`FeynMG`: Automating particle physics calculations in scalar-tensor theories

Sergio Sevillano Muñoz

Supervised by
Prof. Edmund J. Copeland
Dr. Peter Millington
Dr. Adam Moss

Thesis submitted to the University of Nottingham
for the degree of **Doctor of Philosophy, August 2023.**

Para Nacho y la One

Abstract

This thesis explores the automation of the analysis of scalar-tensor theories at subatomic scales. For this, we make use of the fact that, when appended to the Standard Model, these theories can be expressed as standard gravity plus a Beyond the Standard Model theory. Therefore, studying the modifications that scalar-tensor theories have on a matter sector in this description requires the use of quantum field theory.

For this, we first investigate the origin of long-range interactions (fifth forces) in scalar-tensor theories of gravity both working in the Einstein and the Jordan-frames. We focus on theories of Brans-Dicke type in which an additional scalar field is coupled directly to the Ricci scalar of General Relativity. In our exploration of the Jordan frame calculation, we find that a specific gauge choice called scalar-harmonic gauge is convenient to perform a consistent linearization of the gravitational sector in the weak-field limit, which gives rise to a kinetic mixing between the non-minimally coupled scalar field and the graviton. It is through this mixing that a fifth force can arise between matter fields. We are then able to compute the matrix elements for fifth-force exchanges obtaining frame-independent results. Moreover, we also show the pivotal role that sources of explicit scale symmetry breaking in the matter sector play in admitting fifth-force couplings.

Irrespectively of the selected frame, we find the calculation to be very time-consuming and model dependent, motivating the development of computational tools for these derivations. The ability to represent perturbative expansions of interacting quantum field theories in terms of simple diagrammatic rules has revolutionized calculations in particle physics (and elsewhere). Moreover, these rules are readily automated, a process that has catalysed the rise of symbolic algebra packages. However, in the case of extended theories of gravity, such as scalar-tensor theories, it is necessary to precondition the Lagrangian to apply this automation or, at the very least, to take advantage of existing software pipelines.

In this context, we present the `Mathematica` package `FeynMG`, which works in conjunction with the well-known package `FeynRules`. `FeynMG` takes as inputs the `FeynRules` model file for a non-gravitational theory and a user-supplied gravitational Lagrangian. `FeynMG` provides functionality that inserts the minimal gravitational couplings of the degrees of freedom specified in the model file, determines the couplings of the additional tensor and scalar degrees of freedom (the metric and the scalar field from the gravitational sector), and preconditions the resulting Lagrangian so that it can be passed to `FeynRules`, either directly or by outputting an updated `FeynRules` model file. The Feynman rules can then be determined and output through `FeynRules`, using existing universal output formats and interfaces to other analysis packages, such as `MadGraph`. Therefore, in combination with these additional analysis packages, `FeynMG` will make possible to test for modifications to the Standard Model due to scalar-tensor theories in particle colliders.

Acknowledgements

I would like to use this opportunity to express my sincere gratitude to the people who made this thesis and this whole journey possible. Firstly, I would like to thank my family, starting with my parents, Montse and José Luis, who have supported me since the very first day and have always taught me the virtues of hard work while caring for me in difficult and stressful periods. Also, I want to especially thank Nacho, who has always led the way by showing me the wonders of Physics and Mathematics.

The biggest thanks go to Ed Copeland and Pete Millington, who made all this work possible. Ed, you have been one of my role models since I met you. I have always appreciated the trust you have put in me since the first day, giving me the freedom to discover my path in this process and supporting me in every decision I make. Pete, I have learnt a lot from you during my PhD, both from Physics and about all the essential qualities a scientist must have. I can't emphasize enough how grateful I am for every guidance you have given me, every opportunity I have been able to chase thanks to you, and all the countless times that you didn't lose your patience and kept showing me the correct (or precise) way of doing things. However, from all the things you both taught me, the one I will hold closest is that in this competitive environment, there is space for joy and kindness, and we can always choose to spread them with our actions.

Additionally, I am deeply grateful to Laura Santos and Moustafa Gharamti. You both gave me a great push in this learning journey, making it possible for me to be here right now. From both of you I hope to learn your passion for teaching and its power for changing people's lives.

I would also like to thank Clare Burrage, Christoph Englert, Adam Moss, Michael Spannowsky, and David Stefanyszyn for very valuable conversations over the years that have been crucial for completing this thesis and will guide my future steps.

I owe a lot to many people I have met along the way. I want to give a special thanks to Rosh, Zosh, André, and all the Spanish people from Erasmus and Part III coleguitas for making my undergrad period so great. From my PhD life, I want to thank all the wonderful people from the investigating team for all the laughs and help: Chad Briddon, Kieran Wood, Oliver Gould, Lauren Gaughan, Matt Gorton, Swagat Mishra, and my friend Bradley March. I am also very grateful to all the Astro people for including me in the fun plans that the particle people were too lazy to do. Similarly, I want to thank M. Tamayo, Chema, Belén, Mar, Edmundo, Pedro, Pepe, Alfonso, Tomás R., and all the medics (especially Arantxa, Isa, and Pablo Nuevo) for keeping me feeling very close to home throughout all these years.

Finally, I would like to thank Carla. You have been my constant support during my PhD since the very first day. You are always my best source of strength and stability, showing me how to be kind to myself and always be hopeful for the future. Every problem becomes easy, and every decision feels right when you are around.

List of Publications

I hereby declare that this dissertation is all my own work, except as indicated in the text. No part of this thesis has previously been submitted for a degree or other qualification at this or any other university.

The published original research is contained in the following publications:

[1] E. J. Copeland, P. Millington and S. S. Muñoz, "Fifth forces and broken scale symmetries in the Jordan frame," JCAP **02** (2022) no.02, 016 doi:10.1088/1475-7516/2022/02/016. arXiv: arXiv:2111.06357 [hep-th].

[2] S. Sevillano Muñoz, E. J. Copeland, P. Millington and M. Spannowsky, "FeynMG: a FeynRules extension for scalar-tensor theories of gravity," Submitted for publication in Computer Physics Communications. arXiv: arXiv:2211.14300 [gr-qc].

This work is distributed as follows throughout the thesis: both Chapters 2 and 3 contain work from [1] and [2]. Chapter 4 is uniquely contained in [2], although it has been considerably extended for this thesis to provide the reader with more context and examples on `FeynMG`. Access for the full package with guiding notebooks can be found here.

Contents

Conventions

Throughout this work, while it is a convention that is uncommon in the gravitation and cosmology literature, we use the "mostly minus" metric signature convention $(+,-,-,-)$, in which timelike four-momenta p^μ have $p^2 > 0$, since this is the convention commonly used by existing particle physics software packages. We use lower-case Greek characters for the Lorentz indices of the curved spacetime and lower-case Roman characters for the Lorentz indices of the flat, tangent space necessary for writing the Dirac Lagrangian in a generally covariant form.

Throughout this thesis we encounter different covariant derivatives, which can be distinguished as follows: D denotes gauge covariant derivatives in flat-spacetime; when including gravity, the gravitational and gauge covariant derivatives are denoted by ∇; and an update to the general and gauge covariant derivative that is useful for scalar-tensor theories of Brans-Dicke type is represented by $\mathcal{D}$. We work in natural units, but *do not* set Newton's gravitational constant to unity.

Chapter 1

Introduction

The leading framework to describe the physics of our Universe is currently provided by two different theories (or formalisms), each applicable in their respective regimes. Both work at a high level of detail, but operate on different energy scales: At large scales we have General Relativity (GR) with ΛCDM, while the physics of the small scales is best described by quantum field theory (QFT) and the Standard Model (SM) of particle physics.

As is well known, the biggest problem in physics comes from the fact that these two theories are difficult to reconcile, as the Standard Model of particle physics does not provide all the ingredients needed for a microscopic basis of the cosmological model. However, even within their own scales, there are still many unknowns and theoretical challenges that the theories, or their respective models, cannot fully explain. In order to understand some of these limitations of the framework, we will first give a brief introduction to both theories, stating their main features and aspects.

General Relativity describes the gravitational interactions between matter by introducing a dynamical spacetime, defined by the metric $g_{\mu\nu}$. The action used in

ΛCDM to generate these interactions is the so-called Einstein-Hilbert action:

$$S_{\rm EH} = \int {\rm d}^4x\sqrt{-g}\left[-\frac{M_{\rm Pl}^2}{2}R - \Lambda\right], \tag{1.1}$$

where we have defined g to be the determinant of the metric and $M_{\rm Pl}$ is the Planck mass, determining the strength of the gravitational interactions. Additionally, we have included the Cosmological Constant, Λ, which may account for the current accelerated expansion of the Universe, and the Ricci scalar, R, contributing to the kinetic energy term for gravity.

In order to arrive at a metric expression of the Ricci scalar, we first need to define the Riemann tensor, as they are very closely related. Formally, the Riemann tensor measures the curvature of the spacetime, and it is defined by its operation on a vector field by

$$R^{\rho}_{\mu\sigma\nu}Y^{\mu} = \nabla_\sigma\nabla_\nu Y^\rho - \nabla_\nu\nabla_\sigma Y^\rho, \tag{1.2}$$

where ∇_μ is the gravitational covariant derivative, used to protect diffeomorphism invariance in the action. There are multiple choices for this covariant derivative, but the most common is the one satisfying the metricity condition, $\nabla_\mu g_{\rho\sigma} = 0$. In this case, when applied on a vector field, it is defined by

$$\nabla_\mu Y_\nu = \partial_\mu Y_\nu + \Gamma^{\rho}_{\mu\nu}Y_\rho, \tag{1.3}$$

where $\partial_\mu = \partial/\partial x^\mu$ and

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) \tag{1.4}$$

are the Christoffel symbols. Therefore, substituting this expression for the covariant derivative into the Riemann curvature tensor in Eq. (1.2), we obtain

$$R^{\rho}_{\mu\sigma\nu} = \partial_\sigma \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu}. \tag{1.5}$$

From this, the Ricci tensor, $R_{\mu\nu}$, is defined by taking a contraction of the first and third index of the Riemann tensor, such as $R^{\rho}_{\mu\rho\nu} = R_{\mu\nu}$. This already takes us to the definition of the Ricci scalar as the trace of the Ricci tensor, $R \equiv g^{\mu\nu}R_{\mu\nu}$.

Once we have a metric definition for the Ricci scalar, we can vary the action with respect to the metric to find the equations of motion of the system. In this way, appending a matter sector $S_{\rm m}[g_{\mu\nu}]$ to the Eintein-Hilbert action in Eq. (1.1), we obtain

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{M_{\rm Pl}^2}\Lambda g_{\mu\nu} = \frac{1}{M_{\rm Pl}^2}T_{\mu\nu} \tag{1.6}$$

commonly called the Einstein field equations, where

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta g^{\mu\nu}} \tag{1.7}$$

is the energy-momentum tensor of the matter action (for a review, see Ref. [3] or any other classic GR textbook.). From this derivation we can already see that the gravitational interaction of matter is related to the curvature of the spacetime; in the words of J. A. Wheeler: *spacetime tells matter how to move; matter tells spacetime how to curve* [4]. This specific action with the addition of Cold Dark Matter adds up to the ΛCDM model of cosmology, which has been extremely successful when describing the physics of our Universe through cosmological observations.

Quantum field theory arises from the unification of Einstein's theory of special relativity and quantum mechanics. In simple terms, the main objects of study in this theory are fields that interact with each other, and, once quantized, their excitations give rise to the known fundamental particles. When describing the Universe we live in, the set of fields and interactions that explain the experimental phenomenology are encoded in the Standard Model, whose underlying set of symmetries is defined by the gauge group SU(3)×SU(2)×U(1) [5]. In this model, the fields are conveniently split into the strong interactions, described by quantum-chromodynamics, and the electro-weak sector, whose symmetries are broken by the Higgs mechanism. In this way, the Standard Model is able to explain the electromagnetic interaction as well as the strong and weak nuclear forces, and was completed by the discovery of the Higgs boson in 2012 [6].

Even so, there are many open questions about the Standard Model, starting with the fact that the structure of the Higgs potential remains to be probed experimentally [7–14]. Similarly, the origin of the observed neutrino masses [15] is still unknown, requiring an extension to the Standard Model. Moreover, since neutrinos are electrically neutral, it should be possible for them to be either Majorana or Dirac fermions. In both cases, this would have strong associated implications on the Standard Model [16]: For Majorana fermions, lepton number conservation would be violated unless protected by an additional symmetry, while for Dirac neutrinos, one would have to worry about such accidental symmetries arising in the Standard Model. Other problems related to the naturalness of the Standard Model include the large difference between the Higgs and the Planck mass, the so-called hierarchy problem, or the extremally small neutron electric dipole moment, related to the Strong-CP problem. Furthermore, we can add recent tensions in experimental tests that may lead to a mismatch between theory and experiment, such as the anomalous magnetic moment of the muon [17]. To address all these different existing open problems, many extensions and modifications of the Standard Model have been proposed, known collectively as Beyond Standard Model (BSM) theories (for a review on this topic, see Ref. [18]).

General Relativity also has open problems involving naturalness issues. In chronological order, to explain the initial conditions of our Universe after the Big Bang, the most accepted explanation introduces an early inflationary period (a period of accelerated expansion of the scale factor) to flatten any initial curvature and produce the quantum fluctuations that would later seed the formation of structure in the Universe. With regard to inflation, we do not know the exact physics that took place during this period of time, including whether inflation describes it at all [19, 20]. In addition, although through many measurements we have a good understanding of where dark matter is located and how it has behaved throughout the history of the Universe, we are still far away from understanding its origin and nature. To these unknowns we can add the cosmological constant problem [21, 22], which when calculated using the ground state vacuum energy of the Standard Model is off by ~ 120 orders of magnitude. Combining these three aspects of cosmology, we find the high energies from inflation must fade away fast enough to allow for a reheating period (leading to the standard early radiation domination) before both dark matter and the cosmological constant dominate the energy content of the Universe at the same time, creating another problem called the *coincidence problem* [23]. There are also current tensions on different observations in cosmology, such as the Hubble tension [24], where the measured value for the expansion rate of the Universe using nearby galaxies [25] does not match the theoretical value derived from the Cosmic Microwave Background [26].

When suggesting possible solutions to the mentioned limitations of ΛCDM, the usual direction is to choose a model that can solve both cosmological and Standard Model problems. For example, the axion field [27] arises naturally from the Peccei-Quinn solution to the Strong-CP problem [28–30], but its ultra-light mass also makes it a very good candidate for fuzzy dark matter [31]. Similarly, some extensions of the axion can address the hierarchy problem, for example via the relaxation mechanism [32], in which the Higgs mass is controlled by the vacuum expectation value of an axion-like field. Furthermore, the addition of an extra scalar field can explain the expansion of the Universe through Quintessence [33, 34] or K-Essence [35, 36] models, having also a possible connection with the Hubble tension [24, 37].

In this thesis, we will focus on a different type of naturalness problem in General Relativity that goes beyond the ΛCDM model, and this is the choice for the gravitational action. In this way, although the Einstein-Hilbert action from Eq. (1.1) is the standard choice when model-building theories of gravity, there may be other choices that can lead to consistent and physically interesting phenomena. Now, when building new theories of gravity, one might think that there are as many options to choose from as one can imagine. However, depending on the assumptions and symmetries we impose on the system, these choices will be very constrained.

The most reasonable set of assumptions is given by Lovelock's theorem [38, 39], which reduces notably our freedom when creating a gravitational sector. This theorem states the following: *In 4-D the only divergence-free rank-2 tensor constructed from only the metric $g_{\mu\nu}$ and its derivatives up to second order, and preserving diffeomorphism invariance, is the Einstein tensor with a cosmological constant term.* Therefore, given any 4-dimensional gravitational two-form, $L^{\mu\nu}$, appearing in our equations of motion and satisfying the following conditions:

i) $L^{\mu\nu} = L^{\mu\nu}(g_{\alpha\beta}, \partial_\sigma g_{\alpha\beta}, \partial_\sigma \partial_\rho g_{\alpha\beta})$;

ii) $\nabla_\mu L^{\mu\nu} = 0$,

it will necessarily be defined as

$$L^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu}, \tag{1.8}$$

where a and b are constants and $G^{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. As we can see, this equation has a very close resemblance with Einstein's field equations in Eq. (1.6), where $L^{\mu\nu} \propto T^{\mu\nu}$. Therefore, in terms of the action, this implies that any such terms satisfying those conditions will always be produced by varying the Einstein-Hilbert action with respect to the metric.

The nice thing, however, about Lovelock's theorem is that it also gives us the formula to produce modified theories of gravity. For example, we may do it by increasing the number of spacetime dimensions, as with Kaluza-Klein or braneworld models [40]; by including higher order derivatives of the metric; by considering non-local terms [41], such as inverse d'Alembertian operators acting on the Ricci scalar (i.e., $\Box^{-1}R = (\nabla_\mu\nabla^\mu)^{-1}R$); or even by using emergent theories of gravity that are not directly derived by varying the action with respect to the metric, such as those where the dynamics of the system depend on the entropy of the Universe [42]. However, one of the main options for modified theories of gravity are those that retain Lovelock's assumptions, but allow for the inclusion of additional degrees of freedom in the gravitational sector.

In particular, we will work on scalar-tensor theories [43], which couple a scalar degree of freedom to the curvature objects in the action, and can lead to modifications of the matter sector with strengths that need not be Planck-suppressed. Since any inclusion of a scalar field into the action can also be studied from the perspective of particle physics, we will pay special attention to the description of these theories as Beyond the Standard Model physics.

1.1 Scalar-tensor theories: A natural extension for gravity

When considering scalar-tensor theories of gravity, it is important to emphasize that we are still working within the formalism of General Relativity, since we have a metric and the equations of motion are derived from the action. For example, their main motivation is usually their close connection with string theory, as such couplings in the gravitational action appear when compactifying the extra dimensions [44]. However, such theories arise more generally, as we expect them to emerge from any theory that aims to describe the physics at ultra-high energy scales, where additional couplings to curvature will occur.

In particular, any interacting quantum field theory, such as the Standard Model, is only defined up to a particular energy scale, with loop corrections and renormalization of couplings creating different effective theories. Therefore, without symmetries to prevent it, any quantum field theory containing both gravity and a scalar field should lead to couplings between these two fields at higher scales [45–47]. This can be illustrated as

$$\int_{\Lambda_0} \mathrm{d}^4x\,\sqrt{-g}[aR+\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi] \to \int_{\Lambda_1} \mathrm{d}^4x\,\sqrt{-g}[a'R+F(\varphi)R+bR^2+\frac{B(\varphi)}{2}\partial_\mu\varphi\partial^\mu\varphi+\ldots], \tag{1.9}$$

where Λ_0 and Λ_1 correspond to the cut-off energy scales, $F(\varphi)$ and $B(\varphi)$ are generic functions of the scalar field φ, $\{a, a', b\}$ are constants, and we have omitted in the ellipsis all possible couplings between the scalar field and gravity allowed by the symmetries of GR. However, we do not need to worry about the generation of the infinite number of possible operators, since we can always reduce the number of terms by imposing necessary assumptions, as in Lovelock's theorem. In this way, considering only up to second-order derivatives in the equations of motion and in 4-dimensional spacetime, the most generic ghost-free scalar-tensor theory is described by Horndeski's theory [48, 49], defined via

$$S_{\mathrm{H}} = \int \mathrm{d}^4x\,\sqrt{-g}\,[\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4+\mathcal{L}_5] \tag{1.10}$$

where the Lagrangian densities are given by

$$\mathcal{L}_2 = G_2(\varphi, X) \tag{1.11}$$

$$\mathcal{L}_3 = G_3(\varphi, X)\Box\varphi \tag{1.12}$$

$$\mathcal{L}_4 = G_4(\varphi, X)R + G_{4,X}(\varphi, X)\left[(\Box\varphi)^2 - (\nabla_\mu\nabla_\nu\varphi)^2\right] \tag{1.13}$$

$$\begin{aligned}\mathcal{L}_5 &= G_5(\varphi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi \\ &\quad - \frac{1}{6}G_{5,X}(\varphi, X)\left[(\Box\varphi)^3 - 3\Box\varphi(\nabla_\mu\nabla_\nu\varphi)^2 + 2(\nabla_\mu\nabla_\nu\varphi)^3\right],\end{aligned} \tag{1.14}$$

where $G_i(\varphi, X)$ are generic functions of the scalar field φ and its canonical kinetic term $X = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$, and $G_{i,X} = \partial G_i/\partial X$. These types of direct couplings of the scalar field to the curvature terms are called *non-minimal*, as opposed to the usual minimal couplings of a quantum field theory in a curved background. Additionally, one can further extend these constraints to build more generic scalar-tensor theories, examples are beyond Horndeski [50, 51] and DHOST [52, 53] theories.

In this thesis, we will focus on the simplest scalar-tensor theory, called the Brans-Dicke theory [54], building up our intuition such that our results can be extended to the full family of Horndeski theory and beyond. In this model, the dynamical scalar field is set to replace the Planck mass, and in its generic form is described by setting

$$G_2(\varphi, X) = Z(\varphi)X - U(\varphi) \qquad\qquad G_4(\varphi, X) = F(\varphi), \tag{1.15}$$

with vanishing $G_3(\varphi, X)$ and $G_5(\varphi, X)$. The action then takes the following form:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{F(\varphi)}{2}R + \frac{1}{2}Z(\varphi)\partial_\mu\varphi\partial^\mu\varphi - U(\varphi)\right] + S_\mathrm{m}[g_{\mu\nu}]. \tag{1.16}$$

Therefore, the real scalar field, φ, is subject to the self-interaction potential $U(\varphi)$ and coupled non-minimally to the Ricci scalar R through the function $F(\varphi)$. From a phenomenological perspective, this model can be thought of as an Einstein-Hilbert action with a time-varying Planck mass, such that $\langle F(\varphi)\rangle = M_{\rm Pl}^2$ (where the brakets refer to the expectation value of the argument). Notice that the field φ is not or, at least, does not appear to be canonically normalized, by virtue of the function $Z(\varphi)$ included in its kinetic term. In fact, additional contributions to the kinetic energy of the field φ arise through the coupling to the scalar curvature.

1.1.1 Modifications to the dynamics of the matter sector

As one might expect, the modification of gravity will lead to new interactions in the matter sector when compared to the usual Einstein-Hilbert action. However, as we will show throughout this thesis, we can treat these modifications as being independent of gravity itself, making it possible to isolate them into a Beyond the Standard Model-like description. Therefore, appending a modified theory of gravity to the Standard Model would be equivalent to a BSM theory plus standard gravity.

In this section, we will demonstrate that this conversion is possible by working directly with the classical action. For the Brans-Dicke example from Eq. (1.16), there are two ways the Beyond Standard Model description can be obtained:

Going to the Einstein frame: We can make a Weyl rescaling of the metric to remove the non-minimal gravitational coupling of the field φ to the Ricci scalar, taking us to the so-called *Einstein frame*.

For the Brans-Dicke action in Eq. (1.16), we just need to perform the following Weyl rescaling

$$g_{\mu\nu} \to \frac{\tilde{M}_{\rm Pl}^2}{F(\varphi)}\tilde{g}_{\mu\nu}, \qquad\qquad g^{\mu\nu} \to \frac{F(\varphi)}{\tilde{M}_{\rm Pl}^2}\tilde{g}^{\mu\nu}. \tag{1.17}$$

where $\tilde{g}_{\mu\nu}$ and $\tilde{M}_{\rm Pl}$ are the metric and Planck mass defined in the Einstein frame, respectively. Making use of these transformation rules in the Ricci scalar dependent term, we obtain

$$\sqrt{-g}\frac{F(\varphi)}{2}R \to \sqrt{-\tilde{g}}\left(\frac{\tilde{M}_{\rm Pl}^2}{2}\tilde{R} - \frac{3\tilde{M}_{\rm Pl}^2 F'(\varphi)^2}{4F(\varphi)^2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right). \tag{1.18}$$

Therefore, the whole action in the Einstein frame is given by

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[-\frac{\tilde{M}_{\rm Pl}^2}{2}R + \frac{\tilde{M}_{\rm Pl}^2}{2}\left[\frac{Z(\varphi)}{F(\varphi)} + \frac{3F'(\varphi)^2}{2F(\varphi)^2}\right]\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\tilde{M}_{\rm Pl}^4}{F(\varphi)^2}U(\varphi)\right] \\ + S_{\rm m}\left[\frac{\tilde{M}_{\rm Pl}^2}{F(\varphi)}\tilde{g}_{\mu\nu}\right], \end{aligned} \tag{1.19}$$

in which we can see that the modifications of gravity appear directly in the matter action through the minimal couplings to gravity, while recovering an Einstein-Hilbert gravitational sector. Thus, this theory is equivalent to a BSM theory. In this way, the ability to make Weyl rescalings of the metric and so-called disformal transformations [55–58] of the form

$$g_{\mu\nu} \to A(\varphi)\tilde{g}_{\mu\nu} + B(\varphi)\partial_\mu\varphi\partial_\nu\varphi, \tag{1.20}$$

allows us to make connections between scalar-tensor theories of gravity and gauge-singlet, scalar extensions of the Standard Model of particle physics, such as Higgs- or neutrino-portal theories [59–66], as we will demonstrate in Chapter 2. However, as will be shown below, the Einstein frame calculation has limitations when considering full-generic scalar-tensor theories, motivating the use of the Jordan frame.

Staying in the Jordan frame: We can continue in the *Jordan frame* (where the curvature couplings are manifest), by analyzing how the metric degrees of freedom mediate interactions between the field φ and our would-be Standard Model fields.

From a classical field theory standpoint, this requires us to study the equations of motion of the system. For the Brans-Dicke action in Eq. (1.16), they are given by

$$\frac{1}{2}F(\varphi)G_{\mu\nu} + \nabla_\mu\nabla_\nu F(\varphi) - g_{\mu\nu}\Box F(\varphi) = \frac{1}{2}T^{(\mathrm{m})}_{\mu\nu} + \frac{1}{2}T^{(\varphi)}_{\mu\nu} \tag{1.21}$$

$$Z(\varphi)\Box\varphi + \frac{1}{2}Z'(\varphi)\partial_\mu\varphi\partial_\nu\varphi + U'(\varphi) + \frac{F'(\varphi)}{2}R = 0, \tag{1.22}$$

where the first equation corresponds to the Einstein field equations of the modified gravitational action, and

$$T^{(\mathrm{m})}_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} \tag{1.23}$$

is the energy-momentum tensor of the matter fields. In this way, the energy-momentum tensor for the φ field is given by

$$T^{(\varphi)}_{\mu\nu} = -Z(\varphi)\partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}\left(\frac{1}{2}Z(\varphi)\partial_\sigma\varphi\partial^\sigma\varphi - U(\varphi)\right). \tag{1.24}$$

Now, looking at these equations, the only reason why we assume that gravity in the Jordan frame is modified is because the left-hand side of the Einstein field equations, containing the gravitational sector, has extra dependencies on the non-minimally coupled field. However, we can trivially rearrange the expression such that the modifications on gravity appear in the right-hand side of the equation, such as

$$G_{\mu\nu} = \frac{2}{F(\varphi)}\left(\frac{1}{2}T^{(\mathrm{m})}_{\mu\nu} + \frac{1}{2}T^{(\varphi)}_{\mu\nu} - \nabla_\mu\nabla_\nu F(\varphi) + g_{\mu\nu}\Box F(\varphi)\right). \tag{1.25}$$

In this way, the system now looks like a Beyond the Standard Model theory with an Einstein-Hilbert gravitational action. Furthermore, to eliminate the dependence on the Ricci scalar in the equations of motion of the φ field in Eq. (1.22), we can just substitute in the trace of Eq. (1.25). In this way, we obtain

$$Z(\varphi)\Box\varphi + \frac{1}{2}Z'(\varphi)\partial_\mu\varphi\partial^\mu\varphi - U'(\varphi) + \frac{F'(\varphi)}{F(\varphi)}\left(T_\mu^{\mu(\mathrm{m})} + T_\mu^{\mu(\varphi)} + 3\Box F(\varphi)\right) = 0, \tag{1.26}$$

where we find that the coupling of the extra degree of freedom depends on the trace of the energy momentum of the matter fields, which will play an important role in Chapter 3. Finally, considering Eqs. (1.25) and (1.26), we find that there is not a hint of a modified gravitational sector in the equations of motion.

Independently of the frame, we should always find the same observables.[1] Although going to the Einstein frame is the most common choice, we will pay special attention to performing all the calculations in the Jordan frame. The merits of dealing with scalar-tensor theories in the Jordan frame are threefold:

1. In the presence of additional non-minimal gravitational couplings, for example, $R_{\mu\nu}\nabla^\mu\nabla^\nu\varphi$ (as arises in the Horndeski class of scalar-tensor theories in Eq. (1.10)), the Weyl rescaling of the metric from Eq. (1.17) (or more generally a disformal transformation of the metric in Eq. (1.20)) may not be able to remove all non-minimal couplings simultaneously. In these cases, we may not be able to transform into an Einstein frame and will have little choice but to continue in the Jordan frame, working with non-minimal interactions with gravity.

2. As discussed at the beginning of this section, in an interacting quantum field

[1] There are multiple proofs of this statement in classical physics, for example, see Ref. [67]. However, interpreting the anomalies arising from quantum field theory when performing the conformal transformation is still an open problem in physics [68, 69].

theory, the Einstein frame may exist only at a particular energy scale, with loop corrections and the renormalization of couplings regenerating Jordan frame-like interactions.

3. The conformal transformation to the Einstein frame, and the subsequent field redefinitions needed to bring the theory as close to being canonically normalized as possible (notwithstanding any curvature of the field-space metric) must be performed on a model-by-model basis and may not be easily automated.

Thus far, we have chosen to isolate the non-minimally coupled field, φ, from the matter sector. This is a widely accepted choice, which is present since the definition of what is called the *prototypical* Brans-Dicke theory [54], defined by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{\varphi}{2}R + \frac{w(\varphi)}{2\varphi}\partial_\mu\varphi\partial^\mu\varphi - U(\varphi)\right] + S_\mathrm{m}[g_{\mu\nu}], \tag{1.27}$$

where $\omega(\varphi)$ is usually taken to be constant. However, it is possible to include additional couplings between φ and the Standard Model, at the expense of breaking the Weak Equivalence Principle [43]. Popular examples of such models include those in which the Higgs is non-minimally coupled to gravity, as is required in Higgs inflation or the Higgs-dilaton theory [70–84]. Therein, both the Higgs field and an additional gauge-singlet scalar are non-minimally coupled to the Ricci scalar, a model that will play an important role in Chapter 3.

Independently of the specific couplings, however, we can already infer from Eqs. (1.21) and (1.22) that the matter sector will be sensitive (in most cases) to the modifications of the gravitational sector. Moreover, depending on the exact form of these new interactions, we expect them to have implications on a wide range of scales, both for GR and QFT. In the next section, we will give a brief description of some of the main tests and subsequent constraints imposed on these theories.

1.2 Implications and constraints on scalar-tensor theories

When considering modifications due to extensions of gravity, the thing that comes to mind are the possible existence of long-range interactions, commonly referred to in cosmology as *fifth forces.* However, as we will see in this section, the increasing complementarity of high precision data from cosmological observations and high energy physics experiments makes it possible to test for deviations from standard gravity that probe different phenomena complementary to long-range interactions.

In what follows, we provide a brief description of some of the main constraints and expected effects that these models may have at different scales in our Universe:

- **Cosmological scale tests:**

The main constraints on cosmological scales are not only directly given by fifth forces themselves, but also by the evolution of the gravitational coupling strength, given by the Planck mass, $M_{\rm Pl}$. As we mentioned earlier, one model that produces a time-varying Planck mass was the generic Brans-Dicke theory in Eq. (1.16), since this parameter is produced by the vacuum expectation value (vev) of the scalar field φ via $\langle F(\varphi)\rangle = M_{\rm Pl}^2$. In this way, in Ref. [85], the authors analysed the implications of a prototype Brans-Dicke theory (Eq. (1.27)) on the Cosmic Microwave Background (CMB). This is a very good place to test modified theories of gravity because of the excellent agreement between ΛCDM and the CMB power spectrum. For this particular model, they found that the Brans-Dicke parameter ω is constrained to $\omega > 890$, and also an upper bound on the time evolution of the Planck Mass of $\dot{M}_{\rm Pl}/M_{\rm Pl} \sim 10^{-13}$/year, where $\dot{M}_{\rm Pl} = \frac{dM}{dt}$.

In addition, a modified theory of gravity will affect the natural expansion of the Universe. We have already mentioned that scalar field dynamics can lead to an accelerated expansion of the Universe through models such as Quintessence [33, 34] or K-Essence [35, 36], which can be thought of as scalar-tensor theories without a non-minimal coupling to gravity (i.e., setting $G_4 = 0$ and $G_5 = 0$ in Eq. (1.10)). However, focusing on the Brans-Dicke theory from Eq. (1.16), the non-trivial coupling of the scalar field to the trace of the energy-momentum tensor in Eq. (1.26) will have important implications for the dynamics of this scalar field. In particular, as described in Refs. [86, 87], every time a matter field becomes non-relativistic due to the expansion of the Universe, the scalar field gets an influx of kinetic energy that increases temporarily the sound horizon of the Universe. In this paper, the authors relate these peaks to the possible Hubble tension [24], although they require the presence of heavy sterile neutrinos.

- **Compact object tests:**

The modification of gravity can also affect the formation of compact objects, such as black holes or neutron stars, and their interaction. In terms of the formation of these objects, the non-minimal coupling of the extra scalar degree of freedom can lead to what are known as *tachyonic instabilities*. They get their name from the fact that the metric solutions for these objects derived from standard gravity are no longer stable, leading to tachyonic-like perturbations of the field when expanding the fields around these solutions. This is a very active area of research, where the modification of the standard gravity solutions due to non-minimal couplings to scalar fields is called *spontaneous scalarization* [88]. The first such model was the Damour-Esposito-Farèse (DEF) model [89], which has been tightly constrained by pulsar timing techniques [90]. Although this effect does not take place for all scalar-tensor theories, such as for the generic Brans-Dicke theory from Eq. (1.16), it occurs for a wide variety within the Horndeski theory, including those involving the Gauss-Bonet tensor [91], $\mathcal{G}_{\mu\nu} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. However, even in the models in which scalarization is not present, the accretion disk around compact objects can lead to big deviations from standard General Relativity because of the effect that the high energy densities have in the non-minimally coupled field's profile (i.e., see Eq. (1.31)) [92]. Thus, although the astrophysical signatures of most of the extensions of the DEF model are still being developed, the tachyonic instability usually leads to large scalar field amplitudes near very massive objects, which could lead to important signatures in future observations [88].

However, the main source of constraints from these objects comes from the radiation they emit. For example, one of the most important observations came from the detection of gravitational waves generated by the neutron star merger event GW170817/GRB170817A [93]. The key aspect of this event is the simultaneous emission of gravitational and electromagnetic radiation from the collision, which allowed us to place very tight constraints on the speed of gravity, c_T, given by $-3 \cdot 10^{-15} \leq c_T/c - 1 \leq 7 \cdot 10^{-16}$. For Horndeski theories producing the late time accelerated expansion of the Universe (i.e., the scalar field is the dominant component of dark energy), the tensor speed defined in an expanding FLRW-Universe is given by [94]

$$\frac{c_T^2}{c^2} = \frac{G_4 - XG_{5,\varphi} - \ddot{\varphi}G_{5,\varphi}}{G_4 - 2XG_{4,\varphi} + XG_{5,\varphi} - \dot{\varphi}HXG_{5,X}} \tag{1.28}$$

where H is the Hubble parameter indicating the speed of expansion of the Universe. Therefore, the observational requirement $c_T = c$ reduces the original Horndeski action definition in Eq. (1.10) to the following simplified form

$$\mathcal{L}_{\mathrm{H}}^{(c)} = G_4(\varphi)R + G_2(\varphi, X) - G_3(\varphi, X)\Box\varphi. \tag{1.29}$$

Therefore, it admits non-minimal couplings to the Ricci scalar that do not depend on a kinetic coupling to φ, such as Quintessence, K-essence and generic Brans-Dicke theories, Eq. (1.16). It is important to point out that these mentioned bounds do not apply to subdominant Horndeski theories in our Universe, allowing for a less constrained set of G_i functions [95]. For a full study of the validity of other modified theories of gravity, see Refs. [96, 97].

• **Solar System tests:**

Solar system tests have historically been the main source of observational tests of gravity, going all the way back to Newton's theory of gravitation through the confirmation of Kepler's laws, up to Einstein's theory of gravity predicting the precession of Mercury's orbit and the deflection of light by the Sun. Similar experiments are currently being carried out to further constrain the nature of the gravitational interaction, focusing on different aspects than modified gravity.

