Equations of length five over groups

by

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Abstract

This work considers the problem of Equations Over Groups and settles the KL-conjecture for equations of length five.

Firstly, the problem of equations over groups is stated and discussed and the results, which were up to now obtained, are presented. Then, by way of contradiction, it is assumed that for the remaining cases of equations of length five a solution does not exist. The methodology adopted uses the combinatorial and topological arguments of relative diagrams. If D is a relative diagram representing the counter example, all types of interior regions of positive curvature are listed for each type of equation of length five. For each interior region of positive curvature, one region of negative curvature is found and the positive curvature is added to it to obtain the total curvature in the interior of diagram D.

In the final chapter the curvature of the interior of D is added to the curvature of the boundary regions to obtain the total curvature of the diagram. It is proved that the total curvature of 4π cannot be achieved, our desired contradiction, and therefore equations of length five have a solution.

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Chapter 1

Introduction

Equations over groups entered Combinatorial Group Theory in a paper of B.H. Neumann [N] in 1943, where he studied the process of embedding a given group in another group. The problem of equations over groups was then defined and it was shown that, for any positive integer n and any element g of a group G the equation $x^n = g$ always has a solution in some overgroup H containing G. In 1949, Higman, Neumann and Neumann [HNN], showed that two elements a and b of a group G are conjugate in a suitable extension of G if and only if they have the same order. In terms of equations over groups this would mean that the equation $t^{-1}at = b$ for a, $b \in G$ has a solution in a group containing G if and only if a and b have the same order. In 1962, Levin [L] showed that a solution to an equation exists if t occurs in $r(t) = 1$ only in positive exponent.

The central problem concerning equations over groups was stated by Lyndon [Ly] in 1987, (problem2) as the Kervaire-Laudenbach problem, one of the unsolved problems of Combinatorial Group Theory, as follows:

> If G has a presentation $G =< X|R >$ and $H =< X \cup t|R \cup w >$ is obtained by adding one new generator and one defining relator, when does the inclusion $X \to X \cup t$ induce an injection of G into H?

In simpler language, for $g_1, ..., g_n$ in G, when does an equation $w = w(g_1, ..., g_n, t) =$

1 have a solution t in some group H containing G?

The Kervaire-Laudenbach (KL) conjecture asserts that if the exponent sum of t in w is non-zero then the equation $\mathbf{w = 1}$ always has a solution.

Today, the KL-conjecture still remains one of the unsolved problems in combinatorial group theory. There were also several other conjectures [St], [G3] which would have implied that the KL-conjecture is true, but a counter example was produced in [G2].

There were several results concerning the KL-conjecture, when the problem was restricted to a type of equation or a class of groups, but no universal proof of it is yet known. One of the approaches was proving the conjecture for equations of certain length. The conjecture was proved for equations of length up to four (for all types of groups and all types of equations) and equations of length five but with certain restrictions held on the group. This thesis proposes that these restrictions may be removed and therefore, settles the conjecture for equations of length up to five.

Chapter two of this thesis gives some background theory and basic concepts of combinatorial group theory. The concepts of free groups and group presentations are defined. Several definitions and results on the theory of free products, free products with amalgamation and HNN-extensions are stated. Also, some basic definitions and results are given from the geometrical point of group theory. More specifically, some definitions from the theory of complexes and fundamental groups are presented. Last, the problem of equations over groups and systems of equations over groups are defined and their geometrical realisation is described.

In chapter three, the results that have been obtained up to now on equations over groups and systems of equations over groups are stated and discussed. Starting from the early results of Neumann [N] and the embedding theorems of Higman, Neumann and Neumann [HNN], we then present the results obtained in the 60s and 70s for certain types of groups. Then the results on the equations of certain length are presented since these methods are used in the present study. Finally, the latest results of Edjvet and Juhasz [EJ(1)] on equations of length five are discussed, as we claim that their restrictions held on equations of length five can be removed.

In chapter four we sketch the method of proof adopted in this study and in the next three chapters the three different types of equations of length five are discussed and the combinatorial calculations performed on each are presented. By way of contradiction it is assumed that any equation of length five which is not covered by the results of [EJ(1)]does not have a solution. Using the geometrical realisation of equations over groups we investigate whether it is possible to obtain a diagram D with interior regions of positive curvature. This is achieved by calculating the possible labelling since the labels on interior vertices need to be closed paths on the star graph Γ of the equations. An overview of the proof of Chapters 5, 6 and 7 is also given in this chapter to make it easier to follow the details in the main text.

In the last chapter we state and prove our final result by calculating the total curvature of the diagram. This is the curvature of interior regions and the curvature of the boundary regions at the distinguished vertex v_0. It is proved that this curvature cannot be 4π and this proves our that the KL-conjecture is true for equations of length up to five.

Chapter 2

Basic Concepts

In this chapter some basic concepts of the combinatorial group theory are presented. We start with the definitions of free groups and group presentations, the theory of free products, free products with amalgamation and HNN-extensions.

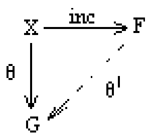
Also, some basic definitions and results from the geometric methods used in group theory are presented. Namely, some definitions from the theory of complexes and fundamental groups are stated as these are used later in the geometrical realisation of equations and systems of equations over groups.

Last, the problem of equations over groups and systems of equations over groups are defined and their geometrical realisation is described.

2.1 Free Groups and Group Presentations

The following definitions, theorems and remarks are quoted from [J].

Definition 2.1 (J) *A group F is said to be free on a subset X of F if, given any group G and a map $\theta : X \to G$, there is a unique homomorphism $\theta' : F \to G$ extending θ. That is, having the property that $x\theta' = x\theta$ for all $x \in X$, or that the diagram*



is commutative. Then X is called a basis of F and $|X|$ the rank of F, written $r(F)$.

The following, are some basic results:

1. If F is free on X, then X generates F. If the elements of X did not generate F, the extension of θ would only be defined as far as the subgroup $\langle X \rangle$ of F generated by X and thereafter would be arbitrary, violating the uniqueness.

2. There are no non-trivial relations in $X^{\pm}$. If there is a relation $w = w'$ among the members of $X^{\pm} = \{x, x^{-1} | x \in X\}$, then we could find a group G, with corresponding elements (under θ) for which that relation does not hold; then θ' would have to map $e = w'w^{-1} \in F$ to an element of G other than the identity.

3. Every element of F can be written uniquely as a product of elements of X and their inverses.

4. Given a any set X the free group $F(X)$ can be constructed by considering copies of X and using "juxtaposition plus cancellation" as the group binary operation.

5. Every group is isomorphic to a factor group of some free group (i.e. it can be realized in the form F/N where F is a free group).

Given a group G let X be a set of generators for G (which always exists: take for example $X = G$). Then let $\theta' : F(X) \to G$ be the extension of the inclusion $\theta : X \to G$. Now $\operatorname{Im} \theta' = G$, since $< X >= G$, so that if $K = \ker \theta'$, then, by the first isomorphism theorem $G = \operatorname{Im} \theta' \cong F(X)/K$.

The above result leads to the definition of Free Presentations of Groups.

Definition 2.2 (J) *A group free presentation or simply a* presentation *is a pair* $< X|R >$ *where* X *is a set, called the generators of* G*, and* R *is a subset of* $F(X)$*, called the set of (defining) relators for* g*, so that* $G = F(X)/N$*. (i.e.* G *is a factor group of* $F(X)$*, the free group on* X*, by the normal closure* $N = \bar{R}$ *of* R *in* F*.)*

A group is called *finitely presented* if it has a presentation with both X and R finite sets.

2.2 Free Products, Free Products with Amalgamation and HNN Extensions

In this section we quote from [LyS] some basic definitions and theorems on the theory of Free Products and HNN extensions. Apart from the definitions of the concepts we quote the normal form theorems, the torsion free theorems and the conjugacy theorems.

2.2.1 Free Products

Definition 2.3 (LyS) *Given groups* $G =< X|R >$ *and* $H =< Y|S >$*, with disjoint sets of generators, their free product is given by the presentation* $G*H =< Y, X|R, S >$*. Groups* G *and* H *are called the (free) factors of* $G * H$*.*

Lemma 2.4 (LyS) *The free product* $G*H$ *is uniquely determined by the groups* G *and* H*. Moreover* $G*H$ *is generated by two subgroups* G' *and* H' *which are isomorphic to* G *and* H *respectively such that* $G' \cap H' = 1$*.*

Definition 2.5 (LyS) *A reduced sequence (or normal form) is a sequence* $g_1, .., g_n$ $n \geq 0$ *of elements of* $G * H$ *such that* $g_i \neq 1$*, each* g_i *is in one of the factors* G *or* H *and successive* $g_i g_{i+1}$ *are not in the same factor.*

Theorem 2.6 (LyS) *(The Normal Form Theorem for Free Products) Consider the free product $G * H$. Then the following two equivalent statements hold: (i) If $w = g_1 ... g_n$ $n > 0$, where $g_1, .., g_n$ is a reduced sequence, then $w \neq 1$ in $G * H$. (ii) Each element w of $G * H$ can be uniquely expressed as a product $g = g_1 .. g_n$, where $g_1 .. g_n$ is a reduced sequence.*

Corollary 2.7 (LyS) *If G and H are both finitely generated groups with solvable word problem, then $G * H$ has solvable word problem.*

Theorem 2.8 (LyS) *(The Conjugacy Theorem for Free Products) Each element of $G*H$ is conjugate to a cyclically reduced element. If $u = g_1 ... g_n$ and $\upsilon = h_1 ... h_m$ are cyclically reduced elements which are conjugates in $G * H$ and $n > 1$ then $m = n$ and the sequences $g_1, .., g_n$ and $h_1, ..., h_m$ are cyclic permutations of each other. If $n \leq 1$ then $m = n$ and $u = \upsilon$ in the same factor and are conjugates in the same factor.*

Theorem 2.9 (LyS) *(The Torsion Theorem for Free Products) An element of finite order in $G * H$ is a conjugate of an element of finite order in G or H.*

Theorem 2.10 (LyS) *(Grushko - Neumann Theorem) Let $A = G * H$ and $g_{1,...,} g_\kappa$, κ finite, a set of generators for A. Then $g_{1,...,} g_\kappa$ can be obtained by Nielsen transformation form a set of generators, part of which lie in G and the rest of which lie in H. (These generators in G or H need not be minimal in number, and may include 1).*

Corollary 2.11 (LyS) *The rank (i.e. the minimum number of generators) of $A = G * H$ is the rank of G added to the rank of H.*

Remark 2.12 *The free product of groups G and H as a binary operation of G and H is commutative i.e. $G * H \backsimeq H * G$. The free product is also associative $(G * H) * K \backsimeq G * (H * K)$.*

Remark 2.13 *The direct product $G \times H$ of two groups G and H is the homomorphic image of the free product under a mapping which maps the replica of G in $G * H$ onto the replica G in $G \times H$ (and similar for H). The kernel of this mapping is called the Cartesian subgroup of $G * H$.*

2.2.2 Free Products with Amalgamation and HNN-Extensions

Definition 2.14 (LyS) *Let $G =< x_1, ...; r_1, ... >$, $H =< y_1, ...; s_1, ... >$ be groups. Let $A \subseteq G$, $B \subseteq H$ be subgroups of G and H respectively such that there exists an isomorphism $\phi : A \to B$. Then the free product of G and H amalgamating the subgroups A and B is the group $< x_1, ..., y_1, ...; r_1, ..., s_1, ..., a = \phi(a), a \in A >$, also noted as $< G * H; a = \phi(a) >$ or $< G * H; A = B, \phi >$. G and H are called the free factors with amalgamation and A and B are called the amalgamated subgroups.*

Definition 2.15 (LyS) *Let G be a group and A and B be subgroups of G, $\phi : A \to B$ an isomorphism. The HNN extension of G relative to A and B is the subgroup $G^* =< G, t; t^{-1}at = \phi(a), a \in A >$. The group G is called the base of G^*, t is called the stable letter and A and B are called the associated subgroups.*

The following definitions, lemmas and theorems refer to the HNN extensions:

Definition 2.16 (LyS) *Let G be the HNN extension of G relative to its isomorphic subgroups A, B as in the above definition. A sequence $g_0, t^{\varepsilon_1}, g_1, ..., g_{n-1}, t^{\varepsilon_n}, g_n$, $n \geq 0$ is said to be reduced if there is no consecutive subsequence t^{-1}, g_i, t with $g_i \in A$ or t, g_j, t^{-1} with $g_j \in B$.*

Lemma 2.17 (LyS) *(Brittons Lemma) If the sequence $g_0, t^{\varepsilon_1}, g_1, ..., g_{n-1}, t^{\varepsilon_n}, g_n$ is reduced and $n \geq 1$ then $g_0.t^{\varepsilon_1}.g_1.....t^{\varepsilon_n} \neq 1$ in G^*.*

Choose a set of representatives of the right cosets of A in G, and a set of representatives for the right cosets of B in G. It is assumed that 1 is the representative of both A and B. If $g \in G$, $\bar{g}$ will denote the representative of the coset Ag and $\hat{g}$ will denote the representative of the coset Bg.

Definition 2.18 (LyS) *A normal form is a sequence* $g_0, t^{\varepsilon_1}, g_1, ..., t^{\varepsilon_n}, g_n$ $(n \geq 0)$ *where*

1. g_0 *is an arbitrary element of* G*,*
2. *If* $\varepsilon_i = -1$*, then* g_i *is a representative of a coset of* A *in* G*,*
3. *If* $\varepsilon_i = +1$*, then* g_i *is a representative of a coset of* B *in* G*, and*
4. *there is no consecutive subsequence* t^{ε} $, 1, t^{-\varepsilon}$.

Theorem 2.19 (LyS) *(The Normal Form Theorem for HNN-Extensions) Let* $G^* =< G, t; t^{-1}at = \phi(a), a \in A >$ *be and HNN extension. Then:*

(i) The group G *is embedded in* G^* *by the map* $g \to g$*. If* $g_0t^{\varepsilon_1}...t^{\varepsilon_n}g_n = 1$ *in* G^* *where* $n \geq 1$ *then* $g_0, t^{\varepsilon_1}, ..., t^{\varepsilon_n}, g_n$ *is not reduced.*

(ii) Every element w *of* G^* *has a unique representative as* $w = g_0t^{\varepsilon_1}...t^{\varepsilon_n}g_n$ *where* $g_0, t^{\varepsilon_1}, ..., t^{\varepsilon_n}, g_n$ *is a normal form.*

Corollary 2.20 (LyS) *Let* $G^* =< G, t; t^{-1}At = B, \phi >$ *be an HNN extension. If* G *has solvable word problem and the generalised word problems of* A *and* B *in* G *is solvable, and* ϕ *and* ϕ^{-1} *are effectively calculable then* G^* *has solvable word problem.*

Lemma 2.21 (LyS) *Let* $u = g_0t^{\varepsilon_1}...t^{\varepsilon_n}g_n$ *and* $\upsilon = h_0t^{\delta_1}...t^{\delta_n}h_m$ *be reduced words and suppose* $u = \upsilon$ *in* G^**. Then* $m = n$ *and* $\varepsilon_i = \delta_i$ $i = 1, .., n$.

Theorem 2.22 (LyS) *(The torsion Theorem for HNN Extensions) Let* $G^* =< G, t; t^{-1}At = B, \phi >$ *be an HNN extension. Then every element of finite order in* G^* *is a conjugate of an element of finite order in the base* G*. Thus* G^* *has elements of finite order* n *only if* G *has elements of order* n*.*

Theorem 2.23 (LyS) *(The Conjugacy Theorem for HNN Extensions) Let* $G^* =< G, t; t^{-1}At = B, \phi >$ *be an HNN extension. Let* $u = g_0 t^{\varepsilon_1} ... t^{\varepsilon_n} g_n$, $n \geq 1$ *and* υ *be conjugate cyclically reduced elements of* G*. Then* $|u| = |\upsilon|$*, and* u *can be obtained from* υ *by taking a suitable cyclic permutation* υ^* *of* υ*, which ends in* t^{ε_n}*, and then conjugating by an element* z*, where* $z \in A$ *if* $\varepsilon_n = -1$ *and* $z \in B$ *if* $\varepsilon_n = +1$*.*

The following definitions and theorems refer to free products with amalgamation.

Let $P =< G * H; a = \phi(a), a \in A >$ be the free product of G and H amalgamating the subgroups $A \subseteq G$ and $B \subseteq H$, with $\phi : A \rightarrow B$ an isomorphism. P can be viewed as the quotient of the free product $G * H$ by the normal subgroup $\{a\phi(a)^{-1}, a \in A\}$. A sequence of elements $c_1, ..., c_n$ $n \geq 0$, of elements of $G * H$ will be called reduced if:

1. Each c_i is one of the factors of G or H.
2. Successive c_i, c_{i+1} come from different factors.
3. If $n > 1$, no c_i is in A or B.
4. If $n = 1$, $c_i \neq 1$.

Theorem 2.24 (LyS) *(Normal Form Theorem for Free Products with Amalgamation) If* $c_1, ..., c_m$ *is a reduced sequence,* $n \geq 1$*, then the product* $c_1 ... c_m \neq 1$ *in* P*. In particular,* G *and* H *are embedded in* P *by the maps* $g \rightarrow g$ *and* $h \rightarrow h$*.*

Theorem 2.25 (LyS) *(Torsion Theorem) Every element of finite order in* $P =< G * H; a = \phi(a), a \in A >$ *is a conjugate of an element of finite order in* H *or* G*.*

Theorem 2.26 (LyS) *(The conjugacy theorem for Free Products with Amalgamation) Let* $P =< G * H; a = \phi(a), a \in A >$ *be a free product with amalgamation. Let* $u = c_1 ... c_n$ *be a cyclically reduced element of* P *where* $n \geq 2$*. Then every cyclically reduced conjugate of* u *can be obtained by cyclically permuting* $c_1 ... c_n$ *and then conjugating by an element of the amalgamated part* A*.*

2.3 Complexes and Fundamental Groups

The following basic definitions and results about complexes and fundamental groups are quoted from Lyndon and Schupp (1977).

Definition 2.27 (LyS) *A* 1-complex *or* graph *is two disjoint sets V and E, together with three functions $\alpha : E \to V$, $\omega : E \to V$ and $n_1 : E \to E$. The elements of V are called* vertices *(or points) and the elements of E* edges. *For $e \in E$, $\alpha(e)$ is called the initial point of e and $\omega(e)$ the terminal point and we say that e runs from $\alpha(e)$ to $\omega(e)$. $n_1(e)$ is called the inverse of e, or the oppositely oriented edge and is written $n_1(e) = e^{-1}$. We impose on these functions the conditions that n_1 should be an involution without fixed elements and that e^{-1} run from $\omega(e)$ to $\alpha(e)$.*

Since it is required that C contains together with each edge its inverse, a graph is an undirected graph i.e. the pair $\{e, e^{-1}\}$ of opposite edges is considered. If we want to orient an undirected graph, this can be done by choosing between one from each pair of inverse edges.

Definition 2.28 *A path in C is a finite sequence of edges for which we write $\rho = e_1 ... e_n$, $n \geq 1$, such that, for $1 \leq i \leq n$, e_{i+1} begins were e_i ends, that is $\alpha(e_{i+1}) = \omega(i)$. The length of the path ρ is $|\rho| = n$. The path begins with $a(\rho) = \alpha(e_1)$ and ends at $\omega(\rho) = \omega(e_n)$. If $a(e_1) = \omega(e_n)$ then ρ is a loop. For each vertex υ, 1_υ represents a path without edges beginning and ending with υ i.e. 1_υ is a loop and has length 0. The inverse ρ^{-1} to path ρ is the path $\rho^{-1} = e_n^{-1} ... e_1^{-1}$. If $\rho = e_1 ... e_n$ is a loop then every cyclic permutation $\rho' = \rho_i ... \rho_n \rho_1 ... \rho_{i-1}$ is also a loop. We call the set of all cyclic permutations of a loop ρ a cyclic path or a cycle. A path is reduced if it contains no part ee^{-1}; A loop or the corresponding cycle is cyclically reduced if it is reduced and $e_1 \neq e_n^{-1}$. A path is called simple if for $i \neq j$ $\alpha(e_i) \neq \alpha(e_j)$ and $\omega(e_i) \neq \omega(e_j)$ i.e. two edges do not start or end at the same point.*

A 2-complex consists of a 1-complex C^1, its 1-skeleton, together with a set F of 2-cells or faces and two functions ϑ and n_2 defined on F. To each D in F the function ϑ assigns a cyclically reduced cycle ϑD in C^1, the boundary of D, and the function n_2 assigns to D a second face $n_2(D) = D^{-1}$, the inverse of D; we require that $\vartheta(D^{-1})$ be the inverse of D in the obvious sense. A vertex v is on a face D if it is the initial point on some edge on ϑD; then a boundary path for D at v is any loop in the cycle ϑD that begins at v. In the most interesting cases ϑD will be simple, whence there is a unique boundary path for D at each vertex on D). Note that every 1-complex is also a 2-complex with an empty set of faces. A path in a 2-complex C will mean a path in its 1-skeleton.

Observe that the set $\pi(C)$ of all the paths has a certain amount of algebraic structure.

The product $p.q$ of two paths can be defined, provided $\omega(p) = \alpha(q)$ by juxtaposition. This multiplication is associative. Also $1_{\alpha(p)}.p = p$ and $p.1_{\alpha(p)} = p$ and if pq is defined then $(pq)^{-1} = q^{-1}p^{-1}$. We define 1-equivalence between paths p and p' and write pp' which holds if one can pass from p and p' by a succession of steps that consists of insertion or deletion of parts of the form ee^{-1}. This is an equivalence relation so we can pass to the quotient structure $\pi'(C)$ of $\pi(C)$. Since $pp^{-1} = 1_{\alpha(p)}$, $\pi'(C)$ is a structure with unique inverses. It can been seen (in the same way as treating reduced words in a free group) that every path is equivalent to a unique reduced path. In particular a non-trivial reduced loop in $\pi(C)$ will not map into one of the elements in $\pi'(C)$ which equals $1_{\alpha(p)}$.

A relation of 2-equivalence can be defined among paths p and p', and write $p \sim p'$, if it is possible to pass from one to the other by succession of insertions and deletions of the parts of the form ee^{-1} or of the form q where q is a boundary path at some point for a face D. This too is an equivalence relation of $\pi(C)$. The quotient of $\pi(C)$ by this relation is a groupoid, the fundamental groupoid $\pi(C)$ of complex C.

Observe that for any vertex v, the subset $\pi(C, v)$ of $\pi(C)$ consisting of all *loops at υ is a semi-group and that its image $\pi(C, \upsilon)$ in $\pi(C)$ is a group,* the fundamental group of C at point υ.

Proposition 2.29 (LyS) *If C is a 1-complex and υ any vertex of C, then $\pi(C, \upsilon)$ is a free group.*

Proposition 2.30 (LyS) *Let C be a finite connected 1-complex and υ any vertex. Let γ_0 be the number of vertices in C and γ_1 the number of undirected edges (that is, of the unordered pairs $\{e, e^{-1}\}$ of inverse edges). Then $\pi(C, \upsilon)$ is a free group of rank $\gamma_1 - \gamma_0 + 1$.*

With every presentation $G =< X|R >$ where all r in R are cyclically reduced we associate a special complex $K(X|R)$ with a single vertex. First, K has a single vertex υ, and an edge χ(from υ to υ) for each element χ of X, together with its inverse χ^{-1}. Now every path in K is a loop. Second if $r = x_1^{e_1}...x_n^{e_n}$ is in R where all $x_i \in X$, $e_i = \pm 1$, we introduce a face D with boundary path (at υ) $x_1^{e_1}...x_n^{e_n}$ together with D^{-1}.

Proposition 2.31 (LyS) *If $G =< X|R >$ and $K(X|R)$ is associated with the presentation as above, then $\pi(K, \upsilon) \simeq G$.*

2.4 Equations and Systems of Equations Over Groups

In this section the problem of Equations and Systems of Equations Over Groups is going to be defined. Also basic definition about their geometric realisation, relative diagrams, the star graph and the weight test are included.

2.4.1 Definition of Equations Over Groups

We have already given in the introduction the definition of the problem of equations over groups as stated by Lyndon [Ly]. In terms of presentations of groups we will use the following definitions for an equation or a system of equations over a group G:

Definition 2.32 *Let G be group and t an unknown. Then an equation in t over G is an expression*

$r(t) = g_0 t^{e_1} g_1 ... t^{e_n} g_n \in G * \langle t \rangle \qquad (g_i \in G, e_i = \pm 1)$

and a solution to it over G is an embedding ϕ of G into a group H, together with an element h of H, such that $\phi(g_0)h^{e_1}\phi(g_1)...\phi(g_n)h^{e_n} = 1$ in H.

The integer n is called the t-length of the equation and $\sigma = e_1 + ... + e_n$ the exponent sum.

The KL-conjecture asserts that any equation over any group G, with exponent sum $\sigma \neq 0$, has a solution over G.

Definition 2.33 *A system W of equations over a group G in the set X of indeterminates is a collection of expressions $w = 1$ where*

$w(x_1, x_2, ..., x_n) \qquad (w \in W) \qquad (x_i \in X)$

w's are elements of the free product $G * F(X)$, (where $F(X) = F$ is the free group with basis X).

A solution for the system W in G is the map $\theta : X \to G$ such that $(id_G * F(\theta))(w) = 1$.

Considering a finite system of equations $w_1(x_1, x_2, ..., x_n) = 1$, $w_2(x_1, x_2, ..., x_n) = 1$, ..., $w_m(x_1, x_2, ..., x_n) = 1$, let M be the $(m \times n)$ matrix whose $(i.j)$ entry is the sum of the exponents of x_j occurring in the word w_i. The system is called ***independent or nonsingular*** *if the rows of M are linearly independent, that is if M has rank m.*

An infinite system of equations over G is defined to be independent if each finite subsystem is independent.

Howie's conjecture [Ho1], [Ho4], generalises the KL-conjecture for systems of equations over a group G and asserts that any independent system of equations over G has a solution.

2.4.2 Systems of equations and their geometric realizations

These results are quoted from [Ho1] and were used to prove that the (KL) conjecture holds for locally indicable groups. A presentation $\langle G, x_1, x_2, ... | w_1, w_2, ... \rangle$ of a group H, the universal solution group of a system W of equations over groups (the group containing all the solutions of the system, see Theorem 3.1) is shown to correspond via geometric realization, to a pair (L, K) of connected $CW-$complexes, with $L \backslash K$ 2-dimensional, and $\pi_1(L) \cong H$. This correspondence is also used for the definition of the relative diagrams and star graphs.

Given a polynomial $w \in G * F(X)$, and a homomorphism $\phi : G \to H$ of groups, let w^ϕ denote the polynomials $(\phi * id_F)(w)$ in $H * F(X)$. Given a set $W \subset G * F(X)$ of polynomials, let W^ϕ denote the set $\left\{w^\phi, w \in W\right\}$ of polynomials in $H * F(X)$, and also the associated system of equations over H.

Among all pairs (ϕ, θ) where $\phi : G \to H$ is a homomorphism, and $\theta : X \to H$ is a solution for W^ϕ in H, there is a universal one, defined as follows. Let N be the normal closure of W in $G * F$. Then ϕ, θ are the canonical maps

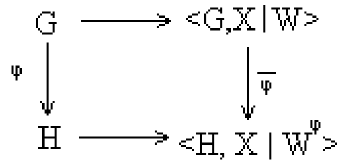
$\phi : G \hookrightarrow G * F \longrightarrow (G * F)/N$, $\theta : X \hookrightarrow F \hookrightarrow G * F \longrightarrow (G * F)/N$ respectively. The universal solution of $(G * F)/N$ is given by the relative presentation $\langle G, X | W \rangle$.

If such an H exists and there is an embedding $\phi : G \hookrightarrow H$ the canonical map $G \to \langle G, X | W \rangle$ is injective.

Remark 2.34 *Let $\phi : G \to H$ be any homomorphism, and let W be a system of equations over G.*

1. The induced system W^ϕ is independent if and only if W is. This is because the exponent-sum matrix of U and U^ϕ coincide for finite subsystem U of W.

2. The square



is a pushout, where the horizontal maps are the canonical ones, and $\bar{\phi}$ *is induced by* ϕ *on* G *and the identity on* X*. In particular if* ϕ *is injective, then* W *has a solution over* G *if and only if* W^{ϕ} *has a solution over* H*.*

In the same way as a group presentation gives rise to a 2-complex, so a relative group presentation $< G, X|W >$, together with a connected $CW-$complex K with fundamental group G, gives rise to a 2-dimensional extension L of K. (One adjoins a 1-cell to K for each element of X to obtain a complex K' with fundamental group $G * F(X)$. One then attaches 2-cells to K', one for each $w \in W$, with attaching map in the class w in $\pi_1(K')$.

The complex L is determined up to homotopy by K, X and W. The pair (L, K) is the *geometric realisation* of the relative presentation $< G, X|W >$, or of the system of equations W.

Conversely, any relative 2-dimensional pair $(L, K)$of connected $CW-$complexes determines a relative presentation

$$\pi_1(L) =< \pi_1(K), X|W >$$

of $\pi_1(L)$ (and so a system W of equations over $\pi_1(K)$) for which (L, K) is a geometric realization.

Proposition 2.35 (Ho1) *Suppose* (L, K) *is a geometric realization of the system* W *of equations over* G*. Then* W *admits a solution over* G *if and only if the inclusion - induced map* $\pi_1(K) \to \pi_1(L)$ *is injective.*

2.4.3 Relative diagrams

The following about relative diagrams are quoted from [Ho2] where the solution of equations of length three is discussed. The same arguments using relative diagrams were used for the solutions for equations of length four by [EH] and the solution of certain sets of equations [E1].

Definition 2.36 (Ho2) *A relative diagram for the equation $r(t) = 1$ over G is a triple (D, υ_0, ϕ) where D is a cellular subdivision of the 2-sphere S^2, with oriented 1-skeleton $D^{(1)}$; υ_0 is a vertex (0-cell) of D; and ϕ is a labelling function which associates to each edge (1-cell) of D the element t, and to each corner of each face (2-cell) of D an element of G; such that the following conditions are satisfied:*

1. *Reading the labels around any face in the clockwise direction from a suitable starting point gives either r or r^{-1} in cyclically reduced form. (Hence an edge is to be read as t or t^{-1} depending on its orientation).*

2. *The product of the labels, read anti-clockwise around any vertex $\upsilon \neq \upsilon_0$ of D (the vertex label of υ), is equal to 1 in G.*

It follows that the label of each corner is one of the coefficients or its inverse. Therefore, the vertex-labels of all the vertices other than υ_0 create relations between the coefficients which hold in G.

The vertex-label of υ_0 is also defined (up to conjugacy) and is an element of the intersection of G with the normal closure of $r(t)$ in $G * \langle t \rangle$. It is therefore a necessary condition for the existence of a solution over G to the equation $r(t) = 1$, that in any relative diagram for $r(t) = 1$, the vertex label of υ_0 is equal to 1 in G.

The following Lemma states that the above condition is also sufficient.

Lemma 2.37 (Ho2) *If the equation $r(t) = 1$ has no solution over a group G, then there exists a relative diagram (D, υ_0, ϕ) for $r(t) = 1$ such that the vertex label of υ_0 is non trivial.*

So if the equation $r(t) = 1$ does not have a solution over G, then by the above lemma there must be a relative diagram (D, υ_0, ϕ) for $r(t) = 1$ such that the vertex label of υ_0 is not trivial in the group G. This last remark is very important as it provides the beginning of the investigation method for the present thesis.

2.4.4 The star graph and the weight test

The results which were obtained up to now on equations or systems of equations over group used the definitions of star graphs and the weight test of [BP] and [G1]. Similar arguments with the use of weight test are used in the present study for the solution of $r_3(t) = 1$. The definitions of star graphs and the weight test are presented.

Definition 2.38 (BP) *Let $P =< H, X; r >$ to be an oriented relative presentation. The elements of r are words in $H \cup X \cup X^{-1}$ which are assumed to be cyclically reduced. r^* is the set of all cyclic permutations of the elements of r and their inverses. The star complex P^{st} of P is a graph whose edges are labelled by elements of the coefficient group H. The vertex and edge set are $X \bigcup X^{-1}$, r^* respectively. For $R \in r^*$ write $R = Sh$ where $h \in H$ and S begins and ends with x-symbols. The initial and terminal functions are given by: $i(R)$ is the first symbol of S, $\tau(R)$ is the inverse of the last symbol of S. The inversion function on edges is given by operator $-$. (Operator $-$ on r^* is defined so that for every $R \in r$, $R = Sh$ where $h \in H$ and S begins and ends with an x-symbols. Set $\bar{R} = S^{-1}h^{-1}$). Since P is oriented $\bar{R} \neq R$ for all $R \in r^*$. The labelling function is defined by $\lambda(R) = h^{-1}$ and it is extended to paths in the obvious way. Note that $\lambda(\bar{R}) = \lambda(R)^{-1}$.*

Definition 2.39 *A weight function θ on P^{st} is a real valued function on the set of edges of P^{st} which satisfies $\theta(\bar{R}) = \theta(R)$ for all $R \in r^*$. The weight of a path is the sum of the weights of the edges appearing.*

For systems of equations over groups the star graph Γ of the system Σ consists of vertices x and x^{-1} for each unknown X of Σ, and one edge (labelled g) from $X^{-\alpha}$ to Y^{β} whenever some cyclic permutation of an equation in Σ begins with $X^{\alpha}gY^{\beta}$. (This is because by the definitions of the initial and terminal functions $i(R)$ and $\tau(R)$).

The weight function ω on the system Σ is the real valued function defined on the edges of Γ, such that, for each equation of Σ, of length n, the sum of weights of the n edges of Γ corresponding to the coefficients appearing in that equation is at most $180(n-2)$.

For a single equation $r(t) = 1$ the star graph Γ is a graph with two vertices t and t^{-1} and edges joining these vertices for each occurrence of t in $r(t)$. It has an edge x connecting $t^{-\varepsilon_i}$ and t^{ε_j} with ε_i, $\varepsilon_j \in \{1, -1\}$ for every cyclic permutation of $r(t)$ which begins with $t^{\varepsilon_i}\ x\ t^{\varepsilon_j}$. The star graphs of equations of length five are presented in Chapter 4 in Figure 4-2.

Chapter 3

Theoretical Background

In this chapter we refer to the most important results that are have been up to now obtained on the problem of equations over groups and the existence of a solution. We start with the early results of Neumann [N] and the embedding theorem of Higman, Neumann & Neumann [HNN] which was the actual introduction of the problem of equations and system of equations over a group G to Combinatorial Group Theory. We continue by quoting further results, into two sections, according to the method adopted: whether the problem was restricted to a type of groups and whether the problem was restricted to a kind of equation over an arbitrary group.

3.1 Early results: Adjuction of elements and embedding theorems

In [N] the process of embedding a given group in another group was studied and the problem of equations over groups was actually introduced into Combinatorial Group Theory by this study. If the equation $\omega(x) = 1$ or the system of equations $\omega_1(x) = 1$, $w_2(x) = 1$, ... does not have a solution in G, it might be possible to adjoin a solution to G that is, find a group H overgroup of G and element(s) h

so that $w(h)$ =1 in H.

It was shown that a system of equations over group $G =< Y|R >$ has a solution (in other words a solution can be adjoined to G) if and only if $L \cap F(Y) = R$ where $H = K/L$ is the solution group with K the group generated by X and h_1, h_2,.... which take the place of the indeterminates $x_1, x_2, ...$ when these have become solutions of the given equations and L is generated by R and $w(h_{1,}h_2, ...)$.

As a corollary to this main result the following Theorem was stated about the universal solution of the equation or the system of equations:

Theorem 3.1 *If one solution to the system of equations* $\omega_1(x, g) = 1$, $w_2(x, g) = 1, ...$ *(in interminates* $x_1, x_2...$*) exists, then all the groups obtainable from* G *by adjoining a solution to the system of equations (i.e. all the solution groups of the equations over* G*) are homomorphic images of the so called* ***universal solution*** *of the system over* G*, namely the free product* $G* < x_1, x_2... >$ *factored by the normal closure of* $\omega_1(x, g) = 1$, $w_2(x, g) = 1, ...$.

Another important result proved in the same paper concerns the solution of the equation $x^m = g$.

Theorem 3.2 (N) *Solutions of the equation* $x^m = g$ *can always be adjoined to the arbitrary group* G*, whatever the element* $g \in G$ *and the integer* $m \neq 0$ *may be.*

This means that *for any positive integer* m *and any element* g *of a group* G*, the equation* $x^m = g$ *always has a solution in some group* H *containing* G*.*

In [HNN] the first problem discussed concerned whether for any group G, with subgroups A and B, there exists a group H containing G, within which A and B are conjugates. Obviously, if such an H exists then an isomorphism can be defined between subgroups A and B. The main theorem proved in this paper asserts that the above condition is not only necessary but also sufficient.

Theorem 3.3 (HNN) *Let μ be an isomorphism of a subgroup A of G onto a second B subgroup of G. Then there exists a group H containing G, and an element t of H, such that the transform by t of any element of a is its image under μ, i.e. $t^{-1}at = \mu(a)$.*

This means that a solution to the equation $t^{-1}at = b$ can be found if and only if there is an HNN-extension of G relative to the subgroups A and B.

Interpreted in terms of systems of equations over groups the following results is proved:

Theorem 3.4 (HNN) *A solution t of the system of equations $t^{-1}a_\sigma t b_\sigma^{-1} = 1$ (where σ ranges over some index set Σ) can be adjoined to the group G, which contains a_σ, b_σ, if and only if the mapping $\mu(a_\sigma) = b_\sigma$ $(\sigma \in \Sigma)$ generates an isomorphism of the group A generated by all a_σ onto the group G generated by all b_σ.*

Considering the index σ taking just one value the following result follows:

Corollary 3.5 (HNN) Two elements a and b of a group G are conjugate in a suitable extension of G if and only if they have the same order.

3.2 The solution of equations in certain types of groups

In this section we present the results obtained when the problem of equations over groups is restricted to a type of group (e.g. residually finite, locally indicable etc.).

In [GR] systems of equations over a group G are considered when G is a compact connected Lie group. As a corollary to the main theorem presented there, it is also proved that equations over a finite group always have a solution.

Theorem 3.6 (GR) *Let H be a compact Lie group of rank m, $W_1, ..., W_r$ be a system of equations over H, and d_{ij} be the degree of W_i in x_j (i.e. the sum of the exponents of x_j occurring in the word W_i). Set $d = \det(d_{ij})$. Then,*

1. *the degree of the mapping $T : H^{(r)} \to H^{(r)}$ defined by $T(x_1, ..., x_r) = T(x) = (W_1(x), ..., W_r(x))$ is d^m, and*

2. *if $d \neq 0$, then there exists $x_1, ..., x_r$ in H such that $W_i(x) = 1$, $i = 1, ..., r$.*

Theorem 3.7 (GR) *(The Extension Theorem) Let G be a finite group, $W_1, ..., W_r$ be a system of equations over H and d_{ij} be the degree of W_i in x_j . Set $d =$ $\det(d_{ij})$. If $d \neq 0$, then there exists a finite extension of G containing elements $x_1, ..., x_r$ such that $W(x_1, ..., x_r) = 1$, $i = 1, ..., r$.*

In simpler language, a system of equations with $d \neq 0$ has a solution over a compact Lie group and more specifically over a finite group.

In [R] the above results are generalized not only for finite groups but locally residually finite groups. A group is locally residually finite group if each one of its finitely generated subgroups is residually finite, i.e. every non identity element of a finitely generated subgroups lives in a finite image of the subgroup. A compact Lie group is locally residually finite since every finitely generated subgroup of complex matrices is residually finite.

Continuing with the notation of the Extension Theorem of [GR] and setting $d \neq 0$ and G being a locally residually finite group, the main theorem of this paper states that if F is the free group generated by $x_1, x_2, ..., x_n$ then the natural homomorphism π of $G * H$ onto $H = G * F/W$ is injective.

Therefore, any systems of equations with $d \neq 0$ has a solution over a residually finite group.

According to Theorem1 of [S] polycyclic-by-finite groups are residually finite so in particular the following statement holds:

Any systems of equations with $d \neq 0$ has a solution over a polycyclic-by-finite group.

In [Ho1] a combination of combinatorial and topological methods was used to investigate the existence of solutions over a group G to an independent system of equations over G and consequently to a single equation. The author introduced the geometrical realisation (L, K) of connected CW-complexes and the system of equations that was presented in the previous chapter. The main result of this study is the following theorem:

Theorem 3.8 (Ho1) Any independent system of equations over a locally indicable group G has a solution over G.

3.3 The solutions of certain types of equations

In this section we present the results obtained on equations of a certain type (e.g. of certain length).

One of the first general results proved in [L] on the type of equation is that if the length of the equation equals the exponent sum of t in $r(t)$ then the equation always has a solution.

In [Ho2] the main theorem proved is about equations of length three as follows:

Theorem 3.9 (Ho2) *Let G be any group, and let $r \in G* < t >$ be an element of t-length 3. Then the equation $r(t) = 1$ has a solution over G.*

The study of equations of length three over any group G is reduced to the study of $r(t) = atbtct^{-1}$ over the subgroup G_0 of G generated by coefficients a, b and c. This is because if the equation has a solution over group G_0, there exists an H overgroup of G_0 and an $h \in H$ such that $ahbhch^{-1} = 1$ in H. But then the free product $G *_{G_0} H$ can be defined and $r(t) = 1$ has a solution over this as well. Also by applying the automorphism $g \mapsto g$ $(g \in G)$, $t \mapsto tb^{-1}$ of $G* < t >$, the equation can be transformed to one of the form $a't^2c't^{-1} = 1$. It can be therefore assumed without loss that $b = 1$.

For the proof of the above theorem it was assumed by way of contradiction an equations $r(t) = atbtct^{-1}$ did not have a solution. Then according to [Ho2] (Lemma 2.37 of Chapter 2), there exists a relative diagram (D, v_0, ϕ) for $r(t) = 1$ such that the vertex-label of v_0 is non-trivial. D is chosen with the smallest possible number of faces. The vertices of a face are labeled with cyclically reduced words in the symbols a, b, c. Reading the labels anticlockwise gives relators in G which force the group to be a homomorphic image of a residually finite group.

In [E1] the same method was used to prove the following theorem extending the types of equations for which it is known that there exists a solution to the following:

Theorem 3.10 (E1) *Let G be a group and a, $d \in G$. Then the equation $at^ndt^{-1} (n \neq 1)$ has a solution over G.*

Also similar methods were used in [E2] to extend this result to any powers of t (other than exponent-sum zero) i.e. to include the case when the equation involves only two elements of G both of order greater than two or three.

Theorem 3.11 (E2) *Let G be a group and $a, d \in G$ such that $\{|a|, |d|\} \neq \{2, 3\}$. The equation $at^ndt^{-m} = 1$ $(n, m \in Z, n \neq m)$ has a solution over G.*

The same methods were used to investigate the equations of non zero exponent-sum and length four. (The equation $at^2bt^{-2} = 1$ has a solution if and only if a and b have the same order.) By conjugation or inversion the problem is reduced to the equation $atbtctdt^{-1} = 1$. The theorem proved in [EH] is the following:

Theorem 3.12 (EH) *Let G be a group, and a,b,c,d $\in G$. Then the equation $atbtctdt^{-1} = 1$ has a solution over G.*

Also the following were proved in [Ho3] in the investigation of solution to systems of equations over a group:

Theorem 3.13 (Ho3) *Let G be a group, and Σ a non singular system of equations of length at most 3 over G. Then Σ can be solved in an overgroup of G.*

Applying these results to equations of length five the following results were obtained:

Theorem 3.14 (Ho3) *Let G be a group, and let x be an unknown. If $b = d$ in G then the equation $axbxcxdxex^{-1} = 1$ has a solution.*

Theorem 3.15 (Ho3) *Let G be a group, and let x be an unknown. If either $be = 1$ or $ce = 1$ in G then the equation $axbxcxdx^{-1}ex^{-1} = 1$ has a solution.*

Theorem 3.16 (Ho3) *Let G be a group, and let x be an unknown. If either $c = e^{\pm 1}$ or $a = d^{\pm 1}$ in G then the equation $axbxcx^{-1}dxex^{-1} = 1$ has a solution.*

The results presented above prove the KL-conjecture for the cases of equations of length at most four and some special cases of length five. In [EJ(1)] the equations of length five and higher are considered, but still with some restrictions imposed on the (associated) star graph of the equation. These restrictions concern the existence of admissible paths of length two on the star graph, i.e. cyclically reduced paths with label equal to 1 in G.

The main theorem proved is the following:

Theorem 3.17 (EJ) *Let G be a group. If $r(t) = 1$ is an equation over G whose associated star graph contains no admissible paths of length less than three then $r(t) = 1$ has a solution over G*

Observe that the above result does not impose the restriction that the exponent sum of t in the equation must be non-zero.

An equation of length five can be put into one of the following forms by cyclic permutation and inversion:

$$r_0(t) = atbtctdtet = 1,$$
$$r_1(t) = atbtctdtet^{-1} = 1,$$
$$r_2(t) = atbtctdt^{-1}et^{-1} = 1,$$
$$r_3(t) = atbtct^{-1}dtet^{-1} = 1,$$

where $a, b, c, d, e \in G$. The first equation $r_0(t) = 1$ has been settled by [L]. It can be seen that the non existence of admissible paths of length two means that no element of G has order two. According to the following corollary if the elements represented by loops on the star graph do not have order two (i.e. they are not square loops) then equations of length five have a solution.

Theorem 3.18 (EJ) *(i) If $|a| > 2$ and $|e| > 2$ in G then $r_1(t) = 1$ has a solution over G; (ii) if $|a| > 2$ and $|d| > 2$ in G then $r_2(t) = 1$ has a solution over G; (iii) if $|a| > 2$, $|c| > 2$, $|d| > 2$ and $|e| > 2$ in G then $r_3(t) = 1$ has a solution over G.*

There were also several other more recent results concerning the solution of equations of length four when the exponent sum of t in the equation is zero (i.e. an equation is singular [E3], [EJ(2)]). There have also been several results on cases of equations over groups when group G is torsion free [K].

[L] settles the KL-conjecture for equations of length two ($atbt = 1$), and Theorem 3.9 settles it for equations of length three ($atbtct^{-1} = 1$). The conjecture for equations of length four ($r(t) = atbtctdt^{-1}$) is settled by Theorem 3.12. Any equation of length five can be put in one of the forms $r_i(t) = 1$, $i = 0, 1, 2, 3$ above. Equation $r_0(t) = 1$ has been settled by [L], and Theorems 3.13, 3.17, 3.18 prove the conjecture for $r_i(t) = 1$, $i = 1, 2, 3$ with certain restrictions imposed on them. These results leave equations of length five still unsolved for (i) $r_1(t) = 1$ when $b \neq d$ and at least one of a and e has order two, (ii) $r_2(t) = 1$ when $be \neq 1$ and $ce \neq 1$ and at least one of a and d has order two and (iii) $r_3(t) = 1$ when $c \neq e^{\pm 1}$ or $a \neq d^{\pm 1}$ and at least one of a, c, d and e has order two.

Chapter 4

Method

This study considers the cases of equations of length five for which it is not known whether a solution exists. By way of contradiction it is assumed that an equation of this type does not have a solution. The non-existence of solution implies by [Ho2] (Lemma2.37 of Chapter 2) the existence of a relative diagram D, or (D, φ, υ_0) representing a counter example. Each vertex label, except the label of υ_0 is an admissible path on the star graph of the equation (i.e. reading the labels around a vertex $\upsilon \neq \upsilon_0$ give us a word that equals one in the group). The label of υ_0 is a non trivial element in G. We first describe the general methodology of proof and in Section 4.2. we present an overview of the proof with reference to the next three Chapters where the complete proof can be found.

4.1 General Methodology

The methodology adopted uses the arguments of [Ho2], [EH] and [EJ(1)]. The relative diagram (D, φ, υ_0) representing the counter example is a spherical diagram, or tessellation of the 2-sphere, D, with labelling function φ on the corners of D, and distinguished vertex υ_0. The vertices of D other than υ_0 are called interior vertices. Since we are examining equations of length five the faces Δ of D

are 5-sided, with edge-orientations and corner labels so that reading clockwise on a face Δ will give a cyclic permutation of the equation or its inverse. Therefore, according to Definition 2.36(1) a face Δ_i appearing in a relative diagram D of the equation $r_i(t) = 1 \;\; i = 1, 2, 3$ will be as in Figure 4-1.

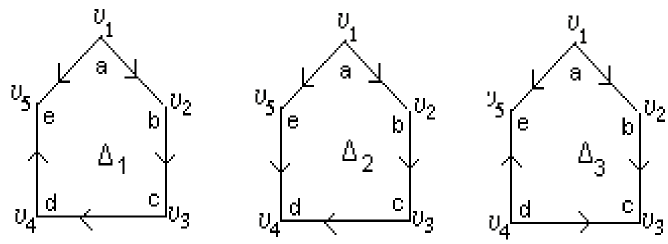


Figure 4-1:

A face is called interior if all its vertices are interior. A face of which at least one of its vertices is v_0 is called a boundary face. The label of a vertex v of D is the list of corner labels at v read in a counterclockwise direction from any starting point to make a word on $\{a, b, c, d, e\}$. These labels on a vertex v, with the unique exception of that of v_0 are relations in the group G. We denote the vertices of Δ by v_i, $i = 1, .., 5$ and the label of v_i by l_i. The vertex-labels give rise to closed paths in the labelled star graph Γ of this 2-complex. According to Definition 2.38 of Chapter 2 the stargraphs for each of equation of length five $r_i(t) = 1$ for $i = 1, 2, 3$ are shown in Figure 4-2.

Using the restrictions imposed by the star graph on the labels of interior vertices, in the following chapters we calculate the possible labelling on interior regions Δ of positive curvature.

The same curvature arguments as those of [EJ(1)] are used . The corner of every face Δ of a given relative diagram is given an angle (in principle any real number). The curvature of a vertex is defined to be 2π less the sum of the angles at v. The curvature of a k-gonal face is the sum of all the angles of the corners of the face, less $(k-2)\pi$. The curvature of a subdiagram K of D is the sum of

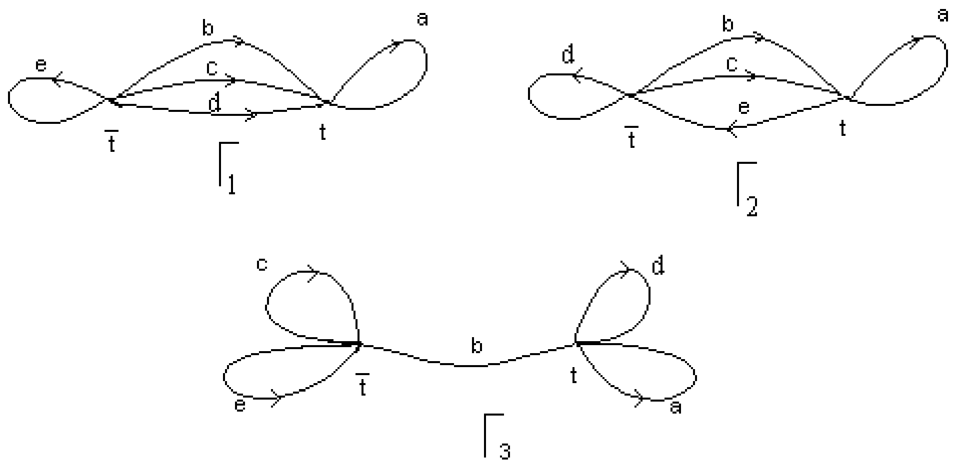


Figure 4-2:

the curvatures of the vertices and faces of that subdiagram and denoted by $c(K)$. The total curvature of diagram D will be the sum of the total curvature of all of its faces plus the curvature of all of its vertices.

Lemma 4.1 *The total curvature of D is precisely 4π.*

Proof. Each corner at a vertex of index l is given an angle of $\frac{2\pi}{l}$. This means that the vertices have zero curvature so the total curvature of D is the sum of the total curvature of the faces. Let V be the number of vertices, E the number edges and F the number of faces in diagram D. Let Δ be a k-gonal face of D (i.e. the degree of the k-gonal face is $d(\Delta) = k$) and the degrees of vertices of Δ are d_i $(1 \leq i \leq k)$. So the total curvature of D is $c(D) = 2\pi V - \sum\limits_{\Delta \in D} (d(\Delta) - 2)\pi$. So $c(D) = 2\pi V - (\sum\limits_{\Delta \in D} d(\Delta) - 2F)\pi = 2\pi(V - \frac{1}{2}\sum\limits_{\Delta \in D} d(\Delta) + F) = 2\pi(V - E + F)$. Now using the Euler's formula it can be concluded that $c(D)$ is precisely 4π. ■

Since every vertex has zero curvature the curvature of each k-gon of D is

$$c(\Delta) = c(d_1, ..., d_k) = c(2-k)\pi + 2\pi \sum_{i=1}^{k} \frac{1}{d_i} \ (*)$$

where d_i, $i = 1, ..k$ is the degree of each vertex. In the case of 5-gons the curvature of each face is $c(\Delta) = c(d_1, d_2, d_3, d_4, d_5) = -3\pi + 2\pi \sum_{i=1}^{5} \frac{1}{d_i}$. In the following three chapters we list all the possible interior faces for which $c(\Delta) > 0$.

Any equation of length five is by cyclic permutation and inversion one of $r_i(t) = 1$ for $i = 0, 1, 2, 3$ where $a, b, c, d, e \in G$. Since the first equation $r_0(t) = 1$ has been settled, the other three equations were discussed by [EJ(1)].

To isolate the cases for which a solution is not known to exist the following assumptions are imposed on $r_i(t) = 1$, for $i = 1, 2, 3$:

1. Group G is neither locally indicable nor residually finite, more specifically it is not a homomorphic image of a polycyclic by finite group. (If G is one of these groups then the equation has a solution by [GR], [Ho1], [R].)

2. The star graph Γ_i for $i = 1, 2, 3$ contains at least one admissible path of length less than three that is a square loop (that is a cyclically reduced path of length two for which the label equals 1 in G and passes only from the same vertex). Since the labels on the faces of Δ are closed paths on Γ this actually means that some interior faces of D are allowed to have vertices of degree two. (If the star graph of the equation does not have any admissible paths of length two, or any square loops then the equation has a solution by [EJ(1)].)

3. In case $r_1(t) = 1$, $b \neq d$. In case $r_2(t) = 1$, neither $be = 1$ nor $ce = 1$. In case $r_3(t) = 1$ neither $c = e^{\pm 1}$ nor $a = d^{\pm 1}$. (If any of these conditions is true the equation would have a solution by [Ho3].)

4. Group G is generated by $\{a, b, c, d, e\}$. This is because if the equation has a solution over the subgroup G_0 of G which is generated by $\{a, b, c, d, e\}$ (in an overgroup H of G_0) then it has a solution over G in $G *_{G_0} H$.

The following Lemma, which makes use of the results of [C], [E1] and [Ho2], shows that because of assumption 1, group G cannot be any of the following groups or their quotients.

Lemma 4.2 *The following groups and all their quotients are residually finite:*

(a) $< a, c | a^2 = [a, c^2] = 1 >$

(b) $< a, c | a^2 = acac^p = 1 >$ *where* p *is any positive integer*

(c) $< a, c | a^2 = c^4 = w(a, c) = 1 >$ *where* $w(a, c)$ *is any of* $(ac^2)^2$, $ac^{q_1}ac^{q_2}ac^{q_3}$ *(with* $|q_1| + |q_2| + |q_3| \leq 5$*),* $ac^{q_4}ac^{q_5}ac^{q_6}ac^{q_7}$ *(*$|q_4| + |q_5| + |q_6| + |q_7| \leq 5$ *) and* q_i *are positive integers.*

(d) $< a, c | a^2 = c^3 = 1 = w(a, c) = 1 >$ *where* $w(a, c)$ *is* $ac^{\delta_1}...ac^{\delta_l}$ *with* $\delta_j = \pm 1$ *for* $1 \leq j \leq l$ *and* $1 \leq l \leq 6$.

In the following chapters if one set of the above set of relators holds in the group (i.e. the group is one of (a), (b), (c) and (d)) then we say that the group has property X.

4.2 An overview of the proof

The total curvature of diagram D is the sum of the curvature of its faces. If the assumption that an equation without a solution exists, the curvature of 4π must be found on the corresponding relative diagram D. In Chapters 5, Chapter 6 and Chapter 7 we list all the interior regions of positive curvature and prove that wherever an interior region Δ of positive curvature exists there is one or more neighbouring regions of negative curvature which when interior compensate for the positive curvature of Δ. In the following three sections we give an overview of this proof with reference to the three Chapters including the full proof of this statement. In the final Chapter we calculate the curvature around the distinguished vertex v_0.The assumptions imposed on each equation are stated at the beginning of each section. At the end of each section the conclusion for each equation is presented.

4.2.1 $r_1(t) = 1$

In this case we will see that the following assumptions may be made; see Chapter 5 for details.

(N1) Group G is generated by $\{a, b, c, d, e\}$

(N2) Group G is neither locally indicable nor residually finite, or a homomorphic image of a polycyclic by finite group. More specifically it does not have property X.

(N3) The star graph Γ_1 *contains at least one admissible path of length less than three and either* $|a| = 2$ *or/and* $|e| = 2$.

(N4) $b \neq d$

(N5) $b = 1$

(N6) $a \neq 1,\ e \neq 1,\ d \neq 1$

(N7) $r_1(t) = 1$ *can be solved modulo the transformation* T_1

$$a \to e^{-1}, b \to d^{-1}, c \to c^{-1}, d \to b^{-1}, e \to a^{-1} (\mathrm{T}_1)$$

(N8) $a^2 = 1$

(N9) Diagram D contains minimal number of faces Δ

(N10) Diagram D contains maximal number of vertices with label a^2

(N11) Diagram D contains maximal number of vertices of degree two.

Lemma 4.3 *If* Δ *has no vertices degree two then* $c(\Delta) \leq 0$ *except for the cases appearing in Figures 5-3 , 5-4, 5-5, 5-8 when* $a = c = d$ *and* $2 = |a| < |e|$.

The positive curvature of Δ in Lemma 4.3 is distributed to a region adjacent to it across a $b - c$ edge.

[Figures 5-3, 5-4, 5-5, 5-8]

Lemma 4.4 *If* Δ *has vertices of degree two but not* a^2 *or* e^2 *then* $c(\Delta) \leq 0$ *except for the cases appearing in Figures 5-13, 5-14, 5-15, 5-16, 5-17, 5-18, 5-19, 5-20, 5-21 when* $a = c = d$ *and* $2 = |a| < |e|$.

The positive curvature of Δ in Lemma 4.3 is distributed to a region adjacent to it across a $b - c$ edge.

[Figures 5-13, 5-14, 5-15, 5-16, 5-17, 5-18, 5-19, 5-20, 5-21]

Lemma 4.5 *If Δ has vertices of degree two and one of them is a^2 and $a = c = d$ and $2 = |a| < |e|$ then $c(\Delta) \leq 0$ except for the case appearing in Figure 5-22*

[Figure 5-22]

The positive curvature of Δ in Lemma 4.5 is distributed to neighbouring regions across a $b - c$ edges.

From now on it is assumed that the relations $a = c = d$ and $a^2 = e^k = 1$, $k \geq 3$ do not all hold in G. If there is an interior region of positive curvature then the label a^2 must appear somewhere in D. Wherever that happens, we delete the vertex and create regions Δ of degree six as in Figure 5-23. To achieve this the new element (eb) is added to the star graph Γ_1.

[Figure 5-23]

First assume that $e = |a| < |e|$. Suppose Δ is one such region of degree six with $c(\Delta) \geq 0$. At least one vertex must have degree two and so one of the following ten mutually exclusive condition must be true:

1. $c = e$

$c(\Delta) \leq 0$ unless it is the region appearing in Figure 5-25. Its positive curvature is distributed to a region of positive curvature adjacent to it across a $d - e$ edge.

[Figure 5-25]

2. $d = c$

$c(\Delta) \leq 0$ unless it is one of the regions appearing in Figures 5-29, 5-31, 5-33. Its positive curvature is distributed to a region of positive curvature adjacent to it across a $d - e$ edge.

[Figures 5-29, 5-31, 5-33]

3. $d = e$ and $c = 1$

$c(\Delta) \leq 0$ unless it is one of the regions appearing in Figures 5-35, 5-38, 5-39, 5-40. Its positive curvature is distributed to a region of positive curvature adjacent to it across a $d - e$ edge.

[Figures 5-35, 5-38, 5-39, 5-40]

4. $d = e$ and $c \neq 1$

$c(\Delta) \leq 0$ unless it is one of the regions appearing in Figures 5-42, 5-43, 5-45, 5-47. Its positive curvature is distributed to a region of positive curvature adjacent to it across a $d - e$ edge.

[Figures 5-42, 5-43, 5-45, 5-47]

5. $d \neq e$ and $c = 1$

$c(\Delta) \leq 0$

Now assume that $2 = |a| = |e|$. Labels a^2 and e^2 must appear somewhere in D. Wherever that happens, we delete them and create regions F_1 and F_2 of degree six as in Figure 5-23 and Figure 5-49. To achieve this two new elements (eb) and (da) are added to the star graph.

[Figures 5-23, 5-49]

In this last case no interior region has positive curvature.

Conclusion 4.6 *If Δ is an interior region with $c(\Delta) > 0$ in a diagram D representing a counter example for $r_1(t) = 1$, then there is always a region with negative curvature which, if interior, uniquely compensates for the positive curvature of Δ. The case when either Δ or the region of negative curvature is a boundary region is considered in the final chapter.*

4.2.2 $r_2(t) = 1$

In this case we will see that the following assumptions may be made; see Chapter 6 for details.

(A1) Group G is generated by $\{a, b, c, d, e\}$

(A2) Group G is neither locally indicable nor residually finite, or a homomorphic image of a polycyclic by finite group. More specifically it does not have property X.

(A3) The star graph Γ_2 contains at least one admissible path of length less than three and either $a^2 = 1$ or/and $d^2 = 1$.

(A4) $be \neq 1$ and $ce \neq 1$

(A5) $b = 1$

(A6) $a \neq 1$, $d \neq 1$, $e \neq 1$

(A7) $r_2(t) = 1$ can be solved modulo the transformation T_2

$$a \to d^{-1}, b \to c^{-1}, c \to b^{-1}, d \to a^{-1}, e \to e^{-1} \ (\mathrm{T}_2)$$

(A8) $a^2 = 1$

(A9) Diagram D contains minimal number of faces Δ

(A10) The total number of sources N_1 is maximal in D.

(A11) The number of vertices of degree two N_2 is maximal in D.

Lemma 4.7 *If Δ has no vertices degree two then $c(\Delta) \leq 0$ except for case 6.1. in Figure 6-2 when $a = c$, $d = e$ and $a^2 = d^3 = 1$.*

The positive curvature of Δ is distributed to region Δ_1 adjacent to it across a $d - e$ edge.

[Figure 6-2]

Lemma 4.8 *If there exists a region Δ with $c(\Delta) > 0$ as in Figure 6-2 then $a = c$, $d = e$ and $a^2 = d^3 = 1$ holds in G and all the other regions of positive curvature in D are of the same type (i.e. no vertices of degree two).*

Lemma 4.9 *If in a region Δ a^2 or d^2 do not appear as labels then $c(\Delta) \leq 0$ except for the case 6.2. of Figure 6-4 when $c = 1$, $d = e$, $a^2 = d^3 = 1$.*

Lemma 4.10 *If $c = 1$, $d = e$, $a^2 = d^3 = 1$ a region Δ with $c(\Delta) > 0$ will be one of 6.2.-6.4. of Figure 6-4 .*

The positive curvature of Δ in Lemmas 4.9 and 4.10 is distributed to regions Δ_1 and Δ_2 always across a $d - e$ edge.

[Figure 6-4]

From now on it is assumed that $a = c$, $d = e$, $a^2 = d^3 = 1$ and that $c = 1$, $d = e$, $a^2 = d^3 = 1$ are not all true. Therefore if there are to be any interior regions of positive curvature then the label a^2 must appear somewhere in D. Wherever that happens, we delete the vertex and create regions Δ of degree six as in Figure 6-6. To achieve this the new edge (eb) is added to the star graph.

[Figure 6-6]

Therefore, possible regions of positive curvature will either be of degree six or one of F_1, F_2, E and D_2 of Figure 6-23.

Suppose Δ is one such region of degree six with $c(\Delta) > 0$. At least one vertex must have degree two and so one of the following ten mutually exclusive condition must be true:

1. $c = 1$ and $a = e$ (so $d^2 \neq 1$ and $e^2 = 1$)

Δ can only be as in Figure 6-7.

[Figure 6-7]

Positive curvature is added to the neighbouring region(s) of negative curvature across a $b^{-1} - a^{-1}$ edge. $c(\Delta) \leq 0$

2. $c = 1$ and $a \neq e$, $e^2 = 1$, $d^2 = 1$

$c(\Delta) \leq 0$

3. $c = 1$ and $a \neq e$, $e^2 = 1$, $d^2 \neq 1$

$c(\Delta) \leq 0$

4. $c = 1$ and $a \neq e$, $e^2 \neq 1$, $d^2 = 1$

$c(\Delta) \leq 0$

5. $c = 1$ and $a \neq e$, $e^2 \neq 1$, $d^2 \neq 1$

$c(\Delta) \leq 0$

6. $c \neq 1$ and $a = e$ (so $e^2 = 1$), $d^2 = 1$

$c(\Delta) \leq 0$

7. $c \neq 1$ and $a = e$ (so $e^2 = 1$), $d^2 \neq 1$

$c(\Delta) \leq 0$

8. $c \neq 1$ and $a \neq e$, $e^2 = 1$, $d^2 = 1$

Δ will be one of the regions 6.6. -6.7. of Figures 6-12 and Figure 6-14.

[Figure 6-12, 6-14]

Positive curvature is distributed to negative across a $d - e$ edge.

9. $c \neq 1$ and $a \neq e$, $e^2 \neq 1$, $d^2 = 1$

(a) If c and d (but not a) are powers of e then Δ is one of regions 6.8., 6.9., 6.11. of Figures 6-16, 6-17, 6-19. and its positive curvature is distributed across a $d - e$ edge.

[Figures 6-16, 6-17, 6-19]

(b) If a and c (but not d) are powers of e then Δ is region 6.10. in Figure 6-18 and its positive curvature is distributed across an $a - e$ edge.

[Figure 6-18]

10. $c \neq 1$ and $a \neq e$, $e^2 = 1$, $d^2 \neq 1$

(a) If $c = d^{-1}$ and $a = d^2$ then Δ will be one of regions 6.12. of Figure 6-21 and positive curvature is distributed across a $b^{-1} - a^{-1}$ edge.

[Figure 6-21]

(b) If $c = d^{-1}e$ and $a = c^{-1}d^2c$ then Δ is region 6.13.(a) of Figure 6-22 (a) and positive curvature is distributed across an $e - a$ edge.

[Figure 6-22(a)]

(c) If $c = d$ and $e = d^2$ then Δ is the region of 6.13. (b) of Figure 6-22 and positive curvature is distributed across a $d - e$ edge.

[Figure 6-22(b)]

Finally, we turn to the interior of D of positive curvature of degree five (F_1, F_2, E and D_2 of Figure 6-23).

(a) If $a = e$ then a region of positive curvature will be one of those shown in Figures 6-24, 6-25, 6-26, 6-27, 6-28, 6-29 and the positive curvature is distributed across a $d - e$ edge.

[Figures 6-24, 6-25, 6-26, 6-27, 6-28, 6-29]

(b) If $c = 1$ then a region of positive curvature will be one of those in Figures 6-30, 6-31 and 6-32. Their positive curvature is always added uniquely to a region of negative curvature across a $d^{-1} - c^{-1}$ edge.

[Figures 6-30, 6-31, 6-32]

(c) If $c \neq 1$ and $a \neq e$ then a region Δ might occur in the following mutually exclusive cases:

1. $c = a$

1.1. $c = a = e^2$

Δ is the region in Figure 6-33. Positive curvature is added across an $e - a$ edge.

[Figure 6-33]

1.2. $c = a$ and one or more of the relations $d = e^{-1}ae$, $d = (ae)^2$, $aeae^{\pm k} = 1$ holds with $k = 1, 2$, $a = e^3$, $e^2 = 1$.

Δ is the region in Figure 6-34. Positive curvature is distributed across a $c^{-1} - b^{-1}$.

[Figure 6-34]

2. and 3. $d = a$

Δ is the region in Figure 6-35. Positive curvature is distributed across an $e - a$ edge.

[Figure 6-35]

4. $c = de^{-1}$

Δ is the region in Figure 6-36. Positive curvature is added across a $d - e$ edge.

[Figure 6-36]

Conclusion 4.11 *If Δ is an interior region with $c(\Delta) > 0$ on a diagram D representing a counter example for $r_2(t) = 1$, then there is always a region with negative curvature which, if interior, compensates for the positive curvature of Δ. The case when Δ or the region of negative curvature is a boundary region is considered in the final chapter.*

4.2.3 $r_3(t) = 1$

In this case we will see that the following assumptions may be made; see Chapter 7 for details.

(H1) Group G is generated by $\{a, b, c, d, e\}$

(H2) Group G is neither locally indicable nor residually finite, or a homomorphic image of a polycyclic by finite group. More specifically it does not have property X.

(H3) The star graph Γ_3 contains at least one admissible path of length less than three and at least one of a, c, d, e has order two in G.

(H4) $c \neq e^{\pm 1}$ and $a \neq d$

(H5) $b = 1$

(H6) $a \neq 1$, $c \neq 1$, $d \neq 1$, $e \neq 1$

(H7) $r_3(t) = 1$ can be solved modulo the transformation T_3

$$a \to c^{-1}, b \to b^{-1}, c \to a^{-1}, d \to e^{-1}, e \to d^{-1} \qquad (T_3)$$

(H8) Diagram D contains minimal number of faces Δ

(H9) The number of vertices in D with label a^2 are of maximal number

(H10) The number of vertices of degree two is maximal

Lemma 4.12 *If Δ is a region of positive curvature then it has at least one vertex of degree two.*

Therefore, the regions with at least one vertex of degree two are further considered. If $a^2 = 1$ then the label a^2 can be deleted as with the previous equations, by adding the edge (eb) on the stargraph. If $a^2 = e^2 = 1$ then all labels a^2 and e^2 can be delete by adding two new edges (eb) and (da) to the star graph. If $a^2 \neq 1$ and $e^2 = d^2 = 1$ then labels e^2 and d^2 can be deleted by adding (eb) and (ce) to the star graph. (See Figures 7-3 and 7-26).

[Figure, 7-3, 7-26]

The cases are examined according to the number of elements of order two:

1. Four elements of order two $(a^2 = c^2 = d^2 = e^2 = 1)$

$c(\Delta) \leq 0$ unless it falls under one of the following cases:

1.1. $d = e$

Δ will be one of the regions of Figures 7-5 and Figure 7-7.

[Figure 7-5, 7-7]

Positive curvature is distributed across a $d - e$ edge or an $a - b$ edge. If a region of negative curvature is an F_1 region (where a^2 has been deleted) then it can only receive positive curvature across its $d - e$ edges and if it is an F_2 then it can only receive positive curvature across its $a - b$ edges.

1.2. $e = a$

Δ will be one of the regions of Figure 7-9 .

[Figure 7-9]

Positive curvature is distributed across a $d-e$ edge.

1.3. $e=da$

Δ will be one of the regions of Figures 7-4, 7-6 and 7-8 .

[Figures 7-4, 7-6 and 7-8]

Positive curvature is distributed across a $d-e$ or $b-c$ or $a-b$ edges. If the region of negative curvature is a region of degree five then it can only receive positive across its $a-b$ edge. If the region of negative curvature is an F_1 region then it can receive curvature either across its $d-e$ edges or across its $b-c$ edges. As proved in section 7.2.1. even if the region of negative curvature receives positive curvature across more than one of its edges, there is always enough negative curvature to compensate for all the positive curvature ($\leq \frac{\pi}{3}$) the region receives.

2. Three elements of order two

2.1. $a^2=c^2=e^2=1$

c(Δ) ≤ 0 unless $d=ea$.

Δ will be one of the regions of Figures 7-11, 7-12, 7-13, 7-18.

[Figures 7-11, 7-12, 7-13]

If a negative region is a region of degree five it will receive the positive curvature only across its $a-b$ edge (Figure 7-11) and if it is an F_1-type region of degree six like in Figure 7-12 and Figure 7-13 it will receive positive curvature across its $d-e$ edges.

2.2. $a^2=d^2=e^2=1$

2.2.1. $e=d=c^2$

Δ will be one of the regions of Figures 7-14, 7-16 .

[Figures $7-14, 7-16$]

Positive curvature is distributed to negative across a $c-d$ edge.

2.2.2. $e=a=c^2$

Δ will be one of the regions of Figure 7-15 .

[Figure 7-15]

Positive curvature is distributed to negative across a $b-c$ edge.

2.2.3. $a \neq e$, $d \neq e$

Δ will be one of the regions of Figure 7-17 .

[Figure 7-17]

Positive curvature is distributed to negative across a $c-d$ edge.

3. Two elements of order two

3.1. $|a|=|c|=2$

c(Δ) ≤ 0 unless it falls under one of the following cases:

3.1.1. $d=e$

Δ will be one of the regions of Figures 7-18 and Figure 7-21.

[Figure 7-18, 7-21]

Positive curvature is distributed either across a $d-e$ edge or a $b-c$ edge. In the case that an F_1 region receives curvature across a $b-c$ edge and a $d-e$ edge the total curvature of the positive regions is not greater than $\frac{\pi}{3}$ while the negative curvature will be at least $-\frac{19\pi}{30}$ which is enough to compensate for all the positive curvature in the neigbourhood.

3.1.2. $d = e^{-1}$

3.1.2.1. $c = e^2$

Δ will be one of the regions of Figures 7-19 .

[Figure 7-19]

Positive curvature is distributed across a $b - c$ edge.

3.1.2.2. $a = e^2$ or $a = e^3$

Δ will be one of the regions of Figures 7-20 .

[Figure 7-20]

Positive curvature is distributed across an $a - b$ edge.

3.2. $|a| = |e| = 2$

$c(\Delta) \leq 0$

3.3. $|a| = |d| = 2$ and $|d|$, $|c| \geq 3$

c(Δ) ≤ 0 unless it falls under one of the following cases:

3.3.1. $cec = 1$

Δ will be one of the regions of Figures 7-23, 7-25.

3.3.2. $ece = 1$

Δ will be one of the regions of Figures 7-22, 7-24.

[Figures 7-23, 7-25, 7-22, 7-24]

Positive curvature is distributed to negative across a $d - e$ edge or an $e - a$ edge. In the case that a region of negative curvature receives positive curvature across more than one type of edge there is another region of negative curvature adjacent to it where the positive curvature is distributed.

3.4. $|e| = |d| = 2$

$c(\Delta) \leq 0$

4. Only one element of order two

$c(\Delta) \leq 0$

Conclusion 4.13 *If Δ is an interior region with $c(\Delta) > 0$ in a diagram D representing a counter example for $r_3(t) = 1$, then there is always a region with negative curvature which, if interior, compensates for the positive curvature of Δ. The case when Δ or the region of negative curvature is a boundary region is considered in the final chapter.*

Chapter 5

$r_1(t) = atbtctdtet^{-1}$

As described in the Method Chapter we assume by way of contradiction that equation $r_1(t) = 1$ does not have a solution. According to the results mentioned in our Theoretical Background the following assumptions must hold.

(N1) Group G is generated by $\{a, b, c, d, e\}$

(N2) Group G is neither locally indicable nor residually finite, or a homomorphic image of a polycyclic by finite group. More specifically it does not have property X.

(N3) The star graph Γ_1 *contains at least one admissible path of length less than three and either* $|a| = 2$ *or/and* $|e| = 2$*.*

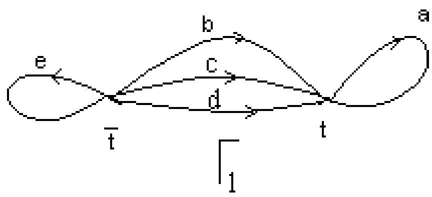


Figure 5-1:

This means that some interior faces may have vertices of degree two. (It is possible that the order of a or/and e is two but vertices with labels a^2 and e^2 do not appear on any interior vertex).

(N4) $b \neq d$

Remark 5.1 *The following additional assumptions can be made for* $r_1(t) = atbtctdtet^{-1}$ *without any loss of generality:*

(N5) $b = 1$

On $r_1(t) = atbtctdtet^{-1}$ we make the substitution $s = tb$. Then $atbtctdtet^{-1} = 1$, if and only if $assb^{-1}csb^{-1}dsb^{-1}ebs^{-1} = 1$. By taking $a' = a$, $b' = 1$, $c' = b^{-1}c$, $d' = b^{-1}d$, $e' = b^{-1}eb$, $r_1(t) = 1$ if and only if $a't^2c'td'te't^{-1} = 1$ has a solution. Therefore b can be taken to equal 1 in G without any loss.

(N6) $a \neq 1,\ e \neq 1,\ d \neq 1$

If any a or e equals 1, then the equation $r_1(t) = 1$ collapses to an equation of smaller length for which a solution is known to exist. Also, since b can be taken to equal 1 it is obvious that $d \neq 1$ (since $b \neq d$).

