Value Distribution of Some Families of Meromorphic Functions

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Contents

Abstract

This thesis is structured as follows.

In Chapter 1, we provide background material about the concepts and techniques which are used in this thesis.

In Chapter 2, we prove results which provide two new criteria for normal families of meromorphic functions, and which extend a recent result of Bergweiler and Langley.

In Chapter 3, we extend a theorem of Bergweiler and Langley, and provide a result regarding the growth of a particular type of meromorphic function in an unbounded annulus.

In Chapter 4, we extend two value distribution theorems of Langley and Zheng.

In Chapter 5, we prove normal families and value distribution results in connection with composite functions.

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Chapter 1

Background material

This chapter is a reference chapter for the concepts and techniques used in this thesis. We provide subsections on each topic that arises, although we emphasise that these are restricted to what is necessary for this thesis, and therefore we provide references for further reading. For brevity, we omit most proofs, and all unattributed results are standard.

1.1 Analytic and meromorphic functions

Let Ω be a domain in $\mathbb{C}$. A function f is *complex differentiable* at a point $a \in \Omega$ if there is a complex number $f'(a)$ such that

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

A function f is *analytic* on Ω if at any point $a \in \Omega$, f is complex differentiable on an open set which contains a. A function f is *entire* if f is analytic in $\mathbb{C}$.

Let f be an analytic function on Ω and let $a \in \Omega$. Then f has a *zero of multiplicity* $m \geq 1$ at a, if $f(a) = 0$ and if there is an analytic function g on Ω such that $f(z) = (z-a)^m g(z)$, where $g(a) \neq 0$. If $m = 1$, then a is a *simple zero* of f.

We define $\log^+ x$ as follows

$$\begin{aligned} \log^+ x &= \begin{cases} \log x & \text{if } x \geq 1, \\ 0 & \text{if } 0 < x < 1. \end{cases} \\ &= \max\{0, \log x\}. \end{aligned}$$

For an entire function f, we define the *order* of f, denoted $\rho(f)$, by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{\log r} \tag{1.1}$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$.

Theorem 1.1.1 (Pólya, [33])**.** *Suppose that f and g are entire functions and that $f \circ g$ has finite order, $\rho(f \circ g) < \infty$. Then either f has zero order, or g is a polynomial. Hence, if f and g are entire functions such that $\rho(f) > 0$ and g is a transcendental function, then $f \circ g$ has infinite order, $\rho(f \circ g) = \infty$.*

The following results from function theory use some of the ideas in this section and provide results which are useful in later chapters. We provide proofs for completeness.

Theorem 1.1.2. *Let f be an entire function of finite order $\rho(f) = \rho < \infty$. Let g be a polynomial of degree d. Then the order of $f \circ g$ is at most $d\rho$, that is, $\rho(f \circ g) \leq d\rho$.*

Proof The result is obvious if g is constant, so assume that g is nonconstant. Let $\sigma > \rho$. Then since $\rho(f) = \rho$, we have that for w large,

$$\log^+ |f(w)| < |w|^\sigma. \tag{1.2}$$

For z large, since g is a polynomial of degree d we have that $|g(z)| \leq c_0|z|^d$, for some positive constant c_0. Then, by (1.2), we have that

$$\log^+ |f(g(z))| < |g(z)|^\sigma < (c_0|z|^d)^\sigma = c_1 r^{d\sigma},$$

where $r = |z|$ and c_1 is a positive constant. Then we have that

$$\begin{aligned} \rho(f \circ g) &= \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r, f \circ g)}{\log r} \\ &\leq \limsup_{r\to\infty} \frac{\log^+ c_1 r^{d\sigma}}{\log r} \\ &\leq \limsup_{r\to\infty} \frac{\log^+ c_1 + d\sigma \log^+ r}{\log r} \\ &= d\sigma. \end{aligned}$$

Theorem 1.1.3. *Let f be an entire function with $\rho(f) < \infty$. Suppose f has m zeros in $\mathbb{C}$, counting multiplicities, for some $m \in \mathbb{N} \cup \{0\}$. Then we can write*

$$f(z) = P_1(z)e^{P_2(z)}$$

where P_1 and P_2 are polynomials of degree m and n respectively, with m as above and $n \in \mathbb{N} \cup \{0\}$.

Proof If $m = 0$ then set $P_1(z) = 1$. Otherwise, since f has $m \geq 1$ zeros on $\mathbb{C}$, it is clear by factorisation that P_1 is a polynomial of degree m, and that f can be written as $P_1(z)e^{P_2(z)}$ for some entire function P_2. Then we have that for z large, $|P_1(z)| \geq c|z|^m$ for some positive constant c. And so, for r large, we have that

$$\begin{aligned}
\log^+ M(r,f) = \log^+ M(r, P_1 e^{P_2}) &= \log^+ \max\{|P_1(z)e^{P_2(z)}| : |z| = r\} \\
&= \log^+ \max\{|P_1(z)||e^{P_2(z)}| : |z| = r\} \\
&\geq \log^+ \max\{c|z|^m |e^{P_2(z)}| : |z| = r\} \\
&= \log^+ (cr^m \max\{|e^{P_2(z)}| : |z| = r\}) \\
&= \log^+ (cr^m M(r, e^{P_2})) \\
&\geq \log^+ M(r, e^{P_2}),
\end{aligned}$$

since c is a positive constant, $m \geq 1$ and r is large. Then we have, whether or not $m = 0$, that

$$\begin{aligned}
\rho(f) = \rho(P_1 e^{P_2}) &= \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r, P_1 e^{P_2})}{\log r} \qquad (1.3) \\
&\geq \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r, e^{P_2})}{\log r} \\
&= \rho(e^{P_2}).
\end{aligned}$$

Suppose now that P_2 is a transcendental function. Then since $\rho(e^z) = 1$ we have, by Theorem 1.1.1, that $\rho(e^{P_2}) = \infty$ which, by (1.3), contradicts the fact that $\rho(f) < \infty$. Therefore P_2 is a polynomial.

A function f has an *isolated singularity* at a point a if f is not defined at a, but there is some $s > 0$ such that f is analytic in the punctured disc $\{z : 0 < |z - a| < s\}$. Then f has *Laurent expansion* at a given by

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k.$$

The function f has a *removable singularity* at a if all the coefficients of negative powers are zero; f has a *pole* at a if all but finitely many of the coefficients of negative powers are 0; and f has an *essential singularity* at a otherwise.

Thus, a function f has a *pole* at a if $|f(z)| \to \infty$ as $z \to a$. The *residue* of f at a, denoted by $\mathrm{Res}(f, a)$, is a_{-1}, the coefficient of the $(z-a)^{-1}$ term in the Laurent expansion of f.

A function f is *meromorphic* on Ω if given any point $a \in \Omega$, f is either analytic at a, or f has a pole at a. A function f is meromorphic at ∞ if $f(1/z)$ is meromorphic at 0.

Let f be a meromorphic function on Ω and let $a \in \Omega$. Then f has a *pole of multiplicity* $m \geq 1$ *at* a if $1/f(z)$ has a zero of multiplicity m at a. If $m = 1$, then a is a *simple pole* of f.

Let f be a meromorphic function and $z_0 \in \mathbb{C}$. If $f(z_0) \neq 0, \infty$ then an analytic branch of $\log f(z)$ may be defined near z_0, with derivative

$$\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)}$$

and $\frac{f'(z)}{f(z)}$ is called the *logarithmic derivative* of f. The branch $\log f(z)$ may be continued along any path which avoids zeros and poles of f. Then if z_0 and z are not zeros or poles of f, we have that

$$\log f(z) = \log f(z_0) + \int_{z_0}^{z} \frac{f'(t)}{f(t)} dt$$

for such a path. Since changing the path adds an integer multiple of $2\pi i$, by the Residue Theorem (Theorem 1.10.5), we get the following identity

$$f(z) = f(z_0) \exp\left(\int_{z_0}^{z} \frac{f'(t)}{f(t)} dt \right). \tag{1.4}$$

We refer the reader to [1] for further reading.

1.2 Elementary Nevanlinna theory

Let f be a meromorphic function in $|z| \leq r$, for some $r > 0$. The *Nevanlinna characteristic* $T(r, f)$ is defined to be

$$T(r, f) = m(r, f) + N(r, f), \tag{1.5}$$

where $m(r, f)$ is the *proximity function* given by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \tag{1.6}$$

and $N(r, f)$ is the *integrated counting function* given by

$$N(r, f) = \int_0^r (n(t, f) - n(0, f)) \frac{dt}{t} + n(0, f) \log r, \tag{1.7}$$

where $n(r, f)$ denotes the number of poles of f in $|z| \leq r$, counting multiplicities.

Theorem 1.2.1. *If f is a rational function,*

$$T(r, f) = O(\log r) \qquad \text{as } r \to \infty,$$

that is, there exists $M > 0$ such that $T(r, f) \leq M \log r$ as $r \to \infty$.

Theorem 1.2.2. *If f is transcendental and meromorphic in $\mathbb{C}$, then*

$$T(r, f)/\log r \to \infty \qquad \text{as } r \to \infty.$$

Thus, $\log r = o(T(r, f))$ as $r \to \infty$ since

$$\frac{\log r}{T(r, f)} \to 0 \qquad \text{as } r \to \infty.$$

Theorem 1.2.3. *Let $f_1, \ldots, f_k$ be meromorphic functions. Then*

$$m(r, \sum_{j=1}^{k} f_j(z)) \leq \sum_{j=0}^{k} m(r, f_j(z)) + \log k, \tag{1.8}$$

$$m(r, \prod_{j=1}^{k} f_j(z)) \leq \sum_{j=1}^{k} m(r, f_j(z)), \tag{1.9}$$

$$N(r, \sum_{j=1}^{k} f_j(z)) \leq \sum_{j=1}^{k} N(r, f_j(z)), \tag{1.10}$$

$$N(r, \prod_{j=1}^{k} f_j(z)) \leq \sum_{j=1}^{k} N(r, f_j(z)), \tag{1.11}$$

$$T(r, \sum_{j=1}^{k} f_j(z)) \leq \sum_{j=1}^{k} T(r, f_j(z)) + \log k, \tag{1.12}$$

$$T(r, \prod_{j=1}^{k} f_j(z)) \leq \sum_{j=1}^{k} T(r, f_j(z)). \tag{1.13}$$

Theorem 1.2.4 (First fundamental theorem)**.** *Let f be meromorphic and nonconstant in $\mathbb{C}$ and let $a \in \mathbb{C}$. Then*

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1). \tag{1.14}$$

We deduce that for $a = 0$,

$$T(r, 1/f) = T(r, f) + O(1). \tag{1.15}$$

We use $(n.e.)$ as an abbreviation for *"nearly everywhere"*, that is, to denote the phrase *"outside a set of finite measure"*.

Lemma 1.2.5 (Lemma of the logarithmic derivative)**.** *Let f be meromorphic and nonconstant in the plane. Then there are positive constants c_1 and c_2 such that*

$$m(r, f'/f) \leq c_1 \log r + c_2 \log T(r, f) = O(\log rT(r, f)) \qquad (n.e.), \tag{1.16}$$

as $r \to \infty$.

Theorem 1.2.6. *Let* f *be a meromorphic function. Then*

$$T(r, f') \le T(r, f) + N(r, f) + O(\log rT(r, f)) = O(T(r, f)) \qquad (n.e.). \tag{1.17}$$

Then using (1.13), (1.15) and (1.17),

$$T(r, f'/f) \le T(r, f') + T(r, f) + O(1) = O(T(r, f)) \qquad (n.e.). \tag{1.18}$$

Theorem 1.2.7. *Let* f *be meromorphic and nonconstant on* $|z| \le R$*. If* f *has no poles and* $0 < r < R$*, then*

$$T(R, f) \le \log^+ M(R, F), \qquad \log M(r, f) \le \left(\frac{R + r}{R - r}\right) T(R, f).$$

For a meromorphic function f on $\mathbb{C}$, we define the *order* of f, $\rho(f)$ to be

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}. \tag{1.19}$$

We note that by Theorem 1.2.7, we have that $T(r, f)$ and $\log M(r, f)$ are comparable for entire functions. Then we note that for entire functions, (1.1) and (1.19) are equivalent.

Theorem 1.2.8. *If* f *is a nonconstant and meromorphic function in* $\mathbb{C}$*, then*

$$\rho(f^{(k)}) \le \rho(f).$$

For a meromorphic function f and $a \in \mathbb{C}$, we define the *deficiency* of f at a, $\delta(a, f)$ to be

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)},$$

and,

$$\delta(\infty, f) = \liminf_{r\to\infty} \frac{m(r,f)}{T(r,f)}.$$

Finally, following the notation in [22], for f meromorphic in the plane, we denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = O(\log r + \log^+ T(r,f)) \tag{1.20}$$

as $r \to \infty$ outside a set of finite measure, not necessarily the same set at each occurrence.

We refer the reader to [22] for further reading.

1.3 Nevanlinna theory on an annulus

For $r \geq 0$, define an annulus $\mathcal{A}(r)$ by

$$\mathcal{A}(r) = \{z : r \leq |z| < \infty\}. \tag{1.21}$$

The previous subsection contains material concerning Nevanlinna theory for functions meromorphic in the plane. This subsection provides a variant of Nevanlinna theory for functions meromorphic in an annulus. A standard reference for this variant of Nevanlinna theory is [7], but we outline the necessary material here.

A point z_0 is called a *limit point* of a set Ω if every neighbourhood of z_0 intersects with Ω other than at the point z_0 itself.

Let $r_0 > 0$ and let f be a function meromorphic in the annulus $\mathcal{A}(r_0)$. That is, we say that a function f is meromorphic in $\mathcal{A}(r_0)$ if f is meromorphic in a domain containing $\mathcal{A}(r_0)$. Then the poles of f cannot have a limit point in $\mathcal{A}(r_0)$. Similarly, for $f \not\equiv 0$, the zeros of f cannot have a limit point in $\mathcal{A}(r_0)$. We then have the following representation.

Theorem 1.3.1 (Valiron, [42]). *Let $r_0 > 0$ and let f be a nonconstant meromorphic function in an annulus $\mathcal{A}(r_0)$, as defined by* (1.21). *Then there exist an integer m, a function F meromorphic in $\mathbb{C}$, and a function ϕ analytic and non-zero in $\mathcal{A}(r_0)$ such that*

$$\phi(\infty) = \lim_{z\to\infty} \phi(z) = 1$$

and

$$f(z) = z^m F(z)\phi(z)$$

on $\mathcal{A}(r_0)$. The zeros and poles of F on $\mathbb{C}$ are precisely the zeros and poles of f on $\mathcal{A}(r_0)$.

We then define for $r \geq r_0$,

$$\begin{aligned} m(r, f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \\ &= m(r, F) + O(\log r). \end{aligned}$$

Next we define

$$N(r, f) = \int_{r_0}^{r} n(t, f)\frac{dt}{t} = N(r, F)$$

where $n(t, f)$ is the number of poles of f in $r_0 \leq |z| \leq t$, counting multiplicities.

We define $T(r, f)$ as in (1.5), which gives

$$\begin{aligned} T(r, f) &= m(r, f) + N(r, f) \\ &= m(r, F) + N(r, F) + O(\log r) \\ &= T(r, F) + O(\log r). \end{aligned} \tag{1.22}$$

Also, $f(z) = z^m F(z)\phi(z)$ gives

$$f'(z) = mz^{m-1}F(z)\phi(z) + z^m F'(z)\phi(z) + z^m F(z)\phi'(z),$$

and so we have

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + \frac{F'(z)}{F(z)} + \frac{\phi'(z)}{\phi(z)}.$$

Then by (1.8), (1.16), (1.22) and Theorem 1.2.1, we have that

$$\begin{aligned}
m(r, f'/f) &\leq m(r, m/z) + m(r, F'/F) + m(r, \phi'/\phi) + \log 3 \\
&\leq O(\log r) + O(\log rT(r, F)) \qquad (n.e.) \\
&\leq O(\log r + \log^+ T(r, F)) \qquad (n.e.) \\
&\leq O(\log r + \log^+ T(r, f)) \qquad (n.e.) \\
&\leq O(\log rT(r, f)) \qquad (n.e.)
\end{aligned}$$

which is the Lemma of the logarithmic derivative (Lemma 1.2.5).

Next, if $T(r, f) = O(\log r)$ as $r \to \infty$ then $T(r, F) = O(\log r)$, and so by Theorem 1.2.1, F is a rational function, and thus $\lim_{z\to\infty} f(z)$ exists.

Finally, we note also that results analogous to those in § 1.2 exist, and in direct analogy with (1.20), for f meromorphic in $\mathcal{A}(r_0)$, we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = O(\log r + \log^+ T(r, f)) \tag{1.23}$$

as $r \to \infty$ outside a set of finite measure, not necessarily the same set at each occurrence.

We refer the reader to [7] for further reading.

1.4 The spherical metric

We note first that the *spherical metric* is also called the *chordal metric*. Also, we use the convention $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ to denote the *extended complex plane*.

A sequence of functions (f_n) *converges (spherically) uniformly on compact subsets* of a domain Ω to a function $f(z)$ if, for any compact subset $K \subseteq \Omega$ and $\epsilon > 0$, there exists a number $n_0 = n_0(K, \epsilon)$ such that $n \geq n_0$ implies

$$|f_n(z) - f(z)| < \epsilon, \qquad (\chi(f_n(z), f(z)) < \epsilon)\,,$$

for all $z \in K$.

Given $z_1, z_2 \in \mathbb{C}$ the *spherical distance*, or *chordal distance* $\chi(z_1, z_2)$ is

$$\chi(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}.$$

If $z_2 = \infty$, then

$$\chi(z_1, \infty) = \frac{1}{\sqrt{1 + |z_1|^2}}.$$

Let f be meromorphic in a domain Ω. If $z \in \Omega$ is not a pole of f, then the *spherical derivative* $f^\#(z)$ of f at z is

$$f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2}. \tag{1.24}$$

If a is a pole of f, define

$$f^\#(a) = \lim_{z \to a} \frac{|f'(z)|}{1 + |f(z)|^2}.$$

We note that $f^\#(z) = (1/f(z))^\#$ for all $z \in \mathbb{C}^*$.

Lemma 1.4.1 ([4]). *Let f be a meromorphic function with bounded spherical derivative, that is $|f^{\#}(z)| \leq M$ for some constant M. Then f is of order at most 2, that is, $\rho(f) \leq 2$. Thus, if f is a meromorphic function with $\rho(f) > 2$, then f has unbounded spherical derivative.*

We refer the reader to [37] for further reading.

1.5 Normal families

A family $\mathcal{G}$ of meromorphic (analytic) functions is a *normal family* on a domain Ω, if every sequence of functions (f_n) in $\mathcal{G}$ contains a subsequence which converges uniformly on compact sub-regions either to a meromorphic (analytic) limit or identically to ∞, with respect to the spherical metric.

We note that if $\{f \in \mathcal{G}\}$ is a normal family, then $\{1/f : f \in \mathcal{G}\}$ is a normal family also.

Example 1.5.1. *The family of functions $\{f_n(z) = \frac{z}{n} : n \in \mathbb{N}\}$ is a normal family on $\mathbb{C}$.*

Example 1.5.2. *The family of functions $\{f_n(z) = nz : n \in \mathbb{N}\}$ is not normal on any domain containing the origin. This is because $f_n(0) = 0$, whereas for $z \in \mathbb{R}^+$ we have that $f_n(z) = nz \to \infty$ as $n \to \infty$.*

The following result is by Pang and Zalcman, and we refer to it as the *Pang-Zalcman lemma.*

Lemma 1.5.3 (Pang-Zalcman lemma)**.** *Let $\mathcal{G}$ be a family of functions meromorphic in the unit disc $B(0,1)$, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{G}$. Then if $\mathcal{G}$ is not normal, there exist for each $-1 < \alpha \leq k$,*

(a) *a number $0 < r < 1$,*

(b) *points z_n with $|z_n| < r$,*

(c) *functions $f_n \in \mathcal{G}$, and*

(d) *positive numbers $\rho_n \to 0$,*

such that

$$g_n(z) = \frac{f_n(z_n + \rho_n z)}{\rho_n^\alpha} \to g(z)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$ such that $g^\sharp(z) \leq g^\sharp(0) = kA + 1$.

As a point of interest, we note that the Pang-Zalcman lemma encompasses a previous result by Zalcman which is referred to as the *Zalcman lemma*, and which can be seen as the case where $\mathcal{G}$ is not normal at 0 and where $z_n \to 0$ with $\alpha = k = 0$. We refer the reader to [30] and [31] for further reading.

A family of functions $\mathcal{G}$ is *locally bounded* on a domain Ω if, for each $z_0 \in \Omega$, there is a positive number $M = M(z_0)$ and a neighbourhood U of z_0 contained in Ω, such that $|f(z)| \leq M$ for all $z \in U$ and for all $f \in \mathcal{G}$.

Theorem 1.5.4 (Montel-Vitali theorem)**.** *If $\mathcal{G}$ is a locally bounded family of analytic functions on a domain Ω, then $\mathcal{G}$ is a normal family in Ω.*

Finally, we include the Bloch principle. We note that this is a heuristic principle, and that counterexamples do exist, see [36]. However, we note that the Bloch principle is true in most important known cases, and is a useful tool for detecting criteria for normal families.

Bloch principle *A family of meromorphic (analytic) functions which have a common property $\mathcal{P}$ on a domain Ω will in general be a normal family if $\mathcal{P}$ reduces a meromorphic (analytic) function in $\mathbb{C}$ to a constant.*

We refer the reader to [30], [31] and [37] for further reading.

1.6 Wronskians

Let $f_1, \ldots, f_k$ be meromorphic functions. We define the *Wronskian* $W(f_1, \ldots, f_k)$ as follows

$$W(f_1, \ldots, f_k) = \begin{vmatrix} f_1 & \ldots & f_k \\ f_1' & \ldots & f_k' \\ \vdots & & \vdots \\ f_1^{(k-1)} & \ldots & f_k^{(k-1)} \end{vmatrix}.$$

Following the notation in [25], we denote by $W_j(f_1, \ldots, f_k)$, for $j = 0, \ldots, k-1$, the determinant which is obtained from $W(f_1, \ldots, f_k)$ by replacing the row $(f_1^{(j)}, \ldots, f_k^{(j)})$ by $(f_1^{(k)}, \ldots, f_k^{(k)})$.

Lemma 1.6.1. *Let $f_1, \ldots, f_k$ be meromorphic functions on a domain Ω. Then $W(f_1, \ldots, f_k)$ vanishes identically on Ω, if and only if $f_1, \ldots, f_k$ are linearly dependent on Ω.*

Lemma 1.6.2. *Let $f_1, \ldots, f_k, g$ be meromorphic functions and $c_1, \ldots, c_k$ be complex numbers. Then*

(a) $W(c_1 f_1, \ldots, c_k f_k) = c_1 \ldots c_k W(f_1, \ldots, f_k)$.

