frac{7}{2860\pi}.$$

- The coefficient of $y_{-4}^2y_4$ in $\dot{y}_{-4}$ of the general equivariant vector field is $2\alpha_2$. The term $y_{-4}^2y_4$ occurs 3 times in the cubed sum of equation (6.4.28) so we have

$$\alpha_2 \quad = \quad -\frac{3}{2}\int_0^{2\pi}\int_0^{\pi} (Y_4^{-4})^2Y_4^{4}\overline{Y_4^{-4}}\sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = -\frac{6615}{9724\pi}.$$

- The coefficient of $y_{-3}^2y_2$ in $\dot{y}_{-4}$ of the general equivariant vector field is $70\sqrt{7}\delta_1$. The term $y_{-3}^2y_2$ occurs 3 times in the cubed sum of equation (6.4.28) so we have

$$\delta_1 \quad = \quad -\frac{3}{70\sqrt{7}}\int_0^{2\pi}\int_0^{\pi} (Y_4^{-3})^2Y_4^{2}\overline{Y_4^{-4}}\sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = -\frac{27}{34034\pi}.$$

- The coefficient of $y_{-4}x_{-2}x_2$ in $\dot{y}_{-4}$ of the general equivariant vector field is $2\beta_2$. The term $y_{-4}x_{-2}x_2$ occurs 6 times in the cubed sum of equation (6.4.28) so we have

$$\beta_2 \quad = \quad -3\int_0^{2\pi}\int_0^{\pi} Y_4^{-4}Y_3^{-2}Y_3^{2}\overline{Y_4^{-4}}\sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = -\frac{315}{572\pi}.$$

- The coefficient of $y_{-3}x_{-3}x_2$ in $\dot{y}_{-4}$ of the general equivariant vector field is $70\sqrt{3}\delta_2$. The term $y_{-3}x_{-3}x_2$ occurs 6 times in the cubed sum of equation (6.4.28) so we have

$$\delta_2 \quad = \quad -\frac{6}{70\sqrt{3}}\int_0^{2\pi}\int_0^{\pi} Y_4^{-3}Y_3^{-3}Y_3^{2}\overline{Y_4^{-4}}\sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = -\frac{3}{286\pi}.$$

- The coefficient of $y_2x_{-3}^2$ in $\dot{y}_{-4}$ of the general equivariant vector field is $10\sqrt{7}\delta_3$. The term $y_2x_{-3}^2$ occurs 3 times in the cubed sum of equation (6.4.28) so we have

$$\delta_3 \quad = \quad -\frac{3}{10\sqrt{7}}\int_0^{2\pi}\int_0^{\pi} Y_4^{2}(Y_3^{-3})^2\overline{Y_4^{-4}}\sin\theta \,\mathrm{d}\theta\mathrm{d}\phi = \frac{3}{143\pi}.$$

- The coefficient of $y_{-3}x_{-2}x_1$ in $\dot{y}_{-4}$ of the general equivariant vector field is $14\sqrt{5}\delta_4$. The term $y_{-3}x_{-2}x_1$ occurs 6 times in the cubed sum of equation (6.4.28) so we have

$$\delta_4 \quad = \quad -\frac{6}{14\sqrt{5}}\int_0^{2\pi}\int_0^{\pi} Y_4^{-3} Y_3^{-2} Y_3^{1} \overline{Y}_4^{-4} \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi = 0.$$

It is now possible to use these values of the coefficients in the general form of the equivariant vector field (6.3.24)–(6.3.25) to study the stability and bifurcations of the various branches of solutions of the Swift–Hohenberg equation. We will do this in Chapter 7.

CHAPTER 7

SYMMETRIC SPIRAL PATTERNS ON SPHERES

7.1 Introduction

In Section 1.2.2 we discussed the fact that spiral patterns (both rotating spiral waves and stationary spirals) have been found in numerical simulations of reaction–diffusion systems, Rayleigh–Bénard convection and other pattern forming systems on the sphere. In addition to spiral waves with trivial isotropy (no symmetry), spirals with certain symmetries can exist on the sphere. The fact that the one-armed spiral patterns found by Calhoun et al. [16], Li et al. [62], Matthews [65] and Zhang et al. [90] all have a rotation-through-π symmetry in some axis in the equatorial plane has, until now, not been noted or utilised. The spiral pattern found by Calhoun et al. [16] is given in Figure 7.1.

In this chapter we will study the generic existence of spiral patterns with symmetry on spheres using equivariant bifurcation theory methods. We will investigate the spiral patterns which can occur as a result of stationary bifurcations on a sphere and subsequent secondary bifurcations. We also consider the stability of these spiral patterns.

7.1.1 Stationary spiral patterns on spheres

In this chapter we will be considering stationary spiral patterns on spheres which have the form of those given in Figure 7.2. These spiral patterns are functions on the sphere, $w(\theta, \phi, t)$, where the areas on which $w > 0$ and $w < 0$ form intertwined spirals. The contours along which $w = 0$ are Archimedean spherical spirals which originate at a single point on the surface of the sphere and terminate at the antipodal point.

We say that the spiral is m-armed if at the tips, or point of origin, there are m areas where $w > 0$. This means that for an m-armed spiral pattern there are $2m$ zero contour Archimedean spirals. The top row of images in Figure 7.2 shows one, two and three armed spirals, looking directly at the point of origin, and the bottom row shows the same patterns from the side. The red areas

Figure 7.1: The one-armed spiral pattern found in numerical simulations of a reaction-diffusion system by Calhoun et al. [16].

show where $w > 0$ and blue areas show where $w < 0$. We make no distinction between patterns which spiral clockwise or anticlockwise from the north pole since the symmetries of the pattern are the same in either case.

Symmetries of spiral patterns on spheres

Consider the spiral patterns with one, two and three arms as in Figure 7.2. We can describe the symmetries of these patterns and also those with larger numbers of arms in terms of rotations and reflections of the sphere. Each of the spirals has a rotation-through-π symmetry, R^e_π, in an axis in the plane of the equator as viewed in the images in the top row of Figure 7.2. In addition, the m-armed spiral has rotation-through-$2\pi/m$ symmetry, $R^t_{2\pi/m}$ in the axis through the spiral tips. Thus the symmetry group of a one-armed spiral pattern is $\mathbb{Z}_2$ and an m-armed spiral for $m \geq 2$ has symmetry group $\mathbf{D}_m$ as a subgroup of $\mathbf{O(3)}$. Furthermore, for any choice of generators, the group of symmetries of a one-armed spiral is contained in a copy of the symmetry group of an m-armed spiral for any value of m.

Notice that inversion in the origin, $-I \in \mathbf{O(3)}$ does not act as the identity or minus the identity on any spiral pattern. If a pattern, $w(\theta, \phi, t)$, can be made with a linear combination of spherical harmonics of even degree then, since $-I$ acts as the identity on all spherical harmonics of even degree, $-I$ must act as the identity on $w(\theta, \phi, t)$. Similarly if $w(\theta, \phi, t)$ can be made with a linear combination of spherical harmonics of odd degree then, since $-I$ acts as minus the identity on all spherical harmonic of odd degree, $-I$ must act as minus the identity on $w(\theta, \phi, t)$. However, $-I$ acts as neither plus nor minus the identity on spiral patterns so we conclude that spiral patterns can only be made through linear combinations of spherical harmonics of odd and even degrees.

Indeed, we find that spiral patterns such as those in Figure 7.2 can be made with linear combi-

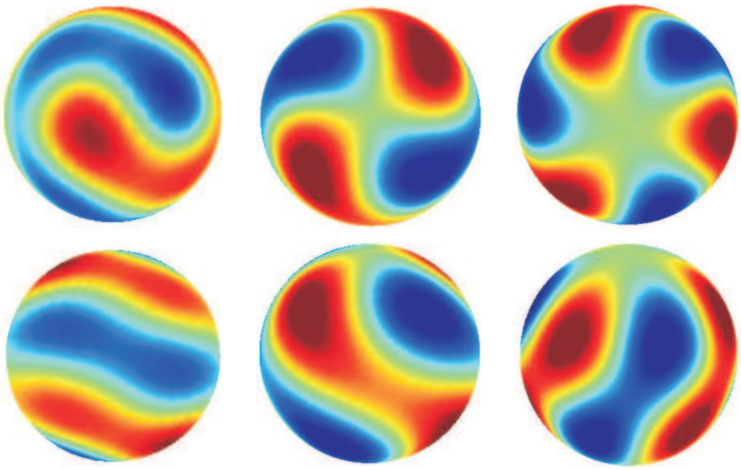


Figure 7.2: Images of one-, two- and three-armed spirals. The top row shows the pattern looking directly at the tips and the bottom row of images shows the patterns viewed from the side. The red areas show where the functions are positive and blue areas show where they are negative.

nations of spherical harmonics of degrees ℓ and $\ell+1$ and hence are patterns of the form

$$w(\theta,\phi,t) = \sum_{m=-\ell}^{\ell} x_m(t)Y_\ell^m(\theta,\phi) + \sum_{n=-\ell-1}^{\ell+1} y_n(t)Y_{\ell+1}^n(\theta,\phi)$$

where $x_{-m} = (-1)^m\overline{x}_m$ and $y_{-n} = (-1)^n\overline{y}_n$ since $w(\theta,\phi,t)$ must be real. Hence, for it to be possible for spiral patterns to exist as a result of a stationary bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry, the representation of $\mathbf{O}(\mathbf{3})$ must be a reducible representation on $V_\ell \oplus V_{\ell+1}$.

One of the conditions which must be satisfied in order for the m-armed spirals described above to exist as a result of a stationary bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry in a reducible representation on $V_\ell \oplus V_{\ell+1}$ is that its symmetry group ($\mathbb{Z}_2$ in the case of a one-armed spiral and $\mathbf{D}_m$ for an m-armed spiral where $m \geq 2$) must be an isotropy subgroup of $\mathbf{O}(\mathbf{3})$ in this reducible representation. Recall that in Section 6.2.2 we saw that any pattern which is a combination of spherical harmonics of degrees ℓ and $\ell+1$ cannot have an isotropy subgroup which is axial. Thus $\mathbb{Z}_2$ and $\mathbf{D}_m$ are never axial isotropy subgroups. However, they may be isotropy subgroups which fix a subspace of $V_\ell \oplus V_{\ell+1}$ of dimension greater than one.

This means that the existence of spiral patterns at a stationary bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry is never guaranteed by the equivariant branching lemma. To determine whether spiral patterns can exist at such a bifurcation in the representation on $V_\ell \oplus V_{\ell+1}$ for a particular value of ℓ we must compute the $\mathbf{O}(\mathbf{3})$ equivariant vector field for this representation (which is $4\ell+4$ dimensional) and find solutions with the relevant symmetries directly. Since $\mathbb{Z}_2 \subset \mathbf{D}_m$ for all values of m, if any spiral solutions with symmetry $\mathbb{Z}_2$ or $\mathbf{D}_m$ exist then they can be found in the restriction of the $\mathbf{O}(\mathbf{3})$ equivariant vector field to the $2\ell+2$ dimensional subspace $\mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(\mathbb{Z}_2)$. Even for low values of ℓ this vector space is large, so to find one-armed spirals with $\mathbb{Z}_2$ symmetry in this space is not a simple task.

We should also note that since

$$\begin{aligned} \operatorname{rank}(N(\mathbb{Z}_2)/\mathbb{Z}_2) &= \operatorname{rank}(\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c/\mathbb{Z}_2) = 1 &\quad (7.1.1)\\ \operatorname{rank}(N(\mathbf{D}_m)/\mathbf{D}_m) &= \operatorname{rank}(\mathbf{D}_{2m}/\mathbf{D}_m) = 0, &\quad (7.1.2)\end{aligned}$$

by Theorem 2.4.9, generically solutions with $\mathbf{D}_m$ which exist in $\mathbf{O}(\mathbf{3})$ equivariant vector fields are stationary, whereas solutions with $\mathbb{Z}_2$ symmetry are generically relative equilibria with one period i.e. they rotate. This only makes it more difficult to establish the existence of one-armed spiral patterns with such symmetry.

The most symmetric spiral patterns on spheres

Since finding one-armed spirals with $\mathbb{Z}_2$ symmetry requires us to solve a large number of equations we make the following simplification which forms the basis for most of the work in this chapter. Rather than look for the spiral patterns with symmetries contained in $\mathbf{O}(\mathbf{3})$, we consider the most symmetric spiral patterns on spheres, which, in addition to the symmetries contained in $\mathbf{O}(\mathbf{3})$, have the symmetry,

$$\left(R^t_{\pi/m}, -1\right) \in \mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2. \quad (7.1.3)$$

Here, $R^t_{\pi/m}$ is a rotation through π/m in the axis through the spiral tips and -1 is the non-identity element in $\mathbb{Z}_2$ which acts (in all representations) as multiplication by -1 thus sending red areas to blue and vice versa in the images given in Figure 7.2. This means that the spiral pattern is such that the areas where $w(\theta, \phi, t) > 0$ and $w(\theta, \phi, t) < 0$ are of identical size and shape. Such spiral patterns have symmetry groups which are subgroups of the larger group $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$.

Recall that in Chapter 6 we studied the twisted subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$. With the additional symmetry (7.1.3) the symmetries of a one-armed spiral form the twisted subgroup

$$\widetilde{\mathbf{D}_2} = (\mathbf{D}_2, \mathbb{Z}_2) = \left\langle (R^e_\pi, 1), (R^t_\pi, -1) \right\rangle, \quad (7.1.4)$$

and the symmetries of an m-armed spiral for $m \geq 2$ form the twisted subgroup

$$\widetilde{\mathbf{D}_{2m}} = (\mathbf{D}_{2m}, \mathbf{D}_m) = \left\langle (R^e_\pi, 1), (R^t_{\pi/m}, -1) \right\rangle. \quad (7.1.5)$$

Using the methods given in Chapter 6 (the information in Table 6.7 and the 'massive chain criterion', Theorem 3.4.1) we can determine when these twisted subgroups are isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$.

Proposition 7.1.1. *The subgroup $\widetilde{\mathbf{D}_{2m}}$ is an isotropy subgroup of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ in the representation on $V_\ell \oplus V_{\ell+1}$ when $\ell \geq m$ for $m \geq 1$.*

Proof. By the massive chain criterion, Theorem 3.4.1, $\widetilde{\mathbf{D}_{2m}}$ is an isotropy subgroup iff for each strictly larger and adjacent group Δ,

$$\dim \operatorname{Fix}_{V_\ell \oplus V_{\ell+1}}(\Delta) - r(\Delta) < \dim \operatorname{Fix}_{V_\ell \oplus V_{\ell+1}}(\widetilde{\mathbf{D}_{2m}}) - r(\widetilde{\mathbf{D}_{2m}}). \quad (7.1.6)$$

Notice that in reducible representations on $V_\ell \oplus V_{\ell+1}$, $r(\Sigma) = q(\Sigma)$ for all twisted subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$. For each twisted subgroup Σ, the values of $\dim \mathrm{Fix}(\Sigma)$ and $q(\Sigma)$ are given in Table 6.7. We must show that (7.1.6) holds for all strictly larger and adjacent groups Δ when $\ell \geq m$.

Case 1: $m = 1$**.** The twisted subgroup $\widetilde{\mathbf{D}_2}$ given by the pair $(H, K) = (\mathbf{D}_2, \mathbb{Z}_2)$ has

$$\dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(\widetilde{\mathbf{D}_2}) = \ell + 1 \qquad \text{and} \qquad q(\widetilde{\mathbf{D}_2}) = 0.$$

It is contained in strictly larger groups given by pairs of types

$$(H,K) = \begin{array}{llll} (\mathbf{D}_{2m}, \mathbf{D}_m) & (\mathbf{D}_{2m}, \mathbb{Z}_{2m}) & (\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbb{Z}_{2m} \times \mathbb{Z}_2^c) & (\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_{2m}^z) \\ (\mathbf{D}_{4m}^d, \mathbf{D}_{2m}^z) & (\mathbf{D}_{4m}^d, \mathbb{Z}_{4m}^-) & (\mathbf{D}_{4m}^d, \mathbf{D}_{2m}). & \end{array}$$

All of the twisted subgroups, H^θ, given by these pairs have $q(H^\theta) = 0$, so we need only show that for the pairs which give twisted subgroups adjacent to $\widetilde{\mathbf{D}_2}$, $\dim \mathrm{Fix}(H^\theta) < \ell + 1$ when $\ell \geq 1$ to demonstrate that $\widetilde{\mathbf{D}_2}$ is an isotropy subgroup. From Table 6.7, we see that for each of the pairs listed above, $\dim \mathrm{Fix}(H^\theta)$ is an decreasing function of m. In other words, the larger the group H^θ, the smaller the value of $\dim \mathrm{Fix}(H^\theta)$. Thus, for each pair above, we need only consider the smallest value of m which gives a twisted subgroup which is strictly larger and adjacent to $\widetilde{\mathbf{D}_2}$. Then we are left with the pairs

$$(H,K) = \begin{array}{llll} (\mathbf{D}_4, \mathbf{D}_2) & (\mathbf{D}_4, \mathbb{Z}_4) & (\mathbf{D}_2 \times \mathbb{Z}_2^c, \mathbb{Z}_2 \times \mathbb{Z}_2^c) & (\mathbf{D}_2 \times \mathbb{Z}_2^c, \mathbf{D}_2^z) \\ (\mathbf{D}_4^d, \mathbf{D}_2^z) & (\mathbf{D}_4^d, \mathbb{Z}_4^-) & (\mathbf{D}_4^d, \mathbf{D}_2). & \end{array}$$

Each of these pairs has $\dim \mathrm{Fix}(H^\theta) < \ell + 1$ when $\ell \geq 1$. Hence $\widetilde{\mathbf{D}_2}$ is an isotropy subgroup when $\ell \geq 1$.

Case 2: $m \geq 2$**.** The twisted subgroup $\widetilde{\mathbf{D}_{2m}}$ where $m \geq 2$ is given by the pair $(H, K) = (\mathbf{D}_{2m}, \mathbf{D}_m)$. It has

$$\dim \mathrm{Fix}_{V_\ell \oplus V_{\ell+1}}(\widetilde{\mathbf{D}_{2m}}) = [(\ell + m)/2m] + [(\ell + m + 1)/2m] \qquad \text{and} \qquad q(\widetilde{\mathbf{D}_2}) = 0.$$

It is contained in the strictly larger and adjacent groups given by the pairs

$$(H,K) = \begin{cases} (\mathbf{D}_{2pm}, \mathbf{D}_{pm}) & \text{for } p \geq 3 \text{ and prime} \\ (\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_m \times \mathbb{Z}_2^c) & \\ (\mathbf{D}_{2m} \times \mathbb{Z}_2^c, \mathbf{D}_{2m}^d) & \end{cases}$$

Using Table 6.7 we see that all of the twisted subgroups, H^θ, given by these pairs have $q(H^\theta) = 0$ and $\dim \mathrm{Fix}(H^\theta) < \dim \mathrm{Fix}(\widetilde{\mathbf{D}_{2m}})$ when $\ell \geq m$. Hence $\widetilde{\mathbf{D}_{2m}}$ is an isotropy subgroup when $\ell \geq m$.

Having considered all possible cases, this concludes the proof. $\square$

We have shown that the twisted subgroups $\widetilde{\mathbf{D}_{2m}}$ can be isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in some representations on $V_\ell \oplus V_{\ell+1}$. Recall from Section 6.2.2 that these isotropy subgroups are never axial so solutions with these symmetries are never guaranteed to exist at a stationary bifurcation with $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry by the equivariant branching lemma. We must look for these solutions in the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field.

Notice that the symmetry group of a one-armed spiral, $\widetilde{\mathbf{D}_2}$, is still contained in that of an m-armed spiral, $\widetilde{\mathbf{D}_{2m}}$, for any value of $m \geq 2$ when the additional symmetry (7.1.3) is included. When m is odd this containment is obvious, however when m is even it is less so. Suppose that we have an m-armed spiral, for m even, with tips at the poles. This has a symmetry group which contains the symmetries of a one-armed spiral with tips on the equator. Since $\widetilde{\mathbf{D}_2} \subset \widetilde{\mathbf{D}_{2m}}$ for all values of m, all m-armed spiral patterns which exist for any given value of ℓ can be found in the restriction of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field to the subspace $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$. This space is $\ell + 1$ dimensional. By looking for spiral solutions with the additional symmetry (7.1.3) we have halved the number of equations which we must solve in order to find such solutions.

Another benefit of considering spiral patterns with additional symmetry (7.1.3) is that since

$$\mathrm{rank}(N_{\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2}(\widetilde{\mathbf{D}_{2m}}) / \widetilde{\mathbf{D}_{2m}}) = 0, \tag{7.1.7}$$

for all values of m, by Theorem 2.4.9, all spiral patterns with $\widetilde{\mathbf{D}_{2m}}$ symmetry are (generically) stationary patterns and are therefore easier to find as equilibria of the $\ell + 1$ equations in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$.

In Section 7.2 we look for spiral patterns with $\widetilde{\mathbf{D}_{2m}}$ symmetry in the restriction to $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field on $V_\ell \oplus V_{\ell+1}$ when $\ell = 2$ and $\ell = 3$. For the representation on $V_2 \oplus V_3$, where $\widetilde{\mathbf{D}_{2m}}$ is an isotropy subgroup for $m = 1$ and $m = 2$, we find conditions on the values of the coefficients in the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field for one- and two-armed spiral patterns to exist and consider the specific case of the set of coefficients which occur for the Swift–Hohenberg equation which were computed in Section 6.4. We also consider the one-, two- and three-armed spiral patterns which can exist in the Swift–Hohenberg equation for the representation on $V_3 \oplus V_4$.

In Section 7.3 we consider the effect of breaking the symmetry from $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ to $\mathbf{O}(\mathbf{3})$ on the spiral pattern solutions with $\widetilde{\mathbf{D}_{2m}}$ symmetry i.e. breaking the symmetry (7.1.3). We investigate whether the spiral patterns found in Section 7.2 can persist under this forced symmetry breaking.

7.2 Spiral patterns with symmetries contained in $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$

In this section we look for solutions with symmetry $\widetilde{\mathbf{D}_{2m}}$ in $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector fields for representations on $V_\ell \oplus V_{\ell+1}$ for small values of ℓ. Recall that $\widetilde{\mathbf{D}_{2m}}$ is the symmetry group of the most symmetric m-armed spiral pattern on a sphere. We show that it is possible for such patterns to exist and demonstrate how they bifurcate from other solutions.

Consider the system of ODEs

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = f(\mathbf{z}, \lambda), \tag{7.2.1}$$

where $\lambda \in \mathbb{R}$ is a bifurcation parameter and the mapping $f : V_\ell \oplus V_{\ell+1} \times \mathbb{R} \to V_\ell \oplus V_{\ell+1}$ commutes with the action of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ on $V_\ell \oplus V_{\ell+1}$. Suppose that $\mathbf{z} = (\mathbf{x}; \mathbf{y})$ where $\mathbf{x} \in V_\ell$

and $\mathbf{y} \in V_{\ell+1}$. We saw in Section 6.3.4 that the mapping $f(\mathbf{z}, \lambda)$ is of the form

$$f(\mathbf{z}, \lambda) = (g(\mathbf{z}, \lambda); h(\mathbf{z}, \lambda))^{\mathbf{T}}$$

and the two linear equivariant mappings are $\mu_x(\lambda)(\mathbf{x}; \mathbf{0})$ and $\mu_y(\lambda)(\mathbf{0}; \mathbf{y})$ where $\mu_x(\lambda)$ and $\mu_y(\lambda)$ are real valued functions of λ.

As in Chapter 6, since f is equivariant with respect to $-1 \in \mathbb{Z}_2$, f is odd in $\mathbf{z}$ and hence (7.2.1) has a trivial equilibrium $f(\mathbf{0}, \lambda) = \mathbf{0}$ for all values of λ. This undergoes stationary bifurcations when $\mu_x(\lambda) = 0$ and $\mu_y(\lambda) = 0$. At these stationary bifurcations, branches of equilibrium solutions with certain symmetries are guaranteed to be created. These solution branches have the symmetries of the axial isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in the representation on $V_\ell \oplus V_{\ell+1}$ by the equivariant branching lemma. These isotropy subgroups were computed in Section 6.2 for all values of ℓ. Subsequent bifurcations of these solution branches can lead to solution branches with the symmetries of an isotropy subgroup of $\mathbf{O(3)} \times \mathbb{Z}_2$ which fixes a subspace of $V_\ell \oplus V_{\ell+1}$ of dimension larger than one. We are particularly interested in the existence of such submaximal solutions with $\widetilde{\mathbf{D}_{2m}}$ symmetry and how they can bifurcate from other solution branches.

Remark 7.2.1. Recall from Section 2.4.3 that it is possible for secondary steady-state bifurcations from group orbits or equilibria to lead to relative equilibria as well as new equilibria. By Theorem 2.4.9, the number of frequencies of a relative equilibrium $(\mathbf{O(3)} \times \mathbb{Z}_2)\, \mathbf{z}_0$ with isotropy subgroup $\widetilde{\mathbf{D}_{2m}}$, (the symmetries of an m-armed spiral) is

$$k = \operatorname{rank}\left(N_{\mathbf{O(3)}\times\mathbb{Z}_2}(\widetilde{\mathbf{D}_{2m}})/\widetilde{\mathbf{D}_{2m}}\right) = \operatorname{rank}\left(\mathbf{D}_{4m} \times \mathbb{Z}_2^c \times \mathbb{Z}_2^c/\widetilde{\mathbf{D}_{2m}}\right) = 0 \qquad \text{for all } m \geq 1,$$

and hence all spiral solutions resulting from secondary stationary bifurcations after a stationary bifurcation with $\mathbf{O(3)} \times \mathbb{Z}_2$ symmetry are equilibria with zero frequencies (i.e. stationary patterns).

We now look for stationary solutions with $\widetilde{\mathbf{D}_{2m}}$ symmetry in the restriction to $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ of the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector fields on $V_2 \oplus V_3$ and $V_3 \oplus V_4$. We also consider the stability of such patterns when the coefficients in the vector field are those we computed in Section 6.4 for the Swift–Hohenberg equation.

7.2.1 Spiral patterns in the representation on $V_2 \oplus V_3$

Recall that in Example 6.2.10 we computed all isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in the representation on $V_2 \oplus V_3$. Among these isotropy subgroups were $\widetilde{\mathbf{D}_2}$ and $\widetilde{\mathbf{D}_4}$, the symmetry groups of one- and two-armed spiral respectively, which fix three- and two- dimensional subspaces of $V_2 \oplus V_3$ respectively. By Proposition 7.1.1 these are the only isotropy subgroups of the form $\widetilde{\mathbf{D}_{2m}}$ in this representation. We now show that it is possible for stationary solutions with these symmetries to exist in the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field on $V_2 \oplus V_3$.

Since $\widetilde{\mathbf{D}_2} \subset \widetilde{\mathbf{D}_4}$ we must have that $\mathrm{Fix}(\widetilde{\mathbf{D}_4}) \subset \mathrm{Fix}(\widetilde{\mathbf{D}_2})$, i.e. for every choice of generators of $\widetilde{\mathbf{D}_2}$, $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ contains a copy of $\mathrm{Fix}(\widetilde{\mathbf{D}_4})$.

Suppose that we choose the copy of $\widetilde{\mathbf{D}_2}$ which is given by

$$\widetilde{\mathbf{D}_2} = \langle\ (R^y_\pi, -1),\ (R^z_\pi, 1)\ \rangle. \tag{7.2.2}$$

The one-armed spiral with this symmetry spirals between the two points on the surface of the sphere on the y-axis. This copy of $\widetilde{\mathbf{D}_2}$ is contained within the copy of $\widetilde{\mathbf{D}_4}$ which is given by

$$\widetilde{\mathbf{D}_4} = \langle\ (R^z_{\pi/2}, -1),\ (R^{x=-y}_\pi, 1)\ \rangle$$

where $R^{x=-y}_\pi$ is the rotation through π in the line $x = -y, z = 0$ which sends $x \to -y, y \to -x$ and $z \to -z$. The two-armed spiral with this symmetry group is oriented as in the top row of Figure 7.2, spiralling from the north to south poles.