(N7) $r_1(t) = 1$ *can be solved modulo the transformation* T_1

$r_1(t) = 1$ has a solution if and only if the equation $e^{-1}t^{-1}d^{-1}t^{-1}c^{-1}t^{-1}b^{-1}t^{-1}a^{-1}t = 1$ has a solution over G. Replacing t by t^{-1} means that we only need to solve the equation modulo the transformation (T_1).

$$a \to e^{-1}, b \to d^{-1}, c \to c^{-1}, d \to b^{-1}, e \to a^{-1} (\mathrm{T}_1)$$

(N8) $a^2 = 1$

The case for which $e^2 = 1$ and $a^2 \neq 1$ is equivalent modulo (T_1) to the case when $a^2 = 1$ and $e^2 \neq 1$

Since the equation does not have a solution there is a relative diagram D with a non-trivial label on the distinguished vertex v_0 representing a counter example. The following three assumptions are further imposed on D.

(N9) Diagram D *contains minimal number of faces* Δ

This assumptions points to the fact that the labels are cyclically reduced paths on Γ (if for example a and a^{-1} were adjacent cancelation of faces would be possible).

(N10) Diagram D contains maximal number of vertices with label a^2

(N11) Diagram D contains maximal number of vertices of degree two.

Lemma 5.2 *$r_1(t) = 1$ has a solution in the following cases: (i) $b = c = 1$ and $ad = 1$ (ii) $c = d = e$ (iii) $c = a^{-1}$ and $d = e^{-1}$ (iv) $c = a$, $d = e^{-1}a$ and $a^2 = 1$ (v) $d = a$, $c = e^{-1}a$ and $a^2 = 1$*

Proof. i) If $b = c = 1$ and $ad = 1$, $r_1(t) = 1$ takes the form $d^{-1}t^3dtet^{-1} = 1$ and further $t^3 = (dt)\,e^{-1}\,(dt)^{-1}$. By [L] the equation $x^3 = e^{-1}$ has a solution in an overgroup of G so it is enough to consider the equation $t^3 = ((dt)\,x\,(dt)^{-1})^3$. But the equation $t = (dt)\,x\,(dt)^{-1}$ is an equation of length three and therefore has a solution by [Ho2], and this solution will be a solution to the original equation.(ii) If $c = d = e$ then as with (i) $r_1(t) = 1$ takes the form $at^2ctctct^{-1} = 1$ and further $(tc)^3 = t^{-1}a^{-1}t$. The equation $x^3 = a^{-1}$ has a solution, so it is enough to consider the equation $(tc)^3 = (t^{-1}xt)^3$. Now, the equation $tc = t^{-1}xt$ is an equation of length three and therefore has a solution which will also be a solution of the original equation.(iii) If $c = a^{-1}$ and $d = e^{-1}$ $r_1(t) = 1$ takes the form $at^2a^{-1}tdtd^{-1}t^{-1} = 1$. But this equation has a solution in G, take for example $t = 1$. (iv) If $c = a$, $d = e^{-1}a$ and $a^2 = 1$ then $r_1(t) = 1$ takes the form $at^2ate^{-1}atet^{-1} = 1$. This equation also has a solution in G, take for example $t = a$.(v) If $d = a$, $c = e^{-1}a$ and $a^2 = 1$ then as in (iv), the equation takes the form $at^2e^{-1}atatet^{-1} = 1$. This has a solution, for example $t = a$. ∎

Throughout this thesis we make use of *diamond moves* which were also used in the study of certain sets of equations [E1] and equations of smaller lengths ([EH], [Ho2]). A complete discussion of diamond moves on spherical diagrams and other diagrams is contained in [CH]. The following Lemma is an example of the use of diamond moves.

Lemma 5.3 *(i) a^2 or a^{-2} cannot be a proper sublabel in the diagram D. (ii) D does not contain any labels $xaya^{\pm 1}$ for non empty words $x = y = 1$.*

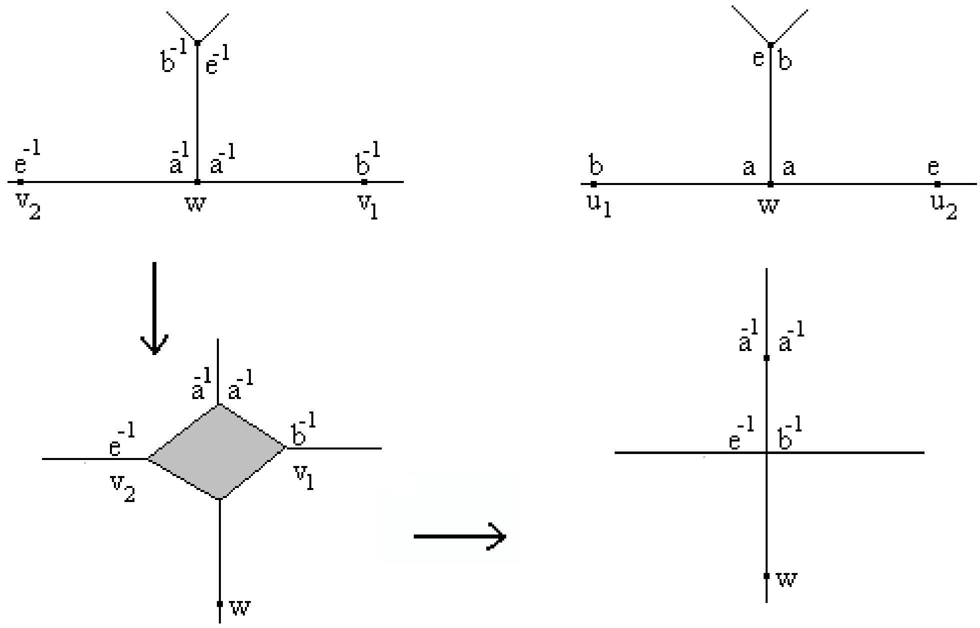


Figure 5-2:

Proof. (i) Suppose a^2 or a^{-2} appears as a proper sublabel on some vertex v in diagram D as in Figure 5-2. It is possible to perform a *diamond move* on the vertex which contains a^2 or a^{-2} as in Figure 5-2 to increase the number of vertices labeled a^2, contrary to the assumption (N10) of their maximality. (ii) The same is possible if words $xaya^{\pm 1}$ for non empty words $x = y = 1$ appears as a label on a vertex. By the use of diamond moves a non-reduced path is created (and this would reduce the number of faces of the diagram contrary to (N9)) or a^2 becomes a proper sublabel and a second diamond move as in (i) would increase the number of labels a^2 contrary to (N10). ■

Note that diamond moves can also be applied in the case any other word $\omega = 1$ of length two is contained as proper sublabel on a vertex, if the two adjacent vertices do not have label $a^{\pm 2}$. Performing a diamond move would create a new vertex of degree two (contrary to the assumption of their maximality (N11) without altering the number of faces or the number of vertices with label $a^{\pm 2}$.

The possible cases of an interior region Δ with $c(\Delta) > 0$ are considered according to whether Δ contains or does not contain vertices of degree 2. Let Δ be the region given by Figure 4-1. Let l_i denote the label of v_i $(1 \leq i \leq 5)$ and $|v|$ denote the degree of vertex v. In Section 5.1. the cases of a region Δ that does not have any vertices of degree two are considered. In Section 5.2. the cases of regions which contain at least one vertex of degree two are considered into to two subsections: In Subsection 5.2.1. the regions that contain a vertex of degree two but not a^2 or e^2 are listed and in Subsection 5.2.2. the regions that contain a^2 or e^2 are listed.

5.1 Δ has no vertices of degree 2

It follows from the curvature formula (*) that at least four vertices have degree 3 and the remaining vertex has degree 3, 4 or 5 otherwise $c(\Delta) \leq 0$ since $c(3,3,3,4,4) = 0$ and $c(3,3,3,3,6) = 0$. We proceed by finding the individual cases of labels on regions Δ of positive curvature.

5.1.1 $|v_1| = |v_2| = |v_3| = |v_4| = |v_5| = 3$

$|v_1| = 3$ implies $l_1 \in \{ab^{-1}c, ab^{-1}d, ac^{-1}b, ac^{-1}d, ad^{-1}b, ad^{-1}c\}$. We check the case for each possible label of v_1.

$l_1 = ab^{-1}c$

Now $a = c$, $l_5 = edw$ and therefore $l_5 \in \{edb^{-1}, edc^{-1}\}$. In the first case $c = a$ and $d = e^{-1}$ so by Lemma 5.2(iii), the equation has a solution. So $l_5 = edc^{-1}$ and

$d = e^{-1}a$. Again by Lemma 5.2(iv) the equation has a solution. So if $l_1 = ab^{-1}c$ then $|v_5| \in \{4, 5\}$.

$l_1 = ab^{-1}d$

Now $a = d$, $l_5 = e^3$ and so $l_4 \in \{dab^{-1}, dac^{-1}\}$. The second label forces $c = 1$ and by Lemma 5.2(i) the equation has a solution. So $l_4 = dab^{-1}$ and so $l_3 = ca^{-1}w$. The only possible label for v_3 is $l_3 = ca^{-1}b^{-1}$ and so $l_2 = c^{-1}ba^{-1}$. This case appears in Figure 5-3.

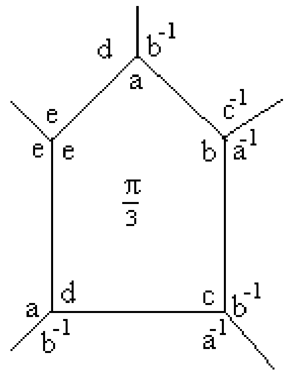


Figure 5-3:

Lemma 5.4 *If Δ is the above region then there exist regions Δ_1 and Δ_2 adjacent to Δ such that $c(\Delta) + c(\Delta_1) \leq 0$ or $c(\Delta) + c(\Delta_1) + c(\Delta_2) \leq 0$.*

Proof. Consider the neighbouring region Δ_1 of Δ as in Figure 5-4.

It can be seen that $v_{4\Delta_1}$ cannot have degree three or four as this would make e belong to the subgroup generated by the remaining elements, or its order to be two and thus the group to be generated by two elements of order two (i.e. $G =< a, e | a^2 = e^2 = 1 >$), but then G has property X, a contradiction. If $|v_{4\Delta_1}|$ $= 5$ then $l_{4\Delta_1} = d^{-1}e^{\pm 3}c$. But in this case diamond moves can be performed on v_5 to create one more vertex of degree two, contrary to assumption (N11) of

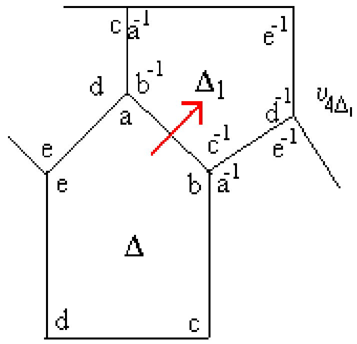


Figure 5-4:

maximality of vertices of degree two, without altering the number of faces or the number of vertices with label a^2. So the degree of $v_{4\Delta_1}$ must be six or greater. Now if both $v_{1\Delta_1}$ and $v_{5\Delta_1}$have degree four or greater then $c(\Delta_1) \leq c(3,3,4,4,6) = -\frac{\pi}{3}$ and we are done.

Suppose $|v_{1\Delta_1}| = 3$ and so $l_{1\Delta_1} = ca^{-1}b^{-1}$. Regions Δ and Δ_1 are as in the Figure 5-5.

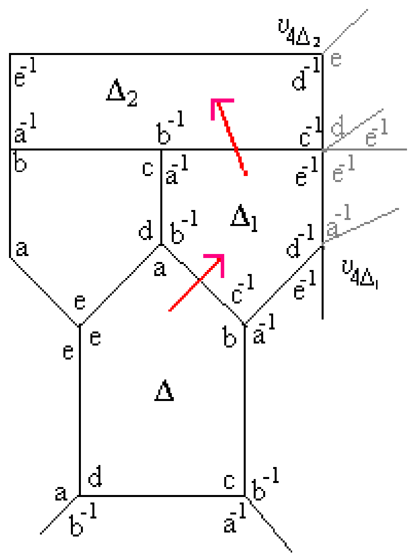


Figure 5-5:

Observe that $|v_{4\Delta_1}|$ will be six or greater and if $|v_{5\Delta_1}| \geq 6$ the curvature of Δ_1 would be $c(\Delta_1) \leq c(3,3,3,6,6) = -\frac{\pi}{3}$. So the only case to be checked is for $l_{5\Delta_1} = c^{-1}e^{-3}d$. In this case we can look for negative curvature in region Δ_2. Now $v_{4\Delta_2}$ would have degree greater than six and therefore $c(\Delta_2) \leq c(3,3,3,5,6) = -\frac{4\pi}{15}$ and $c(\Delta) + c(\Delta_1) + c(\Delta_2) < 0$.

Now, suppose $|v_{5\Delta_1}| = 3$ and so $l_{5\Delta_1} = e^{-3}$. If the degree of $v_{1\Delta_1}$ is six or greater then $c(\Delta_1) \leq c(3,3,3,6,6) = -\frac{\pi}{3}$ and we are done. So let the degree of $v_{1\Delta_1}$ be five and so $l_{1\Delta_1} = ca^{-1}d^{-1}cb^{-1}$. In this case it is possible to perform diamond moves without decreasing the number of faces, or the number of vertices with label a^2 or the number of vertices of degree two to make the degree of $v_{1\Delta_1}$ to be three and the degree of $v_{5\Delta 1}$ to be five as in Figure 5-6. This is the same situation as in the previous paragraph and negative curvature may be found in Δ_1 and Δ_2.

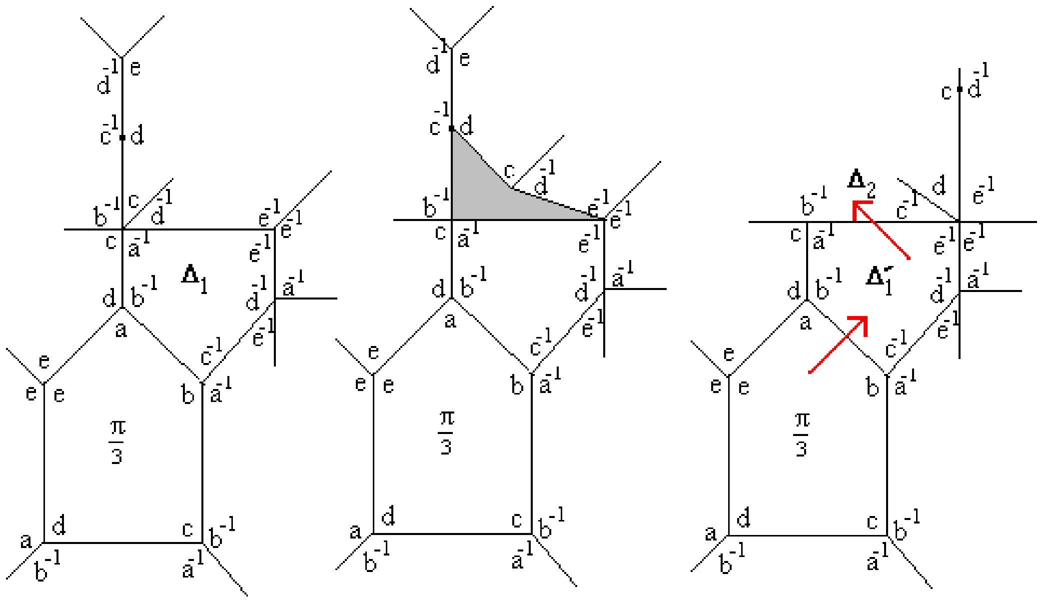


Figure 5-6:

In this way the positive curvature of Δ is added to the negative curvature of Δ_1 and Δ_2. Since in this case a region of negative curvature always receives positive curvature across a $b-c$ or a $c^{-1}-b^{-1}$ edge, it is not possible for such a region to receive positive curvature from more than one of its neighbours. (This idea of unique compensation will be used throughout the thesis). ∎

Lemma 5.5 *If* $a^2 = 1$, $a = c = d$ *and* $l_1 = dab^{-1}$, $l_2 = c^{-1}ba^{-1}w$ *where* w *is any word in* G, *including the empty word, then there exist regions* Δ_1 *and* Δ_2 *adjacent to* Δ *such that* $c(\Delta) + c(\Delta_1) \leq 0$ *or* $c(\Delta) + c(\Delta_1) + c(\Delta_2) \leq 0$.

Proof. This is a generalisation of Lemma 5.4, for the case that the region Δ contains a vertex of degree two (such cases will appear in the next section). A vertex of degree two in this case can only be v_4 with label dc^{-1}. In that case the degree of v_5 will have to be six or greater and the degree of v_3 four or greater. Therefore the maximum curvature that Δ can have is again $\frac{\pi}{3}$. Also, during the Lemma 5.4 we did not use the labels of v_3 and v_4 or v_5 to obtain the result and thus it applies without any restrictions on the labels of l_3, l_4 and l_5. ∎

$l_1 = ac^{-1}b$

Now $a = c$, $l_5 \in \{ecb^{-1}, ecd^{-1}\}$ and $a = c = e$ or $a = c$ and $d = ea$. Also $l_2 = d^{-1}ba^{\pm 1}$ and $d = a$, which makes G cyclic. Thus $l_1 = ac^{-1}b$ then $|v_1| = |v_2| = |v_5| = 3$ is not possible.

$l_1 = ac^{-1}d$

Now $d = ca$. Now, $l_2 = d^{-1}ba^{\pm 1}$ and $d = a$. This implies that $c = b = 1$ and therefore by Lemma1(i) the equation has a solution. So if $l_1 = ac^{-1}d$ then $|v_2| \in \{4, 5\}$.

$l_1 = ad^{-1}b$

Then $a = d$ and $l_2 \in \{e^{-1}bc^{-1}, e^{-1}bd^{-1}\}$ and $e = c^{-1}$ or $e = d^{-1}$.If $l_2 = e^{-1}bd^{-1}$ then $e = a = d$ and any attempt to label v_5 makes the group cyclic. So $l_2 =$

$e^{-1}bc^{-1}$ and $c = e^{-1}$. Also $l_5 \in \{ecb^{-1}, ecd^{-1}\}$. If $l_5 = ecd^{-1}$ then $d = 1$, a contradiction, so $l_5 = ecb^{-1}$. The label of v_3 is $l_3 = d^{-1}cw$ and any effort of labeling with degree three makes the group cyclic.

$l_1 = ad^{-1}c$

Then $d = ca$ and $l_2 \in \{e^{-1}bc^{-1}, e^{-1}bd^{-1}\}$ and $e = c^{-1}$ or $e = d^{-1}$. Let $l_2 = e^{-1}bc^{-1}$ and $e = c^{-1}$, $d = e^{-1}a$. Now $l_3 = d^{-1}ca^{\pm 1}$ and so $l_4 \in \{edb^{-1}, edc^{-1}, b^{-1}da^{\pm 1}\}$. The first two make group G cyclic, generated by e, while the last one forces $c = e = 1$, a contradiction. Let $l_2 = e^{-1}bd^{-1}$ and so $e = d^{-1}$. Now $l_3 \in \{e^{-1}cb^{-1}, e^{-1}cd^{-1}\}$ and so $c = e$ or $c = e^2$ which makes G cyclic in any case. So $|v_1| = |v_2| = |v_3| = |v_4| = 3$ is not possible in this case.

5.1.2 $|v_1| = |v_2| = |v_3| = |v_4| = 3$ and $|v_5| \in \{4, 5\}$

From section 5.1.1. it can be seen that if $l_1 \in \{ac^{-1}d, ad^{-1}c\}$ then $|v_1| = |v_2| = |v_3| = |v_4| = 3$ is not possible so we check for the remaining possible labels of v_1 i.e. $l_1 \in \{ab^{-1}c, ab^{-1}d, ac^{-1}b, ad^{-1}b\}$.

$l_1 = ab^{-1}c$

Now, $a = c$. So $l_2 = c^{-1}ba^{\pm 1}$ which does not yield any other relator. So the label of v_3 can be $l_3 \in \{ecb^{-1}, ecd^{-1}, b^{-1}ca^{\pm 1}\}$. We check for each of these labels individually.

$l_3 = ecb^{-1}$ The relators holding in the group are $e = c = a$ and v_4 can have label $l_4 = c^{-1}da^{\pm 1}$ and then $d = 1$, a contradiction.

$l_3 = ecd^{-1}$ Now $d = ea$ and $l_4 \in \{e^{-1}db^{-1}, e^{-1}dc^{-1}\}$. The first label causes a contradiction by making $a = 1$ while $l_4 = e^{-1}dc^{-1}$ is possible. This forces $l_5 = d^{-1}edw$ where w is a word of length one or two. So $l_5 \in \{d^{-1}eda^{\pm 1}, d^{-1}edb^{-1}c, d^{-1}edc^{-1}b\}$.

The first of these labels makes $e^2 = 1$ and therefore group G is generated by two elements of degree two and so it has property X, while the other two force $d = 1$, a contradiction.

$l_3 = b^{-1}ca$ Now $l_4 \in \{edb^{-1}, edc^{-1}\}$. The first makes the equation have a solution by Lemma 5.2 (iii), so $l_4 = edc^{-1}$ and $d = ea$. Now, any possible label for v_5 will be as in the previous paragraph, and so we get a contradiction.

$l_3 = b^{-1}ca^{-1}$ Then $l_4 = b^{-1}da^{\pm1}$ and $d = a = c$. Now $l_5 \in \{e^2dw, b^{-1}edw\}$ where w is of length one or two i.e. $l_5 \in \left\{ \begin{array}{c} e^2db^{-1}, e^2dc^{-1}, e^2da^{\pm1}b^{-1}, e^2da^{\pm1}c^{-1}, \\ e^2da^{\pm1}d^{-1}, e^2db^{-1}e, e^2dc^{-1}e, b^{-1}eda^{\pm1}, \\ b^{-1}edc^{-1}d, b^{-1}edb^{-1}c, b^{-1}edb^{-1}d \end{array} \right\}$.
Each of these labels, either makes the order of e to be two and therefore, group G to have property X, therefore forcing a contradiction except for $l_5 = e^2dc^{-1}e$. But in this case a *diamond move* may be performed at v_5 to increase the number of labels of degree two, contrary to the assumption (N11) of their maximality, without altering the number of vertices with label a^2.

$l_1 = ab^{-1}d$

Now $a = d$ and $l_2 = c^{-1}ba^{\pm1}$ and $a = c = d$. If $l_2 = c^{-1}ba^{-1}$ by Lemma 5.5, there are negative regions Δ_1 and Δ_2 with enough negative curvature such that $c(\Delta) + c(\Delta_1) + c(\Delta_2) \leq 0$. So we can take $l_2 = c^{-1}ba$ and so $l_3 \in \{ecb^{-1}, ecd^{-1}\}$. Any of these labels make G cyclic, a contradiction.

$l_1 = ac^{-1}b$

Now $a = c$ and $l_2 = d^{-1}ba^{\pm1}$, so $d = c = a$. This forces $l_3 = b^{-1}ca^{-1}$ and $l_4 = b^{-1}da^{\pm1}$. Now $l_5 \in \{b^{-1}ecw, e^2cw\}$. So $l_5 \in \left\{ \begin{array}{c} b^{-1}eca^{\pm1}, b^{-1}eca^{\pm1}c, b^{-1}eca^{\pm1}d, \\ b^{-1}ecb^{-1}c, b^{-1}ecb^{-1}d, b^{-1}ecd^{-1}c, \\ e^2cb^{-1}, e^2cd^{-1}, e^2ca^{\pm1}b^{-1}, e^2ca^{\pm1}c^{-1}, \\ e^2ca^{\pm1}d^{-1}, e^2cb^{-1}e, e^2cd^{-1}e \end{array} \right\}$.
The only label not forcing a contradiction, is $l_5 = e^2cd^{-1}e$. This case is shown in Figure 5-7 and the region has curvature $c(\Delta) = c(3,3,3,3,5) = \frac{\pi}{15}$.

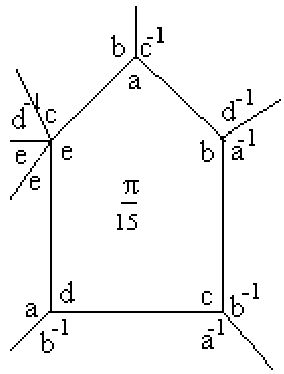


Figure 5-7:

It can be seen that it is again possible to find a region Δ_1 with negative curvature as in Figure 5-8.

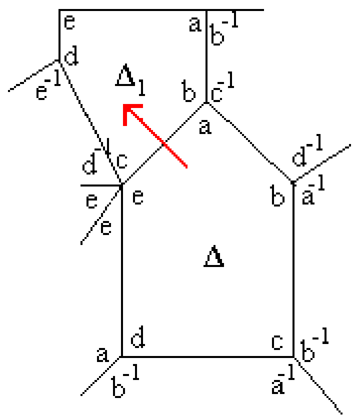


Figure 5-8:

The curvature of Δ_1 is again negative and enough to compensate for the positive curvature of Δ as $v_{4\Delta_1}$ must have degree six or greater which makes $c(\Delta_1) \leq c(3,3,3,5,6) = -\frac{4\pi}{15}$. The positive curvature of Δ is again added to that of Δ_1 across a $b-c$ edge of Δ_1 so it is not possible for a negative region to receive curvature from more than one of its neighbours.

$l_1 = ad^{-1}b$

Now $a = d$ and $l_2 \in \{e^{-1}bc^{-1}, e^{-1}bd^{-1}\}$. If $l_2 = e^{-1}bd^{-1}$ then $d = e = a$ and any labelling on υ_3 makes the group cyclic. If $l_2 = e^{-1}bc^{-1}$ and $c = e^{-1}$. Now $l_3 = d^{-1}ca^{\pm1}$ and so $c = e = 1$, a contradiction.

5.1.3 $|\upsilon_1| = |\upsilon_2| = |\upsilon_4| = |\upsilon_5| = 3$ and $|\upsilon_3| \in \{4, 5\}$

From 5.1.1. it is known that this is only possible for $l_1 \in \{ab^{-1}d, ad^{-1}c\}$. We check each individual case.

$l_1 = ab^{-1}d$

Now $a = d$ and $l_2 = c^{-1}ba^{\pm1}$ and so $a = c = d$. In the case that $l_2 = c^{-1}ba^{-1}$ any region Δ of positive curvature will have by Lemma 5.5 neighbouring regions Δ_1 and Δ_2 with enough negative curvature such that $c(\Delta) + c(\Delta_1) + c(\Delta_2) \leq 0$. So $l_2 = c^{-1}ba$ and $l_5 = e^3$. So $l_4 \in \{dab^{-1}, dac^{-1}\}$. The second one forces $a = 1$, a contradiction so $l_4 = dab^{-1}$ and thus $l_3 = eca^{-1}w$. So $l_3 \in \{eca^{-1}b^{-1}, eca^{-1}c^{-1}, eca^{-1}d^{-1}, eca^{-1}b^{-1}e, eca^{-1}c^{-1}e, eca^{-1}d^{-1}e\}$. Any of these labels forces relators that make the group to have property X, a contradiction.

$l_1 = ad^{-1}c$

Now $d = ca$ and $l_2 \in \{e^{-1}bc^{-1}, e^{-1}bd^{-1}\}$. Let $l_2 = e^{-1}bc^{-1}$ and $e = c^{-1}$, $d = e^{-1}a$. Also $l_5 \in \{edb^{-1}, edc^{-1}\}$. The first of these make $a = 1$ and the second forces G to be cyclic, a contradiction in either case. Let $l_2 = e^{-1}bd^{-1}$ and so $e = d^{-1}$ and $c = e^{-1}a$. Now υ_5 may have degree three with possible label $l_5 = edb^{-1}$. Now $l_4 = dc^{-1}e^{\pm1}$ which makes G cyclic.

5.1.4 $|v_1| = |v_3| = |v_4| = |v_5| = 3$ and $|v_2| \in \{4, 5\}$

From 5.1.1. we know that this is not possible for $l_1 = abc^{-1}$. Therefore we check the individual cases for $l_1 \in \{ab^{-1}d, ac^{-1}b, ac^{-1}d, ad^{-1}b, ad^{-1}c\}$.

$l_1 = ab^{-1}dq$

From 5.1.1. it can be seen that this label on v_1 implies the following labels on v_3, v_4 and v_5: $l_5 = e^3$, $l_4 = dab^{-1}$, $l_3 = ca^{-1}b^{-1}$. The relations holding in G are $a^2 = e^3 = 1, d = a = c$ and $l_2 = c^{-1}ba^{-1}w$. So $l_2 \in \left\{ \begin{array}{c} c^{-1}ba^{-1}b^{-1}d, c^{-1}ba^{-1}c^{-1}b, \\ c^{-1}ba^{-1}c^{-1}d, c^{-1}ba^{-1}d^{-1}b \end{array} \right\}$. From theses labels the only one not forcing $a = 1$ is $l_2 = c^{-1}ba^{-1}c^{-1}d$ but in this case a *diamond move* may be performed to increase the number of vertices of degree two, without altering the number of faces or the number of vertices with label a^2, a contradiction to assumption (N11).

$l_1 = ac^{-1}b$

Now $l_5 \in \{ecb^{-1}, ecd^{-1}\}$. We look at each case in turn. If $l_5 = ecb^{-1}$ then the relations holding are $a = c = e$ and $l_4 \in \{da^{-1}b^{-1}, da^{-1}c^{-1}\}$. Any of these labels makes group G cyclic, a contradiction. If $l_5 = ecd^{-1}$ then the relators holding are $a = c$ and $d = ea$ and $l_4 \in \{dc^{-1}e^{-1}, dc^{-1}e\}$. If $l_4 = dc^{-1}e$ then the order of e is two and therefore group G has property X. So $l_4 = dc^{-1}e^{-1}$ and $l_3 \in \{cd^{-1}e, cd^{-1}e^{-1}\}$. As before, the second one forces $e^2 = 1$ and the group is forced to by cyclic so the label of v_3 must be $l_3 = cd^{-1}e$. Therefore, $l_2 = d^{-1}baw$ and since a^2 cannot be a proper sublabel the degree of v_2 must be five. So $l_2 \in \{d^{-1}bab^{-1}c, d^{-1}bac^{-1}b, d^{-1}bad^{-1}b, d^{-1}bad^{-1}c\}$. The first three labels force $a = 1$ while the last forces $e^2 = 1$, a contradiction in each case.

$l_1 = ac^{-1}d$

Now, $l_5 = e^3$ and $l_4 \in \{dab^{-1}, dac^{-1}\}$. The first label makes the equation have a solution by forcing $b = c = 1$ and $d = a$ (Lemma 5.2(i)). So $l_4 = dac^{-1}$ and the relations holding are $d = ca$ and $e^3 = 1$. Now $l_3 = cb^{-1}e^{\pm 1}$ and so $c = e^{\pm 1}$. We check the label of υ_2 for each case.

$l_3 = cb^{-1}e$ Now, $c = e^{-1}$ and $d = e^{-1}a$. Also $l_2 = d^{-1}baw$ and since a^2 is not allowed as a proper sublabel $l_2 \in \{d^{-1}bab^{-1}c, d^{-1}bac^{-1}b, d^{-1}bad^{-1}b, d^{-1}bad^{-1}c\}$. Thus we have $aeae^{\pm 1} = 1$ or $a = e$. The first relator makes the group residually finite, while the second makes the group collapse to the trivial group, a contradiction.

$l_3 = cb^{-1}e^{-1}$ Now $c = e$, $d = ea$ and $l_2 = dad^{-1}w$. The label of υ_2 will be $l_2 \in \{dad^{-1}e^{\pm 1}, dad^{-1}e^{\pm 2}, dad^{-1}bc^{-1}, dad^{-1}cb^{-1}\}$. The first two labels make the group cyclic and the rest force $a = e$, a contradiction.

$l_1 = ad^{-1}b$

Now $d = a$ and $l_5 \in \{ecb^{-1}, ecd^{-1}\}$. In the second case $c = e^{-1}a$ and so the equation has a solution by Lemma 5.2(vi). So $l_5 = ecb^{-1}$ and $c = e^{-1}$. This forces $l_4 \in \{da^{-1}b, da^{-1}c\}$. If $l_4 = da^{-1}c$ then $c = e = 1$, a contradiction. So $l_4 = da^{-1}b^{-1}$ and $l_3 \in \{ca^{-1}, ca^{-1}d\}$. Each of these labels forces a contradiction, so no cases are left here.

5.1.5 $|\upsilon_1| = |\upsilon_2| = |\upsilon_3| = |\upsilon_5| = 3$ and $|\upsilon_4| \in \{4, 5\}$

From 5.1.1. we know that this is possible for $l_1 \in \{ab^{-1}c, ab^{-1}d, ad^{-1}c\}$. We check each case:

$l_1 = ab^{-1}c$

Now $a = c$ and $l_2 = c^{-1}ba^{\pm 1}$. Also $l_5 \in \{edb^{-1}, edc^{-1}\}$. The first label implies $c = a$ and $d = e^{-1}$ and so the equation has a solution by Lemma 5.2(iii). Also $l_5 = edc^{-1}$ implies $d = e^{-1}a$ and $c = a$ and so again the equation has a solution by Lemma 5.2(iv).

$l_1 = ab^{-1}d$

Now $a = d$ and $l_5 = e^3$. If $l_2 = c^{-1}ba^{\pm 1}$ and $a = c = d$ any possible region of positive curvature will have by Lemma 5.5 neighbouring regions Δ_1 and Δ_2 with enough negative curvature such that $c(\Delta) + c(\Delta_1) + c(\Delta_2) \leq 0$. So $l_2 = c^{-1}ba$ and $l_3 \in \{ecb^{-1}, ecd^{-1}\}$. Any of these labels forces a contradiction by making the group to have property X or by making $e = 1$.

$l_1 = ad^{-1}c$

Now $d = ca$ and $l_5 \in \{edb^{-1}, edc^{-1}\}$. The first label forces $d = e^{-1}$ and $c = e^{-1}a$ and so the equation has a solution by Lemma 5.2(v). So $l_5 = edc^{-1}$ and so $ecac^{-1} = 1$, and so $e^2 = 1$. Now $l_2 \in \{e^{-1}bc^{-1}, e^{-1}bd^{-1}\}$ and $e = c$ or $e = d$. In the first case $a = 1$, a contradiction, so $l_2 = e^{-1}bd^{-1}$. This makes the label of v_3 to be $l_3 \in \{e^{-1}cb^{-1}, e^{-1}cd^{-1}\}$. Each of these labels forces a contradiction, so no case is left here.

5.1.6 $|v_2| = |v_3| = |v_4| = |v_5| = 3$ and $|v_1| \in \{4, 5\}$

We check each case starting from the possible labels of l_5. Those label starting with eb would force a^2 to be a proper sublabel on l_1 which we do not allow by Lemma 5.3, so we check the remaining cases i.e. $l_5 \in \{ecb^{-1}, ecd^{-1}, edb^{-1}, edc^{-1}, e^3\}$.

$l_5 = ecb^{-1}$

Now $c = e^{-1}$ and $l_4 \in \{da^{-1}b^{-1}, da^{-1}c^{-1}\}$. If $l_4 = da^{-1}b^{-1}$then $d = a$ and $c = e^{-1}$. Also $l_3 \in \{ca^{-1}b^{-1}, ca^{-1}d^{-1}\}$. The first makes G cyclic while the second makes $c = e = 1$, a contradiction. So $l_4 = da^{-1}c^{-1}$ and so $d = e^{-1}a$ and $l_3 = cb^{-1}e^{\pm 1}$. If the order of e is two, then the group has property X, so $l_3 = cb^{-1}e$. So $l_2 \in \{bac^{-1}, bad^{-1}\}$ but each of these vertices forces a contradiction.

$l_5 = ecd^{-1}$

Now, $d = ec$ and $l_4 \in \{dc^{-1}e, dc^{-1}e^{-1}\}$. We check these two cases:

$l_4 = dc^{-1}e$ Now, then the order of e is two and $l_3 \in \{cab^{-1}, cad^{-1}\}$. The first one forces the group to be generated by e and a of degree two, so it has property X. So $l_3 = cad^{-1}$ and $l_2 = bc^{-1}e^{\pm 1}$ which forces $d = 1$, a contradiction.

$l_4 = dc^{-1}e^{-1}$ Now, no new relator emerges and $l_3 = cd^{-1}e^{\pm 1}$. If $l_3 = cd^{-1}e^{-1}$ then $e^2 = 1$ and $l_2 = bd^{-1}e^{-1}$, so $d = e^{-1}$ and $c = 1$. Now the label of v_1 would involve letter a just once and therefore, a is a power of e and so G is cyclic. If $l_3 = cd^{-1}e$ then $l_2 \in \{bac^{-1}, bad^{-1}\}$.If $l_2 = bad^{-1}$ then $d = a$ and $c = e^{-1}a$ then the equation has a solution by Lemma 5.2(v). If $l_2 = bac^{-1}$ then $a = c$ and $d = ea$ and $l_1 = bab^{-1}w$. So $l_1 \in \{bab^{-1}e^{\pm 1}, bab^{-1}e^{\pm 2}, bab^{-1}cd^{-1}, bab^{-1}dc^{-1}\}$. Each of these labels forces a contradiction by making the group to have property X.

$l_5 = edb^{-1}$

Now $d = e^{-1}$ and $l_4 \in \{dc^{-1}e, dc^{-1}e^{-1}\}$. If $l_4 = dc^{-1}e$ then $c = 1$ and $l_3 \in \{cab^{-1}, cad^{-1}\}$. Each of these forces a contradiction, so it must be $l_4 = dc^{-1}e^{-1}$, and so $c = e^{-2}$. $l_3 \in \{cd^{-1}e, cd^{-1}e^{-1}\}$. The second label makes $e^2 = 1$ and so G to have property X. So the label of v_3 must be $l_3 = cd^{-1}e$ and the label of v_2 must be $l_2 \in \{bac^{-1}, bad^{-1}\}$ which in any case makes G cyclic, a contradiction.

$l_5 = edc^{-1}$

Now $d = e^{-1}c$ and $l_4 \in \{db^{-1}e, db^{-1}e^{-1}\}$. If $l_4 = db^{-1}e$ then $d = e^{-1}$, $c = 1$ and $l_3 \in \{cab^{-1}, cad^{-1}\}$ which make G cyclic, a contradiction. So $l_4 = db^{-1}e^{-1}$ and so $d = e$ and $c = e^2$. Now $l_3 \in \{cd^{-1}e, cd^{-1}e^{-1}\}$. The second one makes $e^2 = 1$ and so G to have property X. So $l_3 = cd^{-1}e^{-1}$ and $l_2 = bd^{-1}e^{-1}$. Now the label of v_1 will involve letter a just once and therefore, make group G cyclic, generated by e, a contradiction.

$l_5 = e^3$

Now $l_4 \in \{dab^{-1}, dac^{-1}\}$. We examine these two cases individually:

$l_4 = dab^{-1}$ Now the relators holding in G are $e^3 = 1$ and $d = a$ and $l_3 \in \{ca^{-1}b^{-1}, ca^{-1}d^{-1}\}$. The second label would make $c = 1$ and so by Lemma 5.2(i) the equation would have a solution. So $l_3 = ca^{-1}b^{-1}$ and $a = c = d$. Now, $l_2 \in \{ba^{-1}c^{-1}, bad^{-1}\}$ and both labels are possible. So $l_1 \in \{dab^{-1}w, dac^{-1}w\}$ where w is a word of length one or two. So $l_1 \in \left\{ \begin{array}{c} dab^{-1}e^{\pm1}, dab^{-1}e^{\pm2}, dab^{-1}cb^{-1}, dab^{-1}db^{-1}, \\ dab^{-1}dc^{-1}, dac^{-1}e^{\pm1}, dac^{-1}e^{\pm2}, \\ dac^{-1}bc^{-1}, dac^{-1}db^{-1}, dac^{-1}dc^{-1} \end{array} \right\}$.
Out of these labels those not forcing any contradiction by making the group to have property X, are $l_1 \in \{dab^{-1}dc^{-1}, dac^{-1}db^{-1}, dac^{-1}bc^{-1}\}$. In the first two cases, it is possible to perform *a diamond move* to create one more vertex of degree two, with label dc^{-1} or its inverse without interfering with the number of faces or the number of labels a^2, contrary to (N11). So the only case which remains to be considered is for $l_1 = dac^{-1}bc^{-1}$. But in this cases a *diamond move* may also be performed as in Figure 5-9 to turn the particular case to that of Figure 5-7 for which enough negative curvature was found to exist to one of its neighbouring regions.

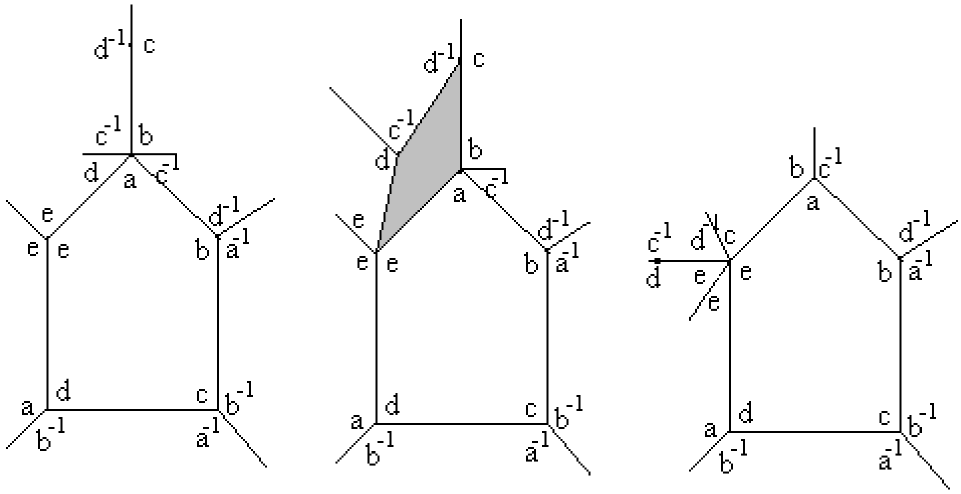


Figure 5-9:

$l_4 = dac^{-1}$ The relators holding in G are $e^3 = 1$ and $d = ca$ and now, $l_3 \in \{cb^{-1}e, cb^{-1}e^{-1}\}$. If $l_3 = cb^{-1}e$ then $c = e^{-1}$ and $d = e^{-1}a$. So $l_2 \in \{bac^{-1}, bad^{-1}\}$ and either $a = c$ or $a = d$ which either makes G cyclic or $e = 1$, a contradiction. If $l_3 = cb^{-1}e^{-1}$ then $c = e$ and $d = ea$. So $l_2 = bd^{-1}e^{\pm 1}$ which makes again G cyclic or $a = 1$, a contradiction.

In conclusion we have shown that if D contains an interior region Δ of positive curvature none of whose vertices have degree two then $a^2 = 1$ and $a = c = d$. Moreover it can be assumed without any loss that Δ is given by the region in Figure 5-4 or Figure 5-5. If the regions Δ_1 and Δ_2 of Figures 5-4 or 5-5 are both interior then we have shown that they have sufficient negative curvature to compensate for the positive curvature of Δ. We have observed also that there is uniqueness of compensation since any positive curvature is added across a $b - c$ or a $c^{-1} - b^{-1}$ edge of D. The cases when either Δ_1 or Δ_2 is not interior will be considered in the final chapter.

5.2 Δ has vertices of degree 2

It turns out that any vertex of Δ can have degree two with possible labels a^2, bc^{-1}, cd^{-1}, e^2 or their inverses. This is because the labels of length two must be paths of length two on the star graph Γ_1 (Figure 5-1). Such paths on the stargraph can only be square loops i.e. a^2 and e^2 and bc^{-1}, cd^{-1} and bd^{-1} or the inverses. As bd^{-1} or the inverse would mean $b = d$, a contradiction to (N4), the remaining labels are considered. These labels and the relations imposed by the existence of such a vertex appear on Figure 5-10.

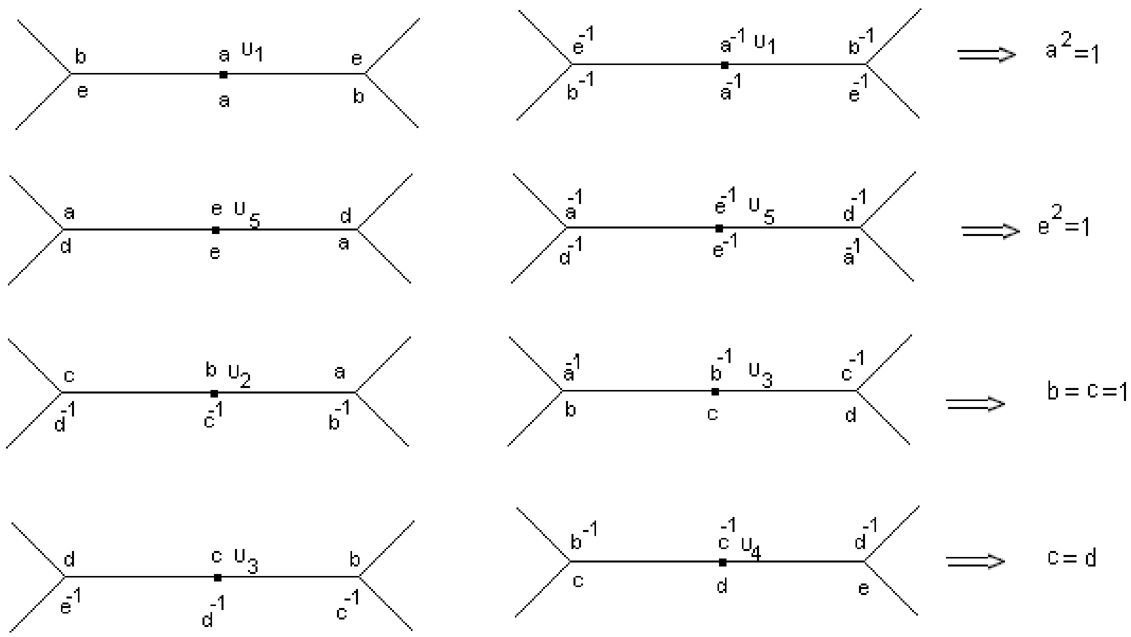


Figure 5-10:

The next follows from observation of Figure 5-10.

Lemma 5.6 *If Δ is an interior region of D then two adjacent vertices of Δ cannot both have degree two. (Therefore, Δ can have at most two vertices of degree two).*

Proof. Let $|v_1| = 2$. Then $a^2 = 1$. The two adjacent vertices must have $(eb)^{\pm 1}$ as a sublabel and therefore, cannot have degree two.

Let $|v_2| = 2$. Then $l_2 = bc^{-1}$ and $l_1 = ab^{-1}w$ and $l_3 = cd^{-1}w$. Vertex v_1 cannot have degree two and $|v_3| = 2$ implies $b = c = d = 1$, a contradiction.

Let $|v_3| = 2$. The label of v_3 will be $l_3 \in \{cb^{-1}, cd^{-1}\}$. If $l_3 = cb^{-1}$ then $c = b = 1$ and $l_2 = ba^{-1}w$, $l_4 = c^{-1}dw$. Vertex v_2 cannot have degree two and $|v_4| = 2$ implies $b = c = d = 1$, a contradiction. If $l_3 = cd^{-1}$ then$l_4 = e^{-1}dw$ and it cannot have degree two while $l_2 = bc^{-1}w$ and $|v_2| = 2$ implies $b = c = d = 1$, a contradiction

Let $|v_4| = 2$. Then $l_4 = dc^{-1}$ and $d = c$. The labels of the adjacent vertices are $l_3 = cb^{-1}w$ but if this is of degree two then $b = c = d = 1$, a contradiction then l_5 and $l_5 = d^{-1}ew$ and cannot have degree two.

Let $|v_5| = 2$. Then $e^2 = 1$ and both of the adjacent vertices must have da as a sublabel and therefore cannot have degree two. ■

It follows from the above Lemma that a region Δ may have at most two vertices of degree two.

The type of regions Δ containing exactly one vertex of degree two are shown in Figure 5-11. (The unlabeled vertices have degree at least three.)

The types of regions for which exactly two vertices have degree two are shown in Figure 5-12. It can be seen that if Δ has two vertices of degree two then one of them must be a^2 or e^2.

Types of regions are examined in two sections depending whether they contain a vertex labeled a^2 or e^2 or not.

5.2.1 Δ does not contain the label a^2 or e^2

If Δ is an interior region of positive curvature with at least one vertex of degree two but not a^2 or e^2 then it will be one of regions B_1-B_4 in Figure 5-11. Recall that we assume that $2 = |a| \leq |e|$ since the cases for which $2 = |e| < |a|$ are

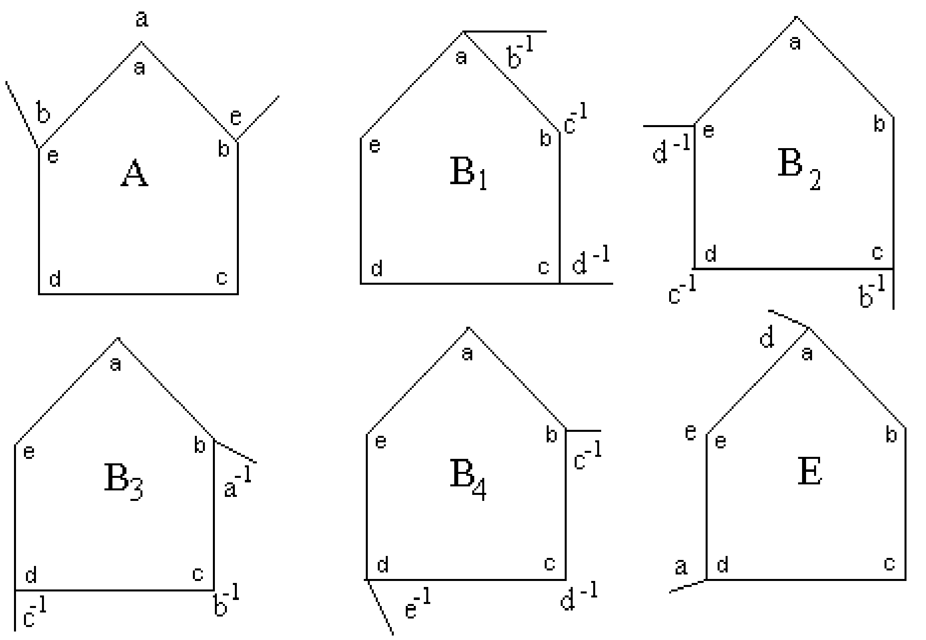


Figure 5-11:

equivalent to these modulo T_1. We examine each case in turn.

$\mathbf{B}_1$

Now the relators holding in the group are $a^2 = 1$ and $c = b = 1$. If also $e^2 = 1$ then none of the vertices can have degree three and so $c(\Delta) \leq c(2,4,4,4,4) = 0$. Recall it can be assumed by Lemma 5.3 that a^2 does not appear as proper sublabel anywhere in D. If $|v_1| = 3$ then $l_1 \in \{ab^{-1}c, ab^{-1}d\}$ and so $a = d$. Also if $|v_3| = 3$ then $l_3 = d^{-1}ca^{\pm 1}$ and then $d = a$. But in the case $b = c = 1$ and $d = a$ by Lemma 5.2(i) $r_1(t) = 1$ has a solution, so it can be assumed that $|v_1|, |v_3| \geq 4$. So at least one of v_4, v_5 must have degree three otherwise the region cannot have positive curvature as $c(\Delta) \leq c(2,4,4,4,4) = 0$. If $|v_5| = 3$ then $l_5 \in \{e^3, ebd^{-1}, ecd^{-1}, ebd^{-1}, ecd^{-1}\}$. Observe that if $l_5 = ebd^{-1}$ then a^2 is proper sublabel on v_1 which we do not allow. So we look at the form of region Δ for each of the other possible labels of v_5 in turn:

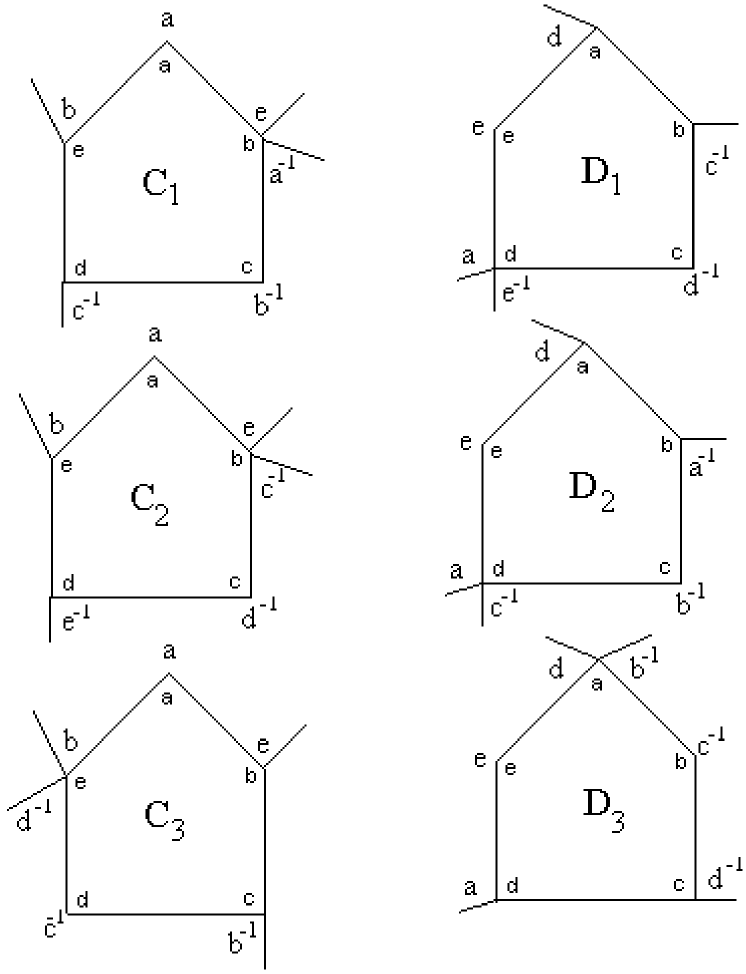


Figure 5-12:

$l_5 = e^3$ Now there must be at least two vertices of degree four otherwise $c(\Delta) \leq c(2,3,4,5,5) < 0$. If $|v_4| = 3$ then $l_4 \in \{dab^{-1}, dac^{-1}\}$ and so $da = 1$ and by lemma 1(i) $r_1(t) = 1$ would have a solution. If $|v_4| = 4$ then $l_4 \in \{dab^{-1}e^{\pm 1}, dac^{-1}e^{\pm 1}, dad^{-1}e^{\pm 1}\}$. The first two vertices are possible and make $d = e^{\pm 1}a$ while the last one makes $e^2 = 1$ and therefore, it is not possible. $|v_3| = 4$ then $l_3 \in \{d^{-1}cb^{-1}c, d^{-1}cd^{-1}b, d^{-1}cd^{-1}c\}$. The first label forces $d = 1$, a contradiction while the last two are possible and force $d^2 = 1$. If $|v_1| = 4$ then $l_1 = dab^{-1}e^{\pm 1}$ and $d = e^{\pm 1}a$. It can be seen that $|v_3| = |v_1| = 4$ or $|v_3| = |v_4| = 4$

make the group to have property X and we are done, so it must be $|v_1| = |v_4| = 4$ and $|v_3| = 5$.

If $l_1 = dab^{-1}e^{-1}$ then $l_4 \in \{dab^{-1}e^{-1}, dac^{-1}e^{-1}\}$ otherwise the degree of e is forced to be two which is not possible. So $l_3 \in \left\{ \begin{array}{c} d^{-1}cd^{-1}ba^{\pm1}, d^{-1}cd^{-1}ca^{\pm1}, \\ d^{-1}cd^{-1}e^{\pm1}b, d^{-1}cd^{-1}e^{\pm1} \end{array} \right\}$. Any of these vertices makes the group cyclic.

If $l_1 = dab^{-1}e$ then $l_4 \in \{dab^{-1}e, dac^{-1}e\}$ for the same reason as before. Now $l_3 \in \{d^{-1}cab^{-1}c, d^{-1}cac^{-1}b, d^{-1}cad^{-1}b, d^{-1}cad^{-1}c\}$. Again none of these is possible, so we conclude that if $l_5 = e^3$ no interior region Δ can have positive curvature.

$l_5 = ecd^{-1}$ Now $d = e$. If $|v_1| = 4$ then $l_1 = bab^{-1}e^{\pm1}$ and so $e = a$ which makes the group cyclic, so it must be $|v_1| \geq 5$. If $|v_3| = 4$ then $d^2 = 1$ and so the group has property X. If $|v_3| \geq 5$ then $c(\Delta) \leq c(2,3,4,5,5) < 0$ so v_3 must have degree three. So $l_3 = dc^{-1}e^{-1}$. But now any possible labeling of v_1 for a degree less than eight makes the group have property X and since $c(\Delta) \leq c(2,3,3,5,8) < 0$ no further positive regions Δ are possible for $l_5 = ecd^{-1}$.

$l_5 = edc^{-1}$ Now $d = e^{-1}$. As before it must be $|v_1| \geq 8$ and $|v_3| \geq 4$, then $l_1 = bab^{-1}e^{\pm1}$ and so $e = a$ which makes the group cyclic, so it must be $|v_4| = 3$ and so $l_3 = db^{-1}e$. But now the label of v_3 would also involve letter a and so $|v_3| \geq 8$. Therefore, the curvature of Δ will be $c(\Delta) \leq c(2,3,3,8,8) < 0$.

We conclude that if the degree of v_5 is three then region Δ cannot have positive curvature. So the degree of v_4 must be three and so $l_4 \in \left\{ \begin{array}{c} da^{\pm1}b^{-1}, da^{\pm1}c^{-1}, \\ db^{-1}e^{\pm1}, dc^{-1}e^{\pm1} \end{array} \right\}$. In the case of the first two labels $da = 1$ and therefore the equation has a solution by Lemma 5.2(i). So it must be $d = e^{\pm1}$. In this case the degree of v_1 cannot have degree less than eight because it involves letter a, and any labelling would make the group residually finite. Also the degree of v_3 in any case cannot be less than four so $c(\Delta) \leq c(2,3,4,4,8) < 0$.

It is concluded that there are no regions of positive curvature of B_1 type.

B_2

In this case $c = d$. Again we look for any possible vertices of degree three. $|v_5| = 3$ implies $l_5 \in \{d^{-1}ec, d^{-1}eb\}$ and $d = c = e$ and so the equation would have a solution by Lemma 5.2(ii). So it must be $|v_5| \geq 4$. If $|v_3| = 3$ then $l_3 = cb^{-1}e$ and so $c = e^{-1}$. But in this case the degree of v_2 involves a and any labelling on v_1 and v_2 with length less than eight would make the group to have property X by creating a relator between the two elements. So $c(\Delta) \leq c(2, 3, 3, 8, 8,) < 0$. So it must be $|v_3| \geq 4$. If $|v_2| = 3$ then $l_2 \in \{bc^{-1}e^{-1}, bd^{-1}e^{-1}, ba^{\pm 1}c^{-1}, ba^{\pm 1}d^{-1}\}$. In the first two case the degree of v_1 must be greater or equal to eight and the degree of v_5 equal or greater to four so $c(\Delta) \leq c(2, 3, 4, 4, 8) < 0$. So if $|v_2| = 3$ then $l_2 \in \{ba^{\pm 1}c^{-1}, ba^{\pm 1}d^{-1}\}$ and $d = a = c$. If $|v_1| = 3$ then $l_1 \in \{ab^{-1}c, ab^{-1}d, ac^{-1}b, ad^{-1}b\}$ and in any case $a = c = d$. In the case that $l_2 = ba^{-1}c^{-1}$ and $l_1 = dab^{-1}$ by Lemma 5.5 there is enough negative curvature to compensate for any possible positive curvature.

We look at the remaining cases. We check for the possible labels for or v_3 having degree three.

First let $l_2 \in \{bac^{-1}, bad^{-1}\}$. Now $l_3 = ecb^{-1}w$. If the degree of v_3 if four or five its label will be $l_3 \in \{ecb^{-1}e, ecb^{-1}e^2, ecb^{-1}cb^{-1}, ecb^{-1}cd^{-1}, ecb^{-1}db^{-1}, ecb^{-1}dc^{-1}\}$. Any of these labels forces the group to be cyclic and therefore it must be $|v_3| \geq 6$. Also $|v_5| \in \{4, 5\}$ implies $l_5 \in \left\{ \begin{array}{c} d^{-1}e^2c, d^{-1}e^2b, d^{-1}eba^{\pm 1}, d^{-1}eca^{\pm 1}, d^{-1}eda^{\pm 1}, \\ d^{-1}e^3b, d^{-1}e^3c, d^{-1}e^2ba^{\pm 1}, d^{-1}e^2ca^{\pm 1}, d^{-1}e^2da^{\pm 1} \end{array} \right\}$. Any of these labels makes the group to have property X except $l_5 = d^{-1}e^3c$. In this case it is not possible for the degree of v_3 to be six or seven so $c(\Delta) \leq c(2, 3, 3, 5, 7) < 0$.

If $l_2 \in \{ba^{-1}c^{-1}, bad^{-1}\}$ then as before the only possible label for v_5 is $l_5 = d^{-1}e^3c$. If $|v_3|$ and $|v_5|$ are greater than six the region cannot have positive curvature, so at least one of them has degree four or five. If $|v_3| \in \{4, 5\}$

then its label becomes $l_3 \in \left\{ \begin{array}{c} b^{-1}cb^{-1}c, b^{-1}cb^{-1}d, b^{-1}cb^{-1}ca^{\pm 1}, \\ b^{-1}cb^{-1}da^{\pm 1}, b^{-1}cb^{-1}e^{\pm 1}c, b^{-1}cb^{-1}e^{\pm 1}d \end{array} \right\}$. Out of these the only possible ones are $l_3 \in \{b^{-1}cb^{-1}c, b^{-1}cb^{-1}d\}$ while the rest force a contradiction. It can be seen that if $|\upsilon_1| \neq 3$ then $|\upsilon_1| \geq 5$ and then $c(\Delta) \leq c(2,3,4,5,5) < 0$ so the degree of υ_1 must always be three. Since we have excluded the cases for which $l_2 = ba^{-1}c^{-1}$ and $l_1 = dab^{-1}$ we are left with two specific cases (i) $l_1 = bac^{-1}$, $l_2 = d^{-1}ba^{-1}$, $l_3 \in \{b^{-1}cb^{-1}c, b^{-1}cb^{-1}d\}$ and $l_5 = d^{-1}ecw$ and (ii) $l_1 = cab^{-1}$, $l_2 = c^{-1}ba^{-1}$, $l_3 \in \{b^{-1}cb^{-1}c, b^{-1}cb^{-1}d\}$ and $l_5 = d^{-1}edw$.

(i) Region Δ will be as in Figure 5-13. We look for negative curvature in region Δ_1.

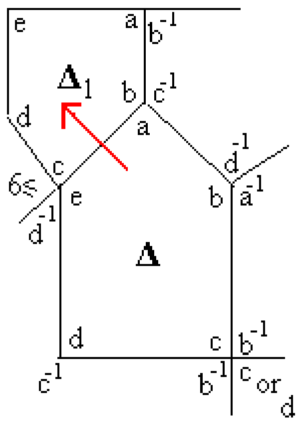


Figure 5-13:

We first check the possible labeling on υ_5. In the case that $|\upsilon_5| = 6$ the labels force the group to be cyclic or residually finite. It is possible that $|\upsilon_5| = 7$ with $l_5 \in \{d^{-1}eca^{\pm 1}b^{-1}e^{-1}c, d^{-1}ecd^{-1}e^2c\}$. In the case of $l_5 = d^{-1}eca^{\pm 1}b^{-1}e^{-1}c$ it is possible to perform a *diamond move* to a create a sublabel ee^{-1} and we can reduce the number of faces, a contradiction (N9). So if $|\upsilon_5| = 7$ then $l_5 = d^{-1}ecd^{-1}e^2c$. We look at the form of Δ_1. In this case the label of $\upsilon_{4\Delta_1}$ is $l_{4\Delta_1} = e^{-1}dw$ and therefore $|\upsilon_{4\Delta_1}| \geq 5$ and so $c(\Delta_1) \leq c(3,3,3,5,7) < -\frac{\pi}{6}$. If $|\upsilon_5| \geq 8$ then $c(\Delta) \leq$

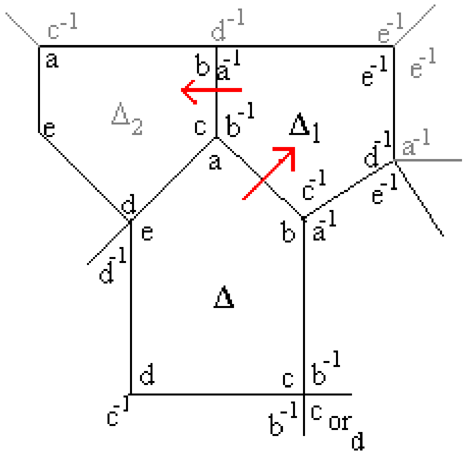


Figure 5-14:

$c(2,3,3,4,8) = \frac{\pi}{12}$. Still Δ_1 has enough negative curvature. If $|v_{4\Delta_1}| \geq 3$ and so $c(\Delta_1) \leq c(3,3,3,3,8) = -\frac{\pi}{12}$. Even in the worse case that $|v_{4\Delta_1}| = 2$ the order of $v_{5\Delta_1}$ is forced to be greater or equal to six and so $c(\Delta_1) \leq c(2,3,3,6,8) = -\frac{\pi}{12}$. It is concluded that Δ_1 always has enough negative curvature to compensate for any positive of Δ and the positive curvature is added across a $b-c$ edge.

(ii) The region appears in Figure 5-14 and we again look for negative curvature in the neighbouring regions Δ_1 and Δ_2. First we check the possible labeling on v_5. It can be seen that if $|v_5| = 6$ then $l_5 = d^{-1}edc^{-1}e^{-1}c$ and in this case it is possible to perform a *diamond move* to decrease the number of faces. If $|v_5| = 7$ the only possible label is $l_5 = d^{-1}eda^{\pm1}b^{-1}e^{-1}c$ and it is again possible to perform *diamond moves* to create a non reduced path. So it can be assumed that $|v_5| \geq 8$ and so $c(\Delta) \leq c(2,3,3,4,8) = \frac{\pi}{12}$.

In the case that at least one of $v_{1\Delta_1}$ and $v_{5\Delta_1}$ does not have degree three, it would have degree at least five and the curvature of Δ_1 would be $c(\Delta_1) \leq c(3,3,3,5,6) = -\frac{4\pi}{15}$. Suppose both of these vertices have degree three. Then it might not be possible to find enough negative curvature in Δ_1 as if $|v_4| = 6$ then $c(\Delta_1) = c(3,3,3,3,6) = 0$. In any case the curvature of Δ_1 cannot be positive.

We look at region Δ_2. The curvature of Δ_2 is $c(\Delta_2) \leq \max\{c(3,3,3,5,8), c(3,3,4,4,8)\} = -\frac{\pi}{3}$. In this case $c(\Delta) + c(\Delta_1) + c(\Delta_2) < 0$. It should be observed that negative curvature is compensated from both Δ_1 and Δ_2 always through a $b - c$ or a $c^{-1} - b^{-1}$ edge.

Now let $|v_1| = 3$ and $|v_2| \geq 4$. If $|v_5| \geq 6$ then $c(\Delta) \leq c(2,3,4,4,6) = 0$. We check for possible labels of degree four and five on v_5. If $l_1 = ab^{-1}c$ then $l_5 = d^{-1}edw$ and $|v_5| \in \{4,5\}$ implies $l_5 \in \{d^{-1}eda^{\pm 1}, d^{-1}edb^{-1}c, d^{-1}edc^{-1}b\}$ but each of these labels forces the group to be cyclic. If $l_1 \in \{ac^{-1}b, ad^{-1}b\}$ then $l_5 = d^{-1}ec$ and again any possible labeling for $|v_5| \in \{4,5\}$ forces the group to be cyclic. If $l_1 = ab^{-1}d$ then it is possible for v_5 to have degree five and $l_5 = d^{-1}e^3c$. Now, it must be $|v_2| = |v_3| = 4$ otherwise $c(\Delta) \leq c(2,3,4,5,5) < 0$. So $l_2 \in \{c^{-1}bc^{-1}b, c^{-1}bd^{-1}b\}$. But for each of these labels if the degree of v_3 is four, a contradiction is forced.

In conclusion all the regions of B_2 type which have positive curvature force the same relators as before i.e. $a = c = d$ and $a^2 = e^k = 1$. There are always regions Δ_1 and Δ_2 such that $c(\Delta) + c(\Delta_1) + c(\Delta_2) \leq 0$ and the positive curvature is always 'given' throw a $b - c$ or a $c^{-1} - b^{-1}$edge.

$\mathbf{B}_3$

Now $b = c = 1$. If $|v_2| = 3$ then $l_2 = ba^{-1}d$ and so $ad = 1$. Also $|v_4| = 3$ implies $l_4 = c^{-1}da^{\pm 1}$ and $ad = 1$. If $|v_1| = 3$ then $l_1 \in \{ab^{-1}d, ac^{-1}d, ad^{-1}b, ad^{-1}c\}$ and so $ad = 1$. But in this case $r_1(t) = 1$ has a solution by Lemma 5.2(i) and so it must be $|v_1|, v_2|, |v_4| \geq 4$. So the degree of and v_5 must be three and so $l_5 \in \{e^3, ecd^{-1}, edb^{-1}, edc^{-1}\}$. In any of the last three case the degree of v_1 and v_2 must be equal to or greater than six and $c(\Delta) \leq c(2,3,3,6,6) = 0$. So if $|v_5| = 3$ then $l_3 = e^3$. Now $l_3 = c^{-1}daw$ and so $|v_3| \geq 5$. So it must be $|v_1| = |v_2| = 4$. This implies $l_2 \in \{ba^{-1}b^{-1}e^{\pm 1}, ba^{-1}c^{-1}e^{\pm 1}, ba^{-1}d^{-1}e^{-1}\}$. The first two cases make the order of e to be two, a contradiction. So it must be $l_2 = ba^{-1}d^{-1}e^{-1}$. But then $l_1 = dad^{-1}e^{\pm 1}$ which again force a contradiction. It is concluded that there are no regions of positive curvature of type B_3.

B$_4$

In this case $c = d$. Again we look for any possible vertices of degree three.

$|v_4| = 3$ implies $l_4 \in \{e^{-1}db^{-1}, e^{-1}dc^{-1}\}$ and either $e = c = d$ and the equation has a solution or $e = 1$, a contradiction, so it must be $|v_4| \geq 4$.

If $|v_2| = 3$ then $l_2 = bc^{-1}e^{-1}$ and $c = d = e^{-1}$. But now the degree of v_1 would be greater than or equal to six so the degree of v_5 must be three otherwise $c(\Delta) \leq c(2,3,4,4,6) = 0$. Therefore, $l_5 \in \{e^3, ecb^{-1}, edb^{-1}\}$. But this forces the label of v_4 to involve letter a and therefore its degree would be greater than five (otherwise a will be a power of e and G will be cyclic) and so $c(\Delta) \leq c(2,3,3,6,6) = 0$. So we have $|v_2| \geq 4$ and therefore at least one of v_1, v_5 must have degree three.

If $|v_5| = 3$ then $l_5 \in \{ecb^{-1}, edb^{-1}, e^3\}$. First let $l_5 \in \{ecb^{-1}, edb^{-1}\}$. Once again $c = d = e^{-1}$ and the degree of v_4 must be greater than or equal to six so $c(\Delta) \leq c(2,3,4,4,6) = 0$. So if $|v_5| = 3$ then $l_5 = e^3$. If the v_1 also has degree three then $l_1 = dab^{-1}$ and $c = d = a$. If $|v_4| = 4$ then $l_4 = e^{-1}dab^{-1}$ and $d = ea$ and if $|v_4| = 5$ then $l_4 = e^{-1}dab^{-1}e^{-1}$ and $d = e^2a$ and in either case G is cyclic. So if $|v_1| = 3$ then $|v_4| \geq 6$. If the degree of v_2 is also greater than or equal to six then the region cannot have positive curvature. So it must be $|v_2| \in \{4, 5\}$ and so $l_2 \in \{c^{-1}bc^{-1}b, c^{-1}bc^{-1}da^{\pm 1}\}$. In the case that $l_2 = c^{-1}bc^{-1}da^{-1}$ then by Lemma 5.5 there exist regions Δ_1 and Δ_2 next to Δ such that $c(\Delta) + c(\Delta_1) + c(\Delta_2) \leq 0$. So the case is considered for the two possible labels of v_2 and the degree of v_4 being greater than or equal to six but not greater than twelve: (i) $l_1 = dab^{-1}$, $l_2 = c^{-1}bc^{-1}b$, $l_5 = e^3$ and $6 \leq |v_4| \leq 11$ (ii) $l_1 = dab^{-1}$, $l_2 = c^{-1}bc^{-1}da^{\pm 1}$, $l_5 = e^3$ and $|v_4| \in \{6, 7\}$.

(i) The region will be as in Figure 5-15. We look for negative curvature in Δ_1

next to it.

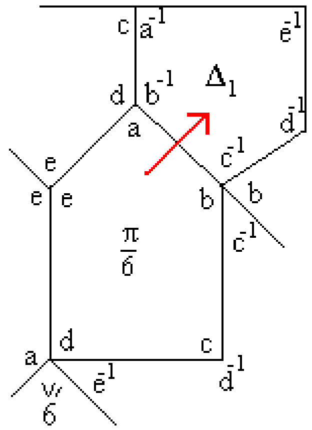


Figure 5-15:

It can be seen that if $|v_{1\Delta_1}| = 3$ then $|v_{5\Delta_1}| \geq 5$ and if $|v_{5\Delta_1}| = 3$ then $|v_{1\Delta_1}| \geq 5$. In the case that $v_{1\Delta_1}$ has degree four or greater then again $c(\Delta) \leq c(3,4,4,5,5)$. So in the case that the degree of $v_{4\Delta_1}$ is not two $c(\Delta_1) \leq$ max $\{c(3,3,4,5,5), c(3,4,4,4,5)\} < -\frac{\pi}{3}$ which is enough negative curvature to compensate for Δ. So the case when $|v_{4\Delta_1}| = 2$ is further considered. In this case $v_{5\Delta_1}$ has degree five or greater while it is possible for $v_{1\Delta_1}$ to have degree three or five. If both $v_{5\Delta_1}$ and $v_{1\Delta_1}$ have degree greater than or equal to six then $c(\Delta_1) \leq c(2,3,4,6,6) = -\frac{\pi}{6}$ which is enough to compensate for the curvature of Δ. We examine the following remaining cases:

1. $|v_{5\Delta_1}| = 5$ and so $l_{5\Delta_1} = e^{-1}dc^{-1}e^{-2}$

2. $|v_{1\Delta_1}| = 5$ and so $l_{1\Delta_1} = ca^{-1}d^{-1}cb^{-1}$ and $|v_{5\Delta_1}| \geq 6$

3. $|v_{1\Delta_1}| = 3$ and so $l_{1\Delta_1} = ca^{-1}b^{-1}$ and $|v_{5\Delta_1}| \geq 6$

1. $|v_{5\Delta_1}| = 5$ and so $l_{5\Delta_1} = e^{-1}dc^{-1}e^{-2}$

Now $l_{1\Delta_1} = ca^{-1}d^{-1}w$. If $|v_{1\Delta_1}| \geq 8$ then $c(\Delta_1) \leq c(2,3,4,5,8) = -\frac{11\pi}{60}$ which is enough to compensate for the curvature of Δ. The degree of $v_{1\Delta_1}$ cannot be seven, so we examine the case for $|v_{1\Delta_1}| = 5$ and $|v_{1\Delta_1}| = 6$.

If $|v_{1\Delta_1}| = 5$ then $l_{1\Delta_1} = ca^{-1}d^{-1}cb^{-1}$. In this case it is possible to perform *diamond moves* (see Figure 5-16) to create one vertex of degree two, in the same time breaking one (not labeled a^2), like this keeping the total number of vertices of degree two the same. In the new situation the curvature of Δ_1' is also positive

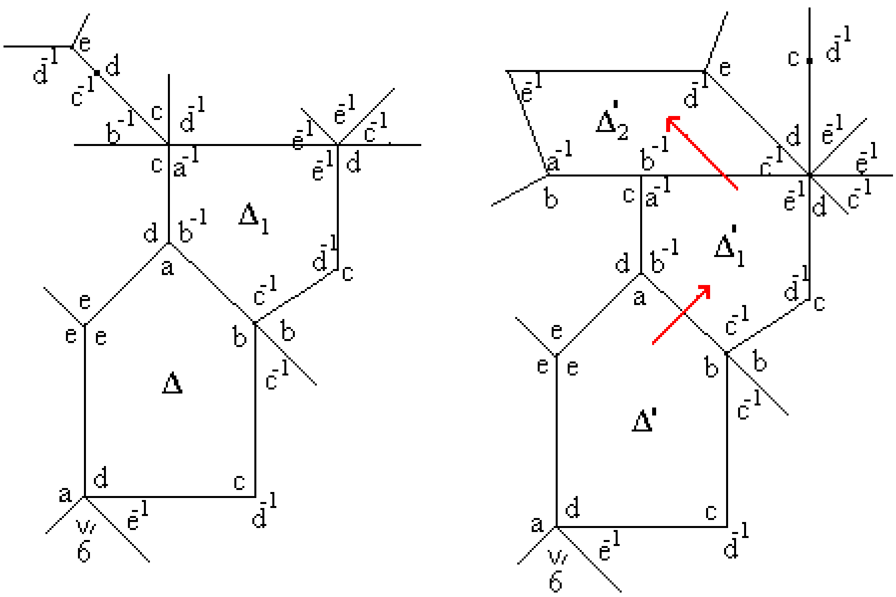


Figure 5-16:

$c(\Delta_1') \leq c(2,3,3,4,7) = \frac{5\pi}{42}$ and so the total positive curvature in the area is $c(\Delta') + c(\Delta_1') = \frac{\pi}{6} + \frac{5\pi}{42} = \frac{2\pi}{7}$ and $c(\Delta_2') \leq c(3,3,3,6,7) = -\frac{8\pi}{21}$. The curvature of Δ_2' is enough to compensate for the total positive curvature in the area with positive curvature added to it across a $c^{-1} - b^{-1}$ edge.