(b) $W(f_1, \ldots, f_k, 1) = (-1)^k W(f_1', \ldots, f_k')$.

(c) $W(gf_1, \ldots, gf_k) = g^k W(f_1, \ldots, f_k)$.

(d) $W(f_1, \ldots, f_k) = f_1^k W\left(\left(\frac{f_2}{f_1}\right)', \ldots, \left(\frac{f_k}{f_1}\right)'\right)$.

For a function f and functions $a_0, \ldots, a_{k-1}$ we define a homogeneous linear differential operator L by

$$L(f) = f^{(k)} + a_{k-1} f^{(k-1)} + \ldots + a_0 f. \tag{1.25}$$

The following lemma combines results from [17] and [25].

Lemma 1.6.3. *Let $k \geq 1$. Let $a_0, \ldots, a_{k-1}$ be meromorphic functions in a domain Ω. Let $f_1, \ldots, f_k$ be linearly independent meromorphic functions in Ω that satisfy the homogeneous linear differential equation $L(w) = 0$, where L is defined by (1.25). Then the a_j can be written in the form*

$$a_j = -\frac{W_j(f_1, \ldots, f_k)}{W(f_1, \ldots, f_k)} \tag{1.26}$$

for $j = 0, \ldots, k-1$ and, in particular,

$$a_{k-1} = -\frac{W(f_1, \ldots, f_k)'}{W(f_1, \ldots, f_k)}. \tag{1.27}$$

The poles of a_j in Ω have multiplicity at most $k-j$ and can only arise among the poles of $f_1,\dots,f_k$ and the zeros of $W(f_1,\dots,f_k)$. Furthermore, if f is meromorphic in Ω then

$$W(f_1,\dots,f_k,f) = W(f_1,\dots,f_k)L(f). \tag{1.28}$$

Thus we have that there is a link between the k linearly independent solutions $f_1,\dots,f_k$ of a differential equation

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \dots + a_0 f = 0$$

and the coefficients $a_0,\dots,a_{k-1}$ of that equation. Thus examination of the a_j leads to results about the f_j. This approach was used in [19] by Frank and Hellerstein.

We refer the reader to [25] for further reading.

1.7 Analytic continuation

Analytic continuation is where we extend a given analytic or meromorphic function to a larger domain, or where we continue a function along a curve. Here we use definitions and information from [14] to illustrate this method.

A *function element* is a pair (f,G) where G is a domain and f is an analytic function in G. For a given function element (f,G) define the *germ of f at a*, denoted $[f]_a$, to be the collection of all function elements (g,D) such that $a \in D$ and $f(z) = g(z)$ for all z in a neighbourhood of a.

Let $\gamma : [0,1] \to \mathbb{C}$ be a path and suppose that for each $t \in [0,1]$ there is a function element (f_t, D_t) such that

(a) $\gamma(t) \in D_t$;

(b) for each $t \in [0,1]$ there is a $\delta > 0$ such that $|s-t| < \delta$ implies $\gamma(s) \in D_t$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$.

Then we say that (f_1, D_1) is the *analytic continuation of* (f_0, D_0) *along the path* γ, or that (f_1, D_1) is *obtained from* (f_0, D_0) *by analytic continuation along* γ.

We note that (b) gives

$$f_s(z) = f_t(z)$$

on a neighbourhood of $\gamma(s)$, whenever $|s-t| < \delta$.

If $\gamma : [0,1] \to \mathbb{C}$ is a path from a to b and $\{(f_t, D_t) : 0 \leq t \leq 1\}$ is an analytic continuation along γ, then the germ $[f_1]_b$ is the *analytic continuation of* $[f_0]_a$ *along* γ.

If (f, G) is a function element then the *complete analytic function obtained from* (f, G) is the collection $\mathcal{C}$ of all germs $[g]_b$ for which there is a point a in G and a path γ from a to b such that $[g]_b$ is the analytic continuation of $[f]_a$ along γ.

A collection of germs $\mathcal{C}$ is called a *complete analytic function* if there is a function element (f, G) such that $\mathcal{C}$ is the complete analytic function obtained from (f, G).

Let (f, D) be a function element and let G be a domain which contains D. Then (f, D) *admits unrestricted analytic continuation in* G if for any path γ in G with initial point in D there is an analytic continuation of (f, D) along γ.

A curve $\gamma : [a, b] \to \mathbb{C}$ is *rectifiable* if

$$\sup \left\{ \sum_{k=1}^{n} |\gamma(x_k) - \gamma(x_{k+1})| : n \in \mathbb{N},\, a = x_0 < x_1 < \ldots < x_n = b \right\} < \infty.$$

If $\gamma_0, \gamma_1 : [0, 1] \to G$ are two rectifiable curves in G such that $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$ then γ_0 and γ_1 are *fixed end point homotopic* if there is a continuous map $\Gamma : I^2 \to G$ such that

$$\begin{aligned} \Gamma(s, 0) &= \gamma_0(s) & \qquad \Gamma(s, 1) &= \gamma_1(s) \\ \Gamma(0, t) &= a & \qquad \Gamma(1, t) &= b \end{aligned}$$

for $0 \leq s, t \leq 1$.

Theorem 1.7.1 (Monodromy theorem). *Let (f, D) be a function element and let G be a domain containing D such that (f, D) admits unrestricted analytic continuation in G. Let $a \in D$, $b \in G$ and let γ_0 and γ_1 be paths in G from a to b. Let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ and $\{(g_t, D_t) : 0 \leq t \leq 1\}$ be analytic continuations of (f, D) along γ_0 and γ_1 respectively. If γ_0 and γ_1 are fixed end point homotopic in G then*

$$[f_1]_b = [g_1]_b.$$

Corollary 1.7.2. *Let (f, D) be a function element which admits unrestricted analytic continuation in a simply connected domain G. Then there is an analytic function $F : G \to \mathbb{C}$ such that $F(z) = f(z)$ for all $z \in D$.*

We refer the reader to [14] for further reading.

1.8 Iteration theory

Let f be a meromorphic function. A point z is called a *multiple point* of f if $f'(z) = 0$ or if z is a pole of f of multiplicity two or higher.

Let f be a meromorphic function. A point $z \in \mathbb{C}$ is called a *critical point* of f if f fails to be injective in any neighbourhood of z. These critical points consist of the multiple points of f. The image of a critical point, $f(z) = w$, is called a *critical value*.

Let f be a meromorphic function and let $w \in \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. If $f(z) \to w$ as $|z| \to \infty$ along some continuous path γ, then w is called an *asymptotic value* of f and γ is called a *path of determination*.

Let f be a meromorphic function. We denote by

$$f^0 = \mathrm{id}, \quad f^1 = f, \quad f^2 = f \circ f, \quad \ldots, \quad f^{n+1} = f^n \circ f = f \circ f^n, \quad \ldots$$

the *sequence of iterates* of f.

A point $z \in \mathbb{C}$ is called *normal* if the sequence (f^n) is defined and forms a normal family in some neighbourhood of z. (See § 1.4).

The *Fatou set* of f is the set of normal points of f and is denoted by $\mathcal{F}_f$.

The *Julia set* of f is the complement of the Fatou set of f and is denoted by

$$\mathcal{J}_f = \mathbb{C}^* \setminus \mathcal{F}_f.$$

A solution z_0 of the equation $f(z) = z$ is called a *fixpoint* of f and $f'(z_0)$ is called its *multiplier*. We say that ∞ is a fixpoint of f, if 0 is a fixpoint of

$1/f(1/z)$, and we define its multiplier to be the multiplier of the fixpoint 0 of the conjugate $z \mapsto 1/(f(1/z))$.

If z_0 is a fixpoint of f^p, but not a fixpoint of f^n for any n, $0 < n < p$, then

$$\alpha = \{z_0, f(z_0), \dots, f^{p-1}(z_0)\}$$

is called a *cycle of length* p. Then elements of α are called *periodic points*. The *multiplier* $\lambda = \lambda(\alpha)$, is defined to be the multiplier of the fixpoint z_0 of f^p, $\lambda = (f^p)'(z_0)$. By the Chain Rule, we have

$$(f^p)'(z_0) = \prod_{j=0}^{p-1} f'(f^j(z_0)).$$

Then if $f^j(z_0) \neq \infty$ for $0 \leq j < p$, the value of λ depends only on α and not on the particular periodic point z_0.

The cycle α is called

(i) *superattracting* if $|\lambda| = 0$.

(ii) *attracting* if $0 < |\lambda| < 1$.

(iii) *indifferent* if $|\lambda| = 1$.

(iv) *repelling* if $|\lambda| > 1$.

We also classify periodic points in this way.

Theorem 1.8.1. *The Julia set of a function f is the closure of the set of repelling periodic points of f.*

We refer the reader to [5] and [40] for further reading.

1.9 Univalent functions

A single-valued function f is said to be *univalent* on a domain Ω if it never takes the same value twice, that is, if $f(z_1) \neq f(z_2)$ for z_1 and z_2 in Ω with $z_1 \neq z_2$.

The function f is said to be *locally univalent* at a point $z_0 \in \Omega$ if it is univalent in some neighbourhood of z_0.

An analytic univalent function f is also called a *conformal mapping*, and we say that f maps a domain Ω *conformally*.

Theorem 1.9.1 (Koebe distortion theorem). *Let f be a conformal mapping in the open unit disc $B(0,1)$. Then for $0 < r < 1$,*

$$\frac{1-r}{(1+r)^3}|f'(0)| \leq \max_{|z|\leq r} |f'(z)| \leq \frac{1+r}{(1-r)^3}|f'(0)|.$$

We refer the reader to [15] and [34] for further reading.

1.10 Miscellaneous theorems

The following is an alphabetical list of standard theorems from complex analysis which are used in this thesis. We refer the reader to [1] and [37] for further reading.

Theorem 1.10.1 (Argument principle (Special case)). *Let γ be a circle, described once counter-clockwise, and f be a meromorphic function on a domain containing γ and its interior, with no zeros or poles of f on γ.*

Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$$

is equal to the number of zeros minus the number of poles of f, inside γ, where the zeros and poles are counted according to multiplicity.

Theorem 1.10.2 (Hurwitz' theorem (Special case))**.** *Let (f_n) be a sequence of meromorphic functions on a domain Ω which converge spherically uniformly on compact subsets to a function f (which may be $\equiv \infty$). If each $f_n \neq 0$ on Ω then either $f \neq 0$ on Ω, i.e. f has no zeros on Ω, or $f \equiv 0$.*

Theorem 1.10.3 (Maximum principle)**.** *Let f be an analytic and nonconstant function in a region Ω. Then $|f(z)|$ has no maximum in Ω. Furthermore, if Ω is a closed bounded region, then the maximum of $|f(z)|$ is taken on the boundary of Ω.*

Theorem 1.10.4 (Picard's theorem)**.** *Let $a, b, c \in \mathbb{C}^*$ be distinct points. Let f be a meromorphic function which omits a, b, c on $\mathbb{C}$. Then f is constant.*

Theorem 1.10.5 (Residue theorem)**.** *Let f be analytic in a domain Ω, apart from isolated singularities z_j. Let γ be a cycle in Ω avoiding the z_j and such that $n(\gamma, a) = 0$ for all $a \in \mathbb{C} \setminus \Omega$ where $n(\gamma, a)$ is the winding number of γ about a. Then there are just finitely many z_j for which $n(\gamma, z_j) \neq 0$, and*

$$\int_\gamma f(z)dz = 2\pi i \sum n(\gamma, z_j)\mathrm{Res}(f, z_j).$$

Theorem 1.10.6 (Rouché's theorem)**.** *Let Ω be a simply connected domain containing a simple closed curve Γ and its interior. Let f and g be analytic* *functions on* Ω *such that* $|g|$ *strictly dominates* $|f|$*, i.e.* $|g| > |f|$*, on* Γ*. Then* g *and* $f + g$ *have the same number of zeros inside* Γ*.*

Theorem 1.10.7 (Taylor's theorem)**.** *Let* f *be an analytic function on a domain* Ω *and let* $a \in \Omega$*. Then*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k$$

for z *near* a*.*

Theorem 1.10.8 (Weierstrass theorem)**.** *Let* (f_n) *be a sequence of analytic functions on a domain* Ω *which converges uniformly on compact subsets of* Ω *to a function* $f : \Omega \to \mathbb{C}$*. Then* f *is analytic in* Ω *and, for* $k \in \mathbb{N}$*, the sequence of derivatives* $(f_n^{(k)})$ *converges uniformly on compact subsets to* $f^{(k)}$*.*

Chapter 2

Two new criteria for normal families

In this chapter, we present results which provide two new criteria for normal families of meromorphic functions.[1]

Let $a_0, \ldots, a_{k-1}$ be analytic functions on a domain Ω. Let $\mathcal{F}$ be a family of meromorphic functions f on Ω such that $f \neq 0$ and $f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_0 f \neq 0$ on Ω, for all $f \in \mathcal{F}$. Then $\{f'/f : f \in \mathcal{F}\}$ is a normal family. Furthermore, let $a_0, \ldots, a_{k-1}$ be meromorphic functions on a domain Ω. Let $\mathcal{F}$ be a family of meromorphic functions f on Ω such that $f \neq 0$, $f' \neq 0$ and $f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_0 f \neq 0$ on Ω, for all $f \in \mathcal{F}$. Then $\{f'/f : f \in \mathcal{F}\}$ is a normal family. These two new criteria for normal families extend a recent result of Bergweiler and Langley, [6, Corollary 1.1].

[1] The results in this chapter have been published by *Computational Methods and Function Theory*, see [11].

2.1 Introduction

A function f is *nonvanishing* on a domain Ω if it is without zeros there, that is, if $f(z) \neq 0$ for all $z \in \Omega$.

Example 2.1.1. *Let* $f(z) = e^{2z+3}$. *Then* $f(z) \neq 0$ *on* $\mathbb{C}$ *and also* $f^{(k)}(z) = 2^k e^{2z+3} \neq 0$ *on* $\mathbb{C}$ *for* $k \in \mathbb{N}$. *That is,* f *and* $f^{(k)}$ *are nonvanishing on* $\mathbb{C}$.

Since at least as far back as Pólya [32], interest has been shown in determining meromorphic functions f such that f and $f^{(k)}$ are nonvanishing in $\mathbb{C}$ for $k \in \mathbb{N}$. In 1959, Hayman [21] conjectured that the following result would be true.

Theorem 2.1.2. *Let* f *be a meromorphic function in* $\mathbb{C}$ *and let* $k \geq 2$ *be an integer. Suppose* f *and* $f^{(k)}$ *are nonvanishing in* $\mathbb{C}$. *Then* f *has the form* $f(z) = e^{az+b}$ *or* $f(z) = (az+b)^{-n}$ *where* $a, b \in \mathbb{C}$, $a \neq 0$ *and* $n \in \mathbb{N}$.

Hayman [21] proved the $k = 2$ case for entire functions, and Clunie [13] proved the general case for entire functions. In 1976, Frank [18] proved the $k \geq 3$ case of Theorem 2.1.2. In 1977, Frank, Hennekemper and Polloczek [20] considered the case where $k \geq 3$ and $ff^{(k)}$ has finitely many zeros. In 1993, Langley [28] proved the $k = 2$ case of Theorem 2.1.2.

Interest has also been shown in determining meromorphic functions f such that f and $L(f)$ are nonvanishing on $\mathbb{C}$, where L is defined, as in (1.25), by

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1 f' + a_0 f,$$

for $k \in \mathbb{N}$. Steinmetz [39], following the work of Frank and Hellerstein [19], proved results for the case where the a_j are constants and $k \geq 3$.

Brueggemann [8] proved results for the case where the a_j are polynomials, not all constant. Langley ([26], [27]) proved results for the case where the a_j are rational functions. We prove results for the cases where the a_j are analytic functions and meromorphic functions.

We note that by Theorem 2.1.2, if f is an entire function such that f and $f^{(k)}$ are nonvanishing on $\mathbb{C}$ for some $k \geq 2$, then we have that $f(z) = e^{az+b}$ and so $f'(z)/f(z)$ is constant. Then by the Bloch Principle (see § 1.5), we see that Theorem 2.1.2 may provide a criterion for normal families. This is in fact the case. Schwick [38] proved this for families of analytic functions, and Bergweiler and Langley [6] proved it for families of meromorphic functions, their result being stated as follows.

Theorem 2.1.3 (Bergweiler and Langley, [6])**.** *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions that are meromorphic in a domain Ω. Suppose that f and $f^{(k)}$ are nonvanishing in Ω, for all $f \in \mathcal{F}$. Then $\mathcal{G} = \{f'/f : f \in \mathcal{F}\}$ is a normal family in Ω.*

We include the following example to show that Theorem 2.1.3 does not hold for $k = 1$.

Example 2.1.4. *Let $\mathcal{F} = \{f_n(z) = \frac{1}{e^{nz}-1} : n \in \mathbb{N}\}$. Then $f_n(z)$ and $f_n'(z) = -\frac{ne^{nz}}{(e^{nz}-1)^2}$ are nonvanishing in $\mathbb{C}$, for all $n \in \mathbb{N}$. However, $\mathcal{G} = \{f_n'(z)/f_n(z) = \ -ne^{nz}/(e^{nz} - \ 1) : n \in \mathbb{N}\}$ is not a normal family in $\mathbb{C}$ since $f_n'(0)/f_n(0) = \infty$ whereas $f_n'(x)/f_n(x) \to 0$ for $x \in \mathbb{R}^-$, as $n \to \infty$.*

We first extend Theorem 2.1.3 to the following result.

Theorem 2.1.5. *Let $k \geq 2$ and let $\mathcal{F}$ be a family of meromorphic functions in a domain Ω. Let $a_0, \ldots, a_{k-1}$ be analytic functions in Ω. For each $f \in \mathcal{F}$ define $L(f)$, as in* (1.25)*, by*

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1 f' + a_0 f.$$

Suppose that f and $L(f)$ are nonvanishing in Ω for each $f \in \mathcal{F}$. Then $\mathcal{G} = \{f'/f : f \in \mathcal{F}\}$ is a normal family in Ω.

The following example shows that Theorem 2.1.5 cannot be extended to the case where $a_0, \ldots, a_{k-1}$ are meromorphic functions, in the $k = 2$ case.

Example 2.1.6. *Let $\mathcal{F} = \{f_n(z) = e^{nz}/z : n \in \mathbb{N}\}$. Set $a_0(z) = 0$ and $a_1(z) = 2/z$. Then for $k = 2$, we have*

$$L(f_n) = f_n^{(2)} + a_1(z)f_n'(z) + a_0(z)f(z) = \frac{n^2}{z}e^{nz}.$$

Thus for all $n \in \mathbb{N}$ we have that $f_n(z)$ and $L(f_n)$ are nonvanishing in $\mathbb{C}$. However $\mathcal{G} = \{f_n'(z)/f_n(z) = (z - \frac{1}{n})/(z.\frac{1}{n}) : n \in \mathbb{N}\}$ is not a normal family in $\mathbb{C}$, since $f_n'(0)/f_n(0) = \infty$, whereas $f_n'(\frac{1}{n})/f_n(\frac{1}{n}) = 0$.

Nevertheless, by including the extra condition that $f' \neq 0$ on Ω, for all $f \in \mathcal{F}$, we can extend Theorem 2.1.5 to the case where $a_0, \ldots, a_{k-1}$ are meromorphic functions. We state the result as follows.

Theorem 2.1.7. *Let $k \geq 2$ and let $\mathcal{F}$ be a family of meromorphic functions in a domain Ω. Let $a_0, \ldots, a_{k-1}$ be meromorphic functions in Ω. For each $f \in \mathcal{F}$ define $L(f)$, as in* (1.25)*, by*

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1 f' + a_0 f.$$

Suppose that f, f' and $L(f)$ are nonvanishing in Ω for each $f \in \mathcal{F}$. Then $\mathcal{G} = \{f'/f : f \in \mathcal{F}\}$ is a normal family on Ω.

The proof of Theorem 2.1.5 depends on Theorem 2.2.5 below, which in turn depends on several lemmas. For this reason, we provide the preliminaries of the proof of Theorem 2.1.5 in § 2.2. In § 2.3, we prove Theorem 2.2.5, and in § 2.4, we provide an example concerning Theorem 2.2.5. In § 2.5, we provide the main part of the proof of Theorem 2.1.5. In § 2.6, we prove Theorem 2.1.7.

2.2 Proof of Theorem 2.1.5: preliminaries

We follow a similar method of proof to that used by Bergweiler and Langley in their proof of Theorem 2.1.3, see [6, Corollary 1.1].

The following assertion is stated in [6], and we provide a proof here for completeness.

Lemma 2.2.1 (Bergweiler and Langley, [6])**.** *Let F be meromorphic in a simply connected domain Ω. Then there exists a function f meromorphic in Ω such that $F = f'/f$, if and only if all poles of F are simple with integer residues. Furthermore, f is nonvanishing on Ω if and only if all residues of F are negative integers.*

Proof Suppose first that $F = f'/f$ for some function f meromorphic in Ω. Let $a \in \Omega$ and suppose that f has a zero at a of order $m > 0$, or a pole at a of order $-m > 0$. Then near a we have

$$f(z) = (z-a)^m h(z)$$

for some function h meromorphic near a with $h(a) \neq 0, \infty$. Then we have that $f'(z) = m(z-a)^{m-1}h(z) + (z-a)^m h'(z)$ and so

$$F(z) = \frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{h'(z)}{h(z)}.$$

Then a is a simple pole of F and $\mathrm{Res}(F, a) = m$ for some integer m.

Suppose now that F is meromorphic in Ω and that all poles of F in Ω are simple poles with integer residues. Let E be the set of poles of F in Ω. Choose $z_0 \in \Omega \setminus E$ and define f initially on $\Omega \setminus E$ by

$$f(z) = \exp\left(\int_{z_0}^{z} F(t)dt\right)$$

in which the integration is along any piecewise smooth path from z_0 to z in $\Omega \setminus E$.

We show first that the value of $f(z)$ does not vary with choice of path. Let γ_1 and γ_2 be paths from z_0 to z in $\Omega \setminus E$. Set $\gamma = \gamma_1\gamma_2^{-1}$, that is, γ_1 followed by γ_2 backwards. Then γ is a closed curve and since Ω is a simply connected domain we have by the Residue Theorem (Theorem 1.10.5) that

$$\begin{aligned}\int_{\gamma_1} F(t)dt - \int_{\gamma_2} F(t)dt &= \int_{\gamma} F(t)dt \\ &= 2\pi i \sum_{\alpha \in E} \mathrm{Res}(F, \alpha) n(\gamma, \alpha) \\ &= 2\pi i m\end{aligned}$$

for some $m \in \mathbb{Z}$, where $n(\gamma, \alpha)$ is the winding number of γ about α.