For these choices of generators the fixed-point subspaces are

$$\begin{aligned} \mathrm{Fix}(\widetilde{\mathbf{D}_2}) &= \{(\mathrm{i}a, 0, 0, 0, -\mathrm{i}a\ ;\ 0, b, 0, c, 0, b, 0)\} &(7.2.3)\\ \mathrm{Fix}(\widetilde{\mathbf{D}_4}) &= \{(\mathrm{i}a, 0, 0, 0, -\mathrm{i}a\ ;\ 0, b, 0, 0, 0, b, 0)\} &(7.2.4)\end{aligned}$$

where a, b, $c \in \mathbb{R}$. Notice that $\mathrm{Fix}(\widetilde{\mathbf{D}_4}) \subset \mathrm{Fix}(\widetilde{\mathbf{D}_2})$. Hence both one- and two-armed spirals (if they exist) can be found in the restriction of the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field to $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ as above. We computed this vector field to cubic order in Section 6.3.5 where we found that it is as in (6.3.22)–(6.3.23). The restriction to $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ is given by

$$\begin{aligned} \dot{a} &= \mu_x a + 2\alpha_1 a^3 + (2\beta_1 + 8\gamma_1 - 6\gamma_2)\, ab^2 + (\beta_1 + \gamma_2)\, ac^2 &(7.2.5)\\ \dot{b} &= \mu_y b + (2\beta_2 - 50\delta_1)\, b^3 + (2\alpha_2 - 30\delta_2 + 10\delta_3)\, ba^2 + (\beta_2 + 15\delta_1)\, bc^2 &(7.2.6)\\ \dot{c} &= \mu_y c + (\beta_2 - 9\delta_1)\, c^3 + (2\alpha_2 - 30\delta_2 + 18\delta_3)\, ca^2 + (2\beta_2 + 30\delta_1)\, cb^2 &(7.2.7)\end{aligned}$$

where μ_x, μ_y, α_1, β_1, γ_1, γ_2, α_2, β_2, δ_1, δ_2 and δ_3 are real functions of λ. These equations have residual symmetry

$$N(\widetilde{\mathbf{D}_2})/\widetilde{\mathbf{D}_2} = \mathbf{D}_4 \times \mathbb{Z}_2^c \times \mathbb{Z}_2/\widetilde{\mathbf{D}_2} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

so if solutions with $\widetilde{\mathbf{D}_2}$ symmetry exist then there are $|N(\widetilde{\mathbf{D}_2})/\widetilde{\mathbf{D}_2}| = 8$ equivalent solutions within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$.

The trivial solution $\mathbf{z} = \mathbf{0}$ undergoes stationary bifurcations when $\mu_x = 0$ and $\mu_y = 0$. To investigate the interactions between the $\ell = 2$ and $\ell = 3$ modes, we assume that $\mu_x = \lambda$ and $\mu_y = \lambda + \rho$. Then the trivial solution is stable when $\lambda < \min(0, -\rho)$. At $\lambda = 0$ the $\ell = 2$ modes become unstable and the equivariant branching lemma guarantees that the unrestricted system (6.3.22)–(6.3.23) has stationary solution branches with the symmetries of axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ in the representation on V_2 which bifurcate at $\lambda = 0$. Similarly, at $\lambda = -\rho$ the $\ell = 3$ modes become unstable and give rise to solution branches with the symmetries of axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ in the representation on V_3.

Branches of solutions with the symmetries of the axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ in the representation on $V_2 \oplus V_3$ which contain $\widetilde{\mathbf{D}_2}$ are guaranteed to exist in (7.2.5)–(7.2.7). The other equilibrium solutions which it may be possible to find in these equations have the symmetries of the isotropy subgroups which contain $\widetilde{\mathbf{D}_2}$ and fix a subspace of dimension greater than one.

All isotropy subgroups containing $\widetilde{\mathbf{D}_2}$ are listed in Table 7.1 along with the generators of a copy of the group which contains the copy of $\widetilde{\mathbf{D}_2}$ given by (7.2.2). The fixed-point subspace which lies inside $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ given by (7.2.3) is also listed for each isotropy subgroup. These isotropy subgroups were computed in Example 6.2.10.

Isotropy subgroup	Generators	Fixed-point subspace
$\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$	$(R^z_{\pi/2}, -1)$, $(R^{x=-y}_\pi, 1)$, $(-I, 1)$	$\{(\mathrm{i}a, 0, 0, 0, -\mathrm{i}a\,;\, 0, 0, 0, 0, 0, 0, 0)\}$
$\widetilde{\mathbf{O(2)} \times \mathbb{Z}_2^c}$	$(R^z_\theta, 1)$, $(\kappa_{xz}, 1)$, $(-I, -1)$	$\{(0, 0, 0, 0, 0\,;\, 0, 0, 0, c, 0, 0, 0)\}$
$\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$	$(-R^y_{\pi/3}, 1)$, $(R^z_\pi, 1)$, $(-I, -1)$	$\left\{\left(0, 0, 0, 0, 0\,;\, 0, \sqrt{\frac{3}{10}}c, 0, c, 0, \sqrt{\frac{3}{10}}c, 0\right)\right\}$
$\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$	$(-R^z_{\pi/2}, 1)$, $(\mathcal{R}_{2\pi/3}, 1)$, $(\kappa_{xz}, 1)$, $(-I, -1)$	$\{(0, 0, 0, 0, 0\,;\, 0, b, 0, 0, 0, b, 0)\}$
$\widetilde{\mathbf{D}_4}$	$(R^z_{\pi/2}, -1)$, $(R^{x=-y}_\pi, 1)$	$\{(\mathrm{i}a, 0, 0, 0, -\mathrm{i}a\,;\, 0, b, 0, 0, 0, b, 0)\}$
$\widetilde{\mathbf{D}_4^d}$	$(-R^z_{\pi/2}, -1)$, $(R^{x=-y}_\pi, 1)$	$\{(\mathrm{i}a, 0, 0, 0, -\mathrm{i}a\,;\, 0, 0, 0, c, 0, 0, 0)\}$
$\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c}$	$(R^z_\pi, 1)$, $(\kappa_{xz}, 1)$, $(-I, -1)$	$\{(0, 0, 0, 0, 0\,;\, 0, b, 0, c, 0, b, 0)\}$
$\widetilde{\mathbf{D}_2}$	$(R^y_\pi, -1)$, $(R^z_\pi, 1)$	$\{(\mathrm{i}a, 0, 0, 0, -\mathrm{i}a\,;\, 0, b, 0, c, 0, b, 0)\}$

Table 7.1: Isotropy subgroups for the representation of $\mathbf{O(3)} \times \mathbb{Z}_2$ on $V_2 \oplus V_3$ which contain $\widetilde{\mathbf{D}_2}$. Also shown is the form of the fixed-point subspace which lies inside $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$. The generators which give this subspace are also listed. Here κ_{xz} is a reflection in the xz plane sending $y \to -y$, $\mathcal{R}_{2\pi/3}$ is a rotation through $2\pi/3$ in the line $x = y = z$ sending $x \to y \to z$ and $R^{x=-y}_\pi$ is rotation through π in the line $x = -y$, $z = 0$ sending $x \to -y$, $y \to -x$ and $z \to -z$.

The subsection of the lattice of isotropy subgroups (Figure 6.5) which shows only the isotropy subgroups containing $\widetilde{\mathbf{D}_2}$ is given in Figure 7.3.

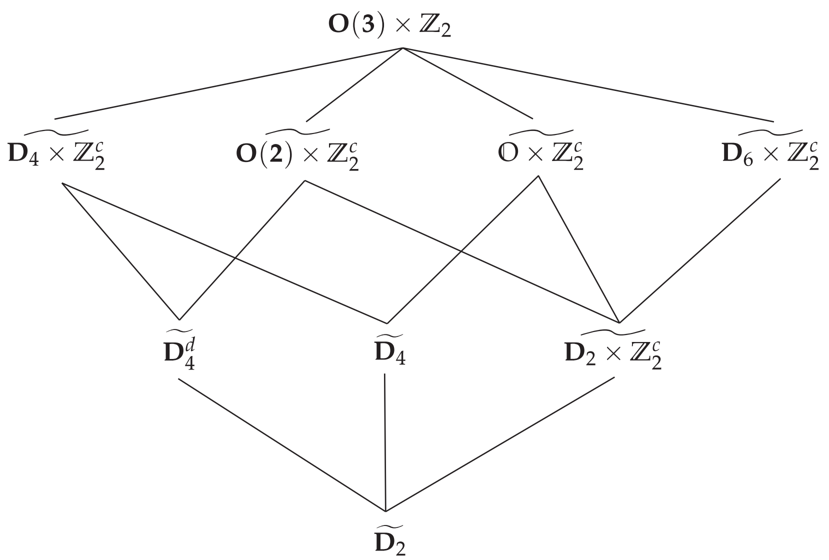


Figure 7.3: Subsection of lattice of isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in the representation on $V_2 \oplus V_3$ including only those isotropy subgroups which contain $\widetilde{\mathbf{D}_2}$.

We now investigate whether it is possible for solutions of (7.2.5)–(7.2.7) with non-axial symmetry to exist and, if so, the conditions on the coefficients α_1, β_1, γ_1, γ_2, α_2, β_2, δ_1, δ_2 and δ_3 which must be satisfied for the solutions to exist and be stable. We are most interested in the existence of solutions with $\widetilde{\mathbf{D}_2}$ and $\widetilde{\mathbf{D}_4}$ symmetry since these are the stationary spiral solutions.

Existence of equilibria in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$

The non-trivial equilibrium solutions of (7.2.5)–(7.2.7) are as follows.

1. If $b = c = 0$ then $\dot{a} = 0$ when
$$a = \pm\sqrt{-\frac{\lambda}{2\alpha_1}}.$$
This solution branch has $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry and bifurcates from the trivial solution at $\lambda = 0$. It is an $\ell = 2$ axial solution.

2. If $a = b = 0$ then $\dot{c} = 0$ when
$$c = \pm\sqrt{\frac{-\lambda-\rho}{\beta_2 - 9\delta_1}}.$$
This solution branch has $\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2}$ symmetry and bifurcates from the trivial solution at $\lambda = -\rho$. It is an $\ell = 3$ axial solution.

3. If $a = c = 0$ then $\dot{b} = 0$ when
$$b = \pm\sqrt{\frac{-\lambda-\rho}{2\beta_2 - 50\delta_1}}.$$
This solution branch has $\widetilde{\mathbb{O} \times \mathbb{Z}_2}$ symmetry and bifurcates from the trivial solution at $\lambda = -\rho$. It is an $\ell = 3$ axial solution.

4. If $a = 0$ but $b \neq 0$ and $c \neq 0$ then $\dot{b} = \dot{c} = 0$ when
$$b = \pm\sqrt{\frac{-3(\lambda+\rho)}{16\beta_2}} \qquad \text{and} \qquad c = \pm\sqrt{\frac{-5(\lambda+\rho)}{8\beta_2}} \quad \text{so} \quad c^2 = \frac{10}{3}b^2.$$
These solutions have $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2}$ symmetry and bifurcate from the trivial solution at $\lambda = -\rho$. They are $\ell = 3$ axial solutions.

Remark 7.2.2. There is no solution with $\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2}$ symmetry since any solution in $\mathrm{Fix}(\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2})$ with $b \neq 0$ and $c \neq 0$ satisfies
$$\begin{cases} \lambda + \rho + (2\beta_2 - 50\delta_1)\, b^2 + (\beta_2 + 15\delta_1)\, c^2 = 0 \\ \lambda + \rho + (\beta_2 - 9\delta_1)\, c^2 + (2\beta_2 + 30\delta_1)\, b^2 = 0. \end{cases}$$
The only solution is $c^2 = \frac{10}{3}b^2$ and so the fixed-point solution has $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2}$ symmetry.

5. If $b = 0$ but $c \neq 0$ and $a \neq 0$ then $\dot{a} = \dot{c} = 0$ when
$$\begin{cases} \lambda + 2\alpha_1 a^2 + (\beta_1 + \gamma_2)\, c^2 = 0 \\ \lambda + \rho + (\beta_2 - 9\delta_1)\, c^2 + (2\alpha_2 - 30\delta_2 + 18\delta_3)\, a^2 = 0. \end{cases}$$
Generically these equations have one solution for (a^2, c^2) which is given by
$$\begin{aligned} a^2 &= \frac{-\lambda\,(\beta_2 - 9\delta_1) + (\lambda+\rho)\,(\beta_1+\gamma_2)}{2\alpha_1\,(\beta_2 - 9\delta_1) - (2\alpha_2 - 80\delta_2 + 18\delta_3)\,(\beta_1+\gamma_2)} \\ c^2 &= \frac{-2\alpha_1(\lambda+\rho) + \lambda\,(2\alpha_2 - 30\delta_2 + 18\delta_3)}{2\alpha_1\,(\beta_2 - 9\delta_1) - (2\alpha_2 - 30\delta_2 + 18\delta_3)\,(\beta_1+\gamma_2)}. \end{aligned}$$

Hence solutions for (a, c) with $a \neq 0$ and $c \neq 0$ exist when $a^2 > 0$ and $c^2 > 0$. These solutions have $\widetilde{\mathbf{D}_4^d}$ symmetry and result from secondary bifurcations from $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry. They bifurcate from the solution branch with $\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c}$ symmetry when $a = 0$, i.e.

$$\lambda = \frac{(\beta_1 + \gamma_2)\rho}{\beta_2 - 9\delta_1 - \beta_1 - \gamma_2},$$

and from the solution with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry when $c = 0$, i.e.

$$\lambda = \frac{2\alpha_1\rho}{2\alpha_2 - 30\delta_2 + 18\delta_3 - 2\alpha_1}.$$

6. If $c = 0$ and $a \neq 0$, $b \neq 0$ then $\dot{a} = \dot{b} = 0$ when

$$\begin{cases} \lambda + 2\alpha_1 a^2 + (2\beta_1 + 8\gamma_1 - 6\gamma_2)\, b^2 = 0 \\ \lambda + \rho + (2\beta_2 - 50\delta_1)\, b^2 + (2\alpha_2 - 30\delta_2 + 10\delta_3)\, a^2 = 0. \end{cases}$$

Generically these equations have one solution for (a^2, b^2) which is given by

$$\begin{aligned} a^2 &= \frac{-\lambda\,(2\beta_2 - 50\delta_1) + (\lambda + \rho)\,(2\beta_1 + 8\gamma_1 - 6\gamma_2)}{2\alpha_1\,(2\beta_2 - 50\delta_1) - (2\alpha_2 - 30\delta_2 + 10\delta_3)\,(2\beta_1 + 8\gamma_1 - 6\gamma_2)} \\ b^2 &= \frac{-2\alpha_1(\lambda + \rho) + \lambda\,(2\alpha_2 - 30\delta_2 + 10\delta_3)}{2\alpha_1\,(2\beta_2 - 50\delta_1) - (2\alpha_2 - 30\delta_2 + 10\delta_3)\,(2\beta_1 + 8\gamma_1 - 6\gamma_2)}. \end{aligned}$$

Hence solutions for (a, b) with $a \neq 0$ and $b \neq 0$ exist when $a^2 > 0$ and $b^2 > 0$. These solutions have $\widetilde{\mathbf{D}_4}$ symmetry which is the symmetry group of a two-armed spiral. They result from secondary bifurcations from $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry. They bifurcate from the solution branch with $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ symmetry when $a = 0$, i.e.

$$\lambda = \frac{(2\beta_1 + 8\gamma_1 - 6\gamma_2)\rho}{2\beta_2 - 50\delta_1 - 2\beta_1 - 8\gamma_1 + 6\gamma_2},$$

and from the solution with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry when $b = 0$, i.e.

$$\lambda = \frac{2\alpha_1\rho}{2\alpha_2 - 30\delta_2 + 10\delta_3 - 2\alpha_1}.$$

7. If $a \neq 0$, $b \neq 0$ and $c \neq 0$ then $\dot{a} = \dot{b} = \dot{c} = 0$ when

$$\begin{cases} \lambda + 2\alpha_1 a^2 + (2\beta_1 + 8\gamma_1 - 6\gamma_2)\, b^2 + (\beta_1 + \gamma_2)\, c^2 = 0 \\ \lambda + \rho + (2\beta_2 - 50\delta_1)\, b^2 + (2\alpha_2 - 30\delta_2 + 10\delta_3)\, a^2 + (\beta_2 + 15\delta_1)\, c^2 = 0 \\ \lambda + \rho + (\beta_2 - 9\delta_1)\, c^2 + (2\alpha_2 - 30\delta_2 + 18\delta_3)\, a^2 + (2\beta_2 + 30\delta_1)\, b^2 = 0. \end{cases}$$

Generically these equations have one solution for (a^2, b^2, c^2) which is given by

$$a^2 = \frac{M}{N} \qquad b^2 = \frac{P}{N} \qquad c^2 = \frac{Q}{N},$$

where

$$\begin{aligned} M &= -128\lambda\beta_2\delta_1 + 64(\lambda + \rho)\delta_1\,[2\beta_1 + 3\gamma_1 - \gamma_2] \\ P &= 8\lambda\,[\delta_3\beta_2 + 6\alpha_2\delta_1 - 90\delta_1\delta_2 + 45\delta_1\delta_3] - 8(\lambda + \rho)\,[\beta_1\delta_3 + \gamma_2\delta_3 + 6\alpha_1\delta_1] \\ Q &= 16\lambda\,[10\alpha_2\delta_1 - 150\delta_1\delta_2 - \beta_2\delta_3 + 75\delta_1\delta_3] + 16(\lambda + \rho)\,[\delta_3\beta_1 + 4\gamma_1\delta_3 - 3\gamma_2\delta_3 - 10\alpha_1\delta_1] \\ N &= -64\,[(2\beta_1 + 3\gamma_1 - \gamma_2)\,(2\alpha_2\delta_1 - 30\delta_1\delta_2 + 15\delta_1\delta_3) + \delta_3\beta_2\,(\gamma_1 - \gamma_2) - 4\alpha_1\beta_2\delta_1]. \end{aligned}$$

Solutions for (a, b, c) with $a \neq 0$, $b \neq 0$ and $c \neq 0$ exist when $a^2 > 0$, $b^2 > 0$ and $c^2 > 0$. This occurs when

$$MN > 0, \qquad PN > 0 \qquad \text{and} \qquad QN > 0.$$

Any such solution would have $\widetilde{\mathbf{D}_2}$ symmetry which is the group of symmetries of a one-armed spiral on a sphere. They bifurcate from the solution branch with $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$ symmetry when $a = 0$, i.e.

$$\lambda = \frac{(2\beta_1 + 3\gamma_1 - \gamma_2)\rho}{2\beta_2 - 2\beta_1 - 3\gamma_1 + \gamma_2}$$

from the solution with $\widetilde{\mathbf{D}_4^d}$ symmetry when $b = 0$, i.e. when $P = 0$ and from the solution with $\widetilde{\mathbf{D}_4}$ symmetry when $c = 0$, i.e. when $Q = 0$.

Images of solutions with each of these symmetry types are shown in Figure 7.4.

Expressions for the eigenvalues corresponding to a linearisation of each of these solutions can be found. However, due to the large number of coefficients involved, the exact form of the eigenvalues of all but the axial solutions (solutions 1–4) is algebraically very messy. Rather than computing the stability of the solution branches and the bifurcation structure for general values of the coefficients α_1, β_1, γ_1, γ_2, α_2, β_2, δ_1, δ_2 and δ_3 we consider instead specific values of these coefficients, where solutions with $\widetilde{\mathbf{D}_2}$ (one-armed spiral) symmetry exist.

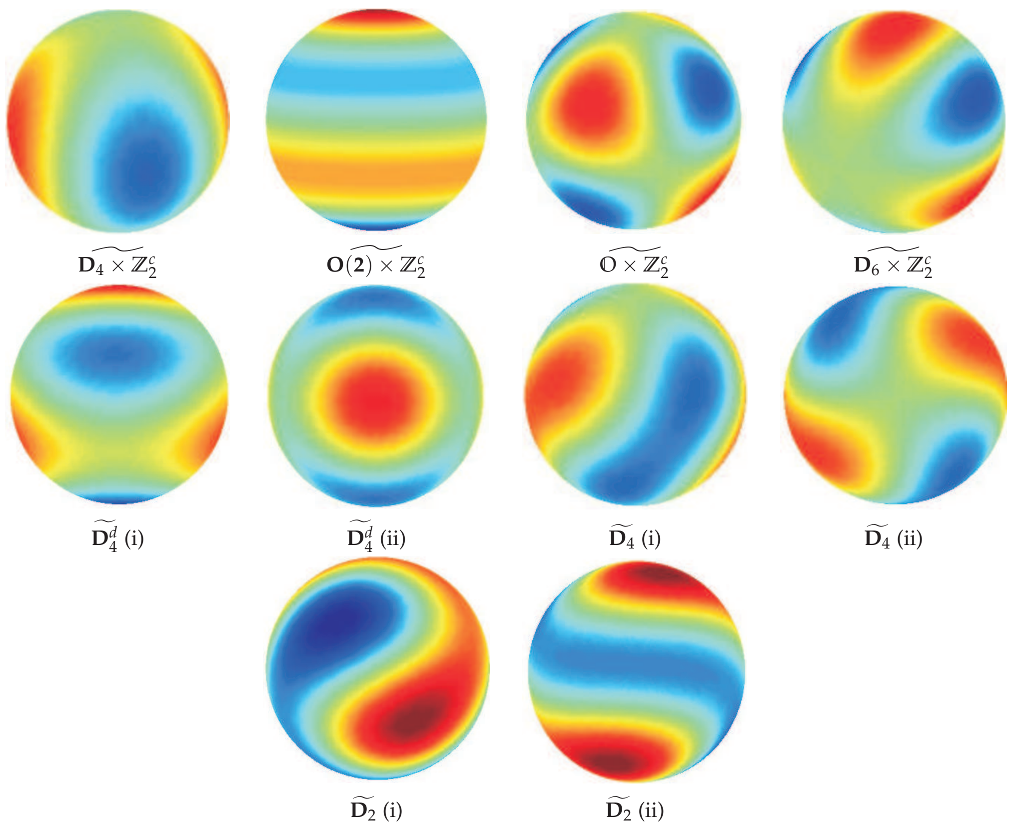


Figure 7.4: Images of solutions to (7.2.5)–(7.2.7). These solutions all have symmetry groups containing $\widetilde{\mathbf{D}_2}$. In some cases two views of the solutions are given to fully describe the symmetries.

Example 7.2.3 (Solutions with $\widetilde{\mathbf{D}_2}$ symmetry exist but are unstable). Suppose that

$$\begin{gathered}\alpha_1 = -1, \quad \alpha_2 = -1, \quad \beta_1 = -1, \quad \beta_2 = -1, \\ \gamma_1 = -\frac{1}{2}, \quad \gamma_2 = -\frac{1}{2}, \quad \delta_1 = -\frac{1}{60}, \quad \delta_2 = -\frac{1}{2}, \quad \delta_3 = -\frac{1}{5}.\end{gathered} \tag{7.2.8}$$

In the restriction of the equivariant vector field to $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ we have

$$\dot{a} = \lambda a - 2a^3 - 3ab^2 - \frac{3}{2}ac^2 \tag{7.2.9}$$

$$\dot{b} = (\lambda+\rho)b - \frac{7}{6}b^3 + 11ba^2 - \frac{5}{4}bc^2 \tag{7.2.10}$$

$$\dot{c} = (\lambda+\rho)c - \frac{17}{20}c^3 + \frac{47}{5}ca^2 - \frac{5}{2}cb^2. \tag{7.2.11}$$

The branch of solutions with $\ell = 2$ axial isotropy (with symmetry $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$) bifurcate at $\lambda = 0$ and those with $\ell = 3$ axial isotropy (those with symmetries $\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2}$, $\widetilde{\mathbb{O} \times \mathbb{Z}_2}$ and $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2}$) bifurcate at $\lambda = -\rho$. We use the results of Section 7.2.1 to see where the solutions with submaximal symmetry can exist and we investigate the stability within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ of all solution branches.

1. **(The solution branch with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry)** Since $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ is an axial isotropy subgroup there is always a branch of solutions with this symmetry bifurcating from $\lambda = 0$. With the values of the coefficients (7.2.8) the solution branch bifurcates supercritically so exists when $\lambda > 0$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -2\lambda \qquad \xi_2 = \frac{13}{2}\lambda + \rho \qquad \xi_3 = \frac{57}{10}\lambda + \rho.$$
Thus the solution branch is stable when $\lambda < -\frac{10}{57}\rho$ and undergoes stationary bifurcations at $\lambda = -\frac{10}{57}\rho$ and $\lambda = -\frac{2}{13}\rho$.

2. **(The solution branch with $\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c}$ symmetry)** With the values of the coefficients (7.2.8), this solution branch bifurcates supercritically from $\lambda = -\rho$ so exists when $\lambda > -\rho$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -2(\lambda+\rho) \qquad \xi_2 = -\frac{1}{17}(13\lambda + 30\rho) \qquad \xi_3 = -\frac{8}{17}(\lambda+\rho).$$
It undergoes a stationary bifurcation at $\lambda = -\frac{30}{13}\rho$ and is stable when $\lambda > -\frac{30}{13}\rho$.

3. **(The solution branch with $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ symmetry)** With the values of the coefficients (7.2.8), this solution branch bifurcates supercritically from $\lambda = -\rho$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -2(\lambda+\rho) \qquad \xi_2 = -\frac{1}{7}(11\lambda + 18\rho) \qquad \xi_3 = -\frac{8}{7}(\lambda+\rho).$$
It undergoes a stationary bifurcation at $\lambda = -\frac{18}{11}\rho$ and is stable when $\lambda > -\frac{18}{11}\rho$.

4. **(The solution branch with $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$ symmetry)** With the values of the coefficients (7.2.8), this solution branch bifurcates supercritically from $\lambda = -\rho$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -2(\lambda+\rho) \qquad \xi_2 = -\frac{1}{2}(\lambda + 3\rho) \qquad \xi_3 = \frac{1}{2}(\lambda+\rho).$$
This branch of solutions always has at least one positive eigenvalue so it is always unstable. It undergoes a stationary bifurcation at $\lambda = -3\rho$ where $\xi_2 = 0$.

5. **(The solution branch with $\widetilde{\mathbf{D}_4^d}$ symmetry)** With the values of the coefficients (7.2.8) this solution branch bifurcates from the solution with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = -\frac{10}{57}\rho$ and joins the branch with $\widetilde{\mathbf{O(2)} \times \mathbb{Z}_2}$ symmetry at $\lambda = -\frac{30}{13}\rho$. Thus it exists only when $\rho < 0$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues

$$\xi_1 = -\frac{4}{79}\,(7\lambda + 4\rho)$$

and ξ_2 and ξ_3 are the roots of

$$\xi^2 + \frac{1}{790}\,(839\lambda - 130\rho)\,\xi - \frac{1}{395}\,(13\lambda + 30\rho)\,(57\lambda + 10\rho) = 0.$$

They have negative real part for all values of λ and ρ where the solution exists and hence the solution branch is stable when $\lambda > -\frac{4}{7}\rho$.

6. **(The solution branch with $\widetilde{\mathbf{D}_4}$ symmetry)** With the values of the coefficients (7.2.8) this solution branch with two-armed spiral symmetry bifurcates from the solution with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = -\frac{2}{13}\rho$ and joins the branch with $\widetilde{\mathbf{O} \times \mathbb{Z}_2}$ symmetry at $\lambda = -\frac{18}{11}\rho$. Thus it exists only when $\rho < 0$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues

$$\xi_1 = -\frac{4}{265}\,(27\lambda - 4\rho)$$

and ξ_2 and ξ_3 are the roots of

$$\xi^2 + \frac{1}{106}\,(69\lambda - 22\rho)\,\xi - \frac{1}{53}\,(11\lambda + 18\rho)\,(13\lambda + 2\rho) = 0.$$

They have negative real part for all values of λ and ρ where the solution exists and hence the solution branch is stable when $\lambda > \frac{4}{27}\rho$. Since the solution only exists when $\lambda > -\frac{2}{13}\rho$ and $\rho > 0$ the solution branch is always stable.

