# Matchings, factors and cycles in graphs.

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# Abstract

A matching in a graph is a set of pairwise nonadjacent edges, a $k$-factor is a $k$-regular spanning subgraph, and a cycle is a closed path.

This thesis has two parts. In Part I (by far the larger part) we study sufficient conditions for structures involving matchings, factors and cycles. The three main types of conditions involve: the minimum degree; the degree sum of pairs of non-adjacent vertices (Ore-type conditions); and the neighbourhoods of independent sets of vertices. We show that most of our theorems are best possible by giving appropriate extremal graphs.

We study Ore-type conditions for a graph to have a Hamilton cycle or 2-factor containing a given matching or path-system, and for any matching and single vertex to be contained in a cycle. We give Ore-type and neighbourhood conditions for a matching $L$ of $l$ edges to be contained in a matching of $k$ edges ($l < k$). We generalise two different aspects of this result: the $l = 0$ case with an Ore-type condition for a heavy matching in an edge-weighted graph; and the conditions for a perfect matching containing $L$ with degree and neighbourhood conditions for a $k$-factor ($k \geqslant 2$) containing a given set of edges. We also establish neighbourhood conditions for the existence of a cycle of length at least $k$.

A list-edge-colouring of a graph is an assignment of a colour to each edge from its own list of colours. In Part II we study edge colourings of powers of cycles, and prove the List-Edge-Colouring Conjecture for squares of cycles of odd length.

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# Contents

# Part I

# Degree and neighbourhood conditions for matchings, factors and cycles in graphs

# Chapter 1

# Introduction to Part I

Section 1.1 contains most of the graph-theoretic definitions used in Part I. More specialised terminology is given in the introduction to each chapter. In Section 1.2 we give some background to each chapter in Part I.

## 1.1 Terminology and notation

Throughout Part I, $G$ will denote a simple graph (without loops or multiple edges). Write $V(G)$ and $E(G)$ for the vertex-set and edge-set of $G$, and $n = |V(G)|$ for the *order* of $G$. Two vertices $u, v$ are *adjacent* if there exists an edge $e = uv$ joining them, and the vertices $u, v$ are *incident* with the edge $e$. Two edges are adjacent if they share an incident vertex.

The *degree* $d(v) = d_G(v)$ of a vertex $v \in V(G)$ is the number of vertices adjacent to $v$ in $G$. We use $\delta(G)$ to denote the *minimum degree* of $G$. Define

$$\sigma_2(G) = \min\{d(u) + d(v)\ : u, v \in V(G),\, u \neq v,\, uv \notin E(G)\}$$

(interpreted as $+\infty$ if $G$ is complete). Clearly $\sigma_2(G) \geqslant 2\delta(G)$. If $X \subseteq V(G)$, the *neighbourhood* of $X$ in $G$ is $N(X) = N_G(X) := \bigcup_{v \in X} N(v)$, where $N(v)$ is the set of vertices adjacent to $v$. A set $X \subseteq V(G)$ is *independent* if there are no edges of $G$ incident with two vertices in $X$. The *independence number* of $X$, denoted by $\alpha(X)$, is the number of vertices in a largest independent set contained in $X$. Write $\mathcal{I}(G)$ for the family of nonempty independent subsets of $V(G)$.

Our conditions in Part I mostly involve lower bounds on $\delta(G)$, $\sigma_2(G)$, or $|N(X)|$ whenever $X \in \mathcal{I}(G)$. In the latter case, we often need to assume a bound on the minimum degree that is just slightly stronger than would be obtained by putting $|X| = 1$ in the neighbourhood condition.

A subgraph $H$ of $G$ is a graph whose vertex-set $V(H)$ and edge-set $E(H)$ are subsets of $V(G)$ and $E(G)$ respectively. If $v \in V(H)$, we write $d_H(v)$ for the degree of $v$ in $H$. If $V(H) = V(G)$ then $H$ is a *spanning subgraph* of $G$. If $X \subset V(G)$, we write $G - X$ for the subgraph of $G$ obtained by removing the vertices of $X$ and their incident edges; if $H$ is a subgraph of $G$, we write $G - H$ for $G - V(H)$. If $X$ and $Y$ are sets or lists of vertices in $G$, we write $e(X : Y)$ for the number of edges of $G$ incident with a vertex in $X$ and a vertex in $Y$.

Let $f : V(G) \to \mathbb{N} \cup \{0\}$ be a function. An *$f$-factor* of $G$ is a spanning subgraph $H$ of $G$ such that $d_H(v) = f(v)$ for all $v \in V(H) = V(G)$. A *$k$-factor* is an $f$-factor such that $f(v) = k$ for all $v \in V(G)$ (that is, a *$k$-regular* spanning subgraph of $G$).

A *matching* is a set of pairwise nonadjacent edges, a *perfect matching* is the edge-set of a 1-factor, and a $k$-matching is a matching with $k$ edges. Abusing the notation slightly, we write $V(M)$ for the set of vertices incident with the edges of a matching $M$.

A *path* $P : u_1u_2 \ldots u_m$ is the graph with vertex-set $\{u_1, u_2, \ldots, u_m\}$ (no repeated vertices) and edge-set $\{u_1u_2, u_2u_3, \ldots, u_{m-1}u_m\}$. The *end vertices* of $P$ are $u_1$ and $u_m$, and all other vertices of $P$ are *interior vertices.* A *cycle* $C : v_0v_1 \ldots v_{n-1}v_0$ is a graph with vertex set $\{v_0, \ldots, v_{n-1}\}$ in which two vertices $v_i$ and $v_j$ are adjacent if and only if $j \in \{i-1, i+1\}$ (modulo $n$). The *length* of a path or cycle is the number of edges it contains. A path is *nontrivial* if it has length at least one. A graph $G$ is *Hamiltonian* if it contains a *Hamilton cycle*, which is a cycle of length $n = |V(G)|$; a *Hamilton path* has length $n - 1$.

A graph $G$ is *connected* if there exists a path between every pair of its vertices; it is *$t$-connected* if connected and there does not exist a set of $t - 1$ vertices whose removal disconnects $G$. The *components* of a graph are the maximal connected subgraphs. An *odd* (*even*) *component* is a component with an odd (even) number of vertices. A *path-system* in $G$ is a subgraph $F$ of $G$ in which every component of $F$ is a nontrivial path.

The union $G \cup H$ of two graphs $G$ and $H$ is the graph whose components are those of $G$ and $H$. The *join* $G + H$ is the graph obtained from $G \cup H$ by adding an edge between every vertex in $G$ and every vertex in $H$.

We use $K_n$ to denote the *complete graph* on $n$ vertices, and $K_{r,s}$ to denote the *complete bipartite graph* with partite sets with $r$ and $s$ vertices. The *cycle power* $C_n^p$ is the graph with vertex set $\{v_0, \ldots, v_{n-1}\}$ in which two vertices $v_i$ and $v_j$ are adjacent if and only if $j \in \{i - p, \ldots, i - 1, i + 1, \ldots, i + p\}$ (modulo $n$).

## 1.2 Background

**Chapter 2 : Ore-type conditions for a Hamilton cycle or 2-factor containing a given matching**

A well-known result of Ore [34] states that if $\sigma_2(G) \geqslant n$ then a graph $G$ of order $n$ is Hamiltonian. In Chapter 2 we establish sharp Ore-type conditions for a graph to have a Hamilton cycle or 2-factor containing all the edges of a given matching or path-system. Our results, summarised in Table 1.1, improve upon those of Kronk [30] and Häggkvist [21]; see Section 2.1 for more details. Generalising a result of Berman [6], we also prove that if $\sigma_2(G) \geqslant n + 1$ then any matching and single vertex are contained in a cycle in $G$.

Table 1.1: Sharp lower bounds on $\sigma_2(G)$ for a matching or path-system with $k$ edges and $c$ components to be contained in a 2-factor with at most two cycles.

| Range of $k, n$, and $c$ | 2-factor | Hamilton cycle |
|---|---|---|
| Matching: | | |
| $n = 2k$ ($k$ even) or $n = 2k+1$ ($k$ odd) | $n$ | $n+1$ |
| $n = 2k$ ($k$ odd) or $n = 2k+1$ ($k$ even) | $n+1$ | $n+1$ |
| $2k+2 \leqslant n \leqslant 3k+1$ | $2n-2k-1$ | $2n-2k-1$ |
| Matching or path-system: | | |
| $n \geqslant k+2c+1$, $n+k$ odd | $n+k$ | $n+k$ |
| $n \geqslant k+2c+2$, $n+k$ even, $c \geqslant 2$ | $n+k-1$ | $n+k-1$ |
| $c=1$, $n=k+2$ | $n+k-1$ | $n+k-1$ |
| $c=1$, $n=k+4$ | $n+k$ | $n+k$ |
| $c=1$, $n \geqslant k+6$ | $n+k-1$ | $n+k$ |

**Chapter 3 : Extension of matchings to larger matchings**

In 1947, Tutte [41] characterised graphs without a 1-factor by showing that such graphs never contain a set $S$ of vertices whose removal results in more than $|S|$ odd components. This result has the following useful generalisation. (A matching of *defect* $d$ is one in which the edges cover all except $d$ of the vertices of the graph.)

**Theorem 1.1.** (Defect form of Tutte's theorem, Berge [5]) *If* $S \subseteq V(G)$, *let* $o(S)$ *denote the number of odd components of* $G - S$. *Then the maximum size of a matching in* $G$ *is exactly*

$$\min_{S \subseteq V(G)} \tfrac{1}{2}(n - o(S) + |S|). \qquad \square$$

A matching $L$ *extends* to a matching $M$ if $M$ contains $L$. In particular, a graph is $l$-*extendable* if it has an $l$-matching and every such matching can be extended to a 1-factor. Plummer [35] showed that if $\delta(G) \geqslant \frac{1}{2}n + l$ then a graph $G$ of even order $n$ is $l$-extendable.

In Chapter 3 we consider a defect form of $l$-extendability, giving conditions which suffice to ensure that an $l$-matching extends to a $k$-matching, where $l < k$; the defect here is $n-2k$. Our main result generalises both Plummer's result, and the following result of Robertshaw and Woodall, which gives sharp neighbourhood conditions for a graph to have a 1-factor. A graph $G$ is *sesquiconnected* if it is connected and, for each vertex $v \in V(G)$, $G-\{v\}$ has at most two components.

**Theorem 1.2.** (Robertshaw and Woodall [40]) *Let $G$ be a sesquiconnected graph with even order $n$ and minimum degree $\delta(G) \geqslant \frac{1}{4}n+1$. Suppose that*
either (i)

$$|N(X)| \geqslant \tfrac{1}{4}(2|X|+n-4)$$

*whenever $X \in \mathcal{I}(G)$ and $5 \leqslant |X| \leqslant \frac{1}{2}n+1$,*
or (ii)

$$|N(X)| \geqslant \tfrac{1}{2}(4|X|-n-4)$$

*for every set $X \subset V(G)$ such that $\alpha(X) \geqslant n-|X|+2$ and $\frac{1}{2}n+1 \leqslant |X| \leqslant \frac{3}{4}n-1$.*

*Then $G$ has a 1-factor.* $\square$

**Chapter 4 : Ore-type conditions for a heavy matching**

A *weighted graph* is a graph $G$ in which every edge $e$ is assigned a nonnegative number $w(e)$, called the *weight* of $e$. The *weighted degree* of a vertex $v \in V(G)$ is the sum of the weights of the edges incident with $v$. The weight of a matching, cycle or path is the sum of the weights of its edges.

Bondy and Fan [8] gave weighted degree conditions for heavy cycles and paths in weighted graphs. Bondy et al. [7] gave an Ore-type theorem for heavy cycles, and Enomoto, Fujisawa and Ota [16] proved a similar result about heavy paths between specified vertices (these two results are stated in Section 4.1). In Chapter 4 we an give an Ore-type condition for heavy matchings in weighted graphs. We also consider the case when all edge-weights are integers.

**Chapter 5 : $k$-factors containing a given set of edges**

Let $k \geqslant 2$ be an integer and $G$ a graph of order $n$, where $kn$ is even. (If $kn$ is odd, then $G$ cannot have a $k$-factor $H$, since each edge of $H$ contributes 2 to $\sum_{v \in V(G)} d_H(v) = kn$.) In 1985, Katerinis [25] showed that if $n \geqslant 4k-5$ and $\delta(G) \geqslant n/2$, then $G$ has a $k$-factor. Woodall then established neighbourhood conditions for $k$-factors:

**Theorem 1.3.** (Woodall [47]) *Let $k \geqslant 2$ be an integer. Let $G$ be a graph of order $n$, and suppose that, if $k$ is odd, then $n$ is even and $G$ is connected. Suppose that*

$$|N(X)| \geqslant \frac{1}{2k-1}(|X| + (k-1)n - 1) \quad \textit{whenever } X \in \mathcal{I}(G),$$

*and $G$ has minimum degree*

$$\delta(G) \geqslant \frac{(k-1)(n+2)}{2k-1}.$$

*Suppose further that if $n < 4k-6$ then $\delta(G) > n + 2k - 2\sqrt{kn+2}$.*
*Then $G$ has a $k$-factor.*

In Chapter 5 we generalise these results of Katerinis and Woodall, giving degree and neighbourhood conditions for a set of $l$ edges (not necessarily a matching) to be contained in a $k$-factor ($k \geqslant 2$). Poole [36] gave neighbourhood conditions for a $k$-factor containing a set of $l$ edges in a bipartite graph.

**Chapter 6 : Circumference**

The *circumference* of a graph $G$ is the length of a longest cycle in $G$. In 1978, Woodall [46, 47] proved that if $G$ is a 2-connected graph of order $n$ such that $\delta(G) \geqslant \frac{1}{3}(n+2)$, and $|N(X)| \geqslant \frac{1}{3}(|X| + n - 1)$ whenever $X \in \mathcal{I}(G)$, then

$G$ is Hamiltonian. This result is sharp, and contains Dirac's result [13] that if $\delta(G) \geqslant \frac{1}{2}n$ then $G$ is Hamiltonian.

In Chapter 6 we generalise Woodall's result, giving neighbourhood conditions for a graph to have circumference at least $k$. On a similar theme, Robertshaw and Woodall [39] showed that if $|N(X)| > \frac{1}{3}(n + |X|)$ whenever $X \in \mathcal{I}(G)$, then $G$ contains a triangle; and, if we also assume that $G$ is 2-connected, then $G$ is *pancyclic* (that is, $G$ contains cycles of all lengths between 3 and $|V(G)|$).

Variants of Woodall's Hopping Lemma [45] have been used to prove many results about long cycles: for example, see [3, 10, 27]. In Section 6.2 we prove a version of the Hopping Lemma, using a similar result of Min Aung [3].

# Chapter 2

# Ore-type conditions for a Hamilton cycle or $2$-factor containing a given matching

## 2.1 Introduction

Recall that a 2-factor in a graph $G$ is a 2-regular spanning subgraph, that is, a union of vertex-disjoint cycles spanning $V(G)$. A connected 2-factor is therefore a Hamilton cycle. A *$k$-matching* is a set of $k$ pairwise nonadjacent edges, and a *path-system* is a subgraph $F$ of $G$ in which every component of $F$ is a nontrivial path.

In this chapter we establish sharp Ore-type conditions for a graph $G$ to have a Hamilton cycle or 2-factor containing all the edges of a given path-system or matching. Our results are summarised in Table 1.1 (see section 1.2). We also show that if $\sigma_2(G) \geqslant n + 1$ then any matching and single vertex are contained in a cycle in $G$.

Let $F$ be a path system of $k \geqslant 1$ edges in a graph $G$ of order $n$. In 1963, Pósa [37] proved that if $\delta(G) \geqslant \frac{1}{2}(n+k)$ then $G$ has a Hamilton cycle containing $F$. Then Kronk [30] showed that the same conclusion holds if $\sigma_2(G) \geqslant n+k$; this is sharp when $n+k$ is odd. Our first result, proved in Section 2.2, shows that the bound on $\sigma_2(G)$ can, in most cases, be weakened to $n+k-1$ when $n+k$ is even.

**Theorem 2.1.** *Let $G$ be a graph of order $n$ and let $F$ be a path-system with $k$ edges in total. Suppose that*

$$\sigma_2(G) \geqslant 2\lceil \tfrac{1}{2}(n+k) \rceil - 1. \tag{2.1}$$

*Then $G$ has a Hamilton cycle containing $F$, except possibly when $F$ is a single path, $n \geqslant k+4$ and $\sigma_2(G) = n+k-1$ (so that $n+k$ is even), in which case $G$ has a 2-factor of at most two cycles containing $F$ if $n \geqslant k+6$.*

Now suppose that $F$ has $c$ components. In Section 2.5 we show that Theorem 2.1 is sharp whenever $n \geqslant k+2c+1$, and when $c = 1$ and $n = k+2$. If $n = k+1$ then the Theorem is vacuous, since (2.1) then becomes $\sigma_2(G) \geqslant 2n-1$, which is impossible. For completeness, note that if $n = k+1$ then the condition $\sigma_2(G) \geqslant 2n-3$, which forces $G$ to be complete, is sufficient and sharp. We also show that $\sigma_2(G) \geqslant n+k$ is needed to force a Hamilton cycle containing $F$ when $c = 1$ and $n \geqslant k+4$. If $c = k$ and $F$ is a matching, then Theorem 2.1 is only sharp when $n \geqslant 3k+1$. The case $n \leqslant 3k$ was investigated by Häggkvist [21], who proved the following result about perfect matchings, and deduced Theorem 2.3 using an inductive argument.

**Theorem 2.2.** (Häggkvist [21]) *Let $G$ be a graph of even order $n$ such that $\sigma_2(G) \geqslant n+1$, and let $M$ be a perfect matching in $G$. Then $G$ has a Hamilton cycle containing $M$.*

**Theorem 2.3.** (Häggkvist [21]) *Let $G$ be a graph of order $n$, and let $M$ be a $k$-matching in $G$. Suppose that $\sigma_2(G) \geqslant \min\{n+k, 2n-2k+1\}$. Then $G$ has a Hamilton cycle containing $M$.*

Häggkvist also conjectured the following result, which contains Theorem 2.2.

**Theorem 2.4.** (Berman [6]) *Let $G$ be a graph of order $n$ such that $\sigma_2(G) \geqslant n+1$. Then every matching in $G$ lies in a cycle.*

Berman's proof of Theorem 2.4 uses a counting argument involving a *theta-graph*, that is, two vertices of degree three joined by three paths which share no interior vertices. In Section 2.7 we adopt the same approach to prove the following result.

**Theorem 2.5.** *Let $G$ be a graph of order $n$ such that $\sigma_2(G) \geqslant n+1$. Let $M$ be a matching in $G$, and let $v \in V(G)$. Then there exists a cycle in $G$ containing $M$ and $v$.*

In Section 2.3 we use Theorem 2.5 in a similar way to Häggkvist's use of Theorem 2.2 to obtain Theorem 2.6, an improvement on Theorem 2.3 when $2k+1 \leqslant n \leqslant 3k$.

**Theorem 2.6.** *Let $G$ be a graph of order $n$ and let $M$ be $k$-matching in $G$. Suppose that $\sigma_2(G) \geqslant \max\{n+1, 2n-2k-1\}$. Then $G$ has a Hamilton cycle containing $M$.*

It is easy to check that if $(n,k) = (3,1)$ or $(4,2)$, then the weaker condition $\sigma_2(G) \geqslant n$ suffices to ensure the existence of a Hamilton cycle containing $M$. In Section 2.6 we give examples to demonstrate that, except in these cases, Theorem 2.6 is sharp for $n \leqslant 3k+1$. Moreover, for such values of $n$ and $k$, Theorem 2.6 is also sharp for the existence of a 2-factor containing $M$, except in the cases given by the following result.

**Theorem 2.7.** *Let $G$ be a graph of order $n = 2k$, where $k$ is even, or order $n = 2k+1$, where $k$ is odd, and suppose that $\sigma_2(G) \geqslant n$. Then any $k$-matching in $G$ is contained in a 2-factor consisting of at most two cycles.*

Theorem 2.7 is proved, by an extension of Berman's method, in Section 2.8. Since the proof is quite long, in Section 2.4 we also give a short proof, using Tutte's Theorem [41], of this result stated without the requirement "consisting of at most two cycles". Jackson and Wormald [23] gave the extremal graphs to Theorems 2.2 and 2.4. Their results imply Theorem 2.7 in the case when $n = 2k$ and $k$ is even; in fact, they also show that if $\delta(G) \geqslant \frac{1}{2}n$ and $n \equiv 0 \pmod 4$ then any 1-factor is contained in a Hamilton cycle in $G$.

## 2.2 Path-systems - proof of Theorem 2.1

Suppose that there is no Hamilton cycle containing $F$. Clearly $G$ is not complete; but we may assume that the number of edges in $G$ is maximal, so that the addition of any further edge would create such a Hamilton cycle. Thus if $u_1$ and $u_n$ are two nonadjacent vertices of $G$ then there exists a Hamilton path $P : u_1u_2 \dots u_n$ that contains $F$. [Note that the following three claims rely upon the existence of the Hamilton path $P$; they do not use the edge-maximality of $G$.]

**Claim 2.8.** *$d(u_1) + d(u_n) = \sigma_2(G) = n + k - 1$ (so that $n + k$ is even). Moreover, $E(F) = \{u_{i-1}u_i : u_1u_i$ and $u_{i-1}u_n \in E(G)\}$, and if $u_{j-1}u_j \notin E(F)$ then exactly one of $u_1u_j$ and $u_{j-1}u_n$ is an edge of $G$.*

**Proof.** If $u_{j-1}u_j \notin E(F)$ then $u_1u_j$ and $u_{j-1}u_n$ are not both edges of $G$, since otherwise

$$u_1u_iu_{j+1} \dots u_nu_{j-1}u_{j-2} \dots u_1$$

is a Hamilton cycle containing $F$. Since $|E(F)| = k$ and $|E(P) \setminus E(F)| = n-k-1$, it follows that

$$\begin{aligned} n+k-1 \leqslant \sigma_2(G) \leqslant d(u_1)+d(u_n) &\leqslant \sum_{j=2}^{n}[e(u_1:u_j)+e(u_{j-1}:u_n)] \\ &\leqslant 2k+(n-k-1) \\ &= n+k-1. \end{aligned} \tag{2.2}$$

Thus equality holds throughout (2.2), and the claim follows. □

This proves that if $\sigma_2(G) \geqslant n+k$ then $G$ has a Hamilton cycle containing $F$. In particular this holds when $n+k$ is odd, by (2.1); so suppose that $n+k$ is even. Note that Claim 2.8 applies to every Hamilton path containing $F$.

**Claim 2.9.** *Every end vertex of a path in $F$ is adjacent to both $u_1$ and $u_n$.*

**Proof.** Let $u_r \dots u_s$ $(r < s)$ be any path in $F$. Since $u_{s-1}u_s \in E(F)$, Claim 2.8 implies that $u_1u_s \in E(G)$; hence $s < n$. Suppose that $u_su_n \notin E(G)$. Then $u_1u_{s+1} \in E(G)$ by Claim 2.8, since $u_su_{s+1} \notin E(F)$. Hence

$$P' : u_su_{s-1}\dots u_1u_{s+1}u_{s+2}\dots u_n$$

is a Hamilton path which contains $F$ and has nonadjacent end vertices. Since $u_su_{s-1} \in E(F)$, applying Claim 2.8 to $P'$ implies $u_su_n \in E(G)$, a contradiction. Thus $u_s$ is adjacent to both $u_1$ and $u_n$, and by symmetry $u_r$ is too. □

**Claim 2.10.** *There does not exist $r$ $(2 \leqslant r \leqslant n-1)$ such that neither $u_1u_r$ nor $u_ru_n$ is an edge of $G$.*

**Proof.** Suppose that such an $r$ exists. Then Claim 2.8 implies that $u_1u_{r+1}$ and $u_{r-1}u_n$ are both edges of $G$. Hence there are Hamilton paths

$$P_1 : u_ru_{r-1}u_{r-2}\dots u_1u_{r+1}\dots u_n \text{ and } P_2 : u_1u_2\dots u_{r-1}u_nu_{n-1}\dots u_r,$$

where each path contains $F$ and has nonadjacent end vertices. Applying Claim 2.8 to $P_1$ and $P_2$ implies that $d(u_r)+d(u_n)=n+k-1$ and $d(u_1)+d(u_r)=n+k-1$. Hence

$$2(d(u_1)+d(u_r)+d(u_n))=3(n+k-1),$$

contradicting the supposition that $n+k$ is even. $\square$

Suppose that $F$ has at least two components and let $u_{s-1}u_s$, $u_tu_{t+1}$ $(s<t)$ be edges of $F$ such that $u_{j-1}u_j \notin E(F)$ for $j=s+1,\ldots,t$. By Claim 2.9, $u_1u_t \in E(G)$ and $u_su_n \in E(G)$. Since $u_su_{s+1} \notin E(F)$, Claim 2.8 implies that $u_1u_{s+1} \notin E(G)$; hence $t \geqslant s+2$. Let $r$ be minimal such that $s+2 \leqslant r+1 \leqslant t$ and $u_1u_{r+1} \in E(G)$. Then $u_1u_r \notin E(G)$ by minimality, and $u_ru_n \notin E(G)$ by Claim 2.8, since $u_ru_{r+1} \notin E(F)$. This contradicts Claim 2.10.

So suppose that $F$ is a single path of length $k$. If $n=k+2$ then $d(u_1)+d(u_n) \leqslant 2k$, but this contradicts the fact that $\sigma_2(G)=n+k-1=2k+1$; so we may assume that $n \geqslant k+4$. In this case we cannot prove the existence of a Hamilton cycle containing $F$, so suppose instead that $F$ is not contained in any 2-factor with at most two cycles.

Define $G' := G+K_1$. Then (2.1) gives $\sigma_2(G')=\sigma_2(G)+2 \geqslant n+k+1=|V(G')|+k$. The remark after Claim 2.8 shows that $G'$ has a Hamilton cycle containing $F$; hence $G$ has a Hamilton path $P: u_1u_2\ldots u_n$ containing $F$. By supposition there is no Hamilton cycle in $G$ containing $F$, hence $u_1u_n \notin E(G)$. Also, by the existence of $P$, Claims 2.8 to 2.10 hold as before.

Let $u_{s-k}$ and $u_s$ be the end vertices of the path $F$, and suppose that $s \leqslant n-3$. Claim 2.9 implies that $u_1u_s$ and $u_su_n \in E(G)$. Since $u_su_{s+1} \notin E(F)$, Claim 2.8 implies that $u_1u_{s+1} \notin E(G)$. It follows by Claim 2.10 that $u_{s+1}u_n \in E(G)$, hence the cycles

$$C_1: u_1u_su_{s-1}\ldots u_1 \text{ and } C_2: u_nu_{s+1}u_{s+2}\ldots u_n$$

form a 2-factor containing $F$, which is a contradiction. Hence $s \geqslant n-2$, and by symmetry $s-k \leqslant 3$. Combining these inequalities we see that if there is no 2-factor with at most two cycles which contains $F$, then $n \leqslant k+5$. This completes the proof of Theorem 2.1. □

## 2.3 Matchings - proof of Theorem 2.6

We prove the result by induction on $n-2k$, assuming the truth of Theorem 2.5, which we will prove in section 2.7. There are four cases to consider.

*Case* 1 : $n = 2k$. Then the result is just Theorem 2.2 (Häggkvist [21]).

*Case* 2 : $n = 2k+1$. Then $G - V(M)$ is a single vertex, and the result follows from Theorem 2.5.

*Case* 3 : $n = 2k+2$. Let $v$ and $w$ be the two vertices outside $V(M)$, where $d(v) \leqslant d(w)$. If $vw \in E(G)$ then Theorem 2.4 gives a Hamilton cycle containing $M \cup \{vw\}$. So we may assume $vw \notin E(G)$, hence

$$d(v) + d(w) \geqslant \sigma_2(G) \geqslant n+1 = 2k+3,$$

which implies $d(w) \geqslant k+2$.

By Theorem 2.5 there exists a cycle $C$ containing $M$ and $v$; we may assume that $w \notin V(C)$ and $C$ has length $n-1 = 2k+1$. Since there are $k+1$ edges in $E(C) - M$, and $d(w) \geqslant k+2$, $w$ must be adjacent to both end vertices of some such edge; hence we can add $w$ to $C$ giving a Hamilton cycle containing $M$, as required.

*Case* 4 : $n \geqslant 2k+3$. Then, by hypothesis, $\sigma_2(G) \geqslant 2n-2k-1$. Since $G$ is connected by Ore's Theorem, and $n > 2k$, there exists a path $xyz$ in $G$ for which $xy \in M$ and $z$ is a vertex outside $V(M)$. Let $G'$ denote the graph obtained from $G-y$ by adding the edge $xz$ if it is not already present, and define $M' := (M \setminus \{xy\}) \cup \{xz\}$. Then $M'$ is a $k$-matching in $G'$, and

$$\sigma_2(G') \geqslant \sigma_2(G) - 2 \geqslant 2n - 2k - 3 = 2|V(G')| - 2k - 1 \geqslant n = |V(G')| + 1.$$

By the inductive hypothesis, it follows that $G'$ has a Hamilton cycle $C'$ containing $M'$. Replacing the edge $xz$ in $C$ by the path $xyz$ gives a Hamilton cycle in $G$ containing $M$. $\square$

## 2.4 2-factor containing a perfect or near-perfect matching

In this section we prove the following weaker version of Theorem 2.7.

**Theorem 2.11.** *Let $G$ be a graph of order $n = 2k$, where $k$ is even, or of order $n = 2k + 1$, where $k$ is odd, and let $M$ be a $k$-matching in $G$. Suppose that $\sigma_2(G) \geqslant n$. Then $G$ has a 2-factor containing $M$.*

**Proof.** Suppose first that $n = 2k + 1$, where $k$ is odd, and let $k' = k + 1$. Let $v$ be the vertex of $G$ that is outside $V(M)$, and form a graph $G'$ from $G$ by adding a vertex $w$ that is adjacent to $v$ and all $G$-neighbours of $v$. Let $M' := M \cup \{vw\}$. Then $M'$ is a perfect matching in $G'$, which has order $2k'$, where $k'$ is even. Moreover, $G'$ has a 2-factor containing $M'$ if and only if $G$ has a 2-factor containing $M$. Thus both parts of Theorem 2.11 will follow from the following Lemma.

**Lemma 2.12.** *Let $G$ be a graph of order $n = 2k$, where $k$ is even, and let $M$ be a perfect matching in $G$. Suppose that $\sigma_2(G) \geqslant n-1$, and that there exists $v \in V(G)$ such that if either $x = v$ or $xv \in E(G)$ then*

$$d(x) + d(y) \geqslant n \text{ for all } y \in V(G) \text{ such that } xy \notin E(G). \tag{2.3}$$

*Then $G$ has a 2-factor containing $M$.*

**Proof.** Suppose that $G$ has no 2-factor containing $M$. Then there is no 1-factor in $G - M$, and so by Tutte's Theorem [41] there exists $S \subseteq V(G)$ such that $o(S) \geqslant |S| + 2$, where $o(S)$ is the number of *odd components* (components of odd order) of $G - M - S$. Choose $|S| + 2$ such odd components $C_1, C_2, \ldots, C_{|S|+2}$ in such a way that there exist vertices $u_1 \in C_1$ and $u_2 \in C_2$ such that $u_1u_2 \notin E(G)$. (This is possible since the only edges of $G$ between components of $G - M - S$ are in $M$: if we choose any vertex $u_1$ in a smallest component $C_1$, then there must be at least two vertices in $G - S - V(C_1)$, because if $o(S) = 2$ and $S = \emptyset$ then $|V(C_2)| \geqslant 3$, since $n = 2k \geqslant 4$ as $k$ is even.) Then

$$d_G(u_i) \leqslant |S| + |V(C_i)| \quad (i = 1, 2).$$

Since $\sigma_2(G) \geqslant n - 1$, it follows that

$$\begin{aligned} |S| + \sum_{j=1}^{|S|+2} |V(C_j)| - 1 \leqslant n - 1 &\leqslant d_G(u_1) + d_G(u_2) \\ &\leqslant 2|S| + |V(C_1)| + |V(C_2)|. \end{aligned} \tag{2.4}$$

At least one of the last two inequalities in (2.4) must be strict, because $n$ and $2|S| + |V(C_1)| + |V(C_2)|$ are both even; hence

$$|S| + \sum_{j=1}^{|S|+2} |V(C_j)| \leqslant n \leqslant 2|S| + |V(C_1)| + |V(C_2)|. \tag{2.5}$$

Since $|V(C_i)| \geqslant 1$ for each $i$ ($3 \leqslant i \leqslant |S| + 2$), it follows that equality holds throughout (2.5). Hence each component $C_j$ ($3 \leqslant j \leqslant |S| + 2$) consists of a single vertex, $o(S) = |S| + 2$ and

$$n = 2|S| + |V(C_1)| + |V(C_2)|. \tag{2.6}$$

We now show that we can choose $u_1$ so that it is matched by an edge of $M$ to a vertex $x \in S \cup V(C_1)$. This is clear if $o(S)$ is odd, since then there must be an edge $u_1 x \in M$, where $u_1 \in C_1$, say, and $x \in S$. If $o(S) \geqslant 4$ then two applications of the above argument shows that every component $C_j$ $(1 \leqslant j \leqslant |S|+2)$ must consist of a single vertex. Then by (2.6) it follows that $|S| = \frac{1}{2}(n-2) = k-1$ and $o(S) = k+1$, which is odd by hypothesis. The only remaining case is that $S = \emptyset$ and $o(S) = 2$; then let $C_1$ be the larger odd component. Since $k$ is even, $|V(C_1)| \geqslant k+1$; hence there exists an edge $u_1 x \in M$ with $u_1$ and $x \in V(C_1)$.