First, since all the planets orbit the Sun (and therefore are in free fall), the Solar System is the perfect laboratory for studying violations of the equivalence principles. In particular, it is possible to impose tight constraints on the Strong Equivalence Principle (SEP), which states that all laws of physics should be independent of location and velocity (i.e., spacetime is locally flat), implying an equality between the gravitational and inertial masses on free-falling bodies. This affects any modified theory of gravity introducing long-range forces [43], including the generic Brans-Dicke theory, as they would change the effective value for the gravitational mass of an object. Thus, using the Earth and the Moon as test objects, it is possible to measure any SEP deviation with lunar laser ranging experiments, finding a ratio between gravitational and inertial mass of $\Delta(M_G/M_I) = (-2.0 \pm 2.0) \cdot 10^{-13}$ [98]. Moreover, these experiments also help us to constrain the evolution of the non-minimally coupled field φ through its dependence on $M_{\rm Pl}$, setting the rate of change of the gravitational constant at $\dot{M}_{\rm Pl}/M_{\rm Pl} = (-2 \pm 7) \cdot 10^{-13}$/year [99], in agreement with the previously mentioned result from the CMB.

Second, by measuring the frequency shift of photons as they travel large distances through the Solar System, we can directly test for modifications on the gravitational potential because of the non-minimal coupling of φ to the curvature created by the Sun. This experiment, which has placed the tightest constraints yet on possible modifications of gravity, was carried out by the Cassini spacecraft, which was set to send and receive radio wave signals on its way to Saturn. For the prototypical Brans-Dicke theory, these observations constrain the ω parameter to be bounded by $\omega > 40,000$ [100, 101].

- **Laboratory tests:**

Perhaps one of the most famous tests of gravity is the Cavendish experiment, which was the first laboratory experiment that measured the gravitational force between two lead spheres. This provided the first accurate value for the gravitational constant.

In this line of work, there have been numerous improvements of similar experiments that test the nature of gravity at very small scales, each finding perfect agreement with Newton's inverse square law at ranges $> 52.6\mu$m [102] and with masses down to (90.7 ± 0.1)mg [103]. In addition, the Eöt-Wash Group uses torsion-balance experiments which, when their results are combined with the lunar laser ranging measurements [98], give an allowed deviation from the SEP no more than 0.04% [104]. In Figure 1.1, we show the existing bounds on Yukawa-type modifications to Newton's Law of the form

$$V(r) = \frac{Gm_1m_2}{r}\left(1 + \alpha e^{-r/\lambda}\right), \tag{1.30}$$

where α is a coupling constant and λ corresponds to the Compton wavelength for the mediating field. From these experiments, we can constrain a wide class of modified theories of gravity as any deviation from Newton's law introduced through a Yukawa-type coupling will have this specific form, as will be shown in Chapter 3

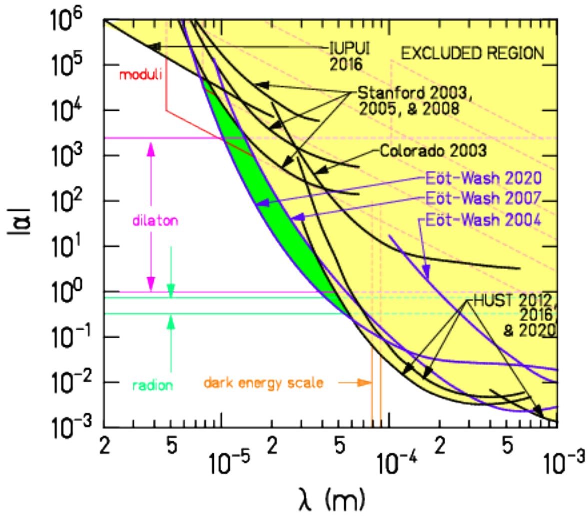


Figure 1.1: *Figure reproduced from Ref. [102], where torsion-balance experiments are used to constrain Yukawa-type modifications to Newton's law. These constraints can be avoided either by increasing the mass of the field or decreasing the coupling constant. We can see that the Eöt-Wash experiments impose the tightest bounds on the parameter space; for a detailed description of all the different lines and experiments in the plot, see Ref. [102].*

when considering the generic Brans-Dicke theory from Eq. (1.16). In particular, we find that fifth forces require a high mass for the non-minimally coupled field (as $\lambda \propto m_\varphi^{-1}$) or/and a weak coupling to matter to satisfy the constraints.

Another experimental probe of modified gravity is provided by atom interferometry experiments [105, 106], which test for new forces acting on individual atoms. These experiments are similar to the double-slit experiment, but the different paths are determined by whether an atom absorbs a photon or not. If it does, that atom absorbs the photon's momentum and is given an upward velocity, being later recombined with the other atoms as it falls back down. Since there is a probability that the atoms will absorb the photon, they will be in a superposition of states in the absence of observation depending on the path taken. In the detector, the interference between the states can be used to infer the acceleration experienced by the excited atom and thus constrain its gravitational interaction with the Earth. These and future experiments, such as the AION project [107], can be used to constrain (or ideally detect) dark energy, ultra-light dark matter and modified theories of gravity [108–110], among others.

In particular, we can see that Solar System and laboratory scale tests impose the tightest constraints on any deviation from standard gravity. However, some modified theories of gravity present so-called *screening mechanisms*, which naturally lead to the suppression of their fifth forces in high density backgrounds (such as the Solar System) without requiring fine tuning of the model parameters. In the next subsection, we will demonstrate these mechanisms by focusing on the chameleon, leaving the subnuclear tests of scalar-tensor theories for Chapter 1.3.

1.2.1 Screening mechanisms: The chameleon field

So far we have assumed the non-minimally coupled field propagates in a spatially homogeneous and empty background, with the exception of the spontaneous scalarization process. Although this effect was only described in the surroundings of very massive compact objects, similar physics contribute to screening mechanisms that may lead to vanishing fifth forces within the Solar System.

The relation between the fifth forces and the local density of the matter sector can be seen directly from the equations of motion. Recalling Eq (1.26), we obtained the following equation of motion for the non-minimally coupled field

$$Z(\varphi)\Box\varphi + \frac{1}{2}Z'(\varphi)\partial_\mu\varphi\partial^\mu\varphi - U'(\varphi) + \frac{F'(\varphi)}{F(\varphi)}\left(T^{\mu\,(\mathrm{m})}_{\mu} + T^{\mu\,(\varphi)}_{\mu} + 3\Box F(\varphi)\right) = 0, \quad (1.31)$$

which has a clear dependence on the background distribution of the matter fields through the trace of the energy-momentum tensor $T^{\mu\,(\mathrm{m})}_{\mu}$. There are multiple screening mechanisms that take advantage of this coupling to suppress the fifth forces. In this section, we will focus on the chameleon mechanism [109, 111, 112], in which the scalar field's effective mass depends on the local density, affecting the interactions mediated by this field.

Although this model is usually defined in the Einstein frame, here we will show its effect working directly in the Jordan frame [113]. The Chameleon action is based on the prototypical Brans-Dicke action from Eq. (1.27), defined as

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{\varphi}{2}R + \frac{w}{2\varphi}\partial_\mu\varphi\partial^\mu\varphi - U_{\mathrm{ch}}(\varphi)\right] + S_{\mathrm{m}}[g_{\mu\nu}]. \tag{1.32}$$

where w is treated as a constant, and the generic Chameleon potential is defined as

$$U_{\mathrm{ch}}(\varphi) = \frac{\Lambda^{4+n}}{\varphi^n}, \tag{1.33}$$

where Λ is a coupling constant not related to the cosmological constant. Substituting this model's parameters into Eq. (1.31), we find the equations of motion for this case are

$$\frac{3+2w}{2}\Box\varphi = \varphi U'_{\mathrm{ch}}(\varphi) - 2U_{\mathrm{ch}}(\varphi) + \frac{T^{\mu(\mathrm{m})}_{\mu}}{2}. \tag{1.34}$$

Here, we can see that the right-hand side can be treated as an effective potential that will depend on the matter density background. In this way, we can define the simple Klein-Gordon equation

$$\Box\varphi - \tilde{U}'_{\mathrm{ch}}(\varphi) = 0, \tag{1.35}$$

with

$$\tilde{U}'_{\mathrm{ch}}(\varphi) = \frac{2}{3+2w}\left(\varphi U'_{\mathrm{ch}}(\varphi) - 2U_{\mathrm{ch}}(\varphi) + \frac{T^{\mu(\mathrm{m})}_{\mu}}{2}\right). \tag{1.36}$$

Perturbing this field about the local minimum, such that $\varphi \to \varphi + \bar{\varphi}(\rho)$, this equation can be represented as

$$\Box\varphi - \hat{m}^2_{\varphi}(\rho)^2\varphi = 0, \tag{1.37}$$

where

$$\hat{m}_{\varphi}^{2}(\rho)^{2} = n(n+2)^{\frac{1}{n}} \frac{\rho^{\frac{n+1}{n}}}{(2\Lambda^{n+4})^{\frac{n+1}{n}}} \tag{1.38}$$

is the effective mass of the non-minimally coupled field, and we have assumed $T_{\mu}^{\mu(\mathrm{m})} = \rho$ to be the energy density of the matter fields. Notice that the allowed limits of the exponent n for this equation differ from the ones in the literature as we carried out the calculation directly in the Jordan frame, instead of going to the Einstein frame, meaning that the original definition of the potentials differ by the rescaling of the action. From this, we can calculate the Yukawa potential just by solving for φ in Eq. (1.37). After some algebra, we find

$$V_5(r) = -\frac{1}{4\pi r}\frac{M^2}{M_{\mathrm{Pl}}^2} e^{-\hat{m}(\rho) r}, \tag{1.39}$$

where M is a constant mass scale. Thus, substituting in the value for the chameleon field's mass in Eq. (1.38), we see that fifth forces will be suppressed in high-density backgrounds, and so naturally satisfying bounds such as those in Figure 1.1. In this derivation, we assumed ρ to be very slowly varying in space. However, obtaining a precise estimate of the screening profile of the field in realistic regions, such as galaxies, clusters or cosmic voids [114], is an active area of research.

Other screening mechanisms also take advantage of the coupling to the energy-momentum tensor of the matter fields to suppress the fifth forces in high-density regimes. However, they differ from the chameleon mechanism by the fact that they do not increase the mass of the scalar field to do so. For example, the symmetron mechanism [115, 116] uses a double-well potential that sets $\omega \to \infty$ in Eq. (1.27) in high-density regimes. Before canonically normalizing the non-minimally coupled field, this limit for ω freezes the scalar field, recovering a standard Einstein-Hilbert action. However, in terms of the Yukawa potential in Eq. (1.39), the $\omega \to \infty$ limit is equivalent to a suppression of the coupling constant to matter (i.e., taking

$M \to 0$). Another example is the Vainshtein mechanism [117–119], which takes place in massive theories of gravity, such as dRGT [120, 121]. In these theories, the non-linear self-couplings of gravity become important in high density backgrounds, eliminating any modification to the dynamics of the matter sector.

In summary, screening mechanisms allow for certain scalar-tensor theories to have effects on cosmological scales while suppressing any modification to standard gravity in local environments. For recent reviews on experimental and observational constraints on screened fifth forces, see Refs. [122, 123]. In the next section, we will explore the possibility of further constraining extensions of gravity by studying them on subatomic scales, where some processes exist that can be used to *see through* screening mechanisms.

1.3 Testing scalar-tensor theories in particle physics

Previously in Chapter 1.1.1 we described how the effect of a modified gravity on the matter sector can be expressed in terms of a Beyond the Standard Model theory. As such, we may also set constraints by directly studying the effects that the extra scalar field φ has on particle physics, thus testing scalar-tensor theories of gravity using subatomic scale experiments.

In recent years, there have been many proposals with different implications that extended theories of gravity would have on these tiny scales. For example, in Ref. [124], the authors consider the effect that chameleons (the associated particle with the chameleon field) produced in the Sun by strong magnetic fields would have on dark matter detectors such as the XENON1T [125, 126], finding that for a wide range of masses they could detect chameleons. It is important to clarify that most dark matter experiments testing for new couplings to the photon are in principle not viable for the generic Brans-Dicke theory [Eq. (1.16)], including the chameleon

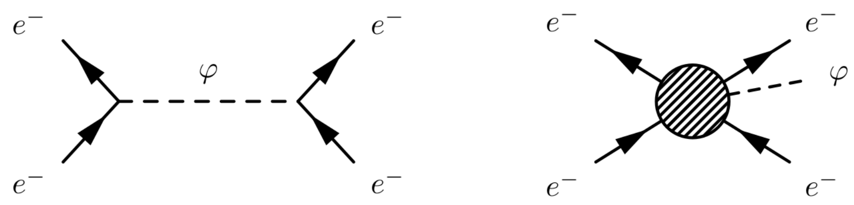


Figure 1.2: *Examples for modifications due to BSM physics in a Møller scattering. On the left, we can see that the new physics (φ) introduces a new channel mediated by the additional scalar. On the right, the new field only appears through a modification to the vertices, such that, while all the internal lines (hidden in the blob) are SM fields, the modifications lead to an extra particle being produced in the final state, producing missing energy signals.*

field presented in Chapter 1.2.1. This is because for these models the coupling to the matter sector depends on the trace of its energy-momentum tensor, which classically vanishes for the photon[2] (and also for any scale invariant quantity, as we will show at the end of Chapter 2.2). However, any scalar-tensor theory connected to the Einstein frame through disformal transformations of the form of Eq. (1.20), as considered in Ref. [124], has a non-vanishing coupling to gauge fields. Furthermore, the non-minimally coupled field could affect some of the experiments used to measure fundamental properties of the Standard Model particles. For example, the chameleon field may affect the measured magnetic moment of fermions [128] (linking it to the Fermilab (g-2) experiment [17]) or shift the energy levels of light atoms, such as hydrogen or muonium [129].

However, when studying modifications of the Standard Model through BSM theories, the most definitive way to test for them is to study their effect in particle colliders [130, 131]. When considering scalar-tensor theories, we do not expect them to generate direct evidence for the extra scalar field, as it cannot interact with a detector due to its neutrality. However, there are still ways to infer its existence indirectly from the data [132, 133]. Depending on the role that the extra field plays in the scattering process, we can distinguish two different types of signatures:

[2] However, such interactions can arise through quantum anomalies [127].

Internal line: These processes correspond to those in which the extra field appears as a virtual particle (i.e., as a mediator), providing a new channel of interaction through which Standard Model particles can interact. Therefore, such BSM interactions would have an impact on the resonances of the differential cross sections of the Standard Model scattering amplitudes, which is an active field of study of the LHC [134]. In Figure 1.2, we show an example of such a modification in a Møller scattering ($e^- + e^- \to e^- + e^-$), which in the non-relativistic limit would lead to a Yukawa potential of the form of the one obtained in Eq. (1.39).

External line: In this case, the additional field of the BSM theory is only present in the final particle excitation states, while all interactions (and hence virtual lines) are mediated by Standard Model fields. This corresponds to a modification on the external vertices of the scattering amplitude. As an example, we can consider an extension of the Møller scattering mentioned above, where an extra particle is produced in the final state, such as ($e^- + e^- \to e^- + e^- + \varphi$). Since the extra scalar particle will not be detected, such processes would indicate a mismatch between the incoming and outgoing total energy and momentum. Finding modifications of the Standard Model through these processes requires very precise theoretical predictions, and is common practice when testing for supersymmetric particles [135] and dark matter candidates [136]. In addition, depending on the mass of the scalar field, once the extra particle has been produced, it may later decay into other Standard Model particles that may be detected. These are called *long-lived particles*, and are the main object of study for the FASER detector [137] in CERN.

In a real process we expect both types of modification to occur simultaneously. However, while the chameleon type of screening mechanism would affect the internal lines (as the effective mass contributes to the field's propagator), it would not modify the interaction couplings with the Standard Model. Therefore, the described missing energy signals would not be screened by chameleon mechanisms, given that the effective mass of the produced particle in the final state would not affect the scattering amplitude itself. Similar arguments can be applied to Higgs-portal theories [138, 139].

In this thesis, we will focus on formalising and automating the process of studying scalar-tensor theories in particle colliders (although our work will be also beneficial for many areas of subatomic tests). In order to calculate the exact amplitudes resulting from scattering processes (such as those in Figure 1.2), we must first obtain the exact modifications and new interactions appearing in the Standard Model Lagrangian by using quantum field theory. So far, such calculations for testing modified gravity effects at colliders have already been considered, but using simplified models that focus on reduced sectors of the Standard Model [140–142]. We are still lacking a systematic approach that allows us to test the implications of any scalar-tensor theory on the whole Standard Model.

Notice that, even without modifications of the gravitational sector, generating accurate predictions involving the whole Standard Model is computationally demanding. Therefore, to automate these types of calculations, we introduce `FeynMG` [2], a Mathematica package that, working within the environment of `FeynRules` [143], helps the user to perform all the necessary algebra to obtain the BSM description from any scalar-tensor theory. Furthermore, being within `FeynRules` makes possible the connection of scalar-tensor theories with the rest of particle phenomenology software analysis packages, allowing the user to obtain precise predictions of the modifications on the Standard Model at colliders. Furthermore, once the Lagrangian for the BSM description of a scalar-tensor theory is obtained, we can also use theoretical arguments to test the consistency of the model, such as checking whether unitarity and causality are maintained [144].

Outline

This thesis is intended to be both explicit and pedagogical. As such, we begin in Chapter 2 with a review of how to separate the mixing between gravity and the additional scalar field, and so obtain the Lagrangian for the BSM description of the scalar-tensor theory, working both in the Einstein and the Jordan frames. In this calculation, we will pay special attention to identifying the relationship between the scale invariance of the matter sector and the subsequent modifications due to the extension of gravity. Then, in Chapter 3, we consider the long-range modifications to the Møller scattering due to the modification of gravity. For that, we obtain the Yukawa potential in the non-relativistic limit, concentrating on systems where the scale symmetry is broken either explicitly or dynamically. Finally, in Chapter 4, after describing the state of the art for using symbolic algebra to do particle phenomenology, we present `FeynMG` [2] with detailed examples on its main functions and routines. Our conclusions are presented in Chapter 5. Additional details are provided in the Appendices.

Chapter 2

Scalar-tensor theories as BSM physics

When considering extensions of the Standard Model using particle theory, we are used to working directly in Minkowski spacetime since the corrections from gravity are highly suppressed. However, throughout this chapter, we will make the case that even a Minkowski quantum field theory that is solely minimally coupled to gravity can give rise to new interactions with additional scalar fields that are non-minimally coupled to the scalar curvature of the gravity sector.

In this way, once the BSM description is obtained, the new dynamics appearing in the matter sector because of the modification of gravity can be calculated by using field theory to compute scattering amplitudes. For simplicity, we will work with a toy model of QED plus a real scalar prototype of the Higgs sector, such that our arguments extend to fields with spins 0, 1/2 and 1. Generalizing to a complex scalar field that is charged under U(1) would be a technical complication that does not add to the main points that we wish to illustrate below. The action of this model in Minkowski spacetime is given by

$$S_{\mathrm{m}} = \int \mathrm{d}^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right.$$
$$\left. + \, i\bar{\psi}\gamma^\mu D_\mu \psi - y\bar{\psi}\phi\psi - V(\phi) \right], \tag{2.1}$$

where we have introduced a would-be Higgs field ϕ, a Dirac fermion ψ, which will later be chosen as a proxy for the electron, and the U(1) gauge field A_μ, which corresponds to the photon, with its usual field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Notice that the potential for the ϕ field, $V(\phi)$, is not specified at this stage. We will keep it generic until the next section, in which we will show that the potential new interactions will be sensitive to the exact form of this function.

The Dirac fermion is charged under U(1), and it is minimally coupled to the photon field via the gauge covariant derivative

$$D_\mu \psi = \partial_\mu \psi + iqA_\mu \psi, \tag{2.2}$$

where q is the would-be electromagnetic coupling.

Before analyzing the interactions induced by extending the gravitational sector beyond the usual Einstein-Hilbert action, we first need to insert all the minimal gravitational couplings that have so far been ignored by working in Minkowski spacetime. This means that, for every pair of contracted Lorentz indices, we must include a factor of the metric $g^{\mu\nu}$. Additionally, for every γ matrix appearing in the Dirac Lagrangian, we must include a vierbein e^μ_a, which satisfies $\eta^{ab} e^\mu_a e^\nu_b = g^{\mu\nu}$, where η^{ab} is the flat spacetime metric. (We remind the reader that the flat-space indices of the vierbein are raised and lowered with the flat-space metric.) The latter is necessary since the algebra of the γ matrices is defined with respect to the Minkowski metric, i.e., $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$; the vierbeins relate the curved and flat, tangent spaces. By this means, we obtain the minimally coupled action

$$S_{\mathrm{m}}[g_{\mu\nu}] = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{4}g^{\alpha\mu}g^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \right.$$
$$\left. + i\bar{\psi}e^\mu_a\gamma^a\nabla_\mu\psi - y\bar{\psi}\Phi\psi - V(\Phi)\right], \tag{2.3}$$

where we have also included a factor of $\sqrt{-g}$ in the spacetime volume element. Herein, the Minkowski gauge covariant derivative has been promoted to the general covariant derivative.

For scalar fields, the gravitational covariant derivative just trivially reduces to a partial derivative, such that $\nabla_\mu\Phi \to \partial_\mu\Phi$. However, when acting on a covector Y_ρ, the covariant derivative takes the form

$$\nabla_\mu Y_\nu = \partial_\mu Y_\nu - \Gamma^\rho_{\mu\nu}Y_\rho, \tag{2.4}$$

where $\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$ are the Christoffel symbols introduced in Chapter 1. So far, this definition for the covariant derivative has been chosen such that $\nabla_\rho g_{\mu\nu} = 0$, which ensures that the connection vanishes in the absence of a gravitational force, but it can take many other forms. For instance, we will later introduce a different choice that will be more convenient for the specific case of Brans-Dicke theories [1]. However, it does not matter which definition one uses in this action, given that the following property will always hold

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.5}$$

since the curvature-dependent terms are symmetric under the permutation of μ and ν. Finally, the covariant derivative acting on a fermion field, including the dependence on the gauge field, is given by

$$\nabla_\mu \psi = \partial_\mu \psi + iqA_\mu \psi - \frac{i}{2}\Omega_\mu \psi, \tag{2.6}$$

where

$$\Omega_\mu = (\Gamma_{ab})_\mu S^{ab} \tag{2.7}$$

is the spin connection, crucial for keeping the fermionic kinetic energy Hermitian and scale-invariant on curved backgrounds. The latter is defined by

$$(\Gamma_{ab})_\mu = e_{a\alpha} e_b^\beta \Gamma^\alpha_{\mu\beta} + e_{a\alpha}\partial_\mu e_b^\alpha, \qquad \text{where} \qquad S^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]. \tag{2.8}$$

We can now proceed to append the gravitational sector. For this, we will choose the generic Brans-Dicke action from Eq. (1.16), for which we obtain

$$\begin{aligned} S = \int \mathrm{d}^4x \sqrt{-g} \Big[&-\frac{F(\varphi)}{2} R + \frac{1}{2} Z(\varphi) g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \\ &- \frac{1}{4} g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta}F_{\mu\nu} + \frac{1}{2} g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \\ &+ i\bar{\psi} e^\mu_a \gamma^a \partial_\mu\psi + \frac{1}{2}\bar{\psi} e^\mu_a \gamma^a \Omega_\mu \psi \\ &- q\bar{\psi} e^\mu_a \gamma^a A_\mu \psi - y\bar{\psi}\Phi\psi - U(\Phi,\varphi) \Big], \end{aligned} \tag{2.9}$$

where the potentials for φ and Φ have been absorbed into $U(\Phi, \varphi)$ to account for possible interactions between the scalar fields. However, even in the case where there is not an explicit coupling between the scalar fields, we will find that new interactions of this kind might appear in the Beyond the Standard Model description.

In this chapter, we will show this in two different ways. First by performing a Weyl transformation to the Einstein frame, in which the modifications to the gravitational sector will be present on the matter Lagrangian, leaving a canonical Einstein-Hilbert gravitational sector. Then, we will replicate the calculation working directly in the Jordan frame, where we will have to perturb the modified theory of gravity around Minkowski spacetime.

2.1 Weyl transforming to the Einstein frame

This method consists of a redefinition of the curvature-dependent objects (i,e,. the Weyl transformation from Eq. (1.17)) such that the resulting gravitational action does not present any non-minimal couplings. For the Lagrangian defined in Eq. (2.9), this transformation will take the following form

$$g_{\mu\nu} \rightarrow \frac{\tilde{M}_{\rm Pl}^2}{F(\varphi)}\tilde{g}_{\mu\nu}, \qquad g^{\mu\nu} \rightarrow \frac{F(\varphi)}{\tilde{M}_{\rm Pl}^2}\tilde{g}^{\mu\nu}, \tag{2.10a}$$

$$e^a_\mu \rightarrow \frac{\tilde{M}_{\rm Pl}}{\sqrt{F(\varphi)}}\tilde{e}^a_\mu, \qquad e^\mu_a \rightarrow \frac{\sqrt{F(\varphi)}}{\tilde{M}_{\rm Pl}}\tilde{e}^\mu_a, \tag{2.10b}$$

where $\tilde{g}_{\mu\nu}$, $\tilde{e}^\mu_a$ and $\tilde{M}_{\rm Pl}$ are the metric, vierbein and Planck mass defined in the Einstein frame, respectively. To get through the algebra, the following two transformations will be useful:

$$\sqrt{-g}\frac{F(\varphi)}{2}R \rightarrow \sqrt{-\tilde{g}}\left(\frac{\tilde{M}_{\rm Pl}^2}{2}\tilde{R} - \frac{3\tilde{M}_{\rm Pl}^2 F'(\varphi)^2}{4F(\varphi)^2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right), \tag{2.11a}$$

$$e^\mu_a\gamma^a\Omega_\mu \rightarrow \frac{\sqrt{\varphi}}{\tilde{M}_{\rm Pl}}\tilde{e}^\mu_a\gamma^a\left(\tilde{\Omega}_\mu - \frac{3i}{2}\frac{F'(\varphi)}{F(\varphi)}\partial_\mu\varphi\right), \tag{2.11b}$$

where $F'(\varphi) = \partial F(\varphi)/\partial\varphi$ and all the curvature-dependent quantities with a tilde are built with the Einstein-frame metric $\tilde{g}_{\mu\nu}$ or vierbein $\tilde{e}^\mu_a$.

Applying the transformations in Eq. (2.10) to the Jordan-frame action, we obtain

$$S = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\,\Bigg[-\frac{\tilde{M}_{\mathrm{Pl}}^2}{2}R + \frac{\tilde{M}_{\mathrm{Pl}}^2}{2}\left[\frac{Z(\varphi)}{F(\varphi)} + \frac{3F'(\varphi)^2}{2F(\varphi)^2}\right]\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$$
$$- \frac{1}{4}\tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} + \frac{\tilde{M}_{\mathrm{Pl}}^2}{2F(\varphi)}\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - q\frac{\tilde{M}_{\mathrm{Pl}}^3}{F(\varphi)^{3/2}}\bar{\psi}\tilde{e}^\mu_a\gamma^a A_\mu\psi$$
$$+ i\frac{\tilde{M}_{\mathrm{Pl}}^3}{F(\varphi)^{3/2}}\bar{\psi}\tilde{e}^\mu_a\gamma^a\partial_\mu\psi + \frac{1}{2}\frac{\tilde{M}_{\mathrm{Pl}}^3}{F(\varphi)^{3/2}}\bar{\psi}\tilde{e}^\mu_a\gamma^a\psi\left(\tilde{\Omega}_\mu - \frac{3i}{2}\frac{F'(\varphi)}{F(\varphi)}\partial_\mu\varphi\right)$$
$$- \frac{\tilde{M}_{\mathrm{Pl}}^4}{F(\varphi)^2}\Big(y\bar{\psi}\Phi\psi + U(\Phi,\varphi)\Big)\Bigg], \tag{2.12}$$

wherein we have recovered a canonical Einstein-Hilbert term for the gravitational action. However, all the couplings of the Brans-Dicke scalar arising from the modification of gravity now appear explicitly in the matter Lagrangian. Notice, in particular, that most of the kinetic energies of the fields are not canonically normalized due to these new couplings.

As a starting point, one can canonically normalize the field φ by solving the integral

$$\tilde{\chi}(\varphi) \equiv \tilde{M}_{\mathrm{Pl}}\int_{\varphi_0}^{\varphi}\mathrm{d}\hat{\varphi}\sqrt{\frac{Z(\hat{\varphi})}{F(\hat{\varphi})} + \frac{3F'(\hat{\varphi})^2}{2F(\hat{\varphi})^2}}, \tag{2.13}$$

where φ_0 is taken to be zero without loss of generality. For the rest of the fields, we rescale them according to their classical scaling dimension, i.e.,

$$\psi \to \sqrt{\frac{\tilde{F}(\tilde{\chi})^{3/2}}{\tilde{M}_{\mathrm{Pl}}^3}}\tilde{\psi}, \qquad \Phi \to \frac{\sqrt{\tilde{F}(\tilde{\chi})}}{\tilde{M}_{\mathrm{Pl}}}\tilde{\Phi}, \tag{2.14}$$

where $\tilde{F}(\tilde{\chi}) \equiv F(\varphi)$. With this, the Lagrangian takes the following form:

$$\mathcal{L} = -\frac{\tilde{M}_{\mathrm{Pl}}^2}{2}R + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi} - \frac{1}{4}\tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}F_{\alpha\beta}F_{\mu\nu}$$
$$+ i\bar{\tilde{\psi}}\tilde{e}^\mu_a\gamma^a\tilde{\nabla}_\mu\tilde{\psi} + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi} - y\bar{\tilde{\psi}}\tilde{\Phi}\tilde{\psi} - \frac{\tilde{M}_{\mathrm{Pl}}^4}{\tilde{F}(\tilde{\chi})^2}\tilde{U}(\tilde{\Phi},\tilde{\chi})$$
$$+ \frac{1}{2}\frac{\tilde{F}'(\tilde{\chi})}{\tilde{F}(\tilde{\chi})}\tilde{\Phi}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\chi} + \frac{1}{8}\left(\frac{\tilde{F}'(\tilde{\chi})}{\tilde{F}(\tilde{\chi})}\right)^2\tilde{\Phi}^2\tilde{g}^{\mu\nu}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi}, \tag{2.15}$$

where $\tilde{U}(\tilde{\Phi},\tilde{\chi}) \equiv U(\Phi,\varphi)$, $\tilde{F}'(\tilde{\chi}) = \partial\tilde{F}(\tilde{\chi})/\partial\tilde{\chi}$ and, for simplicity, we have regrouped the covariant derivative,

$$\tilde{\nabla}_\mu\tilde{\psi} = \partial_\mu\tilde{\psi} + iqA_\mu\tilde{\psi} - \frac{i}{2}\tilde{\Omega}_\mu\tilde{\psi}. \tag{2.16}$$

Thus, one of the main inconveniences of working in the Einstein frame is that it loses the simplicity of the Lagrangian defined in the Jordan frame. This is because the Weyl transformation and the redefinition of the fields introduces factors of $\tilde{F}(\tilde{\chi})$ throughout the Lagrangian, which, in regimes where we can make a series expansion of $\tilde{F}(\tilde{\chi})$, will introduce infinite towers of operators that involve the SM fields and increasing powers of the scalar field $\tilde{\chi}$.

Let us now draw our attention to the potential term for the would-be Higgs field, $\tilde{\Phi}$. As we can see, the rescaling from Eq. (2.14) introduces a factor of $\sqrt{\tilde{F}(\tilde{\chi})}$ on each Φ-dependent term. However, this dependence on $\tilde{F}(\tilde{\chi})$ is exactly inversely proportional to the prefactor of $\tilde{U}(\tilde{\Phi},\tilde{\chi})$ in the second line of Eq. (2.15), which arose from the Weyl transformation. Therefore, depending on our choice for the potential, this rescaling might lead to a total cancellation of this coupling between $\tilde{\chi}$ and $\tilde{\Phi}$. For instance, let's study the case for the well-known Higgs-like double-well potential

$$V(\Phi) = -\frac{1}{2}\mu^2\tilde{\Phi}^2 + \frac{\lambda}{4!}\tilde{\Phi}^4 - \frac{3}{2}\frac{\mu^4}{\lambda}, \tag{2.17}$$

where we can see that the constant term ensures that the vacuum has zero energy density in the symmetry-broken phase. This function will transform under the Weyl transformation and Eq. (2.14) into

$$\frac{\tilde{M}_{\rm Pl}^4}{\tilde{F}(\tilde{\chi})^2} V\left(\frac{\sqrt{\tilde{F}(\tilde{\chi})}}{\tilde{M}_{\rm Pl}}\tilde{\Phi}\right) = -\frac{1}{2}\mu^2\frac{\tilde{M}_{\rm Pl}^2}{\tilde{F}(\tilde{\chi})}\tilde{\Phi}^2 + \frac{\lambda}{4!}\tilde{\Phi}^4 - \frac{3}{2}\frac{\mu^4}{\lambda}\frac{\tilde{M}_{\rm Pl}^4}{\tilde{F}(\tilde{\chi})^2}, \tag{2.18}$$

so that only the quartic term, $\lambda\tilde{\Phi}^2/4!$, stays invariant.

At this point, we can already make an important observation: The couplings between the SM fields and the scalar field $\tilde{\chi}$ arise only through the scalar kinetic terms and terms with dimensionful parameters, i.e., those terms that are not invariant under Weyl transformations. Thus, for the Standard Model (illustrated already by the toy model described here with the double-well potential from Eq. (2.17)), the modifications to the dynamics from the new scalar field $\tilde{\chi}$ are, in the Einstein frame, communicated by the Higgs sector, both via kinetic and mass mixings. However, new interactions arising through kinetic mixings when these involve a field with non-zero mass are suppressed due to the additional momentum dependence ($\propto q^2$) that occurs for each insertion into the matrix element of the kinetic mixing operator. As a result, the mass mixing will provide the dominant fifth force. In this way, there are strong parallels between the Brans-Dicke-type scalar-tensor theories and Higgs portal theories (see Ref. [138]).

Even if the original matter Lagrangian is only minimally coupled to gravity in the Jordan frame, there can be experimentally testable modifications to the force laws that depend on the dynamics of the new scalar field that need not be Planck suppressed. However, before calculating the exact form of these modifications, we will first replicate the transformation of a scalar-tensor theory into its equivalent BSM description by working directly in the Jordan frame.

2.2 Staying in the Jordan frame

In this frame, the modifications to the interactions between the fields of the matter sector arise through the modified gravitational sector itself, so we proceed by perturbing the metric around a flat spacetime [145–147]. Expanding the metric up to leading order in perturbations corresponds to

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.19a}$$
$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \ldots, \tag{2.19b}$$

where $\eta_{\mu\nu}$ is the usual flat spacetime metric and $h_{\mu\nu}$ is the perturbation in the metric, which, once quantized, corresponds to the graviton. The higher order terms in the expansion of $g_{\mu\nu}$ are necessary to satisfy $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$ to all orders.

Note that, although quantum field theory and General Relativity cannot be reconciled, it is possible to quantize gravity as long as we work in the gravitational weak-field limit, where we can ignore the infinite tower of higher order terms that lead to the non-renormalization of gravity. In this way, we arrive at the following weak-field expansions:

$$\sqrt{-g}^{(1)} \approx 1 + \frac{1}{2}\eta_{\mu\nu}h^{\mu\nu}, \tag{2.20a}$$
$$\Gamma^{\mu(1)}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\lambda}(\partial_\alpha h_{\lambda\beta} + \partial_\beta h_{\alpha\lambda} - \partial_\lambda h_{\alpha\beta}), \tag{2.20b}$$
$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\partial^\rho\partial_\mu h_{\nu\rho} + \partial^\rho\partial_\nu h_{\mu\rho} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right), \tag{2.20c}$$
$$\begin{aligned} R^{(2)}_{\mu\nu} = {} & \frac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\mu\partial_{(\nu}h_{\rho)\sigma} + \frac{1}{4}\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} \\ & + \partial^\sigma h^\rho_\nu \partial_{[\sigma}h_{\rho]\mu} + \frac{1}{2}\partial_\sigma(h^{\sigma\rho}\partial_\rho h_{\mu\nu}) - \frac{1}{4}\partial^\rho h \partial_\rho h_{\mu\nu} \\ & - (\partial_\sigma h^{\sigma\rho} - \frac{1}{2}\partial^\rho h)\partial_{(\mu}h_{\nu)\rho}, \end{aligned} \tag{2.20d}$$

where the exponent in parenthesis shows the order in the metric fluctuations h, and we have used the following index symmetrization and antisymmetrization notation

$$\begin{aligned} h_{(\mu\nu)} = {} & \frac{1}{2}(h_{\mu\nu} + h_{\nu\mu}) \\ h_{[\mu\nu]} = {} & \frac{1}{2}(h_{\mu\nu} - h_{\nu\mu}). \end{aligned} \tag{2.21}$$

For completeness, we will first show how to perturb around a Minkowski background the standard Einstein-Hilbert gravitational action, so that we can later compare it with the Brans-Dicke case. Starting with

$$S_{\mathrm{EH}} = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{M_{\mathrm{Pl}}^2}{2}R\right], \tag{2.22}$$

and perturbing up to second order in $h_{\mu\nu}$, we find

$$\mathcal{L}_{\mathrm{EH}} = \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial^\mu h_{\mu\nu}\partial^\nu h - \frac{1}{2}\partial^\mu h_{\mu\rho}\partial_\nu h^{\nu\rho}\right). \tag{2.23}$$

We can see that the Ricci scalar corresponds to the kinetic energy source for the graviton, and its non-kinetic interactions will be given by the minimal couplings of the matter sector to gravity. It still remains to fix a gauge, and one choice is the harmonic gauge, which satisfies the following condition:

$$\nabla_\mu\nabla^\mu = \partial_\mu\partial^\mu \to g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0. \tag{2.24}$$

As for any quantum field theory, one cannot directly substitute this constraint into the Lagrangian, as this would eliminate all possible off-shell effects. In order to consistently introduce a gauge-fixing condition into the Lagrangian we need to add a term to the action that encodes the desired constraint in its equations of motion. For the harmonic gauge in Eq. (2.24), the best choice for such a term (which preserves the Lorentz invariance of the action) is

$$\mathcal{L}_{\mathrm{GF}} = \frac{M_{\mathrm{Pl}}^2}{4}\xi g_{\mu\nu}\Gamma^\mu\Gamma^\nu, \tag{2.25}$$

where $\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}$ and the prefactor is chosen such that the effects of this term are of the order of the Ricci scalar. Herein, ξ is a constant Lagrange multiplier that is introduced so that, when varying the action with respect to it, we obtain the desired gauge constraint in the equations of motion. However, we will set $\xi = 1$ without loss of generality.

