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Dirac Operators and Batalin-Vilkovisky Quantisation in Noncommutative Geometry

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Till Louise och Florence

Abstract

The underlying theme of this thesis is noncommutative geometry, with a particular focus on Dirac operators. In the first part of the thesis, we investigate through a module theoretic approach to noncommutative Riemannian (spin) geometry how one can induce differential, Riemannian and spinorial structures from a noncommutative ambient space to an appropriate notion of a noncommutative hypersurface, thus providing a framework for constructing Dirac operators on noncommutative hypersurfaces from geometrical data on the embedding space. This is applied to the sequence $\mathbb{T}^2_\theta \hookrightarrow \mathbb{S}^3_\theta \hookrightarrow \mathbb{R}^4_\theta$ of noncommutative hypersurface embeddings. The obtained Dirac operators agree with ones found in the literature obtained by other means.

The second part of the thesis deals with BV quantisation of finite dimensional noncommutative field theories. The modern formulation of the BV formalism of Costello and Gwilliam is adapted to fit our setting. The formalism is illustrated through the computation of correlation functions for scalar field theories and Chern-Simons theories on the fuzzy 2-sphere. The techniques are generalised to accommodate theories with symmetries encoded by triangular Hopf algebras. We use this to compute correlation functions for braided scalar field theories on the fuzzy 2-torus. The BV formalism is also used to study gauge-theoretic aspects of dynamical fuzzy spectral triple models of quantum gravity. Perturbations around the trivial Dirac operator $D_0 = 0$ and an example of perturbations around a non-trivial Dirac operator $D_0 \neq 0$ in the quartic $(0,1)$-model are investigated. From our analysis, we conclude that the gauge-theoretical effects on the correlation functions depend strongly on the amount of gauge symmetry that is broken by the background Dirac operator one chooses to perturb around.

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CONTENTS

INTRODUCTION

One of the greatest challenges of modern theoretical physics is the development of a quantum theory for gravity. So far, the attempts at unravelling the microscopic nature of gravity have been met with resistance. Notable approaches include the likes of string theory, loop quantum gravity, causal set theory, etc.

It is a well known fact that quantum physics is in general noncommutative in nature. Therefore, it is not far-fetched to investigate the possibility if spacetime itself could be described as a noncommutative space in an attempt at understanding quantum gravity. This leads to one of the main themes underlying this thesis: *noncommutative geometry*. In ordinary commutative geometry, there are in many cases a one-to-one correspondence between the (commutative) algebras of functions and the spaces themselves. Thus, one may often reconstruct the spaces from their algebra of functions. A seminal result is that of the commutative *Gelfand-Naimark theorem* (or *Gelfand duality*) [GN43], which essentially states that the category of commutative (and possibly non-unital) $C*$-algebras is equivalent to the category of locally compact Hausdorff spaces. Noncommutative geometry takes this idea and generalises it to also include noncommutative algebras to be thought of as the algebra of functions on some quantum space. One may then construct analogues of geometric objects found in regular commutative geometry.

Noncommutative geometry is also interesting from a physical standpoint apart from quantum gravity. A remarkable example is that the standard model can be expressed in terms of noncommutative geometry [CL91, Con96, CC97, Bar06, Con06, CCM07], see also [vanS15]. Another example is the phase space of a quantum mechanical particle, which is known to be noncommutative. Indeed, the presence of Heisenberg's uncertainty principle is due to the noncommutativity of the position and momentum operators. In particular, this means that the notion of point is absent in the quantum phase space - the uncertainty principle states that one cannot determine the position and momentum of a particle to arbitrary accuracy. In fact, this is a generic feature in noncommutative geometry.

There are several approaches to noncommutative geometry. We will in particular focus on two formulations of noncommutative Riemannian (spin) geometry. The first follows a module theoretic approach akin to the formalisms described in e.g. [Lan97, D-V01, BM20]. The other concept we will focus on is that of *real spectral triples*, conceived and vastly developed by Alain Connes [Con80, Con85, Con94]. Since its inception, this subject has been widely researched and there is an abundance of literature on spectral triples, see e.g. [Con94, G-BVF01, CM08, vanS15]. It is today considered to be one of the main cornerstones of noncommutative geometry. Other approaches for encoding the geometry on a noncommutative space are for instance the vielbein approaches of [AC09] or deformations of commutative Riemannian geometry as in [ADMW06]. There are also other metric based works (which are different but share some conceptual resemblances to the ones we consider in this thesis) with roots in traditional Riemannian geometry such as [AN19, Nor21, AI22].

We will put a particular emphasis on Dirac operators, which play a significant role in the description of the noncommutative version of Riemannian (spin) geometry. A Dirac operator is thought of as encoding the geometric/metric structure of a noncommutative space; in ordinary commutative Riemannian spin geometry, one can reconstruct the geodesic distance from the Dirac operator via the Connes distance formula [Con94]. In Part I of the thesis, we will approach this from a perspective close to that of classical differential geometry. There, we will provide a framework for inducing Dirac operators on a certain notion of noncommutative hypersurfaces from geometrical data on some ambient noncommutative embedding space. Afterwards, Dirac operators will make a return in the spectral triples approach in Part II, Chapter 6, where we will see Dirac operators regarded as dynamical fields of a field theory, describing (toy) models for quantum gravity.

This thesis is based on the three papers [NS20, NSS21, GNS22], and is divided in two parts. Part I treats [NS20] which is joint work of the author and Alexander Schenkel. First out in Chapter 1 is Section 1.1, which aims to introduce the underlying geometrical concepts for building Dirac operators from a module theoretical perspective of noncommutative Riemannian (spin) geometry. This entails to introducing a notion of differential calculus, Riemannian structure and spinorial structure, encoded in a layer by layer fashion. We also introduce in Section 1.1.1 the embedding space $\mathbb{R}^4_\theta$ which serves as the embedding space from which we will induce the geometric structures for the examples later. The following section, Section 1.2, is dedicated to reviewing the relevant Hopf algebra theory used throughout the thesis. In there, we will first introduce the concept of Hopf algebras and present some basic properties before moving on to treat modules over them. There, we will in particular meet the important concept of a (quasi)triangular structure which amongst other things defines a braiding on the category of (left) Hopf algebra modules. We then translate the discussion to the case of Hopf algebra comodules, which carries equivalent formulations of the concepts introduced for modules.

In Chapter 2 we present the central results of [NS20], i.e. the framework for inducing the differential, Riemannian and spinorial structures to noncommutative hypersurfaces (in the sense of Definition 2.1.2) in Section 2.1. For the construction to work, we require that certain hypotheses should be satisfied, introduced in Assumptions 2.1.6, 2.1.11 and 2.1.13. This leads to the main outcome of the section: an explicit formula for the Dirac operator on noncommutative hypersurfaces. In order to illustrate the formalism, we apply in Section 2.2 the construction to the sequence of hypersurface embeddings $\mathbb{T}^2_\theta \hookrightarrow \mathbb{S}^3_\theta \hookrightarrow \mathbb{R}^4_\theta$ studied by Arnlind and Norkvist [AN19]. This is performed stepwise, first for the embedding of the Connes-Landi sphere $\mathbb{S}^3_\theta$ into $\mathbb{R}^4_\theta$, and then for the embedding of the noncommutative Clifford torus $\mathbb{T}^2_\theta$ into $\mathbb{S}^3_\theta$ using the results from the previous case. The induced hypersurface Dirac operators on $\mathbb{S}^3_\theta$ and $\mathbb{T}^2_\theta$ turn out to be isospectral deformations of the commutative ones. In fact, they coincide with the Dirac operators obtained from toric deformations in [CL01, CD-V02, BLvS13].

We close Part I of the thesis in Chapter 3 with some concluding remarks around possible applications and future directions regarding the framework we built in Chapter 2.

The main topic of Part II is that of quantisation of noncommutative field theories, in particular *Batalin-Vilkovisky* (*BV*) *quantisation*. This part is based on [NSS21] which is joint work with Alexander Schenkel and Richard J. Szabo, and [GNS22] which is joint work with James Gaunt and Alexander Schenkel. It is a well-known fact that noncommutative field theories display several features not present in ordinary commutative quantum field theory, see e.g. [Sza03]. One of the challenges up to date is to understand quantisation of noncommutative gauge theories, see e.g. [BKSW10] for an overview of different approaches to noncommutative gauge theory. In this part of the thesis we present our approach to quantisation of noncommutative gauge theories based on the modern formulation of the BV formalism of Costello and Gwilliam [CG16, CG21, Gwi12]. The BV formalism, pioneered by Batalin and Vilkovisky [BV81], is a procedure for defining gauge theoretic path integrals through homological methods. In this thesis, we only consider finite dimensional systems, which significantly simplifies the discussion without impairing the central ideas. In particular, the issues surrounding functional analytical technicalities otherwise found in continuum field theories do not arise in such models. We also would like to mention that similar work has been performed in [Ise19a, Ise19b] for the finite dimensional *classical* BV formalism on certain matrix models, and in [IvS17] in the context of spectral triples.

We begin in Chapter 4 by presenting the underlying mathematical concepts of the finite dimensional BV formalism. In Section 4.1, we review the language in which the finite dimensional BV formalism is expressed, namely that of cochain complexes. There, we give the basic definitions and conventions and provide some important examples. We then introduce the concept of L_∞-*algebras* in Section 4.2. These are higher generalisations of ordinary Lie algebras where the Jacobi identity only holds up to homotopy. We also discuss so called *cyclic structures* on L_∞-algebras, which are the appropriate notion of inner products in this setting. In fact, there is a relation between L_∞-algebras, classical field theories and the BV formalism, see [JRSW19]. Having set up the mathematical background, we then outline the finite dimensional BV formalism, which constructs from the data of a *free BV theory* (which is the cochain complex of the fields, ghosts, and their respective antifields, together with a pairing playing the role of a shifted symplectic structure, see Definition 4.3.1) the free classical observables, which one then deforms in order to obtain the interacting case and the quantum observables. We also provide a method for constructing interaction terms using cyclic L_∞-algebras based on the so called *homotopy Maurer-Cartan action*, see e.g. [JRSW19]. Finally, we describe how one can obtain and compute correlation functions using homological perturbation theory. In particular, the n-point correlation functions can be computed perturbatively order by order in coupling constant or Planck's constant (here treated as formal parameters).

The main content of [NSS21] is presented in Chapter 5. Section 5.1 is an application of the BV formalism as described in Section 4.3 to field theories on the fuzzy 2-sphere. In particular, we consider scalar field theory and Chern-Simons theory on the fuzzy sphere. For the scalar field theory, we study in detail the example of Φ^4-theory where we compute the 2-point correlation function at 1-loop order, reproducing the results obtained through more traditional means as in [CMS01]. From the calculations, one observes the known distinction between planar and non-planar loop corrections in the standard noncommutative field theories, see e.g. [Sza03]. The scalar field theory however has trivial gauge symmetries. To have an example with non-trivial gauge group, we consider Chern-Simons theory (see [ARS00, GMS01]). Note that the fuzzy 2-sphere has a 3-dimensional differential calculus, which explains why we may define Chern-Simons theory on it. Quantisation of noncommutative Chern-Simons theory was mentioned in [GMS02]. Here, we provide a full framework for the quantisation and tools for a perturbative computation of the correlation functions on the fuzzy 2-sphere. The results share many similarities with the study in [CM10] on the Chern-Simons model on finite dimensional commutative dg Frobenius algebras. Next, in Section 5.2, we begin the analysis of so called *braided field theories*, studied in [DCGRS20, DCGRS21]. In there, they defined the notion of *braided L_∞ algebras* which lead to the construction of braided field theories. In short, these are field theories equivariant under the action of a triangular Hopf algebra and whose fields are braided commutative. The first account of standard noncommutative field theories formulated in terms of L_∞-algebras was presented in [BBKL18]. We would like to stress that the terminology "braided" used in this context in fact refers to algebraic structures which are defined in a *symmetric* braided monoidal category, whose symmetric braiding is however defined through a non-trivial triangular R-matrix (i.e. it is not the identity). The more general situation of non-symmetric braidings are significantly more complicated because it obstructs the formulation of certain key properties such as a braided version of symmetry and Jacobi identity for the antibracket. In order to accommodate the braided field theories, a braided version of the considerations in Section 4.3 is developed. This includes a braided BV formalism, a method for constructing interaction terms using braided L_∞-algebras and an adaptation of homological perturbation theory for computing correlation functions to this setting. For scalar field theories, this agrees with the earlier accounts of [Oec00, Oec01] of (symmetric) braided quantum field theory, based on a braided generalisation of Wick's theorem and Gaussian integration. However, in contrast to our framework, theories with gauge symmetries where not treated there. In order to illustrate the braided formalism, we consider scalar field theories on the fuzzy 2-torus in Section 5.3. Due to the braided symmetry, the planar and non-planar loop corrections coincide. This was pointed out for twist deformed theories in [Oec00], and later by [Bal+07]. We will also through an explicit example see that there are non-trivial effects from the braiding in the correlation functions.

The final topic of this thesis is treated in Chapter 6 and is based on [GNS22], which is a contribution to the spectral triples approach to noncommutative quantum gravity. We will mostly work with so called *fuzzy spectral triples*, which are a type of matrix geometries developed by Barrett [Bar15]. We review the relevant concepts related to fuzzy spectral triples in Section 6.1. Roughly speaking, a spectral triple $(A, \mathcal{H}, D)$ consists of a potentially noncommutative algebra A represented on a Hilbert space $\mathcal{H}$, together with a Dirac operator D. The algebra A should be interpreted as the algebra of functions of some noncommutative space and $\mathcal{H}$ should be taken as the space of spinors on which A act. The Dirac operator should, as mentioned above, be thought of as encoding the metric structure of noncommutative Riemannian geometry. For real spectral triples, a so called real structure J and chirality operator Γ are further required, which are related to spinorial structures. In fact, ordinary Riemannian spin geometry can be reformulated in terms of real spectral triples. We are interested in so called *fuzzy spectral triples*, which are finite dimensional versions of real spectral triples where $A = \mathrm{Mat}_N(\mathbb{C})$ is the algebra of complex $(N \times N)$-matrices, together with a fixed (p, q)-Clifford module V such that $\mathcal{H} = A \otimes V$. We also use the terminology *(p, q)-fermion space* $(A, \mathcal{H}; \Gamma, J)$ for a fuzzy spectral triple without the data of a Dirac operator.

Finite spectral triples were classified in [Kra98]. Based on this, the full classification of fuzzy spectral triples was described in [Bar15], where in particular an explicit form of the Dirac operators in terms of Hermitian and anti-Hermitian matrices was presented. This sets up a good foundation for building a theory of *random noncommutative geometry*, see [BG16] for the original account. For a fixed (p, q)-fermion space $(A, \mathcal{H}; \Gamma, J)$, we may consider the *finite dimensional* vector space of Dirac operators $\mathcal{D}$, the so-called *Dirac ensemble* of $(A, \mathcal{H}; \Gamma, J)$, parametrising the fuzzy spectral triples with (p, q)-fermion space $(A, \mathcal{H}; \Gamma, J)$. For a suitable action $S : \mathcal{D} \to \mathbb{R}$, the partition function is defined as the integral (which exists rigorously due to the finite dimensionality of $\mathcal{D}$)

$$Z = \int_{D \in \mathcal{D}} e^{-S(D)} \, \mathrm{d}D \tag{0.0.1a}$$

and the expectation value of an observable $\mathcal{O} : \mathcal{D} \to \mathbb{C}$ by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{D \in \mathcal{D}} \mathcal{O}(D)\, e^{-S(D)}\, \mathrm{d}D \quad . \tag{0.0.1b}$$

Through the classification of finite spectral triples [Kra98], together with the explicit characterisation of Dirac operators [Bar15], the path integrals can be rewritten in terms of random multi-matrix models. There is evidence that random noncommutative geometry exhibit properties also expressed by random matrix theory, for instance the Wigner semicircle law and phase transitions. This was studied using Monte-Carlo simulations in [BG16]. Other works on the numerical side include [BDG19, D'Ar22]. Analytical considerations of examples of such models can be found in e.g. [BDG19, AK19, HKP21, KP21a, KP21b, PS19, PS21a, PS21b]. Random noncommutative geometry of fuzzy spectral triples directly links to the study of (toy-)models of quantum gravity on noncommutative spaces [BG16]. The idea is to consider Dirac operators D, which encode the metric structure, as dynamical variables getting quantised through the path integral (0.0.1). This is where the terminology of dynamical spectral triples stem from. Another reason for considering this approach to quantum gravity is that these Dirac operators have a maximum eigenvalue, which can be interpreted as a natural cutoff to gravitational phenomena at the Planck scale [BG16].