Since $f(z) = \exp\left(\int_{z_0}^{z} F(t)dt\right)$ and $\exp(2\pi i\, m) = 1$ we have that the value of $f(z)$ does not vary with choice of path. Also we note that f is analytic and nonvanishing in $\Omega \setminus E$.

Next we investigate $\lim_{z\to a} f(z)$ for some $a \in E$. Since F is meromorphic on Ω and all poles of F are simple with integer residues, we have the following near a,

$$F(t) = H(t) + \frac{m}{t-a}$$

where m is an integer and H is an analytic function near a, and thus bounded near a. Choose $z_1 \in \Omega \setminus E$ with z_1 near a, but not equal to a. Then as $z \to a$ we have by (1.4) that

$$\begin{aligned} f(z) &= \exp\left(\int_{z_0}^{z} \frac{f'(t)}{f(t)}dt\right) \\ &= f(z_1)\exp\left(\int_{z_1}^{z} F(t)dt\right) \\ &= f(z_1)\exp\left(\int_{z_1}^{z} H(t)dt + m\log\frac{z-a}{z_1-a}\right) \\ &= f(z_1)\exp\left(\int_{z_1}^{z} H(t)dt\right)\left(\frac{z-a}{z_1-a}\right)^m \\ &= \phi(z)(z-a)^m \end{aligned}$$

where $\phi = f(z_1)\exp\left(\int_{z_1}^{z} H(t)dt\right)/(z_1-a)^m$ is an analytic function at a with $\phi(a) \neq 0$ since f is nonvanishing in $\Omega \setminus E$. Hence f has a zero or a pole at a, depending on the sign of m. In particular, f is nonvanishing on Ω if each m is a negative integer.

Next, following the method used by Bergweiler and Langley in [6], we define differential polynomials $\Psi_k(F)$ for $k \in \mathbb{N}$ by

$$\Psi_1(F) = F, \qquad \Psi_{k+1}(F) = F\Psi_k(F) + (\Psi_k(F))'. \tag{2.1}$$

Here is an example to illustrate this.

Example 2.2.2. *Let* $F(z) = 2/z$. *Then by (2.1) we have that*

$$\begin{aligned}
\Psi_1(F)(z) &= F(z) = 2/z, \\
\Psi_2(F)(z) &= F^2(z) + F'(z) = 4/z^2 - 2/z^2 = 2/z^2, \\
\Psi_3(F)(z) &= F^3(z) + 3F(z)F'(z) + F^{(2)}(z) = 8/z^3 + 3(2/z)(-2/z^2) + 4/z^3 = 0.
\end{aligned}$$

The link between these operators and nonvanishing derivatives is given by the following lemma from [6], which can be easily proved by induction.

Lemma 2.2.3 (Bergweiler and Langley, [6])**.** *Let* f *be meromorphic in a domain* Ω *and let* $F = f'/f$. *Then for each* $k \in \mathbb{N}$ *we have* $\Psi_k(F) = f^{(k)}/f$.

Example 2.2.4. *Let* $F(z) = 2/z$ *as in Example 2.2.2. Then we note that* $F = f'/f$ *where* $f(z) = z^2$. *Then* $f'(z) = 2z$, $f^{(2)}(z) = 2$ *and* $f^{(3)}(z) = 0$ *and so we have*

$$\begin{aligned}
\Psi_1(F)(z) &= f'/f = 2/z, \\
\Psi_2(F)(z) &= f^{(2)}/f = 2/z^2, \\
\Psi_3(F)(z) &= f^{(3)}/f = 0.
\end{aligned}$$

Next, let $a_0, \ldots, a_{k-1}$ be analytic functions on a domain Ω, and define differential polynomials $\Lambda_k(F)$ for $k \in \mathbb{N}$ by

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_1\Psi_1(F) + a_0. \tag{2.2}$$

Finally, we use the following theorem, noting that it is an extension of [6, Theorem 1.3]. We note also that we use $B(a, r)$ to denote the open disc $B(a, r) = \{z : |z - a| < r\}$.

Theorem 2.2.5. *Let $k \geq 2$ and let $\mathcal{G}$ be a family of functions meromorphic in a domain Ω. Let $a_0, \ldots, a_{k-1}$ be analytic functions on Ω. For $F \in \mathcal{G}$ define $\Lambda_k(F)$, as in* (2.2)*, by*

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_1\Psi_1(F) + a_0.$$

Suppose that there exists $\delta \in (0, 1]$ such that the following conditions hold for all $F \in \mathcal{G}$:

(i) $\Lambda_k(F)$ *has no zeros.*

(ii) if a is a simple pole of F then $|\mathrm{Res}(F, a) - j| \geq \delta$ for $j \in \{0, 1, \ldots, k-1\}$.

(iii) if $c \in \Omega$ and $R > 0$ with $B(c, R) \subset \Omega$, if $B(c, \delta R)$ contains two poles of F, counting multiplicities, and if $B(c, R) \setminus B(c, \delta R)$ contains no poles of F, then

$$\left| \sum_{a \in B(c,\delta R)} \mathrm{Res}(F, a) - (k-1) \right| \geq \delta.$$

Then $\mathcal{G}$ is a normal family.

We prove this theorem in § 2.3, and provide an example in § 2.4 to show that Theorem 2.2.5 cannot be extended to the case where $a_0, \ldots, a_{k-1}$ are meromorphic functions.

2.3 Proof of Theorem 2.2.5

We need several lemmas for the proof of Theorem 2.2.5. The first assertion in the following lemma is proved in [6], and the second is an extension which follows immediately.

Lemma 2.3.1 (Bergweiler and Langley, [6]). *Let $k \geq 2$ be an integer. Let y be meromorphic in a domain Ω, such that if a is a simple pole of y then* $\mathrm{Res}(y, a) \notin \{1, \ldots, k-1\}$. *Let $n \in \mathbb{N}$ be such that $n \leq k$. If y has a pole at a of multiplicity m then $\Psi_n(y)$ has a pole at a of multiplicity nm, and $\Lambda_n(y)$ has a pole at a of multiplicity nm, where $\Lambda_n(y)$ is defined as in* (2.2) *by*

$$\Lambda_n(y) = \Psi_n(y) + a_{n-1}\Psi_{n-1}(y) + \ldots + a_1\Psi_1(y) + a_0$$

where $a_0, \ldots, a_{n-1}$ are analytic functions on Ω.

We need the following theorems of Bergweiler and Langley.

Theorem 2.3.2 (Bergweiler and Langley, [6]). *Let $k \geq 3$ be an integer, and let F be meromorphic and nonconstant in the plane and satisfy both of the following conditions:*

(i) $\Psi_k(F)$ has no zeros.

(ii) if a is a simple pole of F then $\mathrm{Res}(F, a) \notin \{1, \ldots, k-1\}$.

Then F has the form

$$F(z) = \frac{(k-1)z + \alpha}{z^2 + \beta z + \gamma}, \tag{2.3}$$

or

$$F(z) = \frac{1}{\alpha z + \beta}. \tag{2.4}$$

Here $\alpha, \beta, \gamma \in \mathbb{C}$ *with* $\alpha \neq 0$ *in* (2.4).
Conversely, if F *has the form* (2.3) *or* (2.4), *and if* (*ii*) *holds, then* $\Psi_k(F)$ *has no zeros. If* F *has the form* (2.3) *or* (2.4), *but* (*ii*) *does not hold, then* $\Psi_k(F) \equiv 0$.

Theorem 2.3.3 (Bergweiler and Langley, [6])**.** *Let* F *be meromorphic and nonconstant in the plane, such that*

(i) $\Psi_2(F) = F' + F^2$ *has no zeros.*

(ii) if a *is a simple pole of* F, *then* $\mathrm{Res}(F, a) \neq 1$.

(iii) there exists $\delta > 0$ *such that, if* a *is a simple pole of* F, *then* $|\mathrm{Res}(F, a)| \geq \delta$.

Then either F *has the form* (2.3) *with* $k = 2$, *or the form* (2.4).

In the proof of Theorem 2.2.5, we will use the Pang-Zalcman Lemma (Lemma 1.5.3). In particular, we will apply it to the family of all functions $1/f$ with $f \in \mathcal{G}$, for the case where $\alpha = k = 1$. We obtain the following lemma, noting that this approach was also used by Bergweiler and Langley [6, Lemma 4.2].

Lemma 2.3.4. *Let* $\mathcal{G}$ *be a family of functions meromorphic in the unit disc* $B(0, 1)$. *Suppose that there exists* $\delta > 0$ *such that if* $f \in \mathcal{G}$ *has a simple pole* a, *then* $|\mathrm{Res}(f, a)| \geq \delta$. *Then if* $\mathcal{G}$ *is not normal, there exist* $r \in (0, 1)$, *points z_n with $|z_n| < r$, functions $F_n \in \mathcal{G}$ and positive numbers ρ_n tending to zero such that*

$$\rho_n F_n(z_n + \rho_n z) \to F(z)$$

locally uniformly in $\mathbb{C}$, where F is a nonconstant meromorphic function on $\mathbb{C}$ such that $F^\sharp(z) \leq F^\sharp(0) = 1 + 1/\delta$ for all $z \in \mathbb{C}$.

The proof of Theorem 2.2.5 will involve rescaling. We therefore need the following results, and include the proof for completeness.

Lemma 2.3.5. *Let F and g be functions such that*

$$g(z) = \rho F(a + \rho z)$$

where $\rho > 0$, $a \in \mathbb{C}$. Then the following statements are true:

(a) If g has a pole at b, then F has a pole at $a + \rho b$, and $\mathrm{Res}(g, b) = \mathrm{Res}(F, a + \rho b)$.

(b) For each $j \in \mathbb{N}$, we have

$$\Psi_j(g)(z) = \rho^j \Psi_j(F)(a + \rho z)$$

where Ψ_j is defined as in (2.1).

(c) If $a_0, \ldots, a_{k-1}$ are analytic functions and

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_0$$

as defined by (2.2)*, then*

$$\Lambda_k(F)(a+\rho z) = \rho^{-k}\left[\Psi_k(g)(z) + \sum_{j=1}^{k-1} a_j(a + \rho z)\rho^{k-j}\Psi_j(g)(z) + \rho^k a_0(a + \rho z)\right]. \tag{2.5}$$

Proof (a) It is evident that $g(b) = \infty$ implies $F(a + \rho b) = \infty$ also. Let r be small and positive. Then integrating once counter-clockwise and setting $w = a + \rho z$ gives

$$\begin{aligned} 2\pi i \operatorname{Res}(g, b) &= \int_{|z-b|=r} g(z)dz \\ &= \int_{|w-(a+\rho b)|=\rho r} F(w)dw \\ &= 2\pi i \operatorname{Res}(F, a + \rho b) \end{aligned}$$

as required.

(b) We use a proof by induction. For $j = 1$, we have by (2.1),

$$\Psi_1(g)(z) = g(z) = \rho F(a + \rho z) = \rho \Psi_1(F)(a + \rho z).$$

Now assume that $\Psi_j(g)(z) = \rho^j \Psi_j(F)(a + \rho z)$ for all $1 \le j \le n$. Then by (2.1), we have

$$\begin{aligned} \Psi_{n+1}(g)(z) &= g(z)\Psi_n(g)(z) + (\Psi_n(g)(z))' \\ &= g(z)[\rho^n \Psi_n(F)(a + \rho z)] + [\rho^n \Psi_n(F)(a + \rho z)]' \end{aligned}$$

by the inductive hypothesis. Then since $g(z) = \rho F(a + \rho z)$ we have

$$\begin{aligned} \Psi_{n+1}(g)(z) &= \rho^{n+1} F(a + \rho z)\Psi_n(F)(a + \rho z) + \rho^n (\Psi_n(F)'(a + \rho z))\rho \\ &= \rho^{n+1}[F(a + \rho z)\Psi_n(F)(a + \rho z) + \Psi_n(F)'(a + \rho z))] \\ &= \rho^{n+1}\Psi_{n+1}(F)(a + \rho z). \end{aligned}$$

And so $\Psi_j(g)(z) = \rho^j \Psi_j(F)(a + \rho z)$ for all $j \in \mathbb{N}$.

(c) We have by (2.2) and by (b) that

$$\begin{aligned}\Lambda_k(F)(a+\rho z) &= \Psi_k(F)(a+\rho z) + \sum_{j=1}^{k-1} a_j(a+\rho z)\Psi_j(F)(a+\rho z) + a_0(a+\rho z) \\ &= \rho^{-k}\Psi_k(g)(z) + \sum_{j=1}^{k-1} a_j(a+\rho z)\rho^{-j}\Psi_j(g)(z) + a_0(a+\rho z) \\ &= \rho^{-k}\left[\Psi_k(g)(z) + \sum_{j=1}^{k-1} a_j(a+\rho z)\rho^{k-j}\Psi_j(g)(z) + \rho^k a_0(a+\rho z)\right]\end{aligned}$$

as required.

We proceed to the proof of Theorem 2.2.5. We note that it uses essentially the same methods as [6, Theorem 1.3].

Proof of Theorem 2.2.5 Since normality is a local property we can assume, without loss of generality, that Ω is a disc, and that $a_0, \ldots, a_{k-1}$ are bounded on Ω since they are analytic functions. Using a linear change of variables $g(z) = \rho F(a+\rho z)$, for suitable choice of $\rho > 0$ and $a \in \mathbb{C}$, we may assume that Ω is the open unit disc $B(0,1)$ since Lemma 2.3.5 (a) shows that the residues of $g(z)$ are unaltered, and Lemma 2.3.5 (c) shows that if $\Lambda_k(F)$ is nonvanishing then $\tilde{\Lambda}_k(g)$ is nonvanishing, where the coefficients of $\tilde{\Lambda}_k$ are given by (2.5).

Suppose now that $\mathcal{G}$ is not normal. Then by condition (ii) of Theorem 2.2.5, with $j = 0$, we can apply Lemma 2.3.4. Let r, z_n, F_n, ρ_n and F be as in Lemma 2.3.4, so that,

$$g_n(z) = \rho_n F_n(z_n + \rho_n z) \to F(z)$$

locally uniformly in $\mathbb{C}$ as $n \to \infty$.

Let a be a simple pole of F. Then, by Hurwitz' Theorem (Theorem 1.10.2), if n is sufficiently large, g_n has a simple pole at a_n with $a_n \to a$ as $n \to \infty$. By Lemma 2.3.5 (a), $z_n + \rho_n a_n$ is a simple pole of F_n with $\text{Res}(F_n, z_n + \rho_n a_n) = \text{Res}(g_n, a_n)$. Hence with $\delta \in (0,1]$, we deduce from condition (ii) of Theorem 2.2.5 that $|\text{Res}(g_n, a_n) - j| \geq \delta$ for $j \in \{0, 1, \ldots, k-1\}$. Then we have that $|\text{Res}(F, a) - j| \geq \delta$ for $j \in \{0, 1, \ldots, k-1\}$, and so $\text{Res}(F, a) \notin \{1, \ldots, k-1\}$. Then, by Lemma 2.3.1, every pole of F is a pole of $\Psi_k(F)$.

Next, by Lemma 2.3.5 (c), we have that

$$\Lambda_k(F_n)(z_n+\rho_n z) = \rho_n^{-k}\left[\Psi_k(g_n)(z) + \sum_{j=1}^{k-1} a_j(z_n + \rho_n z)\rho_n^{k-j}\Psi_j(g_n)(z) + \rho_n^k a_0(z_n + \rho_n z)\right].$$

By condition (i) of Theorem 2.2.5, this is nonvanishing. Hence,

$$\tilde{\Lambda}_k(g_n)(z) = \Psi_k(g_n(z)) + \sum_{j=1}^{k-1} a_j(z_n + \rho_n z)\rho_n^{k-j}\Psi_j(g_n(z)) + \rho_n^k a_0(z_n + \rho_n z)$$

is nonvanishing, since ρ_n is a sequence of positive numbers. We know by (2.1) that $\Psi_j(g_n)$ is a linear combination of products of g_n and its derivatives. Let E be the set of poles of F. Then, by the Weierstrass Theorem (Theorem 1.10.8), we have that

$$\tilde{\Lambda}_k(g_n)(z) \to \Psi_k(F)(z)$$

as $n \to \infty$, locally uniformly on $\mathbb{C}\setminus E$, since $\rho_n \to 0$ and the a_j are bounded.

By Hurwitz' Theorem (Theorem 1.10.2), either $\Psi_k(F) \equiv 0$ or $\Psi_k(F) \neq 0$ on $\mathbb{C} \setminus E$. In the latter case, we deduce that $\Psi_k(F) \neq 0$ on $\mathbb{C}$ since every pole of F is a pole of $\Psi_k(F)$.

Case 1. $\Psi_k(F) \equiv 0$.

Since $|\mathrm{Res}(F,a) - j| \geq \delta$ for $j \in \{0,1,\ldots,k-1\}$, if a is a simple pole of F, we deduce that F has no poles, since by Lemma 2.3.1, every pole of F is a pole of $\Psi_k(F)$. Thus F is entire and so is the function f defined by setting $f(z) = \exp(\int_0^z F(t)dt)$. Then $F = f'/f$ and thus $f^{(k)}/f = \Psi_k(F) \equiv 0$ by Lemma 2.2.3. Hence f is a polynomial. Then, by the way that we have defined f, we must have that f is constant. Hence $F = f'/f \equiv 0$, which is a contradiction since F is nonconstant.

Case 2. $\Psi_k(F) \neq 0$ *on* $\mathbb{C}$.

It follows from Theorem 2.3.2 for $k \geq 3$ and from Theorem 2.3.3 for $k = 2$ that F has the form (2.3) or (2.4).

Suppose first that F has the form (2.4). Then $1/|\alpha| = |\mathrm{Res}(F, -\beta/\alpha)| \geq \delta$ so that $|\alpha| \leq 1/\delta$. On the other hand, $|\alpha| \geq |\alpha|/(1+|\beta|^2) = F^\sharp(0) = 1+1/\delta$ by Lemma 2.3.4. This is a contradiction.

Suppose second that F has the form (2.3) but is not of the form (2.4). Then F has two poles, counting multiplicities. We also observe that if F is of the form (2.3), then

$$\sum_{a \in F^{-1}(\{\infty\})} \mathrm{Res}(F,a) = k - 1 \tag{2.6}$$

by the Residue Theorem (Theorem 1.10.5). Next, choose $R > 0$ such that these poles are contained in $B(0, \delta R)$. Since F has no other poles we deduce from Hurwitz' theorem (Theorem 1.10.2) that for n sufficiently large, g_n has two poles in $B(0, \delta R)$, but no poles in $B(0, R) \setminus B(0, \delta R)$. Thus F_n has two poles in $B(z_n, \delta \rho_n R)$, but no poles in $B(z_n, \rho_n R) \setminus B(z_n, \delta \rho_n R)$. By Lemma

2.3.5 (a) and condition (iii) of Theorem 2.2.5, we deduce that

$$\left|\sum_{a\in B(0,\delta R)} \mathrm{Res}(g_n,a)-(k-1)\right| = \left|\sum_{a\in B(z_n,\delta\rho_n R)} \mathrm{Res}(F_n,a)-(k-1)\right| \geq \delta.$$

But this gives

$$\left|\sum_{a\in B(0,\delta R)} \mathrm{Res}(F,a)-(k-1)\right| \geq \delta,$$

which contradicts (2.6).

2.4 An example concerning Theorem 2.2.5

We use an example to show that Theorem 2.2.5 cannot be extended to the case where $a_0,\ldots,a_{k-1}$ are meromorphic functions, in the $k=2$ case. Similar counterexamples can be constructed in the general case.

Example 2.4.1. *Let $k=2$ and let $\mathcal{G}$ be the family of meromorphic functions $\mathcal{G}=\{F_n(z)=\frac{1}{nz^3} : n\in\mathbb{N}\}$. Let $a_0(z)=0$ and $a_1(z)=3/z$. Then a_0 and a_1 are meromorphic functions and for $F_n\in\mathcal{G}$ we have that*

$$\begin{aligned}\Lambda_2(F_n)(z) &= \Psi_2(F_n)(z)+a_1(z)\Psi_1(F_n)(z)+a_0(z)\\ &= \frac{1}{n^2z^6}-\frac{3}{nz^4}+\frac{3}{z}\frac{1}{nz^3}=\frac{1}{n^2z^6}.\end{aligned}$$

This is nonvanishing on $\mathbb{C}$, and so condition (i) is satisfied.
Also, each $F_n\in\mathcal{G}$ has a triple pole at $z=0$, and no other poles. Thus conditions (ii) and (iii) are trivially satisfied. Thus all the conditions of Theorem 2.2.5 are satisfied except for the analyticity of the a_j. However $\mathcal{G}$ is not a normal family since $F_n(0)=\infty$ while $F_n(z)=\frac{1}{nz^3}\to 0$ as $n\to\infty$ for $z\in\mathbb{R}^+$. Therefore Theorem 2.2.5 cannot be extended to the case where $a_0,\ldots,a_{k-1}$ are meromorphic functions.

It is interesting to note here, with regard to Theorem 2.1.7, that although the F_n in Example 2.4.1 have no zeros, they cannot be written in the form $F_n = f_n'/f_n$ where f_n is a nonvanishing meromorphic function.

2.5 Proof of Theorem 2.1.5: main part

Proof First we let $F \in \mathcal{G}$. Then $F = f'/f$ for some meromorphic function f in $\mathcal{F}$ such that f and $L(f)$ are nonvanishing on Ω. By Lemma 2.2.3, we have that

$$\begin{aligned} L(f)/f &= f^{(k)}/f + a_{k-1}f^{(k-1)}/f + \ldots + a_1 f'/f + a_0 \qquad (2.7) \\ &= \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_1\Psi_1(F) + a_0 \\ &= \Lambda_k(F). \end{aligned}$$

Then since f and $L(f)$ are nonvanishing on Ω for each $f \in \mathcal{F}$, we see that $\Lambda_k(F)$ is nonvanishing also, and so condition (i) of Theorem 2.2.5 is satisfied.

Next, by Lemma 2.2.1, we have that the poles of F are all simple poles with negative integers as residues. Then, for $\delta \in (0, 1]$, if a is a pole of F, then for $j \in \{0, 1, \ldots, k-1\}$ we have that

$$|\text{Res}(F, a) - j| \geq |-1-0| = 1 \geq \delta.$$

Then condition (ii) of Theorem 2.2.5 is satisfied.

Finally, if $B(c, \delta R)$ contains two poles of F, say a_1 and a_2 with $a_1 \neq a_2$, then we have that $\text{Res}(F, a_1)$ and $\text{Res}(F, a_2)$ are negative integers, and since

k is an integer with $k \geq 2$, we have that

$$\left| \sum_{a \in B(c,\delta R)} \mathrm{Res}(F, a) - (k-1) \right| \geq |-1-1-(2-1)| = 3 > \delta.$$

Then condition (iii) of Theorem 2.2.5 is satisfied.

Therefore, by Theorem 2.2.5, we have that $\mathcal{G}$ is a normal family.