In all cases, choose $u_2$ to be any vertex in $C_2$; then $u_1 u_2 \notin E(G)$, and (2.6) gives

$$d_G(u_1) + d_G(u_2) \leqslant (|S| + |V(C_1)| - 1) + (|S| + |V(C_2)|) = n - 1. \tag{2.7}$$

Since $\sigma_2(G) \geqslant n-1$, equality holds throughout (2.7); hence $u_1$ is adjacent to every other vertex in $S \cup V(C_1)$. Thus if $v \in S \cup V(C_1)$ then (2.7) contradicts (2.3). So we may assume that $v \notin S \cup V(C_1)$. But now we can rechoose $u_2$ if necessary so that $v = u_2$, and then (2.7) again contradicts (2.3). This completes the proof of Lemma 2.12, and also that of Theorem 2.11. □ □

## 2.5 Sharpness of Theorem 2.1

First let $G$ be the complete graph $K_{k+2}$ with an edge $uv$ removed, and let $P$ be a path of length $k$ in $G - v$ with $u$ as one end vertex. Then $\sigma_2(G) = 2k = n + k - 2$, but $P$ is not contained in any Hamilton cycle in $G$. Hence (2.1) is sharp when $F$ is a single path and $n = k+2$. From now on we assume that $n \geqslant k+3$.

Given $c \geqslant 1$, $k \geqslant c$, and $n \geqslant k + 2c + 1$, let $n = k + 2(c+r) + \zeta$, where $r \geqslant 0$ and

$\zeta \in \{1, 2\}$. Define $G := A + B$, where $A := K_{c+k+r}$ and $B := (c + r + \zeta)K_1$. Then

$$\sigma_2(G) = 2(c + k + r) = n + k - \zeta = 2\lceil \tfrac{1}{2}(n + k)\rceil - 2,$$

since $\zeta \equiv k + n \pmod 2$. Let $F$ be a path-system with $c$ components and $k$ edges inside $A$, and suppose that $G$ has a 2-factor containing $F$, with $q$ of its edges between $A$ and $B$. Since there are $c + r + \zeta$ components in $B$, and $k$ edges of $F$ in $A$ we obtain

$$2|V(A)| = 2(c + r + \zeta) = q \leqslant 2|V(B)| - 2k = 2(c + r),$$

which is impossible, since $\zeta > 0$. Thus $G$ has no 2-factor containing $F$. It follows that (2.1) is sharp when $n \geqslant k + 2c + 1$. Thus if $F$ is (the path-system induced by) a $k$-matching (i.e., $k = c$) then (2.1) is sharp when $n \geqslant 3k + 1$.

Lastly, consider the graph $G$ of order $n = k + 2s + 4$ ($s \geqslant 0$) consisting of two complete graphs $K_{k+s+2}$ and $K_{k+s+3}$ sharing exactly $k + 1$ vertices. Then

$$\sigma_2(G) = (k + s + 1) + (k + s + 2) = n + k - 1,$$

but a single path $P$ of length $k$ within the intersection of the two complete graphs is not contained in a Hamilton cycle, since $G - V(P)$ has two components and a cycle containing $P$ can only pass through one of them. Hence we need $\sigma_2(G) \geqslant n + k$ to force a Hamilton cycle containing $F$ in the case when $c = 1$, $n + k$ is even and $n \geqslant k + 4$. If $n = k + 4$ (i.e., $s = 0$) then $P$ is not contained in any 2-factor, since neither component of $G - V(P)$ contains enough vertices to form a cycle. Thus the condition $n \geqslant k + 6$ in the last sentence of Theorem 2.1 is sharp.

## 2.6 Sharpness of Theorems 2.6 and 2.7

Let $G_1$ be a graph of order $n$ $(2k \leqslant n \leqslant 2k+2)$ consisting of two complete graphs, each of order at least 3, sharing exactly two vertices and the edge $e$ joining them. Then $\sigma_2(G_1) = n$. Let $M$ be a matching of $k$ edges in $G_1$ which includes $e$. Then $M$ is not contained in a Hamilton cycle, since $G_1 - V(e)$ has two components and a cycle containing $e$ can only pass through one of them. Thus the condition $\sigma_2(G) \geqslant n+1$ in Theorem 2.6 is sharp if $n \geqslant 5$, or if $n = 4$ and $k = 1$. Later we will consider the cases in which this condition is needed to force a 2-factor containing $M$.

Suppose that $2k \leqslant n \leqslant 3k+1$, and let $k = m+s$, where $m = n-2k-1 \geqslant -1$ and $s = 3k+1-n \geqslant 0$. Consider the graph of order $n = 2k+m+1 = 3m+2s+1$ defined by

$$G_2 := K_{2m+s} + (\tfrac{1}{2}(s-t)K_2 \cup (m+t+1)K_1),$$

where $t \in \{0,1\}$ and $t \equiv s \pmod 2$. Let $A := K_{2m+s}$ and $B := G_2 - A$. Let $M$ be a $k$-matching consisting of $\frac{1}{2}(2m+s-t)$ edges inside $A$, $\frac{1}{2}(s-t)$ edges inside $B$ and, if $s$ is odd, one edge from the unmatched vertex in $A$ to a copy of $K_1$ in $B$. Suppose that $G_2$ has a 2-factor containing $M$, with $q$ of its edges between $A$ and $B$. Then $q \leqslant 2m+s+t$, because $M$ contains a matching of $\frac{1}{2}(2m+s-t)$ edges inside $A$. On the other hand, since $B$ has $m+t+1+\frac{1}{2}(s-t)$ components,

$$q \geqslant 2(m+t+1) + (s-t) > 2m+s+t.$$

This contradiction shows that $G_2$ has no Hamilton cycle, or even 2-factor, containing $M$.

If $n \geqslant 2k+2$ (i.e., $2n-2k-1 \geqslant n+1$) then $m \geqslant 1$. Then $G_2$ has at least two nonadjacent vertices of degree $2m+s$, and $\sigma_2(G_2) = 2(2m+s) = 2(n-k-1)$. This proves that the condition $\sigma_2(G) \geqslant 2n-2k-1$ in Theorem 2.6 is sharp even for the existence of a 2-factor containing $M$.

If $n = 2k$ and $k$ is odd then $m = -1$ and $s$ is even; then it follows that $t = 0$ and $\sigma_2(G_2) = 2(s-1) = n$. If $n = 2k+1$ and $k$ is even then $m = 0$ and $s$ is even, so $t = 0$ and $\sigma_2(G_2) = 2s+1 = n$. Thus if $n = 2k$, where $k$ odd, or if $n = 2k+1$, where $k$ even, then the condition $\sigma_2(G_2) \geqslant n+1$ in Theorem 2.6 is also sharp for 2-factors.

Lastly, we use the same example to show that Theorem 2.7 is sharp. If $n = 2k$ and $k$ is even then $m = -1$ and $s$ is odd, so $t = 1$ and $\sigma_2(G_2) = 2s-3 = n-1$. If $n = 2k+1$ and $k$ is odd then $m = 0$ and $s$ is odd, so $t = 1$ and $\sigma_2(G_2) = 2s = n-1$. Thus the condition $\sigma_2(G) \geqslant n$ in Theorem 2.7 is sharp.

## 2.7 Cycle containing a matching and a vertex - proof of Theorem 2.5

Suppose that there is no cycle in $G$ containing $M$ and $v$. By Theorem 2.4 there exists a cycle $C$ containing $M$, and we may suppose that $v$ lies outside $C$ (in particular, $v \notin V(M)$). Also, $v$ cannot be adjacent to any vertex $w$ outside $V(M)$, else Theorem 2.4 gives a cycle containing $M \cup \{vw\}$. Let $k$ denote the number of edges in $M$. If $d(v) > k$ then it follows that we can add $v$ to $C$ giving a cycle containing $M$ and $v$; hence

$$d(v) \leqslant k < \tfrac{1}{2}(n+1). \tag{2.8}$$

By Ore's Theorem, $G$ is connected. Hence $v$ is adjacent to some vertex of $C$, and there exists a path containing $M$ and $v$. Theorem 2.5 will follow from the next lemma, which is also used in the proof of Theorem 2.7.

**Lemma 2.13.** *Let $G$ be a graph of order $n$ such that $\sigma_2(G) \geqslant n$. Let $M_v$ consist of a $k$-matching $M$ in $G$ and, if $n > 2k$, also a vertex $v$ outside $V(M)$. Suppose that there is a path containing $M_v$ but no cycle containing $M_v$. Then*

(a) *there exists a theta-graph containing $M_v$ in $G$, $\Theta$ say, such that every edge of $M$ incident with a vertex of degree three in $\Theta$ lies in the same path, $R$ say.*

(b) *Among all theta-graphs satisfying the conditions of* (a), *choose $\Theta$ so that $R$ has greatest length. Denote the paths of $\Theta$ as follows (see Figure* 2.1);

$$P : xp_1p_2 \dots p_\alpha y, \;\; Q : xq_1q_2 \dots q_\beta y, \;\; R : xr_1r_2 \dots r_\gamma y.$$

*Then*

$$d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) \leqslant 2n + \epsilon_p + \epsilon_q, \tag{2.9}$$

*where*

$$\epsilon_p = \begin{cases} 1 & \text{if } n > 2k,\ v \in R,\ vp_1 \notin E(G) \text{ and } vp_\alpha \notin E(G), \\ 0 & \text{otherwise,} \end{cases}$$

*and $\epsilon_q$ is defined similarly, but with $p_1$ and $p_\alpha$ replaced by $q_1$ and $q_\beta$.*

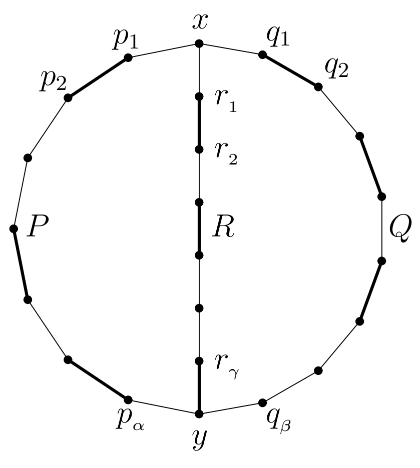



Figure 2.1

We now return to the proof of Theorem 2.5, assuming the truth of Lemma 2.13.

Applying Lemma 2.13(a) shows $G$ has a theta-graph containing $M$ and $v$, and if we choose $\Theta$ as in 2.13(b) then (2.9) holds. It is easy to see (and will be shown in the proof of Lemma 2.13(b)) that $p_1 q_\beta \notin E(G)$; similarly $p_\alpha q_1 \notin E(G)$. Since $\sigma_2(G) \geqslant n+1$ by assumption in Theorem 2.5, it follows that

$$d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) \geqslant 2n+2, \tag{2.10}$$

which contradicts (2.9) unless $\epsilon_p = \epsilon_q = 1$. In this case, since $\sigma_2(G) \geqslant n+1$ and $v$ is not adjacent to any of $p_1$, $p_\alpha$, $q_1$, and $q_\beta$, (2.8) implies

$$d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) > 4(\tfrac{1}{2}(n+1)) = 2n+2, \tag{2.11}$$

again contradicting (2.9). This proves Theorem 2.5. $\square$

**Proof of Lemma 2.13(a).** Let $\Pi : u_1 u_2 \ldots u_m$ be a longest path among those containing $M_v$. Then the vertices $u_1$ and $u_m$ are only adjacent to vertices in $\Pi$. Also $u_1 u_m \notin E(G)$, since there is no cycle containing $M_v$ by hypothesis. Now there exists $i$ $(3 \leqslant i \leqslant m-1)$ such that both $u_1 u_i$ and $u_{i-1} u_m$ are edges in $G$, since otherwise

$$\begin{aligned} d(u_1) + d(u_m) &= \sum_{j=2}^{m} [e(u_1 : u_j) + e(u_{j-1} : u_m)] \\ &\leqslant m-1 \leqslant n-1, \end{aligned}$$

contradicting the hypothesis that $\sigma_2(G) \geqslant n$. If $u_{i-1} u_i \notin M$ then

$$u_1 u_i u_{i+1} \ldots u_m u_{i-1} u_{i-2} \ldots u_1$$

is a cycle containing $M_v$, which is a contradiction. Therefore $u_{i-1} u_i \in M$, and the three paths

$$u_{i-1} u_{i-2} \ldots u_1 u_i, \quad u_{i-1} u_m u_{m-1} \ldots u_i \ \text{ and } \ u_{i-1} u_i$$

together form a theta-graph with the required properties. □

**Proof of Lemma 2.13(b).** Since there is no cycle containing $M_v$, paths $P$ and $Q$ must each contain either an edge of $M$ or the vertex $v$. Vertices $p_1$ and $q_\beta$ are not adjacent since otherwise

$$p_1 \dots p_\alpha y r_\gamma \dots r_1 x q_1 \dots q_\beta p_1 \tag{2.12}$$

is a cycle containing $M_v$, a contradiction. Let $Q_1, Q_2, \dots, Q_\xi$ be the sequence of paths in $Q - M$, with $x \in V(Q_1)$ and $y \in V(Q_\xi)$. Suppose that $1 < l < \xi$ and $V(Q_l) = \{q_s, q_{s+1}, \dots, q_{s+t}\}$. Then $p_1 q_i \notin E(G)$ for $i = s, s+1, \dots, s+t-1$, since otherwise the paths

$$P' : q_i p_1 p_2 \dots p_\alpha y, \quad Q' : q_i q_{i+1} \dots q_\beta y \;\text{ and }\; R' : q_i q_{i-1} \dots q_1 x r_1 \dots r_\gamma y$$

form a theta-graph satisfying the conditions of Lemma 2.13(a), contradicting the choice of $\Theta$ with $R$ longest. Now if $p_1 q_{s+t} \in E(G)$ then $q_\beta q_{s+t-1} \notin E(G)$ since otherwise

$$p_1 q_{s+t} \dots q_\beta q_{s+t-1} \dots q_1 x r_1 \dots r_\gamma y p_\alpha \dots p_1$$

is a cycle containing $M_v$, again a contradiction. It follows that

$$e(p_1, q_\beta : V(Q_l)) \leqslant |V(Q_l)| \text{ if } 1 < l < \xi. \tag{2.13}$$

If $\xi > 1$ then the same argument applies to the path $Q_1$ except that $p_1 x \in E(G)$, and to $Q_\xi - y$ except that $p_1 q_\beta \notin E(G)$ and $e(q_\beta : q_\beta) = 0$. Thus

$$e(p_1, q_\beta : V(Q_1)) \leqslant |V(Q_1)| + 1, \tag{2.14}$$

$$e(p_1, q_\beta : V(Q_\xi) - y) \leqslant |V(Q_\xi)| - 2. \tag{2.15}$$

If $\xi > 1$ then $V(Q)-y$ is the disjoint union of $V(Q_1),\dots,V(Q_{\xi-1})$ and $V(Q_\xi)-y$, and so inequalities (2.13)–(2.15) give

$$e(p_1, q_\beta : V(Q)-y) \leqslant |V(Q)|-1 = \beta+1. \tag{2.16}$$

If $\xi = 1$ then $v \in Q$ and $Q = Q_1 = Q_\xi$. Then the same reasoning implies that (2.16) holds. Similarly

$$e(q_1, p_\alpha : V(Q)-x) \leqslant \beta+1, \tag{2.17}$$

$$e(q_1, p_\alpha : V(P)-y) \leqslant \alpha+1, \tag{2.18}$$

$$e(p_1, q_\beta : V(P)-x) \leqslant \alpha+1. \tag{2.19}$$

Now define $R'' := V(R) - \{r_1, r_\gamma, x, y\}$. Consider the sequence of subpaths in $R-x-y-M$. If $n > 2k$ and $v = r_s \in R''$, then replace the subpath $r_i \dots r_s \dots r_j$ by the two paths $r_i r_{i+1} \dots r_s$ and $r_s r_{s+1} \dots r_j$. (Here $i < s < j$, since $v$ lies outside $V(M)$.) Label the resulting paths $R_1, R_2, \dots, R_\rho$ and let $\mathcal{R} = \{R_1, \dots, R_\rho\}$. Each path in $\mathcal{R}$ has length at least one except possibly $R_1$ or $R_\rho$, which consist of a single vertex if $r_1 r_2 \in M$ or $r_{\gamma-1} r_\gamma \in M$ respectively.

**Claim 2.14.** *Let $r_i$, $r_j$ be any two distinct vertices of a path $R_l \in \mathcal{R}$. Then at most one of $p_1 r_i$ and $p_\alpha r_j$, and at most one of $q_1 r_i$ and $q_\beta r_j$, is an edge of $G$.*

**Proof.** Since $p_1 x \notin M$ and $p_\alpha y \notin M$ by Lemma 2.13(a), the two paths $P-x-y$ and $(Q \cup R) - \{r_i \dots r_j\}$ together contain $M_v$. Hence if $p_1 r_i$ and $p_\alpha r_j \in E(G)$ then there is a cycle containing $M_v$, which is a contradiction. A similar argument holds if $q_1 r_i$ and $q_\beta r_j$ are both edges of $G$. $\square$

If path $R_1$ is the single vertex $r_1$ then $r_1 r_2 \in M$ and $x r_1 \notin M$. Then $p_\alpha r_1 \notin E(G)$ since otherwise

$$p_\alpha r_1 \dots r_\gamma y q_\beta \dots q_1 x p_1 \dots p_\alpha \tag{2.20}$$

is a cycle containing $M_v$, a contradiction. Similarly, if $R_\rho$ is a single vertex $r_\gamma$ then $p_1 r_\gamma \notin E(G)$. Suppose first that $\alpha \geqslant 2$. Then it follows from Claim 2.14 and the the above observations that

$$e(p_1, p_\alpha : V(R_l)) \leqslant |V(R_l)| \text{ for all } R_l \in \mathcal{R}. \tag{2.21}$$

If $v \in R''$, say $v = r_s = R_\tau \cap R_{\tau+1}$, then it follows from Claim 2.14 that the only way in which $e(p_1, p_\alpha : V(R_\tau \cup R_{\tau+1}))$ can exceed $|V(R_\tau \cup R_{\tau+1})|$ is if

$$V(R_\tau \cup R_{\tau+1}) = \{r_{s-1}, v, r_{s+1}\}$$

and

$$e(p_1, p_\alpha : r_{s-1}, v, r_{s+1}) = 4 = |\{r_{s-1}, v, r_{s+1}\}| + 1.$$

In this case neither $p_1$ nor $p_\alpha$ is adjacent to $v$, so that $\epsilon_p = 1$. Thus

$$e(p_1, p_\alpha : V(R_\tau \cup R_{\tau+1})) \leqslant |V(R_\tau \cup R_{\tau+1})| + \epsilon_p. \tag{2.22}$$

Since $V(R) - x - y = \bigcup_{i=1}^{\rho} V(R_i)$, (2.21) and (2.22) give

$$e(p_1, p_\alpha : V(R) - x - y) \leqslant \gamma + \epsilon_p \text{ if } \alpha \geqslant 2. \tag{2.23}$$

Similarly,

$$e(q_1, q_\beta : V(R) - x - y) \leqslant \gamma + \epsilon_q \text{ if } \beta \geqslant 2. \tag{2.24}$$

Now let $S$ be the set of vertices of $G$ not belonging to $\Theta$, and let $\delta = |S|$. Then

$$\alpha + \beta + \gamma + \delta + 2 = n. \tag{2.25}$$

If $w \in S$ then $p_1$ and $q_\beta$ cannot both be adjacent to $w$, since otherwise there is a cycle containing $M_v$ (obtained from the cycle in (2.12) by inserting the vertex $w$).

Therefore

$$e(p_1, q_\beta : S) \leqslant \delta. \tag{2.26}$$

Similarly,

$$e(q_1, p_\alpha : S) \leqslant \delta. \tag{2.27}$$

If $\alpha, \beta \geqslant 2$ then by (2.16)–(2.19) and (2.23)–(2.27) we obtain

$$\begin{aligned}
&d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) \\
= {}& e(p_1, q_\beta : V(Q) - y) + e(q_1, p_\alpha : V(Q) - x) + e(q_1, p_\alpha : V(P) - y) \\
&+ e(p_1, q_\beta : V(P) - x) + e(p_1, p_\alpha : V(R) - x - y) + e(q_1, q_\beta : V(R) - x - y) \\
&+ e(p_1, q_\beta : S) + e(q_1, p_\alpha : S) \\
\leqslant {}& (\beta + 1) + (\beta + 1) + (\alpha + 1) + (\alpha + 1) + (\gamma + \epsilon_p) + (\gamma + \epsilon_q) + \delta + \delta \\
\leqslant {}& 2(\alpha + \beta + \gamma + \delta + 2) + \epsilon_p + \epsilon_q = 2n + \epsilon_p + \epsilon_q.
\end{aligned} \tag{2.28}$$

Thus (2.9) holds when $\alpha, \beta \geqslant 2$. Now suppose that $n > 2k$ and without loss of generality that $\alpha = 1$ (i.e., $v = p_1 = p_\alpha$, so that $\epsilon_p = \epsilon_q = 0$. Then $v$ is adjacent to at most one vertex of each path $R_l \in \mathcal{R}$, else $v$ can be added to the cycle $Q \cup R$. For the same reason, if $R_1$ or $R_\rho$ is a single vertex then $v$ is not adjacent to this vertex. Hence

$$2e(v : V(R) - x - y) \leqslant \gamma. \tag{2.29}$$

Applying the above calculation (2.28) with $2e(v : V(R) - x - y)$ in place of $e(p_1, p_\alpha : V(R) - x - y)$, we see that (2.9) holds in this case. This completes the proof of Lemma 2.13(b). $\square$

We note for future reference that if there is strict inequality in any of (2.13)–(2.19), or in any of (2.21)–(2.24) when $\alpha, \beta \geqslant 2$, or in (2.29) when $\alpha = 1$, then there is strict inequality in (2.9) as well.

## 2.8 2-factor with at most two cycles - proof of Theorem 2.7

Suppose that some $k$-matching $M$ is not contained in any 2-factor consisting of at most two cycles, and define $G' := G + K_1$. Then

$$\sigma_2(G') = \sigma_2(G) + 2 \geqslant n + 2 = |V(G')| + 1,$$

and so $G'$ has a Hamilton cycle containing $M$ by Theorem 2.6. Hence $G$ has a Hamilton path containing $M$. By Lemma 2.13(a) it follows that there exists a theta-graph $\Theta$ spanning $G$ and containing $M$. Choose $\Theta$ so that the length of $R$ is longest, then (2.9) holds by Lemma 2.13(b). Adopt the same notation as in Lemma 2.13 and its proof (see Figure 2.1). In particular, if $n = 2k + 1$ let $v$ denote the vertex of $G$ which lies outside $V(M)$. Note that $v \notin \{r_1, r_\gamma\}$ by Lemma 2.13(a), since if $v = r_1$, say, then $x$ is not covered by $M \cup \{v\}$, and so $n > 2k + 1$, a contradiction. It is easy to see (as in the proof of Lemma 2.13(b)) that $p_1 q_\beta \notin E(G)$; similarly $p_\alpha q_1 \notin E(G)$. Since $\sigma_2(G) \geqslant n$ it follows that

$$d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) \geqslant 2n. \tag{2.30}$$

Hence equality holds in (2.9) when $\epsilon_p = \epsilon_q = 0$. To show that equality holds in general, we now prove three claims, using the assumption that $M$ is not contained in a 2-factor with one or two cycles. Define the set $\mathcal{R} = \{R_1, \ldots, R_\rho\}$ as in the proof of Lemma 2.13(b). Since $\Theta$ has order $n = 2k$ or $2k + 1$, each path in $\mathcal{R}$ has length exactly one, except possibly $R_1$ if $v = x$, or $R_\rho$ if $v = y$.

**Claim 2.15.** *Let $R_l \in \mathcal{R}$ be a nontrivial path with $V(R_l) = \{r_i, r_j\}$. Then at most one of $p_1 r_i$ and $q_\beta r_j$, and at most one of $p_\alpha r_i$ and $q_1 r_j$, is an edge of $G$.*

**Proof.** Suppose that $p_1 r_i$ and $q_\beta r_j$ are both edges of $G$. Note that the two paths

$$p_1 \dots p_\alpha y r_\gamma \dots r_{\max\{i,j\}} \text{ and } r_{\min\{i,j\}} \dots r_1 x q_1 \dots q_\beta$$

together contain $M_v$ and span $V(G)$. Hence if $i < j$ then there is a Hamilton cycle containing $M$, while if $i > j$ there are two cycles forming a 2-factor containing $M$; a contradiction in either case. A similar argument holds if $p_\alpha r_i$ and $q_1 r_j$ are both edges of $G$. $\square$

**Claim 2.16.** *$\epsilon_p = \epsilon_q =: \epsilon$, say.*

**Proof.** Suppose without loss of generality that $\epsilon_p = 1$ and $\epsilon_q = 0$, so that $p_1 v, p_\alpha v \notin E(G)$ and either $q_1 v \in E(G)$ or $q_\beta v \in E(G)$. Let $v = r_s = R_\tau \cap R_{\tau+1}$, and recall that $v \notin \{r_1, r_\gamma\}$. Applying Claim 2.15 to $R_\tau$ and $R_{\tau+1}$, it follows that $e(p_1, p_\alpha : r_{s-1}, v, r_{s+1}) \leqslant 2$. This is 2 less than the upper bound obtained in (2.22); hence (2.23) becomes

$$e(p_1, p_\alpha : V(R) - x - y) \leqslant \gamma - 1,$$

and (2.28) becomes

$$d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) \leqslant 2n - 1, \tag{2.31}$$

contradicting (2.30). $\square$

**Claim 2.17.** *If $\epsilon = 1$, or if $p_1 = p_\alpha = v$, then $d(v) \leqslant k$.*

**Proof.** First suppose that $\epsilon = 1$ and let $v = r_s \in R''$. (Recall that $v \notin \{r_1, r_\gamma\}$). If $e(p_1, q_1, p_\alpha, q_\beta : r_{s-1}, v, r_{s+1}) \leqslant 5$, which is 3 less than the total upper bound obtained from (2.22) and its analogue for $q_1, q_\beta$, then

$$e(p_1, q_1, p_\alpha, q_\beta : V(R) - x - y) \leqslant 2\gamma - 1.$$

Then (2.28) becomes (2.31), contradicting (2.30) as before. Thus we may assume without loss of generality that $e(p_1, p_\alpha : r_{s-1}, r_{s+1}) \geqslant 3$. Since the two paths $P - x - y$ and $Q \cup R - v$ together contain $M$, it follows that there is a cycle $C$ of length $2k$ containing $M$. The same holds if $p_1 = p_\alpha = v$, when $C := Q \cup R$ is a cycle of length $2k$ containing $M$. Now $v$ cannot be adjacent to both end vertices of any edge of $C - E(M)$, else $G$ has a Hamilton cycle containing $M$; hence $d(v) \leqslant k$. $\square$

If $\epsilon = 1$, then Claim 2.17 and the fact that $\sigma_2(G) \geqslant n = 2k+1$ imply that

$$d(p_1) + d(p_\alpha) + d(q_1) + d(q_\beta) \geqslant 4(k+1) = 2n+2. \tag{2.32}$$

Hence equality holds in (2.9) if $\epsilon = 1$. Recall from (2.30) that equality holds in (2.9) if $\epsilon = 0$. It follows (by the remark at the end of section 2.7) that equality must hold in (2.13)–(2.19), and in (2.21)–(2.24) if $\alpha, \beta \geqslant 2$, or in (2.29) when $\alpha = 1$. This allows us to make the following claim about the structure of $\Theta$.

**Claim 2.18.** *$\alpha \leqslant 2$ and $\beta \leqslant 2$.*

**Proof.** Since equality holds in (2.14), the argument preceding (2.14) implies $q_1 q_\beta \in E(G)$. If $\beta \geqslant 3$ then $(Q - x - y) \cup \{q_1 q_\beta\}$ and $P \cup R$ are two cycles forming a 2-factor containing $M$, a contradiction. Hence $\beta \leqslant 2$; similarly $\alpha \leqslant 2$. $\square$

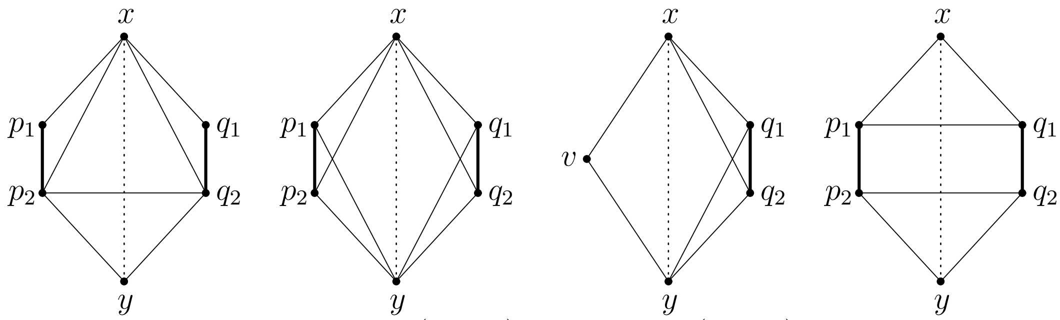
Assume without loss of generality that $\beta = 2$, and either $\alpha = 1$ (when $n = 2k+1$ and $v = p_1 = p_\alpha$) or $\alpha = 2$. We now examine the neighbourhoods in $G$ of the vertices $p_1, p_\alpha, q_1$ and $q_2$.

First consider $P \cup Q$. By equality in (2.16)–(2.19) we obtain

$$e(p_1, q_2 : V(Q) - y) = e(q_1, p_\alpha : V(Q) - x) = 3, \tag{2.33}$$

$$e(q_1, p_\alpha : V(P) - y) = e(p_1, q_2 : V(P) - x) = \alpha + 1. \tag{2.34}$$

Since $V(Q)-y=\{x,q_1,q_2\}$, $p_1x$ and $q_1q_2 \in E(G)$, and $p_1q_2 \notin E(G)$, (2.33) implies that exactly one of $p_1q_1$ and $q_2x$ is an edge of $G$. From this and similar observations it follows that there are essentially three possibilities for the neighbours of $p_1, p_\alpha, q_1$ and $q_2$ in $P \cup Q$ (see Figure 2.2). If $\alpha = 1$ (i.e., $v = p_1 = p_\alpha$) then we must be in Case 2, since if $vq_1$ or $vq_2 \in E(G)$ then $G$ has a Hamilton cycle containing $M_v$.



Case 1 Case 2 ($\alpha = 2$) Case 2 ($\alpha = 1$) Case 3

Figure 2.2

Now consider $V(R) - x - y$. Recall that $\mathcal{R} = \{R_1, \ldots, R_\rho\}$, as in the proof of Lemma 2.13(b), and let $n = 2k + t$ ($t \in \{0, 1\}$). Since there is no Hamilton cycle containing $M$, and $\alpha, \beta \leqslant 2$, there must be exactly one element of $M_v$ in the interior of each of $P$ and $Q$, and at least one element of $M_v$ in $R$. Hence there are $k+t-2$ elements of $M_v$ inside $R$, and $\rho = k+t-3 \geqslant 0$. Since $\rho$ is always odd by hypothesis, it follows that $\mathcal{R} \neq \emptyset$. By Claims 2.14 and 2.15 and equality in (2.21) when $\alpha \geqslant 2$, or equality in (2.29) when $\alpha = 1$, we obtain

$$N(p_i) \cap R = N(q_i) \cap R \quad (i = 1, 2) \quad \text{if } \alpha = 2, \tag{2.35}$$

$$N(p_1) \cap R = N(q_1) \cap R = N(q_2) \cap R \quad \text{if } \alpha = 1. \tag{2.36}$$

In particular, the paths $R_l \in \mathcal{R}$ of length one can be partitioned into four sets, $A$, $B$, $C$ and $D$, defined as follows. (See Figure 2.3 for the case where $\alpha = 2$ and

$v \notin \{r_{j-1}, r_j\}$.)

$$A := \{R_l : V(R_l) = \{r_{j-1}, r_j\} \text{ and } p_1, p_\alpha, q_1 \text{ and } q_2 \text{ are all adjacent to } r_{j-1}\},$$

$$B := \{R_l : V(R_l) = \{r_{j-1}, r_j\} \text{ and } p_1, p_\alpha, q_1 \text{ and } q_2 \text{ are all adjacent to } r_j\},$$

$$C := \{R_l : V(R_l) = \{r_{j-1}, r_j\} \text{ and } p_1 \text{ and } q_1 \text{ are both adjacent to } r_{j-1} \text{ and } r_j\},$$

$$D := \{R_l : V(R_l) = \{r_{j-1}, r_j\} \text{ and } p_\alpha \text{ and } q_2 \text{ are both adjacent to } r_{j-1} \text{ and } r_j\}.$$

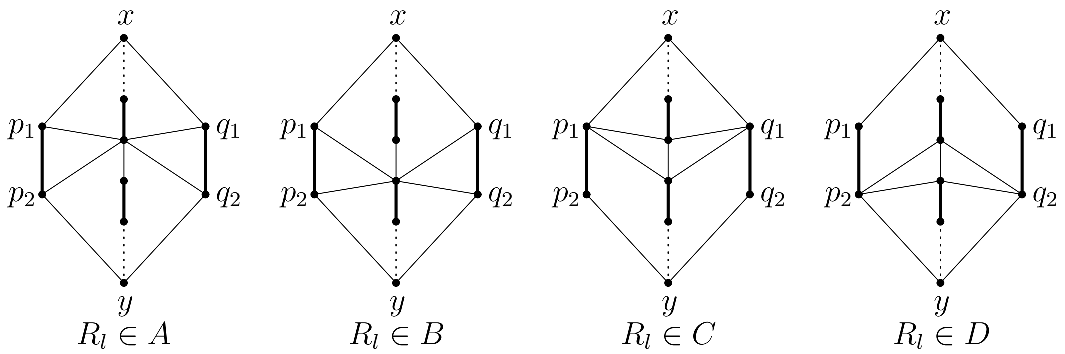



Figure 2.3

Suppose that $n = 2k + 1$. If $v = p_1 = p_\alpha$, then (2.36) implies that each path $R_l \in \mathcal{R}$ must either be in $A$ or $B$. If $v = x$ then $p_\alpha r_1 \notin E(G)$, else $G$ has a Hamilton cycle containing $M$; hence equality in (2.21) forces both $p_1 r_1$ and $q_1 r_1$ to be edges of $G$. If $v \in R''$, say $v = R_\tau \cap R_{\tau+1}$, then equality in (2.22) gives three possibilities for the neighbours of $p_i, q_i$ ($i \in \{1, 2\}$) in $R_\tau \cup R_{\tau+1}$. (See Figure 2.4.)