Recall that the study of ordinary gravity is considered up to diffeomorphism gauge symmetries. However, so far, gauge-theoretic considerations for dynamical spectral triple models have not been presented in the literature yet. We present in Section 6.2 a natural noncommutative analogue of these diffeomorphism gauge symmetries carried by the dynamical spectral triple models. The gauge transformations are described by certain unitary operators U on $\mathcal{H}$ and act on the Dirac ensemble $\mathcal{D}$ via the adjoint action $D \mapsto U\,D\,U^*$. We are mainly concerned with the infinitesimal versions of the gauge transformations. This is because we use the BV formalism (described in Section 4.3), which works in the perturbative realm, to define and compute the gauge theoretic path integrals (0.0.1). The actions we consider are invariant under these infinitesimal gauge transformations. Furthermore, we will work with formal perturbations around exact solutions $D_0 \in \mathcal{D}$ of the classical equations of motion of a given action S. This is one of the focus points of [GNS22]. From these considerations, we define an action $\tilde{S}$ for the perturbation around D_0, which by construction is invariant under the infinitesimal gauge transformations.

The principal accomplishment of [GNS22] is an explicit and computationally accessible account of both the classical and quantum BV formalism for the dynamical spectral triple models introduced in [BG16]. The *classical* BV formalism, described in Section 6.3, takes for a fixed fermion space as input the Dirac ensemble $\mathcal{D}$, the infinitesimal gauge symmetries and the action $\tilde{S}$, and returns the classical observables. This is achieved by adapting the systematic techniques from [BSS21], which are based on the modern language of derived algebraic geometry. The perspective on the BV formalism of starting from (in some sense) the more primitive data of a space of fields, infinitesimal gauge symmetries and a gauge invariant action can also often be found in the physics part of the literature, see e.g. [HT92]. Next, in Section 6.4, we briefly recall the BV quantisation and framework for computing correlation functions from Section 4.3 in order to adapt the notation to the current setting.

One of the goals of [GNS22] was to understand whether the gauge symmetries play any role for the path integrals of dynamical spectral triples or not. The final two sections of Chapter 6 aim to answer this question. It turns out that it depends strongly on the amount of gauge symmetry that is broken by the chosen background Dirac operator D_0 one perturbs around. The analysis is split in two parts: one for investigating the instance of perturbations around a vanishing background Dirac operator $D_0 = 0$ and one for the non-zero case $D_0 \neq 0$. The former, $D_0 = 0$, is covered in Section 6.5. We show in Proposition 6.5.3 that the correlation functions for D_0 do not receive any contributions from the ghosts or antifield for ghosts. In other words, there are no gauge-theoretic alterations to the path integrals (0.0.1) in this case. On the contrary, for the case of non-zero backgrounds $D_0 \neq 0$ that break some of the gauge symmetry, the quantum correlation functions indeed receive gauge-theoretic contributions to the quantum correlation functions. This is observed in Section 6.6 through the study of the explicit example of the so called quartic $(0, 1)$-model from [BG16]. Because this model inhabits a "symmetry-breaking potential", it displays a behaviour akin to the Higgs mechanism. In fact, this is identified as the reason for the non-trivial gauge-theoretic adjustments to the correlation functions. This is in particular displayed in Examples 6.6.2 and 6.6.3, where we compute to leading order in coupling constant the 1-point and 2-point functions respectively. Hence, we may conclude that when considering quantum fluctuations localised around a non-trivial classical solution $D_0 \neq 0$

that breaks the gauge symmetry, the path integrals (0.0.1) require gauge theoretic corrections.

In Chapter 7, we summarise the conclusions of the material in Part II and give some indications of possible future projects to consider. That is the final chapter, thus marking the end of the thesis.

conventions and notation: In the following, we list the conventions and notations used in this thesis.

- All vector spaces, algebras, etc., are over a field $\mathbb{K}$ of characteristic 0.
- All algebras are unital and associative unless stated otherwise.
- For an algebra A, we denote the category of left A-modules by ${}_A\mathsf{Mod}$ and the category of A-bimodules by ${}_A\mathsf{Mod}_A$. Furthermore, ${}_A\mathsf{Mod}_A$ is monoidal with respect to the relative tensor product $V \otimes_A W \in {}_A\mathsf{Mod}_A$ of bimodules $V, W \in {}_A\mathsf{Mod}_A$ and the monoidal unit is given by the 1-dimensional free A-bimodule $A \in {}_A\mathsf{Mod}_A$. We denote by ${}_A\mathsf{Mod}$ the left module category over the monoidal category $({}_A\mathsf{Mod}_A, \otimes_A, A)$ with left action given by the relative tensor product $V \otimes_A \mathcal{E} \in {}_A\mathsf{Mod}$ for all $V \in {}_A\mathsf{Mod}_A$ and $\mathcal{E} \in {}_A\mathsf{Mod}$.
- We denote the dual of a finite dimensional vector space V by $V^\vee := \underline{\hom}(V, \mathbb{K})$.
- We use the notation m^* for Hermitian conjugation.
- The natural numbers $\mathbb{N}$ include 0.
- The Levi-Civita symbol ϵ_{ijk} is defined by $\epsilon_{123} = 1$ and total antisymmetry in the exchange of indices.
- The Kronecker delta symbol is denoted by δ_{ij}.
- Summation over repeated indices is always understood, unless stated otherwise.

We will also frequently use the language of monoidal categories and other related concepts. There is a plethora of literature treating this subject, see e.g. [Mac78, Kel05]. For a quick survey of the fundamental definitions surrounding monoidal categories, see e.g. [Bae04]. We will also need the concept of closed monoidal categories and internal homs, see e.g. [Kel05].

Part I: Dirac operators and modules in noncommutative geometry

1

PRELIMINARIES

The first part of the thesis is built around the paper [NS20], which aims to describe a framework for inducing Dirac operators from a noncommutative embedding space to a suitable notion of noncommutative hypersurface. In this first chapter, we will review the underlying mathematical notions required for the construction. This includes an account of the module theoretic approach to noncommutative Riemannian (spin) geometry in Section 1.1 and some theory surrounding Hopf algebras in Section 1.2. The theory reviewed in Section 1.2 will furthermore be relevant for Chapter 5, when we in Section 5.2 and Section 5.3 consider noncommutative field theories with symmetry encoded by a so called triangular Hopf algebra.

1.1 MODULES AND NONCOMMUTATIVE RIEMANNIAN (SPIN) GEOMETRY

The approach to noncommutative geometry to be described in the following is similar to textbook commutative geometry in the sense that familiar structures such as differential calculi, metrics, connections and spinorial structures are encoded layer by layer in a rather explicit fashion. In contrast to the more implicit nature of spectral triples, intuition from ordinary geometry is thus more directly transferable. This is useful for constructing and interpreting examples of noncommutative spaces. In fact, under certain circumstances, this approach is shown by Beggs and Majid [BM17] to give rise to spectral triples.

We now review some relevant concepts, see e.g. [Lan97, D-V01, BM20] and [NS20, Section 2] (which this section is based on) for more details. For an (unital and associative) algebra A, which plays the role of the function algebra for some noncommutative space, we begin by providing a concept of a differential calculus. The goal is to arrive to a notion of a Dirac operator.

Definition 1.1.1. A (first-order) differential calculus on an algebra A is a pair (Ω^1_A, d) consisting of an A-bimodule $\Omega^1_A \in {}_A\mathsf{Mod}_A$ and a linear map $\mathrm{d} : A \to \Omega^1_A$ (called *differential*), such that

(i) $\mathrm{d}(a\,a') = (\mathrm{d}a)\,a' + a\,(\mathrm{d}a')$, for all $a, a' \in A$,

(ii) $\Omega^1_A = A\,\mathrm{d}(A) := \big\{ \sum_i a_i\,\mathrm{d}a'_i \,:\, a_i, a'_i \in A \big\}$.

We call Ω^1_A the A-bimodule of 1-*forms* on A.

Definition 1.1.2. Let (Ω^1_A, d) be a differential calculus on an algebra A.

(i) A *connection* on a left A-module $\mathcal{E} \in {}_A\mathsf{Mod}$ is a linear map $\nabla : \mathcal{E} \to \Omega^1_A \otimes_A \mathcal{E}$ that satisfies the left Leibniz rule

$$\nabla(a\,s) \;=\; a\,\nabla(s) + \mathrm{d}a \otimes_A s \quad , \tag{1.1.1}$$

for all $a \in A$ and $s \in \mathcal{E}$.

(ii) A *bimodule connection* on an A-bimodule $V \in {}_A\mathsf{Mod}_A$ is a pair (∇, σ) consisting of a linear map $\nabla : V \to \Omega^1_A \otimes_A V$ and an A-bimodule isomorphism $\sigma : V \otimes_A \Omega^1_A \to \Omega^1_A \otimes_A V$, such that the following left and right Leibniz rules

$$\nabla(a\,v) \;=\; a\,\nabla(v) + \mathrm{d}a \otimes_A v \quad , \tag{1.1.2a}$$

$$\nabla(v\,a) \;=\; \nabla(v)\,a + \sigma(v \otimes_A \mathrm{d}a) \quad , \tag{1.1.2b}$$

are satisfied, for all $a \in A$ and $v \in V$.

Note that the map σ in the definition for a bimodule connection is required to really make ∇ a map into $\Omega^1_A \otimes_A V$. Simply flipping the tensor factors only works when A is commutative. The following proposition motivates further why the definition of bimodule connections in Definition 1.1.2 (ii) is sensible, see e.g. [D-V01, Section 10].

Proposition 1.1.3. *Let (Ω^1_A, d) be a differential calculus on an algebra A.*

(i) Let $\nabla^{\mathcal{E}}$ be a connection on a left A-module $\mathcal{E} \in {}_A\mathsf{Mod}$ and (∇^V, σ^V) a bimodule connection on an A-bimodule $V \in {}_A\mathsf{Mod}_A$. Then

$$\nabla^{\otimes}(v \otimes_A s) \;:=\; \nabla^V(v) \otimes_A s + (\sigma^V \otimes_A \mathrm{id})\big(v \otimes_A \nabla^{\mathcal{E}}(s)\big) \quad , \tag{1.1.3}$$

for all $v \in V$ and $s \in \mathcal{E}$, defines a connection on the tensor product module $V \otimes_A \mathcal{E} \in {}_A\mathsf{Mod}$.

(ii) Let (∇^V, σ^V) and (∇^W, σ^W) be bimodule connections on two A-bimodules $V, W \in {}_A\mathsf{Mod}_A$. Then

$$\nabla^{\otimes}(v \otimes_A w) := \nabla^V(v) \otimes_A w + (\sigma^V \otimes_A \mathrm{id})\big(v \otimes_A \nabla^W(w)\big) \quad , \tag{1.1.4a}$$

for all $v \in V$ and $w \in W$, and the composite A-bimodule isomorphism

$$\sigma^{\otimes} \; : \; V \otimes_A W \otimes_A \Omega^1_A \xrightarrow{\mathrm{id} \otimes_A \sigma^W} V \otimes_A \Omega^1_A \otimes_A W \xrightarrow{\sigma^V \otimes_A \mathrm{id}} \Omega^1_A \otimes_A V \otimes_A W \tag{1.1.4b}$$

defines a bimodule connection on the tensor product bimodule $V \otimes_A W \in {}_A\mathsf{Mod}_A$.

Since we are interested in noncommutative Riemannian geometry, we need a concept of a metric. Additionally, we would like a symmetry property of metrics and some sort of compatibility between metrics and connections, generalising Levi-Civita connections in the commutative world.

Definition 1.1.4. A (generalised) *metric* on Ω^1_A is an A-bimodule map $g : A \to \Omega^1_A \otimes_A \Omega^1_A$ for which there exists an A-bimodule map $g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A$, such that the two compositions

$$\begin{aligned} \Omega^1_A \;\cong\; \Omega^1_A \otimes_A A \xrightarrow{\mathrm{id} \otimes_A g} \Omega^1_A \otimes_A \Omega^1_A \otimes_A \Omega^1_A \xrightarrow{g^{-1} \otimes_A \mathrm{id}} A \otimes_A \Omega^1_A \;\cong\; \Omega^1_A \\ \Omega^1_A \;\cong\; A \otimes_A \Omega^1_A \xrightarrow{g \otimes_A \mathrm{id}} \Omega^1_A \otimes_A \Omega^1_A \otimes_A \Omega^1_A \xrightarrow{\mathrm{id} \otimes_A g^{-1}} \Omega^1_A \otimes_A A \;\cong\; \Omega^1_A \end{aligned} \tag{1.1.5}$$

are the identity morphisms. We call g^{-1} the *inverse metric*.

Remark 1.1.5. The algebra A itself is a free A-bimodule with basis given by the unit element $1 \in A$. Therefore, a bimodule map $g : A \to \Omega^1_A \otimes_A \Omega^1_A$ is completely specified by its image $g(1) \in \Omega^1_A \otimes_A \Omega^1_A$. Furthermore, $g(1)$ is central, i.e. $a\, g(1) = g(1)\, a$ for all $a \in A$. By writing $g(1) = \sum_\alpha g^\alpha \otimes_A g_\alpha$, the two conditions in Equation (1.1.5) can be recast as

$$\sum_\alpha g^{-1}(\omega \otimes_A g^\alpha)\, g_\alpha = \omega = \sum_\alpha g^\alpha\, g^{-1}(g_\alpha \otimes_A \omega) \quad , \tag{1.1.6}$$

for all $\omega \in \Omega^1_A$. This alternative form is useful later when proving that two A-bimodule maps are a metric and inverse metric pair. It also makes a quick proof of the uniqueness of g^{-1}. △

Proposition 1.1.6. *If it exists, the A-bimodule map $g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A$ in Definition 1.1.4 is unique.*

Proof. Assume there is another A-bimodule map $g'^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A$ such that

$$\sum_\alpha g'^{-1}(\omega \otimes_A g^\alpha)\, g_\alpha = \omega = \sum_\alpha g^\alpha\, g'^{-1}(g_\alpha \otimes_A \omega) \quad , \tag{1.1.7}$$

for all $\omega \in \Omega^1_A$. Then in particular, for any $\omega' \in \Omega^1_A$,

$$\sum_\alpha g^{-1}(\omega \otimes_A g^\alpha)\, g_\alpha \otimes \omega' = \sum_\alpha g'^{-1}(\omega \otimes_A g^\alpha)\, g_\alpha \otimes \omega' \quad . \tag{1.1.8}$$

Applying g^{-1} to both sides and using that both g^{-1} and g'^{-1} are A-bimodule maps together with (1.1.6), we obtain for the left hand side of (1.1.8)

$$\begin{aligned} \sum_\alpha g^{-1}(\omega \otimes_A g^\alpha)\, g^{-1}(\,g_\alpha \otimes \omega') &= g^{-1}(\omega \otimes_A \sum_\alpha g^\alpha\, g^{-1}(\,g_\alpha \otimes \omega')) \\ &= g^{-1}(\omega \otimes_A \omega') \end{aligned} \tag{1.1.9}$$

and for the right hand side,

$$\begin{aligned} \sum_\alpha g'^{-1}(\omega \otimes_A g^\alpha)\, g^{-1}(\,g_\alpha \otimes \omega') &= g'^{-1}(\omega \otimes_A \sum_\alpha g^\alpha\, g^{-1}(\,g_\alpha \otimes \omega')) \\ &= g'^{-1}(\omega \otimes_A \omega') \quad . \end{aligned} \tag{1.1.10}$$

Since $\omega, \omega' \in \Omega^1_A$ are arbitrary, this completes the proof. $\square$

Definition 1.1.7. Let (Ω^1_A, d) be a differential calculus on an algebra A. A *Riemannian structure* on (Ω^1_A, d) is a pair $(g, (\nabla, \sigma))$ consisting of a (generalised) metric g on Ω^1_A and a bimodule connection (∇, σ) on Ω^1_A that satisfies the following properties:

(i) *Symmetry:* The diagram

$$\begin{tikzcd} \Omega^1_A \otimes_A \Omega^1_A \arrow[rr, "\sigma"] \arrow[dr, "g^{-1}"'] & & \Omega^1_A \otimes_A \Omega^1_A \arrow[dl, "g^{-1}"] \\ & A & \end{tikzcd} \tag{1.1.11}$$

commutes.

(ii) *Metric compatibility:* The diagram

$$\begin{tikzcd} \Omega^1_A \otimes_A \Omega^1_A \arrow[r, "\nabla^{\otimes}"] \arrow[d, "g^{-1}"'] & \Omega^1_A \otimes_A \Omega^1_A \otimes_A \Omega^1_A \arrow[d, "\mathrm{id} \otimes_A g^{-1}"] \\ A \arrow[r, "\mathrm{d}"'] & \Omega^1_A \cong \Omega^1_A \otimes_A A \end{tikzcd} \tag{1.1.12}$$

commutes, where $\nabla^{\otimes}$ is the tensor product connection from Proposition 1.1.3.