7. **(The solution branch with $\widetilde{\mathbf{D}_2}$ symmetry)** With the values of the coefficients (7.2.8) the solution branch with one-armed spiral symmetry bifurcates from the solution branch with $\widetilde{\mathbf{D}_4^d}$ symmetry when $\lambda = -\frac{4}{7}\rho$ and joins the branch with $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = -3\rho$. It can be found that this solution branch is always unstable in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ with one positive and two negative eigenvalues for all values of λ and ρ where it exists.

The bifurcation diagram for $\rho < 0$ is as in Figure 7.5. There are no bifurcations when $\rho > 0$ and the only solution branches that exist for these values of ρ are the axial solution branches. This can be seen from the gyratory bifurcation diagram in Figure 7.6. In these bifurcation diagrams, and all subsequent bifurcation diagrams in this thesis, a solid line indicates a solution which is stable within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ and a dashed line indicates an unstable solution branch. The signs (e.g. $++-$) next to the branch indicate the signs of the real parts of the eigenvalues on this branch. In addition, in all bifurcation diagrams in this chapter only one copy of each branch of solutions occurs. This is because all branches with the same symmetry have the same L_2 norm.

Example 7.2.4 (Solutions with $\widetilde{\mathbf{D}_2}$ symmetry exist and are stable). Suppose that

$$\alpha_1 = -1, \quad \alpha_2 = -1, \quad \beta_1 = -\frac{1}{3}, \quad \beta_2 = -1, \tag{7.2.12}$$

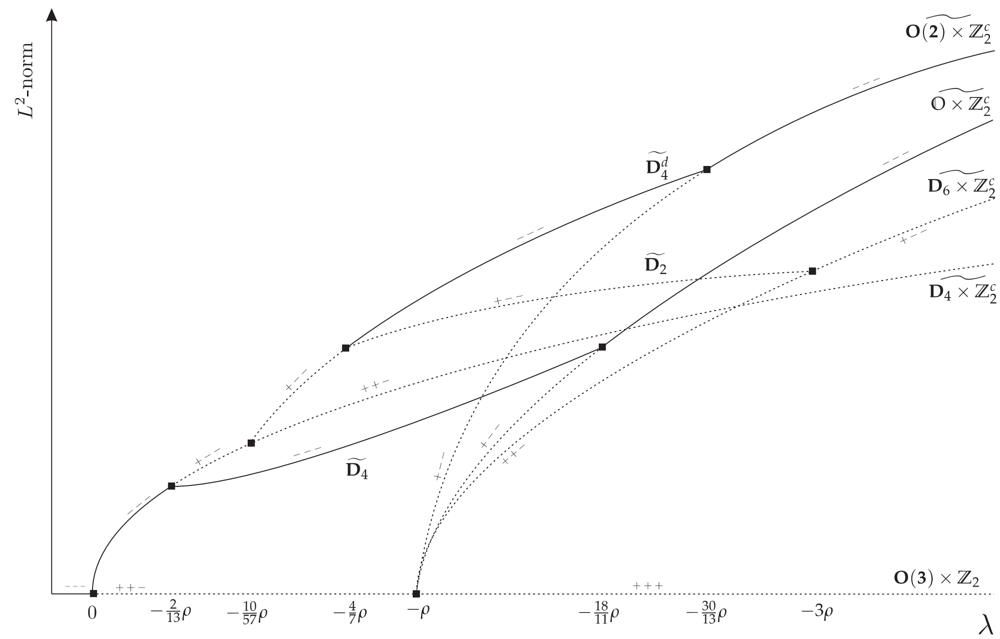


Figure 7.5: Bifurcation diagram for the system (7.2.9)–(7.2.11) when $\rho < 0$. The branch of solutions with $\widetilde{\mathbf{D}_2}$ (one-armed spiral) symmetry is always unstable and the solution with $\widetilde{\mathbf{D}_4}$ (two-armed spiral) symmetry is always stable. All bifurcations are pitchfork bifurcations. Note that the L^2 norm of the solution $x = (a, b, c)$ is given by $\|x\|^2 = a^2 + b^2 + c^2$. The diagram is not to scale but the relative sizes of the L^2 norms are shown.

$$\gamma_1 = \frac{1}{2}, \quad \gamma_2 = \frac{1}{2}, \quad \delta_1 = \frac{1}{60}, \quad \delta_2 = \frac{1}{2}, \quad \delta_3 = -\frac{1}{5}.$$

In the restriction of the equivariant vector field to $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ we have

$$\dot{a} = \lambda a - 2a^3 + \frac{1}{3}ab^2 + \frac{1}{6}ac^2 \tag{7.2.13}$$

$$\dot{b} = (\lambda + \rho)b - \frac{17}{6}b^3 - 19ba^2 - \frac{3}{4}bc^2 \tag{7.2.14}$$

$$\dot{c} = (\lambda + \rho)c - \frac{23}{20}c^3 - \frac{103}{5}ca^2 - \frac{3}{2}cb^2. \tag{7.2.15}$$

As for the previous example, we use the results of Section 7.2.1 to see where the solutions with submaximal symmetry can exist in (7.2.13)–(7.2.15) and we investigate the stability within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ of all solution branches.

1. **(The solution branch with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry)** Since $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ is an axial isotropy subgroup there is always a branch of solutions with this symmetry bifurcating from $\lambda = 0$. With the values of the coefficients (7.2.12) the solution branch bifurcates supercritically so exists when $\lambda > 0$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues

$$\xi_1 = -2\lambda \qquad \xi_2 = -\frac{17}{2}\lambda + \rho \qquad \xi_3 = -\frac{93}{10}\lambda + \rho.$$

Thus the solution branch is stable when $\lambda > \frac{10}{93}\rho$ and undergoes stationary bifurcations at $\lambda = \frac{10}{93}\rho$ and $\lambda = \frac{2}{17}\rho$.

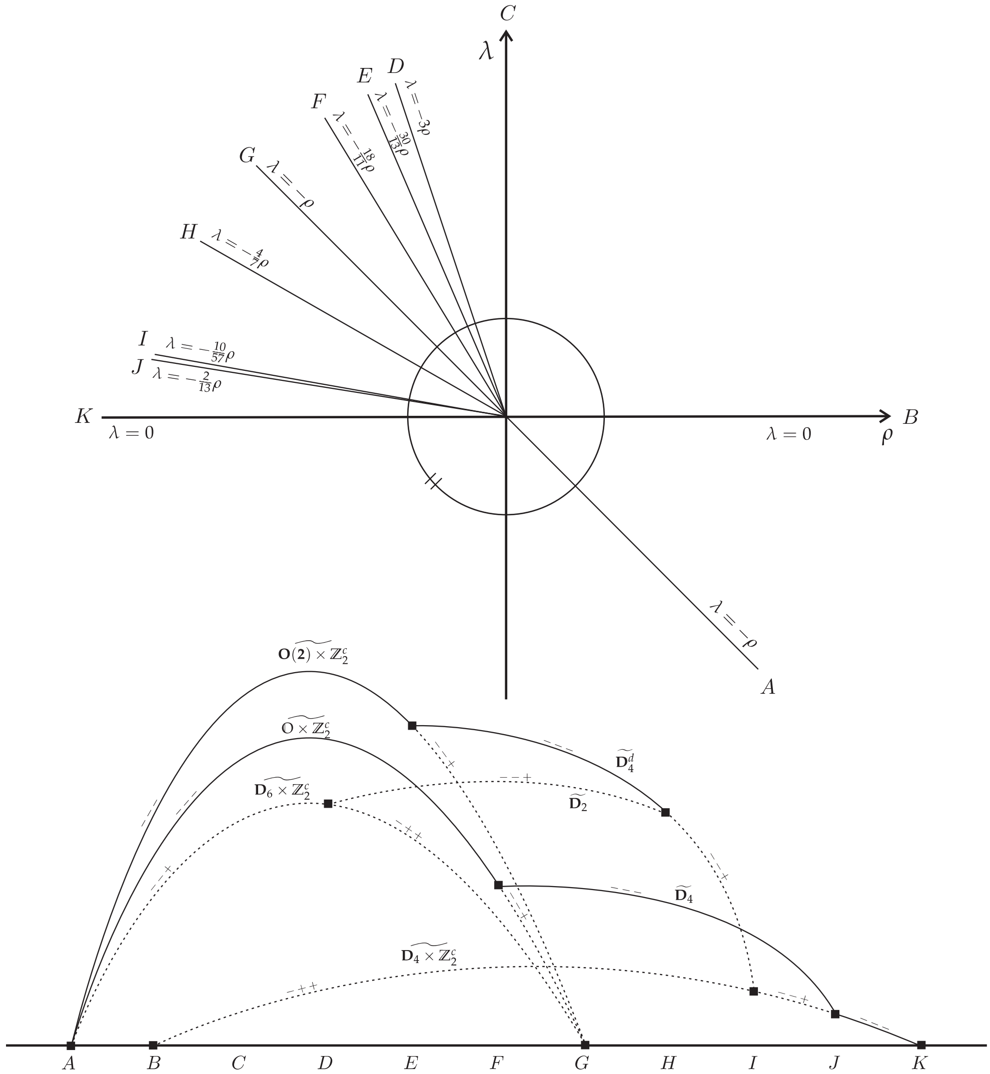


Figure 7.6: The top diagram is an unfolding diagram showing the lines on which bifurcations of the solution branches occur in Example 7.2.3 as the circle around the codimension 2 point, $\lambda = \rho = 0$, is traversed. The gyratory bifurcation diagram at the bottom of this figure shows the solution branches and their stability in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$. All bifurcations are pitchfork bifurcations.

2. **(The solution branch with $\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c}$ symmetry)** With the values of the coefficients (7.2.12), this solution branch bifurcates supercritically from $\lambda = -\rho$ so exists when $\lambda > -\rho$. Within Fix$(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -2(\lambda + \rho) \qquad \xi_2 = \frac{1}{69}(79\lambda + 10\rho) \qquad \xi_3 = \frac{8}{23}(\lambda + \rho).$$
It undergoes a stationary bifurcation at $\lambda = -\frac{10}{79}\rho$ but always has at least one positive eigenvalues and so is always unstable.

3. **(The solution branch with $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ symmetry)** With the values of the coefficients (7.2.12), this solution branch bifurcates supercritically from $\lambda = -\rho$. Within Fix$(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -2(\lambda + \rho) \qquad \xi_2 = \frac{1}{17}(19\lambda + 2\rho) \qquad \xi_3 = \frac{8}{17}(\lambda + \rho).$$
It undergoes a stationary bifurcation at $\lambda = -\frac{2}{19}\rho$ but always has at least one positive eigenvalues and so is always unstable.

4. **(The solution branch with $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$ symmetry)** With the values of the coefficients (7.2.12), this solution branch bifurcates supercritically from $\lambda = -\rho$. Within Fix$(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -2(\lambda + \rho) \qquad \xi_2 = \frac{1}{6}(7\lambda + \rho) \qquad \xi_3 = -\frac{1}{2}(\lambda + \rho).$$
Thus the solution branch is stable when $\lambda < -\frac{1}{7}\rho$.

5. **(The solution branch with $\widetilde{\mathbf{D}_4^d}$ symmetry)** With the values of the coefficients (7.2.12), this solution branch bifurcates from the solution with $\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2}$ symmetry at $\lambda = -\frac{10}{79}\rho$ and joins the branch with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = \frac{10}{93}\rho$. Thus it exists only when $\rho > 0$. Within Fix$(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -\frac{8}{43}(5\lambda - \rho) = 0$$
and ξ_2 and ξ_3 are the roots of
$$\xi^2 - \frac{1}{860}(5627\lambda - 790\rho)\,\xi - \frac{17}{36980}(79\lambda + 10\rho)(93\lambda - 10\rho).$$
They have negative real part for all values of λ and ρ where the solution exists and hence the solution branch has three eigenvalues with negative real part when $\lambda > \frac{1}{5}\rho$. However the branch only exists when $\lambda < \frac{10}{93}\rho$ so the solution branch is always unstable.

6. **(The solution branch with $\widetilde{\mathbf{D}_4}$ symmetry)** With the values of the coefficients (7.2.12), this solution branch with two-armed spiral symmetry bifurcates from the solution with $\widetilde{\mathbb{O} \times \mathbb{Z}_2}$ symmetry at $\lambda = -\frac{2}{19}\rho$ and joins the branch with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = \frac{2}{17}\rho$. Thus it exists only when $\rho > 0$. Within Fix$(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -\frac{8}{45}(13\lambda - \rho)$$
and ξ_2 and ξ_3 are the roots of
$$\xi^2 - \frac{1}{36}(251\lambda - 38\rho)\,\xi - \frac{1}{18}(19\lambda + 2\rho)(17\lambda - 2\rho) = 0.$$

They have negative real part for all values of λ and ρ where the solution exists and hence the solution branch is stable when $\lambda > \frac{1}{13}\rho$.

7. **(The solution branch with $\widetilde{\mathbf{D}_2}$ symmetry)** With the values of the coefficients (7.2.12), the solution branch with one-armed spiral symmetry bifurcates from the solution branch with $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$ symmetry when $\lambda = -\frac{1}{7}\rho$ and joins the branch with $\widetilde{\mathbf{D}_4}$ symmetry at $\lambda = \frac{1}{13}\rho$. It can be found that this solution branch has one real negative eigenvalue and a complex conjugate pair with negative real part in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ for all values of λ and ρ where it exists. Hence the solution is stable in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$.

The bifurcation diagram for $\rho > 0$ is as in Figure 7.7. There are no bifurcations when $\rho < 0$ and the only solution branches that exist for these values of ρ are the axial solution branches. This can be seen from the gyratory bifurcation diagram in Figure 7.8.

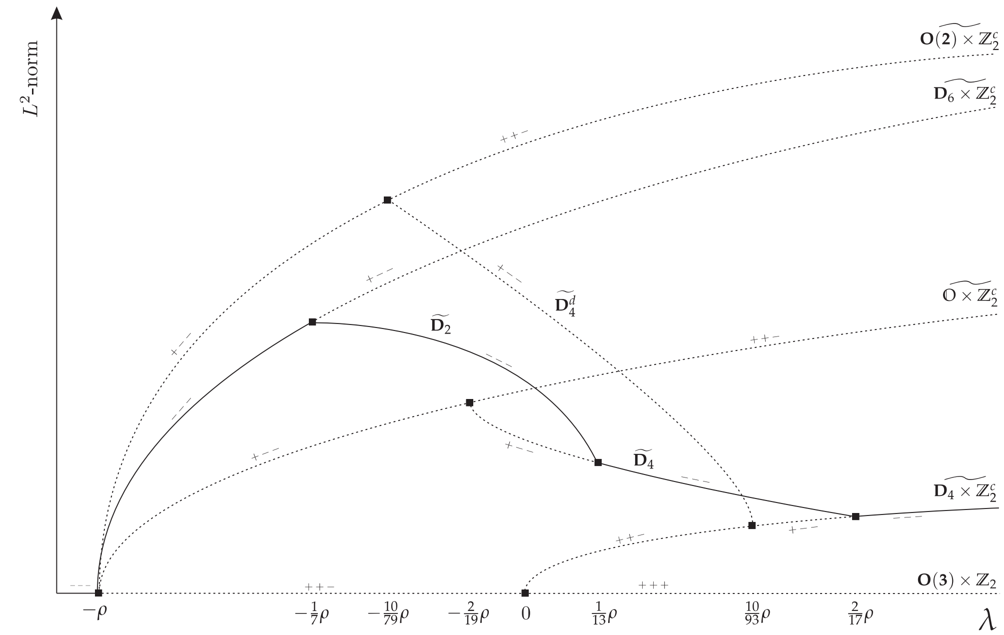


Figure 7.7: Bifurcation diagram for the system (7.2.13)–(7.2.15) when $\rho > 0$. The branch of solutions with $\widetilde{\mathbf{D}_2}$ (one-armed spiral) symmetry is always stable and the solution with $\widetilde{\mathbf{D}_4}$ (two-armed spiral) symmetry is stable for some values of λ and ρ. All bifurcations are pitchfork bifurcations. Note that the L^2 norm of the solution $x = (a, b, c)$ is given by $\|x\|^2 = a^2 + b^2 + c^2$. The diagram is not to scale but the relative sizes of the L^2 norms are shown.

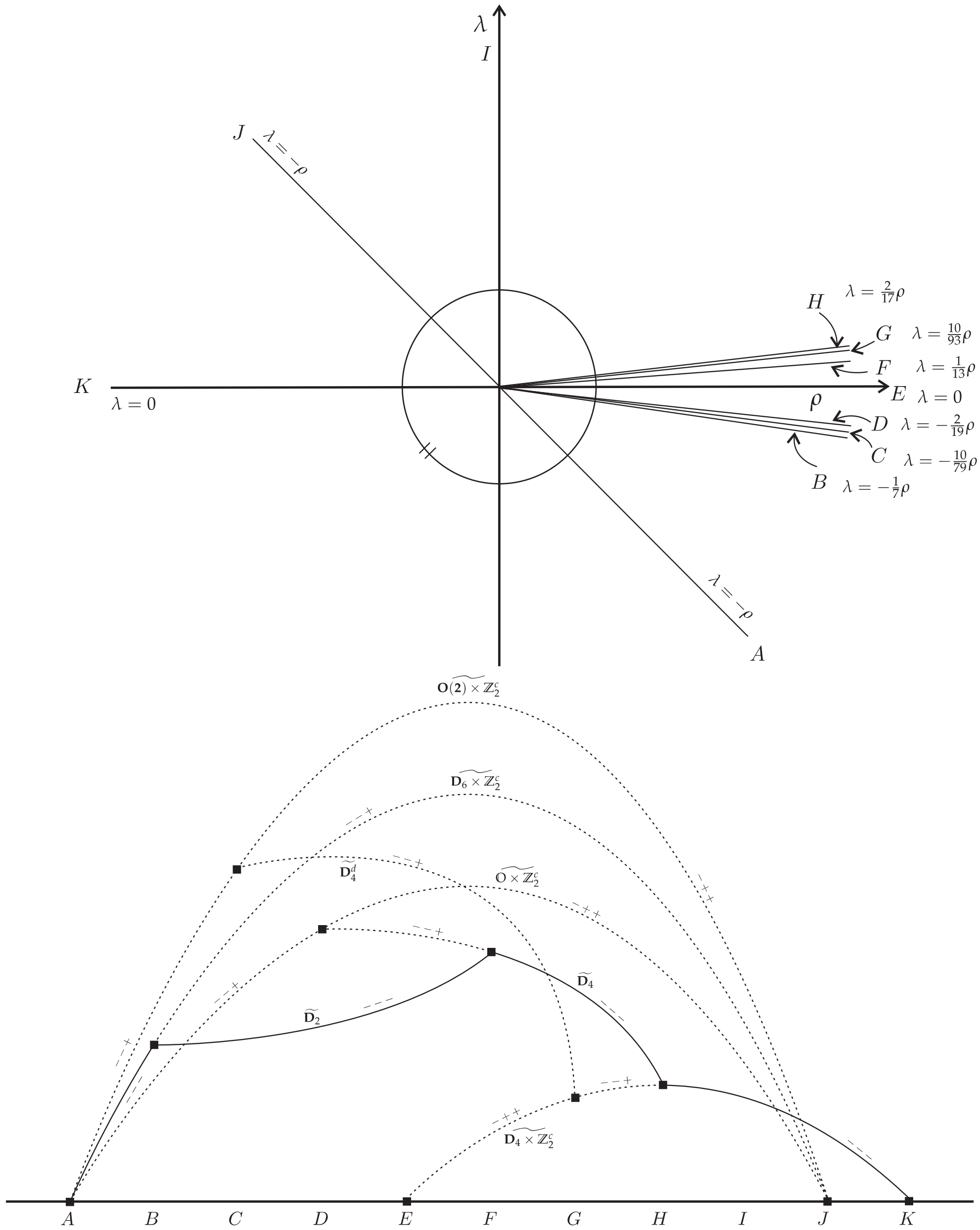


Figure 7.8: The top diagram is an unfolding diagram showing the lines on which bifurcations of the solution branches occur in Example 7.2.4 as the circle around the codimension 2 point, $\lambda = \rho = 0$, is traversed. The gyratory bifurcation diagram at the bottom of this figure shows the solution branches and their stability in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$. All bifurcations are pitchfork bifurcations.

Example 7.2.5 (Solutions with spiral symmetry in the Swift–Hohenberg equation on a sphere of radius near 3). Recall that in Section 6.4.2 we found that for the Swift–Hohenberg equation (6.4.1) on a sphere of radius $R = 3 + \epsilon^2 R_2$ the relevant representation of $\mathbf{O(3)}$ is the representation on $V_2 \oplus V_3$ and the critical value of the parameter μ is $\frac{1}{9}$. We computed that the values of the coefficients in the equivariant vector field are

$$\mu_x = \mu_2 - \frac{8}{27}R_2,\ \alpha_1 = -\frac{15}{28\pi},\ \beta_1 = -\frac{6}{11\pi},\ \gamma_1 = \frac{3}{44\pi},\ \gamma_2 = \frac{1}{44\pi}, \tag{7.2.16}$$

$$\mu_y = \mu_2 + \frac{16}{27}R_2,\ \alpha_2 = -\frac{25}{22\pi},\ \beta_2 = -\frac{175}{286\pi},\ \delta_1 = -\frac{7}{2860\pi},\ \delta_2 = -\frac{3}{44\pi}\ \text{and}\ \delta_3 = -\frac{1}{22\pi}$$

where $\mu = \frac{1}{9} + \epsilon^2 \mu_2$. Let $\mu_x = \lambda$ then $\mu_y = \lambda + \rho$ where $\rho = \frac{8}{9}R_2$. Substituting these values into equations (7.2.5)–(7.2.7) we have

$$\begin{aligned}
\dot{a} &= \lambda a - \frac{15}{14\pi}a^3 - \frac{15}{22\pi}ab^2 - \frac{23}{44\pi}ac^2 && (7.2.17)\\
\dot{b} &= (\lambda+\rho)\,b - \frac{315}{286\pi}b^3 - \frac{15}{22\pi}ba^2 - \frac{371}{572\pi}bc^2 && (7.2.18)\\
\dot{c} &= (\lambda+\rho)\,c - \frac{1687}{2860\pi}c^3 - \frac{23}{22\pi}ca^2 - \frac{371}{286\pi}cb^2. && (7.2.19)
\end{aligned}$$

As for the previous examples, we use the results of Section 7.2.1 to see where the solutions with submaximal symmetry can exist and we investigate the stability within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ of all solution branches.

1. **(The solution branch with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry)** Since $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ is an axial isotropy subgroup there is always a branch of solutions with this symmetry bifurcating from $\lambda = 0$. With the values of the coefficients (7.2.16) the solution branch bifurcates supercritically so exists when $\lambda > 0$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues

$$\xi_1 = -2\lambda \qquad \xi_2 = \frac{4}{11}\lambda + \rho \qquad \xi_3 = \frac{4}{165}\lambda + \rho.$$

When $\rho > 0$ the branch is unstable but when $\rho < 0$ it is stable when $\lambda < -\frac{11}{4}\rho$. There are stationary bifurcations when $\lambda = -\frac{11}{4}\rho$ and $\lambda = -\frac{165}{4}\rho$.

2. **(The solution branch with $\widetilde{\mathbf{O(2)} \times \mathbb{Z}_2^c}$ symmetry)** With the values of the coefficients (7.2.16) this solution branch bifurcates supercritically from $\lambda = -\rho$ so exists when $\lambda > -\rho$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues

$$\xi_1 = -2(\lambda+\rho) \qquad \xi_2 = \frac{1}{1687}\left(192\lambda - 1495\rho\right) \qquad \xi_3 = -\frac{24}{241}\left(\lambda+\rho\right).$$

The eigenvalues ξ_1 and ξ_3 are always negative. There is a stationary bifurcation at $\lambda = \frac{1495}{192}\rho$ when $\rho > 0$ and this solution is stable when $\lambda < \frac{1495}{192}\rho$.

3. **(The solution branch with $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ symmetry)** With the values of the coefficients (7.2.16) this solution branch bifurcates supercritically from $\lambda = -\rho$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues

$$\xi_1 = -2(\lambda+\rho) \qquad \xi_2 = \frac{1}{21}\left(8\lambda - 13\rho\right) \qquad \xi_3 = -\frac{8}{45}\left(\lambda+\rho\right).$$

The eigenvalues ξ_1 and ξ_3 are always negative. There is a stationary bifurcation at $\lambda = \frac{13}{8}\rho$ when $\rho > 0$ and this solution is stable when $\lambda < \frac{13}{8}\rho$.

4. **(The solution branch with $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$ symmetry)** With the values of the coefficients (7.2.16) this solution branch bifurcates supercritically from $\lambda = -\rho$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -2(\lambda+\rho) \qquad \xi_2 = \frac{1}{35}(9\lambda - 26\rho) \qquad \xi_3 = \frac{3}{25}(\lambda+\rho)\,.$$
This branch of solutions always has at least one positive eigenvalue so it is always unstable. It undergoes a stationary bifurcation at $\lambda = \frac{26}{9}\rho$ where $\xi_2 = 0$.

5. **(The solution branch with $\widetilde{\mathbf{D}_4^d}$ symmetry)** With the values of the coefficients (7.2.16) this solution branch bifurcates from the solution with $\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2}$ symmetry at $\lambda = \frac{1495}{192}\rho$ when $\rho > 0$ and the branch with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = -\frac{165}{4}\rho$ when $\rho < 0$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = \frac{4}{269}(18\lambda - 199\rho)$$
and ξ_2 and ξ_3 are the roots of
$$\xi^2 + \frac{2}{9415}(9607\lambda + 7040\rho)\,\xi + \frac{1}{9415}(192\lambda - 1495\rho)(4\lambda + 165\rho) = 0.$$
They have negative real part for all values of λ and ρ where the solution exists and hence the solution branch is stable when $\lambda < \frac{199}{18}\rho$.

6. **(The solution branch with $\widetilde{\mathbf{D}_4}$ symmetry)** With the values of the coefficients (7.2.16) this solution branch with two-armed spiral symmetry bifurcates from the solution with $\widetilde{\mathbb{O} \times \mathbb{Z}_2}$ symmetry at $\lambda = \frac{13}{8}\rho$ when $\rho > 0$ and the branch with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = -\frac{11}{4}\rho$ when $\rho < 0$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -\frac{4}{75}(6\lambda - \rho)$$
and ξ_2 and ξ_3 are the roots of
$$\xi^2 + \frac{2}{35}(43\lambda + 22\rho)\,\xi + \frac{1}{35}(4\lambda + 11\rho)(8\lambda - 13\rho) = 0.$$
They have negative real part for all values of λ and ρ where the solution exists and hence the solution branch is stable when $\lambda > \frac{1}{6}\rho$. Since the branch only exists when $\lambda > \frac{13}{8}\rho$ it is always stable.

7. **(The solution branch with $\widetilde{\mathbf{D}_2}$ symmetry)** With the values of the coefficients (7.2.16) the solution branch with one-armed spiral symmetry bifurcates from the solution branch with $\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c}$ symmetry when $\lambda = \frac{26}{9}\rho$ and the branch with $\widetilde{\mathbf{D}_4^d}$ symmetry at $\lambda = \frac{199}{18}\rho$ when $\rho > 0$. The branch only exists for positive values of ρ. It can be found that this solution branch has one positive and two negative eigenvalues in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ for all values of λ and ρ where it exists. Hence the solution is unstable.

The bifurcation diagram when $\rho > 0$ is as in Figure 7.9 and the bifurcation diagram when $\rho < 0$ is as in Figure 7.10. We can combine these bifurcation diagrams into a gyratory bifurcation diagram as in Figure 7.11. Note that these bifurcation diagrams are qualitatively the same as those for Example 7.2.3.