If $|v_{1\Delta_1}| = 6$ then $l_{1\Delta_1} = ca^{-1}d^{-1}ba^{\pm 1}b^{-1}$ and in this case the curvature of Δ_1 is negative but perhaps not enough to compensate for Δ. In this case the positive curvature could be added to the negative curvature of Δ_2 as in Figure 5-17. Now $c(\Delta_1) \leq c(2,3,4,5,6) = -\frac{\pi}{10}$ and $c(\Delta_2) \leq c(2,3,4,5,6) = -\frac{\pi}{10}$ makes

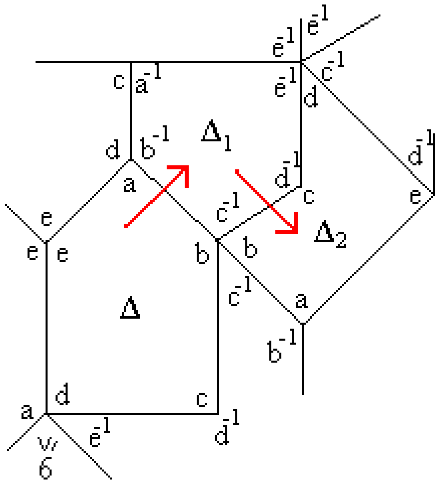


Figure 5-17:

the total negative curvature $-\frac{\pi}{5}$ while the total positive curvature is not greater than $\frac{\pi}{6}$.

2. $|v_{1\Delta_1}| = 5$ and so $l_{1\Delta_1} = cad^{-1}cb^{-1}$ and $|v_{5\Delta_1}| \geq 6$

Now the curvature of Δ_1 is negative but perhaps not enough, $c(\Delta_1) \leq c(2,3,4,5,6) = -\frac{\pi}{10}$. Like in the previous section in Figure 5-16, it is possible to perform *diamond moves* keeping the total number of vertices of degree two the same to create a new situation where $c(\Delta_1') \leq c(2,3,3,4,8) = \frac{\pi}{12}$ and like that making the total positive curvature not greater than $\frac{\pi}{6} + \frac{\pi}{12} = \frac{\pi}{4}$ and the curvature of Δ_2' negative enough to compensate for all, $c(\Delta_2') \leq c(3,3,3,6,8) < -\frac{\pi}{3}$.

3. $|v_{1\Delta_1}| = 3$ and so $l_{1\Delta_1} = ca^{-1}b^{-1}$ and $|v_{5\Delta_1}| \geq 6$

Now the curvature of Δ_1 is $c(\Delta_1) \leq c(2,3,3,4,8) = \frac{\pi}{12}$ and we can have up to total positive curvature of $c(\Delta) + c(\Delta_1) = \frac{\pi}{6} + \frac{\pi}{12} = \frac{\pi}{4}$ and so we have to look for negative curvature in the other neighbouring regions. We look at Δ_2 as in Figure 5-18 and examine the case according to whether Δ_2 contains a vertex of degree two or not.

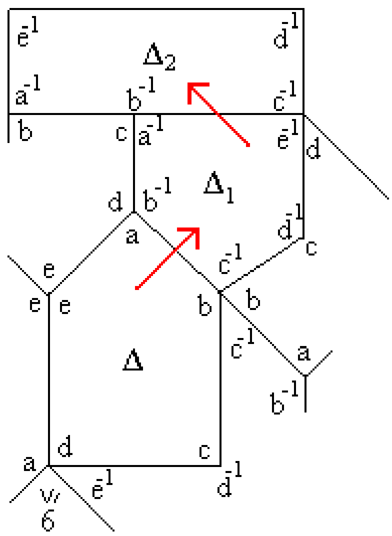


Figure 5-18:

$|v_{4\Delta_2}| \geq 3$ In the case that at least one of the remaining vertices of Δ_2 has degree four or greater then $c(\Delta_2) \leq c(3,3,3,4,8) = -\frac{\pi}{4}$ which is enough to compensate for the total positive curvature. So we assume that $|v_{1\Delta_1}| = |v_{5\Delta_2}| = |v_{4\Delta_2}| = 3$. Now $|v_{3\Delta_2}| \geq 9$ and the regions in the area look as in Figure 5-19.

The curvature of Δ_2 is negative but perhaps not enough to compensate both for the positive curvature of Δ and Δ_1 as now $c(\Delta_1) \leq c(2,3,3,4,9) = \frac{\pi}{18}$ and $c(\Delta_2) \leq c(3,3,3,3,9) = -\frac{\pi}{9}$. We look at region Δ_3. If one of $v_{1\Delta_3}$, $v_{5\Delta_3}$ has degree greater than three then $c(\Delta_3) \leq c(2,3,4,4,9) = -\frac{2\pi}{9}$ and $c(\Delta) + c(\Delta_1) + c(\Delta_2) + c(\Delta_3) \leq 0$. So we take $|v_{1\Delta_3}| = |v_{5\Delta_3}| = 3$. Now the curvature of Δ_3 might also be positive with $c(\Delta_3) \leq c(2,3,3,4,9) = \frac{\pi}{18}$. In such a case we look for negative curvature in Δ_4. If $|v_{1\Delta_4}| \geq 6$ then $c(\Delta_4) \leq c(2,3,4,6,6) = -\frac{\pi}{6}$ and again the total curvature in the area cannot be positive. If $|v_{1\Delta_4}| = 3$ and $l_{1\Delta_4} = ca^{-1}b^{-1}$ then it is possible for the curvature of Δ_4 to be positive. Now $l_{5\Delta_4} = l_{4\Delta} = c^{-1}e^{-1}daw$ and it is possible to find negative curvature in region Δ_5. If the degree of this common vertex is greater than or equal to nine then $c(\Delta) \leq c(2,3,3,4,9) = \frac{\pi}{18}$ and $c(\Delta_4) \leq c(2,3,3,4,9) = \frac{\pi}{18}$

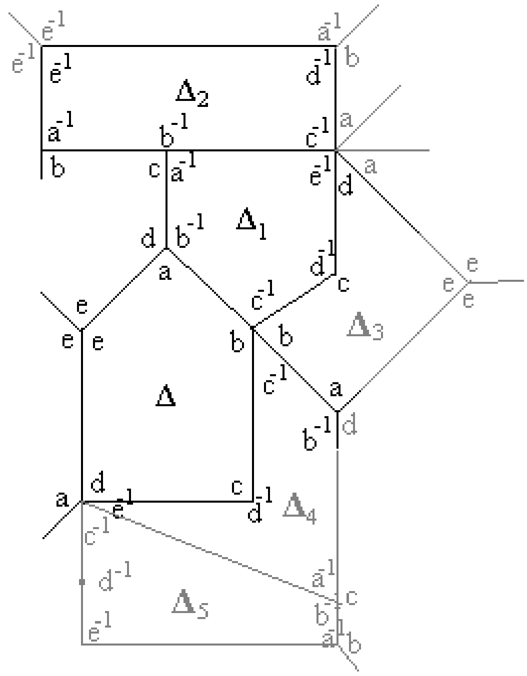


Figure 5-19:

and $c(\Delta_5) \leq \max\{c(2,3,3,6,9), c(3,3,3,3,9)\} = -\frac{\pi}{9}$. If $|v_{4\Delta}| = 8$ and so $l_{4\Delta} = c^{-1}e^{-1}dab^{-1}e^{-1}d$ then *diamond moves* may be performed (without altering the number of faces and keeping the number of vertices with label a^2 and the total number of vertices the same) to create a Δ_4' with curvature $c(\Delta_4') \leq c(2,3,4,6,6) \leq -\frac{\pi}{6}$ and like this the total curvature of the area cannot be positive.

$|v_{4\Delta_2}| = 2$ Once again the curvature of Δ_1 can be positive with $c(\Delta_1) \leq c(c(2,3,3,4,8) = \frac{\pi}{12}$. If $|v_{5\Delta_2}| = 5$ then $l_5 = e^{-1}dc^{-1}e^{-2}$ and it is possible to perform diamond moves to create a situation exactly like the one described above. So we can consider $|v_{5\Delta_2}| \geq 6$ and so $c(\Delta_2) \leq c(2,3,3,6,8) = -\frac{\pi}{12}$. This is negative but not enough to compensate for both the curvature of Δ and Δ_1 so we look for more negative curvature in Δ_3. We use exactly the same process as that used before as shown in Figure 5-20.

Overall Δ has neighbouring regions, so that positive curvature can be added

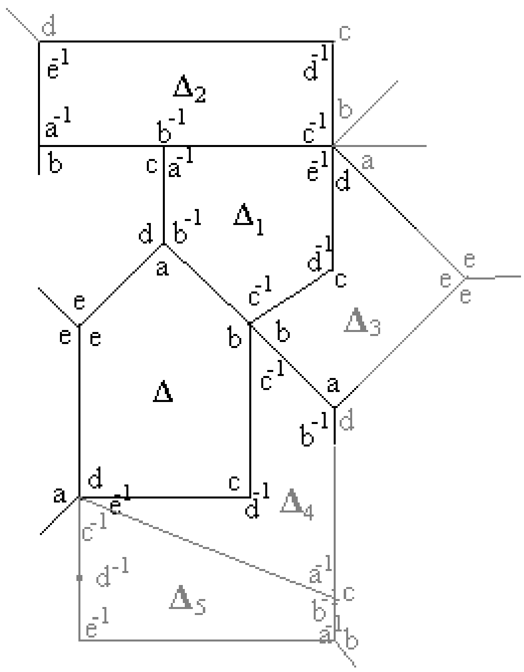


Figure 5-20:

to the negative across $b - c$ or $c^{-1} - b^{-1}$ edges so that the total curvature is not positive.

(ii) In this case region Δ looks as in Figure 5-21 and has curvature of at most $c(\Delta) \leq c(2,3,3,5,6) = \frac{\pi}{15}$. We look for negative curvature in region Δ_1 .

υ_1 and υ_5 of Δ_1 cannot both have degree three at the same time, and if one of them has degree three the other would have degree at least five. So in this case $c(\Delta_1) \leq c(3,3,3,5,5) = -\frac{\pi}{5}$ which is enough to compensate for the positive curvature of Δ.

If $|\upsilon_1| \geq 4$ then their must be at least two vertices of degree four and the remaining one must have degree four or five. If $|\upsilon_1| = 4$ then its label will be $l_4 \in \{dab^{-1}e^{\pm 1}, dac^{-1}e^{\pm 1}, dad^{-1}e^{\pm}\}$. For the last two cases the order of element e is forced to be two and so the group has property X. If $l_4 = dab^{-1}e^{\pm 1}$ then $d = e^{\pm 1}a$ and $|\upsilon_2| \in \{4, 5\}$ implies $l_2 \in \left\{ \begin{array}{c} c^{-1}bc^{-1}b, c^{-1}bc^{-1}d, c^{-1}bc^{-1}ba^{\pm 1}, \\ c^{-1}bc^{-1}da^{\pm 1}, c^{-1}bc^{-1}e^{\pm 1}b, c^{-1}bc^{-1}e^{\pm 1}d \end{array} \right\}$. Each

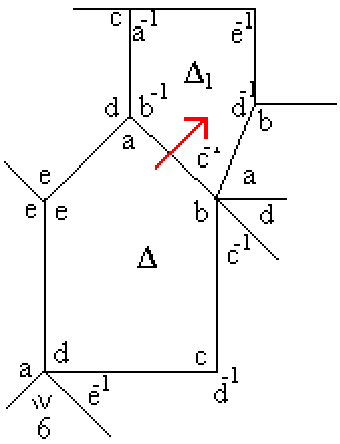


Figure 5-21:

of these labels forces a contradiction, so it must be $|v_2| \geq 6$ and so the curvature of Δ cannot be positive.

Now suppose $|v_5| \geq 4$ and so the only vertex of degree three is v_1. Now $l_1 \in \{ab^{-1}c, ab^{-1}d, ac^{-1}b, ad^{-1}b\}$. We examine each case:

If $l_1 = ab^{-1}c$ then $a = c = d$ and $l_5 = edw$. So $|v_5| \in \{4, 5\}$ implies that its label will be
$$l_5 \in \left\{ \begin{array}{c} edc^{-1}e, edb^{-1}e, eda^{\pm 1}b^{-1}, eda^{\pm}c^{-1}, eda^{\pm 1}d^{-1}, \\ eda^{\pm 1}b^{-1}e, eda^{\pm 1}c^{-1}e, eda^{\pm 1}d^{-1}e, edb^{-1}e^2, edb^{-1}cb^{-1}, \\ edb^{-1}cd^{-1}, edb^{-1}db^{-1}, edb^{-1}dc^{-1}, edc^{-1}e^2, \\ edc^{-1}bc^{-1}, edc^{-1}bd^{-1}, edc^{-1}db^{-1}, edc^{-1}dc^{-1} \end{array} \right\}.$$
Each of these labels forces a contradiction except $l_5 = edc^{-1}e^2$ but in this case *diamond moves* may be performed to increase the number of vertices of degree two (without altering the number of faces or the number of vertices with label a^2), a contradiction to the assumption of their maximality (N3). So $|v_5| \geq 6$ and so the curvature of Δ cannot be positive.

If $l_1 = ab^{-1}d$ then $l_5 = e^2w$ and more precisely the possible labels are $l_5 \in$

$$\left\{\begin{array}{c} e^4, e^2bd^{-1}, e^2bc^{-1}, e^2cd^{-1}, e^2cb^{-1}, e^2db^{-1}, e^2dc^{-1}, \\ e^5, e^3bc^{-1}, e^3bd^{-1}, e^3cb^{-1}, e^3cd^{-1}, e^3db^{-1}, e^3dc^{-1}, \\ e^2ba^{\pm 1}b^{-1}, e^2ba^{\pm 1}c^{-1}, e^2ba^{\pm 1}d^{-1}, e^2ca^{\pm 1}b^{-1}, e^2ca^{\pm 1}c^{-1}, \\ e^2ca^{\pm 1}d^{-1}, e^2da^{\pm 1}b^{-1}, e^2da^{\pm 1}c^{-1}, e^2da^{\pm 1}d^{-1}, e^2cb^{-1}e, \\ e^2cd^{-1}e, e^2db^{-1}e, e^2dc^{-1}e \end{array}\right\}.$$ Out of these labels the only possible ones are $e^4, e^5, e^3cd^{-1}, e^2dc^{-1}e, e^3dc^{-1}$ and $e^2cd^{-1}e$. In the case of the last three labels *diamond moves* may be performed to increase the number of vertices of degree two so $l_5 \in \{e^4, e^5, e^2cd^{-1}e\}$ and the order of e is three, four or five. But in these cases $l_4 = e^{-1}daw$ and cannot have length four or five, so the curvature of Δ cannot be positive.

If $l_1 = ac^{-1}b$ then $l_5 = ecw$ and more precisely for $|v_5| \in \{4, 5\}$, $l_5 = ecd^{-1}e^2$. But in this case $l_4 = e^{-1}daw$ and therefore the degree of v_4 cannot be four, so the region cannot have positive curvature.

If $l_1 = ad^{-1}b$ then the possible labels for v_5 and v_4 are as before, and no region of positive curvature is again possible.

In conclusion when an interior face of positive curvature does not contain the labels a^2 or e^2 then the relators holding in the group are $a = c = d$ and $a^2 = e^k = 1$. For all these regions it is possible to find neigbouring regions of negative curvature so that the total curvature in the area is non-positive. A region of negative curvature which receives this positive curvature always has at most one vertex of degree two and the positive curvature is being received across a $b-c$ or a $c^{-1}-b^{-1}$ edge. The maximum positive curvature a region may receive is $\frac{\pi}{3}$ in the case that all the five vertices of the positive neighbour have degree three.

5.2.2 Δ contains the label a^2 or e^2

Let Δ_1 be an interior region of positive curvature in diagram D having $|v_1| = 2$ with $l_1 = a^2$ or (one of A, C$_1$, C$_2$, C$_3$ of Figures 5-11 and Figure 5-12). We first examine the case where the relators $a = c = d$, $e^k = 1(\ k \geq 3)$ hold since these are case which are possible to arise in a diagram D containing the regions examined in the previous sections (when possible regions of positive curvature exist on which labels a^2 or e^2 do not appear.)

$a = c = d$, $e^k = 1(\ k \geq 3)$

Since $l_1 = a^2$ the labels of v_1 and v_5 begin with eb they must have degree six or greater as any label of shorter length would make the group have property X. So to get a region of positive curvature, at least one of v_2 and v_3 must have degree two otherwise $c(\Delta) \leq c(2, 3, 3, 6, 6) = 0$.

First let $l_3 = cd^{-1}$ and so $l_4 = e^{-1}dw$. This forces the degree of v_4 to be five or greater. If it is greater than or equal to six then $c(\Delta) \leq c(2, 2, 6, 6, 6) = 0$, so the degree of v_4 must actually be five and its label will be $l_4 = e^{-1}dc^{-1}e^{-2}$. But now the label of v_5 will be $l_5 = d^{-1}ebw$ and its degree cannot be six or seven and, therefore, $c(\Delta) \leq c(2, 2, 5, 7, 8) < 0$.

Now, let $l_4 = dc^{-1}$. The label of v_5 is $l_5 = d^{-1}ebw$ so its degree must be eight or greater. Also the degree of v_1 is six or greater. Therefore, the degree of v_3 must be four or less and since it cannot be three -this would make the group cyclic- it is actually four. So $l_3 \in \{cb^{-1}cb^{-1}, cb^{-1}db^{-1}\}$. Region Δ will be as in Figure 5-22 and will have curvature $c(\Delta) \leq c(2, 2, 4, 6, 8) = \frac{\pi}{12}$. We look for negative curvature at its neighbouring region Δ_1.

First suppose that the label of v_3 is $l_3 = cb^{-1}db^{-1}$. At least one of $v_{1\Delta_1}$ and $v_{5\Delta_1}$ has degree four or greater and so $c(\Delta_1) \leq c(2, 3, 4, 4, 8) = -\frac{\pi}{12}$ that is enough negative to compensate for Δ.

Now let $l_3 = cb^{-1}db^{-1}$. It is possible for both $v_{1\Delta_1}$ and $v_{5\Delta_1}$ to have degree with labels $l_{5\Delta_1} = e^{-3}$ and $l_{1\Delta_1} = a^{-1}d^{-1}b$. In this case the curvature of Δ_1 is also positive with $c(\Delta_1) \leq c(2, 3, 3, 4, 8) = \frac{\pi}{12}$. So the total positive curvature is added to $c(\Delta) + c(\Delta_1) = \frac{\pi}{12} + \frac{\pi}{12} = \frac{\pi}{6}$ and we have to look for negative curvature

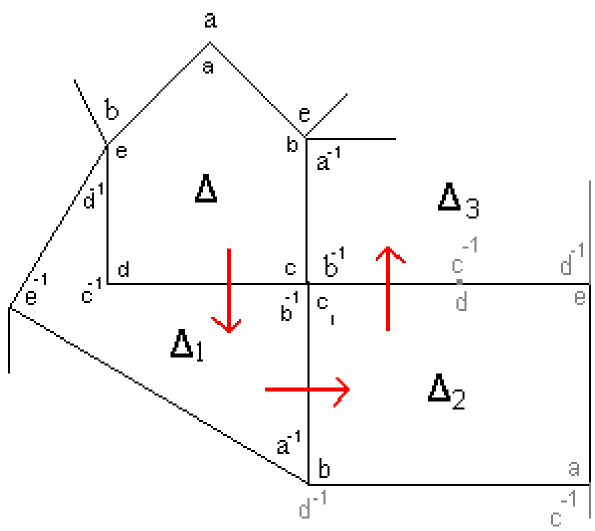


Figure 5-22:

further into Δ_2. Now $v_{2\Delta_2}$ must have degree two, otherwise a *diamond move* may be performed to increase the number of vertices of degree two without breaking any vertices with label a^2. So the label of $v_{5\Delta_2}$ starts with de^{-1} and therefore, it should have degree five or greater. But the degree of $v_{1\Delta_2}$ can be three so the curvature of Δ_2 may be negative but not enough to compensate for the positive curvature of Δ and Δ_1, or it can even be positive. If $|v_{1\Delta_2}| = 3$ then $|v_{5\Delta_2}|$ will be eight or greater and the curvature of Δ_2 will be $c(\Delta_2) \leq c(2,3,3,4,8) = \frac{\pi}{12}$ so the total positive curvature is now added to $\frac{\pi}{4}$. We look for negative curvature in Δ_3 and since now $|v_{4\Delta_3}| \geq 8$, its curvature will be $c(\Delta_3) \leq c(2,3,4,6,8) = -\frac{\pi}{4}$. If $|v_{1\Delta_2}| \neq 3$ then the curvature of Δ_2 might be negative but not enough to compensate for all the positive curvature. Now, $c(\Delta_2) \leq c(2,3,4,5,5) = -\frac{\pi}{60}$ and we can again look in Δ_3 for more negative curvature. This time $c(\Delta_3) \leq \max\{c(2,3,4,6,6,),c(2,4,4,5,6)\} = -\frac{\pi}{6}$ which is enough.

In conclusion if $a = c = d$ and a region Δ has a vertex with label a^2 there are always regions Δ_1, Δ_2 and Δ_3 such that $c(\Delta) + c(\Delta_1) + c(\Delta_2) + c(\Delta_3) \leq 0$.

During this process of adding the positive curvature to the negative one, the positive curvature is always added across a $b-c$ or $c^{-1}-b^{-1}$ edge, as those in the previous section, so a region of negative curvature can only accept positive curvature from at most one of its neighbours.

So for the rest of this section when we consider faces which contain a vertex with label a^2 or e^2, it can be assumed that $a=c=d$ is not true, as these cases were examined separately above. Let Δ_1 be an interior region which has the label a^2 and consider its neighbouring region Δ_2 with common vertex a^2. It is possible to delete vertex a^2 and the two edges incident to it to obtain a new region F_1 with this six vertices as in Figure 5-23.

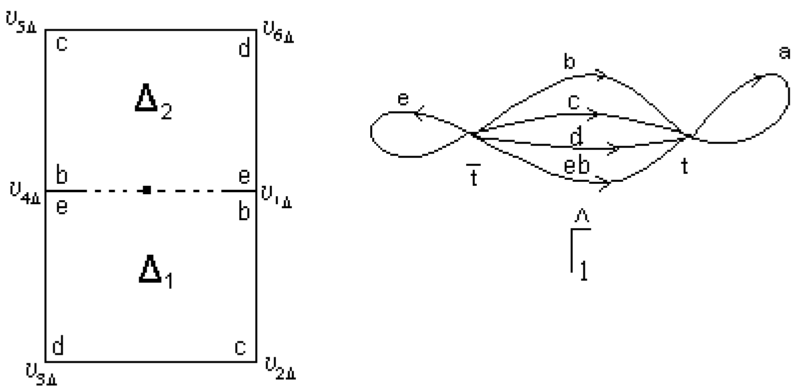


Figure 5-23:

This is achieved by adding the new element eb in the star graph Γ_1 going from $\bar{t}$ to t. We do that for every label a^2 encountered in diagram D. Now the possible paths of length two on new $\hat{\Gamma}$ are $a^2, (eb)d^{-1}$, $(eb)c^{-1}$, dc^{-1}, cb^{-1}, e^2 or their inverses. Since we delete a^2 wherever it is encountered in the diagram vertices of degree two will have one of the labels of degree two. We examine the remaining cases according to whether only a has order two or both a and e have order two.

a has order two and e has order greater than two

If in region Δ where the a^2 was deleted none of the vertices has degree 2 then $c(\Delta) \leq c(3,3,3,3,3,3) = 0$. Since a^2 cannot appear on Δ, it can be assumed that exactly one of the following distinct conditions must be true:

1. $c = e$

2. $d = c$

3. $d = e$ and $c = 1$

4. $d = e$ and $c \neq 1$

5. $d \neq e$ and $c = 1$

We examine the case for each condition holding in group G to see which are the possible types of regions Δ of positive curvature.

$c = e$

The only possible label on vertices of degree two in Δ is $(eb)c^{-1}$ or the inverse so v_1, v_2, v_4, v_5 may have degree two and v_3, v_6 must have degree three or greater. Observe that two neighbouring vertices of Δ cannot both have degree two, so Δ must have at least one and at most two vertices of degree two. We look at the following possible cases:

1. $|v_1| = |v_4| = 2$

2. $|v_1| = |v_5| = 2$

3. $|v_2| = |v_4| = 2$ (This case is by symmetry the same as case 2.)

4. $|v_2| = |v_5| = 2$

5. $|v_1| = 2$ and $|v_2|, |v_4|, |v_5| \geq 3$

6. $|v_4| = 2$ and $|v_1|, |v_2|, |v_5| \geq 3$ (This case is by symmetry the same as case 5.)

7. $|v_2| = 2$ and $|v_1|, |v_4|, |v_5| \geq 3$

8. $|v_5| = 2$ and $|v_1|, |v_2|, |v_4| \geq 3$ (This case is by symmetry the same as case 7.)

1. $|v_1| = |v_4| = 2$ Region Δ will be as in the Figure 5-24.

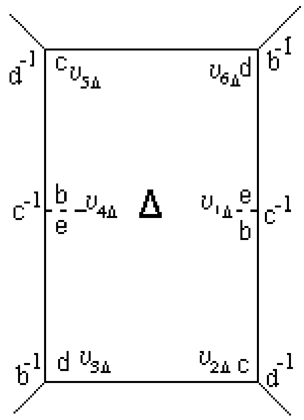


Figure 5-24:

We check for any possible vertices of degree three. If $|v_5| = 3$ then $l_5 = d^{-1}ca^{\pm 1}$ and $d = ea$. In this case $l_6 \in \{b^{-1}db^{-1}w, edb^{-1}w\}$. If $|v_6| = 3$ then $l_6 = edb^{-1}$ and so G is cyclic. If $|v_6| = 4$ then $l_6 \in \left\{ \begin{array}{c} b^{-1}db^{-1}c, b^{-1}db^{-1}d, b^{-1}d(eb)^{-1}c, \\ b^{-1}d(eb)^{-1}d, edb^{-1}e \end{array} \right\}$ but any of these relators makes group G have property X. If $|v_6| = 5$ then

$$l_6 \in \left\{ \begin{array}{c} b^{-1}db^{-1}ec, b^{-1}db^{-1}ed, b^{-1}db^{-1}ca^{\pm 1}, b^{-1}db^{-1}(eb)a^{\pm 1}, b^{-1}db^{-1}da^{\pm 1}, \\ b^{-1}d(eb)^{-1}ca^{\pm 1}, b^{-1}d(eb)^{-1}ba^{\pm 1}, b^{-1}d(eb)^{-1}da^{\pm 1}, edb^{-1}cb^{-1}, \\ edb^{-1}cd^{-1}, edb^{-1}db^{-1}, edb^{-1}dc^{-1}, edb^{-1}(eb)c^{-1}, edb^{-1}(eb)d^{-1}, \\ edb^{-1}e^2, ed(eb)^{-1}bc^{-1}, ed(eb)^{-1}bd^{-1}, ed(eb)^{-1}cd^{-1}, ed(eb)^{-1}cb^{-1}, \\ ed(eb)^{-1}c(eb)^{-1}, ed(eb)^{-1}db^{-1}, ed(eb)^{-1}dc^{-1} \end{array} \right\}.$$ Each of these labels forces a relator that make group G have property X and therefore

it is concluded that if $|v_5| = 3$ then $|v_6| \geq 6$. The same applies for if $|v_2| = 3$ then $|v_3| \geq 6$. Also, if $|v_5| = 3$ and $l_5 = d^{-1}ca^{\pm 1}$, the degree of v_3 cannot be three so in any case the curvature of Δ is $c(\Delta) \leq \max \left\{ \begin{array}{c} c(2,2,3,3,6,6), c(2,2,3,4,4,6), \\ c(2,2,3,4,4,6) \end{array} \right\} = 0$. We conclude that if either $|v_5| = 3$ or $|v_2| = 3$ we cannot have a region of positive curvature.

So either $|v_3| = 3$ and/or $|v_6| = 3$. If $|v_6| = 3$ then $l_6 \in \{db^{-1}e, db^{-1}e^{-1}, d(eb)^{-1}e^{-1}\}$. If $l_6 = db^{-1}e^{-1}$ then $d = c = e$ and the equation has a solution by Lemma 5.2(ii). We check for any possible regions of positive curvature for the remaining two labels.

$l_6 = db^{-1}e$ Now, $d = e^{-1}$ and the label of v_5 is $l_5 = b^{-1}caw$. Since by Lemma 5.3 we do not allow an a to be followed by another, any effort of labelling v_5 with degree 3,4 or 5 makes the group to have property X. Also any effort for a label of length less than 8 creates a relator of the type ae^kae^l where k, $l \in \{\pm 1, \pm 2\}$ and therefore, make the group to have property X. If the order of v_3 is greater than four, then $c(\Delta) \leq c(2,2,3,4,4,8) < 0$. So the order of v_3 must also be three. If $l_3 = db^{-1}e$ then $l_2 = b^{-1}ca$ and so $|v_2| \geq 8$ in the same way as before. So $c(\Delta) \leq c(2,2,3,3,8,8) < 0$. Therefore, the label of v_3 must be $l_3 = d(eb)^{-1}e^{-1}$ and so $e^3 = 1$. Now any effort of labeling v_5 with a label of length 11 or less, would create relator as those before, or a relator of the type $ae^kae^lae^m$ where $k, l, m \in \{\pm 1, \pm 2\}$. The only case that is not true is when we have $l_2 = a^{\pm 1}w_1d^{-1}caw_2$ where $w_1d^{-1}c = w_2 = 1$ in G but by Lemma 5.3 this is not possible. It is therefore, concluded that $c(\Delta) \leq c(2,2,3,3,4,12) = 0$. So no region of positive curvature is actually possible when either $l_3 = db^{-1}e$ or $l_6 = db^{-1}e$.

$l_6 = d(eb)^{-1}e^{-1}$ Now $l_5 = d^{-1}cd^{-1}w$ and it is possible to have $|v_5| \in \{4, 5\}$ or greater. If $|v_5| = 4$ then $l_5 = d^{-1}cd^{-1}b$ and $e^3 = 1$. If $|v_2| = 5$ then $l_5 \in$

$\{d^{-1}cd^{-1}e^{-1}c, d^{-1}cd^{-1}e^{-1}b\}$ and $e^3 = 1$ or $e^4 = 1$ respectively. In this case it is possible to have regions of positive curvature with $c(\Delta) \leq c(2,2,3,3,4,4) = \frac{\pi}{3}$ as in the Figure 5-25.

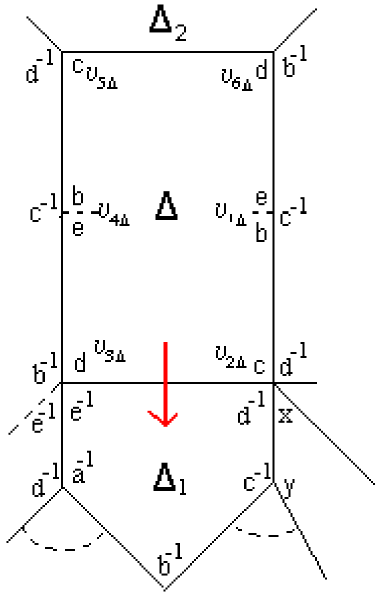


Figure 5-25:

Observe that if $|v_2| \in \{4,5\}$ then $x \in \{e^{-1}, b\}$ and so $y = a^{\pm 1}$ and so $|v_{3\Delta_1}| \geq 6$. Also $|v_{1\Delta_1}| \geq 6$ and since the order of $v_{2\Delta_1}$ cannot be two, (in that case a^{-2} would be a proper sublabel on $v_{1\Delta}$ something we do not allow) the curvature of Δ_1 is $c(\Delta_1) \leq c(3,3,4,6,6) = -\frac{\pi}{2}$. If the order v_2 is greater than or equal to six, then the order of v_6 must be three otherwise the region cannot have positive curvature. And in this case the degree of v_5 must be four or five, otherwise $c(\Delta) \leq c(2,2,3,3,6,6) = 0$. So a Δ_2 exactly with the same labels as with Δ_1 can be found with negative curvature in any case less than $-\frac{\pi}{2}$. In any case there is enough negative curvature so that $c(\Delta) + c(\Delta_1) \leq 0$ or $c(\Delta) + c(\Delta_2) \leq 0$. The positive curvature of Δ is always being added to the negative across an $e^{-1} - d^{-1}$ edge, so a region Δ_1 or Δ_2 could compensate only for one Δ.

The above proves a more general result which is stated in the following

Lemma:

Lemma 5.7 *Let Δ be a region of index six. If $c = e$, $d = e^2$ and the label of v_3 be $l_3 \in \{d(eb)^{-1}e^{-1}, dc^{-1}e^{-1}\}$, $l_2 = d^{-1}cdw$ (or $l_6 \in \{d(eb)^{-1}e^{-1}, dc^{-1}e^{-1}\}$, $l_5 = d^{-1}cdw$) where the first letter of w is b or e^{-1} then there exists a region Δ_1 sharing a $d - e$ edge with Δ with curvature $c(\Delta_1) \leq -\frac{\pi}{2}$.* ■

2. $|v_1| = |v_5| = 2$ Region Δ will be as in Figure 5-26.

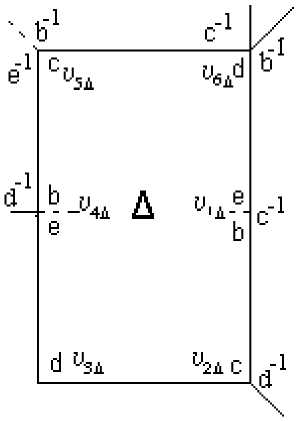


Figure 5-26:

At least one of vertices v_2, v_3, v_4, v_6 must have degree three otherwise the curvature of Δ will be at most $c(\Delta) \leq c(2,2,4,4,4,4) = 0$. Since $l_6 = c^{-1}db^{-1}w$, it is not possible for v_6 to have degree three. We check the possibility of the degrees of v_2, v_3, v_4 being three. $|v_2| = 3$ implies $l_2 = d^{-1}ca^{\pm1}$ and so $d = ea$. $|v_4| = 3$ implies $l_4 = (eb)d^{-1}e$ and so $d = e^2$. Also $|v_3| = 3$ implies $l_3 \in \{da^{\pm1}b^{-1}, da^{\pm1}c^{-1}, da^{\pm1}(eb)^{-1}, db^{-1}e, dc^{-1}e^{-1}, d(eb)^{-1}e^{-1}\}$ and so one of the following is true $d = a$, $d = ea$, $d = e^{-1}$ or $d = e^2$.

Let $|v_2| = 3$ and so $l_2 = d^{-1}ca^{\pm1}$ and $d = ea$. It is obvious that $|v_4| \geq 4$ otherwise G is cyclic. Also $l_3 \in \{b^{-1}dw, edw\}$. If $|v_3| = 3$ then $d = a$ or $d = e^{-1}$ and G

is forced to be cyclic. $|v_3| = 4$ implies $l_3 \in \left\{ \begin{array}{c} b^{-1}db^{-1}c, b^{-1}db^{-1}d, b^{-1}dc^{-1}d, \\ b^{-1}d(eb)^{-1}c, b^{-1}d(eb)^{-1}d, eda^{\pm 1}b^{-1}, \\ eda^{\pm 1}c^{-1}, eda^{\pm 1}d^{-1}, edb^{-1}e, edc^{-1}e \end{array} \right\}$.
Each of these labels forces a contradiction by making group G to have property X. So $|v_3| \geq 5$. Also $|v_6| = 4$ implies $l_4 \in \left\{ \begin{array}{c} c^{-1}db^{-1}(eb), c^{-1}db^{-1}d, c^{-1}dc^{-1}b, \\ c^{-1}dc^{-1}(eb), c^{-1}dc^{-1}d, c^{-1}d(eb)^{-1}d \end{array} \right\}$.
Each of these labels again forces a relation in G that make it to have property X and so $|v_6| \geq 5$. Therefore, the curvature of Δ cannot be positive since $c(\Delta) \leq c(2, 2, 3, 4, 5, 5) < 0$.

Let $|v_4| = 3$ and so $l_4 = (eb)d^{-1}e$ and $d = e^2$. As before $|v_6| \geq 4$ and $|v_2| \geq 4$ and since the label of v_3 now involves letter a any effort of labelling with length less than six would produce a relation which would make the group to have property X. So $c(\Delta) \leq c(2, 2, 3, 4, 4, 6) = 0$.

So it must be $|v_3| = 3$. We check the case for each possible label of v_3.

$l_3 = db^{-1}e$ **and** $d = e^{-1}$ The label of v_4 and v_6 cannot be less than four and the label of v_2 involves letter a so any effort of labeling with degree less than six makes the group have property X. So $c(\Delta) \leq c(2, 2, 3, 4, 4, 6) = 0$.

$l_3 = d(eb)^{-1}e^{-1}$ **and** $d = e^2$ Then $l_2 = d^{-1}cd^{-1}w$ and $|v_2| = 4$ implies $l_2 = d^{-1}cd^{-1}b$ and so $e^3 = 1$. Also $l_6 = c^{-1}db^{-1}w$ and $|v_6| = 4$ implies $l_6 \in \{c^{-1}db^{-1}d, c^{-1}d(eb)^{-1}b\}$. Now, $l_4 = c^{-1}(eb)d^{-1}w$ and so $|v_4| = 4$ implies $l_4 \in \{c^{-1}(eb)d^{-1}b, c^{-1}(eb)d^{-1}(eb)\}$ which forces a contradiction in any case. If $|v_4| \geq 6$ then the curvature of Δ cannot be positive, and so $|v_4| = 5$. So $l_4 \in \{c^{-1}(eb)d^{-1}e^{-1}b, c^{-1}(eb)d^{-1}e(eb)\}$ while the other possible labels force a contradiction. If either v_2 or v_6 have degree greater than four, then $c(\Delta) \leq c(2, 2, 3, 4, 5, 5) < 0$. So it must be $|v_2| = |v_6| = 4$ and $|v_4| = 5$ with the corresponding labels. Now we have $l_2 = d^{-1}cd^{-1}b$ and $l_3 = d(eb)^{-1}e^{-1}$ so there exists a region Δ_1 as in Lemma 5.7. The curvature of Δ is $c(\Delta) = c(2, 2, 3, 4, 4, 5) = \frac{\pi}{15}$ so there is enough negative curvature to compensate it going across a $d^{-1} - e^{-1}$

edge.

$l_3 = dc^{-1}e^{-1}$ **and** $d = e^2$ The label of v_2 is as with previous section and so $|v_2| = 4$ is possible with $l_2 = d^{-1}cd^{-1}b$ and $e^3 = 1$. Also $|v_4| = 4$ is possible with $l_4 = d^{-1}(eb)d^{-1}b$ and $e^3 = 1$. It is also possible for $|v_6|$ to be four with $l_6 \in \{c^{-1}db^{-1}d, c^{-1}d(eb)^{-1}b\}$. We also check for a possible vertex of degree five. $|v_4| = 5$ implies that the label of v_4 is $l_4 \in \{d^{-1}(eb)d^{-1}e^{-1}b, d^{-1}(eb)d^{-1}e^{-1}c\}$. The first label would make the order of e to be four and therefore, the degree of v_2 cannot be four which makes the region to have negative curvature. So if $|v_4| = 5$ then $l_4 = d^{-1}(eb)d^{-1}e^{-1}c$. Checking for $|v_6| = 5$ gives no possible labels. If $|v_2| = 5$ then $l_2 \in \{d^{-1}cd^{-1}e^{-1}b, d^{-1}cd^{-1}e^{-1}c\}$. Again the first label would force the degree of e to be four and therefore $|v_2| \geq 5$ which would make the curvature of Δ non-positive. So if $|v_2| = 5$ then $l_2 = d^{-1}cd^{-1}e^{-1}c$. So the cases left are for when all v_2, v_4, v_6 have degree four or one of them has degree five. But these cases are again covered by Lemma 5.7 since $l_2 \in \{d^{-1}cd^{-1}b, d^{-1}cd^{-1}e^{-1}c\}$, $x \in \{b, e^{-1}\}$ and in any case $c(\Delta) \leq c(2,2,3,4,4,4) = \frac{\pi}{6}$. Therefore, there is a region Δ_1 with curvature $c(\Delta_1) \leq -\frac{\pi}{2}$ which receives the positive curvature across its $e^{-1} - d^{-1}$ edge (see Figure 5-25).

$l_3 = da^{\pm 1}b^{-1}$ **and** $d = a$ Each of the labels of v_2, v_4 and v_6 involves both d and c or e so any effort of labeling them with degree less than five would make the group cyclic. It is concluded than no region of positive curvature is possible here.

$l_3 = da(eb)^{-1}$ **and** $d = ea$ Now $|v_2| = 4$ implies $l_2 \in \{d^{-1}cd^{-1}c, d^{-1}cd^{-1}(eb)\}$. Also $|v_6| = 4$ implies $l_6 = c^{-1}d(eb)^{-1}d$. But $|v_4| = 4$ implies $l_4 = e(eb)d^{-1}e$ which makes the group cyclic so it must be $|v_2| = |v_6| = 4$ and $|v_4| = 5$. Now $|v_4| = 5$ implies $l_4 \in \left\{ \begin{array}{c} e(eb)d^{-1}bc^{-1}, e(eb)d^{-1}bd^{-1}, e(eb)d^{-1}cb^{-1}, \\ e(eb)d^{-1}cd^{-1}, e(eb)d^{-1}(eb)c^{-1}, e(eb)d^{-1}(eb)d^{-1} \end{array} \right\}$. Each of these labels forces a relator that make group G have property X, therefore $|v_4| \geq 6$ and Δ cannot have positive curvature.

$l_3 = da^{-1}(eb)^{-1}$ **and** $d = ea$ As above $|v_2| = 4$ implies $l_2 \in \{d^{-1}cd^{-1}c, d^{-1}cd^{-1}(eb)\}$ and $|v_6| = 4$ implies $l_6 = c^{-1}d(eb)^{-1}d$. Now $|v_4| = 4$ implies $l_4 = b^{-1}(eb)d^{-1}c$ which again makes the group cyclic so it must be $|v_2| = |v_6| = 4$ and $|v_4| = 5$. Now $|v_4| = 5$ implies $l_4 \in \left\{ \begin{array}{c} b^{-1}(eb)d^{-1}e^{\pm 1}c, b^{-1}(eb)d^{-1}e^{\pm 1}d, b^{-1}(eb)d^{-1}ba^{\pm 1}, \\ b^{-1}(eb)d^{-1}ca^{\pm 1}, b^{-1}(eb)d^{-1}(eb)a \end{array} \right\}$. Once more each of these labels forces a contradiction either by making group G have property X. So no regions of positive curvature is possible here.

$l_3 = da^{\pm 1}c^{-1}$ **and** $d = ea$ Now $|v_2| = 4$ implies $l_2 \in \{d^{-1}cb^{-1}(eb), d^{-1}cb^{-1}c\}$ which make the group cyclic in either case. Also the labels of v_4 and v_6 will be as with the previous section so it is concluded that no region of positive curvature is possible here.

4. $|v_2| = |v_5| = 2$ A possible region Δ of positive curvature would be as in Figure 5-27.

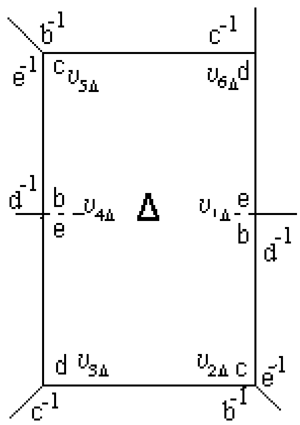


Figure 5-27:

We check the possibility of any vertex having degree 3. If $|v_1| = 3$ then $l_1 = (eb)d^{-1}e$ and $d = e^2$. But now $l_6 = c^{-1}daw$ and any effort of finding an appropriate label for $|v_6| \in \{3, 4, 5\}$ would make the group cyclic. So if $|v_1| = 3$ then $|v_6| \geq 6$. The same applies for $|v_4| = 3$, i.e. this would force $|v_3| \geq 6$. If $|v_6| = 3$ then $l_6 = c^{-1}da^{\pm 1}$ and $d = ea$ while $l_1 \in \{e(eb)d^{-1}w, b^{-1}(eb)d^{-1}w\}$.$|v_1| = 3$ is not possible while $|v_1| = 4$ implies $l_1 \in \{e(eb)d^{-1}e, b^{-1}(eb)d^{-1}c\}$ which make the group cyclic. Also if the degree of $v_1$is five then its label will be $l_1 \in \left\{\begin{matrix} e(eb)d^{-1}bc^{-1}, e(eb)d^{-1}bd^{-1}, e(eb)d^{-1}cb^{-1}, e(eb)d^{-1}cd^{-1}, e(eb)d^{-1}(eb)c^{-1}, \\ e(eb)d^{-1}(eb)d^{-1}, b^{-1}(eb)d^{-1}e^{\pm 1}c, b^{-1}(eb)d^{-1}e^{\pm 1}d, b^{-1}(eb)d^{-1}ba^{\pm 1}, \\ b^{-1}(eb)d^{-1}ca^{\pm 1}, b^{-1}(eb)d^{-1}(eb)a \end{matrix}\right\}$ which in any case make the group have property X. It is therefore, concluded that if $|v_6| = 3$ then $|v_1| \geq 6$ and the same applies for if $|v_3| = 3$ then $|v_4| \geq 6$. Once again, the curvature of Δ cannot be positive.

5. $|v_1| = 2$ **and** $|v_2|, |v_4|, |v_5| \geq 3$ Now we must have at least four of $|v_2|$, $|v_3|$, $|v_4|$, $|v_5|$, $|v_6|$ equal to three otherwise $c(\Delta) \leq c(2, 3, 3, 3, 4, 4) = 0$. $|v_2| = 3$ implies $l_2 = d^{-1}ca^{\pm 1}$ and $d = ea$. $|v_6| = 3$ implies $l_6 = db^{-1}e$ and $d = e^{-1}$ so $|v_2| = |v_6| = 3$ is not possible. Also if $|v_6| = 3$ then $l_5 = caw$ and since letter a is involved any effort of labelling v_6 with degree less than six would make the group to have property X. Therefore, if $|v_6| = 3$ then $c(\Delta) \leq c(2, 3, 3, 3, 3, 6) = 0$. So it must be $|v_2| = 3$, $l_2 = d^{-1}ca^{\pm 1}$ and $d = ea$. But then $|v_3| = 3$ makes $l_3 \in \{edb, ed(eb)^{-1}, edc^{-1}, b^{-1}da^{\pm 1}\}$ which in any case makes G cyclic. So if $|v_2| = 3$ then $|v_3|, |v_6| \geq 4$ and $c(\Delta) \leq c(2, 3, 3, 3, 4, 4) = 0$.

7. $|v_2| = 2$ **and** $|v_1|, |v_4|, |v_5| \geq 3$ Now $|v_3| = 3$ implies $l_3 = c^{-1}da^{\pm 1}$ and $d = ea$. $|v_1| = 3$ implies $l_1 = (eb)d^{-1}e$ and $d = e^2$. So v_3 and v_1 cannot have degree three at the same time. If $|v_1| = 3$ and so $l_1 = (eb)d^{-1}e$ and $d = e^2$ then $l_6 = daw$ and any effort of finding a label of degree less than six fails. So it must be $|v_3| = 3$, $l_3 = c^{-1}da^{\pm 1}$ and $d = ea$. So $|v_4| = 3$ implies

$l_4 \in \{b^{-1}(eb)a^{\pm 1}, e(eb)c^{-1}, e(eb)d^{-1}\}$ which again make the group cyclic. It is therefore concluded that no labeling is possible here to make $c(\Delta)$ positive.

$c = d$

The only possible label on vertices of degree two on Δ is cd^{-1} or the inverse so v_2, v_3, v_5, v_6 may have degree two and v_1, v_4 must have degree three or greater. Again two neighbouring vertices of Δ cannot both have degree two, so Δ may have at most two vertices of degree two. We look at the following possible cases:

1. $|v_2| = |v_5| = 2$

2. $|v_3| = |v_6| = 2$

3. $|v_2| = |v_6| = 2$

4. $v_3| = |v_5| = 2$ (This case is by symmetry the same as case 3)

5. $|v_2| = 2$ and $|v_3|, |v_5|, |v_6| \geq 3$

6. $|v_5| = 2$ and $|v_2|, |v_3|, |v_6| \geq 3$ (This case is by symmetry the same as case 5)

7. $|v_3| = 2$ and $|v_2|, |v_5|, |v_6| \geq 3$

8. $|v_6| = 2$ and $|v_2|, |v_3|, |v_5| \geq 3$ (This case is by symmetry the same as case 7)

1. $|v_2| = |v_5| = 2$ A region Δ of possible positive curvature will be as in Figure 5-28.

If $|v_4| = 3$ then $l_4 = (eb)c^{-1}e$ and so $d = c = e^2$. But then $l_3 = e^{-1}daw$ and any effort of labelling v_4 with a label of length less than six makes the group cyclic. So if $|v_4| = 3$ then $|v_3| \geq 6$. In the same way if $|v_1| = 3$ then $l_1 = (eb)c^{-1}e$ and $|v_6| \geq 6$. If $|v_3| = 3$ then $l_3 \in \{e^{-1}db^{-1}, e^{-1}dc^{-1}, e^{-1}d(eb)^{-1}\}$. The first two

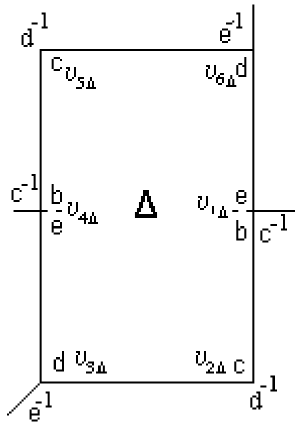


Figure 5-28:

force a contradiction by making $e = d = c$ (and therefore the equation to have a solution by Lemma5.2(i)) or $e = 1$ therefore, the third label is the only possible so $l_3 = e^{-1}d(eb)^{-1}$. Now it is possible for the degree of υ_4 to be four, five or greater than or equal to six. $|\upsilon_4| = 4$ implies $l_4 = c^{-1}(eb)c^{-1}b$ and $e^3 = 1$. $|\upsilon_4| = 5$ implies $l_4 \in \{c^{-1}(eb)c^{-1}e^{-1}b, c^{-1}(eb)c^{-1}ed\}$. In the second case a *diamond move* may be performed to increase the number of vertices of degree two with label $c^{-1}d$, without decreasing the number of vertices with label a^2, so $l_4 = c^{-1}(eb)c^{-1}e^{-1}b$ and $e^4 = 1$. So region Δ must have at least one and at most two vertices of degree three.

If $|\upsilon_1| = |\upsilon_4| = 3$ then $|\upsilon_6|, |\upsilon_3| \geq 6$ and so the curvature of Δ cannot be positive. If $|\upsilon_1| = |\upsilon_3| = 3$ then $|\upsilon_6| \geq 6$. So the degree of υ_4 must be four or five otherwise $c(\Delta) \leq c(2,2,3,3,6,6) = 0$. But $l_6 = e^{-1}daw$ and in the case $e^3 = 1$, $|\upsilon_6| \geq 12$. Also if $e^4 = 1$ then $|\upsilon_6| \geq 8$ and so in any case $c(\Delta) \leq \max\{c(2,2,3,3,4,12), c(2,2,3,3,5,8)\} = 0$. The same exactly applies if $|\upsilon_4| = |\upsilon_6| = 3$.

So if two vertices in Δ have degree three these must be υ_3 and υ_6. If both the remaining two vertices have degree equal to or greater than six then Δ cannot have positive curvature. So at least one of l_1, $l_4 \in \{c^{-1}(eb)c^{-1}b, c^{-1}(eb)c^{-1}e^{-1}b\}$ and $e^3 = 1$ or $e^4 = 1$ respectively. As seen in Figure 5-29 it is then possible to find a region Δ_1 of negative curvature next to Δ.

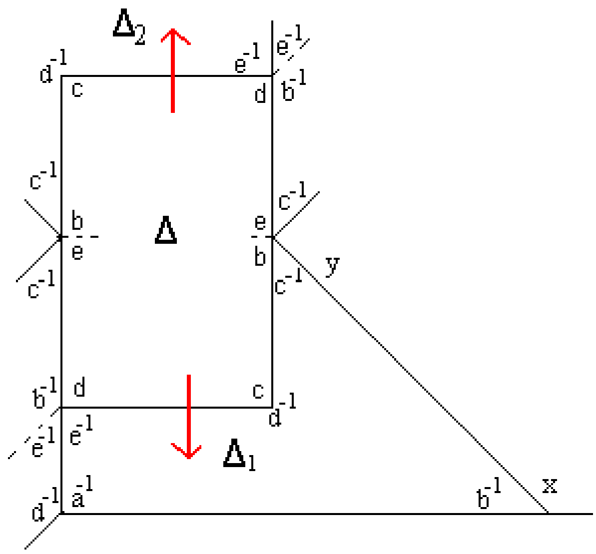


Figure 5-29:

If $y \in \{b, e^{-1}\}$ then $x = a^{\pm 1}$ and so the degree of $v_{2\Delta_1}$ is in greater than or equal to eight. Also $l_{1\Delta_1} = a^{-1}d^{-1}w$ and so the degree of $v_{1\Delta_1}$ is in any case greater than or equal to eight therefore, $c(\Delta_1) \leq c(2, 3, 4, 8, 8) = -\frac{\pi}{3}$. If $y \notin \{b, e^{-1}\}$ then the degree of v_1 cannot be four or five and therefore the degree of v_4 must be four or five. In that case it is possible to find a region Δ_2 as in the Figure 5-29 with $c(\Delta_2) \leq -\frac{\pi}{3}$. In any case the negative curvature of Δ_1 or Δ_2 is enough to compensate for the curvature of Δ which $c(\Delta) \leq c(2, 2, 3, 3, 4, 4) = \frac{\pi}{3}$. The positive curvature is added to Δ_1 across a $d^{-1} - e^{-1}$ edge, so a region a region of negative curvature Δ_1 or Δ_2 can only be given positive curvature from at most one of its neighbours.

If the only vertex of degree three in Δ is v_1 then as before $|v_6| \geq 6$ and so $c(\Delta) \leq c(2, 2, 3, 4, 4, 6) = 0$. So if there is only one vertex of degree 3 in Δ, this must be either v_3 or v_6. Let $|v_3| = 3$ and so $l_3 = e^{-1}d(eb)^{-1}$ and so

$l_4 = c^{-1}(eb)c^{-1}w$. Now at least two of υ_1, υ_4, υ_6 must have degree four.$|\upsilon_6| = 4$ is not possible so it must be $|\upsilon_1| = |\upsilon_4| = 4$. So $l_4 = c^{-1}(eb)c^{-1}b$ and $e^3 = 1$. Also $l_1 \in \{(eb)c^{-1}bc^{-1}, (eb)c^{-1}bd^{-1}\}$. The curvature of Δ is $c(\Delta) = c(2, 2, 3, 4, 4, 5) = \frac{\pi}{15}$. As in Figure 5-28 $y = b$ so it possible to find a region Δ_1 next to Δ with $c(\Delta_1) \leq -\frac{\pi}{3}$. The situation when υ_6 is the only region of degree three in Δ is analogous and in that case again it is possible to find a Δ_2 with enough negative curvature to compensate for the positive curvature of Δ. This can be generalised in the following Lemma.

Lemma 5.8 *Let Δ be a region of degree six and $c = d = e^2$. If $l_1 = c^{-1}(eb)cw$, $l_2 = cd^{-1}$ and $l_3 = e^{-1}d(eb)^{-1}e^{-1}$ (or $l_4 = c^{-1}(eb)cw$, $l_5 = cd^{-1}$ and $l_6 = e^{-1}d(eb)^{-1}e^{-1}$) where y is b or e^{-1} then there exists a Δ_1 adjacent to Δ with curvature $c(\Delta_1) \leq -\frac{\pi}{3}$.* ■

2. $|\upsilon_3| = |\upsilon_6| = 2$ A region Δ of possible positive curvature will be as in Figure 5-30.

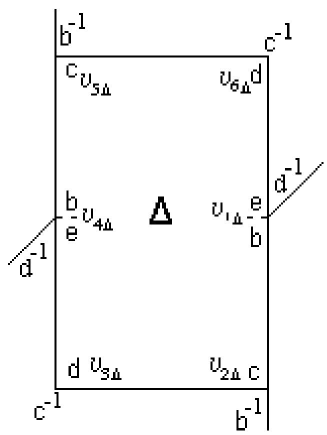


Figure 5-30:

We check for the possibility of any vertex having degree 3.

If $|v_1| = 3$ then $l_1 = d^{-1}(eb)a^{\pm 1}$ and so $d = ea$. In that case $l_2 \in \{ecb^{-1}w, b^{-1}cb^{-1}w\}$. Any effort of finding possible labels for $|v_2| \in \{3, 4, 5\}$ makes the group have property X, so it is concluded that if $|v_1| = 3$ then $|v_2| \geq 6$. Also since $d = ea$ the degree of v_5 cannot be three as this would make the group is cyclic so it is concluded that in the case $|v_1| = 3$, $c(\Delta) \leq \max\{c(2, 2, 3, 3, 6, 6), c(2, 2, 3, 4, 4, 6)\} = 0$. The same applies if $|v_4| = 3$ and so if any of v_1 or v_3 has degree three we cannot get a region of positive curvature.

So $|v_2| = 3$ and/or $|v_5| = 3$. If $|v_2| = 3$ then $l_3 \in \{c(eb)^{-1}e^{-1}, cb^{-1}e\}$ and $c = d = e^2$ or $c = d = e^{-1}$ respectively. If $|v_2| = 3$ and $l_3 = cb^{-1}e$ then $l_1 = d^{-1}(eb)aw$ and any effort of labelling v_1 with degree less than eight makes the group have property X. Since the degree of v_4 cannot be four in this case, the degree of v_5 must be three. If $l_3 = cb^{-1}e$ then the degree of v_4 would be greater than or equal to eight and the curvature of Δ cannot be positive. Also in the case $l_3 = c(eb)^{-1}e^{-1}$ makes $e^3 = 1$ and now any effort at labelling v_1 with degrees less than twelve, makes the group have property X or allows *diamond moves* to be performed in order to increase the number of vertices of degree two or to create a non reduced word. It is concluded than if $|v_2| = 3$ then $l_2 = c(eb)^{-1}e^{-1}$ and the same applies for v_6. Let $l_2 = c(eb)^{-1}e^{-1}$ and the case for $l_4 = c(eb)^{-1}e^{-1}$ is analogous. We check for possible labels of on v_1. The label of v_1 is now $l_1 = d^{-1}(eb)d^{-1}w$. If $|v_1| = 4$ then $l_1 = d^{-1}(eb)d^{-1}b$ and so $e^3 = 1$. If $|v_1| = 5$ then $l_1 \in \{d^{-1}(eb)d^{-1}e^{-1}b, d^{-1}(eb)d^{-1}ec\}$ and in the first case $e^4 = 1$ while in the second case no further relators emerge. We look at region Δ_1 next to Δ as in Figure 5-31.

In the case that $y \in \{b, e^{-1}\}$, then $x = a^{\pm 1}$ and so $|v_{3\Delta_1}| \geq 6$. Also $|v_{1\Delta_1}| \geq 6$ and so $c(\Delta_1) \leq c(3, 3, 4, 6, 6) = -\frac{\pi}{2}$. This is enough negative curvature to compensate for the curvature of Δ which in any case cannot be larger than $c(2, 2, 3, 3, 4, 4) = \frac{\pi}{3}$. If the degree of v_1 is greater than or equal to six then the degree of v_5 must definitely be three and the degree of v_4 four or five. In the cases that the labels of v_1 and/or v_4 is $d^{-1}(eb)d^{-1}b$ or $d^{-1}(eb)d^{-1}e^{-1}b$ then $x = a^{\pm 1}$ on

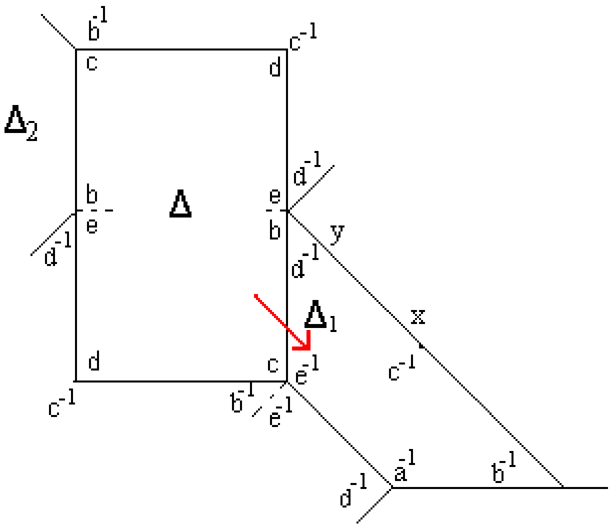


Figure 5-31:

Δ_1 or a Δ_2 which can be found next to Δ. So with enough negative curvature to compensate for any positive curvature of Δ since$c(\Delta_1$ or $\Delta_2) \leq c(3,3,4,6,6) = -\frac{\pi}{2}$.

Therefore, the only cases which remain to be examined is when l_1 and/or l_4 is $d^{-1}(eb)d^{-1}ec$ and an $x \neq a^{\pm 1}$ cannot be found on Δ_1 or Δ_2. Suppose $l_1 = d^{-1}(eb)d^{-1}ec$. If the order of v_4 is greater than three then Δ will not have positive curvature unless $|v_4| = |v_5| = 4$. In this case $c(\Delta) = c(2,2,3,4,4,5) = \frac{\pi}{15}$. Checking for possible labels on v_5 we conclude that $l_5 \in \{cb^{-1}c(eb)^{-1}, cb^{-1}d(eb)^{-1}\}$ and so $e^3 = 1$. In this case $|v_{1\Delta_1}| \geq 12$ and so $c(\Delta_1) \leq c(2,3,3,5,12) = -\frac{\pi}{10}$, that is enough negative curvature to compensate for the positive curvature $\frac{\pi}{15}$ of Δ. So the only case which remains to be investigated is when $|v_2| = |v_6| = 3$ and at least one of v_1 and v_4 has label $d^{-1}(eb)d^{-1}ec$. Now the curvature of Δ is $c(\Delta) \leq c(2,2,3,3,5,5) = \frac{2\pi}{15}$. Now we use the negative curvature of both Δ_1 and Δ_2. If $|v_{2\Delta_1}| = 3$ then $l_{2\Delta_1} = cb^{-1}e$ and $e^3 = 1$. This would make $|v_{1\Delta_1}| \geq 12$ and so $c(\Delta_1) \leq \max\{c(2,3,4,5,6), c(2,3,3,5,12)\} = -\frac{\pi}{10}$. In the same way $c(\Delta_2) \leq -\frac{\pi}{10}$. So overall the negative curvature of $-\frac{\pi}{5}$ is enough to compensate for the positive one of Δ. A region of negative curvature cannot compensate for the positive curvature of more than one regions of positive curvature, since always positive curvature is being compensated through an $d^{-1} - e^{-1}$ edge.

3. $|v_2| = |v_6| = 2$ A region Δ of possible positive curvature will be as in Figure 5-32.

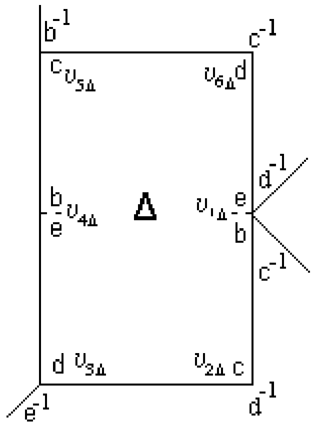


Figure 5-32:

Since $l_1 = d^{-1}(eb)c^{-1}w$ it is obvious that $|v_1| \neq 3$. We check for any other possible vertices of degree 3.

$|v_3| = 3$ implies $l_3 \in \{e^{-1}d(eb)^{-1}, (eb)^{-1}da^{\pm 1}\}$. If $l_3 = (eb)^{-1}da^{\pm 1}$ then $d = ea$ and $l_4 \in \{b^{-1}(eb)w, e(eb)w\}$.$|v_4| \in \{3, 4\}$ implies that its label will be $l_4 \in \left\{ \begin{array}{c} b^{-1}(eb)a^{\pm 1}, e(eb)c^{-1}, e(eb)d^{-1}, b^{-1}(eb)c^{-1}d, \\ b^{-1}(eb)d^{-1}c, e(eb)c^{-1}e, e(eb)d^{-1}e, e(eb)a^{\pm 1}b^{-1}, \\ e(eb)a^{\pm 1}c^{-1}, e(eb)a^{\pm 1}d^{-1}, e(eb)a^{\pm 1}(eb)^{-1}, \end{array} \right\}$. Any of these labels forces a contradiction, so it must be $|v_4| \geq 5$. Also $|v_5| \in \{3, 4\}$ implies $l_5 \in$ $\left\{ \begin{array}{c} cb^{-1}e, c(eb)^{-1}e^{-1}, cb^{-1}e^2, cb^{-1}(eb)d^{-1}, cb^{-1}cb^{-1}, cb^{-1}c(eb)^{-1}, cb^{-1}cd^{-1}, \\ cb^{-1}db^{-1}, cb^{-1}d(eb)^{-1}, c(eb)^{-1}e^{-2}, c(eb)^{-1}cb^{-1}, c(eb)^{-1}c(eb)^{-1}, c(eb)^{-1}cd^{-1}, \\ c(eb)^{-1}db^{-1}, c(eb)^{-1}d(eb)^{-1} \end{array} \right\}$. Out of these labels the only ones not forcing a contradiction are $c(eb)^{-1}d(eb)^{-1}$ and

$c(eb)^{-1}c(eb)^{-1}$. In this case $l_4 \in \{b^{-1}(eb)d^{-1}w, e(eb)d^{-1}w\}$. We check the possibility of this vertex to have degree 5. Now the label of v_4 for degree five will be $l_4 \in \left\{\begin{array}{c} b^{-1}(eb)d^{-1}ec, b^{-1}(eb)d^{-1}ed, b^{-1}(eb)d^{-1}e^{-1}c, b^{-1}(eb)d^{-1}e^{-1}d, \\ b^{-1}(eb)d^{-1}(eb)a^{\pm 1}, b^{-1}(eb)d^{-1}ca^{\pm 1}, e(eb)d^{-1}e^2, e(eb)d^{-1}(eb)c^{-1}, \\ e(eb)d^{-1}(eb)d^{-1}, e(eb)d^{-1}cb^{-1}, e(eb)d^{-1}cd^{-1} \end{array}\right\}$. Any of these labels forces G to have property X, so we conclude that $|v_4| \geq 6$ and so $c(\Delta) \leq c(2,2,3,4,4,6) = 0$. So we conclude that l_3 cannot be $(eb)^{-1}da^{\pm 1}$ and so if $|v_3| = 3$ then $l_3 = e^{-1}d(eb)^{-1}$.

If $|v_5| = 3$ then $l_5 \in \{cb^{-1}e, c(eb)^{-1}e^{-1}\}$. If $l_5 = cb^{-1}e$ then the label of v_4 involves letter a. In that case $|v_3|$ should be three otherwise $c(\Delta) \leq c(2,2,3,4,4,6) = 0$. So $l_3 = e^{-1}d(eb)^{-1}$ and $e^3 = 1$. But in that case $|v_4| \geq 12$ and so $c(\Delta) \leq c(2,2,3,3,4,12) = 0$. We conclude that if $|v_5| = 3$ then $l_5 = c(eb)^{-1}e^{-1}$.

If $|v_4| = 3$ then $l_4 \in \{(eb)a^{\pm 1}b^{-1}, (eb)a^{\pm 1}c^{-1}, (eb)a^{\pm 1}d^{-1}, (eb)c^{-1}e, (eb)d^{-1}e\}$. Let $l_4 = (eb)a^{\pm 1}b^{-1}$ and so $e = a$. Now $l_3 = e^{-1}da^{-1}w$. If $|v_3| = 3$ then $l_3 = (eb)^{-1}da^{-1}$but then $d = 1$, a contradiction. If $|v_3| = 4$ then its label will be $l_3 \in \{e^{-1}da^{-1}(eb)^{-1}, e^{-1}da^{-1}c^{-1}, e^{-1}da^{-1}d^{-1}\}$. Each of these relators makes the group to have property X so it is concluded that $|v_3| \geq 5$. So $|v_5| = |v_1| = 4$ otherwise the region cannot have positive curvature. But if $|v_1| = 4$ then $l_1 \in \{d^{-1}(eb)c^{-1}b, d^{-1}(eb)c^{-1}(eb)\}$ and again the group has property X. Let $l_4 \in \{(eb)a^{\pm 1}c^{-1}, (eb)a^{\pm 1}d^{-1}\}$. Now, $c = d = ea$. Now, $l_3 \in \{e^{-1}dc^{-1}w, e^{-1}db^{-1}w\}$, $l_5 \in \{ecb^{-1}w, b^{-1}cb^{-1}w\}$ and neither $|v_3|$ nor $|v_5|$ cannot be three because a relation is forced to make G have property X. Also $|v_3| = 4$ implies $l_3 \in \{e^{-1}dc^{-1}e^{-1}, (eb)^{-1}dc^{-1}d, (eb)^{-1}db^{-1}c, (eb)^{-1}db^{-1}c,\}$ and $|v_5| = 4$ implies $l_5 \in \{b^{-1}cb^{-1}c, b^{-1}cb^{-1}d\}$ and each of theses labels forces G have X. So $|v_3|, |v_5| \geq 5$ and so the curvature of Δ cannot be positive. Now let $l_4 \in \{(eb)c^{-1}e, (eb)d^{-1}e\}$. Now, $c = d = e^2$ and since the label of v_3 would involve a the degree of v_3 is at least eight. Now if the degree of v_1 is greater than four the curvature of Δ would be $c(\Delta) \leq c(2,2,3,3,5,8) < 0$. But if $|v_1| = 4$ then

$l_1 = d^{-1}(eb)c^{-1}$ and $e^3 = 1$ and so $|v_3| \geq 12$ and $c(\Delta) \leq c(2,2,3,3,4,12) = 0$. It is concluded that $|v_4|$ cannot be three.

So either $|v_3| = 3$ and $l_3 = e^{-1}d(eb)^{-1}$or/and $|v_5| = 3$ and $l_3 = c(eb)^{-1}e^{-1}$.

First suppose that $|v_3| = |v_5| = 3$. We look at the neighbouring regions Δ_1 and Δ_2 as in Figure 5-33.

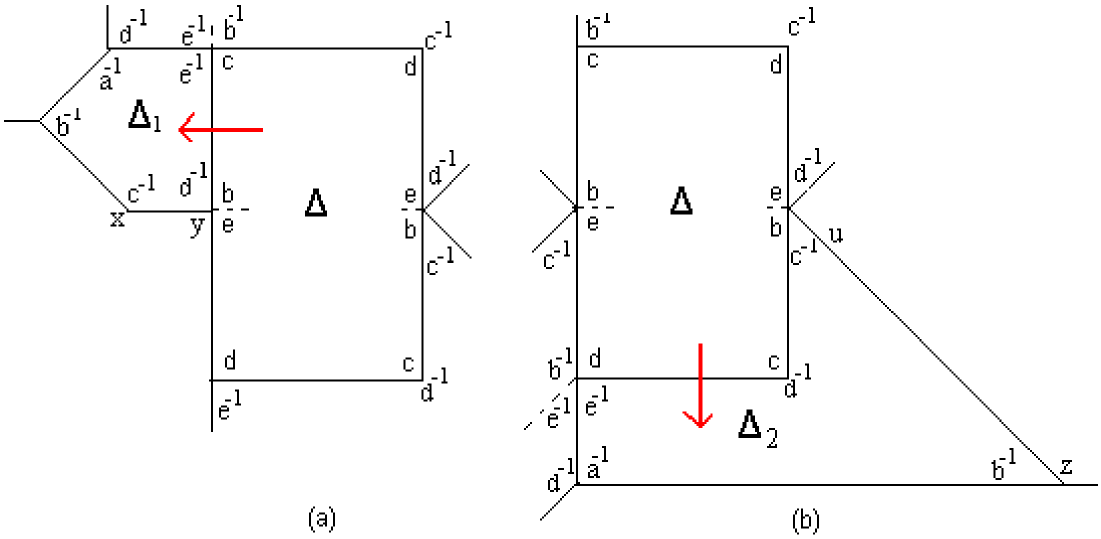


Figure 5-33:

If $|v_4| \in \{4,5\}$ then $l_4 \in \{c^{-1}(eb)d^{-1}b, c^{-1}(eb)d^{-1}e^{-1}b\}$. But in any of these cases $y \in \{b, e^{-1}\}$ and so $x = a^{\pm 1}$ and so $c(\Delta_1) \leq c(3,3,4,6,6) = -\frac{\pi}{2}$. If $|v_4| \geq 6$ then v_1 must have degree four or five otherwise the curvature of Δ cannot be positive. So we assume that $|v_1| \in \{4,5\}$ and so $c(\Delta) \leq c(2,2,3,3,4,6) = \frac{\pi}{6}$. Now $l_1 \in \{d^{-1}(eb)c^{-1}b, d^{-1}(eb)c^{-1}e^{-1}b, d^{-1}(eb)c^{-1}ec\}$. We look at region Δ_2. The first two possible labels of v_1 would make $z = a^{\pm 1}$ and so $c(\Delta_2) \leq c(2,3,4,6,6) = -\frac{\pi}{6}$. So suppose $l_1 = d^{-1}(eb)c^{-1}ec$. Now $c(\Delta) \leq c(2,2,3,3,5,6) = \frac{\pi}{15}$. If $|v_{2\Delta_2}| \geq 4$ then $c(\Delta_2) \leq c(2,3,4,5,6) = -\frac{\pi}{10}$. If $|v_{2\Delta_2}| = 3$ then $l_{2\Delta_2} = b^{-1}de$ and $e^3 = 1$. Now $|v_{1\Delta_2}| \geq 12$ and so $c(\Delta_2) \leq c(2,3,3,5,12) = -\frac{\pi}{10}$ which is enough for compensation.

If the only vertex of degree three is v_5 then $c(\Delta) \leq c(2,2,3,4,4,4) = \frac{\pi}{6}$.

The order of v_4 must be four or five otherwise region Δ cannot have positive curvature. We look at the possible labels of v_4. Having excluded all the cases which force a contradiction, or allow *diamond moves* to be performed in order to increase the number of vertices of degree two without decreasing the number faces or the number of vertices with label a^2, it turns out that $l_4 \in \left\{ \begin{array}{c} (eb)d^{-1}bc^{-1}, (eb)d^{-1}bc^{-1}, (eb)d^{-1}e^{-1}bc^{-1}, \\ (eb)d^{-1}e^{-1}bd^{-1}, (eb)d^{-1}e^{-1}cb^{-1}, (eb)d^{-1}e^{-1}db^{-1} \end{array} \right\}$. Then we can find a region Δ_1 as in Figure 5-33(a). In any of these case $y \in \{e^{-1}, b\}$ and so $x = a^{\pm 1}$. So $c(\Delta_1) \leq c(3,3,4,6,6) = -\frac{\pi}{2}$ which is enough for compensation.

Now, the only case remaining is when v_3 is the only vertex of degree three. If $|v_1| \geq 6$ then region Δ cannot have positive curvature. We look at the possible labels of v_1 for degree four or five. Having excluded all the contradictory cases $l_1 \in \{d^{-1}(eb)c^{-1}b, d^{-1}(eb)c^{-1}e^{-1}b, d^{-1}(eb)c^{-1}ec\}$. We look at region Δ_2 as in Figure 5-33(b). The first two cases make $z = a^{\pm 1}$ and so $c(\Delta_2) \leq c(2,3,4,6,6) = -\frac{\pi}{6}$ which is enough to compensate for $c(\Delta) \leq c(2,2,3,4,4,4) = \frac{\pi}{6}$. The only case left is when $l_1 = d^{-1}(eb)c^{-1}ec$. Now $c(\Delta) \leq c(2,2,3,4,4,5) = \frac{\pi}{15}$ and $c(\Delta_2) \leq \max\{c(2,3,3,5,12), c(2,3,4,5,6)\} = -\frac{\pi}{10}$.