Remark 1.1.8. One might expect there to be an additional torsion-free condition in the definition of a Riemannian structure. Our choice to skip the implementation of such a condition is because the construction to be outlined in Chapter 2 also covers connections with torsion. Moreover, one would need a notion of a second-order differential calculus Ω^2_A in order to describe torsion. In [BM20], the torsion tensor is defined as $T := \wedge \circ \nabla - \mathrm{d} : \Omega^1_A \to \Omega^2_A$, whereby the torsion-free condition simply would be $T = 0$. $\triangle$

The final building block we need for the definition of a Dirac operator is the concept of a spinorial structure, which we base on the module theoretic approach by Beggs and Majid [BM17, BM20]. For an algebra A, let (Ω^1_A, d) be the corresponding differential calculus and $(g, (\nabla, \sigma))$ be a Riemannian structure on (Ω^1_A, d). The spinorial structure should consist of three objects which are interpreted as the module of sections of a spinor bundle (i.e. spinors), together with a spin connection and Clifford multiplication. For the spinors, we take a left A-module $\mathcal{E} \in {}_A\mathsf{Mod}$ and for the spin connection, a connection $\nabla^{\mathrm{sp}} : \mathcal{E} \to \Omega^1_A \otimes_A \mathcal{E}$. Finally, Clifford multiplication is implemented by an A-module map $\gamma : \Omega^1_A \otimes_A \mathcal{E} \to \mathcal{E}$. These objects should furthermore be compatible with the Riemannian structure $(g, (\nabla, \sigma))$. Before we give the full definition of a spinorial structure, with the associated Dirac operator, let us introduce a convenient notation for the A-module map

$$\gamma_{[2]} \ : \ \Omega^1_A \otimes_A \Omega^1_A \otimes_A \mathcal{E} \xrightarrow{\ \mathrm{id} \otimes_A \gamma\ } \Omega^1_A \otimes_A \mathcal{E} \xrightarrow{\quad \gamma \quad} \mathcal{E} \tag{1.1.13}$$

obtained from iteratively applying γ. Analogously, one obtains the n-times applied Clifford multiplication $\gamma_{[n]} : {\Omega^1_A}^{\otimes_A n} \otimes_A \mathcal{E} \to \mathcal{E}$, for all $n \in \mathbb{N}$.

Definition 1.1.9. Let (Ω^1_A, d) be a differential calculus on an algebra A and $(g, (\nabla, \sigma))$ a Riemannian structure on (Ω^1_A, d). A *spinorial structure* on $(g, (\nabla, \sigma))$ is a triple $(\mathcal{E}, \nabla^{\mathrm{sp}}, \gamma)$ consisting of a left A-module $\mathcal{E} \in {}_A\mathsf{Mod}$, a connection ∇^{sp} on $\mathcal{E}$ and an A-module map $\gamma : \Omega^1_A \otimes_A \mathcal{E} \to \mathcal{E}$ that satisfies the following properties:

(i) *Clifford relations:* The diagram

$$\begin{tikzcd} \Omega^1_A \otimes_A \Omega^1_A \otimes_A \mathcal{E} \arrow[r, "-2g^{-1} \otimes_A \mathrm{id}"] \arrow[d, "\gamma_{[2]} + \gamma_{[2]} \circ (\sigma \otimes_A \mathrm{id})"'] & A \otimes_A \mathcal{E} \\ \mathcal{E} \arrow[ur, "\cong"'] & \end{tikzcd} \tag{1.1.14}$$

commutes.

(ii) *Clifford compatibility:* The diagram

$$\begin{tikzcd} \Omega^1_A \otimes_A \mathcal{E} \arrow[r, "\nabla^{\otimes}"] \arrow[d, "\gamma"'] & \Omega^1_A \otimes_A \Omega^1_A \otimes_A \mathcal{E} \arrow[d, "\mathrm{id}\otimes_A \gamma"] \\ \mathcal{E} \arrow[r, "\nabla^{\mathrm{sp}}"'] & \Omega^1_A \otimes_A \mathcal{E} \end{tikzcd} \tag{1.1.15}$$

commutes, where $\nabla^{\otimes}$ is the tensor product connection from Proposition 1.1.3.

We shall call the composite

$$D \,:\, \mathcal{E} \xrightarrow{\nabla^{\mathrm{sp}}} \Omega^1_A \otimes_A \mathcal{E} \xrightarrow{\gamma} \mathcal{E} \tag{1.1.16}$$

the *Dirac operator* associated with the given spinorial structure.

The Dirac operator satisfies a derivation-like property, specified in the proposition below.

Proposition 1.1.10. *The Dirac operator* (1.1.16) *satisfies*

$$D(a\, s) = a\, D(s) + \gamma(\mathrm{d}a \otimes_A s) \quad , \tag{1.1.17}$$

for all $a \in A$ and $s \in \mathcal{E}$.

Proof. This is a direct consequence of the Leibniz rule (1.1.1) for ∇^{sp} and the fact that γ is left A-linear. □

Remark 1.1.11. The definition of spinorial structure by Beggs and Majid [BM17, BM20] is more general since it excludes the Clifford compatibility condition. We made the decision to include it since it serves as an important guiding principle for our construction of Dirac operators on noncommutative hypersurfaces and is satisfied by the examples we consider, see Chapter 2. △

1.1.1 *Example: noncommutative* $\mathbb{R}^4$

To illustrate the formalism, let us study a noncommutative version of $\mathbb{R}^4$, which we will denote by $\mathbb{R}^4_\theta$.[1] The material covered here can be found in [NS20]. In Chapter 2, we will develop a framework for inducing the geometric structures described above to

1 The noncommutative space $\mathbb{R}^4_\theta$ should not be confused with the Moyal plane, which is given by a certain deformation quantisation of the smooth functions on $\mathbb{R}^n$ known as the Moyal (star) product.

noncommutative hypersurfaces. In the example we study there, $\mathbb{R}^4_\theta$ will serve as the embedding space from which the geometric structures are induced.

To describe $\mathbb{R}^4_\theta$, it will be more convenient to work in complex coordinates rather than real ones. If the real coordinates are given by x^μ, for $\mu = 1, \dots, 4$, the corresponding complex coordinates are $z^1 := x^1 + ix^2$ and $z^2 := x^3 + ix^4$, together with their complex conjugates $z^3 := \overline{z^1} = x^1 - ix^2$ and $z^4 := \overline{z^2} = x^3 - ix^4$. Then, the noncommutative algebra for $\mathbb{R}^4_\theta \cong \mathbb{C}^2_\theta$ is

$$A := \frac{\mathbb{C}[z^1, z^2, z^3, z^4]}{(z^i\, z^j - R^{ji}\, z^j\, z^i)} \tag{1.1.18}$$

i.e. the algebra freely generated by the complex coordinates, modulo the ideal generated by $z^i\, z^j - R^{ji}\, z^j\, z^i$, where R^{ji} are the entries of the matrix

$$R := \begin{pmatrix} 1 & e^{-i\theta} & 1 & e^{i\theta} \\ e^{i\theta} & 1 & e^{-i\theta} & 1 \\ 1 & e^{i\theta} & 1 & e^{-i\theta} \\ e^{-i\theta} & 1 & e^{i\theta} & 1 \end{pmatrix} \quad , \quad \theta \in \mathbb{R} \quad . \tag{1.1.19}$$

Here, and in the following, we do *not* sum over repeated indices unless there is a summation symbol displayed. Observe that the entries of R satisfy

$$R^{ij} = \overline{R^{ji}} \tag{1.1.20a}$$

and

$$R^{ij}\, R^{ji} = 1 \quad . \tag{1.1.20b}$$

We begin by introducing a differential calculus on A (see Definition 1.1.1). Consider the free left A-module

$$\Omega^1_A := \bigoplus_{i=1}^{4} A\, \mathrm{d}z^i \quad , \tag{1.1.21a}$$

together with the right A-action defined by

$$\mathrm{d}z^i\, z^j := R^{ji}\, z^j\, \mathrm{d}z^i \quad . \tag{1.1.21b}$$

(compare with the commutation relations in (1.1.18).) One checks that this defines a bimodule $\Omega^1_A \in {}_A\mathsf{Mod}_A$. By equipping it with the differential $\mathrm{d} : A \to \Omega^1_A$, defined by $\mathrm{d} : z^i \mapsto \mathrm{d}z^i$ on the generators and extended by Leibniz rule (Item (i) in Definition 1.1.1), (Ω^1_A, d) becomes a differential calculus on A:

Proposition 1.1.12. *The pair* (Ω^1_A, d) *is a differential calculus on* A*.*

Proof. The statement holds by construction. □

The next step is to introduce a Riemannian structure. The metric of choice is the standard flat Euclidean metric on $\mathbb{R}^4_\theta$

$$g := \sum_{i,j=1}^{4} g_{ij}\, \mathrm{d}z^i \otimes_A \mathrm{d}z^j \in \Omega^1_A \otimes_A \Omega^1_A \quad , \tag{1.1.22a}$$

where g_{ij} are the entries of the matrix

$$(g_{ij}) := \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad . \tag{1.1.22b}$$

By a straightforward calculation using (1.1.21), (1.1.20b) and (1.1.19), one confirms that the metric $g \in \Omega^1_A \otimes_A \Omega^1_A$ is a central element. By Remark 1.1.5, it defines an A-bimodule map $g : A \to \Omega^1_A \otimes_A \Omega^1_A$. The inverse metric $g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A$ is defined on the basis $\{\mathrm{d}z^i \otimes_A \mathrm{d}z^j : i, j = 1, \dots, 4\}$ of $\Omega^1_A \otimes_A \Omega^1_A$ by

$$g^{-1}\big(\mathrm{d}z^i \otimes_A \mathrm{d}z^j\big) = g^{ij} \quad , \tag{1.1.23a}$$

where g^{ij} are the entries of the matrix

$$(g^{ij}) = 2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad . \tag{1.1.23b}$$

Observe that

$$\sum_{j=1}^{4} g_{ij}\, g^{jk} = \delta_i^k \quad . \tag{1.1.24}$$

Lemma 1.1.13. *The element* $g \in \Omega^1_A \otimes_A \Omega^1_A$ *in* (1.1.22) *defines a (generalised) metric with inverse metric* $g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A$ *defined by* (1.1.23).

Proof. It is sufficient to verify the conditions (1.1.6) in Remark 1.1.5 on the basis $\{\mathrm{d}z^k \in \Omega^1_A\}$. Using (1.1.24), we compute

$$\sum_{i,j=1}^{4} g_{ij}\, \mathrm{d}z^i\, g^{-1}(\mathrm{d}z^j \otimes_A \mathrm{d}z^k) = \sum_{i,j=1}^{4} g_{ij}\, \mathrm{d}z^i\, g^{jk} = \mathrm{d}z^k \quad . \tag{1.1.25}$$

The second condition in (1.1.6) is confirmed through a similar calculation. □

The other part of a Riemannian structure is that of a bimodule connection. As a candidate, we consider the standard connection $\nabla : \Omega^1_A \to \Omega^1_A \otimes_A \Omega^1_A$ on Ω^1_A defined on the basis by

$$\nabla(\mathrm{d}z^i) := 0 \tag{1.1.26}$$

and the left Leibniz rule. Furthermore, we define an A-bimodule isomorphism $\sigma : \Omega^1_A \otimes_A \Omega^1_A \to \Omega^1_A \otimes_A \Omega^1_A$ by

$$\sigma(\mathrm{d}z^i \otimes_A \mathrm{d}z^j) := R^{ji}\, \mathrm{d}z^j \otimes_A \mathrm{d}z^i \tag{1.1.27}$$

and left A-linear extension to all of $\Omega^1_A \otimes_A \Omega^1_A$ (compare again with the commutation relations in (1.1.18)).

Lemma 1.1.14. *The pair* (∇, σ) *introduced in* (1.1.26) *and* (1.1.27) *defines a bimodule connection.*

Proof. We only need to confirm the right Leibniz rule from Definition 1.1.2 (ii). For this, it is sufficient to consider homogeneous elements $a = z^{j_1} \cdots z^{j_n} \in A$, for some $n \in \mathbb{N}$. We compute

$$\begin{aligned}
\nabla\big(\mathrm{d}z^i\, z^{j_1} \cdots z^{j_n}\big) &= \nabla\big(R^{j_1 i} \cdots R^{j_n i}\, z^{j_1} \cdots z^{j_n}\, \mathrm{d}z^i\big) \\
&= R^{j_1 i} \cdots R^{j_n i}\, \mathrm{d}(z^{j_1} \cdots z^{j_n}) \otimes_A \mathrm{d}z^i \\
&= \sigma\big(\mathrm{d}z^i \otimes_A \mathrm{d}(z^{j_1} \cdots z^{j_n})\big) \quad ,
\end{aligned} \tag{1.1.28}$$

where in the first equality we used (1.1.21) and in the second equality we used the left Leibniz rule for the connection (1.1.26). The last equality follows by writing $\mathrm{d}(z^{j_1} \cdots z^{j_n}) = \sum_{k=1}^{n} z^{j_1} \cdots z^{j_{k-1}}\, \mathrm{d}z^{j_k}\, z^{j_{k+1}} \cdots z^{j_n}$ utilising the Leibniz rule and then using (1.1.21), (1.1.20b) and the definition of σ in (1.1.27) in order to rearrange these terms. □

Proposition 1.1.15. *The pair* $(g, (\nabla, \sigma))$ *defined above is a Riemannian structure on* (Ω^1_A, d).

Proof. We have to verify the two properties of Definition 1.1.7. The symmetry property (1.1.11) is immediate from the definition of g^{-1} in (1.1.23), σ in (1.1.27) and R in (1.1.19). The metric compatibility property (1.1.12) also holds since

$$(\mathrm{id} \otimes_A g^{-1}) \nabla^{\otimes}\big(\mathrm{d}z^i \otimes_A \mathrm{d}z^j\big) = 0 = \mathrm{d}\big(g^{-1}(\mathrm{d}z^i \otimes_A \mathrm{d}z^j)\big) \otimes_A 1 \quad , \tag{1.1.29}$$

where the first equality follows from (1.1.26) and the second equality from $g^{-1}(\mathrm{d}z^i \otimes_A \mathrm{d}z^j) \in \mathbb{C}$. □

With both a differential calculus and a Riemannian structure defined, we are now ready to introduce a spinorial structure on $\mathbb{R}^4_\theta$. For the spinor module, we choose the 4-dimensional free left A-module

$$\mathcal{E} := A^4 \in {}_A\mathsf{Mod} \quad . \tag{1.1.30}$$

We will use the notation $\{e_\alpha \in \mathcal{E} : \alpha = 1, \dots, 4\}$ for the standard basis for A^4, i.e. e_α is the vector with 1 in the entry α and 0 elsewhere. For spin connection, we take the standard spin connection $\nabla^{\mathrm{sp}} : \mathcal{E} \to \Omega^1_A \otimes_A \mathcal{E}$ defined by

$$\nabla^{\mathrm{sp}}(e_\alpha) := 0 \tag{1.1.31}$$

and the left Leibniz rule. The final structure for a spinorial structure is a Clifford multiplication, which however requires a bit more effort to write down. We begin by considering the commutative case first before moving to the noncommutative one. In our current complex coordinates z^1, z^2, z^3, z^4, the standard Euclidean gamma matrices for $\mathbb{R}^4$ are given by

$$\begin{aligned}
\gamma^1 &= \begin{pmatrix} 0 & -\sigma^1 - i\,\sigma^2 \\ \sigma^1 + i\,\sigma^2 & 0 \end{pmatrix} \quad , \\
\gamma^2 &= \begin{pmatrix} 0 & -\sigma^3 - I_2 \\ \sigma^3 - I_2 & 0 \end{pmatrix} \quad , \\
\gamma^3 &= \begin{pmatrix} 0 & -\sigma^1 + i\,\sigma^2 \\ \sigma^1 - i\,\sigma^2 & 0 \end{pmatrix} \quad , \\
\gamma^4 &= \begin{pmatrix} 0 & -\sigma^3 + I_2 \\ \sigma^3 + I_2 & 0 \end{pmatrix} \quad ,
\end{aligned} \tag{1.1.32}$$

where I_2 is the 2×2 identity matrix and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.1.33}$$

are the Pauli matrices. The gamma matrices satisfy the Clifford relations $\{\gamma^i, \gamma^j\} := \gamma^i\,\gamma^j + \gamma^j\,\gamma^i = -2\,g^{ij}\,I_4$, with g^{ij} given in (1.1.23) and I_4 is the 4×4 identity matrix. Recall now that the Clifford relations (1.1.14) are given by an anticommutator involving the isomorphism σ in (1.1.27). To define the Clifford multiplication for $\mathbb{R}^4_\theta$, we need to deform the gamma matrices. This can be done via the cocycle deformation techniques developed in [BLvS13, AS14, BSS14]. For an explanation, see the term $\gamma_{S^3}(\mathrm{d}a \otimes_{B_{S^3_\theta}})$ in (2.2.26) and surrounding discussion. The deformed gamma matrices are given by

$$
\begin{aligned}
\gamma^1_\theta &= \begin{pmatrix} 0 & e^{\frac{i}{4}\theta}\,(-\sigma^1 - i\,\sigma^2) \\ e^{-\frac{i}{4}\theta}\,(\sigma^1 + i\,\sigma^2) & 0 \end{pmatrix} \quad , \\
\gamma^2_\theta &= \begin{pmatrix} 0 & e^{-\frac{i}{4}\theta}\,(-\sigma^3 - I_2) \\ e^{\frac{i}{4}\theta}\,(\sigma^3 - I_2) & 0 \end{pmatrix} \quad , \\
\gamma^3_\theta &= \begin{pmatrix} 0 & e^{\frac{i}{4}\theta}\,(-\sigma^1 + i\,\sigma^2) \\ e^{-\frac{i}{4}\theta}\,(\sigma^1 - i\,\sigma^2) & 0 \end{pmatrix} \quad , \\
\gamma^4_\theta &= \begin{pmatrix} 0 & e^{-\frac{i}{4}\theta}\,(-\sigma^3 + I_2) \\ e^{\frac{i}{4}\theta}\,(\sigma^3 + I_2) & 0 \end{pmatrix} \quad .
\end{aligned}
\tag{1.1.34}
$$

Finally, we define the associated Clifford multiplication $\gamma : \Omega^1_A \otimes_A \mathcal{E} \to \mathcal{E}$ by

$$\gamma(\mathrm{d}z^i \otimes_A e_\alpha) \;:=\; \gamma^i_\theta\, e_\alpha \tag{1.1.35}$$

and left A-linear extension to all of $\Omega^1_A \otimes_A \mathcal{E}$, where $\gamma^i_\theta\, e_\alpha$ denotes the action of the matrix γ^i_θ on the basis spinors $e_\alpha \in \mathcal{E} = A^4$ (i.e. the usual multiplication of a matrix and a column vector). In the following, we record some identities that will be useful later.