2.6 Proof of Theorem 2.1.7

Proof Let $\{\alpha_j : j \in J\}$ be the set of poles of $a_0, \ldots, a_{k-1}$ in Ω. By Theorem 2.1.5, the result is true in the case where $a_0, \ldots, a_{k-1}$ are analytic functions, and so it is sufficient to prove that $\mathcal{G}$ is normal at the α_j, for $j \in J$.

Suppose there exists α_j such that $\mathcal{G}$ is not normal at α_j. Choose $\delta > 0$ such that the punctured disc $\Omega_j = \{z : 0 < |z - \alpha_j| < \delta\}$ is contained in Ω, and such that Ω_j does not contain any poles of $a_0, \ldots, a_{k-1}$. Then we have that there exists a sequence (F_n) in $\mathcal{G}$, where $F_n = f_n'/f_n$ for some $f_n \in \mathcal{F}$, such that (f_n'/f_n) has no subsequence that converges locally uniformly on $\Omega_j \cup \{\alpha_j\}$. However, since $a_0, \ldots, a_{k-1}$ are analytic in Ω_j, we have by Theorem 2.1.5 that $\mathcal{G}$ is normal on Ω_j. Then there exists a subsequence of (f_n'/f_n), denoted (f_n'/f_n) without loss of generality, which converges uniformly on compact sub-regions of Ω_j, either to a meromorphic limit ϕ, or identically to ∞. Then there are two cases.

Case 1. *(f_n'/f_n) converges uniformly to a meromorphic limit ϕ on compact sub-regions of Ω_j.*

We note first that for $n \in \mathbb{N}$, since $f_n \neq 0$ on Ω, the poles of f_n'/f_n can only arise at poles of f_n', and therefore only at poles of f_n. We note that for the remainder of this proof, we refer to the poles of f_n'/f_n only as the poles of f_n. We note also that f_n'/f_n has no zeros on Ω since $f_n' \neq 0$ there.

Let Γ be a circular contour in Ω_j which goes once anti-clockwise around α_j and which does not pass through any poles of ϕ. Since ϕ is a meromorphic limit, Γ lies in the interior of a closed annulus $\mathcal{A}$ on which ϕ has no poles. Further, since (f_n'/f_n) converges uniformly to ϕ on compact sub-regions of Ω_j, we have that there exists $n_0 \in \mathbb{N}$ such that f_n has no poles on $\mathcal{A}$ for $n \geq n_0$.

Let Ω_Γ be the domain enclosed by Γ. By the Argument principle (Theorem 1.10.1), and since each f_n has no zeros in Ω, we have that for each $n \in \mathbb{N}$,

$$\int_\Gamma \left(\frac{f_n'}{f_n}\right)(z)dz = -2\pi i q_n \tag{2.8}$$

where q_n is the number of poles of f_n in Ω_Γ, counting multiplicities. Then since,

$$\lim_{n\to\infty} \int_\Gamma \left(\frac{f_n'}{f_n}\right)(z)dz = \int_\Gamma \phi(z)dz = \lambda$$

for some $\lambda \in \mathbb{C}$, we have that there exists $n_1 \in \mathbb{N}$ such that $n_1 \geq n_0$ and such that $\int_\Gamma (f_n'/f_n)(z)dz$ is constant for $n \geq n_1$. Then by (2.8), we must have that for $n \geq n_1$, the f_n have the same number of poles, say q, in Ω_Γ.

We list the poles by $\gamma_{n,1}, \ldots, \gamma_{n,q}$ repeating according to multiplicity. Then we can write for $n \geq n_1$,

$$\left(\frac{f_n'}{f_n}\right)(z) = \sum_{l=1}^{q} \left(-\frac{1}{z-\gamma_{n,l}}\right) + \psi_n(z), \tag{2.9}$$

where ψ_n is an analytic function on $\Omega_\Gamma \cup \mathcal{A}$.

Next, we show that the ψ_n are uniformly bounded on Ω_Γ. We note first that $\gamma_{n,1}, \ldots, \gamma_{n,q}$ are not in the closed annulus $\mathcal{A}$, and so $|\phi(z)| \leq C_1$ and $|z - \gamma_{n,l}| \geq c_1$ for some positive constants C_1 and c_1, for $z \in \Gamma$. Since (f_n'/f_n) converges uniformly to ϕ on Γ, there exists $n_2 \in \mathbb{N}$ such that $n_2 \geq n_1$ and such that, on Γ, $|(f_n'/f_n)(z)| \leq C_1 + 1$ for $n \geq n_2$. Since $|z - \gamma_{n,l}| \geq c_1$, we have that $\left|\sum_{l=1}^{q}\left(-\frac{1}{z-\gamma_{n,l}}\right)\right| \leq q/c_1$. Then we have that, on Γ,

$$|\psi_n(z)| \leq \left|\left(\frac{f_n'}{f_n}\right)(z)\right| + \left|\sum_{l=1}^{q}\left(-\frac{1}{z-\gamma_{n,l}}\right)\right| \leq C_1 + 1 + q/c_1,$$

for $n \geq n_2$. Therefore the ψ_n are uniformly bounded on Γ. By the maximum principle (Theorem 1.10.3), the ψ_n are uniformly bounded on Ω_Γ.

Choose a subsequence of $(\gamma_{n,l})$, denoted $(\gamma_{n,l})$ without loss of generality, such that $\gamma_{n,l} \to \gamma_l$ as $n \to \infty$ for $l = 1, \ldots, q$. Then there are two subcases, depending on whether some of $\gamma_1, \ldots, \gamma_q$ are equal to α_j.

Case 1.1. *Some of* $\gamma_1, \ldots, \gamma_q$ *are equal to* α_j.

Rearrange $\gamma_1, \ldots, \gamma_q$ so that $\gamma_1, \ldots, \gamma_p$ are equal to α_j and $\gamma_{p+1}, \ldots, \gamma_q$ are not equal to α_j, for some $p \in \{1, \ldots, q\}$. Then by (2.9), we have for $n \geq n_2$,

$$\left(\frac{f_n'}{f_n}\right)(z) = \sum_{l=1}^{p}\left(-\frac{1}{z-\gamma_{n,l}}\right) + \sum_{l=p+1}^{q}\left(-\frac{1}{z-\gamma_{n,l}}\right) + \psi_n(z). \tag{2.10}$$

Next, since $\gamma_{n,p+1}, \ldots, \gamma_{n,q}$ tend to $\gamma_{p+1}, \ldots, \gamma_q$ as $n \to \infty$, and $\gamma_{p+1}, \ldots, \gamma_q$ are not equal to α_j, we can choose $\delta_1 > 0$ such that $B(\alpha_j, 3\delta_1)$ is contained in $\Omega_\Gamma \backslash \mathcal{A}$ and does not contain $\gamma_{p+1}, \ldots, \gamma_q$. Then there exists $n_3 \in \mathbb{N}$ such that $n_3 \geq n_2$ and such that, for $n \geq n_3$ and $z \in B(\alpha_j, \delta_1)$, we have $|z - \gamma_{n,l}| \geq \delta_1$ for $l = p+1, \ldots, q$. Then for each $n \geq n_3$, we have $\left|\sum_{l=p+1}^{q}\left(-\frac{1}{z-\gamma_{n,l}}\right)\right| \leq (q-p)/\delta_1$ and so $\sum_{l=p+1}^{q}\left(-\frac{1}{z-\gamma_{n,l}}\right)$ is uniformly bounded and analytic in $B(\alpha_j, \delta_1)$. Then since the ψ_n are uniformly bounded on Ω_Γ, there exists a large positive constant M such that for $n \geq n_3$,

$$\left| \sum_{l=p+1}^{q} \left(-\frac{1}{z - \gamma_{n,l}} \right) + \psi_n(z) \right| \leq M \tag{2.11}$$

on $B(\alpha_j, \delta_1)$.

Now choose $\delta_2 > 0$ such that δ_2/δ_1 is small, and consider the circle $S(\alpha_j, \delta_2) = \{z : |z - \alpha_j| = \delta_2\}$. We have that $\gamma_{n,1}, \ldots, \gamma_{n,p}$ each tend to α_j as $n \to \infty$, and so each $\left| -\frac{1}{z-\gamma_{n,l}} \right|$ is large on $S(\alpha_j, \delta_2)$ for $l = 1, \ldots, p$. Then, in particular, we have that

$$\left| \sum_{l=1}^{p} -\frac{1}{z - \gamma_{n,l}} \right| \to \left| \sum_{l=1}^{p} -\frac{1}{z - \gamma_l} \right| = \left| -\frac{p}{z - \alpha_j} \right| = \frac{p}{\delta_2} > 2M,$$

as $n \to \infty$, for a suitable choice of δ_2.

Then by (2.10) and (2.11), there exists $n_4 \in \mathbb{N}$ such that $n_4 \geq n_3$ and such that f_n'/f_n is large on $S(\alpha_j, \delta_2)$ for $n \geq n_4$, and thus f_n/f_n' is small on $S(\alpha_j, \delta_2)$ for $n \geq n_4$. Next, we know that each f_n/f_n' is analytic in Ω since $f_n' \neq 0$ on Ω. Then for $n \geq n_4$, by the maximum principle (Theorem 1.10.3) and since f_n/f_n' is small on $S(\alpha_j, \delta_2)$, each f_n/f_n' is small on $B(\alpha_j, \delta_2)$. Then (f_n/f_n') is a uniformly bounded sequence of analytic functions on $B(\alpha_j, \delta_2)$, and by the Montel-Vitali theorem (Theorem 1.5.4), we have that (f_n/f_n') is normal on $B(\alpha_j, \delta_2)$. Therefore (f_n'/f_n) is normal on $B(\alpha_j, \delta_2)$, and thus, in particular, (f_n'/f_n) is normal at α_j. This is a contradiction.

Case 1.2. *$\alpha_j \neq \gamma_l$ for all $l = 1, \ldots, q$.*

Then we can choose $\delta_3 > 0$ such that $B(\alpha_j, 3\delta_3)$ is contained in $\Omega_\Gamma \setminus \mathcal{A}$, and does not contain $\gamma_1, \ldots, \gamma_q$. Then there exists $n_5 \in \mathbb{N}$ such that $n_5 \geq n_3$ and such that, for $n \geq n_5$ and $z \in B(\alpha_j, \delta_3)$, we have $|z - \gamma_{n,l}| \geq \delta_3$ for $l = 1, \ldots, q$. Then $\left| \sum_{l=1}^{q} -\frac{1}{z-\gamma_{n,l}} \right| \leq \frac{q}{\delta_3}$, and since $\sum_{l=1}^{q} \left(-\frac{1}{z_n - \gamma_{n,l}} \right)$ is uniformly bounded and analytic in $B(\alpha_j, \delta_3)$ and since the ψ_n are uniformly bounded on Ω_Γ, we have that (2.11) holds with $p = 0$, on $B(\alpha_j, \delta_3)$.

Therefore, (f'_n/f_n) is a uniformly bounded sequence of analytic functions on $B(\alpha_j, \delta_3)$. Then by the Montel-Vitali theorem (Theorem 1.5.4), we have that (f'_n/f_n) is normal on $B(\alpha_j, \delta_3)$, and thus, in particular, (f'_n/f_n) is normal at α_j. This is a contradiction.

Case 2. *(f'_n/f_n) converges identically to ∞ on Ω_j.*

Then we have that (f_n/f'_n) converges identically to 0 on Ω_j. We note that for each n, we have that f_n/f'_n is analytic in $\Omega_j \cup \{\alpha_j\}$ since $f'_n \neq 0$ on Ω. Then by the maximum principle (Theorem 1.10.3), we have that $((f_n/f'_n)(\alpha_j))$ converges to 0, and so (f'_n/f_n) converges identically to ∞ on $\Omega_j \cup \{\alpha_j\}$. Therefore (f'_n/f_n) is normal on $\Omega_j \cup \{\alpha_j\}$. This is a contradiction.

Therefore $\mathcal{G}$ is a normal family.

Chapter 3

Extending a theorem of Bergweiler and Langley

In this chapter, we extend some results of Bergweiler and Langley [6].[1] We consider the differential operator Λ_k defined by

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_1\Psi_1(F) + a_0,$$

where $a_0, \ldots, a_{k-1}$ are analytic functions of restricted growth and $\Psi_k(F)$ is defined by (2.1). We suppose that $k \geq 3$, that F is a meromorphic function on an annulus $\mathcal{A}(r_0)$, and that $\Lambda_k(F)$ has all its zeros on a set E such that E has no limit point in $\mathcal{A}(r_0)$. We suppose also that all simple poles a of F in $\mathcal{A}(r_0) \setminus E$ have $\text{Res}(F, a) \notin \{1, \ldots, k-1\}$. We then deduce that F is a function of restricted growth in the Nevanlinna sense. We show also that this result does not hold for $a_0, \ldots, a_{k-1}$ meromorphic functions.

[1] The results in this chapter have been published by *Computational Methods and Function Theory*, see [9].

3.1 Introduction

In the proof of Theorem 2.2.5, Bergweiler and Langley use Theorem 2.3.2, which we restate here for convenience, and we recall that $\Psi_k(F)$ is defined, as in (2.1), by

$$\Psi_1(F) = F, \qquad \Psi_{k+1}(F) = F\Psi_k(F) + (\Psi_k(F))'.$$

Theorem 3.1.1 (Bergweiler and Langley, [6]). *Let $k \geq 3$ be an integer, and let F be a meromorphic and nonconstant function in the plane that satisfies both of the following conditions:*

(i) $\Psi_k(F)$ *is nonvanishing.*

(ii) if a is a simple pole of F then $\mathrm{Res}(F, a) \notin \{1, \ldots, k-1\}$.

Then F has the form

$$F(z) = \frac{(k-1)z + \alpha}{z^2 + \beta z + \gamma}, \tag{3.1}$$

or

$$F(z) = \frac{1}{\alpha z + \beta}. \tag{3.2}$$

Here $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha \neq 0$ in (3.2).
Conversely, if F has the form (3.1) *or* (3.2), *and if* (*ii*) *holds, then* $\Psi_k(F)$ *is nonvanishing. If F has the form* (3.1) *or* (3.2), *but* (*ii*) *does not hold, then* $\Psi_k(F) \equiv 0$.

We note that this theorem implies that F is a rational function, and hence by Theorem 1.2.1,

$$T(r, F) = O(\log r) \qquad \text{as } r \to \infty.$$

Defining an annulus $\mathcal{A}(r_0)$, as in (1.21), by

$$\mathcal{A}(r_0) = \{z : r_0 \leq |z| < \infty\}$$

we extend Theorem 3.1.1 in two ways.

First, we let F be meromorphic and nonconstant on $\mathcal{A}(r_0)$, by which we mean that F is meromorphic in a domain containing $\mathcal{A}(r_0)$.

Second, we weaken condition (i) as follows. We let $a_0, \ldots, a_{k-1}$ be analytic functions on $\mathcal{A}(r_0)$ of restricted growth as $z \to \infty$, such that

$$a_j(z) = O(|z|^{(\lambda-1)(k-j)}) \qquad \text{as } z \to \infty, \tag{3.3}$$

for some fixed $\lambda \geq 0$. Define $\Lambda_k(F)$, as in (2.2), by

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_1\Psi_1(F) + a_0.$$

We then assume for condition (i) that $\Lambda_k(F) = 0$ only on a set E such that E has no limit point in the annulus $\mathcal{A}(r_0)$. This implies that $\Lambda_k(F) = 0$ only on a countable set E.

The new conclusion is that F is a function of restricted growth in the Nevanlinna sense. We state the extended theorem as follows, recalling from (1.20) that we denote by $S(r, F)$ any quantity satisfying

$$S(r, F) = O(\log r + \log^+ T(r, F)),$$

as $r \to \infty$ outside a set of finite measure, not necessarily the same set at each occurrence. We also refer the reader to § 1.3 for background material concerning Nevanlinna theory in an annulus.

Theorem 3.1.2. *Let $k \geq 3$ be an integer and let F be meromorphic and nonconstant in an annulus $\mathcal{A}(r_0)$, as defined by* (1.21)*. Suppose $a_0, \ldots, a_{k-1}$ are analytic functions on $\mathcal{A}(r_0)$ of restricted growth, as in* (3.3)*, that is, such that*

$$a_j(z) = O(|z|^{(\lambda-1)(k-j)}) \qquad \text{as } z \to \infty,$$

for some fixed $\lambda \geq 0$. Let $f_1, \ldots, f_k$ be solutions of $L(w) = 0$ where L is defined, as in (1.25)*, by*

$$L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_0 f,$$

in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$. Let $\Lambda_k(F)$ be defined as in (2.2) *by*

$$\Lambda_k(F) = \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_1\Psi_1(F) + a_0.$$

Suppose there exists a set E, such that E has no limit point in $\mathcal{A}(r_0)$, and such that $\Lambda_k(F)$ has all its zeros in E. Suppose further that all simple poles a of F in $\mathcal{A}(r_0) \setminus E$ have $\mathrm{Res}(F, a) \notin \{1, \ldots, k-1\}$. Set

$$N_E(r) = \int_{r_0}^{r} \frac{n_E(t)}{t} dt$$

where $n_E(t)$ is the number of points in $E \cap \{z : r_0 \leq |z| \leq t\}$.
Then either:

(i) $T(r, F) \leq cN_E(r) + S(r, F), \quad \text{as } r \to \infty,$

where c is a constant depending only on k,

or

(ii) F is a rational function of the f_j and their derivatives, in which case

$$T(r, F) = O(r^{\lambda} + \log r), \quad \text{as } r \to \infty.$$

We note that when $\lambda = 0$, it follows from (ii) that $\lim_{z\to\infty} F(z)$ exists.

Theorem 3.1.2 is deduced from several lemmas, including Lemma 3.2.3, which is in turn deduced from several lemmas. For this reason, we first provide the preliminaries of the proof of Theorem 3.1.2 in § 3.2. In § 3.3, we prove Lemma 3.2.3. In § 3.4, we prove the main part of Theorem 3.1.2. In § 3.5, we state some corollaries of Theorem 3.1.2. Finally, in § 3.6 we provide an example showing that Theorem 3.1.2 cannot be extended to the case where $a_0, \dots, a_{k-1}$ are meromorphic functions.

3.2 Proof of Theorem 3.1.2: Preliminaries

Suppose that k, F, $\mathcal{A}(r_0)$ and $a_0, \dots, a_{k-1}$ are as in the statement of Theorem 3.1.2. We may define linearly independent analytic solutions $f_1, \dots, f_k$ of (1.25) in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$. These f_j are analytic in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$, and since the a_j are analytic in $\mathcal{A}(r_0)$, the f_j admit unrestricted analytic continuation in $\mathcal{A}(r_0)$, and so they satisfy the following lemma.

Lemma 3.2.1 (Langley, [27]). *Suppose that $k \geq 1$ and that $a_0, \dots, a_{k-1}$ are analytic in an annulus $\mathcal{A}(r_0)$, as defined by* (1.21)*, such that as in* (3.3)*, for some $\lambda \geq 0$,*

$$a_j(z) = O(|z|^{(\lambda-1)(k-j)}) \qquad \text{as } z \to \infty.$$

Let $f_j(z)$ be a solution of $L(w) = 0$, where L is defined by (1.25)*, in a sectorial region*

$$\mathcal{S} = \{z : |z| > r_0, \alpha < \arg z < \alpha + 2\pi\},$$

where α is real. Then as $z \to \infty$ in $\mathcal{S}$,

$$\log^+ |f_j(z)| = O(|z|^\lambda + \log |z|). \tag{3.4}$$

By Lemma 3.2.1, we have that $\log^+ |f_j(z)| = O(|z|^\lambda + \log |z|)$ for $j = 1, \ldots, k$ and thus the continuations satisfy $\log^+ \log^+ |f_j(z)| = O(\log |z|)$ for $|z| > r_0$, $|\arg z| < 2\pi$. Then we have that $f_1, \ldots, f_k$ also satisfy the conditions of the following lemma.

Lemma 3.2.2 (Frank and Langley, [17])**.** *Suppose that $f_1, \ldots, f_k$ each admit unrestricted analytic continuation in an annulus $\mathcal{A}(r_0)$, as defined by (1.21), and satisfy $\log^+ \log^+ |f_j(z)| = O(\log |z|)$ for z in a sectorial region $\mathcal{S} = \{z : |z| > r_0, |\arg z| < 2\pi\}$. Suppose that F is meromorphic in $\mathcal{A}(r_0)$. Suppose further that, for some non-negative integer M, each of the functions $h_1, \ldots, h_k$ on $\mathcal{S}$ is a polynomial in the $f_j^{(m)}$, $F^{(m)}$, $1 \le j \le k$, $0 \le m \le M$. Suppose finally that $h_1, \ldots, h_k$ are linearly independent solutions in $\mathcal{S}$ of an equation*

$$w^{(k)} + \sum_{j=0}^{k-1} B_j w^{(j)} = 0$$

in which the B_j are meromorphic in $\mathcal{A}(r_0)$. Then we have, for $j = 0, \ldots, k-1$,

$$m(r, B_j) = S(r, F).$$

Choose a simply connected domain $\Omega \subseteq \mathcal{A}(r_0)$, on which F has no poles and $\Lambda_k(F)$ has no zeros. Define functions f, g, and h in Ω by

$$f'/f = F \quad \Lambda_k(F) = g^{-k}, \quad h = -Fg. \tag{3.5}$$

Then f, g and h are analytic in Ω since F has no poles and $\Lambda_k(F)$ has no zeros there.

We next need the following lemma, which is similar to [6, Lemma 2.3], but uses a different method of proof. We refer the reader to § 1.6 and § 1.7 for background material about Wronskians and analytic continuation.

Lemma 3.2.3. *Define functions w_j and h_j, for $j = 1, \ldots, k$, on Ω by*

$$w_j(z) = f_j'(z)g(z) + f_j(z)h(z), \quad h_j(z) = -f_j'(z) + f_j(z)F(z). \tag{3.6}$$

Then the w_j form a fundamental solution set on Ω of the differential equation

$$w^{(k)} + \sum_{j=0}^{k-1} A_j w^{(j)} = 0, \tag{3.7}$$

in which the A_j are meromorphic functions on $\mathcal{A}(r_0)$ with

$$T(r, A_j) \leq cN_E(r) + S(r, F), \quad \text{as } r \to \infty, \tag{3.8}$$

for $j = 0, \ldots, k-1$, where c is a constant depending only on k.

3.3 Proof of Lemma 3.2.3

The following summarises some results from Nevanlinna theory which are used in Lemma 3.2.3. We provide a proof for completeness. For background material about Nevanlinna theory in an annulus, we refer the reader to § 1.3. We also recall that we use $(n.e.)$ as an abbreviation for *"nearly everywhere"*, that is, to denote the phrase *"outside a set of finite measure"*.