Adding the gauge fixing term to the Einstein-Hilbert action in Eq. (2.22) and perturbing gravity up to second order $h_{\mu\nu}$ leads to the familiar Fierz-Pauli Lagrangian [145], given by

$$\mathcal{L}_{\rm FP} = \frac{M_{\rm Pl}^2}{4}\left(\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h\right), \tag{2.26}$$

which needs to be canonically normalized by rescaling the graviton via $h_{\mu\nu} \to (2/M_{\rm Pl})h_{\mu\nu}$.

Lets now turn our attention to the gravitational sector for Brans-Dicke-type theories [Eq. (1.16)], with action

$$S_{\rm G} = \int {\rm d}^4x\sqrt{-g}\left[-\frac{F(\varphi)}{2}R\right]. \tag{2.27}$$

Linearizing gravity, we obtain the following expansion up to second order in $h_{\mu\nu}$:

$$\begin{aligned}\mathcal{L}_{\rm G} =& \frac{F(\varphi)}{2}\left(\frac{1}{4}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial^\mu h_{\mu\nu}\partial^\nu h - \frac{1}{2}\partial^\mu h_{\mu\rho}\partial_\nu h^{\nu\rho}\right)\\ &- \frac{F'(\varphi)}{4}\partial_\mu\varphi\partial^\mu h + \frac{F'(\varphi)}{2}\partial_\mu\varphi\partial_\nu h^{\mu\nu}.\end{aligned} \tag{2.28}$$

We can see that the modification of gravity has not only replaced $M_{\rm Pl}^2 \to F(\varphi)$ in the linearized Eintein-Hilbert action from Eq. (2.23), but it has also introduced two new kinetic interaction terms between h and φ that vanished in the standard gravity case since they were total derivatives.

Now, as for the Einstein-Hilbert action, we need to gauge-fix this linearized theory in Eq. (2.28). At first glance, it is already clear that the harmonic gauge choice from Eq. (2.26) will not take us to the Fierz-Pauli-like kinetic energy for the modified spin-2 degree of freedom. The most straightforward way of solving this issue is to also make the replacement $M_{\rm Pl}^2 \to F(\varphi)$ to the harmonic gauge itself. However, in this calculation, we will use a different gauge that will be proven to be convenient for the calculations in the Jordan frame [see below Eq. (2.33)]: one that maps to the harmonic gauge when performing the Weyl transformation to the Einstein frame.[1] This can be achieved by redefining the covariant derivative such that its action on a covector Y_ν is as follows:

$$\mathcal{D}_\mu Y_\nu = \partial_\mu Y_\nu - \Gamma^\rho_{\mu\nu} Y_\rho - C^\rho_{\mu\nu} Y_\rho, \tag{2.29}$$

where

$$C^\rho_{\mu\nu} = \frac{F'(\varphi)}{2F(\varphi)}(\delta^\rho_\mu \partial_\nu \varphi + \delta^\rho_\nu \partial_\mu \varphi - g_{\mu\nu}\partial^\rho \varphi). \tag{2.30}$$

This modified covariant derivative will map to ∇_μ when going to the Einstein frame and satisfies the identity $\mathcal{D}_\rho(F(\varphi)g_{\mu\nu}) = 0$ while preserving diffeomorphism invariance in the action, as shown in Ref. [1, 69, 148, 149]. We can then define a *scalar-harmonic gauge* condition in terms of the new covariant derivative, namely

$$\mathcal{D}^\mu \mathcal{D}_\mu = \partial^\mu \partial_\mu \to g^{\mu\nu}\Gamma^\rho_{\mu\nu} - \frac{F'(\varphi)}{F(\varphi)}\partial^\rho \varphi = 0. \tag{2.31}$$

Following the same steps as for the Einstein-Hilbert action, we can introduce this condition into the system by adding the following term into the Lagrangian:

$$\mathcal{L}_{\rm gf} = \frac{F(\varphi)}{4}\xi g_{\alpha\beta}\left[g^{\mu\nu}\Gamma^\alpha_{\mu\nu} - \frac{F'(\varphi)}{F(\varphi)}\partial^\alpha\varphi\right]\left[g^{\sigma\rho}\Gamma^\beta_{\sigma\rho} - \frac{F'(\varphi)}{F(\varphi)}\partial^\beta\varphi\right]. \tag{2.32}$$

where once again ξ is a Lagrange multiplier that will generate the scalar-harmonic gauge constraint in the equations of motion when perturbing the action with respect

[1] This argument can also be applied when gauge fixing other fields, making them conformally invariant (and so avoiding the modifications from entering through the gauge fixing term). See Appendix A for a demonstration of working with the U(1) gauge field.

to it.

It is important to point out that this is not the first time this gauge has been used. For instance, it is mentioned in its linearized form by Fuji and Maeda [43] and used in a number of papers where the authors employ the frame-covariant Vilkovisky-DeWitt approach [47, 69, 148–150], which, by making the appropriate set of field redefinitions, an explicit frame invariance is achieved within the action that can be undone once a frame is chosen. However, here, unlike in previous papers, we introduce the gauge fixing term at the action level, using its complete metric form, which allows us to perturb consistently to higher orders in the fluctuations.

Inserting this gauge fixing term (Eq. (2.32)) into the full Brans-Dicke action defined in Eq. (2.9), we obtain the following Lagrangian after linearizing up to first order in $1/\sqrt{F(\varphi)}$ (noting that $M_{\rm Pl}^2 = \langle F(\varphi)\rangle$):

$$\begin{aligned}\mathcal{L} =& \frac{F(\varphi)}{4}\left[\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h\right] - \frac{F'(\varphi)}{4}\partial_\mu h\partial^\mu\varphi \\ &-\frac{1}{2}\left[Z(\varphi) + \frac{F'(\varphi)}{2F(\varphi)}^2\right]\partial_\mu\varphi\partial^\mu\varphi - U(\varphi) + \frac{1}{2}h^{\mu\nu}T_{\mu\nu} + \mathcal{L}_{\rm m}[\eta_{\mu\nu}],\end{aligned} \tag{2.33}$$

where $T_{\mu\nu}$ is the matter energy-momentum tensor as defined in Eq. (1.7).

Herein, we have recovered the usual kinetic energy terms of the graviton, as they appear in the Fierz-Pauli Lagrangian in Eq. (2.26), with the exception that non-minimal couplings to the field φ appear through the overall factor of $F(\varphi)$. One of the benefits of the scalar-harmonic gauge (Eq. (2.32)) is that it has eliminated the kinetic interaction between φ and the non-traced graviton (i.e., the last term on Eq. (2.28)), which will turn out to be very convenient when diagonalizing this kinetic mixing. However, the price to pay is that it has also introduced an additional term that contributes to the kinetic energy of the field φ, which can be canonically normalized by defining

$$\chi(\varphi) = \int_{\varphi_0}^{\varphi} \mathrm{d}\hat{\varphi} \sqrt{Z(\hat{\varphi}) + \frac{F'(\hat{\varphi})^2}{2F(\hat{\varphi})}}, \tag{2.34}$$

where φ_0 is again taken to be zero without loss of generality. Doing so leads to the Lagrangian

$$\begin{aligned}\mathcal{L} =& \frac{\hat{F}(\chi)}{4}\left(\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h \partial^\mu h\right) - \frac{\hat{F}'(\chi)}{4}\partial_\mu \chi \partial^\mu h \\ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\ & + i\bar{\psi}\gamma^\mu\partial_\mu\psi - q\bar{\psi}\gamma^\mu A_\mu\psi - y\bar{\psi}\phi\psi \\ & + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{3\mu^4}{2\lambda} - \hat{U}(\chi) + \frac{1}{2}h_{\mu\nu}T^{\mu\nu}\Big] + \cdots , \end{aligned} \tag{2.35}$$

where $\hat{F}(\chi) \equiv F(\varphi)$, $\hat{F}'(\chi) = \partial\hat{F}(\chi)/\partial\chi$ and $\hat{U}(\chi) \equiv U(\varphi)$.

Now, there is only the graviton left to canonically normalize, since it is still non-minimally coupled to the function $\hat{F}(\chi)$. However, as noted previously, the potential $\hat{U}(\chi)$ must lead to a non-vanishing vacuum expectation value for χ at late times so that the theory mimics Einstein gravity.[2] With this in mind, we shift $\chi \to \chi + v_\chi$ to obtain

$$\begin{aligned}\mathcal{L} =& \frac{\hat{F}(v_\chi)}{4}\left(\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h \partial^\mu h\right) - \frac{\hat{F}'(v_\chi)}{4}\partial_\mu \chi \partial^\mu h \\ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \\ & + i\bar{\psi}\gamma^\mu\partial_\mu\psi - q\bar{\psi}\gamma^\mu A_\mu\psi - y\bar{\psi}\phi\psi \\ & + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{3\mu^4}{2\lambda} - \hat{U}(\chi + v_\chi) + \frac{1}{2}h_{\mu\nu}T^{\mu\nu} + \cdots , \end{aligned} \tag{2.36}$$

where higher-order terms in the interactions between χ and $h_{\mu\nu}$ have been omitted

[2] We might expect v_χ to be evolving on cosmological timescales, but these timescales are long compared to the those relevant for elementary particle interactions.

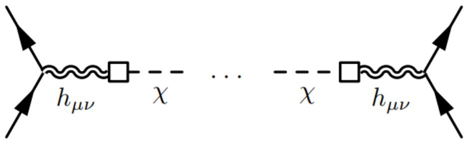


Figure 2.1: *Series of diagrams contributing to the fifth force between two fermions, and arising from the kinetic mixing between the graviton $h_{\mu\nu}$, and the scalar field χ. In these diagrams, the straight solid lines correspond to fermions, the double wavy lines to gravitons, the dashed line to the non-minimally coupled field and the boxes a kinetic mixing oscillation. Additionally, the ellipsis represents the series summing over all insertions of the kinetic mixing.*

in the ellipsis. Therefore, modification of gravity leads to a kinetic mixing between the trace of the graviton h and the χ field; the last term in the first line.

In contrast to the Einstein frame, the main contribution to the fifth-force coupling, as analyzed in the Jordan frame, is via the kinetic mixing and not a mass mixing, as illustrated in Figure 2.1 (see Ref. [1]). This comes from the fact that the graviton propagator ($\propto 1/q^2$) cancels the momentum dependence of the mixing vertex ($\propto q^2$) in every oscillation between the field φ and the graviton, such that, unlike the case of a massive field, there is no additional momentum suppression in the non-relativistic limit. Although it is already viable to work with the kinetic mixing present in the Lagrangian, we can remove it by the following transformation of the graviton and χ field:[3]

$$h_{\mu\nu} \to \frac{2}{M_{\rm Pl}} h_{\mu\nu} + \frac{1}{M_{\rm Pl}} \frac{\hat{F}'(v_\chi)}{\sqrt{M_{\rm Pl}^2 + \hat{F}'(v_\chi)^2}} \sigma \eta_{\mu\nu}, \tag{2.37a}$$

$$\chi \to -\frac{1}{\sqrt{1 + \left(\frac{\hat{F}'(v_\chi)}{M_{\rm Pl}}\right)^2}} \sigma, \tag{2.37b}$$

where $\hat{F}(v_\chi) = M_{\rm Pl}^2$ has been substituted and σ corresponds to the canonically normalized scalar field. This amounts to a perturbative implementation of the Weyl

[3]This transformation can be calculated in multiple ways. In Ref. [1], this was achieved by solving the system of equations that left the action diagonalized. Alternatively, in Appendix B, we show how to achieve this directly by transforming the kinetic matrix.

transformation, as is clear when one considers the resulting Lagrangian

$$\begin{aligned}\mathcal{L} =&\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ i\bar{\psi}\gamma^\mu\partial_\mu\psi - q\bar{\psi}\gamma^\mu A_\mu\psi - y\bar{\psi}\phi\psi - \hat{U}(\chi(\sigma)+v_\chi) \\ &+\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{3\mu^4}{2\lambda} \\ &+\frac{1}{M_{\rm Pl}}h_{\mu\nu}T^{\mu\nu} + \frac{1}{2M_{\rm Pl}}\frac{\hat{F}'(v_\chi)}{\sqrt{M_{\rm Pl}^2+\hat{F}'(v_\chi)^2}}\sigma T^\mu_\mu + \cdots\,,\end{aligned} \tag{2.38}$$

where T^μ_μ is the trace of the matter energy-momentum tensor. Using the definition from Eq. (1.7), we obtain

$$T^\mu_\mu = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}g^{\mu\nu}, \tag{2.39}$$

which vanishes for scale-invariant sectors. To understand this, let us consider a generic action $\hat{S}$ that presents a symmetry under conformal transformations of the form

$$g_{\mu\nu} \to e^\Omega g_{\mu\nu}. \tag{2.40}$$

In this way, the variation of $\hat{S}$ with respect to this transformation is

$$\delta\hat{S} = \int\frac{\delta\hat{S}}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \int\frac{\delta\hat{S}}{\delta g^{\mu\nu}}\Omega g^{\mu\nu} = 0, \tag{2.41}$$

where we have made the infinitesimal expansion of $e^\Omega \approx 1+\Omega+\mathcal{O}(\Omega^2)$. Therefore, for any generic Ω, this equality can only be satisfied by a traceless $\delta\hat{S}/\delta g^{\mu\nu}$. Substituting this result into Eq. (2.39) we can deduce that for any scale-invariant sector, the trace of the energy-momentum tensor vanishes.

Therefore, noticing that the new interactions in Eq. (2.38) depend on the trace of the energy-momentum tensor of the interacting particles, we obtain that the σ field will not couple to scale-invariant sectors [82], agreeing with our Einstein frame result. Since the only explicit scale breaking term in the Standard Model is the mass of the Higgs, we can express Brans-Dicke gravity equivalently as Higgs portal theories. However, notice that to obtain this result we used the equations of motion, meaning that it only holds classically. In this way, quantum effects may be able to introduce modifications on the type of couplings arising in the matter sector, for example by breaking the scale invariance spontaneously a la Coleman-Weinberg [151] or through conformal anomalies [152].

Chapter 3

Fifth forces and dynamical scale symmetry breaking

In Chapter 2, we focused on establishing the field transformations and redefinitions that are needed to express a generic Brans-Dicke scalar-tensor theory as a Beyond the Standard Model theory. In the process, we showed that the dynamics arising from this type of modification of gravity only couple to scale-dependent sectors.

This brings us to the following dilemma: As illustrated by the SM toy model example from Chapter 2.1 in which we considered the usual Higgs double-well potential, the scale-breaking terms are the ones allowing the fermions to acquire a Yukawa mass (and also introducing the new dynamics in the matter sector). Therefore, it might seem that it is not possible to both have massive fermionic fields while avoiding new interactions in the matter sector.

However, as with any symmetry, there can be different sources of scale breaking in a given theory, such as via quantum effects [151] or the dynamical emergence of scales. In this chapter, we will focus on the latter case, showing how it is possible to avoid new dynamics in the matter sector, while allowing for massive fermions.

3.1 Introduction to dynamically broken scale symmetries

The only term in the potential for the would-be Higgs field, $\tilde{\Phi}$, that did not couple to the additional scalar $\tilde{\chi}$ was $\frac{\lambda}{4!}\tilde{\Phi}^4$. This is because its scaling dimension agrees with the dimension of the space-time, canceling the contribution from the conformal transformation after redefining the fields as in Eq. (3.26). In this way, we should not expect the generation of new couplings in a potential built uniquely from quartic scalar terms, leaving any scattering process unmodified by the Brans-Dicke-type extension of gravity.

A generalized proof for this was given in Ref. [82], where the authors introduce the most generic multi-scalar action

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\sum_{i=1}^{N}\alpha_i\varphi_i^2 R + \frac{1}{2}\sum_{i=1}^{N}\partial_\mu\varphi_i\partial^\mu\varphi_i - W(\vec{\varphi}) + \mathcal{L}_F\{\psi,\varphi_1\}\right], \tag{3.1}$$

with α_i being dimensionless constants that allow for non-minimal couplings between the scalar fields, φ_i, and gravity. The potential $W(\vec{\varphi})$ contains the following possible interactions between the scalar fields:

$$W(\vec{\varphi}) = \sum_{i=1}^{N}\sum_{j=1}^{N}\varphi_i^2 W_{ij}\varphi_j^2, \tag{3.2}$$

which are weighted by the dimensionless matrix W_{ij}. Additionally, $\mathcal{L}_F\{\psi,\varphi_1\}$ encodes the fermionic sector of the action, such that

$$\mathcal{L}_F\{\psi,\varphi_1\} = \bar{\psi}i\gamma^\mu\nabla_\mu\psi - y\bar{\psi}\varphi_1\psi, \tag{3.3}$$

where

$$\nabla_\mu\psi = \partial_\mu\psi - \frac{i}{2}\Omega_\mu\psi \tag{3.4}$$

contains the spin connection. We have chosen, without loss of generality, φ_1 to give mass to the fermion after the would-be Higgs undergoes spontaneous symmetry breaking.

We can easily see that, since all the operators in W are dimensionless, the whole matter action is scale invariant. Therefore, from Noether's theorem, there should be a conserved current from this continuous Weyl symmetry of the theory. A straightforward way of finding this conserved current is by deriving the equations of motion of the theory. After varying the action with respect to the matter fields and the metric, we find

$$\Box\varphi_i - \alpha_i\varphi_i R - W_{\varphi_i} - y\bar{\psi}\psi = 0, \tag{3.5a}$$

$$\gamma^\mu \nabla_\mu \psi - y\varphi_1\psi = 0, \tag{3.5b}$$

$$-\sum_{i=1}^{N} \alpha_i \varphi_i^2 R = \sum_{i=1}^{N} \left[(6\alpha_i - 1)\ \partial_\mu\varphi_i \partial^\mu \varphi_i + 6\alpha_i \varphi_i \Box\varphi_i\right] + 4W + T^\mu_\mu\{\psi\}, \tag{3.5c}$$

where $W_{\varphi_i} = \partial W/\partial\varphi_i$, and

$$T^\mu_\mu\{\psi\} = \bar{\psi}\gamma^\mu\nabla_\mu\psi - 4\left(\bar{\psi}\gamma^\mu\nabla_\mu\psi - y\bar{\psi}\varphi_1\psi\right) \tag{3.6}$$

is the trace of the energy-momentum tensor for the fermionic sector. Substituting Eq. (3.5b) into Eq. (3.6), it simplifies into

$$T^\mu_\mu\{\psi\} = y\bar{\psi}\varphi_1\psi. \tag{3.7}$$

Thus, substituting each of the field equations into the Einstein's equation we get

$$\sum_{i=1}^{N} \left[\varphi_i W_{\varphi_i} - \varphi_i \Box\varphi_i\right] = \sum_{i=1}^{N} \left[(6\alpha_i - 1)\, \partial_\mu\varphi_i \partial^\mu\varphi_i + 6\alpha_i\varphi_i\Box\varphi_i\right] + 4W. \tag{3.8}$$

Then, given that $\sum_{i=1}^{N} \varphi_i W_{\varphi_i} = 4W$, we finally get

$$\sum_{i=1}^{N} \left[(6\alpha_i - 1)\, \partial_\mu \varphi_i \partial^\mu \varphi_i + (6\alpha_i - 1)\varphi_i \Box \varphi_i \right] = 0, \tag{3.9}$$

which can be easily rearranged into a conservation law of the form $\partial_\mu K^\mu = 0$, where $K^\mu = \partial^\mu K$ corresponds to the Noether's current, with

$$K = \frac{1}{2} \sum_{i=1}^{N} (1 - 6\alpha_i)\varphi_i^2. \tag{3.10}$$

Moreover, once we include the expansion of the Universe, the Hubble friction will slow down the scalar fields, such that K ends up in a constant value $K \to K_0$, constraining the fields to lie on the ellipse given by Eq. (3.10) (and so leading to a vanishing Noether's current). This leads to a classical scale symmetry breaking in the theory, given that at least one of the scalar fields has obtained a non-vanishing vev. Since the scale appears indirectly through the stabilization of the fields, we will refer to this as dynamical scale breaking.[1]

From Goldstone's theorem, when a continuous symmetry is broken, a massless mode is generated in the theory. In this case, it will correspond to a massless scalar field, σ, that will introduce the long-range interactions we referred to as *fifth forces* in Chapter 1.1.1. Such long-range forces arising from the additional scalar fields in scalar-tensor theories cannot, in principle, be avoided without screening mechanisms, and they will be the main focus of this section.

However, as anticipated in Chapter 2, the classical scale symmetry from Eq. (3.1) will make this massless mode decouple from the rest of the fields. To show this, we

[1] Although in the literature this mechanism of scale breaking is commonly known as spontaneous scale symmetry breaking [153], we use the term 'dynamical' to differentiate it from the spontaneous scale symmetry breaking à la Coleman-Weinberg through field self-interactions [151].

can always perform the following generic field redefinitions

$$\varphi_i = e^{-\frac{\sigma}{f}}\tilde{\varphi}_i \tag{3.11}$$

$$\psi = e^{-\frac{3\sigma}{2f}}\tilde{\psi} \tag{3.12}$$

$$g_{\mu\nu} = e^{2\frac{\sigma}{f}}\tilde{g}_{\mu\nu} \tag{3.13}$$

which is nothing but going to the Einstein frame of this theory (with respect to the σ field), as we did in Chapter 2.1. In this frame, $\tilde{\varphi}_i$ are constrained to lie on the ellipse given by

$$\bar{K} = \frac{1}{2}\sum_{i=1}^{N}(1-6\alpha_i)\,\hat{\varphi}_i^2 = f^2 \tag{3.14}$$

where f^2 is a constant. With this, after some algebra, the action transforms to

$$S = \int d^4x\sqrt{-\tilde{g}}\left[-\frac{1}{2}\sum_i^N \alpha_i\tilde{\varphi}_i^2\tilde{R} + \frac{1}{2}\sum_i^N \partial_\mu\tilde{\varphi}_i\partial^\mu\tilde{\varphi}_i \right. \tag{3.15}$$

$$\left. +\frac{1}{f^2}\bar{K}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{f}\partial_\mu\sigma\partial^\mu\bar{K} - W(\vec{\varphi}) + \mathcal{L}_F\{\tilde{\psi},\tilde{\varphi}_1\}\right]. \tag{3.16}$$

where the tilded quantities are built with the Einstein frame's metric, $\tilde{g}_{\mu\nu}$, and vierbeins, $\tilde{e}_a^\mu$, and

$$\mathcal{L}_F\{\tilde{\psi},\tilde{\varphi}_1\} = \bar{\tilde{\psi}}i\gamma^\mu\tilde{\nabla}_\mu\tilde{\psi} - y\bar{\tilde{\psi}}\tilde{\varphi}_1\tilde{\psi} \tag{3.17}$$

has stayed invariant through the re-scaling. Moreover, the massless mode couples to the rest of the fields only through $\bar{K}$-dependent terms. Thus, given that $\bar{K} = f^2$ is constant, the cross terms between σ and $\tilde{\varphi}_i$ will vanish on-shell, showing the decoupling from the matter fields. We can see that, after the transformation, we are still left with non-minimal couplings to the Ricci scalar. However, as they correspond to massive degrees of freedom, their effect on any non-relativistic potentials will be exponentially suppressed, producing negligible deviations from any test of gravity. We can already see the positive aspects of such models: The dynamical breaking of the scale symmetry leads to a suppression of the fifth forces while presenting non-vanishing vevs for the scalar fields. Therefore, they will still provide a Yukawa mass to the fermion.

One of the most well-studied of such models is the Higgs-dilaton [70–84], in which the non-minimally coupled field is the would-be Higgs field. In this way, the action is defined by

$$\begin{aligned} S_{\mathrm{HD}} = \int d^4x\sqrt{-g}\Big[&-\frac{1}{2}\left(\xi_\varphi\varphi^2+\xi_\Phi\Phi^2\right)R+\frac{1}{2}\partial_\mu\Phi\partial^\nu\Phi+\frac{1}{2}\partial_\mu\varphi\partial^\nu\varphi \\ &-\frac{\lambda}{4}\left(\Phi^2-\frac{\alpha}{\lambda}\varphi^2\right)^2-\beta\varphi^4+\mathcal{L}_F\{\tilde{\psi},\Phi\}\Big]\,, \end{aligned} \tag{3.18}$$

where the set of parameters $\{\xi_\varphi, \xi_\Phi, \lambda, \alpha, \beta\}$ are constant and the fermionic sector, $\mathcal{L}_F\{\tilde{\psi}, \Phi\}$, has the same form as in Eq. (3.3) with $\phi_1 \to \Phi$. Notice that, since all the terms in the potential are dimensionless, the same arguments described above hold, leading to the suppression of additional long-range interactions.

3.1.1 Combined scale symmetry breakings

As stated before, dynamical scale breakings have been widely studied mainly through the Higgs-Dilaton models, and extensive literature covers its cosmological and phenomenological implications [70–84]. However, the mid-point between both kinds of scale symmetry breaking (explicit and dynamical) has not yet been fully explored. For that reason, in this chapter, we will consider a model in which the scale symmetry is broken both explicitly and dynamically.

Starting from the gravitational sector, we will now specialize to the Brans-Dicke theory [54], whose Jordan-frame action is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{\varphi}{2}R+\frac{\omega(\varphi)}{2\varphi}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right]+S_{\mathrm{m}}[g_{\mu\nu},\{\psi\}], \tag{3.19}$$

where we will take $\omega(\varphi) = \omega$ to be a dimensionless constant. For the matter content, a good choice is the model introduced in Ref. [138], which introduces a new scalar field (Θ) into the system to allow for dynamical scale symmetry breaking. Written in terms of the Jordan-frame metric $g_{\mu\nu}$, the scalar sector extends to

$$S_{\rm SB} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}g^{\mu\nu}\partial_\mu\Theta\partial_\nu\Theta + \bar{\psi}e^\mu_a\gamma^a\partial_\mu\psi - y\bar{\psi}\Phi\psi \right.$$
$$\left. - U(\Phi,\Theta) + \frac{1}{2}\mu_\theta^2\frac{\tilde{M}_{\rm Pl}^2}{\varphi}\Theta^2 - \frac{\lambda_\theta}{4!}\Theta^4 - \frac{3}{2}\frac{\mu_\theta^4}{\lambda_\theta}\frac{\tilde{M}_{\rm Pl}^4}{\varphi^2}\right], \qquad (3.20)$$

where

$$U(\Phi,\Theta) = \frac{\lambda}{4!}\left(\Phi^2 - \frac{\beta}{\lambda}\Theta^2\right)^2 - \frac{1}{2}\mu^2\left(\Phi^2 - \frac{\beta}{\lambda}\Theta^2\right) + \frac{3}{2}\frac{\mu^4}{\lambda}. \qquad (3.21)$$

Notice that this is not the most general potential allowed by the imposed symmetries, as our aim is only to study the emergence of fifth forces in models where the scale breaking takes place both dynamically and explicitly, as we will see below. In this way, the matter sector still contains the would-be Higgs field Φ that gives mass to the fermion ψ through a Yukawa interaction, plus the additional Θ field that has been introduced so that we can move smoothly between two scenarios of scale breaking. Furthermore, the specific choice of the potential is such between the Θ and φ fields has been tuned so that these fields do not have a mass mixing in the Einstein frame [see, e.g., Eq. (3.30) below], while also allowing us to establish a hierarchy between the masses of the three physical modes (see Ref. [138]).

Independent of the non-minimally coupled field φ and its dynamics, the two limiting cases for scale breaking are the following:

Pure explicit breaking (prototype SM Higgs sector) $\boldsymbol{\beta \to 0}$: In the limit in which $\beta \to 0$, the mixings between Φ and Θ vanish, decoupling Θ from the matter Lagrangian. We are then left with the following potential in the Jordan frame [Eq. (3.21)]:

$$U(\Phi, \Theta) = \frac{\lambda}{4!}\Phi^4 - \frac{1}{2}\mu^2\Phi^2 + \frac{3}{2}\frac{\mu^4}{\lambda}, \tag{3.22}$$

which is just the prototype of the SM Higgs potential. As shown earlier, the term quadratic in Φ provides an explicit source of scale breaking through the dimensionful mass parameter μ. It is this term that plays the key role in allowing fifth forces to couple to the fermion field ψ. Although the constant term also seems to break the scale invariance, it is essential to cancel the vacuum energy density of the Higgs potential when perturbing around its minima. For instance, the absence of this term would change the background from a Minkowski to a de Sitter spacetime, leading to effective mass terms for both the graviton and the non-minimally coupled field when perturbing gravity.

Pure dynamical scale breaking (prototype Higgs-dilaton model) $\mu \to 0$: In this limit, all the sources of explicit scale breaking vanish from $U(\Phi, \Theta)$, leaving a scale-invariant potential. We therefore do not expect the conformal field φ to couple to this potential in the Einstein frame, leaving the fermionic sector free of fifth forces. The potential is reduced to

$$U(\Phi, \Theta) = \frac{\lambda}{4!}\left(\Phi^2 - \frac{\beta}{\lambda}\Theta^2\right)^2, \tag{3.23}$$

which is analogous to the Higgs-dilaton potential from Eq. (3.18). In those scenarios, however, both Φ and Θ are non-minimally coupled to the Ricci scalar in the Jordan frame (and the dilaton is the light degree of freedom with the potential to mediate long-range forces), whereas we will take only the additional field φ to be non-minimally coupled.

In this chapter, making use of the Møller scattering ($e^- + e^- \to e^- + e^-$) for simplicity, we will calculate the possible modifications to its Yukawa potential due to fifth forces. For that, we will make use of and extend both methods introduced in Chapter 2: First, we will calculate the Yukawa potential in the Einstein frame and then replicate the calculation in the Jordan frame. In the process, we will focus on how the fifth forces turn on and off depending on the scale-breaking limit we take.

3.2 Going to the Einstein frame

Let us now turn our attention to an explicit calculation of the Einstein frame description for this model. First, we need to express the matter action in terms of the Einstein-frame metric $\tilde{g}_{\mu\nu}$. To do so, we must perform the conformal transformation defined previously in Eq. (2.10), which for the combined scale symmetry breaking model from Eq. (3.19) implies

$$g_{\mu\nu} \to \frac{\tilde{M}_{\rm Pl}^2}{\varphi}\tilde{g}_{\mu\nu}, \qquad g^{\mu\nu} \to \frac{\varphi}{\tilde{M}_{\rm Pl}^2}\tilde{g}^{\mu\nu}, \tag{3.24a}$$

$$e_a^\mu \to \frac{\tilde{M}_{\rm Pl}}{\sqrt{\varphi}}\tilde{e}_a^\mu, \qquad e_\mu^a \to \frac{\sqrt{\varphi}}{\tilde{M}_{\rm Pl}}\tilde{e}_\mu^a. \tag{3.24b}$$

This transformation takes us to the following Einstein-frame description of the action

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\Bigg[&-\frac{\tilde{M}_{\rm Pl}^2}{2}\tilde{R} + \frac{\varphi^{3/2}}{\tilde{M}_{\rm Pl}^3}\tilde{e}_a^\mu\gamma^a\bar{\psi}i\partial_\mu\psi + \frac{1}{2}\frac{\tilde{M}_{\rm Pl}^3}{\varphi^{3/2}}\bar{\psi}\tilde{e}_a^\mu\gamma^a\psi\left(\tilde{\Omega}_\mu - \frac{3i}{2}\frac{1}{\varphi}\partial_\mu\varphi\right) \\ &+ \frac{\tilde{M}_{\rm Pl}^2(2\omega+3)}{4\varphi^2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}\frac{\varphi}{\tilde{M}_{\rm Pl}^2}\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}\frac{\varphi}{\tilde{M}_{\rm Pl}^2}\tilde{g}^{\mu\nu}\partial_\mu\Theta\partial_\nu\Theta + \\ &-y\frac{\varphi^2}{\tilde{M}_{\rm Pl}^4}\bar{\psi}\Phi\psi - \frac{\varphi^2}{\tilde{M}_{\rm Pl}^4}U(\Phi,\Theta) + \frac{1}{2}\frac{\varphi}{\tilde{M}_{\rm Pl}^2}\mu_\theta^2\Theta^2 - \frac{\lambda_\theta}{4!}\frac{\varphi^2}{\tilde{M}_{\rm Pl}^4}\Theta^4 - \frac{3}{2}\frac{\mu_\theta^4}{\lambda_\theta}\Bigg], \end{aligned} \tag{3.25}$$

where we remind the reader that every tilded quantity is built with the Einstein-frame metric, $\tilde{g}_{\mu\nu}$, and vierbein, $\tilde{e}_a^\mu$. Following the same steps as in Chapter 2.1, to leave the matter sector as close to being canonically normalized as possible, we redefine the fields according to their classical scaling dimensions, such that

$$\tilde{\phi} \equiv \frac{\sqrt{\varphi}}{\tilde{M}_{\mathrm{Pl}}}\Phi, \qquad \tilde{\theta} \equiv \frac{\sqrt{\varphi}}{\tilde{M}_{\mathrm{Pl}}}\Theta, \qquad \tilde{\psi} \equiv \frac{\varphi^{3/4}}{\tilde{M}_{\mathrm{Pl}}^{3/2}}\psi. \tag{3.26}$$

Additionally, the φ field can be canonically normalized through

$$\varphi = \tilde{M}_{\mathrm{Pl}}^2 \exp\left[2\frac{\tilde{\chi}}{\tilde{M}}\right]. \tag{3.27}$$

Herein, we have defined

$$\tilde{M}^2 = 2(2\omega + 3)\tilde{M}_{\mathrm{Pl}}^2. \tag{3.28}$$

It then follows that the Einstein-frame matter sector is

$$\begin{aligned}\tilde{\mathcal{L}}_{\mathrm{m}} =& \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - \frac{\tilde{\phi}}{\tilde{M}}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\chi} + \frac{\tilde{\phi}^2}{2\tilde{M}^2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi}\\ &+\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\theta}\partial_\nu\tilde{\theta} - \frac{\tilde{\theta}}{\tilde{M}}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\theta}\partial_\nu\tilde{\chi} + \frac{\tilde{\theta}^2}{2\tilde{M}^2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi}\\ &-\tilde{U}(\tilde{\phi},\tilde{\theta},\tilde{\chi}) + \bar{\tilde{\psi}}i\tilde{\nabla}\!\!\!/\,\tilde{\psi} - y\bar{\tilde{\psi}}\tilde{\phi}\tilde{\psi} - \frac{1}{2}\mu_\theta^2\tilde{\theta}^2 + \frac{\lambda_\theta}{4!}\tilde{\theta}^4 + \frac{3}{2}\frac{\mu_\theta^4}{\lambda_\theta},\end{aligned} \tag{3.29}$$

where $\tilde{\nabla}$ corresponds to the pure gravitational covariant derivative of the fermionic field, containing the spin connection, and

$$\begin{aligned}\tilde{U}(\tilde{\phi},\tilde{\theta},\tilde{\chi}) =& \frac{\lambda}{4!}\left(\tilde{\phi}^2 - \frac{\beta}{\lambda}\tilde{\theta}^2\right)^2 - \frac{1}{2}\mu^2\left(\tilde{\phi}^2 - \frac{\beta}{\lambda}\tilde{\theta}^2\right)\exp\left[2\frac{\tilde{\chi}}{\tilde{M}}\right]\\ &+\frac{3}{2}\frac{\mu^4}{\lambda}\exp\left[4\frac{\tilde{\chi}}{\tilde{M}}\right].\end{aligned} \tag{3.30}$$

Thus, we can see that the redefinitions from Eq. (3.26) eliminate all the couplings of $\tilde{\chi}$ in the fermionic sector and in the pure Θ potential from the last line of Eq. (3.29). However, the same does not apply to $\tilde{U}(\tilde{\phi},\tilde{\theta},\tilde{\chi})$, since it contains dimensionful parameters. Moreover, as explained earlier in Chapter 3.1, the only terms coupling to $\tilde{\chi}$ in the Einstein frame are the ones that break the scale symmetry explicitly.