Lemma 1.1.16. *Define the θ-anticommutator $\{\gamma^i_\theta, \gamma^j_\theta\}_\theta := \gamma^i_\theta\,\gamma^j_\theta + R^{ji}\,\gamma^j_\theta\,\gamma^i_\theta$ and the θ-commutator $[\gamma^i_\theta, \gamma^j_\theta]_\theta := \gamma^i_\theta\,\gamma^j_\theta - R^{ji}\,\gamma^j_\theta\,\gamma^i_\theta$. Then the following properties hold true:*

(i) $\{\gamma^i_\theta, \gamma^j_\theta\}_\theta = R^{ji}\,\{\gamma^j_\theta, \gamma^i_\theta\}_\theta$

(ii) $[\gamma^i_\theta, \gamma^j_\theta]_\theta = -R^{ji}\,[\gamma^j_\theta, \gamma^i_\theta]_\theta$

(iii) $\{\gamma^i_\theta, \gamma^j_\theta\}_\theta = -2\,g^{ij}\,I_4$

Proof. Items (i) and (ii) follow directly from the definitions and (1.1.20b). Item (iii) is a straightforward calculation. □

Proposition 1.1.17. *The triple $(\mathcal{E}, \nabla^{\mathrm{sp}}, \gamma)$ defined in* (1.1.30), (1.1.31) *and* (1.1.35) *is a spinorial structure on the Riemannian structure $(g, (\nabla, \sigma))$.*

Proof. We have to verify the two properties of Definition 1.1.9. It is sufficient to do the verifications on bases. The Clifford relations (1.1.14) follow directly from Lemma 1.1.16 (iii), because

$$\begin{aligned}\big(\gamma_{[2]} + \gamma_{[2]}\,(\sigma \otimes_A \mathrm{id})\big)\big(\mathrm{d}z^i \otimes_A \mathrm{d}z^j \otimes_A e_\alpha\big) &= \big(\gamma^i_\theta\,\gamma^j_\theta + R^{ji}\,\gamma^j_\theta\,\gamma^i_\theta\big)\,e_\alpha\\ &= \{\gamma^i_\theta, \gamma^j_\theta\}_\theta\,e_\alpha\\ &= -2\,g^{ij}\,e_\alpha\\ &= -2\,g(\mathrm{d}z^i \otimes_A \mathrm{d}z^j)\,e_\alpha \quad .\end{aligned} \tag{1.1.36}$$

Clifford compatibility (1.1.15) follows from

$$(\mathrm{id} \otimes_A \gamma)\nabla^{\otimes}(\mathrm{d}z^i \otimes_A e_\alpha) = 0 = \nabla^{\mathrm{sp}}\gamma(\mathrm{d}z^i \otimes_A e_\alpha) \quad , \tag{1.1.37}$$

where we used (1.1.26), (1.1.31) and (1.1.35). □

We end the section by giving an explicit expression for the Dirac operator (1.1.16) associated with our spinorial structure on $\mathbb{R}^4_\theta$. Using our basis, the spinors can be written as $s = \sum_{\alpha=1}^4 s^\alpha\, e_\alpha \in \mathcal{E}$. Furthermore, let us introduce the notation $\mathrm{d}a =: \sum_{i=1}^4 \partial_i a\, \mathrm{d}z^i$, for all $a \in A$. The Dirac operator on $\mathbb{R}^4_\theta$ associated to the spinorial structure developed in this subsection is

$$\begin{aligned}D(s) = \gamma\big(\nabla^{\mathrm{sp}}(s)\big) &= \sum_{\alpha=1}^4 \gamma(\mathrm{d}s^\alpha \otimes_A e_\alpha) = \sum_{\alpha=1}^4 \sum_{i=1}^4 \partial_i s^\alpha\, \gamma^i_\theta\, e_\alpha\\ &= \sum_{\alpha=1}^4 \sum_{i=1}^4 \gamma^i_\theta\, \partial_i s^\alpha\, e_\alpha = \sum_{i=1}^4 \gamma^i_\theta\, \partial_i s \quad ,\end{aligned} \tag{1.1.38}$$

where in the last equality we used the shorthand notation $\partial_i s := \sum_{\alpha=1}^4 \partial_i s^\alpha\, e_\alpha$.

1.2 HOPF ALGEBRAS AND THEIR (CO)MODULES

Hopf algebras emerged as a noncommutative generalisation of the concept of groups and Lie algebras. In noncommutative geometry, Hopf algebras play an important role in constructing examples of noncommutative spaces from commutative ones. Furthermore, some noncommutative spaces exhibit a natural Hopf algebra (co)action on their function algebras. In this section, we will review the concept of Hopf algebras and (co)modules over them. For more details on this subject, see e.g. [Maj95, BM20]. Much of the material covered here can also be found reviewed in [NS20, NSS21].

In the following, for two vector spaces V and W, we shall denote the *flip map* by $\mathrm{flip} : V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$.

Definition 1.2.1. A *Hopf algebra* is an associative unital algebra H over $\mathbb{K}$ together with two algebra homomorphisms[2] $\Delta : H \to H \otimes H$ (*coproduct*) and $\epsilon : H \to \mathbb{K}$ (*counit*), as well as a $\mathbb{K}$-linear map $S : H \to H$ (*antipode*) satisfying

$$(\Delta \otimes \mathrm{id}_H) \circ \Delta \,=\, (\mathrm{id}_H \otimes \Delta) \circ \Delta \quad , \tag{1.2.1a}$$

$$(\epsilon \otimes \mathrm{id}_H) \circ \Delta \,=\, \mathrm{id}_H \,=\, (\mathrm{id}_H \otimes \epsilon) \circ \Delta \quad , \tag{1.2.1b}$$

$$\mu \circ (S \otimes \mathrm{id}_H) \circ \Delta \,=\, \eta \circ \epsilon \,=\, \mu \circ (\mathrm{id}_H \otimes S) \circ \Delta \quad , \tag{1.2.1c}$$

where $\mu : H \otimes H \to H$ denotes the product and $\eta : \mathbb{K} \to H$ denotes the unit of the algebra H. Property (1.2.1a) is called *coassociativity*.

Following common practice, we use concatenation $h\,h'$ for the product $\mu(h \otimes h')$. We will do this for other product structures as well for the rest of the thesis.

Remark 1.2.2. The coproduct of an element $h \in H$ is given by a sum $\Delta(h) = \sum_{i=1}^{n} h_{1\,i} \otimes h_{2\,i} \in H \otimes H$, for some positive integer $n \in \mathbb{Z}_{\geq 1}$. A standard notational convention for the coproduct is the so called *Sweedler notation*. For $h \in H$, we write

$$\Delta(h) = \sum_{i=1}^{n} h_{1\,i} \otimes h_{2\,i} =: h_{\underline{1}} \otimes h_{\underline{2}} \qquad \text{(summation understood)} \tag{1.2.2a}$$

and for the 2-fold application of the coproduct,

$$\Delta^2(h) = h_{\underline{1}} \otimes h_{\underline{2}} \otimes h_{\underline{3}} \qquad \text{(summation understood)} \quad . \tag{1.2.2b}$$

The notation for the 2-fold application is motivated by the coassociativity property (1.2.1a), $\Delta^2 \,=\, (\Delta \otimes \mathrm{id}_H) \circ \Delta \,=\, (\mathrm{id}_H \otimes \Delta) \circ \Delta$. This extends to the n-fold application of the coproduct

$$\Delta^n(h) = h_{\underline{1}} \otimes \cdots \otimes h_{\underline{n+1}} \qquad \text{(summation understood)} \quad . \tag{1.2.2c}$$

Expressed in Sweedler notation, the second and third properties in (1.2.1) take the form

$$\epsilon(h_{\underline{1}})\, h_{\underline{2}} \,=\, h \,=\, h_{\underline{1}}\, \epsilon(h_{\underline{2}}) \quad , \tag{1.2.3a}$$

$$S(h_{\underline{1}})\, h_{\underline{2}} \,=\, \epsilon(h)\, \mathbb{1} \,=\, h_{\underline{1}}\, S(h_{\underline{2}}) \quad , \tag{1.2.3b}$$

for all $h \in H$, where $\mathbb{1} = \eta(1)$ and 1 is the unit in the field $\mathbb{K}$. △

2 The algebra structure of $H \otimes H$ is given by $(h \otimes g)(h' \otimes g') = h\,h' \otimes g\,g'$ for all $h, h', g, g' \in H$.

The antipode can be thought of as something like an "inversion" map. Let us document the following properties of the antipode in the below proposition, see e.g. [Maj95, Proposition 1.3.1].

Proposition 1.2.3. *Let H be a Hopf algebra with antipode S.*

(i) The antipode of a Hopf algebra is unique; if S' is another $\mathbb{K}$-linear map satisfying the axioms of an antipode for H, then $S = S'$.

(ii) The antipode is an algebra antihomomorphism: $S(h\,h') = S(h')\,S(h)$ and $S(\mathbb{1}) = \mathbb{1}$ for all $h, h' \in H$.

(iii) The antipode is a coalgebra antihomomorphism : $S(h)_{\underline{1}} \otimes S(h)_{\underline{2}} = S(h_{\underline{2}}) \otimes S(h_{\underline{1}})$ and $\epsilon(S(h)) = \epsilon(h)$ for all $h \in H$.

1.2.1 *Hopf algebra modules*

We now introduce the concept of a module over a fixed Hopf algebra H. These modules, together with the appropriate morphisms, form a closed monoidal category. Afterwards, we will encounter so called *quasitriangular structures*, which can be used to define braidings for the monoidal category of H-modules. We will also briefly touch upon H-modules with a compatible algebra structure. We end this subsection by providing two standard examples, which can be seen as the starting point of the study of Hopf algebras.

Definition 1.2.4. Let H be a Hopf algebra. A *left H-module* is a vector space V together with a $\mathbb{K}$-linear map $\triangleright : H \otimes V \to V$, $h \otimes v \mapsto h \triangleright v$ (*left action*) such that

$$(h\,h') \triangleright v \;=\; h \triangleright (h' \triangleright v) \quad , \qquad \mathbb{1} \triangleright v \;=\; v \quad , \tag{1.2.4}$$

for all $h, h' \in H$ and $v \in V$. We denote by ${}_H\mathsf{Mod}$ the category of left H-modules and H-equivariant maps (H-module morphisms), i.e. $\mathbb{K}$-linear maps $f : V \to W$ satisfying $f(h \triangleright v) = h \triangleright f(v)$ for all $h \in H$ and $v \in V$.

The category ${}_H\mathsf{Mod}$ is endowed with a monoidal structure defined by the coproduct and the counit in H:

- The monoidal product $V \otimes W$ of two objects $V, W \in {}_H\mathsf{Mod}$ is given by the tensor product; the left tensor product action on $V \otimes W$ is given by

$$h \triangleright (v \otimes w) := (h_{\underline{1}} \triangleright v) \otimes (h_{\underline{2}} \triangleright w) \quad . \tag{1.2.5}$$

  This action is well-defined because the coproduct Δ is an algebra homomorphism.

- The monoidal unit is given by the one-dimensional vector space $\mathbb{K}$ with the trivial left H-action $h \triangleright c = \epsilon(h)\, c$, for all $h \in H$ and $c \in \mathbb{K}$.

- The associator is the usual one for tensor products of vector spaces, $\alpha : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$.

- The left and right unitors are given by the isomorphisms $\mathbb{K} \otimes V \cong V \cong V \otimes \mathbb{K}$, defined by identifying $1 \otimes v$ and $v \otimes 1$ with v, respectively.

Due to the antipode, the finite dimensional objects $V \in {}_H\mathsf{Mod}$ have duals given by $V^\vee = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ together with the left action

$$h \triangleright f = f \circ (S(h) \triangleright \cdot) \quad , \tag{1.2.6}$$

for all $h \in H$ and $f \in V^\vee$. This action is well-defined because the antipode is an algebra antihomomorphism, see Proposition 1.2.3 (ii).

Furthermore, the antipode ensures that ${}_H\mathsf{Mod}$ is closed monoidal. For $V, W \in {}_H\mathsf{Mod}$, the internal hom from V to W is the vector space $\underline{\mathrm{hom}}(V, W) := \mathrm{Hom}_{\mathbb{K}}(V, W)$ of all (not necessarily H-equivariant) linear maps from V to W with the left H-module structure given by the left adjoint action

$$h \triangleright f := (h_{\underline{1}} \triangleright \cdot) \circ f \circ (S(h_{\underline{2}}) \triangleright \cdot) \quad , \tag{1.2.7}$$

for all $h \in H$ and all linear maps $f : V \to W$. Note that this really is a left action because the coproduct is an algebra homomorphism and the antipode is an algebra antihomomorphism (Proposition 1.2.3 (ii)); for $h, h' \in H$ and $f \in \underline{\mathrm{hom}}(V, W)$, we have

$$\begin{aligned}
(h\, h') \triangleright f &= (h_{\underline{1}}\, h'_{\underline{1}} \triangleright \cdot) \circ f \circ (S(h_{\underline{2}}\, h'_{\underline{2}}) \triangleright \cdot) \\
&= (h_{\underline{1}}\, h'_{\underline{1}} \triangleright \cdot) \circ f \circ (S(h'_{\underline{2}})\, S(h_{\underline{2}}) \triangleright \cdot) \\
&= (h_{\underline{1}} \triangleright \cdot) \circ (h' \triangleright f) \circ (S(h_{\underline{2}}) \triangleright \cdot) \\
&= h \triangleright (h' \triangleright f)
\end{aligned} \tag{1.2.8}$$

and

$$\mathbb{1} \triangleright f = (\mathbb{1} \triangleright \cdot) \circ f \circ (S(\mathbb{1}) \triangleright \cdot) = f \quad . \tag{1.2.9}$$

In the case when $V \in {}_H\mathsf{Mod}$ is finite dimensional and for any $W \in {}_H\mathsf{Mod}$, the left adjoint action can be identified with the left action on $W \otimes V^\vee \cong \underline{\hom}(V, W)$, which is the tensor product action (1.2.5) in conjunction with the dual action (1.2.6).