Lemma 3.3.1. *Let $p \in \mathbb{N}$ and let Ω be a simply connected domain in the annulus $\mathcal{A}(r_0)$, as defined by* (1.21)*. Suppose that w is meromorphic in Ω such that $W = w^p$ is meromorphic in $\mathcal{A}(r_0)$. Then $w^{(k)}/w$ is meromorphic in $\mathcal{A}(r_0)$, for each $k \in \mathbb{N}$, with poles of multiplicity at most k. Further, we have*

$$T(r, w^{(k)}/w) = O(T(r, W)) \qquad (n.e.),$$

and

$$m(r, w^{(k)}/w) = S(r, W) \qquad (n.e.),$$

as $r \to \infty$.

Proof We use a proof by induction. Let the inductive hypothesis be as stated above, and we consider the case $k = 1$. Then $W' = (w^p)' = pw^{p-1}w'$ and so we have

$$\frac{w'}{w} = \frac{W'}{pw^{p-1}w} = \frac{W'}{pW}. \tag{3.9}$$

Since W is meromorphic in $\mathcal{A}(r_0)$, we have that W' is meromorphic in $\mathcal{A}(r_0)$ and thus w'/w is meromorphic in $\mathcal{A}(r_0)$. Also, by (3.9), w'/w has poles of multiplicity at most 1.

Now, by (1.18), we have that $T(r, w'/w) = T(r, W'/pW) = O(T(r, W))$, $(n.e)$ and by (1.16) and (3.9),

$$m(r, w'/w) = m(r, W'/pW) \leq O(\log rT(r, W)) = S(r, W) \quad (n.e.), \tag{3.10}$$

as $r \to \infty$. And so, case $k = 1$ is proved.

Now let $k \geq 2$ and assume the inductive hypothesis holds for $k - 1$. We have

$$\left(\frac{w^{(k-1)}}{w}\right)' = \frac{w^{(k)}}{w} - \frac{w^{(k-1)}}{w}\frac{w'}{w},$$

and so,

$$\frac{w^{(k)}}{w} = \left(\frac{w^{(k-1)}}{w}\right)' + \frac{w^{(k-1)}}{w}\frac{w'}{w}. \tag{3.11}$$

Now by the inductive hypothesis, $w^{(k-1)}/w$ is meromorphic in $\mathcal{A}(r_0)$, and so $(w^{(k-1)}/w)'$ is also. Then the right-hand side of (3.11) is meromorphic in $\mathcal{A}(r_0)$, and thus $w^{(k)}/w$ is meromorphic in $\mathcal{A}(r_0)$.

Now consider the poles of $w^{(k)}/w$. By the inductive hypothesis, we have that $w^{(k-1)}/w$ has poles of multiplicity at most $k-1$, and hence $\left(\frac{w^{(k-1)}}{w}\right)'$ and $\frac{w^{(k-1)}}{w}\frac{w'}{w}$ have poles of multiplicity at most k. Thus $w^{(k)}/w$ has poles of multiplicity at most k.

Next, by (1.12), (1.13), (1.17), (3.11) and the inductive hypothesis, we have

$$\begin{aligned} T(r, \frac{w^{(k)}}{w}) &\leq T(r, \left(\frac{w^{(k-1)}}{w}\right)') + T(r, \frac{w^{(k-1)}}{w}) + T(r, \frac{w'}{w}) + \log 2 \\ &= O(T(r, \frac{w^{(k-1)}}{w})) + O(T(r, W)) \qquad (n.e.) \\ &= O(T(r, W)) \qquad (n.e.). \end{aligned} \tag{3.12}$$

Finally, by (1.8), (1.9), (3.10), (3.11), (3.12) and the inductive hypothesis,

$$\begin{aligned} m(r, \frac{w^{(k)}}{w}) &\leq m(r, \left(\frac{w^{(k-1)}}{w}\right)') + m(r, \frac{w^{(k-1)}}{w}) + m(r, \frac{w'}{w}) + \log 2 \\ &\leq 2m(r, \frac{w^{(k-1)}}{w}) + O(\log rT(r, \frac{w^{(k-1)}}{w})) + m(r, \frac{w'}{w}) + \log 2 \qquad (n.e.) \\ &= O(\log rT(r, W)) = S(r, W) \qquad (n.e.), \end{aligned}$$

as $r \to \infty$. And so, this case satisfies the inductive hypothesis. Thus the inductive hypothesis is satisfied for all $k \in \mathbb{N}$.

We now prove Lemma 3.2.3.

Proof of Lemma 3.2.3 We divide this proof into a number of steps.

Step 1. *The w_j form a fundamental solution set on Ω of the differential equation* (3.7)*, and $A_{k-1} = a_{k-1}$.*

We note first that $w_j = fg(f_j/f)'$ on Ω by (3.5), and by (1.28) and Lemma 1.6.2 we have that

$$\begin{aligned} W(w_1,\ldots,w_k) &= W(fg(f_1/f)',\ldots,fg(f_k/f)') \\ &= f^k g^k W((f_1/f)',\ldots,(f_k/f)') \\ &= f^k g^k (-1)^k W(f_1/f,\ldots,f_k/f,1) \\ &= f^{-1} g^k (-1)^k W(f_1,\ldots,f_k,f) \\ &= f^{-1} g^k (-1)^k L(f) W(f_1,\ldots,f_k). \end{aligned} \tag{3.13}$$

Then since $g^k = (\Lambda_k(F))^{-1}$ by (3.5) and since $L(f)/f = \Lambda_k(F)$ by (2.7), we have from (3.13) that

$$W(w_1,\ldots,w_k) = (-1)^k W(f_1,\ldots,f_k). \tag{3.14}$$

By Lemma 1.6.1, the right-hand side is not identically zero, since the f_j form a linearly independent solution set of (1.25). Thus, again by Lemma 1.6.1, the w_j form a linearly independent solution set for the differential equation (3.7). Also, by (1.27), we have that

$$A_{k-1} = -\frac{W(w_1,\ldots,w_k)'}{W(w_1,\ldots,w_k)} = -\frac{(-1)^k W(f_1,\ldots,f_k)'}{(-1)^k W(f_1,\ldots,f_k)} = a_{k-1}. \tag{3.15}$$

Step 2. *The h_j are linearly independent solutions of the differential equation*

$$w^{(k)} + \sum_{j=0}^{k-1} B_j w^{(j)} = 0, \tag{3.16}$$

in which the B_j are meromorphic in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$.

We have by Lemma 1.6.2, (3.5) and (3.6) that $w_j = -h_j g$, and so by (3.5) we have that

$$\begin{aligned} W(w_1, \ldots, w_k) &= W(-h_1 g, \ldots, -h_k g) \\ &= (-1)^k g^k W(h_1, \ldots, h_k) \\ &= (-1)^k \Lambda_k(F)^{-1} W(h_1, \ldots, h_k). \end{aligned}$$

Then by (3.14), we have that

$$W(h_1, \ldots, h_k) = \Lambda_k(F) W(f_1, \ldots, f_k).$$

The right-hand side is not identically zero on Ω since $\Lambda_k(F) \neq 0$ on Ω and since the f_j form a linearly independent solution set of (1.25) on $\mathcal{A}(r_0) \setminus \mathbb{R}^-$. Thus, by Lemma 1.6.1, the h_j form a linearly independent solution set of the differential equation (3.16) on Ω. By (3.6), we have $h_j = -f_j' + f_j F$ and thus the h_j are meromorphic on $\mathcal{A}(r_0) \setminus \mathbb{R}^-$ since the f_j are analytic there, and F is meromorphic in $\mathcal{A}(r_0)$. Hence the coefficients B_j are meromorphic in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$.

Step 3. *The B_j extend to be meromorphic in $\mathcal{A}(r_0)$.*

Let γ be a path in $\mathcal{A}(r_0)$ that encloses the origin. Continue the f_j along γ, starting and ending at a point z_0 say. Then since the f_j are analytic in $\mathcal{A}(r_0) \setminus \mathbb{R}^-$, and the a_j are analytic in $\mathcal{A}(r_0)$, we have that each f_j continues analytically to $\tilde{f}_j$, a linear combination of $f_1, \ldots, f_k$ near z_0. We also have that F is meromorphic in $\mathcal{A}(r_0)$, and thus since $h_j = -f_j' + f_j F$, each h_j continues analytically to $\tilde{h}_j$, a linear combination of $h_1, \ldots, h_k$ near z_0.

We claim that $\tilde{h}_1, \ldots, \tilde{h}_k$ are linearly independent. Suppose not. Then there exist non-zero constants $c_1, \ldots, c_k$ such that $c_1\tilde{h}_1 + \ldots + c_k\tilde{h}_k \equiv 0$ near z_0. Then, continuing backwards along γ we have that $c_1 h_1 + \ldots + c_k h_k \equiv 0$ near z_0. This is a contradiction since $h_1, \ldots, h_k$ are linearly independent. Thus $\tilde{h}_1, \ldots, \tilde{h}_k$ are linearly independent and form a fundamental solution set to an equation

$$w^{(k)} + \sum_{j=0}^{k-1} \tilde{B}_j w^{(j)} = 0 \tag{3.17}$$

near z_0. The $\tilde{B}_j$ are meromorphic near z_0 since $\tilde{h}_1, \ldots, \tilde{h}_k$ are.

Since $\tilde{h}_1, \ldots, \tilde{h}_k$ are linear combinations of $h_1, \ldots, h_k$ near z_0, they also solve (3.16) near z_0. We then must have that $\tilde{B}_j = B_j$ near z_0 for $j = 0, \ldots, k-1$ since otherwise we could subtract (3.17) from (3.16) to get a differential equation of order at most $k-1$, with k linearly independent solutions $\tilde{h}_1, \ldots, \tilde{h}_k$ near z_0. Therefore the B_j are unchanged by analytic continuations around γ and so extend to be meromorphic in $\mathcal{A}(r_0)$.

Step 4. *The B_j have poles of multiplicity at most $k-j$ on $\mathcal{A}(r_0)$.*

By Step 3 the B_j are meromorphic in $\mathcal{A}(r_0)$. Furthermore, (3.16) has k linearly independent meromorphic solutions on a neighbourhood of each point of $\mathcal{A}(r_0)$, namely $h_1, \ldots, h_k$. Hence by Lemma 1.6.3, the B_j have poles of multiplicity at most $k-j$ on $\mathcal{A}(r_0)$.

Step 5. *Estimate for $m(r, B_j)$.*

We recall by (3.6) that $h_j = -f_j' + f_j F$, and note that all the conditions of Lemma 3.2.2 are satisfied. Then, by Lemma 3.2.2, we have that

$$m(r, B_j) = S(r, F) \tag{3.18}$$

for $j = 0, \dots, k-1$.

Step 6. *Estimate for $m(r, A_j)$.*

First we express A_j in terms of B_j and $g^{(p)}/g$. We recall that the w_j solve the differential equation

$$w^{(k)} + \sum_{j=0}^{k-1} A_j w^{(j)} = 0$$

in Ω. Then since $w_j = -h_j g$ and since $(-h_j g)^{(n)} = -\sum_{p=0}^{n} \binom{n}{p} h_j^{(p)} g^{(n-p)}$, we have

$$-\sum_{p=0}^{k} \binom{k}{p} h_j^{(p)} g^{(k-p)} - \sum_{q=0}^{k-1} A_q \sum_{p=0}^{q} \binom{q}{p} h_j^{(p)} g^{(q-p)} = 0. \tag{3.19}$$

We recall that Ω does not contain poles of F, and so also does not contain poles of $\Lambda_k(F)$. Then since $\Lambda_k(F) = g^{-k}$, we have that Ω also does not contain zeros of g. Then we can divide (3.19) by $-g$ to get

$$\sum_{p=0}^{k} \binom{k}{p} h_j^{(p)} \frac{g^{(k-p)}}{g} + \sum_{q=0}^{k-1} A_q \sum_{p=0}^{q} \binom{q}{p} h_j^{(p)} \frac{g^{(q-p)}}{g} = 0.$$

Since the h_j solve the differential equation

$$w^{(k)} + \sum_{j=0}^{k-1} B_j w^{(j)} = 0$$

we can compare coefficients and find that, for example,

$$\begin{aligned} B_{k-1} &= \binom{k}{k-1} \frac{g'}{g} + A_{k-1} \\ B_{k-2} &= \binom{k}{k-2} \frac{g''}{g} + \binom{k-1}{k-2} \frac{g'}{g} A_{k-1} + A_{k-2}. \end{aligned}$$

In general, for $j = 0, \dots, k-1$, we have that

$$B_j = \binom{k}{j} \frac{g^{(k-j)}}{g} + \sum_{q=j}^{k-1} A_q \binom{q}{j} \frac{g^{(q-j)}}{g}.$$

Then for $j = 0, \dots, k-1$, we have that

$$A_j = B_j - \binom{k}{j} \frac{g^{(k-j)}}{g} - \sum_{q=j+1}^{k-1} A_q \binom{q}{j} \frac{g^{(q-j)}}{g}, \tag{3.20}$$

which is initialised by

$$A_{k-1} = B_{k-1} - \binom{k}{k-1} \frac{g'}{g}.$$

We now note that by Lemma 3.3.1 for all $p \in \mathbb{N}$, we have that $g^{(p)}/g$ is meromorphic in $\mathcal{A}(r_0)$ and $m(r, g^{(p)}/g) = S(r, F)$. Thus we have, by (1.8), (1.9), (3.18) and (3.20), that each A_j is meromorphic in $\mathcal{A}(r_0)$ and

$$m(r, A_j) = S(r, F)$$

for $j = 0, \dots, k-1$.

Step 7. *Estimate for $N(r, A_j)$.*

We show first that the poles of A_j can only arise on E, the set containing all points where $\Lambda_k(F) = 0$.

We know by Lemma 1.6.3, that the poles of A_j can only arise among the zeros of the continuations of $W(w_1, \dots, w_k)$ and the poles of the continuations of $w_1, \dots, w_k$. By (3.15), $a_{k-1} = -W(w_1, \dots, w_k)'/W(w_1, \dots, w_k)$ and since a_{k-1} is analytic in $\mathcal{A}(r_0)$, we have that $W(w_1, \dots, w_k)$ continues without zeros. Thus the poles of A_j can only arise among the poles of the continuations of $w_1, \dots, w_k$. Then, since by (3.5) and (3.6) we have that $w_j = (f_j' - f_j F)/(\Lambda_k(F))^{1/k}$, and since the f_j are analytic, then the poles of A_j can only arise at poles of F and zeros of $\Lambda_k(F)$.

Now let $z_0 \in \mathcal{A}(r_0) \setminus E$ and suppose that a pole of A_j arises at z_0. Since $\Lambda_k(F) \neq 0$ on $\mathcal{A}(r_0) \setminus E$ we must have that z_0 is a pole of F, of multiplicity

m say, and if $m = 1$ then $\text{Res}(F, z_0) \notin \{1, \ldots, k-1\}$ since F satisfies the hypotheses of Theorem 3.1.2. Then by Lemma 2.3.1, $\Lambda_k(F)$ has a pole at z_0 of multiplicity mk, and so since $g^{-k} = \Lambda_k(F)$, we have that g can be analytically continued to a neighbourhood of z_0 and has a zero of multiplicity m there. Thus $h = -Fg$ can be analytically continued to z_0 and since the f_j can be continued analytically in $\mathcal{A}(r_0)$, we have that $w_j = f_j' g + f_j h$ can be analytically continued to z_0. We therefore deduce that the A_j are analytic at z_0. This contradicts our hypothesis, and so the poles of A_j can only arise in E.

We recall from Step 4 that the poles of B_j have multiplicity at most $k-j$ on $\mathcal{A}(r_0)$. We note also by Lemma 3.3.1, that $g^{(p)}/g$ have poles of multiplicity at most p there. Thus we have by (3.20) that the poles of A_j must have multiplicity at most c where c is a constant depending only on k. Therefore we have, for $j = 0, \ldots, k-1$,

$$N(r, A_j) \leq cN_E(r),$$

where N_E is as defined in the statement of Theorem 3.1.2.

Step 8. *Conclusion.*

We have since $T(r, A_j) = N(r, A_j) + m(r, A_j)$ that

$$T(r, A_j) \leq cN_E(r) + S(r, F), \quad \text{as } r \to \infty,$$

for $j = 0, \ldots, k-1$, where c is a constant depending only on k.

3.4 Proof of Theorem 3.1.2: Main Part

There are two final lemmas needed for the proof of Theorem 3.1.2.

Lemma 3.4.1 (Frank and Langley, [17]). *Let $k \geq 1$ be an integer, and let $f_1, \ldots, f_k, G, H$ and $a_0, \ldots, a_{k-1}$ and $A_0, \ldots, A_{k-1}$ be meromorphic in a domain Ω. Suppose that $f_1, \ldots, f_k$ are linearly independent solutions in Ω of $L(w) = 0$, where L is defined as in* (1.25)*. Then the functions $f_1'g + f_1h, \ldots, f_k'g + f_kh$ are solutions in Ω of the equation* (3.7) *if and only if, setting $A_k = 1$ and $A_{-1} = a_{-1} = 0$ and, for $0 \leq q \leq k$,*

$$M_{k,q}(w) = \sum_{m=q}^{k} \binom{m}{q} A_m w^{(m-q)}, \quad M_{k,-1}(w) = 0,$$

we have, for $0 \leq q \leq k-1$,

$$M_{k,q}(h) - a_q h = -M_{k,q-1}(g) + a_q M_{k,k-1}(g) - (a_q a_{k-1} - a_q' - a_{q-1})g. \quad (3.21)$$

The following lemma is proved in [17] for $a_0, \ldots, a_{k-1}$ rational functions, and F meromorphic in the plane. The proof extends without modification to the case where $a_0, \ldots, a_{k-1}$ are analytic functions and F is meromorphic in $\mathcal{A}(r_0)$. This gives the following lemma.

Lemma 3.4.2 (Frank and Langley, [17]). *Let $\lambda \geq 0$ and $k \geq 2$, and let $a_0, \ldots, a_{k-1}$ be analytic functions of restricted growth, as in* (3.3)*, satisfying $a_j(z) = O(|z|^{(\lambda-1)(k-j)})$ as $z \to \infty$. Suppose that F is meromorphic in the annulus $\mathcal{A}(r_0)$, as defined by* (1.21)*, and has in some domain Ω a representation as a rational function in solutions f_j of the equation $L(w) = 0$ and their derivatives, where L is defined by* (1.25)*. If $\lambda > 0$ then $T(r, F) = O(r^{\lambda})$ as $r \to \infty$. If $\lambda = 0$ then $T(r, F) = O(\log r)$ and $\lim_{z\to\infty} F(z)$ exists.*

We are now in a position to complete the proof of Theorem 3.1.2. We note that we use methods found in [17, Theorem 3].

Proof of Theorem 3.1.2 We apply Lemma 3.4.1 to equation (3.7) and to g and h in Ω. The k equations (3.21) can be written in the form

$$T_q(g) = S_q(h) = \sum_{j=0}^{k-q} c_{j,q} h^{(j)}, \qquad 0 \le q \le k-1, \tag{3.22}$$

in which T_q and S_q are homogeneous linear differential operators with coefficients λ_ν which are rational functions in the a_j, A_j and their derivatives. Then by Lemma 3.2.3, we have that

$$T(r, \lambda_\nu) \le cN_E(r) + S(r, F), \qquad \text{as } r \to \infty, \tag{3.23}$$

where c is a constant depending only on k.

We have in particular that $q = k-1$ gives

$$M_{k,k-1}(h) - a_{k-1}h = -M_{k,k-2}(g) + a_{k-1}M_{k,k-1}(g) - (a_{k-1}a_{k-1} - a'_{k-1} - a_{k-2})g.$$

Then since

$$\begin{aligned} M_{k,k-1}(h) &= A_{k-1}h + kh', \\ M_{k,k-1}(g) &= A_{k-1}g + kg', \\ M_{k,k-2}(g) &= A_{k-2}g + (k-1)A_{k-1}g' + k(k-1)g''/2, \end{aligned}$$

we have that

$$\begin{aligned} A_{k-1}h + kh' - a_{k-1}h &= -A_{k-2}g - (k-1)A_{k-1}g' - k(k-1)g''/2 + a_{k-1}A_{k-1}g + \\ &\quad + ka_{k-1}g' - (a_{k-1}a_{k-1} - a'_{k-1} - a_{k-2})g \end{aligned}$$

which gives

$$h' = U(g) = -(k-1)g''/2 + a_{k-1}g'/k + (a'_{k-1} + a_{k-2} - A_{k-2})g/k \quad (3.24)$$

since $a_{k-1} = A_{k-1}$ on Ω by (3.15). We note that we can then write (3.22) in the form

$$T_q(g) = c_{0,q}h + \sum_{j=1}^{k-q} c_{j,q} \frac{d^{j-1}}{dz^{j-1}}(U(g)). \quad (3.25)$$

We distinguish two cases here.

Case 1. *We assume that the coefficient of* h *in at least one* S_q *in* (3.22) *is not identically zero.*

Let ν be the largest integer, $0 \leq \nu \leq k-1$, such that $c_{0,\nu} \not\equiv 0$. Then since $h = -Fg$ by (3.5), equations (3.22) and (3.25) give

$$h = -Fg = (c_{0,\nu})^{-1} \left(T_\nu(g) - \sum_{j=1}^{k-\nu} c_{j,\nu} \frac{d^{j-1}}{dz^{j-1}}(U(g)) \right) = V(g). \quad (3.26)$$

Then by (3.22), (3.24) and (3.26) we have that g solves the system of equations

$$U(g) = \frac{d}{dz}(V(g)), \qquad S_q(V(g)) = T_q(g), \qquad 0 \leq q \leq k-2. \quad (3.27)$$

Here we distinguish two sub-cases.

Case 1.1. *We assume that the solution space of* (3.27) *has dimension 1.*

That is, we have that every common solution of the equations (3.27) is a constant multiple of g. Then (3.23) and a standard reduction procedure, see [24, p.126], give a first order equation

$$p_1 g' + p_0 g = 0, \qquad p_1 \not\equiv 0,$$

where the p_j are rational functions in the λ_ν and their derivatives. It follows by (3.23) that

$$T(r, g'/g) \leq cN_E(r) + S(r, F), \qquad \text{as } r \to \infty,$$

where c is a constant depending only on k. Hence, since $F = -h/g$ and using (3.23) and (3.26),

$$T(r, F) \leq cN_E(r) + S(r, F), \qquad \text{as } r \to \infty.$$

Hence we have conclusion (i) of the theorem.