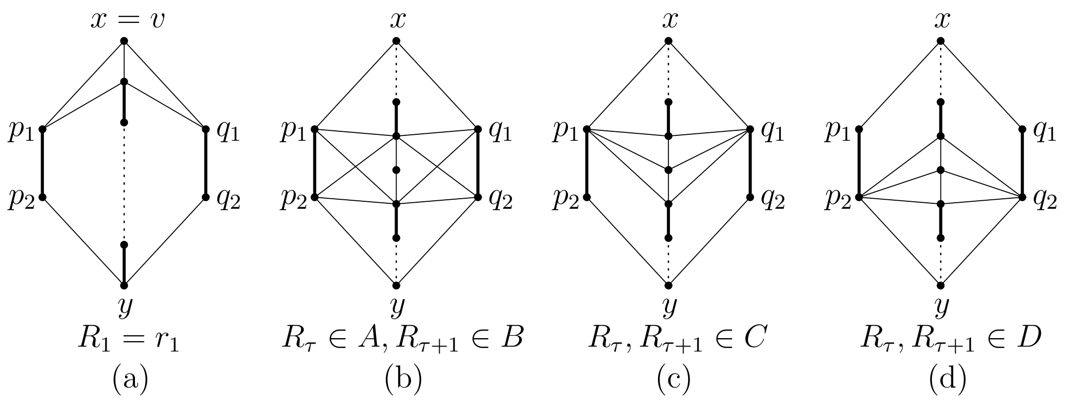



Figure 2.4

We now consider each of the cases shown in Figure 2.2.

**Case 1** : *$\alpha = 2$ and $p_2q_2$, $p_2x$ and $q_2x$ are all edges of $G$.* If $R_l \in C$ for some $l$ $(1 \leqslant l \leqslant \rho)$ and $V(R_l) = \{r_{j-1}, r_j\}$ then

$$xr_1 \dots r_{j-1}p_1p_2yr_\gamma \dots r_jq_1q_2x \tag{2.37}$$

is a Hamilton cycle containing $M$; this contradiction shows that $C = \emptyset$ in this case. Since $\rho = k + t - 3$, it follows that

$$\begin{aligned} d(p_1) + d(q_1) &= \sum_{l=1}^{\rho} e(p_1, q_1 : R_l) + e(p_1, q_1 : P \cup Q) \\ &\leqslant 2\rho + 4 = 2k + 2t - 2 < 2k + t = n, \end{aligned}$$

contradicting the fact that $p_1q_1 \notin E(G)$ and $\sigma_2(G) \geqslant n$. Thus Case 1 is impossible.

The next lemma, examining consecutive sets $R_s, R_{s+1} \in \mathcal{R}$, is a key step towards proving that Cases 2 and 3 of Figure 2.2 are impossible.

**Lemma 2.19.** *If $R_s \in A$ then $1 \leqslant s < \rho$ and $R_{s+1} \in B$. Moreover, if $A \neq \emptyset$ then $C = D = \emptyset$ and, if $n = 2k + 1$, $v \notin \{x, y\}$.*

**Proof.** Let $V(R_s) = \{r_{i-1}, r_i\}$. Since $R_s \in A$ (Figure 2.3), both $r_{i-1}q_1$ and $r_{i-1}q_2 \in E(G)$. Note that $e(r_i : V(R_s)) = 1$. Consider any nontrivial subpath $R_l \in \mathcal{R}$ $(l \neq s)$ and let $V(R_l) = \{r_{j-1}, r_j\}$. If $r_ir_j \in E(G)$ then there exist Hamilton paths

$$\begin{aligned} &r_{j-1} \dots r_ir_j \dots yp_2p_1x \dots r_{i-1}q_1q_2 \ \text{ if } \ i < j, \\ &r_{j-1} \dots xp_1p_2y \dots r_ir_j \dots r_{i-1}q_1q_2 \ \text{ if } \ i > j, \end{aligned}$$

and the same with $q_1$, $q_2$ interchanged; and all these Hamilton paths contain $M$. By assumption, $G$ has no Hamilton cycle containing $M$, and so $r_{j-1}q_1, r_{j-1}q_2 \notin E(G)$. Thus if $r_ir_j \in E(G)$ then $R_l \in B$. Now suppose that $l \neq s + 1$ and $r_ir_{j-1} \in E(G)$.

If $r_j q_1 \in E(G)$ then there are two cycles,

$$r_i r_{i+1} \dots r_{j-1} r_i \text{ and } x \dots r_{i-1} q_2 q_1 r_j \dots y p_2 p_1 x \text{ if } i < j,$$
$$r_i \dots y p_2 p_1 x \dots r_{j-1} r_i \text{ and } r_j \dots r_{i-1} q_2 q_1 r_j \text{ if } i > j,$$

which form a 2-factor containing $M$. Hence $r_j q_1 \notin E(G)$. Similarly $r_j q_2 \notin E(G)$; thus if $l \neq s+1$ and $r_i r_{j-1} \in E(G)$ then $R_l \in A$. It follows that

$$e(r_i : R_l) \leqslant 1 \text{ if } l \neq s+1 \text{ and } R_l \in A \cup B, \tag{2.38}$$
$$e(r_i : R_l) = 0 \text{ if } l \neq s+1 \text{ and } R_l \in C \cup D, \tag{2.39}$$
$$e(r_i : R_{s+1}) = 1 \text{ if } s < \rho \text{ and } R_{s+1} \notin B. \tag{2.40}$$

By (2.38)–(2.40), and since $R_s \in A$ and $\rho = k + t - 3$, we obtain

$$\begin{aligned} d(r_i) &= \sum_{l=1}^{\rho} e(r_i : R_l) + e(r_i : x, y) + e(r_i : p_1, p_2, q_1, q_2) \\ &\leqslant k - 4 + t + e(r_i : R_{s+1}) + 2 + 0 \\ &\leqslant k + t, \end{aligned} \tag{2.41}$$

with strict inequality if $s = \rho$, since then $e(r_i : R_l) \leqslant 1$ for all $l$. There are now six cases to consider in the proof of Lemma 2.19. First we prove a simple claim.

**Claim 2.20.** *If $r \in R$ and $e(r : p_1, p_2, q_1, q_2) = 0$ then $d(r) \geqslant k$.*

**Proof.** Suppose $d(r) \leqslant k - 1$. Then, since $\sigma_2(G) \geqslant n = 2k + t$,

$$d(p_1) + d(p_2) + d(q_1) + d(q_2) \geqslant 4(k + 1 + t) = 2n + 4 + 2t,$$

which contradicts (2.9). $\square$

*Case* 2.19(a) : $n = 2k$ (i.e., $t = 0$). By Claim 2.20, $d(r_i) \geqslant k$, hence equality must hold in (2.41). By the calculation that gave rise to (2.41) it follows that $s < \rho$,

$e(r_i : R_{s+1}) = 2$, and $e(r_i : R_l) = 1$ for all $l \neq s+1$. Hence (2.40) implies $R_{s+1} \in B$ and (2.39) implies $C \cup D = \emptyset$, as required. So we may suppose that $n = 2k+1$ (i.e., $t = 1$).

*Case* 2.19(b) : $v = r_i$. Then $R_{s+1} \in B$ since $R_s \in A$ (Figure 2.4(b)). Also $\epsilon = 1$, and so Claims 2.17 and 2.20 together imply $d(v) = k$. Since $e(r_i : R_{s+1}) = 1$, the calculation that gave rise to (2.41) implies $e(v : R_l) = 1$ for all $l$ ($1 \leqslant l \leqslant \rho$). Then (2.39) implies $C = D = \emptyset$ as required. So we may suppose that $v \neq r_i$.

*Case* 2.19(c) : $v = R_\tau \cap R_{\tau+1} \in R''$ *and* $\epsilon = 0$. Then $R_\tau, R_{\tau+1} \in C \cup D$ (Figure 2.4(c), (d)). Then $e(r_i : R_\tau) = e(r_i : R_{\tau+1}) = 0$ by (2.39), and so (2.41) becomes $d(r_i) \leqslant k-1$, contradicting Claim 2.20.

*Case* 2.19(d) : *Either* $\epsilon = 1$ *or* $\alpha = 1$. If $\epsilon = 1$, let $v = R_\tau \cap R_{\tau+1} \in R''$ (Figure 2.4(b)). Then $R_\tau \notin B$, and the argument at the start of the proof of Lemma 2.19 (with $r_j = v$) shows that $r_i v \notin E(G)$. The same holds if $\alpha = 1$ (i.e., $v = p_1 = p_\alpha$) since $R_s \in A$. In both cases, Claim 2.17 implies $d(v) \leqslant k$. Since $\sigma_2(G) \geqslant n = 2k+1$ and $r_i v \notin E(G)$, we obtain $d(r_i) \geqslant k+1$; hence equality must hold in (2.41). As in Case 2.19(a), it follows that $s < \rho$, $R_{s+1} \in B$ and $C = D = \emptyset$ as required.

*Case* 2.19(e) : $v = x$. Then $p_1 r_1 \in E(G)$ (Figure 2.4(a)) and there exist Hamilton paths

$$r_i \ldots r_\gamma y p_2 p_1 r_1 \ldots r_{i-1} q_2 q_1 x \quad \text{and} \quad r_1 \ldots r_{i-1} q_2 q_1 x p_1 p_2 y r_\gamma \ldots r_i,$$

each containing $M$. Hence $r_i x, r_1 r_i \notin E(G)$, else $G$ has a Hamilton cycle containing $M$. Thus $e(r_i : x, y) \leqslant 1$ and $e(r_i : R_1) = 0$, and it follows that (2.41) becomes $d(r_i) \leqslant k-1$, contradicting Claim 2.20. Hence $v \neq x$.

*Case* 2.19(f) : $v = y$. Then $R_\rho = r_\gamma$ (so $s < \rho$) and $p_2 r_\gamma \in E(G)$. Hence

$$r_i r_{i+1} \ldots r_\gamma p_2 p_1 x r_1 \ldots r_{i-1} q_1 q_2 y$$

is a Hamilton path containing $M$. Also, if $s < \rho - 1$ and $r_i r_\gamma \in E(G)$ then two cycles

$$r_i r_{i+1} \dots r_\gamma r_i \text{ and } x p_1 p_2 y q_2 q_1 r_{i-1} \dots r_1 x$$

form a 2-factor containing $M$. Hence $r_i y, r_i r_\gamma \notin E(G)$. Thus $e(r_i : x, y) \leqslant 1$, and either $e(r_i : R_\rho) = 0$, or $s = \rho - 1$ and $e(r_i : R_{s+1}) = 1$. As before it follows that (2.41) becomes $d(r_i) \leqslant k - 1$, contradicting Claim 2.20. Hence $v \neq y$.

Thus if $A \neq \emptyset$ then $v \notin \{x, y\}$. This completes the proof of Lemma 2.19. □

By symmetry we obtain:

**Lemma 2.21.** *If $R_s \in B$ then $1 < s \leqslant \rho$ and $R_{s-1} \in A$. Moreover, if $B \neq \emptyset$ then $C = D = \emptyset$ and if $n = 2k + 1$ then $v \notin \{x, y\}$.* □

The following Corollary will enable us to rule out Case 2 of Figure 2.2. Recall that $\mathcal{R} \neq \emptyset$, by the remarks preceding (2.35).

**Corollary 2.22.** *$A \cup B = \emptyset$; hence $\epsilon = 0$.*

**Proof.** Suppose $A \cup B \neq \emptyset$. Then Lemmas 2.19 and 2.21 imply that $C = D = \emptyset$, and if $n = 2k + 1$ then $v \notin \{x, y\}$; thus all paths $R_l \in \mathcal{R}$ have length one. By Lemma 2.21, $R_1 \notin B$; hence $R_1 \in A$. Now Lemma 2.19 implies $R_2 \in B$, and then Lemma 2.21 gives $R_3 \in A$ (since if $R_3 \in B$ then $R_2 \in A$.) Continuing in this way, since $\rho = k + t - 3$ is odd by hypothesis, we obtain $R_\rho \in A$, which contradicts Lemma 2.19. □

**Case 2** : *$\alpha \in \{1, 2\}$ and $p_1 y$, $q_1 y$, $p_\alpha x$ and $q_2 x$ are all edges of $G$.* If $v = x$ then $p_1 r_1 \in E(G)$ (Figure 2.4(a)) and the two cycles $x q_1 q_2 x$ and $y p_2 p_1 r_1 \dots r_\gamma y$ form a 2-factor containing $M$, which is a contradiction; thus $v \neq x$. Similarly, $v \neq y$. It follows that all paths in $\mathcal{R}$ have length one; let $R_l \in \mathcal{R}$. Since $A \cup B = \emptyset$ by Corollary 2.22, $R_l \in C \cup D$. As noted after Figure 2.3, this is impossible if $\alpha = 1$; so suppose $\alpha = 2$. If $R_l \in C$ and $V(R_l) = \{r_{j-1}, r_j\}$ then, since $p_2 x, q_2 x \in E(G)$ (as in Case 1), the Hamilton cycle in (2.37) contains $M$, which is a contradiction.

By symmetry, since $p_1y, q_1y \in E(G)$, a similar contradiction occurs if $R_l \in D$. Thus Case 2 is impossible.

**Case 3**: $\alpha = 2$ *and* $p_1q_1$ *and* $p_2q_2$ *are both edges of* $G$. In this case we may assume that $xy \notin E(G)$, since otherwise the cycles $p_1p_2q_2q_1p_1$ and $xr_1 \dots r_\gamma yx$ form a 2-factor containing $M$. Hence

$$d(x) + d(y) \geqslant \sigma_2(G) \geqslant n = 2k + t. \tag{2.42}$$

Consider any nontrivial path $R_l \in \mathcal{R}$, and let $V(R_l) = \{r_{j-1}, r_j\}$. Note that $R_l \in C \cup D$ by Corollary 2.22. If $xr_j$ and $p_2r_{j-1}$ are both edges of $G$ then there is a Hamilton cycle

$$xr_j \dots r_\gamma yq_2q_1p_1p_2r_{j-1} \dots r_1x$$

containing $M$. If $l > 1$ and $xr_{j-1}, p_2r_j \in E(G)$ then the two cycles

$$xr_1 \dots r_{j-1}x \text{ and } yq_2q_1p_1p_2r_j \dots r_\gamma y$$

form a 2-factor containing $M$. It follows that

$$e(x, p_2 : R_l) \leqslant 2 \text{ if } 1 < l \leqslant \rho, \tag{2.43}$$

$$e(x, p_2 : R_1) \leqslant 3. \tag{2.44}$$

Similarly,

$$e(y, p_1 : R_l) \leqslant 2 \text{ if } 1 \leqslant l < \rho, \tag{2.45}$$

$$e(y, p_1 : R_\rho) \leqslant 3. \tag{2.46}$$

Since $e(p_1, p_2 : R_l) = 2$ for all subpaths $R_l \in C \cup D$ (Figure 2.3), it follows from

(2.43)–(2.46) that

$$e(x, y : R_l) \leqslant 2 \ \text{ if } 1 < l < \rho, \tag{2.47}$$

$$e(x, y : R_l) \leqslant 3 \ \text{ if } l = 1 \text{ or } \rho. \tag{2.48}$$

By (2.47)–(2.48) and since $\rho = k + t - 3$,

$$\begin{aligned} d(x) + d(y) &= \sum_{l=1}^{\rho} e(x, y : R_l) + e(x, y : p_1, p_2, q_1, q_2) \\ &\leqslant 2(k + t - 3) + 2 + 4 = 2k + 2t. \end{aligned} \tag{2.49}$$

There are now three subcases to consider.

**Case 3(a)** : $n = 2k$ (i.e., $t = 0$). By (2.42), equality holds in (2.49). Hence equality also holds in (2.43)–(2.48). Since $A \cup B = \emptyset$, it follows by equality in (2.44) and (2.46) respectively that

$$R_1 \in D \ \text{ and } \ R_\rho \in C. \tag{2.50}$$

**Claim 2.23.** *If $R_s \in D$ then $1 \leqslant s < \rho$ and $R_{s+1} \in C$.*

**Proof.** Let $V(R_s) = \{r_{i-1}, r_i\}$. Since $R_s \in D$, $e(p_1 : R_s) = 0$ (Figure 2.3). Thus equality in (2.45) implies $yr_{i-1} \in E(G)$, and the paths

$$P' : xp_1p_2r_i, \ \ Q' : xq_1q_2r_i \ \text{ and } \ R' : xr_{i-1}yr_\gamma \dots r_i$$

form a theta-graph satisfying the conditions of Lemma 2.13(a), and $R'$ has the same length as $R$. Since the path $R_{s+1}$ is in the same position in $R'$ as that of $R_\rho$ in $R$, it follows from (2.50) that $R_{s+1} \in C$. $\square$

By symmetry we also obtain:

**Claim 2.24.** *If $R_{s+1} \in C$ then $1 < s \leqslant \rho$ and $R_s \in D$.* $\square$

Claims 2.23 and 2.24 imply that, moving from $x$ to $y$ along $R$, the sets $R_l$ must alternate between $C$ and $D$. Since $\rho$ is odd, this contradicts (2.50), and so Case 3(a) is impossible.

So we may suppose that $n = 2k+1$ (i.e., $t = 1$). Since $\alpha = 2$ by assumption in Case 3, $v \in R$; recall that $v \notin \{r_1, r_\gamma\}$.

**Case 3(b)**: $v \in \{x, y\}$. If $v = x$ then $R_1$ is the single vertex $r_1$. Then $r_1 y \notin E(G)$, else the two cycles $xp_1p_2q_2q_1x$ and $r_1 \ldots r_\gamma y r_1$ form a 2-factor containing $M$. Hence $e(x, y : R_1) = 1$, which is 2 less than the bound given in (2.48). It follows by the calculation that gave rise to (2.49) that $d(x) + d(y) \leqslant 2k$, contradicting (2.42); thus $v \neq x$. Similarly $v \neq y$; thus Case 3(b) is impossible.

**Case 3(c)** : $v \in R''$. Let $v = r_s = R_\tau \cap R_{\tau+1}$. Note that $\rho \geqslant 3$, since $\rho$ is odd by hypothesis, and $\rho \geqslant 2$ here. Also, since $\epsilon = 0$ by Corollary 2.22, either $vp_1$ or $vp_2$ is an edge of $G$. Suppose without loss of generality that $vp_1 \in E(G)$ (i.e., $R_\tau, R_{\tau+1} \in C$; Figure 2.4(c)). Then $e(p_1 : R_\tau \cup R_{\tau+1}) = 3$, and so (2.45) and (2.46) imply that $e(y : R_\tau \cup R_{\tau+1}) = 0$, unless $\tau = \rho - 1$, in which case $e(y : R_\tau \cup R_{\tau+1}) = 1$. (In particular, $vy \notin E(G)$.) Hence

$$e(x, y : R_\tau \cup R_{\tau+1}) \leqslant 3 \ \text{ if } 1 \leqslant \tau < \rho - 1, \tag{2.51}$$

$$e(x, y : R_\tau \cup R_{\tau+1}) \leqslant 4 \ \text{ if } \tau = \rho - 1. \tag{2.52}$$

Since these upper bounds are 1 less than the totals for $R_\tau, R_{\tau+1}$ given by (2.47) and (2.48), it follows that (2.49) becomes $d(x) + d(y) \leqslant 2k + 1$. Then equality holds in (2.42) and, by the calculation that gave rise to (2.49), equality also holds in (2.43)–(2.48).

Since equality holds in (2.43) and (2.44), $xr_{s+1} \in E(G)$, and the paths

$$P' : vp_1p_2y, \;\; Q' : vq_1q_2y \;\text{ and }\; R' : vr_{s-1} \ldots r_1 x r_{s+1} \ldots r_\gamma y.$$

form a theta-graph, $\Theta'$ say, which satisfies the conditions of Lemma 2.13(a). Since $R'$ has the same length as $R$, the existence of $\Theta'$ leads to a contradiction as in Case 3(b). Thus Case 3(c) is impossible.

Hence all of the cases in Figure 2.2 are impossible. This completes the proof of Theorem 2.7. $\square$

# Chapter 3

# Extension of matchings to larger matchings

## 3.1 Introduction

A *k-matching* is a set of $k$ pairwise nonadjacent edges, and a *perfect matching* is the edge set of a 1-factor. A matching $L$ *extends* to a matching $M$ if $M$ contains $L$. In this chapter we give conditions which suffice to ensure that an $l$-matching extends to a $k$-matching in a graph of order $n$ $(0 \leqslant l < k \leqslant \frac{1}{2}n)$.

Say that a graph $G$ is *suitable for a* 1*-factor* if every component of $G$ is even. Say that $G$ is *very suitable for a* 1*-factor* if $G$ is suitable for a 1-factor and $G - v$ has exactly one odd component, for each vertex $v \in V(G)$. (Note that it makes no difference in this context if we replace 'exactly one odd component' by 'at most two odd components', since if $G$ is suitable for a 1-factor then every component of $G$ is even and so the number of odd components of $G - v$ is odd.) Clearly if $G$ has a perfect matching containing and $l$-matching $L$, then $G - V(L)$ is very suitable for a 1-factor, and if $G$ has a $\frac{1}{2}(n-1)$-matching containing $L$, then $G - V(L)$ has exactly one odd component.

Recall that $\mathcal{I}(G)$ denotes the family of nonempty independent subsets of $V(G)$, and if $X \subseteq V(G)$ then the *independence number* of $X$, denoted by $\alpha(X)$, is the number of vertices in a largest independent set contained in $X$. We can now state the main result of this chapter; we will prove it in Section 3.2, and give examples to demonstrate that the bounds are sharp in Section 3.3.

**Theorem 3.1.** *Suppose that $0 \leqslant l < k \leqslant \frac{1}{2}n$, and let $L$ be an $l$-matching in a graph $G$ of order $n$. Then each of the following conditions suffices to ensure that $L$ can be extended to a $k$-matching in $G$.*

(a) $\sigma_2(G) \geqslant 2(k+l)-1$.

(b) $l \geqslant 3k-n-2$ *and either*

(i)

$$|N(X)| \geqslant \tfrac{1}{2}(|X| + 3(k+l) - n - 2) \tag{3.1}$$

*whenever $X \in \mathcal{I}(G)$ and $n - 3k + l + 3 \leqslant |X| \leqslant n - k - l + 1$, or*

(ii)

$$|N(X)| \geqslant 2|X| + 3(k+l) - 2n - 2 \tag{3.2}$$

*whenever $X \subseteq V(G)$, $\alpha(X) \geqslant 2(n-k-l+1) - |X|$ and*

$$n - k - l + 1 \leqslant |X| \leqslant n - 2l.$$

(c) *$G - V(L)$ is very suitable for a 1-factor if $k = \frac{1}{2}n$, $G - V(L)$ has exactly one odd component if $k = \frac{1}{2}(n-1)$, $\delta(G) \geqslant \frac{1}{2}(3(k+l) - n) + 1$ and either*

(i) (3.1) *holds whenever $X \in \mathcal{I}(G)$ and $5 \leqslant |X| \leqslant n - k - l + 1$, or*

(ii) (3.2) *holds whenever $X \subseteq V(G)$, $\alpha(X) \geqslant 2(n-k-l+1) - |X|$ and*

$$n - k - l + 1 \leqslant |X| \leqslant \tfrac{3}{2}(n-k-l) - 1.$$

The case $k = \frac{1}{2}n$ of Theorem 3.1 yields sufficient conditions for a 1-factor containing a given $l$-matching; see Theorem 5.1 in Section 5.1. In particular, Theorem 5.1(a) improves slightly upon Plummer's result [35] that if $\delta(G) \geqslant \frac{1}{2}n + l$ then a graph $G$ of even order $n$ is $l$-extendable.

Theorem 3.1(c) contains Robertshaw and Woodall's result on 1-factors (stated as Theorem 1.2 in Section 1.2) since the degree and neighbourhood conditions agree when $l = 0$ and $k = \frac{1}{2}n$, and any sesquiconnected graph of even order is clearly suitable for a 1-factor. (Recall that a graph $G$ is *sesquiconnected* if it is connected and, for each vertex $v$ in $G$, $G - \{v\}$ has at most two components.)

Restating Theorem 3.1 with $l = 0$ yields the following defect form of Theorem 1.2.

**Theorem 3.2.** *Let $G$ of even order $n$. If $0 < k \leqslant \frac{1}{2}n$ then each of the following conditions suffices to ensure that $G$ has a $k$-matching.*

(a) $\sigma_2(G) \geqslant 2k - 1$.

(b) $k \leqslant \frac{1}{3}(n + 2)$ *and*

$$|N(X)| \geqslant \tfrac{1}{4}(2|X| + n + 6l - 4) \quad \text{whenever } X \in \mathcal{I}(G). \tag{3.3}$$

(c) *$G$ is very suitable for a 1-factor if $k = \frac{1}{2}n$, $G$ has exactly one odd component if $k = \frac{1}{2}(n-1)$, $\delta(G) \geqslant \frac{1}{2}(3k - n + 2)$ and (3.3) holds.*

In section 3.4 we establish further sufficient conditions for matching extension, generalising Theorem 2.2 of [40].

## 3.2 Extending matchings - proof of Theorem 3.1

Our proof of (a) is straightforward, exploring the consequences of choosing a longest path in $G - V(L)$. Next we prove (b) and (c) together, using Berge's defect form of Tutte's criterion (stated as Theorem 1.1 in Section 1.2); this proof generalises Robertshaw and Woodall's proof of Theorem 1.2. We also give an alternative proof of Theorem 3.1(c), showing that this result is essentially a corollary of Theorem 1.2.

Let $H := G - V(L)$, and for convenience write $k' := k - l$ and $h := |V(H)| = n - 2l$. Note for future reference that

$$n - 2k = h - 2k', \tag{3.4}$$

$$n - k - l = h - k', \tag{3.5}$$

$$\text{and } n - 3k + l = h - 3k'. \tag{3.6}$$

In each case (a), (b) and (c) we will prove that $H$ contains a $k'$-matching.

**Proof of Theorem 3.1(a).** First note that

$$\sigma_2(H) \geqslant \sigma_2(G) - 4l \geqslant 2(k + l) - 1 - 4l = 2k' - 1. \tag{3.7}$$

If $k' = 1$ then (3.7) clearly forces $H$ to contain an edge; so suppose $k' \geqslant 2$, and $h \geqslant 2k' \geqslant 4$. Let $P : u_1u_2 \ldots u_m$ be a longest path in $H$. Note that $m \geqslant 3$, since otherwise there must be two nonadjacent vertices in $H$, whose degree sum is at most 2, contradicting (3.7). If the length of $P$ is at least $2k' - 1$ then $P$ contains a $k'$-matching; so we may suppose that $P$ has length at most $2k' - 2$, i.e., $m \leqslant 2k' - 1$. We now show that there is a cycle $C$ with $V(C) = V(P)$. This is clear if $u_1u_m \in E(G)$, so suppose that $u_1u_m \notin E(H)$. Then (3.7) gives

$$d(u_1) + d(u_m) = \sum_{i=2}^{m} [e(u_1 : u_i) + e(u_{i-1} : u_m)] \geqslant 2k' - 1. \tag{3.8}$$

Since $m \leqslant 2k' - 1$, there exists $i$ $(3 \leqslant m - 1)$ such that $u_1u_i$ and $u_{i-1}u_m$ are both edges in $H$. Then

$$C : u_1 u_i u_{i+1} \dots u_m u_{i-1} \dots u_1$$

is a cycle with $V(C) = V(P)$. Since $P$ is maximal, the vertices of $P$ form a component, $C_1$ say, of $H$. Since $h \geqslant 2k' > m$, it follows that $H$ is disconnected. Consider any other component $C_2$ of $H$, and let $c_i := |V(C_i)|$ $(i = 1, 2)$. If $u \in C_1$ and $v \in C_2$ then $d(u) + d(v) \geqslant 2k' - 1$ by (3.7); hence

$$c_1 + c_2 \geqslant 2k' + 1. \tag{3.9}$$

Also, since $d(u) \leqslant m-1 \leqslant 2k'-2$, it follows that $d(v) \geqslant 1$ for all $v \in V(C_2)$. Thus either $C_2 = K_2$ or applying the above argument to $C_2$ shows that it must have a spanning cycle. It follows that $C_1 \cup C_2$ contains a matching with $\lfloor \frac{c_1}{2} \rfloor + \lfloor \frac{c_2}{2} \rfloor$ edges. If $c_1$ and $c_2$ are both odd then (3.9) implies $c_1 + c_2 \geqslant 2k' + 2$, and so $\lfloor \frac{c_1}{2} \rfloor + \lfloor \frac{c_2}{2} \rfloor = \frac{1}{2}(c_1 + c_2) - 1 \geqslant k'$. Otherwise, $\lfloor \frac{c_1}{2} \rfloor + \lfloor \frac{c_2}{2} \rfloor \geqslant \frac{1}{2}(c_1 + c_2 - 1) \geqslant k'$, again by (3.9). This completes the proof of Theorem 3.1(a). $\square$

**Proof of Theorem 3.1(b) and (c).** Suppose that $H$ does not contain a $k'$-matching. Then by the defect form of Tutte's theorem (stated as Theorem 1.1 in Section 1.2), $H$ contains a set $S$ of vertices such that $\frac{1}{2}(h - o(S) + |S|) < k'$, where $o(S)$ denotes the number of odd components of $H - S$. Since $h - o(S) + |S|$ is even, it follows that

$$o(S) \geqslant |S| + h - 2k' + 2. \tag{3.10}$$

Choose $|S| + h - 2k' + 2$ of the odd components of $H - S$, let $x$ of the chosen components consist of a single vertex, and let $X_1$ be the set of these vertices. Each odd component of $H - S$ which is not in $X_1$ has at least three vertices; hence

$$h \geqslant |S| + x + 3(|S| + h - 2k' + 2 - x),$$

which implies

$$|S| \leqslant \tfrac{1}{2}(3k' - h + x - 3). \tag{3.11}$$

Since $|S| \geqslant 0$, (3.11) and (3.6) imply

$$x \geqslant h - 3k' + 3 = n - 3k + l + 3. \tag{3.12}$$

**Claim 3.3.** $X_1$ *is nonempty* (*i.e.*, $x > 0$); *hence* $X_1 \in \mathcal{I}(G)$.

**Proof.** If (b) holds then $l \geqslant 3k - n - 2$, and so (3.12) implies $x > 0$ as required. So suppose (c) holds. Then

$$\delta(H) \geqslant \delta(G) - 2l \geqslant \tfrac{1}{2}(3(k+l) - n) + 1 - 2l = \tfrac{1}{2}(3k' - h + 2), \tag{3.13}$$

by (3.6). It follows that each of the chosen odd components must have at least $\frac{1}{2}(3k' - h + 2) + 1 - |S|$ vertices, and so

$$h \geqslant |S| + (|S| + h - 2k' + 2)(\tfrac{1}{2}(3k' - h + 4) - |S|).$$

Hence

$$\begin{aligned}(|S| - (2k' - h + 2))(|S| - \tfrac{1}{2}(3k' - h + 4)) &\geqslant |S| - h - 4(|S| - \tfrac{1}{2}(3k' - h + 4)) \\ &= 6k' - 3h - 3|S| + 8,\end{aligned}$$

so that

$$\begin{aligned}(|S| - (2k' - h + 2))(|S| - \tfrac{1}{2}(3k' - h - 2)) &\geqslant 6k' - 3h - 3|S| + 8 \\ &\quad + 3(|S| - (2k' - h + 2)) = 2. \end{aligned} \tag{3.14}$$

Note that (3.14) is impossible if

$$2k' - h + 2 \leqslant |S| \leqslant \tfrac{1}{2}(3k' - h - 2).$$

To show that $X_1$ is nonempty in (c) it remains to show that $|S| \geqslant 2k' - h + 2$, since then (3.14) implies $|S| > \frac{1}{2}(3k' - h - 2)$, and then (3.11) implies $x > 0$ as required.

First suppose $k = \frac{1}{2}n$, so that $k' = \frac{1}{2}h$ by (3.4), and (3.10) gives $o(S) \geqslant |S| + 2$. Then since $H$ is very suitable for a 1-factor by hypothesis in (c), it follows that $|S| \geqslant 2 = 2k' - h + 2$. If $k = \frac{1}{2}(n-1)$ then $k' = \frac{1}{2}(h-1)$ by (3.4), and (3.10) gives $o(S) \geqslant |S| + 3$. Then $S$ must be nonempty, since $H$ has exactly one odd component by hypothesis in (c); thus $|S| \geqslant 1 = 2k' - h + 2$. Finally, if $k \leqslant \frac{1}{2}(n-2)$ then $k' \leqslant \frac{1}{2}(h-2)$ by (3.4), and so $|S| \geqslant 0 \geqslant 2k' - h + 2$. Thus in all cases $|S| \geqslant 2k' - h + 2$; this completes the proof of Claim 3.3. $\square$

Now $h \geqslant |S| + o(S) \geqslant |S| + (|S| + h - 2k' + 2)$ by (3.10), and so

$$|S| \leqslant k' - 1. \tag{3.15}$$

Since $x \leqslant |S| + h - 2k' + 2$ (the number of odd components chosen), (3.15) and (3.5) imply

$$x = |X_1| \leqslant h - k' + 1 = n - k - l + 1. \tag{3.16}$$

If (c) holds then by (3.13), since $X_1$ is nonempty, we obtain

$$|S| \geqslant \delta(H) \geqslant \tfrac{1}{2}(3k' - h) + 1. \tag{3.17}$$

Together with (3.11), this gives

$$x = |X_1| \geqslant 2|S| + h - 3k' + 3 \geqslant 5. \tag{3.18}$$

Hence, by (3.12) and (3.16) if (b)(i) holds, or by (3.18) and (3.16) if (c)(i) holds, we can apply (3.1) to $X_1 \in \mathcal{I}(G)$; thus

$$\begin{aligned}|S| \geqslant |N_H(X_1)| \geqslant |N_G(X_1)| - 2l \geqslant \tfrac{1}{2}(|X_1| + 3(k+l) - n - 2) - 2l \\ = \tfrac{1}{2}(x + 3k' - h - 2)\end{aligned}$$

by (3.6), which contradicts (3.11).