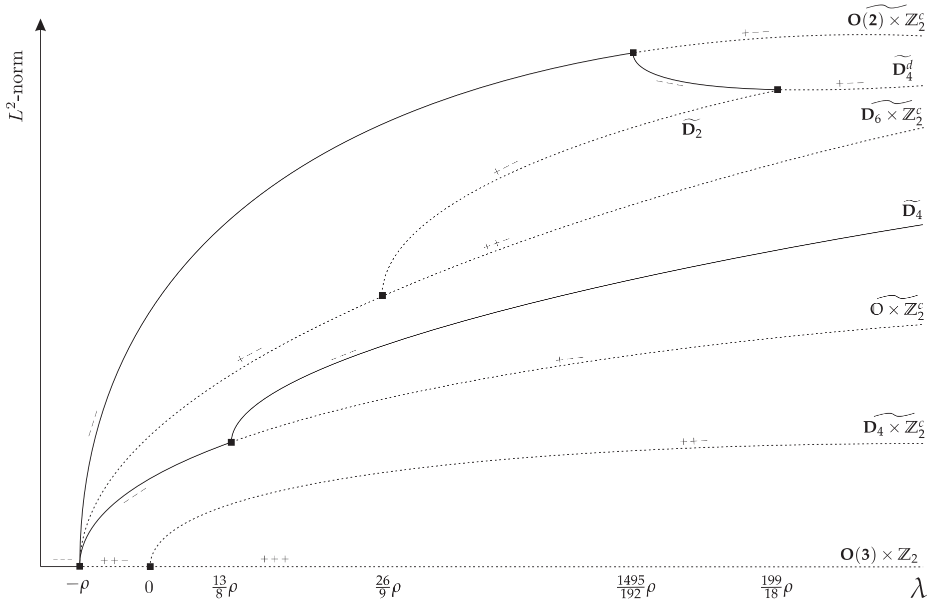


Figure 7.9: Bifurcation diagram for the Swift–Hohenberg equation in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ for the representation on $V_2 \oplus V_3$ when $\rho > 0$. We find that the solution with the symmetry group of a one-armed spiral, $\widetilde{\mathbf{D}_2}$, is unstable where it exists. The solution with the symmetry group of a two-armed spiral, $\widetilde{\mathbf{D}_4}$, is always stable. All bifurcations are pitchfork bifurcations. Note that the L^2 norm of the solution $x = (a, b, c)$ is given by $\|x\|^2 = a^2 + b^2 + c^2$. The diagram is not to scale but the relative sizes of the L^2 norms are shown.

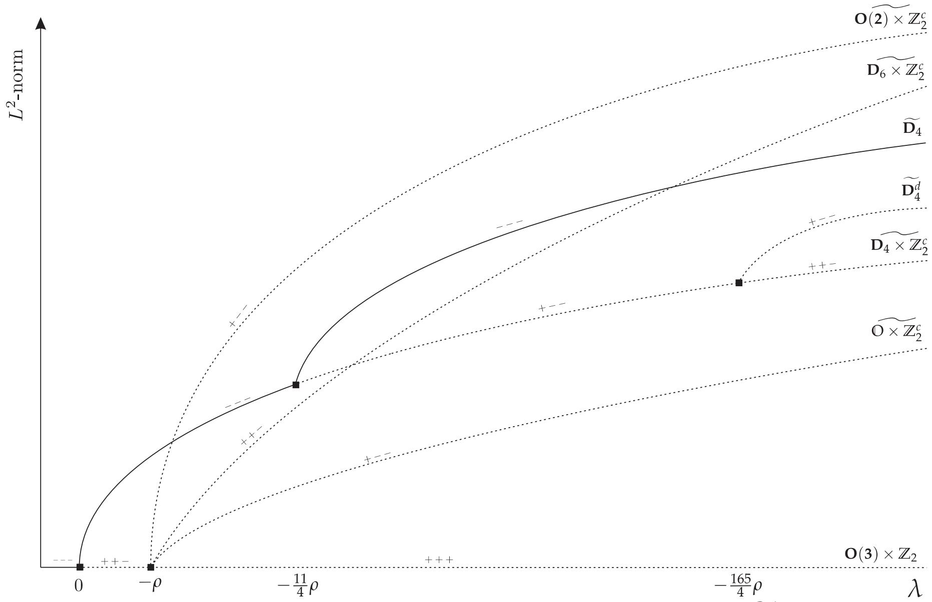


Figure 7.10: Bifurcation diagram for the Swift–Hohenberg equation in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ for the representation on $V_2 \oplus V_3$ when $\rho < 0$. We find that the solution with the symmetry group of a one-armed spiral, $\widetilde{\mathbf{D}_2}$, does not exist but a stable solution with $\widetilde{\mathbf{D}_4}$ does exist when $\lambda > -\frac{11}{4}\rho$ and is always stable. All bifurcations are pitchfork bifurcations. Note that the L^2 norm of the solution $x = (a, b, c)$ is given by $\|x\|^2 = a^2 + b^2 + c^2$. The diagram is not to scale but the relative sizes of the L^2 norms are shown.

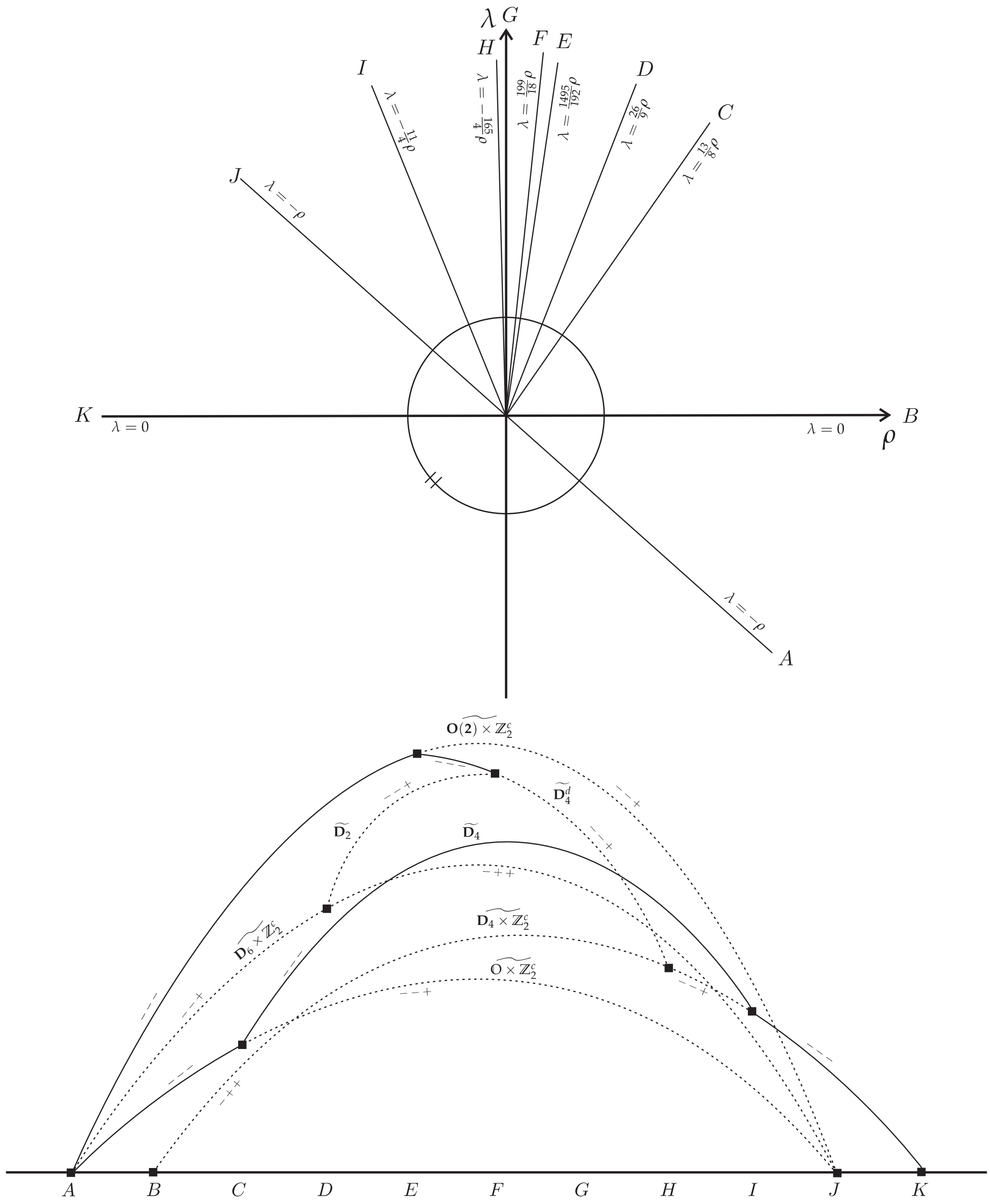


Figure 7.11: The top diagram is an unfolding diagram showing the lines on which bifurcations of the solution branches occur for the Swift–Hohenberg equation as the circle around the codimension 2 point, $\lambda = \rho = 0$, is traversed. The gyratory bifurcation diagram at the bottom of this figure shows the solution branches and their stability in Fix($\widetilde{\mathbf{D}_2}$). All bifurcations are pitchfork bifurcations.

Remark 7.2.6. We have found that for the Swift–Hohenberg equation on a sphere of radius near 3, and with the parameter μ near $\frac{1}{9}$, solutions with the symmetries of one- and two-armed spirals can exist for some values of λ and $\rho = \frac{8}{9}R_2$. Recall that the relationships between the parameters λ and ρ and the radius of the sphere R and the parameter μ in the Swift–Hohenberg equation (6.4.1) are given by

$$R = 3 + \epsilon^2 R_2 \qquad \text{and} \qquad \mu = \frac{1}{9} + \epsilon^2 \mu_2 = \frac{1}{9} + \epsilon^2 \left(\lambda + \frac{8}{27}R_2\right) = \frac{1}{9} + \epsilon^2 \left(\lambda + \frac{1}{3}\rho\right).$$

By restricting the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field for the representation of $\mathbf{O(3)} \times \mathbb{Z}_2$ on $V_2 \oplus V_3$ to the invariant subspace $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ we have been able to find explicit expressions for the seven branches of solutions which exist within this subspace and determine their stability within this subspace. We have found that although a one-armed spiral pattern with $\widetilde{\mathbf{D}_2}$ symmetry can exist for some values of the parameters λ and ρ, the solution is not stable in this subspace. We have also found that two-armed spiral patterns with $\widetilde{\mathbf{D}_4}$ symmetry can exist for some values of the parameters λ and ρ and moreover these solutions are stable within the subspace $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$.

Sufficiently close to the codimension 2 point $\lambda = \rho = 0$, numerical simulations of the Swift–Hohenberg equation (6.4.1) in the subspace $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ on a sphere of radius near 3 agree with the analytical results above. With initial conditions within the invariant subspace $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ we can find the stable solution branches and bifurcation points as in Figures 7.9 – 7.11 by varying the values of ρ and λ.

However, if the initial conditions are random in the whole 12-dimensional space $V_2 \oplus V_3$ then the simulations lead us to believe that a solution with symmetry group $\mathbf{D}_4^d$ (see Table 6.9) may be stable in the whole space. These solutions have the symmetries of the 'tennis ball' pattern as discovered in the numerical simulations of Matthews [65]. We now investigate the stability of this solution analytically in the whole space, $V_2 \oplus V_3$ and also compute all of the eigenvalues of the solution with $\widetilde{\mathbf{D}_4}$ symmetry to determine if two-armed spirals can be stable in the whole space for the Swift–Hohenberg equation.

Stability of two-armed spirals in $V_2 \oplus V_3$ for the Swift–Hohenberg equation

In Example 7.2.5 we found that in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ the solution with the symmetries of a two-armed spiral on a sphere, $\widetilde{\mathbf{D}_4}$, is stable in the Swift–Hohenberg equation for all values of the bifurcation parameters λ and ρ where it exists. We now investigate whether this solution is stable in the whole space $V_2 \oplus V_3$ for any values of λ and ρ.

To do this we must compute the values of the nine eigenvalues in the complement of $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$. We find that four of these eigenvalues are the roots of

$$\left[425250\xi^2 + 405\,(2124\lambda + 451\rho)\,\xi + 126\,(\lambda - \rho)\,(2376\lambda + 4329\rho)\right]^2 = 0$$

so the eigenvalues are double and for all values of λ and ρ where the solution exists $2124\lambda + 451\rho > 0$ and $126\,(\lambda - \rho)\,(2376\lambda + 4329\rho) > 0$ so the eigenvalues always have negative real part.

Another of the eigenvalues is

$$\xi = -\frac{2}{35}(27\lambda + 13\rho)$$

which is negative for all values of λ and ρ where the solution exists. The four other eigenvalues are zero. Of these only three are forced to be zero by symmetry so one of them would be found to be nonzero if we were to consider a higher order truncation of the equivariant vector field. Hence it may be possible for this solution to be stable in the Swift–Hohenberg equation on a sphere with radius near 3, depending on the values of coefficients of fifth order terms in the equivariant vector field. However, numerical simulations suggest that the solution is in fact unstable. Small perturbations from this solution lead the system to prefer a solution with $\mathbf{D}_4^d$ symmetry. We now consider the stability of this 'tennis ball solution' with symmetry group $\mathbf{D}_4^d$.

Stability of tennis ball pattern in $V_2 \oplus V_3$ for the Swift–Hohenberg equation

Numerical simulations of the Swift–Hohenberg on a sphere of radius approximately 3 with random initial conditions i.e. starting at any point in $V_2 \oplus V_3$, lead us to believe that the solution with $\mathbf{D}_4^d$ symmetry is stable when it exists. This solution has the symmetries of a tennis ball as found in the numerical simulations of Matthews [65]. Such a pattern is shown in Figure 7.12.

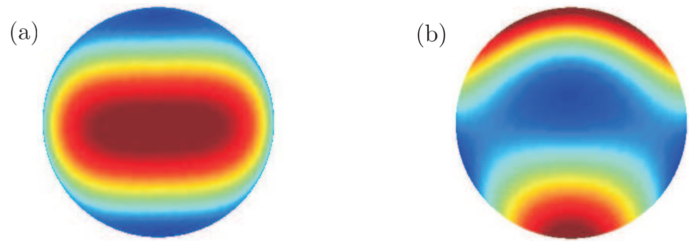


Figure 7.12: Images of a pattern with $\mathbf{D}_4^d$ symmetry viewed (a) from the top and (b) from the side.

The subgroup $\mathbf{D}_4^d \subset \mathbf{O(3)} \times \mathbf{Z}_2$ is certainly an isotropy subgroup in this representation (see Table 6.9). We now confirm analytically, up to a degeneracy in one eigenvalue, that this solution is indeed stable in $V_2 \oplus V_3$ at cubic order.

Recall from Table 6.9 that

$$\mathrm{Fix}(\mathbf{D}_4^d) = \{(0, 0, a, 0, 0\,;\, 0, ib, 0, 0, 0, -ib, 0)\} \tag{7.2.20}$$

where $a, b \in \mathbb{R}$. We find that in the subspace $\mathrm{Fix}(\mathbf{D}_4^d)$ the equivariant vector field on $V_2 \oplus V_3$ with the Swift–Hohenberg coefficients (7.2.16) reduces to

$$\dot{a} = \lambda a - \frac{15}{28\pi}a^3 - \frac{15}{22\pi}ab^2 \tag{7.2.21}$$

$$\dot{b} = (\lambda + \rho)\, b - \frac{315}{286\pi}b^3 - \frac{15}{44\pi}a^2 b. \tag{7.2.22}$$

These equations have a fixed-point solution with $\mathbf{D}_4^d$ symmetry given by

$$a^2 = \frac{11}{75}\pi\,(8\lambda - 13\rho) \qquad b^2 = \frac{286}{525}\pi\,(4\lambda + 11\rho)$$

which exists when $\rho > 0$ and $\lambda > \frac{13}{8}\rho$ or $\rho < 0$ and $\lambda > -\frac{11}{4}\rho$.

The nonzero eigenvalues of this solution in $V_2 \oplus V_3$ are the roots of quadratic equations

$$\xi^2 + B\xi + C = 0$$

and so have negative real part when $B > 0$ and $C > 0$. The quadratic equations are as follows.

- Two eigenvalues are the roots of

$$1575\xi^2 + 30\,(109\lambda + 18\rho)\,\xi + 26\,(4\lambda + 11\rho)\,(10\lambda - 11\rho) = 0.$$

Since when the solution with $\mathbf{D}_4^d$ symmetry exists

$$\lambda > -\frac{18}{109}\rho \qquad \text{and} \qquad (4\lambda + 11\rho)\,(10\lambda - 11\rho) > 0,$$

the roots always have negative real parts.

- Another two eigenvalues are the roots of

$$35\xi^2 + 2\,(43\lambda + 22\rho)\,\xi + (4\lambda + 11\rho)\,(8\lambda - 13\rho) = 0.$$

Since when the solution exists

$$\lambda > -\frac{22}{43}\rho \qquad \text{and} \qquad (4\lambda + 11\rho)\,(8\lambda - 13\rho) > 0,$$

the roots always have negative real parts.

- There are double eigenvalues which are the roots of

$$15750\xi^2 + 15\,(2012\lambda + 633\rho)\,\xi + 2\,(376\lambda + 299\rho)\,(13\lambda - 8\rho) = 0.$$

Since when the solution exists

$$\lambda > -\frac{299}{376}\rho \qquad \text{and} \qquad (376\lambda + 299\rho)\,(13\lambda - 8\rho) > 0,$$

the roots always have negative real parts.

Hence all of the non-zero eigenvalues have negative real part for all values of λ and ρ where the solution exists. We find that there are four zero eigenvalues using the cubic order truncation of the equivariant vector field on $V_2 \oplus V_3$. However only three of the four zero eigenvalues are forced to be zero by symmetry. The other eigenvalue would be found to be non-zero if we were to consider a higher order truncation of the equivariant vector field. Hence, as for the solution with $\widetilde{\mathbf{D}_4}$ symmetry, it may be that this solution is stable in the Swift–Hohenberg equation on a sphere with radius near 3, depending on the values of coefficients of fifth order terms in the equivariant vector field. Numerical simulations indicate that this solution is stable to any perturbation in $V_2 \oplus V_3$. This suggests that the final eigenvalue which depends on the fifth order coefficients has negative real part when the solution exists.

7.2.2 Spiral patterns in the representation on $V_3 \oplus V_4$

We next consider spiral patterns which can exist in the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field for the representation on $V_3 \oplus V_4$ as computed in Section 6.3.6. The existence of solutions with the symmetries of the axial isotropy subgroups in this representation is guaranteed by the equivariant branching lemma (Theorem 2.4.6) and solutions with the symmetries of the other isotropy subgroups may exist depending on the values of the coefficients in the equivariant vector field. By Proposition 7.1.1 the twisted subgroups $\widetilde{\mathbf{D}_2}$, $\widetilde{\mathbf{D}_4}$ and $\widetilde{\mathbf{D}_6}$ are isotropy subgroups in this representation. These are the symmetry groups of the most symmetric one-, two- and three-armed spiral patterns on the sphere respectively. We will determine if patterns with these symmetries can exist for any values of the coefficients in the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field and if so, we compute how these solutions bifurcate from other solutions. To do this we will study the equilibria in the restriction of the vector field to $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$. These solutions have the symmetries of the isotropy subgroups which contain $\widetilde{\mathbf{D}_2}$. We begin by computing these isotropy subgroups.

Isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in the representation on $V_3 \oplus V_4$ which contain $\widetilde{\mathbf{D}_2}$

As discussed in Section 6.2.2, for the reducible representation of $\mathbf{O(3)} \times \mathbb{Z}_2$ on $V_3 \oplus V_4$ the axial isotropy subgroups are precisely the axial isotropy subgroups for the irreducible representations of $\mathbf{O(3)} \times \mathbb{Z}_2$ on V_3 and V_4 as found in Examples 6.2.6 and 6.2.7. Using Table 6.7 and the massive chain criterion (Theorem 3.4.1) we can compute that the isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in the representation on $V_3 \oplus V_4$ which contain $\widetilde{\mathbf{D}_2}$ are as in Table 7.2. Note that $\widetilde{\mathbf{D}_2}$ itself is an isotropy subgroup with

$$\mathrm{Fix}(\widetilde{\mathbf{D}_2}) = \{(0, a, 0, b, 0, a, 0\,;\ \mathrm{i}c, 0, \mathrm{i}d, 0, 0, 0, -\mathrm{i}d, 0, -\mathrm{i}c)\} \tag{7.2.23}$$

for the copy of $\widetilde{\mathbf{D}_2}$ which is given by

$$\widetilde{\mathbf{D}_2} = \langle\ (R^y_\pi, -1),\ (R^z_\pi, 1)\ \rangle.$$

The one-armed spiral with this symmetry spirals between the two points on the surface of the sphere which lie on the y-axis. In Table 7.2, for each isotropy subgroup H^θ, we give the form of the invariant subspace $\mathrm{Fix}(H^\theta)$ which is contained in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ as above. The copy of $\mathrm{Fix}(\widetilde{\mathbf{D}_4})$ given in Table 7.2 is that of a two-armed spiral with tips at the poles whereas the copy of $\mathrm{Fix}(\widetilde{\mathbf{D}_6})$ contained in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ as in (7.2.23) defines a three-armed spiral which, like the one-armed spiral, has its tips on the y-axis.

The section of the lattice of isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in this representation including only those isotropy subgroups with contain $\widetilde{\mathbf{D}_2}$ is as in Figure 7.13.

The restriction of the equivariant vector field to $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$

To discover whether it is possible for solutions with the symmetries of one-, two- and three-armed spirals ($\widetilde{\mathbf{D}_2}$, $\widetilde{\mathbf{D}_4}$ and $\widetilde{\mathbf{D}_6}$ respectively) to exist in the equivariant vector field (6.3.24)–

Isotropy subgroup H^θ	H	K	Fixed-point subspace
$\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{O}^-$	$\{(0,a,0,0,0,a,0\,;\,0,0,0,0,0,0,0,0,0)\}$
$\widetilde{\mathbf{O(2)} \times \mathbb{Z}_2^c}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbf{O(2)}^-$	$\{(0,0,0,b,0,0,0\,;\,0,0,0,0,0,0,0,0,0)\}$
$(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c})_{\mathbf{D}_6^d}$	$\mathbf{D}_6 \times \mathbb{Z}_2^c$	$\mathbf{D}_6^d$	$\left\{\left(0,\sqrt{\frac{3}{10}}b,0,b,0,\sqrt{\frac{3}{10}}b,0\,;\,0,0,0,0,0,0,0,0,0\right)\right\}$
$\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$	$\mathbf{D}_4 \times \mathbb{Z}_2^c$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\{(0,0,0,0,0,0,0\,;\,0,0,\mathrm{i}d,0,0,0,-\mathrm{i}d,0,0)\}$
$(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c})_{\mathbf{D}_3 \times \mathbb{Z}_2^c}$	$\mathbf{D}_6 \times \mathbb{Z}_2^c$	$\mathbf{D}_3 \times \mathbb{Z}_2^c$	$\left\{\left(0,0,0,0,0,0,0\,;\,\mathrm{i}c,0,\sqrt{7}\mathrm{i}c,0,0,0,-\sqrt{7}\mathrm{i}c,0,-\mathrm{i}c\right)\right\}$
$\widetilde{\mathbf{D}_8 \times \mathbb{Z}_2^c}$	$\mathbf{D}_8 \times \mathbb{Z}_2^c$	$\mathbf{D}_4 \times \mathbb{Z}_2^c$	$\{(0,0,0,0,0,0,0\,;\,\mathrm{i}c,0,0,0,0,0,0,0,-\mathrm{i}c)\}$
$(\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c})_{\mathbf{D}_2^z}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbf{D}_2^z$	$\{(0,a,0,b,0,a,0\,;\,0,0,0,0,0,0,0,0,0)\}$
$(\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c})_{\mathbb{Z}_2 \times \mathbb{Z}_2^c}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2 \times \mathbb{Z}_2^c$	$\{(0,0,0,0,0,0,0\,;\,\mathrm{i}c,0,\mathrm{i}d,0,0,0,-\mathrm{i}d,0,-\mathrm{i}c)\}$
$\widetilde{\mathbf{D}_6}$	$\mathbf{D}_6$	$\mathbf{D}_3$	$\left\{\left(0,\sqrt{\frac{3}{10}}b,0,b,0,\sqrt{\frac{3}{10}}b,0\,;\,\mathrm{i}c,0,\sqrt{7}\mathrm{i}c,0,0,0,-\sqrt{7}\mathrm{i}c,0,-\mathrm{i}c\right)\right\}$
$\widetilde{\mathbf{D}_4}$	$\mathbf{D}_4$	$\mathbf{D}_2$	$\{(0,a,0,0,0,a,0\,;\,0,0,\mathrm{i}d,0,0,0,\mathrm{i}d,0,0)\}$
$\widetilde{\mathbf{D}_8^d}$	$\mathbf{D}_8^d$	$\mathbf{D}_4^z$	$\{(0,0,0,b,0,0,0\,;\,\mathrm{i}c,0,0,0,0,0,0,0,-\mathrm{i}c)\}$
$(\widetilde{\mathbf{D}_4^d})_{\mathbf{D}_2^z}$	$\mathbf{D}_4^d$	$\mathbf{D}_2^z$	$\{(0,0,0,b,0,0,0\,;\,0,0,\mathrm{i}d,0,0,0,-\mathrm{i}d,0,0)\}$
$(\widetilde{\mathbf{D}_4^d})_{\mathbb{Z}_4^-}$	$\mathbf{D}_4^d$	$\mathbb{Z}_4^-$	$\{(0,a,0,0,0,a,0\,;\,\mathrm{i}c,0,0,0,0,0,0,0,-\mathrm{i}c)\}$
$\widetilde{\mathbf{D}_2}$	$\mathbf{D}_2$	$\mathbb{Z}_2$	$\{(0,a,0,b,0,a,0\,;\,\mathrm{i}c,0,\mathrm{i}d,0,0,0,-\mathrm{i}d,0,-\mathrm{i}c)\}$

Table 7.2: Isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in the representation on $V_3 \oplus V_4$ which contain $\widetilde{\mathbf{D}_2}$.

(6.3.25) we consider the restriction to the subspace $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ given by (7.2.23) where

$$\begin{aligned}\dot{a} &= \mu_x a + (2\alpha_1 - 50\gamma_1)a^3 + (\alpha_1 + 15\gamma_1)ab^2 + (2\beta_1 - 420\gamma_2 + 56\gamma_3 - 140\gamma_4)ac^2 \\ &\quad + (2\beta_1 - 320\gamma_2 + 26\gamma_3 - 140\gamma_4)ad^2 + (2\gamma_2 - \gamma_3 + 4\gamma_4)\sqrt{210}bcd \qquad (7.2.24)\\ \dot{b} &= \mu_x b + (2\alpha_1 + 30\gamma_1)ba^2 + (\alpha_1 - 9\gamma_1)b^3 + (2\beta_1 - 336\gamma_2 + 42\gamma_3 - 252\gamma_4)bc^2 \\ &\quad + (2\beta_1 - 324\gamma_2 + 24\gamma_3 - 108\gamma_4)bd^2 + 2(2\gamma_2 - \gamma_3 + 4\gamma_4)\sqrt{210}acd \qquad (7.2.25)\\ \dot{c} &= \mu_y c + 2\beta_2 ca^2 + (\beta_2 + 7\delta_2 + 7\delta_3 + 21\delta_4)cb^2 + 2\alpha_2 c^3 + (2\alpha_2 - 280\delta_1)cd^2 \\ &\quad + (\delta_2 + \delta_3 - \delta_4)\sqrt{210}abd \qquad (7.2.26)\\ \dot{d} &= \mu_y d + (2\beta_2 + 60\delta_2 + 30\delta_3 + 10\delta_4)da^2 + (\beta_2 + 16\delta_2 + 16\delta_3)db^2 + (2\alpha_2 - 280\delta_1)dc^2 \\ &\quad + (2\alpha_2 - 240\delta_1)d^3 + (\delta_2 + \delta_3 - \delta_4)\sqrt{210}abc \qquad (7.2.27)\end{aligned}$$

As for the representation on $V_2 \oplus V_3$, these equations have residual symmetry

$$N(\widetilde{\mathbf{D}_2})/\widetilde{\mathbf{D}_2} = \mathbf{D}_4 \times \mathbb{Z}_2^c \times \mathbb{Z}_2/\widetilde{\mathbf{D}_2} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

so if solutions with $\widetilde{\mathbf{D}_2}$ symmetry exist then there are $|N(\widetilde{\mathbf{D}_2})/\widetilde{\mathbf{D}_2}| = 8$ equivalent solutions within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$.