After including the original kinetic energy term for $\tilde{\chi}$ from Eq. (3.25), the non-gravitational part of the Einstein-frame Lagrangian can be written up to first order in $\tilde{M}^{-1}$ as

$$\begin{aligned}\tilde{\mathcal{L}} =& \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\chi}\partial_\nu\tilde{\chi} + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - \frac{\tilde{\phi}}{\tilde{M}}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\chi} \\ &+\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\theta}\partial_\nu\tilde{\theta} - \frac{\tilde{\theta}}{\tilde{M}}\tilde{g}^{\mu\nu}\partial_\mu\tilde{\theta}\partial_\nu\tilde{\chi} - \tilde{U}(\tilde{\phi},\tilde{\theta},\tilde{\chi}) \\ &+\bar{\tilde{\psi}}i\tilde{\nabla}\tilde{\psi} - y\bar{\tilde{\psi}}\tilde{\phi}\tilde{\psi} - \frac{1}{2}\mu_\theta^2\tilde{\theta}^2 + \frac{\lambda_\theta}{4!}\tilde{\theta}^4 + \frac{3}{2}\frac{\mu_\theta^4}{\lambda_\theta}\cdots\,, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned}\tilde{U}(\tilde{\phi},\tilde{\theta},\tilde{\chi}) =& \frac{\lambda}{4!}\left(\tilde{\phi}^2 - \frac{\beta}{\lambda}\tilde{\theta}^2\right)^2 - \frac{1}{2}\mu^2\left(\tilde{\phi}^2 - \frac{\beta}{\lambda}\tilde{\theta}^2\right)\left(1 + 2\frac{\tilde{\chi}}{\tilde{M}}\right) \\ &+\frac{3}{2}\frac{\mu^4}{\lambda}\left(1 + 4\frac{\tilde{\chi}}{\tilde{M}}\right). \end{aligned} \tag{3.32}$$

The fields acquire the vevs:

$$v_{\tilde{\phi}} = \pm\left(\frac{6\mu^2 + \beta v_{\tilde{\theta}}^2}{\lambda}\right)^{1/2}, \qquad v_{\tilde{\theta}} = \pm'\left(\frac{6\mu_\theta^2}{\lambda_\theta}\right)^{1/2}, \qquad v_\chi = 0, \tag{3.33}$$

where the $'$ indicates that the choice of sign for the two non-vanishing vevs is independent. Expanding the scalar fields around their vevs ($\tilde{\phi} \to v_{\tilde{\phi}} + \tilde{\phi}$, $\tilde{\theta} \to v_{\tilde{\theta}} + \tilde{\theta}$ and $\tilde{\chi} \to v_{\tilde{\chi}} + \tilde{\chi}$) will introduce both kinetic and mass mixings of $\tilde{\phi}$ and $\tilde{\theta}$ with $\tilde{\chi}$. However, as explained in Chapter 2.1, the mass mixing will provide the dominant fifth force, given by the operator

$$\tilde{\mathcal{L}} \supset \alpha_{\mathrm{M}}\tilde{\phi}\tilde{\chi} = -2\mu^2\frac{v_{\tilde{\phi}}}{\tilde{M}}\tilde{\phi}\tilde{\chi}. \tag{3.34}$$

Thus, as illustrated in Figure 3.1, when two fermions interact via their Yukawa coupling and exchange a would-be Higgs boson ($\tilde{\phi}$) in the t channel, there are two contributions to the central potential: a short-range interaction due to the heavy

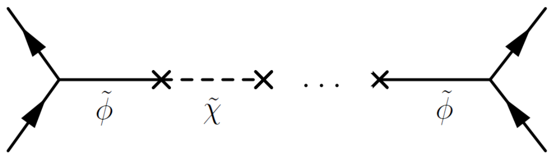


Figure 3.1: *Diagrammatic representation of the infinite series of diagrams contributing to the Møller scattering in the Einstein frame.*

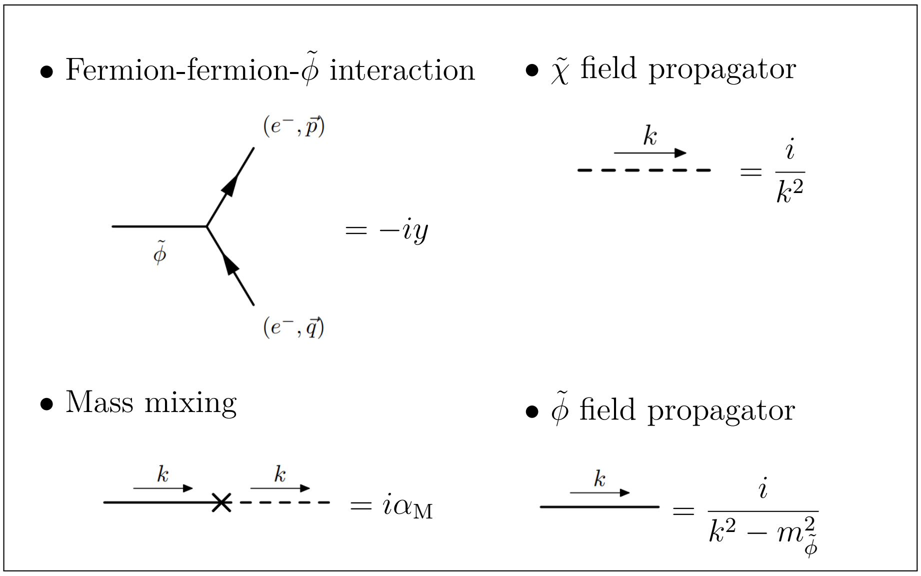


Figure 3.2: *Feynman rules necessary for the Møller scattering in Figure 3.1.*

mode (the Higgs boson) and a long-range interaction due to the light mode (the light, additional scalar boson), corresponding to the fifth forces, see Ref. [138].

3.2.1 Møller scattering

We proceed by considering the scalar contributions to the Møller scattering ($e^-e^- \to e^-e^-$) for our fermion ψ. These arise from the series of diagrams shown in Fig. 3.1. The external fermions couple only to the would-be Higgs field, represented by a continuous line, which then oscillates into a $\tilde{\chi}$ particle (dashed line) via the mass mixing term from the effective Lagrangian [Eq. (3.34)]. The ellipsis represents the infinite series of insertions of the mass mixing.

The resulting matrix element is given by

$$
\begin{aligned}
i\mathcal{M}\left(e^-e^- \to e^-e^-\right) \supset\ & \bar{u}\left(\mathbf{p}_1, s_1\right)(-iy)u\left(\mathbf{p}_3, s_3\right) \\
& \times\left(-\frac{i}{t+m_{\tilde{\phi}}^2}\right)\left[\sum_{n=0}^{\infty}\left(i\alpha_{\mathrm{M}}\right)^{2n}\left(-\frac{i}{t+m_{\tilde{\phi}}^2}\right)^n\left(-\frac{i}{t}\right)^n\right] \\
& \times \bar{u}\left(\mathbf{p}_2, s_2\right)(-iy)u\left(\mathbf{p}_4, s_4\right).
\end{aligned}
\tag{3.35}
$$

Since we assume the scattering fermions to be distinguishable, we need only consider the t-channel exchange, where $t = -(p_1 - p_3)^2$ is the usual Mandelstam variable. Also, $u(\mathbf{p}, s)$ and $\bar{u}(\mathbf{p}, s)$ are respectively the Dirac four-spinor and its Dirac conjugate, with spin projection s.

To extract the non-relativistic potential, we take $t = \mathbf{Q}^2$ (where $\mathbf{Q}$ is the exchange momentum), and the contribution to the Yukawa potential is

$$
\tilde{V}(r) = -y^2\int\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3}e^{i\mathbf{Q}\cdot\mathbf{x}}\frac{\mathbf{Q}^2}{\mathbf{Q}^2\left(\mathbf{Q}^2+m_{\tilde{\phi}}^2\right)-\alpha_{\mathrm{M}}^2} \approx -\frac{y^2}{4\pi}\left(1-\frac{\alpha_{\mathrm{M}}^2}{m_{\tilde{\phi}}^4}\right)\frac{e^{-m_{\mathrm{h}}r}}{r} - \frac{y^2}{4\pi}\frac{\alpha_{\mathrm{M}}^2}{m_{\tilde{\phi}}^4}\frac{1}{r},
\tag{3.36}
$$

where m_{h} is the mass of the would-be Higgs boson and the potential has been expanded to leading order in α_{M}^2. Plugging in α_{M}, as extracted from Eq. (3.34), we can distinguish two different contributions to the Yukawa potential: one strong but short-ranged interaction, corresponding to the Higgs field, and one weak but long-ranged contribution corresponding to the fifth forces. Isolating this fifth-force contribution, we find

$$
\tilde{V}_5(r) = -\frac{1}{4\pi r}\frac{m_e^2}{\tilde{M}_{\mathrm{Pl}}^2 2(2\omega+3)}\frac{4\mu^4}{m_{\tilde{\phi}}^4},
\tag{3.37}
$$

where we have chosen the fermions to represent electrons with mass m_e. Notice that, since the fifth-force mediator is massless, the potential has a similar form to the usual Newtonian gravitational potential.

To study how the different mechanisms of scale breaking affect the modification to the Yukawa potential [Eq. (3.37)], we need only recall that the mass of the $\tilde{\phi}$ field is given by

$$m_{\tilde{\phi}}^2 = 2\mu^2 + \frac{\beta v_{\tilde{\theta}}^2}{3}. \tag{3.38}$$

From this, we can distinguish the two extreme cases:

Pure explicit scale breaking (SM toy model) $\boldsymbol{\beta \to 0}$: In this limit, the mass of the $\tilde{\phi}$ field reduces to

$$m_{\tilde{\phi}}^2 = 2\mu^2, \tag{3.39}$$

agreeing with the numerator of the fraction in the potential (3.37). Hence, the modification to the Yukawa potential becomes independent of the Higgs mass, and we find

$$\tilde{V}_5(r) = -\frac{1}{4\pi r}\frac{m_e^2}{\tilde{M}_{\rm Pl}^2 2(2\omega+3)}. \tag{3.40}$$

Such a contribution to the non-relativistic potential can lead to significant deviations in the inferred gravitational force. As mentioned in Chapter 1, the most stringent constraint at Solar System scales is given by the Cassini spacecraft [100], setting a bound on $\omega \gg 40,000$. Bounds at cosmological scales are less stringent[2], such as those based on Cosmic Microwave Background data from Planck [85], which are consistent with $\omega > 692$ at the 99% confidence level. Therefore, in the absence of any screening mechanism, we can see that in the case of pure explicit scale breaking it is necessary to fine tune the value of ω to achieve an agreement with experiments.

Pure dynamical scale breaking (Higgs-dilaton model) $\boldsymbol{\mu \to 0}$: In this case, the numerator of the modified Yukawa potential [Eq. (3.37)] tends to zero, whereas

[2]Even though Solar System scale tests are more constraining than cosmological ones, they are more affected by higher-order terms, making it possible to avoid the bounds through screening mechanisms.

the denominator tends to

$$m_{\tilde{\phi}}^2 = \frac{\beta v_{\tilde{\theta}}^2}{3}. \tag{3.41}$$

Hence, even though classically scale-invariant theories might break the scale symmetry dynamically, the fifth forces still do not couple to the fermionic sector. It is important to remark that the vev of the would-be Higgs field $\tilde{\phi}$ field [Eq. (3.33)] does not vanish in the limit $\mu \to 0$, such that the mass-generation mechanism for the elementary fermions is preserved (with $m_e = yv_{\tilde{\phi}}$). More generally, we see that the fifth-force coupling is proportional to the ratio $\mu/m_{\tilde{\phi}}$, such that we can suppress fifth forces by combining explicit and dynamical scale-breaking mechanisms [138].

For this tree-level example, the transformation to the Einstein frame and the subsequent calculation of the matrix elements were easily tractable. This may not be the case, in general, however. In the next section, we will describe in detail how we can proceed directly in the Jordan frame (or Jordan-like frames), without performing the conformal transformation and subsequent field rescalings.

3.3 Staying in the Jordan frame

Having derived the general expression for the Lagrangian up to second order in the metric fluctuations for the Brans-Dicke-type scalar-tensor theories in Chapter 2.2, we now turn our attention to see how the fifth forces arise from the kinetic mixings between the graviton and the non-minimally coupled field φ. We will show that the results agree with those obtained previously in the Einstein frame.

We remind the reader that the Jordan-frame action corresponding to Eq. (3.19), with a matter sector given by Eqs. (3.20) and (3.21), is

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-g}\bigg[&-\frac{\varphi}{2}R + \mathcal{L}'_{\mathrm{gf}} + \frac{\omega(\varphi)}{2\varphi}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \\ &-\frac{1}{2}g^{\mu\nu}\partial_\mu\Theta\partial_\nu\Theta + \frac{1}{2}\mu_\theta^2\frac{\varphi}{\tilde{M}_{\mathrm{Pl}}}\Theta^2 - \frac{\lambda_\theta}{4!}\Theta^4 - \frac{3}{2}\frac{\mu_\theta^4}{\lambda_\theta}\frac{\varphi^2}{\tilde{M}_{\mathrm{Pl}}^2} \\ &+\bar{\psi}i\not{\nabla}\psi - y\bar{\psi}\Phi\psi - U(\Phi,\Theta)\bigg], \end{aligned} \tag{3.42}$$

where U is given by

$$U(\Phi,\Theta) = \frac{\lambda}{4!}\left(\Phi^2 - \frac{\beta}{\lambda}\Theta^2\right)^2 - \frac{1}{2}\mu^2\left(\Phi^2 - \frac{\beta}{\lambda}\Theta^2\right) + \frac{3}{2}\frac{\mu^4}{\lambda}. \tag{3.43}$$

We now proceed to linearize the Lagrangian, making use of the results from the preceding chapter. Using the scalar-harmonic gauge condition from Eq. (2.32), we thus find

$$\begin{aligned} \mathcal{L} =&\frac{\varphi}{4}\left(\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h\right) + \frac{1}{2}\frac{2\omega+1}{2\varphi}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi \\ &-\frac{1}{4}\partial_\mu h\partial^\mu\varphi - U(\Phi,\Theta) + \frac{1}{2}\partial_\mu\Theta\partial^\mu\Theta + \frac{1}{2}\mu_\theta^2\frac{\varphi}{\tilde{M}_{\mathrm{Pl}}}\Theta^2 - \frac{\lambda_\theta}{4!}\Theta^4 - \frac{3}{2}\frac{\mu_\theta^4}{\lambda_\theta}\frac{\varphi^2}{\tilde{M}_{\mathrm{Pl}}^2} \\ &+\frac{1}{2}h^{\mu\nu}T_{\mu\nu} + \bar{\psi}i\not{\partial}\psi - y\bar{\psi}\Phi\psi + \cdots, \end{aligned} \tag{3.44}$$

where the ellipsis indicates terms higher than second order in $h_{\mu\nu}$. The φ field can be canonically normalized via

$$\varphi = \frac{\chi^2}{2(2\omega+1)}. \tag{3.45}$$

Therefore, we are now just left with the non-canonical graviton kinetic energy terms. To solve this issue, we just need to linearize around the background solution for χ, namely v_χ. To do so, we assume the background value of χ to vary very slowly compared to the interaction time. After some algebra, we obtain the following vevs for the scalar fields

$$v_\Phi^2 = \frac{6\mu^2 + \beta {v_\Theta}^2}{\lambda}, \qquad\qquad v_\chi^2 = \frac{\lambda_\theta \tilde{M}'^2}{6\mu_\theta^2} v_\Theta^2, \tag{3.46}$$

where

$$\tilde{M}'^2 = \tilde{M}_{\rm Pl}^2 2(2\omega + 1). \tag{3.47}$$

We can see that the system is open because v_Θ is indeterminate, which is an artefact of the specific choice of the scale-invariant sector of the potential. However, as in Chapter 3.1, the vacuum expectation values will end up taking an arbitrary constant value due to the Hubble friction acting on the evolution of the fields. Expanding now the scalar fields in Eq. (3.44) around their vevs (such as $\Phi \to \Phi + v_\Phi$, $\Theta \to \Theta + v_\Theta$ and $\chi \to \chi + v_\chi$), leads to

$$\begin{aligned}\mathcal{L} =& \frac{v_\chi^2}{8(2\omega+1)}\left[\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h \partial^\mu h\right] + \frac{1}{2}\partial_\mu\Theta\partial^\mu\Theta + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi \\ &+ \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{v_\chi}{4(2\omega+1)}\partial_\mu h\partial^\mu\chi - V(\Phi,\Theta,\chi) \\ &+ \frac{1}{M_{\rm Pl}} h^{\mu\nu}T_{\mu\nu} + \bar{\psi} i \overleftrightarrow{\not{\partial}}\psi - y v_\Phi \bar{\psi}\psi - y\bar{\psi}\Phi\psi + \cdots \,.\end{aligned} \tag{3.48}$$

As expected, the fermion obtains a non-vanishing mass via the vev of the Φ field in the Yukawa coupling. The potential terms are defined within $V(\Phi,\Theta,\chi)$ for simplicity; considering just up to second-order in the interactions of the shifted scalar fields, we obtain

$$\begin{aligned}V(\Phi,\Theta,\chi) =& \frac{m_\Phi^2}{2}\Phi^2 + \frac{m_\Theta^2}{2}\Theta^2 + \frac{m_\chi^2}{2}\chi^2 \\ & - A m_\Phi \Phi\Theta - B m_\chi \Theta\chi,\end{aligned} \tag{3.49}$$

where

$$m_\Phi^2 = \frac{\lambda v_\Phi^2}{3}, \qquad m_\Theta^2 = \frac{\beta^2}{2\lambda} v_\Theta^2 + \frac{\mu_\theta^2}{\tilde{M}'^2} v_\chi^2, \qquad m_\chi^2 = \frac{\mu_\theta^2}{\tilde{M}'^2} v_\Theta^2, \tag{3.50a}$$

$$A^2 = \frac{\beta^2}{2\lambda} v_\Theta^2, \qquad B^2 = \frac{\mu_\theta^2}{\tilde{M}'^2} v_\chi^2. \tag{3.50b}$$

Thus, canonically normalizing the graviton now just means making the replacement $h_{\mu\nu} \to 2h_{\mu\nu}/M_{\text{Pl}}$, where

$$M_{\text{Pl}}^2 = \frac{v_\chi^2}{2(2\omega + 1)}, \tag{3.51}$$

so that the Lagrangian takes the form

$$\begin{aligned} \mathcal{L} = &\frac{1}{4}\partial_\mu h \partial^\mu h - \frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + \frac{1}{2}\partial_\mu \Theta \partial^\mu \Theta + \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi \\ &+ \frac{1}{2}\partial_\mu \chi \partial^\mu \chi + \frac{v_\chi}{4(2\omega+1)}\partial_\mu h \partial^\mu \chi - V(\Phi, \Theta, \chi) \\ &+ \frac{1}{M_{\text{Pl}}} h^{\mu\nu} T_{\mu\nu} + \bar{\psi} i \overset{\leftrightarrow}{\not{\partial}} \psi - y v_\Phi \bar{\psi}\psi - y \bar{\psi}\Phi\psi + \cdots . \end{aligned} \tag{3.52}$$

Note that M_{Pl} is the effective gravitational coupling in the Jordan frame, while the $\tilde{M}_{\text{Pl}}$ parameter appearing in the potential $V(\Phi, \Theta, \chi)$ in Eq. (3.49) is the one defined in the Einstein frame. Even though they belong to different frames, the conformal transformations appear to have forced them into the same Lagrangian. However, as we will see, this is not the case as the $\tilde{M}_{\text{Pl}}^2$ dependence cancels out, leading to a final result in terms of M_{Pl}^2, as expected.

3.3.1 Diagonalizing the mass matrix

Although in the original matter sector we did not include explicit mass mixing between the scalar fields, the expansion around the fields' background led to the potential in Eq. (3.49), where the three scalars not only present non-vanishing masses, but also mass mixing terms. Although, a priori, this gives the impression that there are no long-range interactions in the system, we know that the dynamical breaking of the scale symmetry must have created a massless Goldstone mode that will mediate the fifth forces (as seen in Chapter 3.1).

To isolate this massless mode, we will need to diagonalize the mass matrix of the theory, which is given by

$$m^2 = \begin{pmatrix} m_\Phi^2 & -Am_\Phi & 0 \\ -Am_\Phi & m_\Theta^2 & -Bm_\chi \\ 0 & -Bm_\chi & m_\chi^2 \end{pmatrix}. \tag{3.53}$$

After the diagonalization, we thus obtain a new set of fields (corresponding to the eigenvectors) ϕ, θ and σ, whose squared mass eigenvalues are

$$m_{\phi,\theta}^2 = \frac{m_\Phi^2 + m_\Theta^2 + m_\chi^2 \pm \sqrt{(-m_\Phi^2 - A^2 + B^2 + m_\chi^2)^2 + 4A^2B^2}}{2}, \qquad m_\sigma^2 = 0, \tag{3.54}$$

wherein we see the anticipated massless mode σ. As discussed in Chapter 3.1, this massless field corresponds to the Goldstone mode arising from the dynamical scale symmetry breaking. Although it may seem counterintuitive that the generation of the massless mode holds even when there is an explicit scale-breaking term in the potential (i.e., when $\mu \neq 0$), this occurs because the χ dependent terms are always scale-invariant in this model, generating dynamically its vev (even when $\beta \to 0$, as found in Refs. [1, 138]). Had the scale symmetries been explicitly broken in the χ sector (e.g., via the inclusion of a cosmological constant), we would find that the massless mode would not be present except in the limit $\mu \to 0$, where the scale symmetry is recovered in the Φ sector.

To determine how the original fields depend on these three modes, we need to find the eigenvectors of the mass matrix (3.53). After some algebra, we can show that

$$\phi = N_\phi \begin{pmatrix} \frac{\beta v_\Theta v_\Phi}{3(m_\Phi^2 - C + D)} \\ 1 \\ \frac{\mu_\theta^2 v_\Theta v_\chi}{\tilde{M}'^2 (m_\chi^2 - C + D)} \end{pmatrix} \quad \theta = N_\theta \begin{pmatrix} \frac{\beta v_\Theta v_\Phi}{3(m_\Phi^2 - C - D)} \\ 1 \\ \frac{\mu_\theta^2 v_\Theta v_\chi}{\tilde{M}'^2 (m_\chi^2 - C - D)} \end{pmatrix} \quad \sigma = N_\sigma \begin{pmatrix} \frac{\beta v_\Theta}{\lambda v_\Phi} \\ 1 \\ \frac{v_\chi}{v_\Theta} \end{pmatrix}, \tag{3.55}$$

where N_ϕ, N_θ and N_σ are normalization factors, and

$$C = \frac{m_\Phi^2 + m_\Theta^2 + m_\chi^2}{2}, \qquad D = \frac{\sqrt{(-m_\Phi^2 - A^2 + B^2 + m_\chi^2)^2 + 4A^2 B^2}}{2}. \tag{3.56}$$

For the fifth-force contribution to the Møller scattering, we need only expand the χ and Φ fields in terms of the massless eigenmode, since they are the only ones coupling to the fermion and graviton directly. The relevant expansions take the forms

$$\chi = \frac{a}{N_\phi}\phi + \frac{b}{N_\theta}\theta + \frac{c}{N_\sigma}\sigma, \qquad \Phi = \frac{a'}{N_\phi}\phi + \frac{b'}{N_\theta}\theta + \frac{c'}{N_\sigma}\sigma, \tag{3.57}$$

where $\{a, b, ...\}$ are constant coefficients. Since we are interested in the massless mode, we only need to determine c and c', and, after some algebra, we have

$$c = \frac{\theta_3 - \phi_3}{(\theta_1 - \phi_1)(\sigma_3 - \phi_3) + (\sigma_1 - \phi_1)(\phi_3 - \theta_3)}, \tag{3.58a}$$

$$c' = -\frac{\theta_1 - \phi_1}{(\theta_1 - \phi_1)(\sigma_3 - \phi_3) + (\sigma_1 - \phi_1)(\phi_3 - \theta_3)}, \tag{3.58b}$$

where the subscripts refer to each component of the eigenvectors defined in Eq. (3.55), without the corresponding normalizing factor $N_{\{\phi,\theta,\sigma\}}$. We are now in a position to derive an expression for the effective Lagrangian in terms of the massless mode and subsequently calculate its contribution to the Møller scattering.

3.3.2 Møller scattering

After diagonalizing the mass terms in the Lagrangian, we have found all the different ways that the long-range fifth forces can couple to the matter fields. From the linearization of scalar-tensor gravity, the fifth forces arise through the kinetic mixing between the graviton and the σ field. In addition, after diagonalizing the mass terms, a new coupling between the massless mode and the fermion field appears as a result of their Yukawa interaction with the Φ field.

The terms in the Lagrangian relevant to the fifth force are as follows:

$$\begin{aligned}\mathcal{L}_{\mathrm{JF}} =& \frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma \\ &- \frac{cN_\sigma^{-1}}{\sqrt{2(2\omega+1)}}\partial_\mu h\partial^\mu\sigma - yc'N_\sigma^{-1}\bar{\psi}\sigma\psi + \frac{1}{M_{\mathrm{Pl}}}h^{\mu\nu}T_{\mu\nu} + \ldots,\end{aligned} \tag{3.59}$$

which will introduce long-range fifth forces to the Møller scattering through four distinct Feynman diagrams shown in Fig. 3.3. It is possible to calculate the matrix element for this process directly by calculating the Feynman rules for this Lagrangian and adding up the infinite series created by each diagram. However, it is more convenient to diagonalize the kinetic mixing between the scalar field σ and the trace of the graviton, h, such that the new dynamics are isolated from the gravitational interaction. In this chapter, we will take the latter approach for efficiency, although a full evaluation of the diagrams in Figure 3.3 can be found in Appendix C.

In Chapter 2.2, we already performed a diagonalization of a kinetic mixing with gravity, so we won't go through the details again this time. However, we remind the reader that a detailed derivation of the necessary transformations can be found in

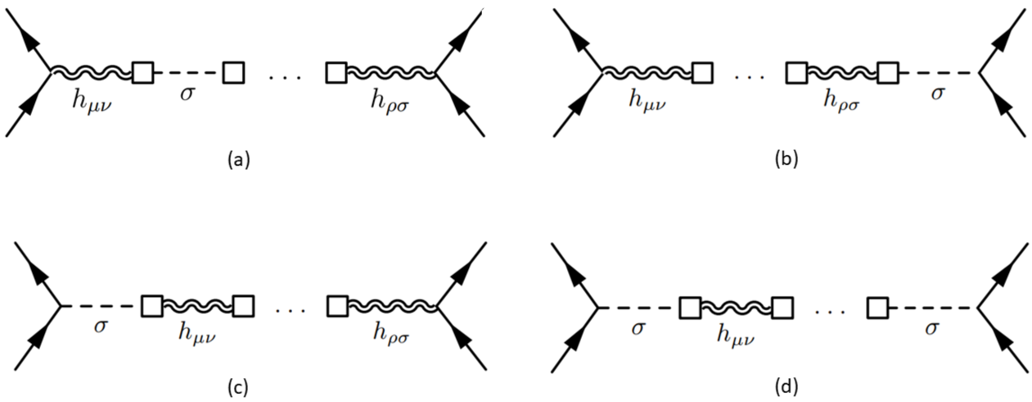


Figure 3.3: *The diagrams that contribute to the Møller scattering in the Jordan frame.*

Appendix B. For the specific Lagrangian in Eq. (3.59), we obtain

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{cN_\sigma^{-1}}{\sqrt{2(2\omega+1+2cN_\sigma^{-1})}}\sigma\eta_{\mu\nu}, \tag{3.60a}$$

$$\chi \to -\frac{\sqrt{2(2\omega+1)}}{\sqrt{2(2\omega+1+2cN_\sigma^{-1})}}\sigma. \tag{3.60b}$$

Recalling that $M_{\mathrm{Pl}} = v_\chi/\sqrt{2(2\omega+1)}$, we obtain the following diagonalized Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{JF}} =& \frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma \\ &+ \frac{yv_\chi c' N_\sigma^{-1}}{M_{\mathrm{Pl}}\sqrt{2(2\omega+1+2cN_\sigma^{-1})}}\bar{\psi}\sigma\psi \\ &+ \frac{1}{M_{\mathrm{Pl}}}\frac{cN_\sigma^{-1}}{\sqrt{2(2\omega+1+2cN_\sigma^{-1})}}\sigma T^\mu_\mu + \ldots,\end{aligned} \tag{3.61}$$

where once again we have just focused on the relevant terms for the long-range fifth forces. Therefore, to compute the matrix element for the Møller scattering, we just need to compute the Feynman diagram from Figure 3.4. Considering the Feynman rules from Figure 3.5, we obtain

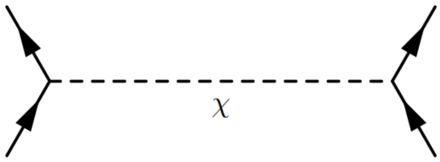
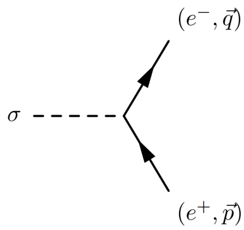


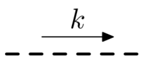
Figure 3.4: *Diagram for the fifth force modification to the Møller scattering in the Jordan frame after diagonalizing the kinetic mixing between the additional scalar field and the graviton.*

- Fermion-fermion-σ interaction



$$= i\frac{(yc'v_\chi + c\tau_\mu^\mu)}{cM_{\mathrm{Pl}}\sqrt{2(2\omega+3)}}; \quad \text{where} \quad \tau_\mu^\mu = \frac{1}{2}\Big(8m_e - 3(\not{q} + \not{p})\Big)$$

- σ field propagator:



$$= \frac{i}{k^2}$$

Figure 3.5: *Feynman rules for the diagonalized Lagrangian [Eq.* (3.61)*], where* $\not{p} = p_\mu \gamma^\mu$*, and* γ_μ *are the gamma matrices. To a good approximation, we can take* $cN_\sigma^{-1} \approx 1$*, since* $\tilde{M} \gg 1$.

$$i\mathcal{M} = \bar{u}(\mathbf{p}_1, s_1)\,(i\alpha_{\rm K})\,u(\mathbf{p}_3, s_3)\left(-\frac{i}{t}\right)\bar{u}(\mathbf{p}_2, s_2)\,(i\alpha_{\rm K})\,u(\mathbf{p}_4, s_4), \tag{3.62}$$

where, as before, $t = -(p_1 - p_3)^2$, $u(\mathbf{p}, s)$ and $\bar{u}(\mathbf{p}, s)$ are respectively the Dirac four-spinor and its Dirac conjugate, with spin projection s. Note that, for clarity, we have isolated each vertex and propagator with parentheses. For convenience, we have also defined the parameter

$$\alpha_{\rm K} = \frac{(yc'v_\chi + c\tau^\mu_\mu)}{cM_{\rm Pl}\sqrt{2(2\omega+3)}}. \tag{3.63}$$

Working in the non-relativistic limit and choosing the fermions to represent electrons with mass m_e, such that $p^\mu \sim q^\mu \approx (m_e, \vec{0})$, the spinors satisfy

$$\begin{aligned} \bar{u}(\mathbf{p}, s)u(\mathbf{q}, s') &= 2m_e\delta_{ss'} \\ \bar{u}(\mathbf{p}, s)\gamma_\mu u(\mathbf{q}, s') &= 2m_e\delta_{\mu 0}\delta_{ss'}, \end{aligned} \tag{3.64}$$

in which case, using the expression for τ^μ_μ extracted from Fig. 3.5, we have

$$\bar{u}(\mathbf{p}, s)\tau^\mu_\mu u(\mathbf{q}, s') = 2m_e^2\delta_{ss'}. \tag{3.65}$$

Inserting this result into Eq. (3.63), we obtain

$$\bar{u}(\mathbf{p}, s)\alpha_{\rm K}u(\mathbf{q}, s') = \frac{2m_e^2\,(\gamma+1)}{M_{\rm Pl}\sqrt{2(2\omega+3)}}\delta_{ss'}, \tag{3.66}$$

where we have defined $\gamma = (v_\chi c')/(v_\Phi c)$ and used $m_e^2 = yv_\Phi$. The matrix element then reduces to

$$\mathcal{M} = \frac{4m_e^4\,(\gamma+1)^2}{2M_{\rm Pl}^2 t(2\omega+3)}\delta_{s1s3}\delta_{s2s4}. \tag{3.67}$$

As for the Einstein frame, to extract the non-relativistic potential, we take $t = \mathbf{Q}^2$ (where $\mathbf{Q}$ is the exchange momentum), and the contribution to the Yukawa potential is

$$V_5(r) = -\frac{1}{M_{\rm Pl}^2}\frac{m_e^2\,(\gamma+1)^2}{2(2\omega+3)}\int\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3}e^{i\mathbf{Q}\cdot\mathbf{x}}\frac{1}{\mathbf{Q}^2} = -\frac{1}{4\pi r}\frac{m_e^2\,(\gamma+1)^2}{M_{\rm Pl}^2 2(2\omega+3)}. \tag{3.68}$$

After some algebra, we can show that

$$\gamma = \frac{c' v_\chi}{c v_\Phi} = -\frac{\beta v_\Theta^2}{\lambda v_\Phi^2}, \tag{3.69}$$

and, using the fact that $v_\Theta^2 = (\lambda v_\Phi^2 - 6\mu^2)/\beta$, we obtain the following final expression

$$V_5(r) = -\frac{1}{4\pi r}\frac{m_e^2}{M_{\rm Pl}^2 2(2\omega+3)}\frac{4\mu^4}{m_\Phi^4}, \tag{3.70}$$

where we recall that

$$m_\Phi^2 = 2\mu^2 + \frac{\beta v_\Theta^2}{3}. \tag{3.71}$$

This is in perfect agreement with the result in the Einstein frame[3] [Eq. (3.37)]. Notice therefore that we also find that the fifth force vanishes in the absence of explicit scale breaking ($\mu \to 0$), as we did in the Einstein frame.

3.4 Going beyond toy models: Should we get some help?

So far, we have studied the fifth forces that can arise in scalar-tensor theories of gravity by considering the tree-level matrix elements working both in the Einstein and the Jordan frame. For this, using field-theory techniques, we calculated the

[3]Since conformal transformations modify the rulers used to measure distances, we must compare dimensionless quantities, which are unaffected by coordinate transformations. This could, e.g., be the ratio of the fifth-force potential to the standard Newtonian potential. This is to say that the expressions for the potentials should match but with $\tilde{M}_{\rm Pl}$ for the Einstein frame and $M_{\rm Pl}$ for the Jordan frame.

Beyond Standard Model description of Brans-Dicke theories of gravity.

Depending on the frame of choice, we find that the new dynamics due to the modification of gravity have a different origin. In the Einstein frame, they couple to all scale-invariant quantities, meaning that for the Standard Model, they will leak through the Higgs field. Similar results can also be obtained working directly in the Jordan frame, where the trace of the graviton mixes with the non-minimally coupled scalar field. This implies that the fifth forces will exclusively depend on the trace of the energy-momentum tensor, which vanishes for scale-invariant terms. Moreover, we also studied this relation between scale breaking and the fifth force strength by considering a more complicated scale-breaking mechanism in which the scale invariance is broken both explicitly and dynamically.

Throughout this calculation, we have seen that working in the Jordan frame requires us to linearize the gravitational sector and to diagonalize the fields, while in the Einstein frame, we had to perform the Weyl transformation and various rescalings of the matter fields, losing the simplicity of the original action in the process. For the specific models considered in this thesis, the resulting fifth forces were always weaker than the gravitational interaction. However, this might not be the case for more generic scalar-tensor theories, for which calculating the equivalent BSM description will be considerably more difficult.

Additionally, we have consistently ignored all modifications to the dynamics of the matter sector that did not categorize into fifth forces (i.e., any coupling to kinetic terms or second-order interactions). Considering all these non-trivial operators for the Standard Model would be an impossible task to do by hand on a model-by-model basis. Thus, whichever approach we take, the overall message of this section is not a discussion on which frame is best for calculations, as it is a matter of preference, but the fact that deriving Feynman rules for scalar-tensor theories is a tedious and time-consuming task, even for the simplest models.

This begs for a tool that helps us automate this process so that it can be applied beyond toy models. In the rest of this thesis, we will introduce `FeynMG`, a `Mathematica` package that efficiently helps the user to perform the necessary manipulations to express any scalar-tensor theories as their equivalent BSM theory, tracking all the new interactions appearing in a given matter sector.

Chapter 4

FeynMG: Using symbolic algebra to study scalar-tensor theories

Using computational assistance to study the implications of modified gravity is a common practice due to the complex nature of these theories. The most common line of programming is to numerically solve Einstein's equations, which has been used to simulate phenomena such as screening effects on cosmic voids [114], modifications to black hole dynamics [154], gravitational waves [155] or galaxy clusters [156, 157], among others. However, since we aim to automate the manipulation and interpretation of a Lagrangian, it will be best to use functional languages.

In this context, `Mathematica` stands out as the best programming language, in which fields can be represented by functions and the inbuilt replacement rules allow us to do symbolic algebra. There are already packages built to deal with the complex tensor algebra that arises in General Relativity. `xAct` [158] is perhaps the most well-known package, having already been followed by multiple compatible packages that allow the study of gravity in different cosmological scenarios. In particular, the package `xIST/COPPER` [159] extends `xAct` for general scalar-tensor theories, and it was used in Ref. [160] to calculate the effect of modified gravity on cosmological perturbations.