If (b)(ii) or (c)(ii) holds, define $X_2 := V(H) \setminus S$. Then by (3.5) and (3.15), and since $X_2 \subseteq H$, we obtain

$$n - k - l + 1 = h - k' + 1 \leqslant h - |S| = |X_2| \leqslant h = n - 2l.$$

Also, since taking one vertex from each odd componnent of $H - S$ gives an independent set, (3.10) and (3.5) imply

$$\begin{aligned}\alpha(X_2) \geqslant |S| + h - 2k' + 2 &= 2(h - k') - |X_2| + 2 \\ &= 2(n - k - l) - |X_2| + 2. \end{aligned} \tag{3.19}$$

If (c)(ii) holds then (3.17) and (3.5) give

$$|X_2| \leqslant \tfrac{3}{2}(h - k') - 1 = \tfrac{3}{2}(n - k - l) - 1.$$

Hence in both (b)(ii) and c(ii) we can apply (3.2) to $X_2$; thus

$$\begin{aligned}h - x \geqslant |N_H(X_2)| \geqslant |N_G(X_2)| - 2l \geqslant 2|X_2| + 3(k+l) - 2n - 2 - 2l \\ = 2|X_2| + 3k' - 2h - 2 \\ = 3k' - 2|S| - 2,\end{aligned}$$

which rearranges to $|S| \geqslant \frac{1}{2}(x + 3k' - h - 2)$, contradicting (3.11) as before. This completes the proof of Theorem 3.1(b) and (c). $\square$

**Alternative proof of Theorem 3.1(c).** Recall that the *join* $G_1 + G_2$ of two graphs $G_1$ and $G_2$ is the graph obtained from their disjoint union by adding an edge between every vertex in $G_1$ and every vertex in $G_2$. Define $J := H + K_{n-2k}$ so that

$$|V(J)| = h + n - 2k = 2(n - k - l).$$

Then $H$ has $k'$-matching if and only if $J$ has a 1-factor. We want to apply Theorem 1.2 (see Section 1.2) to $J$. Firstly, $J$ has minimum degree

$$\begin{aligned}\delta(J) \geqslant \delta(H) + n - 2k &\geqslant \tfrac{1}{2}(3(k+l) - n) + 1 - 2l + n - 2k \\ &= \tfrac{1}{2}(n - k - l) + 1 \\ &= \tfrac{1}{4}|V(J)| + 1.\end{aligned}$$

If (c)(i) holds then

$$\begin{aligned}|N_J(X)| &\geqslant \tfrac{1}{2}(|X| + 3(k+l) - n - 2) - 2l + n - 2k \\ &= \tfrac{1}{2}(|X| + n - k - l - 2) \\ &= \tfrac{1}{4}(2|X| + |V(J)| - 4)\end{aligned}$$

for every set $X \in \mathcal{I}(J)$ such that $5 \leqslant |X| \leqslant \frac{1}{2}|V(J)| + 1$. If (c)(ii) holds then

$$\begin{aligned}|N_J(X)| &\geqslant 2|X| + 3(k+l) - 2n - 2 - 2l + n - 2k \\ &= 2|X| + k + l - n - 2 \\ &= \tfrac{1}{2}(4|X| - |V(J)| - 4)\end{aligned}$$

for every set $X \subseteq V(J)$ such that $\alpha(X) \geqslant |V(J)| - |X| + 2$ and

$$\tfrac{1}{2}|V(J)| + 1 \leqslant |X| \leqslant \tfrac{3}{4}|V(J)| - 1.$$

If $n - 2k \geqslant 2$, then $J$ is 2-connected. Thus $J$ satisfies the hypotheses of Theorem 1.2, except perhaps sesquiconnectedness when $n - 2k \in \{0, 1\}$. The only place where the fact that the graph $G$ is sesquiconnected is used in the proof of Theorem 1.2 is to make the following deduction: *if $S \subseteq V(G)$ and $G - S$ has at least $|S| + 2$ odd components, then $|S| \geqslant 2$.* The same deduction also follows here (with $J$ in place of $G$) when $n - 2k \in \{0, 1\}$: since if $n = 2k$ then $J = H$ is very suitable for a 1-factor by hypothesis in (c), and if $n = 2k + 1$ then $J = H + K_1$, and $H$ has exactly one odd component by hypothesis in (c).

Hence a slight modification of the proof of Theorem 1.2 shows that $J$ has a 1-factor in all cases, and so $L$ extends to a $k$-matching in $G$, as required. □

## 3.3 Sharpness of Theorem 3.1

We now give examples to demonstrate the sharpness of the bounds in Theorem 3.1. Our examples take the general form $K_{2l} + H$. If $L$ is an $l$-matching inside the $K_{2l}$, then $L$ extends to a $k$-matching in $G$ if and only if $H$ has a $(k - l)$-matching.

To see that the bound on $\sigma_2(G)$ in (a) is sharp, let $t \geqslant 0$ and define the graph $G_1 := K_{2l} + K_{k-l-1,k-l+t}$. Then $\sigma_2(G_1) = 2(k + l - 1) = 2(k + l) - 2$, but the largest matching in $H = K_{k-l-1,k-l+t}$ has $k - l - 1$ edges.

We now show that conditions (3.1) and (3.2) are best possible in parts (b) and (c) of Theorem 3.1. Given $k$, $l$ and $n$ such that $0 \leqslant l < k \leqslant \frac{1}{2}n$, let $b$ be an integer satisfying $b \geqslant 0$ and

$$b \leqslant \begin{cases} k - l - 1 & \text{if (b) holds,} \\ \frac{1}{2}(k - l) & \text{if (c) holds.} \end{cases}$$

Define

$$a := n - 2b - k - l + 1. \tag{3.20}$$

If (b) holds then $l \geqslant 3k - n - 2$ by hypothesis, and so

$$a \geqslant n - 3k + l + 3 \geqslant 1, \tag{3.21}$$

and if (c) holds then $a \geqslant n - 2k + 1 \geqslant 1$, since $2k \leqslant n$. Consider the graph

$$G_2 := K_{2l} + ((aK_1 \cup bK_3) + K_{k-l-b-1}),$$

which has order $n = a + 2b + k + l - 1$. Let $S = V(K_{k-l-b-1}) \subset V(H)$, so that $H - S$ has $o(S) = a + b = |S| + n - 2k + 2$ odd components. Then by Theorem 1.1, the largest matching in $H$ has at most

$$\tfrac{1}{2}(n - 2l - o(S) + |S|) = k - l - 1$$

edges. Set $X_1 := V(aK_1)$ and $X_2 := V(aK_1 \cup bK_3) = V(H) \setminus S$. Then

$$\alpha(X_2) = a + b = 2(a + 2b) - (a + 3b) = 2(n - k - l + 1) - |X_2|, \tag{3.22}$$

by (3.20), and

$$|N_{G_2}(X_1)| = 2l + (k - l - b - 1) = \tfrac{1}{2}(|X_1| + 3(k + l) - n - 3), \tag{3.23}$$
$$|N_{G_2}(X_2)| = 2l + (k - l + 2b - 1) = 2|X_2| + 3(k + l) - 2n - 3. \tag{3.24}$$

(The right term of (3.23) is obtained by adding $\frac{1}{2}y$ to the middle term, where

$$y = a + 2b + k + l - n - 1 = 0, \tag{3.25}$$

and the right term of (3.24) is obtained by adding $2y$ to the middle term.)

By (3.23), $X_1$ only just fails to satisfy (3.1). However, if $X \in \mathcal{I}(G_2)$ and $X \neq X_1$, then $|N(X)| \geqslant |N(X_1)|$ and $X$ satisfies (3.1): if $|X| < |X_1|$ then this follows directly from (3.23); while if $|X| \geqslant |X_1|$ and $X$ includes any vertices from outside $X_1$ then clearly $|N(X)| - |N(X_1)|$ exceeds $\frac{1}{2}(|X| - |X_1|)$.

By (3.24), $X_2$ only just fails to satisfy (3.2). Let $X \subseteq V(G_2)$ with $X \neq X_2$. If $|X| < |X_2|$, then $\alpha(X) \leqslant \alpha(X_2)$, and (3.22) implies that $X$ does not satisfy $\alpha(X) \geqslant 2(n - k - l + 1) - |X|$. If $|X| \geqslant |X_2|$ and $X$ includes any vertices from outside $X_2$, then $|N(X)| - |N(X_2)| = a$. Hence, by (3.24), $X$ satisfies (3.2) if $a > 2(|X| - |X_2|)$. If (b) holds then $l > 3k - n - 3$ and $|X| \leqslant n - 2l$, and so by (3.20) we obtain

$$\begin{aligned} a = n - 2b - k - l + 1 &> 2(k - l - b - 1) \\ &= 2(n - 2l - a - 3b) \\ &\geqslant 2(|X| - |X_2|), \end{aligned}$$

as required. If (c) holds and $|X| \leqslant \frac{3}{2}(n - k - l) - 1$, then (3.20) gives

$$\begin{aligned} 2(|X| - |X_2|) &\leqslant 3(n - k - l) - 2 - 2(a + 3b) \\ &= 3(n - k - l - 2b) - 2a - 2 \\ &= 3(a - 1) - 2a - 2 = a - 5 < a, \end{aligned}$$

as required. It remains to check that $G_2$ satisfies the other hypotheses of Theorem 3.1(c). By (3.23), $G_2$ has minimum degree

$$\delta(G_2) = 2l + (k - l - b - 1) = \tfrac{1}{2}(3(k + l) - n) + 1 + \tfrac{1}{2}(a - 5).$$

Thus if $a \geqslant 5$ then the bound on $\delta(G)$ in Theorem 3.1(c) is satisfied. Also, if $a \geqslant 5$ then $0 \leqslant 2b \leqslant n - k - l - 4$ by (3.20). It follows that $n - l \geqslant k + 4$, and

$$k - l - b - 1 \geqslant k - l - \tfrac{1}{2}(n - k - l - 4) - 1 = \tfrac{1}{2}(3k - n - l) + 1. \qquad (3.26)$$

Hence if $k \geqslant \frac{1}{2}(n-1)$ then $n - l \geqslant \frac{1}{2}(n-1) + 4$, which implies $n - 2l \geqslant 7$. Then by (3.26) with $k \geqslant \frac{1}{2}(n-1)$ we obtain

$$k - l - b - 1 \geqslant \lceil \tfrac{1}{4}(n - 2l - 6) \rceil + 1 \geqslant 2.$$

Therefore $H = G_2 - V(L)$ is 2-connected if $k \geqslant \frac{1}{2}(n-1)$. Hence if $k = \frac{1}{2}n$ then $H$ is very suitable for a 1-factor (since $n$ is then even), and if $k = \frac{1}{2}(n-1)$ then $H$ has exactly one odd component. Thus all of the hypotheses of Theorem 3.1(b) and (c) are satisfied in $G_2$, except that (3.1) fails when $X = X_1$ and (3.2) fails when $X = X_2$. Hence the bounds given in (3.1) and (3.2) are sharp.

We now consider the ranges of $|X|$ in Theorem 3.1. For (c), note that

$$n - k - l + 1 = a + 2b \geqslant a = |X_1| \geqslant 5$$

and

$$n - k - l + 1 = a + 2b \leqslant a + 3b = |X_2| \leqslant \tfrac{3}{2}a + 3b - \tfrac{5}{2} = \tfrac{3}{2}(n - k - l) - 1,$$

with equality on the left in each case if $b = 0$ and on the right if $a = 5$. If (b) holds, then $b \leqslant k - l - 1$ and so $|X_1| = a \geqslant n - 3k + l + 3$ by (3.21); also $|X_2| \leqslant |V(H)| = n - 2l$. It follows that the ranges of values of $|X|$ in Theorem 3.1(b) and (c) cannot be reduced.

When $k = \frac{1}{2}n$, the assumption in (c) that $H$ is very suitable for a 1-factor is needed to eliminate the graphs $K_{2l} + (K_p \cup K_q)$ and $K_{2l} + (K_1 + (K_r \cup K_s \cup K_t))$, where $p + q = n - 2l$ and $r + s + t = n - 2l - 1$; also $p, q, r, s$ and $t$ are all odd (thus in both graphs $H$ has no perfect matching) and, to ensure that the bound on $\delta(G)$ holds, $\frac{1}{4}(n - 2l) + 2 \leqslant p \leqslant q$ and $\frac{1}{4}(n - 2l) + 1 \leqslant r \leqslant s \leqslant t$.

When $k = \frac{1}{2}(n-1)$, the assumption in (c) that $H$ has at most one odd component is needed is needed to eliminate graphs such as $K_{2l} + (K_p \cup K_q \cup K_r)$ where $p + q + r = n - 2l$; also $p, q$ and $r$ are all odd (so that $H$ does not contain a $(k - l)$-matching), and $\frac{1}{4}(n - 2l + 5) \leqslant p \leqslant q \leqslant r$, so that the bound on $\delta(G)$ holds.

If $3k - l - n \leqslant 2$ then we know from (b) that the neighbourhood conditions suffice, without assuming an additional bound on $\delta(G)$. We now prove that the bound $\delta(G) \geqslant \frac{1}{2}(3(k+l)-n)+1$ in (c) is sharp when $3k - l - n \geqslant 3$, except if $k \geqslant \frac{1}{2}(n-1)$ then it is sharp when $3k - l - n \geqslant 7$. Given $k$, $l$ and $n$ satisfying these bounds, let $r \in \{0, 1\}$ such that $n + k + l + 1 \equiv r \pmod 2$, and define

$$p := \tfrac{1}{2}(3k - l - n - r - 3) \geqslant 0 \tag{3.27}$$

and $$q := k - l - p - 2r - 1.$$

Adding $2p - 3k + l + n + r + 3 = 0$ to the definition of $q$ gives

$$q = n - 2k + p - r + 2 \geqslant 1, \tag{3.28}$$

since $2k \leqslant n$ and $r \in \{0, 1\}$. Consider the graph

$$G_3 := K_{2l} + (K_p + (qK_3 \cup rK_5)), \tag{3.29}$$

which has order

$$n = 2l + p + 3q + 5r = 3k - l - 2p - r - 3.$$

Let $S = V(K_p) \subset V(H)$. Then $H - S$ has $o(S) = q + r = |S| + n - 2k + 2$ odd components by (3.28). Hence, by Theorem 1.1, the largest matching in $H$ has at most

$$\tfrac{1}{2}(n - 2l - o(S) + |S|) = k - l - 1$$

edges. Note that $G_3$ does not satisfy the bound on $\delta(G)$ in (c), since (3.27) gives

$$\delta(G_3) = 2l + p + 2 = \tfrac{1}{2}(3(k+l) - n - r + 1),$$

and $r \in \{0, 1\}$. It remains to check that $G_3$ satisfies the other hypotheses of Theorem 3.1(c). If $3k - l - n \geqslant 7$ then $p \geqslant 2$, hence $H$ is 2-connected. Thus $H$ is very suitable for a 1-factor if $k = \frac{1}{2}n$, and $H$ has exactly one odd component if $k = \frac{1}{2}(n-1)$. If $X \in \mathcal{I}(G_3)$ with $|X| \geqslant 5$ then $X \subset H - S = qK_3 \cup rK_5$, and (3.27) gives

$$\begin{aligned} |N_{G_3}(X)| \geqslant 2|X| + 2l + p &= 2|X| + \tfrac{1}{2}(3(k+l) - n - r - 3) \\ &= \tfrac{1}{2}(|X| + 3(k+l) - n - 2) + \tfrac{1}{2}(3|X| - r - 1). \end{aligned}$$

Hence $X$ satisfies (3.1), since $|X| \geqslant 1$ and $r \in \{0, 1\}$. Lastly, if $X \subseteq V(G_3)$ with $|X| \leqslant \frac{3}{2}(n - k - l) - 1$, then

$$2(n - k - l + 1) - |X| \geqslant \tfrac{1}{2}(n - k - l) + 3.$$

But $X$ does not satisfy the bound on $\alpha(X)$ in (c)(ii), since

$$\alpha(X) \leqslant \alpha(V(G_3)) = q + r = \tfrac{1}{2}(n - k - l - r + 1),$$

by (3.28) and (3.27). Thus $G_3$ satisfies all of the other hypotheses of Theorem 3.1(c), and the bound on $\delta(G)$ in (c) is sharp.

## 3.4 Further conditions for extending matchings

In this section we prove Theorem 3.4, generalising Theorem 2.2 of [40]. By defining $J := H + K_{n-2k}$, as in the alternative proof of Theorem 3.1(c), it is possible to show that Theorem 3.4 follows from Theorem 2.2 of [40]. Instead, we adapt Robertshaw and Woodall's proof of Theorem 2.2 of [40] to show that Theorem 3.4 is a corollary of Theorem 3.1(c).

**Theorem 3.4.** *Let $L$ be an $l$-matching in a graph $G$ of order $n$, and suppose that $0 \leqslant l < k \leqslant \frac{1}{2}n$. Each of the following conditions suffices to ensure that $L$ extends to a $k$-matching in $G$.*

(i) $|N(X)| \geqslant \frac{1}{2}(|X| + 3(k+l) - n)$ *for every set* $X \subset V(G)$ *such that* $1 \leqslant |X| \leqslant n-k-l$, *with strict inequality when* $|X| = 1$ *or* $n-k-l$.

(ii) $|N(X)| \geqslant 2|X| + 3(k+l) - 2n$ *for every set* $X \subset V(G)$ *such that*

$$n-k-l \leqslant |X| \leqslant \tfrac{3}{2}(n-k-l),$$

*with strict inequality when* $|X| = n-k-l$ *or* $\frac{3}{2}(n-k-l) - \frac{1}{2}$.

(iii) $G - V(L) \ncong tK_1 + 3K_5$ *or* $tK_1 + 3K_7$ *if* $n - 2k = 1 - t$ $(t \in \{0,1\})$, *and if* $X \subset V(G)$ *then* $|N(X)| >$

$$\begin{cases} \text{(a) } \frac{1}{2}(3(k+l) - n + 1) & \text{if } |X| = 1, \\ \text{(b) } \frac{1}{2}(|X| + 3(k+l) - n - 3) & \text{if } n-k-l \geqslant 4 \text{ and} \\ & \quad 5 \leqslant |X| < \max\{\frac{1}{2}(n-k-l)+2, n-k-l-2\}, \\ \text{(c) } |X| + 2(k+l) - n & \text{if } n-k-l \geqslant 4 \text{ and} \\ & \max\{\frac{1}{2}(n-k-l)+2, n-k-l-2\} \leqslant |X| \leqslant n-k-l. \end{cases}$$

(iv) $G - V(L) \ncong tK_1 + 3K_5$ *or* $tK_1 + 3K_7$ *if* $n - 2k = 1 - t$ $(t \in \{0,1\})$, *and if* $X \subset V(G)$ *then* $|N(X)| >$

$$\begin{cases} \text{(c) } |X| + 2(k+l) - n & \textit{if } n-k-l \geqslant 4 \textit{ and} \\ & n-k-l \leqslant |X| \leqslant \min\{n-k-l+2, \frac{3}{2}(n-k-l)-2\}, \\ \text{(b) } 2|X|+3(k+l)-2n-3 & \textit{if } n-k-l \geqslant 4 \textit{ and} \\ & \min\{n-k-l+2, \frac{3}{2}(n-k-l)-2\} < |X| \leqslant \frac{3}{2}(n-k-l)-1, \\ \text{(a) } n-1 & \textit{if } \frac{3}{2}(n-k-l) - \frac{1}{2} \leqslant |X| \leqslant \frac{3}{2}(n-k-l). \end{cases}$$

Theorem 3.4 contains Theorem 2.2 of [40], since the degree and neighbourhood conditions agree when $l = 0$ and $k = \frac{1}{2}n$.

**Proof of Theorem 3.4.** First we show the following.

**Claim 3.5.** (i)⇒(iii) and (ii)⇒(iv)

**Proof.** Note that

$$\tfrac{1}{2}(|X| + 3(k+l) - n) > |X| + 2(k+l) - n \quad \text{if} \quad |X| < n-k-l,$$

and $$2|X| + 3(k+l) - 2n > |X| + 2(k+l) - n \quad \text{if} \quad |X| > n-k-l.$$

Also, if $2k = n + t - 1$ ($t \in \{0, 1\}$) and $G - V(L) \cong tK_1 + 3K_r$ ($r \in \{5, 7\}$), then $2k = 2l + 3r + 2t - 1$, so that

$$k + l = 2l + \tfrac{1}{2}(3r - 1) + t.$$

If $X = V(K_r)$ then $|N(X)| \leqslant 2l + r + t$, and we get a contradiction to (i), since

$$\tfrac{1}{2}(5 + 3(2l + 7 + t) - (2l + 15 + t)) > 2l + 5 + t,$$

and $$\tfrac{1}{2}(7 + 3(2l + 10 + t) - (2l + 21 + t)) > 2l + 7 + t.$$

If $X = V(2K_r)$ then $|N(X)| \leqslant 2l + 2r + t$, and we get a contradiction to (ii), since

$$2 \times 10 + 3(2l + 7 + t) - 2(2l + 15 + t) > 2l + 10 + t,$$

and $$2 \times 14 + 3(2l + 10 + t) - 2(2l + 21 + t) > 2l + 14 + t. \quad \square$$

Thus it suffices to prove the results for (iii) and (iv), which we do together. Suppose that $G$ satisfies (iii) or (iv).

**Claim 3.6.** $\delta(G) \geqslant \frac{1}{2}(3(k+l) - n) + 1$.

**Proof.** This is clear if (iii)(a) holds; so suppose that (iv)(a) holds. If $v$ is a vertex with degree less than $\frac{1}{2}(3(k+l) - n) + 1$, let $X$ be a set of $\frac{3}{2}(n-k-l) - \frac{1}{2}$ or $\frac{3}{2}(n-k-l)$ vertices that are not adjacent to $v$. Then $|N(X)| \leqslant n-1$, contradicting (iv)(a). $\square$

Thus $H := G - V(L)$ has minimum degree

$$\delta(H) \geqslant \tfrac{1}{2}(3(k+l) - n) + 1 - 2l \geqslant \tfrac{1}{2}(3k' - h) + 1, \tag{3.30}$$

by (3.6) (where $h$ and $k'$ are defined as in Section 3.2). To show that $H$ is very suitable for a 1-factor if $k = \frac{1}{2}n$, and that $H$ has exactly one odd component if $k = \frac{1}{2}(n-1)$, we prove something stronger:

**Claim 3.7.** *$H$ is sesquiconnected if $k = \frac{1}{2}n$, and $H$ has at most two components if $k = \frac{1}{2}(n-1)$.*

**Proof.** First let $k = \frac{1}{2}n$, so that $k' = \frac{1}{2}h$ by (3.4), and suppose that $H$ is not connected. Let $X_1$ be the vertex-set of a smallest component of $H$, and let $X_2 := V(H) \setminus X_1$. Since $\delta(H) \geqslant \frac{1}{4}h + 1$ by (3.30), we obtain $\frac{1}{4}h + 2 \leqslant |X_1| \leqslant \frac{1}{2}h$, which implies $h \geqslant 8$. Moreover, if $h = 8$ and $k' = 4$ then (3.30) forces $H = 2K_4$, in which case $H$ has a 1-factor. So we may assume $h \geqslant 10$, so that

$$5 \leqslant \lceil \tfrac{1}{4}h + 2 \rceil \leqslant |X_1| \leqslant \tfrac{1}{2}h \text{ and } \tfrac{1}{2}h \leqslant |X_2| \leqslant \tfrac{3}{4}h - 2.$$

Also $|N(X_1)| \leqslant |X_1|$ and $|N(X_2)| \leqslant |X_2|$. We get an immediate contradiction to (iii)(c) or (iv)(c) if $|X_1| \geqslant \frac{1}{2}h - 2$ or $|X_2| \leqslant \frac{1}{2}h + 2$, respectively. But if $|X_1| \leqslant \frac{1}{2}n - 3$

or $|X_2| \geqslant \frac{1}{2}h+3$ then

$$|N(X_1)| \leqslant |X_1| \leqslant \tfrac{1}{4}(2|X|+h-6)$$

and

$$|N(X_2)| \leqslant |X_2| \leqslant \tfrac{1}{2}(4|X|-h-6),$$

so we get a contradiction from (iii)(b) or (iv)(b) instead. Thus we may assume that $H$ is connected if $k=\frac{1}{2}n$.

Suppose that $k = \frac{1}{2}n$ and $H$ is not sesquiconnected. Let $v$ be a vertex such that $H-v$ has at least three components, let $Y_1$ be the vertex-set of a smallest component of $H-v$, and let $Y_2 := V(H) \setminus (Y_1 \cup \{v\})$. Since $\delta(H) \geqslant \frac{1}{4}h+1$, $\frac{1}{4}h+1 \leqslant |Y_1| \leqslant \frac{1}{3}(h-1)$. This implies $n \geqslant 16$, so that

$$5 \leqslant \tfrac{1}{4}h+1 \leqslant |Y_1| \leqslant \tfrac{1}{3}(h-1) \quad \text{and} \quad \tfrac{2}{3}(h-1) \leqslant |Y_2| \leqslant \tfrac{3}{4}h-2.$$

Also $|N(Y_1)| \leqslant |Y_1|+1$ and $|N(Y_2)| \leqslant |Y_2|+1$. Thus $|Y_1| \geqslant \frac{1}{2}h-4$ if (iii)(b) holds, since

$$|Y_1|+1 \leqslant \tfrac{1}{4}(2|Y_1|+h-6) \text{ if } |Y_1| \leqslant \tfrac{1}{2}h-5,$$

and $|Y_2| \geqslant \frac{1}{2}h-3$ if (iv)(b) holds, since

$$|Y_2|+1 \leqslant \tfrac{1}{2}(4|Y_2|-h-6) \text{ if } |Y_2| \leqslant \tfrac{1}{2}h+4.$$

Therefore,

$$\max\{\tfrac{1}{4}h+1, \tfrac{1}{2}h-4\} \leqslant |Y_1| \leqslant \tfrac{1}{3}(h-1)$$

and

$$\tfrac{2}{3} \leqslant |Y_2| \leqslant \min\{\tfrac{1}{2}h+3, \tfrac{3}{4}h-2\}.$$

These inequalities force $(h, |Y_1|, |Y_2|) = (16,5,10)$, $(20,6,13)$ or $(22,7,14)$. In these three cases, the components of $H - v$ have orders $(5,5,5)$, $(6,6,7)$ and $(7,7,7)$ respectively, and the lower bound on $\delta(H)$ ensures that all components are complete except possibly for the component of order 7 when $h = 20$, which however is certainly Hamiltonian. Thus if $H$ is not sesquiconnected then either $H \cong K_1 + 3K_5$ or $K_1 + 3K_7$, which are ruled out by the hypotheses of the theorem, or else $h = 20$ and it is easy to see that $H$ has a 1-factor. So we may assume that $H$ is sesquiconnected if $k = \frac{1}{2}n$.

If $k = \frac{1}{2}(n-1)$ then $k' = \frac{1}{2}(h-1)$ by (3.4). Define $H^+ := H + \{v\}$; then $H^+$ is connected and (3.30) implies

$$\delta(H^+) = \delta(H) + 1 \geqslant \tfrac{1}{4}(h+1) + 1 = \tfrac{1}{4}|V(H^+)| + 1.$$

By the preceding argument, it follows that if $H^+$ is not sesquiconnected then either $H \cong 3K_r$ $(r \in 5,7)$, which is ruled out by hypothesis, or $H \cong 2K_6 \cup K_7$, in which case $H$ clearly has a $k'$-matching ($k' = 9$). It follows that $H$ has at most two components if $k = \frac{1}{2}(n-1)$. This completes the proof of Claim 3.7. $\square$

If $n - k - l \leqslant 3$ then (3.4) and (3.5) give $2k' \leqslant h \leqslant k' + 3$. It follows that $(h, k')$=$(6,3)$, $(5,2)$, $(4,2)$, $(4,1)$, $(3,1)$ or $(2,1)$. In these cases it is easy to check that (3.30) ensures that $H$ contains a $k'$-matching. (For example, in the case when $h = 6$, (3.30) implies that $\delta(H) \geqslant 3$ and so $H$ is Hamiltonian by Dirac's theorem.) So we may assume $n - k - l \geqslant 4$. Note that, in both (iii) and (iv), condition (c) is stronger than (b) in the range of values of $|X|$ in which (c) applies. Hence if (iv) holds then condition (c)(ii) of Theorem 3.1 holds, and if (iii) holds then condition (c)(i) of Theorem 3.1 holds; the only case where this is not obvious is if $|X| = n - k - l + 1$ in (3.1). But then we can choose $X' \subset X$ with

$|X'| = n - k - l \geqslant \frac{1}{2}(n-k-l)+2$, and it follows from (iii)(c) that

$$\begin{aligned} |N(X)| \geqslant |N(X')| &> |X'| + 2(k+l) - n \\ &= k + l \\ &> \tfrac{1}{2}(2(k+l)-2) \\ &= \tfrac{1}{2}(|X| + 3(k+l) - n - 3). \end{aligned}$$

Thus in all cases the hypotheses of Theorem 3.1(c) are satisfied; hence $L$ extends to a $k$-matching in $G$. This completes the proof of Theorem 3.4. □

# Chapter 4

# Ore-type conditions for a heavy matching

## 4.1 Introduction

A *weighted graph* is a simple graph $G$ in which every edge $e$ is assigned a nonnegative number $w(e)$, called the *weight* of $e$. If $uv \notin E(G)$, we define $w(uv) = 0$. The *weighted degree* of a vertex $v \in V(G)$, denoted by $d^w(v)$, is the sum of the weights of the edges incident with $v$. For a matching $M$, the *weight of $M$* is defined by

$$w(M) = \sum_{e \in M} w(e).$$

The weight of a cycle or path is defined similarly. Bondy and Fan [8] gave weighted degree conditions for heavy cycles and paths in weighted graphs. Bondy et al. [7] gave an Ore-type theorem for heavy cycles:

**Theorem 4.1.** (Bondy et al. [7]) *Let $G$ be a 2-connected weighted graph and let $d > 0$ be a real number. If $d^w(u)+d^w(v) \geqslant 2d$ for every pair of nonadjacent vertices $u$ and $v$, then $G$ contains either a cycle of weight at least $2d$ or a Hamilton cycle.*

Enomoto, Fujisawa and Ota [16] proved a similar result about heavy paths between specified vertices. An $(x, y)$-path is a path with end vertices $x$ and $y$.

**Theorem 4.2.** (Enomoto et al. [16]) *Let $G$ be a 2-connected weighted graph and let $d$ be a real number. Let $x$ and $y$ be distinct vertices in $G$. If $d^w(u)+d^w(v) \geqslant 2d$ for every pair of nonadjacent vertices $u$ and $v$ in $V(G) \setminus \{x, y\}$, then $G$ contains an $(x, y)$-path of weight at least $d$ or a Hamilton $(x, y)$-path.*

We will prove the following analogous result for heavy matchings. A *near-perfect matching* in a graph of odd order $n$ is a matching with $\frac{1}{2}(n-1)$ vertices.

**Theorem 4.3.** *Let $G$ be a weighted graph in which $d^w(u) + d^w(v) \geqslant 2d$ for every pair of nonadjacent vertices $u$ and $v$. Then either $G$ contains a matching of weight at least $d$ or every heaviest matching in $G$ is contained in a perfect or near-perfect matching.*

We also prove a version of this result for integer edge-weights, Theorem 4.4. This generalises our result on $k$-matchings (Theorem 3.2(a)); since if all edges have weight 1 then a matching of weight $d$ is a $d$-matching, and the weighted degree sum condition in Theorem 4.4 becomes $\sigma_2(G) \geqslant 2d - 1$.

**Theorem 4.4.** *Let $G$ be an weighted graph in which all of the edge-weights are nonnegative integers. If $d^w(u) + d^w(v) \geqslant 2d - 1$ for every pair of nonadjacent vertices $u$ and $v$, then either $G$ contains a matching of weight at least $d$ or every heaviest matching in $G$ is contained in a perfect or near-perfect matching.*

## 4.2 Proof and sharpness of Theorems 4.3 and 4.4

**Proof.** We prove both theorems simultaneously. Let $M$ be a heaviest matching in $G$, so that $w(e) = 0$ for all edges $e$ in $G - V(M)$. Suppose that $w(M) < d$, and choose a matching $M^+$ with as many edges as possible among matchings containing $M$. If $M^+$ is not a perfect or near-perfect matching then there exist distinct vertices, $x$ and $y$ say, in $V(G) \setminus V(M^+)$. Since there are no edges in $G - V(M^+)$, $xy \notin E(G)$; thus by hypothesis

$$d^w(x) + d^w(y) \geqslant \begin{cases} 2d & \text{in Theorem 4.3,} \\ 2d-1 & \text{in Theorem 4.4.} \end{cases} \tag{4.1}$$

Denote the edges of $M$ by $u_1v_1$, $u_2v_2$, $\ldots$, $u_tv_t$. If $w(u_jx) + w(v_jy) > w(u_jv_j)$ for some $j$ then $(M - \{u_jv_j\}) \cup \{u_jx, v_jy\}$ (with a nonedge omitted if necessary) is a heavier matching than $M$, which is a contradiction. It follows that

$$w(u_ix) + w(v_iy) \leqslant w(u_iv_i) \;\; (1 \leqslant i \leqslant t). \tag{4.2}$$

Similarly,

$$w(u_iy) + w(v_ix) \leqslant w(u_iv_i) \;\; (1 \leqslant i \leqslant t). \tag{4.3}$$

Since $M$ is a heaviest matching, any edges of nonzero weight incident with $x$ or $y$ must be incident with $V(M)$. Hence by (4.2) and (4.3) we obtain

$$\begin{aligned} d^w(x) + d^w(y) &= \sum_{i=1}^{t} [w(u_ix) + w(v_iy) + w(u_iy) + w(v_ix)] \\ &\leqslant 2\sum_{i=1}^{t} w(u_iv_i) = 2w(M) < 2d, \end{aligned}$$

and if all weights are integers, then $w(M) < d$ implies

$$d^w(x) + d^w(y) \leqslant 2w(M) \leqslant 2(d-1) < 2d-1.$$

In either case we obtain a contradiction to (4.1). Hence either $w(M) \geqslant d$, or $M^+$ is a perfect or near-perfect matching. This completes the proof of Theorems 4.3 and 4.4. □

If all edge-weights in $G$ are strictly positive, then $e(G-V(M)) = 0$ for any heaviest matching $M$, hence $M^+ = M$; thus in this case the words "contained in" can be removed from the statements of Theorems 4.3 and 4.4.