As in the case for the representation on $V_2 \oplus V_3$, we assume that $\mu_x = \lambda$ and $\mu_y = \lambda + \rho$. Then the trivial solution is stable when $\lambda < \min(0, -\rho)$. At $\lambda = 0$ the $\ell = 3$ modes become unstable and the equivariant branching lemma guarantees that the unrestricted system (6.3.24)–(6.3.25) has solution branches with the symmetries of axial isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in the representation on V_3 which bifurcate at $\lambda = 0$. Similarly at $\lambda = -\rho$ the $\ell = 4$ modes become unstable and solution branches with the symmetries of axial isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ in the representation on V_4 bifurcate.

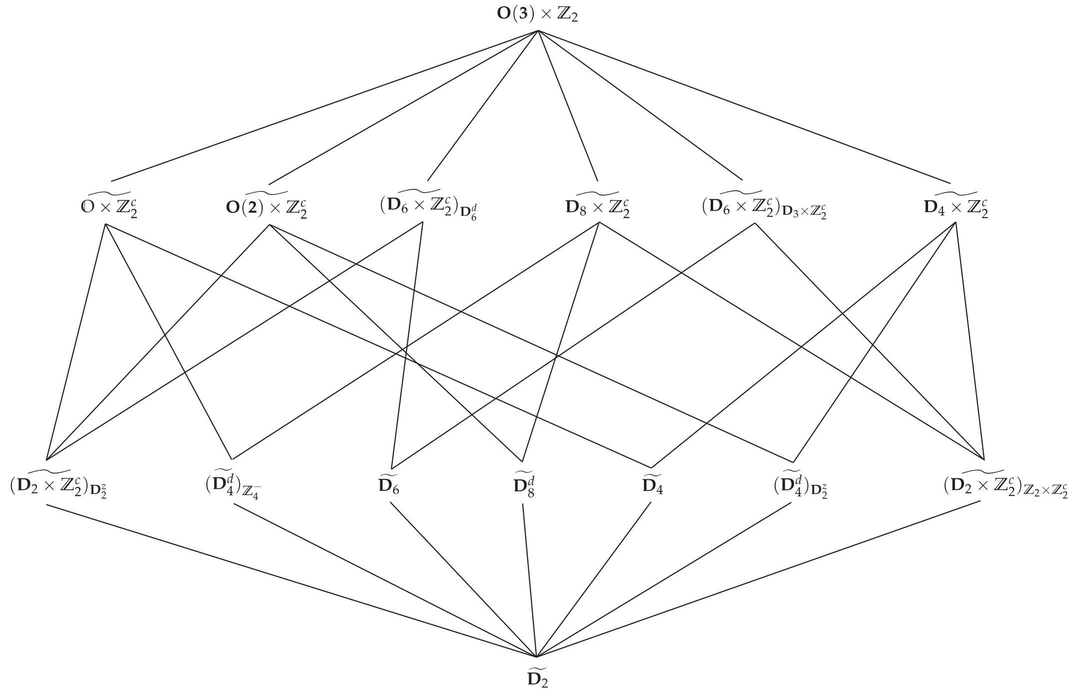


Figure 7.13: The lattice of isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ which contain $\widetilde{\mathbf{D}_2}$ for the representation on $V_3 \oplus V_4$.

It is possible to find analytic expressions for solutions to (7.2.24)–(7.2.27) with the symmetries of all isotropy subgroups in Table 7.2 with the exceptions of $(\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c})_{\mathbb{Z}_2 \times \mathbb{Z}_2^c}$, $(\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c})_{\mathbf{D}_2^z}$ and $\widetilde{\mathbf{D}_2}$. Solutions with $(\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c})_{\mathbb{Z}_2 \times \mathbb{Z}_2^c}$ or $(\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c})_{\mathbf{D}_2^z}$ symmetry do not exist; every solution in $\mathrm{Fix}((\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c})_{\mathbb{Z}_2 \times \mathbb{Z}_2^c})$ has $(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c})_{\mathbf{D}_3 \times \mathbb{Z}_2^c}$ symmetry and every solution in $\mathrm{Fix}((\widetilde{\mathbf{D}_2 \times \mathbb{Z}_2^c})_{\mathbf{D}_2^z})$ has $(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c})_{\mathbf{D}_6^d}$ symmetry. Solutions with $\widetilde{\mathbf{D}_2}$ symmetry do exist but unfortunately, unlike the case for the representation on $V_2 \oplus V_3$, we are not able to find an expression for the equilibrium solutions to (7.2.24)–(7.2.27) with $\widetilde{\mathbf{D}_2}$ symmetry. To find solutions with $\widetilde{\mathbf{D}_2}$ symmetry and their stability within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it is necessary to use a numerical branch continuation package such as AUTO. This requires us to give values for the coefficients α_1, α_2, β_1, β_2, γ_1, γ_2, γ_3, γ_4, δ_1, δ_2, δ_3 and δ_4.

In Example 7.2.7 we will consider the system (7.2.24)–(7.2.27) with the values of the coefficients which we computed in Section 6.4 for the Swift–Hohenberg equation on a sphere of radius approximately 4.

Example 7.2.7 (Solutions with spiral symmetry in the Swift–Hohenberg equation on a sphere of radius near 4). Recall that in Section 6.4.2 we found that for the Swift–Hohenberg equation (6.4.1) on a sphere of radius $R = 4 + \epsilon^2 R_2$ the relevant representation of $\mathbf{O}(\mathbf{3})$ is the representation on $V_3 \oplus V_4$ and the critical value of the parameter μ is $\frac{1}{16}$. We computed that the values of the coefficients in the equivariant vector field are

$$\mu_x = \mu_2 - \frac{3}{16}R_2, \ \ \alpha_1 = -\frac{175}{286\pi}, \ \ \beta_1 = -\frac{735}{572\pi}, \ \ \gamma_1 = -\frac{7}{2860\pi}, \tag{7.2.28}$$

$$\gamma_2 = -\frac{21}{2860\pi}, \ \ \gamma_3 = -\frac{5}{143\pi}, \ \ \gamma_4 = -\frac{7}{2860\pi}, \ \ \mu_y = \mu_2 + \frac{5}{16}R_2, \ \ \alpha_2 = -\frac{6615}{9724\pi},$$

$$\beta_2 = -\frac{315}{572\pi}, \quad \delta_1 = -\frac{27}{34034\pi}, \quad \delta_2 = -\frac{3}{286\pi} \quad \delta_3 = \frac{3}{143\pi} \quad \text{and} \quad \delta_4 = 0.$$

Let $\mu_x = \lambda$ then $\mu_y = \lambda + \rho$ where $\rho = \frac{1}{2}R_2$. Substituting these values into equations (7.2.24)–(7.2.27) we have

$$\dot{a} = \lambda a - \frac{315}{286\pi}a^3 - \frac{371}{572\pi}ab^2 - \frac{315}{286\pi}ac^2 - \frac{225}{286\pi}ad^2 + \frac{3\sqrt{210}}{286\pi}bcd \quad (7.2.29)$$

$$\dot{b} = \lambda b - \frac{1687}{2860\pi}b^3 - \frac{371}{286\pi}ba^2 - \frac{21}{22\pi}bc^2 - \frac{219}{286\pi}bd^2 + \frac{3\sqrt{210}}{143\pi}acd \quad (7.2.30)$$

$$\dot{c} = (\lambda + \rho)\,c - \frac{6615}{4862\pi}c^3 - \frac{315}{286\pi}ca^2 - \frac{21}{44\pi}cb^2 - \frac{5535}{4862\pi}cd^2 + \frac{3\sqrt{210}}{286\pi}abd \quad (7.2.31)$$

$$\dot{d} = (\lambda + \rho)\,d - \frac{39825}{34034\pi}d^3 - \frac{225}{286\pi}da^2 - \frac{219}{572\pi}db^2 - \frac{5535}{4862\pi}dc^2 + \frac{3\sqrt{210}}{286\pi}abc. \quad (7.2.32)$$

We now find the solutions of (7.2.29)–(7.2.32) for which there are explicit expressions. We compute their stability in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ and expect to find bifurcations where it is possible that a solution with $\widetilde{\mathbf{D}_2}$ symmetry may be created.

1. **(The solution branch with $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ symmetry)** Since $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ is an axial isotropy subgroup, a branch of solutions with this symmetry bifurcating from $\lambda = 0$ is guaranteed to exist by the equivariant branching lemma. It is given by $b = c = d = 0$ and
$$a^2 = \frac{286}{315}\pi\lambda.$$
The solution branch bifurcates supercritically so exists when $\lambda > 0$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -2\lambda \qquad \xi_2 = -\frac{8}{45}\lambda \qquad \xi_3 = \rho \qquad \xi_4 = \frac{(2\lambda + 7\rho)}{7}.$$
When $\rho > 0$ the branch is unstable but when $\rho < 0$ it is stable when $\lambda < -\frac{7}{2}\rho$. There are stationary bifurcations when $\rho = 0$ and also when $\lambda = -\frac{7}{2}\rho$ for $\rho < 0$.

2. **(The solution branch with $\widetilde{\mathbf{O}(2) \times \mathbb{Z}_2^c}$ symmetry)** Since $\widetilde{\mathbf{O}(2) \times \mathbb{Z}_2^c}$ is also an axial isotropy subgroup a branch of solutions with this symmetry bifurcating from $\lambda = 0$ is guaranteed to exist. It is given by $a = c = d = 0$ and
$$b^2 = \frac{2860}{1687}\pi\lambda.$$
The solution branch bifurcates supercritically so exists when $\lambda > 0$. Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues
$$\xi_1 = -2\lambda \qquad \xi_2 = -\frac{24}{241}\lambda \qquad \xi_3 = \frac{(46\lambda + 241\rho)}{241} \qquad \xi_4 = \frac{(592\lambda + 1687\rho)}{1687}.$$
When $\rho > 0$ the branch is unstable but when $\rho < 0$ it is stable when $\lambda < -\frac{1687}{592}\rho$. There are stationary bifurcations when $\lambda = -\frac{241}{46}\rho$ and $\lambda = -\frac{1687}{592}\rho$ for $\rho < 0$.

3. **(The solution branch with $(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c})_{\mathbf{D}_6^d}$ symmetry)** A branch of solutions with this symmetry is guaranteed to bifurcate from $\lambda = 0$ since $(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c})_{\mathbf{D}_6^d}$ is an axial isotropy subgroup. It is given by $c = d = 0$, $a = \sqrt{\frac{3}{10}}b$ and
$$b^2 = \frac{143}{140}\pi\lambda.$$

The solution branch bifurcates supercritically so exists when $\lambda > 0$. Within Fix($\widetilde{\mathbf{D}_2}$) it has eigenvalues

$$\xi_1 = -2\lambda \qquad \xi_2 = \frac{3}{25}\lambda \qquad \xi_3 = \frac{(2\lambda + 5\rho)}{5} \qquad \xi_4 = \frac{(\lambda + 7\rho)}{7}.$$

Since ξ_1 and ξ_2 have opposite signs this solution can never be stable. When $\rho < 0$ there are stationary bifurcations when $\lambda = -\frac{5}{2}\rho$ and $\lambda = -7\rho$.

4. **(The solution branch with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry)** A branch of solutions with this symmetry is guaranteed to bifurcate from $\lambda = -\rho$ since $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ is an axial isotropy subgroup. It is given by $a = b = c = 0$ and
$$d^2 = \frac{34034}{39825}\pi\left(\lambda + \rho\right).$$
The solution branch bifurcates supercritically so exists when $\lambda > -\rho$. Within Fix($\widetilde{\mathbf{D}_2}$) it has eigenvalues
$$\xi_1 = -2\left(\lambda + \rho\right) \qquad \xi_2 = \frac{8}{295}\left(\lambda + \rho\right) \qquad \xi_3 = \frac{(58\lambda - 119\rho)}{177} \qquad \xi_4 = \frac{(4588\lambda - 8687\rho)}{13275}.$$
Since ξ_1 and ξ_2 have opposite signs this solution can never be stable. When $\rho > 0$ there are stationary bifurcations of this solution when $\lambda = \frac{119}{58}\rho$ and $\lambda = \frac{8687}{4588}\rho$.

5. **(The solution branch with $(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c})_{\mathbf{D}_3 \times \mathbb{Z}_2^c}$ symmetry)** Since $(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c})_{\mathbf{D}_3 \times \mathbb{Z}_2^c}$ is an axial isotropy subgroup, a branch of solutions with this symmetry bifurcating from $\lambda = -\rho$ is guaranteed to exist. It is given by $a = b = 0$, $d = \sqrt{7}c$ and
$$c^2 = \frac{2431}{22680}\pi\left(\lambda + \rho\right).$$
The solution branch bifurcates supercritically so exists when $\lambda > -\rho$. Within Fix($\widetilde{\mathbf{D}_2}$) it has eigenvalues
$$\xi_1 = -2\left(\lambda + \rho\right) \qquad \xi_2 = -\frac{1}{21}\left(\lambda + \rho\right) \qquad \xi_3 = \frac{(11\lambda - 34\rho)}{45} \qquad \xi_4 = \frac{(10\lambda - 17\rho)}{27}.$$
When $\rho < 0$ the branch is unstable but when $\rho > 0$ it is stable when $\lambda < \frac{17}{10}\rho$. There are stationary bifurcations of this solution when $\lambda = \frac{17}{10}\rho$ and $\lambda = \frac{34}{11}\rho$ for $\rho < 0$.

6. **(The solution branch with $\widetilde{\mathbf{D}_8 \times \mathbb{Z}_2^c}$ symmetry)** Since $\widetilde{\mathbf{D}_8 \times \mathbb{Z}_2^c}$ is an axial isotropy subgroup, a branch of solutions with this symmetry which bifurcates from $\lambda = -\rho$ is guaranteed to exist. It is given by $a = b = d = 0$ and
$$c^2 = \frac{4862}{6615}\pi\left(\lambda + \rho\right).$$
The solution branch bifurcates supercritically so exists when $\lambda > -\rho$. Within Fix($\widetilde{\mathbf{D}_2}$) it has eigenvalues
$$\xi_1 = -2\left(\lambda + \rho\right) \qquad \xi_2 = \frac{8}{49}\left(\lambda + \rho\right) \qquad \xi_3 = \frac{(4\lambda - 17\rho)}{21} \qquad \xi_4 = \frac{(94\lambda - 221\rho)}{315}.$$
Since ξ_1 and ξ_2 have opposite signs this solution can never be stable. When $\rho > 0$ there are stationary bifurcations of this solution when $\lambda = \frac{17}{4}\rho$ and $\lambda = \frac{221}{94}\rho$.

7. **(The solution branch with $\widetilde{\mathbf{D}_6}$ symmetry)** This solution with the symmetries of a three-armed spiral is given by $a = \sqrt{\frac{3}{10}}b$ and $d = \sqrt{7}c$ where

$$b^2 = \frac{143\pi\,(10\lambda - 17\rho)}{2352} \qquad \text{and} \qquad c^2 = \frac{2431\pi\,(2\lambda + 5\rho)}{70560}$$

and so it exists when $\rho > 0$ and $\lambda > \frac{17}{10}\rho$ and also when $\rho < 0$ and $\lambda > -\frac{5}{2}\rho$. It bifurcates from the solution with $(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c})_{\mathbf{D}_3\times\mathbb{Z}_2^c}$ symmetry at $\lambda = \frac{17}{10}\rho$ when $\rho > 0$ and from the solution with $(\widetilde{\mathbf{D}_6 \times \mathbb{Z}_2^c})_{\mathbf{D}_6^d}$ symmetry at $\lambda = -\frac{5}{2}\rho$ when $\rho < 0$.

Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues

$$\xi_1 = -\frac{9}{49}\,(\lambda - \rho) \qquad \xi_2 = -\frac{1}{105}\,(\lambda + 34\rho)$$

and ξ_3 and ξ_4 are the roots of

$$21\xi^2 + 4\,(13\lambda + 2\rho)\,\xi + 4\,(10\lambda - 17\rho)\,(2\lambda + 5\rho) = 0.$$

Since $13\lambda + 2\rho > 0$ and $(10\lambda - 17\rho)\,(2\lambda + 5\rho) > 0$ for all values of λ and ρ where the solution exists, these eigenvalues always have negative real parts. We can see that $\xi_1 < 0$ when the solution exists and when $\rho < 0$ there is a stationary bifurcation at $\lambda = -34\rho$ where it may be possible that a branch of solutions with $\widetilde{\mathbf{D}_2}$ symmetry bifurcates. This solution branch is always stable when $\rho > 0$ and also when $\rho < 0$ and $\lambda > -34\rho$.

8. **(The solution branch with $\widetilde{\mathbf{D}_4}$ symmetry)** This solution with the symmetries of a two-armed spiral is given by $b = c = 0$,

$$a^2 = \frac{143\pi\,(58\lambda - 119\rho)}{14490} \qquad \text{and} \qquad d^2 = \frac{2431\pi\,(2\lambda + 7\rho)}{10350}$$

and so it exists when $\rho > 0$ and $\lambda > \frac{119}{58}\rho$ and also when $\rho < 0$ and $\lambda > -\frac{7}{2}\rho$. It bifurcates from the solution with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = \frac{119}{58}\rho$ when $\rho > 0$ and from the solution with $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ symmetry at $\lambda = -\frac{7}{2}\rho$ when $\rho < 0$.

Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues which are the roots of the two quadratic equations

$$161\xi^2 + (380\lambda + 203\rho)\,\xi + (58\lambda - 119\rho)\,(2\lambda + 7\rho) = 0 \tag{7.2.33}$$

$$595125\xi^2 + 115\,(1384\lambda - 3551\rho)\,\xi + \left(2656\lambda^2 - 62368\lambda\rho + 119476\rho^2\right) = 0. \tag{7.2.34}$$

The roots of (7.2.33) always have negative real part for the values of ρ and λ where the solution exists. The eigenvalues resulting from (7.2.34) are zero when

$$\lambda = \left(\frac{1949}{166} \pm \frac{345}{332}\sqrt{86}\right)\rho.$$

Hence when $\rho > 0$ there are stationary bifurcations at these points where it is possible for solutions with $\widetilde{\mathbf{D}_2}$ symmetry to bifurcate. This solution branch is always stable where exists for $\rho < 0$ and when $\rho > 0$ it is stable when $\lambda > \left(\frac{1949}{166} + \frac{345}{332}\sqrt{86}\right)\rho$.

9. **(The solution branch with $\widetilde{\mathbf{D}_8^d}$ symmetry)** This solution is given by $a = d = 0$,

$$b^2 = \frac{143\pi\,(94\lambda - 221\rho)}{11487} \qquad \text{and} \qquad c^2 = \frac{2431\pi\,(46\lambda + 241\rho)}{344610}$$

and so it exists when $\rho > 0$ and $\lambda > \frac{221}{94}\rho$ and also when $\rho < 0$ and $\lambda > -\frac{241}{46}\rho$. It bifurcates from the solution with $\widetilde{\mathbf{D}_8 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = \frac{221}{94}\rho$ when $\rho > 0$ and from the solution with $\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c}$ symmetry at $\lambda = -\frac{241}{46}\rho$ when $\rho < 0$.

Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues which are the roots of the two quadratic equations

$$8205\xi^2 + (18572\lambda + 11327\rho)\,\xi + (94\lambda - 221\rho)\,(46\lambda + 241\rho) = 0 \tag{7.2.35}$$

$$6283389\xi^2 - (415720\lambda - 187621\rho)\,\xi - \left(188640\lambda^2 + 346464\lambda\rho - 613836\rho^2\right) = 0. \tag{7.2.36}$$

The roots of (7.2.35) always have negative real part for the values of ρ and λ where the solution exists. Equation (7.2.36) gives one positive and one negative eigenvalue for all values of λ and ρ where the solution exists and hence this solution branch is never stable and undergoes no bifurcations.

10. **(The solution branch with $(\widetilde{\mathbf{D}_4^d})_{\mathbf{D}_2^z}$ symmetry)** This solution is given by $a = c = 0$,

$$b^2 = \frac{143\pi\,(4588\lambda - 8687\rho)}{644133} \qquad \text{and} \qquad d^2 = \frac{2431\pi\,(592\lambda + 1687\rho)}{2760570}$$

and so it exists when $\rho > 0$ and $\lambda > \frac{8687}{4588}\rho$ and also when $\rho < 0$ and $\lambda > -\frac{1687}{592}\rho$. It bifurcates from the solution with $\widetilde{\mathbf{D}_4 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = \frac{8687}{4588}\rho$ when $\rho > 0$ and from the solution with $\widetilde{\mathbf{O}(\mathbf{2}) \times \mathbb{Z}_2^c}$ symmetry at $\lambda = -\frac{1687}{592}\rho$ when $\rho < 0$.

Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues which are the roots of the two quadratic equations

$$3220665\xi^2 + (7799378\lambda + 3869978\rho)\,\xi + (592\lambda + 1687\rho)\,(4588\lambda - 8687\rho) = 0 \tag{7.2.37}$$

$$\begin{aligned} 2822498787\xi^2 &- 61346\,(6919\lambda - 14324\rho)\,\xi \\ &- \left(18727920\lambda^2 + 97350552\lambda\rho - 239976828\rho^2\right) = 0 \end{aligned} \tag{7.2.38}$$

The roots of (7.2.37) always have negative real part for the values of ρ and λ where the solution exists. The eigenvalues resulting from (7.2.38) are zero when

$$\lambda = \left(-\frac{36543}{14060} \pm \frac{1}{14060}\sqrt{3868479589}\right)\rho = \lambda^*_\pm\rho.$$

Of these two points, the solution only exists at $\lambda = \lambda^*_-\rho$ when $\rho < 0$. There is a stationary bifurcation at this point where it is possible for solutions with $\widetilde{\mathbf{D}_2}$ symmetry to bifurcate. This solution branch is stable when $\rho < 0$ and $\lambda < \lambda^*_-\rho$.

11. **(The solution branch with $(\widetilde{\mathbf{D}_4^d})_{\mathbb{Z}_4^-}$ symmetry)** This solution is given by $b = d = 0$,

$$a^2 = \frac{143\pi\,(4\lambda - 17\rho)}{630} \qquad \text{and} \qquad c^2 = \frac{2431\pi\rho}{2520}$$

and so it exists when $\rho > 0$ and $\lambda > \frac{17}{4}\rho$. It bifurcates from the solution with $\widetilde{\mathbf{D}_8 \times \mathbb{Z}_2^c}$ symmetry at $\lambda = \frac{17}{4}\rho$ when $\rho > 0$ and from the solution with $\widetilde{\mathbb{O} \times \mathbb{Z}_2^c}$ symmetry at $\rho = 0$.

Within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ it has eigenvalues which are the roots of the two quadratic equations

$$\xi^2 + 2\,(\lambda + \rho)\,\xi + \rho\,(4\lambda - 17\rho) = 0 \tag{7.2.39}$$

$$315\xi^2 - 2\left(17\lambda + 152\rho\right)\xi - \left(16\lambda^2 - 88\lambda\rho - 68\rho^2\right) = 0. \tag{7.2.40}$$

The roots of (7.2.39) always have negative real part for the values of ρ and λ where the solution exists. The eigenvalues resulting from (7.2.40) are zero when

$$\lambda = \left(\frac{11}{4} \pm \frac{3}{4}\sqrt{21}\right)\rho.$$

Of these two points, the solution only exists at $\lambda = \left(\frac{11}{4} \pm \frac{3}{4}\sqrt{21}\right)\rho$ when $\rho > 0$. There is a stationary bifurcation at this point where it is possible for solutions with $\widetilde{\mathbf{D}_2}$ symmetry to bifurcate. This solution branch is never stable.

Images of solutions with each of these symmetry types and also a solution with $\widetilde{\mathbf{D}_2}$ symmetry are shown in Figure 7.14.

We have found seven stationary bifurcations from which it may be possible for a solution with $\widetilde{\mathbf{D}_2}$ symmetry to bifurcate. Four of these bifurcations occur for $\rho > 0$ and the other three occur when $\rho < 0$. We can now use AUTO to locate these branches of solutions and compute their stability within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$. We find that the bifurcation diagram for $\rho > 0$ is as in Figure 7.15 and the bifurcation diagram for $\rho < 0$ is as in Figure 7.16. We can combine these bifurcation diagrams into a gyratory bifurcation diagram as in Figures 7.17 and 7.18.

Remark 7.2.8. We have found that for the Swift–Hohenberg equation on a sphere of radius near 4 and with the parameter μ near $\frac{1}{16}$, solutions with the symmetries of one-, two- and three-armed spirals can exist for some values of λ and $\rho = \frac{1}{2}R_2$. Recall that the relationships between the parameters λ and ρ and the radius of the sphere R and the parameter μ in the Swift–Hohenberg equation (6.1.1) are given by

$$R = 4 + \epsilon^2 R_2 \qquad \text{and} \qquad \mu = \frac{1}{16} + \epsilon^2\mu_2 = \frac{1}{16} + \epsilon^2\left(\lambda + \frac{3}{16}R_2\right) = \frac{1}{16} + \epsilon^2\left(\lambda + \frac{3}{8}\rho\right).$$

By restricting the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field for the representation of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ on $V_3 \oplus V_4$ to the invariant subspace $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ we have been able to find explicit expressions for eleven types of solution branches which exist within this subspace and determine their stability within this subspace. Solutions with $\widetilde{\mathbf{D}_2}$ symmetry and their stability were found using the numerical branch continuation package AUTO.

We have found that it is possible for solutions with the symmetries of one, two and three-armed spiral patterns to exist and that each of these solution types can be stable within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ for some values of the parameters λ and ρ.

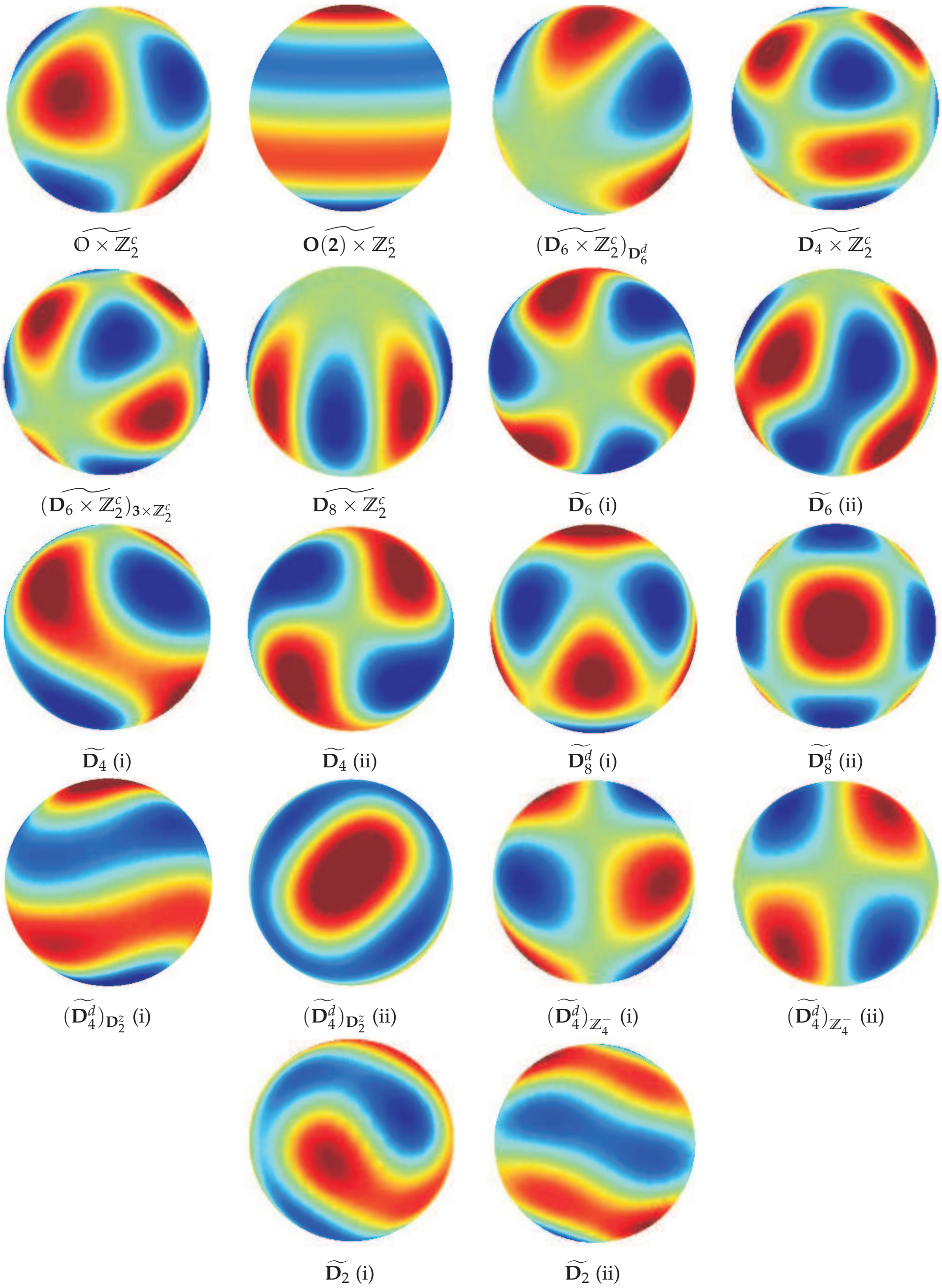


Figure 7.14: Images of solutions to (7.2.24)–(7.2.27). These solutions all have symmetry groups containing $\widetilde{\mathbf{D}_2}$. In some cases two views of the solution are given to fully describe the symmetries.