Case 1.2. *We assume that there is a solution G for the system (3.27) such that G/g is nonconstant.*

In particular, we note that this will be the case if the system (3.27) is trivial. Define H by $H = V(G)$. Then, by (3.27),

$$H' = U(G), \quad S_q(H) = T_q(G), \quad 0 \leq q \leq k-2.$$

In particular, the equations (3.22) hold with g and h replaced by G and H respectively. And so, by Lemma 3.4.1, the functions $f_j H + f_j' G$ are solutions of (3.7) and thus are linear combinations of $w_1, \ldots, w_k$. Hence, there are solutions g_j of $L(f) = 0$ where L is defined by (1.25), such that

$$f_j H + f_j' G - g_j h - g_j' g = 0, \qquad 1 \leq j \leq k. \tag{3.28}$$

We regard the equations in (3.28) as a system of k equations in H, G, h, g, over the field $\mathbb{F}$ of functions meromorphic in Ω, with coefficients f_j, f_j', g_j, g_j'.

Next, we note that the rank of the coefficient matrix is ≤ 3, since there is a non-trivial solution for the system. We claim that the rank is precisely

3. Suppose not. Then there are functions $\phi_m \in \mathbb{F}$, with $1 \leq m \leq 3$, not all identically zero, and functions $\psi_m \in \mathbb{F}$, with $1 \leq m \leq 3$, again not all identically zero, such that

$$\phi_1 f_j' + \phi_2 f_j = \phi_3 g_j, \qquad \psi_1 f_j' + \psi_2 f_j = \psi_3 g_j',$$

for $1 \leq j \leq k$. Since we know that $W(f_1, \ldots, f_k) \not\equiv 0$, neither ϕ_3 nor ψ_3 can be identically zero. We may assume therefore that $\phi_3 \equiv \psi_3 \equiv 1$. Thus

$$\phi_1 f_j'' + f_j'(\phi_1' + \phi_2 - \psi_1) + f_j(\phi_2' - \psi_2) = 0,$$

for $1 \leq j \leq k$. Again, since $W(f_1, \ldots, f_k) \not\equiv 0$, we must have

$$\phi_1 \equiv \phi_1' + \phi_2 - \psi_1 \equiv \phi_2' - \psi_2 \equiv 0,$$

which gives $g_j = \phi_2 f_j$. Then, by Lemma 1.6.2,

$$W(g_1, \ldots, g_k) = W(\phi_2 f_1, \ldots, \phi_2 f_k) = (\phi_2)^k W(f_1, \ldots, f_k). \tag{3.29}$$

Since $g_1, \ldots, g_k$ and $f_1, \ldots, f_k$ are solution sets of $L(f) = 0$ where L is defined as in (1.25), we have by (1.27) that

$$\frac{W(g_1, \ldots, g_k)'}{W(g_1, \ldots, g_k)} = -a_{k-1} = \frac{W(f_1, \ldots, f_k)'}{W(f_1, \ldots, f_k)}. \tag{3.30}$$

However, by (3.29), we have that

$$\begin{aligned} \frac{W(g_1, \ldots, g_k)'}{W(g_1, \ldots, g_k)} &= \frac{((\phi_2)^k W(f_1, \ldots, f_k))'}{(\phi_2)^k W(f_1, \ldots, f_k)} \\ &= \frac{((\phi_2)^k)' W(f_1, \ldots, f_k) + (\phi_2)^k W(f_1, \ldots, f_k)'}{(\phi_2)^k W(f_1, \ldots, f_k)} \\ &= \frac{((\phi_2)^k)'}{(\phi_2)^k} + \frac{W(f_1, \ldots, f_k)'}{W(f_1, \ldots, f_k)}. \end{aligned}$$

Then, by (3.30), we have that $\frac{((\phi_2)^k)'}{(\phi_2)^k} = 0$, and so ϕ_2 must be constant.

Now by (3.28), for $1 \leq j \leq k$, we have that

$$f_j(H - \phi_2 h) + f_j'(G - \phi_2 g) = 0,$$

and since $W(f_1, \ldots, f_k) \not\equiv 0$ we must have $H = \phi_2 h$ and $G = \phi_2 g$. This contradicts the assumption that G/g is non-constant. Hence the rank of the system (3.28) is precisely 3.

We can then solve for $-F = h/g$ as a quotient of determinants in f_j, f_j', g_j, g_j'. Thus F is a rational function of the f_j and their derivatives. Then by Lemma 3.4.2, we have that

$$T(r, F) = O(r^\lambda + \log r), \qquad \text{as } r \to \infty,$$

and so we have conclusion (ii) of the theorem.

Case 2. *We assume that* $c_{0,q} \equiv 0$ *for* $0 \leq q \leq k-1$ *in* (3.22).

We then have that the equations (3.22) are satisfied when g and h are replaced by 0 and 1 respectively, and thus so are the equations (3.21). Then, by Lemma 3.4.1, the f_j are solutions of (3.7). Thus the equations $L(f) = 0$ and (3.7) are the same, where L is defined by (1.25), and for $1 \leq q \leq k$ we may write

$$f_j h + f_j' g = g_j, \tag{3.31}$$

in which each g_j is a solution of $L(f) = 0$. Then since f_1 and f_2 are linearly independent, we have $f_1 f_2' - f_1' f_2 \not\equiv 0$ and so

$$F = -h/g = (f_1' g_2 - f_2' g_1)/(f_1 g_2 - f_2 g_1),$$

which gives F as a quotient of determinants in f_j, f_j', g_j, g_j'. Then by Lemma 3.4.2 we have that

$$T(r, F) = O(r^\lambda + \log r), \qquad \text{as } r \to \infty,$$

and so we have conclusion (ii) of the theorem.

3.5 Corollaries of Theorem 3.1.2

The following corollaries are deduced directly from Theorem 3.1.2.

Corollary 3.5.1. *Let $k \geq 3$ be an integer and let F be meromorphic and nonconstant in an annulus $\mathcal{A}(r_0)$, as defined by* (1.21)*. Suppose there exists a set E, such that E has no limit point in $\mathcal{A}(r_0)$, and such that $\Psi_k(F)$ has all its zeros in E. Suppose further that all simple poles a of F, such that $a \notin E$, have* $\text{Res}(F, a) \notin \{1, \ldots, k-1\}$*. Then either:*

(i) $T(r, F) \leq cN_E(r) + O(\log r + \log^+ T(r, F))$ *(n.e.)*,
where c is a constant depending only on k,

or

(ii) $\lim_{z\to\infty} F(z)$ *exists.*

Corollary 3.5.2. *Let $k \geq 3$ be an integer, and let F be meromorphic and nonconstant in an annulus $\mathcal{A}(r_0)$, as defined by* (1.21)*. Suppose F satisfies both of the following conditions:*

(i) $\Psi_k(F)$ *is nonvanishing.*

(ii) if a is a simple pole of F then $\text{Res}(F, a) \notin \{1, \ldots, k-1\}$.

Then $\lim_{z\to\infty} F(z)$ *exists.*

Corollary 3.5.3. *Let $k \geq 3$ be an integer, and let F be meromorphic in $\mathbb{C}$ and satisfy both of the following conditions:*

(i) $\Psi_k(F)$ *has finitely many zeros.*

(ii) for all but finitely many simple poles of F, we have $\mathrm{Res}(F, a) \notin \{1, \ldots, k-1\}$.

Then F is a rational function.

3.6 An example concerning Theorem 3.1.2

The following example shows that Theorem 3.1.2 cannot be extended to the case where $a_0, \ldots a_{k-1}$ are meromorphic functions.

Example 3.6.1. *Let $k \geq 3$ and let $F = f'/f$ where f is a meromorphic function which is nonvanishing. Define $a_0, \ldots, a_{k-1}$ by*

$$a_j = \begin{cases} -\frac{f^{(k)}}{f} & \text{if } j = 0\,, \\ \frac{f}{f^{(j)}} & \text{if } j = 1, \ldots, k-1\,. \end{cases}$$

Then $a_0, \ldots, a_{k-1}$ are meromorphic functions and we have, using Lemma 2.2.3, that

$$\begin{aligned} \Lambda_k(F) &= \Psi_k(F) + a_{k-1}\Psi_{k-1}(F) + \ldots + a_1\Psi_1(F) + a_0 \\ &= \frac{f^{(k)}}{f} + \frac{f}{f^{(k-1)}}\frac{f^{(k-1)}}{f} + \ldots + \frac{f}{f'}\frac{f'}{f} - \frac{f^{(k)}}{f} \\ &= k - 1. \end{aligned}$$

Thus $\Lambda_k(F)$ is nonvanishing and so the set E of points where $\Lambda_k(F) = 0$ is such that $E = \emptyset$. And so, all the hypotheses of Theorem 3.1.2 are satisfied *except for the analyticity and growth of the* a_j*. However, since* f *may be any nonvanishing meromorphic function, no conclusions may be drawn about the growth of* F*.*

Chapter 4

Extending two theorems of Langley and Zheng

In this chapter, we prove results which extend some results of Langley and Zheng.[1]

We extend two theorems on fixpoints of $f(z)$ by Langley and Zheng [29] to the consideration of points where $f(z) = Q(z)$ for some rational function Q such that $Q(\infty) = \infty$. In addition, we extend the class of functions f from transcendental entire functions to meromorphic functions with relatively few poles.

[1]The results in this chapter have been published by *Resultate der Mathematik*, see [10].

4.1 Introduction

Let $\mathcal{B}$ denote the class of functions f meromorphic in the plane, for which the set of finite singular values of the inverse function f^{-1} is bounded, that is, the class of all meromorphic functions f whose set of finite asymptotic and critical values is bounded. This class $\mathcal{B}$ has been considered extensively in iteration theory, see [3], [16], [29]. We refer the reader to § 1.8 for background material on iteration theory.

Example 4.1.1. *Let $f(z) = \frac{1}{z} + e^z$. Then $f'(z) = -\frac{1}{z^2} + e^z$. If z is a critical point of f we have $f'(z) = 0$ and so $e^z = \frac{1}{z^2}$ which gives the critical value $f(z) = \frac{1}{z} + \frac{1}{z^2}$. Since $\frac{1}{z} + \frac{1}{z^2} \to 0$ as $z \to \infty$, we see that the set of critical values of f is bounded. Also, 0 is the only finite asymptotic value of f. Therefore, $f \in \mathcal{B}$.*

In [29], Langley and Zheng prove the following theorem, for transcendental entire functions in the class $\mathcal{B}$.

Theorem 4.1.2 (Langley and Zheng, [29])**.** *Let $0 < \alpha < 1$. There is a positive constant c, depending only on α, such that if f is a transcendental entire function in the class $\mathcal{B}$, then there are infinitely many fixpoints z satisfying*

$$f(z) = z, \qquad |f'(z)| > c \log M(\alpha|z|, f). \tag{4.1}$$

We recall that $M(\alpha|z|, f) = \max\{|f(w)| : |w| = \alpha|z|\}$. Then for z large, we have that $c \log M(\alpha|z|, f) > 1$ and so $|f'(z)| > 1$, which gives that the fixpoint z in Theorem 4.1.2 is a *repelling fixpoint*, see § 1.8. Then Theorem

4.1.2 is of interest since by Theorem 1.8.1, the Julia set of f is the closure of the set of repelling periodic points of f.

Let f be a transcendental function and Q be any rational function. We define a *Q-point* of f to be a solution z_0 of the equation $f(z) = Q(z)$.

We extend Theorem 4.1.2 in two ways. First, we extend the result to transcendental meromorphic functions f in $\mathcal{B}$ such that $\delta(\infty, f) > 0$, see § 1.2. Since

$$\delta(\infty, f) = \liminf_{r\to\infty} \frac{m(r, f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, f)}{m(r, f) + N(r, f)},$$

we have that these functions f are meromorphic functions with relatively few poles. Second, we extend (4.1) to Q-points, in particular, to where $f(z)$ is equal to a rational function $Q(z)$ such that $Q(\infty) = \infty$. We state the extended theorem as follows.

Theorem 4.1.3. *Let $0 < \alpha < 1$. There is a positive constant c, depending only on α, such that if f is a transcendental meromorphic function in the class $\mathcal{B}$ with $\delta(\infty, f) > 0$, and Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞, then there are infinitely many Q-points z satisfying*

$$f(z) = Q(z), \qquad |f'(z)| > c\,\delta(\infty, f)\frac{|Q(z)|}{|z|}T(\alpha|z|, f). \tag{4.2}$$

In [29], Langley and Zheng also prove the following theorem.

Theorem 4.1.4 (Langley and Zheng, [29])**.** *Let f be a meromorphic function in the class $\mathcal{B}$, with order $\infty \geq \rho(f) > \mu > 0$. Then f has infinitely many fixpoints z with*

$$f(z) = z, \qquad |f'(z)| > |z|^{\mu/2}.$$

This result is of interest since it relates the multipliers of fixpoints of f to the order of f. We extend this theorem in a similar way to Theorem 4.1.2 in that we extend it to Q-points, that is, the case where $f(z) = Q(z)$ where Q is a rational function such that $Q(\infty) = \infty$. We state the extended theorem as follows.

Theorem 4.1.5. *Let f be a meromorphic function in the class $\mathcal{B}$, with order $\infty \geq \rho(f) > \mu > 0$. Let Q be a rational function with a pole of multiplicity $p \geq 1$ at ∞. Then f has infinitely many Q-points z with*

$$f(z) = Q(z), \qquad |f'(z)| > |z|^{\mu/2+p-1}.$$

In § 4.2, we prove a lemma which is used in the proof of Theorem 4.1.3 and Theorem 4.1.5. In § 4.3 and § 4.4 we prove Theorem 4.1.3 and Theorem 4.1.5 respectively.

4.2 A useful lemma

The following lemma is used in the proof of Theorem 4.1.3 and Theorem 4.1.5. We note that it is an extension of [29, Lemma 1], and we will provide the proof for completeness. We note that $B(a, r)$ denotes the open disc given by $B(a, r) = \{z : |z - a| < r\}$. Also, we refer the reader to § 1.9 for background material about univalent functions and the Koebe distortion theorem (Theorem 1.9.1).

Lemma 4.2.1. *Suppose that f is a transcendental meromorphic function in the class $\mathcal{B}$. Let Q be a rational function such that $Q(\infty) = \infty$. Define* *a function G by*

$$G(z) = f(z)/Q(z), \qquad G'(z)/G(z) = f'(z)/f(z) - Q'(z)/Q(z). \tag{4.3}$$

Suppose that δ is a positive constant. Then there exists a positive constant ϵ such that the following is true. If $|z_1|$ is large and $|G(z_1) - 1| < \frac{1}{4}\epsilon$, then z_1 lies in a component C of the set $\{z : |G(z) - G(z_1)| < \frac{1}{2}\epsilon\}$, such that C is contained in $B(z_1, \delta|z_1|)$. Define a function H by

$$H(z) = \frac{2}{\epsilon}(G(z) - G(z_1)). \tag{4.4}$$

Then C is mapped conformally onto $B(0,1)$ by H. Furthermore, $|Q(z)G'(z)|$ is large on C, and given any $z_0 \in C$ such that $|G(z_0)-G(z_1)| < \frac{1}{4}\epsilon$, we have that

$$\frac{1}{12}|G'(z_1)| \leq |G'(z_0)| \leq \frac{27}{4}|G'(z_1)|. \tag{4.5}$$

We need the following lemma.

Lemma 4.2.2 (Eremenko, Lyubich and Bergweiler, [3], [5], [16], [35])**.** *Suppose that f is a transcendental meromorphic function in the class $\mathcal{B}$. Then there are positive constants R, S and c such that*

$$|zf'(z)/f(z)| \geq c \log^+ |f(z)/R|$$

for $|z| > S$ and $|f(z)| > S$. Here, R and S depend on f, and c does not.

Proof of Lemma 4.2.1 First we choose a sufficiently large positive R_1 and a small positive constant ϵ such that $|G(z) - 1| > \epsilon$ on $|z| = R_1$. Suppose that $|z| > R_1$, $|f(z)| > R_1$ and that $|G(z) - 1| < \epsilon$. Then we have that $|G(z)| > \frac{1}{2}$, and so by (4.3), $|f(z)| > \frac{1}{2}|Q(z)|$. Then by Lemma 4.2.2 and the fact that $Q(\infty) = \infty$, there is a positive constant R such that

$$|zf'(z)/f(z)| \geq c_1 \log^+ |f(z)/R| > c_2 \log |Q(z)|, \tag{4.6}$$

where c_i denotes a positive constant which does not depend on R_1 or ϵ.

For such z, since R_1 is sufficiently large, we have by (4.3) that

$$|zG'(z)/G(z)| = |zf'(z)/f(z) - zQ'(z)/Q(z)| \geq |zf'(z)/f(z)| - |zQ'(z)/Q(z)|. \tag{4.7}$$

We recall that $|G(z)-1| < \epsilon$ and so $|G(z)|$ is close to 1, and since $Q(\infty) = \infty$ and z is large, we have that $|zQ'(z)/Q(z)| = O(1)$ as $z \to \infty$. Then by (4.6) and (4.7) we have that

$$|zG'(z)| > R_2 = c_3 \log |Q(R_1)|. \tag{4.8}$$

Then since $Q(\infty) = \infty$ and $|z| > R_1$ with R_1 sufficiently large, we have that $|Q(z)| \geq c_4|z|$ and so,

$$|Q(z)G'(z)| > R_3 = c_5 R_2. \tag{4.9}$$

Suppose now that δ is a positive constant and that z_1 is as in the statement of the lemma, with $|z_1| > 2R_1$. That is, $|G(z_1) - 1| < \frac{1}{4}\epsilon$, and so z_1 lies in a component C of the set $\{z : |G(z) - G(z_1)| < \frac{1}{2}\epsilon\}$. Let H be defined as in the statement of the lemma, that is, $H(z) = \frac{2}{\epsilon}(G(z) - G(z_1))$. Then $H(z_1) = 0$ and $|H'(z_1)| = \frac{2}{\epsilon}|G'(z_1)| > 2R_2/\epsilon|z_1|$ by (4.8), and so $H'(z_1) \neq 0$. Then H is a conformal mapping in a neighbourhood of z_1.

Next define

$$h(w) = \sum_{k=0}^{\infty} \alpha_k w^k, \qquad \alpha_0 = z_1, \tag{4.10}$$

to be that branch of the inverse function H^{-1} which maps 0 to z_1, and let r_1 be the radius of convergence of the series. Then, by a standard compactness argument, there is some w^* with $w^* = r_1 e^{i\theta^*}$, for some real θ^*, such that h has no analytic continuation to a neighbourhood of w^*. Thus the image of the path $w = te^{i\theta^*}$, $0 \leq t < r_1$, under h must, as $t \to r_1$, either tend to ∞ or to a multiple point z^* of H, with $H(z^*) = w^*$.

Set $r_2 = \min\{r_1, 1\}$. Let γ be the path $\gamma(t) = te^{i\theta^*}$, $0 \leq t < r_2$. For $|z| = R_1$, we have that

$$|H(z)| = \frac{2}{\epsilon}|G(z) - G(z_1)| \geq \frac{2}{\epsilon}(|G(z) - 1| - |G(z_1) - 1|) > \frac{2}{\epsilon}(\epsilon - \frac{1}{4}\epsilon) = \frac{3}{2},$$

and since $h(0) = z_1$ we must have that the image path $h(\gamma)$ lies in $|z| > R_1$. Now $H(h(w)) = w$, which by differentiation gives $(H(h(w)))' = H'(z)h'(w) = 1$ where $z = h(w)$. Then,

$$h(w)/h'(w) = zH'(z) = \frac{2}{\epsilon}zG'(z) \tag{4.11}$$

and this is large on γ by (4.8). Then for w on γ we have by (4.8) and (4.11) that

$$\begin{aligned} \log|h(w)/z_1| &\leq \int_0^{|w|} |h'(se^{i\theta^*})/h(se^{i\theta^*})|ds \\ &\leq \frac{\epsilon}{2}\frac{|w|}{\inf_{z\in h(\gamma)}|zG'(z)|} \leq \frac{\epsilon}{2R_2}. \end{aligned}$$

Then if ϵ was chosen small enough, the path $h(\gamma)$ lies in $B(z_1, \delta|z_1|)$, and replacing θ^* by any $\theta \in [0, 2\pi]$ we see that $h(B(0, r_2)) \subseteq B(z_1, \delta|z_1|)$.

On the path $h(\gamma)$ we have

$$|G(z) - 1| \leq |G(z) - G(z_1)| + |G(z_1) - 1| < \frac{1}{2}\epsilon + \frac{1}{4}\epsilon = \frac{3}{4}\epsilon,$$

and, using (4.8), we have that $|H'(z)| = \frac{2}{\epsilon}|G'(z)| \geq 2R_2/\epsilon|z|$. In particular, we have that $h(\gamma)$ is bounded and does not tend to a critical point of H. Since $r_2 = \min\{r_1, 1\}$, we must then have that $r_1 \geq 1$. Thus, C is contained in $B(z_1, \delta|z_1|)$ and is mapped conformally onto $B(0,1)$ by H. Furthermore, by (4.9), $|Q(z)G'(z)|$ is large on C.

Next, let $z_0 \in C$ such that $|G(z_0) - G(z_1)| < \frac{1}{4}\epsilon$. Then $|H(z_0)| = \frac{2}{\epsilon}|G(z_0) - G(z_1)| < \frac{1}{2}$. Then since h is univalent on $B(0,1)$, there exists some $w_0 \in B(0,1)$, such that $h(w_0) = z_0$. Further, $|w_0| < \frac{1}{2}$, and so by the Koebe distortion theorem (Theorem 1.9.1) we have,

$$\frac{4}{27}|h'(0)| \leq |h'(w_0)| \leq 12|h'(0)|. \tag{4.12}$$

Then since $H'(z)h'(w) = 1$ we have $h'(w) = 1/H'(z)$ for $w \in B(0,1)$, and in particular, $h'(0) = 1/H'(z_1)$ and $h'(w_0) = 1/H'(z_0)$. Also, by (4.11), $H'(z) = \frac{2}{\epsilon}G'(z)$, and so by (4.12) we have

$$\frac{1}{12}|G'(z_1)| \leq |G'(z_0)| \leq \frac{27}{4}|G'(z_1)|.$$

4.3 Proof of Theorem 4.1.3

We need the following theorem, where $f^{\#}(z) = \frac{|f'(z)|}{1+|f(z)|^2}$ is the spherical derivative of f.

Theorem 4.3.1 (Toppila and Winkler, [41]). *Let f be a transcendental meromorphic function of order λ such that $\delta(\infty, f) > 0$. Then*

$$\limsup_{z\to\infty,\, z\in E(f)} \frac{|z|f^{\sharp}(z)}{T(|z|, f)} \geq A_0\delta(\infty, f)(1+\lambda)$$

where $E(f) = \{z : |f(z)| = 1\}$ and A_0 is a positive absolute constant.

We now prove Theorem 4.1.3.

Proof of Theorem 4.1.3 Let $0 < \alpha < 1$ and let R be a large positive constant. Let δ and ϵ be as in Lemma 4.2.1, with $\frac{1}{1+\delta} > \alpha$. Define functions G and f_1 by

$$G(z) = \frac{f(z)}{Q(z)} = 1 + \frac{\epsilon f_1(z)}{16}. \tag{4.13}$$

Then $f_1(z) = \frac{16}{\epsilon}(\frac{f(z)}{Q(z)} - 1) = c_1\frac{f(z)}{Q(z)} - c_2$ where c_j denotes a positive constant not depending on R. Then since f is a transcendental meromorphic function and since Q is a rational function, we have that f_1 is a transcendental meromorphic function.