**Sharpness of Theorems 4.3 and 4.4.**

To see that the condition $d^w(u) + d^w(v) \geqslant 2d$ (or $2d - 1$) is sharp, consider the complete bipartite graph $G_1 = K_{r,s}$, where $1 \leqslant r \leqslant s - 2$. Clearly $G_1$ has no perfect or near-perfect matching.

First, assign the same weight $\omega$ to all edges of $G_1$. Then $d^w(u) + d^w(v) \geqslant 2r\omega$ for all pairs $u, v$ of nonadjacent vertices in $G_1$, with equality if $u$ and $v$ are in the larger partite set of $G_1$. The heaviest matching in $G_1$ has weight $r\omega$. Thus, for a given $d$, if $2r\omega < 2d$ then the heaviest matching in $G_1$ has weight strictly less than $d$.

Next, we show that the weighted degree sum condition is still sharp if we allow edges to have different weights. Let $v_0, \ldots, v_{r-1}$ denote the $r$ vertices in the smaller partite set of $G_1$, and assign weight $\omega + i$ to each edge incident with the vertex $v_i$ $(0 \leqslant i \leqslant r-1)$. Then the heaviest matching in $G_1$ has weight

$$a = \sum_{i=0}^{r-1}(\omega + i) = r\omega + \tfrac{1}{2}r(r-1).$$

Note that

$$d^w(u) + d^w(v) = 2r\omega + r(r-1) = 2a$$

for every pair $u, v$ of vertices in the larger partite set, and

$$\begin{aligned} d^w(v_0) + d^w(v_1) = s\omega + s(\omega+1) &= s(2\omega+1) \\ &\geqslant (r+2)(2\omega+1) = 2r\omega + 4\omega + r + 2. \end{aligned}$$

Thus if $4\omega + r + 2 \geqslant r(r-1)$ then $d^w(u) + d^w(v) \geqslant 2a$ for all pairs $u, v$ of nonadjacent vertices in $G_1$. If $2a < 2d$ then the heaviest matching in $G_1$ has weight strictly less than $d$; so again the bound on $d^w(u) + d^w(v)$ is sharp.

Finally, we give two graphs to demonstrate that the clause about perfect or near-perfect matchings is required in Theorems 4.3 and 4.4. Let $G_2 := K_n$ and, for $n \geqslant 5$, let $G_3 := K_n - \{xy\}$. In each graph, assign the same weight $\omega$ to every edge. Then in both $G_2$ and $G_3$ the heaviest matching has weight $\lfloor \frac{n}{2} \rfloor \omega$. However, $G_2$ satisfies the condition of Theorem 4.3 for all positive $d$, while in $G_3$,

$$d^w(x) + d^w(y) = 2(n-2)\omega > 2\lfloor \tfrac{n}{2} \rfloor \omega,$$

since $n \geqslant 5$. So neither graph has a heavy enough matching. But any heaviest matching $M$ in $G_2$ or $G_3$ must be a perfect or near-perfect matching: if there were at least two vertices outside $V(M)$ then any edge of $M$ could be replaced by two independent edges (giving a matching heavier than $M$, a contradiction).

# Chapter 5

# $k$-factors containing a given set of edges

## 5.1 Introduction

The main result of Chapter 3 (Theorem 3.1) contains the following sharp conditions for an $l$-matching to be contained in a 1-factor.

**Theorem 5.1.** *Let $L$ be an $l$-matching in a graph $G$ of even order $n$, where $0 \leqslant l < \frac{1}{2}n$. Then each of the following conditions suffices to ensure that $L$ is contained in a 1-factor in $G$.*

(a) $\sigma_2(G) \geqslant n + 2l - 1$.

(b) $l \geqslant \frac{1}{2}n - 2$ *and*

$$|N(X)| \geqslant \tfrac{1}{4}(2|X| + n + 6l - 4) \quad \text{whenever } X \in \mathcal{I}(G). \tag{5.1}$$

(c) $G - V(L)$ *is very suitable for a* 1*-factor,* $\delta(G) \geqslant \frac{1}{4}(n + 6l) + 1$*, and* (5.1) *holds.*

In this chapter we give degree and neighbourhood conditions for a set of edges to be contained in a $k$-factor (i.e., a $k$-regular spanning subgraph of $G$) when $k \geqslant 2$. We need to make some assumptions about the suitability of a set of edges to be contained in a $k$-factor, analogous to the assumption in Theorem 5.1(c) that $G - V(L)$ is very suitable for a 1-factor.

Let $L \subseteq E(G)$, and write $d_L(v)$ for the number of edges in $L$ that are incident with a vertex $v$ of $G$. We say that $L$ is *weakly $k$-good* if it satisfies condition (L1) below, and *strongly $k$-good* if it satisfies (L1) and (L2):

(L1) $d_L(v) \leqslant k$ for every vertex $v \in V(G)$;

(L2) if $Z := \{v \in V(G) : d_L(v) = k\}$, then $\sum_{v \in V(C)}(k - d_L(v))$ is even for every component $C$ of $G - L - Z$.

It is not difficult to see that if $G$ has a $k$-factor that includes all edges in $L$, then $L$ must be strongly $k$-good. Say that $G$ is *suitable for a $k$-factor* if $k$ is even, or if $k$ is odd and every component of $G$ is even. It is also not difficult to see that if there exists a strongly $k$-good set of edges in $G$, then $G$ must be suitable for a $k$-factor.

In Section 5.3 we will prove the main result of this chapter, Theorem 5.2, and in Section 5.4 we discuss its sharpness. Theorem 5.2(a) generalises the result of Katerinis [25] mentioned in Section 1.2; parts (b) and (c) generalise Woodall's result on $k$-factors [47], stated as Theorem 1.3 in Section 1.2.

[When $L = \emptyset$ and $k$ is odd, the assumption that $L$ is strongly $k$-good implies that every component of $G$ is even. This replaces the assumption in Theorem 1.3 that $G$ is connected when $k$ is odd, and eliminates the possibility that $G$ is the disjoint union of two odd complete graphs. When $L = \emptyset$ the remaining hypotheses of Theorem 5.2(b) and (c) are the same as those of Theorem 1.3; thus (b) and (c) together contain Theorem 1.3.]

Recall that $\mathcal{I}(G)$ denotes the family of nonempty independent subsets of $V(G)$.

**Theorem 5.2.** *Let $k \geqslant 2$. Let $G$ be a graph of order $n$ such that $kn$ is even, and let $L$ be a weakly $k$-good set of $l$ edges in $G$. Then each of the following conditions suffices to ensure that $G$ has a $k$-factor that contains all the edges of $L$.*

(a) $2l < kn - 4k^2 + 6k$ *and* $\delta(G) \geqslant \frac{1}{2}n + l/k$.

(b) $\delta(G) > n + 2k - 2\sqrt{kn - 2l + 2}$, *and if* $2l > kn - 4k^2 + 6k$ *then $L$ is strongly $k$-good.*

(c) *$L$ is strongly $k$-good,* $2l \leqslant kn - 4k^2 + 6k$,

$$\delta(G) \geqslant \frac{(k-1)(n+2) + 2l}{2k-1}, \tag{5.2}$$

*and*

$$|N(X)| > \frac{|X| + (k-1)n + 2l - 2}{2k-1} \quad \text{whenever } X \in \mathcal{I}(G). \tag{5.3}$$

In section 5.2 we prove a technical lemma, generalising to $f$-factors a result of Katerinis and Woodall [26] about $k$-factors. This will be useful in the proof of Theorem 5.2.

## 5.2 A lemma about $f$-factors

Let $G$ be a graph of order $n$ and let $f : V(G) \to \mathbb{N} \cup \{0\}$ be a function. If $A$ and $B$ are disjoint subsets of $V(G)$, let $H := G - (A \cup B)$ and write $\omega(A, B; f)$ for the number of components $C$ of $H$ such that $e(B : C) + \sum_{v \in V(C)} f(v)$ is odd.

**Lemma 5.3.** *Using the notation of the previous paragraph, suppose* $\sum_{v \in V(G)} f(v)$ *is even, and there exist disjoint subsets* $A, B$ *of* $V(G)$ *such that*

$$\omega(A, B; f) + \sum_{v \in B} (f(v) - d_{G-A}(v)) \geqslant \sum_{v \in A} f(v) + 2. \tag{5.4}$$

*Let* $u \in V(H)$. *If* $|A \cup B|$ *is maximal with respect to* (5.4), *then*

(i) $d_{G-A}(u) \geqslant f(u) + 1$;

(ii) $|N(u) \cap B| \leqslant f(u) - 1$;

(iii) $|V(C)| \geqslant 3$ *for every component* $C$ *of* $H$.

**Proof.** We will need the following observation of Tutte [42]:

$$\omega(A, B; f) + \sum_{v \in B} (f(v) - d_{G-A}(v)) - \sum_{v \in A} f(v) \equiv \sum_{v \in V(G)} f(v) \pmod 2. \tag{5.5}$$

(i) Suppose $d_{G-A}(u) \leqslant f(u)$, and define $B' = B \cup \{u\}$. Then

$$\sum_{v \in B'} (f(v) - d_{G-A}(v)) \geqslant \sum_{v \in B} (f(v) - d_{G-A}(v)),$$

and $\omega(A, B'; f) \geqslant \omega(A, B; f) - 1$, so that (5.4) implies

$$\omega(A, B'; f) + \sum_{v \in B'} (f(v) - d_{G-A}(v)) \geqslant \sum_{v \in A} f(v) + 1. \tag{5.6}$$

By (5.5), using the fact that $\sum_{v \in V(G)} f(v)$ is even, it follows from (5.6) that (5.4) holds with $B'$ in place of $B$, which contradicts the maximality of $|A \cup B|$. Thus $d_{G-A}(u) \geqslant f(u) + 1$.

(ii) Now suppose $|N(u) \cap B| \geqslant f(u)$, and define $A' = A \cup \{u\}$. Then

$$\sum_{v \in B} (f(v) - d_{G-A'}(v)) \geqslant \sum_{v \in B} (f(v) - d_{G-A}(v)) + f(u),$$

and $\omega(A', B; f) \geqslant \omega(A, B; f) - 1$, so that (5.4) implies

$$\omega(A', B; f) + \sum_{v \in B} (f(v) - d_{G-A'}(v)) \geqslant \sum_{v \in A'} f(v) + 1.$$

By (5.5), again using the fact that $\sum_{v \in V(G)} f(v)$ is even, it follows that (5.4) holds with $A'$ in place of $A$, contradicting the maximality of $|A \cup B|$ as before. Thus $|N(u) \cap B| \leqslant f(u) - 1$.

(iii) Let $C$ be any component of $H$. If $u \in V(C)$ then by (i) and (ii) we obtain

$$|N(u) \cap V(C)| = d_{G-A}(u) - |N(u) \cap B| \geqslant 2;$$

hence $|V(C)| \geqslant 1 + |N(u) \cap V(C)| \geqslant 3$, as required. $\square$

## 5.3 Proof of Theorem 5.2

We prove parts (a), (b) and (c) of the theorem simultaneously.

**Claim 5.4.** *The hypotheses of Theorem* 5.2 *are related as follows:*

(i) *If* (a) *holds then* (5.2) *holds.*

(ii) *If* (b) *holds then* $\delta(G) \geqslant \frac{1}{2}n + l/k$.

(iii) *If* (b) *holds then* (5.2) *holds.*

**Proof.** (i) Suppose (a) holds. If

$$\frac{kn + 2l}{2k} \leqslant \delta(G) \leqslant \frac{(k-1)(n+2) + 2l - 1}{2k - 1}$$

then

$$(2k-1)(kn + 2l) \leqslant 2k(k-1)(n+2) + 2k(2l-1),$$

which implies $2l \geqslant kn - 4k^2 + 6k$, a contradiction.

(ii) Write $\epsilon = kn - 2l + 2$. If

$$n + 2k - 2\sqrt{\epsilon} < \delta(G) \leqslant \frac{kn + 2l - 2}{2k}$$

then multiplying through by $2k$ and rearranging gives

$$kn - 2l + 2 - 4k\sqrt{\epsilon} + 4k^2 = (2k - \sqrt{\epsilon})^2 < 0,$$

a contradiction. Since $kn$ is even by hypothesis, this implies (ii).

(iii) Suppose that

$$n + 2k - 2\sqrt{\epsilon} < \delta(G) \leqslant \frac{(k-1)(n+2) + 2l - 1}{2k - 1}.$$

Then

$$(2k-1)n + (2k-1)2k - 2(2k-1)\sqrt{\epsilon} < (k-1)n + 2l + 2k - 3,$$

and so

$$kn - 2l + 2 + 4k^2 - 4k + 1 - 2(2k-1)\sqrt{\epsilon} = (2k - 1 - \sqrt{\epsilon})^2 < 0,$$

a contradiction. □

Suppose now that $G$ satisfies (a), (b) or (c).

**Claim 5.5.** *$L$ is strongly $k$-good.*

**Proof.** Since $L$ is weakly $k$-good (i.e., (L1) holds) by hypothesis, it is enough to show that (L2) holds if either (a) holds, or (b) holds and $2l \leqslant kn - 4k^2 + 6k$. By Claim 5.4(ii), we may assume that $\delta(G) \geqslant \frac{1}{2}n + l/k$ and $2l \leqslant kn - 4k^2 + 6k$. Recall that $Z = \{v \in V(G) : d_L(v) = k\}$. If $G - L - Z$ is connected, then

$$\sum_{v \in V(G-L-Z)} (k - d_L(v)) = \sum_{v \in V(G)} (k - d_L(v)) = kn - 2l, \tag{5.7}$$

which is even since $kn$ is even by hypothesis. Then (L2) holds, and $L$ is strongly $k$-good. So suppose $G - L - Z$ is not connected. Let $C$ be the smallest component of $G - L - Z$, so that $|V(C)| \leqslant \frac{1}{2}(n - |Z|)$, and let $v \in V(C)$. Then

$$d_{G-L-Z}(v) \leqslant \tfrac{1}{2}(n - |Z|) - 1. \tag{5.8}$$

Let $D := V(G - Z)$ and write $e_L(v : D)$ for the number of edges of $L$ incident with $v$ and a vertex in $D$. Then

$$d_{G-L-Z}(v) = d_{G-Z}(v) - e_L(v : D) \geqslant \tfrac{1}{2}n + l/k - |Z| - e_L(v : D). \tag{5.9}$$

Comparing (5.8) and (5.9) gives

$$e_L(v : D) \geqslant l/k - \tfrac{1}{2}|Z| + 1. \tag{5.10}$$

Now define

$$d_L(G - Z) := \sum_{v \in D} d_L(v) = 2l - k|Z|.$$

Then (5.10) implies

$$e_L(v : D) \geqslant \frac{d_L(G - Z)}{2k} + 1. \tag{5.11}$$

Since (5.11) applies to any vertex in $C$, it follows that

$$\begin{aligned} d_L(G - Z) &\geqslant 2 \sum_{u \in V(C)} e_L(u : D) - 2e_L(C) \\ &\geqslant 2|V(C)| \left( \frac{d_L(G - Z)}{2k} + 1 \right) - 2e_L(C), \end{aligned} \tag{5.12}$$

where $e_L(C)$ denotes number of edges of $L$ with both incident vertices in $C$. Since $|V(C)| - e_L(C) > 0$, it follows from (5.12) that $d_L(G-Z) > 0$, so that $|Z| < 2l/k$; and $|V(C)| < k$, so that

$$d_{G-L-Z}(v) \leqslant |V(C)| - 1 \leqslant k-2.$$

Comparing this with (5.9), we obtain

$$\begin{aligned} k-2 &\geqslant \tfrac{1}{2}n + l/k - |Z| - e_L(v:D) \\ &> \tfrac{1}{2}n + l/k - 2l/k - (k-1), \end{aligned}$$

since $|Z| < 2l/k$, and since $e_L(v:D) \leqslant d_L(v) \leqslant k-1$, by (L2) and because $v \notin Z$. Multiplying through by $2k$ gives $2l > kn - 2k(2k-3)$, contradicting our assumption that $2l \leqslant kn - 4k^2 + 6k$. This completes the proof of Claim 5.5. □

The structure of the proof now follows that of Theorem 1.3, although the details are much more complicated. Suppose that $G$ has no $k$-factor containing $L$. Then $G-L$ does not have an $f$-factor, where $f(v) := k - d_L(v)$ for all $v \in V(G)$. It follows by Tutte's criterion [42] that there exist disjoint sets $A, B \subseteq V(G)$ such that

$$\omega(A,B;f) \geqslant \sum_{v\in A} f(v) + \sum_{v\in B}[d_{G-L-A}(v) - f(v)] + 2, \tag{5.13}$$

where $\omega(A,B;f)$ is the number of components $C$ in $G-L-(A\cup B)$ such that $e_{G-L}(B:C) + \sum_{v\in V(C)} f(v)$ is odd. Since $f(v) = k - d_L(v)$ and

$$d_{G-L-A}(v) \geqslant d_{G-A}(v) - d_L(v) \quad \text{for all } v \in B,$$

it follows from (5.13) that

$$\omega(A,B;f) \geqslant k|A| - k|B| + \sum_{v\in B} d_{G-A}(v) - \sum_{v\in A} d_L(v) + 2. \tag{5.14}$$

Note that

$$\sum_{v \in V(G-L)} f(v) = kn - \sum_{v \in V(G)} d_L(v) \equiv 0 \pmod 2,$$

since $kn$ is even by hypothesis and $V(G-L) = V(G)$. Thus if $A$ and $B$ are chosen such that $|A \cup B|$ is maximal subject to (5.13), then $G-L$ satisfies the hypotheses of Lemma 5.3; it follows that every component of $H-L$ has at least three vertices, where $H := G - (A \cup B)$.

Write $\omega$ for the number of components of $H - L$; clearly $\omega \geqslant \omega(A, B; f)$. Let $a := |A|$, $b := |B|$ and $h := |V(H)|$. If $\omega > 0$ (i.e., $H$ is nonempty), let $c \geqslant 3$ be the order of the smallest component of $H-L$. If $B \neq \emptyset$ let $\beta := \min\{d_{G-A}(v) : v \in B\}$, and if $\beta = 0$ let

$$X := \{v \in B : d_{G-A}(v) = 0\} \in \mathcal{I}(G).$$

Note the following:

$$\delta(G) \leqslant \beta + a \quad \text{if } B \neq \emptyset; \tag{5.15}$$

$$c\omega \leqslant h = n - a - b \leqslant n; \tag{5.16}$$

and, writing $d_L(H) := \sum_{v \in V(H)} d_L(v)$,

$$\sum_{v \in A} d_L(v) \leqslant \sum_{v \in A \cup B} d_L(v) = 2l - d_L(H). \tag{5.17}$$

Since $\sum_{v \in B} d_{G-A}(v) \geqslant \beta b$, it follows from (5.14) and (5.17) that

$$ka + (\beta - k)b - 2l + 2 \leqslant \omega - d_L(H), \tag{5.18}$$

interpreting $(\beta - k)b$ as 0 if $b = 0$, when $\beta$ is not defined. If $\beta \leqslant k$ then since $b \leqslant n - a - \omega$ by (5.16), and $d_L(H) \geqslant 0$, we obtain

$$ka + (\beta - k)(n - a - \omega) - 2l + 2 \leqslant \omega.$$

Since $\omega \geqslant 0$, this implies

$$ka + (\beta - k)(n - a) - 2l + 2 \leqslant 0 \quad \text{if } \beta \leqslant k - 1. \tag{5.19}$$

If $\beta = 0$ then $\sum_{v \in B} d_{G-A}(v) \geqslant b - |X|$; hence (5.18) becomes

$$ka + (1 - k)b - |X| - 2l + 2 \leqslant \omega - d_L(H);$$

and since $k \geqslant 2$, a similar argument to that preceding (5.19) implies

$$ka + (1 - k)(n - a) - |X| - 2l + 2 \leqslant 0 \quad \text{if } \beta = 0. \tag{5.20}$$

There are now five cases to consider.

*Case* 1 : $B \neq \emptyset$ *and* $\beta = 0$. Equations (5.15) and (5.19) give

$$\delta(G) \leqslant a \leqslant \frac{kn + 2l - 2}{2k}, \tag{5.21}$$

which gives an immediate contadiction if (a) holds, or if (b) holds by Claim 5.4(ii). If (c) holds then equation (5.20) gives

$$a \leqslant \frac{1}{2k - 1}(|X| + (k - 1)n + 2l - 2),$$

which contradicts (5.3) since $X \in \mathcal{I}(G)$ and $N(X) \subseteq A$. Thus Case 1 is impossible.

*Case* 2 : $B \neq \emptyset$ *and* $1 \leqslant \beta \leqslant k - 1$. Equations (5.15) and (5.19) give

$$\delta(G) \leqslant \beta + a \leqslant \beta + \frac{(k - \beta)n + 2l - 2}{2k - \beta}. \tag{5.22}$$

Denote the right-hand side of (5.22) by $R(\beta)$. By differentiating $R(\beta)$, it is easy to check that in the range $\beta < 2k$ it is a concave function with a unique maximum value of $n + 2k - 2\sqrt{kn - 2l + 2}$ when $\beta = 2k - \sqrt{kn - 2l + 2}$. This gives an immediate contradiction if (b) holds. If $2l \leqslant kn - 4k^2 + 6k$ then $R(1) \geqslant R(2)$, and so the largest value of $R(\beta)$ for integral $\beta$ satisfying $1 \leqslant \beta \leqslant k - 1$ occurs when $\beta = 1$, so that

$$\delta(G) \leqslant 1 + \frac{(k-1)n + 2l - 2}{2k - 1} = \frac{(k-1)(n+2) + 2l - 1}{2k - 1}. \tag{5.23}$$

If (a) or (c) holds then (5.2) holds by Claim 5.4(i), which contradicts (5.23). Thus Case 2 is impossible.

*Case* $3 : B \neq \emptyset$ *and* $\beta = k$. In this case we first claim that

$$d_{G-A}(v) \geqslant k \text{ for every vertex } v \in V(G - A). \tag{5.24}$$

For, suppose that there exists $v \in V(H)$ with $d_{G-A}(v) \leqslant k - 1$. By (5.18) and (5.16) we obtain

$$ka \leqslant \omega + 2l - 2 \leqslant n - a - b + 2l - 2,$$

which implies that $a(k+1) \leqslant n + 2l - 2$, since $b \geqslant 0$. It follows that

$$\delta(G) \leqslant d_{G-A}(v) + a \leqslant k - 1 + \frac{n + 2l - 2}{k + 1}.$$

This is the same as equation (5.22) with $\beta = k - 1$, which we have already seen to be impossible; hence (5.24) holds as claimed.

We now observe a constraint imposed on the order of $L$ by the minimum degree conditions, namely that

$$2l \leqslant kn - k^2 - k. \tag{5.25}$$

To show this, suppose $2l > kn - k^2 - k$. Then $2l \geqslant kn - k^2 - k + 2$, since $kn$ is even by hypothesis. First suppose that either (a) or (c) holds, so that (5.2) holds by Claim 5.4(i), and $2l \leqslant kn - 4k^2 + 6k$. Comparing the bounds for $2l$ gives

$$3k^2 - 7k + 2 = (3k-1)(k-2) \leqslant 0,$$

which is impossible for $k \geqslant 3$. If $k = 2$ then $2l \geqslant 2n - 4$ by supposition, and (5.2) implies

$$\delta(G) \geqslant \tfrac{1}{3}(n + 2 + 2l) \geqslant \tfrac{1}{3}(n + 2 + 2n - 4) = n - \tfrac{2}{3},$$

a contradiction. If (b) holds then the supposition that $2l \geqslant kn - k^2 - k + 2$ implies

$$\delta(G) \geqslant n + 2k - 2\sqrt{k^2 + k} > n + 2k - 2\sqrt{(k + \tfrac{1}{2})^2} = n - 1,$$

another contradiction. Hence (5.25) must hold. Now for convenience of notation write

$$\epsilon = kn - 2l + 2. \tag{5.26}$$

By (5.16), (5.18) and (5.26) we obtain

$$ka - kn + \epsilon = ka - 2l + 2 \leqslant \omega - d_L(H) \leqslant \frac{n-a-b}{c} - d_L(H), \tag{5.27}$$

interpreting $(n-a-b)/c$ as 0 if $\omega = 0$, when $c$ is not defined. We now use (5.27) to show that Case 3 is impossible.

First suppose that either (a) or (c) holds, so that (5.2) holds by Claim 5.4(i), and $2l \leqslant kn - 4k^2 + 6k$, so that

$$\epsilon \geqslant 4k^2 - 6k + 2 = 2(k-1)(2k-1). \tag{5.28}$$

Since $d_L(H) \geqslant 0$, $b \geqslant 1$ and $c \geqslant 3$ if $\omega > 0$, (5.27) gives

$$ka - kn + \epsilon < \tfrac{1}{3}(n - a).$$

(If $\omega = 0$ this clearly holds, since then $n - a = b \geqslant 1$). Rearranging this yields

$$a < n - \frac{3\epsilon}{3k+1}.$$

Hence by (5.15) with $\beta = k$ we obtain

$$\delta(G) < n + k - \frac{3\epsilon}{3k+1}.$$

On the other hand, by (5.2) and the definition of $\epsilon$ (see (5.26)),

$$\delta(G) \geqslant \frac{1}{2k-1}((2k-1)n - kn + 2(k-1) + 2l) = n + \frac{2k-\epsilon}{2k-1}. \tag{5.29}$$

Comparing these two bounds for $\delta(G)$ gives

$$\frac{2k-\epsilon}{2k-1} < k - \frac{3\epsilon}{3k+1},$$

and so $(3k+1)(2k-\epsilon) < (k(3k+1) - 3\epsilon)(2k-1)$, which implies

$$\epsilon(3k-4) < k(3k+1)(2k-3).$$

Since $\epsilon \geqslant 2(k-1)(2k-1)$ by (5.28), we obtain

$$2(k-1)(2k-1)(3k-4) < k(3k+1)(2k-3). \tag{5.30}$$

If $k = 2$ or $3$ this gives $12 < 10$ or $100 < 90$ respectively, a contradiction. If $k \geqslant 4$ then $k + (3k-1)(k-4) > 0$, which is equivalent to $(2k-1)(3k-4) > k(3k+1)$. Since $2(k-1) > 2k-3$, it follows that (5.30) is impossible; thus Case 3 is impossible if either (a) or (c) holds.

Now suppose that (b) holds. If $\omega = 0$ then (5.27) gives $ka \leqslant kn - \epsilon$, and by (5.15) with $\beta = k$ it follows that $\delta(G) \leqslant n + k - \epsilon/k$. Comparing this with the lower bound for $\delta(G)$ in (b) gives

$$n + 2k - 2\sqrt{\epsilon} < n + k - \epsilon/k,$$

which implies that $(k - \sqrt{\epsilon})^2 < 0$, a contradiction. So we may assume that $\omega \geqslant 1$. If $v$ is a vertex in a smallest component of $H - L$ then

$$d_{G-A}(v) \leqslant d_B(v) + d_{H-L}(v) + d_L(v) \leqslant b + c - 1 + d_L(v).$$

By (5.24), and since $d_L(H) = \sum_{u \in V(H)} d_L(u) \geqslant d_L(v)$, it follows that

$$b \geqslant k - c + 1 - d_L(v) \geqslant k - c + 1 - d_L(H).$$

Substituting for $b$ in the right-hand side of (5.27) gives

$$ka - kn + \epsilon \leqslant \frac{1}{c}[n - a - k + c - 1 + d_L(H)] - d_L(H). \tag{5.31}$$

Since

$$n - a - k + c - 1 + d_L(H) \geqslant n - a - b \geqslant c \geqslant 3,$$

repeated use of the fact the that $\frac{x-1}{y-1} \geqslant \frac{x}{y}$ if $x \geqslant y > 1$ shows that the right-hand side of (5.31) is largest when $c = 3$. We can then replace $d_L(H)$ by 0, since it has negative coefficient on the right-hand side of (5.31). Hence

$$ka \leqslant kn - \epsilon + \tfrac{1}{3}(n - a - k + 2),$$

and (5.15) with $\beta = k$ implies

$$\delta(G) \leqslant k + a \leqslant k + n - \frac{1}{3k+1}(3\epsilon + k - 2). \tag{5.32}$$

Writing $\delta(G) = n - d$, (5.32) becomes $3\epsilon + k - 2 \leqslant (k + d)(3k + 1)$, and so

$$\epsilon \leqslant k^2 + dk + \tfrac{1}{3}(d + 2).$$

On the other hand, the assumption in (b) that $\delta(G) > n + 2k - 2\sqrt{\epsilon}$ becomes

$$\epsilon > \tfrac{1}{4}(d + 2k)^2 = k^2 + dk + \tfrac{1}{4}d^2.$$

Since $d$, $k$ and $\epsilon$ are integers, comparing these bounds for $\epsilon$ gives $\lfloor \frac{1}{3}(d+2) \rfloor > \frac{1}{4}d^2$, a contradiction if $d \geqslant 2$. We may assume that $d \neq 1$, since a complete graph (with $\delta(G) = n - 1$) clearly has a $k$-factor containing $L$ when $kn$ is even. Thus Case 3 is impossible.

The contradictions obtained in Cases 1–3 show:

**Claim 5.6.** *If $B \neq \emptyset$ then $\beta \geqslant k + 1$.* $\square$

Before tackling the remaining two cases in the proof of Theorem 5.2, we establish four more claims, leading to an upper bound on $\omega$. Our next claim gives a lower bound for $d_L(H) = \sum_{v \in V(H)} d_L(v)$. For compactness of notation, write $\delta_H := \delta(H)$. If $v \in V(H)$ then $d_H(v) \leqslant d_{H-L}(v) + d_L(v)$. Since $d_L(v) \leqslant k$ for all $v \in V(G)$ by (L1), and $c$ is the order of the smallest component of $H - L$, it follows that

$$\delta_H + 1 \leqslant c + k. \tag{5.33}$$

**Claim 5.7.** $d_L(H) \geqslant c[\omega(\delta_H + 1) - h]$.

**Proof.** Choose an ordering $C_1, C_2, \ldots, C_\omega$ of the components of $H - L$ so that $c = c_1 \leqslant c_2 \leqslant \ldots \leqslant c_\omega$, where $c_i = |V(C_i)|$. Write $\omega_j := |\{i : c_i = j\}|$. If $v \in V(C_i)$ then $d_L(v) \geqslant \delta_H + 1 - c_i$. Hence

$$\sum_{v \in V(C_i)} d_L(v) \geqslant c_i(\delta_H + 1 - c_i) \ \text{ for each } i \ (1 \leqslant i \leqslant \omega).$$

It follows that

$$d_L(H) = \sum_{i=1}^{\omega} \sum_{v \in V(C_i)} d_L(v) \geqslant \sum_{i=1}^{\omega} g(c_i), \tag{5.34}$$

where the function $g$ is defined by

$$g(x) := \begin{cases} x(\delta_H + 1 - x) & \text{if } c \leqslant x \leqslant \delta_H + 1, \\ 0 & \text{if } x \geqslant \delta_H + 1. \end{cases}$$

Suppose that $c < c_p \leqslant c_q < \delta_H + 1$ for some $p, q$ such that $1 \leqslant p < q \leqslant \omega$. Since $g(x)$ is a concave function for $c \leqslant x \leqslant \delta_H + 1$, we may replace the values of $c_p$ and $c_q$ by $c_p - 1$ and $c_q + 1$, so that (5.34) still holds. Continuing in this way we may assume that there is at most one $j$ with $c < c_j < \delta_H + 1$. We can replace such a $c_j$ by two fractional parts whose sum is $c_j$ (dropping the requirement that they are integers), namely

$$\frac{\delta_H + 1 - c_j}{\delta_H + 1 - c}(c) \quad \text{and} \quad \frac{c_j - c}{\delta_H + 1 - c}(\delta_H + 1).$$

Then (5.34) still holds by the concavity of $g(x)$, and $\sum_{i=1}^{\omega} c_i = h$ still holds at this point. Finally, if $c_j > \delta_H + 1$ then $g(c_j) = 0$, so we may replace any such $c_j$ by $\delta_H + 1$, thereby reducing $\sum_{i=1}^{\omega} c_i$ and leaving (5.34) unchanged. Then

$$\omega_c + \omega_{\delta_H+1} = \omega \text{ and } c\omega_c + (\delta_H + 1)\omega_{\delta_H+1} \leqslant h. \tag{5.35}$$

It follows from (5.34) and (5.35) that

$$\begin{aligned} d_L(H) \geqslant \omega_c\, c(\delta_H + 1 - c) &= c(\delta_H + 1)(\omega - \omega_{\delta_H+1}) - c^2\omega_c \\ &\geqslant c(\delta_H + 1)\omega - ch, \end{aligned}$$

as required. □

By Claim 5.6, if $B \neq \emptyset$ then $\sum_{v \in B} d_{G-A}(v) \geqslant \beta b \geqslant (k+1)b$. Hence (5.14) implies (trivially if $B = \emptyset$) that

$$\sum_{v \in A} [k - d_L(v)] + b \leqslant \omega - 2. \tag{5.36}$$

Since $\sum_{v \in B} [k - d_L(v)] \leqslant kb$ and $k > 1$, and then by (5.36), we obtain

$$\begin{aligned} \sum_{v \in A \cup B} [k - d_L(v)] &\leqslant \sum_{v \in A} [k - d_L(v)] + kb \\ &\leqslant k \left( \sum_{v \in A} [k - d_L(v)] + b \right) \leqslant k(\omega - 2). \end{aligned} \tag{5.37}$$

This is quite small, because $\omega < (2k-1)/3$ by (5.43) with $c \geqslant 3$. Since

$$\sum_{v \in V(G)} [k - d_L(v)] = kn - 2l,$$

and $V(G) = V(H) \cup A \cup B$ (disjoint union), (5.37) gives the lower bound

$$kh - d_L(H) = \sum_{v \in V(H)} [k - d_L(v)] \geqslant kn - 2l - k(\omega - 2). \tag{5.38}$$

Substituting for $d_L(H)$ using Claim 5.7 gives

$$kh \geqslant c[\omega(\delta_H + 1) - h] + kn - 2l - k(\omega - 2). \tag{5.39}$$

**Claim 5.8.** (i) $n - \delta(G) \geqslant h - \delta_H$.

(ii) *If* (5.2) *holds then*

$$(2k-1)\delta_H \geqslant (2k-1)h + 2(k-1) - kn + 2l, \tag{5.40}$$

*and*
$$(2k - 1 - c\omega)(\delta_H + 1) \geqslant (k - c - 1)h + 6k - 3 - k\omega. \tag{5.41}$$

**Proof.**(i) This is true since the maximum degree in the complement of $G$, i.e., $\Delta(\overline{G}) = n - \delta(G) - 1$, is greater than or equal to $\Delta(\overline{H}) = h - \delta_H - 1$.