In any case where a region Δ of negative curvature is encountered it is possible to find a neighbouring region Δ_1 or Δ_2 with enough negative curvature to compensate for the positive. These regions always compensate with negative curvature through a $e^{-1} - d^{-1}$ so it is not possible for the same region of negative curvature to compensate for more than one regions of positive curvature.

5. $|v_2| = 2$ **and** $|v_3|, |v_5|, |v_6| \geq 3$ Now at least four of the other vertices have degree three. We check for their possible labels. $|v_1| = 3$ implies $l_1 = (eb)c^{-1}e$ and $d = c = e^2$. But in this case $l_1 = daw$ and therefore its length cannot be six or less so $c(\Delta) \leq c(2,3,3,3,3,6) = 0$. Therefore v_1 cannot have degree three and so the rest of $|v_3|, |v_5|, |v_6|$ must all be three. $|v_3| = 3$ implies $l_3 \in \{(eb)^{-1}da^{\pm 1}, e^{-1}d(eb)^{-1}\}$. If $l_3 = (eb)^{-1}da^{\pm 1}$ then $l_4 \in \{e(eb)c^{-1}, e(eb)d^{-1}, b^{-1}(eb)a^{\pm 1}\}$

and each of these labels forces a contradiction. If $l_3 = e^{-1}d(eb)^{-1}$ then $l_4 = c^{-1}(eb)a^{\pm 1}$ which again forces a contradiction by making the group cyclic. It is concluded that no cases are possible in this case.

7. $|v_3| = 2$ **and** $|v_2|, |v_5|, |v_6| \geq 3$ Again we look at the possibility of having four vertices of degree three. $|v_2| = 3$ implies $l_2 \in \{c(eb)^{-1}e^{-1}, cb^{-1}e\}$. If $l_2 = cb^{-1}e$ then $l_1 = (eb)aw$ and therefore $|v_1| \geq 6$ and the curvature of Δ cannot be positive. So $l_1 = c(eb)^{-1}e^{-1}$ and $d = c = e^2$. $|v_4| = 3$ implies $l_4 = d^{-1}(eb)a^{\pm 1}$ and $d = ea$, so either $|v_2| = 3$ or $|v_4| = 3$ but not both at the same time have degree three. Let $l_2 = c(eb)^{-1}e^{-1}$. Now $l_1 = (eb)d^{-1}e$ but this label would force $|v_6| \geq 6$ and so the region cannot have positive curvature. So $|v_4| = 3$ and $l_4 = d^{-1}(eb)a^{\pm 1}$. But now $l_5 \in \{b^{-1}ca^{\pm}, ecb^{-1}, ecd^{-1}\}$ which in any case forces a contradiction. It is concluded that no cases are possible as well.

$d = e$ **and** $c = b = 1$

Now the possible labels of degree two are $d(eb)^{-1}$ and bc^{-1} and therefore, every vertex of Δ could have degree two. Two adjacent vertices cannot both have degree two, so it is possible to have up to three vertices of degree two. We check the following individual cases:

1. $|v_2| = |v_4| = |v_6| = 2$
2. $|v_1| = |v_3| = |v_5| = 2$ (This case is symmetric to $|v_2| = |v_4| = |v_6| = 2$)
3. $|v_2| = |v_4| = 2$
4. $|v_1| = |v_5| = 2$ (This case is symmetric to $|v_2| = |v_4| = 2$)
5. $|v_2| = |v_6| = 2$
6. $|v_3| = |v_5| = 2$ (This case is symmetric to $|v_2| = |v_6| = 2$)
7. $|v_1| = |v_3| = 2$

8. $|v_4| = |v_6| = 2$ (This case is symmetric to $|v_1| = |v_3| = 2$)

9. $|v_2| = |v_5| = 2$

10. $|v_3| = |v_6| = 2$

11. $|v_1| = |v_4| = 2$

12. Only $|v_1| = 2$

13. Only $|v_4| = 2$ (This case is symmetric to only $|v_1| = 2$)

14. Only $|v_2| = 2$

15. Only $|v_5| = 2$ (This case is symmetric to only $|v_2| = 2$)

16. Only $|v_3| = 2$

17. Only $|v_6| = 2$ (This case is symmetric to only $|v_3| = 2$)

1. $|v_2| = |v_4| = |v_6| = 2$ Now Δ will be as in Figure 5-34.

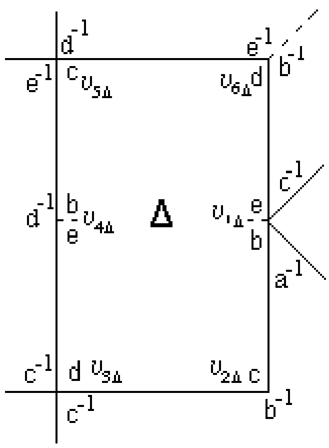


Figure 5-34:

It is obvious that the degree of v_1, v_3 and v_5 cannot be three. We check the possible labels for larger degrees. $|v_3| = 4$ implies $l_3 \in \{c^{-1}dc^{-1}d, c^{-1}dc^{-1}(eb)\}$

but then the order of element e in G is forced to be two, a contradiction. $|v_3| = 5$ implies $l_3 \in \{c^{-1}dc^{-1}e^{-1}b, c^{-1}dc^{-1}ed\}$ or element a is generated by e which makes the group cyclic. $|v_5| \in \{4,5\}$ implies that the label of v_5 is $l_5 \in \{e^{-1}cd^{-1}e^{-1}, e^{-1}cd^{-1}bd^{-1}, e^{-1}cd^{-1}c(eb)^{-1}, e^{-1}cd^{-1}cd^{-1}e^{-1}cd^{-1}e^{-2}, (eb)^{-1}cd^{-1}e^{-1}c\}$.

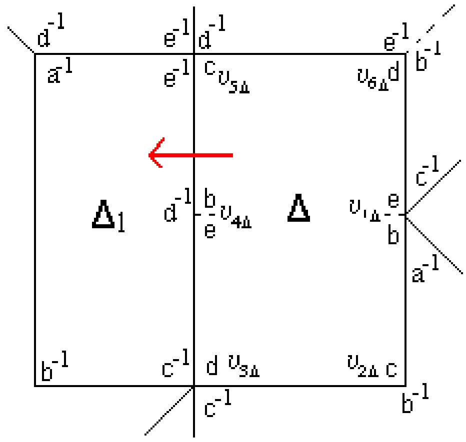


Figure 5-35:

The degree of v_1 must be greater than or equal to ten, since it involves letter a and any effort of finding a label with degree less than ten would force the group to have property X. If both v_3 and v_5 have degree five or greater then $c(\Delta) \leq c(2,2,2,5,5,10) = 0$, so the degree of v_5 should be four and the degree of v_3 must be five or six. So $l_5 = e^{-1}cd^{-1}e^{-1}$ and the order of element e is now three which forces the degree of v_1 to be at least twelve as labels of smaller length would force relators which would make the group to have property X or allow *diamond moves* to reduce the number of faces. Since $c(2,2,2,4,6,12) = 0$ we are only left with the case that the degree of v_3 is five and so $l_3 \in \{c^{-1}dc^{-1}e^{-1}b, c^{-1}dc^{-1}ed\}$. So the curvature of Δ is at most $c(\Delta) \leq c(2,2,2,4,5,12) = \frac{2\pi}{15}$. We look at the neighbouring region Δ_1 of Δ as in Figure 5-35. Observe that the degree of $v_{2\Delta_1}$ cannot be two and so $c(\Delta_1) \leq c(2,3,4,5,10) = -\frac{7\pi}{30}$ which is enough negative curvature to compensate for the positive curvature of Δ. The negative curvature is being compensated through a $e^{-1} - d^{-1}$ edge.

3. $|v_2| = |v_4| = 2$ Now Δ will be as in Figure 5-36.

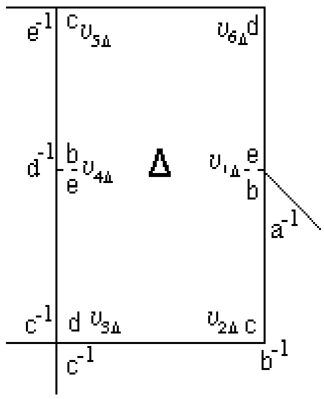


Figure 5-36:

As with previous section $|v_3| \geq 5$ and $|v_1| \geq 6$. Also $|v_4| = 3$ implies that the label of v_4 is $l_4 \in \{e^{-1}cd^{-1}, e^{-1}c(eb)^{-1}, (eb)^{-2}ca^{\pm 1}\}$ which makes the group to have property X. So $c(\Delta) \leq c(2, 2, 3, 4, 4, 6) = 0$.

5. $|v_2| = |v_6| = 2$ The label of v_2 would be cb^{-1} and the label of v_6 would be $d(eb)^{-1}$. Now the label of v_1 would be $l_1 = c^{-1}(eb)a^{-1}w$ and its degree must be equal to or greater than ten. The degree of v_4 should be greater than four and therefore, $c(\Delta) \leq c(2, 2, 3, 4, 4, 10) < 0$.

7. $|v_1| = |v_3| = 2$ Now region Δ would be as in Figure 5-37.

We check the possibility of each other vertex having degree three. It is obvious that v_2 and v_4 cannot have degree three and $|v_6| = 3$ implies $l_6 = dc^{-1}e^{-1}$. If $|v_5| = 3$ then $l_5 = cd^{-1}e$.

If $l_5 = cd^{-1}e$ then the label of v_4 would involve letter a and so its degree would be greater than or equal to eight. If the degree of v_2 is greater than or equal to five

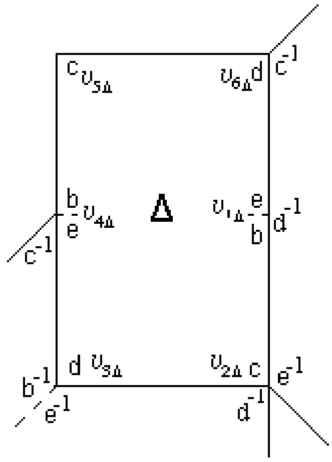


Figure 5-37:

then $c(\Delta) \leq c(2,2,3,3,5,8) < 0$ and if the degree of v_2 is four then $e^3 = 1$ and the degree of v_4 must be twelve or greater and so $c(\Delta) \leq c(2,2,3,3,4,12) = 0$.

If $|v_6| = 3$ and $l_6 = dc^{-1}e^{-1}$, there must be at least two of v_2, v_4 and v_5 of degree four and the remaining one must have degree four or five. $|v_2| = 4$ implies $l_2 = e^{-1}cd^{-1}e^{-1}$. $|v_5| = 4$ implies $l_5 = cd^{-1}e^{-2}$. $|v_4| = 4$ implies $l_4 = c^{-1}(eb)d^{-1}b$ and the order of element e is three. The curvature of Δ is at most $c(\Delta) \leq c(2,2,3,4,4,4) = \frac{\pi}{6}$. Negative curvature can be found in region Δ_1 as in Figure5-38.

If $|v_{1\Delta_1}| \neq 2$ then $c(\Delta_1) \leq c(2,3,4,6,6) = -\frac{\pi}{6}$ which is enough for compensation. If $|v_{1\Delta_1}| = 2$ then it is one of those vertices which was deleted and so in Δ $l_2 = (eb)^{-1}cd^{-1}w$. Now the degree of v_2 cannot be four so the degree of v_4 and v_5 must be four with $l_4 = c^{-1}(eb)d^{-1}b$ and $l_5 = c^{-1}d^{-1}e^{-2}$. In that case region Δ_2 with curvature $c(\Delta_2) \leq c(2,4,4,6,6) = -\frac{\pi}{3}$ can be used for compensation as in Figure . In any case the positive curvature is given to the negative region across an e^{-1}– d^{-1}edge.

So the only case left to examine is for $l_2 = e^{-1}dc^{-1}e^{-1}$ and $l_5 = cd^{-1}e^{-2}$ and the degree of v_4 must be five. Now $l_4 = c^{-1}(eb)d^{-1}w$ and so $l_4 = c^{-1}(eb)d^{-1}d$.

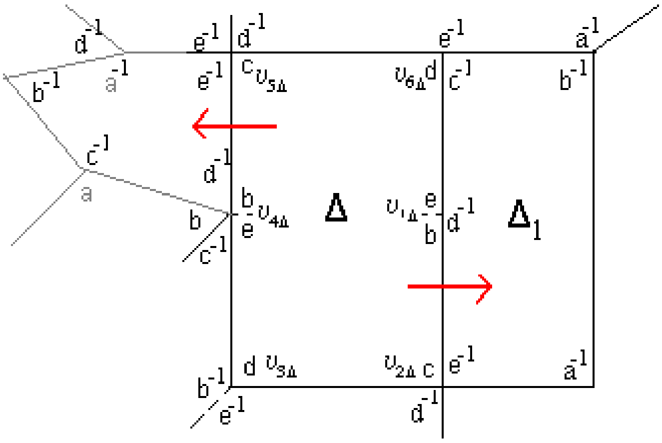


Figure 5-38:

Region Δ will be as in Figure 5-39 below.

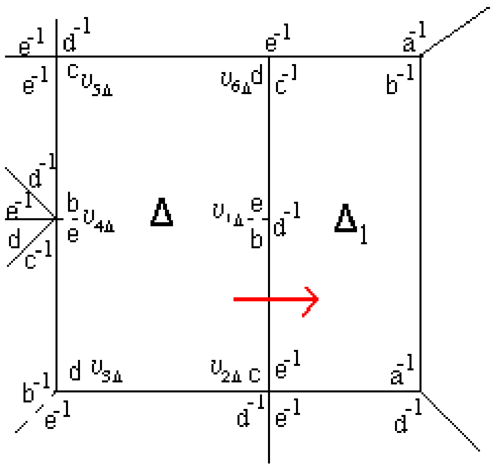


Figure 5-39:

The positive curvature of Δ is $c(\Delta) = c(2,2,3,4,4,5) = \frac{\pi}{15}$. The curvature of Δ_1 next to Δ is $c(\Delta_1) \leq c(2,3,4,6,6) = -\frac{\pi}{6}$ which is enough to compensate for the positive curvature of Δ. The negative curvature is being compensated through an $e^{-1} - d^{-1}$.

9. $|v_2| = |v_5| = 2$ The labels of v_2 and v_5 would be cb^{-1}. Now two of the remaining vertices would involve letter a and therefore have degree at least six. So $c(\Delta) \le c(2,2,3,3,6,6) = 0$.

10. $|v_3| = |v_6| = 2$ Now the label of v_3 and v_6 is $d(eb)^{-1}$ and so $l_1 = c^{-1}(eb)w$, $l_2 = cd^{-1}w$, $l_4 = c^{-1}(eb)w$, $l_5 = cd^{-1}w$. It is obvious that the degree of v_1 and v_4 should be four or greater. If the degree of v_2 or v_5 equals three then $l = cd^{-1}e$ and the degree of $v_1$or v_4 respectively would involve a and therefore have degree six or greater. It is concluded that $c(\Delta) \le max \left\{ \begin{array}{c} c(2,2,3,3,6,6), c(2,2,3,4,4,6), \\ c(2,2,4,4,4,4) \end{array} \right\} = 0$.

11. $|v_1| = |v_4| = 2$ Now $l_1 = l_4 = (eb)d^{-1}$. It can be seen that neither v_2 or v_5 can have degree three, so at least one of v_3 and v_6 must have degree three. Suppose $|v_3| = 3$ and $l_3 = dc^{-1}e^{-1}$. The case is analogous for $l_6 = dc^{-1}e^{-1}$. It is possible for the degree of v_2 to be four, five or greater. We look at neighbouring region Δ_1 as in Figure 5-40.

If $|v_2| = 4$ then $l_2 = e^{-1}cd^{-1}e^{-1}$ and $e^3 = 1$. In this case $x = a^{-1}$ and $e^3 = 1$ which forces $c(\Delta_1) \le c(2,3,4,8,8) = -\frac{\pi}{3}$ which is enough negative curvature to compensate for the positive curvature of Δ which at any time cannot be greater than $c(2,2,3,3,4,4) = \frac{\pi}{3}$.

If $|v_2| = 5$ then $l_5 \in \left\{ \begin{array}{c} e^{-1}cd^{-1}bd^{-1}, e^{-1}cd^{-1}e^{-2}, (eb)^{-1}cd^{-1}e^{-1}c, \\ e^{-1}cd^{-1}c(eb)^{-1}, e^{-1}cd^{-1}cd^{-1}, (eb)^{-1}cd^{-1}ed \end{array} \right\}$ and the curvature of Δ is at most $c(\Delta) \le c(2,2,3,3,4,5) = \frac{7\pi}{30}$. In the case of the first three labels $x = a^{\pm 1}$ and therefore $c(\Delta_1) \le -\frac{\pi}{3}$ as before. We look at region Δ_1 in the case of the last three vertices. If Δ_1 contains no vertices of degree two then its curvature is $c(\Delta_1) \le c(3,3,3,5,6) = -\frac{4\pi}{15}$. If $v_{3\Delta_1}$ has label $c^{-1}b$ then the label of $v_{2\Delta_1}$ involves letter a and therefore has degree greater than six and so $c(\Delta_1) \le c(2,3,5,6,6) = -\frac{4\pi}{15}$. If $v_{2\Delta_1}$ has label $b^{-1}c$ then the degree of $v_{3\Delta_1}$ cannot be four, except perhaps in the case that $l = c^{-1}d(eb)^{-1}b$. But even in

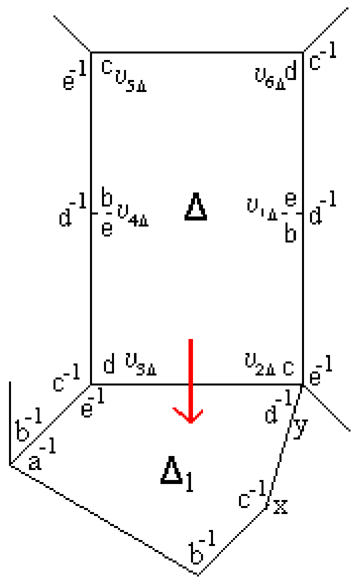


Figure 5-40:

that case a *diamond move* may be performed to increase the number of vertices of degree two by one without decreasing the number of vertices with label a^2 or the number of faces. So $|\upsilon_{3\Delta_1}| \geq 5$ and so $c(\Delta_1) \leq c(2,3,5,5,8) = -\frac{17\pi}{60}$. In any of these cases the curvature of Δ_1 is enough to compensate for the curvature of Δ with the positive curvature given across an $e^{-1} - d^{-1}$ edge.

12. Only $|\upsilon_1| = 2$ Now, at least four of the remaining vertices must have degree three otherwise $c(\Delta) \leq c(2,3,3,3,4,4) = 0$. If $l_1 = (eb)d^{-1}$ then $l_2 = e^{-1}cw$ and it cannot have degree three. Therefore $|\upsilon_3| = |\upsilon_4| = |\upsilon_5| = |\upsilon_6| = 3$. So $l_6 = dc^{-1}e^{-1}$ and $l_5 = cd^{-1}e$. But then l_4 would involve a and therefore cannot have degree three, so no regions of positive curvature can be found here.

14. Only $|\upsilon_2| = 2$ Now $l_2 = cb^{-1}$ and the label of υ_1 involves letter a and therefore would have degree at least six, so $c(\Delta) \leq c(2,3,3,3,3,6) = 0$.

16. Only $|v_3| = 2$ Now $l_3 = d(eb)^{-1}$ and $l_4 = c^{-1}(eb)w$ and so v_4 can have degree four or greater. So the remaining vertices must all have degree three. So $l_2 = cd^{-1}e$ but then the label of v_1 involves letter a and so v_1 has degree six or greater. So $c(\Delta) \leq c(2, 3, 3, 3, 3, 6) = 0$.

$d = e$ **and** $c \neq b = 1$

Now the only possible label of degree two is $d(eb)^{-1}$ or the inverse, so v_2 or v_5 cannot have degree two while it is possible for v_1, v_3, v_4 and v_5 to have degree two. It is not possible for two adjacent vertices both to have degree two, so we consider the following cases:

1. $|v_1| = |v_4| = 2$
2. $|v_3| = |v_6| = 2$
3. $|v_1| = |v_3| = 2$
4. $|v_4| = |v_6| = 2$ (This case is symmetric to 3.)
5. $|v_1| = 2$ and $|v_3|, |v_4|, |v_6| \geq 3$
6. $|v_4| = 2$ and $|v_1|, |v_3|, |v_6| \geq 3$ (This case is symmetric to 5.)
7. $|v_3| = 2$ and $|v_1|, |v_4|, |v_6| \geq 3$
8. $|v_6| = 2$ and $|v_1|, |v_3|, |v_6| \geq 3$ (This case is symmetric to 7.)

1. $|v_1| = |v_4| = 2$ Now region Δ will be as in Figure 5-41.

We check for possible vertices of degree three. If $|v_2| = 3$ then its label will be $l_2 \in \{e^{-1}c(eb)^{-1}, e^{-1}cd^{-1}, (eb)^{-1}ca^{\pm 1}\}$. The first two labels make $c = e^2$ and the last $c = ea$. The same labels are possible for $|v_5| = 2$. If $|v_3| = 3$ then $l_3 = dc^{-1}e$. In the case $l_3 = dc^{-1}e$ the label of v_2 would be $l_2 = e^{-1}caw$ and then $|v_2| \geq 8$

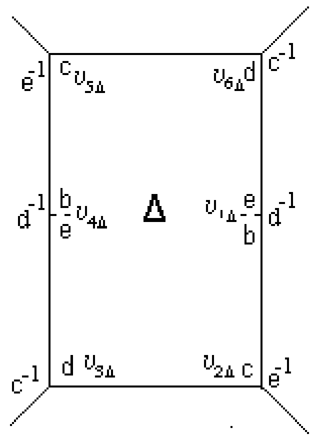


Figure 5-41:

otherwise its label forces a relator that makes the group have property X. In the same way if $|\upsilon_6| = 3$ then $|\upsilon_5| \geq 8$.

First let, $|\upsilon_3| = 3$ with $l_3 = dc^{-1}e$. The order of υ_5 should be three otherwise the curvature or Δ cannot be positive as $c(\Delta) \leq \max\{c(2,2,3,3,8,8),\ c(2,2,3,4,4,8)\} < 0$. So $l_5 \in \{e^{-1}c(eb)^{-1}, e^{-1}cd^{-1}\}$. But then $l_6 \in \{c^{-1}dc^{-1}w, e^{-1}dc^{-1}\}$. If $|\upsilon_6| = 4$ then $l_6 \in \{c^{-1}dc^{-1}b, e^{-1}dc^{-1}e^{-1}\}$ and $e^3 = 1$. But in that case $|\upsilon_2| \geq 12$ and $c(\Delta) \leq c(2,2,3,3,4,12) = 0$. So if any of υ_3 or υ_6 has degree three region Δ cannot have positive curvature.

Now, let $|\upsilon_2| = 3$, $l_2 = (eb)^{-1}ca^{\pm 1}$ and $c = ea$. Now $l_3 \in \{edc^{-1}w, b^{-1}dc^{-1}w\}$. $|\upsilon_3| = 3$ is not possible and $|\upsilon_3| = 4$ implies $l_3 \in \{edc^{-1}e, b^{-1}dc^{-1}d\}$ which force contradiction by making the group cyclic. $|\upsilon_3| = 5$ implies that the label will be

$$l_3 \in \left\{ \begin{array}{c} edc^{-1}e^2, edc^{-1}bc^{-1}, edc^{-1}bd^{-1}, edc^{-1}(eb)c^{-1}, edc^{-1}(eb)d^{-1}, edc^{-1}db^{-1}, \\ edc^{-1}, b^{-1}dc^{-1}ba^{\pm 1}, b^{-1}dc^{-1}(eb)a^{\pm 1}, b^{-1}dc^{-1}da^{\pm 1}, \\ b^{-1}dc^{-1}ec, b^{-1}dc^{-1}ed, b^{-1}dc^{-1}e^{-1}c, b^{-1}dc^{-1}e^{-1}c \end{array} \right\}.$$

Each of these labels makes the group have property X, so it can be assumed that $|\upsilon_3| \geq 6$. Also $|\upsilon_6|$ cannot be four in this case and if $|\upsilon_5| = 3$ then $l_5 = (eb)^{-1}ca^{\pm 1}$. But in this case $|\upsilon_6| \geq 6$ and so it is concluded that if either υ_2 or υ_5 have the label $(eb)^{-1}ca^{\pm 1}$ then $c(\Delta) \leq \max\{c(2,2,3,4,4,6), c(2,2,3,3,6,6)\} = 0$.

So the only possible vertices of degree three are v_2 and v_5 with label $e^{-1}c(eb)^{-1}$ or $e^{-1}cd^{-1}$ and so $d = e$, $c = e^2$.

First let $l_2 = e^{-1}cd^{-1}$. Now $l_3 = e^{-1}dc^{-1}w$ and $|v_3| \neq 3$. If $|v_3| = 4$ then $l_3 \in \{(eb)^{-1}dc^{-1}d, e^{-1}dc^{-1}e^{-1}\}$. The first label forces a contradiction, while the $l_3 = e^{-1}dc^{-1}e^{-1}$ is possible and forces $e^3 = 1$. If $|v_3| = 5$ then its label will be $l_3 \in \{e^{-1}dc^{-1}e^{-2}, e^{-1}dc^{-1}bc^{-1}, e^{-1}dc^{-1}bd^{-1}, e^{-1}dc^{-1}(eb)c^{-1}, (eb)^{-1}dc^{-1}ed\}$. If $l_3 = (eb)^{-1}dc^{-1}ed$ it is possible to perform diamond moves to produce one more vertex of degree two, contrary to the assumption of their maximality. So if $|v_3| \in \{4, 5\}$ then $l_3 \in \{e^{-1}dc^{-1}e^{-1}, e^{-1}dc^{-1}e^{-2}, e^{-1}dc^{-1}bc^{-1}, e^{-1}dc^{-1}bd^{-1}, e^{-1}dc^{-1}(eb)c^{-1}\}$ and $e^3 = 1$ or $e^4 = 1$. If any of these labels occur on v_3 it is possible to find a region Δ_1 as in Figure 5-42 with negative curvature.

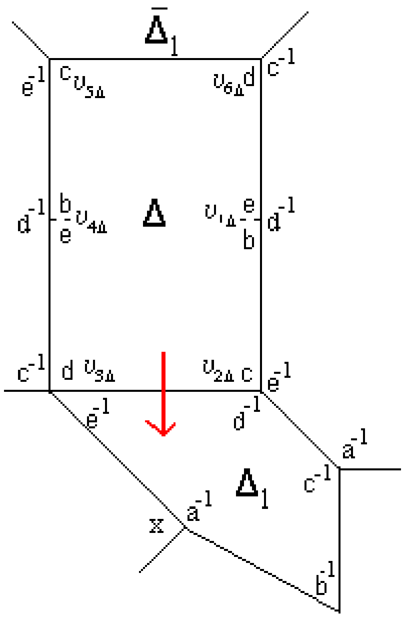


Figure 5-42:

In any case $x \neq a^{-1}$ and since a^{-2} cannot be a proper sublabel on a vertex the degree of $v_{1\Delta_1}$ is equal to or greater than six and so $c(\Delta_1) \leq c(3, 3, 4, 6, 6) = -\frac{\pi}{2}$. This is enough negative curvature to compensate for any possible positive of Δ

which is $c(\Delta) \leq c(2,2,3,3,4,4) = \frac{\pi}{3}$. If $|v_3| \geq 6$ then $|v_5|$ must also be three and $|v_6| \in \{4,5\}$. In the case that $l_5 = e^{-1}cd^{-1}$ then it is possible to find a $\bar{\Delta}_1$ as before, on the other side of Δ.

So at least one of v_2 or v_5 has label $e^{-1}c(eb)^{-1}$. Let $l_2 = e^{-1}c(eb)^{-1}$ (and the case for $l_5 = e^{-1}c(eb)^{-1}$ is analogous). If $|v_3| \in \{4,5\}$ then its label will be $l_3 \in \{c^{-1}dc^{-1}b, c^{-1}dc^{-1}e^{-1}b, c^{-1}dc^{-1}e^{-1}d\}$ and either $e^3 = 1$ or $e^4 = 1$. We look at region Δ_2 next to Δ as in Figure 5-43.

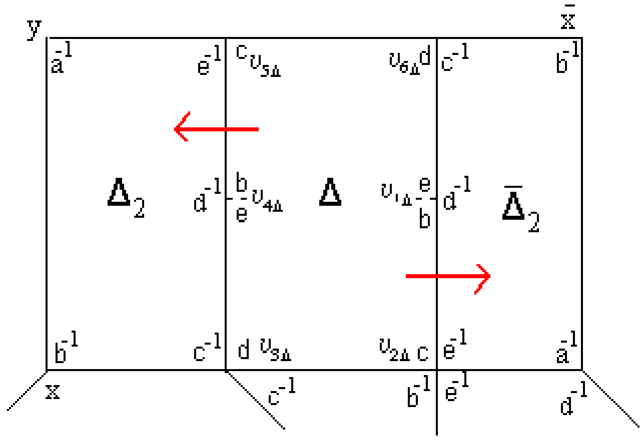


Figure 5-43:

If $|v_3| \in \{4,5\}$ then $x = a^{\pm 1}$. If at the same time also $|v_5| = 3$ then $c(\Delta_2) \leq c(2,3,4,8,8) = -\frac{\pi}{3}$ which is enough to compensate for the positive curvature of Δ. If $|v_3| \geq 6$ then it must be $|v_5| = 3$ and $|v_6| \in \{4,5\}$ so it is possible to find a similar $\bar{\Delta}_2$ or a $\bar{\Delta}_1$ as before.

So the only case left to see is when $l_2 = e^{-1}c(eb)^{-1}$ and $|v_3|, |v_5|, |v_6| \in \{4,5\}$.Now $c(\Delta) \leq c(2,2,3,4,4,4) = \frac{\pi}{6}$. We look at Δ_2 next to Δ as in Figure 5-41. Since $|v_3| \in \{4,5\}$ we have $x = a^{\pm 1}$. If $y \neq a^{-1}$ then $c(\Delta_2) \leq c(2,4,4,6,6) = -\frac{\pi}{3}$ which is enough to compensate for the positive curvature of Δ. We also look at $\bar{\Delta}_2$. If $\bar{x} = a^{\pm 1}$ then $c(\bar{\Delta}_2) \leq c(2,3,4,8,8)$. Now suppose $y = a^{-1}$ and $\bar{x} \neq a^{\pm 1}$. In this case $l_6 \in \{dc^{-1}dw, dc^{-1}ew\}$ and $l_5 = (eb)^{-1}cw$. It turns out that since $|v_6| \in$

$\{4,5\}$ then $l_6 \in \{dc^{-1}db^{-1}, dc^{-1}dc^{-1}e^{-1}, dc^{-1}ed(eb)^{-1}, dc^{-1}(eb)c^{-1}e^{-1}, dc^{-1}(eb)d^{-1}e^{-1}\}$. The first of these labels would force $l_5 = (eb)^{-1}ca^{-1}w$ and any possible labeling on v_5 would force a contradiction by creating a relator to make the group have property X. In the other cases $l_5 = (eb)^{-1}cd^{-1}w$ and its degree should be four. But any effort of finding such a label fails, so it is concluded that in any case either a Δ_2 or a $\bar{\Delta}_2$ can be found with enough negative curvature to compensate for the positive curvature of Δ.

In each case there is enough negative curvature always receiving the positive curvature across an $e^{-1} - d^{-1}$. Thus a negative region can only compensate for the positive curvature of at most one region.

2. $|v_3| = |v_6| = 2$ Now region Δ will be as in Figure 5-44.

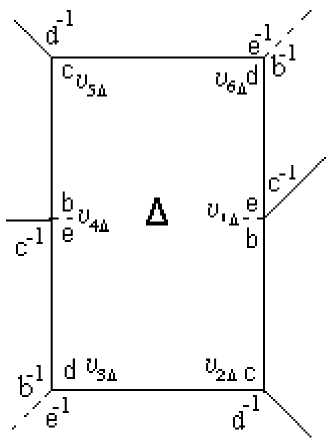


Figure 5-44:

Now if $|v_2| = 3$ then $l_2 \in \{cd^{-1}e, cd^{-1}e^{-1}\}$. The first label forces $c = 1$, a contradiction. Suppose $l_2 = cd^{-1}e$ and so $c = e^2$ and $l_1 = a(eb)c^{-1}w$. Therefore, $|v_1| \geq 6$ otherwise the group is forced to have property X. Also the degree of v_1 cannot be three and therefore, $c(\Delta) \leq max\,\{c(2,2,3,4,4,6), c(2,2,3,3,6,6)\} = 0$. The same applies for v_5 we can, therefore, assume that if one of $|v_2|, |v_5|$ has degree three then they have label $cd^{-1}e^{-1}$ and $c = e^2$.

$|v_1| = 3$ implies $l_1 = c^{-1}(eb)a^{\pm 1}$ and so $c = ea$. Now $l_2 \in \{ecd^{-1}w, b^{-1}cd^{-1}w\}$. It is obvious that $|v_2| \neq 3$. If v_2 has degree four or five then its label will be $l_2 \in$
$$\left\{\begin{array}{c} ecd^{-1}e, ecd^{-1}e^2, ecd^{-1}bc^{-1}, ecd^{-1}bd^{-1}, ecd^{-1}(eb)c^{-1}, ecd^{-1}(eb)d^{-1}, \\ ecd^{-1}cb^{-1}, ecd^{-1}c(eb)^{-1}, ecd^{-1}cd^{-1}, b^{-1}cd^{-1}c, b^{-1}cd^{-1}ba^{\pm 1}, b^{-1}cd^{-1}(eb)a^{\pm 1}, \\ b^{-1}cd^{-1}ca^{\pm 1}, b^{-1}cd^{-1}ec, b^{-1}cd^{-1}ed, b^{-1}cd^{-1}e^{-1}c, b^{-1}cd^{-1}e^{-1}d \end{array}\right\}.$$
Any of these labels forces a contradiction by making the group have property X. So it must be $|v_2| \geq 6$. In the same way if $l_4 = c^{-1}(eb)a^{\pm 1}$ then $|v_5| \geq 6$. Since in this case none of the other vertices can have degree three, a region of positive curvature cannot be obtained.

So at least one of v_2, v_5 has degree three with label $cd^{-1}e^{-1}$. Suppose $l_2 = cd^{-1}e^{-1}$. So $l_1 = c^{-1}(eb)d^{-1}w$ and it can be seen that the order of v_1 must be greater than five. We look at region Δ_1 as in Figure 5-45.

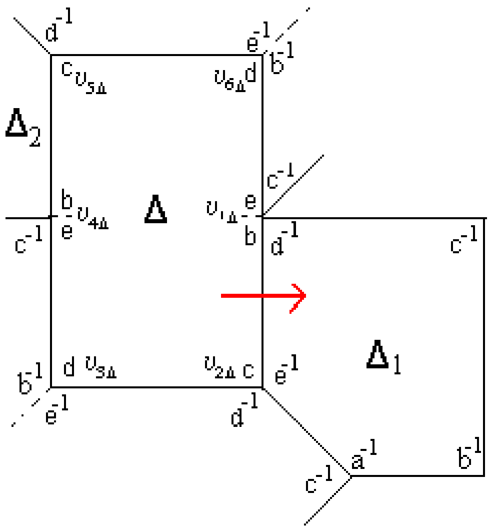


Figure 5-45:

Now $c(\Delta) \leq \max\{c(2,2,3,4,4,5), c(2,2,3,3,5,5)\} = \frac{2\pi}{15}$ and the curvature of Δ_1 is $c(\Delta_1) \leq c(3,3,3,5,6) = -\frac{4\pi}{15}$ which is enough negative curvature to compensate for $c(\Delta)$. If $l_5 = cd^{-1}e^{-1}$ then in the same way it is possible to find a Δ_2 with $c(\Delta_2) \leq -\frac{4\pi}{15}$. Positive curvature is always added to the negative curvature across an $e^{-1} - d^{-1}$ edge.

3. $|v_1| = |v_3| = 2$ Now region Δ will be as in Figure 5-46.

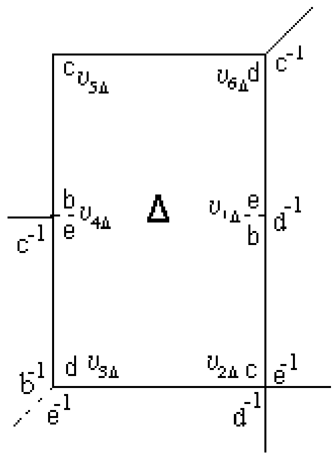


Figure 5-46:

Now if $|v_6| = 3$ then $l_6 = dc^{-1}e$ and $c = e^2$. This forces $|v_5| \geq 8$ and $|v_4| \geq 4$. If $|v_4| \geq 5$ then $c(\Delta) \leq c(2,2,3,3,5,8) < 0$ so it must be $|v_4| = 4$. So the label of v_4 is $l_4 \in \{c^{-1}(eb)c^{-1}b, c^{-1}(eb)c^{-1}(eb), c^{-1}(eb)c^{-1}d, c^{-1}(eb)d^{-1}b, c^{-1}(eb)d^{-1}(eb)\}$. The only possible label is $l_4 = c^{-1}(eb)c^{-1}b$ but this one forces $e^3 = 1$ and makes $|v_5| \geq 12$ otherwise relators will arise to make the group have property X or allow diamond moves to reduce the number of faces. So $c(\Delta) \leq c(2,2,3,3,4,12) = 0$. So if Δ is a region of positive curvature then $|v_6| \geq 4$.

If $|v_4| = 3$ then $l_3 = c^{-1}(eb)a^{\pm 1}$ and so $c = ea$ and $l_5 \in \{ecb^{-1}w, b^{-1}cb^{-1}w\}$. Any effort of finding labels for l_5 for $|v_5| = 4$ fails, so it must be $|v_2| = |v_6| = 4$ and $|v_5| = 5$ otherwise $c(\Delta) \leq c(2,2,3,4,4,6) = 0$. So $l_2 = (eb)^{-1}cdc^{-1}$ and $l_6 = dc^{-1}(eb)c^{-1}$. This makes $l_5 \in \{b^{-1}cb^{-1}w, ecb^{-1}w\}$. Any effort of finding labels for $|v_5| = 5$ makes the group to have property X. So it can be assumed that $|v_4| \geq 4$.

If $|v_5| = 3$ then $l_3 \in \{cb^{-1}e, ca^{\pm 1}b^{-1}, ca^{\pm 1}(eb)^{-1}, cad^{-1}, cd^{-1}e^{-1}, c(eb)^{-1}e^{-1}\}$. It turns out that with the first four labels it is not possible to obtain a region of positive curvature. So $l_5 \in \{cd^{-1}e^{-1}, c(eb)^{-1}e^{-1}\}$ and $c = e^2$.

To have a region of positive curvature it must be either $|v_2| = 3$ and/or $|v_5| = 3$. If $|v_5| = 3$ then Δ will be as in Figure 5-47 and in any case $x \neq a^{-1}$ and since a^2 cannot be a proper sublabel none of the vertices of Δ_1 can have degree two.

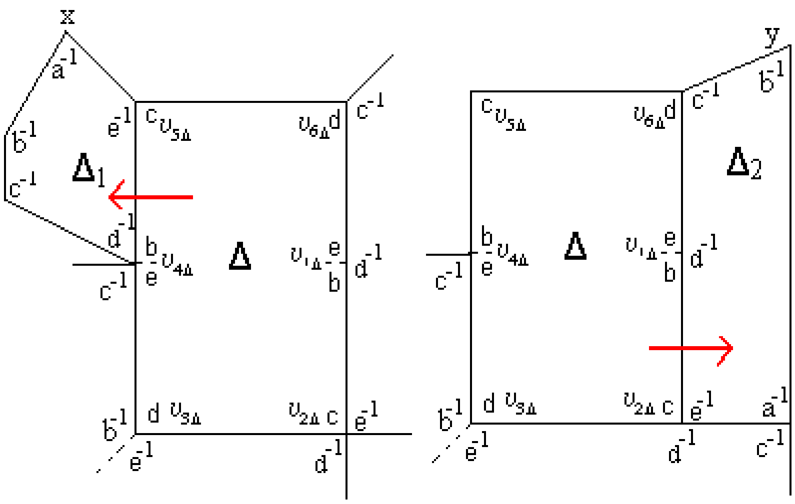


Figure 5-47:

Also, $l_4 = c^{-1}(eb)d^{-1}w$ and $|v_4| \neq 4$. So it $c(\Delta) \leq c(2,2,3,3,4,5) = \frac{7\pi}{30}$. The curvature of neighbouring Δ_1 would be $c(\Delta_1) \leq c(3,3,3,5,6) = -\frac{4\pi}{15}$ which is enough negative curvature to compensate for the positive one of Δ.

Now suppose the degree of v_5 is four or greater. The degree of v_2 must be three and then $c(\Delta) \leq c(2,2,3,4,4,4) = \frac{\pi}{6}$. If $|v_6| = 4$ then $l_6 \in \{dc^{-1}bc^{-1}, dc^{-1}db^{-1}\}$. If $l_6 = dc^{-1}bc^{-1}$ then $y = a$ and so $c(\Delta_2) \leq c(2,3,4,6,6) = -\frac{\pi}{6}$ (Figure 5-47). If $l_6 = dc^{-1}db^{-1}$ then $|v_5| \geq 6$ and the curvature of Δ cannot be positive. If $|v_6| = 5$ then $|v_4| = |v_5| = 4$ and so $l_4 = c^{-1}(eb)c^{-1}b$ and $e^3 = 1$. Also it must be $l_5 = d^{-1}cb^{-1}c$ and $l_6 = cd^{-1}ce^{-1}d$. So $c(\Delta) = c(2,2,3,4,4,5) = \frac{\pi}{15}$ and $c(\Delta_2) \leq c(2,3,3,5,12) = -\frac{\pi}{10}$.

Therefore, in any case there is enough negative curvature so that the positive curvature of Δ is added to it across an $e^{-1} - d^{-1}$ and the total curvature is non-positive.

5. Only $|v_1| = 2$ Now $|v_6| = 3$ implies $l_3 = dc^{-1}e$ and $c = e^2$. But like this the degree of v_5 will involve letter a and since a^2 cannot be a proper sublabel by Lemma 5.3 any effort of labelling v_5 with degree less than six, would make the group to have property X. So $|v_5| \geq 6$ but then $c(\Delta) \leq c(2,3,3,3,3,6) = 0$. So it can be assumed that $|v_6| \geq 4$ and so the remaining four vertices must have degree three. $|v_2| = 3$ implies $l_2 \in \{e^{-1}c(eb)^{-1}, e^{-1}cd^{-1}, (eb)^{-1}ca^{\pm1}\}$. If $l_2 = (eb)^{-1}ca^{\pm1}$ then the degree of v_3 cannot be three and therefore region Δ cannot have positive curvature. So it must be one of the other two vertices and $c = e^2$. But in this case as well, v_3 cannot have degree three so the region cannot have positive curvature.

7. Only $|v_3| = 2$ Now $|v_2| = 3$ implies $l_2 = cd^{-1}e^{-1}$ and $c = e^2$. But like this the degree of v_4 will involve letter a and therefore will have degree six or greater. So $|v_2| \geq 4$. If $|v_4| = 3$ then $l_4 = c^{-1}(eb)a^{\pm1}$ and $c = ea$. But now the degree of v_5 cannot be four. So the region cannot have positive curvature.

It is concluded that if only one vertex of region Δ has degree two, then it cannot have positive curvature.

$d \neq e$ **and** $c = b = 1$

Now the only possible label of degree two is cb^{-1} or the inverse, so v_2 or v_5 are the only possible vertices of degree two.

First let v_2 be the only vertex of degree two (and the case is analogous when v_5 is the only vertex of degree two). We look for possible vertices of degree three. $|v_1| = 3$ implies $l_1 = (eb)a^{-1}b^{-1}$and so $d = ea$. Also $|v_3| = 3$ implies $l_3 = c^{-1}da^{\pm1}$ and so $d = a$. It is obvious that $|v_1| = |v_3| = 3$ is not possible. Also if $l_1 = (eb)a^{-1}b^{-1}$ the label of v_6 will be $l_6 = dc^{-1}w$ and if it of length three the group turns out to be cyclic. In the same way if $l_3 = c^{-1}da^{\pm1}$ the label of v_4 will be $l_4 = e(eb)w$ and any label of length three forces a contradiction. Therefore the region will have at least two vertices of degree three and so $c(\Delta) \leq c(2,3,3,3,4,4) = 0$.

From the above it is concluded that in order to have a region of positive curvature both v_2 and v_5 must have degree two. So region Δ will be as in Figure 5-48.

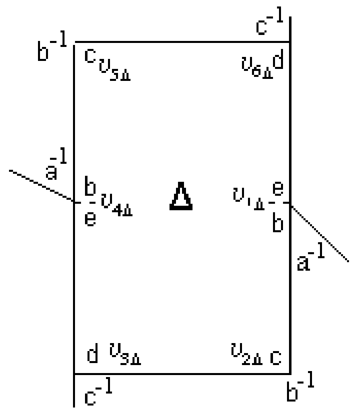


Figure 5-48:

There must be at least one vertex of degree three otherwise, the curvature of Δ cannot be positive. First suppose $|v_1| = 3$ and so$l_1 = (eb)a^{-1}d^{-1}$ and $d = ea$. The label of v_6 will be and $l_6 = c^{-1}dc^{-1}w$. The degree of v_6 cannot be three and $|v_6| \in \{4,5\}$ implies $l_6 \in \left\{ \begin{array}{c} c^{-1}dc^{-1}b, c^{-1}dc^{-1}(eb), c^{-1}dc^{-1}d, c^{-1}dc^{-1}ba^{\pm 1}, c^{-1}dc^{-1}(eb)a^{\pm 1}, \\ c^{-1}dc^{-1}da^{\pm 1}, c^{-1}dc^{-1}e(eb), c^{-1}dc^{-1}ed, c^{-1}dc^{-1}e^{-1}b, c^{-1}dc^{-1}e^{-1}d \end{array} \right\}$. These vertices again force a contradiction, and so $|v_6| \geq 6$. So for every possible label of $v_1$for its degree being three it must be $|v_6| \geq 6$. The same applies for $|v_4| = 3$ then $|v_4| \geq 6$. It is therefore concluded that $c(\Delta) \leq \min\{c(2,2,3,4,4,6), c(2,2,3,3,6,6), c(2,2,4,4,4,4)\} = 0$.

So in this case there cannot be any regions of positive curvature.

a and e both have order two in G

Now consider the case for which the order of e is two and the order of a is two. In the same way as in the previous section we can delete vertices with label a^2 wherever it appears in the diagram. But it is also possible to delete vertices with label e^2 wherever they appear in D. By deleting the vertex with label e^2 it is possible to obtain a region of degree six where e does not appear. This implies that a new element is added in the star graph, element (da). The new star graph and the new regions of degree six will be as in Figure 5-49. We call regions of degree six where an a^2 was deleted F_1-regions and those where an e^2 was deleted an F_2-regions.

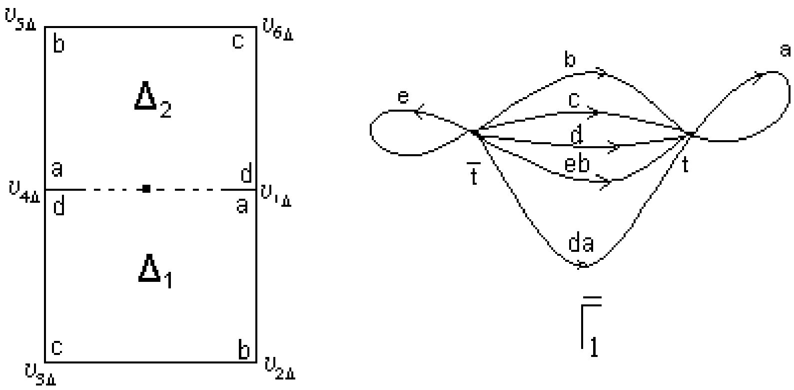


Figure 5-49:

With the new element added, possible vertices of degree two on F_1 or F_2-regions are bc^{-1}, $b(da)^{-1}$, cd^{-1}, $c(da)^{-1}$, $c(eb)^{-1}$, $d(eb)^{-1}$, $(da)(eb)^{-1}$ or their inverses. If none of the above labels appears on an F region then this region cannot have positive curvature. So one of the following distinguished conditions must be true:

1. $c = 1$

2. $c = d$

3. $c = e$

4. $d = ea$

5. $c = da$

6. $d = a$

7. $d = e$

It can be seen that a label $b(da)^{-1}$ cannot appear on an F_1 region and a label $d(eb)^{-1}$ cannot appear on an F_2 region.
We examine what the curvature of an F region can be under each condition and show that $c(\Delta) \leq 0$ in each case.

$c = 1$

F_1 Let Δ be an F_1-region of positive curvature. The vertices that can have degree two are υ_2 or/and υ_5 with label cb^{-1}.

First let υ_2 be the only vertex of degree two and the case is analogous for υ_5 being the only vertex of degree two. The label of υ_1 will be $l_1 = (eb)a^{-1}w$ and the label of υ_3 will be $l_3 = c^{-1}dw$. The degree of υ_3 cannot be three and if $|\upsilon_1| = 3$ then $l_1 \in \{(eb)a^{-1}b^{-1}, (eb)a^{-1}c\}$ and $a = e$. Since $c(2,3,3,3,4,4) = 0$ the remaining vertices except υ_3 must have degree three. Now because of the label of υ_1 the label of υ_6 will be $\{db^{-1}e, d(eb)^{-1}e^{-1}, da^{-1}b^{-1}, da^{-1}c^{-1}, da^{-1}(eb)^{-1}, d(da)^{-1}e^{\pm 1}\}$but any of this forces a contradiction so the region cannot have positive curvature unless two vertices have degree two.

So $|\upsilon_2| = |\upsilon_5| = 2$ and at least one of υ_1 and υ_4 must have degree three otherwise $c(\Delta) \leq c(2,2,4,4,4,4) = 0$. Let $l_1 \in \{(eb)a^{-1}b^{-1}, (eb)a^{-1}c\}$ and so $a = e$. Now the label of υ_6 is $l_6 \in \{c^{-1}db^{-1}w, c^{-1}da^{-1}w\}$ and it can be seen that its degree must be at least six. The same applies for υ_4 i.e. $|\upsilon_4| = 3$ implies $|\upsilon_3| \geq 6$. Therefore, the curvature of the region is at most $c(\Delta) \leq \{c(2,2,3,4,4,6), c(2,2,3,3,6,6)\} = 0$.

F_2 Let Δ be an F_2-region of positive curvature. Now it is possible for v_2, v_3, v_5 and v_6 to have degree two but two adjacent vertices cannot have degree two at the same time. So Δ has at most two vertices of degree two. First suppose $|v_2| = 2$ with label $bc^{-1}$and so the label of v_1 becomes $l_1 = (da)b^{-1}w$ and the label of v_3 becomes $l_3 = d^{-1}cw$. It can be seen that none of these labels can be of length three so if v_2 is the only vertex of degree two $c(\Delta) \leq c(2,3,3,3,4,4) = 0$. Now suppose that $|v_3| = 2$ with label cb^{-1} and so $l_4 = c^{-1}(da)w$ and $l_2 = ba^{-1}w$. It can be seen that the degree of v_4 cannot be three while if $|v_2| = 3$ then $l_2 = ba^{-1}(eb)^{-1}$ and $a = e$. But in such a case $l_1 = (da)d^{-1}w$ and its degree is at least four. So if v_3 is the only vertex of degree two $c(\Delta) \leq c(2,3,3,3,4,4) = 0$. So Δ must have two vertices of degree two.

If $|v_2| = |v_5| = 2$ then v_1, v_3, v_4 and v_6 must all have degree at least four and so $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

If $|v_2| = |v_6| = 2$ then v_1 and v_3 have degree four or greater so at least one of v_3 and v_4 must have degree three. First let $|v_5| = 3$ and so $l_5 = ba^{-1}(eb)^{-1}$ and $a = e$. Now the label of v_5 becomes $l_4 = (da)d^{-1}w$ and it can be seen that its degree at least six otherwise the group has property X. Now let $|v_4| = 3$ and so $l_4 = (ad)d^{-1}e^{\pm 1}$ and $l_3 \in \{d^{-1}caw, d^{-1}cd^{-1}w\}$ and $l_5 = e^{-1}ba^{-1}w$. It can be seen that any labelling of length four on v_1 and v_5 forces a contradiction so $c(\Delta) \leq c(2,2,3,4,5,5) < 0$.

If $|v_3| = |v_6| = 2$ then v_1 and v_4 have degree at least four and so one of v_2 and v_5 must have degree three with label $ba^{-1}(eb)^{-1}$. But then the vertex adjacent to it must have degree at least six and so a region of positive curvature cannot exist.

$c = d$

F_1 Now vertices v_2, v_3, v_5 and v_6 can have degree two but two adjacent vertices cannot have degree two at the same time, so Δ has at most two vertices of degree two. If $|v_2| = 2$ then $l_1 = (eb)c^{-1}w$ and $l_3 = e^{-1}dw$ and it can be seen that v_1 cannot have degree three but v_3 may do with label $l_3 = e^{-1}d(ad)^{-1}$ and then $a = e$. In the case that v_3 has degree three the label of v_4 becomes $l_4 = b^{-1}(eb)w$ and this will be of length greater than three. So if v_2 is the only vertex of degree three then $c(\Delta) \leq 0$. If $|v_3| = 2$ then the labels of v_2 and v_4 become $l_2 = cb^{-1}w$ and $l_4 = d^{-1}(eb)w$ and none of them can have degree three. So if v_3 is the only vertex of degree two in Δ then $c(\Delta) \leq c(2, 3, 3, 3, 4, 4) = 0$. The same applies for v_6 i.e. if v_6 is the only vertex of degree two then $c(\Delta) \leq c(2, 3, 3, 3, 4, 4) = 0$.

If $|v_3| = |v_6| = 2$ then their adjacent vertices cannot have degree three and so $c(\Delta) \leq c(2, 2, 4, 4, 4, 4) = 0$.

If $|v_2| = |v_5| = 2$ then v_1 and v_4 have degree at least four and so at least one of v_3 and v_5 must have degree three. If $|v_3| = 3$ then $l_3 = e^{-1}d(da)^{-1}$ and $e = dad^{-1}$. Now the degree of v_4 becomes $l_4 = b^{-1}(eb)c^{-1}w$ and it cannot have degree four or five. The same applies for v_6 i.e. if $|v_6| = 3$ then $|v_1| \geq 6$. Therefore, $c(\Delta) \leq \{c(2, 2, 3, 4, 4, 6), c(2, 2, 3, 3, 6, 6)\} = 0$.

If $|v_2| = |v_6| = 2$ then $l_5 = cb^{-1}w$ and $l_1 = d^{-1}(eb)c^{-1}w$ and none of these two vertices can have degree two, so at least one of v_3 and v_4 must have degree three. If $|v_3| = 3$ then $l_3 = e^{-1}d(da)^{-1}$ and $e = dad^{-1}$ and if $|v_4| = 3$ then $l_4 = (eb)a^{\pm 1}(eb)$ and $e = a$. So v_3 and v_4 cannot have degree three at the same time. If $|v_3| = 3$ and $l_3 = e^{-1}d(da)^{-1}$ then the label of v_4 becomes $l_4 = b^{-1}(eb)w$ and it can not have degree four either. Also in this case any labelling of length four on v_5 forces a contradiction so $c(\Delta) \leq c(2, 2, 3, 4, 5, 5) < 0$. If $|v_4| = 3$ and $l_4 = (eb)a^{\pm 1}(eb)$ then the label of v_1 becomes $l_1 = d^{-1}(eb)c^{-1}w$ and it cannot have degree four. Also v_3 cannot have degree three or four and again $c(\Delta) \leq c(2, 2, 3, 4, 5, 5) < 0$.

F_2 Let Δ be an F_2-region of positive curvature when $c = d$. Vertices v_3 and v_6 can have degree two. If $|v_3| = 2$ then $l_2 = bc^{-1}w$ and its degree cannot be three. If v_4 has degree three then $l_4 \in \{e^{-1}(da)c^{-1}, e^{-1}(da)d^{-1}\}$ and $e = dad^{-1}$.

If v_5 has degree three then $l_5 \in \{ba^{\pm 1}(eb)^{-1}, b(da)^{-1}e^{-1}\}$ and $a = e$. Therefore v_4 and v_5 cannot have degree three at the same time. So if v_3 is the only vertex of degree two then $c(\Delta) \leq c(2,3,3,3,4,4) = 0$. The same applies with v_6 i.e. if v_6 is the only vertex of degree two a region of positive curvature cannot exist.

If $|v_3| = |v_6| = 3$ then v_2 and v_5 have degree greater or equal to four so at least one of v_1 and v_4 must have degree three with label $e^{-1}(da)c^{-1}$ or $e^{-1}(da)d^{-1}$ and so $e = dad^{-1}$. The vertex adjacent to such a vertex has label $d^{-1}bc^{-1}w$ or $e^{-1}bc^{-1}w$ and it can be seen that it cannot have degree four or five. Therefore, $c(\Delta) \leq \{c(2,2,3,4,4,6), c(2,2,3,3,6,6)\} = 0$.

$c = e$

F_1 Let Δ be an F_1-region of positive curvature when $c = e$. Now it is possible for v_1, v_3, v_4 and v_6 to have degree two but since two adjacent vertices cannot both have degree two Δ has at most two vertices of degree two. If $|v_2| = 2$ then $l_1 = (eb)d^{-1}w$ and it cannot have degree three. Also $l_3 = c^{-1}dw$ and it cannot have degree three either. If $|v_1| = 2$ then $l_2 = d^{-1}cw$ and $l_6 = db^{-1}w$ and none of them can have degree three. So Δ must have two vertices of degree two.

If $|v_1| = |v_4| = 2$ or if $|v_2| = |v_5| = 2$ then $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

If $|v_2| = |v_4| = 2$ then v_1, v_3 and v_5 have degree at least four so v_6 must have degree three. If $|v_6| = 3$ then $l_6 = d(da)^{-1}e^{\pm 1}$ and $e = dad^{-1}$. But the if $e = dad^{-1}$ then the degree of v_1 and v_3 must be greater than or equal to five and so $c(\Delta) \leq c(2,2,3,4,5,5) < 0$.

F_2 Let Δ be an F_2 region of positive curvature. Its vertices that can have degree four are v_3 and v_6. If $|v_6| = 3$ and so $l_6 = c(eb)^{-1}$ then $l_1 = c^{-1}(da)w$ and $l_5 = bd^{-1}w$ and it can be seen that none of them can have degree three. So it v_6 is the only vertex of degree two in Δ then $c(2,3,3,3,4,4) = 0$ and if both v_3 and v_6 have degree two $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

$d = ea$

F_1 Let Δ be an F_1-region of positive curvature when $d = ea$. Vertices v_1 and v_4 can have degree two with label $(eb)(da)^{-1}$. If $|v_4| = 2$ then $l_3 = dc^{-1}w$ and $l_5 = b^{-1}cw$. It can be seen that none of these two can have degree four and so v_4 is the only vertex of degree two in Δ then $c(2,3,3,3,4,4) = 0$ and if both v_4 and v_1 have degree two $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

F_2 Let Δ be an F_2-region of positive curvature when $d = ea$. Once again vertices v_1 and v_4 can have degree two with label $(da)(eb)^{-1}$. The labels are the same as with previous paragraph and the adjacent vertices cannot have degree less than four $c(\Delta) \leq c(2,3,3,3,4,4) = 0$ or $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

$c = ad$

F_1 Let Δ be an F_1-region of positive curvature when $c = ad$. Vertices v_2 and v_5 can have degree two with label $c(da)^{-1}$. If $|v_2| = 2$ then $l_1 = (eb)c^{-1}w$ and $l_3 = b^{-1}cw$. It can be seen that none of these two can have degree three and so if v_4 is the only vertex of degree two in Δ then $c(2,3,3,3,4,4) = 0$ and if both v_4 and v_1 have degree two $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

F_2 Let Δ be an F_2-region of positive curvature when $c = ad$. Now vertices v_1, v_3 v_4 and v_6 can have degree two with label $(da)c^{-1}$ or the inverse. If $|v_2| = 2$ then $l_1 = (eb)c^{-1}w$ and $l_3 = b^{-1}dw$ and none of these can have degree less than four. The same applies if $|v_6| = 2$ then v_1 and v_5 have degree at least four. If $|v_1| = 2$ then $l_2 = d^{-1}bw$ and $l_5 = cb^{-1}w$ and none of these can degree less than four. In any case the curvature of Δ is at most $c(\Delta) \leq c(2,3,3,3,4,4) = 0$ or $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

$d = a$

F_1 It can be seen that F_1 cannot have the label $b(da)^{-1}$ and no other label of length two can be found on any of its vertices, so $c(\Delta) \leq c(3, 3, 3, 3, 3, 3) = 0$.

F_2 Let Δ be an F_2-region of positive curvature when $a = d$. Now vertices v_1, v_2 v_4 and v_5 can have degree two with label $(da)b^{-1}$ or the inverse. If $|v_1| = 2$ then $l_2 = c^{-1}bw$ and $l_6 = ca^{-1}w$ and none of these can have degree less than four. The same applies if $|v_6| = 2$ then v_1 and v_5 have degree at least four. So if only one vertex has degree three the curvature of Δ is at most $c(\Delta) \leq c(2, 3, 3, 3, 4, 4) = 0$ and if $|v_1| = |\ v_4| = 2$ or $|v_3| = |v_6| = 2$ the region has curvature at most $c(\Delta) \leq c(2, 2, 4, 4, 4, 4) = 0$.

Let $|v_1| = |v_5| = 2$. Since v_2, v_4 and v_6 cannot have degree three, the degree of v_3 must be three. But this is not possible as the label would involve only one c and make it generated by the remaining elements. So $c(\Delta) \leq c(2, 2, 4, 4, 4, 4) = 0$.

$d = e$

F_1 Let Δ be an F_1-region of positive curvature when $d = e$. Now vertices v_1, v_3 v_4 and v_6 can have degree two with label $d(eb)^{-1}$ or the inverse. If $|v_1| = 2$ then $l_2 = e^{-1}cw$ and $l_6 = dc^{-1}w$ and none of these can have degree less than four. If $|v_3| = 2$ then $v_1 = cd^{-1}w$ and $l_4 = c^{-1}(eb)w$ and again none of these can have degree two. So if only one vertex has degree three the curvature of Δ is at most $c(\Delta) \leq c(2, 3, 3, 3, 4, 4) = 0$ and if $|v_1| = |\ v_4| = 2$ or $|v_3| = |v_6| = 2$ the region has curvature at most $c(\Delta) \leq c(2, 2, 4, 4, 4, 4) = 0$. Let $|v_1| = |v_3| = 2$. Since v_2, v_4 and v_6 cannot have degree three, the degree of v_5 must be three. Again, any labelling of length three would involve only one c and make it to be generated by the remaining elements. So $c(\Delta) \leq c(2, 2, 4, 4, 4, 4) = 0$.

F_2 It can be seen that F_2 cannot have the label $d(eb)^{-1}$ and no other label of length two can be found on any of its vertices, so $c(\Delta) \leq c(3, 3, 3, 3, 3, 3) = 0$.

It is concluded that if $a^2 = e^2 = 1$ there are no regions of positive curvature.

Overall, interior regions of positive curvature are encountered when one of the following mutually exclusive type of relators holds in group G.

1. $a^2 = e^k = 1$, $a = c = d$ with $k \geq 3$ (positive curvature is added across a $b - c$ edge.)

2. $a^2 = e^k = 1$, $c = e$, $d = e^2$ with $k \geq 3$ (positive curvature is added across a $d - e$ edge.)

3. $a^2 = e^k = 1$, $c = d = e^2$ with $k \geq 3$ (positive curvature is added across a $d - e$ edge.)

4. $a^2 = e^k = 1$, $d = e$ and $c = b = 1$ with $k \geq 3$ (positive curvature is added across a $d - e$ edge.)

5. $a^2 = e^k = 1$, $d = e$, $c = e^2$ with $k \geq 3$ (positive curvature is added across a $d - e$ edge.)

It was assumed that the order of a is two and it can be seen that if diagram D has an interior region of positive curvature the order of e is at least three i.e. no vertices e^2 can be found any where in D. It was proved that for each of these interior regions there is enough negative curvature to compensate for it. In the worst case the positive curvature can be given to more than one of the neighbours of the positive region, but during this process a region of negative curvature would always receives the positive curvature through the same edge. Therefore, a region of negative curvature can only receive positive curvature from at most one region. It should also be observed that the regions of negative curvature used for compensation of positive are always regions having five vertices (no deletion of vertex has been performed) and they have at most one vertex of degree two. So the total positive curvature of a diagram representing a counter example must be concentrated in the boundary regions.

Chapter 6

$r_2(t) = atbtctdt^{-1}et^{-1}$

It is assumed that equation $r_2(t) = 1$ does not have a solution. As with $r_1(t) = 1$ we first state the assumptions holding in this case:

(A1) Group G is generated by $\{a, b, c, d, e\}$

(A2) Group G is neither locally indicable nor residually finite, or a homomorphic image of a polycyclic by finite group. More specifically it does not have property X.

(A3) The star graph Γ_2 *contains at least one admissible path of length less than three and either* $a^2 = 1$ *or/and* $d^2 = 1$.

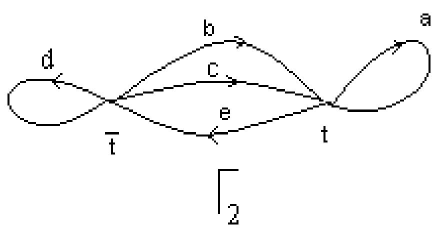


Figure 6-1:

(A4) $be \neq 1$ *and* $ce \neq 1$

Remark 6.1 *The following additional assumptions can be made for* $r_2(t) =$

$atbtctdt^{-1}et^{-1}$ *without any loss of generality:*

(A5) $b = 1$

As with $r_1(t) = 1$ we can make the substitution $s = tb$. Then $atbtctdt^{-1}et^{-1} = 1$, if and only if $assb^{-1}csb^{-1}dbs^{-1}ebs^{-1} = 1$. By taking $a' = a$, $b' = 1$, $c' = b^{-1}c$, $d' = b^{-1}db$, $e' = eb$, $r_2(t) = 1$ if and only if $a't^2c'td't^{-1}e't^{-1} = 1$ has a solution. Therefore b can be taken to equal 1 in G without any loss.

(A6) $a \neq 1,\ d \neq 1,\ e \neq 1$

If any a or d equals 1, then the equation $r_2(t) = 1$ collapses to an equation of smaller length for which a solution is known to exist. Since b can be taken to equal 1 and $be \neq 1$ it is obvious that $e \neq 1$.

(A7) $r_2(t) = 1$ *can be solved modulo the transformation* T_2

$r_2(t) = 1$ has a solution if and only if the equation $e^{-1}td^{-1}tc^{-1}t^{-1}b^{-1}t^{-1}a^{-1}t^{-1} = 1$ has a solution over G. Replacing t by t^{-1} means that we only need to solve the equation modulo the transformation (T_2).

$$a \to d^{-1}, b \to c^{-1}, c \to b^{-1}, d \to a^{-1}, e \to e^{-1} \quad (\mathrm{T}_2)$$

(A8) $a^2 = 1$

The case for which $d^2 = 1$ and $a^2 \neq 1$ is equivalent modulo (T_2) to the case when $d^2 = 1$ and $a^2 \neq 1$

Using the definitions of [E1] we can impose further assumptions on D with a non-trivial label on the distinguished vertex v_0 representing a counter example. A source is a vertex of D whose label involves only a and a sink is a vertex whose label involves only d. Let N_1 be the total number of sources and sinks in D and let N_2 denote the total number of vertices of degree two. The following assumption can also be imposed on diagram

(A9) Diagram D *contains minimal number of faces* Δ

(A10) The total number of sources N_1 *is maximal in* D.

(A11) The number of vertices of degree two N_2 is maximal in D.

These assumptions imply that if the order of a (or/and d) is two in G then the number of vertices with label a^2 (or/and d^2) is maximal. Also by the use of diamond moves it can be seen as for equation $r_1(t) = 1$ that, if the order of a is two there are no vertices with sublabel $(awa)^{\pm 1}$ where $w = 1$ in G. If the order of d is k, the number of vertices with label d^k is maximal. In the case that the order of d is three then there is no sublabel $(dw_1dw_2dw_3)^{\pm 1} = 1$ with $w_1 = w_2 = w_3 = 1$ in G.

Lemma 6.2 *$r_2(t) = 1$ has a solution in the following cases: (i) $abcde = 1$ (for example $a = e^{-1}$ and $d = c^{-1}$ or $c = a^{-1}$ and $e = d^{-1}$) (ii) $c = d$ and $e = a^{-1}d^{-1}$ (iii) $c = a$, $e = ad^{-1}$ and $a^2 = 1$ (iv) $a = e$, $c = d^{-1}a$ (v) $d = e$, $d = cac^{-1}$ and $a^2 = 1$*

Proof. i) In the case that $abcde = 1$ a solution of $r_2(t) = 1$ is $t = 1$ in G (ii) A solution to the equation would be $t = d^{-1}$ For (iii) and (iv) a solution to the equation is $t = a$ (v) A solution to the equation is $t = ac^{-1}$. ■

Now suppose that Δ is an interior region of positive curvature. The only vertex of Δ which cannot have degree two is v_5 since it was assumed that $ce \neq 1$ and $be \neq 1$. Possible labels on vertices of degree two are a^2, cb^{-1}, d^2 or their inverses. In the case that v_2 or v_3 has degree two the neighbouring two vertices cannot have degree two and so two adjacent vertices of Δ cannot both have degree two. (Therefore, Δ can have at most two vertices of degree two). The possible cases of an interior region Δ with $c(\Delta) > 0$ are considered according to whether Δ contains or does not contain vertices of degree 2. In Section 6.1. the cases of a region Δ that does not have any vertices of degree two are considered. In Section 6.2. the cases of regions which contain at least one vertex of degree two are considered into to two subsections: In Subsection 6.2.1. the regions that contain a vertex of degree two but not a^2 or d^2 are listed and in Subsection 6.2.2. the regions that contain a^2 or d^2 are listed.

6.1 Δ has no vertices of degree two

It follows from the curvature formula (*) that at least four vertices have degree 3 and the remaining vertex has degree 3, 4 or 5, otherwise $c(\Delta) \leq 0$ since $c(3,3,3,4,4) = 0$ and $c(3,3,3,3,6) = 0$.

6.1.1 $|v_1| = |v_2| = |v_3| = |v_4| = 3$, $|v_5| \in \{3,4,5\}$

The label of v_1 will be $l_1 \in \{aec, aeb, ac^{-1}b, ac^{-1}e^{-1}, ab^{-1}c, ab^{-1}e^{-1}\}$. The label $l_1 = a^3$ is not possible since it was assumed that $a^2 = 1$. We examine each possible case in turn.

$l_1 = aec$

Now $c = e^{-1}a$ and $l_2 = dbw$ where w is either e or c^{-1}. If $l_2 = dbc^{-1}$ then $d = c$ and by Lemma 6.2(ii) the equation has a solution. So $l_2 = dbe$ and $d = e^{-1}$, $c = d^{-1}a$ and $l_3 = dc\omega$ where ω is either b^{-1} or e. In any of this cases the group is forced to be cyclic, a contradiction. So if $l_1 = aec$ then $|v_1| = |v_2| = |v_3| = 3$ is not possible.

$l_1 = aeb$

Now $a = e$ and $l_2 \in \{dbc^{-1}, dbe\}$. We check for each case. First let $l_2 = dbc^{-1}$. Now $d = c$ and the order of d must be greater than two, otherwise G has property X, a contradiction. Also, $l_3 \in \{d^{-1}cb^{-1}, d^{-1}ce\}$. In the case that $l_3 = d^{-1}ce$ G is forced to be cyclic, a contradiction. So $l_3 = d^{-1}cb^{-1}$ and $l_4 \in \{c^{-1}db, c^{-1}de^{-1}\}$. Once again the second label forces the group to be cyclic so it must be $l_4 = c^{-1}db$. So $l_5 = aecw$ and the order of v_5 cannot be three (G would be forced to be cyclic) or four (this would force a^2 to be a proper sublabel) so it must be five. So $l_5 \in \{aecb^{-1}c, aecb^{-1}e^{-1}, aeceb, aecec\}$. Any of these labels forces G to be have property X, a contradiction. So $l_2 = db^{-1}c$ is not possible. Now let $l_2 = db^{-1}e^{-1}$

and so $d = a = e$ and $l_3 \in \{dce, dcb^{-1}\}$. In any case d is generated by the other elements and G is forced to be cyclic, a contradiction.

$l_1 = ac^{-1}b$

Now $a = c$ and $l_2 \in \{d^{-1}bc^{-1}, d^{-1}be\}$. In the case $l_2 = d^{-1}bc^{-1}$, $a = c = d$ the degree of v_5 would be greater than five as any effort of labelling it with degree three, four or five would make e to be generated by the rest of the elements and therefore group G will be cyclic. In the case $l_2 = d^{-1}be$ we have the relators $a = c$ and $d = e$ holding in G so the order of d must be greater two. But $l_3 = dcw$ where w is e or b^{-1} and any of these makes the group cyclic, a contradiction.

$l_1 = ac^{-1}e^{-1}$

Now $c = e^{-1}a$ and $l_2 \in \{d^{-1}bc^{-1}, d^{-1}be\}$. In the case $l_2 = d^{-1}be$ we have $d = e$ and $l_3 = dcw$ where w is either e or b^{-1}. If $l_3 = dce$ then G is forced to be cyclic and if $l_3 = dcb^{-1}$ then $ec = 1$, a contradiction. In the case $l_2 = d^{-1}bc^{-1}$ we have $d = c^{-1} = ae$. Also $l_3 \in \{d^{-1}ce, d^{-1}cb^{-1}\}$. If $l_3 = d^{-1}ce$ then $e = c^{-2}$ and G is forced to be cyclic. So $l_3 = d^{-1}cb^{-1}$ and $d = c = ae$ with $d^2 = 1$ and the group is forced to have property X. So $|v_1| = |v_2| = |v_3| = 3$ is not possible when $l_1 = ac^{-1}e^{-1}$.

$l_1 = ab^{-1}c$

Now $a = c$ and $l_2 \in \{c^{-1}ba, c^{-1}ba^{-1}\}$. Any of these labels is possible and we check each one in turn. If $l_2 = c^{-1}ba$ then $l_3 = eca^{\pm 1}$ and this would force $e = 1$, a contradiction. If $l_2 = c^{-1}ba^{-1}$, $l_3 = b^{-1}ca^{\pm 1}$. If $l_3 = b^{-1}ca$ then $l_4 \in \{edb, edc\}$. If $l_4 = edb$ then $e = d^{-1}$ and by Lemma 6.2(i) the equation has a solution. If $l_4 = edc$ then $e = ad^{-1}$ and again by Lemma 6.2(ii) the equation has a solution. So $l_3 = b^{-1}ca^{-1}$ and $l_4 \in \{b^{-1}dc, b^{-1}de^{-1}\}$. We check for each of these two labels in turn. If $l_4 = b^{-1}dc$ then $c = a = d$ and effort of labelling v_5 with degree three, four or five would cause a contradiction by making the group to have property X. If $l_4 = b^{-1}de^{-1}$ then $d = e$ and $l_5 = a^{-1}edw$. It is obvious that $|\upsilon_5| \neq 3$ and if $|\upsilon_5| \in \{4, 5\}$ then $l_5 \in \{a^{-1}edb, a^{-1}edc, a^{-1}ede^{-1}, a^{-1}ed^2b, a^{-1}ed^2c, a^{-1}ed^2e^{-1}\}$. From these labels the only one not forcing G to have property X or probably to collapse to the trivial group is $l_5 = a^{-1}ed^2c$. This region is presented and discussed at the end of this section.

$l_1 = ab^{-1}e^{-1}$

Now $a = e$ and $l_2 = c^{-1}ba^{\pm 1}$ so $a = c = e$. But then $ce = 1$, a contradiction. So $|\upsilon_1| = |\upsilon_2| = 3$ is not possible for $l_1 = ab^{-1}e^{-1}$.

6.1.2 $|\upsilon_2| = |\upsilon_3| = |\upsilon_4| = |\upsilon_5| = 3$, $|\upsilon_1| \in \{4, 5\}$

Since $|\upsilon_2| = 3$ then $l_1 \in \{ba^{\pm 1}c^{-1}, bc^{-1}d^{\pm 1}, bed^{\pm 1}\}$. The label $ba^{\pm 1}e$ is not possible since it forces a^2 to be a proper sublabel on υ_1, a contradiction. We examine each of the remaining cases in turn.

$l_2 = bac^{-1}$

Now $a = c$ and $l_3 = eca^{\pm 1}$ and so $e = 1$, a contradiction. It is concluded that if $l_2 = bac^{-1}$ then $|\upsilon_2| = |\upsilon_3| = 3$ is not possible.