Additionally, the utilization of `Mathematica` for understanding Lagrangians and performing phenomenological analysis is not an original idea, and this is precisely the main purpose of the renowned package `FeynRules` [143], whose original aim was to derive the Feynman rules from any given Lagrangian (allowing a consistent study of the Standard Model and BSM theories). However, such was the impact of this package that it led to the creation of high-energy physics analysis software that takes the output from `FeynRules` to do phenomenology associated with any given Lagrangian. Considering also previously existing packages that are also compatible with `FeynRules`, we highlight `CalcHep/CompHEP` [161, 162], `FeynArts` [163], `FeynCalc` [164], `FormCalc` [165], `MadGraph` [166, 167], `Sherpa` [168], `Whizard/Omega` [169] and `ASperge` [170].

Therefore, using and expanding the functionality provided by `FeynRules`, we present `FeynMG`[1] [2], a `Mathematica` package that helps the user to derive the BSM description from any scalar-tensor theory, allowing their definition within `FeynRules`. This, in combination with the just-mentioned software analysis packages, makes it possible to study the particle phenomenology arising from these kinds of modifications of gravity. In this sense, `FeynMG` extends `FeynRules` as `xIST/COPPER` extends `xAct`.

This chapter is structured as follows: In Section 4.1, we will introduce the current state of the art for using symbolic algebra to do field theory, and highlight what are the main problems we have to consider when building `FeynMG`. Then, we will focus on building up our intuition on `FeynMG` in Section 4.2. For this, we will demonstrate with a brief set of examples some of the basic functions introduced by `FeynMG`. Finally, in Section 4.3, we will present the more sophisticated routines by replicating the calculations shown in Chapters 2 and 3, where we will also show the

[1] The full available package can be found in here.

compatibility with other software analysis packages.

4.1 State of the art

FeynRules' well-deserved success can be attributed to the fact that using Mathematica to properly handle the different fields and parameters, while keeping track of the different algebras is a challenging task. For this, FeynRules uses as an input what is known as a "model file", which is a ".fr" document that contains all the information about the defined gauge groups, parameters, fields (or classes, as they call it) and Lagrangians. For an extensive description on how to build these files, see Ref. [143].

All of this together, allows FeynRules to understand the Lagrangian as our scientific convention dictates, but with an efficiency that only a computer can achieve. In return, for FeynRules to correctly evaluate a given Lagrangian, it needs the following two conditions to be satisfied:

1. The Lagrangian must be in a flat spacetime background, and
2. All fields must be canonically normalized, and both kinetic and mass matrices must be diagonal. [2]

It is important to point out that FeynRules will not crash in the event of a violation of either of these two conditions. Instead, when generating the Feynman rules or producing the output for other software analysis packages, FeynRules will only consider terms of higher order than quadratic and assume that all fields are canonically normalized. Thus, it ignores any information about mass and kinetic

[2] There is an exception with mass mixings, in which FeynRules evaluates them correctly as long as they are properly specified in the model file. However, this is not usually the case for our type of calculations.

mixing matrices, which may lead to incorrect evaluation of any phenomena. Therefore, it is necessary to ensure the satisfaction of both conditions to obtain accurate results.

In this way, we encounter the main problem when trying to extend `FeynRules` to scalar-tensor theories, as these theories are classically formulated within the framework of General Relativity, making it inconsistent to take the flat spacetime limit right away in the classical action. However, here is where the benefits of the BSM description become apparent. By using this approach, the modified Lagrangian satisfies both conditions, allowing us to use the functionality of `FeynRules` to study the phenomenological implications of scalar-tensor theories.

The challenge, however, lies in extracting the BSM theory from the modified theory of gravity, while accurately accounting for the various new couplings appearing in the matter sector. As seen previously, dealing with this requires linearization of the extended gravitational sector, transformations of the metric, expansion around non-trivial vacuum configurations, the diagonalization of kinetic and mass mixings, and the truncation of infinite series of operators [1, 138].

Doing this is the aim of `FeynMG` [2], which provides the functionality to implement the minimal gravitational couplings to the Lagrangian and append any desired extended gravitational sector. Then, `FeynMG` linearizes gravity and performs the necessary redefinitions to the fields such that the BSM description is obtained and thus both mentioned conditions are satisfied. Once this is achieved, the generated model can be processed using the existing `FeynRules` package and its interfaces.

In what follows, we will provide a number of examples, some simple and some advanced, to show how to use `FeynMG` and in what situations each function must be used. Although we will be working with specific models, the functions presented apply to any type of Lagrangian, making them generic to any scalar-tensor theory of choice. Moreover, these examples will allow us to check the validity of the code against our analytical understanding, as we will point out throughout the rest of this Chapter.

4.2 The basics of FeynMG

FeynMG has dependencies on FeynRules, so both packages need to be loaded into Mathematica to make use of FeynMG. This can be done by running

```
In[1]:= << FeynRules`;
        << FeynMG`;
```

within the appropriate working directory (set via SetDirectory[]). The next step is then to load a model file that is compatible with FeynRules using the FeynRules function LoadModel[].

We aim to make the code as easy to use as possible without losing the generality in the model files and desired gravitational actions. For that, FeynMG provides a wide functionality that will help the user to get the classical Lagrangian to the point in which it can be consistently understood and used by FeynRules. In this section, we will present some of the basic functions, so that we gradually build up intuition for FeynMG before diving into more sophisticated routines with generic examples. For clarity, each family of functions will be presented in a separate subsection.

4.2.1 InsertCurv: Inserting minimal couplings to gravity

Following the steps from Chapter 2, the first thing to do is to ensure that all the minimal couplings to gravity in the matter sector are properly incorporated. In FeynMG, this can be easily achieved by manually introducing the metric or vierbeins using the functions **gUp**, **gDown**, **vierUp** and **vierDown**, in which the suffixes "**Up**" and "**Down**" refer to the position (upper- or lower-indices) of the Lorentzian indices in the curvature objects. Similarly, it is possible to add gravitational covariant derivatives by using **CovDev**.

Due to the extensive FeynRules usage, many users may already have created model files that contain a high number of terms in them, making it a tedious task to manually add every gravitational coupling where necessary. To address this, FeynMG introduces a function called **InsertCurv[]**, allowing to use an action defined in flat spacetime as an input, and so reuse a FeynRules model file without modifying it. For this, **InsertCurv** inserts a metric $g^{\mu\nu}$ or vierbein $e^{\mu a}$, as appropriate, at every pair of contracted indices and promotes partial derivatives to covariant derivatives. As an example, we will consider the following subset of terms from a flat-spacetime Lagrangian:

$$\texttt{I1} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\partial^\mu A_\mu\partial^\nu A_\nu - eA_\mu\gamma^\mu\bar{\psi}\psi, \tag{4.1}$$

where ϕ, A_μ and ψ are a generic set of scalar, U(1) gauge and fermionic fields, respectively. Commonly, we would expand it into

$$\texttt{I1} = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g^{\mu\sigma}g^{\nu\rho}\partial_\sigma A_\mu\partial_\rho A_\nu - eA_\mu\gamma^a e^\mu_a\bar{\psi}\psi, \tag{4.2}$$

where we have not specified the Jacobian prefactor, $\sqrt{-g}$, because the standard input in FeynMG's functions is the Lagrangian density. However, unless explicitly specified otherwise, this term will be automatically taken into account when applying any function to a given Lagrangian. Using FeynMG, this operation is satisfied via

```
In[2]:= InsertCurv[I1]
```

Out[2]= $\frac{1}{2}\partial_{\text{a2}}\texttt{[phi]}\,\partial_{\text{mu}}\texttt{[phi]}\,\texttt{gUp[a2,mu]} - \frac{1}{4}\texttt{D}^{\texttt{Grav}}{}_{\texttt{nu}}\texttt{[A}_{\texttt{a3}}\texttt{]}\,\texttt{D}^{\texttt{Grav}}{}_{\texttt{a4}}\texttt{[A}_{\texttt{mu}}\texttt{]}$
$\texttt{gUp[a3,mu]}\,\texttt{gUp[a4,nu]} - \texttt{e}\,\texttt{A}_{\texttt{mu}}\,\bar{\texttt{psi}}_{\texttt{i1,i2}}.\texttt{psi}_{\texttt{j1,i2}}\gamma_{\texttt{v1}}{}^{\texttt{i1,j1}}\,\texttt{VUp[mu,v1]}$

agreeing exactly with Eq. (4.2). Looking closely at the last term, corresponding to the usual electromagnetic interaction, we find multiple indices associated to the fermionic field. These indices correspond to the spin and generation of the fermionic fields. To consistently track these distinct indices, `FeynRules` allows us to specify them using the function **`Index[X,Y]`**, where **`Y`** represents the index and **`X`** denotes the type (either **`Spin`**, **`Generation`** or **`Lorentz`**, in this context). Thus, it is worth noting that **`InsertCurv`** only considers and expands Lorentzian indices, those written as **`Index[Lorentz,Y]`**.

Furthermore, notice that the covariant derivative only operates on the gauge field and has not been expanded in terms of the metric. This deliberate choice is for later convenience, allowing the user to choose the modified covariant derivative from Eq. (2.29), which can only be constructed once the gravitational sector is appended. Similarly, had we considered a fermionic kinetic energy term in our previous example, **`InsertCurv`** would conveniently insert the spin connection as follows:

```
In[3]:= InsertCurv[barpsi.del[psi,Index[Lorentz,mu]]
```

$$\text{Out[3]=}\quad \mathtt{p\bar{s}i}_{\mathtt{i1,i2}}.\partial_{\mathtt{mu}}[\mathtt{psi}_{\mathtt{j1,i2}}]\,\gamma_{\mathtt{i1,j1}}{}^{\mathtt{v2}}\,\mathtt{VUp[mu,v2]} + \ldots\ldots + \frac{1}{8}\mathtt{i}\,\partial_{\mathtt{mu}}[\mathtt{VUp[d1,c1]}]\,\mathtt{p\bar{s}i}_{\mathtt{i1,i2}}.\mathtt{psi}_{\mathtt{j1,i2}}\,\mathtt{VDown[c2,c1]}\ \mathtt{VUp[mu,v3]}\,\gamma^{\mathtt{c2}}.\gamma^{\mathtt{d1}}.\gamma^{\mathtt{v3}}{}_{\mathtt{i1,j1}}$$

where the ellipsis contains the rest of the terms inside the spin connection. Readers might wonder why we opted not to keep the spin connection as a generic term, similar to the covariant derivative of the gauge field. In this case, the kinetic energy for the fermion is scale invariant, meaning that there is no need for a modified spin connection since any new dynamics would not couple to this sector. However, it is still possible to use a different choice for the spin connection by manually making the necessary replacement.

4.2.2 AddParameter/AddScalar: Creating a gravitational sector

After incorporating all the necessary minimal curvature dependencies into the Lagrangian, the next step is to append a gravitational action, wherein, e.g., the Ricci scalar can be specified using **RScalar** (see Appendix D.1 for the list of defined curvature objects).

Given the available set of curvature objects, it now becomes possible to define any kind of scalar-tensor theory by including non-minimal couplings to any scalar present in the model file. However, as is the case for FeynRules, it is necessary to identify any set of new fields and parameters in the model file before loading it. In order to stick to the principle of reusing old model files, FeynMG provides two additional functions, **AddScalar[]** and **AddParameter[]**. These functions allow the user to properly define new scalar fields and parameters, respectively, once the package is running, providing complete freedom when creating the gravitational sector.

As an example, let's consider that the user wants to define a new massless scalar field, **S1**, with a vanishing decay width, and a new parameter **cx**, with a value of 0.5 eV and an **InteractionOrder** of **{MG,2}** (this will be important later for compatibility with MadGraph [166]).[3] The user can do this by using

```
In[4]:= AddScalar[S1];
```

We can verify that these functions are properly defined within FeynRules by checking if they appear in the loaded list of fields and parameters. This is done as follows

```
In[5]:= M$Parametters[[-1]]
```

[3]See Appendix D.8 for the available options on these functions.

```
Out[5]= {cx=={ParameterType→Internal, Value→0.5,
        InteractionOrder→{MG,2}}}
        {S[N]=={ClassName→S1,SelfConjugate→True,Mass→{MS1,0},
        Width→{WS1,0},
        PropagatorLabel→S1,PropagatorType→D,PropagatorArrow→None,
        ParticleName→S1,FullName→S1}}
```

where the `N` in `S[N]` refers to the total number of scalar fields in the theory. Once the variables have been properly defined, they can be treated consistently by `FeynRules` and `FeynMG`. One way to see this is using the following functions:

```
In[6]:= numQ[cx]

Out[6]= True
```

meaning that `FeynRules` ensures that `cx` is number-like (meaning that its derivative vanishes), and

```
In[7]:= {FieldQ[S1],ScalarFieldQ[S1],FermionQ[S1]}

Out[7]= {True,True,FermionQ[S1]}
```

which shows that it will be treated as a scalar field, and not like a fermionic field. This is crucial since certain routines should only apply to the scalar fields (such as vev calculations and expansions).

4.2.3 ToEinsteinFrame/LinearizeGravity: Going to the Einstein frame or staying in the Jordan frame

The most common way to deal with the particular case of Brans-Dicke gravity is to perform a Weyl transformation (see Eq. (1.17)) such that the gravitational sector is of Einstein-Hilbert form and the matter action is instead dressed with additional scalar interactions. This is implemented in `FeynMG` by the function **`ToEinsteinFrame`**, for which, as an example, we will demonstrate its action on the following simple Lagrangian:

$$\mathtt{I2} = -\frac{\chi^2}{2}R + \frac{w}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{1}{4}g^{\mu\sigma}g^{\nu\rho}\nabla_\sigma A_\mu \nabla_\rho A_\nu, \tag{4.3}$$

where we have introduced a Brans-Dicke type theory with a non-minimally coupled scalar field, χ, and a Weyl invariant gauge fixing term for a U(1) gauge field, A_μ. The physical consistency of this Lagrangian is not important in this example, as our focus lies on the effect of the conformal transformation on `I2`. As already covered in Chapter 2, this Lagrangian in the Einstein frame is transformed into

$$\tilde{\mathtt{I2}} = -\frac{\tilde{M}_{\rm Pl}^2}{2}\tilde{R} + \frac{(2w+3)\tilde{M}_{\rm Pl}^2}{4\chi^2}\tilde{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \frac{1}{4}\tilde{g}^{\mu\sigma}\tilde{g}^{\nu\rho}\tilde{\nabla}_\sigma A_\mu \tilde{\nabla}_\rho A_\nu, \tag{4.4}$$

where once again every tilded quantity is constructed with the Einstein frame metric, $\tilde{g}^{\mu\nu}$. After all fields and parameters are properly defined in the model file, we may replicate this operation using `FeynMG` by running

In[8]:= $\tilde{\mathtt{I2}}$**`=ToEinsteinFrame[I2]`**

Out[8]= $-\frac{1}{2}\mathtt{M^2{}_{pl}R_{Sc}} + \frac{\mathtt{M^2{}_{pl}\ w\ \partial_{mu}[chi]\partial_{nu}[chi]\ gUp[mu,nu]}}{\mathtt{2chi^2}} + \frac{\mathtt{3M^2{}_{pl}\ \partial_{mu}[chi]\partial_{nu}[chi]\ gUp[mu,nu]}}{\mathtt{4chi^2}} - \frac{1}{4}\mathtt{D^{Grav}{}_{a1}[A_{mu}]D^{Grav}{}_{a2}[A_{nu}]gUp[a1,mu]gUp[a2,nu]}$

where we can see that `FeynMG` has identified the non-minimal coupling and rescaled the whole action by applying the corresponding set of transformation rules on every gravitational object present in the Lagrangian. In the process, it has also introduced the Einstein frame's Planck mass $\mathtt{M_{pl}}$, which is automatically added and treated as a constant parameter (i.e., `FeynMG` has directly executed **`AddParameter[Mpl]`**).

Notice that we have not included the $\sqrt{-g}$ factor in the Lagrangian; this is because, for simplicity, `FeynMG` always assumes this term to be present, although it could be omitted using the option **`{Jacobian->Off}`**. At this point, all gravitational contributions to the Lagrangian can be ignored by running

```
In[9]:= GravityOff[I2]
```

$$\text{Out[9]=}\quad \frac{M^2{}_{pl}\, w\, \partial_{mu}[chi]\partial_{mu}[chi]}{2chi^2} + \frac{3M^2{}_{pl}\, \partial_{mu}[chi]\partial_{mu}[chi]}{4chi^2} - \frac{1}{4}\partial_{mu}[A_{mu}]\partial_{nu}[A_{nu}]$$

which imposes the set of rules $\{h_{\mu\nu} \to 0, h^{\mu\nu} \to 0\}$, so that the modifications can be directly studied as a BSM theory.

More general scalar-tensor theories may not have an Einstein frame, forcing us to stay in the Jordan frame and proceed by linearizing gravity. This is implemented by the function **`LinearizeGravity`**, where the gravitational sector is expanded up to second order, generating the kinetic energy for the graviton, and the matter sector gets expanded up to linear order in the interactions with the metric perturbation $h_{\mu\nu}$. Since providing a brief and simple example for this routine is not feasible, we will dedicate Chapter 4.3 to explain in detail this specific process of linearising gravity and canonically normalizing the gravitational sector.

4.2.4 `CanonScalar/MassDiagMG/KineticDiagMG`: Canonicalizing the scalar sector

So far, the mentioned functions address the first of the `FeynRules` conditions (stated on Page 75), allowing for the creation of a scalar-tensor theory Lagrangian and the expansion around a flat spacetime background. However, the second condition is yet to be satisfied by these kinds of models, as most fields still need to be canonically normalized. In this section, we will show how to achieve canonical normalization for scalar fields using `FeynMG`, leaving the case for the graviton for Chapter 4.3.

Independent of our choice of frame, we encounter that the non-minimally coupled field becomes non-canonical due to either the conformal transformation to the Einstein frame or by the linearization of gravity in the Jordan frame. For scalar fields, the canonical normalization is implemented by the function **`CanonScalar`**, which will find and normalize the lowest-order derivative term of every field. In the case where the lowest order is too complicated, one can use the in-built `Mathematica` function **`Series`** to perform a series expansion up to the required order term. As an example, let's consider the following Lagrangian in flat space

$$\mathtt{I3} = \frac{3}{\chi}\partial_\mu\chi\partial^\mu\chi + \frac{3}{\phi}\partial_\mu\phi\partial^\mu\phi + \phi^2 + \chi\phi. \tag{4.5}$$

At first glance, it might appear that this Lagrangian presents a mass mixing between the scalar fields. However, for this to be true, we first need to canonically normalize the scalar fields. This is achieved by running

```
In[10]:= CanonScalar[I3]
```

$$\text{Out[10]=}\quad \frac{\mathtt{chi}^2\,\mathtt{phi}^2}{576} + \frac{\mathtt{phi}^4}{576} + \frac{1}{2}\partial_{\mathtt{mu}}[\mathtt{chi}]^2 + \frac{1}{2}\partial_{\mathtt{mu}}[\mathtt{phi}]^2$$

where it is now apparent that there is no mass or kinetic mixing between the scalar fields. However, if there were any mixing, **`MassDiagMG`** and **`KineticDiagMG`** can be used to diagonalize the scalar field masses and kinetic energies, respectively. Consider the following Lagrangian that includes an explicit kinetic mixing

$$\mathtt{I4} = \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\phi - 3\phi^2 - \frac{1}{2}\chi^2 \tag{4.6}$$

`FeynMG` will diagonalize the kinetix matrix via

```
In[11]:= I5=KineticDiagMG[I4]
```

Out[11]= $-\frac{7\,\text{chi}^2}{6}-\frac{5\,\text{chi}\,\text{phi}}{\sqrt{3}}-\frac{7\,\text{phi}^2}{2}+\frac{1}{2}\partial_{\text{mu}}[\text{chi}]^2+\frac{1}{2}\partial_{\text{mu}}[\text{phi}]^2$

Now, we can observe that the kinetic diagonalization has introduced a mass mixing due to the transformation we just applied, which also affects the mass matrix. This can be solved by running the following function

In[12]:= `MassDiagMG[I5]`

Out[12]= $-\frac{7\,\text{chi}^2}{3}-\frac{\sqrt{31}\,\text{chi}^2}{3}-\frac{7\,\text{phi}^2}{3}+\frac{\sqrt{31}\,\text{phi}^2}{3}+\frac{1}{2}\partial_{\text{mu}}[\text{chi}]^2+\frac{1}{2}\partial_{\text{mu}}[\text{phi}]^2$

Note that while the transformations to the kinetic matrix affected the diagonalized mass matrix of the theory, the opposite is not true, since a diagonalized and canonical kinetic matrix is proportional to the identity matrix.

`FeynMG` uses a similar procedure for **`KineticDiagMG`** and **`MassDiagMG`**. First, it isolates the quadratic couplings in the Lagrangian and extracts the kinetic/mass matrix for the theory. Then, it calculates the eigenvectors of the corresponding matrix and applies the subsequent rules on the existing fields so that the resulting kinetic and mass matrices in the Lagrangian are diagonal. Thus, using this set of functions the user can manipulate the scalar sector to leave it canonically normalized and diagonalized, such that it can be consistently evaluated by `FeynRules` functions. In cases where the expressions are too long and for which it might be challenging to determine whether the scalar fields are canonically normalized, `FeynMG` introduces the following checking functions: **`CheckCanonScalar`**, **`CheckKineticMatrix`** and **`CheckMassMatrix`**.[4]

4.2.5 `IndexSimplify`: Introducing Einstein's index notation

When dealing with tensor algebra, we are used to working with Einstein's index notation, for which the following identity holds: $A_\mu A^\mu = A_\rho A^\rho$. However, `Mathematica`

[4]See Appendix D.6 for a brief description on these functions.

will treat both terms $A_\mu A^\mu$ and $A_\rho A^\rho$ as distinct, since their indices are not represented by the same variable. A simple example for that may be that

$$\texttt{I6} = A_\sigma A^\sigma \partial_\mu A^\nu \partial_\nu A^\mu - A_\rho A^\rho \partial_\alpha A^\beta \partial_\beta A^\alpha, \tag{4.7}$$

trivially cancels due to the fact that repeated indices get summed over. However, `Mathematica` will just output

```
In[13]:= I6
```

```
Out[13]= AσAσ∂μAν∂νAμ-AρAρ∂αAβ∂βAα
```

This problem appears when linearizing gravity, as it leads to an overly complicated and long expression filled with repeated terms. The good news, however, is that these expressions will simplify when performing any phenomenological calculation since the software automatically performs the explicit summation over repeated indices. Nevertheless, reducing the length of the expressions can still be important for the simplicity and efficiency of the output.

To address this issue, `FeynMG` introduces the function **`IndexSimplify`**, which tackles the problem by replacing indices term by term from a given set of indices. By applying this function on **`I3`**, we obtain

```
In[14]:= IndexSimplify[I6]
```

```
Out[14]= 0
```

To understand what exactly occurs when using **`IndexSimplify`**, let's consider the scenario where we have two indexed terms that are not equivalent, such as

$$\texttt{I7} = A_\sigma A^\sigma \partial_\mu A^\nu \partial_\nu A^\mu + \partial_\rho \chi \partial^\rho \chi. \tag{4.8}$$

Then, **IndexSimplify** replaces every Lorentz-index in all the terms using the same set of unique variables, such that

```
In[15]:= IndexSimplify[I7]
```

```
Out[15]= A_MG1 A_MG1 ∂_MG2 A_MG3 ∂_MG3 A_MG2 + ∂_MG1 chi ∂_MG1 chi
```

We can see that if both terms were equivalent, they would get the same set of indices and thus combine. Additionally, we can also choose which set of indices **IndexSimplify** uses as a substitution. This can be done by specifying

```
In[16]:= IndexSimplify[I7, {a,b}]
```

```
Out[16]= A_a A_a ∂_b A_MG1 ∂_MG1 A_b + ∂_a chi ∂_a chi
```

where we can see that once the provided indices run out, FeynMG automatically uses unique variables (**MG1** in this case). **IndexSimplify** uses Mathematica's automatic alphabetical ordering of products of functions, so that similar terms are always ordered the same. However, even in the case where a subset of terms in a Lagrangian don't add up perfectly or cancel, this function would still have improved the efficiency of the code by simplifying the expression. Furthermore, as mentioned above, once any Lagrangian is used to perform a phenomenological calculation, the software automatically performs the summation over repeated indices, leading to an explicit cancellation of the repeated terms.

In the next section, we will see how this function is crucial for simplifying the linearized gravitational action, recovering the usual Fierz-Pauli terms and noticeably optimizing the time consumption of any further operation on the Lagrangian.

4.2.6 GiveMpl/InsertMpl: Finding the strength of gravity

In scalar-tensor theories, it is important to track the effective Planck mass of the theory due to the possible constraints that it imposes on a given model. For this, `FeynMG` presents **`GiveMpl`**, a function that automatically infers the value of this parameter from any given Lagrangian. Before showing the usage of this function, let us recall how the Planck mass can be extracted in the different stages of the calculations shown in Chapters 2 and 3.

For the classical action, the effective Planck mass will always be trivially inferred via the constant prefactor of the Ricci scalar in the gravitational action. For the following theory:

$$\texttt{I8} = -\frac{(a+b\chi)^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{4.9}$$

the effective Planck mass will be defined as $M_{\rm Pl} = a + bv_\chi$, where v_χ is the vacuum expectation value for the χ scalar field. However, when considering linearized theories of modified gravity, the Ricci scalar might not be as traceable as in the non-perturbed theory. For example, for the action described above, we would find the following Lagrangian

$$\begin{aligned}\texttt{I9} =& \frac{(a+bv_\chi)^2}{4}\left(\frac{1}{2}\partial_\mu h_{\sigma\rho}\partial^\mu h^{\sigma\rho} - \frac{1}{4}\partial_\mu h\partial^\mu h\right) - \frac{b(a+bv_\chi)}{2}\partial_\mu h\partial^\mu\chi \\ &+ \frac{1+2b^2}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}h_{\mu\nu}T^{\mu\nu},\end{aligned} \tag{4.10}$$

where, for this model,

$$T^{\mu\nu} = \partial^\mu\chi\partial^\nu\chi + \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}(\partial^\mu\chi\partial_\mu\chi + \partial^\mu\phi\partial_\mu\phi). \tag{4.11}$$

In this case, we will have to locate the Fierz-Pauli terms, corresponding to the first term of Eq. (4.10), in order to trace the Ricci scalar, and so the effective Planck mass. If the graviton is not yet canonically normalized, the constant prefactor to the graviton's kinetic energy will be the Planck mass squared. However, if the graviton has already been canonically normalized, notice that the action will take the following form:

$$\texttt{I10} = \frac{1}{2}\partial_\mu h_{\sigma\rho}\partial^\mu h^{\sigma\rho} - \frac{1}{4}\partial_\mu h\partial^\mu h - b\partial_\mu h\partial^\mu\chi + \frac{1+2b^2}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{(a+bv_\chi)}h_{\mu\nu}T^{\mu\nu}, \tag{4.12}$$

and so we will have to extract the effective Planck mass from the energy-momentum prefactor.

Therefore, using **GiveMpl**, FeynMG will be able to recognize each of the cases (**I5,I6** or **I7**), and extract the effective Planck mass, making it usable at any point in the calculation. Moreover, using the function **InsertMpl**, it will both calculate the effective $M_{\rm Pl}$ from the action and substitute it into the expression. Using the same cases as before, we can see that for the non-perturbed action, it trivially replaces

```
In[17]:= InsertMpl[I8]

Out[17]= Using the values defined in the parameter and mass classes, the
         effective value for Mpl in this Lagrangian is a + b vevchi GeV.
         It can be changed by modifying the Model File obtained after
         using OutputModelMG.
```

$$-\frac{1}{2}\,\mathrm{M_{pl}}^2\,\mathrm{R_{Sc}} + \frac{1}{2}\partial_{\mathrm{mu}}[\mathrm{chi}]^2 + \frac{1}{2}\partial_{\mathrm{mu}}[\mathrm{phi}]^2$$

Similarly, if the linearized theory is not canonically normalized, it will track the Planck mass from the Fierz-Pauli kinetic terms and replace

```
In[18]:= InsertMpl[I9]

Out[18]= Using the values defined in the parameter and mass classes, the
         effective value for Mpl in this Lagrangian is a + b vevchi GeV.
         It can be changed by modifying the Model File obtained after
         using OutputModelMG.
```

$$\frac{1}{2}\partial_{\text{mu}}[\text{chi}]^2 + b^2\partial_{\text{mu}}[\text{chi}]^2 + \frac{1}{8}M_{\text{pl}}{}^2\partial_{\text{rho}}[h_{\text{mu,nu}}]^2 - \frac{1}{2}b\,M_{\text{pl}}\partial_{\text{mu}}[\text{chi}]\partial_{\text{mu}}[h]$$
$$-\frac{1}{16}M_{\text{pl}}{}^2\partial_{\text{rho}}[h]\partial_{\text{rho}}[h] + \ldots + \frac{1}{4}\partial_{\text{mu}}[\text{phi}]^2 h$$

Finally, once the graviton is canonically normalized, `FeynMG` will trace the prefactor of the graviton interaction to the energy-momentum tensor, leading to

In[19]:= **InsertMpl[I10]**

Out[19]=
```
Using the values defined in the parameter and mass classes, the
effective value for Mpl in this Lagrangian is a + b vevchi GeV.
It can be changed by modifying the Model File obtained after
using OutputModelMG.
```

$$\frac{1}{2}\partial_{\text{mu}}[\text{chi}]^2 + b^2\partial_{\text{mu}}[\text{chi}]^2 + \frac{1}{8}\partial_{\text{rho}}[h_{\text{mu,nu}}]^2 - b\partial_{\text{mu}}[\text{chi}]\partial_{\text{mu}}[h]$$
$$-\frac{1}{4}\partial_{\text{rho}}[h]\partial_{\text{rho}}[h] + \ldots + \frac{1}{2M_{\text{pl}}}\partial_{\text{mu}}[\text{phi}]^2 h$$

As expected, for the same theory we find the same effective M_{Pl} at different stages of the calculation, which serves as a direct proof of the validity of the function.

4.3 Advanced `FeynMG` routines and examples

In this guide through `FeynMG`, there are some key routines that cannot be fully demonstrated with basic examples. These functions primarily address the canonical normalization of the graviton kinetic energy. However, before diving into the more complicated examples, let us provide a brief overview of the functions that will play an important role.

Depending upon the gravitational action, we might need to expand the scalar fields around their vacuum expectation values. This is possible using **`VevExpand`**, which first calculates all the possible values for the vevs, and then shifts all the fields around the user's chosen branch of solutions. Once the graviton kinetic energy has a constant prefactor, we can then use **CanonGravity**, leaving all the fields canonically normalized with derivative interactions.

When proceeding in the Jordan frame, as we saw in the last section, the dominant modifications to the dynamics arise through kinetic mixing between the additional scalar field and the trace of the graviton (cf., e.g., Figure 2.1). The function **GravKinMixing** will calculate and substitute into the Lagrangian the field redefinitions that diagonalizes this kinetic mixing, i.e., the equivalent of Eq. (2.37). With this, the Lagrangian should be in a form ready to be used by FeynRules.

Finally, linearizing gravity and manipulating the Lagrangian into a form amenable to FeynRules can take significant computing time for extensive or complicated models. So that this process does not need to be repeated each time, the user can use the function **OutputModelMG** to create a new model file from the final form of the Lagrangian produced by FeynMG, which includes all the information about the redefined fields, the parameters of the extended model and the effective Lagrangian itself. This model file can then be used directly in FeynRules without the need to rerun the routines implemented by FeynMG.

In this section, we will combine all the functions mentioned thus far to offer a complete demonstration of how to use FeynMG to perform the manipulations described in the preceding chapters. In Appendix D, we provide a summary of the tools provided by FeynMG.

4.3.1 Example 1: Defining a gravitational action and transforming to the Einstein Frame

Throughout this section, we will work with the same Lagrangian from Eq. (2.9), whose matter sector is defined via

$$\begin{aligned}\texttt{LQED} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \alpha\partial^\mu A_\mu\partial^\nu A_\nu \\ &+ i\bar{\psi}\gamma^\mu\partial_\mu\psi - q\bar{\psi}\gamma^\mu A_\mu\psi - y\bar{\psi}\phi\psi \\ &+ \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{3\mu^4}{2\lambda},\end{aligned} \tag{4.13}$$

where we have substituted a double-well potential for the would-be Higgs field and introduced a generic covariant gauge-fixing term for the U(1) gauge field.

The first thing to do is to introduce the minimal gravitational couplings of this matter Lagrangian. This amounts to inserting metrics or vierbeins, as appropriate, for each pair of contracted indies, and promoting all partial derivatives to covariant ones. To implement this in `FeynMG`, we run

```
In[20]:= LCurv=InsertCurv[LQED]
```

$$\text{Out[20]=}\; -\frac{3\,\mu^4}{2\,\mathtt{lam}} + \frac{\mu^2\,\mathtt{phi}^2}{2} - \frac{\mathtt{lam\ phi}^4}{24} - \mathtt{y\,phi}\,\bar{\mathtt{psi}}_{\mathtt{i1,i2}}\mathtt{psi}_{\mathtt{i1,i2}}$$
$$+\frac{1}{2}\partial_{\mathtt{a2}}[\mathtt{phi}]\,\partial_{\mathtt{mu}}[\mathtt{phi}]\,\mathtt{gUp[a2,mu]} - \frac{1}{4}\mathtt{D^{Grav}}_{\mathtt{nu}}[\mathtt{A_{a3}}]\,\mathtt{D^{Grav}}_{\mathtt{a4}}[\mathtt{A_{mu}}]$$
$$\mathtt{gUp[a3,mu]\;gUp[a4,nu]} + \;.....\; - \mathtt{e}\,\mathtt{A_{mu}}\,\bar{\mathtt{psi}}_{\mathtt{i1,i2}}\cdot\mathtt{psi}_{\mathtt{j1,i2}}\gamma_{\mathtt{i1,j1}}{}^{\mathtt{v1}}$$
$$\mathtt{VUp[mu,v1]} + \mathtt{i}\,\bar{\mathtt{psi}}_{\mathtt{i1,i2}}\cdot\partial_{\mathtt{mu}}[\mathtt{psi}_{\mathtt{j1,i2}}]\;\gamma_{\mathtt{i1,j1}}{}^{\mathtt{v2}}\;\mathtt{VUp[mu,v2]}$$
$$+\frac{1}{8}\mathtt{i}\;\partial_{\mathtt{mu}}[\mathtt{VUp[d1,c1]}]\,\bar{\mathtt{psi}}_{\mathtt{i1,i2}}\cdot\mathtt{psi}_{\mathtt{j1,i2}}\;\mathtt{VDown[c2,c1]}$$
$$\mathtt{VUp[mu,v3]}\;\gamma^{\mathtt{c2}}\cdot\gamma^{\mathtt{d1}}\cdot\gamma^{\mathtt{v3}}{}_{\mathtt{i1,j1}}\;+\;.....$$

We remind that `gUp[a,b]` and `gDown[a,b]` are upper- and lower-indexed metrics, respectively, `VUp[a,b]` and `VDown[a,b]` are upper- and lower-indexed vierbeins, respectively, and $\mathtt{D^{Grav}}_{\mathtt{a}}\mathtt{[]}$ is the gravitational covariant derivative.

Since the expressions can be long, we will show only the main sections of the output that motivate the next step in the calculation and represent the rest of the terms by ellipses.

For this example, we will introduce a Brans-Dicke gravitational sector of the form of Eq. (1.16), such that

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{\chi}{2}R + \frac{\omega}{2\chi}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + \frac{1}{2}\mu_\chi^2\chi - \frac{\lambda_\chi}{4!}\chi^2 - \frac{3\mu_\chi^4}{2\lambda_\chi} + \texttt{LCurv}\right], \quad (4.14)$$

where the χ field should not be confused with the one defined in Eq. (2.34). Before defining the gravitational part of the Lagrangian within `FeynMG`, we need to give appropriate attributes to the additional field χ and the additional parameters $(\{\omega, \mu_\chi, \lambda_\chi\})$. In principle, these can be directly added by updating the model file itself (which should be done before loading it into `FeynRules`). However, we will use the `FeynMG` functions **`AddScalar[]`** and **`AddParameter[]`**,[5] which allow the new scalar fields and parameters to be defined after the model file has been loaded into `FeynRules`. For the specific case of Eq. (4.14), we need to execute the following:

```
In[21]:=  AddScalar[chi];
          AddParameter[muC];
          AddParameter[lamC];
          AddParameter[w];
```

The full Lagrangian can then be defined via

```
In[22]:= LJordan= LCurv - chi RScalar/2 + (w/(2chi))
         gUp[Index[Lorentz,mu],Index[Lorentz,nu]]
         del[chi,Index[Lorentz,mu]] del[chi,Index[Lorentz,nu]]
         + (muC^2chi)/2 - (lamC(chi^2))/(4!)
         - (3muC^4)/(2 lamC);
```

[5] For more information on these functions, see Chapter 4.2.2.