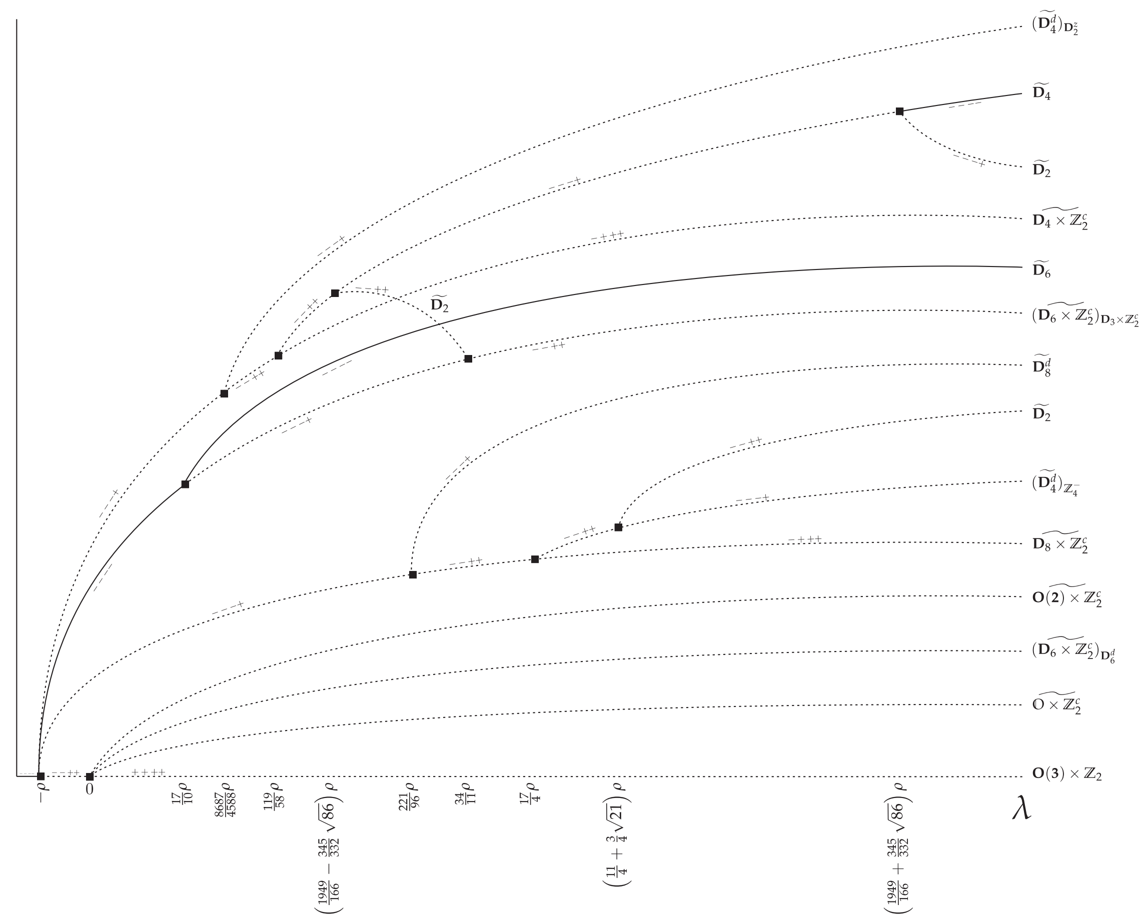


Figure 7.15: Bifurcation diagram for the Swift–Hohenberg equation in $\mathrm{Fix}(\widetilde{\mathbf{D}}_2)$ for the representation on $V_3 \oplus V_4$ when $\rho > 0$. All bifurcations are pitchfork bifurcations.

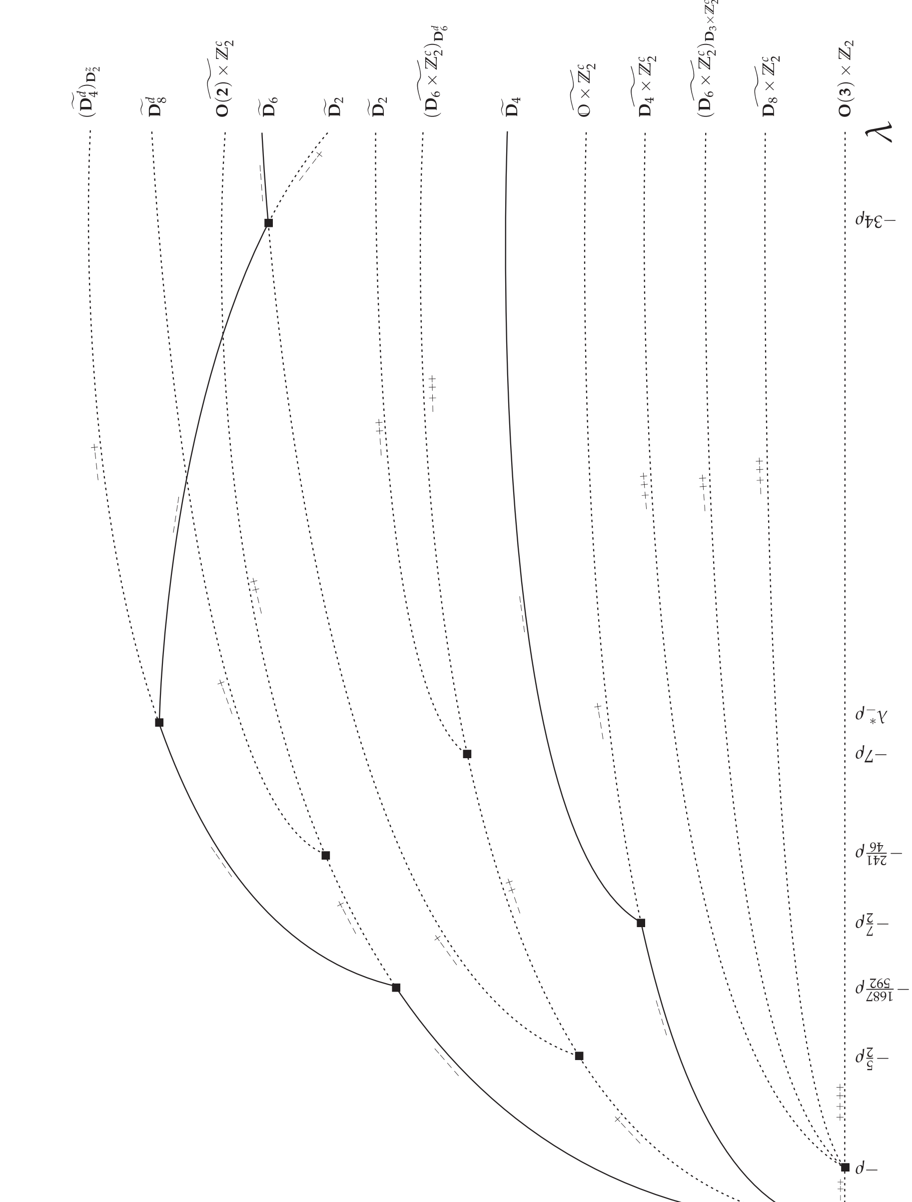


Figure 7.16: Bifurcation diagram for the Swift–Hohenberg equation in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ for the representation on $V_3 \oplus V_4$ when $\rho < 0$. All bifurcations are pitchfork bifurcations with the exception of the bifurcation at $\lambda = -34\rho$ which is transcritical.

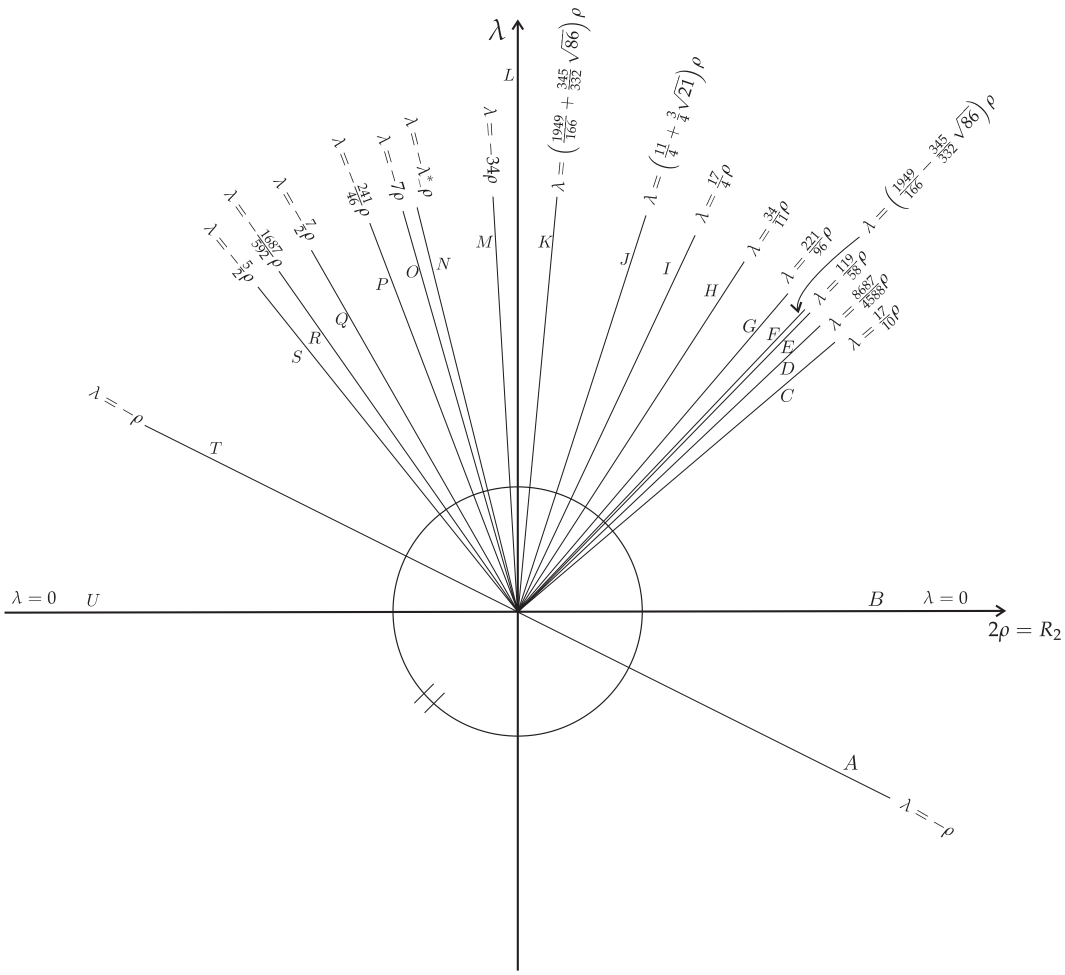


Figure 7.17: Unfolding diagram for the Swift–Hohenberg equation in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ for the representation on $V_3 \oplus V_4$. This diagram shows the lines on which bifurcations of the solution branches occur as the circle around the codimension 2 point, $\lambda = \rho = 0$, is traversed.

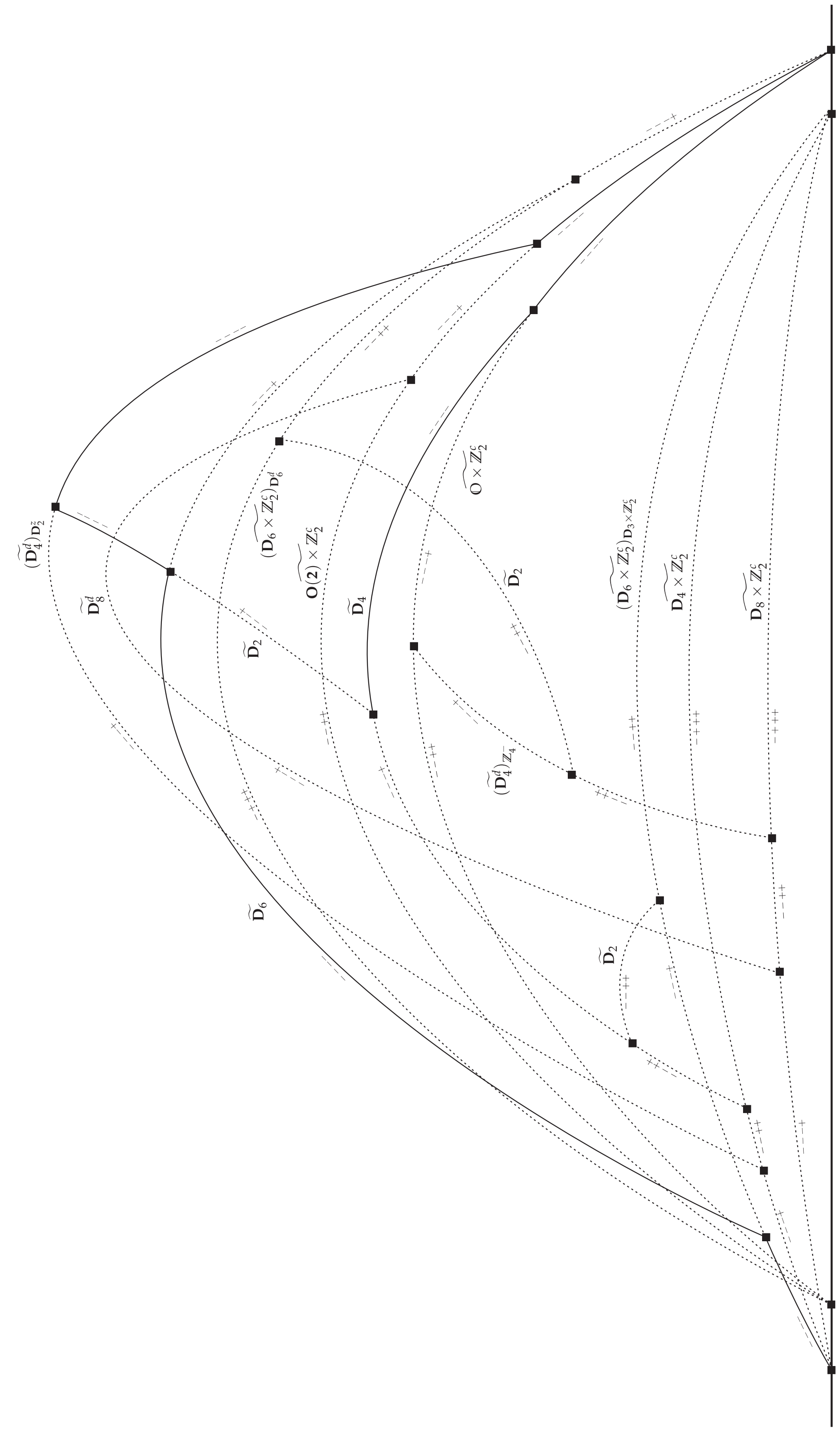


Figure 7.18: Gyratory bifurcation diagram which shows the solution branches and their stability as the path around the codimension 2 point in Figure 7.17 is traversed. All bifurcations are pitchfork bifurcations with the exception of the bifurcation at M which is transcritical.

Sufficiently close to the codimension 2 point $\lambda = \rho = 0$, numerical simulations of the Swift–Hohenberg equation (6.1.1) in the subspace $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ on a sphere of radius near 4 agree with the analytical results above. With initial conditions within the invariant subspace $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ we can find the stable solution branches and bifurcation points as in Figures 7.15 – 7.18 by varying the values of ρ and λ.

7.2.3 Conclusions on stationary spirals in the Swift–Hohenberg equation

In Examples 7.2.5 and 7.2.7 we have seen that it is possible for spiral patterns with symmetries $\widetilde{\mathbf{D}_{2m}}$ contained in the group $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ to exist in the Swift–Hohenberg equation on spheres of radius near 3 and 4 where the relevant representations of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ are on $V_2 \oplus V_3$ and $V_3 \oplus V_4$ respectively.

We have found that in the representation on $V_2 \oplus V_3$, two-armed spirals can exist and are stable to perturbations within the subspace $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ for all values of λ and ρ where the solution exists. We have also seen that it may be possible for these solutions to be stable in the whole space $V_2 \oplus V_3$ depending on the values of coefficients of order 5 terms in the equivariant vector field. In addition we found that one-armed spirals can exist, although they are never stable.

In the representation on $V_3 \oplus V_4$ we found that it is possible for solutions with the symmetries of one, two and three-armed spiral patterns to exist. We have found that each of these solution types can be stable within $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ for some values of the parameters λ and ρ.

Numerical simulations in MATLAB of the Swift–Hohenberg equation (6.1.1) on spheres of radii near 3 and 4 agree with these analytical results sufficiently close to the codimension 2 point $\lambda = \rho = 0$. Furthermore, simulations on spheres of larger radii suggest that one-armed spirals in particular exist and may be stable to any small perturbations. For example, the pattern shown in Figure 7.19 results from a simulation with random initial conditions on a sphere of radius $R = 6.01$ and $\mu = 0.25$.

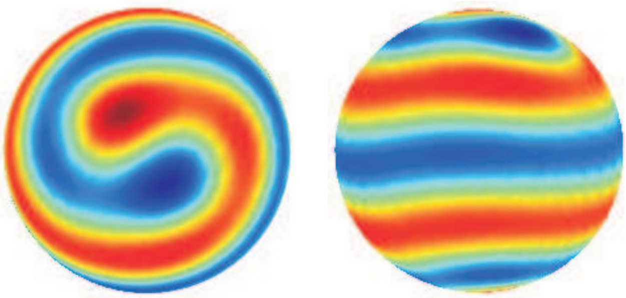


Figure 7.19: One-armed spiral solution resulting from numerical simulation of Swift–Hohenberg equation (6.1.1) with $R = 6.01$ and $\mu = 0.25$.

We now wish to discover whether the spiral patterns with symmetry groups $\widetilde{\mathbf{D}_{2m}}$ contained in $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ which we have found in the representations on $V_2 \oplus V_3$ and $V_3 \oplus V_4$ can persist as solutions with less symmetry when the symmetry is weakly broken from $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ to $\mathbf{O}(\mathbf{3})$.

7.3 Persistence of symmetric spiral patterns under forced symmetry breaking

In Section 7.2 we found several spiral patterns with symmetries $\widetilde{\mathbf{D}_{2m}} \subset \mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ which can exist on the sphere, both generically and in specific examples including the Swift–Hohenberg equation. These patterns all have the symmetry (7.1.3) which means that the areas on the surface of the sphere where the pattern function, $w(\theta, \phi, t)$, is positive and negative are of identical size and shape. We now consider what happens to spiral patterns with $\widetilde{\mathbf{D}_{2m}}$ symmetry when the overall symmetry of the system is slightly broken from $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ to $\mathbf{O}(\mathbf{3})$. Can spiral patterns with $\widetilde{\mathbf{D}_{2m}}$ symmetry persist as spiral patterns without the symmetry (7.1.3)?

If a system has overall symmetry $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$, this can be weakly broken to $\mathbf{O}(\mathbf{3})$ by introducing small terms which are only equivariant with respect to $\mathbf{O}(\mathbf{3})$. This means adding small even order terms to vector fields which are equivariant with respect to $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ or equivalently adding small quadratic terms to PDEs such as the Swift–Hohenberg equation (6.1.1) which break the $w \to -w$ symmetry.

Suppose that $\mathbf{z}_0$ is an equilibrium solution of a $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field for some representation on $V_\ell \oplus V_{\ell+1}$ which has isotropy subgroup H^θ. The isotropy subgroup is uniquely determined by the pair of subgroups of $\mathbf{O}(\mathbf{3})$, (H, K) where $H/K = \mathbb{Z}_2$ or $\mathbb{1}$. Suppose now that small even order terms which commute only with $\mathbf{O}(\mathbf{3})$ are added to the vector field. Solutions in this vector field have the symmetries of isotropy subgroups of $\mathbf{O}(\mathbf{3})$ in the representation on $V_\ell \oplus V_{\ell+1}$. Should the solution $\mathbf{z}_0$ with H^θ symmetry persist, it would have only symmetry K. This means that if $\mathbf{z}_0$ is a one-armed spiral solution with $\widetilde{\mathbf{D}_2}$ symmetry then, if it persists after symmetry breaking terms are added, it would have only $\mathbb{Z}_2$ symmetry. Similarly an m-armed spiral with $\widetilde{\mathbf{D}_{2m}}$ symmetry, if it persists, would have $\mathbf{D}_m$ symmetry. We now investigate whether it is possible for these solutions to persist.

Remark 7.3.1. Recall from Section 2.4.3 that it is possible for secondary steady-state bifurcations from group orbits of equilibria to lead to relative equilibria as well as new equilibria. By Theorem 2.4.9, the number of frequencies of a relative equilibrium $(\mathbf{O}(\mathbf{3}))\,\mathbf{z}_0$ with isotropy subgroup $\mathbf{D}_m$, (the symmetries of an m-armed spiral without symmetry (7.1.3)) when $m \geq 2$ is

$$k = \operatorname{rank}\left(N_{\mathbf{O}(\mathbf{3})}(\mathbf{D}_m)/\mathbf{D}_m\right) = 0 \qquad \text{for all } m \geq 2$$

and hence when breaking symmetry from $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ to $\mathbf{O}(\mathbf{3})$ m-armed spiral patterns for $m \geq 2$ (if they persist) remain stationary. In contrast, since

$$k = \operatorname{rank}\left(N_{\mathbf{O}(\mathbf{3})}(\mathbb{Z}_2)/\mathbb{Z}_2\right) = 1,$$

in general, one-armed spiral patterns with $\mathbb{Z}_2$ symmetry (if they exist) are singly periodic i.e. they are forced to rotate.

7.3.1 Persistence of m-armed spirals for $m \geq 2$

Throughout this section the representation of $\mathbf{O}(\mathbf{3})$ is assumed to be the reducible representation on $V_\ell \oplus V_{\ell+1}$.

In order to show that m-armed spirals for $m \geq 2$ with $\widetilde{\mathbf{D}_{2m}}$ symmetry persist as m-armed spirals with $\mathbf{D}_m$ symmetry when $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry is broken to $\mathbf{O}(\mathbf{3})$ we use the Implicit Function Theorem:

Theorem 7.3.2 (Implicit Function Theorem). *Suppose that $F : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^p$ is differentiable with $F(\mathbf{a}, \mathbf{b}) = 0$ and $\det(\mathrm{d}F)|_{(\mathbf{a},\mathbf{b})} \neq 0$ then for $(\mathbf{c}, \mathbf{d})$ in a neighbourhood of $(\mathbf{a}, \mathbf{b})$ the system $F(\mathbf{c}, \mathbf{d}) = 0$ has the unique solution $\mathbf{c} = g(\mathbf{d})$ and g is differentiable.*

Suppose that $F : \mathbb{R}^{4(\ell+1)} \times \mathbb{R}^n \to \mathbb{R}^{4(\ell+1)}$ is the $\mathbf{O}(\mathbf{3})$ equivariant vector field for the representation on $V_\ell \oplus V_{\ell+1}$ to cubic order where the vector of amplitudes of the spherical harmonics is $\mathbf{z} \in \mathbb{R}^{4(\ell+1)}$. Suppose further that $(\alpha, \beta) \in \mathbb{R}^n$ is a vector containing all of the coefficient values of each of the terms in the vector field where $\alpha \in \mathbb{R}^m$ is the vector of coefficients of the quadratic terms in $\mathbf{z}$ and $\beta \in \mathbb{R}^{n-m}$ is the vector of coefficients of the cubic terms. When $\alpha = 0$, F is equivariant with respect to $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ since there are no quadratic terms.

Suppose that $F(\mathbf{z}_0, 0, \beta_0) = 0$ where $\mathbf{z}_0$ is a solution with symmetry $H^\theta = (H, K) \subset \mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ and $\det(\mathrm{d}F)|_{(\mathbf{z}_0,0,\beta_0)} \neq 0$ then by the Implicit Function Theorem the system $F(\mathbf{z}, \alpha, \beta) = 0$ has a unique solution $\mathbf{z}_1 = g(\alpha, \beta)$ in a neighbourhood of $(\mathbf{z}_0, 0, \beta_0)$ (i.e. α near 0, β near β_0 and $\mathbf{z}_1$ near $\mathbf{z}_0$) and this solution has symmetry K.

For a stationary solution $\mathbf{z}_0$ in the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field with symmetry H^θ to persist as a stationary solution with K symmetry in the $\mathbf{O}(\mathbf{3})$ equivariant vector field by the Implicit Function Theorem we require that $(\mathrm{d}F)_{(\mathbf{z}_0,0,\beta)}$ have no zero eigenvalues. In other words, the solution with symmetry H^θ has no zero eigenvalues in $\mathrm{Fix}(K)$.

Theorem 7.3.3. *Stationary solutions $\mathbf{z}_0$ with symmetry $\widetilde{\mathbf{D}_{2m}}$ ($m \geq 2$) which exist within $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector fields persist as stationary solutions with $\mathbf{D}_m$ symmetry when the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry is broken to $\mathbf{O}(\mathbf{3})$ by adding small even order terms to the vector field.*

Proof. We must show that the solution $\mathbf{z}_0$ with $\widetilde{\mathbf{D}_{2m}}$ symmetry has no zero eigenvalues within $\mathrm{Fix}(\mathbf{D}_m)$ in order to use the Implicit Function Theorem to show the persistence of the solution.

Recall that generically the solution with $\widetilde{\mathbf{D}_{2m}}$ symmetry has

$$\dim(\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2) - \dim(\widetilde{\mathbf{D}_{2m}}) = 3$$

zero eigenvalues in $V_\ell \oplus V_{\ell+1}$. In the restriction of the $\mathbf{O}(\mathbf{3})$ equivariant vector field to $\mathrm{Fix}(\mathbf{D}_m)$ the equations are equivariant with respect to $N_{\mathbf{O}(\mathbf{3})}(\mathbf{D}_m)/\mathbf{D}_m$. Since

$$\dim(N_{\mathbf{O}(\mathbf{3})}(\mathbf{D}_m)/\mathbf{D}_m) = 0 \qquad \forall m \geq 2$$

the group orbit of *any* solution in $\mathrm{Fix}(\mathbf{D}_m)$ is zero dimensional and hence generically no solution which exists in $\mathrm{Fix}(\widetilde{\mathbf{D}_{2m}})$ has a zero eigenvalue in $\mathrm{Fix}(\mathbf{D}_m)$. The three eigenvalues of the solution with $\widetilde{\mathbf{D}_{2m}}$ symmetry which are forced to be zero by symmetry must lie in the complement of $\mathrm{Fix}(\mathbf{D}_m)$. $\square$

Furthermore, in a small enough neighbourhood of the solution with $\widetilde{\mathbf{D}_{2m}}$ symmetry, solutions with $\mathbf{D}_m$ symmetry will have the same stability properties.

7.3.2 Persistence of one-armed spiral solutions

Since $\dim(N_{\mathbf{O(3)}}(\mathbb{Z}_2)/\mathbb{Z}_2) = 1$ every solution in $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ has a zero eigenvalue in $\mathrm{Fix}(\mathbb{Z}_2)$ and hence the Implicit Function Theorem cannot be used to show the persistence of one-armed spiral solutions with $\widetilde{\mathbf{D}_2}$ symmetry. These one-armed spiral patterns cannot be shown to persist generically. We must consider each one-armed spiral on a case by case basis.