$l_2 = ba^{-1}c^{-1}$

Once again $a = c$ and the label of υ_3 is $l_3 = b^{-1}ca^{\pm 1}$. If $l_3 = b^{-1}ca$ then $l_4 \in \{edc^{-1}, edb^{-1}\}$. In the first case $d = e^{-1}$ and by Lemma 6.2(i) the equation has a solution. So $l_4 = edc^{-1}$ and $l_5 = bed^{\pm 1}$ and G is cyclic, a contradiction. If $l_3 = b^{-1}ca^{-1}$ then $l_4 \in \{b^{-1}dc, b^{-1}de^{-1}\}$. In the first case $a = c = d$ and any label on υ_5 makes e to be generated by the other elements and thus the group cyclic. In the second case $d = e$ and $l_5 \in \{a^{-1}eb, a^{-1}ec\}$ and any of these forces a contradiction by making the group cyclic or by forcing $c = 1$.

$l_2 = bc^{-1}d$

Now $c = d$ and $l_3 \in \{d^{-1}ce, d^{-1}cb^{-1}\}$. The first label makes $e = 1$, a contradiction and so $l_3 = d^{-1}cb^{-1}$. So $l_4 = c^{-1}db$ so $l_5 \in \{aeb, aec\}$. The first label would force a^2 being a proper sublabel on v_1, a contradiction while the second would make $e = ad^{-1}$ and then by Lemma 6.2(ii) the equation has a solution.

$l_2 = bc^{-1}d^{-1}$

Now $c = d^{-1}$ and $l_3 \in \{d^{-1}cb^{-1}, d^{-1}ce\}$. Let $l_3 = d^{-1}cb^{-1}$, so $d^2 = 1$ and $l_4 \in \{c^{-1}db, c^{-1}de^{-1}\}$. The second label forces a contradiction by making $e = 1$ so $l_4 = c^{-1}db$ and $l_5 = aec$. Now e is generated by a and d and the group has property X. Let $l_3 = d^{-1}ce$ and so $e = d^2$. Now $l_4 = d^3$ and $l_5 \in \{ced^{-1}, ced\}$. The second label forces $e = 1$ while the second one is possible. So $l_1 = e^{-1}ac^{-1}w$ and $|v_1| \in \{4, 5\}$. Since a^2 cannot be a proper sublabel the label of v_1 would contain letter a just once and therefore would force a to be a power of d and thus make G cyclic, a contradiction.

$l_2 = bed$

Now $e = d^{-1}$ and $l_3 \in \{dcb^{-1}, dce\}$. Let $l_3 = dcb^{-1}$ and so $c = e = d^{-1}$. Now $l_4 \in \{c^{-1}de^{-1}, c^{-1}db\}$. The second label forces the order of d to be two and therefore $ec = 1$, a contradiction. So $l_4 = c^{-1}de^{-1}$ and $d^3 = 1$. So $l_5 = a^{-1}ew$ and any label on v_5 would involve a^{-1} once and thus make a to be generated a power of d and so the group cyclic, a contradiction.

$l_2 = bed^{-1}$

Now $d = e$ and $l_3 \in \{dce, dcb^{-1}\}$. The second label forces a contradiction by making $ce = 1$ and the second makes $c = e^{-2}$. Now, any effort at labelling v_1 with a label of length less than six, forces the group to be cyclic, since it would involve a just once.

6.1.3 $|\upsilon_1| = |\upsilon_3| = |\upsilon_4| = |\upsilon_5| = 3$, $|\upsilon_2| \in \{4, 5\}$

Now $l_1 \in \{aec, aeb, ab^{-1}c, ab^{-1}e^{-1}, ac^{-1}b, ac^{-1}e\}$. We check for each possible case:

$l_1 = aec$

Now $c = e^{-1}a$ and $l_5 = edw$ and so $l_5 \in \{edb, edc\}$. If $l_5 = edb$ then $e = d^{-1}$ and $l_4 \in \{dcb^{-1}, dce\}$. Any of these words makes c to be generated by e and so a is also generated by e so the group is cyclic, a contradiction. If $l_5 = edc$ then $d = e^{-1}ae$ and the order of d must be two. But $l_4 = d^3$ which forces $d = 1$, a contradiction. So $|\upsilon_1| = |\upsilon_4| = |\upsilon_5| = 3$ is not possible when $l_1 = aec$.

$l_1 = aeb$

Now $a = e$ and $l_5 = eca^{\pm 1}$ so $b = c = 1$. Now any effort of labelling υ_4 makes the group cyclic, as l_4 would involve letter d only once and thus making d generated by a. So $|\upsilon_1| = |\upsilon_4| = |\upsilon_5| = 3$ is not possible when $l_1 = aeb$.

$l_1 = ab^{-1}c$

Now $a = c$ and $l_5 \in \{edb, edc\}$. In the first case $e = d^{-1}$ and in the second case $d = e^{-1}a$ and by Lemma 6.2(i)and (ii) the equation has a solution. So $|\upsilon_1| = |\upsilon_5| = 3$ is not possible when $l_1 = ab^{-1}c$.

$l_1 = ab^{-1}e^{-1}$

Now $a = e$ and $l_5 \in \{ed^{-1}b, ed^{-1}c\}$. If $l_5 = ed^{-1}b$ then $a = d = e$ and $l_4 = dcw$, therefore any labelling of degree three on υ_3 would make the group cyclic by forcing c to be generated by the other elements. If $l_5 = ed^{-1}c$ then $c = da$ and $l_4 = d^3$. The label of υ_3 would be $l_3 = ced^{\pm 1}$ and this is possible only for the negative power of d, i.e. $l_3 = ced^{-1}$ since in the opposite case $d = 1$. The label of υ_2 would be $l_2 = c^{-1}bc^{-1}w$ and $|\upsilon_2| \in \{4, 5\}$ implies $l_2 \in$

$\{c^{-1}bc^{-1}b, c^{-1}bc^{-1}e^{-1}, c^{-1}bc^{-1}d^{\pm1}b, c^{-1}bc^{-1}d^{\pm1}e^{-1}, c^{-1}bc^{-1}ba^{\pm1}, c^{-1}bc^{-1}e^{-1}a^{\pm1}\}$. Any of these labels makes the group to have property X, a contradiction.

$l_1 = ac^{-1}b$

Now $a = c$ and $l_5 = eca$ and so $e = 1$, a contradiction. So $|v_1| = |v_5| = 3$ is not possible when $l_1 = ac^{-1}b$.

$l_1 = ac^{-1}e^{-1}$

Now $c = e^{-1}a$ and $l_5 \in \{ed^{-1}b, ed^{-1}c\}$. If $l_5 = ed^{-1}b$ then $d = e$ and $l_4 \in \{dce, dcb^{-1}\}$. Any of these labels forces a contradiction by making the group cyclic. If $l_5 = ed^{-1}c$ then $d = e^{-1}ae$ and the order of d must be two. But $l_4 = d^3$, a contradiction. Once again $|v_1| = |v_4| = |v_5| = 3$ is not possible for $l_1 = ac^{-1}e^{-1}$.

6.1.4 $|v_1| = |v_2| = |v_4| = |v_5| = 3$, $|v_3| \in \{4, 5\}$

It can be seen from the previous section that the only possible cases for $|v_1| = |v_5| = |v_4| = 3$ is $l_1 = ab^{-1}e^{-1}$, $l_5 = ed^{-1}c$ and $l_4 = d^3$ and the relators holding in the group are $a = e$, $c = da$ and $d^3 = 1$. Now, $|v_2| = 3$ and so the label of v_2 must be $l_2 = c^{-1}ba^{\pm1}$ which forces a contradiction by making $d = 1$.

6.1.5 $|v_1| = |v_2| = |v_3| = |v_5| = 3$, $|v_4| \in \{4, 5\}$

It can be seen from 6.1.1. and 6.1.3. that this is only possible for $l_1 = aeb$. Let $l_1 = aeb$. Now $a = e$ and $l_2 \in \{dbc^{-1}, dbe\}$ and $c = d$ or $d = e = a$. But $l_5 = eca$ which implies $c = 1$ and forces a contradiction in either case.

Overall, if a region does not have any vertices of degree two, it can only have positive curvature in the case $l_1 = ab^{-1}c$, $l_2 = c^{-1}ba^{-1}$, $l_3 = b^{-1}ca^{-1}$, $l_4 = b^{-1}de^{-1}$ and $l_5 = a^{-1}ed^2c$. The relators holding in G are $a = c$, $d = e$ and $a^2 = d^3 = 1$.

This region has curvature $c(\Delta) = c(3,3,3,3,5) = \frac{\pi}{15}$. We look for negative curvature for this region in its neighbouring region Δ_1 as in Figure 6-2.

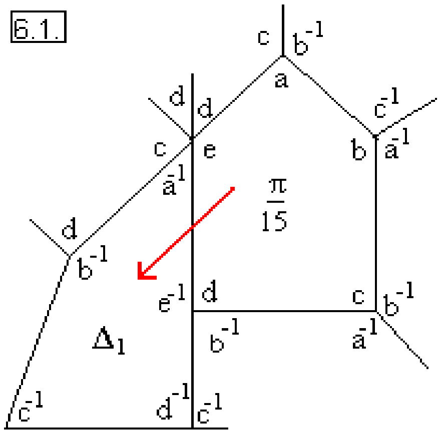


Figure 6-2:

The label of $v_{4\Delta_1}$ starts with $c^{-1}d^{-1}$ and any effort of finding a label of length three or four makes the group to have property X. So the curvature of Δ_1 will be $c(\Delta_1) \leq c(3,3,3,5,5) = -\frac{\pi}{5}$ which is enough to compensate for the positive curvature of Δ. The positive curvature is added to Δ_1 across an $a^{-1} - e^{-1}$ edge.

6.2 Δ has vertices of degree two

The only vertex of Δ which cannot have degree two is v_5 since it was assumed that $ce \neq 1$ and $be \neq 1$. Possible labels on vertices of degree two are a^2, cb^{-1}, d^2 or their inverses. In the case that v_2 or v_3 has degree two the neighbouring two vertices cannot have degree two and so two adjacent vertices of Δ cannot both have degree two. (Therefore, Δ can have at most two vertices of degree two). The type of regions Δ containing at least one vertex of degree two are shown in Figure 6-3.

First we examine whether it is possible to have regions of positive curvature

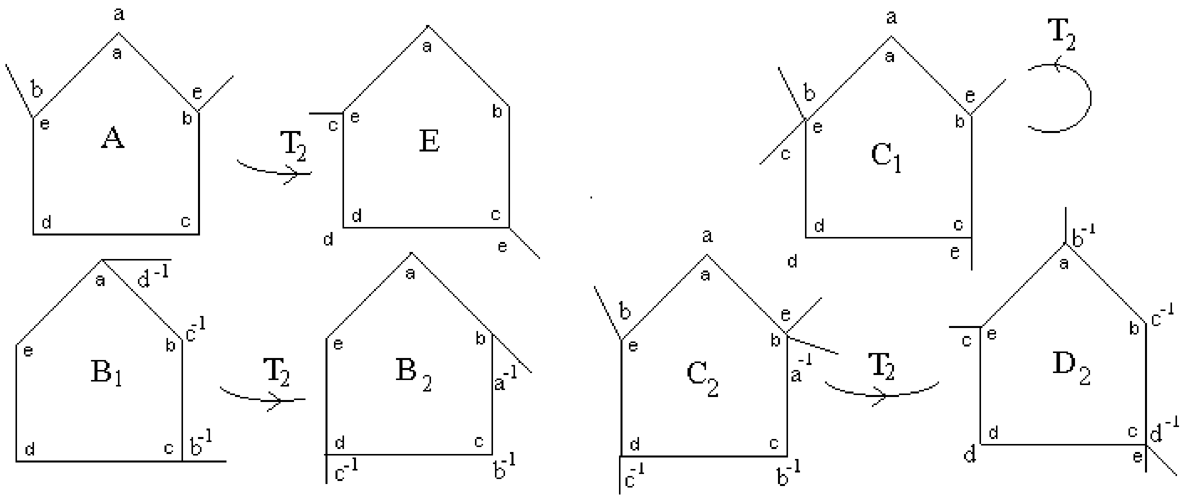


Figure 6-3:

with vertices of degree two in the case that somewhere in diagram D there is a region of positive curvature without vertices of degree two. The relators holding in this case will be $a = c$ and $d = e$, $a^2 = d^3 = 1$ and therefore the only possible vertex of degree two is v_1 with label a^2. So this region can only be of A-type. The labels are $l_2 = ebw$ and $l_5 = ebw$ and so none of v_2 and v_5 can have degree three. So at least one of v_3 and v_4 must have degree three otherwise $c(\Delta) \leq c(2,4,4,4,4) = 0$. If $|v_3| = 3$ then $l_3 = ca^{\pm 1}b^{-1}$ and so $l_2 = eba^{-1}w$. The degree of v_2 cannot be four and any label of length five would make the group to have property X. So in the case that $|v_3| = 3$ the degree of v_2 must be $|v_2| \geq 6$. If $|v_4| = 3$ then $l_4 \in \{d^3, de^{-1}b\}$ and the label of v_5 will be $l_5 \in \{cebw, a^{-1}eb\}$. We check whether it is possible to obtain a label of length four or five on v_5. Now $l_5 \in \left\{ \begin{array}{c} cebe, ceba^{\pm 1}b^{-1}, ceba^{\pm 1}e, cebed^{\pm 1}, cebc^{-1}e^{\pm 1}, \\ a^{-1}ebc^{-1}b, a^{-1}ebc^{-1}e^{-1}, a^{-1}ebeb, a^{-1}ebec^{-1} \end{array} \right\}$. But any of these labels forces a contradiction by making $d = 1$ or the group to be have property X. So if $|v_4| = 3$ then $|v_5| \geq 6$. Therefore, the region will have curvature at most $c(\Delta) \leq \max\{c(2,3,4,4,6), c(2,3,3,6,6)\} = 0$. It is concluded that a region of positive curvature when $a = c$ and $d = e$, $a^2 = d^3 = 1$ cannot have vertices of degree two, and thus it will only be like the one of Figure 6-2 in section 6.1.5.

Therefore, if there is at least one region of positive curvature in the diagram without vertices of degree two, then all the other regions of positive curvature in D do not have vertices of degree two. If there are any vertices of degree two in D then the two adjacent faces must have negative curvature.

For the remaining cases of regions with positive curvature it can be assumed that $a = c$, $d = e$ and $a^2 = d^3 = 1$ is not true. We will consider the two cases of such regions, according to whether they contain a label a^2 or d^2 on Δ or not.

6.2.1 Δ does not contain a^2 or d^2

These cases are B_1 and B_2 of Figure 6-3. It can be assumed without loss that $2 = |a| \leq |d|$ since the regions for which $2 = |d| < |a|$ will be equivalent according to T_2 to those for which $2 = |a| < |d|$ in the other B- type region.

$\mathbf{B}_1$

Now the relators holding in the group are $a^2 = 1$ and $c = b = 1$. We look for possible vertices of degree three. We examine the cases according to the possible number of vertices of degree three that a region may have.

Δ contains at least three vertices of degree three First suppose $|v_1| = 3$ and so $l_1 \in \{ab^{-1}c, ab^{-1}e^{-1}\}$. The first label forces $a = 1$, a contradiction so if $|v_1| = 3$ then $l_1 = ab^{-1}e^{-1}$ and $a = e$. If at the same time $|v_5| = 3$ then $l_5 \in \{ed^{-1}c, ed^{-1}b\}$ which forces the group to be generated by a. So v_1 and v_5 cannot have degree three at the same time. If $|v_3| = 3$ then l_3 cannot be $d^{-1}cb^{-1}$ so it should be $l_3 = d^{-1}ce$ and $d = e$. So it is not possible for v_1 and v_3 to have degree three at the same time.

So it must be $|v_3| = |v_4| = |v_5| = 3$ and the degree of v_1 should be greater than or equal to four. The relations holding in G are $b = c = 1$, $d = e$, $a^2 = d^3 = 1$. The label of υ_1, υ_3, υ_4 and υ_5 are $l_1 = e^{-1}ab^{-1}w$, $l_3 = d^{-1}ce$, $l_4 = d^3$ and $l_5 = ced^{-1}$. If the degree of υ_1 is less than six then element a is contained only once in the label l_1 and this would imply that a is generated by d and therefore G is cyclic. If the label of υ_1 contained another $a^{\pm 1}$ a relator of the type $ad^{\pm 1}ad^{\pm 1}...ad^{\pm 1}$ would emerge. Since we do not allow labels like $a^{\pm 1}w_1a^{\pm 1}w_2$ with $w_1 = w_2 = 1$, the order of υ_1 must be at least 18 as smaller degrees would make the group to have property X. Therefore, a region like that will have curvature at most $c(2, 3, 3, 3, 18) = \frac{\pi}{9}$. This region (6.2.) will be examined with the other regions (6.3. and 6.4.) of positive curvature found under the same type of relators (see Figure 6-4).

Δ contains exactly two vertices of degree three Since $|\upsilon_1| = |\upsilon_3| = 3$ and $|\upsilon_1| = |\upsilon_5| = 3$ are not possible we examine the following remaining case of two vertices having degree two: $|\upsilon_1| = |\upsilon_4| = 3$, $|\upsilon_3| = |\upsilon_4| = 3$, $|\upsilon_3| = |\upsilon_5| = 3$, $|\upsilon_4| = |\upsilon_5| = 3$.

$|\upsilon_1| = |\upsilon_4| = 3$ Now $l_1 = ab^{-1}e^{-1}$ and $a = e$. Also $l_4 = d^3$ otherwise d is generated by a and this would make G cyclic. So $l_4 = d^3$ and $l_3 = d^{-1}cew_1$, $l_5 = ced^{-1}w_2$. We look at the possible labelling of υ_3 and υ_5 for degrees up to five. It turns out that $l_3 \in \{d^{-1}ced^{-1}, d^{-1}ced^{-2}, d^{-1}cebe, d^{-1}cebc^{-1}, d^{-1}cece, d^{-1}cecb^{-1}\}$ but any of this labels forces a contradiction by making d to be generated by a and the group cyclic. So the degree of υ_3 must be at least six . The label of υ_5 for degree four and five is $l_5 \in \{ced^{-2}, ced^{-3}, ced^{-1}be, ced^{-1}cb^{-1}, ced^{-1}ce, ced^{-1}e^{-1}b^{-1}\}$ and again any of these labels forces a contradiction for the same reason as before. Therefore, $|\upsilon_3|$, $|\upsilon_5| \geq 6$ and so $c(\Delta) \leq c(2, 3, 3, 6, 6) = 0$.

$|\upsilon_3| = |\upsilon_4| = 3$ Now $l_3 = d^{-1}ce$ and $l_4 = d^3$. The relators holding in group G are $b = c = 1$, $d = e$ and $a^2 = d^3 = 1$. Now, the label of υ_1 is $l_1 = ab^{-1}w$ and its degree is at least 18, otherwise the group will have property X, and so

$c(2,3,3,4,18) < 0$.

$|v_3| = |v_5| = 3$ $l_3 = d^{-1}ce$ and $d = e$. This forces $l_5 \in \{ed^{-1}b, ed^{-1}c\}$, so $l_1 = e^{-1}ab^{-1}w_1$, and $l_4 \in \{d^2cw_2, d^3w_3\}$. Now the degree of v_1 will have to be at least eight so the degree of v_4 must be four otherwise $c(\Delta) \leq c(2,3,3,5,8) < 0$. So $l_4 \in \{d^2ce, d^4\}$ and the order of d is either three or four. In any case the degree of v_1 is forced to be twelve or greater and so $c(\Delta) \leq c(2,3,3,4,12) = 0$.

$|v_4| = |v_5| = 3$ The label of v_4 would be $l_4 \in \{d^3, dbe, dce, de^{-1}b, de^{-1}c\}$. We look at each case. If $l_4 \in \{dbe, de^{-1}b, de^{-1}c\}$ then $d = e$ or $d = e^{-1}$ and any labelling on v_5 of length three would make the group cyclic. If $l_4 = dce$ then $e = d^{-1}$ and $l_5 = bed^{\pm 1}$. This is not possible for $l_5 = bed^{-1}$ as this would make the order of d to be two and the group to be generated by two elements of degree two. So $l_5 = bed$ and $l_1 = cab^{-1}w_1$ and the degree of v_1 cannot be less than eight. The label of v_3 is $l_3 = d^{-1}caw_2$ and also involves letter a so its degree must be eight or greater so in this case then the region cannot have positive curvature. Therefore the label of v_4 must be $l_4 = d^3$ and $l_5 \in \{ced, ced^{-1}\}$. But now that the order of d is three and v_1 must have degree greater or equal to 18, and so $c(\Delta) \leq c(2,3,3,4,18) < 0$.

Δ contains exactly one vertex of degree three We examine the cases for each of v_1, v_3, v_4 and v_5 being the only vertex of degree three. Since $c(2,3,4,5,5) < 0$ we look for at least two of the other vertices having degree four.

v_1 is the only vertex of degree three The label of v_1 can only be $l_1 = ab^{-1}e^{-1}$ and $a = e$. If $|v_5| = 4$ then $l_5 \in \{ed^{-2}b, ed^{-2}c, ed^{-1}ba^{\pm 1}, ed^{-1}ca^{\pm 1}, ed^{-1}e^{-1}a^{\pm 1}\}$. Any of these labels makes the group to have property X, and therefore $|v_5| \geq 5$. If $|v_3| = 4$ then $l_3 \in \{d^{-1}ca^{\pm 1}b^{-1}, d^{-1}ca^{\pm 1}c^{-1}, d^{-1}ca^{\pm 1}e, d^{-1}cb^{-1}d^{-1}, d^{-1}ced^{-1}\}$.

Once again any of these relators make the group to have property X, so it must be $|v_3| \geq 5$. Therefore, the region cannot have positive curvature.

v_3 is the only vertex of degree three The label of v_3 must be $l_3 = d^{-1}ce$ and $d = e$. The label of v_1 is $l_1 = ab^{-1}w_1$ and its degree cannot be less than six, as this would make the group to have property X. So $c(\Delta) \leq c(2, 3, 4, 4, 6) = 0$.

v_4 is the only vertex of degree three The label of v_4 can be one of $l_4 \in \{d^3, dbe, dce, de^{-1}b^{-1}, de^{-1}c^{-1}\}$. For the last four labels we have the relator $e = d^{\pm 1}$ and as before no label of degree less than six is possible on v_1 and a region of positive curvature cannot be obtained. So the only possible label is $l_3 = d^3$. If $|v_3| = 4$ then $l_3 = d^{-1}ced^{-1}$ and $e = d^2 = d^{-1}$. As before the degree of v_1 would forced to be at least six and $c(\Delta) \leq c(2, 3, 4, 4, 6) = 0$. If $|v_3| = 5$ then $l_5 \in \{d^{-1}cebc^{-1}, d^{-1}cebe, d^{-1}cecb^{-1}, d^{-1}cece\}$. Any of these forces $d = e$ or $d = e^{-2}$ and once again the degree of v_1 must be at least six. Therefore, no regions of positive curvature are possible here.

v_5 is the only vertex of degree three The label of v_5 must be one of $l_5 \in \{eca^{\pm 1}, ed^{\pm 1}b, ed^{\pm 1}c\}$ and $a = e$ or $e = d^{\pm 1}$. If $d = e^{\pm 1}$ the degree of v_1 is forced to be at least six and the region cannot have positive curvature. So $l_5 = eca^{\pm 1}$ and $l_1 = bab^{-1}w$ so its degree cannot be four. The label of v_4 will be $l_4 \in \{dbw, de^{-1}w\}$. If $|v_4| = 4$ and element d is encountered only once in the label of v_4 then it is generated by a and thus the group is cyclic. Otherwise $l_4 \in \{dbc^{-1}d, de^{-1}b^{-1}d, de^{-1}c^{-1}d\}$ and the group is anyway forced to have property X. So no regions of positive curvature can be found.

$\mathbf{B}_2$

Again the relators holding in the group are $a^2 = 1$ and $c = b = 1$. Once more we look for possible vertices of degree three.

Δ contains at least three vertices of degree three If $|v_2| = 3$ then its label will be $l_2 \in \{ba^{-1}e, ba^{-1}c^{-1}\}$. The second label forces $a = 1$, a contradiction, and so it must be $l_2 = ba^{-1}e$ and $a = e$. But this label would force a^2 to be a proper sublabel on v_1 something which we do not allow, so $|v_2| \geq 4$. $|v_4| = 3$ implies that $l_4 \in \{c^{-1}db, c^{-1}de^{-1}\}$. The first label forces a contradiction while the second gives $d = e$. Now, any label of degree three on v_1 would force $a = e$. So it is not possible to have $|v_1| = |v_4| = 3$ because then the group would turn out to be cyclic. It is concluded that a region cannot have more than two vertices of degree three.

Δ contains exactly two vertices of degree three Since v_2 cannot have degree three and $|v_1| = |v_4| = 3$ is not possible, the cases considered are $|v_1| = |v_5| = 3$ and $|v_4| = |v_5| = 3$.

$|v_1| = |v_5| = 3$ The label of v_1 is $l_1 \in \{ab^{-1}e^{-1}, ac^{-1}e^{-1}, aeb, aec\}$ and in any case $a = e$. If $l_1 \in \{ab^{-1}e^{-1}, ac^{-1}e^{-1}, aec\}$ then $l_5 \in \{ed^{\pm1}b, ed^{\pm1}c\}$ and any of these makes the group cyclic. So it must be $l_1 = aeb$ and $l_5 = eca^{\pm1}$. Now the label of v_2 involves letter a and d and therefore must have degree at least eight. So the degree of v_4 must be four and cannot involve $a^{\pm1}$ since this would make the group cyclic. But this makes labelling of v_4 impossible.

$|v_4| = |v_5| = 3$ The label of v_4 is $l_4 = c^{-1}de^{-1}$ and $d = e$. The label of v_5 will be $l_5 \in \{a^{-1}eb, a^{-1}ec\}$ which makes $a = d = e$ and therefore the group cyclic, a contradiction.

Δ contains only one vertex of degree three Since v_2 cannot have degree three we examine the cases for the remaining vertices being the only vertex of degree three.

v_1 **is the only vertex of degree three** $l_1 \in \{ab^{-1}e^{-1}, ac^{-1}e^{-1}, aeb, aec\}$ and in any case $a = e$. If $l_1 \in \{ab^{-1}e^{-1}, ac^{-1}e^{-1}, aec\}$ and $l_5 = ed^{\pm 1}w$. If the degree of v_5 is less than six and the word contains only ones $d^{\pm 1}$ then d is generated by a and G is cyclic. If it is contained twice then $d^2 = 1$ and the group has property X. The only case in which it is contained three times is for $l_5 \in \{ed^{\pm 3}b, ed^{\pm 3}c\}$ but then G becomes cyclic generated by d. If the degree of v_5 is grater than six the region cannot have positive curvature so it must be $l_1 = aeb$. But then the label of v_2 is $l_2 = dba^{-1}w$ and like before its degree cannot be four or five. Therefore, a region of this type cannot have positive curvature.

v_4 **is the only vertex of degree three** The label of v_4 must be $l_4 = c^{-1}de^{-1}$ and $d = e$. But then $l_5 = a^{-1}ew$ and therefore the degree of v_5 cannot be four or five. So it is not possible to obtain a region of positive curvature of this type either.

v_5 **is the only vertex of degree three** The label of v_5 must be one of $l_5 \in \{ed^{\pm 1}b, ed^{\pm 1}c, eca^{\pm 1}\}$ and either $e = d^{\pm 1}$ or $e = a$. If $l_5 = eca^{\pm 1}$ and $e = a$ and if the order of v_4 is four or five it will involve letter d only once and so d will be generated by the remaining elements. If $e = d^{\pm 1}$ then none of v_1 and v_2 could have degree less than six as their labels would involve letter a once. So $c(\Delta) \leq c(2, 3, 4, 6, 6) < 0$.

It is concluded that it is not possible of a region of type B_2 to have positive curvature.

6.2.2 Δ contains the label a^2 or d^2

If Δ_1 is an interior region of positive curvature with at least one vertex of degree two being a^2 or d^2 then it will be one of regions A, E, C_1, C_2 and D_2 of Figure 6-3. As before it can be assumed without loss that $2 = |a| \leq |d|$. We first check whether there are interior regions of positive curvature for any of the type of relators holding for regions found in section 6.2.1. (i.e. $c = 1$, $d = e$, $a^2 = d^3 = 1$). Any such regions of positive curvature will be of type A or C_2 as the order of d cannot be two.

Suppose $c = 1$, $d = e$, $a^2 = d^3 = 1$ hold in the group and let Δ be a region of positive curvature with a^2 being one of its labels. If Δ has only one vertex of degree two then this should be v_1 (i.e. the region is of A-type). The labels of v_2 and v_5 start with eb so it is not possible for them to have degree three or four. This forces both of v_3 and v_4 to have degree three otherwise $c(\Delta) \leq c(2, 3, 4, 5, 5) < 0$. So $l_3 = ced^{-1}$ and $l_4 = d^3$. Now the label of v_2 is $l_2 = ebc^{-1}w$ and its degree cannot be three or four. If $|v_2| = 5$ then $l_2 = ebc^{-1}d^{-1}b$. But in this case *a diamond move* may be performed to increase the number of vertices of degree two without altering the number of faces or the number of sources and links, a contradiction (A11). So $|v_2| \geq 6$. The label of v_5 is $l_5 = cebw$ and so once again cannot be three or four. If $|v_5| = 5$ then $l_5 \in \{cebed, cebc^{-1}d^{-1}\}$. These two cases are possible and are examined with that found at section 6.2.1.

Now, suppose Δ has another vertex of degree two (i.e. it will be of C_2-type). Now the label of v_2 is $l_2 = eba^{-1}w$. If the degree of v_2 is less than six then it contains letter a only once and thus the group is cyclic. If letter a appears more than once and the length of the label is less than 18 then $l_2 = a^{-1}w_1a^{\pm1}w_2...a^{\pm1}w_k$ and G has property X. The degree of v_5 is at least five so the degree of v_4 must be three or four otherwise $c(\Delta) \leq c(2, 2, 5, 5, 18) < 0$. It turns out that the degree of v_4 cannot actually be four so it must be three and $l_4 = c^{-1}de^{-1}$. Now the label of v_5 is $l_5 = a^{-1}ebw$ and its degree must also be at least 18 . So $c(\Delta) < c(2, 2, 3, 18, 18) < 0$.

Therefore, any region of positive curvature that can be found in D under the relators $c = 1$, $d = e$, $a^2 = d^3 = 1$ will be one of regions 6.2.-6.4. in Figure 6-4.

We look for negative curvature in region Δ_1 as in Figure 6-5. If $v_{2\Delta_1}$ does not have degree two or if $v_{3\Delta_1}$ does not have degree three then the curvature of Δ_1

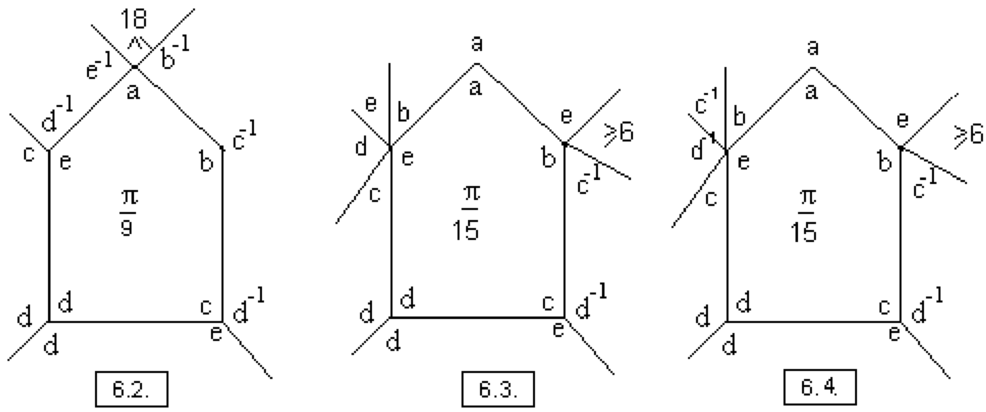


Figure 6-4:

is at most $c(\Delta_1) \leq \max\{c(2,3,3,5,18), c(3,3,3,3,18)\} \leq -\frac{\pi}{9}$ which is enough to compensate for the positive curvature of Δ in any of the cases 6.2.-6.4. Now suppose $v_{2\Delta_1}$ has degree two and $v_{3\Delta_1}$ has degree three. The curvature of Δ_1 is positive and $c(\Delta_1) \leq c(2,3,3,3,18) = \frac{\pi}{9}$ i.e. Δ_1 is a region like 6.2. and cannot be used for compensation of positive curvature. We look at neighbouring region Δ_2 for negative curvature. In the case of 6.2. Δ_2 can have two vertices of degree two but in that case but also two vertices of degree at least 18 and thus, its curvature is at most $c(\Delta_2) \leq c(2,2,3,18,18) = -\frac{\pi}{9}$. In the case of regions 6.3. and 6.4. the curvature of Δ_2 is at most $c(\Delta_2) \leq c(2,3,3,6,18) = -\frac{2\pi}{9}$. A region of negative curvature is always receiving the positive one across a $d - e$ or a $e^{-1} - d^{-1}$ so it is possible to receive negative curvature from at most one region.

Since the cases for which $c = 1$, $d = e$, $a^2 = d^3 = 1$ hold as relators in G where examined above, for the remaining of this chapter it is assumed that these relators do not hold. Suppose Δ_1 is a region of positive curvature with a^2 being one of its labels. Consider its neighbouring region Δ_2 with common the a^2 vertex. By deleting the common vertex and the two incident vertices, a new region Δ may be obtained in a new diagram $\hat{D}$. To achieve this a new element is added,

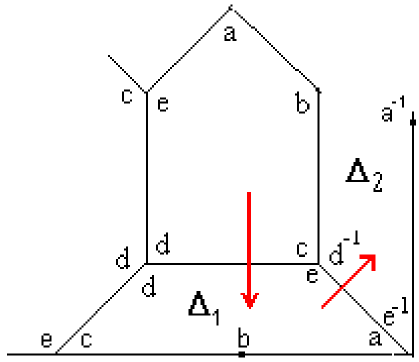


Figure 6-5:

more precisely (eb) and the new star graph $\hat{\Gamma}_2$ will be as in Figure 6-6.

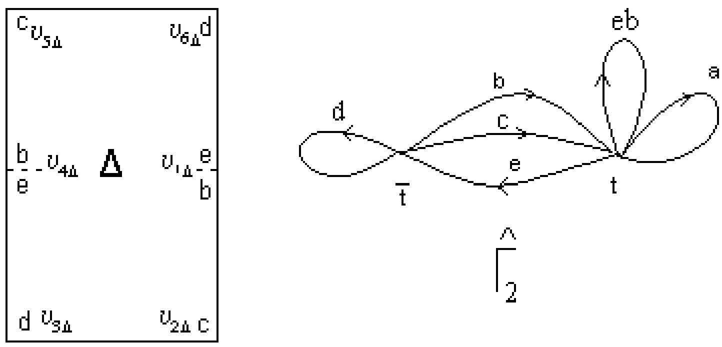


Figure 6-6:

The new element (eb) is a loop so it is possible to have a square loop there if $e^2 = 1$. For the remaining of this chapter it is assumed that a^2 was deleted everywhere in diagram D. Regions of possible positive curvature can be regions of degree six (where an a^2 was deleted) or regions of degree five where a d^2 appears but not an a^2.

First we examine regions Δ of degree six . It is possible for $|v_{1\Delta}| = 2$ or $|v_{4\Delta}| = 2$ with $l \in \{(eb)^2, (eb)a^{\pm 1}\}$ and like this $e^2 = 1$ or $e = a$. If $|v_2| = 2$ or

$|\upsilon_5| = 2$ then the label on such a vertex would be cb^{-1} and $b = c = 1$. It $|\upsilon_3| = 2$ or $|\upsilon_6| = 2$ then the label would be d^2. If none of the vertices of Δ has degree two then $c(\Delta) \le c(3,3,3,3,3,3) = 0$ while it is possible for Δ to have up to four vertices of degree two (see case 1 below). In order for at least one vertex of to have degree two, one of the following mutually exclusive conditions must hold.

1. $c = 1$ and $a = e$ (so $d^2 \neq 1$ and $e^2 = 1$)
2. $c = 1$ and $a \neq e$, $e^2 = 1$, $d^2 = 1$
3. $c = 1$ and $a \neq e$, $e^2 = 1$, $d^2 \neq 1$
4. $c = 1$ and $a \neq e$, $e^2 \neq 1$, $d^2 = 1$
5. $c = 1$ and $a \neq e$, $e^2 \neq 1$, $d^2 \neq 1$
6. $c \neq 1$ and $a = e$ (so $e^2 = 1$), $d^2 = 1$
7. $c \neq 1$ and $a = e$ (so $e^2 = 1$), $d^2 \neq 1$
8. $c \neq 1$ and $a \neq e$, $e^2 = 1$, $d^2 = 1$
9. $c \neq 1$ and $a \neq e$, $e^2 \neq 1$, $d^2 = 1$
10. $c \neq 1$ and $a \neq e$, $e^2 = 1$, $d^2 \neq 1$

1. $c = 1$ **and** $a = e$ **(so** $d^2 \neq 1$ **and** $e^2 = 1$)

Now it is possible to have up to four vertices of degree two as υ_1 and υ_2 may have degree two at the same time, and the same applies for υ_4 and υ_5. Possible labels of length two are cb^{-1} , $(eb)a^{\pm 1}$ or their inverses.

First suppose Δ has four vertices of degree two. The region must be as Figure 6-7. The label of υ_3 and υ_6 start with $c^{-1}de^{-1}$ and therefore, any labelling of length less than six on them would involve d just once and thus make the group cyclic. If letter d is encountered more than once, the degree must be at least nine as labels of smaller length would create a relator to force G to have property X. Furthermore, if the order of d in G is three then any labelling of length less than 18 would make the group to have property X, as sublabels like $(dw_1dw_2dw_3)^{\pm 1}$ where $w_1 = w_2 = w_3 = 1$ in G would allow *diamond moves* to increase the number of sinks, contradiction (N10). Therefore if the order of d is three the maximum curvature Δ can have is $c(\Delta) \leq c(2, 2, 2, 2, 18, 18) = \frac{2\pi}{9}$. If the order of d is four in G then labels of length less than twelve on v_3 or v_6 would make the group to have property X and sublabels like $(dw_1dw_2dw_3dw_4)^{\pm 1}$ where $w_1 = w_2 = w_3 = w_2 = 1$ in G are not allowed for the same reason as before. The curvature of Δ can be at most $c(\Delta) \leq c(2, 2, 2, 2, 12, 12) = \frac{\pi}{3}$. If the order of d in G is five then the degree of v_3 or v_6 will be at least ten and the region would have curvature at most $c(\Delta) \leq c(2, 2, 2, 2, 10, 10) = \frac{2\pi}{5}$. If the order of d is six or greater the curvature of Δ is at most $c(\Delta) \leq c(2, 2, 2, 2, 9, 9) = \frac{4\pi}{9}$. In any case enough negative curvature can be found in regions Δ_1 and Δ_2 to compensate for the positive one of Δ as shown in Figure 6-7.

If the order of d is three then each one of Δ_i $i = 1, 2$ will have curvature at most $c(\Delta_i) \leq c(2, 2, 3, 18, 18) = -\frac{\pi}{9}$. If the order of d is four then $c(\Delta_i) \leq c(2, 2, 4, 12, 12) = -\frac{\pi}{6}$. If the order of d is five then $c(\Delta_i) \leq c(2, 2, 5, 10, 10) = -\frac{\pi}{5}$. If the order of Δ is six or greater then $c(\Delta) \leq c(2, 2, 6, 9, 9) = -\frac{2\pi}{9}$. It can be seen that in any case $c(\Delta) + c(\Delta_1) + c(\Delta_2) \leq 0$. A region of negative curvature compensates for at most $\frac{2\pi}{9}$.

If the region has three vertices of degree two then the two of them must be v_1 and v_2 or v_4 and v_5. In any case the labels of v_3 and v_6 will start with de^{-1} or $c^{-1}d$. Their degree will have to be at least eight as labels of smaller length would create relators that would force the group to have property X. The remaining vertex (the one that does not have degree two and does not involve a d) will have to have degree at least four. The region cannot have positive curvature as $c(2, 2, 2, 4, 8, 8) = 0$.

If any of v_1, v_2, v_4 or v_5 does not have degree two then it must have degree at

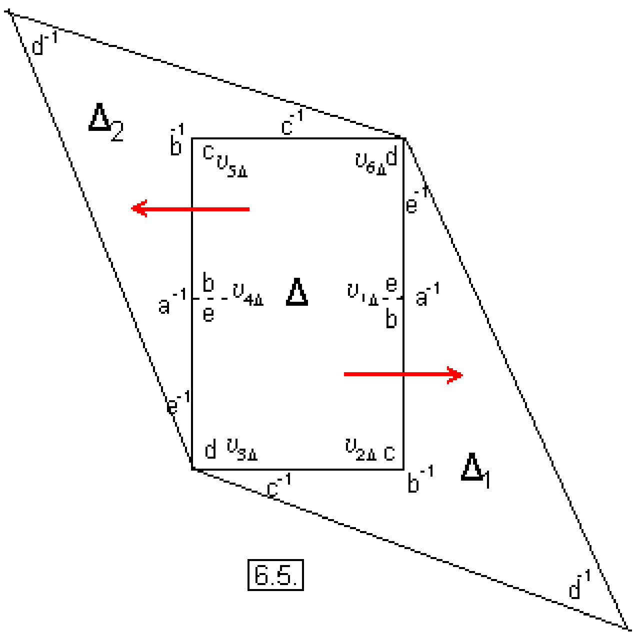


Figure 6-7:

least four. If Δ has two vertices of degree two then its curvature will be $c(\Delta) \leq \max\{c(2,2,3,4,4,6), c(2,2,4,4,4,4)\} = 0$. If region Δ has only one vertex of degree two in any case its curvature will be at most $c(\Delta) \leq c(2,3,3,4,4,4) < 0$.

2. $c = 1$ **and** $a \neq e$**,** $e^2 = 1$**,** $d^2 = 1$

Every vertex of Δ may have degree two with possible labels of length two being $(eb)^2$, cb^{-1}, d^2 or there inverses. It can be seen than two adjacent vertices cannot both have degree two, so we can have at most three vertices of degree two. Since neither $a^{\pm 2}$ or $d^{\pm 2}$ can appear as proper sublabels, if a label contains $d^{\pm 1}$ then it should either have degree two or greater than six in order to contain another $d^{\pm 1}$. In the same way a vertex whose label contains more than one $a^{\pm 1}$ should be of degree four or greater. If $|v_1| = 2$ and so $l_1 = (eb)^2$ the labels of v_5 and v_1 start with dc and therefore must have degree six or greater. The same is true for v_4

so if $|v_4| = 2$ then $|v_3|$, $|v_5| \geq 6$. If $|v_2| = 2$ then $l_2 = cb^{-1}$ and so $l_1 = (eb)a^{-1}$, $l_3 = c^{-1}d$ and as before $|v_1| \geq 4$ and $|v_3| \geq 6$. If $|v_3| = 2$ then $l_3 = d^2$ and $l_2 = cew$ and $l_4 = c(eb)w$. These vertices may have degree three with label $c(eb)e$ but in the case then the label adjacent to it must have length six or greater.

It is concluded that no interior regions of positive curvature is possible under these conditions as in any case the curvature of Δ is at most $c(\Delta) \leq \max\{c(2,2,2,6,6,6), c(2,2,3,3,6,6), c(2,2,3,4,4,6), c(2,3,3,3,4,4)\} = 0$.

3. $c = 1$ **and** $a \neq e$**,** $e^2 = 1$**,** $d^2 \neq 1$

Now the possible vertices of degree two are v_1, v_2, v_4 and v_5 with possible labels $(eb)^2$, cb^{-1} or their universes. Two adjacent vertices cannot both have degree two, so there can be at most two vertices of degree two.

If $|v_1| = 2$ then $l_1 = (eb)^2$ and $l_2 = dcw$, $l_6 = dcw$. It is obvious that in this case v_2 and v_6 will have degree four or greater. The same applies for $|v_4| = 2$. It can be concluded that if v_1 or v_4 is the only vertex of degree two then Δ cannot have positive curvature as $c(\Delta) \leq c(2,3,3,3,4,4) = 0$. Also if there are two vertices of degree two these cannot be v_1 and v_4 at the same time as in that case $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

If $|v_2| = 2$ then $l_2 = cb^{-1}$ and $l_1 = (eb)a^{-1}w$ and $l_3 = dc^{-1}w$. Once again the two neighbouring vertices should have degree at least four and therefore v_2 cannot be the only vertex of degree two, neither can be v_2 and v_5 be of degree two at the same time.

Therefore, the only case which remains to be checked is for $|v_2| = |v_4| = 2$ or $|v_5| = |v_1| = 2$ and the two cases are symmetric. Region Δ will be as in Figure 6-8.

Now $l_3 = c^{-1}dcw$. If $|v_3| = 4$ then $l_3 \in \{c^{-1}dca^{\pm 1}, c^{-1}dc(eb)^{\pm 1}\}$ which makes the order of d to be two, a contradiction. If $|v_3| = 5$ then its label will be $l_3 \in \{c^{-1}dc(eb)^{\pm 2}, c^{-1}dcb^{-1}e^{-1}, c^{-1}dceb\}$ which forces a contradiction in any case.

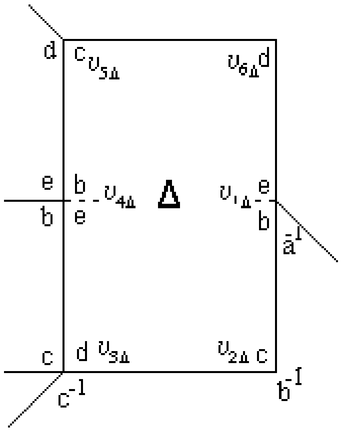


Figure 6-8:

Therefore $|v_3| \geq 6$ and the curvature of Δ is $c(\Delta) \leq c(2,2,3,4,4,6) = 0$.

4. $b = c = 1$ **and** $a \neq e$**,** $e^2 \neq 1$**,** $d^2 = 1$

v_2, v_3, v_5 and v_6 may have degree two with labels cb^{-1} and d^2. If the degree of v_3 is not two then it should be four or greater.

First let $|v_2| = 2$ with label $l_2 = cb^{-1}$. The label of v_1 will be $l_1 = (eb)a^{-1}w$ and the label of v_3 will be $l_3 = c^{-1}dw$. It can be seen that the degree of v_3 cannot be three while it is possible for the degree to v_1 to be three with label $l_1 = (eb)a^{-1}(eb)$ which makes $a = e^2$. In the case that v_2 is the only vertex of degree two $c(\Delta) \leq c(2,3,3,3,4,4) = 0$. Therefore, if $|v_2| = 2$ there is another vertex of degree two. If $|v_2| = |v_5| = 2$ then at least one of v_1 and the v_4 must have degree three otherwise $c(\Delta) \leq c(2,2,4,4,4,4) = 0$. If $l_1 = (eb)a^{-1}(eb)$ then $l_6 = c^{-1}dcw$. But any effort of finding a label of degree four or five for v_6 fails and the same applies in the case of $|v_4| = 3$ for the degree of v_3 (i.e. if $|v_4| = 3$ then $|v_3| \geq 6$). It is concluded that if $|v_2| = |v_5| = 2$ the curvature of Δ will be $c(\Delta) \leq \max\{c(2,2,4,4,4,4), c(2,2,3,3,6,6,), c(2,2,3,4,4,6)\} = 0$.

Now, suppose $|v_3| = 2$ with label $l_3 = d^2$. The label of v_2 is $l_2 = cew$ and the label of v_4 is $l_4 = c(eb)w$. If $|v_2| = 3$ then $l_2 \in \{ced^{\pm 1}, c(eb)e\}$ which in any case makes the order of e to be two, a contradiction. So it should be $|v_2| \geq 4$. If $|v_4| = 3$ then $l_4 = c(eb)e$ which again forces a contradiction, and so $|v_4| \geq 4$. This shows that v_3 cannot be the only vertex of degree two as this makes the curvature of Δ non-positive. Also if $|v_3| = |v_6| = 2$ the remaining vertices would have degree four or greater and the curvature of Δ cannot be positive.

It is concluded that there must be two vertices of degree two and $|v_2| = |v_6| = 2$ or $|v_3| = |v_5| = 2$ and the two cases are symmetric . Region Δ will be as in Figure 6-9.

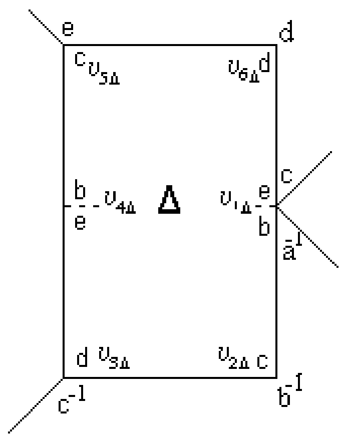


Figure 6-9:

From the discussion above it follows that v_3, v_5 and v_1 will have degree at least four. If the degree of v_4 is also greater than four, the region will not have positive curvature so it must be $|v_4| = 3$ and so $l_4 \in \{(eb)^3, (eb)^2a^{\pm 1}, (eb)a^{\pm 1}(eb)\}$ which make the order of e to be three or four. The degree of v_1 should be four or five since if it is six or greater the curvature of Δ will be $c(\Delta) \leq c(2, 2, 3, 4, 4, 6) = 0$. So $l_1 \in \left\{ \begin{array}{c} c(eb)a^{-1}b^{-1}, c(eb)a^{-1}e, c(eb)a^{-1}b^{-1}d^{\pm 1}, c(eb)a^{-1}c^{-1}d^{\pm 1}, \\ c(eb)a^{-1}ed^{\pm 1}, c(eb)a^{-1}(eb)^{-1}b^{-1}, c(eb)a^{-1}(eb)e, \end{array} \right\}$. From these labels the only one which does not force a contradiction is $l_1 = c(eb)a^{-1}e$ and so the order of e must be four and the label of v_4 must be $l_4 \in \{(eb)^2a^{\pm1}, (eb)a^{\pm1}(eb)\}$. We look at the possible labels of v_3 and v_5. At least one of them should have degree four and the other must have degree at least five as $c(2,2,3,4,5,5) < 0$. In the label of v_3 letter d must be followed by a b, e^{-1} or c, because of the label of v_4, and therefore a label of length four or five would involve d just once. This makes d to be generated by the remaining elements and the group is forced to be cyclic. It is concluded that a region cannot have positive curvature under these conditions.

5. $b = c = 1$ **and** $a \neq e$**,** $e^2 \neq 1$**,** $d^2 \neq 1$

Now the only possible vertices of degree two are v_2 and v_5 with label cb^{-1}. If $|v_2| = 2$ then $l_1 = (eb)a^{-1}w$ and $l_3 = c^{-1}dw$. If $|v_1| = 3$ then $l_1 = (eb)a^{-1}(eb)$ and $a = e^2$. If $|v_3| = 3$ then $l_3 = c^{-1}de^{-1}$ and $d = e$. So v_1 and v_3 cannot have degree three at the same time. If $|v_1| = 3$ then $l_6 = dcw$ and if $|v_6| = 3$ then $l_6 \in \{dcb^{-1}, dce\}$ which forces a contradiction in either case. If $|v_3| = 3$ then $l_4 = a^{-1}(eb)w$ and then $|v_4| = 3$ implies $l_4 = a^{-1}(eb)^2$ which forces a contradiction by making the group cyclic. So at least two vertices of Δ have degree four or greater and therefore a region of positive curvature cannot occur.

From the above paragraph it follows that in order to have a region of positive curvature it must be $|v_2| = |v_5| = 2$ and Δ must have at least one vertex of degree three. Let $|v_1| = 3$ with label $l_1 = (eb)a^{-1}(eb)$. Now the label of v_6 is $l_6 = c^{-1}dcw$ and any effort of labelling it with degree four or five will give a label which involves letter d just once and therefore make the group cyclic. The same applies for the case that $|v_4| = 3$. So at least one of v_3 or v_6 must have degree three. Let $|v_3| = 3$ with label $l_3 = c^{-1}de^{-1}$. The group is now generated by a and e and the label of v_4 is $l_4 = a^{-1}(eb)a^{-1}w$. $|v_4| \in \{4, 5\}$ implies

$$l_4 \in \left\{ \begin{array}{c} a^{-1}(eb)a^{-1}(eb)^{\pm1}, a^{-1}(eb)a^{-1}(eb)^{\pm2}, a^{-1}(eb)a^{-1}b^{-1}c, a^{-1}(eb)a^{-1}b^{-1}e^{-1}, \\ a^{-1}(eb)a^{-1}c^{-1}b, a^{-1}(eb)a^{-1}c^{-1}e^{-1}, a^{-1}(eb)a^{-1}eb^{-1}, a^{-1}(eb)a^{-1}ec^{-1} \end{array} \right\}.$$

Any of these labels forces a contradiction either by making $e = 1$ or by creating a relation of the type $aeae^k = 1$ which makes the group to have property X. The same applies in the case that $|v_6| = 3$, i.e. this forces $|v_1| \geq 6$. Therefore, a region of positive curvature cannot occur.

6. $c \neq 1$ and $a = e$ (so $e^2 = 1$), $d^2 = 1$

Possible labels of degree two are now v_1, v_3, v_4 and v_5 with possible labels being $(eb)^2$, $(eb)a^{\pm 1}$and d^2. Region Δ can have up to two vertices of degree two as two adjacent vertices cannot both have degree two.

First suppose $l_3 = d^2$ and so $l_2 = cew$ and $l_4 = c(eb)w$. If $|v_2| = 3$ then $l_2 \in \{ced^{\pm 1}, c(eb)e\}$. In the first case $c \in < a, d >$ and the group has property X, while the second label forces a contradiction by making $c = 1$. If $|v_4| = 3$ then $l_3 = c(eb)e$ which again forces a contradiction. So $|v_2|$, $|v_4| \geq 4$. So v_3 cannot be the only vertex of degree two. The same applies for v_6 as in that case $c(\Delta) \leq c(2, 3, 3, 3, 4, 4) = 0$. Also if $|v_3| = |v_6| = 2$ then $c(\Delta) \leq c(2, 2, 4, 4, 4, 4) = 0$.

Now suppose $|v_1| = 2$ with $l_1 = (eb)^2$. The labels of both v_6 and v_2 start with dc and if either of them has degree three then the equivalent label will be dcb^{-1} or dce. Once again c is generated by the remaining two elements which both have order two and so G has property X. So $|v_2|$, $|v_6| \geq 4$. So if $l_1 = (eb)^2$ there must be another vertex of degree two (but not v_4 with label $(eb)^2$ as this would force $|v_2|$, $|v_4| \geq 4$ and so $c(\Delta) \leq c(2, 2, 4, 4, 4, 4) = 0$).

Suppose $|v_1| = 2$ with $l_1 = (eb)a$. The labels of v_2 and v_6 will be $l_2 = cew$ and $l_6 = dbw$. As before the degree of v_6 cannot be three while $|v_6| = 3$ is possible with $l_6 = dbe$ and so $d = e = a$. But in this case the label of v_5 will be $l_5 = caw$. It is obvious that the degree of v_5 cannot be three. If $|v_5| \in \{4, 5\}$ and letter c is encountered only once in the label of v_5 then c is generated by the remaining elements and G must be cyclic. So $l_5 \in \{cac^{-1}d^{\pm 1}, ca(eb)^{\pm 1}c^{-1}d^{\pm 1}, cab^{-1}cb^{-1}, cab^{-1}ce, cac^{-1}be, cac^{-1}e^{-1}b, caecb^{-1}, caece\}$. Each of these labels forces a contradiction by making the group cyclic or to collapse to the trivial group or in some cases by forcing a relator of the type $acac^k = 1$ and make the group to have property X. Therefore, it must be $|v_5| \geq 6$. So if v_1 is the only vertex of degree two with label $l_1 = (eb)a$ the curvature of Δ will be $c(\Delta) \leq \{c(2,3,3,3,4,4), c(2,3,3,3,3,6)\} = 0$.

Let $|v_1| = 2$ with $l_1 = (eb)a^{-1}$. Now $l_2 = b^{-1}cw$ and $l_6 = de^{-1}w$. The degree of v_2 cannot be three and if $|v_6| = 3$ then $l_6 = de^{-1}b^{-1}$ which makes $d = e = a$. Then the label of v_5 will be $l_5 = ca^{-1}w$ and as with the previous paragraph its degree must be six or greater so if v_1 is the only vertex of degree two with label $l_2 = (eb)a^{-1}$ a region of positive curvature may not be obtained.

It is concluded that in order to obtain a region of positive curvature there must be two vertices of degree two. If v_1 and v_4 have degree two then a region of positive curvature cannot be obtained since in that case the curvature of Δ will be $c(\Delta) \leq \max\{c(2,2,3,4,4,6), c(2,2,4,4,4,4), c(2,2,3,3,6,6)\} = 0$. So either $|v_1| = |v_3| = 2$ or $|v_4| = |v_6| = 2$ and the two cases are symmetric. If a label contains letter c then the degree of the vertex will be at least four as any label of degree three will make c to be generated by the remaining elements and therefore, the group to have property X. From the above discussion for the individual labels of v_1 it follows that if $|v_1| = 2$ either all the remaining vertices will have degree four or greater or if a vertex has degree three then a vertex adjacent to it will have degree six or greater. Therefore, $c(\Delta) \leq \max\{c(2,2,4,4,4,4), c(2,2,3,4,4,6), c(2,2,3,3,6,6)\} = 0$ and a region of positive curvature may not occur.

7. $c \neq 1$ and $a = e$ (so $e^2 = 1$), $d^2 \neq 1$

Now the possible vertices of degree two are v_1 and v_4 with labels $(eb)^2$ or $(eb)a^{\pm 1}$.

First let $l_1 = (eb)^2$ and so the labels of v_2 and v_6 are dcw. If $|v_2| = 3$ or $|v_6| = 3$ then the label will be dcb^{-1} or dce and so $c = d^{-1}$ or $c = d^{-1}a$. From Lemma 6.2(i)and (iv) it follows that in either case the equation has a solution.

So $|v_2|$, $|v_6| \geq 4$.

Now let $l_1 = (eb)a$ and so $l_2 = ecw$ and $l_6 = dbw$. If $|v_2| = 3$ then $l_2 \in \{eca^{\pm1}, ec(eb)\}$ but any of these labels forces $c = 1$, a contradiction. So it must be $|v_2| \geq 4$. If $|v_6| = 3$ then $l_6 \in \{dbe, dbc^{-1}\}$. The first label forces a contradiction by making the order of d two, while $l_6 = dbc^{-1}$ implies $d = c$ which is possible. If letter c is encountered only once in the label of v_2 the group becomes cyclic. The same is true if an e is encountered only once and no a is encountered. So if $|v_2| \in \{4, 5\}$ then its label will be

$$l_2 \in \left\{ \begin{array}{c} ecb^{-1}c, ecec, eca^{\pm1}b^{-1}c, eca^{\pm1}c^{-1}b, eca^{\pm1}ec, ec(eb)^{-1}b^{-1}c, ec(eb)^{\pm1}c^{-1}b, \\ ec(eb)ec, ecb^{-1}c(eb), eced^{\pm1}b, eced^{\pm1}c, ecec(eb), ececa^{\pm1} \end{array} \right\}.$$

But any of these labels forces the group to be cyclic or to have property X. So if $|v_6| = 3$ then $|v_2| \geq 6$.

Now let $l_1 = (eb)a^{-1}$ and so $l_2 = b^{-1}cw$ and $l_6 = e^{-1}bw$. If $|v_2| = 3$ then $l_2 \in \{b^{-1}ca^{\pm1}, b^{-1}c(eb)^{-1}\}$ but any of these cases forces $ce = 1$, a contradiction. If $|v_6| = 3$ then $l_3 \in \{de^{-1}b^{-1}, de^{-1}c^{-1}\}$. The first case forces the order of d to be two, a contradiction, while the second case is possible and $c = da$. But in that case $|v_2| \in \{4, 5\}$ gives labels that force a contradiction by making the group cyclic, trivial or to have property X. So if $|v_6| = 3$ then $|v_2| \geq 6$.

It is concluded that if a region Δ has only one vertex of degree two then positive curvature cannot be obtained as in any case the curvature will be at most $c(\Delta) \leq \max\{c(2,3,3,3,4,4), c(2,3,3,3,3,6)\} = 0$. If the region Δ has two vertices of degree two then again a region of positive curvature cannot occur as $c(\Delta) \leq \max\{c(2,2,4,4,4,4), c(2,2,3,3,6,6), c(2,2,3,4,4,6)\} = 0$. Therefore, under these conditions, a region cannot have positive curvature.

8. $c \neq 1$ and $a \neq e$, $e^2 = 1$, $d^2 = 1$

Now the possible vertices of degree two are v_1, v_3, v_4 and v_5 with possible labels $(eb)^2$ and d^2. Two adjacent vertices cannot both have degree two, so Δ has at most two vertices of degree two.

First let $|v_1| = 2$ with label $l_1 = (eb)^2$ and so the labels of v_1 and v_6 start with dc. If $|v_2| = 3$ or $|v_6| = 3$ then l_2 or $l_6 \in \{dce, dcb^{-1}\}$. Both of these labels are possible and we check the label of the neighbouring vertex for each one. If $l_6 = dce$ then $c = de$ and $l_5 = caw$. If only one a is encountered in the label of v_5 then a and c are generated by d and e which both have degree two and therefore the group has property X. So $l_5 \in \{ca(eb)^{\pm 1}a^{\pm 1}b^{-1}, ca(eb)^{\pm 1}a^{\pm 1}b^{-1}\}$ but any of these labels makes d (and so c as well) to be generated by a and e and once again the group must have property X. So if $l_6 = dce$ then $|v_5| \geq 6$. If $l_2 = dce$ then $l_3 = d^2w$ and the degree of v_3 can only be two (i.e. w is the empty word) as we do not allow d^2 to be a proper sublabel in the case that $d^2 = 1$. So l_2 cannot be dce in the case that v_3 does not have degree two. If $l_6 = dcb^{-1}$ then $d = c$ and $l_5 = ca^{-1}w$. Once again if only one a is encountered in the label of v_5 then a is generated by d and e which both have degree two and the group has property X. So if $|v_5| \in \{4, 5\}$ then $l_5 \in \{ca^{-1}(eb)^{\pm 1}a^{\pm 1}b^{-1}, ca^{-1}(eb)^{\pm 1}a^{\pm 1}b^{-1}\}$. But in this case again $c = d$ is generated by a and e which both have degree two and the group has property X. So if $l_6 = dcb^{-1}$ then $|v_5| \geq 6$. If $l_2 = dcb^{-1}$ then $l_3 = c^{-1}dw$ and it is possible for v_3 also to have degree three. In the case that $|v_1| = 2$ is the only vertex of degree two then it should contain at least four vertices of degree three. If $|v_6| = 3$ then $|v_5| \geq 6$ and the curvature of the region is $c(\Delta) \leq c(2, 3, 3, 3, 3, 6) = 0$. So the degree of v_6 must be four or greater. So the degree of the remaining vertices must be three and so $l_2 = dcb^{-1}$ and $l_3 = c^{-1}db$. But in that case the label of v_4 is $l_4 = a(eb)w$ and therefore its degree cannot be three. Therefore in the case that $l_1 = (eb)^2$ there must be another vertex of degree two i.e. v_3 with label d^2 or v_4 with label $(eb)^2$.

Now let $|v_3| = 2$ with $l_3 = d^2$. The label of v_4 is $l_4 = c(eb)w$. If the degree of v_4 is three then $l_4 = c(eb)e$ and so $c = 1$, a contradiction. So the degree of v_4 must be four or greater. It is possible for the degree of v_2 to be three with label $ced^{\pm 1}$ and so $c = de$ and $l_1 = (eb)ew$ or $l_1 = (eb)c^{-1}w$. In this case it is possible for the degree of v_1 to be two with label $(eb)^2$ while $|v_1| = 3$

implies $l_1 \in \{(eb)^3, (eb)^2a^{\pm 1}, (eb)ec, (eb)c^{-1}b\}$ but each of these labels forces a contradiction. If the only vertex of degree two in Δ is v_3 with $l_3 = d^2$ there are at least two vertices of degree four and a region of positive curvature cannot be obtained.

It is concluded that there must be exactly two vertices of degree two. We look at the possible cases:

$|v_1| = |v_4| = 2$

Now region Δ will be as in Figure 6-10.

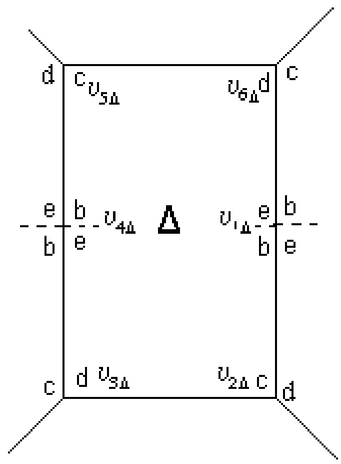


Figure 6-10:

It can be seen that it is possible for v_2, v_3, v_4 and v_5 to have degree three.

From the first paragraph of this section if follows that if $|v_6| = 3$ then $|v_5| \geq 6$, therefore in the case that v_3 has degree three then there must be another vertex of degree three otherwise $c(\Delta) \leq c(2,2,3,4,4,6) = 0$. If v_3 is another vertex of degree three then its label is forced to be the same as that of v_6 but then the degree of v_2 will be six or greater and a region of positive curvature can not be obtained either. So if $|v_6| = 3$ then $|v_2| = 3$ and $l_6 = l_2 = dcb^{-1}$ and $c = d$. From the previous paragraph it is known that in that case $|v_5| \geq 6$ and we look for possible labels of degree four or five on v_3. $l_3 = c^{-1}dcw$ but if the label of

v_3 contains only one $a^{\pm 1}$ and $e^{\pm 1}$ then the group will have property X since it will be generated by the remaining two elements of degree two. So $|v_3| \in \{4, 5\}$ implies $l_3 = c^{-1}dc(eb)^{\pm 2}$ but this forces $d = 1$ a contradiction. So $|v_3| \geq 6$ and once more a region of positive curvature may not be obtained. The same applies if $|v_3| = 3$, i.e. $c(\Delta) \leq \max\{c(2,2,3,3,6,6), c(2,2,3,4,4,6)\} = 0$.

Now we assume that $|v_3|$, $|v_6| \geq 4$ and so at least one of v_2 and v_5 must have degree three with label dcb^{-1}. But if v_2 has degree three then $|v_3| \geq 6$ and the same applies for v_6 i.e. if $|v_6| = 3$ then $|v_6| \geq 6$. Therefore a region of positive curvature may not obtained in this case.

$|v_3| = |v_6| = 2$

Now region Δ will be as in Figure 6-11.

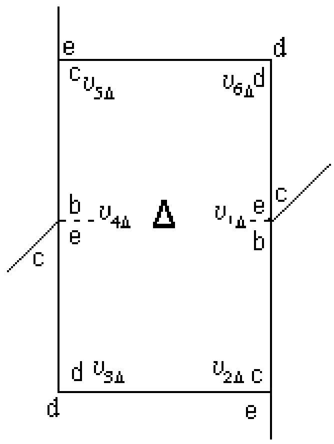


Figure 6-11:

The label of v_1 and v_4 starts with $c(eb)$ and any effort of finding a label for degree three forces a contradiction by making $c = 1$, so it must be $|v_1|$, $|v_4| \geq 4$. So at least one of v_2 and v_5 must have degree three. The label can be $ced^{\pm 1}$ that gives the relator $c = de$. Suppose $l_2 = ced^{\pm 1}$ then $l_1 \in \{c(eb)c^{-1}w, c(eb)ew\}$. We check for possible labels of degree less than six on v_1. If the label l_1 contains exactly one $a^{\pm 1}$ or $d^{\pm 1}$ the group is generated by two elements of degree two and has property X. So it must contain another $a^{\pm 1}$ or a $d^{\pm 1}$, so will be greater than five. In the case $|v_1| = 5$ its label can only be $l_1 \in \{c(eb)c^{-1}be, c(eb)c^{-1}e^{-1}b^{-1}, c(eb)ecb^{-1}\}$. These labels create an additional relator $(de)^2 = 1$. The same applies for v_4. The degree of v_2 cannot be four in the case that $|v_5| = 3$ so both v_2 and v_5 must have degree three and at least one of v_1 and v_4 must have degree five with one of the labels above. The other vertex must have degree five, six or seven.

Region Δ will have curvature at most $c(\Delta) \leq c(2,2,3,3,5,5) = \frac{2\pi}{15}$ and will be as in Figure 6-12. We calculate the curvature of its neighbouring regions Δ_1 and Δ_2.

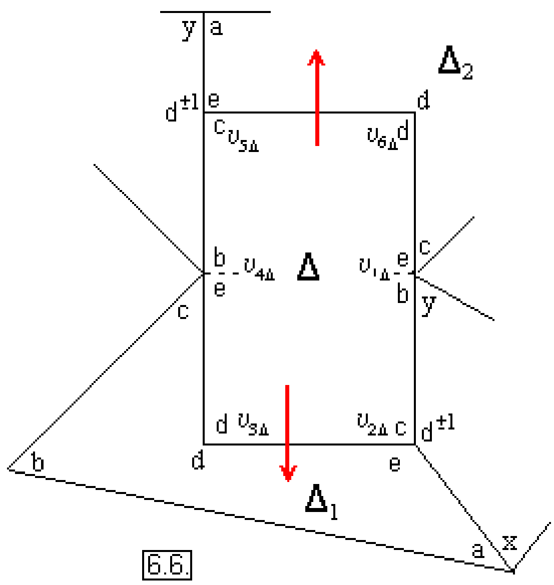


Figure 6-12:

The degree of $v_{2\Delta_1}$ cannot be two or three. The degree of $v_{1\Delta_1}$ is at least six and the degree or $v_{3\Delta_1}$ at least five. This makes the curvature of Δ_1 at most $c(\Delta_1) \leq c(2,3,4,5,6) = -\frac{\pi}{10}$. The same applies for the curvature of Δ_2, so overall $c(\Delta) + c(\Delta_1) + c(\Delta_2) \leq \frac{2\pi}{15} - \frac{\pi}{10} - \frac{\pi}{10} < 0$. Since Δ_1 and Δ_2 are regions of degree five (i.e. no deletion of a^2 was performed) they uniquely receive positive curvature across a $d-e$ vertex.

$|v_1| = |v_3| = 2$

Now the region will be as in Figure 6-13.

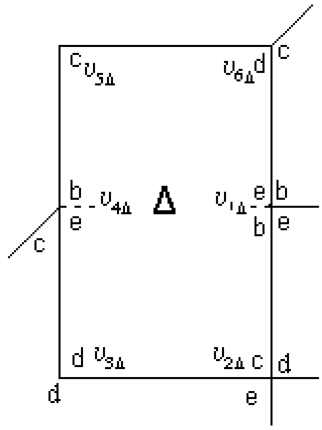


Figure 6-13:

We examine the possible labelling on v_2 for degree three or greater.

Fist let $|v_2| = 3$ so $l_2 = dce$ and $c = de$. We check the possible degrees of the remaining vertices. The degree of v_4 cannot be three or four but it is possible for it to have degree five with possible labels $l_4 \in \left\{ \begin{array}{c} c(eb)c^{-1}be, c(eb)c^{-1}e^{-1}b^{-1}, \\ c(eb)ecb^{-1}, c(eb)^2ed^{\pm 1} \end{array} \right\}$. The degree of v_5 cannot be three as the label $ced^{\pm 1}$ would force d^2 to be a proper sublabel on v_6. If the degree of v_6 is five or greater then $c(\Delta) \leq c(2,2,3,4,5,5) < 0$. We check for possible labels for degree three or four. $|v_6| = 4$ implies $l_6 \in \{dc(eb)^{-1}b^{-1}, dc(eb)^{\pm 1}c^{-1}, dc(eb)e\}$ otherwise an $a^{\pm 1}$ is encountered in the label which would force a to be generated by d and e and therefore the group to have property X. The only of these labels not forcing a contradiction is $l_6 = dc(eb)^{-1}b^{-1}$ which forces $l_4 = c(eb)a^{-1}w$ and therefore the degree of v_5 is forced to be five or greater and once again $c(\Delta) \leq c(2,2,3,4,5,6) < 0$. So the degree of v_6 must be three with label $l_6 = dce$. The label of v_5 is $l_5 = caw$ and from the first paragraph we see that the degree of v_5 must be at least six. If the degree of v_4 is also six or greater then the curvature of the region cannot be positive, so the degree of v_4 must be five and its label will be $l_4 \in \{c(eb)c^{-1}be, c(eb)c^{-1}e^{-1}b^{-1}, c(eb)ecb^{-1}, c(eb)^2ed^{\pm 1}\}$. Region Δ will be as in Figure 6-14 and its curvature is at most $c(\Delta) \leq c(2,2,3,3,5,6) = \frac{\pi}{15}$. Neighbouring region Δ_1 has enough negative curvature to compensate for the positive of Δ, more precisely $c(\Delta_1) \leq c(2,3,4,5,6) = -\frac{\pi}{10}$. Positive curvature is uniquely received across a $d-e$ edge.

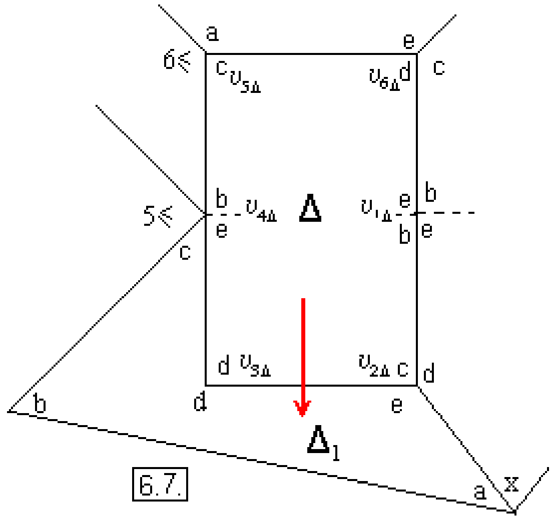


Figure 6-14:

Now suppose the degree of v_2 is four or greater. If $|v_6| = 3$ the degree of v_5 is $|v_5| \geq 6$ and the curvature of Δ cannot be positive as in that case $c(\Delta) \leq c(2,2,3,4,4,6) = 0$. So the degree of v_6 is at least four. The degree of v_4 cannot be three so the degree of v_5 must be three otherwise a region of positive curvature cannot exist. So $l_5 \in \{cb^{-1}d^{\pm 1}, ca^{\pm 1}b^{-1}, ca^{\pm 1}e\}$. We check the possible neighbouring vertices for each equivalent label of v_5. In the case of $l_5 = cb^{-1}d^{\pm 1}$ the label of v_5 will be $l_5 = c^{-1}dcw$ and its degree will be at least six. In the case that $l_5 = ca^{\pm 1}b^{-1}$ $a = c$ and any label containing d (except d^2) with length less than six would make d to be generated by a and e and therefore the group to have property X, so v_2 and v_6 must have degree at least six. If $l_5 = ca^{\pm 1}e$ $c = ea$ and once again since the labels of v_2 and v_5 contain letter d any effort of finding labels of degree less than six makes the group to be generated by two elements of degree two and thus to have property X. In any case $c(\Delta) \leq c(2, 2, 3, 4, 5, 5) < 0$.

9. $c \neq 1$ **and** $a \neq e$ **,** $e^2 \neq 1$**,** $d^2 = 1$

Now the only possible vertices of degree two are v_3 and v_6 with label d^2. There might be only one vertex of degree two or both v_3 and v_6 could have degree two. We check each case.

Only one vertex of degree two The vertex of degree two is one of v_3 and v_6 and the two cases are symmetric. Let $|v_3| = 2$. To obtain a region of positive curvature there must be at least four vertices of degree three. One of the following must hold:

1. $|v_1| = |v_2| = |v_5| = |v_6| = 3$ and $|v_4| \geq 3$.
2. $|v_1| = |v_4| = |v_5| = |v_6| = 3$ and $|v_2| \geq 4$.
3. $|v_1| = |v_2| = |v_4| = |v_6| = 3$ and $|v_5| \geq 4$.
4. $|v_2| = |v_4| = |v_5| = |v_6| = 3$ and $|v_1| \geq 4$.
5. $|v_1| = |v_2| = |v_4| = |v_5| = 3$ and $|v_6| \geq 4$.

$|v_1| = |v_2| = |v_5| = |v_6| = 3$ **and** $|v_4| \geq 3$ The label of v_2 is $l_2 \in \{c(eb)e, ced^{\pm 1}\}$.

First let $l_2 = ced^{-1}$ and so $c = de^{-1}$. The labels of v_1 is $l_1 = (eb)c^{-1}w$ and so $l_1 = (eb)c^{-1}b$ and so $c = e$ and $d = e^2$. The label of v_6 will be $l_6 = dcw$ and so $l_6 \in \{dcb^{-1}, dce\}$. The first label forces a contradiction by making $d = c$ and so $e = 1$, therefore $l_6 = dce$. The label of v_5 is $l_5 = caw$ and since it involves an a

any effort of labelling it with degree less than six would make the group to have property X, so in this case positive curvature cannot be obtained.