Observe that the H-invariants of $\underline{\hom}(V, W)$, i.e. the linear maps $f : v \to W$ such that

$$h \triangleright f = \epsilon(h) f \quad , \tag{1.2.10}$$

are precisely the H-equivariant maps. Indeed, if $f : V \to W$ is H-equivariant, then it is by (1.2.3b) also H-invariant under the left adjoint action (1.2.7). Conversely, let $f : V \to W$ be a H-invariant linear map. Then

$$\begin{aligned} f(h \triangleright v) &= \epsilon(h_{\underline{1}}) f(h_{\underline{2}} \triangleright v) \\ &= h_{\underline{1}} \triangleright f(S(h_{\underline{2}}) \triangleright (h_{\underline{3}} \triangleright v)) \\ &= h_{\underline{1}} \triangleright f(\epsilon(h_{\underline{2}}) \triangleright v) \\ &= h \triangleright f(v) \quad , \end{aligned} \tag{1.2.11}$$

for all $h \in H$ and $v \in V$, i.e. f is H-equivariant. We used (1.2.3a) in the first equality, and in the second one, we utilised the H-invariance of f in tandem with coassociativity of the coproduct (1.2.1a) (using the notation (1.2.2c)) to write out the left adjoint action (1.2.7). In the third equality, we used (1.2.3b) and in the final row (1.2.3a) again. Thus, we have established that the H-invariant maps of $\underline{\hom}(V, W)$ are precisely the morphisms from V to W in ${}_H\mathsf{Mod}$.

A *left H-module algebra* is a monoid in the monoidal category ${}_H\mathsf{Mod}$, i.e. an object $A \in {}_H\mathsf{Mod}$ together with H-equivariant maps $\mu_A : A \otimes A \to A$ and $\eta_A : \mathbb{K} \to A$ such that the diagrams

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\mu_A"] \arrow[d, "\mu_A\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "\mu_A"] \\ A \otimes A \arrow[r, "\mu_A"'] & A \end{tikzcd} \tag{1.2.12a}$$

$$\begin{tikzcd} \mathbb{K} \otimes A \arrow[r, "\eta_A\otimes\mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu_A"] & A \otimes \mathbb{K} \arrow[l, "\mathrm{id}\otimes\eta_A"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd} \tag{1.2.12b}$$

commute. Explicitly, the H-equivariance of the product μ_A and the unit η_A can be written as

$$h \triangleright (a\, a') = (h_{\underline{1}} \triangleright a)\, (h_{\underline{2}} \triangleright a') \tag{1.2.13a}$$

$$h \triangleright \mathbb{1}_A = \epsilon(h)\, \mathbb{1}_A \tag{1.2.13b}$$

for all $h \in H$ and $a, a' \in A$. As before, we write $\eta_A(1) = \mathbb{1}_A$.

Hopf algebras can also be equipped with additional structure that makes their corresponding categories of left (or right) modules closed *braided* or *symmetric* monoidal. Let $\Delta^{\mathrm{op}}(h) = h_{\underline{2}} \otimes h_{\underline{1}}$ be the opposite coproduct. For an element $R^\alpha \otimes R_\alpha \in H \otimes H$ (summation understood), we use the notation

$$R_{21} := R_\alpha \otimes R^\alpha \tag{1.2.14a}$$

and for particular related elements in $H \otimes H \otimes H$,

$$R_{12} := R^\alpha \otimes R_\alpha \otimes \mathbb{1} \quad , \qquad R_{13} := R^\alpha \otimes \mathbb{1} \otimes R_\alpha \quad , \qquad R_{23} := \mathbb{1} \otimes R^\alpha \otimes R_\alpha \quad . \tag{1.2.14b}$$

Definition 1.2.5. A *quasitriangular structure* for a Hopf algebra H is an invertible element $R \in H \otimes H$ (*universal* R*-matrix*) satisfying

$$\Delta^{\mathrm{op}}(h) \;=\; R\, \Delta(h)\, R^{-1} \quad , \tag{1.2.15a}$$

$$(\mathrm{id}_H \otimes \Delta)(R) \;=\; R_{13}\, R_{12} \quad , \tag{1.2.15b}$$

$$(\Delta \otimes \mathrm{id}_H)(R) \;=\; R_{13}\, R_{23} \quad , \tag{1.2.15c}$$

for all $h \in H$. A *triangular structure* is a quasitriangular structure $R \in H \otimes H$ which additionally satisfies $R_{21} = R^{-1}$.

Let us also write down a couple of useful standard identities for the R-matrix, see e.g. [Maj95, Lemma 2.1.2]

Proposition 1.2.6. *Let R be a quasitriangular structure on a Hopf algebra H. Then, R as an element in $H \otimes H$ satisfies*

$$(\epsilon \otimes \mathrm{id})\, R = (\mathrm{id} \otimes \epsilon)\, R = 1 \tag{1.2.16a}$$

$$(S \otimes \mathrm{id})\, R = R^{-1} \; , \;\; (\mathrm{id} \otimes S)\, R^{-1} = R \quad . \tag{1.2.16b}$$

Assuming that $R = R^\alpha \otimes R_\alpha \in H \otimes H$ is a quasitriangular structure on H, we can define a braiding for the closed monoidal category ${}_H\mathsf{Mod}$

$$\tau_R : V \otimes W \longrightarrow W \otimes V \ , \ \ v \otimes w \longmapsto (R_\alpha \triangleright w) \otimes (R^\alpha \triangleright v) \quad , \tag{1.2.17}$$

for every pair of objects $V, W \in {}_H\mathsf{Mod}$. Due to (1.2.15a), the braiding τ_R is a morphism in ${}_H\mathsf{Mod}$ (i.e. τ_R is H-equivariant). If in addition R is triangular, this braiding is symmetric (i.e. $\tau_R = \tau_R^{-1}$), meaning ${}_H\mathsf{Mod}$ is a closed symmetric monoidal category. Using the braiding, we can obtain a notion of commutativity for H-module algebras: we say that a left H-module algebra $A \in {}_H\mathsf{Mod}$ is (*braided*) *commutative* if the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "\tau_R"] \arrow[dr, "\mu_A"'] & A \otimes A \arrow[d, "\mu_A"] \\ & A \end{tikzcd} \tag{1.2.18}$$

commutes. The adjective braided will be used when we need to distinguish from strict commutativity (i.e. $\mu \circ \mathrm{flip} = \mu$).

As mentioned in the beginning of this section, Hopf algebras are related to certain constructions on groups and Lie algebras. We will now outline these examples.

Example 1.2.7. Let $\mathfrak{g}$ be a finite dimensional Lie algebra with Lie bracket $[\,\cdot\,,\,\cdot\,]$. The *universal enveloping algebra* of $\mathfrak{g}$ is the free algebra of $\mathfrak{g}$ modulo the relation encoding the Lie bracket as a commutator,

$$U\mathfrak{g} := T(\mathfrak{g}) / (x \otimes y - y \otimes x - [x, y]) \tag{1.2.19}$$

with $T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k}$ where $\mathfrak{g}^{\otimes 0} = \mathbb{K}$. The universal enveloping algebra is a Hopf algebra with coproduct, counit and antipode

$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad , \quad \epsilon(x) = 0 \quad , \quad S(x) = -x \tag{1.2.20}$$

for all $x \in \mathfrak{g}$ and extended as algebra homomorphisms (in the case of the coproduct Δ and counit ϵ) and algebra antihomomorphism (in the case of the antipode S) to the entirety of $U\mathfrak{g}$. This gives rise to a closed monoidal category ${}_{U\mathfrak{g}}\mathsf{Mod}$ of left $U\mathfrak{g}$-modules. In order to understand (left) $U\mathfrak{g}$-modules more concretely, we check that the closed monoidal category ${}_{U\mathfrak{g}}\mathsf{Mod}$ is in fact isomorphic to the closed monoidal category $\mathrm{Rep}_{\mathbb{K}}(\mathfrak{g})$ of Lie algebra representations of $\mathfrak{g}$ and equivariant maps. The monoidal product of objects $(V, \pi_V), (W, \pi_W) \in \mathrm{Rep}_{\mathbb{K}}(\mathfrak{g})$ is just the tensor product representation $(V \otimes W, \pi_{V \otimes W})$ with $\pi_{V \otimes W}(x) = \pi_V \otimes \mathrm{id} + \mathrm{id} \otimes \pi_W$. The internal hom between representations $(V, \pi_V), (W, \pi_W) \in \mathrm{Rep}_{\mathbb{K}}(\mathfrak{g})$ is the vector space $\underline{\mathrm{hom}}(V, W) :=$

$\mathrm{Hom}_{\mathbb{K}}(V,W)$ of all linear maps between V and W together with the Lie algebra homomorphism $\pi_{\underline{\mathrm{hom}}(V,W)} : \mathfrak{g} \to \mathrm{End}(\underline{\mathrm{hom}}(V,W))$ defined by $x \mapsto \pi_W(x) \circ f - f \circ \pi_V(x)$. By comparing these with the analogous structures of ${}_{U\mathfrak{g}}\mathsf{Mod}$, and the fact that there is a one-to-one correspondence between Lie algebra homomorphisms $\mathfrak{g} \to \mathrm{End}(V)$ and algebra homomorphisms $U\mathfrak{g} \to \mathrm{End}(V)$, it is not far to conclude that ${}_{U\mathfrak{g}}\mathsf{Mod}$ and $\mathrm{Rep}_{\mathbb{K}}(\mathfrak{g})$ are isomorphic as closed monoidal categories.

From (1.2.13) and (1.2.20), a $U\mathfrak{g}$-module algebra is an algebra A on which $\mathfrak{g}$ acts as

$$x \triangleright (a\,a') = (x \triangleright a)\,a' + a\,(x \triangleright a') \quad , \quad x \triangleright \mathbb{1}_A = 0 \tag{1.2.21}$$

for all $x \in \mathfrak{g}$ and $a, a' \in A$, i.e. $\mathfrak{g}$ acts on A as derivations. $\triangledown$

Example 1.2.8. Let G be a finite group. The *group Hopf algebra* $\mathbb{K}[G]$ is the free vector space spanned by the elements $g \in G$. The elements $h \in \mathbb{K}[G]$ can thus be written uniquely as $h = \sum_{g\in G} h_g\, g$ for some $h_g \in \mathbb{K}$. The product in $\mathbb{K}[G]$ is given by bilinear extension of the group operation in G (which we here denote by $\cdot$), i.e. for two elements $h = \sum_{g\in G} h_g\, g$ and $h' = \sum_{g'\in G} h'_{g'}\, g'$ in $\mathbb{K}[G]$, their product is

$$h\,h' = \sum_{g,g'\in G} h_g\, h'_{g'}\, g \cdot g' \quad . \tag{1.2.22}$$

The unit in $\mathbb{K}[G]$ is given by the identity element $e \in G$. The coproduct, counit and antipode are given by

$$\Delta(g) = g \otimes g \quad , \qquad \epsilon(g) = 1 \quad , \qquad S(g) = g^{-1} \quad , \tag{1.2.23}$$

for all $g \in G$ and extending linearly to all of $\mathbb{K}[G]$. Note that the concept of group Hopf algebras also makes sense for non-finite groups, in which case the elements can be written as a sum where all but finitely many of the h_g vanish. However, we will keep to the finite case for simplicity.

The (quasi)triangular structures that a group Hopf algebra may be equipped with depends on what the underlying group G is. One triangular structure which works for all G is the trivial one, $R_{\mathrm{triv}} = e \otimes e \in \mathbb{K}[G] \otimes \mathbb{K}[G]$. The trivial triangular structure R_{triv} gives rise to a trivial symmetric braiding, i.e. the flip map. The corresponding closed symmetric monoidal category ${}_{\mathbb{K}[G]}\mathsf{Mod}$ is nothing but the closed symmetric monoidal category $\mathrm{Rep}_{\mathbb{K}}(G)$ of $\mathbb{K}$-linear representations of G and equivariant maps. The monoidal product of two objects $(V, \rho_V), (W, \rho_W) \in \mathrm{Rep}_{\mathbb{K}}(G)$ is the representation $(V \otimes W, \rho_{V\otimes W})$ with $\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g)$. The symmetric braiding in $\mathrm{Rep}_{\mathbb{K}}(G)$

is the flip map. The internal hom between two G-representations $(V,\rho_V),(W,\rho_W) \in \mathrm{Rep}_{\mathbb{K}}(G)$ is the vector space $\underline{\mathrm{hom}}(V,W) := \mathrm{Hom}_{\mathbb{K}}(V,W)$ of all linear maps between V and W together with the monoid map $\rho_{\underline{\mathrm{hom}}(V,W)} : G \to \mathrm{End}(\underline{\mathrm{hom}}(V,W))$ defined by conjugation $g \mapsto \rho_W(g) \circ f \circ \rho_V(g^{-1})$. Since there is a one-to-one correspondence between monoid maps $G \to \mathrm{End}(V)$ and algebra homomorphisms $\mathbb{K}[G] \to \mathrm{End}(V)$, one readily obtains the isomorphism between the closed symmetric monoidal categories ${}_{\mathbb{K}[G]}\mathsf{Mod}$ and $\mathrm{Rep}_{\mathbb{K}}(G)$.

The conditions (1.2.13) together with (1.2.23) implies that $\mathbb{K}[G]$-module algebras are algebras on which G acts by algebra automorphisms.

The following class of (non-trivial) examples of triangular group Hopf algebras will be useful for describing the symmetries of the fuzzy torus. This is relevant for Chapter 5. In the following, let $\mathbb{K} = \mathbb{C}$. For two positive integers $n, N \in \mathbb{Z}_{\geq 1}$, consider the Abelian group

$$\mathbb{Z}_N^n := \underbrace{\mathbb{Z}_N \times \cdots \times \mathbb{Z}_N}_{n\text{-times}} \tag{1.2.24}$$

consisting of an n-fold product of the cyclic group $\mathbb{Z}_N$ of order N. We will use the notation

$$\underline{k} := (k_1, \dots, k_n) \in \mathbb{Z}_N^n \quad , \tag{1.2.25}$$

for the elements of $\mathbb{Z}_N^n$ and the group operation is given by addition modulo N in each entry. A quasitriangular structure on the group Hopf algebra $\mathbb{C}[\mathbb{Z}_N^n]$ is defined for any N-th root of unity $q \in \mathbb{C}$ and any $n \times n$ integer matrix $\Theta \in \mathrm{Mat}_n(\mathbb{Z})$ by the element

$$\begin{aligned} R &:= \frac{1}{N^n} \sum_{\underline{s},\underline{t} \in \mathbb{Z}_N^n} q^{\underline{s}\Theta\underline{t}}\ \underline{s} \otimes \underline{t} \\ &:= \frac{1}{N^n} \sum_{\underline{s},\underline{t} \in \mathbb{Z}_N^n} q^{\sum_{i,j=1}^n \Theta^{ij} s_i t_j}\ \underline{s} \otimes \underline{t} \ \in\ \mathbb{C}[\mathbb{Z}_N^n] \otimes \mathbb{C}[\mathbb{Z}_N^n] \quad . \end{aligned} \tag{1.2.26}$$

Using the standard identity

$$\sum_{\underline{t} \in \mathbb{Z}_N^n} q^{\underline{s}\Theta\underline{t}} = N^n\, \delta_{\underline{s},\underline{0}} = \sum_{\underline{t} \in \mathbb{Z}_N^n} q^{\underline{t}\Theta\underline{s}} \quad , \tag{1.2.27}$$

where $\delta_{\underline{s},\underline{0}}$ denotes the Kronecker delta-symbol and $\underline{0} := (0, \dots, 0) \in \mathbb{Z}_N^n$ is the identity element, one can verify that R is indeed a quasitriangular structure, see Definition 1.2.5. If the matrix Θ is in addition antisymmetric, the quasitriangular structure R on $\mathbb{C}[\mathbb{Z}_N^n]$ is in fact a triangular structure. $\triangledown$

1.2.2 *Hopf algebra comodules*

The constructions built on Hopf algebras above can also be viewed in the dual picture of comodules over H, where the arrows are reversed. In some examples, the dual picture is more natural, e.g. in the context of spaces with group actions, as we shall see below in Example 1.2.11. In fact, concepts and theorems hold for the arrow-reversed situation (replacing modules with comodules), as well as for the left-right reversal (switching left (co)modules for right (co)modules) of the axioms. We will in the following repeat the preceding constructions for the case of left comodules.

Definition 1.2.9. Let H be a Hopf algebra. A *left H-comodule* is a vector space V together with a $\mathbb{K}$-linear map $\delta : V \to H \otimes V$ (*left coaction*) such that

$$(\mathrm{id} \otimes \delta) \circ \delta = (\Delta \otimes \mathrm{id}) \circ \delta \quad , \qquad (\epsilon \otimes \mathrm{id}) \circ \delta = \mathrm{id} \quad , \tag{1.2.28}$$

for all $h, h' \in H$ and $v \in V$. We denote by ${}^H\mathsf{Mod}$ the category of left H-comodules and H-coequivariant maps (*H-comodule morphisms*), i.e. $\mathbb{K}$-linear maps $f : V \to W$ satisfying $(\mathrm{id} \otimes f) \circ \delta = \delta \circ f$.