Further, since Q is a rational function and f_1 is a transcendental function, we have by Theorem 1.2.1 and Theorem 1.2.2 that $\delta(\infty, f_1) > 0$ since $f(z) = (c_3 f_1(z) + c_4)Q(z)$ and by Theorem 1.2.3, we have

$$\begin{aligned} 0 < \delta(\infty, f) &= \delta(\infty, (c_3 f_1 + c_4)Q) \\ &= \liminf_{r\to\infty} \frac{m(r, (c_3 f_1 + c_4)Q)}{T(r, (c_3 f_1 + c_4)Q)} \\ &\le \liminf_{r\to\infty} \frac{m(r, c_3 f_1 + c_4) + m(r, Q) + O(1)}{T(r, c_3 f_1 + c_4) - T(r, Q) - O(1)} \\ &= \liminf_{r\to\infty} \frac{m(r, f_1) + O(\log r) + O(1)}{T(r, f_1) - O(\log r) - O(1)} \\ &= \delta(\infty, f_1). \end{aligned}$$

Let λ be the order of f_1, and let $E(f_1) = \{z : |f_1(z)| = 1\}$. Then by Theorem 4.3.1, we have that

$$\limsup_{z\to\infty,\, z\in E(f_1)} \frac{|z| f_1^\sharp(z)}{T(|z|, f_1)} \ge A_0 \delta(\infty, f_1)(1 + \lambda),$$

where A_0 is a positive absolute constant. We note that for $z \in E(f_1)$ we have that $f_1^\sharp(z) = \frac{|f_1'(z)|}{1+|f_1(z)|^2} = \frac{|f_1'(z)|}{2}$. Then we have

$$\limsup_{z\to\infty,\, z\in E(f_1)} \frac{|z||f_1'(z)|}{T(|z|, f_1)} > d_1 = A_0\delta(\infty, f_1),$$

and so there exists a sequence (ζ_n) in $E(f_1)$, $\zeta_n \to \infty$ as $n \to \infty$, such that

$$\frac{|\zeta_n||f_1'(\zeta_n)|}{T(|\zeta_n|, f_1)} > d_1.$$

Thus we can choose z_1 in $E(f_1)$ arbitrarily large, and in particular such that $|z_1| > R$, with

$$|z_1||f_1'(z_1)| > d_1 T(|z_1|, f_1). \tag{4.14}$$

Next, since $f(z) = (c_3 f_1(z) + c_4)Q(z)$ we have that

$$\begin{aligned} T(|z_1|, f) &= T(|z_1|, (c_3 f_1 + c_4)Q) \\ &\leq T(|z_1|, c_3 f_1 + c_4) + T(|z_1|, Q) \\ &\leq T(|z_1|, f_1) + O(\log |z_1|), \end{aligned}$$

and so, if R is large enough,

$$T(|z_1|, f_1) \geq T(|z_1|, f) - O(\log |z_1|) > \frac{1}{2} T(|z_1|, f), \tag{4.15}$$

since f is a transcendental function. Then since $|G'(z)| = \frac{\epsilon}{16}|f_1'(z)|$, we have by (4.14) and (4.15) that

$$|G'(z_1)| > \frac{\epsilon}{16}\frac{d_1}{|z_1|} T(|z_1|, f_1) > \frac{d_2}{|z_1|} T(|z_1|, f), \tag{4.16}$$

where $d_2 = \frac{\epsilon}{32} d_1$.

We recall that $G(z) = 1 + \frac{\epsilon f_1(z)}{16}$, and thus since $z_1 \in E(f_1)$, we have that $|G(z_1) - 1| = \frac{\epsilon}{16} < \frac{\epsilon}{4}$. Then by Lemma 4.2.1, z_1 lies in a component C of the set $\{z : |G(z) - G(z_1)| < \frac{1}{2}\epsilon\}$ such that C is contained in $B(z_1, \delta|z_1|)$. Also, by Lemma 4.2.1, C is mapped conformally onto $B(0,1)$ by the function $H(z) = \frac{2}{\epsilon}(G(z) - G(z_1))$. Then we can choose a point z_2 in C such that $G(z_2) = 1$ and $|G(z_2) - G(z_1)| < \frac{1}{4}\epsilon$. Furthermore, we have by (4.5) that,

$$\frac{1}{12}|G'(z_1)| \leq |G'(z_2)| \leq \frac{27}{4}|G'(z_1)|,$$

and so by (4.16) we have

$$|G'(z_2)| > \frac{d_3}{|z_1|}T(|z_1|, f), \tag{4.17}$$

where $d_3 = \frac{d_2}{12}$. Then since we have by (4.3) that,

$$z_2\frac{f'(z_2)}{f(z_2)} = z_2\frac{G'(z_2)}{G(z_2)} + z_2\frac{Q'(z_2)}{Q(z_2)},$$

and since $f(z_2) = Q(z_2)$ and $G(z_2) = 1$, we have that

$$\left|z_2\frac{f'(z_2)}{Q(z_2)}\right| \geq |z_2G'(z_2)| - \left|z_2\frac{Q'(z_2)}{Q(z_2)}\right|. \tag{4.18}$$

Now since Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞, we have that $\left|z\frac{Q'(z)}{Q(z)}\right| = O(1)$ as $z \to \infty$. Since $z_2 \in B(z_1, \delta|z_1|)$, we have that $|z_1| \leq |z_1 - z_2| + |z_2| \leq \delta|z_1| + |z_2|$, which gives that $\left|\frac{z_2}{z_1}\right| \geq 1 - \delta$. Then, since $\frac{1}{1+\delta} > \alpha$, we also have that $|z_1| \geq \frac{1}{1+\delta}|z_2| > \alpha|z_2|$, and so by (4.17) and (4.18), we have that

$$\begin{aligned}
\left|z_2\frac{f'(z_2)}{Q(z_2)}\right| &\geq |z_2G'(z_2)| - O(1) \\
&\geq \frac{1}{2}\left|\frac{z_2}{z_1}\right| d_3T(|z_1|, f) \\
&\geq \frac{1-\delta}{2}d_3T(|z_1|, f) \\
&\geq \frac{1-\delta}{2}d_3T(\alpha|z_2|, f).
\end{aligned}$$

Then since $d_3 = \frac{\epsilon}{384} A_0 \delta(\infty, f)$ and A_0 is an absolute constant, we have that

$$|f'(z_2)| > c\delta(\infty, f) \frac{|Q(z_2)|}{|z_2|} T(\alpha |z_2|, f).$$

where c is a positive constant depending only on α.

4.4 Proof of Theorem 4.1.5

We need the following theorem.

Theorem 4.4.1 (Langley and Zheng, [29]). *Let f be a transcendental meromorphic function in the class $\mathcal{B}$. If Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞, then*

$$m(r, \frac{1}{f - Q}) = O(\log r T(r, f)) \tag{4.19}$$

as $r \to \infty$ outside a set of finite measure.

We note that the Nevanlinna counting function $N(r, \frac{1}{f-z})$ counts the poles of $\frac{1}{f-z}$, that is, the fixpoints of f. In [29], the following implication of Theorem 4.4.1 is noted, namely, that $N(r, \frac{1}{f-z})$ cannot satisfy $N(r, \frac{1}{f-z}) = o(T(r, f))$ as $r \to \infty$. We extend this to a rational function Q such that $Q(\infty) = \infty$ and state the result as a corollary. We provide a proof for completeness.

Corollary 4.4.2. *Let f and Q be as in the statement of Theorem* 4.1.5*. Then the Nevanlinna counting function $N(r, \frac{1}{f-Q})$, of points where $f(z) = Q(z)$, cannot satisfy $N(r, \frac{1}{f-Q}) = o(T(r, f))$ as $r \to \infty$. Also, for $\sigma > 0$, we may choose arbitrarily large r such that there are at least $2r^{\sigma}$ Q-points z_j, such that $f(z_j) = Q(z_j)$, in the annulus $\frac{1}{2} r \leq |z| \leq r$.*

Proof Since Q is a rational function, we have by (1.12) and Theorem 1.2.1, that

$$T(r, f-Q) \leq T(r, f) + T(r, Q) + O(1) = T(r, f) + O(\log r) + O(1),$$

and also that

$$T(r, f) = T(r, f-Q+Q) \leq T(r, f-Q)+T(r, Q)+O(1) = T(r, f-Q)+O(\log r)+O(1).$$

Then we have that

$$|T(r, f) - T(r, f-Q)| \leq O(\log r) + O(1). \tag{4.20}$$

Next, by (1.15), $T(r, \frac{1}{f-Q}) = T(r, f-Q) + O(1)$ and so (4.20) gives us that

$$|T(r, f) - T(r, \frac{1}{f-Q}) - O(1)| \leq O(\log r) + O(1). \tag{4.21}$$

Then since $T(r, \frac{1}{f-Q}) = N(r, \frac{1}{f-Q}) + m(r, \frac{1}{f-Q})$, we have that

$$|T(r, f)-T(r, \frac{1}{f-Q})-O(1)| \geq |T(r, f)-N(r, \frac{1}{f-Q})|-|m(r, \frac{1}{f-Q})+O(1)|.$$

We may assume by (4.19) that $m(r, \frac{1}{f-Q}) = O(\log rT(r, f))$ as $r \to \infty$, with $r \notin E$, for some set E of finite measure. Then by (4.21) and Theorem 4.4.1 we have

$$\begin{aligned} |T(r, f) - N(r, \frac{1}{f-Q})| &\leq O(\log r) + O(1) + |m(r, \frac{1}{f-Q}) + O(1)| \\ &= O(\log r) + O(1) + O(\log rT(r, f)), \qquad r \notin E. \end{aligned}$$

Then since f is a transcendental function, we have that $N(r, \frac{1}{f-Q})$ cannot satisfy $N(r, \frac{1}{f-Q}) = o(T(r, f))$ as $r \to \infty$.

Next we show that all large z_j such that $f(z_j) = Q(z_j)$ are simple zeros of $f - Q$. Let z_0 be large and suppose $f(z_0) = Q(z_0)$. Then since f and

Q satisfy the hypotheses of Lemma 4.2.1 and Lemma 4.2.2, we have as in (4.6) that

$$|z_0 f'(z_0)/f(z_0)| > R_1 = c \log |Q(z_0)|, \tag{4.22}$$

where c is a positive constant. Also, since Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞, we have that $\left|\frac{Q'(z_0)}{Q(z_0)}\right| \leq \frac{p}{|z_0|}(1 + o(1))$. Then, by (4.22) and since $f(z_0) = Q(z_0)$, we have that

$$|f'(z_0)| > \frac{R_1}{|z_0|}|Q(z_0)| \geq \frac{p}{|z_0|}(1 + o(1))|Q(z_0)| \geq |Q'(z_0)|,$$

since we can choose z_0 so large that p/R_1 is very small. Then this gives that $f'(z_0) - Q'(z_0) \neq 0$ and so z_0 is a simple zero of $f - Q$.

Suppose now that $r_0 \geq 0$ is such that for $r \geq r_0$, there are less than $2r^\sigma$ points z_j, such that $f(z_j) = Q(z_j)$, in $\frac{r}{2} \leq |z| \leq r$. Then for $m \in \mathbb{N}$ we have that

$$n\left(2^m r_0, \frac{1}{f-Q}\right) < n\left(r_0, \frac{1}{f-Q}\right) + 2\left[(2^m r_0)^\sigma + (2^{m-1} r_0)^\sigma + \ldots + (2r_0)^\sigma\right]. \tag{4.23}$$

Let r be large, with, in particular, $r > r_0$. Then there exists $m \in \mathbb{N}$ such that $2^{m-1} r_0 \leq r < 2^m r_0$. Then by (4.23) we have

$$\begin{aligned} n\left(r, \frac{1}{f-Q}\right) &\leq n\left(2^m r_0, \frac{1}{f-Q}\right) \\ &< n\left(r_0, \frac{1}{f-Q}\right) + 2\left[(2^m r_0)^\sigma + (2^{m-1} r_0)^\sigma + \ldots + (2r_0)^\sigma\right] \\ &= c_1 + 2r_0^\sigma \left[\frac{2^\sigma(2^{\sigma m} - 1)}{2^\sigma - 1}\right] \\ &\leq c_1 + c_2 r^\sigma, \end{aligned}$$

where $c_1 = n(r_0, \frac{1}{f-Q})$ and c_2 is a positive constant. Then we have $N(r, \frac{1}{f-Q}) = O(r^\sigma)$.

Next, by (4.19), we may choose $r \leq r_1 \leq 2r$ such that $m(r_1, \frac{1}{f-Q}) = O(\log r_1 T(r_1, f))$. Then

$$\begin{aligned} T(r_1, \frac{1}{f-Q}) &= N(r_1, \frac{1}{f-Q}) + m(r_1, \frac{1}{f-Q}) \\ &= O(r_1^\sigma) + O(\log r_1 T(r_1, f)), \end{aligned}$$

and so,

$$T(r, \frac{1}{f-Q}) \leq T(r_1, \frac{1}{f-Q}) = O(r^\sigma).$$

Therefore we have that $T(r, f) = O(r^\sigma)$, and thus,

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r} \leq \sigma.$$

This is a contradiction since $\rho(f) > \sigma$. Therefore, we may choose arbitrarily large r such that there are at least $2r^\sigma$ points z_j such that $f(z_j) = Q(z_j)$ in the annulus $\frac{1}{2}r \leq |z| \leq r$.

We now prove Theorem 4.1.5.

Proof of Theorem 4.1.5 Let f and Q be as in the statement of Theorem 4.1.5. Choose σ with $\rho(f) > \sigma > \mu$. Since $\rho(f) > 0$, we have that f is a transcendental function, and so by Corollary 4.4.2, we can choose arbitrarily large r such that f has at least $2r^\sigma$ points z_j such that $f(z_j) = Q(z_j)$ in the annulus $\frac{1}{2}r \leq |z| \leq r$.

Suppose δ is a small positive constant. Let $G(z) = f(z)/Q(z)$ be as defined in Lemma 4.2.1. Then for each z_j we have that $G(z_j) = 1$. Then by Lemma 4.2.1, we have that to each z_j there corresponds a component C_j of the set $\{z : |G(z) - 1| < \frac{1}{2}\epsilon\}$, and that each C_j is contained in $B(z_j, \delta|z_j|)$. Then choosing δ small enough gives that each C_j lies in the annulus $\frac{1}{4}r \leq |z| \leq 2r$. These C_j are disjoint simple islands since, by Lemma 4.2.1, they are each mapped conformally onto $B(0,1)$ by the function $H_j(z) = \frac{2}{\epsilon}(G(z) - G(z_j))$.

Let $h_j : B(0,1) \to C_j$ be the inverse function of H_j. Then h_j is a univalent function on B(0,1) and by the Koebe distortion theorem (Theorem 1.9.1) we have that for $0 < r_0 < 1$,

$$c_1|h_j'(0)| \leq \max_{|w|\leq r_0} |h_j'(w)| \leq c_2|h_j'(0)|,$$

where c_1 and c_2 are positive constants, depending only on r_0.

In particular, choose $r_0 = \frac{1}{2}$ and let $\hat{C}_j = h_j(B(0, \frac{1}{2}))$. Then $|h_j'(w)| \geq c_1|h_j'(0)|$ for $|w| \leq \frac{1}{2}$. Then since $h_j'(w) = 1/H_j'(z)$ where $z = h_j(w)$, and in particular since $h_j'(0) = 1/H_j'(z_j) = \epsilon/2G'(z_j)$ we have that

$$|h_j'(w)| \geq c_3/|G'(z_j)|, \qquad \text{for } |w| \leq \frac{1}{2}, \tag{4.24}$$

where, from here on, c_i denotes a positive constant which does not depend on r or σ.

Next, since the area of the annulus $\frac{1}{4}r \leq |z| \leq 2r$ is $c_4 r^2$, and since there are at least $2r^\sigma$ disjoint components $\hat{C}_j$ in the annulus, we have that at least r^σ of these $\hat{C}_j$ have area at most $c_5 r^{2-\sigma}$. Then, for these z_j, we have

$$z_j \frac{f'(z_j)}{f(z_j)} = z_j \frac{G'(z_j)}{G(z_j)} + z_j \frac{Q'(z_j)}{Q(z_j)},$$

and since $f(z_j) = Q(z_j)$ and $G(z_j) = 1$, we have that

$$\left| z_j \frac{f'(z_j)}{Q(z_j)} \right| \geq |z_j G'(z_j)| - \left| z_j \frac{Q'(z_j)}{Q(z_j)} \right|. \tag{4.25}$$

Now since Q is a rational function with a pole of multiplicity $p \geq 1$ at ∞, we have that $\left|z\frac{Q'(z)}{Q(z)}\right| = O(1)$ as $z \to \infty$. Therefore, (4.25) gives that

$$\left|z_j \frac{f'(z_j)}{Q(z_j)}\right| \geq |z_j G'(z_j)| - O(1). \tag{4.26}$$

Now by [34, p.4] and (4.24) we have that

$$\text{area of } \hat{C}_j = \int\int_{B(0,\frac{1}{2})} |h'(w)|^2 du\, dv \geq c_3/|G'(z_j)|^2.$$

And so, since the area of $\hat{C}_j$ is at most $c_5 r^{2-\sigma}$ we have that $1/|G'(z_j)|^2 \leq c_5 r^{2-\sigma}$, which gives $|G'(z_j)| \geq c_6 r^{\sigma/2-1}$. Then by (4.26) and since $r \geq |z_j|$, and z_j is large, and $p \geq 1$, we have

$$\begin{aligned} \left|z_j \frac{f'(z_j)}{Q(z_j)}\right| &\geq c_7 r^{\sigma/2} - O(1) \\ &\geq c_8 |z_j|^{\sigma/2}. \end{aligned}$$

Then since $|Q(z_j)| > c_9 |z_j|^p$, we have that

$$\begin{aligned} |f'(z_j)| &> c_{10} |z_j|^{\sigma/2+p-1} \\ &\geq |z_j|^{\mu/2+p-1}. \end{aligned}$$

Chapter 5

Some results in connection with composite functions

In this chapter, we prove a result concerning normal families in connection with composite functions. We also prove several results concerning the value distribution of composite functions.[1]

Let $k \in \mathbb{N}$ and let f be a transcendental entire function f with $\rho(f) < 1/k$. First, we prove a normal families result, namely, the family of analytic functions g such that $(f \circ g)^{(k)}(z) + \sum_{j=0}^{k-1} a_j(z)(f \circ g)^{(j)}(z) \neq a(z)$ in a domain Ω, where $a_0, \dots, a_{k-1}, a$ are analytic functions in Ω, is a normal family. Second, we prove several value distribution results for $(f \circ g)^{(k)}$, where f and k are as above, and g is a nonconstant entire function, and for $(f \circ g)^{(k)} - Q$, where f and k are as above, and g and Q are polynomials, g nonconstant.

[1]The results in this chapter have been accepted for publication by *Journal of Mathematical Analysis and Applications*, see [12].

5.1 Introduction

In [23], Hinchliffe proves the following result which provides a criterion for normal families in connection with composite functions.

Theorem 5.1.1 (Hinchliffe, [23]). *Let f be a transcendental meromorphic function in the plane, and let Ω be a domain in $\mathbb{C}$. If $\mathbb{C}^* \setminus f(\mathbb{C}) = \emptyset, \{\infty\}$ or $\{\alpha, \beta\}$, where α and β are two distinct values in $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, then the family*

$$\mathcal{G} = \{g : g \, \textit{is analytic in} \, \Omega, \; f \circ g \, \textit{has no fixpoints in} \, \Omega\}$$

is a normal family in Ω.

We note that this criterion is that $(f \circ g)(z) \neq z$ in Ω, or that $(f \circ g)^{(0)}(z) - a(z)$ is nonvanishing in Ω, where $a(z) \equiv z$, for $g \in \mathcal{G}$. Theorem 5.1.1 then motivates the idea of a criterion for normal families in connection with composite functions involving $(f \circ g)^{(k)}(z) \neq 0$, for $k \in \mathbb{N}$. This idea is reinforced by the following theorem and corollary by Langley and Zheng. We provide a proof of the corollary for completeness.

Theorem 5.1.2 (Langley and Zheng, [29]). *Let $k \in \mathbb{N}$. Suppose that f and g are transcendental entire functions of finite order. Suppose also that*

$$\overline{N}(r, 1/(f \circ g)^{(k)}) = O(T(r, g)) \qquad (\text{n.e.})\,. \tag{5.1}$$

Then

$$T(r, f) \neq o(r^{1/k}) \qquad \text{as} \quad r \to \infty\,.$$

Corollary 5.1.3. *Let $k \in \mathbb{N}$. Suppose that f is a transcendental entire function such that $\rho(f) < 1/k$. Suppose that g is an entire function of finite order such that*

$$(f \circ g)^{(k)}(z) \neq 0$$

on $\mathbb{C}$. Then g is a polynomial.

Proof Since

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r} < \frac{1}{k}$$

we know that for some $\epsilon > 0$ and r_0 we have,

$$\frac{\log^+ T(r,f)}{\log r} < \frac{1}{k} - \epsilon, \qquad r \geq r_0,$$

which gives that $T(r,f) = o(r^{1/k})$ as $r \to \infty$.

Next, since $(f \circ g)^{(k)}(z) \neq 0$ on $\mathbb{C}$, we have that $\overline{N}(r, 1/(f \circ g)^{(k)}) = 0$. Then since (5.1) is trivially satisfied, we must have that g is not a transcendental function. Since g is entire, g is a polynomial.

We note that the example $f(z) = e^z$ shows that Corollary 5.1.3 cannot be strengthened to $\rho(f) \leq 1/k$, since $\rho(f) = 1$ and $(f \circ g)'(z) = f'(g(z)).g'(z) = e^{g(z)}.g'(z)$ which is nonvanishing for many entire functions of finite order, for example $g(z) = e^z$.

Thus, given a transcendental function f with $\rho(f) < 1/k$ for some $k \in \mathbb{N}$, the Bloch Principle (see § 1.5), Theorem 5.1.1 and Corollary 5.1.3 motivate the question whether the family $\mathcal{G}$ of analytic functions g in a domain Ω, such that $(f \circ g)^{(k)}(z) \neq 0$ in Ω, or more generally, $(f \circ g)^{(k)}(z) \neq Q(z)$ for some analytic function Q, is a normal family. This is true, and is a special case of the following result.