Let $l_2 = ced$ and once again $c = de^{-1}$. The label of v_1 will be $l_1 = (eb)ew$ and so $l_1 \in \{(eb)^3, (eb)^2a^{\pm 1}, (eb)ec\}$. The third label forces a contradiction by making $e^2 = 1$ and $c = 1$, so we check the other two labels. If $l_1 = (eb)^3$ then $e^3 = 1$ the label of v_6 will be $l_6 = dce$. The label of v_5 will be $l_5 \in \{cab^{-1}, cae\}$ and the first one forces a contradiction by making the group to have property X. The second one makes $d = e^{-1}ae$ and the label of v_4 will be $l_4 = c(eb)aw$. We check for possible labels of length four and five on v_4, as it is obvious that length three is not possible . It turns out that $l_4 \in \left\{ \begin{array}{c} c(eb)ab^{-1}, c(eb)ae, c(eb)a(eb)^{-1}b^{-1}, c(eb)a(eb)e, \\ c(eb)ab^{-1}d^{\pm 1}, c(eb)ac^{-1}d^{\pm 1}, c(eb)aed^{\pm 1} \end{array} \right\}$ but any of these labels forces a contradiction either by making $e = 1$ or by forcing a relator of the type $aeae^k = 1$ which makes the group to have property X. $l_1 = (eb)^2a^{\pm 1}$ makes $a = e^2$ and the label of v_6 to be $l_6 \in \{dbw, de^{-1}w\}$. Particularly if v_6 has degree three $l_6 \in \{dbc^{-1}, de^{-1}b^{-1}, de^{-1}c^{-1}\}$ and the only label not forcing a contradiction is $l_6 = de^{-1}c^{-1}$. The label of v_5 will then be $l_5 = cb^{-1}w$ and if it has degree three then $c = d$ which forces a contradiction.

Let the $l_2 = c(eb)e$ and so $c = e^{-2}$. In this case the label of v_1 will be $l_1 = (eb)aw$ and so $|v_1| = 3$ implies $l_1 = (eb)a(eb)$ which gives $c = a = e^2$. The label of v_6 will be $l_6 = dcw$ and any effort of finding a label of degree less for v_6 makes the group to have property X.

$|v_1| = |v_4| = |v_5| = |v_6| = 3$ **and** $|v_2| \geq 4$ In order for v_1 and v_4 to have degree three at the same time it must be $l_1 = l_4 = c(eb)e$ and $c = e^{-2}$. The label of v_5 will be $l_5 = dcw$ and so $l_5 \in \{dcb^{-1}, dce\}$. The second label forces a contradiction and so $d = c = e^2$. But the label of v_1 involves and a so any effort of finding a label for degree less than six would make the group to have property X.

$|v_1| = |v_2| = |v_4| = |v_6| = 3$ and $|v_5| \geq 4$ The label of v_1 and v_4 will be $c(eb)e$ and $l_2 = (eb)aw$, so $l_2 = (eb)a(eb)$ and $c = a = e^2$. The label of v_6 must have length at least six since it involves d and therefore positive curvature cannot be obtained.

$|v_2| = |v_4| = |v_5| = |v_6| = 3$ and $|v_1| \geq 4$ Now $l_4 = c(eb)e$ and $l_5 \in \{dcb^{-1}, dce\}$. The second label forces a contradiction so it should be $l_5 = dcb^{-1}$ and so $d = c = e^2$. But in that case any effort of labelling v_1 with degree less than six forces the group to have property X and so positive curvature cannot be obtained.

$|v_1| = |v_2| = |v_4| = |v_5| = 3$ and $|v_6| \geq 4$ From the above paragraph it can be see that $|v_1| = |v_4| = |v_5| = 3$ is not possible.

Two vertices of degree two Now $|v_3| = |v_6| = 2$ and Δ must be as in Figure 6-15. We check for possible vertices of degree three.

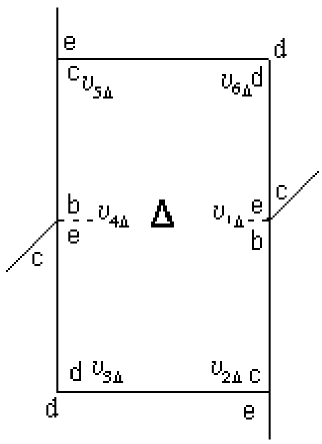


Figure 6-15:

If v_4 has degree three then $l_4 = c(eb)e$ and $c = e^{-2}$. The label of v_5 will be $l_5 = dcew$. It obvious that v_5 cannot have degree three and we look for possible labels of degree six or less i.e.$|v_5| \in \{4,5\}$. This implies $l_5 \in \left\{ \begin{array}{c} dc(eb)c^{-1}, dc(eb)e, dcece, dcecb^{-1}, dc(eb)a^{\pm 1}b^{-1}, \\ dc(eb)a^{\pm 1}c^{-1}, dc(eb)a^{\pm 1}e, dc(eb)^2c^{-1}, dc(eb)^2e \end{array} \right\}$. Out of these labels the only ones not forcing a contradiction are $l_5 \in \{dcece, dc(eb)^2c^{-1}, dcecb^{-1}\}$ and either $d = c = e^2$ or $d = e^2$. Therefore, if v_4 is the only vertex of degree three the remaining two vertices v_1 and v_2 must have degree four. If $|v_1| = 4$ then $l_1 \in \{c(eb)^2e, c(eb)a^{\pm 1}b^{-1}, c(eb)a^{\pm 1}e, c(eb)c^{-1}d^{\pm 1}, c(eb)ed^{\pm 1}\}$ but any of these labels forces a contradiction so if v_4 has degree three there must be another vertex of degree three.

If v_2 has degree three with label $l_2 = c(eb)e$ and $c = e^{-2}$ then $l_1 = c(eb)aw$ and it is obvious that its degree cannot be three or four. We check for the degree of v_1 being five. That implies $l_1 \in \left\{ \begin{array}{c} c(eb)a(eb)^{-1}b^{-1}, c(eb)a(eb)e, \\ c(eb)ab^{-1}d^{\pm 1}, c(eb)ac^{-1}d^{\pm 1}, c(eb)ae^{-1}d^{\pm 1} \end{array} \right\}$. Out of these labels the only one not forcing a contradiction is $l_1 = c(eb)a(eb)^{-1}b^{-1}$ which forces $a = c = e^2$. If v_2 is the only vertex of degree three then v_4 and v_5 must have degree four. If $|v_4| = 4$ then $l_4 \in \left\{ \begin{array}{c} c(eb)^2e, c(eb)a^{\pm 1}b^{-1}, \\ c(eb)a^{\pm 1}e, c(eb)c^{-1}d^{\pm 1}, c(eb)ed^{\pm 1} \end{array} \right\}$ and a contradiction is forced in any of these cases.

If v_2 has degree three with label $l_2 = ced$ then $c = de^{-1}$ and $l_1 = c(eb)ew$. It is obvious that the degree of v_1 cannot be three. We check for possible vertices of degree four and five. In that case the label of v_1 will be $l_1 \in \left\{ \begin{array}{c} c(eb)^2e, c(eb)ed^{\pm 1}, c(eb)^3e, c(eb)^2a^{\pm 1}b^{-1}, c(eb)^2a^{\pm 1}e, \\ c(eb)^2c^{-1}d^{\pm 1}, c(eb)ed^{\pm 1}, c(eb)ecb^{-1}, c(eb)ece \end{array} \right\}$. The labels that are not forcing a contradiction are (i) $l_1 \in \{c(eb)^2e, c(eb)^2c^{-1}d^{\pm 1}\}$ which give the relators $c = e$ and $d = e^2$, (ii) $l_1 = c(eb)ecb^{-1}$ which gives the relator $c^2e^2 = 1$ and (ii) $l_1 = c(eb)^3e$ which give the relator $c = e^2$, $d = e^3$. If v_2 is the only vertex of degree three then there must be at least two vertices of degree four. We check the possible labels of v_4 and v_5 having degree four. $|v_4| = 4$ implies $l_4 \in \{c(eb)^2e, c(eb)a^{\pm 1}b^{-1}\}$ and $|v_5| = 4$ implies $l_5 \in \{c(eb)^2e, c(eb)a^{\pm 1}b^{-1}, cece\}$. If the label of v_1 is $l_1 = c(eb)^2e$ and so $c = e$ and $d = e^2$ then it is possible to obtain a region of positive curvature with one of the other two labels having degree four and the other having degree five. In the other cases the two adjacent vertices cannot both have degree four and so $c(\Delta) \leq c(2,2,3,4,5,5) < 0$. So there is only one remaining case for which it turns out that $l_4 = c(eb)^2e$ and $l_5 = dc(eb)^2c^{-1}$. Region Δ has curvature $c(\Delta) = c(2,2,3,4,4,5) = \frac{\pi}{15}$ and looks like Figure 6-16. Negative curvature may be found at its neighbouring region Δ_1 and this can compensate the positive curvature of Δ. The curvature of Δ_1 is at most $c(\Delta_1) \leq c(2,3,4,6,6) = -\frac{\pi}{6}$ and negative curvature is always given across a $d - e$ edge.

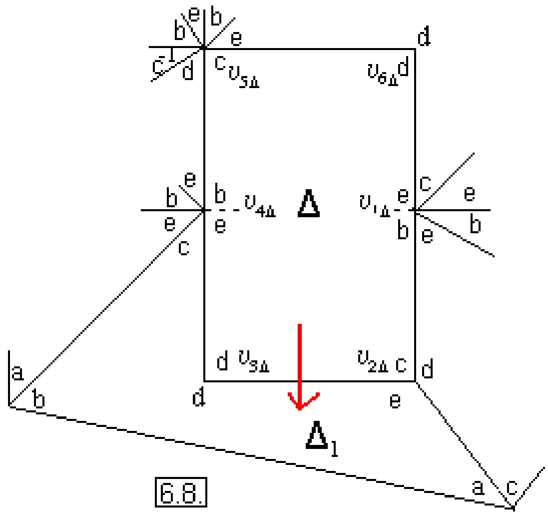


Figure 6-16:

If v_2 has degree three with label $l_2 = ced^{-1}$ then $c = de^{-1}$ and $l_1 = c(eb)c^{-1}w$. We check for the possibility of v_1 to have degree four or five. If $|v_1| = 4$ then $l_1 = c(eb)c^{-1}d^{\pm 1}$ which makes $d = e$ a contradiction, since it was assumed that the order of e is not two in this case. If v_2 is the only vertex of degree two and $|v_1| = 5$ then in order to have a region of positive curvature the degree of v_4 and v_5 must be four at the same time, something which is not possible.

It is concluded that if Δ is not the region of Figure 6-16 then it should contain at least two vertices of degree three. Since two adjacent vertices cannot both have degree three at the same time the following cases are examined:

$|v_1| = |v_4| = 3$ Now $l_1 = l_4 = c(eb)e$ and $c = e^{-2}$ and the label of v_2 and v_5 starts with dce. The two vertices cannot have degree four, while if both have degree six or greater the region cannot have positive curvature. So at least one of them has degree five and we can see from the previous paragraphs that this is only possible for $dcece$, $dc(eb)^2c^{-1}$, $dcecb^{-1}$ and it turns out that either $d = e^2$ or $d = e^3$. A region Δ of positive curvature will be as in Figure 6-17 and has curvature of at most $c(\Delta) \leq c(2, 2, 3, 3, 5, 5) = \frac{2\pi}{15}$.

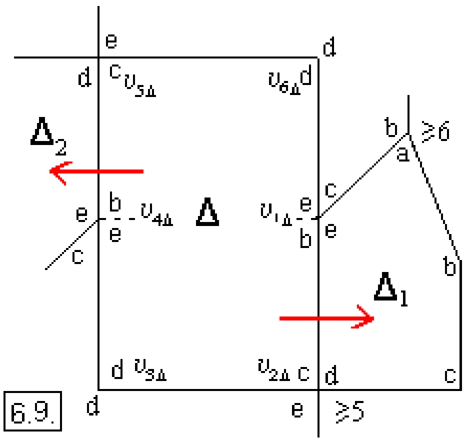


Figure 6-17:

Negative curvature may be found both in Δ_1 and Δ_2. Each of these regions has curvature $c(\Delta) \leq c(3, 3, 3, 5, 6) = -\frac{4\pi}{15}$ and can only receive positive curvature from at most one of its neighbouring regions across a $d - e$ edge.

$|v_1| = |v_5| = 3$ This case is symmetric to $|v_4| = |v_2| = 3$. The label of v_1 will be $l_1 = c(eb)e$ and this forces the label of v_5 to be the same. The degree of v_2 cannot be four and $|v_2| = 5$ makes $l_2 \in \{dcece, dc(eb)^2c^{-1}, dcecb^{-1}\}$ as in the previous paragraph. Any of these labels makes $d = e^2$ or $d = e^3$. The label of v_4 is $l_4 = c(eb)aw$ and this cannot be of degree four. If $|v_4| = 5$ then $l_4 = c(eb)a(eb)^{-1}b^{-1}$ and $c = a = e^2$. The two remaining vertices cannot have degree five at the same time. If one of them has degree five then the other cannot have degree six or seven either as any such label would make the group to have property X. So a region of positive curvature may not occur here.

$|v_2| = |v_5| = 3$ The labels of v_2 and v_5 can be one of $ced^{\pm 1}$, $c(eb)e$ and either both of them are $ced^{\pm 1}$ or both of them are $c(eb)e$ (the opposite would make the order of e to be two).

First let $l_2 = l_5 = c(eb)e$ and so $c = e^{-2}$. The label of v_1 and v_4 cannot have length four but for length five we can have the label $c(eb)a(eb)^{-1}b^{-1}$ and so $a = c = e^2$. If both of them have degree six or greater then positive curvature cannot be obtained so the label $c(eb)a(eb)^{-1}b^{-1}$ is encountered at least once. The curvature of Δ is at most $\frac{2\pi}{15}$ and negative curvature may be found in neighbouring regions Δ_1 or Δ_2 as in Figure 6-18.

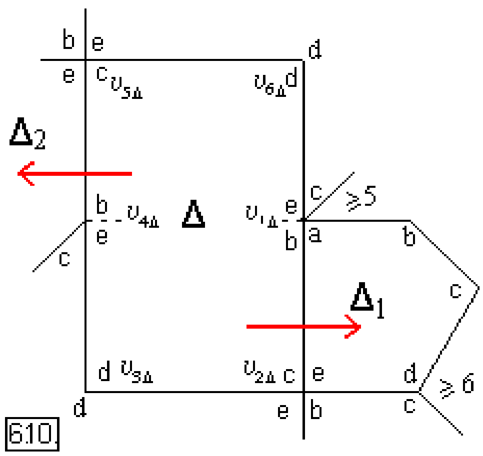


Figure 6-18:

The curvature of Δ_1 or Δ_2 is $c(\Delta^-) \leq c(3, 3, 3, 5, 6) = -\frac{4\pi}{15}$ and any of these regions will receive positive curvature from at most one of its neighbours across an $e-a$ edge.

Now let $l_2 = l_5 = ced$. It is possible for v_1 and v_4 to have degree four so Δ can have curvature of up to $c(2,2,3,3,4,4) = \frac{\pi}{3}$. Region Δ is as in Figure 6-19.

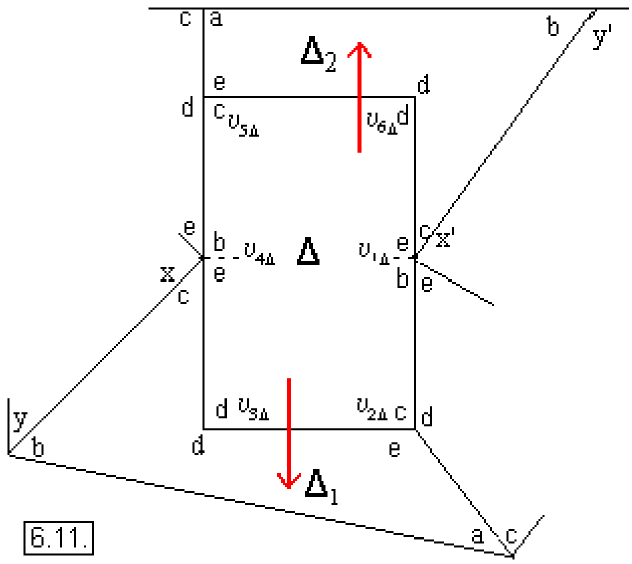


Figure 6-19:

In the case that v_4 has degree four $l_4 = c(eb)^2e$ and so $y = a$. The curvature of Δ_1 will be $c(\Delta_1) \leq c(2,3,4,6,6) = -\frac{\pi}{6}$ and the case is symmetric for Δ_2. In the worse case that both v_1 and v_4 have degree four and so positive curvature of $\frac{\pi}{3}$ is achieved this can be compensated with negative curvature from Δ_1 and Δ_2 together. If the degree of v_4 is four and the degree of v_1 is greater or equal to six (or vice versa) then the curvature of Δ is $c(\Delta) \leq c(2,2,3,3,4,6) = \frac{\pi}{6}$ and the curvature of Δ_1 (or Δ_2) is enough to compensate for the positive curvature on its own. If the degree of v_4 is four and the degree of v_1 is five then it should be $l_1 = c(eb)c^{-1}d^{\pm 1}$ and the curvature of Δ is $c(\Delta) \leq c(2,2,3,3,4,5) = \frac{7\pi}{30}$. Now $y' \in \{e, c^{-1}\}$ and the degree of $v_{2\Delta_2}$ must be greater or equal to four. So $c(\Delta_2) \leq c(2,3,4,5,6) = -\frac{\pi}{10}$. So the negative curvature from Δ_1 and Δ_2 is added up to $c(\Delta_1) + c(\Delta_2) = -\frac{\pi}{6} - \frac{\pi}{10} = -\frac{4\pi}{15}$ that is again enough to compensate for $c(\Delta)$. If the degree of v_1 and v_4 is equal to five then $c(\Delta) \leq c(2,2,3,3,5,5) = \frac{2\pi}{15}$ and $c(\Delta_1) \leq c(2,3,4,5,6) = -\frac{\pi}{10}$ and the same applies for $c(\Delta_2)$. This makes the total negative curvature in the area equal to $-\frac{\pi}{5}$ that is enough to compensate for $c(\Delta)$. If there is only one vertex of degree five then $c(\Delta) \leq c(2,2,3,3,5,6) = \frac{\pi}{15}$ and $c(\Delta_1)$ or $c(\Delta_2)$ is enough to compensate for it on its own.

Now let $l_2 = l_5 = ced^{-1}$. The remaining two vertices can have degree at least five and so the curvature of Δ can be up to $c(\Delta) \leq c(2,2,3,3,5,5) = \frac{2\pi}{15}$. If we consider Δ_1 as with the previous paragraph $v_{1\Delta_1}$ will be $c(eb)ew$ and Δ_1 will have curvature at most $c(\Delta_1) \leq c(2,3,5,5,5) = -\frac{2\pi}{15}$ which is enough to compensate for Δ.

The same applies if one of v_2 and v_5 has label ced and the other has label ced^{-1}. There are always Δ_1 and Δ_2 with enough negative curvature to compensate for any positive one arising from Δ. The positive curvature is given to the region of negative curvature across a $d - e$ edge.

In all of the above case, for the same type of groups, negative curvature is being compensated through the same vertices (i.e. if $a = c = e^2$ across an $a - e$ vertex and across a $d - e$ vertex in all the other cases). So a negative region can only compensate for one region of positive curvature.

10. $c \neq 1$ **and** $a \neq e$ **,** $e^2 = 1$**,** $d^2 \neq 1$

The only possible vertices of degree two are v_1 and v_4 with label $(eb)^2$.

First suppose there is only one vertex of degree two. Let that be v_1 (a the case when that is v_4 is the only vertex of degree two is symmetric). If $|v_4| = 3$ then $l_4 \in \{(eb)^3, (eb)^2a^{\pm1}, (eb)a^{\pm1}(eb), (eb)c^{-1}b, (eb)ec\}$. The first label forces $e = 1$ and the next two force $a = 1$, since the order of e is now two. The next label makes $ec = 1$ and the last label forces $c = 1$, a contradiction in any case. So v_4 cannot have degree three and so the remaining four vertices should. It is possible for v_2 and v_6 to have degree three with label dcb^{-1} or dce. First suppose $l_6 = dcb^{-1}$ and so $d = c^{-1}$ and $l_5 = ca^{-1}w$. If $|v_5| = 3$ then $l_5 \in \{ca^{-1}b^{-1}, ca^{-1}e\}$ and any of these labels forces a contradiction. If $l_6 = dce$ then $c = d^{-1}e$ and $l_5 = caw$ so $l_5 \in \{cab^{-1}, cae\}$. Any of these makes the group to be generated by a and e which both have order two and therefore has property X.

Therefore, to obtain a region of positive curvature both v_1 and v_4 must have degree two and region Δ will be as in Figure 6-20.

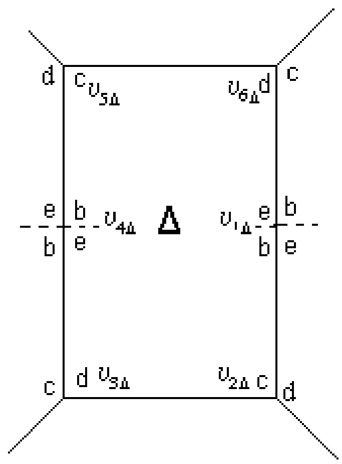


Figure 6-20:

All of the remaining vertices v_2, v_3, v_5 and v_6 may have degree three with labels dcb^{-1} or dce. We check the label of the vertex adjacent to one of degree three.

If $|v_6| = 3$ and $l_6 = dcb^{-1}$ then $c = d^{-1}$ and $l_5 = dca^{-1}w$. $|v_5| = 4$ implies $l_5 \in \{dca^{-1}b^{-1}, dca^{-1}c^{-1}, dca^{-1}e\}$. The fist label forces a contradiction by making $a = 1$ and the second one by making $a = c = d$ and so the order of d to be two. In the case of the third label $a = e$ and by Lemma 6.2(i) the equation has a solution. $|v_5| = 5$ implies $l_5 \in \left\{ \begin{array}{c} dca^{-1}(eb)^{-1}b^{-1}, dca^{-1}(eb)e, dca^{-1}(eb)^{\pm 1}c^{-1}, \\ dca^{-1}b^{-1}d, dca^{-1}c^{-1}d, dca^{-1}ed \end{array} \right\}$. The only of these labels which does not force a contradiction is $l_5 = dca^{-1}c^{-1}d$ which makes

$a = d^2$. The same applies for v_3 i.e. if $l_3 = dcb^{-1}$ then $l_2 = dca^{-1}c^{-1}d$ or $|v_2| \geq 6$.

If $|v_6| = 3$ and $l_6 = dce$ then $c = d^{-1}e$ and $l_5 = dcaw$. It turns out that the degree of v_5 cannot be four and if $|v_5| = 5$ then $l_5 = dcac^{-1}d$ and so $a = ed^2e$. The same applies if $l_3 = dce$ i.e. either $l_2 = dcac^{-1}d$ or $|v_2| \geq 6$.

If $|v_5| = 3$ and $l_5 = dcb^{-1}$ then $d = c^{-1}$ and $l_6 = c^{-1}dcw$. If $|v_6| = 4$ then $l_6 \in \{c^{-1}dca^{\pm1}, c^{-1}dc(eb)^{\pm1}\}$ but any of these forces a contradiction by making the order of d to be two. If $|v_6| = 5$ then $l_6 \in \{c^{-1}dca^{\pm1}(eb)^{\pm1}, c^{-1}dc(eb)^{\pm1}a^{\pm1}, c^{-1}dc(eb)^{\pm2}\}$ but any of these labels forces a contradiction. So if $l_5 = dcb^{-1}$ then $|v_6| \geq 6$ and if $l_2 = dcb^{-1}$ then $|v_3| \geq 6$.

If $|v_5| = 3$ and $l_5 = dce$ then $c = d^{-1}e$ and $l_6 = d^2cw$. If $|v_6| = 4$ then $l_6 \in \{d^2cb^{-1}, d^2ce\}$ and each of these labels forces a contradiction. If $|v_6| = 5$ then its label will be $l_6 \in \{d^2ca^{\pm1}b^{-1}, d^2ca^{\pm1}c^{-1}, d^2ca^{\pm1}e, d^2cb^{-1}d, d^2ced\}$. The only of these labels which does not force a contradiction is $l_6 = d^2cb^{-1}d$ which makes $c = d$ and $e = d^2$. Also if $l_2 = dce$ then either $|v_3| \geq 6$ or $l_3 = d^2cb^{-1}d$.

Now suppose there is only one vertex of degree three in Δ. Then from the above paragraphs we see that the adjacent vertex to it will have degree at least five. That means that to obtain a region of positive curvature the other two vertices should have degree four. If $l_6 = dcb^{-1}$ then $l_5 = dca^{-1}c^{-1}d$ and $c = d^{-1}$, $a = d^2$ and v_2 and v_3 must have degree four. The only label starting with dc which does not force a contradiction under these conditions is $dc(eb)e$ but in this case $l_3 = d^2cw$ and therefore cannot have degree four. So positive curvature may not be obtained like this. If $|v_6| = 3$ and $l_6 = dce$ then $c = d^{-1}e$ and $l_5 = dcac^{-1}d$ and $a = ed^2e$. For v_2 having degree four the label should be $l_2 = dc(eb)^{-1}b^{-1}$ but then the degree of v_3 cannot be four. If $|v_5| = 3$ and $l_5 = dcb^{-1}$ then $|v_6| \geq 6$ so if this is the only vertex of degree three positive curvature may not be obtained. If $|v_5| = 3$ and $l_5 = dce$ then $c = d^{-1}e$ and so $l_6 = d^2cb^{-1}d$ which makes $c = d$ and $e = d^2$. The only possible label for v_2 with length four is $l_2 = dc(eb)^{-1}b^{-1}$ but as before the degree of v_3 cannot be four this time. It is concluded that there must be two vertices of degree three.

The two vertices of degree three must have the same label. Fist suppose $l_6 = dcb^{-1}$ and $l_2 = dcb^{-1}$. The degree of v_3 is six or greater so the degree of v_5 must be five otherwise positive curvature cannot be obtained. So $l_5 = dca^{-1}c^{-1}d$ and region Δ can have positive curvature at most $c(\Delta) \leq c(2,2,3,3,5,6) = \frac{\pi}{15}$ and will be as in Figure 6-21(a). This case is symmetric to $l_3 = dcb^{-1}$ and $l_5 = dcb^{-1}$. If $l_6 = dcb^{-1}$ and $l_3 = dcb^{-1}$ then one of v_5 and v_2 must have the label $dca^{-1}c^{-1}d$ otherwise both labels will be grater to six and so positive curvature may not be obtained. A possible region will be as in Figure 6-21(b). If $l_2 = l_5 = dcb^{-1}$ then v_3 and v_6 have degree six at least six so the region cannot have positive curvature. In any of the above that a region of positive curvature exist one of the remaining vertices has the label $dca^{-1}c^{-1}d$ and the a region Δ_1 can be found as in Figure 6-21 with that $c(\Delta_1) \leq c(3,3,4,4,5) = -\frac{4\pi}{15}$.

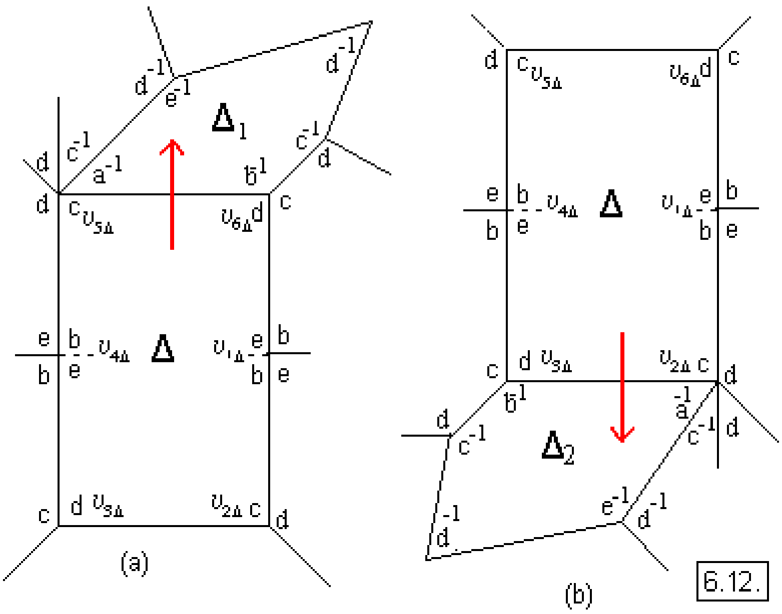


Figure 6-21:

If the labels of degree three have label dce then $c = d^{-1}e$ and there must be at least one label of the form $dcac^{-1}d$ which gives $a = ed^2e^{-1}$ or $d^2cb^{-1}d$ and so

$c = d$ and $e = d^2$. The two labels of degree five cannot occur at the same time. Negative curvature can be found in Δ_1 or Δ_2 as in Figure 6-22.

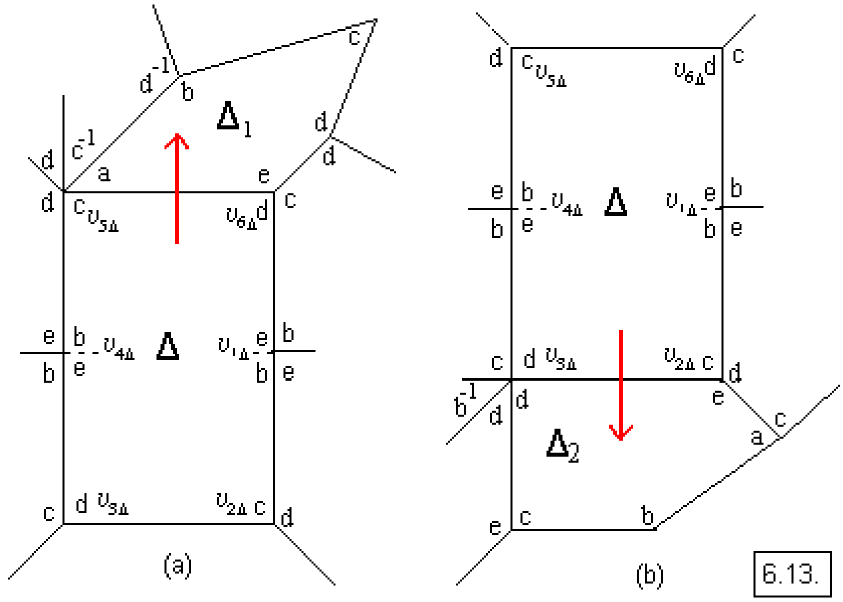


Figure 6-22:

While the curvature of Δ in the worse case is $c(\Delta) \le c(2,2,3,3,5,5) = \frac{2\pi}{15}$ then negative curvature of Δ_1 or Δ_2 will be $c(\Delta^-) \le c(3,3,3,5,5) = -\frac{\pi}{5}$ in any way enough to compensate for any positive curvature arising. Observe that in the first case the relators are $c = d^{-1}e$ and $a = ed^2e^{-1}$, the positive curvature is given to Δ_1 across a $d - e$ edge, while on the second case the relators are $c = d$ and $e = d^2$ and the positive curvature is given to Δ_2 across an $e - a$ edge. The two cases cannot occur in the same diagram as this would make the group generated by d and thus cyclic.

Overall in this section, regions of positive curvature were found in the cases 8, 9 and 10 when one of the following type of relators holds.

- $c \neq 1$, $a \neq e$, $e^2 = 1$ and $d^2 = 1$
- $c \neq 1$, $a \neq e$, $e^2 \neq 1$ and $d^2 = 1$

- $c \neq 1$, $a \neq e$, $e^2 = 1$ and $d^2 \neq 1$

Now suppose that Δ is a region of positive curvature in a diagram D that a^2 appears at least once in the diagram but not on Δ. So d^2 appears on Δ and therefore $a^2 = d^2 = 1$. The following labels or their inverses may appear on vertices of degree two: d^2, cb^{-1}, $a(eb)^{\pm 1}$. Therefore Δ will be one of the regions F_1, F_2, D_2 and E of Figure 6-23. We examine each case.

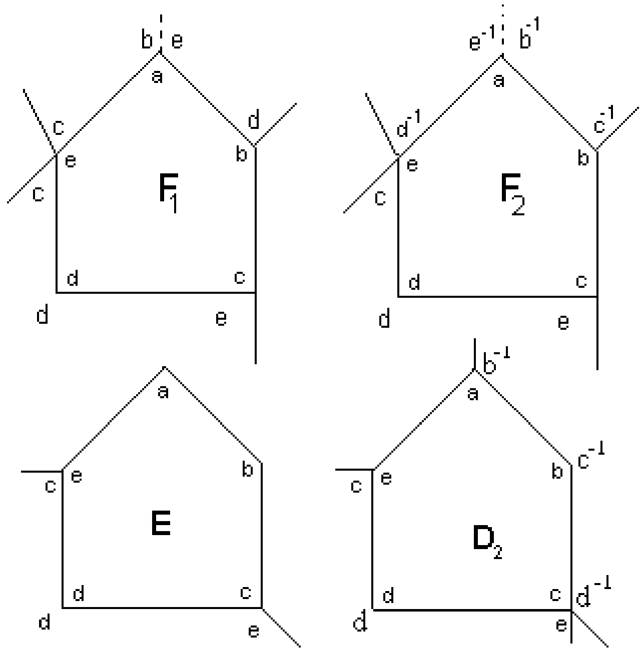


Figure 6-23:

F_1

Now the relators holding in G are $a = e$ and $a^2 = d^2 = 1$ and the group is generated by the three elements a, d and c. If a label contains letter c just once, then the it will be generated by the other two elements of order two and thus the group would have property X. So if a label has a $c^{\pm 1}$ then it should have it more than once and so its length must be at least four. We check for possible labels of degree three. If $|v_2| = 3$ then $l_2 = dbe$ and $d = a = e$. If $|v_2| = 4$ then $l_2 = dba^{\pm 1}b^{-1}$ and again $d = a = e$. If $|v_2| = 4$ then $l_2 = dbc^{-1}bc^{-1}$ and $d = c^2$. The degree of v_5 cannot be three and if $|v_5| = 4$ then $l_5 \in \{cecb^{-1}, cece\}$ and either $a = e = c^2$ or $(ca)^2 = 1$. If $|v_5| = 5$ then $l_5 \in \{ceca^{\pm 1}e, cec(eb)e, ceca^{\pm 1}b^{-1}, cec(eb)^{-1}b^{-1}, ceced^{\pm 1}\}$ and $a = e = c^2$, $(ca)^2 = 1$ or $d = (ca)^2$. The degree of v_3 cannot be three and if $|v_3| = 4$ then $l_3 \in \{cecb^{-1}, c(eb)c^{-1}d^{\pm 1}\}$ and either $a = e = c^2$ or $d = cac^{-1}$.If $|v_3| = 5$ then $l_5 \in \left\{ \begin{array}{c} ceca^{\pm 1}e, cec(eb)e, c(eb)ece, ceca^{\pm 1}b^{-1}, cec(eb)^{-1}b^{-1}, \\ ceced^{\pm 1}, c(eb)c^{-1}be, c(eb)c^{-1}e^{-1}b^{-1}, c(eb)ecb^{-1} \end{array} \right\}$ and $a = e = c^2$, $(ca)^2 = 1$, $d = (ca)^2$, $caca^{-1} = 1$ or $c^2 = 1$. In the case that $a = e = c^2$ if a label involves d and it is not of degree two then it must have length at least twelve otherwise G has property X or *diamond moves* can be performed to increase the total number of sources and sinks, a contradiction (A2). Therefore if a region does not have a vertex of degree four $c(\Delta) \leq c(2, 2, 5, 5, 12) = 0$. Vertices of degree four are possible to exist in this case and so the curvature of such a region is at most $c(\Delta) \leq c(2, 2, 4, 4, 12) = \frac{\pi}{6}$.

In the case of the other relators arising if the degree of v_2 is less than six, then v_3 and v_5 must have degree at least six, otherwise the group is forced to have property X. If v_2 has degree six or greater then v_3 and v_5 can be four or five. Therefore a region can have curvature $c(\Delta) \leq c(2, 2, 4, 4, 6) \leq \frac{7\pi}{30}$ or in the case that $c = d = a$ the curvature must be $c(\Delta) \leq c(2, 2, 3, 6, 6)\frac{\pi}{3}$. The possible regions of positive curvature are categorised according to the relators holding in the group and are examined at the end of this section with those found when a region is F_2.

F_2

The relators holding in the group are again $a = e$ and $a^2 = d^2 = 1$. We look for possible degree of v_2, v_3 and v_5.

The degree of v_5 cannot be three or four and if $|v_5| = 5$ then $l_5 = ced^{-1}ce$. The degree of v_2 cannot be three and if the degree of v_2 is four then its label is $l_2 \in \{c^{-1}bc^{-1}b, c^{-1}bc^{-1}e^{-1}\}$ and either $c^2 = 1$ or $a = c^2$. If $|v_2| = 5$ then its label is $l_2 \in \left\{\begin{array}{c} c^{-1}ba^{\pm 1}c^{-1}b, c^{-1}b(eb)c^{-1}b, c^{-1}bc^{-1}b(eb), c^{-1}ba^{\pm 1}c^{-1}e^{-1}, c^{-1}b(eb)c^{-1}e^{-1}, \\ c^{-1}bc^{-1}e^{-1}a^{\pm 1}, c^{-1}bc^{-1}e^{-1}(eb)^{-1}, c^{-1}beca^{\pm 1}, c^{-1}beca^{\pm 1}, c^{-1}bec(eb)^{\pm 1}, c^{-1}bc^{-1}d^{\pm 1}b \end{array}\right\}$ and the possible relators holding in the group are $a = c^2$, $d = c^2$, $c^2 = 1$ or $acac^{\pm 1}$. The degree of v_3 cannot be three and if $|v_3| = 4$ then $l_3 \in \{cecb^{-1}, c(eb)c^{-1}d^{\pm 1}\}$ and either $a = e = c^2$ or $d = cac^{-1}$. If $|v_3| = 5$ then $l_5 \in \left\{\begin{array}{c} ceca^{\pm 1}e, cec(eb)e, c(eb)ece, ceca^{\pm 1}b^{-1}, cec(eb)^{-1}b^{-1}, \\ ceced^{\pm 1}, c(eb)c^{-1}be, c(eb)c^{-1}e^{-1}b^{-1}, c(eb)ecb^{-1} \end{array}\right\}$ and $a = e = c^2$, $(ca)^2 = 1$, $d = (ca)^2$, $caca^{-1} = 1$ or $c^2 = 1$.

It is therefore possible to have regions with curvature $c(\Delta) \leq c(2, 2, 4, 5, 6) = \frac{7\pi}{30}$. These are categorised in terms of relators holding in the group each time and examined with those found when a region is F_1.

First, it should be noted that the regions of positive curvature found in this section cannot occur on the same diagram with regions of degree six with positive curvature. This would be allowed only for case 6 ($c \neq 1$, $a = e$, $d^2 = 1$) but no regions of degree six were found to have positive curvature in that case. The possible F_1 and F_2 regions of positive curvature can occur when one of the following mutually exclusive type of relators holds in group G:

1. $d = a$
2. $d = c^2$
3. $d = (ac)^2$
4. $c^2 = 1$ or $acac^{\pm 1} = 1$
5. $d = cac^{-1}$
6. $a = c^2$

1. $d = a$ A possible region of positive curvature will be one of 6.14, 6.15 as in Figure 6-24.

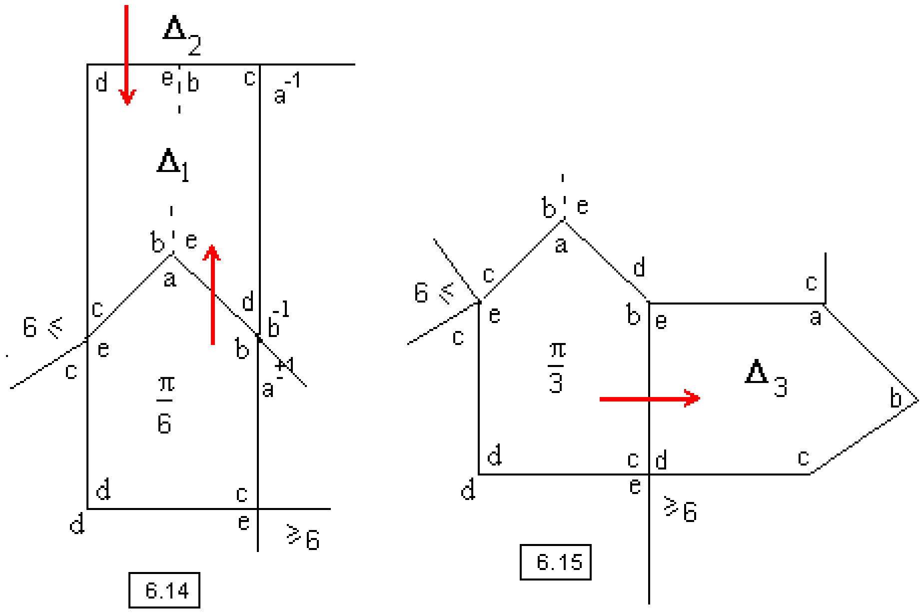


Figure 6-24:

In the case of 6.14. Δ_1 which is a region of degree six can be used to compensate the positive curvature of Δ. Region Δ_1 has curvature at most $c(\Delta) \leq c(2, 2, 3, 4, 6, 6) = -\frac{\pi}{6}$. In the case that Δ_1 is also used to compensate for possible positive curvature coming from Δ_2, this Δ_2 region must again be one of 6.14. type. So $v_{3\Delta_1}$ is forced to have degree four and so $c(\Delta_1) \leq c(2, 2, 4, 4, 6, 6) = -\frac{\pi}{3}$ which is enough to compensate for both $c(\Delta)$ and $c(\Delta_2)$. In the case of 6.15. Δ_3 can be used to compensate for the positive curvature of Δ. The curvature of Δ_3 is at most $c(\Delta_3) \leq c(2, 3, 6, 6, 6) = -\frac{\pi}{3}$. Since Δ_3 is a region of degree five it can only receive negative curvature from at most one of its neighbours. Observe, that in any of these case the positive curvature is added to the negative across a $d - e$ edge.

2. $d = c^2$ Any possible region of positive curvature will be one of 6.16 and 6.17 in Figure 6-25.

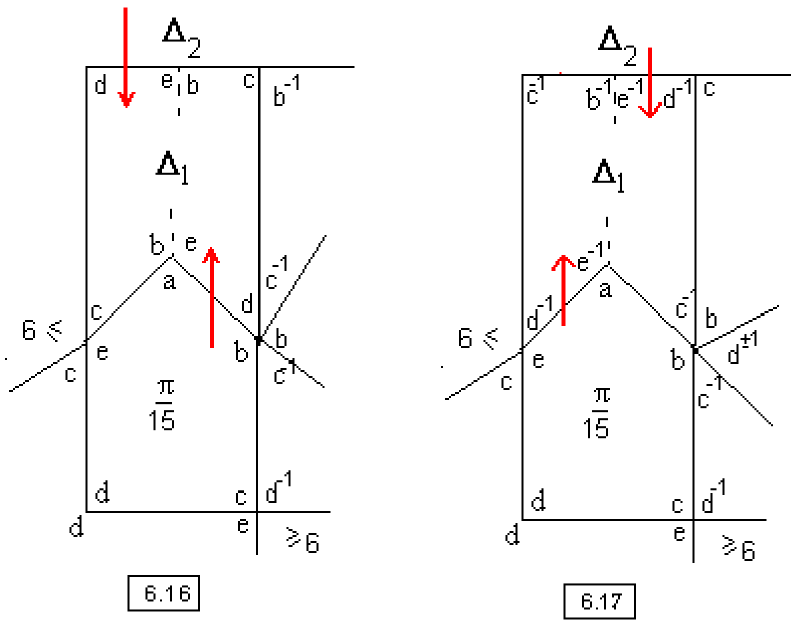


Figure 6-25:

As only one vertex can have degree five, any of these regions has curvature at most $c(\Delta) \leq c(2,2,5,6,6) = \frac{\pi}{15}$. In this case a vertex which involves a $d^{\pm 1}$ can have degree two or five or greater. Therefore, the curvature of a region like Δ_1 adjacent to Δ will be at most $c(\Delta_1) \leq c(2,2,5,5,5,6) = -\frac{7\pi}{15}$ which is enough negative curvature to compensate for any positive of Δ and perhaps another region Δ_2 of positive curvature (not greater than $\frac{\pi}{15}$) with the positive curvature always added to the negative across a $d - e$ edge.

3. $d = (ac)^2$ In any case at most two of the remaining vertices will have degree at least five so a region can have curvature at most $c(2,2,5,5,6) = \frac{2\pi}{15}$ and will be one of regions 6.17. and 6.18. in Figure 6-26.

Now the curvature of region Δ_1 is at most $c(\Delta_1) \leq c(2,2,5,5,5,6) = -\frac{7\pi}{15}$ which is enough to compensate for the positive curvature of Δ and perhaps a Δ_2

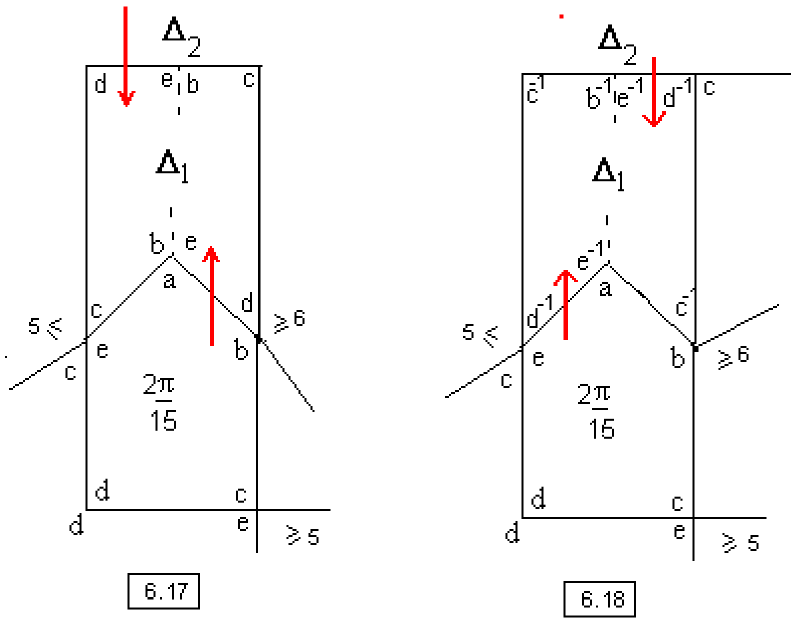


Figure 6-26:

that can occur. The positive curvature is again added across a $d-e$ edge.

4. $c^2 = 1$ **or** $acac^{\pm 1} = 1$ Now any possible region of positive curvature will be as in Figure 6-27.

The curvature of Δ is at most $c(\Delta) \leq c(2,2,4,5,6) = \frac{7\pi}{30}$. The curvature of region Δ_1 is $c(\Delta_1) \leq c(2,2,4,4,6,6) = -\frac{\pi}{3}$ which is enough to compensate for Δ and perhaps another region Δ_2 of positive curvature across its other $d-e$ edge when the degree of v_5 is five or greater. If the degree of v_5 is four i.e. $l_5 = cece$ then region Δ_2 is either has non-positive curvature or can add its positive curvature across another $d-e$ edge adjacent to its $a-b$ edge.

5. $d = cac^{-1}$ Both in the case of F_1 and F_2 the only possible vertex of degree less than six is v_3 with label $l_3 = c(eb)c^{-1}d^{\pm 1}$ and so a region can have curvature at most $c(2,2,4,6,6) = \frac{\pi}{6}$. The neighbouring region of degree six is again used to compensate for the positive one, as before. Its curvature will be at least

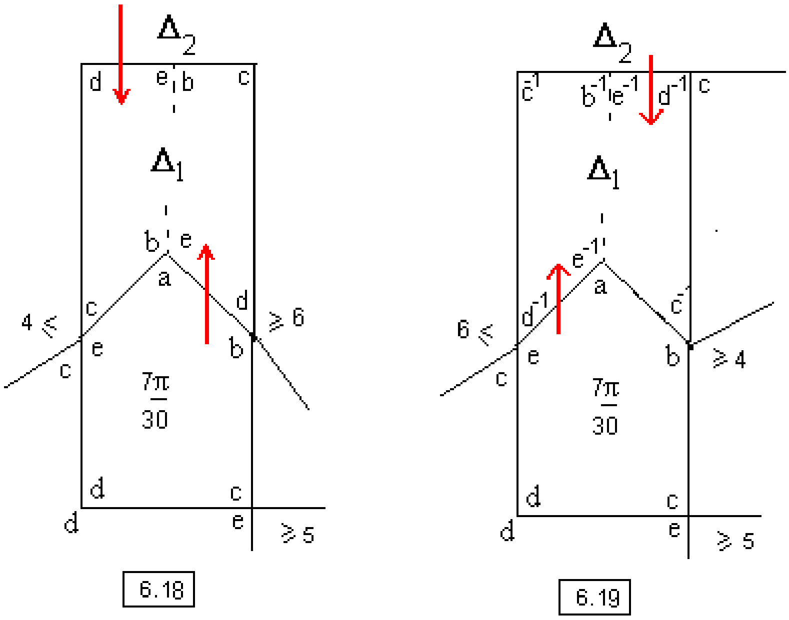


Figure 6-27:

$c(\Delta_1) \leq c(2,2,4,4,6,6) = -\frac{\pi}{3}$ which is enough to compensate for the positive curvature even if there is another region of negative curvature adding it to it (see Figure 6-28). Once again the positive curvature is added to the negative across a $d-e$ edge.

6. $a = c^2$ The degrees of two vertices can be four at the same time, but in such case the degree of the remaining vertex will be six. So a region can have curvature at most $c(\Delta) \leq \{c(2,2,4,4,6), c(2,2,4,5,5)\} = \frac{\pi}{3}$. As before a region of degree six can be used to compensate for positive curvature in the same way, i.e. curvature added to the region of negative curvature across a $d-e$ edge as in Figure 6-29

Such a Δ_1 will have curvature at most $c(\Delta) \leq c(2,2,4,4,6,6) = -\frac{\pi}{3}$. In the case that there is another region of positive curvature adjacent to Δ_1 this must be of the same type and the curvature of Δ_1 becomes $c(\Delta_1) \leq c(2,2,4,4,12,12) =$

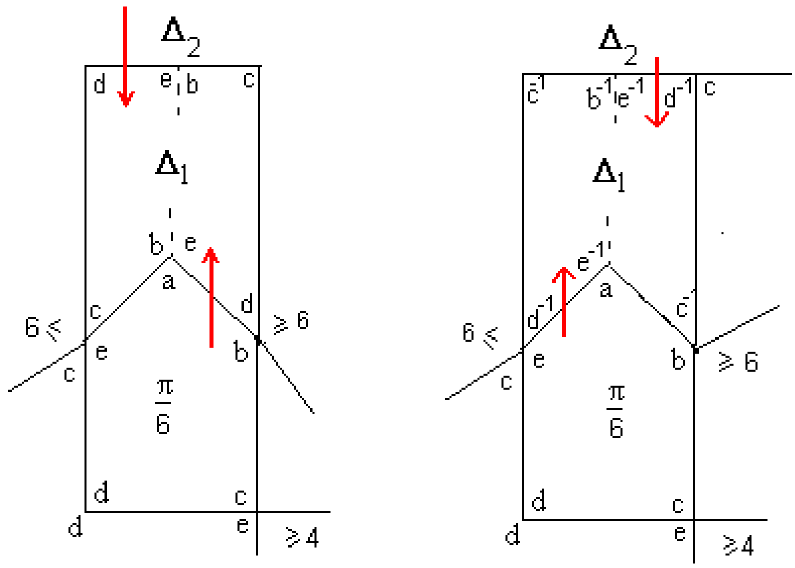


Figure 6-28:

$-\frac{2\pi}{3}$ which is enough for both regions of positive curvature.

D_2

Now the relators holding in the group are $c = 1$ and $a^2 = d^2 = 1$. It is not possible that an F_1 or an F_2- region is encountered in the same diagram as D_2 regions as the group would then have property X. Also no interior regions of positive curvature with degree six are encountered since for the cases $c = 1$ and $d^2 = 1$ no regions of positive curvature with degree six were found.

The label of v_3 is $l_3 = d^{-1}cew$ and its degree cannot be three. If the degree of v_3 is four then $l_3 = d^{-1}c(eb)e$ and $d = e^2$. If its degree is five then $l_3 \in \{dcece, ce(eb)^2e\}$ and either $d = e^2$ or $d = e^3$.

The label of v_1 is abw. If $|v_1| = 3$ then $l_1 = a(eb)^{-2}$ and $a = e^2$. If $|v_1| = 4$ then $l_1 \in \{ab^{-1}d^{\pm1}b, ab^{-1}d^{\pm1}c, a(eb)^{-3}, a(eb)^{-1}c^{-1}e^{-1}, a(eb)^{-1}a^{\pm1}(eb)^{\pm1}\}$ and $a = d$, $a = e^2$, $a = e^3$ or $ae^{-1}ae^{\pm1} = 1$. If $|v_1| = 5$ then its label can be $l_1 \in$

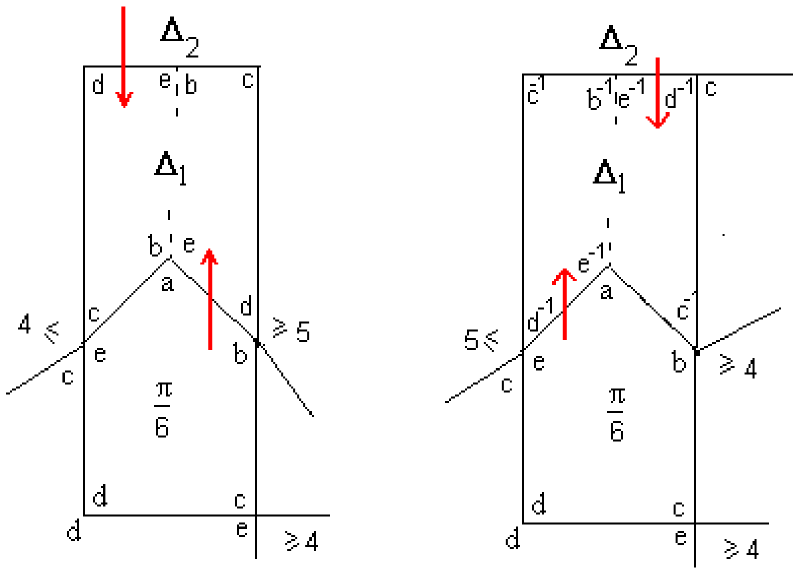


Figure 6-29:

$\left\{ \begin{array}{c} ab^{-1}c(eb)^{-2}, a(eb)^{-2}c^{-1}b, a(eb)^{-2}c^{-1}e^{-1}, a(eb)^{-4}, \\ a(eb)^{-1}a^{\pm1}(eb)^{\pm2}, a(eb)^{-1}a^{\pm1}c^{-1}e^{-1}, a(eb)^{-1}a^{\pm1}ec \end{array} \right\}$ and $a = e^2$, $a = e^3$, $a = e^4$, $aeae^{\pm2}$, $aeae^{\pm1}$.

The degree of v_5 cannot be three and $|v_5| = 4$ implies $l_5 = cece$ and $e^2 = 1$. If $|v_5| = 5$ then $l_5 \in \{ced^{\pm1}be, ced^{\pm1}ce, ceca^{\pm1}e, cec(eb)e, ceced^{\pm1}\}$ and one of the following relators holds $d = e^2$, $a = e^2$ or $e^3 = 1$.

If the $v_1 \in \{a(eb)^{-2}, a(eb)^{-1}c^{-1}e^{-1}, ab^{-1}c(eb)^{-2}, a(eb)^{-2}c^{-1}b, ab^{-1}c(eb)^2\}$ then $a = e^2$. The label of v_3 is $l_3 = d^{-1}cew$ and if d^{-1} is encountered only once the group becomes cyclic. If it is encountered more than once then $|v_3| \geq 12$ otherwise the group will have property X or *diamond moves* may be performed to increase the number of sinks, which would contradict (A10). In the cases that the degree of v_1 is three or four the label of v_5 is $l_5 = ced^{-1}w$ and it will also have degree at least twelve. In the cases that the degree of v_1 is five, the degree of v_5 must also be five or greater. So in any of these cases $c(\Delta) \leq \max\{c(2,2,3,12,12), c(2,2,5,5,12)\} = 0$.

If $l_1 \in \{ab^{-1}d^{\pm 1}b, ab^{-1}d^{\pm 1}c\}$ and $a = d$ then $l_5 \in \{cecw, cedw\}$. If a label involves an e and a c then it should have degree six or greater (the order of e is at least three) and if an $a^{\pm 1}$ or a $d^{\pm 1}$ is further involved then the length of this label must be at least seven. So the degree of v_3 in this case is at least seven and the degree of v_5 at least six. A possible region of positive curvature will have curvature at most $c(\Delta) \leq c(2,2,4,6,7) = \frac{5\pi}{42}$. If the degree of e is four or greater then both the degree of v_5 and v_3 are at least eight and a region of positive curvature cannot exist. So in this case $a = d$ and $e^3 = 1$.

If $l_1 \in \{a(eb)^{-3}, a(eb)^{-2}c^{-1}e^{-1}, a(eb)^{-4}\}$ and $a = e^3$ or $a = e^4$ then $l_5 = ced^{-1}w$ and the degree of v_5 must be at least eight. The degree of v_3 will also be at least eight so a region of positive curvature may not be achieved as $c(\Delta) \leq c(2,2,4,8,8) = 0$.

If $l_1 \in \{a(eb)^{-1}a^{\pm 1}(eb)^{\pm 1}, a(eb)^{-1}a^{\pm 1}(eb)^{\pm 2}, a(eb)^{-1}a^{\pm 1}c^{-1}e^{-1}, a(eb)^{-1}a^{\pm 1}ec\}$ then $aeae^{\pm k} = 1$ with $k \in \{1,2\}$. If d is encountered only once in a label then it must be generated by the other elements and the group would have property X. The same applies if e is encountered only once, so the degree of v_3 and v_5 is at least six except for the case $l_1 \in \{a(eb)^{-1}a^{\pm 1}(eb), a(eb)^{-1}a^{\pm 1}(eb)^2\}$ and $l_5 \in \{cece, cec(eb)e\}$. The group is now generated by elements a, d and e and the order of e can be two or three. The degree of v_5 must be six or greater and this region may have curvature $c(\Delta) \leq c(2,2,4,4,6) = \frac{\pi}{3}$.

Now suppose the degree of v_1 is greater than six. The degree of at least on of v_3 and v_5 must be four or five. The degree of v_3 cannot be three and if its four then $l_3 = d^{-1}c(eb)e$ and $d = e^2$. This forces the degree of v_1 to be at least twelve as labels of smaller length would create relators that make the group to have property X. The degree of v_5 cannot be four and if it is six or greater then region cannot have positive curvature. So it must be $|v_5| = 5$ and so $l_5 \in \{ced^{\pm 1}be, ced^{\pm 1}ce, ceced^{\pm 1}\}$. A possible region of positive curvature has curvature at most $c(\Delta) \leq c(2,2,4,5,12) = \frac{\pi}{15}$. Now if the label of v_3 is $l_3 = d^{-1}cece$ the same relators as before apply and $c(\Delta) \leq c(2,2,5,5,12) < 0$.

Now if the label of v_3 is $l_3 = dc(eb)^2e$ and $d = e^3$ then the degree of v_1 must be at least ten and the degree of v_3 cannot be four or five, so $c(\Delta) \leq c(2,2,5,6,10) < 0$.

Now suppose that v_1 and v_3 both have degree six or greater. Then the degree of v_5 must be four or five. If $l_5 \in \{ced^{\pm 1}be, ced^{\pm 1}ce, ceced^{\pm 1}\}$ and $d = e^2$ then the degree of v_1 is at least twelve and thus a region of positive curvature cannot exist. If $l_5 = ceca^{\pm 1}e$ then $a = e^2$ and the degree of v_3 is at least twelve. So again a region cannot have positive curvature. The only other cases if for $l_5 \in \{cece, cec(eb)e\}$ and the group is then generated by three elements, the order of e can be two or three. This region may have curvature at most $C(\Delta) \leq c(2,2,4,6,6) = \frac{\pi}{6}$.

Overall D_2 regions of positive curvature can exist when at least one of the following type or relators hold in G:

1. $d = e^2$
2. $a = d$ and $e^3 = 1$
3. $a \neq d$ and $aeae^{\pm k} = 1$, $k = 1, 2$ or/and $e^k = 1$, $k = 2, 3$

1. $d = e^2$ A region of positive curvature will be as in Figure 6-30.

We look for negative curvature in region Δ_1 next to it. If Δ_1 is a region of degree five then its vertex which involves the a^{-1} will have to have degree twelve or greater. The vertex which involves the e^{-1} has degree at least four. So its curvature will be $c(\Delta_1) \leq c(2,4,4,4,12) = -\frac{\pi}{3}$ which is enough to compensate for the curvature of Δ that is only $c(2,2,4,5,12) = \frac{\pi}{15}$. If Δ_1 is a region of degree six then another one of its vertices may have degree two (a d^{-2} or $c^{-1}b$) and so $c(\Delta_1) \leq c(2,2,4,4,4,12) = -\frac{\pi}{3}$ that is enough to compensate the curvature of Δ even if Δ_1 receives positive curvature from another region adjacent to it.

2. $a = d$ Now a region of positive curvature will be one the regions of Figure 6-31.

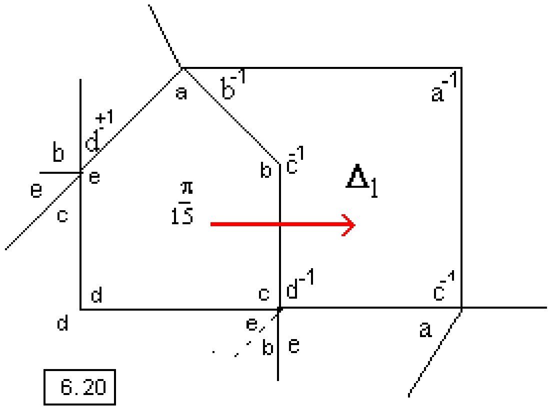


Figure 6-30:

If a vertex involves an a and it is not of degree two then its degree must be at least four. If an $e^{\pm 1}$ is involved then its degree is at least seven and if a $d^{\pm 1}$ is involved and the label is not $d^{\pm 2}$ then the degree of that vertex is at least four. If Δ_1 is a region of degree five then its curvature is at most $c(\Delta_1) \leq c(2,4,4,4,7) = -\frac{3\pi}{14}$. If it is a region of degree six then $c(2,2,4,4,6,6) = -\frac{\pi}{3}$ which is enough curvature to compensate for Δ even if it is receiving positive curvature from another region Δ_2.

3. $aeae^{\pm k} = 1$**,** $k = 1, 2$ **or/and** $e^k = 1$**,** $k = 2, 3$ Now if a vertex involves a $d^{\pm 1}$ then its degree is at least six and if a vertex involves an $a^{\pm 1}$ its degree is at least four. A region of positive will be one of regions 6.23.-6.28. in Figure 6-32.

As before we look for negative curvature in region Δ_1. In these case the label of v_1 can be $l_1 \in \{a(eb)^{-1}a^{\pm 1}(eb)^{\pm 1}, a(eb)^{-1}a^{\pm 1}(eb)^{\pm 2}, a(eb)^{-1}a^{\pm 1}c^{-1}e^{-1}, a(eb)^{-1}a^{\pm 1}ec\}$ Δ_1 is a region of degree six and its curvature is $c(\Delta_1) \leq c(2,2,4,4,6,6) = -\frac{\pi}{3}$. This is enough for cases negative curvature to compensate the positive one of Δ and any possible Δ_2 of positive curvature that can be adjacent to it except perhaps in the case of 6.25. In that case if a region of negative region receives

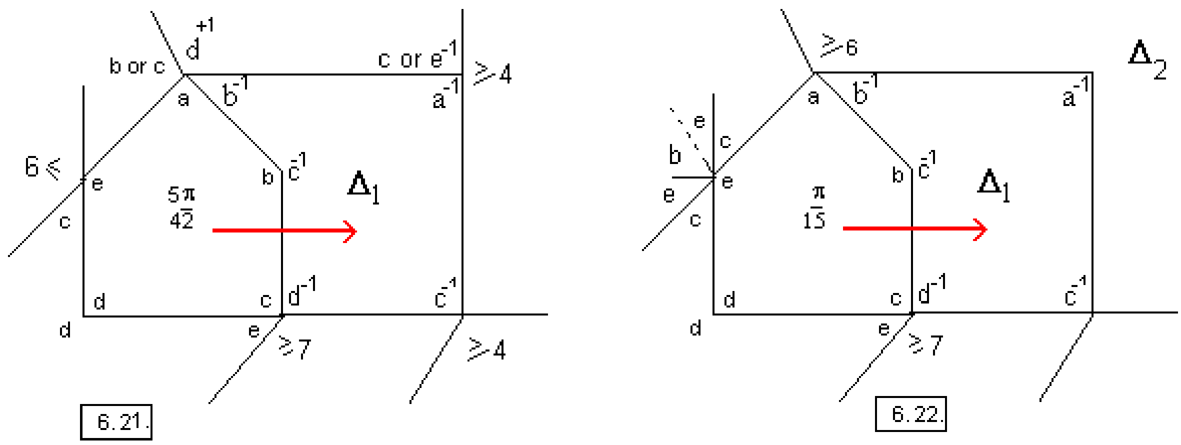


Figure 6-31:

curvature from two regions of positive regions at least one of which is 6.25. a vertex which involves d will have degree eight or greater and the curvature of Δ_1 becomes $c(\Delta_1) \leq c(2, 2, 4, 4, 8, 8) = -\frac{\pi}{2}$ which is enough.

If Δ_1 is a region of degree five then the degree of v_1 is at least six and so $c(\Delta_1) \leq c(2, 4, 4, 6, 6) = -\frac{\pi}{3}$ which is enough to compensate for the curvature of Δ.

E

First we check whether there are any possible regions of positive curvature when $a = e$ or $c = 1$.

Suppose $a = e^{\pm 1}$. The only possible vertex of degree three is v_2 with label $l_2 = bed^{\pm 1}$ and so $a = e = d$ and the label of v_3 is $l_3 = dcew$ and its degree cannot be four. Also the label of v_5 is $l_5 = cew$ and thus its degree cannot be four either. Any region would then have curvature $c(\Delta) \leq c(2, 3, 4, 5, 5) < 0$.

Now suppose $c = 1$. The only possible vertices of degree three are v_1 or v_3 with label $l_1 = a(eb)^{\pm 2}$, $l_3 = c(eb)e$ respectively, but the two of them cannot have

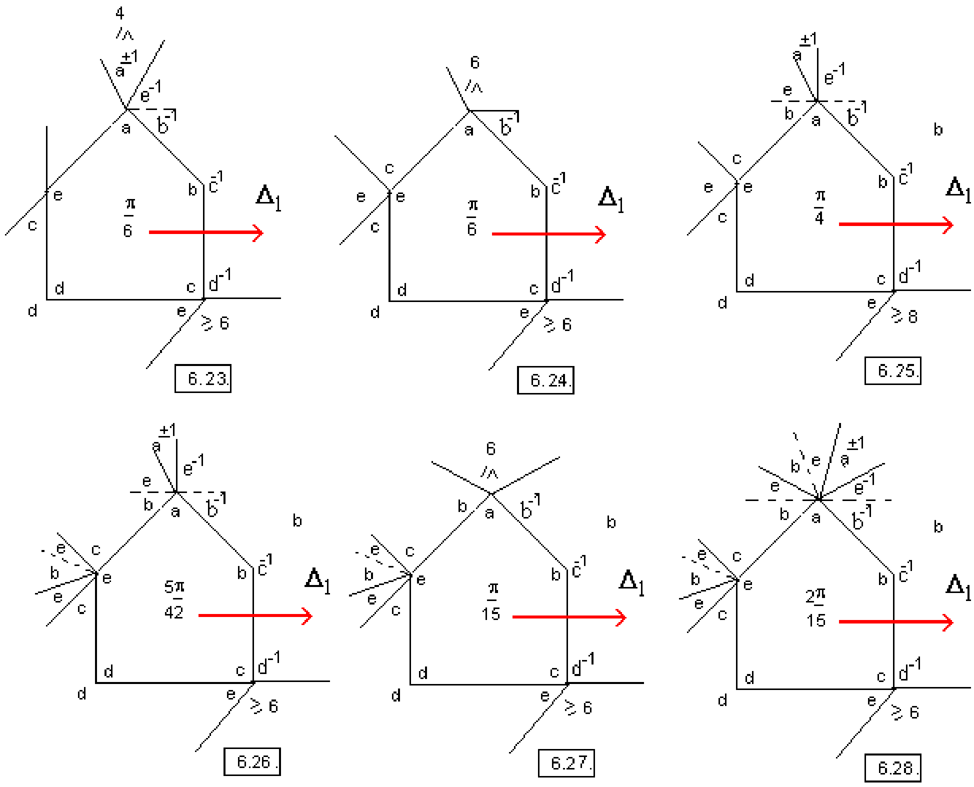


Figure 6-32:

degree three at the same time. First suppose $l_1 = a(eb)^2$ and so $a = e^2$. The labels of v_5 and v_2 are $l_5 = cew$ and so they cannot have degree four. If the label of v_2 does not end with an e^{-1} a *diamond move* can be performed to increase the number of vertices of degree two without altering the number of faces or the number of sinks, a contradiction (A11). Suppose $l_2 = e^{-1}c^{-1}bw$ and v_2 cannot have degree four. So the region cannot have positive curvature. If $l_1 = a(eb)^{-2}$ then $l_5 = ced^{-1}w$ and $l_2 = c^{-1}bw$. The degree of v_5 cannot be four. So the label of v_2 must actually be $l_2 = e^{-1}c^{-1}bw$ and it cannot have length four either and the region cannot have positive curvature. Now let $l_3 = c(eb)e$ and so $e^2 = 1$. The label of v_1 is now $l_1 = baw$ and it can be seen that it cannot have degree four. If either v_1 or v_5 have degree five or greater then positive curvature cannot be achieved and therefore both of them should have degree four. So $l_5 = cece$

and $l_1 = baw$ but then υ_1 cannot have degree four.

Therefore, if there is an $E-$region of positive curvature in a diagram D there are not vertices with labels bc^{-1} or $a(eb)^{\pm 1}$ anywhere on it. So for the remaining case we assume that $c \neq 1$ and $a \neq e$.

Suppose Δ is a region of positive curvature. We look for possible vertices of degree three. If $|\upsilon_5| = 3$ then $l_5 = ced^{\pm 1}$ and $c = de^{-1}$. If $|\upsilon_3| = 3$ then $l_3 = \{ced^{\pm 1}, c(eb)e\}$ and either $c = de^{-1}$ or $c = e^{-2}$. If $|\upsilon_1| = 3$ then $l_1 \in \{a(eb)^{\pm 2}, ab^{-1}c, ac^{-1}b, ac^{-1}e^{-1}, aec\}$ and $a = e^2$, $a = c$ or $c = e^{-1}a$. If $|\upsilon_2| = 3$ then $l_2 \in \{bc^{-1}d^{\pm 1}, bed^{\pm 1}, ba^{\pm 1}c^{-1}\}$ and $c = d$, $e = d$ or $a = c$. It can be seen that υ_5 and υ_2 cannot have degree two at the same time, so it is possible for a region Δ to have at most three vertices of degree three. We check according to the number of vertices of degree three that a region may have.

Δ has three vertices of degree three Theses should be υ_1, υ_2 and υ_3 with labels $l_3 = c(eb)e$, $l_2 = bac^{-1}$ and $l_1 \in \{a(eb)^{-2}, ab^{-1}c\}$. The label of υ_5 is now $l_5 = ced^{\pm 1}w$ and any label of length less than six would make the group cyclic as it would involve only on d and thus make it generated by e. If more than one $d^{\pm 1}$ are involved then the degree of υ_5 must at least twelve otherwise *diamond moves* are allowed to increase the number of d^2 or the group has property X. A region of this type has curvature at most $\frac{\pi}{6}$.

Δ has exactly two vertices of degree three

$|\upsilon_1| = |\upsilon_2| = 3$ If $l_2 = bed^{\pm 1}$ then υ_1 cannot have degree three as this would make the group to be generated by two elements of degree two and so to have property X.

If $l_2 = bc^{-1}d$ then $l_1 = a(eb)^2$ and the relators holding are $c = d$ and $a = e^2$. But now $l_3 = d^{-1}cew$ and $l_5 = cecw$ and any labelling of length less than six would make the group to have property X. So $c(\Delta) \leq c(2, 3, 3, 6, 6) = 0$.

If $l_2 = bc^{-1}d^{-1}$ the label of v_1 must be $l_1 = ac^{-1}b$ and $a = c = d$ and once again the degree of the two remaining vertices must be six or greater, and a region of positive curvature cannot be achieved.

If $l_2 = ba^{\pm 1}c^{-1}$ then $l_1 \in \{a(eb)^{-2}, ab^{-1}c\}$. First let $l_1 = a(eb)^{-2}$ and so $a = c = e^2$. The label of v_5 is $l_5 = ced^{-1}w$ and its degree cannot be less than twelve. So the curvature of the region will be at most $c(\Delta) \le c(2,3,3,4,12) = 0$. Now let $l_1 = ab^{-1}c$ and the only relator is $a = c$. The label of v_5 is $l_5 = cedw$ and its degree cannot be four. If $|v_5| = 5$ then $l_5 \in \{cedce, cede^{-1}b^{-1}\}$ and either $d = (ae)^2$ or $d = e^{-1}ae$. The label of v_3 is $l_3 \in \{ecew, b^{-1}cew\}$. This can be of length four or five with $l_3 \in \left\{ \begin{array}{c} ecec, ec(eb)^2, ec(eb)a^{\pm 1}, b^{-1}cec(eb)^{-1}, ec(eb)^2a^{\pm 1}, b^{-1}ced^{\pm 1}e^{-1}, \\ b^{-1}c(eb)^2a^{\pm 1}, b^{-1}c(eb)a^{\pm 1}(eb)^{-1}, b^{-1}c(eb)c^{-1}e^{-1}, b^{-1}c(eb)ec \end{array} \right\}$ and possible relators are $e^2 = 1$, $aeae^{\pm k} = 1$, $k = 1, 2$ or $d = e^{-1}ae$. In the case that $l_3 = b^{-1}ced^{\pm 1}e^{-1}$ and $d = e^{-1}ae$ it is possible for the v_5 also to have degree five with $l_5 = cede^{-1}b^{-1}$ and the region will have curvature $c(\Delta) \le c(2,3,3,5,5) = \frac{2\pi}{15}$. Otherwise only one of them can have degree four or five and the curvature of Δ will be at most $c(\Delta) \le c(2,3,3,4,6) = \frac{\pi}{6}$.

$|v_1| = |v_3| = 3$ First let $l_1 = a(eb)^2$ and so $a = e^2$. If the label of v_3 is $l_3 = ced$ and $c = de^{-1}$ then $l_2 = dbew$ and it cannot have degree four or five. The label of v_5 is $l_5 = cecw$. If its degree is also greater than five then positive curvature cannot be achieved. It can be seen that v_5 can have degree four with label $l_5 = cece$ but it cannot have degree five. Therefore a region can only have positive curvature of at most $c(2,3,3,4,6) = \frac{\pi}{6}$ when $l_5 = cece$. If the label of v_3 is $l_3 = ced^{-1}$ and so $c = de^{-1}$, then the label of v_2 is $l_2 = dbc^{-1}w$ and it cannot have degree four or five. So again a region positive curvature can only be achieved when $l_5 = cece$ and it has curvature at most $\frac{\pi}{6}$. Now suppose the label of v_3 is $l_3 = c(eb)e$ and so $a = c = e^2$. Then $l_2 = dbaw$ and $l_5 = cecw$ but neither of them can have degree four or five and a region cannot have positive curvature.

If the label of v_1 is $l_1 = a(eb)^{-2}$ and $l_3 = ced$ the labels of v_2 and v_5 are

$l_2 = c^{-1}bew$ and $l_5 = ced^{-1}w$ but neither of them can have degree four or five and a region of positive curvature cannot be achieved. If $l_3 = ced^{-1}$ then the label of v_2 is $l_2 = c^{-1}bc^{-1}w$ and it cannot have degree four or five and v_5 cannot have degree four or five. If the label of v_3 is $l_3 = c(eb)e$ then $l_2 = c^{-1}baw$ and it cannot have degree four or five. Therefore a region cannot have positive curvature when $l_1 = a(eb)^{-2}$.

Now let $l_1 = ab^{-1}c$ and so $a = c$. The only possible label on v_3 is $l_3 = c(eb)e$ and so $a = c = e^2$. The labels of v_2 and v_5 will be $l_2 = c^{-1}baw$ and $l_5 = cedw$ and none of them can have degree less than six.