We will use a Sweedler-like notation for the coaction; for $v \in V$, we write

$$\delta(v) = v_{\underline{-1}} \otimes v_{\underline{0}} \quad \text{(summation understood).} \tag{1.2.29}$$

In this notation, (1.2.28) takes the form

$$v_{\underline{-1}} \otimes (v_{\underline{0}})_{\underline{-1}} \otimes (v_{\underline{0}})_{\underline{0}} = (v_{\underline{-1}})_{\underline{1}} \otimes (v_{\underline{-1}})_{\underline{2}} \otimes v_{\underline{0}} \quad , \qquad \epsilon(v_{\underline{-1}})\, v_{\underline{0}} = v \quad . \tag{1.2.30}$$

As in the case of the category of H-modules ${}_H\mathsf{Mod}$, the category of H-comodules ${}^H\mathsf{Mod}$ is monoidal. The monoidal product is the tensor product $V \otimes W$ of vector spaces $V, W \in {}^H\mathsf{Mod}$ with left tensor product coaction given by

$$\delta(v \otimes w) = v_{\underline{-1}}\, w_{\underline{-1}} \otimes v_{\underline{0}} \otimes w_{\underline{0}} \quad . \tag{1.2.31}$$

The monoidal unit is the one-dimensional vector space $\mathbb{K}$ with trivial left coaction $\delta(c) = \eta(1) \otimes c = \mathbb{1} \otimes c$ for all $c \in \mathbb{K}$. The associator and unitors are the same as for ${}_H\mathsf{Mod}$.

The category ${}^H\mathsf{Mod}$ is closed monoidal, see [CaGu05]. For $V, W \in {}^H\mathsf{Mod}$, consider again the vector space of all linear maps $\mathrm{Hom}_{\mathbb{K}}(V, W)$. However, in contrast to the case of modules, this does *not* define an internal hom from V to W in general. We define the map $\widetilde{\delta} : \mathrm{Hom}_{\mathbb{K}}(V, W) \to \mathrm{Hom}_{\mathbb{K}}(V, H \otimes W)$ by

$$\widetilde{\delta}(f)(v) = S(v_{\underline{-1}})\,(f(v_{\underline{0}}))_{\underline{-1}} \otimes (f(v_{\underline{0}}))_{\underline{0}} \quad . \tag{1.2.32}$$

Using that the counit ϵ is an algebra homomorphism and that S is a coalgebra antihomomorphism (see Proposition 1.2.3 (iii)), together with the coaction properties of the coaction on W and V (1.2.30) respectively, we compute

$$\begin{aligned} ((\epsilon \otimes \mathrm{id}) \circ \widetilde{\delta})(f)(v) &= \epsilon(S(v_{\underline{-1}})\,(f(v_{\underline{0}}))_{\underline{-1}})\,(f(v_{\underline{0}}))_{\underline{0}} \\ &= \epsilon(v_{\underline{-1}})\,\epsilon(f(v_{\underline{0}}))_{\underline{-1}})\,(f(v_{\underline{0}}))_{\underline{0}} \\ &= \epsilon(v_{\underline{-1}})\,f(v_{\underline{0}}) = f(\epsilon(v_{\underline{-1}})\,v_{\underline{0}}) = f(v) \quad . \end{aligned} \tag{1.2.33}$$

The map $\widetilde{\delta}$ is however not a coaction in general because that would require $\widetilde{\delta}(f) \in H \otimes \mathrm{Hom}_{\mathbb{K}}(V, W)$. As $H \otimes \mathrm{Hom}(V, W)_{\mathbb{K}} \subset \mathrm{Hom}_{\mathbb{K}}(V, H \otimes W)$, let

$$\underline{\mathrm{hom}}(V, W) := \{f \in \mathrm{Hom}_{\mathbb{K}}(V, W) : \widetilde{\delta}(f) \in H \otimes \mathrm{Hom}_{\mathbb{K}}(V, W)\} \quad . \tag{1.2.34}$$

Maps $f \in \underline{\mathrm{hom}}(V, W)$ are so called *rational morphisms*. In the case when H is finite dimensional, all morphisms are rational. It is shown in [Ulb90, Lemma 2.2] that $\underline{\mathrm{hom}}(V, W)$ is a H-comodule with coaction $\delta_{\underline{\mathrm{hom}}} : \underline{\mathrm{hom}}(V, W) \to H \otimes \underline{\mathrm{hom}}(V, W)$ given by restricting $\widetilde{\delta}$ to $\underline{\mathrm{hom}}(V, W)$. Finally, by [CaGu05, Proposition 1.2], $\underline{\mathrm{hom}}(V, W)$ is an internal hom from V to W in ${}^{H}\mathsf{Mod}$.

For finite dimensional objects $V \in {}^{H}\mathsf{Mod}$, all linear maps $\phi \in \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}) = V^{\vee}$ are rational because there is a linear isomorphism $H \otimes \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}) \cong \mathrm{Hom}_{\mathbb{K}}(V, H)$. From (1.2.32), we hence obtain a left coaction $\delta^{\vee} : V^{\vee} \to H \otimes V^{\vee}$ defined by $\delta^{\vee}(\phi)(v) = S(v_{\underline{-1}})\,\phi(v_{\underline{0}})$. More generally, again for finite dimensional V, and for any $W \in {}^{H}\mathsf{Mod}$, we have a linear isomorphism $H \otimes \mathrm{Hom}_{\mathbb{K}}(V, W) \cong H \otimes W \otimes V^{\vee} \cong \mathrm{Hom}_{\mathbb{K}}(V, H \otimes W)$. Hence all maps in $\mathrm{Hom}_{\mathbb{K}}(V, W)$ in this case are rational. Similarly to the case of modules, the coaction from (1.2.32) on $\underline{\mathrm{hom}}(V, W) = \mathrm{Hom}_{\mathbb{K}}(V, W)$ can be identified with the coaction on $V^{\vee} \otimes W$ ($\cong W \otimes V^{\vee}$) consisting of the tensor product coaction in conjunction with the coaction on $V^{\vee}$.

The $\delta_{\underline{\mathrm{hom}}}$-coinvariants of $\underline{\mathrm{hom}}(V, W)$, i.e. the linear maps $f : V \to W$ such that

$$\delta_{\underline{\mathrm{hom}}}(f) = \mathbb{1} \otimes f \quad , \tag{1.2.35}$$

are precisely the left H-comodule morphisms. To see that, assume first that $f : V \to W$ is a H-comodule morphism, i.e. $v_{\underline{-1}} \otimes f(v_{\underline{0}}) = (f(v))_{\underline{-1}} \otimes (f(v))_{\underline{0}}$ (see Definition 1.2.9). Then

$$\begin{aligned}
\delta_{\underline{\mathrm{hom}}}(f)(v) &= S(v_{\underline{-1}})\,(f(v_{\underline{0}}))_{\underline{-1}} \otimes (f(v_{\underline{0}}))_{\underline{0}} \\
&= S(v_{\underline{-1}})\,(v_{\underline{0}})_{\underline{-1}} \otimes f((v_{\underline{0}})_{\underline{0}}) \\
&= S((v_{\underline{-1}})_{\underline{1}})\,(v_{\underline{-1}})_{\underline{2}} \otimes f(v_{\underline{0}}) \\
&= \epsilon(v_{\underline{-1}})\,\mathbb{1} \otimes f(v_{\underline{0}}) \\
&= \mathbb{1} \otimes f(v)
\end{aligned} \tag{1.2.36}$$

where in the second equality we used that f is a comodule morphism, in the third the properties of a coaction (1.2.30), in the fourth (1.2.3b) and finally, in the fifth equality, we used that f is a $\mathbb{K}$-linear map in conjunction with (1.2.3a). For the converse, let $f : V \to W$ be $\delta_{\underline{\mathrm{hom}}}$-coinvariant. Then we have

$$\begin{aligned}
v_{\underline{-1}} \otimes f(v_{\underline{0}}) &= (v_{\underline{-1}})\,S((v_{\underline{0}})_{\underline{-1}})\,(f((v_{\underline{0}})_{\underline{0}}))_{\underline{-1}} \otimes (f((v_{\underline{0}})_{\underline{0}}))_{\underline{0}} \\
&= (v_{\underline{-1}})_{\underline{1}}\,S((v_{\underline{-1}})_{\underline{2}})\,(f(v_{\underline{0}}))_{\underline{-1}} \otimes (f(v_{\underline{0}}))_{\underline{0}} \\
&= \epsilon(v_{\underline{-1}})\,(f(v_{\underline{0}}))_{\underline{-1}} \otimes (f(v_{\underline{0}}))_{\underline{0}} \\
&= (f(v))_{\underline{-1}} \otimes (f(v))_{\underline{0}} \quad .
\end{aligned} \tag{1.2.37}$$

In the first row we used coinvariance, the second row (1.2.30) and in the third equality (1.2.3b). In the last step, we used (1.2.3a) together with $\mathbb{K}$-linearity of f. We have thus verified the claim that the $\delta_{\underline{\mathrm{hom}}}$-coinvariants of $\underline{\mathrm{hom}}(V, W)$ are the exactly the morphisms in ${}^{H}\mathsf{Mod}$.

Continuing with the discussion, a *left H-comodule algebra* is a monoid in the monoidal category ${}^{H}\mathsf{Mod}$, i.e. an object $A \in {}^{H}\mathsf{Mod}$ together with left H-comodule morphisms $\mu_A : A \otimes A \to A$ and $\eta_A : \mathbb{K} \to A$ such that the diagrams in (1.2.12) commute. Explicitly, H-coequivariance of the product μ_A and the unit η_A is expressed as

$$\delta(a\,a') = a_{\underline{-1}}\,a'_{\underline{-1}} \otimes a_{\underline{0}}\,a'_{\underline{0}} \tag{1.2.38a}$$

$$\delta(\mathbb{1}_A) = \mathbb{1} \otimes \mathbb{1}_A \tag{1.2.38b}$$

for all $a, a' \in A$.

Following the procedure, a braided or symmetric monoidal structure on ${}^{H}\mathsf{Mod}$ can be obtained by considering additional datum on the Hopf algebra H.

Definition 1.2.10. A *coquasitriangular structure* on H is a convolution-invertible[3] linear map $\mathcal{R} : H \otimes H \longrightarrow \mathbb{K}$, i.e. there exists a linear map $\mathcal{R}^{-1} : H \otimes H \longrightarrow \mathbb{K}$ such that

$$\mathcal{R}(h_{\underline{1}} \otimes h'_{\underline{1}})\, \mathcal{R}^{-1}(h_{\underline{2}} \otimes h'_{\underline{2}}) = \epsilon(h)\, \epsilon(h') \quad , \tag{1.2.39}$$

satisfying

$$\mathcal{R}(h_{\underline{1}} \otimes h'_{\underline{1}})\, h_{\underline{2}}\, h'_{\underline{2}} = h'_{\underline{1}}\, h_{\underline{1}}\, \mathcal{R}(h_{\underline{2}} \otimes h'_{\underline{2}}) \tag{1.2.40a}$$

$$\mathcal{R}(h\, h' \otimes h'') = \mathcal{R}(h \otimes h''_{\underline{1}})\, \mathcal{R}(h' \otimes h''_{\underline{2}}) \tag{1.2.40b}$$

$$\mathcal{R}(h \otimes h'\, h'') = \mathcal{R}(h_{\underline{1}} \otimes h'')\, \mathcal{R}(h_{\underline{2}} \otimes h') \tag{1.2.40c}$$

for all $h, h', h'' \in H$. A *cotriangular structure* is a coquasitriangular structure which additionally satisfies $\mathcal{R} \circ \mathrm{flip} = \mathcal{R}^{-1}$.

Given a coquasitriangular structure $\mathcal{R} : H \otimes H \longrightarrow \mathbb{K}$ on H, a braiding can be defined for ${}^H\mathsf{Mod}$,

$$\tau_{\mathcal{R}} : V \otimes W \to W \otimes V \ , \quad v \otimes w \mapsto \mathcal{R}(w_{\underline{-1}} \otimes v_{\underline{-1}})\, w_{\underline{0}} \otimes v_{\underline{0}} \quad , \tag{1.2.41}$$

for every pair of objects $V, W \in {}^H\mathsf{Mod}$. If $\mathcal{R}$ is also cotriangular, $\tau_{\mathcal{R}}$ is symmetric, making ${}^H\mathsf{Mod}$ a (closed) symmetric monoidal category. Finally, using the symmetric braiding, one says that a H-comodule algebra $A \in {}^H\mathsf{Mod}$ is (braided) commutative if the diagram (1.2.18) (with τ_R replaced with $\tau_{\mathcal{R}}$) commutes.

Example 1.2.11. The following class of examples are given by the so called *affine group schemes*, which can be identified with the commutative (but not necessarily cocommutative) Hopf algebras. Let us give a rough explanation. Consider the category $\mathsf{Aff} := \mathsf{CAlg}_{\mathbb{K}}^{\mathrm{op}}$ of affine schemes, which per definition is the opposite of the category of commutative unital associative $\mathbb{K}$-algebras $\mathsf{CAlg}_{\mathbb{K}}$. An affine group scheme is a group object in Aff, i.e. an object $G \in \mathsf{Aff}$ together with maps $m : G \times G \to G$, $e : \mathrm{pt} \to G$ and $(\,\cdot\,)^{-1} : G \to G$, with $\mathrm{pt} \in \mathsf{Aff}$ the terminal object. These maps are required to satisfy the commutative diagrams encoding the group axioms. By reversing the arrows of these maps and their associated commutative diagrams, the function algebra $H := \mathcal{O}(G) \in \mathsf{CAlg}_{\mathbb{K}}$ of G will thus be equipped with maps $\Delta := \mathcal{O}(m) : H \to H \otimes H$ (recall that the coproduct in $\mathsf{CAlg}_{\mathbb{K}}$ is the tensor product), $\epsilon := \mathcal{O}(e) : H \to \mathbb{K}$ ($\mathbb{K}$

3 One can endow $\mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$ with an algebra structure using the coproduct of H. The product is called the *convolution product* and is given by $f * g := (f \otimes g) \circ \Delta \in \mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$ for all $f, g \in \mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$. The left hand side of (1.2.39) is nothing but the convolution product $\mathcal{R} * \mathcal{R}^{-1}$ in $\mathrm{Hom}(H \otimes H, \mathbb{K})$.

is the initial object in $\mathsf{CAlg}_{\mathbb{K}}$) and $S := \mathcal{O}((\,\cdot\,)^{-1}) : H \to H$ such that the conditions for H being a (commutative) Hopf algebra are satisfied.[4] In a similar fashion, Hopf algebras $H \in \mathsf{CAlg}_{\mathbb{K}}$ correspond to affine group schemes $G := \mathrm{Spec}(H) \in \mathsf{Aff}$, hence the identification. The Examples 1.2.12 and 1.2.13 below are examples of affine group schemes.

In the context of affine group schemes, the notion of Hopf algebra coactions emerge naturally. A (left) group action $l : G \times X \to X$ of an affine group scheme $G = \mathrm{Spec}(H) \in \mathsf{Aff}$ on some space $X = \mathrm{Spec}(A) \in \mathsf{Aff}$ in Aff corresponds to a (left) Hopf algebra coaction $\delta := \mathcal{O}(l) : A \to H \otimes A$ in $\mathsf{CAlg}_{\mathbb{K}}$ by reversing the arrows of the group action l and the relevant diagrams. Indeed, the commutative diagrams for the group action l

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mathrm{id}\times l"] \arrow[d, "m\times\mathrm{id}"'] & G \times X \arrow[d, "l"] \\ G \times X \arrow[r, "l"'] & X \end{tikzcd} \tag{1.2.42a}$$

$$\begin{tikzcd} \mathrm{pt} \times X \arrow[r, "e\times\mathrm{id}"] \arrow[dr, "p_2"'] & G \times X \arrow[d, "l"] \\ & X \end{tikzcd} \tag{1.2.42b}$$

where p_2 is the canonical projection, correspond to commutative diagrams for the coaction δ in $\mathsf{CAlg}_{\mathbb{K}}$

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\delta"'] & H \otimes A \arrow[d, "\mathrm{id}\otimes\delta"] \\ H \otimes A \arrow[r, "\Delta\otimes\mathrm{id}"'] & H \otimes H \otimes A \end{tikzcd} \tag{1.2.43a}$$

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[dr, "\cong"'] & H \otimes A \arrow[d, "\epsilon\otimes\mathrm{id}"] \\ & A \end{tikzcd} \tag{1.2.43b}$$

which translate to the conditions (1.2.28) in Definition 1.2.9, making A a commutative H-comodule algebra. $\triangledown$

The following two examples are heavily inspired by [BSS17] (in particular, Example 1.2.13 is covered there). In these examples, we set the underlying field $\mathbb{K}$ to be the field of complex numbers $\mathbb{C}$.