In the case of Brans-Dicke-type scalar-tensor theories, it may be convenient to transform to the Einstein frame (see Chapter 2.1). This is achieved in FeynMG by executing

```
In[23]:= LEinstein=ToEinsteinFrame[LJordan]
```

$$\text{Out[23]=}\ -\frac{1}{24}\,\mathtt{lamC}\,M_{pl}^4+\frac{M_{pl}^4\,\mathtt{muC}^2}{2\,\mathtt{chi}}+\ .....\ -\frac{1}{2}M_{pl}^2R_{Sc}-\frac{M_{pl}^4\,\mathtt{y}\,\mathtt{phi}\,\bar{\mathtt{psi}}_{i1,i2}\mathtt{psi}_{i1,i2}}{\mathtt{chi}^2}$$
$$+\frac{3\,M_{pl}^2\,\partial_{a1}[\mathtt{chi}]\partial_{mu}[\mathtt{chi}]\mathtt{gUp}[a1,mu]}{4\mathtt{chi}^2}+\frac{M_{pl}^2\,\mathtt{w}\,\partial_{a1}[\mathtt{chi}]\partial_{mu}[\mathtt{chi}]\mathtt{gUp}[a1,mu]}{2\mathtt{chi}^2}$$
$$+\ .....\ -\alpha\,D^{Grav}{}_{a3}[A_{mu}]D^{Grav}{}_{a4}[A_{nu}]\ \mathtt{gUp}[a3,mu]\ \mathtt{gUp}[a4,nu]-\frac{e\,M_{pl}^3\,A_{mu}}{\mathtt{chi}^{3/2}}$$
$$\bar{\mathtt{psi}}_{i1,i2}.\mathtt{psi}_{j1,i2}\gamma_{i1,j1}{}^{v1}\,\mathtt{VUp}[mu,v1]+\frac{i\,M_{pl}^3}{\mathtt{chi}^{3/2}}\,\bar{\mathtt{psi}}_{i1,i2}.\partial_{mu}[\mathtt{psi}_{j1,i2}]$$
$$\gamma_{i1,j1}{}^{v2}\,\mathtt{VUp}[mu,v2]\ +\ .....\ +\frac{iM_{pl}^3}{16\mathtt{chi}^{5/2}}\,\partial_{d2}[\mathtt{chi}]\ \bar{\mathtt{psi}}_{i1,i2}.\mathtt{psi}_{j1,i2}$$
$$\mathtt{VDown}[c2,mu]\ \mathtt{VUp}[d3,d2]\ \mathtt{VUp}[mu,v3]\ \gamma^{v3}.\gamma^{d3}.\gamma^{c2}{}_{i1,j1}$$

The output agrees with the result from Eq. (2.12), including the last term, which comes from the fermion spin-connection [Eq. (2.11)]. As mentioned before, the Jacobian factor $\sqrt{-g}$ is assumed in the calculation (although it can be omitted by specifying the option **{Jacobian→Off}**, see Appendix D.2 for further details). The gravitational sector is now of canonical Einstein-Hilbert form, and we can take the flat-spacetime (Minkowski) limit by calling

```
In[24]:= GravityOff[LEinstein]
```

$$\text{Out[24]=}\ -\frac{1}{24}\,\mathtt{lamC}\,M_{pl}^4+\frac{M_{pl}^4\,\mathtt{muC}^2}{2\,\mathtt{chi}}+\ .....\ +\frac{M_{pl}^2\,\mathtt{w}\,\partial_{mu}[\mathtt{chi}]^2}{2\mathtt{chi}^2}+\frac{3\,M_{pl}^2\,\partial_{mu}[\mathtt{chi}]^2}{4\mathtt{chi}^2}$$
$$+\frac{M_{pl}^2\,\partial_{mu}[\mathtt{phi}]^2}{2\mathtt{chi}}+\frac{1}{2}\partial_{a3}[A_{a4}]\partial_{a4}[A_{a3}]\ -\ .....$$

wherein the couplings of the additional scalar field to the matter fields are manifest. The remaining fields are, however, not canonically normalized, which can be used to check the validity of FeynMG as follows: According to the calculations in Chapter 2.1, in the Einstein frame we expect different couplings to χ in each term depending on the number of metrics or vierbeins that it contains. Therefore, with this in mind, we can test whether both the introduction of the minimal couplings to gravity with **InsertGravity[]** and the conformal transformations are correct, since we only need to compare the minimal couplings in a term with its corresponding coupling to χ. However, further manipulations are still needed to make the Lagrangian compatible with `FeynRules`. These are the focus of the next subsection, in which we will repeat the calculation working directly in the Jordan frame.

4.3.2 Example 2: Brans-Dicke theory for FeynRules in the Jordan frame

The calculation in the Jordan frame repeats the same steps as in Section 4.3.1 up to and including In[22], starting with loading a model file. We then insert the curvature dependence using **InsertCurv[]** with the Lagrangian as the argument and provide a gravitational sector for the theory. The next step is to expand the metric about a flat spacetime background. This can be done by using **LinearizeGravity[]**, which expands every gravitational object in terms of the metric, and then, following the same steps as in Chapters 2 and 3, replaces every metric and vierbein by their linearized form (e.g., $g_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$) around Minkowski background up to the desired order. Acting on **LJordan** (defined previously in In[22]), we obtain,

```
In[25]:= E1=LinearizeGravity[LJordan,{SHGauge->On,UpdDevs->On}]
```

$$\text{Out[25]=} \quad -\frac{\text{chi}^2\,\text{lamC}}{24} + \;.....\; + \frac{1}{4}\partial_{\lambda 1}[\text{chi}]\,\partial_{\text{b1}}[\text{h}_{\text{b1},\lambda 1}] - \frac{1}{8}\,\text{chi}\,\partial_{\text{mu1}}[\text{h}_{\text{b1},\lambda 1}]^2 - \frac{1}{8}\,\text{chi}\,\partial_{\lambda 1}[\text{h}_{\text{b1},\lambda 2}]^2 + \;.....$$

where the provided **{SHGauge->On}** option specifies that the scalar-harmonic gauge from Eq. (2.32) is automatically determined and appends it to the Lagrangian, depending on the specific coupling function $F(\varphi)$. Additionally, **{UpdDevs->On}** updates all covariant derivatives to the modified form from Eq. (2.29). As for the Einstein-frame transformation in Section 4.3.1, the Jacobian $\sqrt{-g}$ has been included when linearizing gravity by default, but it can be omitted using **{Jacobian→Off}** (see Appendix D.3).

As we can see in Out[25], we have repeated terms since `Mathematica` does not use Einstein's index notation, for which two repeated indices are summed over. Consequently, various terms in the output will be equivalent, differing only in their index labels (e.g., $A_\mu A^\mu = A_\rho A^\rho$). As introduced in Section 4.2.1, in order to force `Mathematica` to combine these terms, we have to use the same set of indices for all the terms. This problem is solved by the function **IndexSimplify**:

In[26]:= `E2=IndexSimplify[E1,{mu,nu,rho}]`

Out[26]= $\ldots\ldots -\alpha\, \mathtt{A_{mu} A_{nu}\, CMod[chi]^{\{1\}mu}{}_{rho,rho}\, CMod[chi]^{\{1\}nu}{}_{MG1,MG1}} + \frac{\partial_{\mathtt{mu}}\mathtt{[chi]}^2}{4\,\mathtt{chi}} + \frac{\mathtt{w}\,\partial_{\mathtt{mu}}\mathtt{[chi]}^2}{2\,\mathtt{chi}} + \frac{1}{2}\partial_{\mathtt{mu}}\mathtt{[phi]}^2 - \frac{1}{2}\partial_{\mathtt{nu}}\mathtt{[A_{mu}]}^2 + \ldots\ldots + \frac{1}{8}\mathtt{chi}\,\partial_{\mathtt{rho}}\mathtt{[h_{mu,nu}]}^2 + \ldots\ldots - \frac{1}{4}\partial_{\mathtt{mu}}\mathtt{[chi]}\,\partial_{\mathtt{mu}}\mathtt{[h]} + \ldots\ldots - \frac{1}{16}\mathtt{chi}\,\partial_{\mathtt{mu}}\mathtt{[h]}^2 + \ldots\ldots$

The optional argument **{mu,nu,rho}** allows the user to choose a set of n indices from which the first n replacements will be chosen. The output of **E2** contains significantly fewer terms than **E1**. As expected, this function is able to reproduce the standard gravity result, producing the Fierz-Pauli terms for the graviton. Moreover, **E2** already contains the same graviton kinetic energy and the kinetic mixing between its trace and the scalar field `chi` that we found in Eq. (3.44) for **chi** $\to \varphi$..

Notice the appearance of **CMod** terms in the linearized Lagrangian. As pointed out when linearizing gravity in In[25], we updated the covariant derivatives in the Lagrangian to their modified forms by specifying the option **{UpdDevs->On}**. This choice is the source of these **CMod** terms in the linearized Lagrangian, and they correspond to the modification of the Christoffel symbols (see Eq. (2.30))

$$C^{\rho}_{\mu\nu} = \frac{F'(\varphi)}{2F(\varphi)}(\delta^{\rho}_{\mu}\partial_\nu\varphi + \delta^{\rho}_{\nu}\partial_\mu\varphi - g_{\mu\nu}\partial^\rho\varphi), \tag{4.15}$$

where the $\frac{F'(\varphi)}{2F(\varphi)}$ prefactor will have to be expanded in terms of φ. Once this expansion is truncated at some order in φ, we can no longer make a non-linear redefinition of the φ field (such as $\varphi \to \varphi^2$), since the ignored higher-order terms will give contributions at lower orders. To avoid this problem, **CMod** won't be expanded until all the kinetic energies of the scalar fields have been made canonical.

We can check that the kinetic energies appearing in **E2** are not canonically normalized by running

```
In[27]:= CheckCanonScalar[E2]
```

$$\text{Out[27]=}\quad \frac{(1+2\,\mathtt{w})\ \partial_{\mathtt{mu}}[\mathtt{chi}]^2}{4\ \mathtt{chi}} + \frac{1}{2}\partial_{\mathtt{mu}}[\mathtt{phi}]^2$$

```
There are one or more non-canonical kinetic energies.
Use CanonScalar.
```

As the output indicates, we can execute

```
In[28]:= E3=CanonScalar[E2]
```

$$\begin{aligned}\text{Out[28]=}\quad & \ldots\ldots + \frac{\mathtt{chi}^4\,\mathtt{lamC}}{96(1+2\mathtt{w})^2} + \frac{\mathtt{chi}^2\,\mathtt{muC}^2}{4(1+2\mathtt{w})^2} + \ldots\ldots + \frac{\mathtt{chi}^2\,\mathtt{muC}^2\mathtt{w}}{2(1+2\mathtt{w})^2} + \ldots\ldots + \frac{1}{2}\partial_{\mathtt{mu}}[\mathtt{chi}]^2 \\ & - \frac{4\alpha \mathtt{A}_{\mathtt{mu}}\mathtt{A}_{\mathtt{nu}}\,\partial_{\mathtt{mu}}[\mathtt{chi}]\,\partial_{\mathtt{nu}}[\mathtt{chi}]}{\mathtt{v}_{\mathtt{chi}}^2} + \frac{1}{2}\partial_{\mathtt{mu}}[\mathtt{phi}]^2 + \ldots\ldots + \frac{\mathtt{chi}^2\,\partial_{\mathtt{rho}}[\mathtt{h}_{\mathtt{mu,nu}}]^2}{16(1+2\,\mathtt{w})} \\ & + \ldots\ldots - \frac{\mathtt{chi}\,\partial_{\mathtt{mu}}[\mathtt{chi}]\,\partial_{\mathtt{mu}}[\mathtt{h}]}{4(1+2\mathtt{w})} + \ldots\ldots - \frac{\mathtt{chi}^2\,\partial_{\mathtt{mu}}[\mathtt{h}]^2}{32(1+2\mathtt{w})^2} - \frac{\mathtt{chi}^2\mathtt{w}\,\partial_{\mathtt{mu}}[\mathtt{h}]^2}{16(1+2\mathtt{w})^2} \\ & + \ldots\ldots\end{aligned}$$

The kinetic energies of the scalar fields are now canonically normalized, leading to the expansion of every **CMod[]** (where present). This expansion is performed in terms of the scalar field `chi`, where $\mathtt{v}_{\mathtt{chi}}$ is a placeholder for its vacuum expectation value.

At this stage, the kinetic energy of the graviton is composed of multiple terms. These could be simplified by means of `Mathematica`'s **FullSimplify** command, but this will often prove time-consuming, and it is not necessary, except for aesthetic reasons. From here, the only thing left to do is to canonically normalize the graviton kinetic energy. To this end, we need to perturb the fields around their vevs, so the graviton kinetic energy acquires a constant prefactor. This can be achieved by running

```
In[29]:= E4=VevExpand[E3]
```

$$\left\{1 == \{v_{chi} \to v_{chi}, v_{phi} \to v_{phi}\},\ 2 == \left\{v_{chi} \to -\frac{2\sqrt{3}\sqrt{muC^2 + 2\,muC^2 w}}{\sqrt{lamC}}, v_{phi} \to 0\right\},\ \ldots\ldots,\ 7 == \left\{v_{chi} \to -\frac{2\sqrt{3}\sqrt{muC^2 + 2\,muC^2 w}}{\sqrt{lamC}}, v_{phi} \to \frac{\sqrt{6}\,\mu}{\sqrt{lam}}\right\},\ \ldots\ldots\right\}$$

Out[30]=
$$-\mu^2 phi^2 - \frac{\sqrt{lam}\,\mu\, phi^3}{\sqrt{6}} + \ldots\ldots - \frac{chi^3\sqrt{lamC}\,muC}{4\sqrt{3}\,(1+2w)^{3/2}} + \ldots\ldots + \frac{1}{2}\partial_{mu}[chi]^2 - \frac{\alpha\, lamC\, A_{mu} A_{nu}\, \partial_{mu}[chi]\,\partial_{nu}[chi]}{3\,muC^2(1+2w)} + \frac{1}{2}\partial_{mu}[phi]^2 + \ldots\ldots + \frac{3\,muC^2\partial_{rho}[h_{mu,nu}]^2}{4\,lamC} + \ldots\ldots + \frac{\sqrt{3}\,muC\,\partial_{mu}[chi]\,\partial_{mu}[h]}{2\sqrt{lamC}\sqrt{1+2w}} + \ldots\ldots$$

Note that this function calculates and displays all the extrema in the potential, with the additional option of leaving the Lagrangian with the vevs undefined (option 1). Since there may be multiple minima, the function allows the user to choose which vev (or set of vevs) will be used by a dialogue window prompt (in this case, we choose option 7). Notice that the $\mathtt{v_{chi}}$ dependence already present from the expansion of the `CMod` functions has also been replaced by the user-selected vev in **E4**.

Once we have a constant prefactor to the graviton kinetic energy, we can canonically normalize it, using

```
In[31]:= E5=CanonGrav[E4]
```

Out[31]=
$$-\mu^2 phi^2 - \frac{\sqrt{lam}\,\mu\, phi^3}{\sqrt{6}} + \ldots\ldots - \frac{chi^3\sqrt{lamC}\,muC}{4\sqrt{3}\,(1+2w)^{3/2}} + \ldots\ldots + \frac{1}{2}\partial_{mu}[chi]^2 - \frac{\alpha\, lamC\, A_{mu} A_{nu}\, \partial_{mu}[chi]\,\partial_{nu}[chi]}{3\,muC^2(1+2w)} + \frac{1}{2}\partial_{mu}[phi]^2 + \ldots\ldots + \frac{1}{2}\partial_{rho}[h_{mu,nu}]^2 + \ldots\ldots - \frac{\partial_{mu}[chi]\,\partial_{mu}[h]}{\sqrt{2}\sqrt{1+2w}} + \ldots\ldots - \frac{1}{4}\partial_{mu}[h]^2 + \ldots\ldots$$

We have recovered the usual canonically normalized Fierz-Pauli kinetic energy terms from Eq. (2.26). We also see the expected kinetic mixing between the scalar field and the graviton, which can be identified by executing

```
In[32]:= CheckGravityMixing[E5]
```

Out[32]= $-\frac{\partial_{\text{mu}}[\text{chi}]\,\partial_{\text{mu}}[\text{h}]}{\sqrt{2+4\,\text{w}}}$

```
There are kinetic mixing terms for gravity.
Use GravKinMixing.
```

The final manipulation is to diagonalize this kinetic mixing. This can be achieved by running)

```
In[33]:= E6=GravKinMixing[E5,{OutSimplify->On}]
```

Out[33]=
$$
\begin{aligned}
&-\mu^2\text{phi}^2 - \frac{\sqrt{\text{lam}}\,\mu\,\text{phi}^3}{\sqrt{6}} + \;.....\; + \frac{1}{2}\partial_{\text{mu}}[\text{chi}]^2 + \;.....\\
&+ \frac{1}{2}\partial_{\text{mu}}[\text{phi}]^2 - \frac{\text{chi}\sqrt{\text{lamC}}\,\partial_{\text{mu}}[\text{phi}]^2}{2\sqrt{3}\,\text{muC}\sqrt{3+2\text{w}}} + ...+ \frac{1}{2}\partial_{\text{rho}}[\text{h}_{\text{mu,nu}}]^2\\
&+ \;.....\; - \frac{1}{4}\partial_{\text{mu}}[\text{h}]^2 + ...\; - \frac{\sqrt{6}\,\mu\,\text{y}\,\bar{\text{psi}}_{\text{i1,i2}}.\text{psi}_{\text{i1,i2}}}{\sqrt{\text{lam}}}\\
&+ .....+ \frac{2\sqrt{2}\,\text{chi}\,\text{y}\sqrt{\text{lamC}}\,\mu\,\bar{\text{psi}}_{\text{i1,i2}}.\text{psi}_{\text{i1,i2}}}{\sqrt{\text{lam}}\,\text{muC}\sqrt{3+2\text{w}}} - \;.....\\
&- \frac{\text{i}\sqrt{3}\,\text{chi}\sqrt{\text{lamC}}\,\bar{\text{psi}}_{\text{i1,i2}}.\partial_{\text{mu}}[\text{psi}_{\text{j1,i2}}]\;\gamma^{\text{mu}}_{\text{i1,j1}}}{2\,\text{muC}\sqrt{3+2\,\text{w}}} + \;.....
\end{aligned}
$$

which first identifies the kinetic mixings between the graviton with one or multiple scalar fields to define the kinetic matrix, and then diagonalizes it following the steps shown in Appendix B. Herein, the argument **`{OutSimplify->On}`** applies **`FullSimplify[]`** up to quadratic terms, so that the kinetic energy terms appear explicitly canonicalized. Note that this simplification is not a prerequisite to further processing of the output with `FeynRules`.

As seen in Chapter 4.2.6, `FeynMG` can extract the effective M_{Pl} at any point in the calculations (before or after linearizing gravity or canonically normalizing the kinetic energies). For example, for the diagonalized Lagrangian from Out[33] (corresponding to **E6**), we obtain

In[34]:= `GiveMpl[E6]`

Out[34]=

```
Using the values defined in the parameter and mass classes, the
effective value for Mpl in this Lagrangian is Sqrt[6] GeV.
It can be changed by modifying the Model File obtained after
using OutputModelMG.
```

$$-\frac{\sqrt{6}\,\mathtt{muC}}{\sqrt{\mathtt{lamC}}}$$

The effective value for the Planck mass in the inputted Lagrangian is automatically calculated and printed using the defined values of the loaded parameters. In this case, since both parameters {`muC,lamC`} were defined using **`AddParameter[]`** in In[21] and no value was specified in the options[6], they were set by default to 1. Moreover, we can substitute the calculated value for $M_{\rm Pl}$ into the Lagrangian by calling

In[35]:= `E7=InsertMpl[E6]`

Out[35]=

```
Using the values defined in the parameter and mass classes, the
effective value for Mpl in this Lagrangian is Sqrt[6] GeV.
It can be changed by modifying the Model File obtained after
using OutputModelMG.
```

$$-\mu^2\mathtt{phi}^2-\frac{\sqrt{\mathtt{lam}}\,\mu\,\mathtt{phi}^3}{\sqrt{6}}+\;.....\;+\frac{1}{2}\partial_{\mathtt{mu}}[\mathtt{chi}]^2+\;.....$$
$$+\frac{1}{2}\partial_{\mathtt{mu}}[\mathtt{phi}]^2-\frac{\mathtt{chi}\,\partial_{\mathtt{mu}}[\mathtt{phi}]^2}{\sqrt{2}\,\mathtt{M_{pl}}\sqrt{3+2w}}+\;.....\;+\frac{1}{2}\partial_{\mathtt{rho}}[\mathtt{h_{mu,nu}}]^2$$
$$+\;.....\;-\frac{1}{4}\partial_{\mathtt{mu}}[\mathtt{h}]^2+\;.....\;-\frac{\sqrt{6}\,\mu\,\mathtt{y}\,\bar{\mathtt{psi}}_{\mathtt{i1,i2}}.\mathtt{psi}_{\mathtt{i1,i2}}}{\sqrt{\mathtt{lam}}}$$
$$+\;.....\;+\frac{4\sqrt{3}\,\mathtt{chi}\,\mu\,\mathtt{y}\,\bar{\mathtt{psi}}_{\mathtt{i1,i2}}.\mathtt{psi}_{\mathtt{i1,i2}}}{\sqrt{\mathtt{lam}}\,\mathtt{M_{pl}}\sqrt{3+2w}}$$
$$-\;.....\;+i\,\bar{\mathtt{psi}}_{\mathtt{i1,i2}}.\partial_{\mathtt{mu}}[\mathtt{psi}_{\mathtt{j1,i2}}]\;\gamma^{\mathtt{mu}}_{\mathtt{i1,j1}}$$

[6] In Chapter 4.2.2, we showed that, when using `AddParameter[]`, the value of the parameter can be specified using the option `Value→X`

$$-\frac{3\,\mathrm{i}\,\mathrm{chi}\,\bar{\mathrm{psi}}_{\mathrm{i1,i2}}.\partial_{\mathrm{mu}}[\mathrm{psi}_{\mathrm{j1,i2}}]\ \gamma^{\mathrm{mu}}_{\mathrm{i1,j1}}}{\sqrt{2}\,\mathrm{M}_{\mathrm{pl}}\sqrt{3+2\mathrm{w}}} - \ .....$$

This function also automatically adds M_{Pl} to the list of defined parameters, with its corresponding value ($\sqrt{6}$ GeV for this specific case) and `InteractionOrder→-1`. Notice that a Yukawa coupling between the fermion fields and the `chi` field has appeared in the fourth line, as expected. However, a closer look at this term shows that the coupling constant is four times larger than the result $m_\psi/\sqrt{2M_{\text{Pl}}^2(2+3w)}$ from Chapter 3.3.2. This is because of the last term in the expression, which will also contribute to the tree-level interactions between the fermion and the scalar field, leading then to the same results as in Chapter 3.3.2.

Finally, we can get the BSM description of our modified theory of gravity by ignoring all the couplings to gravity. As in the Einstein frame in In[24], this can be done by calling

In[36]:= **E8=GravityOff[E7]**

Out[36]=

$$-\mu^2\mathrm{phi}^2 - \frac{\sqrt{\mathrm{lam}}\,\mu\,\mathrm{phi}^3}{\sqrt{6}} + \ ..... \ + \frac{1}{2}\partial_{\mathrm{mu}}[\mathrm{chi}]^2 + \ ..... \ + \frac{1}{2}\partial_{\mathrm{mu}}[\mathrm{phi}]^2$$
$$-\frac{\mathrm{chi}\,\partial_{\mathrm{mu}}[\mathrm{phi}]^2}{\sqrt{2}\,\mathrm{M}_{\mathrm{pl}}\sqrt{3+2\mathrm{w}}} + ... \ - \frac{\sqrt{6}\,\mu\,\mathrm{y}\,\bar{\mathrm{psi}}_{\mathrm{i1,i2}}.\mathrm{psi}_{\mathrm{i1,i2}}}{\sqrt{\mathrm{lam}}} + \ .....$$
$$+\frac{4\sqrt{3}\,\mathrm{chi}\,\mu\,\mathrm{y}\,\bar{\mathrm{psi}}_{\mathrm{i1,i2}}.\mathrm{psi}_{\mathrm{i1,i2}}}{\sqrt{\mathrm{lam}}\,\mathrm{M}_{\mathrm{pl}}\sqrt{3+2\mathrm{w}}} + \mathrm{i}\,\bar{\mathrm{psi}}_{\mathrm{i1,i2}}.\partial_{\mathrm{mu}}[\mathrm{psi}_{\mathrm{j1,i2}}]\ \gamma^{\mathrm{mu}}_{\mathrm{i1,j1}}$$
$$-\frac{3\,\mathrm{i}\,\mathrm{chi}\,\bar{\mathrm{psi}}_{\mathrm{i1,i2}}.\partial_{\mathrm{mu}}[\mathrm{psi}_{\mathrm{j1,i2}}]\ \gamma^{\mathrm{mu}}_{\mathrm{i1,j1}}}{\sqrt{2}\,\mathrm{M}_{\mathrm{pl}}\sqrt{3+2\mathrm{w}}} - \ .....$$

At this point, all the interactions up to second order in the fields have been canonically normalized and diagonalized, so there are no kinetic or mass mixings. Therefore, the updated Lagrangian for the matter fields with the additional scalar field couplings is now in a form that can be processed further by `FeynRules` and compatible packages for phenomenological studies.

4.3.3 Example 3: Outputting a model file

FeynMG allows the user to create a new model file with the Lagrangian of their choice, in which all the introduced particles (such as the graviton and additional scalar fields) and new parameters (such as $M_{\rm Pl}$) will be incorporated and properly defined.[7] This can be done by running

```
In[37]:= OutputModelMG["Oldmodelfile.fr", "Newmodelfile.fr", Lagrangian];
```

where "Oldmodelfile.fr" is the name of the original FeynRules model file that the user loaded, "Newmodelfile.fr" is the chosen name of the new model file, and Lagrangian is the final Lagrangian, as prepared with FeynMG.

The upgraded model file can be read directly into FeynRules without needing to load or rerun FeynMG. Starting a new session on Mathematica (or quitting the kernel using Quit[]), we can load the outputted model just by running

```
In[38]:= << FeynRules`
         LoadModel["Newmodelfile.fr"];
```

In this way, we recover all the fields and parameters defined previously, plus the chosen Lagrangian which is now saved in a function called L. Had we used E8 from Out[36], we would find

```
In[39]:= L
```

$$
\text{Out[39]=}\ -\mu^2\text{phi}^2-\frac{\sqrt{\text{lam}}\,\mu\,\text{phi}^3}{\sqrt{6}}+\ .....\ +\frac{1}{2}\partial_{\text{mu}}[\text{chi}]^2+\ .....\ +\frac{1}{2}\partial_{\text{mu}}[\text{phi}]^2
-\frac{\text{chi}\,\partial_{\text{mu}}[\text{phi}]^2}{\sqrt{2}\,\text{M}_{\text{pl}}\sqrt{3+2\text{w}}}+\ .....\ -\frac{\sqrt{6}\,\mu\,\text{y}\,\bar{\text{psi}}_{\text{i1,i2}}.\text{psi}_{\text{i1,i2}}}{\sqrt{\text{lam}}}+\ .....
+\frac{4\sqrt{3}\,\text{chi}\,\mu\,\text{y}\,\bar{\text{psi}}_{\text{i1,i2}}.\text{psi}_{\text{i1,i2}}}{\sqrt{\text{lam}}\,\text{M}_{\text{pl}}\sqrt{3+2\text{w}}}+\text{i}\,\bar{\text{psi}}_{\text{i1,i2}}.\partial_{\text{mu}}[\text{psi}_{\text{j1,i2}}]\ \gamma^{\text{mu}}_{\text{i1,j1}}
-\frac{3\,\text{i}\,\text{chi}\,\bar{\text{psi}}_{\text{i1,i2}}.\partial_{\text{mu}}[\text{psi}_{\text{j1,i2}}]\ \gamma^{\text{mu}}_{\text{i1,j1}}}{\sqrt{2}\,\text{M}_{\text{pl}}\sqrt{3+2\text{w}}}-\ .....
$$

taking us back to where we left it before logging off Mathematica (or quitting the kernels).

[7] New particles and parameters created using AddScalar[] and AddParameter[] will also be added, see Appendix D.8 for more information.

4.3.4 Example 4: Feynman rules and diagrams for Brans-Dicke theories

A positive aspect of working within `FeynRules` is that its inbuilt functions can be directly applied to any output model produced by `FeynMG`. Thus, with the Lagrangian from Out[39] already in its canonical form, we will focus in this subsection on this compatibility using `FeynRules` and `MadGraph` [166, 167].

It is important to point out that the manipulation of the Lagrangian into its canonical form may introduce some inconsistencies in the model file. For example, the mass of the non-minimally coupled field will generally differ from the value initially specified with `AddScalar[]`. Fortunately, we can verify the consistency of the model file within `FeynRules` just by running

```
In[40]:= CheckMassSpectrum[L]
         Neglecting all terms with more than 2 particles.
         All mass terms are diagonal.
         Getting mass spectrum.
         Checking for less than 0.1% agreement with model file values.
```

Out[40]//TableForm=

Particle	Analytic value	Numerical value	Model-file value
phi	$\sqrt{2\,\mu^2}$	125.	125.
chi	$\sqrt{\frac{\mathtt{muC}^2}{3+2\mathtt{w}}}$	0.447214	0. Discrepancy!
e	$\frac{\sqrt{6}\mu\,\mathtt{y}}{\sqrt{\mathtt{lam}}}$	0.000511	0.000511

where we can see that there is a mismatch between the model file and the Lagrangian. To solve the discrepancy, we just need to change the mass of the `chi` field by directly editing the model file. In addition, we also recommend to amend the value for the Planck mass, as it may differ from the one calculated and inserted by `InsertMpl[]`, which was obtained using the rest of the parameters' values. For example, in Out[35], we found $M_{\rm Pl} = \sqrt{6}$ Gev, which is in clear discrepancy with the real value for the Planck mass.[8]

Once all the fields and parameters are correctly defined, we can proceed with studying the phenomenology of the model. One of the most straightforward applications of **FeynRules** is calculating the Feynman rules of a given Lagrangian. For this, using **FeynmanRules** on a Lagrangian will provide us its complete set of Feynman rules, such as

```
In[41]:= FeynmanRules[L]

         Starting Feynman rule calculation.

         Expanding the Lagrangian...

         Collecting the different structures that enter the vertex.

         14 possible nonzero vertices have been found →

         starting the computation: 14 / 14.

         14 vertices obtained.
```

Out[41]= $\{\{\{\{\mathrm{phi},1\},\{\mathrm{phi},2\},\{\mathrm{phi},3\}\},-i\sqrt{6}\sqrt{\mathrm{lam}}\,\mu\}\,,\ \ldots\,,$
$\{\{\{\bar{\mathrm{psi}},1\},\{\mathrm{psi},2\},\{\mathrm{chi},3\}\},\dfrac{4\,i\,\sqrt{3}\,\mu\,y\,\delta_{s_1,s_2}}{\sqrt{\mathrm{lam}}\,\mathrm{Mpl}\sqrt{3+2w}}-\dfrac{3\,i\,\gamma.p_{2s_1,s_2}}{\sqrt{2}\,\mathrm{Mpl}\sqrt{3+2w}}$
$+\dfrac{3\,i\,\gamma.p_{3s_1,s_2}}{\sqrt{2}\,\mathrm{Mpl}\sqrt{3+2w}}\}\,,\ \ldots\,,\{\{\{\bar{\mathrm{psi}},1\},\{\mathrm{psi},2\},\{A,3\}\},-i\,e\,\gamma_{s_1,s_2}{}^{\mu_3}\}\}$

In this example, we have highlighted a few terms, including the Yukawa coupling of χ with the fermions and the usual QED interaction, but the function will display all the different allowed interactions by the Lagrangian **L**. To interpret the output, notice that every element of the list is composed of an array with two list elements: the first specifies the fields taking part in the corresponding Feynman rule, which is given in the second element. Notice that the fields are numbered to keep track of the indices within the rules. One of the best tests for **FeynMG** comes from this function, since it allows us to study directly the couplings of the fifth forces to the matter Lagrangian. In particular, aside from the aforementioned Yukawa coupling to the fermions, we find that there is no net interaction of the gauge field with χ. In order to obtain this result, we had to update all covariant derivatives as specified in In[25], keep track of the arising **CMod** terms, and correctly evaluate the diagonalization of the kinetic mixing between the graviton and χ,

[8]See below Out[34] for a discussion on the origin of this discrepancy.

which lead to the cancellation of fifth forces shown in Appendix A. All this together shows that the different routines within `FeynMG` work in combination, allowing us to perform complicated calculations with confidence.

Given the set of rules, the natural step forward is to consider the new Feynman diagrams that can be constructed due to the modification of gravity. There are two main approaches to tackle this with `FeynRules`: either using `FeynArts` [163] or `MadGraph` [166, 167]. While it is true that `FeynArts` allows for the calculation of loop diagrams, which can be further evaluated using `FeynCalc` or `FormCalc`, here we will use `MadGraph`. This is because `MadGraph` not only allows for the computation of the Feynman rules, but can be further used to calculate the cross-sections for any given process, making it a valuable tool for our purposes, since it serves as a connection between our examples and collider data.

To use `MadGraph`, we will need to export a `".UFO"` (Universal FeynRules Output) file from `FeynRules` (for more information, see Ref. [143, 171]). This can be done by running the following command

```
In[42]:= WriteUFO[L]
         --- Universal FeynRules Output (UFO) v 1.1 ---
         Starting Feynman rule calculation.
         ...
         Computing the squared matrix elements relevant for the 1→2 decays:
         ...
         Done!
```

which will generate the desired `".UFO"` file in the last designated directory (for brevity, we have omitted multiple lines of printed text in the ellipsis). The next step is to load the file into `MadGraph`, which will allow us to compute all the different diagrams for a given process.[9] In this example, we will concentrate on the extension of Møller scattering considered in Chapter 3, where in addition to the two incoming and outgoing electrons, a

[9] In this thesis, the focus lies on the motivation and usage of `FeynMG`, rather than on the details of doing phenomenology with `MadGraph`. For a comprehensive introduction, see Refs. [166, 172].

photon and a chi particle are produced in the outer state (i.e., $e^- + e^- \to e^- + e^- + A_\mu + \chi$). As introduced in Chapter 1.3, such processes would lead to missing energy signals at particle colliders due to the neutrality of the external scalar, and can avoid chameleon type of screening mechanisms. To obtain the full list of diagrams, we write:

```
MG5_aMC>generate e- e- > e- e- a chi WEIGHTED<=4
INFO: Trying process: e- e- > e- e- a chi WEIGHTED<=4 @1
INFO: Process has 212 diagrams
1 processes with 212 diagrams generated in 0.344 s
Total: 1 processes with 212 diagrams
```

where the `WEIGHTED` option specifies the number of interactions in the entire diagram. As we can see, the large number of different consistent events makes it impractical for a human to manually evaluate them all. Furthermore, `MadGraph` can display all distinct Feynman diagrams just by running

```
MG5_aMC>display diagrams
Drawing Process: e- e- > e- e- a chi WEIGHTED<=4 @1
Wrote file /tmp/diagrams_1_emem_ememachi.eps
open /tmp/diagrams_1_emem_ememachi.eps
time to draw 0.09345197677612305
```

A subset of the generated diagrams is displayed in Figure 4.1. It is worth noting the efficiency of the code, which has successfully generated a total of 212 diagrams in just ~ 0.45 seconds. From here on, one can keep using the functionality of `MadGraph` to calculate the cross-sections from this process or employ other software analysis tools to test and compare modified theories of gravity against experimental data. Alternatively, one can also choose to stay at the Lagrangian level, and study the consistency of the theory using quantum field theory techniques.