Recall that in Example 7.2.5 we found an (unstable) one-armed spiral pattern with $\widetilde{\mathbf{D}_2}$ symmetry in the $\mathbf{O(3)} \times \mathbb{Z}_2$ equivariant vector field for the representation on $V_2 \oplus V_3$ with the coefficient values for the Swift–Hohenberg equation. We now demonstrate that this solution can persist as a solution with $\mathbb{Z}_2$ symmetry when small quadratic terms are added to the Swift–Hohenberg equation.

Persistence of one-armed spiral solutions in the Swift–Hohenberg equation

Breaking the symmetry from $\mathbf{O(3)} \times \mathbb{Z}_2$ to $\mathbf{O(3)}$ in the case of the Swift–Hohenberg equation (6.1.1) on the sphere is equivalent to adding a small even order term to the equation which breaks the $w \to -w$ symmetry. The most obvious term to add is a quadratic nonlinearity giving

$$\frac{\partial w}{\partial t} = \mu w - (1 + \nabla^2)^2 w + s w^2 - w^3, \tag{7.3.1}$$

where s is small. The aim is to discover whether spiral solutions with $\widetilde{\mathbf{D}_2}$ symmetry which exist when $s = 0$, and are stationary, can persist as solutions with $\mathbb{Z}_2$ symmetry when s is nonzero. Recall that, in general, spiral patterns with $\mathbb{Z}_2$ symmetry, if they exist, are periodic. However, (7.3.1) is variational and as such cannot have periodic solutions. Thus one-armed spiral solutions of (7.3.1) will remain stationary when $s \neq 0$.

Example 7.3.4 (Persistence of one-armed spirals in the Swift–Hohenberg equation on a sphere of radius near 3). Recall from Section 6.4.2 that for a sphere of radius near 3 the relevant representation of $\mathbf{O(3)}$ is the reducible representation on $V_2 \oplus V_3$. To cubic order the general $\mathbf{O(3)}$ equivariant vector field for this representation is

$$f(\mathbf{z}, \lambda) = (g(\mathbf{z}, \lambda); h(\mathbf{z}, \lambda))$$

where

$$g(\mathbf{z}, \lambda) = \mu_x \mathbf{x} + \eta \mathbf{U}(\mathbf{x}) + \nu \mathbf{V}(\mathbf{y}) + \alpha_1 \mathbf{x}|\mathbf{x}|^2 + \beta_1 \mathbf{x}|\mathbf{y}|^2 + \gamma_1 \mathbf{P}(\mathbf{x}, \mathbf{y}) + \gamma_2 \mathbf{Q}(\mathbf{x}, \mathbf{y}) \tag{7.3.2}$$

$$h(\mathbf{z}, \lambda) = \mu_y \mathbf{y} + \zeta \mathbf{W}(\mathbf{x}, \mathbf{y}) + \alpha_2 \mathbf{y}|\mathbf{x}|^2 + \beta_2 \mathbf{y}|\mathbf{y}|^2 + \delta_1 \mathbf{R}(\mathbf{y}) + \delta_2 \mathbf{S}(\mathbf{x}, \mathbf{y}) + \delta_3 \mathbf{T}(\mathbf{x}, \mathbf{y}) \tag{7.3.3}$$

in which the cubic equivariant mappings $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{T}$ are as in Section 6.3.5 and the quadratic equivariant mappings are

- $\mathbf{U}(\mathbf{x}) = (U_{-2}, U_{-1}, U_0, U_1, U_2)$ where $U_{-k} = (-1)^k \overline{U}_k$ and

$$\begin{aligned} U_{-2}(\mathbf{x}) &= 4x_{-2}x_0 - \sqrt{6}x_{-1}^2 \\ U_{-1}(\mathbf{x}) &= 2\sqrt{6}x_{-2}x_1 - 2x_{-1}x_0 \\ U_0(\mathbf{x}) &= 4x_{-2}x_2 + 2x_{-1}x_1 - 2x_0^2 \end{aligned}$$

- $\mathbf{V}(\mathbf{y}) = (V_{-2}, V_{-1}, V_0, V_1, V_2)$ where $V_{-k} = (-1)^k \overline{V}_k$ and

$$\begin{aligned} V_{-2}(\mathbf{y}) &= -\sqrt{10}y_{-3}y_1 + 2\sqrt{5}y_{-2}y_0 - \sqrt{6}y_{-1}^2 \\ V_{-1}(\mathbf{y}) &= -5y_{-3}y_2 + \sqrt{15}y_{-2}y_1 - \sqrt{2}y_{-1}y_0 \\ V_0(\mathbf{y}) &= -5y_{-3}y_3 + 3y_{-1}y_1 - 2y_0^2 \end{aligned}$$

- $\mathbf{W}(\mathbf{x}, \mathbf{y}) = (W_{-3}, W_{-2}, W_{-1}, W_0, W_1, W_2, W_3)$ where $W_{-k} = (-1)^k \overline{W_k}$ and

$$\begin{aligned} W_{-3}(\mathbf{x},\mathbf{y}) &= 5y_{-3}x_1 - 5y_{-2}x_{-1} + \sqrt{10}y_{-1}x_{-2} \\ W_{-2}(\mathbf{x},\mathbf{y}) &= 2\sqrt{5}y_0x_{-2} - \sqrt{15}y_{-1}x_{-1} + 5y_{-3}x_1 \\ W_{-1}(\mathbf{x},\mathbf{y}) &= 2\sqrt{6}y_1x_{-2} - \sqrt{2}y_0x_{-1} + \sqrt{10}y_{-3}x_2 + \sqrt{15}y_{-2}x_1 - 3y_{-1}x_0 \\ W_0(\mathbf{x},\mathbf{y}) &= 2\sqrt{5}\,(y_{-2}x_{-2} + y_2x_{-2}) + \sqrt{2}\,(y_{-1}x_1 + y_1x_{-1}) - 4y_0x_0. \end{aligned}$$

For the Swift–Hohenberg equation we computed in Section 6.4.2 that the values of the coefficients of the odd order terms in (7.3.2)–(7.3.3) are

$$\mu_x = \mu_2 - \frac{8}{27}R_2, \;\; \alpha_1 = -\frac{15}{28\pi}, \;\; \beta_1 = -\frac{6}{11\pi}, \;\; \gamma_1 = \frac{3}{44\pi}, \;\; \gamma_2 = \frac{1}{44\pi}, \tag{7.3.4}$$

$$\mu_y = \mu_2 + \frac{16}{27}R_2, \;\; \alpha_2 = -\frac{25}{22\pi}, \;\; \beta_2 = -\frac{175}{286\pi}, \;\; \delta_1 = -\frac{7}{2860\pi}, \;\; \delta_2 = -\frac{3}{44\pi} \text{ and } \delta_3 = -\frac{1}{22\pi}$$

where $\mu = \frac{1}{9} + \epsilon^2\mu_2$. By assuming that the $\mathbb{Z}_2$ symmetry is weakly broken so that $s = \epsilon s_1$ we can compute, using the same method as Section 6.4.2, that the values of the quadratic coefficients are

$$\eta = -\frac{1}{14}\sqrt{\frac{5}{\pi}}s_1 \qquad \nu = -\frac{1}{15}\sqrt{\frac{5}{\pi}}s_1 \qquad \zeta = -\frac{1}{15}\sqrt{\frac{5}{\pi}}s_1. \tag{7.3.5}$$

Recall from Example 7.2.5 that when $s_1 = 0$, (7.3.2)–(7.3.3) with the coefficients above have a stationary solution with $\widetilde{\mathbf{D}_2}$ symmetry. Using the restriction to

$$\mathrm{Fix}(\widetilde{\mathbf{D}_2}) = \{(\mathrm{i}a, 0, 0, 0, -\mathrm{i}a\,;\, 0, b, 0, c, 0, b, 0)\}$$

we found that this solution exists when $\rho = \frac{8}{9}R_2 > 0$ and $\lambda = \mu_x$ is such that $\frac{26}{9}\rho < \lambda < \frac{199}{18}\rho$. Moreover, this solution is always unstable.

When the symmetry is broken from $\mathbf{O}(3) \times \mathbb{Z}_2$ to $\mathbf{O}(3)$ (when $s_1 \neq 0$), $\mathrm{Fix}(\widetilde{\mathbf{D}_2})$ is no longer an invariant subspace. The solution with $\widetilde{\mathbf{D}_2}$ symmetry which exists when $s_1 = 0$ is no longer a solution to (7.3.2)–(7.3.3) but may become a (stationary) solution with $\mathbb{Z}_2$ symmetry which is contained in the invariant subspace

$$\mathrm{Fix}(\mathbb{Z}_2) = \{(d + \mathrm{i}a,\, 0,\, e,\, 0,\, d - \mathrm{i}a\,;\, 0,\, b + \mathrm{i}f,\, 0,\, c,\, 0,\, b - \mathrm{i}f,\, 0)\}. \tag{7.3.6}$$

To discover whether a solution with $\mathbb{Z}_2$ symmetry can bifurcate at $s_1 = 0$ from the solution with $\widetilde{\mathbf{D}_2}$ symmetry we can expand in powers of s_1 for small s_1 about the $\widetilde{\mathbf{D}_2}$ symmetric solution.

In the restriction of (7.3.2)–(7.3.3) to $\mathrm{Fix}(\mathbb{Z}_2)$ we let

$$\begin{aligned} a &= a_0 + s_1 a_1 + s_1^2 a_2 + \dots \\ b &= b_0 + s_1 b_1 + s_1^2 b_2 + \dots \\ \vdots & \quad \vdots \\ f &= f_0 + s_1 f_1 + s_1^2 f_2 + \dots \end{aligned}$$

where $d_0 = e_0 = f_0 = 0$ and (a_0, b_0, c_0) is the solution with $\widetilde{\mathbf{D}_2}$ symmetry which exists when $s_1 = 0$. At order s_1^0 we have the solution with $\widetilde{\mathbf{D}_2}$ symmetry. At order s_1^1 we find that

$$a_1 = b_1 = c_1 = 0 \qquad d_1 = d_1(\lambda, \rho, f_1), \qquad e_1 = e_1(\lambda, \rho)$$

and f_1 is arbitrary. If we choose a value for f_1 then at order s_1^2 we find that

$$a_2 = a_2(\lambda, \rho) \qquad b_2 = b_2(\lambda, \rho) \qquad c_2 = c_2(\lambda, \rho) \qquad d_2 = d_2(\lambda, \rho, f_2), \qquad e_1 = 0$$

and f_2 arbitrary. Hence we can see that a whole family of solutions (depending on f) exist as stationary solutions with $\mathbb{Z}_2$ symmetry. This is to be expected since (7.1.1) implies that any rotation in the z-axis of a solution in $\mathrm{Fix}(\mathbb{Z}_2)$ given by (7.3.6) is also a solution with $\mathbb{Z}_2$ symmetry which lies in $\mathrm{Fix}(\mathbb{Z}_2)$ but has a different value of f. These individual solutions are stationary since (7.3.1) is variational.

In conclusion, the one-armed spiral pattern with $\widetilde{\mathbf{D}_2}$ symmetry which we found in Section 7.2.5 persists as one of a family of one-armed spiral patterns with $\mathbb{Z}_2$ symmetry when a quadratic term is included in the Swift–Hohenberg equation which breaks the symmetry from $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ to $\mathbf{O}(\mathbf{3})$.

Persistence of one-armed spiral solutions in a non-variational Swift–Hohenberg equation

Suppose that instead of adding the term sw^2 to (6.4.1) we instead add a nonlinear quadratic term which renders the equation non-variational. Then generically we expect any solution with $\mathbb{Z}_2$ symmetry to drift. To make (6.4.1) non-variational and break the $w \to -w$ symmetry we can add terms such as $|\nabla w|^2$ and $w\nabla^2 w$ so that we have

$$\frac{\partial w}{\partial t} = \mu w - (1 + \nabla^2)^2 w - w^3 + p|\nabla w|^2 + q w \nabla^2 w. \tag{7.3.7}$$

which is non-variational so long as $q \neq 2p$ (see [60]). The coefficients of the even order terms in the amplitude equations will now depend on the values of p and q. We consider the case of the representation of $\mathbf{O}(\mathbf{3})$ on $V_2 \oplus V_3$ where the amplitude equations are given by (7.3.2)–(7.3.3).

Example 7.3.5 (Coefficient values for a non-variational Swift–Hohenberg equation on a sphere of radius near 3). Suppose that in (7.3.7) $p = 0$ and $q = \epsilon q_1$. Then the equation is non-variational and we expect the values of the coefficients η, ν and ζ in the $\mathbf{O}(\mathbf{3})$ equivariant vector field on $V_2 \oplus V_3$, (7.3.2)–(7.3.3), to depend on the value of q_1.

Recall that in this representation, $\mu_c = \frac{1}{9}$ and $R_c = 3$. By letting

$$\mu = \frac{1}{9} + \epsilon^2 \mu_2, \quad R = 3 + \epsilon^2 R_2, \quad T = \epsilon^2 t, \quad w = \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3,$$

we see that the linear differential operator $L = \mu - (1 + \nabla^2)^2$ acts on the spherical harmonics of degree ℓ as in (6.4.18). The operator $M = \nabla^2$ acts as

$$M = \nabla^2 = -\frac{\ell(\ell+1)}{R_c^2} + \frac{2\ell(\ell+1)}{R_c^3} R_2 \epsilon^2 = M_0 + \epsilon^2 M_2$$

on the spherical harmonics of degree ℓ.

At orders ϵ^1 and ϵ^2 in (7.3.7) we find that

$$w_1 = \sum_{m=-2}^{2} x_m(T) Y_2^m(\theta, \phi) + \sum_{n=-3}^{3} y_n(T) Y_3^n(\theta, \phi) \qquad \text{and} \qquad w_2 = 0$$

respectively so that at order ϵ^3,

$$\frac{\partial w_1}{\partial T} = L_0 w_3 + L_2 w_1 - w_1^3 + q_1 w_1 M_0 w_1. \tag{7.3.8}$$

Multiplying (7.3.8) by $\overline{Y}_2^p$ and integrating over the sphere we find that

$$\begin{aligned} \dot{x}_p \quad = \quad & \left(\mu_2 - \frac{8}{27} R_2\right) x_p - \int_0^{2\pi} \int_0^{\pi} w_1^3 \overline{Y}_2^p \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ & -\frac{2q_1}{3} \int_0^{2\pi} \int_0^{\pi} w_1 \left(\sum_{m=-2}^{2} x_m Y_2^m \right) \overline{Y}_2^p \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ & -\frac{4q_1}{3} \int_0^{2\pi} \int_0^{\pi} w_1 \left(\sum_{n=-3}^{3} y_n Y_3^n \right) \overline{Y}_2^p \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \end{aligned} \tag{7.3.9}$$

and by multiplying (7.3.8) by $\overline{Y}_3^p$ and integrating over the sphere we find that

$$\begin{aligned} \dot{y}_p \quad = \quad & \left(\mu_2 + \frac{16}{27} R_2\right) y_p - \int_0^{2\pi} \int_0^{\pi} w_1^3 \overline{Y}_3^p \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ & -\frac{2q_1}{3} \int_0^{2\pi} \int_0^{\pi} w_1 \left(\sum_{m=-2}^{2} x_m Y_2^m \right) \overline{Y}_3^p \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ & -\frac{4q_1}{3} \int_0^{2\pi} \int_0^{\pi} w_1 \left(\sum_{n=-3}^{3} y_n Y_3^n \right) \overline{Y}_3^p \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi. \end{aligned} \tag{7.3.10}$$

Using the same method as Section 6.4.2, we compute that the values of the quadratic coefficients in the $\mathbf{O}(\mathbf{3})$ equivariant vector field, (7.3.2)–(7.3.3), in the non-variational case are

$$\eta = \frac{1}{21}\sqrt{\frac{5}{\pi}} q_1 \qquad \nu = \frac{4}{45}\sqrt{\frac{5}{\pi}} q_1 \qquad \zeta = \frac{1}{15}\sqrt{\frac{5}{\pi}} q_1. \tag{7.3.11}$$

We now demonstrate that with these values of the quadratic coefficients, and the values of the coefficients of the odd order terms given by (7.3.4), the stationary one-armed spiral solution with $\widetilde{\mathbf{D}_2}$ symmetry which exists in (7.3.2)–(7.3.3) when $q_1 = 0$ persists as a solution with $\mathbb{Z}_2$ symmetry and that this solution drifts.

Recall that

$$\mathrm{Fix}(\mathbb{Z}_2) = \{(d + \mathrm{i}a,\ 0,\ e,\ 0,\ d - \mathrm{i}a\ ;\ 0,\ b + \mathrm{i}f,\ 0,\ c,\ 0,\ b - \mathrm{i}f,\ 0)\}.$$

If we try to expand these variables in powers of q_1 as in Example 7.3.4 we cannot find solutions at order q_1. We must change coordinates and let

$$d + \mathrm{i}a = Re^{\mathrm{i}\Phi}, \qquad b + \mathrm{i}f = Se^{\mathrm{i}\Psi} \qquad \text{and} \qquad \Theta = \Phi - \Psi,$$

then the equations in the restriction of (7.3.2)–(7.3.3) to $\mathrm{Fix}(\mathbb{Z}_2)$ can be reduced to five equations for $\dot{R}$, $\dot{S}$, $\dot{\Theta}$, $\dot{c}$ and $\dot{e}$. We then expand

$$\begin{aligned}
R(t) &= r_0 + q_1 r_1 + q_1^2 r_2 + \dots \\
S(t) &= s_0 + q_1 s_1 + q_1^2 s_2 + \dots \\
\Theta(t) &= \Theta_0 + q_1 \Theta_1 + q_1^2 \Theta_2 + \dots \\
c(t) &= c_0 + q_1 c_1 + q_1^2 c_2 + \dots \\
e(t) &= e_0 + q_1 e_1 + q_1^2 e_2 + \dots .
\end{aligned}$$

In these coordinates the stationary solution with $\widetilde{\mathbf{D}_2}$ symmetry which exists when $q_1 = 0$ is given by $(r_0, s_0, \Theta_0, c_0, 0)$ where $\Theta_0 = \pi/2$. When $q_1 \neq 0$, stationary solutions in the coordinates (R, S, Θ, c, e) correspond to periodic solutions of (7.3.2)–(7.3.3). We find that there is a stationary solution with $\mathbb{Z}_2$ symmetry given by

$$\begin{aligned}
R &= r_0(\lambda,\rho) + q_1^2 r_2(\lambda,\rho) + \dots \\
S &= s_0(\lambda,\rho) + q_1^2 s_2(\lambda,\rho) + \dots \\
\Theta &= \pi/2 + q_1 \Theta_1(\lambda,\rho) + \dots \\
c &= c_0(\lambda,\rho) + q_1^2 c_2(\lambda,\rho) + \dots \\
e &= q_1 e_1(\lambda,\rho) + \dots .
\end{aligned}$$

This corresponds to a periodic solution in the original coordinates $a, \dots, f$, which is a one-armed spiral with $\mathbb{Z}_2$ symmetry that drifts with speed

$$\dot{\Phi} = \dot{\Psi} = v_1 q_1 + v_3 q_1^3 + \dots,$$

where $v_1 = v_1(r_0, s_0, \Theta_1, c_0, e_1)$, about the z-axis. Hence we can see that for $p = 0$ and small q, solutions of (7.3.7) with $\mathbb{Z}_2$ symmetry rotate at speed proportional to q.

The rotating spiral solution on a sphere of radius near 3 for the non-variational Swift-Hohenberg equation, found analytically in Example 7.3.5 cannot be found in numerical simulations since it is unstable. However, numerical simulations of (7.3.7), with $p = 0$ and q small and positive, on a sphere of radius $R \approx 4$ starting from the stable single armed spiral found in Example 7.2.7 result in a single armed spiral pattern with $\mathbb{Z}_2$ symmetry which rotates at a rate proportional to q in the direction indicated in Figure 7.20.

We have been able to demonstrate analytically the persistence of one single armed spiral pattern in the Swift–Hohenberg equation in the representation on $V_2 \oplus V_3$. Numerically we find that other single armed spiral patterns in representations on $V_\ell \oplus V_{\ell+1}$ for larger values of ℓ can also persist.

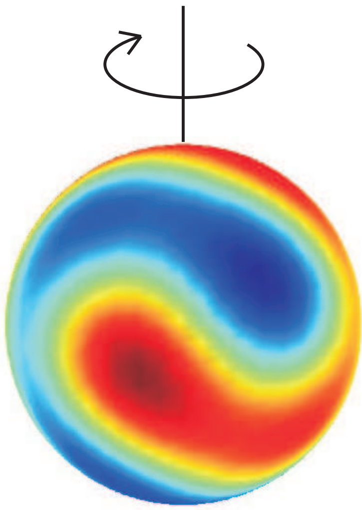


Figure 7.20: The rotating single armed spiral which results from numerical simulations of (7.3.7), with $p = 0$ and q small and positive, on a sphere of radius near 4. The initial condition is a stationary single armed spiral with $\widetilde{\mathbf{D}_2}$ symmetry. The arrow indicates the direction and axis of rotation. The speed of rotation is proportional to q. Note that this is not the rotating spiral found in Example 7.3.5.

CHAPTER 8

CONCLUSIONS

In this thesis we have used the techniques of equivariant bifurcation theory to describe various patterns which can exist on spheres as a result of a bifurcation from a spherically symmetric state. The group theoretical methods of equivariant bifurcation theory have allowed us to describe the symmetries of the solutions which are created at a bifurcation with spherical symmetry using the only action of the group $\mathbf{O(3)}$, of rotations and reflections of the sphere, on the spaces V_ℓ, of spherical harmonics of degree ℓ. Not only have we used symmetries to describe the existence properties of certain patterns on spheres, we have also used symmetries to compute the stability of these patterns. All of this can be done without reference to any particular governing partial differential equation and hence the solution types we have found are generic, i.e. we can expect to find solutions with these symmetries in any system which undergoes a bifurcation from a spherically symmetric state.

In this thesis we have considered two different types of patterns which can exist on the sphere. These were the time periodic solutions which can exist as a result of a Hopf bifurcation from spherical symmetry and the spiral patterns with symmetry which can exist on a sphere as a result of a stationary bifurcation with spherical symmetry.

In Chapters 4 and 5 we investigated the time-periodic solutions which can exist as a result of a Hopf bifurcation from a spherically symmetric steady state. The main result relating to Hopf bifurcations with symmetry, the equivariant Hopf theorem, guarantees that at a Hopf bifurcation with $\mathbf{O(3)}$ symmetry, branches of periodic solutions with the symmetries of the $\mathbb{C}$-axial isotropy subgroups of $\mathbf{O(3)} \times S^1$ are created.

In Chapter 4 we computed these $\mathbb{C}$-axial isotropy subgroups for every representation of $\mathbf{O(3)} \times S^1$ on $V_\ell \oplus V_\ell$ for every value of ℓ. This involved first enumerating the conjugacy classes of twisted subgroups of $\mathbf{O(3)} \times S^1$. Then, using a group theoretical result known as the chain criterion, we determined which of these twisted subgroups are $\mathbb{C}$-axial for the representations on $V_\ell \oplus V_\ell$. Although these computations had been carried out before (see [43, 44, 46]) all previously published lists of the $\mathbb{C}$-axial isotropy subgroups of $\mathbf{O(3)} \times S^1$ contained some errors.

Many of these errors stemmed from incorrect assumptions about the containment relations between the twisted subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ of dihedral type. We have now corrected these errors and presented a revised list of the $\mathbb{C}$-axial isotropy subgroups in Section 4.3.2. From this list one can identify for any representation on $V_\ell \oplus V_\ell$, for any value of ℓ, the symmetry groups of the time-periodic solutions which are guaranteed by the equivariant Hopf theorem to exist at a Hopf bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry for this representation of $\mathbf{O}(\mathbf{3})$.

The corrections which we made to the list of $\mathbb{C}$-axial isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ did not affect the much studied example of the Hopf bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry for the representation on $V_2 \oplus V_2$ [51, 54]. However, our revised list allowed us to see that in the previously unstudied case of the natural representation on $V_3 \oplus V_3$ there are six branches of periodic solutions guaranteed to exist by the equivariant Hopf theorem; one less branch than had previously been predicted [46]. Three of these solutions are travelling waves and the other three are standing waves. By computing to cubic order the general form of a vector field which commutes with the action of $\mathbf{O}(\mathbf{3}) \times S^1$ on $V_3 \oplus V_3$, we were able in Chapter 5 to find conditions on the coefficients in this vector field for each of these solution branches to bifurcate supercritically and be stable. Using these conditions, we found that for each of the six $\mathbb{C}$-axial solution branches it is possible to give a set of coefficient values such that the periodic solution is stable. We also saw that it is possible for all six solution branches to simultaneously bifurcate supercritically and be unstable. This could indicate that it is possible for heteroclinic cycles or chaotic behaviour to be present within the system of equivariant differential equations. The time-dependent behaviour of the system under these circumstances would be an interesting avenue for future investigation.

The branches of periodic solutions guaranteed by the equivariant Hopf theorem to exist at a Hopf bifurcation with $\mathbf{O}(\mathbf{3})$ symmetry are not the only solutions which can exist. A slightly stronger result than the equivariant Hopf theorem guarantees the existence of periodic solution branches with the symmetries of all maximal isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$. These are isotropy subgroups which are not contained in any larger isotropy subgroups except $\mathbf{O}(\mathbf{3}) \times S^1$. Note that $\mathbb{C}$-axial isotropy subgroups are automatically maximal. It is also possible, depending on the values of coefficients in the $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant vector field, for solutions with the symmetries of other (submaximal) isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ to exist. There is no result in equivariant bifurcation theory regarding such solutions. They must be found directly in the equivariant vector field.

By computing the isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ which fix a subspace of $V_\ell \oplus V_\ell$ of dimension greater than 2 it is possible to determine the symmetry groups of solutions which may exist in the $\mathbf{O}(\mathbf{3}) \times S^1$ equivariant vector field. In Section 4.3.3 we used the same method as for the $\mathbb{C}$-axial isotropy subgroups to compute the isotropy subgroups of $\mathbf{O}(\mathbf{3}) \times S^1$ which fix a four-dimensional subspace of $V_\ell \oplus V_\ell$ for all values of ℓ. For those which are maximal, a solution branch with this symmetry is guaranteed to exist.

Establishing the existence properties of solutions with submaximal isotropy involves much more computation, although we have been able to find several such solutions for one particular representation. In the natural representation of $\mathbf{O}(\mathbf{3}) \times S^1$ on $V_3 \oplus V_3$ we found that all six isotropy subgroups which fix a subspace of dimension four are submaximal. In Section 5.5 we investigated whether solutions with these symmetries can exist in the $\mathbf{O(3)} \times S^1$ equivariant vector field. We found, both analytically and by using the numerical branch continuation package AUTO, that for four out of the six isotropy subgroups, Σ, submaximal periodic or quasiperiodic solutions can exist for some values of the coefficients in the $\mathbf{O(3)} \times S^1$ equivariant vector field. Some of these solutions can even be stable within the four-dimensional subspace $\mathrm{Fix}(\Sigma)$.

The second topic considered by this thesis was the existence of symmetric spiral patterns on spheres. In contrast to the single armed spiral patterns which can be found in the plane, one-armed spiral patterns on the sphere can have symmetries. The spiral pattern must have two tips. If these lie at antipodal points on the sphere (say the north and south poles) then it is possible for the spiral pattern to have a rotation symmetry about an axis in the plane of the equator. Patterns with such symmetries have been found in numerous experiments [64] as well as numerical simulations of Rayleigh–Bénard convection [62, 90] and other pattern forming systems such as the Swift–Hohenberg model [65] and reaction–diffusion systems [16]. Until this thesis, no analytical study of the generic existence and stability properties of single or multi-armed symmetric spiral patterns on spheres had been undertaken. Using techniques from equivariant bifurcation theory we investigated whether symmetric spiral patterns on spheres can result from a stationary bifurcation with spherical symmetry and subsequent secondary bifurcations.

To simplify the problem, we began by studying the most symmetric spiral patterns on spheres. In addition to symmetries contained in the group $\mathbf{O(3)}$, these spirals have a symmetry corresponding to a change in sign of the solution function, w, combined with a rotation in $\mathbf{O(3)}$. For these patterns, the areas where $w > 0$ and $w < 0$ are of identical size and shape. The symmetries of such spiral patterns are subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$. Group theoretical results tell us that spiral solutions with these symmetries (if they exist) are generically stationary. Our aim was to demonstrate that stationary single and multi-armed spirals with these symmetries can result from an initial stationary bifurcation with $\mathbf{O(3)} \times \mathbb{Z}_2$ symmetry.