4 In the affine world, $\mathrm{Spec} : \mathsf{CAlg}_{\mathbb{K}}^{\mathrm{op}} \to \mathsf{Aff}$ and $\mathcal{O} : \mathsf{Aff}^{\mathrm{op}} \to \mathsf{CAlg}_{\mathbb{K}}$, are the identity functors. In other words, they are mere labels in the present setting. They are kept here to reminisce about the geometric nature of the involved concepts.

Example 1.2.12. Consider the finite subgroup of the n-torus $\mathbb{T}^n = U(1)^n$

$$\mathbb{T}^n_N = \left\{ \left(\exp\left(\frac{2\,\pi\,\mathrm{i}\,k_1}{N}\right), \dots, \exp\left(\frac{2\,\pi\,\mathrm{i}\,k_n}{N}\right) \right) \in U(1)^n : k_i \in \{0, \dots, N-1\} \right\} \tag{1.2.44}$$

consisting of the N-th roots of unity. Note that the groups $\mathbb{Z}^n_N \cong \mathbb{T}^n_N$ are isomorphic. Hence, the (commutative) Hopf algebra $\mathbb{C}[\mathbb{Z}^n_N]$ in Example 1.2.8 can be recognised as the Hopf algebra of functions $\mathcal{O}(\mathbb{T}^n_N)$ on $\mathbb{T}^n_N$ by identifying $\underline{k}$ with the exponential function $t_{\underline{k}}(\underline{\xi}) = \exp\left(\frac{2\,\pi\,\mathrm{i}}{N}(k_1\,\xi_1 + \cdots + k_n\,\xi_n)\right)$, where $\underline{\xi}$ takes value in $\mathbb{Z}^n_N \cong \mathbb{T}^n_N$. The Hopf algebra structure is given by

$$t_{\underline{k}}\, t_{\underline{k}'} = t_{\underline{k}+\underline{k}'} \quad , \quad \mathbb{1} = t_{\underline{0}} \quad , \quad \Delta(t_{\underline{k}}) = t_{\underline{k}} \otimes t_{\underline{k}} \quad , \quad \epsilon(t_{\underline{k}}) = 1 \quad , \quad S(t_{\underline{k}}) = t_{-\underline{k}} \quad . \tag{1.2.45}$$

A coquasitriangular structure on $\mathcal{O}(\mathbb{T}^n_N)$ is defined on generators by

$$\mathcal{R}(t_{\underline{k}} \otimes t_{\underline{k}'}) = \exp\left(\frac{2\,\pi\,\mathrm{i}}{N}\,\underline{k}\,\Theta\,\underline{k}'\right) = \exp\left(\frac{2\,\pi\,\mathrm{i}}{N}\sum_{l,m}^{n} k_l\,\Theta^{l\,m}\,k'_m\right) \tag{1.2.46}$$

where $\Theta \in \mathrm{Mat}_n(\mathbb{Z})$ is an $n \times n$ integer matrix. If Θ is antisymmetric, $\mathcal{R}$ defines a cotriangular structure. Note that the same could have been defined in the $\mathbb{C}[\mathbb{Z}^n_N]$ picture as well. $\triangledown$

Example 1.2.13. We will now consider the full *algebraic* n-torus $\mathbb{T}^n$, which means the Hopf algebra $\mathcal{O}(\mathbb{T}^n)$ of functions on the n-torus is given by

$$\mathcal{O}(\mathbb{T}^n) = \mathrm{span}_{\mathbb{C}}\{t_{\underline{k}} \,:\, \underline{k} \in \mathbb{Z}^n\} \tag{1.2.47}$$

where $t_{\underline{k}}(\phi_1, \dots, \phi_n) = \exp\left(\mathrm{i}\,(k_1\,\phi_1 + \cdots k_n\,\phi_n)\right)$, with $0 \leq \phi_i < 2\,\pi$, is the exponential function with momentum $\underline{k}$. The (commutative) Hopf algebra structure is given by

$$t_{\underline{k}}\, t_{\underline{k}'} = t_{\underline{k}+\underline{k}'} \quad , \quad \mathbb{1} = t_{\underline{0}} \quad , \quad \Delta(t_{\underline{k}}) = t_{\underline{k}} \otimes t_{\underline{k}} \quad , \quad \epsilon(t_{\underline{k}}) = 1 \quad , \quad S(t_{\underline{k}}) = t_{-\underline{k}} \quad . \tag{1.2.48}$$

As we can see, this case is very similar to the previous case and we may identify $\mathcal{O}(\mathbb{T}^n)$ with the group algebra $\mathbb{C}[\mathbb{Z}^n]$ defined similarly to $\mathbb{C}[\mathbb{Z}^n_N]$ in Example 1.2.8. (In general however, the identification of the function Hopf algebra of a non finite group with a group Hopf algebra is generally not possible.)

The Hopf algebra $\mathcal{O}(\mathbb{T}^n)$ can be endowed with a coquasitriangular structure given by

$$\mathcal{R}(t_{\underline{k}} \otimes t_{\underline{k}'}) = \exp(\mathrm{i}\, \underline{k}\, \Theta\, \underline{k}') = \exp\left(\mathrm{i} \sum_{l,m}^{n} k_l\, \Theta^{l\,m}\, k'_m\right) \tag{1.2.49}$$

with $\Theta \in \mathrm{Mat}_n(\mathbb{R})$ a real $n \times n$ matrix. As before, by choosing Θ to be antisymmetric, the coquasitriangular structure is a triangular structure.

We will now describe an example of a coaction of $\mathcal{O}(\mathbb{T}^2)$. Let $A_{\mathbb{R}^4}$ be the commutative algebra of functions on $\mathbb{R}^4$. In complex coordinates $\{z^i\}_{i=1}^4$, with $z^3 = \overline{z^1}$ and $z^4 = \overline{z^2}$, the algebra of functions $A_{\mathbb{R}^4} = \mathrm{Pol}_{\mathbb{C}}(z^1, z^2, z^3, z^4)$ is given by the (commutative) complex polynomials in $\{z^i\}_{i=1}^4$. Denoting the Hopf algebra of functions on $\mathbb{T}^2$ by $H := \mathcal{O}(\mathbb{T}^2)$ and given any $k_1, k_2 \in \mathbb{Z}$, we define the left H-coaction $\delta : A_{\mathbb{R}^4} \to H \otimes A_{\mathbb{R}^4}$ by

$$\begin{aligned} \delta(z^1) = t_{(k_1,0)} \otimes z^1 \ , \ \ \delta(z^2) = t_{(0,k_2)} \otimes z^2 \\ \delta(z^3) = t_{(-k_1,0)} \otimes z^3 \ , \ \ \delta(z^4) = t_{(0,-k_2)} \otimes z^4 \end{aligned} \tag{1.2.50}$$

such that the conditions in (1.2.38) are satisfied. This makes $A_{\mathbb{R}^4}$ an H-comodule algebra. In the real coordinates $x^1 = \frac{1}{2}\,(z^1 + z^3)$, $x^2 = \frac{1}{2\,\mathrm{i}}\,(z^1 - z^3)$, $x^3 = \frac{1}{2}\,(z^2 + z^4)$, $x^4 = \frac{1}{2\,\mathrm{i}}\,(z^2 - z^4)$, this would correspond to (k_1-fold covers of) rotations in the (x^1, x^2)-plane and (k_2-fold covers of) rotations in the (x^3, x^4)-plane. $\triangledown$

2

DIRAC OPERATORS ON NONCOMMUTATIVE HYPERSURFACES

This chapter is based on Section 3 and 4 of the paper [NS20]. Here, we will develop an appropriate notion of hypersurface, on which we will describe a framework for inducing the differential, Riemannian and spinorial structures from noncommutative embedding spaces (outlined in Section 1.1). The framework is a generalisation of well-known results from classical differential geometry, see e.g. [Bur93, Tra95, Bär96, HMZ02]. From the induced geometric structures, one obtains an explicit expression for the induced hypersurface Dirac operator. This process is described in Section 2.1. In particular, the framework can be used to construct examples of curved noncommutative hypersurfaces and associated Dirac operators from simple flat noncommutative embedding spaces. In Section 2.2., we will investigate the particular example of the sequence of noncommutative hypersurface embeddings $\mathbb{T}^2_\theta \hookrightarrow \mathbb{S}^3_\theta \hookrightarrow \mathbb{R}^4_\theta$ studied in [AN19] (which is another work on noncommutative embeddings different from our approach, but with conceptual similarities).

2.1 INDUCED GEOMETRIC STRUCTURES ON NONCOMMUTATIVE HYPERSURFACES

For the remainder of this section, let A be an algebra together with a differential calculus (Ω^1_A, d) on A (see Definition 1.1.1), a Riemannian structure $(g, (\nabla, \sigma))$ on (Ω^1_A, d) (see Definition 1.1.7) and a spinorial structure $(\mathcal{E}, \nabla^{\mathrm{sp}}, \gamma)$ on $(g, (\nabla, \sigma))$ (see Definition 1.1.9). The algebra A is interpreted as the algebra of functions on a noncommutative embedding space, which is equipped with a differential, Riemannian and spinorial structure. The purpose of this section is to provide a framework for induction of these geometric structures to a suitable class of hypersurfaces.

2.1.1 *Noncommutative hypersurfaces*

In this subsection, we describe the particular class of hypersurfaces for which we build the framework. For a 2-sided ideal $I \subset A$, we consider the quotient algebra

$$B := A/I \tag{2.1.1}$$

together with its quotient algebra map $q : A \to B$. We would like to obtain a differential calculus on B using the data from (Ω^1_A, d). Associated with the quotient map is a change of base functor $q_! : {}_A\mathsf{Mod}_A \to {}_B\mathsf{Mod}_B$ for bimodules, which is given by $q_!(V) = B \otimes_A V \otimes_A B \in {}_B\mathsf{Mod}_B$, for all $V \in {}_A\mathsf{Mod}_A$. Observe that since $B = A/I$ with quotient map $q : A \to B$, the map

$$q_!(V) \overset{\cong}{\longrightarrow} \frac{V}{IV \cup VI}\ ,\ \ [a] \otimes_A v \otimes_A [a'] \longmapsto [a\, v\, a'] \tag{2.1.2}$$

is a natural B-module isomorphism, where $IV := \{a\, v \ : \ a \in I \text{ and } v \in V\} \subseteq V$ and $VI := \{v\, a \ : \ a \in I \text{ and } v \in V\} \subseteq V$. Note that it is not enough to only apply the change of base functor to the bimodule of 1-forms $\Omega^1_A \in {}_A\mathsf{Mod}_A$; in general, the differential $\mathrm{d} : A \to q_!(\Omega^1_A)$ does not descend to the quotient $B = A/I$. Instead, one considers the quotient (see e.g. [BM20, Exercise E1.4])

$$\Omega^1_B := \frac{q_!(\Omega^1_A)}{B[\mathrm{d}I]B} \in {}_B\mathsf{Mod}_B \quad , \tag{2.1.3}$$

where $B[\mathrm{d}I]B := \{\sum_i b_i\, [\mathrm{d}a_i]\, b'_i \ : \ b_i, b'_i \in B \text{ and } a_i \in I\}$ is the B-subbimodule generated by $[\mathrm{d}I] \subseteq q_!(\Omega^1_A)$. In this case, the differential $\mathrm{d} : A \to \Omega^1_A$ descends to a linear map

$$\mathrm{d}_B \, : \, B \longrightarrow \Omega^1_B\ ,\ \ [a] \longmapsto [\mathrm{d}a] \quad . \tag{2.1.4}$$

Proposition 2.1.1. $(\Omega^1_B, \mathrm{d}_B)$ *is a differential calculus on the quotient algebra* $B = A/I$.

Proof. The necessary properties of Definition 1.1.1 are inherited from the differential calculus (Ω^1_A, d) on A, see e.g. [BM20, Exercise E1.4]. □

This is the first step in the definition of our class of hypersurfaces. However, we still need to encode a few properties before $q : A \to B$ can be properly interpreted as (the dual of) an embedding of a noncommutative hypersurface B into the noncommutative embedding space A. In particular, we need to make precise the statement that B should be of "codimension 1" and the existence of a "normalised normal vector field" for B. The noncommutative generalisations of these concepts[1] come about by observing that the canonical quotient map $q_!(\Omega^1_A) \twoheadrightarrow \Omega^1_B$ (see (2.1.3)) gives rise to the short exact sequence of B-bimodules

$$0 \longrightarrow N^1_B := B[\mathrm{d}I]B \longrightarrow q_!(\Omega^1_A) \longrightarrow \Omega^1_B \longrightarrow 0 \quad , \tag{2.1.5}$$

where $N^1_B \in {}_B\mathsf{Mod}_B$ is a noncommutative analogue of the conormal bundle.

Definition 2.1.2. We say that $B = A/I$ is a (metrically co-orientable) *noncommutative hypersurface* if the B-bimodule $N^1_B \in {}_B\mathsf{Mod}_B$ admits a 1-dimensional basis $[\nu] \in N^1_B$ with $\nu \in \Omega^1_A$ a central 1-form, i.e. $a\,\nu = \nu\,a$ for all $a \in A$, that satisfies the normalization condition

$$\big[g^{-1}(\nu \otimes_A \nu)\big] \;=\; 1 \;\in\; B \quad . \tag{2.1.6}$$

Remark 2.1.3. The "codimension 1" property in Definition 2.1.2 is given by the statement that N^1_B has rank 1. The existence of a "normalised normal vector field" is encoded, in our dual language of forms, by the normalised 1-form $\nu \in \Omega^1_A$. The adjective *metrically co-orientable* is used to emphasise the existence of such a form. However, for the rest of the chapter, we will simplify the language by using the term noncommutative hypersurface to refer to metrically co-orientable noncommutative hypersurfaces. △

Example 2.1.4. The *noncommutative level set hypersurfaces* constitute a class of examples of noncommutative hypersurfaces in the sense of Definition 2.1.2. These are determined by 2-sided ideals $I = (f) \subset A$ generated by a central element $f \in \mathcal{Z}(A) \subseteq A$ such that $\nu := \mathrm{d}f \in \Omega^1_A$ is central and satisfies the normalization condition (2.1.6). In this case, $[\nu] = [\mathrm{d}f]$ defines a basis of $N^1_B = B[\mathrm{d}I]B = B[\mathrm{d}f]B = [\mathrm{d}f]B = B[\mathrm{d}f]$, where in the last two steps we used that $\mathrm{d}f$ is central. All examples in Section 2.2 are of this type. ▽

In the following, let $B = A/I$ be a noncommutative hypersurface in the sense of Definition 2.1.2. It will be useful to introduce a projector that produces a splitting of the sequence (2.1.5) which determines a B-bimodule isomorphism between $\Omega^1_B \in {}_B\mathsf{Mod}_B$ and a certain B-subbimodule of $q_!(\Omega^1_A) \in {}_B\mathsf{Mod}_B$. To construct a candidate

1 We would like to thank Branimir Ćaćić for suggesting Definition 2.1.2 to us. This allowed us to generalise our results for noncommutative level set hypersurfaces (cf. Example 2.1.4) from the first version of [NS20].

for such a projector, we first use the inverse metric $g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A$ and the normalised 1-form $\nu \in \Omega^1_A$ to define the A-bimodule endomorphism

$$\Pi : \Omega^1_A \longrightarrow \Omega^1_A \ , \ \ \omega \longmapsto \omega - g^{-1}(\omega \otimes_A \nu)\, \nu \quad . \tag{2.1.7a}$$

The A-linearity follows from the centrality of ν. Then, by using the change of base functor on Π, we obtain a B-bimodule endomorphism

$$\Pi : q_!(\Omega^1_A) \longrightarrow q_!(\Omega^1_A) \ , \ \ [\omega] \longmapsto \Pi\big([\omega]\big) := \big[\Pi(\omega)\big] \quad . \tag{2.1.7b}$$

Proposition 2.1.5. *The B-bimodule endomorphism Π from (2.1.7b) satisfies the following properties:*

(i) $\Pi\big([\nu]\big) = 0$.