(ii) By Claim 5.8(i) and (5.2),

$$n \geqslant \frac{1}{2k-1}((k-1)(n+2)+2l) + h - \delta_H. \tag{5.42}$$

Multiplying through by $(2k-1)$ and rearranging yields

$$(2k-1)\delta_H \geqslant (2k-1)h + 2(k-1) - kn + 2l,$$

which is the same as (5.40). Now replacing $kh$ using (5.39) gives

$$(2k-1)\delta_H \geqslant (k-c-1)h + c\omega(\delta_H+1) + 4k - 2 - k\omega,$$

so that (5.41) holds as claimed. □

**Claim 5.9.** *The following two inequalities cannot hold simultaneously:*

$$2k-1 > c\omega \quad \textit{and} \quad \omega \geqslant 3. \tag{5.43}$$

**Proof.** Suppose that (5.43) holds. Then $2k > 3c+1 = 2(c+1)+c-1$, and so $c \geqslant 3$ implies

$$k \geqslant 6, \tag{5.44}$$

and

$$k - c - 1 > 0. \tag{5.45}$$

First suppose that either (a) or (c) holds, so that (5.2) holds by Claim 5.4(i), and $2l \leqslant kn - 4k^2 + 6k$. Then (5.41) holds by Claim 5.8(ii). By supposition, $2k - 1 - c\omega > 0$ by (5.43), and so (5.33) and (5.41) imply

$$(2k - 1 - c\omega)(c + k) \geqslant (k - c - 1)h + 6k - 3 - k\omega.$$

Since $2l \leqslant kn - 4k^2 + 6k$ by assumption in (a) or (c), and $d_L(H) \geqslant 0$, (5.38) implies that $h \geqslant 4k - 4 - \omega$. Since $k - c - 1 > 0$ by (5.45), it follows that

$$(2k - 1 - c\omega)(c + k) \geqslant (k - c - 1)(4k - 4 - \omega) + 6k - 3 - k\omega,$$

which rearranges to

$$0 \geqslant 2k^2 - 6ck - k + 5c + 1 + \omega(ck - 2k + c^2 + c + 1).$$

Since $w \geqslant 3$ by (5.43), and the coefficient of $\omega$ is positive since $c \geqslant 3$,

$$\begin{aligned} 0 &\geqslant 2k^2 - 6ck - k + 5c + 1 + 3(ck + c^2 + c + 1 - 2k) \\ &= 2k^2 - 3ck - 7k + 3c^2 + 8c + 4 \\ &\geqslant 2k^2 + 3c(c - k) - 7k + 28. \end{aligned}$$

Regarding $3c(c-k)$ as a quadratic in $c$, we see that the minimum occurs at $c = k/2$; hence

$$0 \geqslant 2k^2 - \tfrac{3}{4}k^2 - 7k + 28,$$

and so $0 \geqslant 5k^2 - 28k + 112$. This is clearly impossible, since $k \geqslant 6$ by (5.44). This proves Claim 5.9 if either (a) or (c) holds.

Now suppose (b) holds, so that $\delta(G) > n + 2k - 2\sqrt{kn - 2l + 2}$. Rearranging squaring gives

$$4(kn - 2l + 2) > (n - \delta(G) + 2k)^2;$$

thus Claim 5.8(i) implies

$$4(kn - 2l + 2) > (h - \delta_H + 2k)^2. \tag{5.46}$$

On the other hand, rearranging (5.39) yields

$$kn - 2l + 2 \leqslant kh - c\omega(\delta_H + 1) + ch + k(\omega - 2) + 2.$$

Substituting this into (5.46) gives

$$4(kh - c\omega(\delta_H + 1) + ch + k(\omega - 2) + 2) > (h - \delta_H + 2k)^2,$$

which implies

$$0 > (h - \delta_H)^2 + 4[k^2 + 2k - 2 - k\delta_H + c\omega(\delta_H + 1) - ch - k\omega]. \tag{5.47}$$

Consider the terms involving $c$ and $\omega$. Recall that $c \geqslant 3$, $c \geqslant \delta_H + 1 - k$ by (5.33), and $\omega \geqslant 3$ and certainly $c\omega \leqslant 2k$ by (5.43). We claim that the minimum of $c\omega(\delta_H + 1) - ch - k\omega$, subject to the inequalities of the previous sentence, must occur in one of the following three cases:

(i) $c = \omega = 3$,

(ii) $\omega = 3$ and $3 < c = \delta_H + 1 - k$,

(iii) $c\omega = 2k$.

If $c(\delta_H + 1) \leqslant k$ (or $\omega(\delta_H + 1) \leqslant h$) then the coefficient of $\omega$ (respectively $c$) in (5.47) must be non-positive, so the minimum occurs when $\omega$ (or $c$) is maximised, and (iii) holds. Otherwise we must have $c(\delta_H + 1) > k$ and $\omega(\delta_H + 1) > h$, and so the minimum occurs when $c$ and $\omega$ are as small as possible, and either (i) or (ii) holds.

Fix $c$ and $\omega$ so that the minimum occurs (i.e., either (i), (ii) or (iii) holds). Differentiating the RHS of (5.47) with respect to $h$ shows that the minimum occurs when $h = 2c + \delta_H$ ; thus (5.47) implies, after cancelling a factor of 4,

$$0 > k^2 + 2k - 2 - k\delta_H + c\omega(\delta_H + 1) - c^2 - c\delta_H - k\omega. \tag{5.48}$$

We now consider each of the three possibilities.

(i) $w = c = 3$. Then (5.48) becomes

$$\begin{aligned} 0 &> k^2 + 2k - 2 - k\delta_H + 9(\delta_H + 1) - 9 - 3\delta_H - 3k \\ &= k^2 - k - 2 + \delta_H(6 - k). \end{aligned}$$

Since $\delta_H \leqslant k + 2$ by (5.33), and $k \geqslant 6$ by (5.44), it follows that

$$0 > k^2 - k - 2 + (k + 2)(6 - k) = 3k + 10 > 0,$$

a contradiction.

(ii) $\omega = 3$ and $3 < c = \delta_H + 1 - k$. Substituting into (5.48) gives

$$\begin{aligned} 0 &> k^2 - k - 2 - k\delta_H + 2\delta_H(\delta_H + 1 - k) + 3(\delta_H + 1 - k) - (\delta_H + 1 - k)^2 \\ &= k^2 - k - 2 - k\delta_H + (\delta_H + 2 + k)(\delta_H + 1 - k) \\ &= \delta_H(\delta_H + 1 - k) + 2(\delta_H - k) > 0, \end{aligned}$$

another contradiction.

(iii) $c\omega = 2k$. First suppose that $c = 3$. Then (5.48) gives

$$\begin{aligned} 0 &> k^2 + 4k - 11 + k\delta_H - 3\delta_H - k\omega \\ &= k(k - \omega) + \delta_H(k - 3) + 4k - 11, \end{aligned}$$

a contradiction since $k > \omega$, and $k \geqslant 6$ by (5.44). Thus we may assume that $c \geqslant 4$. Note that $2k \geqslant 3c$ (since $\omega \geqslant 3$) and so $k > c$ and $k^2 > 2c^2$. Hence (5.48) implies

$$\begin{aligned} 0 &> k^2 + 4k - 2 - k\omega - c^2 + \delta_H(k-c) \\ &\geqslant \tfrac{1}{2}k^2 + 4k - 1 - k\omega \\ &= 4k - 1 + k^2(\tfrac{1}{2} - \tfrac{2}{c}) \\ &\geqslant 4k - 1 > 0, \end{aligned}$$

since $c \geqslant 4$. This contradiction shows that (5.43) is impossible when (b) holds; this completes the proof of Claim 5.9. □

**Claim 5.10.** *$\omega \leqslant 2$.*

**Proof.** We prove this claim for (a), (b) and (c) together, since (5.2) holds by Claim 5.4(i) and (iii). Suppose $\omega \geqslant 3$, so that $cw \geqslant 2k - 1$ by Claim 5.9. First suppose that

$$\delta_H + 1 \leqslant \frac{h}{\omega}. \tag{5.49}$$

Since (5.2) holds, Claim 5.8(ii) implies that (5.40) holds, and we can rewrite this as

$$0 \geqslant (2k-1)h - (2k-1)\delta_H + 2l - kn + 2k - 2. \tag{5.50}$$

Relacing $kh$ using (5.38) and $d_L(H) \geqslant 0$ gives

$$\begin{aligned} 0 &\geqslant (k-1)h - (2k-1)\delta_H + 2k - 2 - k(\omega - 2) \\ &= (k-1)h - (2k-1)(\delta_H + 1) + 6k - 3 - k\omega, \end{aligned}$$

and so by (5.49) we obtain

$$0 \geqslant (k\omega - \omega - 2k + 1)\frac{h}{\omega} + 6k - 3 - k\omega. \tag{5.51}$$

Note that $k\omega - \omega - 2k + 1 = (k-1)(\omega - 2) - 1 \geqslant 0$, since $k \geqslant 2$ and $\omega \geqslant 3$. Thus (5.51) and $h \geqslant c\omega$ imply

$$\begin{aligned} 0 &\geqslant c(k\omega - \omega - 2k + 1) + 6k - 3 - k\omega \\ &= (2k-1)(3-c) + \omega(ck - c - k). \end{aligned}$$

Now $ck - c - k = (c-1)(k-1) - 1 > 0$, since $c \geqslant 3$ and $k \geqslant 2$. Hence $\omega \geqslant 3$ gives

$$\begin{aligned} 0 &\geqslant (2k-1)(3-c) + 3(ck - c - k) \\ &= 3(k-1) + c(k-2) > 0. \end{aligned}$$

This contradiction shows that (5.49) cannot hold here, so we may assume

$$\delta_H + 1 > \frac{h}{\omega}. \tag{5.52}$$

This time we note that (5.2) and Claim 5.8(ii) imply (5.41), which rearranges to

$$0 \geqslant (k - c - 1)h + [c\omega - 2k + 1](\delta_H + 1) + 6k - 3 - k\omega.$$

Recall that $cw \geqslant 2k - 1$ by Claim 5.9, since we are assuming $\omega \geqslant 3$. Thus (5.52) implies

$$\begin{aligned} 0 &\geqslant (k - c - 1)h + [c\omega - 2k + 1]\frac{h}{\omega} + 6k - 3 - k\omega \\ &= (k\omega - \omega - 2k + 1)\frac{h}{\omega} + 6k - 3 - k\omega, \end{aligned}$$

which is the same as (5.51), and leads to a contradiction as before. This completes the proof of Claim 5.10. □

Together with (5.14), Claims 5.6 and 5.10 imply

$$ka + b + 2 - \sum_{v \in A} d_L(v) \leqslant \omega(A, B; f) \leqslant \omega \leqslant 2. \tag{5.53}$$

We can now deal with the remaining two cases in the proof of Theorem 5.2.

*Case* $4 : B \neq \emptyset$ *and* $\beta \geqslant k+1$. Since $b \geqslant 1$, (5.53) implies that

$$\sum_{v \in A} d_L(v) \geqslant ka + 1.$$

Hence some vertex $v \in A$ must have $d_L(v) > k$, contradicting (L1), the assumption that $L$ is weakly $k$-good.

*Case* $5 : B = \emptyset$. Since $b = 0$, if $\omega(A, \emptyset; f) \leqslant 1$ then we obtain a contradiction from (5.53) as in Case 4. Hence $\omega(A, \emptyset; f) = \omega = 2$ and $\sum_{v \in A} d_L(v) \geqslant ka$. By (L1) it follows that $\sum_{v \in A} d_L(v) = ka$ and so

$$A \subseteq Z = \{v \in V(G) : d_L(v) = k\}.$$

Hence any component of $H - L = G - L - A$ is the union of some components of $G - L - Z$, together with vertices in $Z - A$. Since $L$ is strongly $k$-good by Claim 5.5, it follows by (L2) that $\sum_{v \in V(C)} (k - d_L(v))$ is even for every component $C$ of $H - L$. This implies that $\omega(A, \emptyset; f)$, the number of components of $H - L$ such that $\sum_{v \in V(C)} (k - d_L(v))$ is odd, is zero; but we have just shown that $\omega(A, \emptyset; f) = 2$. This contradiction completes the proof of Theorem 5.2. $\square$

## 5.4 Sharpness of Theorem 5.2

The examples are adapted from those given in [14] and [47], and take the general form $G = K_a + H$. (Recall that the *join* $G + H$ of two graphs $G$ and $H$ is the graph obtained from their disjoint union by adding an edge between every vertex in $G$ and every vertex in $H$.) Note that if $ka \geqslant 2l$ and $a \geqslant k+1$ then it is possible to fit a weakly $k$-good set $L$ of $l$ edges inside $K_a$. Also, if $ka > 2l$ and $kn$ is even, then such a set $L$ is strongly $k$-good, since then $G - L - Z$ is connected, and so (L2) holds (see (5.7)).

If $G$ has a $k$-factor $K$ containing $L$, let $e_K(K_a : H)$ denote the number of edges of $K$ between the $K_a$ and $H$. Then $e_K(K_a : H) \geqslant \sum_{v \in V(H)}(k - d_H(v))$, and $e_K(K_a : H) \leqslant ka - 2l$. Thus if $G$ has a $k$-factor containing $L$ then

$$ka - 2l - \sum_{v \in V(H)} (k - d_H(v)) \geqslant 0. \tag{5.54}$$

First we show that the bound $\delta(G) \geqslant \frac{1}{2}n + l/k$ in (a) is sharp when $k$ divides $2l$. Let $m > 2k - 4$, and consider the graph $G_1 := K_{m+2l/k} + (m+2)K_1$, which has order $n = 2(m + l/k + 1)$. Then $kn$ is even,

$$2l = kn - 2k(m+1) < kn - 4k^2 + 6k$$

and

$$\delta(G_1) = m + 2l/k = \tfrac{1}{2}n + l/k - 1.$$

But if $L$ is a weakly $k$-good set of $l$ edges inside the $K_{m+2l/k}$, then since

$$k(m + 2l/k) - 2l - k(m+2) < 0,$$

it follows by (5.54) that $G_1$ has no $k$-factor containing $L$.

Our next two examples show that the bound $2l < kn - 4k^2 + 6k$ in (a) is sharp when $k$ divides $2l$. Let

$$G_2 = K_{2k-5+2l/k} + C_{2k-1} \quad \text{and} \quad G_3 = K_{2k-4+2l/k} + (k-1)K_2.$$

Then both graphs have order $n = 4k - 6 + 2l/k$, so that $2l = kn - 4k^2 + 6k$ and $kn$ is even. Also,

$$\delta(G_2) = \delta(G_3) = 2k - 3 + 2l/k = \tfrac{1}{2}n + l/k.$$

But if $L$ is a weakly $k$-good set of $l$ edges inside the $K_{2k-t+2l/k}$ ($t \in \{4, 5\}$), then since

$$k(2k - 5 + 2l/k) - 2l - (k - 2)(2k - 1) = -2$$

and

$$k(2k - 4 + 2l/k) - 2l - 2(k - 1)(k - 1) = -2,$$

it follows by (5.54) that neither $G_2$ nor $G_3$ has a $k$-factor containing $L$.

We now examine the sharpness of the bound $\delta(G) > n + 2k - 2\sqrt{n - 2l + 2}$ in (b). Recall by Claim 5.4(ii) that if (b) holds then $\delta(G) \geqslant \frac{1}{2}n + l/k$; we have already seen that this is sharp when $2l < kn - 4k^2 + 6k$. Given $k$, $l$, and $n$ such that $kn$ is even and

$$kn - 4k^2 + 6k \leqslant 2l \leqslant kn - k^2 - k,$$

let $b := 2\lceil \frac{1}{2}\sqrt{(kn - 2l + 2)}\rceil$. Then $b \leqslant n$ (since $k \leqslant n - 1$),

$$k < \sqrt{k^2 + k + 2} \leqslant \sqrt{kn - 2l + 2} \leqslant b, \tag{5.55}$$

and

$$b \leqslant 2\lceil \tfrac{1}{2}\sqrt{(4k^2 - 6k + 2)}\rceil \leqslant 2\lceil \tfrac{1}{2}\sqrt{(2k - \tfrac{3}{2})^2}\rceil = 2\lceil k - \tfrac{3}{4}\rceil = 2k. \tag{5.56}$$

Thus we can define the graph $G_4 := K_a + C_b^t$, where $a := n - b$ and $t := k - \frac{1}{2}b$, so that $0 \leqslant t < \frac{1}{2}k < \frac{1}{2}b$ by (5.55) and (5.56). Then

$$\delta(G_4) = a + 2t = n + 2k - 2b = n + 2k - 4\lceil \tfrac{1}{2}\sqrt{(kn - 2l + 2)}\rceil.$$

But if $L$ is a set of $l$ edges inside the $K_a$, then $G_4$ has no $k$-factor containing $L$ by (5.54) since

$$ka - 2l - b(k - 2t) = k(n - b) - 2l - b(b - k) = kn - 2l - b^2 \leqslant -2.$$

Thus, provided it is possible to fit a strongly $k$-good set $L$ of $l$-edges inside the $K_a$ (which is the case whenever $a \geqslant k + 1$ and $a > 2l$, since $kn$ is even), the bound $\delta(G) > n + 2k - 2\sqrt{kn - 2l + 2}$ is roughly best possible, and it is exact whenever $kn - 2l + 2$ is an even square. For example, it is best possible for an infinite number of triples $(k, l, n)$ where $k$ is odd, $n$ is even, $kn - 2l + 2 = (k + 1)^2$ and $2k + 2 \leqslant n$; then $b = k + 1$, so $a \geqslant k + 1$, and $ka = kn - k(k + 1) > 2l$, so we can fit a strongly $k$-good set $L$ of $l$ edges inside the $K_a$.

Our last example demonstrates the sharpness of (5.2) and (5.3). Given $k$, $l$ and $n$ such that $n$ is even and $2l \leqslant kn - 4k^2 + 6k$, let $r$ be an integer such that $2(2k - 1)$ divides $n - 2r - 4l + 4$, and $2kr < kn - 2l + 2$. [Such an $r$ almost always exists: let $r$ satisfy $0 \leqslant r \leqslant 2k - 2$ and $\frac{1}{2}n - r - 2l + 2 \equiv 0 \pmod{2k - 1}$; then either

$$2kr \leqslant 2k(2k - 3) < 4k^2 - 6k + 2 \leqslant kn - 2l + 2,$$

or $r = 2k - 2$, in which case we need to suppose that $2l < kn - 4k^2 + 4k + 2$]. Define the integer

$$s := \frac{1}{2(2k - 1)}(n - 2r - 4l + 4).$$

Now define the graph

$$G_5 := K_a + (rK_1 \cup bK_2),$$

where

$$a := 2(k - 1)s + r + 2l - 2 = \frac{1}{2k - 1}((k - 1)n + r + 2l - 2), \tag{5.57}$$

and

$$b := ks + l - 1 = \frac{1}{2(2k-1)}(kn - 2kr - 2l + 2).$$

Then $G_5$ has order

$$n = a + 2b + r = 2(2k-1)s + 2r + 4l - 4.$$

Note that $b > 0$ since $2kr < kn - 2l + 2$. Also, (5.57) implies

$$\begin{aligned} ka(2k-1) &= (k-1)kn + kr + 2kl - 2k \\ &> (k-1)(2kr + 2l - 2) + kr + 2kl - 2k \\ &= (2k-1)(kr + 2l - 2). \end{aligned}$$

Since $a - r$ is even, it follows that $ka \geqslant kr + 2l$. If $r = 0$ and $2l \geqslant k(k+1)$, or if $r \geqslant 2k$, then $a \geqslant k+1$ and $ka \geqslant 2l$; then it is possible to have a weakly $k$-good set $L$ of $l$ edges inside the $K_a$. Let $L$ be such a set. In fact, $L$ must be strongly $k$-good: this is true if $ka > 2l$ since $kn$ is even; and if $r = 0$ and $ka = 2l$, then every component $C$ of $G - L - Z = bK_2$ has $\sum_{v \in V(C)}(k - d_L(v)) = 2k$ which is even, so (L2) holds. Also, since

$$\begin{aligned} &ka - 2l - kr + 2(k-1)b \\ &= k(2(k-1)s + r + 2l - 2) - 2l - 2(k-1)(ks + l - 1) = -2, \end{aligned}$$

it follows by (5.54) that $G_5$ does not have a $k$-factor containing $L$.

If $r = 0$ then (5.57) gives

$$\delta(G_5) = a + 1 = \frac{1}{2k-1}((k-1)(n+2) + 2l - 1);$$

thus (5.2) is sharp when $k(k+1) \leqslant 2l \leqslant kn - 4k^2 + 6k$ and $2(2k-1)$ divides

$n - 4l + 4$.

If $r \geqslant 2k$ and $X := V(rK_1)$ then

$$|N_{G_5}(X)| = a = \frac{1}{2k-1}((k-1)n + r + 2l - 2), \tag{5.58}$$

so that (5.3) just fails to hold. Also, $G_5$ satisfies (5.2), since $\delta(G_5) = a$ and $r \geqslant 2k$. Moreover, any other set $Y \in \mathcal{I}(G_5)$ has $|N(Y)| \geqslant |N(X)|$ and satisfies (5.3): if $|Y| < |X|$ then this follows directly from (5.58), while if $|Y| \geqslant |X|$ and $Y$ includes any vertices from outside $X$, then clearly $|N(Y)| - |N(X)|$ exceeds $\frac{1}{2k-1}(|Y| - |X|)$. Thus (5.3) is sharp.

# Chapter 6

# Circumference

## 6.1 Introduction

The *circumference* $c(G)$ of a graph $G$ is the length of a longest cycle in $G$. In this chapter we give neighbourhood conditions which suffice to ensure that $c(G) \geqslant k$. We generalise Woodall's result on Hamiltonicity [46, 47] (see Section 1.2) with Theorem 6.1, which is proved in Section 6.3 and shown to be sharp in Section 6.4. A graph is $t$*-connected* if it is connected and there does not exist a set of $t-1$ vertices whose removal disconnects the graph. Recall that $\mathcal{I}(G)$ denotes the family of nonempty independent subsets of $V(G)$.

**Theorem 6.1.** *Let $G$ be a graph of order $n$ and let $k$ be an integer, with $3 \leqslant k \leqslant n$. If $k \geqslant 5$ suppose that $G$ is 2-connected, and if $k = 9$ or $11 \leqslant k \leqslant n-2$, suppose further that $G$ is 3-connected. Suppose that $G$ has minimum degree*

$$\delta(G) \geqslant \tfrac{1}{3}k + 1, \tag{6.1}$$

*and if $k = 9$ or $k \geqslant 11$ suppose that*

$$|N(X)| \geqslant \tfrac{1}{3}(|X| + 2k - n - 1) \quad \textit{whenever } X \in \mathcal{I}(G). \tag{6.2}$$

*Then $c(G) \geqslant k$.*

In Section 6.2 we consider some lemmas to be used in the proof of Theorem 6.1. In particular, using a result of Min Aung [3], we prove Lemma 6.6, a generalisation of Woodall's Hopping Lemma [45]. Examples to demonstrate the sharpness of Lemma 6.6 are given in Section 6.4.

If $H$ is a subgraph of $G$, write $e(H)$ for $|E(H)|$. If $C : a_1 a_2 \dots a_m a_1$ is a cycle in $G$, where the suffices of the $a_i$ are reduced modulo $m$, write $C[a_i, a_j]$ for the path $a_i a_{i+1} \dots a_j$.

## 6.2 Preliminary Lemmas

The proof of Lemma 4 of [33] implies the following lemma, which we will use in our proof of Theorem 6.1 in the case when $k$ is close to $n$.

**Lemma 6.2.** (Nash-Williams [33]) *Let $G$ be a 2-connected graph of order $n$, with $\delta(G) \geqslant \frac{1}{3}(n+2)$. If $C$ is a cycle of maximum length in $G$ then $e(G-C) = 0$.* $\square$

If $H$ is a subgraph of $G$, the *average degree of $H$ in $G$* is

$$\frac{1}{|V(H)|} \sum_{z \in V(H)} d_G(z).$$

Now let $C$ be a cycle and let $H$ be a component of $G-C$. We say that $C$ is *locally longest with respect to $H$ in $G$* if we cannot obtain a cycle longer longer than $C$ by replacing a segment $C[u, v]$ by a $(u, v)$-path all of whose interior vertices lie in $H$. If $x \in V(G-C)$, an *$(x, C)$-path* is a path connecting $x$ to some vertex $v \in C$ such that $v$ is the only vertex of $C$ on the path. Two $(x, C)$-paths are said to be *disjoint* if they have only the vertex $x$ in common. A component $H$ of $G-C$ is *locally 3-connected* to $C$ in $G$ if for every vertex $x \in V(H)$, there are three pairwise disjoint $(x, C)$-paths in $G$.

The following lemma is a special case of a theorem of Fan ([18], Theorem 2); Nagayama remedied a gap in the proof of this theorem in [32].

**Lemma 6.3.** (Fan [18]) *Let $C$ be a cycle of length $c$ in $G$, and let $H$ be a component of $G - C$ with $|V(H)| \geqslant 2$. Suppose that $C$ is locally longest with respect to $H$, and $H$ is locally 3-connected to $C$. Suppose further that the average degree of $H$ in $G$ is at least $r$. Then $c \geqslant 3(r-1)$.* □

The following corollary of Lemma 6.3 will be used in the proof of Theorem 6.1.

**Corollary 6.4.** *Let $G$ be a 3-connected graph with minimum degree $\delta(G) \geqslant \frac{1}{3}k+1$. If there exists a cycle $C$ of maximal length in $G$ such that $e(G-C) > 0$, then $c(G) \geqslant k$.*

**Proof.** Choose a component $H$ of $G - C$ such that $|V(H)| \geqslant 2$. Clearly the average degree of $H$ in $G$ is at least as large as $\delta(G)$; $C$ is locally longest with respect to $H$ since it has maximal length in $G$; and $H$ is locally 3-connected to $C$, since $G$ is 3-connected. Hence we can apply Lemma 6.3 to obtain

$$c(G) \geqslant 3(r-1) \geqslant 3(\delta(G)-1) \geqslant k. \qquad \square$$

Next we state a lemma of Woodall [45].

**Lemma 6.5.** (Woodall [45]) *Let $C$ be a cycle of length $m$. Let $X$ be a set of vertices of $C$ that contains no two consecutive vertices of $C$. Let $Y$ be the set of vertices of $C$ whose two neighbours round $C$ are both in $X$. Then $|Y| \geqslant 3|X|-m$.*
□

The remainder of this section concerns the proof of Lemma 6.6, a generalisation of Woodall's Hopping Lemma [45]. The application of Lemmas 6.5 and 6.6 (in the same fashion as the original Hopping Lemma was used in [46]) will be a key step in the proof of Theorem 6.1.

**Lemma 6.6.** (Adapted Hopping Lemma) *Let $G$ be a graph of order $n$, and let $C : a_1a_2 \dots a_ma_1$ be a cycle of maximum length in $G$. Suppose that $e(G-C) = 0$ for every cycle of length $m$ in $G$. If $n \geqslant m+2$, suppose that $\delta(G) \geqslant \frac{1}{4}m+1$. Let $Y_0 := \emptyset$ and, for $j \geqslant 1$, define*

$$X_j := N(Y_{j-1} \cup V(G-C)),$$

*and*

$$Y_j := \{a_i \in C : a_{i-1} \in X_j \text{ and } a_{i+1} \in X_j\}.$$

*(So $N(V(G-C)) = X_1 \subseteq X_2 \subseteq \dots$ and $\emptyset = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$.) Then, for all $j \geqslant 1$, $X_j \subseteq C$ and $X_j$ does not contain two consecutive vertices of $C$. (Hence the same is true for $Y_j$, and $X_j \cap Y_j = \emptyset$.) Also, $Y_j$ is an independent set for each $j$.*

[We first proved a weaker version of Lemma 6.6, with minimum degree condition $\delta(G) \geqslant \frac{1}{3}(m+2)$, by showing that the the vertices in $C$ cannot be arranged to form any path $P : b_1b_2 \dots b_m$ with both end vertices in $X_1 = V(G-C)$. This weaker lemma was enough for the purposes of proving Theorem 6.1.

While trying to obtain a sharp bound on $\delta(G)$ for the Lemma, we found examples showing that $\delta(G) \geqslant \frac{1}{4}m+1$ is best possible (see Section 6.4), but we had not quite proved that this bound on $\delta(G)$ is sufficient. We then discovered that, under the same hypotheses, a similar hopping lemma had already been proved by Min Aung [3], using Claim 6.7 below, and the observation that it is enough to consider just the good paths. Say that an edge of $C$ is *bad* if it is incident with two vertices in $N(v)$ for some $v \in V(G) \setminus V(C)$; a path $P$ is *good* if $V(P) = V(C)$ and it contains no bad edge.]

**Proof of Lemma 6.6.** The result is obvious when $G - C = \emptyset$, so suppose that $n \geqslant m+1$. We follow the proof of the original Hopping Lemma [45]. Since $e(G-C) = 0$, $X_1 \subseteq C$ by definition. To prove the rest of the Lemma, we establish that the following statement $A(j)$ holds for $j \geqslant 1$.

$A(j)$. There is no good path $P_j : b_1 b_2 \dots b_m$ with both end vertices in $X_j$ such that, if $b_i \in Y_{j'}$ for some $j' \leqslant j-1$, and $b_i \notin \{b_1, b_m\}$, then $b_{i-1} \in X_{j'}$ and $b_{i+1} \in X_{j'}$.

[It follows from $A(j)$ that $X_j$ does not contain two consecutive vertices of $C$. For, if $a_i$ and $a_{i+1} \in X_j$, then the path $C[a_{i+1}, a_i]$ is good, since the length of $C$ is maximal in $G$, and it contradicts $A(j)$. Hence $Y_j$ does not contain two consecutive vertices of $C$ and $X_j \cap Y_j = \emptyset$.

Clearly $N(Y_j) \subseteq C \cup V(G-C)$. But if $a \in N(Y_j)$ for some $a \in V(G-C)$ then $N(a) \cap Y_j \neq \emptyset$; whereas in fact $N(a) \cap Y_j \subseteq X_j \cap Y_j$, which we have just shown to be empty. Hence $N(Y_j) \subseteq C$ and so $X_{j+1} \subseteq C$.

It also follows from $A(j)$ that $Y_j$ is an independent set ($j = 1, 2, \dots$), since if $a_i \in Y_j \cap N(Y_j)$ then

$$a_i \in Y_j \cap X_{j+1} \subseteq Y_{j+1} \cap X_{j+1} = \emptyset,$$

a contradiction.]

We prove $A(j)$ by induction on $j$. If $n = m+1$, then $A(1)$ clearly holds: since if the path $P_1$ existed, with both end vertices in $X_1 = N(v)$, then we could add $v$ to $P_1$ to form a cycle of length $m+1$, contradicting the fact that $C$ has maximum length. If $n \geqslant m+2$ then $\delta(G) \geqslant \frac{1}{4}m + 1$, and $A(1)$ follows from the next claim.

**Claim 6.7.** (Min Aung, [3]) *If $G$ satisfies the hypotheses of Lemma* 6.6, *with $\delta(G) \geqslant \frac{1}{4}m + 1$, then there is no good path with both end vertices in $X_1$.* □

The inductive step follows [45], with the modification that paths are good. Let $j \geqslant 2$, and suppose that $P_j : b_1 b_2 \dots b_m$ is a good path as described in $A(j)$, with $b_1$ and $b_m \in X_j$. Then we must have one of the following situations:

(a) $b_1$ and $b_m$ are both in $X_{j-1}$: this contradicts $A(j-1)$ directly.

(b) $b_1$ is in $X_{j-1}$ but $b_m$ is not (or *vice versa*): then $b_m \in N(b_r)$ for some $b_r$ in $Y_{j-1} \backslash Y_{j-2}$. Then $b_r \notin N(G-C)$ since $X_{j-1} \cap Y_{j-1} = \emptyset$ by the induction hypothesis; hence the edge $b_r b_m$ is not bad. Also $b_r \neq b_1$ (again since $X_{j-1} \cap Y_{j-1} = \emptyset$) and clearly $b_r \neq b_m$. By the hypothesis of $A(j)$, $b_{r+1} \in X_{j-1}$, and so the good path

$$P : b_1 b_2 \dots b_r b_m b_{m-1} \dots b_{r+1}$$

connects two vertices in $X_{j-1}$. This contradicts $A(j-1)$ since the only vertices whose neighbours have altered from what they were in $P_j$ are $b_r$ , $b_{r+1}$ and $b_m$, none of which is in $Y_{j'}$ for any $j' \leqslant j-2$ (since $b_r \in Y_{j-1} \backslash Y_{j-2}$; $b_{r+1} \notin Y_{j-1}$, since $b_{r+1} \in X_{j-1}$ and $X_{j-1} \cap Y_{j-1} = \emptyset$; and $b_m \notin Y_{j-2}$, since, if it were, $b_r$ would be in $X_{j-1}$ (since $b_r$ and $b_m$ are adjacent), and $b_r \in Y_{j-1}$, not $X_{j-1}$).