Since the problem of a stationary bifurcation with $\mathbf{O(3)} \times \mathbb{Z}_2$ symmetry had not previously been studied, Chapter 6 of this thesis was devoted to this subject. Bifurcations with $\mathbf{O(3)} \times \mathbb{Z}_2$ symmetry can occur in the numerous systems on a sphere which are invariant under a change in sign of the physical variable. One such example is the Swift–Hohenberg model. At a stationary bifurcation with $\mathbf{O(3)} \times \mathbb{Z}_2$ symmetry, where $\mathbf{O(3)}$ acts on the vector space V, the equivariant branching lemma guarantees that branches of equilibria with the symmetries of the axial isotropy subgroups of $\mathbf{O(3)} \times \mathbb{Z}_2$ bifurcate. The axial isotropy subgroups fix a one-dimensional subspace of V. Here V is either V_ℓ, the space of spherical harmonics of degree ℓ or $V_\ell \oplus V_{\ell+1}$ when there is a mode interaction between the spherical harmonics of degrees ℓ and $\ell + 1$. In Section 6.2 we computed the axial isotropy subgroups in both cases and all of the isotropy subgroups for several examples. We noted that in representations on $V_\ell \oplus V_{\ell+1}$ there are many more isotropy subgroups. This means that mode interactions can result in a much wider range of possible solution patterns than can be found with a single mode.

In Chapter 7 we observed that spiral patterns can only result from mode interactions – such patterns can only be made with a combination of spherical harmonics of odd and even degrees. Indeed, we found that they can exist in an interaction between the spherical harmonics of degrees ℓ and $\ell + 1$. We also observed that the symmetry groups of symmetric spiral patterns are never axial isotropy subgroups so spiral patterns are never guaranteed by the equivariant branching lemma to exist at a stationary bifurcation from spherical symmetry. Thus, to determine when symmetric spiral patterns can exist we had to find them directly in the equivariant vector field.

Through the study of the stationary bifurcation with $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry we found that the symmetry group of the most symmetric m-armed spiral (where $m \geq 1$) is a submaximal isotropy subgroup of $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ in the representation on $V_\ell \oplus V_{\ell+1}$ for $\ell \geq m$. Hence stationary spiral solutions with these symmetries may exist depending on the values of coefficients in the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field. In Section 7.2.1 we determined the conditions which must be satisfied for one- and two-armed spirals to exist in the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariant vector field for the representation on $V_2 \oplus V_3$. We demonstrated that it is possible to find a set of coefficient values which allows a single armed spiral pattern to exist and be stable in its fixed-point subspace. By computing the values of the coefficients for the specific example of the Swift–Hohenberg equation we were able to show that both the one- and two- armed solutions can exist, but that the one-armed spiral pattern is never stable.

A similar treatment of the representation on $V_3 \oplus V_4$ using the values of the vector field coefficients arising from the Swift–Hohenberg equation showed that the most symmetric spiral patterns with one, two and three arms can exist as a result of a stationary bifurcation with $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry and subsequent bifurcations. Furthermore, in this case the stationary one-armed spiral pattern can be stable for some values of the bifurcation parameters.

Finally, in Section 7.3 we considered the effect of weakly breaking the overall symmetry of the system from $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ to $\mathbf{O}(\mathbf{3})$ on the spirals found in previous sections. We saw that any m-armed spiral pattern for $m \geq 2$ will always persist as a stationary spiral pattern without the 'red to blue' symmetries of the most symmetric spiral patterns. The case of the one-armed spiral is not so simple. If a stationary one-armed spiral solution does persist under this weak symmetry breaking then generically it is forced to rotate. This can be demonstrated directly for the most symmetric unstable one-armed spiral which exists in the representation on $V_2 \oplus V_3$ in the Swift–Hohenberg equation. This spiral solution loses symmetry and begins to rotate when non-variational terms which break the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry to $\mathbf{O}(\mathbf{3})$ are added.

Thus we have shown that stationary spiral patterns on spheres can exist generically as a result of a stationary bifurcation with $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry (and subsequent secondary bifurcations) in the case of a mode interaction. Indeed, they do exist in the Swift–Hohenberg equation. These spiral patterns can be stable (in some subspaces) and we have seen how they bifurcate from other solution branches. Furthermore, these spiral patterns can persist when the $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ symmetry is broken to $\mathbf{O}(\mathbf{3})$.

The two types of patterns on spheres which we have considered in this thesis, although quite different, have both been studied using the generic framework of equivariant bifurcation theory. This powerful tool has enabled us to discover new and interesting possible behaviours of whole classes of systems with underlying spherical symmetry. In the case of the Hopf bifurcation with $\mathbf{O(3)}$ we have corrected errors in previous results and added to the range of known solutions which can exist at such a bifurcation. We have also shown the possibility for the dynamics, at certain coefficient values, to contain heteroclinic cycles or even for chaotic behaviour to occur. By studying stationary bifurcations with $\mathbf{O(3)} \times \mathbb{Z}_2$ symmetry we have been able to show both generically, and in specific case of the Swift–Hohenberg model, that symmetric spiral patterns on spheres can exist. These patterns, previously only found in experiments and numerical simulations, have been shown to be potential solutions of many dynamical systems with underlying spherical symmetry.

APPENDIX A

SPHERICAL HARMONICS

A.1 Spherical harmonics of degrees $\ell = 2, 3$ and 4

Here we list the spherical harmonics of degrees $\ell = 2$, 3 and 4 in both spherical polar coordinates, (θ, ϕ) and Cartesian coordinates (x, y, z) where the spherical-to-Cartesian coordinate transformation is given by

$$\begin{aligned} x &= \sin\theta\cos\phi \\ y &= \sin\theta\sin\phi \\ z &= \cos\theta. \end{aligned}$$

Recall that the spherical harmonics satisfy

$$Y_\ell^{-m}(\theta,\phi) = (-1)^m \overline{Y_\ell^m}(\theta,\phi),$$

where the bar denotes complex conjugate. They also satisfy the orthogonality condition

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\phi)\,\overline{Y_{\ell'}^{m'}}(\theta,\phi)\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \delta_{\ell,\ell'}\,\delta_{m,m'}.$$

Here we list the functions for negative values of m only.

Spherical harmonics of degree $\ell = 2$

$$\begin{aligned} Y_2^{-2}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\,\mathrm{e}^{-2\mathrm{i}\phi} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,(x-\mathrm{i}y)^2 \\ Y_2^{-1}(\theta,\phi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\,\sin\theta\,\cos\theta\,\mathrm{e}^{-\mathrm{i}\phi} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,(x-\mathrm{i}y)z \\ Y_2^{0}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\,(3\cos^2\theta-1) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\,(2z^2-x^2-y^2). \end{aligned}$$

Spherical harmonics of degree $\ell = 3$

$$
\begin{aligned}
Y_3^{-3}(\theta,\phi) &= \frac{1}{8}\sqrt{\frac{35}{\pi}}\,\sin^3\theta\,\mathrm{e}^{-3\mathrm{i}\phi} \;=\; \frac{1}{8}\sqrt{\frac{35}{\pi}}\,(x-\mathrm{i}y)^3 \\
Y_3^{-2}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\,\sin^2\theta\,\cos\theta\,\mathrm{e}^{-2\mathrm{i}\phi} \;=\; \frac{1}{4}\sqrt{\frac{105}{2\pi}}\,(x-\mathrm{i}y)^2 z \\
Y_3^{-1}(\theta,\phi) &= \frac{1}{8}\sqrt{\frac{21}{\pi}}\,\sin\theta\,(5\cos^2\theta-1)\,\mathrm{e}^{-\mathrm{i}\phi} \;=\; \frac{1}{8}\sqrt{\frac{21}{\pi}}\,(x-\mathrm{i}y)(4z^2-x^2-y^2) \\
Y_3^{0}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{7}{\pi}}\,(5\cos^3\theta-3\cos\theta) \;=\; \frac{1}{4}\sqrt{\frac{7}{\pi}}\,z(2z^2-3x^2-3y^2).
\end{aligned}
$$

Spherical harmonics of degree $\ell = 4$

$$
\begin{aligned}
Y_4^{-4}(\theta,\phi) &= \frac{3}{16}\sqrt{\frac{35}{2\pi}}\,\sin^4\theta\,\mathrm{e}^{-4\mathrm{i}\phi} \;=\; \frac{3}{16}\sqrt{\frac{35}{2\pi}}\,(x-\mathrm{i}y)^4 \\
Y_4^{-3}(\theta,\phi) &= \frac{3}{8}\sqrt{\frac{35}{\pi}}\,\sin^3\theta\,\cos\theta\,\mathrm{e}^{-3\mathrm{i}\phi} \;=\; \frac{3}{8}\sqrt{\frac{35}{\pi}}\,(x-\mathrm{i}y)^3 z \\
Y_4^{-2}(\theta,\phi) &= \frac{3}{8}\sqrt{\frac{5}{2\pi}}\,\sin^2\theta\,(7\cos^2\theta-1)\,\mathrm{e}^{-2\mathrm{i}\phi} \\
&= \frac{3}{8}\sqrt{\frac{5}{2\pi}}\,(x-\mathrm{i}y)^2(6z^2-x^2-y^2) \\
Y_4^{-1}(\theta,\phi) &= \frac{3}{8}\sqrt{\frac{5}{\pi}}\,\sin\theta\,(7\cos^3\theta-3\cos\theta)\,\mathrm{e}^{-\mathrm{i}\phi} \\
&= \frac{3}{8}\sqrt{\frac{5}{\pi}}\,(x-\mathrm{i}y)z(4z^2-3x^2-3y^2) \\
Y_4^{0}(\theta,\phi) &= \frac{3}{16}\sqrt{\frac{1}{\pi}}\,(35\cos^4\theta-30\cos^2\theta+3) \\
&= \frac{3}{16}\sqrt{\frac{1}{\pi}}\,(3(x^2+y^2)^2+8z^2(z^2-3x^2-3y^2)).
\end{aligned}
$$

APPENDIX B

DETAILS OF COMPUTATIONS

B.1 Proof that (3.2.5) holds

Equation (3.2.5) asserts that in the limit $\theta' \to 0$

$$\begin{aligned} Y_\ell^m(\theta+\theta',0) &= -\frac{1}{2}\sqrt{(\ell+m)(\ell-m+1)}\,\theta'\,Y_\ell^{m-1}(\theta,0) + Y_\ell^m(\theta,0) \\ &\quad +\frac{1}{2}\sqrt{(\ell-m)(\ell+m+1)}\,\theta'\,Y_\ell^{m+1}(\theta,0). \end{aligned}$$

Since this is not immediately obvious from the definition of the spherical harmonics (3.2.1) we show here that this equality holds.

From (3.2.1) we have that

$$Y_\ell^m(\theta+\theta',0) = (-1)^m \left(\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right)^{1/2} \mathcal{P}_\ell^m(\cos(\theta+\theta')).$$

We can expand the associated Legendre function $\mathcal{P}_\ell^m(\cos(\theta+\theta'))$ as follows, discarding powers of θ' greater than one since θ' is infinitesimal:

$$\begin{aligned} \mathcal{P}_\ell^m(\cos(\theta+\theta')) &= \frac{(1-\cos^2(\theta+\theta'))^{m/2}}{2^\ell\,\ell!}\left[\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell\right]_{x=\cos(\theta+\theta')} \\ &= \frac{\sin^m(\theta+\theta')}{2^\ell\,\ell!}\left[\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell\right]_{x=\cos(\theta+\theta')} \\ &= \frac{(\sin\theta+\theta'\cos\theta)^m}{2^\ell\,\ell!}\left[\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell\right]_{x=\cos\theta-\theta'\sin\theta} \\ &= \frac{\sin^m\theta + m\theta'\cos\theta\sin^{m-1}\theta}{2^\ell\,\ell!}\left[\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell\right]_{x=\cos\theta-\theta'\sin\theta} \end{aligned}$$

Here $\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell$ is a polynomial in x of degree less than or equal to $\ell - m$. Define

$$\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}(x^2-1)^\ell = \sum_{k=0}^{\ell-m} a_k x^k = p(x)$$

Then

$$\begin{aligned}
p(\cos\theta - \theta'\sin\theta) &= \sum_{k=0}^{\ell-m} a_k(\cos\theta - \theta'\sin\theta)^k \\
&= \sum_{k=0}^{\ell-m} a_k(\cos^k\theta - k\theta'\sin\theta\cos^{k-1}\theta) \\
&= \sum_{k=0}^{\ell-m} a_k\cos^k\theta - \theta'\sin\theta\sum_{k=0}^{\ell-m} ka_k\cos^{k-1}\theta \\
&= p(\cos\theta) - \theta'\sin\theta\, p'(\cos\theta)
\end{aligned}$$

and so

$$\begin{aligned}
\mathcal{P}_\ell^m(\cos(\theta+\theta')) &= \frac{\sin^m\theta + m\theta'\cos\theta\sin^{m-1}\theta}{2^\ell\ \ell!}\, p(\cos\theta - \theta'\sin\theta) \\
&= \frac{\sin^m\theta + m\theta'\cos\theta\sin^{m-1}\theta}{2^\ell\ \ell!}\, \big(p(\cos\theta) - \theta'\sin\theta\, p'(\cos\theta)\big) \\
&= \frac{\sin^m\theta}{2^\ell\ \ell!}p(\cos\theta) + \theta'\left(\frac{m\cos\theta\sin^{m-1}\theta}{2^\ell\ \ell!}p(\cos\theta) - \frac{\sin^m\theta}{2^\ell\ \ell!}\sin\theta\, p'(\cos\theta)\right) \\
&= \mathcal{P}_\ell^m(\cos\theta) - \theta'\mathcal{P}_\ell^{m+1}(\cos\theta) + \theta'\frac{m\cos\theta}{\sin\theta}\mathcal{P}_\ell^m(\cos\theta)
\end{aligned}$$

Thus we have

$$Y_\ell^m(\theta+\theta',0) = Y_\ell^m(\theta,0) + \sqrt{(\ell-m)(\ell+m+1)}\,\theta'\,Y_\ell^{m+1}(\theta,0) + \theta'\frac{m\cos\theta}{\sin\theta}Y_\ell^m(\theta,0). \quad \text{(B.1.1)}$$

Similarly

$$Y_\ell^{-m}(\theta+\theta',0) = Y_\ell^{-m}(\theta,0) + \sqrt{(\ell+m)(\ell-m+1)}\,\theta'\,Y_\ell^{-m+1}(\theta,0) - \theta'\frac{m\cos\theta}{\sin\theta}Y_\ell^{-m}(\theta,0). \quad \text{(B.1.2)}$$

By (3.2.2) the spherical harmonics satisfy

$$Y_\ell^{-m}(\theta,0) = (-1)^m Y_\ell^m(\theta,0)$$

and so (B.1.2) becomes

$$Y_\ell^m(\theta+\theta',0) = Y_\ell^m(\theta,0) - \sqrt{(\ell+m)(\ell-m+1)}\,\theta'\,Y_\ell^{m-1}(\theta,0) - \theta'\frac{m\cos\theta}{\sin\theta}Y_\ell^m(\theta,0). \quad \text{(B.1.3)}$$

Adding together (B.1.1) and (B.1.3) and dividing by 2 we arrive at (3.2.5) as required.

B.2 Details of computations required to find (6.3.21)

Here we give the details of the computations of the integrals I_1 to I_6 given by (6.3.15)–(6.3.20) which are required to compute the number of cubic $\mathbf{O}(\mathbf{3}) \times \mathbb{Z}_2$ equivariants in the representation on $V_\ell \oplus V_{\ell+1}$.

The integral I_1 given by (6.3.15)

$$
\begin{aligned}
I_1 &= \int_0^\pi (1-\cos(\theta))\chi(R_\theta)^4 \mathrm{d}\theta \\
&= \int_0^\pi 2(1-\cos(\theta)) \left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\theta} + \sum_{m=-\ell-1}^{\ell+1} \mathrm{e}^{\mathrm{i}m\theta} \right)^3 \left(\frac{\cos(\ell\theta) - \cos(\theta)\cos((\ell+1)\theta)}{1-\cos(\theta)} \right) \mathrm{d}\theta \\
&= \int_0^\pi (\cos(\ell\theta) - \cos((\ell+2)\theta)) \left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\theta} + \sum_{m=-\ell-1}^{\ell+1} \mathrm{e}^{\mathrm{i}m\theta} \right)^3 \mathrm{d}\theta \\
&= \int_0^\pi (\cos(\ell\theta) - \cos((\ell+2)\theta)) \left(2\sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\theta} + \mathrm{e}^{-\mathrm{i}\theta(\ell+1)} + \mathrm{e}^{\mathrm{i}\theta(\ell+1)} \right)^3 \mathrm{d}\theta \\
&= \int_0^\pi (\cos(\ell\theta) - \cos((\ell+2)\theta)) \left[8\left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\theta} \right)^3 + 12 \left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\theta} \right)^2 \left(\mathrm{e}^{-\mathrm{i}\theta(\ell+1)} + \mathrm{e}^{\mathrm{i}\theta(\ell+1)} \right) \right. \\
&\qquad \left. + \ 6 \left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\theta} \right) \left(\mathrm{e}^{-\mathrm{i}\theta(\ell+1)} + \mathrm{e}^{\mathrm{i}\theta(\ell+1)} \right)^2 + \left(\mathrm{e}^{-\mathrm{i}\theta(\ell+1)} + \mathrm{e}^{\mathrm{i}\theta(\ell+1)} \right)^3 \right] \mathrm{d}\theta \\
&= \int_0^\pi (\cos(\ell\theta) - \cos((\ell+2)\theta)) \left[8S_1 + 12S_2 + 6S_3 + S_4 \right] \mathrm{d}\theta
\end{aligned}
$$

where the only terms in S_1, S_2, S_3 and S_4 which contribute to the integral are those in $\cos(\ell\theta)$ and $\cos((\ell+2)\theta)$ since

$$
\int_0^\pi \cos(m\theta)\cos(n\theta)\mathrm{d}\theta = \begin{cases} \pi/2 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{B.2.1}
$$

We now compute the coefficients of $\cos(\ell\theta)$ and $\cos((\ell+2)\theta)$ for S_1, S_2, S_3 and S_4 in turn.

S_1: Let the coefficient of $\cos(\ell\theta)$ in S_1 be given by β_ℓ^1. Then β_ℓ^1 is twice the number of triples (m,n,p) such that $m+n+p=\ell$ where $m,n,p \in -\ell,\dots,\ell$. Thus

$$
\beta_\ell^1 = 2\sum_{j=1}^{2\ell+1} j = (2\ell+1)(2\ell+2).
$$

Similarly, if the coefficient of $\cos((\ell+2)\theta)$ in S_1 is $\beta_{(\ell+2)}^1$ then $\beta_{(\ell+2)}^1$ twice the number of triples (m,n,p) such that $m+n+p=\ell+2$ where $m,n,p \in -\ell,\dots,\ell$ and thus

$$
\beta_{(\ell+2)}^1 = 2\sum_{j=1}^{2\ell-1} j = 2\ell(2\ell-1).
$$

S_2: Notice that

$$S_2 = 2\beta_0 \cos((\ell+1)\theta) + 2\sum_{m=1}^{2\ell} \beta_m \left(\cos((\ell+1+m)\theta) + \cos((\ell+1-m)\theta)\right)$$

where β_m is the number of pairs (p,q) such that $p+q=m$ where $p,q \in -\ell,\ldots,\ell$. The coefficients of $\cos(\ell\theta)$ and $\cos((\ell+2)\theta)$ in S_2 are both given by $2\beta_1$ where $\beta_1 = 2\ell$.

S_3: Notice that

$$S_3 = 4 + 4\sum_{m=1}^{\ell} \cos(m\theta) + 2\sum_{m=1}^{2\ell} \left(\cos((2\ell+2+m)\theta) + \cos((2\ell+2-m)\theta)\right).$$

From this we see that the coefficient of $\cos(\ell\theta)$ in S_3 is 4 and the coefficient of $\cos((\ell+2)\theta)$ is 2.

S_4: There are no terms in $\cos(\ell\theta)$ or $\cos((\ell+2)\theta)$ in S_4.

Hence

$$\begin{aligned} I_1 &= \int_0^{\pi} (\cos(\ell\theta) - \cos((\ell+2)\theta)) \left[(8(2\ell+1)(2\ell+2) + 48\ell + 24)\cos(\ell\theta)\right. \\ &\qquad \left. + (16\ell(2\ell-1) + 48\ell + 12)\cos((\ell+2)\theta)\right] \mathrm{d}\theta \\ &= \frac{\pi}{2}\left[(8(2\ell+1)(2\ell+2) + 48\ell + 24) - (16\ell(2\ell-1) + 48\ell + 12)\right] \\ &= 2\pi(16\ell+7) \end{aligned}$$

The integral I_2 given by (6.3.16)

$$I_2 = \int_0^{\pi} (1-\cos(\theta))\chi(-R_\theta)^4 \mathrm{d}\theta = \int_0^{\pi} 16(1-\cos(\theta))\cos^4((\ell+1)\theta)\mathrm{d}\theta$$

Now

$$8\cos^4((\ell+1)\theta) = \cos(4(\ell+1)\theta) + 4\cos(2(\ell+1)\theta) + 3$$

so

$$\begin{aligned} I_2 &= \int_0^{\pi} 2(1-\cos(\theta))\left(\cos(4(\ell+1)\theta) + 4\cos(2(\ell+1)\theta) + 3\right)\mathrm{d}\theta \\ &= 6\pi. \end{aligned}$$

The integral I_3 given by (6.3.17)

$$\begin{aligned} I_3 &= \int_0^{\pi} (1-\cos(\theta))\chi(R_\theta)^2\chi(R_{2\theta})\mathrm{d}\theta \\ &= \int_0^{\pi} (\cos(\ell\theta) - \cos((\ell+2)\theta)) \left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{\mathrm{i}m\theta} + \sum_{m=-\ell-1}^{\ell+1} \mathrm{e}^{\mathrm{i}m\theta}\right)\left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{2\mathrm{i}m\theta} + \sum_{m=-\ell-1}^{\ell+1} \mathrm{e}^{2\mathrm{i}m\theta}\right)\mathrm{d}\theta \\ &= \int_0^{\pi} (\cos(\ell\theta) - \cos((\ell+2)\theta))P(\theta)\mathrm{d}\theta \end{aligned}$$

where

$$
\begin{aligned}
P(\theta) &= \left(\sum_{m=-\ell}^{\ell} e^{im\theta} + \sum_{m=-\ell-1}^{\ell+1} e^{im\theta}\right)\left(\sum_{m=-\ell}^{\ell} e^{2im\theta} + \sum_{m=-\ell-1}^{\ell+1} e^{2im\theta}\right) \\
&= \left(2\sum_{m=-\ell}^{\ell} e^{im\theta} + e^{-i\theta(\ell+1)} + e^{i\theta(\ell+1)}\right)\left(2\sum_{m=-\ell}^{\ell} e^{2im\theta} + e^{-2i\theta(\ell+1)} + e^{2i\theta(\ell+1)}\right) \\
&= 4\left(\sum_{m=-\ell}^{\ell} e^{im\theta}\right)\left(\sum_{m=-\ell}^{\ell} e^{2im\theta}\right) + 2\sum_{m=-\ell}^{\ell} e^{im\theta}\left(e^{-2i\theta(\ell+1)} + e^{2i\theta(\ell+1)}\right) \\
&\qquad +2\sum_{m=-\ell}^{\ell} e^{2im\theta}\left(e^{-i\theta(\ell+1)} + e^{i\theta(\ell+1)}\right) + 2\left(\cos(3(\ell+1)\theta) + \cos((\ell+1)\theta)\right) \\
&= 4S_1 + 2S_2 + 2S_3 + 2S_4.
\end{aligned}
$$

We want to compute the coefficients of $\cos(\ell\theta)$ and $\cos((\ell+2)\theta)$ in $P(\theta)$. There are no such terms in S_3 or S_4. We consider the other terms S_j for $j = 1, 2$ in turn.

S_1: The coefficient of $\cos(\ell\theta)$ in S_1 is twice the number of pairs (m, p) such that $m + 2p = \ell$ where $m, p \in -\ell, \ldots, \ell$. There are $\ell + 1$ such pairs. The coefficient of $\cos((\ell+2)\theta)$ in S_1 is twice the number of pairs (m, p) such that $m + 2p = \ell + 2$ where $m, p \in -\ell, \ldots, \ell$. There are ℓ such pairs.

S_2: Notice that

$$
\begin{aligned}
S_2 &= \sum_{m=-\ell}^{\ell} e^{im\theta}\left(e^{-2i\theta(\ell+1)} + e^{2i\theta(\ell+1)}\right) = \sum_{m=-\ell}^{\ell} e^{i\theta(m-2\ell-2)} + \sum_{p=-\ell}^{\ell} e^{i\theta(p+2\ell+2)} \\
&= 2 + 2\sum_{m=1}^{\ell}\left(\cos((m-2\ell-2)\theta) + \cos((m+2\ell+2)\theta)\right).
\end{aligned}
$$

The coefficient of $\cos(\ell\theta)$ in S_2 is zero and the coefficient of $\cos((\ell+2)\theta)$ is 2.

Hence

$$
I_3 = \int_0^{\pi} (\cos(\ell\theta) - \cos((\ell+2)\theta))\left[8(\ell+1)\cos(\ell\theta) + (8\ell+4)\cos((\ell+2)\theta)\right] d\theta = 2\pi.
$$

The integral I_4 given by (6.3.18)

$$
\begin{aligned}
I_4 &= \int_0^{\pi} (1-\cos(\theta))\chi(-R_\theta)^2\chi(R_{2\theta})d\theta \\
&= \int_0^{\pi} 4(1-\cos(\theta))\cos^2((\ell+1)\theta)\left(4\sum_{m=1}^{\ell}\cos(2m\theta) + 2 + 2\cos(2(\ell+1)\theta)\right) d\theta \\
&= \int_0^{\pi} \left(2\cos(2(\ell+1)\theta) + 2 - \cos((2\ell+3)\theta) - \cos((2\ell+1)\theta) - 2\cos(\theta)\right) \\
&\qquad\qquad \left(4\sum_{m=1}^{\ell}\cos(2m\theta) + 2 + 2\cos(2(\ell+1)\theta)\right) d\theta \\
&= \int_0^{\pi} 4(\cos^2(2(\ell+1)\theta) + 1)d\theta = 6\pi.
\end{aligned}
$$

The integral I_5 given by (6.3.19)

$$
\begin{aligned}
I_5 &= \int_0^\pi (1-\cos(\theta))\chi(R_\theta)\chi(R_{3\theta})\mathrm{d}\theta \\
&= \int_0^\pi (\cos(\ell\theta)-\cos((\ell+2)\theta)) \left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{3\mathrm{i}m\theta} + \sum_{m=-\ell-1}^{\ell+1} \mathrm{e}^{3\mathrm{i}m\theta}\right)\mathrm{d}\theta \\
&= \int_0^\pi (\cos(\ell\theta)-\cos((\ell+2)\theta)) \left(4\sum_{m=-\ell}^{\ell} \cos(3m\theta) + 2 + 2\cos(3(\ell+1)\theta)\right)\mathrm{d}\theta \\
&= \int_0^\pi 4(\cos(\ell\theta)-\cos((\ell+2)\theta)) \left(\sum_{m=-\ell}^{\ell} \cos(3m\theta)\right)\mathrm{d}\theta \\
&= \begin{cases} 2\pi & \text{if } \ell = 0 \bmod 3 \\ -2\pi & \text{if } \ell = 1 \bmod 3 \\ 0 & \text{if } \ell = 2 \bmod 3. \end{cases}
\end{aligned}
$$

The integral I_6 given by (6.3.20)

$$
\begin{aligned}
I_6 &= \int_0^\pi (1-\cos(\theta))\chi(-R_\theta)\chi(-R_{3\theta})\mathrm{d}\theta \\
&= \int_0^\pi 4(1-\cos(\theta))\cos(3(\ell+1)\theta)\cos((\ell+1)\theta)\mathrm{d}\theta \;=\; 0.
\end{aligned}
$$

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