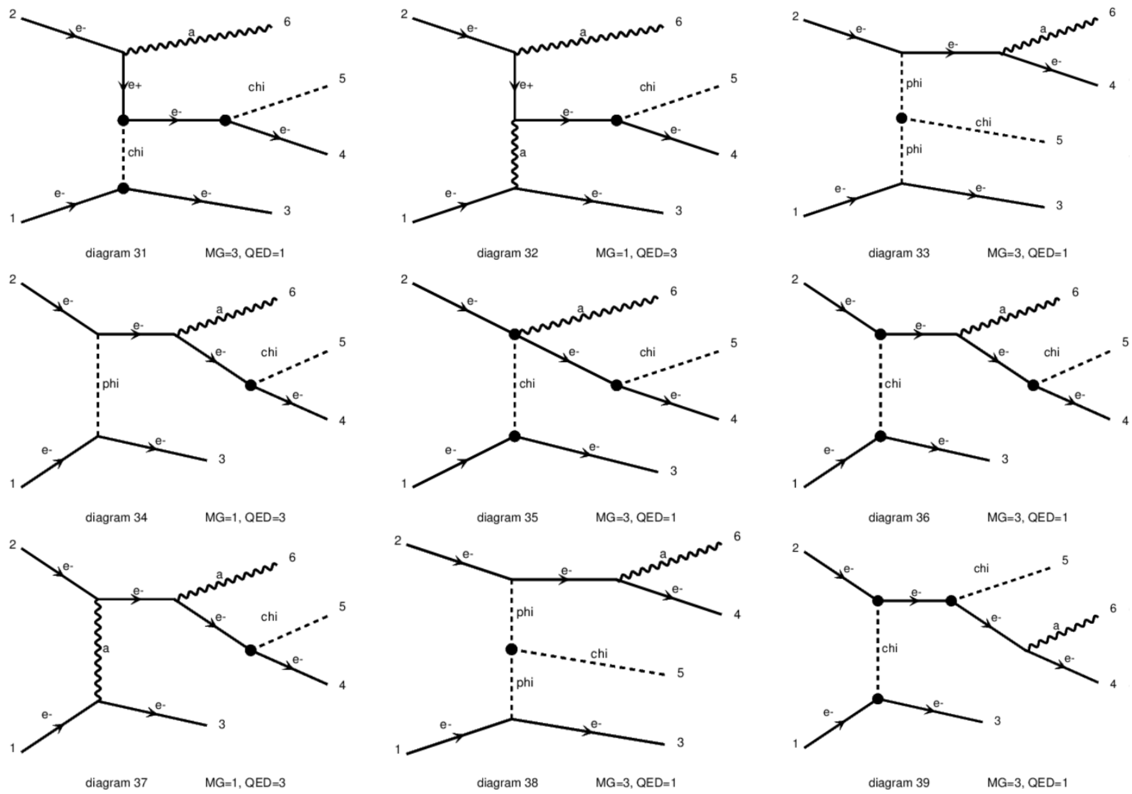


Figure 4.1: *Sample of Feynman diagrams appearing due to the modification of gravity. They represent a process with two incoming electrons (e^-) and two outgoing electrons with an additional photon (`a`) and a `chi` particle, corresponding to the non-minimally coupled field. These diagrams have been generated by `MadGraph` after using both `FeynRules` and `FeynMG` to generate the Lagrangian.*

Chapter 5

Conclusion and outlooks

In this thesis, we began with a review on the main open questions and naturalness issues with respect to the Standard Models of particle physics and cosmology. Among these, we have focused on the possibility of having a different gravitational sector other than the Einstein-Hilbert action, in which an extra scalar degree of freedom non-minimally couples to curvature objects, forming so-called scalar-tensor theories.

Perhaps one of the most characteristic features of these theories are the screening mechanisms, for which they experience a suppression of their effect on matter in high density regimes. As we have shown in Chapter 1, screening mechanisms thus allow these theories to avoid the tightest constraints from Solar System and laboratory scales. However, it is possible to see through some of these mechanisms by studying their effect on particle physics, via what are known as missing energy signals. For a consistent calculation of these scattering processes, it is necessary to work with quantum field theory. To do this, we can take advantage of another feature of these theories, which is that they can be described as Beyond the Standard Model theories.

Therefore, in Chapter 2, we have demonstrated how to consistently derive this BSM description from a scalar-tensor theory with a toy model for QED+Higgs using quantum field theory. We proceeded both working in the Einstein frame, by performing the appropriate Weyl transformation to the fields, and staying in the Jordan frame, where we had to work with linearized gravity. In the course of the calculation, we found that the modifications of the matter sector arise in the Einstein frame through a mass mixing with the would-be Higgs field, while in the Jordan frame they arise through a kinetic mixing with gravity itself. To solve this kinetic mixing, it was convenient to use a special gauge fixing choice, called the scalar-harmonic gauge. Furthermore, in both frames, we found that, at least classically, the new dynamics on the matter sector couples exclusively to the scale-dependant terms, making the modifications to the Standard Model equivalent to Higgs-portal theories.

In Chapter 3, we further explored the relationship between scale breaking and fifth-force strengths by considering two different sources of scale breaking. The first was via an explicit scale-invariant parameter in the action, coming from the would-be Higgs field mass, while the second was a dynamical breaking through the stabilization of multiple coupled scalar fields, which has a close connection with Higgs-Dilaton models. We thus obtained the Yukawa potential for the long-range interactions, specifically for the Møller scattering ($e^- + e^- \to e^- + e^-$), and found that the fifth forces vanish when the scale symmetry is dynamically broken. Apart from the interesting results regarding the scale breaking mechanisms, this calculation motivated the need for computational assistance to predict the full implications of scalar-tensor theories for the whole Standard Model.

To this end, in Chapter 4, we introduced the `Mathematica` package `FeynMG`, which can manipulate scalar-tensor theories of gravity into a format that can be processed by `Feyn-Rules`, and thus be connected to the rest of the available analysis software pipelines. After giving a detailed description of the state of the art in using symbolic algebra to do quantum field theory, we provided a number of examples. Some of these examples were dedicated to specific functions, while others replicated calculations of the type described in Chapter 2 and 3. We concluded this chapter by showing the compatibility of `FeynMG` and `FeynRules` with analysis software packages that allow to perform phenomenological studies on scalar-tensor theories of gravity. In particular, we used `MadGraph` to generate 212 diagrams for an extension of the Møller scattering where a photon and the extra non-minimally coupled field are produced in the final state: the kind of diagrams that would lead to missing energy signals at colliders. The most surprising aspect of this calculation was that we were able to produce all these diagrams in less than a half of a second, demonstrating the key element that `FeynMG` will be in exploring scalar-tensor theories in particle colliders.

The main extension of this work would be to consider a generic scalar-tensor theory and test for its implications for the whole Standard Model using data from colliders, for example by generating the cross-sections using `MadGraph`. However, since `FeynMG` provides the user with a Lagrangian form for the BSM description of a scalar-tensor theory, we can proceed in different directions when studying its phenomenology. For example, we can consider quantum effects, which allows us to calculate loop corrections in String Theory due its aforementioned relation to modified gravity, or test thermal corrections, which may have important implications in the early universe. Similarly, given the mass range and the nature of the extra scalar field, we also find similarities with axion-like particles. In future work, we would like to explore the physics arising from the couplings between these two fields, and whether some relation can be established to the coincidence problem mentioned in Chapter 1.

In conclusion, we are reaching a point where cosmology and particle theory can be studied simultaneously through their effect on each other's scale. It is important to note that, apart from the discovery of quantum field theory itself, the best advance in the development of the Standard Model of particle physics has been the use of computational tools. In this way, `FeynMG` extends this progress to scalar-tensor theories, which, once combined with additional packages, will make it possible to test for modifications of gravity with all the collider data already available. This widens an existing path for phenomenology so far hampered by very long, time-consuming and algorithmic calculations, presenting a bright future for testing scalar-tensor theories at subatomic scales.

Appendix A

Modified covariant derivative and U(1) Feynman gauge

In Chapter 2.2, we described an update to the covariant derivative in the Jordan frame, based on Ref. [1], that proves convenient for Brans-Dicke-type theories with only a non-minimal coupling to the Ricci scalar. This modified covariant derivative $\mathcal{D}_\mu$ reduces to the usual ∇_μ when Weyl transformed to the Einstein frame, and is given by

$$\mathcal{D}_\mu Y_\nu = \nabla_\mu Y_\nu - C^\rho_{\mu\nu} Y_\rho, \tag{A.1}$$

where

$$C^\rho_{\mu\nu} = \frac{1}{2\varphi}(\delta^\rho_\mu \partial_\nu \varphi + \delta^\rho_\nu \partial_\mu \varphi - g_{\mu\nu}\partial^\rho \varphi). \tag{A.2}$$

It allows us to define the so-called scalar-harmonic gauge, which maps to the usual harmonic gauge in the Einstein frame.

We can proceed similarly with other gauge fixing terms, such as the one for the U(1) gauge field. For instance, defined in the Einstein frame where gravity is canonical, the Feynman gauge fixing action takes the following form:

$$S \supset \int d^4x \sqrt{-\tilde{g}}\; \tilde{g}^{\mu\nu} \tilde{g}^{\sigma\rho} \tilde{\nabla}_\mu A_\nu \tilde{\nabla}_\sigma A_\rho, \tag{A.3}$$

where all the tilded objects are built with the Einstein frame metric $\tilde{g}_{\mu\nu}$. On transforming to the Jordan frame, we would find that the gauge fixing term has to be written as

$$S \supset \int d^4x\sqrt{-g}\; g^{\mu\nu}g^{\sigma\rho}\mathcal{D}_\mu A_\nu \mathcal{D}_\sigma A_\rho, \tag{A.4}$$

where the covariant derivatives have transformed to their modified forms from (A.1), which will introduce new couplings between the gauge field A_μ and the scalar field that appears in the Weyl rescaling of the metric. These new couplings are encoded in the $C^\rho_{\mu\nu}$ terms. In what follows, we will show that these new interactions are those that ensure there are no interactions between the scalar field and the gauge field at dimension four, as we would expect from general arguments based on Weyl invariance.

Let us first define the object $\Delta_{\lambda\mu\nu}$ via

$$\frac{1}{2}g^{\rho\lambda}\Delta_{\lambda\mu\nu} = \Gamma^\rho_{\mu\nu} + C^\rho_{\mu\nu}, \tag{A.5}$$

where we have taken the common factor of the upper-indexed metric so the result can be generalized to any order of its expansion in $h^{\mu\nu}$. For the case of Brans-Dicke theory (Eq. (1.16)) with a coupling function $F(\varphi)$, we find

$$\Delta_{\lambda\mu\nu} = \partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} + \frac{F'(\varphi)}{F(\varphi)}(g_{\lambda\nu}\partial_\mu\varphi + g_{\mu\lambda}\partial_\nu\varphi - g_{\mu\nu}\partial_\lambda\varphi), \tag{A.6}$$

which reduces to

$$\Delta_{\lambda\mu\nu} = \partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu} + \frac{F'(\varphi)}{F(\varphi)}(\eta_{\lambda\nu}\partial_\mu\varphi + \eta_{\mu\lambda}\partial_\nu\varphi - \eta_{\mu\nu}\partial_\lambda\varphi), \tag{A.7}$$

on perturbing the metric around a flat background (Eq. (2.19)). After canonically normalizing the φ field through the redefinition from Eq. (2.34), i.e.,

$$\chi(\varphi) = \int_0^\varphi \mathrm{d}\hat{\varphi}\sqrt{Z(\hat{\varphi}) + \frac{F'(\hat{\varphi})^2}{2F(\hat{\varphi})}}, \tag{A.8}$$

this leads to

$$\Delta_{\lambda\mu\nu} = \partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu} + \frac{\hat{F}'(\chi)}{\hat{F}(\chi)}(\eta_{\lambda\nu}\partial_\mu\chi + \eta_{\mu\lambda}\partial_\nu\chi - \eta_{\mu\nu}\partial_\lambda\chi), \tag{A.9}$$

where we have defined $\hat{F}(\chi) \equiv F(\varphi)$ and $\hat{F}'(\chi) \equiv \partial\hat{F}(\chi)/\partial\chi$. As described in Chapter 2.2, we now expand the scalar field around its vev, so that the graviton can also be canonically normalized (see Eq. (2.36), where the kinetic mixing between the graviton and χ is manifest). At this point, $\Delta_{\lambda\mu\nu}$ has the form

$$\begin{aligned}\Delta_{\lambda\mu\nu} =& \partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu} \\ &+ \frac{\hat{F}'(v_\chi)}{\hat{F}(v_\chi) + \chi\hat{F}'(v_\chi)}(\eta_{\lambda\nu}\partial_\mu\chi + \eta_{\mu\lambda}\partial_\nu\chi - \eta_{\mu\nu}\partial_\lambda\chi).\end{aligned} \tag{A.10}$$

The kinetic mixing between the graviton and the scalar can be removed (see Eq.(2.37)) by means of the transformations in Eq. (2.37). With this, we obtain Eq. (2.38) and

$$\begin{aligned}\Delta_{\lambda\mu\nu} =& \frac{2}{M_{\text{Pl}}}\Big(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\Big) \\ &+ \frac{1}{M_{\text{Pl}}}\frac{\hat{F}'(v_\chi)}{\sqrt{M_{\text{Pl}}^2 + \hat{F}'(v_\chi)^2}}\left(\eta_{\lambda\nu}\partial_\mu\sigma + \eta_{\mu\lambda}\partial_\nu\sigma - \eta_{\mu\nu}\partial_\lambda\sigma\right) \\ &- \frac{\hat{F}'(v_\chi)}{\hat{F}'(v_\chi)\sigma + M_{\text{Pl}}\sqrt{M_{\text{Pl}}^2 + \hat{F}'(v_\chi)^2}}(\eta_{\lambda\nu}\partial_\mu\sigma + \eta_{\mu\lambda}\partial_\nu\sigma - \eta_{\mu\nu}\partial_\lambda\sigma),\end{aligned} \tag{A.11}$$

where $\hat{F}(v_\chi) = M_{\text{Pl}}^2$ has been substituted and σ corresponds to the canonically normalized additional scalar field. We can now expand the denominator in the third line up to first order in M_{Pl}^{-1} to give

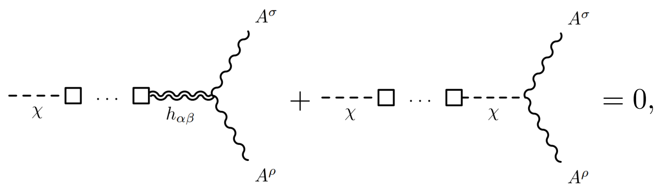
$$\Delta_{\lambda\mu\nu} = \frac{2}{M_{\text{Pl}}}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right), \tag{A.12}$$

showing a perfect cancellation of the couplings to the additional scalar. Thus, after diagonalizing, the covariant derivative takes the following form

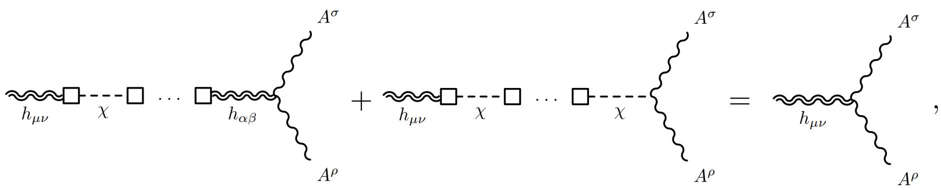
$$\mathcal{D}_\mu A_\nu = \partial_\mu A_\nu - \frac{2}{M_{\text{Pl}}}\Gamma^\rho_{\mu\nu}A_\rho, \tag{A.13}$$

which is nothing but the standard covariant derivative $\nabla_\mu A_\nu$ from Einstein gravity. This is as we would expect, since the diagonalization is essentially a perturbative implementation of the Weyl transformation to the Einstein frame.

We can obtain the same result without diagonalizing and instead summing over all insertions of the graviton-scalar kinetic mixing. Our calculations have shown that the following two series of diagrams cancel with each other:



where the ellipsis contains the sum over the infinite series of insertions of mixings (where zero kinetic mixing is also included for the diagram on the right). Similarly, from the diagrams above, we can calculate the incoming graviton amplitude by inserting an additional kinetic mixing to the left of the χ propagators. Thus, we find that all the diagrams containing kinetic mixings will end up cancelling each other, leaving just the diagram with no kinetic mixings. Diagrammatically, this implies that



which corresponds to the Feynman diagram for the coupling between the gauge field and gravity through the usual Chistoffel symbols.

In either case, we see that the role of the additional terms arising from $C^\rho_{\mu\nu}$ in the updated covariant derivatives is to maintain the Weyl invariance of the Maxwell Lagrangian (at dimension four) once gauge fixing terms are included in the Jordan frame.

Appendix B

Diagonalizing graviton-scalar kinetic mixing

A convenient way to eliminate all the kinetic mixings is to find the matrix transformation that diagonalizes the kinetic terms. However, creating a kinetic mixing matrix between 2-forms (the graviton) and scalar fields is not straightforward. In this appendix, we describe a method for determining the transformation and diagonalizing the kinetic terms.

The main obstacle is that the graviton kinetic term contains both $h_{\mu\nu}$ and its trace h. For example, we might have a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\rho h\partial^\rho h - C\partial_\rho h\partial^\rho \chi + \frac{1}{2}\partial_\rho \chi\partial^\rho \chi, \tag{B.1}$$

where both the graviton and the scalar field have already been canonically normalized, but there remains a kinetic mixing proportional to C (which for the calculation from Chapter 2.2 corresponds to $C = \hat{F}(v_\chi)/4$).

Since the graviton has two kinetic terms, it is unclear how to construct a matrix that encapsulates all the kinetic couplings between distinct fields. Therefore, we proceed by redefining $h_{\mu\nu}$ so that its kinetic energy contains only one term. To do so, we perform an analytic continuation of the graviton into the complex plane, redefining

$$h_{\mu\nu} \to \tilde{h}_{\mu\nu} - \frac{1}{4}(1+i)\tilde{h}\eta_{\mu\nu}. \tag{B.2}$$

This transformation will allow us to easily define a kinetic matrix for our Lagrangian. Once it is diagonalized, we just need to reverse this redefinition. Substituting Eq. (B.2) into Eq. (B.1), we obtain

$$\mathcal{L} = \frac{1}{2}\partial_\rho \tilde{h}_{\mu\nu}\partial^\rho \tilde{h}^{\mu\nu} + Ci\partial_\rho \tilde{h}\partial^\rho \chi + \frac{1}{2}\partial_\rho \chi \partial^\rho \chi, \tag{B.3}$$

which contains only one kinetic energy term for the graviton. The kinetic matrix is then defined straightforwardly as

$$K = \begin{pmatrix} \frac{1}{2} & i\frac{C}{2} \\ i\frac{C}{2} & \frac{1}{8} \end{pmatrix}, \tag{B.4}$$

with partial derivatives of the fields collected into the vector

$$F_{\rho\mu\nu} = \begin{pmatrix} \partial_\rho \tilde{h}_{\mu\nu} \\ \eta_{\mu\nu}\partial_\rho \chi \end{pmatrix}, \tag{B.5}$$

such that the Lagrangian Eq. (B.1) can be written in the form $\mathcal{L} = (F^{\rho\mu\nu})^\mathsf{T} K F_{\rho\mu\nu}$, where T denotes matrix transposition.

We want a transformation W of the matrix K such that

$$W^\mathsf{T} K W = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}. \tag{B.6}$$

The transformations for the fields are as follows:

$$(F^{\rho\mu\nu})^\mathsf{T} K F_{\rho\mu\nu} = (F^{\rho\mu\nu} W^{-1} W)^\mathsf{T} K W W^{-1} F_{\rho\mu\nu} = (\tilde{F}^{\rho\mu\nu})^\mathsf{T} W^\mathsf{T} K W \tilde{F}_{\rho\mu\nu}, \tag{B.7}$$

since, by defining $\tilde{F}_{\rho\mu\nu} = W^{-1} F_{\rho\mu\nu}$, we would get a Lagrangian free of kinetic mixings.

B. *Diagonalizing graviton-scalar kinetic mixing*

For the generic kinetic mixing, where K is defined by Eq. (B.4), the transformation matrix is

$$W = \begin{pmatrix} 1 & \frac{-1}{\sqrt{1+4C^2}} \\ 0 & \frac{-iC}{\sqrt{1+4C^2}} \end{pmatrix}. \tag{B.8}$$

The scalar fields transform through $F_{\rho\mu\nu} = W\tilde{F}_{\rho\mu\nu}$ and therefore

$$\tilde{h}_{\mu\nu} \to \tilde{h}_{\mu\nu} - \frac{iC}{\sqrt{1+4C^2}}\sigma\eta_{\mu\nu}, \tag{B.9a}$$

$$\chi \to \frac{-1}{\sqrt{1+4C^2}}\sigma. \tag{B.9b}$$

Undoing the complexification in Eq. (B.2), we obtain the transformations of the original fields that diagonalize the kinetic terms:

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{C}{\sqrt{1+4C^2}}\sigma\eta_{\mu\nu}, \tag{B.10a}$$

$$\chi \to \frac{-1}{\sqrt{1+4C^2}}\sigma. \tag{B.10b}$$

For the specific case of the Lagrangian in Out[32] from Chapter 4.3.2, for which $C = 1/\sqrt{2+4\omega}$, we get

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{1}{\sqrt{2(2\omega+3)}}\sigma\eta_{\mu\nu}, \tag{B.11a}$$

$$\chi \to -\frac{\sqrt{2\omega+1}}{\sqrt{2\omega+3}}\sigma. \tag{B.11b}$$

Appendix C

Evaluation of Feynman diagrams with a kinetic mixing with gravity

In Chapter 3.3, we calculated the modification to the Yukawa potential for the Møller scattering in the Jordan frame. In the process, we decided that for simplicity, it would be best to diagonalize the kinetic mixing between the graviton and the non-minimally coupled field (Eq. (3.60)), such that the fifth forces could be isolated from the gravitational interaction. Nevertheless, in this appendix, we will repeat this calculation without performing the diagonalization of the kinetic matrix, showing an agreement between both procedures.

As introduced in Eq. (3.59), the terms in the Lagrangian relevant to the fifth force are as follows:

$$\begin{aligned}\mathcal{L}_{\mathrm{JF}} =& \frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\mu h \partial^\mu h + \frac{1}{2}\eta^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma \\ & - \frac{cN_\sigma^{-1}}{\sqrt{2(2\omega+1)}}\eta^{\mu\nu}\partial_\mu h \partial_\nu \sigma - yc'N_\sigma^{-1}\bar{\psi}\sigma\psi + \frac{1}{M_{\mathrm{Pl}}}h^{\mu\nu}T_{\mu\nu} + \mathcal{L}_{\mathrm{m}}(\eta_{\mu\nu}),\end{aligned} \tag{C.1}$$

which will introduce long range fifth forces to the Møller scattering through four distinct Feynman diagrams shown in Fig. C.1.

Since the structure of all the diagrams is very similar, we will describe only the con-

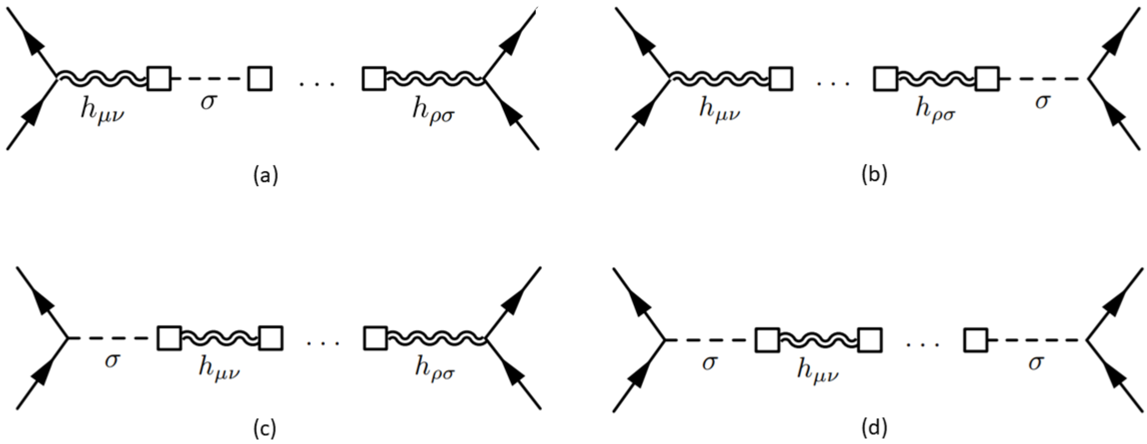


Figure C.1: *The diagrams that contribute to the Møller scattering in the Jordan frame.*

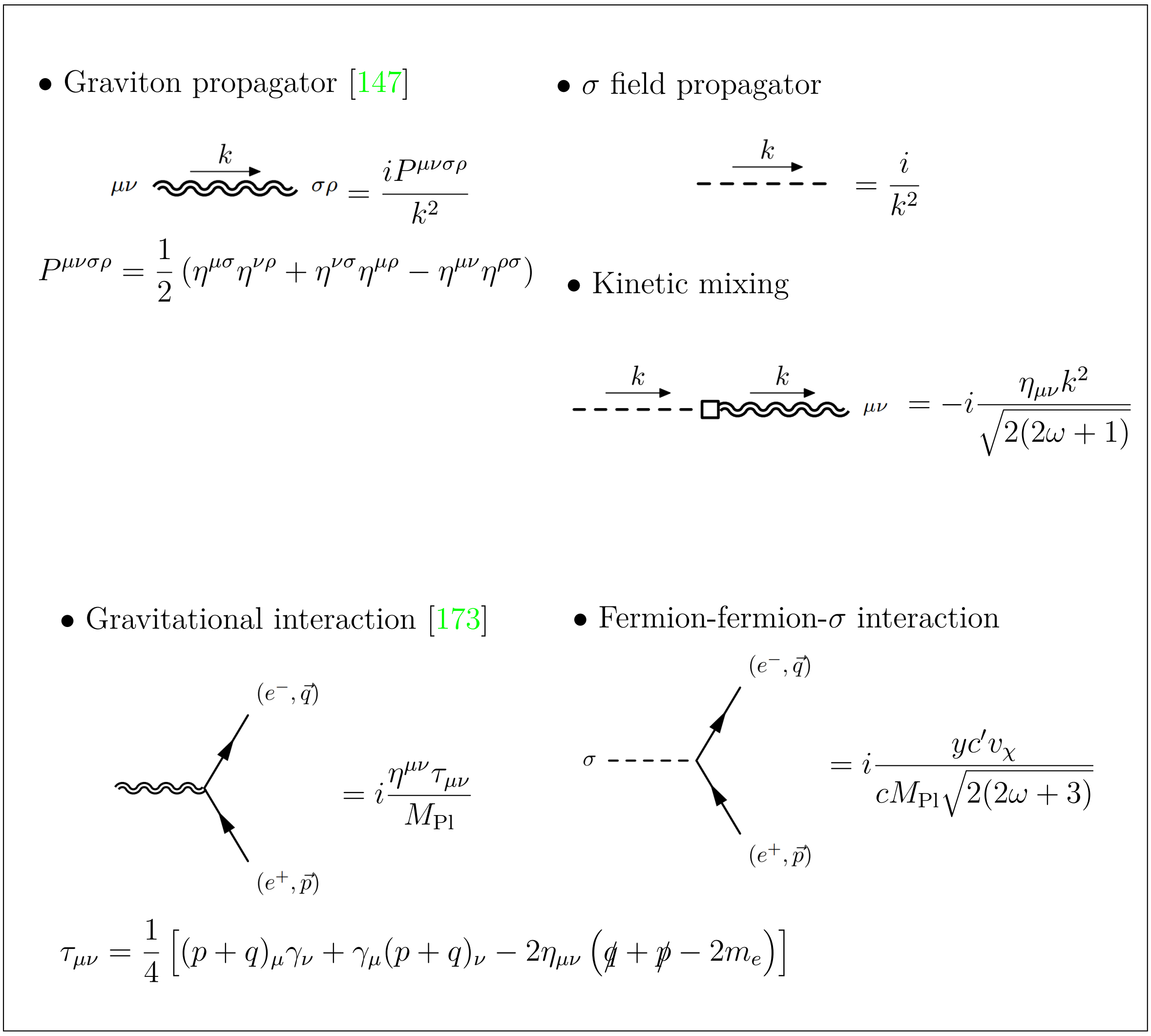


Figure C.2: *Feynman rules for the Lagrangian (Eq.* (C.1)*) with an explicitly broken scale symmetry, where γ_μ are the gamma matrices. To a good approximation, we can take $cN_\sigma^{-1} \approx 1$, since $\tilde{M} \gg 1$.*

tribution from Fig. 3.3(a) in detail. The matrix element for this process is

$$\begin{aligned}
i\mathcal{M}_{(\mathbf{a})} = \bar{u}(\mathbf{p}_1, s_1)\left(i\frac{\tau_{\mu\nu}}{M_{\rm Pl}}\right)u(\mathbf{p}_3, s_3)\left(-i\frac{P^{\mu\nu ab}}{t}\right)(i\eta_{ab}\alpha_{\rm K}t)\left(-\frac{i}{t}\right)\\
\times\left[\sum_{n=0}^{\infty}(i\alpha_{\rm K}t)^n\left(-\frac{i\eta_{cd}P^{cdef}\eta_{ef}}{t}\right)^n(i\alpha_{\rm K}t)^n\left(-\frac{i}{t}\right)^n\right]\\
\times(i\alpha_{\rm K}\eta_{gh}t)\left(-\frac{iP^{gh\sigma\rho}}{t}\right)\bar{u}(\mathbf{p}_2, s_2)\left(i\frac{\tau_{\sigma\rho}}{M_{\rm Pl}}\right)u(\mathbf{p}_4, s_4),
\end{aligned} \tag{C.2}$$

where, as before, $t = -(p_1 - p_3)^2$, $u(\mathbf{p}, s)$ and $\bar{u}(\mathbf{p}, s)$ are respectively the Dirac four-spinor and its Dirac conjugate, with spin projection s. Note that, for clarity, we have isolated each vertex and propagator with parentheses. For convenience, we have also defined the parameter

$$\alpha_{\rm K} = \frac{1}{\sqrt{2(2\omega+1)}}. \tag{C.3}$$

Equation (C.2) can be simplified by making use of the following identities for $P^{\mu\nu\sigma\rho}$:

$$\eta_{\mu\nu}P^{\mu\nu\sigma\rho} = -\eta^{\sigma\rho}, \qquad \eta_{\mu\nu}P^{\mu\nu\sigma\rho}\eta_{\sigma\rho} = -4, \tag{C.4}$$

and we find that we only have vertices involving the trace of $\tau_{\mu\nu}$, as we would have expected from Eq. (3.61).

Working in the non-relativistic limit and choosing the fermions to represent electrons with mass m_e, such that $p^\mu \sim q^\mu \approx (m_e, \vec{0})$, the spinors satisfy

$$\bar{u}(\mathbf{p}, s)\gamma_\mu u(\mathbf{q}, s') = 2m_e\delta_{\mu 0}\delta_{ss'}, \tag{C.5}$$

in which case, using the expression for $\tau = \eta^{\mu\nu}\tau_{\mu\nu}$ extracted from Fig. C.2, we have

$$\bar{u}(\mathbf{p}, s)\tau u(\mathbf{q}, s') = -2m_e^2. \tag{C.6}$$

The matrix element then reduces to

$$\mathcal{M}_{(\mathbf{a})} = \frac{1}{M_{\rm Pl}^2}\frac{4m_e^4\alpha_{\rm K}^2}{t}\left[\sum_{n=0}^{\infty}\left(-4\alpha_{\rm K}^2\frac{1}{t}\right)^n\right]\delta_{s_1s_3}\delta_{s_2s_4} = \frac{1}{M_{\rm Pl}^2}\frac{4m_e^4\alpha_{\rm K}^2}{(1+4\alpha_{\rm K}^2)t}\delta_{s_1s_3}\delta_{s_2s_4}. \tag{C.7}$$

C. Evaluation of Feynman diagrams with a kinetic mixing with gravity

To extract the non-relativistic potential, we take $t = \mathbf{Q}^2$ (where $\mathbf{Q}$ is the exchange momentum), and the contribution to the Yukawa potential is

$$V_{(\mathbf{a})}(r) = -\frac{1}{M_{\text{Pl}}^2}\frac{m_e^2}{(4+\alpha_{\text{K}}^{-2})}\int\frac{\mathrm{d}^3\mathbf{Q}}{(2\pi)^3}e^{i\mathbf{Q}\cdot\mathbf{x}}\frac{1}{\mathbf{Q}^2} = -\frac{1}{4\pi r}\frac{m_e^2}{M_{\text{Pl}}^2 2(2\omega+3)}. \tag{C.8}$$

The contributions from the remaining processes in Fig. 3.3 are

$$V_{(\mathbf{b})}(r) = V_{(\mathbf{c})}(r) = -\frac{1}{4\pi r}\left(\frac{c' v_\chi}{c v_\Phi}\right)\frac{m_e^2}{M_{\text{Pl}}^2 2(2\omega+3)}, \tag{C.9a}$$

$$V_{(\mathbf{d})}(r) = -\frac{1}{4\pi r}\left(\frac{c' v_\chi}{c v_\Phi}\right)^2\frac{m_e^2}{M_{\text{Pl}}^2 2(2\omega+3)}, \tag{C.9b}$$

and the sum of all the contributions to the Yukawa potential is

$$V_5(r) = -\frac{m_e^2}{4\pi r M_{\text{Pl}}^2}\frac{\left(1+\frac{v_\chi\gamma}{v_\Phi}\right)^2}{2(2\omega+3)}. \tag{C.10}$$

After some algebra, we can show that

$$\gamma = \frac{c'}{c} = -\frac{\beta v_\Theta^2}{\lambda v_\chi v_\Phi}, \tag{C.11}$$

and, using the fact that $v_\Theta^2 = (\lambda v_\Phi^2 - 6\mu^2)/\beta$, we obtain the following final expression

$$V_5(r) = -\frac{1}{4\pi r}\frac{m_e^2}{M_{\text{Pl}}^2 2(2\omega+3)}\frac{4\mu^4}{m_\Phi^4}, \tag{C.12}$$

where we recall that

$$m_\Phi^2 = 2\mu^2 + \frac{\beta v_\Theta^2}{3}. \tag{C.13}$$

As we can see, this result perfectly agrees with both the Einstein-frame calculation from Eq. (3.37) and the diagonalized Jordan-frame calculation in Eq. (3.70)

Appendix D

Functions of `FeynMG`

D.1 Curvature building blocks

`gUp[i1,i2]` — Spacetime metric with raised indices, which must be specified as Lorentz, i.e., `gUp[Index[Lorentz, i1],Index[Lorentz, i2]]` leads to an upper-indexed metric g^{i1i2}. For more information on the function `Index`, see `FeynRules` manual [143].

`gDown[i1,i2]` — Spacetime metric with lowered indices, which must be specified as Lorentz, i.e., `gDown[Index[Lorentz, i1],Index[Lorentz, i2]]` leads to an upper-indexed metric g_{i1i2}.

`eta[i1, i2]` — Flat spacetime metric η^{i1i2}. The indices must be Lorentzian, such that `eta[Index[Lorentz,i1],Index[Lorentz,i2]]`. The specification of lower or upper indices is not necessary in this case, since `FeynRules` does not make that distinction.

`Sqrtg` — Square root of minus the determinant of the metric, corresponding to the Jacobian factor, $\sqrt{-g}$, of the volume element. By default it is assumed as a prefactor to any Lagrangian.

D. *Functions of* `FeynMG`

`VUp[mu,a]` — Upper indexed vierbein $e^{\mu a}$. Indices must be Lorentzian, such that `VUp[Index[Lorentz,mu],Index[Lorentz,a]]`.

`VDown[mu,a]` — Lower indexed vierbein $e_{\mu a}$. Indices must be Lorentzian, such that `VDown[Index[Lorentz,mu],Index[Lorentz,a]]`.

`CovDev[A,mu]` — Gravitational covariant derivative. As in General Relativity, it will take a different form depending on which object it is acting on (i.e., a spinor, scalar or vector field).

`ChrisSym[a,b,c]` — Christoffel symbols Γ^a_{bc} of General Relativity.

`RiemannTensor[a,b,c,d]` — Riemann curvature tensor (4-form). It will appear in the Lagrangian as R^a_{bcd} until the function `ExpandGravity` is applied.

`RicciTensor[a,b]` — Ricci tensor (2-form). It will appear as R_{ab} in the Lagrangian until the function `ExpandGravity` is applied.

`RScalar` — Ricci scalar. It will appear in the Lagrangian as R_{Sc} until the function `ExpandGravity` is applied.

`SHGauge[F]` — Generalization of the harmonic gauge — the *scalar-harmonic gauge* [1], see Eq. (2.32) — for Brans-Dicke theories with a curvature term of the form $\mathcal{L}_{\mathrm{G}} = -\sqrt{-g}\mathtt{F}R/2$. It reduces to the usual harmonic gauge for the Einstein-Hilbert action with $\mathtt{F} = M_{\mathrm{Pl}}^2$.

`CheckMetric[L]` — Tests whether every pair of indices in a Lagrangian `L` is contracted with a metric.

`InsertMetric[L]` — Takes a Lagrangian `L` and at every pair of contracted indices inserts an upper indexed metric $g^{\mu\nu}$. Useful for adapting `FeynRules` model files for use in `FeynMG`.

`InsertDevs[L]` — Upgrades all partial derivatives of vector and fermion fields to covariant derivatives. Useful for adapting `FeynRules` model files for used in `FeynMG`.

`InsertCurv[L]` — Applies both `InsertMetric` and `InsertCurv` to the Lagrangian `L`. Useful for adapting `FeynRules` Model Files for used in `FeynMG`.

D.2 Metric transformations

`ToEinsteinFrame[L, Opts]` — Performs a Weyl transformation of the Lagrangian `L` to the Einstein frame, in the case of Brans-Dicke-type scalar-tensor theories. By specifying the options (`Opts`), the user can turn off the default inclusion of the Jacobian $\sqrt{-g}$, using {`Jacobian`→`Off`}.

`WeylTransform[L,`w`]`** — Performs a Weyl transformation of a Lagrangian `L`, such that the metric transforms as $g_{\mu\nu} \to w^2 g_{\mu\nu}$ and $g^{\mu\nu} \to w^{-2} g^{\mu\nu}$ and the vierbeins as $e^a_\mu \to w e^a_\mu$ and $e^\mu_a \to w^{-1} e^\mu_a$.

`GravityOff[L]` Takes the Minkowski limit for all curvature objects and eliminates all gravitational perturbations (graviton) in a Lagrangian `L`.