(c) Neither $b_1$ or $b_m$ is in $X_{j-1}$, but $b_1 \in N(b_r)$ and $b_m \in N(b_s)$ where $b_r$ and $b_s \in Y_{j-1} \backslash Y_{j-2}$; hence $b_r, b_s \notin N(G-C)$ since $X_{j-1} \cap Y_{j-1} = \emptyset$. Note that when we dismissed case (b) we completed the proof of the non-existence of the path $P_j$ with $b_1$ in $X_{j-1}$. It follows that no two vertices in $Y_{j-1}$ are adjacent: for, if

$$a_i \in Y_{j-1} \cap N(Y_{j-1}) \subseteq Y_{j-1} \cap X_j,$$

then $a_{i+1} \in X_{j-1}$ by the definition of $Y_{j-1}$, and so $C[a_{i+1}, a_i]$ would be such a good path $P_j$. In particular, $b_r$ and $b_s$ are non-adjacent, thus $b_1 \neq b_s$ and $b_r \neq b_m$. So there are two possibilities to consider:

(c1) $1 < r \leqslant s < m$. In this case the path

$$b_{r-1} b_{r-2} \dots b_1 b_r b_{r+1} \dots b_s b_m b_{m-1} \dots b_{s+1}$$

connects two vertices in $X_{j-1}$. It is a good path since $b_r, b_s \notin N(G-C)$; and it contradicts $A(j-1)$ for the same reasons as in case (b).

(c2) $1 < s < r < m$. In this case the good path

$$b_{s-1}b_{s-2}\dots b_1 b_r b_{r-1}\dots b_s b_m b_{m-1}\dots b_{r+1}$$

contradicts $A(j-1)$ as before. This completes the proof of Lemma 6.6. $\square$

## 6.3 Proof of Theorem 6.1

There are two main cases to consider.

*Case* 1 : $3 \leqslant k \leqslant 8$ *or* $k = 10$. In this case, we will use the following three statements, proved by Dirac [13], about the circumference of a graph $G$ of order $n \geqslant 3$.

(D1) If $\delta(G) \geqslant t$ then $c(G) \geqslant t+1$.

(D2) If $\delta(G) \geqslant n/2$ then $G$ is Hamiltonian.

(D3) If $G$ is 2-connected and $\delta(G) \geqslant t$, where $n \geqslant 2t$, then $c(G) \geqslant 2t$.

If $k \in \{3, 4\}$ then $\delta(G) \geqslant \lceil \frac{1}{3}k+1 \rceil = k-1$ by (6.1); then Theorem 6.1 follows from (D1). If $k \in \{5, 6, 7, 8, 10\}$ then $G$ is 2-connected by hypothesis, and (6.1) implies $\delta(G) \geqslant \lceil \frac{1}{3}k+1 \rceil \geqslant k/2$; then Theorem 6.1 follows from (D3) if $n \geqslant 2\lceil \frac{1}{3}k+1 \rceil$, and from (D2) otherwise.

*Case* 2 : $k = 9$ *or* $11 \leqslant k \leqslant n$. If $k \leqslant n-2$ then $G$ is 3-connected by hypothesis. Then if there exists a cycle $C$ of maximal length in $G$ such that $e(G-C) > 0$, then $G$ satisfies the hypotheses of Corollary 6.4, and so $c(G) \geqslant k$. If $k \in \{n-1, n\}$ then $G$ satisfies the hypotheses of Lemma 6.2. Thus we may assume that $e(G-C) = 0$ for every cycle $C$ of maximal length in $G$.

Since $n \geqslant k \geqslant 9$ in Case 2, (6.1) and (D1) imply $c(G) \geqslant 5$. Let $C : a_1 a_2 \dots a_m a_1$ be a cycle of maximum length in $G$, so that $e(G-C) = 0$. Define $Y_0 = \emptyset$ and $X_j$

and $Y_j$ $(j = 1, 2, \dots)$ as in Lemma 6.6. Suppose that $m \leqslant k - 1$, so that

$$\delta(G) \geqslant \tfrac{1}{3}(k+1) \geqslant \tfrac{1}{3}(m+2) > \tfrac{1}{4}m + 1,$$

by (6.1) since $m \geqslant 5$. Then Lemma 6.6 implies that the vertices of $X_j$ are nonconsecutive in $C$. Hence Lemma 6.5 implies

$$|Y_j| \geqslant 3|X_j| - m \quad (j = 1, 2, \dots). \tag{6.3}$$

Also, each set $Y_j \cup V(G - C)$ is nonempty (since $m \leqslant k - 1 \leqslant n - 1$) and independent, by Lemma 6.6 and since $e(G - C) = 0$. Hence (6.2) implies

$$\begin{aligned} |X_j| = N(Y_{j-1} \cup V(G - C)) &\geqslant \tfrac{1}{3}((|Y_{j-1}| + n - m) + 2k - n - 1) \\ &= \tfrac{1}{3}(|Y_{j-1}| + 2k - m - 1) \quad (j = 1, 2, \dots). \end{aligned}$$

Substituting this into (6.3) yields

$$|Y_j| \geqslant |Y_{j-1}| + 2k - 2m - 1 \geqslant |Y_{j-1}| + 1. \quad (j = 1, 2, \dots).$$

By induction, it follows that $|Y_j| \geqslant j$ $(j = 0, 1, \dots)$. In particular, $|Y_{n+1}| \geqslant n + 1$, a contradiction since $G$ has order $n$. This completes the proof of Theorem 6.1. $\square$

## 6.4 Sharpness of Lemma 6.6 and Theorem 6.1

We first show that the minimum degree condition $\delta(G) \geqslant \frac{1}{4}m + 1$ in Lemma 6.6 is sharp when $n \geqslant m + 2$. Define the graph $G_1$ of order $n = m + 2 + t$ $(t \geqslant 0)$ as follows. If $m \in \{3, 4\}$, let $G_1$ consist of a cycle $a_1 \dots a_m a_1$, with a vertex $w$ adjacent to $a_2$, and vertices $v, v_1, \ldots, v_t$ each adjacent to $a_3$. If $m \geqslant 5$, let $\alpha := \lfloor \frac{1}{4}(m-1) \rfloor$ and $\gamma := m - 1 - 4\alpha$, so that $\gamma \in \{0, 1, 2, 3\}$. Then define

$$G_1 := \alpha K_1 + (H \cup (\alpha - 1)K_{1,2}),$$

where $H$ is the graph of order $6 + t + \gamma$ consisting of the path

$$a_1 a_2 a_3 a_4 \ldots a_{4+\gamma},$$

with a vertex $w$ adjacent to $a_2$, and vertices $v, v_1, \ldots, v_t$ each adjacent to $a_3$. The longest cycle in $G_1$ has length $2\alpha + 2(\alpha - 1) + 3 + \gamma = m$, and any cycle $C$ of length $m$ in $G_1$ must contain the $(\alpha - 1)$ paths $K_{1,2}$ and a path of length $3 + \gamma$ (with $4 + \gamma$ vertices) inside $H$, and so $e(G_1 - C) = 0$. Thus $G_1$ satisfies all of the hypotheses of Lemma 6.6 except that $\delta(G_1) = \alpha + 1 = \lfloor \frac{1}{4}(m+3) \rfloor$; but if $C$ is a cycle of length $m$ in $G_1$ then $X_1 = N_{G_1}(G_1 - C)$ contains two consecutive vertices of $C$, namely $a_2$ and $a_3$.

**Sharpness of Theorem 6.1.**

In the case $k = n$, the bound $\delta(G) \geqslant \frac{1}{3}(n+2)$ in place of (6.1) is sufficient and sharp, as shown by Woodall [46]. The graphs $aK_2$ and $bK_3$ $(a, b \geqslant 2)$ show that (6.1) is sharp when $k \in \{3, 4\}$. The bipartite graph $K_{r,s}$, with $r = \lceil k/3 \rceil$ and $s \geqslant r$ has circumference $2r$, and is $r$-connected; thus (6.1) is sharp when $5 \leqslant k \leqslant 8$ or $k = 10$. (We have not found a suitable example when $k = 9$.) To see that (6.1) is sharp whenever $11 \leqslant k \leqslant n-1$, define $p := \lceil k/3 \rceil - 1$ and consider the graph

$$G_2 := \begin{cases} pK_1 + \frac{1}{2}(n-p)K_2 & \text{if } n - \lceil k/3 \rceil \text{ is odd,} \\ pK_1 + (K_3 \cup \frac{1}{2}(n-p-3)K_2) & \text{if } n - \lceil k/3 \rceil \text{ is even.} \end{cases}$$

Since $n > k > \max\{2p+1, 3p\}$, we obtain $n - p > \max\{p+1, 2p\}$. It follows that $\delta(G_2) = p + 1 \leqslant \frac{1}{3}(k+2)$; and a longest cycle in $G_2$ must contain the $pK_1$.

Thus if $n - \lceil k/3 \rceil$ is odd then $c(G_2) = 3p < k$, and if $n - \lceil k/3 \rceil$ is even then $c(G_2) = 3p + 1$, which is less than $k$ unless $k \equiv 1 \pmod 3$. In this case, redefine $p := \lceil k/3 \rceil - 2$ so that $\delta(G_2) = p + 1 = \frac{1}{3}(k-1)$ and $c(G_2) = 3p + 1 = k - 3$. It remains to check that $G_2$ satisfies all of the other hypotheses of Theorem 6.1. Firstly, $G_2$ is 3-connected, since $p \geqslant 3$. Now let $X := V(pK_1)$, and let $Y \in \mathcal{I}(G_2)$ consist of $r$ vertices ($2 \leqslant r \leqslant \frac{1}{2}(n-p)$), each from a different copy of $K_2$ (or $K_3$) in $G_2$. Then $|N_{G_2}(X)| = n - p$ and $|N_{G_2}(Y)| \geqslant p + r$. So to see that $X$ and $Y$ satisfy (6.2), we need to check that

$$n - p \geqslant \tfrac{1}{3}(p + 2k - n - 1) \quad \text{and} \quad p + r \geqslant \tfrac{1}{3}(r + 2k - n - 1),$$

that is, $4n \geqslant 4p + 2k - 1$ and $n + 3p + 2r \geqslant 2k - 1$. Both of these inequalities hold, since $n > k > 3p$, $3p \geqslant k - 4$ and $r \geqslant 2$.

Next we show that (6.2) is best possible. Given $n$ and $k$ with $k \leqslant n$ and either $k = 9$ or $k \geqslant 11$, choose $b$ satisfying $0 \leqslant b \leqslant \frac{1}{3}k - 3$ and $b \equiv k + 1 \pmod 2$; then

$$\tfrac{1}{2}(k - b - 1) \geqslant \tfrac{1}{3}k + 1 \geqslant 4. \tag{6.4}$$

Define $a := n - \frac{1}{2}(3b + k - 1)$. Then $a \geqslant n - \frac{1}{2}(2k - 10) \geqslant 5$, since $b \leqslant \frac{1}{3}k - 3$ and $n \geqslant k$. Also, since $3b + 1 < k$ and $k < n + 1 = a + \frac{1}{2}(3b + k + 1)$, it follows that

$$b < \tfrac{1}{2}(k - b - 1) < a + b. \tag{6.5}$$

Consider the graph

$$G_3 := (aK_1 \cup bK_2) + \tfrac{1}{2}(k - b - 1)K_1,$$

which has order $n = a + \frac{1}{2}(3b + k - 1)$. Note that $G_3$ is certainly 3-connected, by (6.4) and since $a \geqslant 5$. By (6.5), the longest cycle in $G_3$ contains the $bK_2$ and the $\frac{1}{2}(k-b-1)K_1$; thus $c(G_4) = (k-b-1)+b = k-1$. Also, it follows from (6.5) and (6.4) that $\delta(G_3) = \frac{1}{2}(k-b-1) \geqslant \frac{1}{3}k+1$; hence $G_3$ satisfies (6.1). If $X = V(aK_1) \in \mathcal{I}(G_3)$ then

$$|N_{G_3}(X)| = \tfrac{1}{2}(k-b-1) = \tfrac{1}{6}(3k-3b-3) = \tfrac{1}{3}(|X|+2k-n-2). \qquad (6.6)$$

Thus $X$ just fails to satisfy (6.2). If $Y \in \mathcal{I}(G_3)$ and $Y \neq X$ then (6.5) implies $|N(Y)| \geqslant |N(X)|$, and $Y$ satisfies (6.2): if $|Y| < |X|$ then this follows directly from (6.6); if $|Y| \geqslant |X|$ and $Y \subseteq V(aK_1 \cup bK_2)$, then clearly $|N(Y)| - |N(X)|$ exceeds $\frac{1}{3}(|Y|-|X|)$; and if $Y = V(\frac{1}{2}(k-b-1)K_1)$ then (6.5) implies

$$\begin{aligned}|N(Y)| = a+2b > \tfrac{1}{2}(k-b-1)+b &\geqslant |N(X)| + \tfrac{1}{3}b \\ &> |N(X)| + \tfrac{1}{3}(\tfrac{1}{2}(k-b-1)-a),\end{aligned}$$

and it follows from (6.6) that $Y$ satisfies (6.2). Hence (6.2) is sharp.

We now show that 2-connectedness is required. Suppose $k \geqslant 5$, and let $G_4$ be a connected graph consisting of at least two copies of $K_{k-1}$ connected in a tree-like structure, that is: any two copies of $K_{k-1}$ share at most one vertex (a *cut-vertex*); any vertex lies in at most two copies of $K_{k-1}$; and any cycle in $G_4$ lies inside one copy of $K_{k-1}$. [Then each copy of $K_{k-1}$ corresponds to a vertex of a tree, with two such vertices joined by an edge if and only if the two corresponding copies of $K_{k-1}$ share a cut-vertex. Note that a tree of order $t$ has $t-1$ edges.]

Then $\delta(G_4) = k-2 > \frac{1}{3}k+1$, since $k \geqslant 5$. Let $X \in \mathcal{I}(G_4)$. Each vertex $x \in X$ has degree at least $k-2$, and no two members of $X$ lie in the same copy of $K_{k-1}$. Also, $|X|$ copies of $K_{k-1}$ can share at most at most $|X|-1$ cut-vertices between them. Hence $|N_{G_4}(X)| \geqslant (k-3)|X|+1$; and $X$ satisfies (6.2) if

$$(k-3)|X|+1 \geqslant \tfrac{1}{3}(|X|+2k-n-1),$$

that is, $(3k-10)|X|+n-2k+4 \geqslant 0$. This is true, since $|X| \geqslant 1$, $n \geqslant k$ and $k \geqslant 5$. Thus $G_4$ satisfies all of the other hypotheses of Theorem 6.1, but $G_4$ is not 2-connected and $c(G_4)=k-1$.

Lastly, we consider 3-connectedness. Given $n$ and $k$ with $k \leqslant n-2$ and either $k=9$ or $k \geqslant 11$, let $q$ be an integer satisfying $\frac{1}{3}k \leqslant q \leqslant \frac{1}{2}(k-3)$. Consider the graph $G_5 := 2K_1 + tK_q$, where $t \geqslant 3$; then $G_5$ has order $n = qt+2$. Also, $G_5$ is 2-connected, and $\delta(G_5) = q+1 \geqslant \frac{1}{3}k+1$. Let $X_1 = V(2K_1)$, and let $X_2 \in \mathcal{I}(G_5)$ consist of $r$ $(2 \leqslant r \leqslant t)$ vertices, each from a different copy of $K_q$. Then

$$|N_{G_5}(X_1)| = qt \geqslant (q-1)r+2 = |N_{G_5}(X_2)|,$$

since $t \geqslant r$ and $q \geqslant \frac{1}{3}k \geqslant 3$; and $|X_1| = 2 \leqslant r = |X_2|$. So to see that both $X_1$ and $X_2$ satisfy (6.2) it suffices to check that

$$(q-1)r+2 \geqslant \tfrac{1}{3}(r+2k-n-1),$$

that is, $n+3qr-4r-2k+7 \geqslant 0$. Since $n = qt+2 \geqslant 3q+2$ and $3q \geqslant k$ we obtain

$$\begin{aligned} n+3qr-4r-2k+7 &\geqslant 6q-2k+3q(r-1)-4r+9 \\ &\geqslant (3q-4)(r-1)+1, \end{aligned}$$

which is positive since $q \geqslant 3$ and $r \geqslant 2$. Thus $G_5$ satisfies all of the hypothesis of Theorem 6.1 except 3-connectedness, but $c(G_5) = 2q+2 < k$. So something stronger than 2-connectedness is needed, but this counter example does suggest a possible refinement of Theorem 6.1:

**Conjecture 6.8.** *The assumption '$G$ is 3-connected' in Theorem 6.1 can be replaced by '$G-v-w$ has at most two components for any two vertices $v, w \in V(G)$'.*

# Part II

# Edge colourings of powers of cycles

# Chapter 7

# Introduction to Part II

## 7.1 Terminology and notation

In Part II we work with simple graphs, but we also refer to the more general notion of *multigraphs*, in which there can be multiple edges between vertices, and *loops* (edges with both ends incident with the same vertex). [The graph colouring definitions in this section apply to any multigraph, not just simple graphs.]

If $n \geqslant 3$ and $p \geqslant 1$, the *cycle power* $C_n^p$ is the graph with vertex-set $\{v_0, \ldots, v_{n-1}\}$ in which two vertices $v_i$ and $v_j$ are adjacent if and only if

$$j \in \{i-p, \ldots, i-1, i+1, \ldots, i+p\} \text{ (modulo n).}$$

Note that if $p \geqslant \frac{1}{2}(n-1)$ then $C_n^p \cong K_n$, the complete graph of order $n$.

A *vertex colouring*, or just *colouring*, is an assignment of a colour to each vertex of a graph $G$. A colouring is *proper* if no two adjacent vertices are given the same colour. If $G$ has a proper colouring using at most $k$ colours, then $G$ is *$k$-colourable.* The smallest integer $k$ such that $G$ is $k$-colourable is called the *chromatic number* $\chi(G)$ of $G$.

A *proper edge colouring* of $G$ is an assignment of a colour to each edge of $G$ such that adjacent edges always get different colours. A graph is *edge-$k$-colourable* if it has a proper edge colouring using at most $k$-colours. The *edge chromatic number* $\chi'(G)$ of $G$ is the smallest number $k$ such that $G$ is edge-$k$-colourable.

A *list-assignment* $L$ to (the vertices of) $G$ is the assignment of a set $L(v)$ of colours to every vertex $v$ of $G$. If $L$ is a list-assignment to $G$, then an *$L$-colouring* of $G$ is a colouring (not necessarily proper) in which each vertex receives a colour from its own list. The graph $G$ is *$k$-choosable* if it is properly $L$-colourable for every list-assignment $L$ such that $|L(v)| \geqslant k$ for all $v \in V(G)$. The *choosability* $\mathrm{ch}(G)$ of $G$ is the smallest integer $k$ such that $G$ is $k$-choosable.

The *edge choosability* $\mathrm{ch}'(G)$ of $G$ is the smallest integer $k$ such that whenever every edge of $G$ is given a list of at least $k$ colours, there exists a proper-edge colouring of $G$ in which every edge receives a colour from its lists.

A *total colouring* of a graph $G$ is an assignment of a colour to every vertex and every edge of $G$ in such a way that no two adjacent or incident objects (edges or vertices) have the same colour. The *total chromatic number* $\chi''(G)$ of $G$ is the smallest integer $k$ such that $G$ has a total colouring using $k$ colours.

The *line graph* $L(G)$ of a graph $G$ has a vertex corresponding to every edge of $G$, with an edge joining two vertices of $L(G)$ whenever the corresponding edges of $G$ are adjacent. Sometimes it is easier to regard a list-edge-colouring of $G$ as a list-vertex-colouring of $L(G)$. We will make use of the fact that $\mathrm{ch}'(G) = \mathrm{ch}(L(G))$ in Chapter 9. Note that the line graphs of a simple graph is the edge-disjoint union of the *cliques* (complete subgraphs) corresponding to the vertices of the original graph, and every vertex in the line graph belongs to exactly two cliques in the clique decomposition.

## 7.2 Background

**Chapter 8 : Edge chromatic number of $C_n^p$**

Write $\Delta = \Delta(G)$ for the maximum degree of a simple graph $G$. By Vizing's Theorem [43], $\chi'(G) \leqslant \Delta+1$. Since clearly $\chi'(G) \geqslant \Delta(G)$, it follows that for every simple graph $G$, either $\chi'(G) = \Delta(G)$ ($G$ is of *class one*) or $\chi'(G) = \Delta(G) + 1$ ($G$ is of *class two*). A graph $G$ is said to be *overfull* if $|E(G)| > \Delta(G)\lfloor\frac{1}{2}|V(G)|\rfloor$. An overfull graph must be of class two, since each colour can be used on at most $\lfloor\frac{1}{2}|V(G)|\rfloor$ edges.

For cycles, it is easy to see that $\chi'(C_n) = 2$ if and only if $n$ is even. Now consider the complete graph $K_n$ ($n \geqslant 3$), and let $V(K_n) = \{v_0, \ldots, v_{n-1}\}$. Assigning colour $i + j \pmod n$ to every edge $v_iv_j$ gives a proper edge-$n$-colouring of $K_n$. If $n$ is odd then $K_n$ is overfull; thus $\chi'(K_n) = n$ if $n$ is odd. If $n$ is even then we can edge-$(n-1)$-colour $K_n - v_{n-1}$ by the above method. Then colouring each edge $v_iv_{n-1}$ with colour $2i \pmod{n-1}$ gives a proper edge-$(n-1)$-colouring of $K_n$. Thus $\chi'(K_n) = n - 1$ if $n$ is even.

A *circulant graph* $G$ is a graph with vertex-set $\{v_0, v_1, \ldots, v_{n-1}\}$ such that two vertices $v_i$ and $v_j$ are adjacent if and only if $|i-j| \in K$, where $K \subset \{1, 2, \ldots, n-1\}$ has the property that $k \in K$ if and only if $n - k \in K$. Sun Liang [31] proved that connected circulant graphs of even order have a 1-factorisation (that is, a partition of the edges into perfect matchings). This proves that $\chi'(C_n^p) = \Delta = 2p$ if $n$ is even and $1 \leqslant p \leqslant \frac{1}{2}n - 1$. Before coming across Sun Liang's paper, we determined $\chi'(C_n^p)$ when $1 \leqslant p \leqslant \frac{1}{2}n - 1$ by finding edge-$2p$-colourings of $C_n^p$ when $n$ is even. We have included our proof in Chapter 8.

**Chapter 9 : Edge choosability of $C_n^2$**

The following conjecture was proposed independently by Vizing, by Gupta, and by Albertson and Collins. It was previously known as the List Colouring Conjecture [1], but is now known as the *List-Edge-Colouring Conjecture* (LECC) [28, 48].

**Conjecture 7.1.** (The LECC) *For every multigraph $G$, $ch'(G) = \chi'(G)$.*

Several previous results, for example, [19, 9, 24], verify the LECC for different classes of graphs. We focus here on results relating to powers of cycles. Both Vizing [44] and Erdös, Rubin and Taylor [17] proved that $\mathrm{ch}(G) = \chi(G) = 2$ if $G$ is an even cycle, and it is easy to see that $\mathrm{ch}(G) = \chi(G) = 3$ if $G$ is in odd cycle. Since $L(C_n) = C_n$, this also proves the LECC for cycles.

In 1996, Ellingham and Goddyn [15] proved the LECC for all $d$-regular edge-$d$-colourable planar multigraphs. In particular, this result proves the LECC for squares of even cycles. (The result of Sun Liang [31] mentioned above shows such graphs have a 1-factorisation, and hence an edge 4-colouring; in Chapter 8 we give explicit edge-4-colourings.) In 1997, Häggkvist and Janssen [22] showed that $\mathrm{ch}'(K_n) \leqslant n$ for every $n$. This proves the LECC for $K_n$ when $n$ is odd, since then $\chi'(K_n) = n$.

Both [15] and [22] use the method of Alon and Tarsi [1, 2]. In Section 9.2.1 we give an outline of this important method. Then in Section 9.2.2 we use it to prove that $\mathrm{ch}'(C_n^2) \leqslant 5$ for every $n \geqslant 3$, which proves the LECC for squares of odd cycles. This completes the proof of the LECC for squares of cycles. In Section 9.3 we suggest ways in which this result might extend to $C_n^p$ when $3 \leqslant p \leqslant \frac{1}{2}n - 1$, and prove that $\mathrm{ch}'(C_8^3) \leqslant 7$.

An equivalent formulation of the LECC is that $\mathrm{ch}(G) = \chi(G)$ for every graph $G$ that is the line graph of a multigraph. Since every line graph is *claw-free* (that is, it does not have $K_{1,3}$ as an induced subgraph), the following conjecture, due to Gravier and Maffray [20], would imply the LECC.

**Conjecture 7.2.** (The List-Colouring Conjecture for Claw-Free Graphs) *For every claw-free graph* $G$, $ch(G) = \chi(G)$.

Kostochka and Woodall [28] conjectured that *squares of graphs* are another class of graphs for which choosablility equals chromatic number:

**Conjecture 7.3.** (The List-Square-Colouring Conjecture) *For every graph* $G$, $ch(G^2) = \chi(G^2)$.

In 2003, Prowse and Woodall [38] used the Alon-Tarsi method to prove that $\chi(C_n^p) = \mathrm{ch}(C_n^p)$. Since $C_n^p$ is claw-free, this proves the List-Colouring Conjecture for Claw-Free Graphs for $C_n^p$; and since $C_n^{2p} = (C_n^p)^2$, it also proves the List-Square-Colouring Conjecture when $G$ is a cycle power.

For total colourings, clearly $\chi''(G) \geqslant \Delta(G) + 1$. The *Total-Colouring Conjecture* (TCC) states that $\chi''(G) \leqslant \Delta(G)+2$, where $\Delta(G)$ denotes the maximum degree of a (finite simple) graph $G$. In 2007, Campos and de Mello [12] proved the TCC for $C_n^p$ when $n$ is even and $2 < p < n/2$. For cycles, Yap [49] proved that $\chi''(C_n) = 3$ if $n \equiv 0 \pmod 3$ and $\chi''(C_n) = 4$ otherwise. For squares of cycles, Campos and de Mello [11] proved that $\chi''(C_n^2) = 4$ if $n \neq 7$, and $\chi''(C_7^2) = 5$. Finally, Behzad et al. [4] proved that $\chi''(K_n) = n$ if $n$ is odd and $\chi''(K_n) = n + 1$ if $n$ is even.

# Chapter 8

# Edge chromatic number of $C_n^p$

Write $\Delta(G)$ for the maximum degree of a simple graph $G$. Recall by Vizing's Theorem [43] that, for any simple graph $G$, $\chi'(G)$ is either $\Delta(G)$ or $\Delta(G)+1$. In this chapter we establish the edge chromatic number of powers of cycles. Sun Liang [31] proved a more general result; see Section 7.2.

**Theorem 8.1.** *Supose that $1 \leqslant p \leqslant \frac{1}{2}n - 1$. Then $\chi'(C_n^p) = \Delta(C_n^p) = 2p$ if and only if $n$ is even.*

Before proving Theorem 8.1, we need an easy lemma. The *d-prism* $\Pi_d$ ($d \geqslant 3$) is the graph with $2d$ vertices consisting of two cycles $u_1u_2\dots u_d$ and $w_1w_2\dots w_d$ with a matching $\{u_iw_i : 1 \leqslant i \leqslant d\}$ of $d$ edges between them. Lemma 8.2 is a special case of a result of Kotzig (Theorem 5 of [29]), since $\Pi_d$ can be regarded as the Cartesian product $C_d \times K_2$, and $K_2$ clearly has a 1-factorisation.

**Lemma 8.2.** $\chi'(\Pi_d) = \Delta(\Pi_d) = 3$.

**Proof.** Assign identical edge-3-colourings $\gamma$ to the two cycles of length $d$, so that $\gamma(u_iu_{i+1}) = \gamma(w_iw_{i+1})$ (modulo $d$) for $1 \leqslant i \leqslant d$. Then, for each $i$, the same two colours have been used on edges meeting at $u_i$ as those meeting at $w_i$, and we can colour $u_iw_i$ with the remaining colour. □

In the following proof, we write $(n,r)$ for the highest common factor of two integers $n$ and $r$.

**Proof of Theorem 8.1.**

Since $1 \leqslant p \leqslant \frac{1}{2}n - 1$, $C_n^p$ is $(2p)$-regular, and $|E(C_n^p)| = np$. If $n$ is odd, then $np > 2p\lfloor\frac{1}{2}n\rfloor = p(n-1)$; thus $C_n^p$ is overfull and $\chi'(C_n^p) = \Delta(C_n^p) + 1 = 2p+1$.

So suppose $n$ is even. We will examine the structure of $C_n^p$ to find an edge-$(2p)$-colouring. For each $i$ $(1 \leqslant i \leqslant p)$, $C_n^p$ contains a 2-regular spanning subgraph consisting of $(n,i)$ cycles, each of length $n/(n,i)$. Label these subgraphs $S_1, \ldots, S_p$. If $p$ is odd, so that $n/(n,p)$ is even, then $S_p$ is the disjoint union of even cycles, and so $S_p$ can be edge-2-coloured. Then there are $2(p-1)$ colours remaining to colour the edges of $G - E(S_p) = C_n^{p-1}$; thus it is sufficient to consider the case when $p$ is even. We will show that the subgraph $S_{p-1} \cup S_p$ is edge-4-colourable; the result follows by induction.

Note that $n/(n,p-1)$ is always even, since $n$ and $p$ are even. If $n/(n,p)$ is also even, then we can edge-2-colour $S_{p-1}$, and then edge-2-colour $S_p$ with 2 new colours, and we are done. So suppose $n = 2dr$, where $d$ is odd, and $(n,p) = 2r$, so that

$$(d, \tfrac{p}{2r}) = (\tfrac{n}{2r}, \tfrac{p}{2r}) = 1. \tag{8.1}$$

Since $p$ is a multiple of $2r$, $dp$ is a multiple of $n$ and

$$\{p, 2p, \ldots, dp\} \subseteq \{2r, 4r, \ldots, 2dr = n\} \pmod{n}.$$

Moreover, if $n$ divides $qp$ then $d = \frac{n}{2r}$ divides $q\frac{p}{2r}$, and so $d$ divides $q$ by (8.1). Thus $dp$ is the smallest multiple of $p$ that is a multiple of $n$, and $p, 2p, \ldots, dp$ are the same as $2r, 4r, \ldots, 2dr = n \pmod{n}$. Thus $S_p$ is the union of $2r$ vertex-disjoint

$d$-cycles:

$$\begin{aligned} Q_1 &: v_1 v_{p+1} \dots v_{(d-1)p+1} \\ Q_2 &: v_2 v_{p+2} \dots v_{(d-1)p+2} \\ &\vdots \\ Q_{2r} &: v_{2r} v_{p+2r} \dots v_{(d-1)p+2r}. \end{aligned}$$

Note that if $v_i \in Q_j$ then $i \equiv j \pmod 2$. For $1 \leqslant k \leqslant r$, let

$$M_k := \{v_i v_{i+p-1} : v_i \in Q_{2k}\}.$$

Then $M_k \subset E(S_{p-1})$, and $M := \cup_{k=1}^r M_k$ is a perfect matching of $S_{p-1}$ (and of $C_n^p$), since every vertex $v_i$ ($i$ even) is in exactly one set $Q_{2k}$ $(1 \leqslant k \leqslant r)$. Thus $S_{p-1} \cup S_p$ is the union of a 1-factor $S_{p-1} \setminus M$ and $r$ disjoint copies of $\Pi_d$, namely $Q_{2k-1} \cup Q_{2k} \cup M_k$ $(1 \leqslant k \leqslant r)$. Hence we can use one colour for the edges of the 1-factor, then (by Lemma 8.2) we can edge-colour each copy of $\Pi_d$ with the remaining 3 colours, giving an edge-4-colouring of $S_{p-1} \cup S_p$. This completes the proof of Theorem 8.1. $\square$

# Chapter 9

# Edge choosability of $C_n^2$

## 9.1 Introduction

The List-Edge-Colouring Conjecture (LECC) states that for every multigraph $H$, $\text{ch}'(H) = \chi'(H)$. In this chapter, we prove the LECC for squares of odd cycles, by giving an upper bound on the edge choosability $\text{ch}'(C_n^2)$ of $C_n^2$. Recall that $C_n^2$ is the graph with vertex-set $\{v_0, \ldots, v_{n-1}\}$ in which two vertices $v_i$ and $v_j$ are adjacent if and only if $j \in \{i-2, i-1, i+1, i+2\}$ (modulo $n$).

**Theorem 9.1.** *For all $n \geqslant 3$, $ch'(C_n^2) \leqslant 5$.*

When $n$ is odd, $\chi'(C_n^2) = 5$, and so Theorem 9.1 proves the LECC for squares of odd cycles. As mentioned in Section 7.2, a result of Ellingham and Goddyn [15] proves the LECC for even cycles.

Häggkvist and Janssen [22] used the method of Alon and Tarsi [1, 2] to prove that $\text{ch}'(K_n) \leqslant n$ for every $n$. In Section 9.2 we adopt a similar approach to prove Theorem 9.1; the Alon-Tarsi method is outlined in Section 9.2.1.

In Section 9.3 we suggest ways in which Theorem 9.1 might extend to $C_n^p$ when $3 \leqslant p \leqslant \frac{1}{2}n - 1$, and prove that $\text{ch}'(C_8^3) \leqslant 7$.

## 9.2 Edge choosability of $C_n^2$

### 9.2.1 Outline of the Alon-Tarsi method

Since we are interested in list-edge-colourings of the graph $C_n^2$, we will be working with list-vertex-colourings of $G := L(C_n^2)$. We shall use the method of Alon and Tarsi [1, 2], with the additional technique of 'blocking out' introduced by Häggkvist and Janssen [22]. First we summarise the general method.

An *orientation* $D$ of a graph is an assignment of exactly one direction to each of its edges. The *outdegree* $d^+(v)$ of a vertex $v$ is the number of edges directed out of $v$ in $D$. A *directed cycle* of $D$ is one in which all the edges have the same orientation. An orientation $D$ is *acyclic* on a subgraph $H$ of $G$ if there are no directed cycles of $D$ within $H$.