Theorem 5.1.4. *Let $k \in \mathbb{N}$. Let f be a transcendental entire function with $\rho(f) < 1/k$. Let $a_0, \ldots, a_{k-1}, a$ be analytic functions in a domain Ω. Then*

$$\mathcal{G} = \{g : g \text{ is analytic in } \Omega,\ (f \circ g)^{(k)}(z) + \sum_{j=0}^{k-1} a_j(z)(f \circ g)^{(j)}(z) \neq a(z) \text{ in } \Omega\}$$

is a normal family in Ω.

In the proof of Theorem 5.1.4, we use the following theorem, which is an interesting value distribution result in its own right.

Theorem 5.1.5. *Let k be an integer, $k \geq 2$. Let f be a transcendental entire function with $\rho(f) < 1/k$. Let g and Q be polynomials, with g nonconstant. Then*

$$(f \circ g)^{(k)} - Q$$

has infinitely many zeros.

We note that in Theorem 5.1.5, the $k = 1$ case is omitted. This is due to the fact that we apply a theorem of $\cos \pi\rho$ type (Theorem 5.2.3), and must have $\rho(f) < 1/2$ in order to do so. However, if $Q \equiv 0$, we can prove Theorem 5.1.5 for $k = 1$ for the extended case where g is a nonconstant entire function. We state the result as follows.

Theorem 5.1.6. *Let f be a transcendental entire function with $\rho(f) < 1$. Let g be a nonconstant entire function. Then $(f \circ g)'$ has infinitely many zeros.*

We note that the example $f(z) = e^z$ and $g(z) = z$ shows that Theorem 5.1.6 cannot be strengthened to $\rho(f) \leq 1/k$, since $\rho(f) = 1$ and $(f \circ g)'(z) = e^z$ which is nonvanishing in $\mathbb{C}$.

From Theorem 5.1.5 and Theorem 5.1.6, we prove the following corollary which strengthens Corollary 5.1.3 and which is used in the proof of Theorem 5.1.4.

Corollary 5.1.7. *Let $k \in \mathbb{N}$. Suppose that f is a transcendental entire function such that $\rho(f) < 1/k$. Suppose that g is an entire function of finite order such that*

$$(f \circ g)^{(k)}(z) \neq 0$$

on $\mathbb{C}$. Then g is constant.

Finally, we note that Theorem 5.1.5 and Theorem 5.1.6 have the following corollaries.

Corollary 5.1.8. *Let k be an integer, $k \geq 2$. Let f be a transcendental entire function with $\rho(f) < 1/k$. Let $\alpha \in \mathbb{C}$. Then for every nonconstant entire function g,*

$$(f \circ g)^{(k)} - \alpha$$

has infinitely many zeros.

Again, although the $k = 1$ case is omitted in Corollary 5.1.8, we can prove the $k = 1$ case when g is a transcendental entire function. We state the result as follows.

Corollary 5.1.9. *Let f be a transcendental entire function with $\rho(f) < 1$. Let $\alpha \in \mathbb{C}$. Then for every transcendental entire function g,*

$$(f \circ g)' - \alpha$$

has infinitely many zeros.

Since the proof of Theorem 5.1.4 depends on Theorem 5.1.5, Theorem 5.1.6 and Corollary 5.1.7, we prove these results in § 5.2, § 5.3 and § 5.4 respectively. We then prove Theorem 5.1.4 in § 5.5. Finally, we prove Corollary 5.1.8 and Corollary 5.1.9 in § 5.6.

5.2 Proof of Theorem 5.1.5

The following lemma is a version of Taylor's theorem (Theorem 1.10.7) and is easily proved by induction.

Lemma 5.2.1. *If f is an entire function and $a \in \mathbb{C}$, then for $k \in \mathbb{N}$ we have*

$$f(z) = f(a)+(z-a)f'(a)+\ldots+\frac{(z-a)^{k-1}}{(k-1)!}f^{(k-1)}(a)+\int_a^z \frac{(z-t)^{k-1}}{(k-1)!}f^{(k)}(t)dt.$$

We also need the following lemma. We include the proof here for completeness.

Lemma 5.2.2. *Let $k \in \mathbb{N}$. Let P_1 and P_2 be polynomials of degree m and n respectively, with $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Then we can choose a straight line Γ from 0 to ∞ such that*

$$I = \left| \int_0^z \frac{(z-t)^{k-1}}{(k-1)!} P_1(t) e^{P_2(t)} dt \right| \leq c|z|^{k+m},$$

as $z \to \infty$ along Γ, for some positive constant c.

Proof Since P_2 is a polynomial of degree n we can write

$$P_2(t) = b_n t^n + b_{n-1} t^{n-1} + \ldots + b_1 t + b_0,$$

where $b_0, \ldots, b_n \in \mathbb{C}$. Then the behaviour of P_2 is dominated by the leading term $b_n t^n$. Setting $t = re^{i\theta}$, we have that $b_n t^n = b_n r^n e^{in\theta}$ and so we have that

$$|e^{b_n t^n}| = |e^{b_n r^n e^{in\theta}}| = e^{\text{Re}(b_n r^n e^{in\theta})} = e^{(\alpha\cos(n\theta)+\beta\sin(n\theta))r^n}$$

for some α, $\beta \in \mathbb{R}$, not both 0.

Choose θ such that $\alpha\cos(n\theta) + \beta\sin(n\theta) = -d < 0$. Let Γ be the straight line $z = re^{i\theta}$, for $0 \leq r < \infty$. Then for t on Γ between 0 and z we have

$$\begin{aligned} |e^{P_2(t)}| &= e^{\text{Re}(b_n r^n e^{in\theta} + b_{n-1} r^{n-1} e^{i(n-1)\theta} + \ldots + b_0)} \\ &= e^{-dr^n + O(r^{n-1})} \\ &\to 0 \end{aligned}$$

as $r \to \infty$, for fixed θ as above. Thus, we have that $|e^{P_2(t)}| \leq c_0$ for some positive constant c_0.

Also, since P_1 has degree m, we have that for t on Γ between 0 and z, $|P_1(t)| \leq c_1(1 + |t|^m)$ for some positive constant c_1. Thus we have that

$$\begin{aligned} I &\leq \int_0^z \frac{|z-t|^{k-1}}{(k-1)!} |P_1(t)| \left|e^{P_2(t)}\right| |dt| \\ &\leq \frac{c_0 c_1 |z|^{k-1}}{(k-1)!} \int_0^z (1 + |t|^m)|dt| \\ &\leq c_2 |z|^{k+m} \end{aligned}$$

as $z \to \infty$ on Γ, for some positive constant c_2, since $|z - t| \leq |z|$ for t on Γ between 0 and z.

Finally, we need a theorem of $\cos\pi\rho$ type, as follows. We refer the reader to [2] for further reading.

Theorem 5.2.3 ([2])**.** *Let f be a nonconstant entire function with $\rho(f) = \rho < 1/2$. For $r > 0$, define $A(r)$ and $B(r)$ as follows*

$$\begin{aligned} A(r) &= \inf\{\log|f(z)| : |z| = r\} \\ B(r) &= \sup\{\log|f(z)| : |z| = r\}\,. \end{aligned}$$

If $\rho < \alpha < 1/2$, then

$$\underline{\log\operatorname{dens}}\{r : A(r) > (\cos\pi\alpha)B(r)\} \geq 1 - \rho/\alpha$$

where if E is a subset of $(1, +\infty)$ the lower logarithmic density of E is defined by

$$\underline{\log\operatorname{dens}}(E) = \liminf_{r\to\infty}\left(\int_1^r \chi(t)dt/t\right)/\log r$$

where $\chi(t)$ is the characteristic function of E.

We now prove Theorem 5.1.5.

Proof of Theorem 5.1.5 We use a proof by contradiction.
Suppose that $(f\circ g)^{(k)} - Q$ has m zeros in $\mathbb{C}$, for some $m \in \mathbb{N}\cup\{0\}$. Then by Theorem 1.1.2, since f is an entire function with $\rho(f) < \infty$, and since g is a polynomial, we have that $\rho(f\circ g) < \infty$. Then, by Theorem 1.2.8, we have that $\rho((f\circ g)^{(k)}) < \infty$ also, and since Q is a polynomial, we have that $\rho((f\circ g)^{(k)} - Q) < \infty$ also. Then by Theorem 1.1.3, we can write

$$(f\circ g)^{(k)}(z) - Q(z) = P_1(z)e^{P_2(z)}$$

for some polynomials P_1 and P_2 of degree m and n respectively, with m as above and $n \in \mathbb{N}$. Then, by Lemma 5.2.1, we have for $a = 0$ that

$$\begin{aligned}(f \circ g)(z) &= (f \circ g)(0) + z(f \circ g)'(0) + \ldots + \frac{z^{k-1}}{(k-1)!}(f \circ g)^{(k-1)}(0) + \\ &\quad + \int_0^z \frac{(z-t)^{k-1}}{(k-1)!}(f \circ g)^k(t)dt \\ &= Q_{k-1}(z) + \int_0^z \frac{(z-t)^{k-1}}{(k-1)!}(Q(t) + P_1(t)e^{P_2(t)})dt,\end{aligned}$$

where Q_{k-1} is a polynomial of degree at most $k-1$. Then

$$|(f \circ g)(z)| \leq |Q_{k-1}(z)| + \left|\int_0^z \frac{(z-t)^{k-1}}{(k-1)!}Q(t)dt\right| + \left|\int_0^z \frac{(z-t)^{k-1}}{(k-1)!}P_1(t)e^{P_2(t)}dt\right|. \tag{5.2}$$

For the remainder of this proof, we use c_j to denote positive constants.

Since Q_{k-1} is a polynomial of degree at most $k-1$, we have that

$$|Q_{k-1}(z)| \leq c_1|z|^{k-1}, \tag{5.3}$$

as $z \to \infty$.

Since Q is a polynomial of degree q say, $q \geq 0$, we have that $|Q(t)| \leq c_2|t|^q$ as $t \to \infty$ on the straight line Γ between 0 and z. Then we have that integrating along the straight line Γ between 0 and z,

$$\left|\int_0^z \frac{(z-t)^{k-1}}{(k-1)!}Q(t)dt\right| \leq \frac{c_2|z|^{k-1}}{(k-1)!}\int_0^z O(|t|^q)dt \leq c_3|z|^{k+q}, \tag{5.4}$$

as $z \to \infty$, since $|z-t| \leq |z|$ for t on the straight line Γ between 0 and z.

In particular, by Lemma 5.2.2, we can choose a straight line path Γ from 0 to ∞ such that

$$\left|\int_0^z \frac{(z-t)^{k-1}}{(k-1)!}P_1(t)e^{P_2(t)}dt\right| \leq c_4|z|^{k+m}$$

as $z \to \infty$ along Γ. Then by (5.2), (5.3) and (5.4), we have that

$$|(f \circ g)(z)| \leq c_5 |z|^{k+q+m}$$

as $z \to \infty$ along Γ. Then we have that

$$\log |(f \circ g)(z)| \leq c_6 \log |z| \tag{5.5}$$

as $z \to \infty$ along Γ.

Since $\rho(f) = \rho < 1/2$, we can apply Theorem 5.2.3 to f. For $r > 0$, define

$$\begin{aligned} A(r) &= \inf\{\log |f(z)| : |z| = r\} \\ B(r) &= \sup\{\log |f(z)| : |z| = r\}. \end{aligned}$$

Then for $\rho < \alpha < 1/2$ we have

$$\underline{\log \text{dens}}\{r : A(r) > (\cos \pi\alpha) B(r)\} \geq 1 - \rho/\alpha. \tag{5.6}$$

Next, since g is a polynomial and is nonconstant, we have that $|g(z)| \geq c_7 |z|$ as $z \to \infty$. Then we have that

$$\log |z| \leq c_8 \log |g(z)|$$

as $z \to \infty$. Then by (5.5), we have that

$$\log |f(g(z))| = \log |(f \circ g)(z)| \leq c_6 \log |z| \leq c_9 \log |g(z)| \tag{5.7}$$

as $z \to \infty$ along Γ. Now choose R large such that $R \in \{r : A(r) > (\cos \pi\alpha) B(r)\}$. Choose w such that $|w| = R$ and $w = g(z)$ for some z on Γ. Then by (5.7), we have that

$$(\cos \pi\alpha) B(R) < A(R) \leq \log |f(w)| \leq c_9 \log R.$$

This is a contradiction since f is a transcendental function, which implies by Theorem 1.2.2 that $B(R)/\log R \to +\infty$ as $R \to \infty$.

5.3 Proof of Theorem 5.1.6

We need the following lemma from Nevanlinna theory.

Lemma 5.3.1. *If f is a transcendental entire function with $\rho(f) < 1$, then f' has infinitely many zeros.*

Proof of Theorem 5.1.6 Since g is a nonconstant entire function, we have by Picard's Theorem (Theorem 1.10.4) that g omits at most one value in $\mathbb{C}$. Since f is a transcendental entire function with $\rho(f) < 1$, we have by Lemma 5.3.1 that f' has infinitely many zeros. Then since g omits at most one of these zeros, we have that $f'(g(z))$ has infinitely many zeros. Therefore, since $(f \circ g)'(z) = f'(g(z)).g'(z)$, we have that $(f \circ g)'$ has infinitely many zeros.

5.4 Proof of Corollary 5.1.7

Proof of Corollary 5.1.7 By Corollary 5.1.3, we have that g is a polynomial. However, by Theorem 5.1.6 for $k = 1$ and by Theorem 5.1.5 for $k \geq 2$, if g is a nonconstant polynomial then $(f \circ g)^{(k)}$ has infinitely many zeros. Therefore g is constant.

5.5 Proof of Theorem 5.1.4

First we state, and prove for completeness, the following result, which is a version of Hurwitz' theorem (Theorem 1.10.2).

Lemma 5.5.1. *Let (f_n) be a sequence of analytic functions on a domain D, which converge spherically uniformly on compact subsets to a function $f : D \to \mathbb{C}$. Let (s_n) be a sequence of analytic functions tending uniformly to 0 on some disc $B(\alpha, \delta) = \{z : |z - \alpha| < \delta\} \subseteq D$, for some $\delta > 0$. If $f \not\equiv 0$ and $f(\alpha) = 0$, then for large n, we have $f_n(z) = s_n(z)$ for some z near α.*

Proof Choose $r > 0$ small, with $r < \delta$, and consider the circle $S(\alpha, r) = \{z : |z - \alpha| = r\}$. Since $f \not\equiv 0$ and $f(\alpha) = 0$, there exists $c > 0$ such that $|f(z)| > c$ on $S(\alpha, r)$. Then since $s_n(z) \to 0$ on $B(\alpha, \delta)$ and $f_n(z) \to f(z)$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $|f_n(z) - f(z) - s_n(z)| < |f(z)|$ on $S(\alpha, r)$ for $n \geq n_0$. Then by Rouché's theorem (Theorem 1.10.6), f and $f + f_n - f - s_n = f_n - s_n$ have the same number of zeros inside $S(\alpha, r)$. Then, since $f(\alpha) = 0$, there exists $z \in B(\alpha, r)$ such that $f_n(z) = s_n(z)$.

Next, we note that Lemma 5.5.1 has the following corollary, which we will use in the proof of Theorem 5.1.4. Again, we provide a proof for completeness.

Corollary 5.5.2. *Let $k \in \mathbb{N}$. Let Ω be the open unit disc $B(0, 1)$. Let a be an analytic function on Ω. Let (f_n) be a sequence of analytic functions on Ω, such that $f_n(z) \neq a(z)$ on Ω. Let (z_n) be a sequence of points tending to $z_0 \in \Omega$, and let (ρ_n) be a positive sequence tending to 0. Suppose g is an entire function such that*

$$\lim_{n \to \infty} \rho_n^k f_n(z_n + \rho_n z) = g(z)$$

locally uniformly on $\mathbb{C}$. Then either $g \equiv 0$ on $\mathbb{C}$, or $g(z) \neq 0$ on $\mathbb{C}$.

Proof Suppose there exists $\alpha \in \mathbb{C}$ such that $g(\alpha) = 0$. If $g \equiv 0$, then we are done. Otherwise, we note that $z_n + \rho_n z \in \Omega$ for n large, and that $(\rho_n^k f_n(z_n + \rho_n z))$ is a sequence of analytic functions which converge to g locally uniformly on $\mathbb{C}$. We note also that since a is analytic, and therefore bounded near z_0, and since (ρ_n^k) is a sequence tending to 0, then $\left(\rho_n^k a(z_n + \rho_n z)\right)$ is a sequence of functions tending to 0, for n large, on $B(\alpha, \delta)$, for some $\delta > 0$. Then by Lemma 5.5.1, we obtain $\rho_n^k f_n(z_n+\rho_n z) = \rho_n^k a(z_n + \rho_n z)$ for n large, for some z near α. Since (ρ_n^k) is a positive sequence, we therefore have that $f_n(z_n + \rho_n z) = a(z_n + \rho_n z)$, which is a contradiction since $z_n + \rho_n z \in \Omega$ for n large.

We now prove Theorem 5.1.4.

Proof of Theorem 5.1.4 Since normality is a local property, we can assume, without loss of generality, that Ω is a disc and $a_0, \ldots, a_{k-1}, a$ are bounded on Ω. Using a linear change of variables $h(z) = g(\alpha + \beta z)$, and the fact that $(f \circ h)^{(j)}(z) = \beta^j (f \circ g)^{(j)}(\alpha + \beta z)$, for a suitable choice of $\alpha, \beta \in \mathbb{C}$, we may assume that Ω is the open unit disc $B(0, 1)$. Suppose that $\mathcal{G}$ is not normal on Ω.

Since $\mathcal{G}$ is a family of analytic functions, we can apply the Pang-Zalcman Lemma (Lemma 1.5.3) taking $\alpha = k = 0$. Then there exist $r \in (0, 1)$, points z_n with $|z_n| < r$, a sequence of functions (g_n) in $\mathcal{G}$, a positive sequence ρ_n tending to 0 and a nonconstant entire function g such that

$$h_n(z) = g_n(z_n + \rho_n z) \to g(z) \tag{5.8}$$

locally uniformly on $\mathbb{C}$, with respect to the spherical metric, with $g^\sharp(z) \leq 1$.

Then since g has bounded spherical derivative, we have by Lemma 1.4.1 that g is a function of finite order.

Next, since f is an entire function, we have that

$$(f \circ h_n)(z) \to (f \circ g)(z),$$

locally uniformly on $\mathbb{C}$. Then by the Weierstrass theorem (Theorem 1.10.8), for $j \in \mathbb{N}$,

$$(f \circ h_n)^{(j)}(z) = \rho_n^j (f \circ g_n)^{(j)}(z_n + \rho_n z) \to (f \circ g)^{(j)}(z), \tag{5.9}$$

locally uniformly on $\mathbb{C}$. However, since each $g_n \in \mathcal{G}$, we have that for $z_n + \rho_n z \in \Omega$,

$$F_n(z) = (f \circ g_n)^{(k)}(z_n + \rho_n z) + \sum_{j=0}^{k-1} a_j(z_n + \rho_n z)(f \circ g_n)^{(j)}(z_n + \rho_n z) \neq a(z_n + \rho_n z).$$

Then we have that

$$\begin{aligned} \rho_n^k F_n(z) &= \rho_n^k (f \circ g_n)^{(k)}(z_n + \rho_n z) + \sum_{j=0}^{k-1} \rho_n^{k-j} a_j(z_n + \rho_n z) \rho_n^j (f \circ g_n)^{(j)}(z_n + \rho_n z) \\ &\neq \rho_n^k a(z_n + \rho_n z). \end{aligned}$$

Next, since $\rho_n^{k-j} \to 0$ for $j = 0, \ldots, k-1$, and since the a_j are assumed bounded on Ω, we have by (5.9) that

$$\lim_{n \to \infty} \rho_n^k F_n(z) = (f \circ g)^{(k)}(z)$$

locally uniformly on $\mathbb{C}$. However, we can write $F_n(z) = G_n(z_n + \rho_n z)$ where $G_n(z) \neq a(z)$ on Ω, and so we have by Corollary 5.5.2, that either $(f \circ g)^{(k)}(z) \equiv 0$ on $\mathbb{C}$, or $(f \circ g)^{(k)}(z) \neq 0$ on $\mathbb{C}$.

Case 1: $(f \circ g)^{(k)}(z) \equiv 0$ *on* $\mathbb{C}$.

Then integrating this equation $k-1$ times, we have that $(f \circ g)'(z) =$

$P_{k-2}(z)$, where P_{k-2} is a polynomial of degree at most $k-2$. Since P_{k-2} has at most $k-2$ zeros, counting multiplicities, $(f \circ g)'(z) = f'(g(z)).g'(z)$ has also. However, f is a transcendental entire function and $\rho(f) < 1$, and so by Lemma 5.3.1, we have that f' has infinitely many zeros on $\mathbb{C}$. Then since g is a nonconstant entire function, we must have that g omits infinitely many zeros of f' on $\mathbb{C}$, which is a contradiction by Picard's theorem (Theorem 1.10.4).

Case 2: $(f \circ g)^{(k)}(z) \neq 0$ *on* $\mathbb{C}$.
Suppose first that $k = 1$. Then by Theorem 5.1.6, we have that $(f \circ g)'$ has infinitely many zeros on $\mathbb{C}$ and so we have a contradiction.
Suppose second that $k \geq 2$. Then since f is a transcendental entire function with $\rho(f) < 1/k$ and g is a nonconstant entire function of finite order, we have by Corollary 5.1.7 that $(f \circ g)^{(k)}$ has at least one zero in $\mathbb{C}$. Therefore we have a contradiction.

Therefore $\mathcal{G}$ is a normal family.

5.6 Proof of Corollary 5.1.8 and Corollary 5.1.9

Proof of Corollary 5.1.8 Suppose that $(f \circ g)^{(k)} - \alpha$ has finitely many zeros.

Suppose first that g is a function of finite order. If g is a polynomial, then we have a contradiction by Theorem 5.1.5. If g is a transcendental function, then since $N(r, 1/((f \circ g)^{(k)} - \alpha)) = O(\log r) = o(T(r, g))$ by (1.7) and

Theorem 1.2.2, and since $T(r, f) = o(r^{1/k})$, then we have a contradiction by Theorem 5.1.2.

Suppose second that g is a function of infinite order. Then by Lemma 1.4.1, we have that g has unbounded spherical derivative. That is, we can choose a sequence of points (α_n) tending to ∞, such that $g^\sharp(\alpha_n) \to \infty$ as $n \to \infty$. Then the family of functions $\{g_n(z) = g(\alpha_n + z) : n \in \mathbb{N}\}$ is not a normal family on the open unit disc $B(0, 1)$. Then by Theorem 5.1.4 we have a contradiction.

Proof of Corollary 5.1.9 Suppose that $(f \circ g)' - \alpha$ has finitely many zeros. If g is a function of finite order, then by the argument in Corollary 5.1.8, we have that g is a polynomial. This is a contradiction since g is a transcendental function. If g is a function of infinite order, then by the argument in Corollary 5.1.8, we have a contradiction by Theorem 5.1.4.

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