D.3 Expansion tools

`LinearizeGravity[L,Opts]` — Linearizes gravity around a flat background metric up to second order in the gravitational sector and first order in the matter sector of a Lagrangian `L`. By specifying the options (Opts), the gravitational sector can be linearized up to third order using {`Grav3pt`→`On`} and the matter sector up to second order using {`Matter2nd`→`On`}. The user can turn off the default inclusion of the Jacobian $\sqrt{-g}$, using {`Jacobian`→`Off`}. Moreover, for Brans-Dicke gravitational sectors, one can choose to automatically use the scalar-harmonic gauge (`SHGauge`) using {`SHGauge`→ `On`}, and update the rest of the covariant derivatives into their modified form from Eq. (2.29) using {`UpdDevs`→ `On`}.

`ExpandGravity[L]` — Expands all the gravitational objects, such as the Ricci scalar, Ricci tensor or Riemann tensor in terms of the metric.

`ExpandCMod[L]` — Expands the `CMod` (modification of the covariant derivatives) in terms of the scalar degree of freedom. This function will be automatically applied once all the scalar fields are canonically normalized.

`Orderh[L,n]` — Truncates a Lagrangian `L` up to the n-th order in the perturbation of the metric perturbation $h_{a,b}$.

`OrderSimplify[L,n]` — Applies the `Mathematica` function `FullSimplify` to all the terms in a Lagrangian `L` of n-th order or lower in the fields.

`IndexSimplify[L,{i1,i2,...}]` — Replaces the Lorentz indices of all the terms of a Lagrangian `L` so that equivalent terms can be combined. The second argument allows the user to specify a set of indices from which the replacements will be chosen.

IndexChange[L,{i1,i2,...}] — Replaces the Lorentz indices of a Lagrangian `L` sequentially from the set of indices `{i1,i2,...}`.

D.4 Tools for canonicalizing fields

CanonScalar[L] — Canonically normalizes the leading kinetic energy terms of the scalar fields of the Lagrangian `L`.

CanonGrav[L] — Canonically normalizes the graviton kinetic energy, assuming that the kinetic terms have a constant prefactor.

MassDiagMG[L] — Diagonalizes the scalar field mass matrix of the Lagrangian `L`.

KineticDiagMG[L,n] — Diagonalizes the kinetic energies of the scalar fields of the Lagrangian `L`.

GravKinMixing[L] — Diagonalizes the kinetic mixings between the trace of the graviton and the scalar fields of the Lagrangian `L`.

D.5 Vacuum expectation values

VevExtract[L, Opts] — Solves for the vacuum expectation values of the real scalar fields in the Lagrangian `L`. By specifying the options {`Fields`→ `{p1,p2,...}`}, the user can choose which fields to expand around their vevs.

VevExpand[L] — Expands and solves for the vacuum expectation values of the real scalar fields in the Lagrangian `L`. The function will output all the different solutions and a dialogue window will prompt the user to select a set of vevs for substitution into the Lagrangian. By specifying the options {`Solution→n`} and {`Fields→{p1,p2,...}`}, the user can choose the n-th solution and the fields to be expanded directly.

D.6 Checking functions

`CheckCanonScalar[L]` — Finds the leading scalar field kinetic energy terms in the Lagrangian `L` and tests whether they are canonically normalized.

`CheckMassMatrix[L]` — Extracts the mass matrix for the scalar fields of the Lagrangian `L` and checks if it is diagonalized.

`CheckKineticMatrix[L]` — Extracts the kinetic energy matrix for the scalar fields of the Lagrangian `L` and checks if it is diagonalized.

`CheckGravityMixing[L]` — Checks whether there is any kinetic mixing between the trace of the graviton h and a scalar field.

D.7 Effective Planck mass

`GiveMpl[L]` — Extracts the effective $M_{\rm Pl}$ from the Lagrangian `L`. It can be used at any stage of the calculation (before or after linearizing gravity or canonically normalizing the kinetic energies).

`InsertMpl[L]` — Extracts and inserts the effective $M_{\rm Pl}$ of the Lagrangian `L`. It can be used at any stage of the calculation (before or after linearizing gravity or canonically normalizing the kinetic energies).

D.8 Output model file

`AddScalar[P,Opts]` — Adds a new massless scalar field named `P` into the loaded set of fields, such that it can be recognized by `FeynRules`. Within the options (`Opts`), the user can choose the mass and width of the associated particle by including `{Mass→X}` or `{Width→X}`, respectively.

`AddParameter[P,Opts]` — Adds a new parameter named `P` into the loaded set of parameters, such that it can be recognized by `FeynRules`. Within the options (`Opts`), the user can choose its value by including `{Value→X}` or its interaction order by including `{InteractionOrder→X}` (where X is set to 1 by default).

`OutputModelMG[OldF,NewF,L,Opts]` — Creates a new model file named `NewF` from an original `FeynRules` model file `OldF`. The new model file will contain the same defined fields and parameters as the original file, with the addition of all the new particles and parameters created using `AddScalar` and `AddParameter`, together with the Lagrangian (`L`), the graviton ($h_{\mu\nu}$) and Planck mass ($M_{\rm Pl}$). By specifying the option `{UpdateMass→True}`, the masses of all scalar fields will be updated.

Bibliography

[1] E. J. Copeland, P. Millington, and S. S. Muñoz, "Fifth forces and broken scale symmetries in the Jordan frame," *JCAP*, vol. 02, no. 02, p. 016, 2022.

[2] S. Sevillano Muñoz, E. J. Copeland, P. Millington, and M. Spannowsky, "FeynMG: a FeynRules extension for scalar-tensor theories of gravity," 2022, [arXiv:2211.14300 [qr-qc]].

[3] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity.* Cambridge University Press, 7 2019.

[4] J. A. Wheeler and K. Ford, *Geons, black holes, and quantum foam: A life in physics*, 1998.

[5] M. D. Schwartz, *Quantum Field Theory and the Standard Model.* Cambridge University Press, 3 2014.

[6] G. Aad *et al.*, "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC," *Phys. Lett. B*, vol. 716, pp. 1–29, 2012.

[7] F. Maltoni, D. Pagani, and X. Zhao, "Constraining the Higgs self-couplings at $e^{+}e^{[-]}$ colliders," *JHEP*, vol. 07, p. 087, 2018.

[8] S. Borowka, C. Duhr, F. Maltoni, D. Pagani, A. Shivaji, and X. Zhao, "Probing the scalar potential via double Higgs boson production at hadron colliders," *JHEP*, vol. 04, p. 016, 2019.

[9] "Constraints on the Higgs boson self-coupling from the combination of single-Higgs and double-Higgs production analyses performed with the ATLAS experiment," 10 2019.

[10] A. M. Sirunyan *et al.*, "Combination of searches for Higgs boson pair production in proton-proton collisions at
$sqrts = 13$ TeV," *Phys. Rev. Lett.*, vol. 122, no. 12, p. 121803, 2019.

[11] J. Tian and K. Fujii, "Measurement of Higgs boson couplings at the International Linear Collider," *Nucl. Part. Phys. Proc.*, vol. 273-275, pp. 826–833, 2016.

[12] H. Abramowicz *et al.*, "Higgs physics at the CLIC electron–positron linear collider," *Eur. Phys. J. C*, vol. 77, no. 7, p. 475, 2017.

[13] P. Agrawal, D. Saha, L.-X. Xu, J.-H. Yu, and C. P. Yuan, "Determining the shape of the Higgs potential at future colliders," *Phys. Rev. D*, vol. 101, no. 7, p. 075023, 2020.

[14] J. Nielsen, "Physics prospects for ATLAS at the HL-LHC," *J. Phys. Conf. Ser.*, vol. 1690, no. 1, p. 012156, 2020.

[15] G. Bellini, L. Ludhova, G. Ranucci, and F. L. Villante, "Neutrino oscillations," *Adv. High Energy Phys.*, vol. 2014, p. 191960, 2014.

[16] R. L. Workman *et al.*, "Review of Particle Physics," *PTEP*, vol. 2022, p. 083C01, 2022.

[17] B. Abi *et al.*, "Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm," *Phys. Rev. Lett.*, vol. 126, no. 14, p. 141801, 2021.

[18] B. C. Allanchach, "Beyond the Standard Model," *CERN Yellow Rep. School Proc.*, vol. 6, pp. 113–144, 2019.

[19] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok, "The Ekpyrotic universe: Colliding branes and the origin of the hot big bang," *Phys. Rev. D*, vol. 64, p. 123522, 2001.

[20] R. H. Brandenberger, "Alternatives to the inflationary paradigm of structure formation," *Int. J. Mod. Phys. Conf. Ser.*, vol. 01, pp. 67–79, 2011.

[21] S. Weinberg, "The cosmological constant problem," *Rev. Mod. Phys.*, vol. 61, pp. 1–23, Jan 1989.

[22] J. Martin, "Everything You Always Wanted To Know About The Cosmological Constant Problem (But Were Afraid To Ask)," *Comptes Rendus Physique*, vol. 13, pp. 566–665, 2012.

[23] H. E. S. Velten, R. F. vom Marttens, and W. Zimdahl, "Aspects of the cosmological "coincidence problem"," *Eur. Phys. J. C*, vol. 74, no. 11, p. 3160, 2014.

[24] M. Kamionkowski and A. G. Riess, "The hubble tension and early dark energy," 2022.

[25] A. G. Riess *et al.*, "A Comprehensive Measurement of the Local Value of the Hubble Constant with 1 km/(s Mpc) Uncertainty from the Hubble Space Telescope and the SH0ES Team," *Astrophys. J. Lett.*, vol. 934, no. 1, p. L7, 2022.

[26] N. Aghanim *et al.*, "Planck 2018 results. VI. Cosmological parameters," *Astron. Astrophys.*, vol. 641, p. A6, 2020, [Erratum: Astron.Astrophys. 652, C4 (2021)].

[27] D. J. E. Marsh, "Axion Cosmology," *Phys. Rept.*, vol. 643, pp. 1–79, 2016.

[28] R. D. Peccei and H. R. Quinn, "CP conservation in the presence of pseudoparticles," *Phys. Rev. Lett.*, vol. 38, pp. 1440–1443, Jun 1977.

[29] S. Weinberg, "A new light boson?" *Phys. Rev. Lett.*, vol. 40, pp. 223–226, Jan 1978.

[30] F. Wilczek, "Problem of strong p and t invariance in the presence of instantons," *Phys. Rev. Lett.*, vol. 40, pp. 279–282, Jan 1978.

[31] W. Hu, R. Barkana, and A. Gruzinov, "Cold and fuzzy dark matter," *Phys. Rev. Lett.*, vol. 85, pp. 1158–1161, 2000.

[32] P. W. Graham, D. E. Kaplan, and S. Rajendran, "Cosmological Relaxation of the Electroweak Scale," *Phys. Rev. Lett.*, vol. 115, no. 22, p. 221801, 2015.

[33] I. Zlatev, L. Wang, and P. J. Steinhardt, "Quintessence, cosmic coincidence, and the cosmological constant," *Physical Review Letters*, vol. 82, no. 5, pp. 896–899, feb 1999.

[34] E. J. Copeland, M. Sami, and S. Tsujikawa, "Dynamics of dark energy," *Int. J. Mod. Phys. D*, vol. 15, pp. 1753–1936, 2006.

[35] C. Armendáriz-Picón, T. Damour, and V. Mukhanov, "k-inflation," *Physics Letters B*, vol. 458, no. 2-3, pp. 209–218, jul 1999.

[36] C. Armendariz-Picon, V. Mukhanov, and P. J. Steinhardt, "Dynamical solution to the problem of a small cosmological constant and late-time cosmic acceleration," *Physical Review Letters*, vol. 85, no. 21, pp. 4438–4441, nov 2000.

[37] L. Knox and M. Millea, "Hubble constant hunter's guide," *Phys. Rev. D*, vol. 101, no. 4, p. 043533, 2020.

[38] D. Lovelock, "The Einstein tensor and its generalizations," *J. Math. Phys.*, vol. 12, pp. 498–501, 1971.

[39] ——, "The four-dimensionality of space and the einstein tensor," *J. Math. Phys.*, vol. 13, pp. 874–876, 1972.

[40] R. Maartens and K. Koyama, "Brane-World Gravity," *Living Rev. Rel.*, vol. 13, p. 5, 2010.

[41] A. S. Koshelev, K. S. Kumar, and A. A. Starobinsky, "Cosmology in nonlocal gravity," 5 2023.

[42] J. Garcia-Bellido and L. Espinosa-Portales, "Cosmic acceleration from first principles," *Phys. Dark Univ.*, vol. 34, p. 100892, 2021.

[43] Y. Fujii and K. ichi Maeda, "The scalar-tensor theory of gravitation," *Classical and Quantum Gravity*, vol. 20, no. 20, p. 4503, oct 2003.

[44] M. Cicoli, J. P. Conlon, A. Maharana, S. Parameswaran, F. Quevedo, and I. Zavala, "String Cosmology: from the Early Universe to Today," 3 2023.

[45] M. Herranen, T. Markkanen, S. Nurmi, and A. Rajantie, "Spacetime curvature and the Higgs stability during inflation," *Phys. Rev. Lett.*, vol. 113, no. 21, p. 211102, 2014.

[46] T. Markkanen, S. Nurmi, A. Rajantie, and S. Stopyra, "The 1-loop effective potential for the Standard Model in curved spacetime," *JHEP*, vol. 06, p. 040, 2018.

[47] C. F. Steinwachs and A. Y. Kamenshchik, "One-loop divergences for gravity non-minimally coupled to a multiplet of scalar fields: calculation in the Jordan frame. I. The main results," *Phys. Rev. D*, vol. 84, p. 024026, 2011.

[48] G. W. Horndeski, "Second-order scalar-tensor field equations in a four-dimensional space," *Int. J. Theor. Phys.*, vol. 10, pp. 363–384, 1974.

[49] T. Kobayashi, "Horndeski theory and beyond: a review," *Rept. Prog. Phys.*, vol. 82, no. 8, p. 086901, 2019.

[50] D. Traykova, E. Bellini, and P. G. Ferreira, "The phenomenology of beyond Horndeski gravity," *JCAP*, vol. 08, p. 035, 2019.

[51] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, "Essential Building Blocks of Dark Energy," *JCAP*, vol. 08, p. 025, 2013.

[52] D. Langlois and K. Noui, "Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability," *JCAP*, vol. 02, p. 034, 2016.

[53] D. Langlois, "Dark energy and modified gravity in degenerate higher-order scalar extendashtensor (DHOST) theories: A review," *Int. J. Mod. Phys. D*, vol. 28, no. 05, p. 1942006, 2019.

[54] C. Brans and R. H. Dicke, "Mach's principle and a relativistic theory of gravitation," *Phys. Rev.*, vol. 124, pp. 925–935, 1961.

[55] D. Bettoni and S. Liberati, "Disformal invariance of second order scalar-tensor theories: Framing the Horndeski action," *Phys. Rev. D*, vol. 88, p. 084020, 2013.

[56] J. D. Bekenstein, "The Relation between physical and gravitational geometry," *Phys. Rev. D*, vol. 48, pp. 3641–3647, 1993.

[57] M. Zumalacárregui and J. García-Bellido, "Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian," *Phys. Rev. D*, vol. 89, p. 064046, 2014.

[58] E. M. Teixeira, A. Nunes, and N. J. Nunes, "Disformally Coupled Quintessence," *Phys. Rev. D*, vol. 101, no. 8, p. 083506, 2020.

[59] V. Silveira and A. Zee, "Scalar phantoms," *Phys. Lett. B*, vol. 161, pp. 136–140, 1985.

[60] J. McDonald, "Gauge singlet scalars as cold dark matter," *Phys. Rev. D*, vol. 50, pp. 3637–3649, 1994.

[61] C. P. Burgess, M. Pospelov, and T. ter Veldhuis, "The Minimal model of nonbaryonic dark matter: A Singlet scalar," *Nucl. Phys. B*, vol. 619, pp. 709–728, 2001.

[62] H. Davoudiasl, R. Kitano, T. Li, and H. Murayama, "The New minimal standard model," *Phys. Lett. B*, vol. 609, pp. 117–123, 2005.

[63] R. M. Schabinger and J. D. Wells, “A Minimal spontaneously broken hidden sector and its impact on Higgs boson physics at the large hadron collider,” *Phys. Rev. D*, vol. 72, p. 093007, 2005.

[64] B. Patt and F. Wilczek, “Higgs-field portal into hidden sectors,” 5 2006.

[65] A. Falkowski, J. Juknevich, and J. Shelton, “Dark Matter Through the Neutrino Portal,” 8 2009.

[66] V. Gonzalez Macias and J. Wudka, “Effective theories for Dark Matter interactions and the neutrino portal paradigm,” *JHEP*, vol. 07, p. 161, 2015.

[67] J. Bamber, “Fifth forces and frame invariance,” *Phys. Rev. D*, vol. 107, no. 2, p. 024013, 2023.

[68] M. Herrero-Valea, “A Path (Integral) to Scale Invariance,” 7 2020.

[69] A. Pilaftsis, K. Finn, V. Gattus, and S. Karamitsos, “Geometrising the Micro-Cosmos on a Supermanifold,” *PoS*, vol. CORFU2021, p. 080, 2022.

[70] C. Wetterich, “Cosmology and the Fate of Dilatation Symmetry,” *Nucl. Phys. B*, vol. 302, pp. 668–696, 1988.

[71] W. Buchmuller and N. Dragon, “Dilatons in Flat and Curved Space-time,” *Nucl. Phys. B*, vol. 321, pp. 207–231, 1989.

[72] M. Shaposhnikov and D. Zenhausern, “Scale invariance, unimodular gravity and dark energy,” *Phys. Lett. B*, vol. 671, pp. 187–192, 2009.

[73] ——, “Quantum scale invariance, cosmological constant and hierarchy problem,” *Phys. Lett. B*, vol. 671, pp. 162–166, 2009.

[74] D. Blas, M. Shaposhnikov, and D. Zenhausern, “Scale-invariant alternatives to general relativity,” *Phys. Rev. D*, vol. 84, p. 044001, 2011.

[75] J. Garcia-Bellido, J. Rubio, M. Shaposhnikov, and D. Zenhausern, “Higgs-Dilaton Cosmology: From the Early to the Late Universe,” *Phys. Rev. D*, vol. 84, p. 123504, 2011.

[76] J. Garcia-Bellido, J. Rubio, and M. Shaposhnikov, “Higgs-Dilaton cosmology: Are there extra relativistic species?” *Phys. Lett. B*, vol. 718, pp. 507–511, 2012.

[77] F. Bezrukov, G. K. Karananas, J. Rubio, and M. Shaposhnikov, “Higgs-Dilaton Cosmology: an effective field theory approach,” *Phys. Rev. D*, vol. 87, no. 9, p. 096001, 2013.

[78] T. Henz, J. M. Pawlowski, A. Rodigast, and C. Wetterich, “Dilaton Quantum Gravity,” *Phys. Lett. B*, vol. 727, pp. 298–302, 2013.

[79] J. Rubio and M. Shaposhnikov, “Higgs-Dilaton cosmology: Universality versus criticality,” *Phys. Rev. D*, vol. 90, p. 027307, 2014.

[80] G. K. Karananas and M. Shaposhnikov, “Scale invariant alternatives to general relativity. II. Dilaton properties,” *Phys. Rev. D*, vol. 93, no. 8, p. 084052, 2016.

[81] P. G. Ferreira, C. T. Hill, and G. G. Ross, “Scale-Independent Inflation and Hierarchy Generation,” *Phys. Lett. B*, vol. 763, pp. 174–178, 2016.

[82] ——, “No fifth force in a scale invariant universe,” *Phys. Rev. D*, vol. 95, no. 6, p. 064038, 2017.

[83] S. Casas, M. Pauly, and J. Rubio, “Higgs-dilaton cosmology: An inflation extendashdark-energy connection and forecasts for future galaxy surveys,” *Phys. Rev. D*, vol. 97, no. 4, p. 043520, 2018.

[84] P. G. Ferreira, C. T. Hill, J. Noller, and G. G. Ross, “Inflation in a scale invariant universe,” *Phys. Rev. D*, vol. 97, no. 12, p. 123516, 2018.

[85] A. Avilez and C. Skordis, “Cosmological constraints on Brans-Dicke theory,” *Phys. Rev. Lett.*, vol. 113, no. 1, p. 011101, 2014.

[86] J. Sakstein and M. Trodden, “Early Dark Energy from Massive Neutrinos as a Natural Resolution of the Hubble Tension,” *Phys. Rev. Lett.*, vol. 124, no. 16, p. 161301, 2020.

[87] M. C. González, Q. Liang, J. Sakstein, and M. Trodden, “Neutrino-assisted early dark energy is a natural resolution of the hubble tension,” 2023.

[88] D. D. Doneva, F. M. Ramazanoğlu, H. O. Silva, T. P. Sotiriou, and S. S. Yazadjiev, “Scalarization,” 2022.

[89] T. Damour and G. Esposito-Farese, “Tensor multiscalar theories of gravitation,” *Class. Quant. Grav.*, vol. 9, pp. 2093–2176, 1992.

[90] J. Zhao, P. C. C. Freire, M. Kramer, L. Shao, and N. Wex, “Closing a spontaneous-scalarization window with binary pulsars,” *Classical and Quantum Gravity*, vol. 39, no. 11, p. 11LT01, may 2022.

[91] D. D. Doneva, S. S. Yazadjiev, N. Stergioulas, and K. D. Kokkotas, “Differentially rotating neutron stars in scalar-tensor theories of gravity,” *Physical Review D*, vol. 98, no. 10, nov 2018.

[92] V. Cardoso, I. P. Carucci, P. Pani, and T. P. Sotiriou, “Matter around Kerr black holes in scalar-tensor theories: scalarization and superradiant instability,” *Phys. Rev. D*, vol. 88, p. 044056, 2013.

[93] B. P. Abbott *et al.*, “GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral,” *Phys. Rev. Lett.*, vol. 119, no. 16, p. 161101, 2017.

[94] A. D. Felice and S. Tsujikawa, “Conditions for the cosmological viability of the most general scalar-tensor theories and their applications to extended galileon dark energy models,” *Journal of Cosmology and Astroparticle Physics*, vol. 2012, no. 02, pp. 007–007, feb 2012.

[95] N. Franchini and T. P. Sotiriou, "Cosmology with subdominant Horndeski scalar field," *Phys. Rev. D*, vol. 101, no. 6, p. 064068, 2020.

[96] J. M. Ezquiaga and M. Zumalacárregui, "Dark energy after GW170817: Dead ends and the road ahead," *Physical Review Letters*, vol. 119, no. 25, dec 2017.

[97] J. Noller, "Cosmological constraints on dark energy in light of gravitational wave bounds," *Physical Review D*, vol. 101, no. 6, mar 2020.

[98] J. G. Williams, S. G. Turyshev, and D. H. Boggs, "Progress in lunar laser ranging tests of relativistic gravity," *Phys. Rev. Lett.*, vol. 93, p. 261101, Dec 2004.

[99] J. Muller and L. Biskupek, "Variations of the gravitational constant from lunar laser ranging data," *Class. Quant. Grav.*, vol. 24, pp. 4533–4538, 2007.

[100] B. Bertotti, L. Iess, and P. Tortora, "A test of general relativity using radio links with the Cassini spacecraft," *Nature*, vol. 425, pp. 374–376, 2003.

[101] J. G. Williams, S. G. Turyshev, and D. H. Boggs, "Lunar laser ranging tests of the equivalence principle with the earth and moon," *Int. J. Mod. Phys. D*, vol. 18, pp. 1129–1175, 2009.

[102] J. G. Lee, E. G. Adelberger, T. S. Cook, S. M. Fleischer, and B. R. Heckel, "New Test of the Gravitational $1/r^2$ Law at Separations down to 52 μm," *Phys. Rev. Lett.*, vol. 124, no. 10, p. 101101, 2020.

[103] T. Westphal, H. Hepach, J. Pfaff, and M. Aspelmeyer, "Measurement of gravitational coupling between millimeter-sized masses," 2021.

[104] S. M. Merkowitz, "Tests of Gravity Using Lunar Laser Ranging," *Living Rev. Rel.*, vol. 13, p. 7, 2010.

[105] C. Burrage, E. J. Copeland, and E. Hinds, "Probing dark energy with atom interferometry," *Journal of Cosmology and Astroparticle Physics*, vol. 2015, no. 03, pp. 042–042, mar 2015.

[106] B. Elder, J. Khoury, P. Haslinger, M. Jaffe, H. Müller, and P. Hamilton, "Chameleon dark energy and atom interferometry," *Physical Review D*, vol. 94, no. 4, aug 2016.

[107] L. Badurina *et al.*, "AION: An Atom Interferometer Observatory and Network," *JCAP*, vol. 05, p. 011, 2020.

[108] P. Brax and A.-C. Davis, "Atomic Interferometry Test of Dark Energy," *Phys. Rev. D*, vol. 94, no. 10, p. 104069, 2016.

[109] C. Burrage and J. Sakstein, "A compendium of chameleon constraints," *Journal of Cosmology and Astroparticle Physics*, vol. 2016, no. 11, pp. 045–045, nov 2016.

[110] D. O. Sabulsky, I. Dutta, E. A. Hinds, B. Elder, C. Burrage, and E. J. Copeland, "Experiment to detect dark energy forces using atom interferometry," *Phys. Rev. Lett.*, vol. 123, no. 6, p. 061102, 2019.

[111] A. Joyce, B. Jain, J. Khoury, and M. Trodden, “Beyond the Cosmological Standard Model,” *Phys. Rept.*, vol. 568, pp. 1–98, 2015.

[112] C. Burrage and J. Sakstein, “Tests of Chameleon Gravity,” *Living Rev. Rel.*, vol. 21, no. 1, p. 1, 2018.

[113] I. Quiros, R. Garcí a-Salcedo, T. Gonzalez, and F. A. Horta-Rangel, “Chameleon effect in the jordan frame of the brans-dicke theory,” *Physical Review D*, vol. 92, no. 4, aug 2015.

[114] A. Tamosiunas, C. Briddon, C. Burrage, A. Cutforth, A. Moss, and T. Vincent, “Chameleon screening in cosmic voids,” *JCAP*, vol. 11, p. 056, 2022.

[115] K. Hinterbichler and J. Khoury, “Symmetron Fields: Screening Long-Range Forces Through Local Symmetry Restoration,” *Phys. Rev. Lett.*, vol. 104, p. 231301, 2010.

[116] K. Hinterbichler, J. Khoury, A. Levy, and A. Matas, “Symmetron Cosmology,” *Phys. Rev. D*, vol. 84, p. 103521, 2011.

[117] A. I. Vainshtein, “To the problem of nonvanishing gravitation mass,” *Phys. Lett. B*, vol. 39, pp. 393–394, 1972.

[118] A. Nicolis, R. Rattazzi, and E. Trincherini, “Galileon as a local modification of gravity,” *Physical Review D*, vol. 79, no. 6, mar 2009.

[119] E. Babichev and C. Deffayet, “An introduction to the vainshtein mechanism,” *Classical and Quantum Gravity*, vol. 30, no. 18, p. 184001, sep 2013.

[120] C. de Rham, G. Gabadadze, and A. J. Tolley, “Resummation of Massive Gravity,” *Phys. Rev. Lett.*, vol. 106, p. 231101, 2011.

[121] C. de Rham, “Massive gravity,” *Living Reviews in Relativity*, vol. 17, no. 1, aug 2014.

[122] R. Alves Batista *et al.*, “EuCAPT White Paper: Opportunities and Challenges for Theoretical Astroparticle Physics in the Next Decade,” 10 2021.

[123] P. Brax, S. Casas, H. Desmond, and B. Elder, “Testing Screened Modified Gravity,” *Universe*, vol. 8, no. 1, p. 11, 2021.

[124] S. Vagnozzi, L. Visinelli, P. Brax, A.-C. Davis, and J. Sakstein, “Direct detection of dark energy: The XENON1T excess and future prospects,” *Phys. Rev. D*, vol. 104, no. 6, p. 063023, 2021.

[125] E. Aprile *et al.*, “The XENON1T Dark Matter Experiment,” *Eur. Phys. J. C*, vol. 77, no. 12, p. 881, 2017.

[126] ——, “Excess electronic recoil events in XENON1T,” *Phys. Rev. D*, vol. 102, no. 7, p. 072004, 2020.

[127] P. Brax, C. Burrage, A.-C. Davis, D. Seery, and A. Weltman, “Anomalous coupling of scalars to gauge fields,” *Phys. Lett. B*, vol. 699, pp. 5–9, 2011.

[128] P. Brax, A.-C. Davis, and B. Elder, “(g-2)μ and screened modified gravity,” *Phys. Rev. D*, vol. 106, no. 4, p. 044040, 2022.

[129] ——, “Screened scalar fields in hydrogen and muonium,” *Phys. Rev. D*, vol. 107, no. 4, p. 044008, 2023.

[130] R. Alemany *et al.*, “Summary Report of Physics Beyond Colliders at CERN,” 2 2019.

[131] J. Jaeckel, M. Lamont, and C. Vallée, “The quest for new physics with the Physics Beyond Colliders programme,” *Nature Phys.*, vol. 16, no. 4, pp. 393–401, 2020.

[132] P. Brax, C. Burrage, A.-C. Davis, D. Seery, and A. Weltman, “Collider constraints on interactions of dark energy with the standard model,” *Journal of High Energy Physics*, vol. 2009, no. 09, pp. 128–128, sep 2009.

[133] S. Argyropoulos, C. Burrage, and C. Englert, “Environmentally aware displaced vertices,” 4 2023.

[134] G. Aad *et al.*, “Anomaly detection search for new resonances decaying into a Higgs boson and a generic new particle X in hadronic final states using $\sqrt{s} = 13$ TeV pp collisions with the ATLAS detector,” 6 2023.

[135] M. Carena, J. Osborne, N. R. Shah, and C. E. M. Wagner, “Supersymmetry and LHC Missing Energy Signals,” *Phys. Rev. D*, vol. 98, no. 11, p. 115010, 2018.

[136] P. J. Fox, R. Harnik, J. Kopp, and Y. Tsai, “Missing Energy Signatures of Dark Matter at the LHC,” *Phys. Rev. D*, vol. 85, p. 056011, 2012.

[137] H. Abreu *et al.*, “The FASER Detector,” 7 2022.

[138] C. Burrage, E. J. Copeland, P. Millington, and M. Spannowsky, “Fifth forces, Higgs portals and broken scale invariance,” *JCAP*, vol. 11, p. 036, 2018.

[139] P. Brax and C. Burrage, “Screening the Higgs portal,” *Phys. Rev. D*, vol. 104, no. 1, p. 015011, 2021.

[140] P. Brax, C. Burrage, and C. Englert, “Disformal dark energy at colliders,” *Phys. Rev. D*, vol. 92, no. 4, p. 044036, 2015.

[141] P. Brax, C. Burrage, C. Englert, and M. Spannowsky, “LHC Signatures Of Scalar Dark Energy,” *Phys. Rev. D*, vol. 94, no. 8, p. 084054, 2016.

[142] M. Aaboud *et al.*, “Constraints on mediator-based dark matter and scalar dark energy models using $\sqrt{s} = 13$ TeV pp collision data collected by the ATLAS detector,” *JHEP*, vol. 05, p. 142, 2019.

[143] A. Alloul, N. D. Christensen, C. Degrande, C. Duhr, and B. Fuks, “FeynRules 2.0 - A complete toolbox for tree-level phenomenology,” *Comput. Phys. Commun.*, vol. 185, pp. 2250–2300, 2014.

[144] C. de Rham, S. Melville, and J. Noller, “Positivity bounds on dark energy: when matter matters,” *JCAP*, vol. 08, p. 018, 2021.

[145] M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," *Proc. Roy. Soc. Lond. A*, vol. 173, pp. 211–232, 1939.

[146] J. F. Donoghue, "Introduction to the effective field theory description of gravity," in *Advanced School on Effective Theories*, 6 1995.

[147] J. F. Donoghue, M. M. Ivanov, and A. Shkerin, "EPFL Lectures on General Relativity as a Quantum Field Theory," 2 2017.

[148] B. S. DeWitt, "Dynamical theory of groups and fields," *Conf. Proc. C*, vol. 630701, pp. 585–820, 1964.

[149] A. O. Barvinsky and G. A. Vilkovisky, "The Generalized Schwinger-Dewitt Technique in Gauge Theories and Quantum Gravity," *Phys. Rept.*, vol. 119, pp. 1–74, 1985.

[150] K. Finn, S. Karamitsos, and A. Pilaftsis, "Frame covariant formalism for fermionic theories," *Eur. Phys. J. C*, vol. 81, no. 7, p. 572, 2021.

[151] E. J. Weinberg, "Radiative corrections as the origin of spontaneous symmetry breaking," Ph.D. dissertation, Harvard U., 1973.

[152] F. Englert, C. Truffin, and R. Gastmans, "Conformal Invariance in Quantum Gravity," *Nucl. Phys. B*, vol. 117, pp. 407–432, 1976.

[153] P. G. Ferreira, C. T. Hill, and G. G. Ross, "Inertial Spontaneous Symmetry Breaking and Quantum Scale Invariance," *Phys. Rev. D*, vol. 98, no. 11, p. 116012, 2018.

[154] S. S. Lahoz and J. Noller, "Testing the speed of gravity with black hole ringdowns," *Phys. Rev. D*, vol. 107, no. 12, p. 124054, 2023.

[155] T. Baker and I. Harrison, "Constraining Scalar-Tensor Modified Gravity with Gravitational Waves and Large Scale Structure Surveys," *JCAP*, vol. 01, p. 068, 2021.

[156] B. S. Haridasu, P. Karmakar, M. De Petris, V. F. Cardone, and R. Maoli, "Testing generalized scalar-tensor theories of gravity with clusters of galaxies," *Phys. Rev. D*, vol. 107, no. 12, p. 124059, 2023.

[157] A. Tamosiunas, "Testing and Emulating Modified Gravity on Cosmological Scales," Ph.D. dissertation, Portsmouth U., 11 2020.

[158] J. M. Martín-García, et al., "xAct: efficient tensor computer algebra for Mathematica," http://xact.es/.

[159] J. Noller, " xIST/COPPER: extension of xAct to investigate scalar-tensor theories at the level of linear perturbations," https://github.com/xAct-contrib/xIST.

[160] M. Lagos, T. Baker, P. G. Ferreira, and J. Noller, "A general theory of linear cosmological perturbations: scalar-tensor and vector-tensor theories," *JCAP*, vol. 08, p. 007, 2016.

[161] A. Belyaev, N. D. Christensen, and A. Pukhov, “CalcHEP 3.4 for collider physics within and beyond the Standard Model,” *Comput. Phys. Commun.*, vol. 184, pp. 1729–1769, 2013.

[162] A. Pukhov, E. Boos, M. Dubinin, V. Edneral, V. Ilyin, D. Kovalenko, A. Kryukov, V. Savrin, S. Shichanin, and A. Semenov, “CompHEP: A Package for evaluation of Feynman diagrams and integration over multiparticle phase space,” 8 1999.

[163] J. Kublbeck, M. Bohm, and A. Denner, “Feyn Arts: Computer Algebraic Generation of Feynman Graphs and Amplitudes,” *Comput. Phys. Commun.*, vol. 60, pp. 165–180, 1990.

[164] V. Shtabovenko, R. Mertig, and F. Orellana, “FeynCalc 9.3: New features and improvements,” *Comput. Phys. Commun.*, vol. 256, p. 107478, 2020.

[165] T. Hahn, S. Passehr, and C. Schappacher, “FormCalc 9 and Extensions,” *PoS*, vol. LL2016, p. 068, 2016.

[166] J. Alwall, M. Herquet, F. Maltoni, O. Mattelaer, and T. Stelzer, “MadGraph 5 : Going Beyond,” *JHEP*, vol. 06, p. 128, 2011.

[167] J. Alwall, R. Frederix, S. Frixione, V. Hirschi, F. Maltoni, O. Mattelaer, H. S. Shao, T. Stelzer, P. Torrielli, and M. Zaro, “The automated computation of tree-level and next-to-leading order differential cross sections, and their matching to parton shower simulations,” *JHEP*, vol. 07, p. 079, 2014.

[168] E. Bothmann *et al.*, “Event Generation with Sherpa 2.2,” *SciPost Phys.*, vol. 7, no. 3, p. 034, 2019.

[169] W. Kilian, T. Ohl, and J. Reuter, “WHIZARD: Simulating Multi-Particle Processes at LHC and ILC,” *Eur. Phys. J. C*, vol. 71, p. 1742, 2011.

[170] A. Alloul, J. D’Hondt, K. De Causmaecker, B. Fuks, and M. Rausch de Traubenberg, “Automated mass spectrum generation for new physics,” *Eur. Phys. J. C*, vol. 73, no. 2, p. 2325, 2013.

[171] C. Degrande, C. Duhr, B. Fuks, D. Grellscheid, O. Mattelaer, and T. Reiter, “UFO - The Universal FeynRules Output,” *Comput. Phys. Commun.*, vol. 183, pp. 1201–1214, 2012.

[172] C. Degrande and O. Mattelaer, “TASI 2013: FeynRules/MadGraph tutorial,” https://indico.cern.ch/event/656211/attachments/1498666/2333189/TASI_feynrules_madgraph_tutorial.pdf.

[173] T. Olyaei and A. Aziziy, “Weak gravitational interaction of fermions: quantum viewpoint,” *Mod. Phys. Lett. A*, vol. 33, no. 37, p. 1850218, 2018.