Let $D_0$ be an arbitrary orientation of an undirected graph $G$, and let $\rho : V(G) \to \mathbb{N}$ satisfy $\rho(v) = d^+(v)$ for all $v \in V(G)$. If $D$ is any other orientation of $G$, let $a(D)$ be the number of edges that have opposite orientations in $D$ and $D_0$, and define $\mathrm{sign}(D)$ to be 1 or $-1$ according as $a(D)$ is even or odd. An orientation of $G$ is said to *obey* $\rho$ if $d^+(v) = \rho(v)$ for all $v \in V(G)$. Let $\mathcal{O}$ denote the set of all such orientations, and let $\sigma(D_0) := \sum_{D\in\mathcal{O}} \mathrm{sign}(D)$. Suppose that every vertex $v$ of $G$ is given a list $L(v)$ of at least $\rho(v) + 1$ colours. The main result of Alon and Tarsi is that, if $\sigma(D_0) \neq 0$, then the vertices of $G$ can be coloured from these lists.

Recall that $G = L(C_n^2)$ is the edge-disjoint union of cliques $Q_0, \ldots, Q_{n-1}$ corresponding to the vertices $v_0, \ldots, v_{n-1}$ of $C_n^2$. Let $\mathcal{O}'$ denote the set of all orientations in $\mathcal{O}$ that are acyclic on every clique $Q_i$. We say that such orientations are *clique-transitive*; note that clique-transitive orientations need not be acyclic on cliques other than $Q_0, \ldots, Q_{n-1}$. A refinement of the basic method (see [1, 22]) is to observe that $\sigma(D_0) = \sum_{D\in\mathcal{O}'} \mathrm{sign}(D)$; that is, in calculating $\sigma(D_0)$, we need only consider clique-transitive orientations.

To further reduce the number of orientations which we need to consider (in fact, to reduce this number to 1), we use the additional refinement of *blocking out* certain values $b_i$ of the possible outdegrees in a clique. This method is described in [22]. The following theorem, a rephrasing of a special case of Proposition 2.4 in [22], encapsulates our use of this technique.

**Theorem 9.2.** (Häggkvist and Janssen [22]) *Let $G$ be the edge-disjoint union of cliques $Q_0, \ldots, Q_{n-1}$, and let $b_0, \ldots, b_{n-1}$ be integers. Let $\rho : V(G) \to \mathbb{N}$ and let $L$ be a list assignment to $V(G)$ such that $|L(v)| > \rho(v)$ for all $v \in V(G)$. Extend each clique $Q_i$ by a new vertex $w_i$, and let $\overline{G}$ be the edge-disjoint union of the extended cliques $\overline{Q}_0, \ldots, \overline{Q}_{n-1}$. Suppose there exists a unique clique-transitive orientation $\overline{D}$ of $\overline{G}$ in which*

(i) *for each $i$, $w_i$ has outdegree $b_i$ in $\overline{D}$, and*

(ii) *$\overline{D}$ obeys $\rho(v)$ at all $v \in V(G)$.*

*Then $G$ is properly $L$-colourable.* □

To use Theorem 9.2 to prove Theorem 9.1, we first define the map $\rho$ (so that $\rho(v) \leqslant 4$ for all $v \in V(G)$) and the blocking-out values $b_0, \ldots, b_{n-1}$. Then we give an algorithm which assigns the outdegrees of an orientation $\overline{D}$ of $\overline{G}$, satisfying (i) and (ii). Working through the cases $n = 8$ and 9 as examples, we then show that $\overline{D}$ is unique. This will prove that $G = L(C_n^2)$ is properly $L$-colourable for any list assignment $L$ satisfying $|L(v)| \geqslant 5$ for all $v \in V(G)$. Thus $\mathrm{ch}'(C_n^2) = \mathrm{ch}(G) \leqslant 5$.

### 9.2.2 Proof of Theorem 9.1

For $3 \leqslant n \leqslant 5$, $C_n^2 = K_n$ and the result is already known (see [22]), so suppose $n \geqslant 6$. By definition, $V(C_n^2) = \{v_0, v_1, \ldots, v_{n-1}\}$. Working modulo $n$, we can label the vertex-set of $G = L(C_n^2)$ by

$$V(G) = \{(i, i+1), (i, i+2) : 0 \leqslant i \leqslant n-1\}.$$

With this notation, vertices $(i, j)$ and $(i', j')$ are joined by an edge in $G$ precisely when $i = i'$, $i = j'$, $j = i'$ or $j = j'$. Also, $G$ is the edge-disjoint union of the cliques $Q_0, \ldots, Q_{n-1}$, where each $Q_i \cong K_4$ is the clique corresponding to the vertex $v_i$ of $C_n^2$, and each vertex $(i, j)$ of $G$ belongs to $Q_i$ and $Q_j$. Define $\rho : V(G) \to \mathbb{N}$ as follows:

$$\rho(i,j) = \begin{cases} 3 & \text{if } i+j < n-1, \text{ or if } n \text{ is even and } (i,j) = (\frac{n}{2}-1, \frac{n}{2}), \\ 4 & \text{otherwise.} \end{cases}$$

We now define blocking-out values $b_i$, $0 \leqslant i < n$. If $n = 6$, let

$$(b_0, b_1, b_2, b_3, b_4, b_5) = (4, 3, 2, 4, 3, 2).$$

For $n \geqslant 7$, define

$$b_i := \begin{cases} 4 & \text{if } 0 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 2 \text{ or } i = n-1, \\ 1 & \text{if } \lfloor \frac{n}{2} \rfloor - 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor, \\ 3 & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor + 2, \\ 2 & \text{if } \lfloor \frac{n}{2} \rfloor + 3 \leqslant i \leqslant n-2. \end{cases}$$

Define the graph $\overline{G}$ as in Theorem 9.2. It is not difficult to see by induction that if the outedges of a clique $K_r$ are a permutation of $\{0, 1, \ldots, r-1\}$, then this corresponds to an acyclic orientation of $K_r$. If this holds in every clique $\overline{Q}_i \cong K_5$ of $\overline{G}$, then the corresponding orientation of $\overline{G}$ is clique-transitive. Since we have already specified the outdegrees $b_0, \ldots, b_{n-1}$ of the vertices $w_0, \ldots, w_{n-1}$, we will show that, for each $i$ $(0 \leqslant i \leqslant n-1)$ there exists a unique assignment of the outdegrees $\{0, 1, 2, 3, 4\} - \{b_i\}$ within clique $Q_i$.

To prove this, we will describe two algorithms (one for $n = 6$, the other for $n \geqslant 7$) which assign two outdegrees to each vertex $v \in V(G)$, one for each clique that contains the vertex. To assign outdegree pair $\{x, y\}$ to vertex $(i, j)$ means that this vertex has outdegree $x$ in clique $\overline{Q}_i$ and outdegree $y$ in clique $\overline{Q}_j$. Since the orientation $\overline{D}$ obeys $\rho$, the outdegree pair must satisfy $x + y = \rho(i, j)$; we say that $y$ is the *complementary degree* to $x$ at vertex $(i, j)$. To show that there is no conflict with the blocked out values, after each assignment at vertex $(i, j)$ we will specify the values $[b_i, b_j]$, as given by the above definition. One can easily check to confirm that $x \neq b_i$ and $y \neq b_j$ at all steps of the algorithms. We also give one of three reasons – outlined below, denoted by (A1), (A2) and (A3) – for the uniqueness of the assignment of $\{x, y\} = \{x, \rho(i, j) - x\}$ at vertex $(i, j) \in V(G)$. Note that some assignments may satisfy more than one of (A1)–(A3).

(A1) : *No other outdegree pair is possible at vertex $(i, j)$. That is, for each $z \neq x$, $0 \leqslant z \leqslant \rho(i, j)$, either $z$ has already been assigned at another vertex in $\overline{Q}_i$ or $\rho(i, j) - z$ has already been assigned at another vertex in $\overline{Q}_j$.*

(A2) : *Vertex $(i, j)$ is the only vertex in clique $\overline{Q}_i$ at which outdegree $x$ can be assigned. That is, for each $k \neq j$ such that either $(i, k) \in V(G)$ or $(k, i) \in V(G)$, it is not possible to assign outdegree $x$ at this vertex in clique $\overline{Q}_i$.*

(A3) : *Vertex $(i, j)$ is the only vertex in clique $\overline{Q}_j$ at which outdegree $y$ can be assigned. That is, for each $k \neq i$ such that either $(k, j) \in V(G)$ or $(j, k) \in V(G)$, it is not possible to assign outdegree $y$ at this vertex in clique $\overline{Q}_j$.*

**Algorithm 1** (for $n = 6$).

| | | Blocked-out | Uniqueness |
|---|---|---|---|
| 1. | Assign $\{4,0\}$ to vertex $(2,4)$. | [2,3] | (A2) |
| 2. | Assign $\{0,4\}$ to vertex $(5,1)$. | [2,3] | (A3) |
| 3. | Assign $\{0,4\}$ to vertex $(3,4)$. | [4,3] | (A3) |
| 4. | Assign $\{4,0\}$ to vertex $(5,0)$. | [2,4] | (A2) |
| 5. | Assign $\{3,1\}$ to vertex $(3,5)$. | [4,2] | (A3) |
| 6 | Assign $\{1,3\}$ to vertex $(4,5)$. | [3,2] | (A1) |
| 7. | Assign $\{2,1\}$ to vertex $(4,0)$. | [3,4] | (A1) |
| 8. | Assign $\{3,0\}$ to vertex $(0,2)$. | [4,2] | (A3) |
| 9. | Assign $\{0,3\}$ to vertex $(1,2)$. | [3,2] | (A3) |
| 10. | Assign $\{2,1\}$ to vertex $(0,1)$. | [4,3] | (A1) |
| 11. | Assign $\{2,1\}$ to vertex $(1,3)$. | [3,4] | (A1) |
| 12. | Assign $\{1,2\}$ to vertex $(2,3)$. | [2,4] | (A1) |

**Algorithm 2** (for $n \geqslant 7$).

| | | Blocked-out | Uniqueness |
|---|---|---|---|
| 1. | Assign $\{4,0\}$ to vertex $(\lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor + 1)$. | [1,3] | (A2) |
| 2. | Assign $\{4,0\}$ to vertex $(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 2)$. | [1,3] | (A2) |
| 3. | Assign $\{2,2\}$ to vertex $(\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2)$. | [3,3] | (A1) |
| 4. | For $i = \lfloor \frac{n}{2} \rfloor - 2, \lfloor \frac{n}{2} \rfloor - 3, \ldots, 0$ : | | |
| | Assign $\{1,2\}$ to vertex $(i, i+2)$. | [4,1 or 4] | (A3) |
| 5. | Assign $\{2,2\}$ to vertex $(n-1, 1)$. | [4,4] | (A3) |
| 6. | Assign $\{1,2\}$ to vertex $(n-2, 0)$. | [2,4] | (A3) |
| 7. | Assign $\{1,3\}$ to vertex $(n-1, 0)$. | [4,4] | (A1) |
| 8. | For $i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$ : | | |
| | Assign $\{0,3\}$ to vertex $(i, i+1)$. | [4 or 1, 4 or 1] | (A1) |
| 9. | Assign $\{0,4\}$ to vertex $(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1)$. | [1,2] | (A1) |
| 10. | For $i = \lfloor \frac{n}{2} \rfloor + 3, \ldots, n-1$ : | | |
| | (a) Assign $\{1,3\}$ to vertex $(i-2, i)$, | [3 or 2, 2 or 4] | (A1) |
| | (b) Assign $\{4,0\}$ to vertex $(i-1, i)$. | [3 or 2, 2 or 4] | (A1) |

Whilst proving that each assignment in Algorithm 2 is unique, we will demonstrate the cases $n = 8$ and $n = 9$ as examples. [The case $n = 7$ would not adequately illustrate step 10 of Algorithm 2.] Afterwards, we will show the orientation $\overline{D}$ of $\overline{G}$ when $n = 6$.

We represent the graph $\overline{G}$ as an $n \times n$ matrix $M$. By cell $(i, j)$ we mean the $1 \times 1$ submatrix in row $i$ and column $j$ of $M$ (regarding the top row as row 0 and leftmost column as column 0). Recall that $\overline{G}$ has vertex-set

$$V(\overline{G}) = \{(i, i+1), (i, i+2) : 0 \leqslant i < n\} \cup \{w_0, w_1, \ldots, w_{n-1}\}.$$

For each vertex $(i, j) \in V(G)$, cells $(i, j)$ and $(j, i)$ of $M$ are identified to represent this vertex. Furthermore, each vertex $w_i$ is represented by cell $(i, i)$. Thus the vertices of $\overline{G}$ correspond to the cells of $M$ which are distance at most 2 along any row or column from the leading diagonal (working modulo $n$); any remaining cells of $M$ are marked with a grey square, "$\blacksquare$", and can be ignored. Hence both row $i$ and column $i$ represent clique $\overline{Q}_i$, and vertices of $\overline{G}$ are connected exactly when the corresponding cells lie in the same row or column. For each $i$, the blocked-out value $b_i$ is shown, encircled, in cell $(i, i)$, and corresponds to the outdegree in $\overline{D}$ of the vertex $w_i$. This vertex lies only in clique $\overline{Q}_i$.

In the following two matrices, representing the cases $n = 8$ and $n = 9$, the blocked-out values $b_i$ are given on the leading diagonal. Also, for each vertex $(i, j) \in V(G)$, the value of $\rho(i, j)$ is represented in cells $(i, j)$ and $(j, i)$ by "$\cdots$" if $\rho(i, j) = 3$ or by " :: " if $\rho(i, j) = 4$.

$$\begin{array}{cc}
 & \begin{array}{cccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \end{array} &
\left[\begin{array}{cccccccc}
4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare & \cdots & :: \\
\cdots & 4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare & :: \\
\cdots & \cdots & 4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \cdots & \cdots & 1 & \cdots & :: & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \cdots & \cdots & 1 & :: & :: & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & :: & :: & 3 & :: & :: \\
\cdots & \blacksquare & \blacksquare & \blacksquare & :: & :: & 3 & :: \\
:: & :: & \blacksquare & \blacksquare & \blacksquare & :: & :: & 4
\end{array}\right]
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{ccccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 & \overline{Q}_8 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \\ \overline{Q}_8 \end{array} &
\left[\begin{array}{ccccccccc}
4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \cdots & :: \\
\cdots & 4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: \\
\cdots & \cdots & 4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \cdots & \cdots & 1 & \cdots & :: & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \cdots & \cdots & 1 & :: & :: & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & :: & :: & 3 & :: & :: & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 3 & :: & :: \\
\cdots & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 2 & :: \\
:: & :: & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 4
\end{array}\right]
\end{array}$$

Next, we give matrices showing the outdegrees in the final orientations $\overline{D}$ for $n = 8$ and $n = 9$. As before, the blocked-out values $b_i$ are given on the leading diagonal. At each cell $(i, j)$ representing a vertex of $G$ (either $(i, j)$ or $(j, i)$), the entry $xy$ means that this vertex has outdegree $x$ in clique $\overline{Q_i}$, and outdegree $y$ in clique $\overline{Q}_j$. For consistency, the entry at cell $(j, i)$ must be $yx$. For example, in step 1 of Algorithm 2 for $n = 8$, we assign $\{4, 0\}$ to vertex $(3, 5)$: this corresponds to entering 40 in cell $(3, 5)$ and also 04 in cell $(5, 3)$. Thus the assigned outdegrees within a clique $\overline{Q}_i$ can be found as the first integer in the entries of row $i$, or as the last integer in the entries of column $i$ (with the encircled value $b_i$ acting in both row $i$ and column $i$). One can check that the set of outdegrees in each clique is some permutation of the set $\{0, 1, 2, 3, 4\}$, thus each of these orientations $\overline{D}$ is clique-transitive. It is also easy to check that this also holds for general $n \geqslant 7$.

$$\begin{array}{cc}
 & \begin{array}{cccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \end{array} &
\left[\begin{array}{cccccccc}
4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare & 21 & 31 \\
30 & 4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare & 22 \\
21 & 30 & 4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & 21 & 30 & 1 & 03 & 40 & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & 21 & 30 & 1 & 04 & 40 & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & 04 & 40 & 3 & 22 & 13 \\
12 & \blacksquare & \blacksquare & \blacksquare & 04 & 22 & 3 & 40 \\
13 & 22 & \blacksquare & \blacksquare & \blacksquare & 31 & 04 & 4
\end{array}\right]
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{ccccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 & \overline{Q}_8 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \\ \overline{Q}_8 \end{array} &
\left[\begin{array}{ccccccccc}
4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 21 & 31 \\
30 & 4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 22 \\
21 & 30 & 4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & 21 & 30 & 1 & 03 & 40 & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & 21 & 30 & 1 & 04 & 40 & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & 04 & 40 & 3 & 22 & 13 & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare & 04 & 22 & 3 & 40 & 13 \\
12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 31 & 04 & 2 & 40 \\
13 & 22 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 31 & 04 & 4
\end{array}\right]
\end{array}$$

**Uniqueness of Algorithm 2**

We now explain why each assignment in Algorithm 2 is unique (and thus the orientation $\overline{D}$ of $\overline{G}$ is unique). To clarify the explanations, we give matrices for $n = 8$ and $n = 9$, showing the assigned outdegrees at various points. As before, we use dots in unassigned cells to represent the value of $\rho$ at the corresponding vertex.

1. *Assign* $\{4, 0\}$ *to vertex* $(\lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor + 1)$.

(A2) : This is the only vertex in clique $\overline{Q}_{\lfloor \frac{n}{2} \rfloor - 1}$ at which $\rho = 4$, and $b_{\lfloor \frac{n}{2} \rfloor - 1} \neq 4$.

2. *Assign* $\{4, 0\}$ *to vertex* $(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 2)$.

(A2) : Outdegree pair $\{4, 0\}$ cannot be assigned at vertex $(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1)$ since outdegree 0 has already been assigned in clique $\overline{Q}_{\lfloor \frac{n}{2} \rfloor + 1}$. The remaining vertices in clique $\overline{Q}_{\lfloor \frac{n}{2} \rfloor}$ have $\rho = 3$.

3. *Assign* $\{2, 2\}$ *to vertex* $(\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2)$.

(A1) : In both cliques $\overline{Q}_{\lfloor \frac{n}{2} \rfloor + 1}$ and $\overline{Q}_{\lfloor \frac{n}{2} \rfloor + 2}$, outdegree 0 has already been assigned, and outdegree 3 is blocked-out.

$$\begin{array}{cc}
 & \begin{array}{cccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \end{array} &
\left[\begin{array}{cccccccc}
4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare & \cdots & :: \\
\cdots & 4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare & :: \\
\cdots & \cdots & 4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \cdots & \cdots & 1 & \cdots & 40 & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \cdots & \cdots & 1 & :: & 40 & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & 04 & :: & 3 & 22 & :: \\
\cdots & \blacksquare & \blacksquare & \blacksquare & 04 & 22 & 3 & :: \\
:: & :: & \blacksquare & \blacksquare & \blacksquare & :: & :: & 4
\end{array}\right]
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{ccccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 & \overline{Q}_8 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \\ \overline{Q}_8 \end{array} &
\left[\begin{array}{ccccccccc}
4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \cdots & :: \\
\cdots & 4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: \\
\cdots & \cdots & 4 & \cdots & \cdots & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \cdots & \cdots & 1 & \cdots & 40 & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \cdots & \cdots & 1 & :: & 40 & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & 04 & :: & 3 & 22 & :: & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare & 04 & 22 & 3 & :: & :: \\
\cdots & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 2 & :: \\
:: & :: & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 4
\end{array}\right]
\end{array}$$

4. *For* $i = \lfloor \frac{n}{2} \rfloor - 2, \lfloor \frac{n}{2} \rfloor - 3, \ldots, 0$ : *Assign* $\{1, 2\}$ *to vertex* $(i, i + 2)$.

(A3) : In all cases, vertex $\{i + 2, i + 4\}$ has already had outdegrees assigned. It is not possible to assign $\{1, 2\}$ at vertex $(i + 1, i + 2)$, since if $i = \lfloor \frac{n}{2} \rfloor - 2$ then $b_{i+1} = 1$, and otherwise outdegree 1 has already been assigned in clique $\overline{Q}_{i+1}$.

If $i = \lfloor \frac{n}{2} \rfloor - 2$ then it is not possible to assign $\{2, 2\}$ at vertex $(i + 2, i + 3)$, since outdegree 2 has already been defined in clique $\overline{Q}_{i+3}$. If $0 \leqslant i < \lfloor \frac{n}{2} \rfloor - 2$ then it is not possible to assign $\{2, 1\}$ at vertex $(i + 2, i + 3)$ since outdegree 1 is either blocked-out, or has already been assigned, in clique $\overline{Q}_{i+3}$.

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \end{array} &
\left[\begin{array}{cccccccc}
4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare & \cdots & :: \\
\cdots & 4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare & :: \\
21 & \cdots & 4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & 21 & \cdots & 1 & \cdots & 40 & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & 21 & \cdots & 1 & :: & 40 & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & 04 & :: & 3 & 22 & :: \\
\cdots & \blacksquare & \blacksquare & \blacksquare & 04 & 22 & 3 & :: \\
:: & :: & \blacksquare & \blacksquare & \blacksquare & :: & :: & 4
\end{array}\right]
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{ccccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 & \overline{Q}_8 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \\ \overline{Q}_8 \end{array} &
\left[\begin{array}{ccccccccc}
4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \cdots & :: \\
\cdots & 4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: \\
21 & \cdots & 4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & 21 & \cdots & 1 & \cdots & 40 & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & 21 & \cdots & 1 & :: & 40 & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & 04 & :: & 3 & 22 & :: & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare & 04 & 22 & 3 & :: & :: \\
\cdots & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 2 & :: \\
:: & :: & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 4
\end{array}\right]
\end{array}
$$

5. *Assign* $\{2,2\}$ *to vertex* $(n-1,1)$.

(A3) : Since outdegree 1 has already been assigned in cliques $\overline{Q}_0$ and $\overline{Q}_2$, it is not possible to assign outdegree pair $\{1,2\}$ at vertex $(0,1)$, or $\{2,1\}$ at $(1,2)$. Also, vertex $(1,3)$ has already been assigned an outdegree pair (in step 4).

6. *Assign* $\{1,2\}$ *to vertex* $(n-2,0)$.

(A3) : It is not possible to assign outdegree pair $\{2,1\}$ at vertex $(0,1)$, since outdegree 1 has already been assigned in clique $\overline{Q}_1$. Also, it is not possible to assign $\{2,2\}$ at vertex $(n-1,0)$ since outdegree 2 has already been assigned in clique $\overline{Q}_{n-1}$ (in step 5).

7. *Assign* $\{1,3\}$ *to vertex* $(n-1,0)$.

(A1) : Outdegrees 1 and 2 have been assigned in clique $\overline{Q}_0$, and $b_0 = b_{n-1} = 4$.

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \end{array} &
\left[\begin{array}{cccccccc}
4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare & 21 & 31 \\
\cdots & 4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare & 22 \\
21 & \cdots & 4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & 21 & \cdots & 1 & \cdots & 40 & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & 21 & \cdots & 1 & :: & 40 & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & 04 & :: & 3 & 22 & :: \\
12 & \blacksquare & \blacksquare & \blacksquare & 04 & 22 & 3 & :: \\
13 & 22 & \blacksquare & \blacksquare & \blacksquare & :: & :: & 4
\end{array}\right]
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{ccccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 & \overline{Q}_8 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \\ \overline{Q}_8 \end{array} &
\left[\begin{array}{ccccccccc}
4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 21 & 31 \\
\cdots & 4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 22 \\
21 & \cdots & 4 & \cdots & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & 21 & \cdots & 1 & \cdots & 40 & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & 21 & \cdots & 1 & :: & 40 & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & 04 & :: & 3 & 22 & :: & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare & 04 & 22 & 3 & :: & :: \\
12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 2 & :: \\
13 & 22 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 4
\end{array}\right]
\end{array}
$$

8. *For* $i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$ : *Assign* $\{0,3\}$ *to vertex* $(i, i+1)$.

(A1) : Every other vertex in clique $\overline{Q}_i$ has already had an outdegree assigned.

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \end{array} &
\begin{bmatrix}
4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare & 21 & 31 \\
30 & 4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare & 22 \\
21 & 30 & 4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & 21 & 30 & 1 & 03 & 40 & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & 21 & 30 & 1 & :: & 40 & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & 04 & :: & 3 & 22 & :: \\
12 & \blacksquare & \blacksquare & \blacksquare & 04 & 22 & 3 & :: \\
13 & 22 & \blacksquare & \blacksquare & \blacksquare & :: & :: & 4
\end{bmatrix}
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{ccccccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 & \overline{Q}_8 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \\ \overline{Q}_8 \end{array} &
\begin{bmatrix}
4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 21 & 31 \\
30 & 4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & 22 \\
21 & 30 & 4 & 03 & 12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & 21 & 30 & 1 & 03 & 40 & \blacksquare & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & 21 & 30 & 1 & :: & 40 & \blacksquare & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & 04 & :: & 3 & 22 & :: & \blacksquare \\
\blacksquare & \blacksquare & \blacksquare & \blacksquare & 04 & 22 & 3 & :: & :: \\
12 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 2 & :: \\
13 & 22 & \blacksquare & \blacksquare & \blacksquare & \blacksquare & :: & :: & 4
\end{bmatrix}
\end{array}
$$

9. *Assign* $\{0,4\}$ *to vertex* $(\lfloor\frac{n}{2}\rfloor, \lfloor\frac{n}{2}\rfloor+1)$.

(A1) : Every other vertex in clique $\overline{Q}_{\lfloor\frac{n}{2}\rfloor}$ has already had an outdegree assigned.

10. *For* $i = \lfloor\frac{n}{2}\rfloor+3, \ldots, n-1$ :

(a) *Assign* $\{1,3\}$ *to vertex* $(i-2,i)$.

(A1) : Every other vertex in clique $\overline{Q}_{i-2}$ has already had an outdegree assigned.

(b) *Assign* $\{4,0\}$ *to vertex* $(i-1,i)$.

(A1) : If $i \leqslant n-1$ then all outdegrees except 0 and 4 have already been assigned in clique $\overline{Q}_i$, and 0 has already been assigned in clique $\overline{Q}_{i-1}$, thus $\{4,0\}$ is the only possible assignment at $(i-1,i)$. If $i = n-1$ then every other vertex in clique $\overline{Q}_i$ has already had an outdegree assigned.

Thus Algorithm 2 gives a unique clique-transtive orientation $\overline{D}$ of $\overline{G}$ for $n \geqslant 7$. Finally, we give matrices showing $\rho(i,j)$, and the final orientation $\overline{D}$ of $\overline{G}$ when $n = 6$.

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \end{array} &
\begin{bmatrix}
4 & \cdots & \cdots & \blacksquare & \cdots & :: \\
\cdots & 3 & \cdots & \cdots & \blacksquare & :: \\
\cdots & \cdots & 2 & \cdots & :: & \blacksquare \\
\blacksquare & \cdots & \cdots & 4 & :: & :: \\
\cdots & \blacksquare & :: & :: & 3 & :: \\
:: & :: & \blacksquare & :: & :: & 2
\end{bmatrix}
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{cccccc} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 \end{array} \\
\begin{array}{c} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \end{array} &
\begin{bmatrix}
4 & 21 & 30 & \blacksquare & 12 & 04 \\
12 & 3 & 03 & 21 & \blacksquare & 40 \\
03 & 30 & 2 & 12 & 40 & \blacksquare \\
\blacksquare & 12 & 21 & 4 & 04 & 31 \\
21 & \blacksquare & 04 & 40 & 3 & 13 \\
40 & 04 & \blacksquare & 13 & 31 & 2
\end{bmatrix}
\end{array}
$$

In the same manner as with Algorithm 2 in the $n \geqslant 7$ case, one can easily verify that Algorithm 1 gives this unique clique-transitive orientation. This completes the proof of Theorem 9.1. □

## 9.3 Possible generalisations to $C_n^p$

The method used for $C_n^2$ (outlined in Section 9.2.1) could in principle be adapted to show that $\mathrm{ch}'(C_n^p) \leqslant 2p+1$ whenever $2 \leqslant p \leqslant \frac{1}{2}n - 1$. This would verify the LECC for all powers of odd cycles. We have had some limited success in this direction, but it is felt that a deeper understanding is required to make further progress. We now give some properties which such a generalisation should possess.

Let $G = L(C_n^p)$. Then $G$ is the disjoint union of cliques $Q_0, \dots, Q_{n-1}$ corresponding to the vertices of $C_n^p$, and each such clique is isomorphic to $K_{2p}$. Extend each clique $Q_i$ by a new vertex $w_i$, and let $\overline{G}$ be the edge-disjoint union of the extended cliques $\overline{Q}_0, \dots, \overline{Q}_{n-1}$. Then

$$V(\overline{G}) = \{(i, i+1), \dots, (i, i+p) : 0 \leqslant i < n\} \cup \{w_0, \dots, w_{n-1}\}.$$

To use Theorem 9.2, we need to define two things: the function $\rho : V(G) \to \mathbb{N}$ so that $\max\{\rho(v) : v \in V(G)\} \leqslant 2p$, and the blocking-out values $b_1, \dots, b_{n-1}$, giving the outdegrees of $w_0, \dots, w_{n-1}$. The number of edges between $V(G)$ and $\{w_0, \dots, w_{n-1}\}$ is $2np$. Counting the number of such edges directed in and out of $V(G)$, we see that the $b_i$ and $\rho$ must satisfy

$$\sum_{i=1}^{n-1} b_i + \sum_{v \in V(G)} \rho(v) - np(2p-1) = 2np,$$

since the number of edges inside each clique $Q_i$ is $p(2p-1)$. As before, one can represent an orientation of $\overline{G}$ with an $n \times n$ matrix. The vertices of $\overline{G}$ correspond to the cells of $M$ which are distance at most $p$ along any row or column from the leading diagonal (working modulo $n$).

As an example, we show that $\mathrm{ch}'(C_8^3) \leqslant 7$. The following two matrices represent $G = L(C_8^3)$. In the first matrix, the blocking-out values $b_i$ are given along the leading diagonal. Also, for each vertex $(i, j) \in V(G)$, the value of $\rho(i, j)$ is represented in cells $(i, j)$ and $(j, i)$ by " ⁙ " if $\rho(i, j) = 5$ or by " ::: " if $\rho(i, j) = 6$. The second matrix gives the outdegree pairs in an orientation $\overline{D}$ of $\overline{G}$; it is easy to verify that $\overline{D}$ is clique-transitive. After these matrices, we give Algorithm 3, a list of assignments of outdegree pairs which produce this orientation.

$$
\begin{array}{cc}
 & \begin{matrix} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 \end{matrix} \\
\begin{matrix} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \end{matrix} &
\begin{bmatrix}
4 & \text{⁙} & \text{⁙} & \text{⁙} & \blacksquare & \text{⁙} & \text{⁙} & \text{:::} \\
\text{⁙} & 4 & \text{⁙} & \text{⁙} & \text{⁙} & \blacksquare & \text{:::} & \text{:::} \\
\text{⁙} & \text{⁙} & 6 & \text{⁙} & \text{⁙} & \text{:::} & \blacksquare & \text{⁙} \\
\text{⁙} & \text{⁙} & \text{⁙} & 5 & \text{:::} & \text{⁙} & \text{:::} & \blacksquare \\
\blacksquare & \text{⁙} & \text{⁙} & \text{:::} & 5 & \text{:::} & \text{:::} & \text{:::} \\
\text{⁙} & \blacksquare & \text{:::} & \text{⁙} & \text{:::} & 3 & \text{:::} & \text{:::} \\
\text{⁙} & \text{:::} & \blacksquare & \text{:::} & \text{:::} & \text{:::} & 3 & \text{:::} \\
\text{:::} & \text{:::} & \text{⁙} & \blacksquare & \text{:::} & \text{:::} & \text{:::} & 6
\end{bmatrix}
\end{array}
\qquad
\begin{array}{cc}
 & \begin{matrix} \overline{Q}_0 & \overline{Q}_1 & \overline{Q}_2 & \overline{Q}_3 & \overline{Q}_4 & \overline{Q}_5 & \overline{Q}_6 & \overline{Q}_7 \end{matrix} \\
\begin{matrix} \overline{Q}_0 \\ \overline{Q}_1 \\ \overline{Q}_2 \\ \overline{Q}_3 \\ \overline{Q}_4 \\ \overline{Q}_5 \\ \overline{Q}_6 \\ \overline{Q}_7 \end{matrix} &
\begin{bmatrix}
4 & 50 & 05 & 23 & \blacksquare & 32 & 14 & 60 \\
05 & 4 & 14 & 32 & 43 & \blacksquare & 60 & 51 \\
50 & 41 & 6 & 14 & 32 & 06 & \blacksquare & 23 \\
32 & 23 & 41 & 5 & 60 & 14 & 06 & \blacksquare \\
\blacksquare & 32 & 23 & 06 & 5 & 60 & 15 & 42 \\
23 & \blacksquare & 60 & 41 & 06 & 3 & 51 & 15 \\
41 & 06 & \blacksquare & 60 & 51 & 15 & 3 & 24 \\
06 & 15 & 32 & \blacksquare & 24 & 51 & 42 & 6
\end{bmatrix}
\end{array}
$$

**Algorithm 3** (for $n = 8$, $p = 3$)

Assign {0,6} at vertex (7,0); {0,6} at (6,1); {6,0} at (3,4); {6,0} at (4,5); {0,6} at (2,5); {0,6} at (3,6); {5,0} at (0,1); {1,5} at (7,1); {0,5} at (0,2); {5,1} at (5,6); {1,5} at (4,6); {1,5} at (5,7); {2,3} at (0,3); {1,4} at (3,5); {2,3} at (5,0); {4,1} at (6,0); {2,4} at (6,7); {1,4} at (1,2); {3,2} at (1,3); {1,4} at (2,3); {2,3} at (1,4); {3,2} at (2,4); {4,2} at (4,7); {3,2} at (7,2).

One can check that these assignments uniquely determine the clique-transitive orientation $\overline{D}$ of $\overline{G}$. By Theorem 9.2, this proves that $\mathrm{ch}'(C_8^3) \leqslant 7$. However, we have not found any definitions of $\rho$ which work for $n > 9$ when $p = 3$, or when $p > 3$.

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