Now suppose the label of v_1 is $l_1 = ac^{-1}b$ and so the label of v_3 must be $l_3 = c(eb)e$ and again $a = c = e^2$. The labels of v_2 and v_5 are $l_2 = c^{-1}baw$ and $l_5 = cedw$ and none of them can have degree less than six.

If $l_1 \in \{ac^{-1}e^{-1}, aec\}$ then label of v_3 can only be $l_3 = ced^{\pm 1}$ and $d = e^{-1}ae$. The label of v_5 is $l_5 = ced^{\pm 1}w$ and it cannot have degree four or five. The label of v_2 is $l_2 \in \{d^{\pm 1}bew, d^{\pm 1}bc^{-1}w\}$ and in either case it cannot have degree less than six.

$|v_1| = |v_5| = 3$ Now the labels of v_1 and v_5 must be either $l_5 = ced$ and $l_1 = cae$ or $l_5 = ced^{-1}$ and $l_1 = e^{-1}ac^{-1}$ and the relators are $c = e^{-1}a$ and $d = e^{-1}ae$. The label of v_2 is $l_2 = d^{\pm 1}bw$ and the label of v_3 is $l_3 = cew$. Neither of them can have degree less than six and therefore the region cannot have positive curvature.

$|v_2| = |v_3| = 3$ The labels now must be $l_3 = c(eb)e$ and $l_2 = bac^{-1}$ and the relators holding are $a = c = e^2$. The label of v_1 is now $l_1 = ad^{-1}w$ and since it involves a d it cannot be of length less than six. The label of v_5 is $l_5 = cew$ and it is possible to have length five with are $l_5 \in \{ceca^{\pm 1}e, cec(eb)^{-1}b^{-1}\}$. But in this case the label of v_1 becomes $l_1 = bad^{-1}w$ and cannot be of length less than eight. Since $c(2, 3, 3, 5, 8) < 0$ the region cannot have positive curvature.

$|v_3| = |v_5| = 3$ The label of both vertices must be $ced^{\pm 1}$ and the only relator holding now is $c = d^{-1}e$. Now it is possible for v_1 and v_2 to have degree four and five. $|v_1| = 4$ implies $l_1 \in \{cac^{-1}d^{\pm 1}, (eb)^{-1}a(eb)^{-2}\}$ and either $d = e^{-1}ae$ or $a = e^3$. If $|v_2| = 4$ then $l_2 \in \{b(eb)c^{-1}d, b(eb)ed, bc^{-1}bc^{-1}\}$ and either $d = e^2$ or $e^2 = 1$or $c^2 = 1$. If $d = e^2$ and $c = e$ the label of v_1 will be $l_1 = acw$ and its degree cannot be less than twelve. So in that case the region cannot have positive curvature. In the other cases v_1 and v_5 cannot have degree four at the same time, so a region of positive curvature will have curvature at most $c(\Delta) \leq c(2, 3, 3, 4, 5) = \frac{7\pi}{30}$.

Δ has exactly one vertex of degree three Two of the remaining vertices must have degree four and the other one at most five otherwise positive curvature cannot be achieved.

v_1 is the only vertex of degree three First suppose that $l_1 = a(eb)^2$. If $|v_5| = 4$ then $l_4 \in \{cecb^{-1}, cece\}$ and either $e = c^{-2}$ or $(ce)^2$. If $|v_3| = 4$ then $l_3 \in \{cece, cecb^{-1}, c(eb)a^{\pm 1}b^{-1}, c(eb)^2e, c(eb)ed^{\pm 1}\}$ and one of $(ce)^2 = 1$, $e = c^{-2}$, $c = e$ or $c = da$. If $|v_2| = 4$ then $l_2 \in \{db(eb)e, dba^{\pm 1}b^{-1}, db(eb)c^{-1}, db(eb)c^{-1}\}$ and $d = a = e^2$, $c = da$ or $c = de$. It turns out that two vertices can have degree four at the same time only for $l_2 = dbab^{-1}$, $l_3 = cece$ and $l_5 = cece$ otherwise $c(\Delta) \leq c(2, 3, 4, 4, 6) = 0$.

Now let $l_1 = a(eb)^{-2}$ and so $a = e^2$. The labels of v_2 and v_3 will be as above and since now $l_5 = ced^{-1}w$ and its degree cannot be four. So $l_2 = dbab^{-1}$and $l_3 = cece$ but the degree of v_5 cannot be less than six and so the region cannot have positive curvature.

Now let $l_1 = ab^{-1}c$. Then the label of v_5 is $l_5 = cedw$ and therefore it cannot have degree four. So v_5 should have degree five and v_2 and v_3 should have degree four. Since only one d can be encountered in the label of v_5 d must be generated by the remaining elements a and e. The label of v_2 must be $l_2 = c^{-1}bc^{-1}b$ and so the label of v_3 will be $l_3 \in \{d^{-1}c(eb)c^{-1}, d^{-1}c(eb)e\}$ and either $e^2 = 1$ or $d = ae^2$ which in any case make the group to have property X.

Now let $l_1 = ac^{-1}b$. If the label of v_5 is four then $l_4 = cece$ and so $(ae)^2 = 1$. But $v_2 = d^{-1}bw$ and any label of length four or five will have d^{-1} only once and so make it generated by the other elements. So if v_4 has degree four the group will have property X. So $|v_4| = 5$ and $|v_2| = |v_3| = 4$. The label of v_2 will be $l_2 \in \{d^{-1}bab^{-1}, db(eb)e\}$ and either $d = a = c$ or $d = e^2$. But any of these labels makes labelling on v_3 for degree four impossible.

Now let $l_1 \in \{ac^{-1}e^{-1}, aec\}$ and so $c = e^{-1}a$. The label of v_5 is $l_5 = ced^{\pm 1}$ and therefore its degree cannot be four. So $|v_5| = 5$ and $|v_2| = |v_3| = 4$. The label of v_2 will be $l_2 \in \{d^{\pm 1}b(eb)c^{-1}, d^{\pm 1}b(eb)e, d^{\pm 1}ba^{\pm 1}b^{-1}\}$ and $d = e^{-1}ae$, $d = e^2$ or $d = a$. If $l_2 \in \{d^{\pm 1}b(eb)c^{-1}, d^{\pm 1}b(eb)e\}$ then the label of v_3 is $l_3 \in \{dc(eb)e, dc(eb)c^{-1}\}$ but in any case these relators make the group to have property X. If $l_2 = d^{\pm 1}ba^{\pm 1}b^{-1}$ then $l_3 = cece$ but any possible labelling on v_5 makes the group to have property X.

v_2 is the only vertex of degree three Let $l_2 = bc^{-1}d$ and so $c = d$. The label of v_3 and so $l_3 = d^{-1}cew$. It v_3 has degree four then $l_3 = d^{-1}c(eb)e$ and $e^2 = 1$. The label of v_1 is $l_1 = aew$ and if $|v_1| = 4$ then $l_1 \in \{aed^{\pm 1}e^{-1}, a(eb)^3, a(eb)a^{\pm 1}(eb)^{\pm 1}, a(eb)c^{-1}e^{-1}\}$ and either $a = e^3$ or $d = e^{-1}ae$ or $aeae^{\pm 1} = 1$. The label of v_5 is $l_5 = cew$ and if $|v_5| = 4$ then $l_5 = cece$ and $(de)^2 = 1$. It can be seen that v_1 and v_5 cannot have degree four at the same time so one of them must have degree five and v_3 has degree four. The only possible label of degree four on v_1 is now $l_1 = a(eb)a^{\pm 1}(eb)^{\pm 1}$ and so $(ae)^2 = 1$. We look for possible labels of length five on v_5. If the label of v_5 involves a $d^{\pm 1}$ then it would be generated by the remaining elements and the group has property X. So $l_5 = cecw$. Any possible label makes the group to have property X. The same applies when the degree of v_5 is four i.e. any possible label of length five on v_1 makes the group to have property X.

Let $l_2 = bc^{-1}d^{-1}$. The label of v_3 and v_5 is the same as with previous paragraph and the if v_1 has degree four then $l_1 = ac^{-1}d^{\pm 1}c$ and $a = c = d$. So if $|v_1| = 4$ none of the other vertices have degree four. So $|v_3| = |v_5| = 4$ and $|v_1| = 5$. The label of v_1 must now be $l_1 = bac^{-1}w$ and any label of degree five makes the group to have property X.

Let $l_2 = bac^{-1}$ and so $a = c$. The label of v_3 is now $l_3 = ecew$ and a label of length four would force $(ae)^2 = 1$ or $a = e^3$. The label of v_1 is $l_1 = ab^{-1}w$ and a label of degree four could be $l_1 \in \{ab^{-1}d^{\pm 1}b, a(eb)^{-3}, a(eb)^{-1}a^{\pm 1}(eb)^{\pm 1}, a(eb)^{-1}c^{-1}e^{-1}\}$ and $a = c = d$, $a = e^3$ or $aeae^{\pm 1} = 1$. If $a = c = d$ then v_3 cannot have degree four or five. In the other cases except for $l_1 = a(eb)^{-1}a^{\pm 1}(eb)$ the label of v_5 involves a d^{-1} and then it cannot have degree four or five. So if $|v_1| = 4$ then $l_1 = a(eb)^{-1}a^{\pm 1}(eb)$ and $l_5 = cecw$. In this case it is possible to have $|v_3| \in \{4, 5\}$ with $l_3 \in \{ecec, eceba^{\pm 1}, eceb(eb), ececa^{\pm 1}, ecec(eb)^{\pm 1}\}$ and $|v_5| \in \{4, 5\}$ with $l_5 \in \{cece, cec(eb)^{\pm 1}, cec(eb)e, ceca^{\pm 1}b^{-1}, ceca^{\pm 1}e\}$. Regions of positive curvature are possible and in the case that all v_1, v_4 and v_5 all have degree four the curvature is $c(\Delta) = c(2, 3, 4, 4, 4) = \frac{\pi}{6}$.

Let $l_2 = ba^{-1}c^{-1}$. The labels of v_1 and v_5 for degree four is the same as with previous paragraph but now $l_3 = b^{-1}cew$ and it cannot have degree four. Now, as with the previous paragraph the degree of v_1 and v_5 can be four at the same time with $l_1 = a(eb)^{-1}a^{\pm 1}(eb)$, $l_5 = cece$ and the degree of v_3 being five. The curvature of a region will be $c(\Delta) = c(2, 3, 4, 4, 5) = \frac{\pi}{15}$.

Let $l_2 = bed$. The label of v_1 for its degree being four is $l_1 \in \{a(eb)a(eb)^{\pm 1}, aede^{-1}\}$ and either $(ae)^2$ or $a = e = d$. The label of v_3 for degree four is $l_3 = dc(eb)c^{-1}$ and $ecec^{-1} = 1$. The label of v_5 for degree four is $l_5 \in \{cece, cecb^{-1}\}$ and either $(ce)^2$ or $e = c^2$. Two of these vertices must have degree four at the same and these should be v_3 and v_5 with labels $l_3 = dc(eb)c^{-1}$ and $l_5 = cece$. But now any labelling of degree less than six on v_1 makes the group to have property X so the region cannot have positive curvature.

Let $l_2 = bed^{-1}$. If $|v_3| = 4$ then $l_4 = dc(eb)c^{-1}$ and $ecec^{-1} = 1$. If $|v_1| = 4$

then $l_1 = ac^{-1}d^{\pm 1}c$ and if $|v_5| = 4$ then $l_5 \in \{cece, cecb^{-1}\}$. It can be seen that the only vertices that are possible to have degree four at the same time are v_3 and v_5 with $l_3 = dc(eb)c^{-1}$ and $l_5 = cecb^{-1}$ and so $e = d = c^2$. The label of v_1 in this case is $l_1 = bac^{-1}w$ and this cannot be of degree four or five. A region cannot have positive curvature.

v_3 is the only vertex of degree three First let the label of v_3 to be $l_3 = c(eb)e$ and so $l_2 = baw$. If $|v_2| = 4$ then $l_2 = bab^{-1}d^{\pm 1}$ and $a = d$. If $|v_5| = 4$ then $l_5 = cecb^{-1}$ and $e^3 = 1$. If $|v_2| = 4$ and so $l_2 = bab^{-1}d^{\pm 1}$ the label of v_1 is $l_1 \in \{aew, ac^{-1}w\}$ and any labelling on it for degree four forces a contradiction. If $|v_5| = 4$ then $l_1 = baw$ and again its label cannot have length four. So it should be $|v_2| = |v_5| = 4$ and so $l_1 \in \{baew, bac^{-1}w\}$ has label of length five. It turns out that the only possible label is $l_1 = ba(eb)c^{-1}d^{\pm 1}$ and so $a = d$ and $c = e$, $e^3 = 1$. This region will have curvature $c(\Delta) = c(2, 3, 4, 4, 5) = \frac{\pi}{15}$.

Now let $l_3 = ced$ and so $c = de^{-1}$. If $|v_5| = 4$ then $l_5 = cece$. If $|v_2| = 4$ then $l_2 \in \{b(eb)ed, b(eb)c^{-1}d^{\pm 1}\}$ and $c = e$ and $d = e^2$ or $e^2 = 1$. If $|v_1| = |v_2| = 4$ then $l_2 = b(eb)c^{-1}d^{\pm 1}$ and $l_1 = a(eb)a^{\pm 1}(eb)^{\pm 1}$ and the group is generated by the three elements three elements of degree two a, e and e and also $(ea)^2 = 1$. The degree of v_5 can be four or five with $l_5 \in \{cece, cec(eb)^{-1}b^{-1}\}$. A region can have curvature $c(\Delta) \leq c(2, 3, 4, 4, 4) = \frac{\pi}{6}$. If $|v_5| = |v_1| = 4$ then $l_5 = cece$ and $l_1 \in \{bab^{-1}d^{\pm 1}, (eb)a(eb)^{\pm 1}a^{\pm 1}, (eb)a(eb)^2\}$and either $a = d$ and $aeae^{\pm 1} = 1$ or $a = e^3$. If $l_1 = bab^{-1}d^{\pm 1}$ and $a = d$ then the degree of v_2 cannot be five. If $l_1 = (eb)a(eb)a^{\pm 1}$ then $|v_2| = 5$ implies $l_2 \in \{db(eb)c^{-1}, dbecb^{-1}, db(eb)^2c^{-1}\}$ and either $e^2 = 1$ or $dede^{-1} = 1$ or $(de)^2 = 1$ and $e^3 = 1$. If $l_1 = (eb)a(eb)^{-1}a^{\pm 1}$ then $|v_2| = 5$ implies $l_2 \in \{c^{-1}bec(eb), c^{-1}bed^{\pm 1}b, c^{-1}bed^{\pm 1}e, c^{-1}b(eb)c^{-1}b\}$ and either $(ed)^2 = 1$ or $dede^2 = 1$. If $l_1 = (eb)a(eb)^2$ then v_2 cannot have degree four or five. Now if the degree of v_1 is five the degree of v_2 and v_5 must be four. In this case it turns out that there is no possible labelling on v_1 with degree five.

Now let $l_3 = ced^{-1}$ and again $c = de^{-1}$. If $|v_5| = 4$ its label will be $l_5 = cece$.

If $|v_2| = 4$ then $l_2 = bc^{-1}bc^{-1}$ and $(de)^2 = 1$. If $|v_5| = |v_1| = 4$ then the only case that v_2 can have degree four or five is for $l_1 = (eb)a(eb)^{-1}a^{\pm 1}$ and $l_2 \in \{c^{-1}bc^{-1}be, c^{-1}bc^{-1}e^{-1}(eb)^{-1}\}$ and $dede^2 = 1$ or $dede^{-1} = 1$ and $l_1 = (eb)a(eb)a$ and $l_2 \in \{dbc^{-1}be, dbc^{-1}e^{-1}b^{-1}\}$ and $dede^{\pm 1} = 1$. If $|v_2| = |v_1| = 4$ and $|v_5| = 5$ then $l_1 = a(eb)^{-1}a^{\pm 1}(eb)^{-1}$ and $aeae^{-1} = 1$. If $|v_2| = |v_5| = 4$ and $|v_1| = 5$ then $l_2 = bc^{-1}bc^{-1}$ and $l_1 \in \{(eb)a(eb)^{-2}a^{\pm 1}, (eb)a(eb)^{-1}a(eb)\}$ and $(de)^2 = 1$ and $(ae)^2 = e^3 = 1$.

v_5 is the only vertex of degree three First let $l_5 = ced$ and so $c = de^{-1}$. The degree of v_1 can not be four so it should be $|v_3| = |v_2| = 4$. The label of v_3 should be $l_3 \in \{c(eb)^2e, cece\}$. If $l_3 = c(eb)^2e$ the degree of v_2 cannot be four and so $l_3 = cece$. The label of v_2 will be $l_2 \in \{bab^{-1}d^{\pm 1}, ba(eb)^{-1}c^{-1}\}$ and $a = d$. Now the label of v_1 is $l_1 \in \{cac^{-1}w, caew, cac^{-1}w\}$ and its degree must be five. It can be seen that any effort of finding a label of length five makes the group to have property X.

Now let $l_5 = ced^{-1}$ and again $c = de^{-1}$. If $|v_3| = 4$ then as above $l_3 \in \{c(eb)^2e, cece\}$. If $|v_1| = 4$ then $l_1 \in \left\{\begin{array}{c} (eb)^{-1}a(eb)^{-2}, (eb)^{-1}a(eb)^{\pm 1}a^{\pm 1}, \\ (eb)^{-1}ac^{-1}b, e^{-1}aed^{\pm 1} \end{array}\right\}$. If $|v_1| = |v_2| = 4$ then $l_1 = (eb)^{-1}a(eb)a$ and $l_2 = abe(eb)$ and $a = e^2$. The degree of v_3 cannot be four but $|v_3| = 5$ implies $l_5 = dc(eb)^2e$. If $|v_3| = |v_2| = 4$ and $l_2 = cece$ and $l_2 \in \{ba(eb)c^{-1}, bab^{-1}d^{\pm 1}\}$ and $a = d$. But then the degree of v_1 is at least six. Finally suppose $|v_1| = |v_3| = 4$ and $|v_2| = 5$. Now $l_2 \in \{c^{-1}baw, d^{\pm 1}baw\}$ and any label of length five on v_2 forces a contradiction.

Overall regions of positive curvature were found when the following three types of relators hold in the group:

1. $c = a$

2. $d = a$ and $c = e$

3. $d = a = e^2$ and $(ce)^2$

4. $c = de^{-1}$

1. In case $c = a$ the other relators can be $c = a = e^2$ or one or more of $d = e^{-1}ae$, $d = (ae)^2$, $aeae^{\pm k} = 1$ with $k = 1, 2$, $a = e^3$, $e^2 = 1$.

If $c = a = e^2$ the region has curvature at most $\frac{\pi}{6}$ and $l_3 = c(eb)e$ and $l_2 = c^{-1}ba$ like in Figure 6-33.

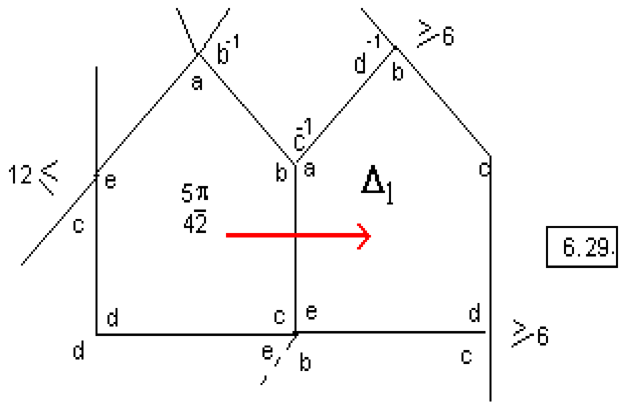


Figure 6-33:

Δ_1 has curvature at most $c(\Delta_1) \leq c(3, 3, 3, 6, 6) = -\frac{\pi}{3}$ enough to compensate for the positive of Δ. It should be observed that regions of degree six which were also found to have positive curvature (6.10. in Figure 6-18) receive positive curvature across an $a - e$ edge.

If $c = a$ and one of the other relators holds a region of positive curvature can have curvature $c(\Delta) \leq \max\{c(2, 3, 3, 5, 5), c(2, 3, 4, 4, 4)\} = \frac{\pi}{6}$. v_2 of Δ has degree three with label $ba^{\pm 1}c^{-1}$ and negative curvature can always be found in region Δ_1 as in Figure 6-34.

If $y \neq e^{-1}$ then Δ_1 is a region of degree five and $c(\Delta_1) \leq c(3, 3, 3, 5, 5) = -\frac{\pi}{5}$. In the case that $l_1 = a(eb)^{-1}a^{\pm 1}(eb)$ then Δ_1 becomes a region of degree six and its curvature will be $c(\Delta_1) \leq c(2, 3, 3, 4, 5, 5) = -\frac{11\pi}{30}$ which is enough to compensate for the curvature of Δ and perhaps another Δ_2 with $c(\Delta_2) > 0$ across its other $c^{-1} - b^{-1}$ edge.

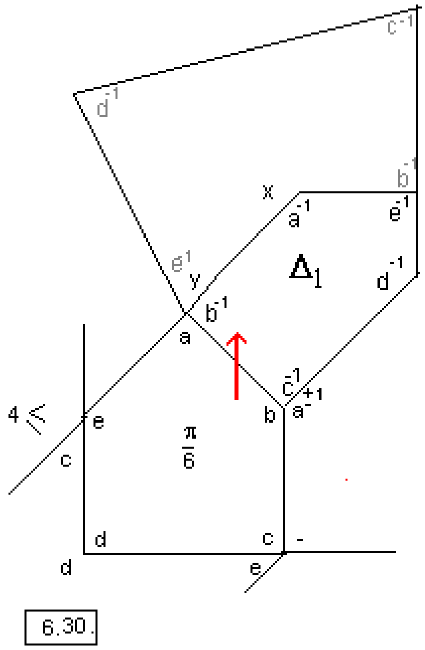


Figure 6-34:

2. and 3. In both cases that $a = d$ the labels are $l_2 = bab^{-1}d^{\pm 1}$ and $l_3 = cece$. The curvature of these region is at most $c(\Delta) \leq c(2,3,4,4,5) = \frac{\pi}{15}$. The curvature of Δ_1 as in Figure 6-35 is $c(\Delta_1) = c(2,4,4,5,5) = -\frac{\pi}{5}$.

4. In the case of $c = de^{-1}$ we look for negative curvature in the region across d^2 as this is across which the positive was compensated for regions of degree six when the $c = de^{-1}$. It is possible now for $l_3 = ced^{\pm 1}$ and/or $l_5 = ced^{\pm 1}$ and the maximum curvature a region Δ can have is $\frac{7\pi}{30}$. If $l_3 = ced$ and $l_5 \in \{cece, cec(eb)^{-1}b^{-1}\}$ the curvature of Δ is at most $\frac{\pi}{6}$ and then $l_{2\Delta_1} = ba^{\pm 1}w$ and $l_{1\Delta_1} = caw$. Since this a region of degree five and a^2 cannot appear as a proper sublabel their degree must be $|v_{1\Delta_1}| \geq 6$ and so $c(\Delta_1) \leq c(2,3,4,6,6) = -\frac{\pi}{6}$ which is enough to compensate for Δ. If $l_3 = ced^{-1}$ and $l_5 = cecw$ or $|v_5| \geq 5$ and $aeae^{\pm k} = 1$ with $k = 1,2$ the curvature of Δ is at most $c(\Delta) \leq c(2,3,4,4,5) = \frac{\pi}{15}$ and $c(\Delta_1) \leq c(2,3,4,5,6) = -\frac{\pi}{10}$ which is enough to compensate for it. If $v_3 = ced^{-1}$ and the curvature of Δ is not enough then negative curvature can be

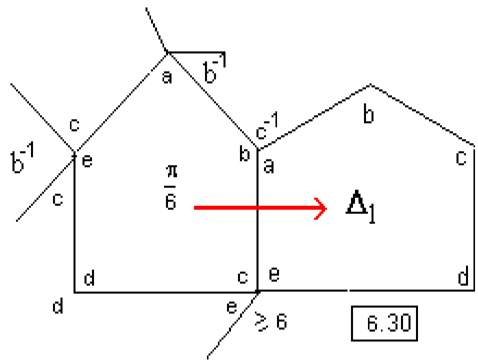


Figure 6-35:

found across a $d - e$ edge in a region of degree six. as in Figure 6-36.

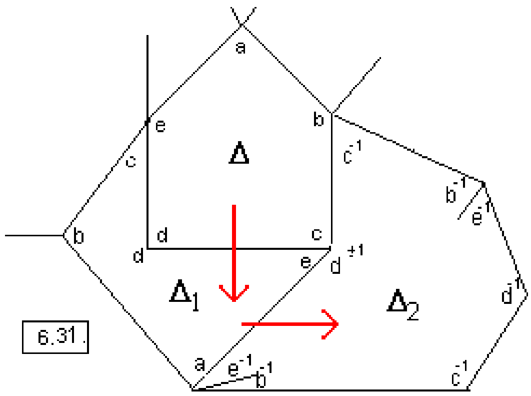


Figure 6-36:

For all the interior regions of positive curvature which were found in this chapter one or more regions of negative curvature were found next to it so that the total curvature of the area is non positive. For the same type of regions negative curvature is always being received across the same edge. Receiving are regions of degree five, or regions of degree six and they receive positive curvature of at most $\frac{\pi}{3}$. It was therefore proved that if there is any positive curvature in a diagram D, this must be concentrated in its boundary.

Chapter 7

$r_3(t) = atbtct^{-1}dtet^{-1}$

It is assumed that equation $r_3(t) = 1$ does not have a solution. As with $r_1(t) = 1$ and $r_2(t) = 1$ we first state the assumptions holding in this case:

(H1) Group G is generated by $\{a, b, c, d, e\}$

(H2) Group G is neither locally indicable nor residually finite, or a homomorphic image of a polycyclic by finite group. More specifically it does not have property X.

(H3) The star graph Γ_3 contains at least one admissible path of length less than three and at least one of a, c, d, e has order two in G.

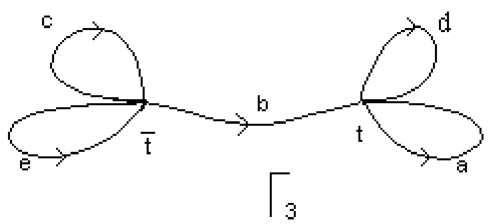


Figure 7-1:

(H4) $c \neq e^{\pm 1}$ and $a \neq d^{\pm 1}$

Remark 7.1 *The following additional assumptions can be made for* $r_3(t) =$

$atbtct^{-1}dtet^{-1} = 1$ *without any loss of generality:*

(H5) $b = 1$

On $r_3(t) = atbtct^{-1}dtet^{-1}$ we make the substitution $s = tb$. Then $atbtct^{-1}dtet^{-1} = 1$, if and only if $assb^{-1}cbsdsb^{-1}ebs^{-1} = 1$. By taking $a' = a$, $b' = 1$, $c' = b^{-1}cb$, $d' = d$, $e' = b^{-1}eb$, $r_3(t) = 1$ if and only if $a't^2c't^{-1}d'te't^{-1} = 1$ has a solution. Therefore b can be taken to equal 1 in G without any loss.

(H6) $a \neq 1,\ c \neq 1,\ d \neq 1,\ e \neq 1$

If any of these elements equals 1, then the equation collapses to an equation of length three for which the existence of a solution is known by [E1].

(H7) $r_3(t) = 1$ *can be solved modulo the transformation* T_3

$r_3(t) = 1$ has a solution if and only if the equation $e^{-1}t^{-1}d^{-1}tc^{-1}t^{-1}b^{-1}t^{-1}a^{-1}t = 1$ has a solution over G. Replacing t by t^{-1} means that we only need to solve the equation modulo the transformation (T_3).

$$a \to c^{-1}, b \to b^{-1}, c \to a^{-1}, d \to e^{-1}, e \to d^{-1} \qquad (T_3)$$

As with the previous two equations we impose further assumptions on D with a non-trivial label on the distinguished vertex v_0 representing a counter example.

(H8) Diagram D *contains minimal number of faces* Δ

(H9) The number of vertices in D *with label* a^2 *are of maximal number*

(H10) The number of vertices of degree two is maximal

Lemma 7.2 *If* $a^2 = 1$ *then* a^2 *or* a^{-2} *do not appear as proper sublabels anywhere on diagram* D*.*

Proof. Suppose a^2 or a^{-2} appears as a proper sublabel on some vertex v in diagram D. As with the example given in Chapter 5 for $r_1(t) = 1$, a *diamond move* may be performed to increase the number of vertices labeled a^2 or the inverse, without altering the number of faces, contrary to the assumption of their maximality. ■

The possible cases of an interior region Δ with $c(\Delta) > 0$ are considered according to whether Δ contains or does not contain vertices of degree 2. In Section 7.1. the cases of a region Δ that does not have any vertices of degree two are considered. In Section 7.2. the cases of regions which contain at least one vertex of degree two are considered in four subsections, according to the number of elements that can have degree two in the group.

7.1 Δ has no vertices of degree two

It follows from the curvature formula (*) that at least four vertices have degree three and the remaining vertex has degree three, four or five, otherwise $c(\Delta) \leq 0$. It can be observed from the star graph Γ_3 that any vertex involving letter b cannot have degree three. Therefore, $|v_2| \geq 4$ and this forces the rest of the vertices to have degree three.

Since $|v_1| = 3$ then $l_1 \in \{a^2d^{\pm 1}, ad^{\pm 2}, ad^{\pm 1}a, a^3\}$. If the label of v_1 ends with letter a then the label l_5 would involve letter b, and therefore have degree greater than three. So we exclude the last two labels and examine the possible labelling on an interior region Δ, for the remaining four possible labels of v_1.

7.1.1 $l_1 = a^2d$

Observe that in this case the order of element a cannot be two or three. (In such a case $d = 1$ or $d = a$, a contradiction). Since $|v_5| = 3$, $l_5 \in \{e^3, e^2c^{\pm 1}\}$. If $l_5 = e^2c^{-1}$ the label of v_4 is forced to contain letter b and therefore cannot be of degree three.

If $l_5 = e^3$ the label of v_4 will be $l_4 \in \{dad, da^2\}$. But in any case the label of v_3 will be forced to involve letter b and therefore will be of degree greater than three.

If $l_5 = e^2c$ element e cannot have order two, (in that case $c = 1$ a contradiction). The label of v_4 will be $l_4 \in \{d^3, d^2a^{-1}, d^2a\}$. In the last case labelling on v_3 will be force to involve letter b, and therefore v_3 will have degree greater than three, a contradiction. The two remaining cases are examined:

$l_4 = d^3$ In this case the order of element d cannot be two and therefore it can be assumed that c has order two, since at least one element must have order two (the star graph must contain at least on square loop). Now, $l_3 \in \{ce^2, cec\}$. The last case forces $e = 1$ a contradiction. Therefore $l_3 = ce^2$. If $|v_2| = 4$ then $l_2 = ebab^{-1}$ and $e = a$ which forces G to be cyclic generated by element a, a contradiction. If $|v_2| = 5$ then $l_5 \in \{eba^2b^{-1}, ebad^{\pm1}b^{-1}, ebab^{-1}e, ebab^{-1}c^{\pm1}\}$. Each of these case forces G to have property X.

$l_4 = d^2a$ In this case the order of elements a and d is 5 and $l_3 \in \{ce^{-2}, ce^{-1}c\}$. If $l_3 = ce^{-2}$, $c^2 = 1$ and $e^4 = 1$. Since the degree of v_2 will be four or five its label will be $l_5 \in \{ebd^{-1}b^{-1}, ebd^{-2}b^{-1}, ebd^{-1}a^{\pm1}b^{-1}, ebd^{-1}b^{-1}e, ebd^{-1}b^{-1}c^{\pm1}\}$. Each of these labels forces a relation that makes G to have property X. If $l_3 = ce^{-1}c$ the order of elements c and e is also forced to be five and the stargraph Γ_3 will not have any square loops, a contradiction.

7.1.2 $l_1 = a^2d^{-1}$

Observe that the order of a cannot be two. The label of v_5 will now be $l_5 \in \{ec^{-2}, ec^{-1}e\}$.

If $l_5 = ec^{-2}$, the label of v_4 is forced to contain letter b and therefore cannot have degree 3. So $l_5 = ec^{-1}e$ and $c = e^2$ and the order of e cannot be two. Since $|v_4| = 3$, then $l_4 \in \{da^2, dad\}$. If $l_4 = da^2$ the label of v_3 is forced to contain letter b and therefore cannot have degree 3. So $l_4 = dad$. This means that the order of a and d in G is five. Also, $l_3 \in \{ce^2, cec\}$. If $l_3 = cec$ the order of e and c is five and so no elements of order two can exist (i.e. no square loops in star graph Γ_3). So it can be assumed that $l_3 = ce^2$. This makes the order of c to be two and the order of element e in G to be four. Since $|v_2| \in \{4, 5\}$ then $l_2 \in \{ebab^{-1}, eba^2b^{-1}, ebab^{-1}c^{\pm 1}, ebab^{-1}e, ebad^{\pm 1}b^{-1}\}$. Each case forces a contradiction by making group G to have property X.

7.1.3 $l_1 = ad^2$

The order of d cannot be two. The label of v_5 will now be $l_5 \in \{e^3, e^2c, e^2c^{-1}\}$. The last case would force the label of v_4 to contain letter b and therefore it could not be of degree three.

If $l_5 = e^3$ the order of element e cannot be two. The label of v_4 will be $l_4 \in \{da^2, dad\}$. If $l_4 = da^2$ the label of v_3 is forced to contain letter b and therefore cannot have degree three. So $l_4 = dad$. The label of v_3 will be $l_3 \in \{ce^2, cec\}$. If $l_3 = ce^2$ then $c = e$ a contradiction. So $l_3 = cec$. Since $|v_2| \in \{4, 5\}$ then $l_2 \in \{cbdb^{-1}, cbd^2b^{-1}, cbda^{\pm 1}b^{-1}, cbdb^{-1}c, cbdb^{-1}e^{\pm 1}\}$. Each case forces a contradiction by making the group to have property X, or every element of G (except $b = 1$) to have order greater than two.

If $l_5 = e^2c$ the order of e cannot be two. Now, $l_4 \in \{d^3, d^2a\}$. If $l_4 = d^3$ then $a = d$, a contradiction. If $l_4 = d^2a$ the label of v_3 is forced to contain letter b and therefore cannot have degree three.

7.1.4 $l_1 = ad^{-2}$

The order of d cannot be two. The label of v_5 will be $l_5 \in \{ec^{-1}e, ec^{-2}\}$.

If $l_5 = ec^{-1}e$ the label of v_4 will be $l_4 \in \{da^2, dad\}$. If $l_4 = da^2$ the label of l_3 will contain letter b and therefore cannot have degree three. So $l_4 = dad$ and element a has order two, element d has order four. Now the label of v_3 will be $l_3 \in \{ce^2, cec\}$. We examine each case:

$l_3 = ce^2$ Since the degree of υ_2 is four or five its label l_2 will be one of the following $l_2 \in \left\{ \begin{array}{c} e^{-1}bab^{-1}, e^{-1}ba^2b^{-1}, e^{-1}bab^{-1}c^{\pm 1}, \\ e^{-1}bab^{-1}e^{-1}, e^{-1}bad^{\pm 1}b^{-1} \end{array} \right\}$. Each of these labels forces a relator that makes the group to have property X, a contradiction.

$l_3 = cec$ The label of υ_2 will be $l_2 \in \left\{ \begin{array}{c} e^{-1}bdb^{-1}, e^{-1}bd^2b^{-1}, e^{-1}bda^{\pm 1}b^{-1} \\ , e^{-1}bdb^{-1}c, e^{-1}bdb^{-1}e^{\pm 1} \end{array} \right\}$. Each case again forces a contradiction by making the group to have property X.

It can be seen that overall no interior regions of positive curvature were found in this section. Therefore, if Δ is an interior region of positive curvature it will have at least one vertex of degree two.

7.2 Δ has vertices of degree two

Each vertex of region Δ except υ_2 can have degree two, but two adjacent vertices cannot have degree two at the same time, so a region Δ may have exactly one or exactly two vertices of degree two, and will be one of regions A_1, A_2, A_3, C_1, C_2, D and E in Figure 7-2.

At least one of elements a, c, d, e must have order two while it is possible that all of them have order two at the same time. We examine the case of this section according to the number of elements of degree two that may have order two in G.

7.2.1 Four elements of order two ($a^2 = c^2 = d^2 = e^2 = 1$)

Now group G is generated by four elements of degree two and a possible interior region of positive curvature can be any of A_1, A_2, A_3, C_1, C_2, D and E of Figure 7-2. As with the previous equations it is possible to delete the vertices with labels a^2 and e^2 wherever they are encountered in diagram D. This is achieved by adding two new elements (eb) and (da) in the star graph to obtain a new

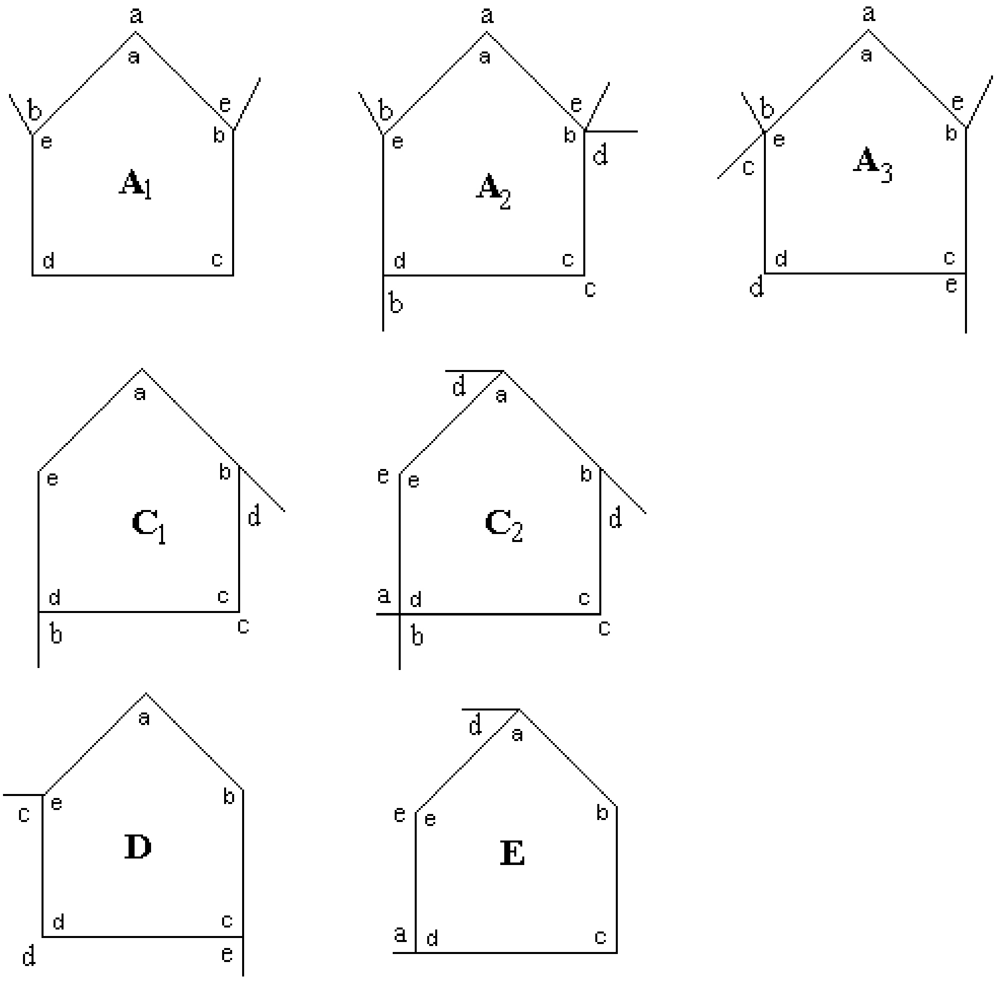


Figure 7-2:

stargraph $\hat{\Gamma}_3$ (Figure 7-3). Now vertices of degree two may have label one of c^2, d^2, $(da)^2$ or the inverses.

Therefore, regions with vertices of degree two may now be of types F_1, F_2 , C_1 or D.

F_1

Let Δ be an F_1-region of positive curvature. υ_2, υ_3, υ_4 and υ_6 may have degree two. Two adjacent vertices of F_1 cannot have degree two so there are at most

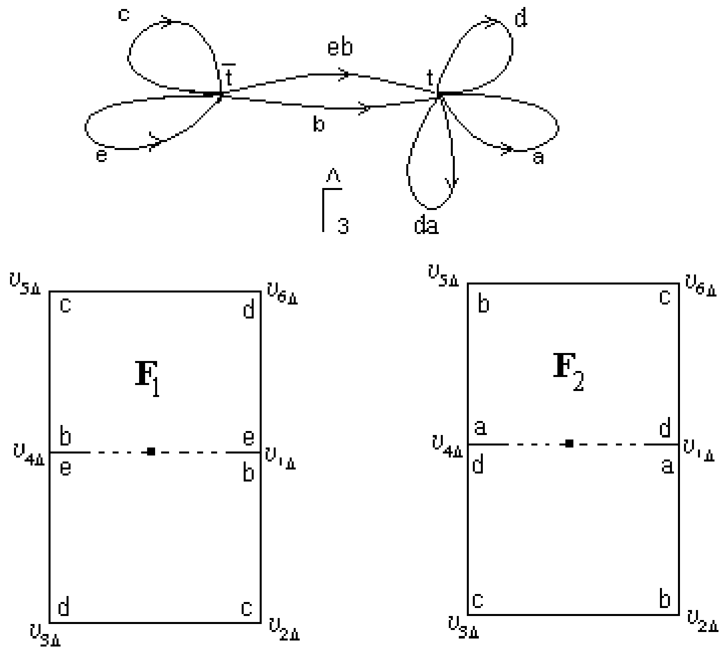


Figure 7-3:

two vertices of degree two. We examine the following cases:

Only $|v_2| = 2$

Only $|v_5| = 2$ (This case is symmetric to the case that only $|v_2| = 2$)

Only $|v_3| = 2$

Only $|v_6| = 2$ (This case is symmetric to the case that only $|v_3| = 2$)

$|v_2| = |v_5| = 2$

$|v_3| = |v_6| = 2$

$|v_2| = |v_6| = 2$

$|v_3| = |v_5| = 2$ (This case is symmetric to the case that only $|v_2| = |v_6| = 2$)

Only $|v_2| = 2$ Region Δ must have at least four vertices of degree three otherwise $c(\Delta) \leq c(2, 3, 3, 3, 4, 4) = 0$.

First let $|v_3| = 3$ and so $l_3 \in \{bd(eb)^{-1}, (eb)db^{-1}\}$ and $d = e$. The label of

υ_4 will be $l_4 = c^{-1}(eb)w$ and therefore it cannot have degree three. So $|\upsilon_1| = |\upsilon_6| = |\upsilon_5| = 3$. So $l_1 = (eb)db^{-1}$ and $e = d$ and $l_6 \in \{d(da)^{-1}a^{\pm 1}, da^{-1}(da)\}$ and $(da)^2 = 1$. The label of υ_5 will be $l_5 \in \{cbw, ce^{-1}w\}$ and its degree cannot be three.

From the above paragraph it follows that if $l_1 = (eb)^{-1}db^{-1}$ $|\upsilon_1| = |\upsilon_6| = |\upsilon_5| = 3$ is not possible. So if $|\upsilon_3| \geq 4$ the label of υ_1 must be $l_1 = (eb)(da)^{\pm 1}b^{-1}$ and so $e = da$. As before $l_6 \in \{d(da)^{-1}a^{\pm 1}, da^{-1}(da)\}$ but any of these labels forces υ_5 to have degree greater than three and therefore, Δ cannot have positive curvature.

Only $|\upsilon_3| = 2$ Now the labels of υ_2 and υ_4 are $l_2 = cew$ and $l_4 = c(eb)w$ and none of these can have degree four. So $c(\Delta) \leq c(2,3,3,3,4,4) = 0$.

$|\upsilon_2| = |\upsilon_5| = 2$ The labels of υ_1 and υ_4 start with $(eb)d$ and the labels of υ_3 and υ_6 with bd. Since there must be at least one vertex of degree three, this must be one of υ_1, υ_3, υ_4 and υ_6.

First let $|\upsilon_1| = 3$ and $l_1 = (eb)(da)b^{-1}$ and so $e = da$. The label of υ_6 becomes $l_6 = bda^{-1}w$ and its degree can be four with label $l_6 = bda^{-1}(eb)^{-1}$. The case is symmetric to the case that the labels of υ_4 and υ_3 are $(eb)(da)b^{-1}$ and $bda^{-1}(eb)^{-1}$. If the degree of υ_6 is not four (it cannot be five so it will be at least six) the degree of υ_4 must be three and the label of υ_3 will be $l_3 = bda^{-1}(eb)^{-1}$. Since the cases are symmetric it can be assumed that $l_1 = (eb)(da)b^{-1}$ and $l_6 = bda^{-1}(eb)^{-1}$. Region Δ will be as in Figure 7-4 and it will have curvature at most $c(\Delta) \leq c(2,2,3,3,4,4) = \frac{\pi}{6}$.

We look for negative curvature in region Δ_1. The curvature of Δ_1 is at most $c(\Delta_1) \leq max\,\{c(2,3,4,6,6), c(3,3,4,4,6)\} = -\frac{\pi}{6}$ which is enough to compensate for the curvature of Δ. Positive curvature is given across a $b^{-1} - a^{-1}$.

Now let $|\upsilon_1| = 3$ and $l_1 = (eb)db^{-1}$ and so $d = e$. The label of υ_6 starts with bda^{-1} and it can be seen that it cannot have degree four. If $|\upsilon_6| = 5$ then

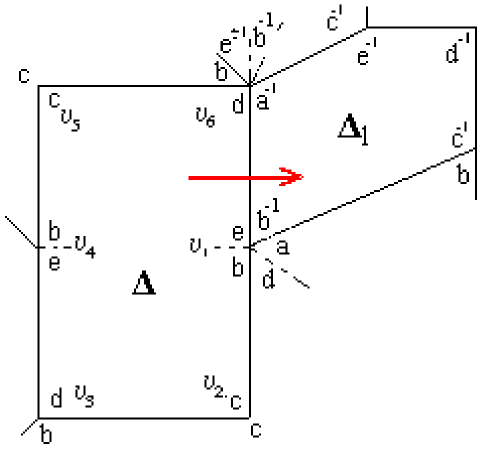


Figure 7-4:

$l_6 \in \{(eb)d(da)^{-2}b^{-1}, bd(da)^{-2}(eb)^{-1}\}$ and $(da)^2$.

If $|v_6| = 3$ then $l_6 \in \{(eb)db^{-1}, bd(eb)^{-1}\}$ and $d = e$. The label of v_1 starts with $c^{-1}(eb)d$ and its degree cannot be four. If $|v_1| = 5$ then $l_1 = c^{-1}(eb)db^{-1}c^{-1}$. But in this case a *diamond move* may be performed to make $|v_1| = 3$ and $l_6 = (eb)d(da)^{-2}b^{-1}$ without breaking a vertex with label a^2 or decreasing the number of vertices of degree two. The cases where v_2 has degree three or v_3 has degree three are symmetric so it can be assumed that if a region has positive curvature then $|v_1| = 3$ and $l_1 = (eb)db^{-1}$, $l_6 = (eb)d(da)^{-2}b^{-1}$. The region will have curvature of at most $c(\Delta) \leq c(2,2,3,3,5,5) = \frac{2\pi}{15}$. The group now is $G =< a, c, d | a^2 = c^2 = d^2 = (da)^2 = 1 >$ and the region will be as in Figure 7-5.

We look for negative curvature in its neighbouring Δ_1. The only vertex of degree two that Δ_1 may have is $v_{1\Delta_1}$ and so its curvature will be at most $c(\Delta_1) \leq c(2,3,3,4,5,5) = -\frac{11\pi}{30}$ which is enough negative curvature to compensate for the positive of Δ even if Δ_1 receives positive curvature from both of its $b^{-1} - a^{-1}$ edges.

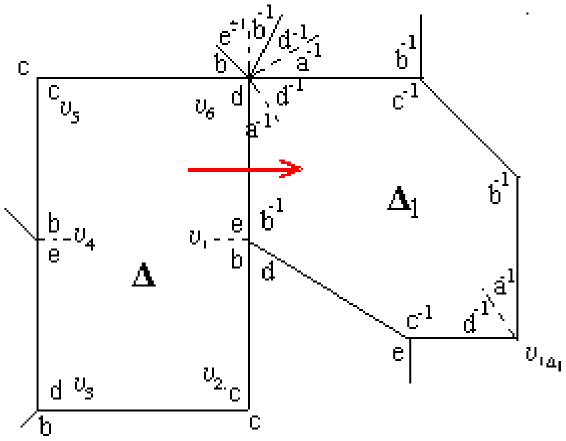


Figure 7-5:

$|v_3| = |v_5| = 2$ Now the labels of the remaining vertices v_1, v_2, v_4 and v_5 all start with ce or $c(eb)$ and thus none of them can have degree three. So $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

$|v_2| = |v_6| = 2$ The labels of v_1 is now $l_1 = c(eb)dw$ and its degree cannot be three. Also $l_5 = cew$ and its degree cannot either be three. So at least one of v_3 and v_4 must have degree three.

If $|v_3| = 3$ then $l_3 \in \{bd(eb)^{-1}, (eb)db^{-1}\}$ and $d = e$. The label of v_4 becomes $c^{-1}(eb)w$ and any label of length less than five will involve letter c only once and make the group generated by only two of the elements of order two. The same applies with v_1 and so its degree must be at least five as well. So $c(\Delta) \leq c(2,2,3,4,5,5) < 0$.

So the degree of v_4 must be three and so its label l_4 can be one of the following labels $l_4 \in \{(eb)a^{\pm 1}b^{-1}, (eb)d^{\pm 1}b^{-1}, (eb)(da)^{\pm 1}b^{-1}\}$. If $l_4 = (eb)a^{\pm 1}b^{-1}$ and $a = e$ the degree of v_1 must be at least six since any label of smaller length will make the group generated by two elements of order two and therefore have property X, a contradiction. So in that case $c(\Delta) \leq c(2,2,3,4,4,6) = 0$. If $l_4 = (eb)d^{\pm 1}b^{-1}$

and $d = e$ the degrees of v_1 and v_3 must be greater than four and $c(\Delta) \leq c(2,2,3,4,5,5) < 0$. Finally, if $l_4 = (eb)(da)^{\pm 1}b^{-1}$ and $e = da$ the degree of v_1 is at least five. The label of v_5 is $e^{\pm 1}cew$ and the only label of degree four possible is $e^{\pm 1}cec = 1$. But then the group is $G =< c, d, e | c^2 = d^2 = e^2 = (ce)^2 = (de)^2 = 1 >$ that is polycyclic-by-finite, a contradiction. So the degree of v_5 must also be five or greater and $c(\Delta)c(2,2,3,4,5,5) < 0$.

Overall the cases that an F_1 region has positive curvature is that of Figures 7-4 and 7-5.

F_2

Let Δ be an F_2-region of positive curvature. v_1, v_3, v_4 and v_5 may have degree two since v_2 and v_5 involve letter b and therefore cannot have degree two. Two adjacent vertices of F_2 cannot have degree two so there are at most two vertices of degree two. We examine the following cases:

Only $|v_1| = 2$

Only $|v_4| = 2$ which is symmetric to the case that only $|v_1| = 2$

Only $|v_3| = 2$

Only $|v_6| = 2$ which is symmetric to the case that only $|v_3| = 2$

$|v_1| = |v_4| = 2$

$|v_3| = |v_6| = 2$

$|v_1| = |v_3| = 2$

$|v_4| = |v_6| = 2$ which is symmetric to the case that $|v_1| = |v_3| = 2$

Only $|v_1| = 2$ The label of v_1 can only be $(da)^2$. In that case $l_2 = l_6 = cbw$ and neither v_2 nor v_6 can have degree three. So $c(\Delta) \leq c(2,3,3,3,4,4) = 0$.

The same applies with v_4 i.e. $|v_4| = 2$ implies $|v_3|$, $|v_5| \geq 4$. This also proves that in the case that $|v_1| = |v_4| = 2$ the curvature of Δ will be at most $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

Only $|v_3| = 2$ The label of v_2 is now $l_2 = bdw$ and the label of v_4 is $l_4 = b(da)w$. If $|v_2| = 3$ then $l_2 \in \{b(da)(eb)^{-1}, bd(eb)^{-1}\}$ and either $e = da$ or $d = e$. If $|v_4| = 3$ then $l_4 \in \{b(da)(eb)^{-1}, (eb)(da)b^{-1}\}$ and $e = da$.

First let $|v_2| = 3$ and $l_2 = bd(eb)^{-1}$. The degree of v_4 cannot be three so the degree of v_1, v_6 and v_5 must be three otherwise $c(\Delta) \leq c(2, 3, 3, 3, 4, 4) = 0$. But the label of v_1 must be $l_1 = (da)d^{-1}a^{\pm 1}$ and so $l_2 \in \{cbw, ce^{-1}w\}$ and in any case it cannot have degree three.

Now let $|v_2| = 3$ and $l_2 = b(da)(eb)^{-1}$. If $|v_1| = 3$ then $l_1 = (da)d^{-1}a^{\pm 1}$ and the degree of v_6 cannot be three. If $|v_4| = 3$ then $l_4 \in \{b(da)(eb)^{-1}, (eb)(da)b^{-1}\}$ and the label of v_5 cannot be three. So in any case $c(\Delta) \leq c(2, 3, 3, 3, 4, 4) = 0$.

$|v_3| = |v_6| = 2$ If $|v_2| = 3$ then $l_2 \in \{b(da)(eb)^{-1}, bd(eb)^{-1}\}$ and either $e = da$ or $d = e$. If $|v_4| = 3$ then $l_4 \in \{b(da)(eb)^{-1}, (eb)(da)b^{-1}\}$ and $e = da$.

If $l_2 = bd(eb)^{-1}$ then $e = d$ and $l_1 = b(da)d^{-1}w$ and v_1 cannot have degree three four or five.

If $l_2 = b(da)(eb)^{-1}$ then $e = da$ and $l_1 = b(da)d^{-1}w$ and its degree cannot be three four or five. The same applies with v_5 i.e. $l_5 = b(da)(eb)^{-1}$ implies $|v_4| \geq 6$.

So v_2 and v_5 cannot have degree three, so at least one of v_1 and v_4 has degree three and the two cases are symmetric. Let $|v_1| = 3$ and so $l_2 = b(da)(eb)^{-1}$ and $e = da$. The label of v_2 is now $l_2 = c^{-1}bdw$ and its degree cannot be three or four. If $|v_2| = 5$ then $l_2 = c^{-2}b(da)(eb)^{-1}$. If the degree of v_2 greater than five and the degree of v_5 is forced to be three. A region can have positive curvature only in the case that at least one of the couples of vertices v_1, v_2 and v_5, v_6 has labels $b(da)(eb)^{-1}$ and $c^{-2}b(da)(eb)^{-1}$. Region Δ will have curvature at most $c(\Delta) \leq c(2, 2, 3, 3, 5, 5) = \frac{2\pi}{15}$ will be as in Figure 7-6.

We look for negative curvature in its neighbouring region Δ_1. The curvature of Δ_1 can be at most $c(\Delta_1) \leq \max\{c(2, 3, 3, 4, 4, 5), c(3, 3, 3, 4, 5, 5)\} = -\frac{4\pi}{15}$ which is enough negative curvature to compensate for the curvature of Δ even in the case that Δ_1 receives positive curvature across its other $c^{-1} - b^{-1}$ edge.

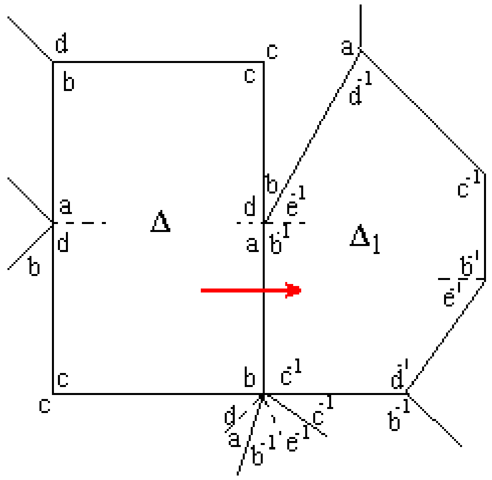


Figure 7-6:

$|v_1| = |v_3| = 2$

The label of v_1 can only be $l_1 = (da)^2$ and the degree of vertices v_2 and v_6 adjacent to it cannot have degree three. If $|v_4| = 4$ then the label of v_4 is $l_4 \in \{(eb)(da)b^{-1}, b(da)(eb)^{-1}\}$ and $e = da$. But in that case the label of v_5 becomes $l_5 = c^{-1}bw$ and its label cannot be four or five as this would c to be generated by the remaining elements and therefore the group to be generated by two elements of order two. For the same reason the degree of v_2 cannot be four and so $c(\Delta) \leq c(2, 2, 3, 4, 5, 5) < 0$.

So the only vertex that can have degree three is v_5. In that case the label of v_5 will be $l_5 \in \{ba^{\pm 1}(eb)^{-1}, bd^{\pm 1}(eb)^{-1}, b(ad)^{\pm 1}(eb)^{-1}\}$ and $e = a$ or $d = e$ or $e = da$. The group is generated by d, a and c and any label of degree three or four on vertices v_2 and v_6 will make c to be generated by the other elements and so the group to be generated by two elements of order two, a contradiction. So $c(\Delta) \leq c(2, 2, 3, 4, 5, 5) < 0$.

Overall the only case that an F_2-region can have positive curvature is that examined above in Figure7-6.

C_1

Let Δ be a region of positive curvature of C_1-type. Δ must have at least one vertex of degree three otherwise $c(\Delta) \leq c(2,4,4,4,4) = 0$. The degree of v_5 cannot be three and if v_2 or v_4 have degree three $d = e$ or $d = ea$. If $|v_1| = 3$ then $e = a$ or $(da)^2$.

$|v_1| = 3$ and $e = a$ Now $l_1 \in \{ab^{-1}(eb), a(eb)^{-1}b\}$ and the label of v_2 is now $l_2 = c^{-1}bdw$ and its degree cannot be three and degree four would make c to be generated by the other elements, and so the group to be generated by two elements of order two. If its degree is five any possible labelling where c is encountered twice is $l_2 \in \{c^{-2}bd(eb)^{-1}, c^{-2}bdb^{-1}, c^{-2}b(da)b^{-1}, c^{-2}b(da)(eb)^{-1}\}$ but any of these labels makes d to be generated by the remaining elements. So it must be $|v_2| \geq 6$ and the curvature of Δ will be at most $c(\Delta) \leq c(2,3,4,5,5) < 0$.

$|v_1| = 3$ and $(da)^2 = 1$ Now the label of v_1 will be $l_1 \in \{a(da)d^{\pm 1}, ad^{\pm 1}(da)^{-1}\}$ and $l_1 \in \{cbdw, e^{-1}bdw\}$ and $l_5 \in \{e^2w, ec^{-1}w\}$.

In the case that in the label of v_1 letter a is followed by letter d the label of v_2 starts with cbd and any label on v_2 of length less than six forces c to be generated by the remaining elements. In the case that v_4 has degree three the degree of v_2 and v_5 must be at least six and the curvature of Δ is at most $c(\Delta) \leq c(2,3,3,6,6) = 0$. In the case that the degree of v_4 is at least four it can be seen that v_4 and v_5 cannot have degree four at the same time. So v_2 must have degree four and one of the other two degree five. This turns out to be impossible and $c(\Delta) \leq c(2,3,4,5,5) < 0$.

So the label of v_1 must be $l_1 = ad^{-1}(da)^{-1}$ and $l_2 = e^{-1}bdw$. Now the degree of v_2 can be three with label $l_2 \in \{(eb)^{-1}bd, (eb)^{-1}b(da)\}$ and either $d = e$ or $e = da$ but in any of these cases the degree of v_5 is at least six. So a region Δ has curvature at most $c(\Delta) \leq c(2,3,3,3,6) = \frac{\pi}{3}$. We look for negative curvature in region Δ_1 as in Figure 7-7.

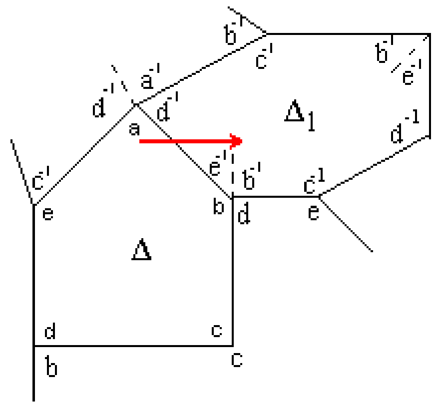


Figure 7-7:

The curvature of Δ_1 is at most $c(\Delta) \leq c(2,3,3,3,6,6) = -\frac{\pi}{3}$ which is enough to compensate for the positive curvature of Δ. In the case that there is another region of positive curvature giving it to Δ_1 across its other $e^{-1} - d^{-1}$ edge Δ_1 will not have any vertices of degree two and $c(\Delta_1) \leq c(3,3,3,3,6,6) = -\frac{2\pi}{3}$ which will be enough for both.

If the degree of υ_2 is four its label can only be $l_2 = (eb)^{-1}bda^{\pm 1}$ and $e = da$. In that case the degree of υ_4 is at least four and the degree of υ_6 at least six and region Δ cannot have positive curvature.

If the degree of υ_2 is five or greater and the degree of υ_4 is three then $d = e$ and the degree of υ_6 becomes at least six. If υ_2 does not have degree five then the region cannot have positive curvature so $l_2 \in \left\{ \begin{array}{c} e^{-1}bd(eb)^{-1}e^{-1}, e^{-1}bd(da)^2, \\ (eb)^{-1}bd(da)^{-2}, (eb)^{-1}(da)d(da)^{-1} \end{array} \right\}$. In the case of the first label a diamond move may be performed to increase the number of vertices of degree two, without breaking any a^2 or d^2 vertices and in the second case to reduce the number of faces, a contradiction. The case of the last two labels starting with an $(eb)^{-1}$ is similar to that of Figure 7-7 (i.e. a region Δ_1 of F_1-type can be found next to Δ in such a way that positive curvature is given to it across an $e^{-1} - d^{-1}$ edge). If the degree of υ_4 is not three a region of positive curvature cannot be obtained as the degree of v_4 and v_5 cannot be four at the same time.

$|v_2| = 3$ **and** $|v_1| \geq 4$ Now the label of v_2 will be $l_2 \in \{(eb)^{-1}bd, (eb)^{-1}b(da)\}$ and either $e = d$ or $e = da$.

First let $l_2 = (eb)^{-1}b(da)$ and $e = da$. The degree of v_4 cannot be three and so the curvature of Δ is at most $c(\Delta) \leq c(2,3,4,4,4) = \frac{\pi}{6}$. We look for negative curvature in region Δ_1 as in Figure 7-8.

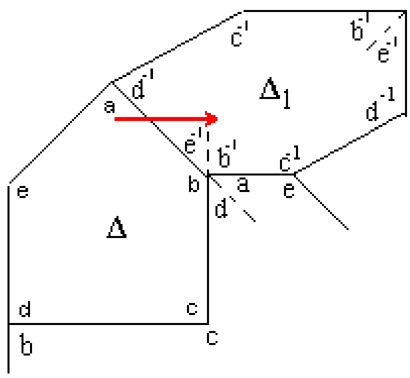


Figure 7-8:

If Δ_1 has no vertices of degree two then its curvature is $c(\Delta_1) \leq c(3,3,3,4,4,4) = -\frac{\pi}{2}$ which is enough to compensate for the positive curvature of Δ and any other possible region of positive curvature giving its positive curvature to Δ_1 across the other $e^{-1} - d^{-1}$ vertex. If Δ_1 has a vertex of degree two with label c^{-2} then v_1 of Δ will have label $l_1 = ad^{-1}b^{-1}w$ and its degree cannot be less than six, therefore $c(\Delta) \leq c(2,3,4,4,6) = 0$. If Δ_1 has a label d^{-2} then its curvature is at most $c(\Delta_1) \leq c(2,3,4,4,6,6) = -\frac{\pi}{3}$ and it is again enough to compensate for the positive curvature of Δ. In that case Δ_1 cannot receive curvature from another region.

If $l_2 = (eb)^{-1}bd$ and $d = e$, vertex v_4 may also have degree three and so the curvature of Δ can be at most $c(\Delta) \leq c(2,3,3,4,4) = \frac{\pi}{3}$. We look for negative curvature in region Δ_1 in the same way as before. The curvature of Δ_1 in this case will be at most $-\frac{\pi}{3}$ and in the case that it receives curvature from more than one regions it will be $-\frac{2\pi}{3}$. In the case that Δ_1 has a vertex c^{-2} the curvature of Δ becomes $c(\Delta) \leq c(2,3,3,4,6) = \frac{\pi}{6}$ and the same applies for any other possible region of positive curvature that is compensated by the curvature of Δ_1. In any case $c(\Delta_1)$ is enough and positive curvature is given across an $e^{-1} - d^{-1}$ edge.

$|v_1|, |v_2| \geq 4$ **and** $|v_4| = 3$ Now $l_4 \in \{(eb)db^{-1}, bd(eb)^{-1}\}$ and $d = e$. The degree of v_5 cannot be four, and if it is greater than five the region will not have positive curvature. So $l_5 = c^{-1}ec^{\pm 1}e^{\pm 1}$. But in that case the degree of v_1 and v_2 cannot be four so $c(\Delta) \leq c(2,3,4,5,5) < 0$.

D

Now v_1 and v_2 are the only possible vertices of degree two. If v_1 has degree three then $l_1 \in \{ab^{-1}(eb), a(eb)^{-1}b, a(da)d^{\pm 1}, ad^{\pm}(da)^{-1}\}$ and $a = e$ or $(da)^2 = 1$. If v_2 has degree three then $l_2 \in \{ba^{\pm 1}(eb)^{-1}, bd^{\pm 1}(eb)^{-1}, b(da)^{\pm 1}(eb)^{-1}\}$.

If v_1 and v_2 have degree three at the same time the label on v_1 must be $l_1 = ad^{-1}(da)^{-1}$. The label on v_2 forces e to be generated by the remaining elements and any label of degree four or five on v_3 or v_5 forces $(ce)^2$ or c to be generated by the remaining elements. The group will either be generated by two elements of degree two or it will be generated by three elements of degree two and the additional relators $(ce)^2$ and $(da)^2 = 1$ hold. This forces the group to be polycyclic-by-finite, a contradiction. So the curvature of Δ will be at most $c(\Delta) \leq c(2,3,3,6,6) = 0$.

If $|v_1| = 3$ and $|v_2| \geq 4$ with $l_1 \in \{a(da)d^{\pm 1}, ad^{\pm}(da)^{-1}\}$ the label on v_2 for degree four or five will again force c or e to be generated by the remaining elements. The labels on v_3 and v_5 force $(ce)^2 = 1$ and the group becomes polycyclic-by-finite, a contradiction.

If $|v_2| = 3$ any of the possible labels force e to be generated by the other three elements and possible labels of degree four and five on v_1, v_3 and v_5 will force the square of two sums to have order two, which makes the group polycyclic-by-finite except in the case $l_2 = ba(eb)^{-1}$, $l_1 = ad^{-2}(eb)^{-1}b$, $l_5 = cece^{\pm 1}$ and $l_3 = ecec^{\pm 1}$.

The other case remaining is for $l_1 \in \{ab^{-1}(eb), a(eb)^{-1}b\}$ and $a = e$. It turns out that the only possible labelling for a region of positive curvature is $l_2 = c^{-2}ba(eb)^{-1}$, $l_3 = ecec^{\pm 1}$ and $l_5 = cece^{\pm 1}$. We consider only this case as with the use of a *diamond move* it is possible to turn the previous case into one of this kind. Region Δ will be as in Figure 7-9.

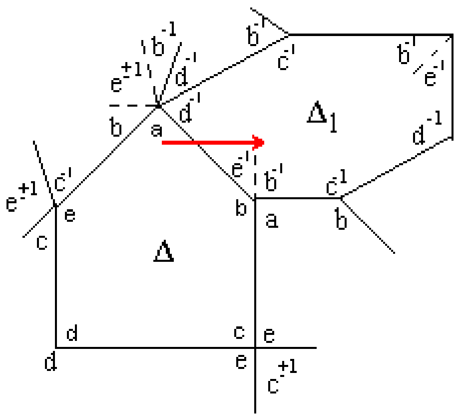


Figure 7-9:

The curvature of Δ is $c(\Delta) = c(2,3,4,4,5) = \frac{\pi}{15}$ and the curvature of Δ_1 at most $c(\Delta_1) \leq c(2,3,3,4,4,5) = -\frac{4\pi}{15}$ which is enough to compensate for the positive curvature of Δ. In the case that there is another region of positive curvature giving its curvature to Δ_1, Δ_1 will not have any vertices of degree two and the total positive curvature given to it is at most $\frac{2\pi}{15}$.

Overall in the case that four elements have order two, regions of positive curvature were found when one of $e = da$, $d = e$, $e = a$ holds. In the case that $e = da$ a region of negative curvature of degree five may receive positive curvature from a $b^{-1} - a^{-1}$ (or an $a - b$) edge. A negative region of F_1-type may receive positive curvature from its $b - c$ edges or its $d - e$ edges. In the case that a region Δ_1 receives positive curvature from both a $b-c$ edge and a $d-e$ (regions as those shown in Figures 7-6 and 7-8) then the curvature on each of its sides at most $c(2,2,3,4,5,5)+c(2,3,4,4,5)$ and so positive curvature cannot be more than $c(\Delta^+) \leq 2(\frac{\pi}{15}+\frac{\pi}{30}) = \frac{\pi}{5}$ and the negative curvature is enough to compensate for both.

7.2.2 Three elements of order two

One of a, c, d, e will have degree greater than two each time. The case when a is the element of degree greater than two is equivalent modulo T_3 to the case that c is the element of degree greater than two and the case that d is the element of degree greater than two is equivalent modulo T_3 to the case that e is the element of degree greater than two.

$a^2 = c^2 = e^2 = 1$ **and** $d^k = 1,\ k \geq 3$

A possible interior region of positive curvature can be any of A_1, A_2, A_3, C_1, C_2 and E. As with the previous section the vertices with labels a^2 and e^2 are deleted wherever they are encountered in diagram D. Now the only possible labels of length two are $(da)^2$, $(da)d$, c^2 or their inverses. Therefore, the regions to be examined are of types F_1, F_2 and C_1 of Figures 7-2 and 7-3 and D_2 of Figure 7-10.

F_1 Vertices which can have degree two are v_2 and v_5 with label c^2 and v_3 and v_6 with label $d(da)$.

First let $|v_3| = 2$ and so $l_3 = d(da)$ and $a = d^2$. The label of v_2 and v_4 are now $l_2 = cbw$ and $l_4 = c(eb)w$ and none of them can have degree three. In the case that v_3 is the only vertex of degree two the region will have curvature at most $c(\Delta) \leq c(2,3,3,3,4,4) = 0$ and in the case that $|v_3| = |v_6| = 2$ $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

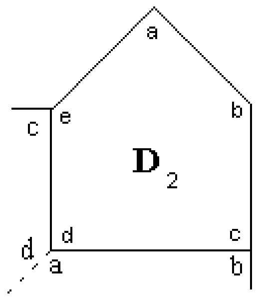


Figure 7-10:

Now let $|v_2| = 2$ and so $l_2 = c^2$. This forces $l_3 = bdw$ and $l_1 = (eb)dw$. Since d does not have order two the degree of v_3 cannot be three. It is possible that the degree of v_1 is three with label $l_1 = (eb)(da)b^{-1}$ and so $d = ea$. If v_2 is the only label of degree two v_1, v_6 and v_5 must have degree three. The label of v_6 will be $l_6 \in \{da^{-1}(da), d(da)^{-1}a^{\pm1}\}$. Now the label of v_5 becomes $l_5 \in \{cbw, ce^{-1}w\}$ and its degree cannot be three. So $c(\Delta) \leq c(2,3,3,3,4,4) = 0$.

Now let $|v_2| = |v_5| = 2$. Since v_3 and v_6 cannot have degree three at least one of v_1 and v_4 must have degree three with label $(eb)da^{\pm1}b^{-1}$ and so $d = ea$. In the case that v_1 is the only vertex of degree three two of the remaining vertices v_6, v_3 and v_4 must have degree four and the other must have degree five, otherwise positive curvature cannot be achieved. If one of them has degree four the label will be $l \in \{bda^{-1}(eb)^{-1}, (eb)da^{-1}b^{-1}\}$. If the degree of v_3 is four the degree of v_4 can be four but not five, and if the degree of v_4 is four the degree of v_5 can be four but not five. Aso the degree of v_6 will be either four or at least six. So the only case that a region will have positive curvature is when all three of these vertices have degree four. Δ will be as in Figure 7-11 and will have curvature $c(\Delta) \leq c(2,2,3,4,4,4) = \frac{\pi}{6}$. Negative curvature may be found in region Δ_1.

The curvature of Δ_1 is at most $c(\Delta_1) \leq c(3,4,4,4,4) = -\frac{\pi}{3}$ which is enough to compensate for the negative curvature of Δ. Positive curvature is given to it across a $b^{-1} - a^{-1}$ edge.

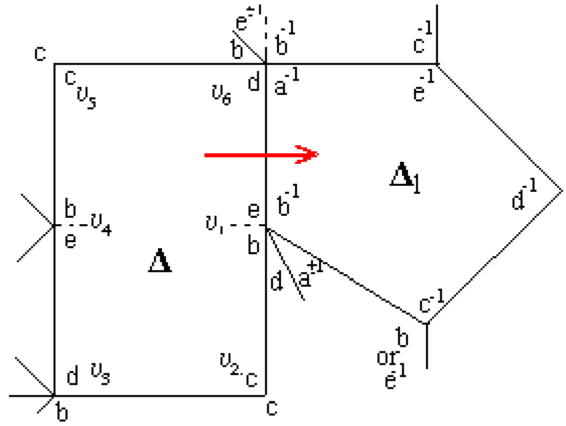


Figure 7-11:

Now suppose that both v_1 and v_4 have degree three. Vertices v_3 and v_4 will have degree four or at least six but if both have degree six or greater the curvature of the region cannot be positive. So the same situation as in Figure 7-11, is encountered either in the couple of vertices v_1 and v_6 or v_4 and v_6. Region Δ may have curvature at most $c(\Delta) \leq c(2,2,3,3,4,4) = \frac{\pi}{3}$ but as before the curvature of Δ_1 is enough to compensate for it.

Now suppose $|v_5| = |v_3| = 2$. (This case is symmetric to the case that $|v_2| = |v_6| = 2$). The only possible label of degree three is v_1 with label $(eb)a^{\pm 1}b^{-1}$ and so $e = a = d^2$. Now the labels of v_2 and v_4 will be $l_1 \in \{ecb, b^{-1}cbw\}$ and $l_4 = c(eb)dw$ and any label of length less than five would make c to be generated by the other elements and the group is forced to be cyclic, a contradiction.

F_2 Now vertices v_1, v_3, v_4 and v_6 can have degree two but two adjacent vertices cannot have degree two at the same time so Δ has at most two vertices of degree two.

First let $|v_1| = 2$ and so $l_1 \in \{(da)d, (da)^2\}$. Now l_2 and l_6 start with cb or ce and cannot have degree three. So if v_1 is the only vertex of degree two the curvature of Δ will be at most $c(\Delta) \leq c(2,3,3,3,4,4) = 0$. Also if $|v_1| = |v_4| = 2$

the curvature will be $c(\Delta) \leq c(2,2,4,4,4,4) = 0$. So at least one vertex of Δ must have label c^2.

Let υ_3 be the only vertex of Δ of degree two. The labels of υ_2 and υ_4 will be $l_2 = bdw$ and $l_4 = b(da)w$ and the these can have degree four with $l \in \{b(da)(eb)^{-1}, (eb)(da)b^{-1}\}$ and so $e = da$. But if υ_2 has degree three υ_1 cannot have degree three and if υ_4 has degree three υ_5 cannot have degree three. Therefore $c(\Delta) \leq c(2,3,3,3,4,4) = 0$.

Now suppose $|\upsilon_3| = |\upsilon_6| = 2$ with label c^2. If υ_2 has degree three with label $l_2 = b(da)(eb)^{-1}$ the label of υ_1 becomes $l_1 = b(da)d^{-1}w$ and it cannot have degree three four or five. If υ_1 has degree three with one of the labels $b(da)(eb)^{-1}, (eb)(da)b^{-1}$ the label of υ_2 becomes $c^{-1}bdw$ and its degree cannot be less than six. So $c(\Delta) \leq \max\{c(2,2,3,3,6,6), c(2,2,3,4,4,6)\} = 0$.

Now suppose $|\upsilon_3| = |\upsilon_1| = 2$ with labels $l_3 = c^2$ and $l_1 \in \{(da)^2, (da)d\}$. The only vertices that can have degree three is υ_5 and υ_4 but not at the same time, with labels $l \in \{(eb)^{-1}ba^{\pm 1}, (eb)^{-1}b(da)^{\pm 1}\}$. But in any case the labels of υ_2 and υ_6 will have to be at least five, as any label of length four makes c to be generated by the other elements or $cec^{\pm 1}e = 1$ which makes the group to have property X. So $c(\Delta) \leq c(2,2,3,4,5,5) < 0$.

C_1 Possible vertices of degree three are υ_1 and υ_2 so at least one of them must have degree three. If $|\upsilon_2| = 3$ then $l_2 = b(da)(eb)^{-1}$ and $e = da$. If $|\upsilon_1| = 3$ then $l_1 \in \{ad^{\pm 2}, a(da)d^{\pm 1}, ad^{\pm 1}(da)^{-1}\}$ and either $a = d^2$ or $adad^{\pm 1} = 1$. The degree of υ_5 and υ_4 cannot be three so at least one of υ_1 and υ_2 must have degree three.

First let $|\upsilon_1| = 3$ and $l_1 = ad^2$. The label of υ_2 becomes $l_2 = cbdw$ and its degree cannot be three or four. So υ_2 must have degree five and υ_4 and υ_5 must have degree four. It turns out that the label of υ_4 must be $l_4 \in \{bd^2(eb)^{-1}, (eb)d^2b^{-1}\}$ and now $e = a = d^2$. The only possible label of degree four on υ_5 is $l_5 = c^2e^2$ but in this case a *diamond move* may be performed to increase the number of vertices of degree two without decreasing the number of vertices with label a^2, a contradiction. So the degree of v_5 cannot be four and the region does not have positive curvature.

Now let $|v_1| = 3$ and $l_1 = ad^{-2}$. The label of v_2 is now $e^{-1}bdw$ and it is possible that it has degree four with label $l_2 \in \{(eb)^{-1}bd^2, (eb)^{-1}b(da)^2, (eb)^{-1}bd(da)^{-1}\}$ and in any case $a = e = d^2$. If the degree of v_5 is four then $l_5 = ec^{-1}e^{\pm 1}c^{\pm 1}$ and $(ce)^2 = 1$ and if v_2 and v_5 have degree four at the same time the group will have property X. Also the label of v_4 is $l_4 = bdw$ and in any case it cannot have degree four. So the region cannot have positive curvature.

Now suppose $|v_1| = 3$ and $l_1 \in \{a(da)d^{\pm 1}, ad^{\pm 1}(da)^{-1}\}$. If $l_1 = a(da)d$ the label of v_5 is $l_5 = e^2w$ and it can only be four in the case $l_5 = e^2c^{\pm 2}$ but in that case with a *diamond move* the number of vertices of degree two can be increased without the number of vertices with label a^2 to be decreased. So the degree of v_5 must be five and the degree of v_2 and v_4 must be four. So $l_4 \in \{bd(eb)^{-1}c^{\pm 1}, (eb)db^{-1}c^{\pm 1}\}$ and in any case the label of v_5 becomes $l_5 = c^{-1}e^2w$ but like that the degree of v_5 is at least six and the region cannot have positive curvature.

So if $|v_1| = 3$ then $l_1 \in \{a(da)d^{-1}, ad(da)^{-1}, ad^{-1}(da)^{-1}\}$ and the degree of v_4 is four the degree of v_5 cannot be four or five. But also if v_2 and v_5 have degree four the degree of v_4 is at least six. So the only case which remains to be considered is for $|v_2| = 3$ which is possible for $l_1 = ad^{-1}(da)^{-1}$. The degree of v_4 and v_5 cannot be four at the same time so the curvature of Δ is at most $c(\Delta) \leq c(2,3,3,4,5) = \frac{7\pi}{30}$. Region Δ will be as in Figure 7-12 and negative curvature may be found in Δ_1.

Vertices of Δ_1 which involve c and b will have degree at least five, as smaller degree would make c to be generated by the other elements and the group to be generated by a and e and so to have property X. The curvature of Δ_1 is at most $c(\Delta_1) \leq c(3,3,3,3,5,5) = -\frac{8\pi}{15}$ which is enough to compensate for the positive curvature of Δ and any other possible neighbouring region of positive curvature.

$|v_1| \geq 4$ and $|v_2| = 3$

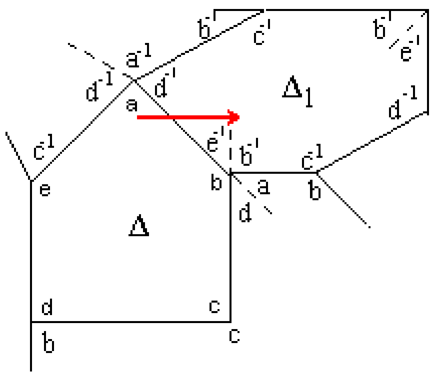


Figure 7-12:

Now the label of v_2 is $l_2 = b(da)(eb)^{-1}$ and $d = ea$. The curvature of Δ is at most $c(\Delta) \leq c(2,3,4,4,4) = \frac{\pi}{6}$ and the positive curvature can again be compensated from the negative curvature found in Δ_1. If Δ_1 does not have a vertex of degree two its curvature will be $c(\Delta_1) \leq c(3,3,3,4,4,4) = -\frac{\pi}{2}$ as with the previous case in Figure 7-12 and this is enough to compensate for the negative curvature of Δ and any other possible positive curvature coming across the other $e^{-1} - d^{-1}$ of region Δ_1. If Δ_1 has a vertex of degree two then the case will be as in Figure 7-13.

If v_1 of Δ has degree four then $v_{1\Delta_1}$ cannot have degree three and so $c(\Delta_1) \leq c(2,3,3,4,4,6) = -\frac{\pi}{3}$ and this is enough to compensate for the curvature of Δ. A possible region Δ_2 giving positive curvature to Δ cannot have two neighbouring regions of degree three so its curvature will also be at most $\frac{\pi}{6}$ and again the curvature of Δ_1 is enough for both. If v_1 of Δ has degree at least five the curvature of Δ will be at most $c(\Delta) \leq c(2,3,4,4,5) = \frac{\pi}{15}$ and the curvature of Δ_1 at most $c(\Delta_1) \leq c(2,3,3,3,5,6) = -\frac{4\pi}{15}$. A possible region Δ_2 giving curvature to Δ_1 will have curvature at most $\frac{\pi}{6}$ so the maximum total positive curvature given to Δ_1 will be at most $\frac{7\pi}{30}$ and so it is negative enough to compensate for all.

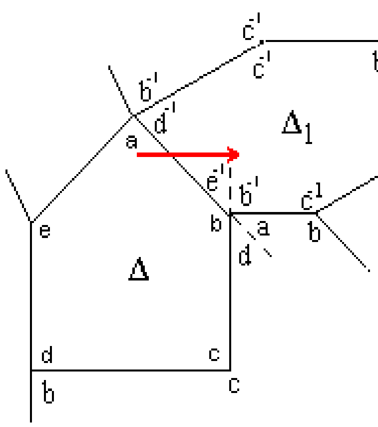


Figure 7-13:

D_2

Vertices v_3 and v_5 of a region of this type cannot have degree three, so at least one of v_1 and v_2 will have degree three. If v_2 has degree three then $l_2 = ba^{\pm 1}(eb)^{-1}$ and $a = e = d^2$ and the labels of v_3 and v_5 will involve letter c and another letter so will have degree at least six and the curvature of the region cannot be positive. If v_1 is the only vertex of degree three the its label will be $l_1 = ad^2$. The label of v_5 is $l_5 = ce(eb)w$ and its degree is at least five. So the degree of v_2 and v_3 must be four. But if one of these vertices has degree four the other has degree at least five and $c(\Delta) \leq c(2, 3, 4, 5, 5) = 0$.

Overall in this section regions of positive curvature where found when $d = ea$. In all of theses cases a neighbouring region of negative was found, with enough curvature to compensate for the curvature of Δ. If a negative region is a region of degree five it will receive the positive curvature across its $a - b$ or $b^{-1} - a^{-1}$edge (Figure 7-11) and if it is an F_1-type region of degree six like in Figure 7-12 and Figure 7-13 it will receive positive curvature across its $d - e$ or $e^{-1} - d^{-1}$ edges.

$a^2 = d^2 = e^2 = 1$ **and** $c^k = 1,\ k \geq 3$

Once again it is possible to delete a^2 and e^2 wherever these are encountered in D so possible regions of positive curvature are of type F_1, F_2 and D. Possible labels of degree two are d^2 and $(da)^2$.

F_1 Let Δ be a region of positive curvature. The only possible vertices of degree two in Δ is now v_3 and v_6 with labels d^2.

First let $|v_3| = 2$ and $l_3 = d^2$ to be the only vertex of degree two. The labels of v_2 and v_4 become $l_2 = cew$ and $l_4 = c(eb)w$. The degree of v_4 cannot be three so the remaining vertices v_1, v_2, v_5 and v_6 must have degree three. So $l_2 = cec$ and $e = c^2$ and $l_1 = (eb)db^{-1}$ so $d = e = c^2$. Now the label of v_6 must be $l_6 = d(da)^{-1}a^{\pm 1}$ and the group will have property X, a contradiction.

Now suppose $|v_3| = |v_6| = 2$. v_1 and v_4 cannot have degree three so at least one of v_2 and v_5 must have degree three. Suppose this is v_3 (and the case when v_6 has degree three is symmetric). So $l_2 = cec$ and $l_1 = c(eb)dw$. The degree of v_1 is forced to be at least six so $c(\Delta) \leq max\,\{c(2,2,3,3,6,6), c(2,2,3,4,4,6)\} = 0$.

F_2

Let Δ be an F_2-region of positive curvature. The only possible vertices of degree two are v_1 and v_4 with label $(da)^2$.

If $|v_1| = 2$ then the labels of v_2 and v_6 become $l_2 = l_6 = cbw$ and none of them can have degree three. If v_1 or v_4 is the only vertex of degree two then $c(\Delta) \leq c(2,3,3,3,4,4) = 0$ and if both of them have degree two then $c(\Delta) \leq c(2,2,4,4,4,4) = 0$.

D

Let Δ be a region of positive curvature. If v_2 and v_5 have degree three then the label will be $l = cec$ and $e = c^2$. If v_1 has degree two then its label l_1 will be

$l_1 \in \{ad^{\pm 1}(da)^{-1}, a(da)d^{\pm 1}, ab^{-1}(eb), a(eb)^{-1}b\}$ and either $(da)^2$ or $a = e$. If v_2 has degree three then its label will be $l_2 \in \{ba^{\pm 1}(eb)^{-1}, bd^{\pm 1}(eb)^{-1}, b(da)^{\pm 1}(eb)^{-1}\}$ and either $a = e$ or $d = e$ or $d = ea$.

First suppose that v_3 and v_5 do not have degree three. At least one of v_1 and v_2 must have degree three. If both v_1 and v_2 have degree three then $l_2 = (eb)^{-1}bw$ and $l_1 = ad^{-1}(da)^{-1}$. The label on v_2 implies that e is generated by d and a and the label on v_1 that $(da)^2 = 1$. But in this case any labelling on v_3 and v_5 for degrees four or five implies a relator which makes the group to have property X, a contradiction. If only v_2 has degree three at least two of the remaining vertices should have degree four. So at least one of v_3 and v_5 must have degree four and the label of v_1 is $ad^{-1}w$. Such labelling would make the group to be polycyclic-by-finite, a contradiction. If v_1 is the only vertex of degree three and then label of v_1 is $l_1 \in \{ab^{-1}(eb), a(eb)^{-1}b\}$. The label of v_2 becomes $c^{-1}bw$ and any label of degree four will make c to be generated by the remaining elements and thus the group to have property X. If the degree of v_2 is five then $a = c^2$ or $d = c^2$ or $d = ac^2a$. The label of v_5 is $cecw$ and its degree cannot be four, so the region will not have positive curvature. If the label of v_1 is $l_1 \in \{ad^{\pm 1}(da)^{-1}, a(da)d^{\pm 1}\}$ then $(da)^2 = 1$ and any label of degree four on v_5 or v_3 gives the relator $cec^{\pm 1}e = 1$. In that case the degree of v_2 cannot be four or five and so $c(\Delta) \leq max\,\{c(2,3,4,5,5), c(2,3,4,4,6)\} = 0$.

If follows from the above paragraph that if Δ is a region of positive curvature then either v_2 or/and v_5 has degree three and $c = e^2$. We check these remaining case according to the number of vertices of degree three that Δ may have. Since it is not possible that all vertices have degree three at the same time, Δ will have at most three vertices of degree three.

Three vertices of degree three If $|v_2| = |v_3| = |v_5| = 3$ then $l_5 = l_3 = cec$ and $l_2 = bd(eb)^{-1}$. Now $d = e = c^2$ and $l_1 = bad^{-1}w$. The degree of v_1 will be at least six so the curvature of the region is at most $c(\Delta) \leq c(2,3,3,3,6) = \frac{\pi}{3}$.

Negative curvature can be found in Δ_1 as in Figure 7-14.

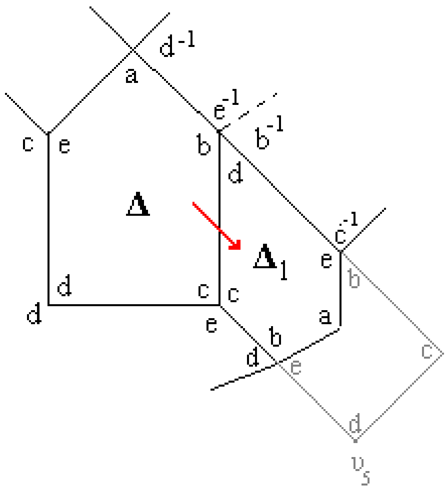


Figure 7-14:

If $v_{5\Delta_1}$ does not have degree two then Δ_1 is a region of degree five and its curvature is at most $c(\Delta_1) \le c(3,3,3,6,6) = -\frac{\pi}{3}$ which is enough to compensate for the positive curvature of Δ. If $v_{5\Delta_1}$ had degree two then it was deleted and Δ_1 is a region of degree six. If it does not have any vertex of degree two then its curvature is at most $c(\Delta_1) \le c(3,3,3,3,5,6) = -\frac{3\pi}{5}$ or $c(3,3,3,3,6,6) = -\frac{2\pi}{3}$ in the case that it receives positive curvature from another region. If it has a vertex of degree two then it cannot be receiving positive curvature from another neighbour and its curvature is at most $c(\Delta_1) \le max\,\{c(2,3,3,3,6,6), c(2,3,3,4,5,6)\} = -\frac{\pi}{3}$. In any case this enough to compensate for the curvature of Δ and perhaps another region of positive curvature also giving its curvature to Δ_1.

If $|v_1| = |v_2| = |v_3| = 3$ then $l_3 = cec$, $l_2 = bd(eb)^{-1}$ and $l_1 = ad^{-1}(da)^{-1}$ and the group is forced to have property X.

If $|v_1| = |v_5| = 3$ then $l_5 = cec$ and $l_1 \in \{ba(eb)^{-1}, (eb)ab^{-1}\}$ and $e = a = c^2$. v_2 cannot have degree three so v_3 has degree three and $l_3 = cec$. The label of v_2 becomes $l_2 = c^{-1}bdw$ and its degree must be at least six so the curvature of Δ is at most $\frac{\pi}{3}$. Now $d = e = c^2$ and $l_1 = bad^{-1}w$. The degree of v_1 will be at least six so the curvature of the region is at most $c(\Delta) \leq c(2, 3, 3, 3, 6) = \frac{\pi}{3}$.

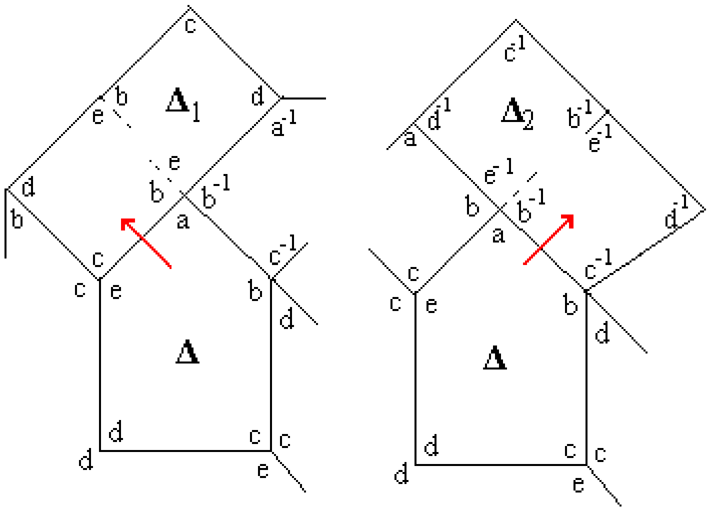


Figure 7-15:

Depending on the label of v_1 negative curvature can be found in Δ_1 or Δ_2 as in Figure 7-15. The curvature of Δ_1 is at most $c(\Delta_1) \leq c(3, 3, 3, 3, 6, 6) = -\frac{2\pi}{3}$ which is enough to compensate for the positive curvature of Δ even if it is receiving positive curvature from another neighbour of it. The curvature of Δ_2 is at most $c(\Delta_2) \leq c(3, 3, 3, 3, 6, 6) = -\frac{2\pi}{3}$ if it does not have any vertices of degree two, while if it has a vertex of degree two it cannot receive negative curvature from another region and its curvature is at most $max\,\{c(2, 3, 3, 3, 6, 6), c(2, 3, 3, 4, 5, 6)\} = -\frac{\pi}{3}$.

Two vertices of degree three Since at least one of v_5 and v_3 should have degree three, we examine the following four cases:

First let $|v_1| = |v_5| = 3$ so $l_5 = cec$ and $l_1 \in \{ab^{-1}(eb), a(eb)^{-1}b\}$ and so $a = e = c^2$. The degree of v_2 and v_3 cannot be four so $c(\Delta) \leq c(2, 3, 3, 5, 5) = \frac{2\pi}{15}$. The relators holding and the labels of v_1 and v_5 are the same as those holding for the region in Figure 7-15 and negative curvature can be found in the same way.

Now let $|v_1| = |v_3| = 3$ so $l_3 = cec$. If $l_1 \in \{ab^{-1}(eb), a(eb)^{-1}b\}$ then $a = e = c^2$ and the label of v_2 becomes $l_2 = c^{-1}bdw$ and the label of v_5 is $l_5 = cecw$ and none of them can have degree four or five. So the curvature of Δ will not be positive. If the label of v_1 is $l_1 \in \{ad^{\pm 1}(da)^{-1}, a(da)d^{\pm 1}\}$ then $e = c^2$ and $(da)^2 = 1$ and again none of v_2 and v_5 can have degree four or five.

Now let $|v_2| = |v_3| = 3$ so $l_3 = cec$ and $l_2 = bd(eb)^{-1}$ and $d = e = c^2$. The degree of v_1 is at least six and the curvature of Δ is at most $c(\Delta) \leq c(2,3,3,4,6) = \frac{\pi}{6}$. The case is exactly as that of Figure 7-14 and enough negative curvature can be found in exactly the same way.

Now let $|v_2| = |v_5| = 3$ so $l_5 = cec$ and $l_2 \in \{ba^{\pm 1}(eb)^{-1}, bd^{\pm 1}(eb)^{-1}, b(da)^{\pm 1}(eb)^{-1}\}$ and either $a = e$ or $d = e$ or $d = ea$. The degree of v_1 is forced to be at least six and the only case that the degree of v_3 is less than six and a region for positive curvature is possible is for $l_3 = cec^{-1}e^{-1}$ and $l_2 = bd^{-1}(eb)^{-1}$ and $d = e = c^2$. The curvature of Δ is at most $c(\Delta) \leq c(2,3,3,4,6) = \frac{\pi}{6}$. We look for negative curvature across its $c - d$ edge. If enough negative curvature cannot be found in region Δ_1 then we continue by looking for negative curvature in region Δ_2 as in Figure 7-16.

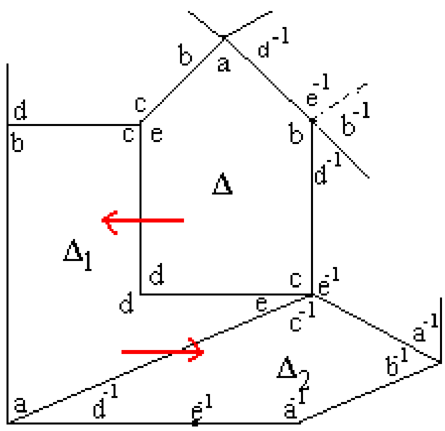


Figure 7-16:

If region Δ_2 is a region of degree five then its curvature is at most $c(\Delta_2) \leq c(3,3,3,6,6) = -\frac{\pi}{3}$ which is enough to compensate for the curvature of Δ. If it is a region of degree six (either an F_1-type or an F_2-type) then again its curvature is $c(\Delta_2) \leq c(2,3,3,4,6,6) = -\frac{\pi}{2}$.

Now suppose $|v_3| = |v_5| = 3$ and the only known relator is now $e = c^2$. If either $a = e$ or $d = e$ vertices v_1 and v_2 will have degree at lest six and $c(\Delta) \leq c(2,3,3,6,6) = 0$. In any other case the degree of v_2 and v_1 is at least five and region Δ will have curvature $c(\Delta) \leq c(2,3,3,5,5) = \frac{2\pi}{15}$. In any of these cases, region Δ_1 can be considered as in Figure 7-17.

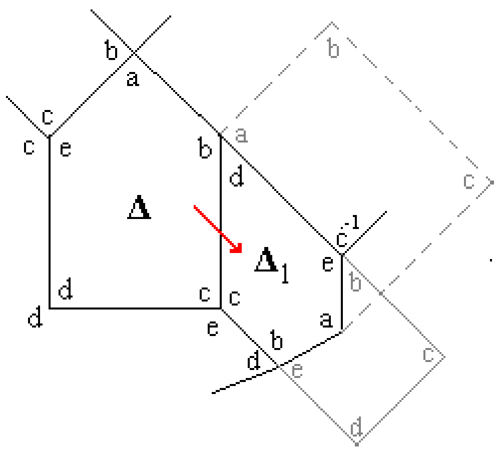


Figure 7-17:

If Δ_1 is a region of degree five then its curvature is at most $c(\Delta_1) \leq c(3,3,3,5,5) = -\frac{\pi}{5}$. If it is an F_1-type region (where a^2 was deleted) then $c(\Delta_1) \leq c(2,3,3,5,5,5) = -\frac{7\pi}{15}$ and if it is an F_2-type region (where e^2 was deleted) then $c(\Delta_1) \leq c(3,3,3,3,5,5) = -\frac{8\pi}{15}$. In any case this enough to compensate for the positive curvature of Δ with positive curvature given across a $c-d$ edge.

One vertex of degree three This must be either v_3 or v_5. It turns out that either two of the remaining vertices will have degree at least five, or if two of the remaining vertices have degree four then the other will have degree at least six. Therefore $c(\Delta) \leq max\,\{c(2,3,4,5,5), c(2,3,4,4,6)\} = 0$.

Overall in this section regions of positive curvature where found when $e = c^2$ and either $a = e$ or $d = e$ or none of the two is true. In the case that $d = e = c^2$ if Δ is a region of negative curvature receiving curvature from a positive neighbour then this is done across a $d^{-1} - c^{-1}$ (or the inverse). In the case that $a = e = c^2$ then negative curvature is always received across a $b - c$ or a $c^{-1} - b^{-1}$ edge and this enough even if the negative region receives positive curvature from more than one of its neighbours. In the cases that none of $a = e$ or $d = e$ is true, positive curvature is received across a $d^{-1} - c^{-1}$ edge.

7.2.3 Two elements of order two

Elements a, c, d, e can have order two. The case when a and e are the elements of order two is equivalent modulo T_3 to the case that c and d are the elements of order two. The case when a and d are the elements of order two is equivalent modulo T_3 to the case that c and e have order two.

$|a| = |c| = 2$ **and** $|d|$**,** $|e| \geq 3$

We can delete vertex a^2 wherever it is encountered in the diagram. Now the only possible label of length two is c^2. A region of positive curvature will be of F_1-type or of C_1-type. It can be seen that if none of conditions $d = e^{\pm 1}$, $e = d^{\pm 2}$, $d = e^{\pm 2}$, $e^3 = 1$, $d^3 = 1$ holds then $\omega(180, 0, 180, 90, 90)$ satisfies the weight test on Γ_3 and therefore $r_3(t) = 1$ will have a solution. So it is assumed that at least one of the above is true.

F_1 Let Δ to be a region of positive curvature. If $|v_2| = 2$ then $l_1 = (eb)dw$ and $l_3 = bdw$. If any of them has degree three then $d = e^{\pm 1}$. First let $|v_1| = 3$ and so $l_1 = (eb)db^{-1}$ and $d = e^{-1}$. The label of v_6 becomes $l_6 = da^{-1}w$ and its degree must be at least four. If $|v_3| = 3$ then $l_3 \in \{(eb)db^{-1}, bd(eb)^{-1}\}$ and the label of v_4 becomes $l_4 = c^{-1}(eb)w$ and so its degree is at least four. Therefore if v_2 is the only vertex of degree two the curvature of Δ will be $c(\Delta) \leq c(2,3,3,3,4,4) = 0$. Therefore Δ has two vertices of degree two.

In the case that v_2 and v_5 have degree two if any of vertices v_1, v_6, v_3 and v_4 has degree three $d = e^{\pm 1}$. We examine the two cases:

$d = e$ In this case v_1 and v_4 cannot have degree three since this would make the order of element e to be two. So at least one of v_3 and v_6 must have degree three with label $bd(eb)^{-1}$. If $|v_3| = 3$ and $l_3 = bd(eb)^{-1}$ then $l_4 = c^{-1}(eb)dw$. If $|v_4| = 4$ then $l_4 \in \{c^{-1}(eb)d(eb)^{-1}, c^{-1}(eb)db^{-1}\}$. The first label forces $c = e$ and the order of e becomes two, while the second label $l_4 = c^{-1}(eb)db^{-1}$ is possible and makes the additional relation of $c = e^2$. If $|v_4| = 5$ then $l_4 \in \{c^{-1}(eb)d^2b^{-1}, c^{-1}(eb)db^{-1}e, c^{-1}(eb)d^2(eb)^{-1}\}$ while the rest of the labels would force a contradiction. The first two labels create the relator $c = e^3$ and the third the relator $c = e^2$. The case is the symmetric for v_6 i.e. if $|v_6| = 3$ then the degree of v_1 can be four, five or greater than six with the corresponding labels. If both the degree of v_1 and v_4 are greater than six then the curvature of Δ cannot be positive. Therefore, the case can be viewed as $|v_3| = 3$ and $|v_4| \in \{4, 5\}$. Region Δ will be as in Figure 7-18 and has curvature at most $c(\Delta) \leq c(2,2,3,3,4,4) = \frac{\pi}{3}$.

We consider the neighbouring region Δ_1. The only possible vertex of degree two in Δ_1 is $v_{3\Delta_1}$ with label c^2 and in this case the degree of $v_{4\Delta_1}$ is forced to be greater than four. Also the degree of $v_{2\Delta_1}$ should be equal to or greater than six since it involves both elements d and a and any label with length less than six would make the group to have property X. Therefore $c(\Delta_1) \leq \max\{c(2,3,3,4,4,6), c(3,3,3,3,3,6)\} = -\frac{\pi}{3}$. If Δ_1 has a vertex of degree two then region Δ_2 adjacent to it cannot be giving positive curvature to Δ_1 and if Δ_1 does not have any vertices of degree two then $c(\Delta_1) \leq \max c(3,3,3,4,6,6) = -\frac{2\pi}{3}$ which is enough to compensate for both regions of positive curvature.

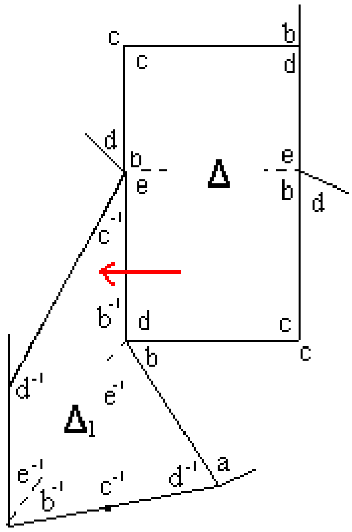


Figure 7-18:

$d = e^{-1}$ Now vertex of v_1, v_3, v_4 and v_6 may have degree 3 with label $(eb)db^{-1}$.

First let $|v_3| = 3$ and $l_3 = (eb)db^{-1}$ the label of v_4 will be $l_4 = c^{-1}(eb)dw$. The degree of v_4 cannot be three or four and $|v_{4\Delta}| = 5$ implies $l_4 = c^{-1}(eb)d^2(eb)^{-1}$ and so $c = e^2$.

Now let $|v_4| = 3$ and $l_4 = (eb)db^{-1}$and so $l_3 = bda^{-1}w$. Therefore, the degree of v_3 cannot be 3. If $|v_3| = 4$ then $l_3 = bda^{-1}(eb)^{-1}$ and $a = e^2$. If $|v_3| = 5$ then $l_3 \in \{bda^{-1}d(eb)^{-1}, bda^{-1}(eb)^{-1}e^{-1}\}$ and $a = e^3$.

The cases for $|v_1| = 3$ and $|v_6| = 3$ is symmetric so one of the following conditions will always hold:

1. $a^2 = c^2 = 1$, $d = e^{-1}$ and $c = e^2$
2. $a^2 = c^2 = 1$, $d = e^{-1}$ and $a = e^2$ or $a = e^3$

Two of the above conditions cannot occur at the same time as these would make the group cyclic. We examine the specific case when each one occurs:

First suppose that $a^2 = c^2 = 1$, $d = e^{-1}$ and $c = e^2$. If $|v_3| = 3$ and

$l_3 = (eb)db^{-1}$ then a possible region Δ of positive curvature will be as in Figure 7-19.

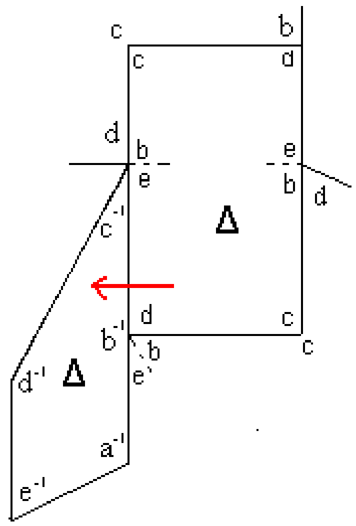


Figure 7-19:

The curvature of Δ can be $c(\Delta) \leq \max\{c(2,2,3,3,5,5), c(2,2,3,4,4,5)\} = \frac{2\pi}{15}$. Consider Δ_1 next to Δ. Then Δ_1 cannot have any vertices of degree 2 and has two vertices of degree at least five so $c(\Delta_1) \leq c(3,3,3,5,5) = -\frac{\pi}{5}$ that is enough to compensate for the positive curvature of Δ.

If $|v_4| = 3$ then $l_4 = (eb)db^{-1}$ and $l_3 = bda^{-1}w$ and so $|v_3| \geq 6$. If no other vertex has degree 3 then $c(\Delta) \leq c(2,2,3,4,4,6) = 0$. If $|v_6| = 3$ and $l_6 = (eb)db^{-1}$ and we can use region Δ_1 incident to $d-e$ edge with $c(\Delta_1) \leq c(3,3,3,5,5) = -\frac{\pi}{5}$. If $|v_1| = 3$ then $|v_6| \geq 6$ and so $c(\Delta) \leq c(2,2,3,3,6,6) = 0$. The cases for $|v_1| = 3$ or $|v_6| = 3$ are symmetric to $|v_4| = 3$ and $|v_3| = 6$ respectively, so there is always a region Δ_1 which can compensate a possible Δ of positive curvature, with positive curvature always given across a $c^{-1} - b^{-1}$ edge.

If $a^2 = c^2 = 1$, $d = e^{-1}$ and $a = e^2$ or $a = e^3$ and $|v_4| = 3$ with $l_4 = (eb)db^{-1}$ then a possible region Δ of positive curvature will be as in Figure 7-19. The curvature of Δ can be $c(\Delta) \leq c(2,2,3,3,4,4) = \frac{\pi}{3}$. Then region Δ_1 cannot have any vertices of degree two and $v_{3\Delta_1}$ and $v_{5\Delta_1}$ of Δ_1 will have degree greater than five. So $c(\Delta_1) \leq c(3,3,4,5,5) = -\frac{11\pi}{30}$ that is enough negative curvature to compensate the positive one of Δ. If $|v_3| = 3$ then $l_3 = (eb)db^{-1}$ and $l_4 = c^{-1}(eb)dw$ and $|v_4| \geq 6$. If none of the other vertices has degree 3 then $c(\Delta) \leq c(2,2,3,4,4,6) = 0$. If $|v_6| = 3$ then $|v_1| \geq 6$ and $c(\Delta) \leq c(2,2,3,3,6,6) = 0$. If $|v_1| = 3$ then we can find a region symmetric to Δ_1 of Figure 7-20.

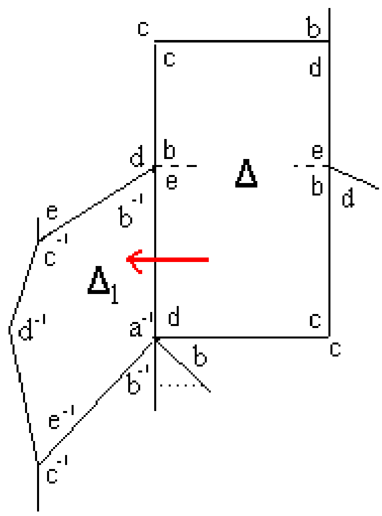


Figure 7-20:

C_1 Now suppose Δ is a region of positive curvature of C_1-type. If v_1 has degree three then $l_1 \in \{ad^2, ad^{-2}\}$ and $a = d^2$. If v_2 has degree three then $l_2 = bd(eb)^{-1}$ and $d = e$. If v_4 has degree three then $l_4 \in \{bd(eb)^{-1}, (eb)db^{-1}\}$ and $d = e^{\pm 1}$. If v_5 has degree three then $l_5 \in \{ce^2, ce^{-2}\}$ and $c = e^2$.

First let $|v_5| = 3$ and so $c = e^2$. Now one of condition $e = d^{\pm 1}$, $e = d^{\pm 2}$ and $d^3 = 1$ must hold otherwise $\omega(180, 0, 180, 90, 90)$ is satisfied on Γ_3 and the equation would have a solution. If $e = d^{\pm 2}$ then none of the remaining vertices can have degree four and actually a label of degree less than six on v_1 makes the group to have property X, so $c(\Delta) \leq c(2,3,4,4,6) = 0$. Also if $d^3 = 1$ then the remaining vertices must all have degree at least five and two of these cannot have degree four at the same time. So $c(\Delta) \leq c(2,3,4,5,5) < 0$. Therefore it must be $d = e^{\pm 1}$ and now the degree of v_1 is forced to be at least ten, since labels of smaller length will force relators that make G to have property X. If the degree of v_4 is greater than four then $c(\Delta) \leq c(2,3,3,5,10) < 0$ so the degree of v_4 must be three or four. If the v_2 is not three then it is at least six and the region will not have positive curvature. It turns out that the only possible case is for $|v_2| = 3$ and $l_2 = (eb)^{-1}bd$ and $|v_4| \in \{3,4\}$ so $l_4 \in \{bd(eb)^{-1}, (eb)db^{-1}c^{\pm 1}\}$. The curvature of Δ is at most $c(\Delta) \leq c(2,3,3,3,10) = \frac{\pi}{5}$.

Now let $|v_5| \geq 4$ and $|v_1| = 3$ so $a = d^2$. If none of conditions $e = d^{\pm 1}$, $d = e^{\pm 2}$ and $e^3 = 1$ holds than the weight test $\omega(180,0,180,90,90)$ is satisfied and the equation has a solution. It turns out that the only possible case is for $d = e$ and $l_1 = ad^{-2}$, $l_2 = (eb)^{-1}bd$ and $|v_4| \in \{3,4\}$ with $l_4 \in \{bd(eb)^{-1}, (eb)da^{\pm 1}b^{-1}\}$. In any case the degree of v_5 must be at least ten as labels of smaller lengths would force relators which would make the group to have property X. The curvature of Δ will be at most $c(\Delta) \leq c(2,3,3,3,10) = \frac{\pi}{5}$.

Now suppose $|v_1|$, $|v_5| \geq 4$. If v_4 is the only vertex of degree two, two of the remaining vertices cannot have degree four at the same time and the region cannot have positive curvature. So both v_2 and v_4 will have degree three and if one of vertices v_1 and v_6 has degree four the other will have degree at least six. So $c(\Delta) \leq \{c(2,3,3,5,5,), c(2,3,3,4,6)\} = \frac{\pi}{6}$.

In any of the case found in this section a region of positive curvature of C_1-type will have $l_2 = bd(eb)^{-1}$ and $d = e$. We look for negative curvature in Δ_1 as in Figure 7-21.

The curvature of Δ_1 is at most $c(\Delta_1) \leq max\,\{c(2,3,3,4,4,6), (3,3,3,3,4,4)\} = -\frac{\pi}{3}$ which is enough to compensate for the positive curvature of Δ. In the case that there is another positive region giving its curvature to Δ_1 the curvature of Δ_1 becomes $c(\Delta_1) \leq max\,\{c(3,3,3,3,4,6), c(2,3,3,4,6,6)\} = -\frac{\pi}{2}$ which will be enough for both.

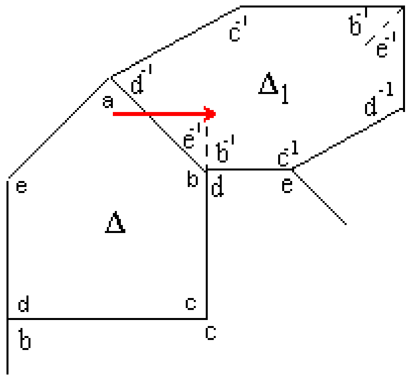


Figure 7-21:

Overall in this case regions of positive curvature were found for $d = e$ or $d = e^{-1}$. In the case that $d = e$ a region of negative curvature which receives positive curvature is always an F_1-type region (Figures 7-18, 7-21). Negative curvature is received either across a $b - c$ or a $d - e$ edge. In the case that a region receives curvature across a $b - c$ edge (from an F_1-type region) and a $d - e$ edge (from a C_1-type region) at the same time the labels on C_1 region become $l_5 = ec^{-1}e$, $l_1 = dad^{-1}$ and $l_4 = (eb)db^{-1}c^{\pm 1}$ and its curvature will be at most $c(\Delta_{C_1}) \leq c(2, 3, 3, 4, 10) = \frac{\pi}{30}$ and the curvature of F_1-type region becomes $c(\Delta_{F_1}) \leq c(2, 2, 3, 4, 4, 4) = \frac{\pi}{6}$. The curvature of Δ_1 becomes now $c(\Delta_1) \leq c(3, 3, 3, 3, 4, 10) = -\frac{19\pi}{30}$ which is enough to compensate for all the positive curvature in the neighbourhood. If region Δ_1 receives positive curvature from another positive neighbour then this will be at most $\frac{\pi}{3}$ and if it is another two positive neighbours their total curvature will be at most $\frac{\pi}{6} + \frac{\pi}{30}$. In any case the curvature of Δ_1 is enough. In the case that $d = e^{-1}$ then if $c = e^2$ negative curvature is received across a $c^{-1} - b^{-1}$ edge (Figures 7-19) and across a $b^{-1} - a^{-1}$edge if $a = e^k$, $k = 2, 3$ (Figure 7-20). These two cases cannot occur at the same time as this would make the group cyclic generated by e.

$|a| = |e| = 2$ **and** $|d|$**,** $|c| \geq 3$ Now it is possible to delete both vertices with labels a^2 and e^2 wherever these are encountered in diagram D. The only vertices of degree two can now be $a^{\pm 1}(da)$ and $(da)^2$ or the inverse. Any possible region with vertices of types F_1, F_2 or D_3 where D_3 is the region of Figure 7-10.

If Δ is of type F_1 then the only possible labels of degree two are υ_3 and υ_6 with label $d(da)$. If υ_3 has degree three then the labels of υ_4 and υ_2 become $l_4 = c(eb)w$ and $l_2 = cbw$ and either of them will have degree at least four. So a region of F_1-type will have curvature at most $c(\Delta) \leq max\,\{c(2,3,3,3,4,4), c(2,2,4,4,4,4)\} = 0$.

If Δ is of type F_2 then the possible labels of degree two are υ_1 and υ_4 with labels $(da)^2$ or $(da)d$ but both labels cannot appear at the same time. If $l_1 = (da)^2$ then the labels of υ_6 and υ_2 become $l_6 = l_2 = cbw$ and none of them can have degree three. Therefore the region cannot have positive curvature. If $l_1 = (da)d$ then $l_2 = cbw$ and $l_6 = cew$. The only possible label of degree three on υ_6 is $l_6 = cec$ and now $e = c^2$ but in this case none of conditions $c = d^{\pm 1}$, $c = d^{\pm 2}$, $d = c^{\pm 2}$, $c^3 = 1$, $d^3 = 1$ can hold and Γ_3 satisfies the weight test with $\omega(180, 0, 90, 90, 180)$. So υ_6 cannot have degree four and a region F_2 will not have positive curvature.

Now suppose Δ is of type D_2. The only vertex of degree two can be υ_4 with label $l_4 = d(da)$ and so $a = d^2$. Vertices υ_3 and υ_6 cannot have degree three so at least one of υ_1 and υ_2 must have degree three. In the case that υ_2 has degree three its labels should be $l_2 = ba^{\pm 1}(eb)^{-1}$ and unless $c^3 = 1$ the star graph Γ_3 will satisfy the weight test $\omega(180, 0, 90, 90, 180)$. But in this case υ_3 is forced to have degree at least six so the υ_1 must also have degree three but this will force the degree of υ_6 to be at least six. So $c(\Delta) \leq max\,\{c(2,3,4,4,6), c(2,3,3,6,6)\} = 0$. If the degree of υ_2 is at least four the degree of υ_1 must be three with label ad^{-2}. In that case if any of υ_2 or υ_3 has degree four then the weight test is satisfied and the equation has a solution.

$|a| = |d| = 2$ **and** $|d|$**,** $|c| \geq 3$ Vertices with labels a^2 are deleted everywhere in D they are encountered. Now the only possible label of degree two is d^2 (or the inverse) and a region of positive curvature can be either an F_1 or a D-type.

F_1 First let Δ be a region of positive curvature of F_1 type. The only possible labels of degree two are v_3 and v_6.

First let v_3 to be the only vertex of degree two. If v_2 has degree two its label will be ce^2 or cec and either $c = e^{-2}$ or $e = c^{-2}$. The label of v_4 is $l_4 = c(eb)w$ and it cannot have degree three. So the remaining vertices v_2, v_1, v_5 and v_6 should have degree three. If $l_2 = ce^2$ then $l_1 = (eb)aw$ and it cannot have degree three. If $l_2 = cec$ then $l_1 = (eb)dw$ and again it cannot have degree three. Therefore, $c(\Delta) \leq c(2,3,3,3,4,4) = 0$.

Now let both v_3 and v_6 have degree three. Since v_1 and v_4 cannot have degree three at least one of v_2 and v_5 will have degree three with label ce^2 or cec. First let $l_2 = ece$ and so $l_1 = c(eb)aw$. The degree of v_1 cannot be four while it is possible that it is five with labels $l_1 \in \{c(eb)ab^{-1}c, c(eb)a(eb)^{-1}e^{-1}\}$ and so $a = e^3$ or $l_1 = c(eb)a(eb)^{-1}c$ and $a = e^4$. It turns out that if v_5 does not have degree three then $|v_4| = |v_5| = 4$ is not possible and the curvature of Δ cannot be positive. So the label of v_5 must also be $l_5 = ce^2$ and the degree of v_4 is at least five. A possible region of positive curvature will have curvature at most $c(\Delta) \leq c(2,2,3,3,5,5) = \frac{2\pi}{15}$. Negative curvature may be given to Δ_1 across its $e - a$ edge as in Figure 7-22.

Δ_1 has curvature at most $c(\Delta_1) \leq c(3,3,4,4,5) = -\frac{11\pi}{30}$ which is enough to compensate for the positive curvature of Δ.

Now $l_2 = cec$ and so $e = c^{-2}$. The only possible regions of positive curvature will have both v_2 and v_3 of degree three and v_1 and v_4 cannot have degree four. They have degree at least five with possible labels $l_1, l_6 \in \{c(eb)db^{-1}e, c(eb)d(eb)^{-1}e^{-1}\}$ and $d = c^3$ or $l_1, l_6 = cede^{-2}$ and $c^6 = 1$. In any case region Δ will have curvature at most $c(\Delta) \leq c(2,2,3,3,5,5) = \frac{2\pi}{15}$ and it negative curvature can be found in

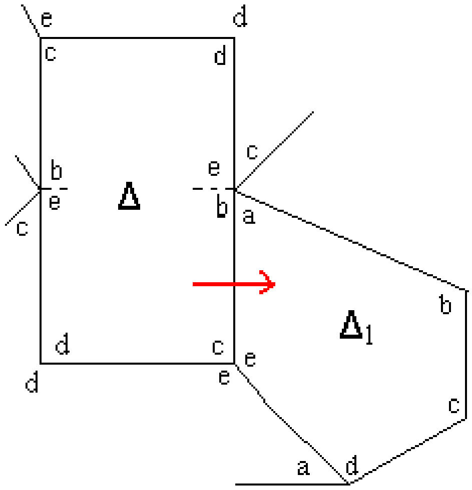


Figure 7-22:

Δ_1 as in Figure 7-23.

The curvature of Δ_1 is at most $c(\Delta_1) \leq c(3,4,4,5,5) = -\frac{11\pi}{30}$ if it is a region of degree five, or $c(\Delta_1) \leq c(2,3,3,5,5,6) = -\frac{8\pi}{15}$ if it is a region of degree six, in any way enough to compensate for the curvature of Δ.

D Let Δ be an interior region of positive curvature of D-type. v_1 and v_2 of Δ cannot have degree three so at least one of v_3 and v_5 will have degree three with label ce^2 or cec.

First let $c = e^{-2}$. If v_1 and/or v_5 have degree three it is possible that the other two vertices have degree four, so a region of positive curvature will have curvature $c(\Delta) \leq c(2,3,3,4,4) = \frac{\pi}{3}$. Negative curvature can be found in the neighbouring regions Δ_1 and Δ_2 as in Figure 7-24. The curvature of each Δ_i $i = 1,2$ is $c(\Delta_i) \leq c(3,4,4,4,4) = -\frac{\pi}{3}$ which is enough to compensate for the curvature of Δ. A difficulty arises when Δ_1 or Δ_2 receive positive curvature from two of their neighbouring regions, but even then the total positive curvature coming across an $e-a$ or a $d-e$ edge will be at most $\frac{\pi}{3}$. Wherever there is a

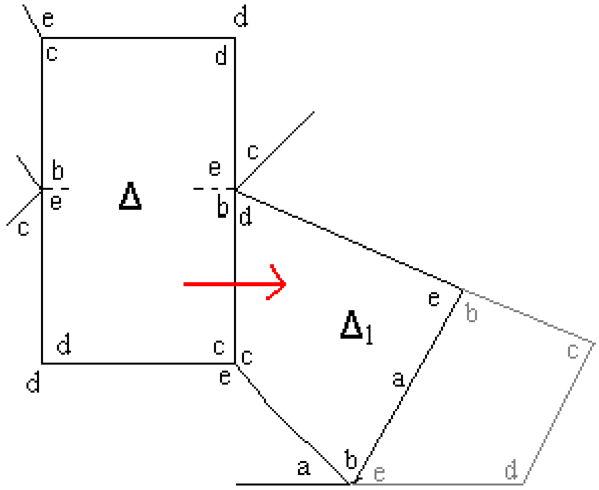


Figure 7-23:

label ce^2, a region of negative curvature may be found.

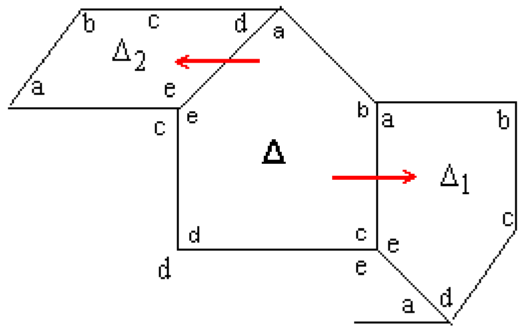


Figure 7-24:

The same arguments apply for $e = c^{-2}$. Now it is possible that υ_2 or/and υ_5 have degree three with label cec and the other vertices can have degree four or greater. The curvature of Δ may be at most $\frac{\pi}{3}$ and this can be given to regions Δ_1 and Δ_2 as in Figure7-25.

The curvature of Δ_1 is at most $c(3, 4, 4, 4, 4) = -\frac{\pi}{3}$ and the curvature of Δ_2

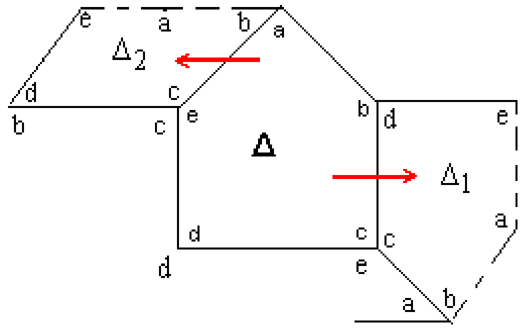


Figure 7-25:

is also at most $c(3,4,4,4,4) = -\frac{\pi}{3}$. If any of them is a region of degree six then $c(2,3,3,4,6,6) = -\frac{\pi}{2}$ and even if a region receives curvature from more than one regions of positive curvature there is enough negative to compensate for than.

Overall in this section regions of positive curvature were found when $c = e^{-2}$ (Figure 7-22, Figure 7-24) or $e = c^{-2}$ (Figure 7-23, Figure 7-25). In the case that a region of negative curvature receives positive curvature from more than one of its neighbours then the two positive regions are adjacent and another region of negative curvature can be found e.g. if region Δ_2 in Figure 7-25 receives curvature from Δ_3 then $l_3 = l_5 = ece$ and the curvature of Δ_1 can also be used to compensate the positive curvature coming from the two regions.

$|e| = |d| = 2$ **and** $|c|$**,** $|a| \geq 3$ Now it can be assumed that the number of vertices of degree two with label $e^{\pm 2}$ is maximal. This implies that $e^{\pm 2}$ does not appear as a proper sublabel anywhere in D. Also, if none of conditions $c = a^{\pm 1}$, $c = a^{\pm 2}$, $a = c^{\pm 2}$, $a^3 = 1$ and $c^3 = 1$ is true the star graph Γ_3 satisfies the conditions of the weight test for $\omega(90,0,90,180,180)$ and the equation will have a solution. So it is further assumed that at least one of these conditions is true.

It can be seen that it is possible to delete labels e^2 and d^2 wherever they are encountered in D. The new stargraph has two new elements (da) and (ce).

Region with vertices of degree two will be of F_2-type or F_3 and C_3 and A_5 as in Figure 7-26.

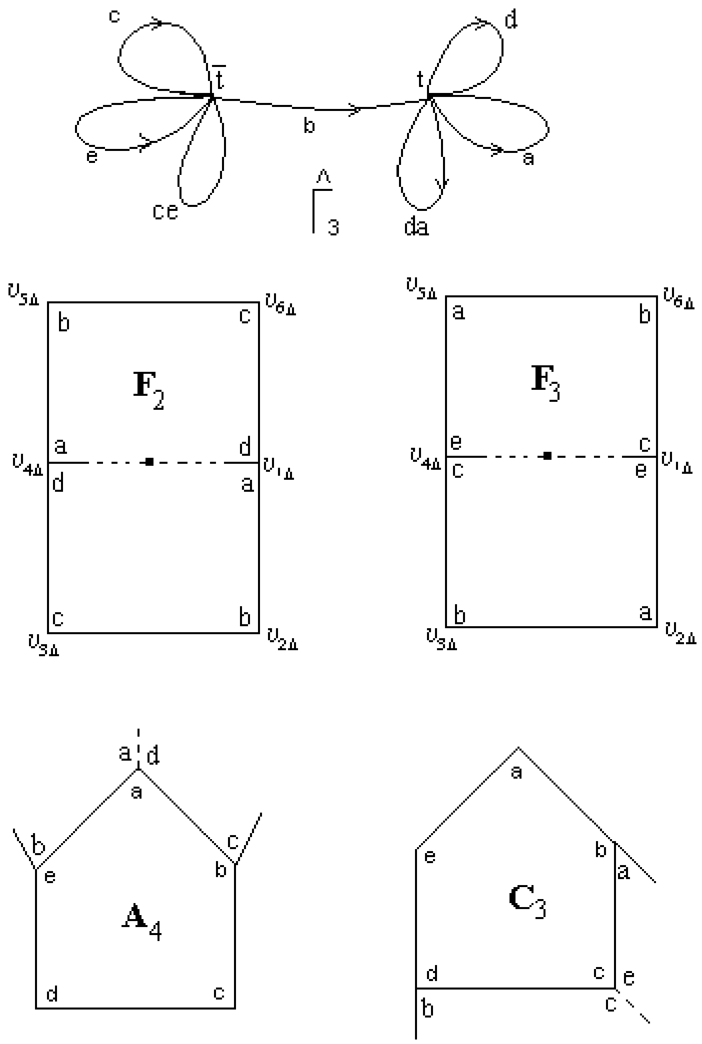


Figure 7-26:

It can be seen that the only possible vertices of degree two are now $d(da)$, $(da)^2$ and $c(ce)$ and it can be seen that any two of them cannot appear on the same region.

If label $(da)^2$ appears this will be on an F_2 region (v_1 or $v_4$will have degree two). The adjacent vertices of the vertex with this label will have degree at least four. The region will have at least two vertices of degree four if there is one vertex of degree two and at least four vertices of degree four if there are two vertices of degree two so its curvature will be at most $c(\Delta) \leq max\{c(2,3,3,3,4,4,), c(2,2,4,4,4,4)\} = 0$. The same applies if the label $d(da)$ appears on an F_2-region. Therefore, a region F_2 cannot have positive curvature. Also, if the label $(ce)^2$ or $c(ce)^2$ appears on an F_3 region the adjacent vertices will have degree at least four so the region cannot have positive curvature. So the only cases to be considered are when $(da)a$ appears on an A_4 region or an a $c(ce)$ label appears on a C_3-region.

A_4 Let Δ be a region of positive curvature of A_4-type so $l_1 = a(da)$. v_2 and v_5 cannot have degree three so at least one of v_3 and v_4 must have degree three. If v_3 has degree three then $l_3 \in \{c^3, c(ce)^2\}$. The degree of v_2 is at least five and the degree of v_5 and v_4 cannot be four at the same time. If v_4 has degree three then $l_4 \in \{d(da)^{-1}a, da^{-1}(da), da^{\pm 2}\}$. The only possible labels for two of the remaining vertices to have degree four are $l_2 = cbab^{-1}$, $l_3 = cbab^{-1}$ and so $c = a^{-1}$. But in that case the degree of v_5 is at least six. In any case the curvature of Δ is at most $c(\Delta) \leq max\{c(2,3,4,5,5), c(2,3,4,4,6)\} = 0$.

C_3 Let Δ be a region of positive curvature of c_3-type so $l_3 = c(ce)$. v_2 and v_4 cannot have degree three so at least one of v_1 and v_5 must have degree three. If v_3 has degree three then $l_1 \in \{ad^{-1}(da)^{-1}, a(da)^2, a^3\}$ and in any case the order of e must be three. In any case the degree of v_2 is at least five. If the degree of v_5 is not three then the region cannot have positive curvature since if v_5 has degree four the degree of v_2 becomes at least six. In the case that the degree of v_5 is three then the degree of the remaining vertices v_1, v_2 and v_4 will have degree at least four and two of them cannot have degree four at the same time. Therefore, a C_3 interior region cannot have positive curvature.

7.2.4 Only one element of order two

Elements a, c, d, e can have order two. The case when a is the only element of order two is equivalent modulo T_3 to the case that d is the only element of order two. The case when d is the only element of order two is equivalent to the case when e is the only case of order two, so we examine the two cases.

If a is the only element of order two then it can be deleted and since none of the other elements has order two regions F_1 cannot have vertices of degree two. So for any Δ, $c(\Delta) \leq c(3,3,3,3,3,3) = 0$.

If e is the only element of order two then we delete all vertices with labels $e^{\pm 2}$. If an $e^{\pm 2}$ appears anywhere as a proper sublabel then diamond moves may be performed to increase the number of vertices of degree two, so it can be assumed that such sublabels do not exist. Vertices with of degree two can only be $d(da)$ or $a(da)$. In such a case either $c = a^{\pm 1}$ or $c = d^{\pm 1}$ is true otherwise the conditions of the weight test are satisfied for $\omega(120, 0, 120, 120, 180)$. Vertices which involve letter e will have degree at least six otherwise the group will have property X. If Δ is a region of degree six then $c(\Delta) \leq c(2,2,3,4,4,6) = 0$ and if it is a region of degree five $c(\Delta) \leq c(2,3,4,4,6) = 0$.

Overall if D is a relative diagram representing a counter example for $r_3(t) = 1$ and Δ is an interior region of positive curvature, a Δ_1 with negative curvature can be found in such a way that $c(\Delta)+c(\Delta_1) \leq 0$. This proves that if there is positive curvature in D this is concentrated in the boundary, around the distinguished vertex v_0.

Chapter 8

Conclusions

For each of the three equations of length five $r_i(t) = 1$, $i = 1, 2, 3$ examined in Chapters 5, Chapter 6 and Chapter 7, it was proved that wherever an interior region Δ of positive curvature exists there is one or more neighbouring regions of negative curvature which when interior compensate for the positive curvature of Δ. This means that if there is positive curvature in D this will be concentrated in the boundary regions around the distinguished vertex v_0. If the assumption that an equation of length five does not have a solution is true, then the curvature of at least 4π must be found around v_0. In this final chapter we calculate the curvature around the distinguished vertex v_0.

Suppose Δ is a boundary region of D (i.e. at least one of its vertices is v_0). We calculate the curvature of Δ whether it is a region of degree six or five according to the number of vertices of Δ that are v_0. We denote the degree of v_0 by k_0 . We will rely heavily on the following important observation.

For all three equation the maximum curvature that a region of negative curvature may receive is $\frac{\pi}{3}$ except for the cases appearing in Figures 7-7, 7-8, 7-12, 7-14, 7-15, 7-18, 7-21, 7-23. In these specific cases which are dealed with separately at the end of this chapter, a region of degree six may receive, in the case of $r_3(t) = 1$, positive curvature up to $\frac{2\pi}{3}$.

For $r_1(t) = 1$ the regions used for compensation of positive curvature are always regions of degree five and they have at most one vertex of degree two. For $r_2(t) = 1$ the regions of degree five used for compensation always have at most one vertex of degree two except perhaps in the cases of the regions of negative curvature shown in Figures 6-5 and 6-7. For $r_3(t) = 1$ the regions of degree five used for compensation of positive always have at most one vertex of degree two. Also, a region of degree six may receive positive curvature when at most two of its vertices have degree two.

8.1 Δ is a boundary region of degree five

8.1.1 All the vertices of Δ are v_0

Let Δ_1 be a boundary region of degree five with all of its vertices being the distinguished vertex v_0. If this is the case the degree k_0 of v_0 is at least 10 since every endpoint of the five edges is v_0 (and no t-edges are identified when obtaining our spherical diagram) and so its curvature is $c(\Delta_1) = c(k_0, k_0, k_0, k_0, k_0) = (\frac{10}{k_0} - 3)\pi$. If this region is receiving positive curvature it will not be from any immediate neighbouring region but even if it does this cannot be more than $\frac{\pi}{3}$. So the total curvature of Δ_1 will be at most $(\frac{10}{k_0} - 3)\pi + \frac{\pi}{3}$ and since the degree of v_0 is at least 10 this amount is always negative.

8.1.2 Four vertices of Δ are v_0

Let Δ_2 be a boundary region of degree five with four of its vertices being the distinguished vertex v_0 and one interior vertex u_1 with degree k_1. Since $k_1 \geq 2$ the curvature of Δ_2 is $c(\Delta_2) = c(k_0, k_0, k_0, k_0, k_1) \leq (\frac{8}{k_0} - 2)\pi$ where k_0 has to be at least eight. So the total curvature of Δ_2 even in the case that it receives positive curvature cannot be greater than $(\frac{8}{k_0} - 2)\pi + \frac{\pi}{3}$ which is always negative.

8.1.3 Three vertices of Δ are v_0

Let Δ_3 be a boundary region of degree five which has three of its vertices being v_0. If at least one of the two remaining vertices has degree greater than or equal to three then $c(\Delta_3) \leq c(2, 3, k_0, k_0, k_0) = (\frac{6}{k_0} - \frac{4}{3})\pi$ and even if $\frac{\pi}{3}$ added the total curvature is less than or equal to zero since $k_0 \geq 6$.

If the two remaining interior vertices each have degree two then Δ_3 does not receive any positive curvature across its edges.(Observe that in Figures 6-5 and 6-7 we are using the fact that three of the vertices of Δ_3 are v_0 to conclude this). Therefore, the total curvature is $c(\Delta_3) \leq c(2, 2, k_0, k_0, k_0) = (\frac{6}{k_0} - 1)\pi \leq 0$ since $k_0 \geq 6$.

8.1.4 Two vertices of Δ are v_0

Let Δ_4 be a boundary region of degree five with two of its vertices being the distinguished vertex v_0.

If out of the remaining vertices at most one has degree two then $c(\Delta_4) \leq c(2, 3, 3, k_0, k_0) = \frac{4\pi}{k_0} - \frac{2\pi}{3}$ which is always less than $\frac{4\pi}{k_0}$ even if positive curvature is added to it.

If out of the remaining three interior vertices two have degree two then region Δ_4 may not receive any positive curvature in the case of $r_1(t) = 1$ and $r_3(t) = 1$. So its total curvature will be $c(\Delta_4) \leq c(2, 2, 3, k_0, k_0) = \frac{4\pi}{k_0} - \frac{\pi}{3}$ which is always less than $\frac{4\pi}{k_0}$. In the case of $r_2(t) = 1$ the regions used for compensation of negative curvature in Figures 6-5 or 6-7, cannot have two vertices being v_0 and two vertices of degree two the same time. So the such a region will not receive negative curvature even in the case of $r_2(t) = 1$ and its curvature is less than $\frac{4\pi}{k_0}$.

8.1.5 One vertex of Δ is v_0

Let Δ_5 be a boundary region of degree five with only one of its vertices being the distinguished vertex v_0.

Δ_5 has no vertices of degree two

In the case that Δ_5 does not contain any vertices of degree two then $c(\Delta_5) \leq c(3,3,3,3,k_0) = (\frac{2}{k_0} - \frac{1}{3})\pi$. Even if there is positive curvature coming into Δ_5 the total curvature of it cannot be greater than $\frac{2\pi}{k_0}$.

Δ_5 has exactly two vertices of degree two

Now suppose two of the vertices of Δ_5 have degree two. We examine this case for each equation separately:

$r_1(t) = 1$

Now Δ_5 does not receive any positive curvature as it has two vertices of degree two. It can be assumed without any loss that one of them is v_1 with label a^2. The only case that vertex a^2 has not been deleted everywhere in D it occurs is when $a = c = d$. In that case one of v_3 and v_4 must have degree two. If v_3 or v_4 have degree two and because of the relators holding in the group now, none of the remaining vertices can have degree three. So whichever vertex is the distinguished vertex v_0 the curvature of the region cannot be greater than $c(2,2,4,4,v_0) = \frac{2\pi}{k_0}$ which is always less than $\frac{4\pi}{k_0}$ since such a regions does not receive any positive curvature.

$r_2(t) = 1$

In this case a^2 can appear as a label only when $a = c$ and $d = e$. But in that case no other vertex of Δ can have degree two.

So in order to have two interior vertices of degree two $a^2 = d^2 = 1$ and the region will be one of regions F_1, F_2 and D_2 of Figure 6-23. In each of these cases either all the remaining vertices will have degree at least four or if one of them has degree three the other will have degree at least six. So $c(\Delta) \leq max\{c(2,2,3,6,k_0), c(2,2,4,4,k_0)\} = \frac{2\pi}{k_0}$. None of these types of regions receives positive curvature from any of its neighbouring regions so the curvature is at most always less than $\frac{4\pi}{k_0}$.

The only case that a region can have two vertices of degree two and not d^2 is that of Figure 8-1.

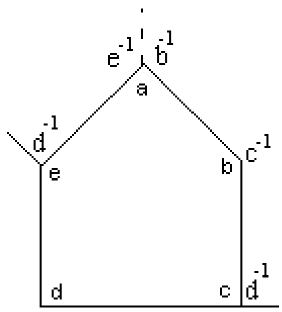


Figure 8-1:

The only vertex of Δ which can have degree three is v_4 while v_3 and v_5 have degree at least six. In the case that the degree of v_4 is three the remaining two vertices will have degree at least 18. In that case $c(\Delta) \leq c(2,2,3,18,k_0) = \frac{2\pi}{k_0} - \frac{2\pi}{9}$ and the curvature of at most $\frac{2\pi}{9}$ may be added to the region (as in Figure 6-7)so the total curvature will be at most $\frac{2\pi}{k_0}$. If the degree of v_4 is four or greater then $c(\Delta) \leq c(2,2,4,6,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{6}$ and the region may be receiving at most $\frac{\pi}{6}$ of positive curvature, so once again the total curvature is less than $\frac{2\pi}{k_0}$.

$r_3(t) = 1$

In any case that two of the vertices have degree two then one of them is deleted so regions of degree five with two vertices of degree two do not exist.

Δ_5 has exactly one vertex of degree two

Now suppose Δ_5 is a boundary region with only vertex of degree two. If the vertex of degree two is v_0 then $c(\Delta) \leq c(3,3,3,3,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{3}$ which is less than $\frac{4\pi}{k_0}$ even if positive curvature is added to it. If the region does receive positive curvature from one of its neighbours then the total curvature is $c(2,3,3,3,k_0) = \frac{2\pi}{k_0}$ which obviously less than $\frac{4\pi}{k_0}$. So it can be assumed that v_0 is not a vertex of degree two and that Δ receives curvature from one of its neighbours.

$r_1(t) = 1$

If the vertex of degree two is a^2 then we know that the relators holding in the group must be $a = c = d$ since all the vertices with label a^2 were deleted in all the other cases. Now since the labels of v_1 and v_5 start with eb both of these vertices will have degree at least six. So in any case the curvature of Δ_5 cannot be greater than $c(2,3,3,6,k_0)$ and even if it receives positive curvature from one of its neighbours the total curvature of Δ_5 is at most $(\frac{2}{k_0} - \frac{1}{3})\pi + \frac{\pi}{3} = \frac{2\pi}{k_0}$ and it is always less than $\frac{4\pi}{k_0}$.

If the vertex of degree two is v_2 with label bc^{-1} then v_1 and v_3 cannot have degree three. If at least one of v_4 and v_5 has degree four or greater then $c(\Delta_5) \leq c(2,3,4,4,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{3}$ which is less than $\frac{4\pi}{k_0}$ even after positive curvature of $\frac{\pi}{3}$ is added to it. So suppose both v_4 and v_5 have degree three. The label of v_4 will be $l_4 \in \{db^{-1}e^{\pm 1}, dc^{-1}e^{\pm 1}\}$ and in that case $d = e^{\pm 1}$ which forces the degree of v_1 to be six or greater. If l_4 is one of $db^{-1}e, dc^{-1}e$ then the degree if v_3 must also be six or greater and in any case $c(\Delta_5) \leq c(2,3,3,6,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{3}$. It turns out that the only other possible case for v_4 and v_5 to have degree three is for l_4 is one of $l_4 = dc^{-1}e^{-1}$ and $l_5 = d^{-1}ec$. Now $d = e$ and the degree of v_3 cannot be four but it is possible that it is five or greater. So the curvature of Δ_5 is at most $c(\Delta_5) \leq c(2,3,3,5,k_0) = \frac{2\pi}{k_0} - \frac{4\pi}{15}$. Now an interior region of positive curvature when this type or relator holds can only have curvature as big as $\frac{7\pi}{30}$ and since a negative region receives positive curvature from at most one of its neighbours the maximum curvature which can be added to it is that of $\frac{7\pi}{15}$. So the total curvature of Δ_5 is at most $\frac{2\pi}{k_0} - \frac{4\pi}{15} + \frac{7\pi}{30} < \frac{2\pi}{k_0}$.

Now, suppose that the vertex of degree two is v_3 with label cb^{-1}. The degrees of v_1, v_2 and v_4 if interior cannot be less than four as this would make the equation have a solution by Lemma 5.2. So $c(\Delta_5) \leq c(2,3,4,4,k_0)$ which is less than $\frac{2\pi}{k_0}$ even if $\frac{\pi}{3}$ is added to it.

Now, suppose that the label of v_3 is cd^{-1} and so $c = d$. The degree of v_4 cannot be three and if two of v_1, v_2 and v_5 have degree four or greater then $c(\Delta_5) \leq c(2,3,4,4,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{3}$. If the degree of v_5 is three then the degree of v_4 and the degree of v_1 must be greater than or equal to six, and so $c(\Delta_5) \leq (2,3,3,6,k_0) \leq \frac{2\pi}{k_0} - \frac{\pi}{3}$. We examine the possibility of two of these vertices having degree three at the same time. If the degree of v_2 is three then the label is $bc^{-1}e^{-1}$ and so $d = c = e^{-1}$. The degree of v_1 cannot be three and if the degree of v_5 is three then $l_5 \in \{ebc^{-1}, ebd^{-1}\}$ but such labels would force a contradiction by creating a^2 as a proper sublabel on v_1. So the case should be that both v_1 and v_5 have degree three. But any label of length three on v_1 would force $a = c = d$ and then any label of degree three on v_5 would make the group to be cyclic.

Now, suppose that v_4 is the only vertex of degree two of the boundary face with label dc^{-1}. The degree of v_5 must be four or greater and if v_3 has degree three then $l_3 = cb^{-1}e$ and $c = d = e^{-1}$. But in case both v_1 and v_2 will have degree six or greater. So $c(\Delta_5) \leq c(2,3,4,6,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{2}$ which is less than $\frac{4\pi}{k_0}$ even if positive curvature of $\frac{\pi}{3}$ is added to it. If v_3 has degree four or grater and at least one of v_1 and v_2 has degree four or greater then $c(\Delta_5) \leq c(2,3,4,4,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{3}$ which less than $\frac{4\pi}{k_0}$ even if positive curvature of $\frac{\pi}{3}$ is added to it. Now if both v_1 and v_2 have degree three then it can be seen from section 5.2.1. that no such region is used for compensation of positive curvature. So the total curvature of the region will be $c(\Delta_5) \leq c(2,3,3,4,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{6}$, less than $\frac{4\pi}{k_0}$.

$r_2(t) = 1$

If $b = c = 1$, $d = e$ and $a^2 = d^3 = 1$ then vertices with labels a^2 are not deleted and one of v_1, v_2 and v_3 may have degree two. The maximum curvature that may be received from a positive neighbour is $\frac{\pi}{9}$. If v_1 has degree two then v_2 and v_5 will have degree at least four and if v_3 has degree two then v_1 and v_2 will have degree at least four and so $c(\Delta) \leq c(2,3,3,4,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{6}$ which is less than $\frac{2\pi}{k_0}$ even if $\frac{\pi}{9}$ is added to it. If v_2 has degree two then v_1 has degree at least six. If v_1 is not the distinguished vertex v_0 then $c(\Delta) \leq c(2,3,3,6,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{3}$ which is less than $\frac{2\pi}{k_0}$ even if positive curvature is added. So suppose that $v_1 = v_0$. If any of the remaining vertices has degree four or greater then $c(\Delta) \leq c(2,3,3,4,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{6}$ which is less than $\frac{2\pi}{k_0}$ even with $\frac{\pi}{9}$ added to it. If all of the remaining vertices have degree three then it is not used for compensation of positive curvature and its curvature is $c(\Delta) \leq c(2,3,3,3,k_0) = \frac{2\pi}{k_0}$, that is always less than $\frac{4\pi}{k_0}$.

In the case that a^2 does not appear in region of degree five, the label of length two must be d^2. The only case that such a region was used for compensation is when v_5 had label $l_5 = ced^{\pm 1}$ and it had at least one vertex of degree four and one of degree at least six and at most $\frac{\pi}{6}$ was added to it. So $c(\Delta) \leq c(2,3,3,4,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{6}$ which is less than $\frac{2\pi}{k_0}$ even if curvature is added to it.

$r_3(t) = 1$

In the case that a region has only one vertex of degree two (not v_0) it will have at least two vertices of degree four and will receive curvature at most $\frac{\pi}{6}$ so $c(\Delta) \leq c(2,3,3,4,k_0) = \frac{2\pi}{k_0} - \frac{\pi}{6}$ which is less than $\frac{2\pi}{k_0}$ even if curvature is added to it. If three of its vertices have degree three then it does not receive positive curvature and its curvature is always at most $\frac{2\pi}{k_0}$.

8.2 Δ is a region of degree six

8.2.1 Δ does not receive more than $\frac{\pi}{3}$ of positive curvature

A region of degree six, in all of the cases examined receives positive curvature when at most two of its vertices have degree two.

If Δ is a boundary region of degree six with all of its vertices being v_0 then the curvature of the region will be at most $c(\Delta) \leq c(k_0, k_0, k_0, k_0, k_0, k_0) = (\frac{12}{k_0} - 4)\pi$. Now k_0 is at least 12 so the curvature in Δ is always negative even if it receives positive curvature from one of its neighbours. If five of the vertices of Δ are v_0 then its curvature is at most $c(\Delta) \leq c(2, k_0, k_0, k_0, k_0, k_0) = (\frac{10}{k_0} - 3)\pi$. The degree of v_0 is at least ten so this amount remains negative even if positive curvature is added to it. If four of the vertices of Δ are v_0 then $c(\Delta) \leq c(2, 2, k_0, k_0, k_0, k_0) = (\frac{8}{k_0} - 2)\pi$ and since the degree of v_0 is at least eight this amount will remain negative even if positive curvature is added to it.

If three of the vertices of Δ are v_0 and the interior vertices all have degree two, then $c(\Delta) \leq c(2, 2, 2, k_0, k_0, k_0) = (\frac{6}{k_0} - 1)\pi$ and this a non positive amount and Δ does not receive any positive curvature as in no case a region with three vertices of degree two was used for compensation. If at most two of the interior vertices have degree two $c(\Delta) \leq c(2, 2, 3, k_0, k_0, k_0) = (\frac{6}{k_0} - 1)\pi - \frac{\pi}{3}$ which still remains a negative amount even if positive curvature is added to the region.

Now let two of the vertices of Δ to be v_0. If the remaining vertices all have degree two then Δ will be the region shown in Figure 6-7 (region 6.5. for $r_2(t) = 1$) but in that case it is not possible for both v_3 and v_6 to be v_0.

If Δ is a boundary region of degree six with at most two vertices of degree two and one or more of its vertices is v_0 its curvature turns out to be always less than $\frac{4\pi}{k_0}$. There is a case for $r_1(t) = 1$ ($c = b = 1$ and $d = e$) when we can have three vertices of degree three. But in that case the remaining vertices must have degree four or more so $c(\Delta) \leq c(2, 2, 2, 4, 4, k_0) = \frac{2\pi}{k_0}$ or $c(\Delta) \leq c(2, 2, 2, 4, k_0, k_0) =$

$\frac{4\pi}{k_0} - \frac{\pi}{2}$, $c(\Delta) \leq c(2, 2, 2, k_0, k_0, k_0) = \frac{6\pi}{k_0} - \pi$ that in any case cannot be equal to or greater than $\frac{4\pi}{k_0}$. Also in the case of $r_2(t) = 1$ if four of the remaining vertices have degree two then the region has curvature $c(\Delta) \leq c(2, 2, 2, 2, 9, k_0) = \frac{2\pi}{k_0} + \frac{2\pi}{9}$. Positive curvature is not added to such a region and negative curvature can be found to compensate this region as with case of the interior region.

8.2.2 Δ receives positive curvature more than $\frac{\pi}{3}$

A region receiving more than $\frac{\pi}{3}$ of positive curvature can only be an F region used for compensation of positive curvature in the case of $r_3(t) = 1$ as shown in Figures 7-7, 7-8, 7-12, 7-14, 7-15, 7-18, 7-21, 7-23. These regions only receive more than $\frac{\pi}{3}$ when two of its adjacent faces are interior and in no case any such vertex has degree two. That means they have at least four interior vertices of degree at least three. If the two remaining faces are all the distinguished vertex v_0 then $c(\Delta) \leq c(3, 3, 3, 3, k_0, k_0) = \frac{4\pi}{k_0} - \frac{4\pi}{3}$ which remains negative even if $\frac{2\pi}{3}$ is added to it. If only one of the remaining vertices is v_0 then $c(\Delta) \leq c(3, 3, 3, 3, 3, k_0) = \frac{2\pi}{k_0} - \frac{2\pi}{3}$ and it is less than $\frac{4\pi}{k_0}$ even if curvature is added to it.

In conclusion if the curvature of all the boundary regions at v_0 is added together with all possible added positive curvature , this will be less than $k_0(\frac{4\pi}{k_0})$ so the total curvature of 4π cannot be achieved. This last conclusion gives the final contradiction and proves that $r_i(t) = 1$ for $e = 1, 2, 3$ always has a solution.

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