(ii) $\Pi^2 = \Pi$, *i.e.* Π *is a projector.*

(iii) The induced B-bimodule map $\Pi : \Omega^1_B \to q_!(\Omega^1_A)$ on Ω^1_B (cf. (2.1.3)*) is a section of the quotient B-bimodule map $k : q_!(\Omega^1_A) \twoheadrightarrow \Omega^1_B$, i.e.* $k \circ \Pi = \mathrm{id}$. *In particular, it defines an isomorphism $\Omega^1_B \cong \Pi q_!(\Omega^1_A)$.*

Proof. Item (i) follows directly from the normalization condition (2.1.6) and item (ii) is a direct consequence of (i). To prove item (iii), note that the induced map $\Pi : \Omega^1_B \to q_!(\Omega^1_A)$ is well-defined because of (i) and the fact that $[\nu]$ is by hypothesis a basis for N^1_B. It is a section of the quotient map because the latter maps $[\nu]$ to 0. This in particular implies that the induced map $\Pi : \Omega^1_B \to q_!(\Omega^1_A)$ is injective, hence it defines an isomorphism onto its image $\Pi q_!(\Omega^1_A)$. □

2.1.2 *Induced Riemannian structure*

We have so far introduced the notion of noncommutative hypersurface $B = A/I$ together with a differential calculus (Ω^1_B, d). The next step is to induce a Riemannian structure $(g_B, (\nabla_B, \sigma_B))$ on (Ω^1_B, d). Applying the change of base functor to the metric $g : A \to \Omega^1_A \otimes_A \Omega^1_A$ on Ω^1_A and the inverse metric $g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A$, we obtain the B-bimodule maps

$$g : B \longrightarrow q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \ , \ \ [a] \longmapsto \big[g(a)\big] \tag{2.1.8a}$$

and

$$g^{-1} \,:\, q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \longrightarrow B \;,\;\; [\omega] \otimes_B [\zeta] \longmapsto \big[g^{-1}(\omega \otimes_A \zeta)\big] \quad . \tag{2.1.8b}$$

This, together with the quotient map $q_!(\Omega^1_A) \twoheadrightarrow \Omega^1_B$ and its section $\Pi : \Omega^1_B \to q_!(\Omega^1_A)$ from Proposition 2.1.5 (see also (2.1.7)), we define the composite B-bimodule maps

$$g_B \;:\; B \xrightarrow{\;g\;} q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \longrightarrow\!\!\!\!\!\rightarrow \Omega^1_B \otimes_B \Omega^1_B \tag{2.1.9a}$$

and

$$g_B^{-1} \;:\; \Omega^1_B \otimes_B \Omega^1_B \xrightarrow{\Pi \otimes_B \Pi} q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \xrightarrow{\;g^{-1}\;} B \quad . \tag{2.1.9b}$$

However, to guarantee that the maps in (2.1.9) define a metric and an inverse metric, we will make the following assumption:

Assumption 2.1.6. The A-bimodule isomorphism $\sigma : \Omega^1_A \otimes_A \Omega^1_A \to \Omega^1_A \otimes_A \Omega^1_A$ associated with the bimodule connection (∇, σ) on Ω^1_A satisfies

$$\sigma(\omega \otimes_A \nu) = \nu \otimes_A \omega \quad , \quad \sigma(\nu \otimes_A \omega) = \omega \otimes_A \nu \quad , \tag{2.1.10}$$

for all $\omega \in \Omega^1_A$.

Lemma 2.1.7. *Assumption 2.1.6 implies the following properties:*

(i) $g^{-1}(\omega \otimes_A \nu) = g^{-1}(\nu \otimes_A \omega)$*, for all* $\omega \in \Omega^1_A$.

(ii) $\big[g^{-1}\big(\Pi(\omega) \otimes_A \nu\big)\big] = 0$ *and* $\big[g^{-1}\big(\nu \otimes_A \Pi(\omega)\big)\big] = 0$ *in* $B = A/I$*, for all* $\omega \in \Omega^1_A$*. This implies that*

$$\big[g^{-1}\big(\Pi(\omega) \otimes_A \Pi(\zeta)\big)\big] = \big[g^{-1}\big(\omega \otimes_A \Pi(\zeta)\big)\big] = \big[g^{-1}\big(\Pi(\omega) \otimes_A \zeta\big)\big] \quad , \tag{2.1.11}$$

for all $\omega, \zeta \in \Omega^1_A$.

(iii) $\big[(\mathrm{id} \otimes_A g^{-1})\big(\nabla(\nu) \otimes_A \nu\big)\big] = 0$.

(iv) The two B-bimodule maps in (2.1.9) *define a metric* g_B *and its inverse* g_B^{-1} *on* Ω^1_B.

Proof. Item (i) is a direct consequence of the symmetry property of g^{-1} (cf. Definition 1.1.7) and Assumption 2.1.6. The first equality of item (ii) follows from a short calculation

$$\begin{aligned}\big[g^{-1}\big(\Pi(\omega) \otimes_A \nu\big)\big] &= \big[g^{-1}\big(\big(\omega - g^{-1}(\omega \otimes_A \nu)\,\nu\big) \otimes_A \nu\big)\big] \\ &= \big[g^{-1}(\omega \otimes_A \nu) - g^{-1}(\omega \otimes_A \nu)\, g^{-1}(\nu \otimes_A \nu)\big] = 0 \quad , \end{aligned} \tag{2.1.12}$$

where we used the normalization condition (2.1.6) for ν. The second equality in item (ii) follows from this and (i).

Item (iii) follows from the calculation

$$\begin{aligned}\big[(\mathrm{id}\otimes_A g^{-1})\big(\nabla(\nu)\otimes_A \nu\big)\big] &= \big[\mathrm{d}\big(g^{-1}(\nu\otimes_A \nu)\big) - (\mathrm{id}\otimes_A g^{-1})(\sigma\otimes_A \mathrm{id})\big(\nu\otimes_A \nabla(\nu)\big)\big] \\ &= -\big[(\mathrm{id}\otimes_A g^{-1})\big(\nabla(\nu)\otimes_A \nu\big)\big] \quad , \end{aligned} \tag{2.1.13}$$

where in the first step we used metric compatibility (1.1.12) and in the second step we used the normalization condition (2.1.6) and item (i).

To prove item (iv), we use the same notations as in Remark 1.1.5 to write $g_B(1) = [g(1)] = [\sum_\alpha g^\alpha \otimes_A g_\alpha] = \sum_\alpha [g^\alpha]\otimes_B [g_\alpha]$ and $g_B^{-1}([\omega]\otimes_B [\zeta]) = \big[g^{-1}\big(\Pi(\omega)\otimes_A \Pi(\zeta)\big)\big]$. We then compute

$$\sum_\alpha g_B^{-1}\big([\omega]\otimes_B [g^\alpha]\big)\,[g_\alpha] = \Big[\sum_\alpha g^{-1}\big(\Pi(\omega)\otimes_A g^\alpha\big)\,g_\alpha\Big] = \big[\Pi(\omega)\big] = [\omega] \quad , \tag{2.1.14}$$

where in the first step we used (ii). The second step follows from g^{-1} being the inverse metric of g and the last step uses that Π is a section of the quotient map (see Proposition 2.1.5). The second property $\sum_\alpha [g^\alpha]\, g_B^{-1}\big([g_\alpha]\otimes_B [\omega]\big) = [\omega]$ follows from a similar calculation. □

Remark 2.1.8. Observe that Lemma 2.1.7 (ii) can be interpreted as self-adjointness of the projector $\Pi : q_!(\Omega^1_A) \to q_!(\Omega^1_A)$ with respect to the inverse metric $g^{-1} : q_!(\Omega^1_A)\otimes_B q_!(\Omega^1_A) \to B$. △

The remaining part of the Riemannian structure on (Ω^1_B, d) is a metric compatible bimodule connection, induced from (∇,σ) on Ω^1_A. We first make the observation that the connection $\nabla : \Omega^1_A \to \Omega^1_A\otimes_A \Omega^1_A$ descends to a connection $\nabla : q_!(\Omega^1_A) \to \Omega^1_B \otimes_B q_!(\Omega^1_A)$ on $q_!(\Omega^1_A) \in {}_B\mathsf{Mod}_B$, which can be seen from how Ω^1_B is defined (2.1.3). This map is indeed well defined: from the left Leibniz rule, we have $[\nabla(a\,\omega)] = [a\,\nabla(\omega) + \mathrm{d}a\otimes_A \omega] = 0$, for all $a\in I$. As in the case for the metric, we use the quotient map $q_!(\Omega^1_A) \twoheadrightarrow \Omega^1_B$ and its section $\Pi : \Omega^1_B \to q_!(\Omega^1_A)$ from Proposition 2.1.5 (see also (2.1.7)) to define the composite linear map

$$\nabla_B\,:\,\Omega^1_B \xrightarrow{\;\Pi\;} q_!(\Omega^1_A) \xrightarrow{\;\nabla\;} \Omega^1_B\otimes_B q_!(\Omega^1_A) \longrightarrow\!\!\!\!\!\rightarrow \Omega^1_B\otimes_B \Omega^1_B \quad . \tag{2.1.15}$$

The A-bimodule isomorphism $\sigma : \Omega^1_A\otimes_A \Omega^1_A \to \Omega^1_A\otimes_A\Omega^1_A$ associated with the bimodule connection (∇,σ) on Ω^1_A descends to the B-bimodule isomorphism

$$\sigma_B\,:\,\Omega^1_B\otimes_B\Omega^1_B \longrightarrow \Omega^1_B\otimes_B\Omega^1_B\,,\ \ [\omega]\otimes_B[\zeta] \longmapsto \big[\sigma(\omega\otimes_A\zeta)\big] \tag{2.1.16}$$

as an immediate consequence of Assumption 2.1.6.

Lemma 2.1.9. *The pair* (∇_B, σ_B) *defined by* (2.1.15) *and* (2.1.16) *is a bimodule connection on* Ω^1_B*. It reads explicitly as*

$$\nabla_B\big([\omega]\big) \;=\; \big[\nabla(\omega) - g^{-1}(\omega \otimes_A \nu)\, \nabla(\nu)\big] \quad , \tag{2.1.17}$$

for all $[\omega] \in \Omega^1_B$.

Proof. The explicit expression (2.1.17) is obtained by a short calculation

$$\begin{aligned} \nabla_B\big([\omega]\big) &= \big[\nabla\big(\omega - g^{-1}(\omega \otimes_A \nu)\, \nu\big)\big] \\ &= \big[\nabla(\omega) - g^{-1}(\omega \otimes_A \nu)\, \nabla(\nu) - \mathrm{d}\big(g^{-1}(\omega \otimes_A \nu)\big) \otimes_A \nu\big] \\ &= \big[\nabla(\omega) - g^{-1}(\omega \otimes_A \nu)\, \nabla(\nu)\big] \quad , \end{aligned} \tag{2.1.18}$$

where in the second step we used the left Leibniz rule for ∇ and in the third step that ν is identified with 0 in Ω^1_B (cf. (2.1.3)). The left Leibniz rule is a direct consequence of this expression and the right Leibniz rule follows from the fact that $\nabla(\nu) \in \Omega^1_A \otimes_A \Omega^1_A$ is a central element. The latter statement is proven as follows

$$a\, \nabla(\nu) = \nabla(a\, \nu) - \mathrm{d}a \otimes_A \nu = \nabla(\nu\, a) - \sigma(\nu \otimes_A \mathrm{d}a) = \nabla(\nu)\, a \quad , \tag{2.1.19}$$

where we used the left and right Leibniz rules for (∇, σ), centrality of ν and Assumption 2.1.6. □

Remark 2.1.10. The formula (2.1.17) is a noncommutative analogue of the usual Gauss formula for connections on Riemannian submanifolds, see e.g. [KN96, Chapter VII.3]. △

In order to ensure that the metric and bimodule connection we introduced in this section really defines a Riemannian structure on (Ω^1_B, d), we need to make an additional assumption:

Assumption 2.1.11. The diagrams

$$\begin{tikzcd} q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \arrow[r, "\Pi \otimes_B \mathrm{id}"] \arrow[d, "\sigma"'] & q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \arrow[d, "\sigma"] \\ q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \arrow[r, "\mathrm{id} \otimes_B \Pi"'] & q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \end{tikzcd} \tag{2.1.20a}$$

and

$$\begin{tikzcd} q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \arrow[r, "\mathrm{id}\otimes_B \Pi"] \arrow[d, "\sigma"'] & q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \arrow[d, "\sigma"] \\ q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \arrow[r, "\Pi\otimes_B \mathrm{id}"'] & q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \end{tikzcd} \tag{2.1.20b}$$

commute.

Proposition 2.1.12. *The pair* $(g_B, (\nabla_B, \sigma_B))$ *defined in* (2.1.9), (2.1.15) *and* (2.1.16) *is a Riemannian structure on* (Ω^1_B, d).

Proof. It remains to prove the symmetry and metric compatibility properties from Definition 1.1.7. The symmetry property (1.1.11) for $(g_B, (\nabla_B, \sigma_B))$ follows immediately from Assumption 2.1.11 and symmetry of g^{-1}. To verify metric compatibility (1.1.12) for $(g_B, (\nabla_B, \sigma_B))$, we compute by using metric compatibility of the original Riemannian structure $(g, (\nabla, \sigma))$

$$\mathrm{d}_B\big(g_B^{-1}([\omega] \otimes_B [\zeta])\big) = \Big[(\mathrm{id} \otimes_A g^{-1})\Big(\nabla\Pi(\omega) \otimes_A \Pi(\zeta) + \sigma_{12}\big(\Pi(\omega) \otimes_A \nabla\Pi(\zeta)\big)\Big)\Big] \quad , \tag{2.1.21}$$

where $\sigma_{12} := \sigma \otimes_A \mathrm{id}$. Using Lemma 2.1.7 (ii), we can in the first term replace $\nabla\Pi(\omega)$ with $(\mathrm{id} \otimes_A \Pi)\nabla\Pi(\omega)$. Using also Assumption 2.1.11, we can in the second term replace $\sigma_{12}\big(\Pi(\omega) \otimes_A \nabla\Pi(\zeta)\big)$ with $(\mathrm{id} \otimes_A \Pi \otimes_A \mathrm{id})\sigma_{12}\big(\omega \otimes_A \nabla\Pi(\zeta)\big)$ and hence, via Lemma 2.1.7 (ii), with $(\mathrm{id} \otimes_A \Pi \otimes_A \Pi)\sigma_{12}\big(\omega \otimes_A \nabla\Pi(\zeta)\big)$. The resulting expression proves metric compatibility for $(g_B, (\nabla_B, \sigma_B))$. □

2.1.3 *Induced spinorial structure*

Having introduced a Riemannian structure on (Ω^1_B, d) (Section 2.1.2), the next task is to induce a spinorial structure $(\mathcal{E}_B, \nabla^{\mathrm{sp}}_B, \gamma_B)$ on the Riemannian structure $(g_B, (\nabla_B, \sigma_B))$. In this section, we use well-known results on spinorial structures in classical, commutative, differential geometry (see e.g. [Bur93, Tra95, Bär96] and also [HMZ02] for a good review) to motivate our definitions and constructions. We begin by using the change of base functor (for left modules) to define the left B-module of spinors

$$\mathcal{E}_B := q_!(\mathcal{E}) \cong \frac{\mathcal{E}}{I\mathcal{E}} \in {}_B\mathsf{Mod} \quad . \tag{2.1.22}$$

For the Clifford multiplication $\gamma_B : \Omega^1_B \otimes_B \mathcal{E}_B \to \mathcal{E}_B$, we turn to the classical case from [HMZ02, Eqn. (3.4)] for inspiration and define

$$\gamma_B \,:\, \Omega^1_B \otimes_B \mathcal{E}_B \;\longrightarrow\; \mathcal{E}_B \;,\quad [\omega] \otimes_B [s] \;\longmapsto\; \big[\gamma_{[2]}\big(\Pi(\omega) \otimes_A \nu \otimes_A s\big)\big] \quad , \tag{2.1.23}$$

where $\gamma_{[2]}$ was defined in (1.1.13). Since the normalised 1-form $\nu \in \Omega^1_A$ is central by Definition 2.1.2, this map is well-defined.

The final structure we need for our framework is the spin connection on $\mathcal{E}_B \in {}_B\mathsf{Mod}$. Due to the relations in (2.1.3) and (2.1.22), note that the spin connection $\nabla^{\mathrm{sp}} : \mathcal{E} \to \Omega^1_A \otimes_A \mathcal{E}$ on $\mathcal{E} \in {}_A\mathsf{Mod}$ descends to a connection $\nabla^{\mathrm{sp}} : \mathcal{E}_B \to \Omega^1_B \otimes_B \mathcal{E}_B$ on $\mathcal{E}_B \in {}_B\mathsf{Mod}$ because $[\nabla^{\mathrm{sp}}(a\, s) = a\, \nabla^{\mathrm{sp}}(s) + \mathrm{d}a \otimes_A s] = 0$, for all $a \in I$. However, this is not the desired induced spin connection on $\mathcal{E}_B \in {}_B\mathsf{Mod}$. To amend this issue, we use this connection together with the classical spinorial Gauss formula from [HMZ02, Eqn. (3.5)] to define

$$\nabla^{\mathrm{sp}}_B \,:\, \mathcal{E}_B \;\longrightarrow\; \Omega^1_B \otimes_B \mathcal{E}_B \;,\quad [s] \;\longmapsto\; \Big[\nabla^{\mathrm{sp}}(s) + \frac{1}{2}(\mathrm{id} \otimes_A \gamma_{[2]})\big(\nabla(\nu) \otimes_A \nu \otimes_A s\big)\Big] \quad . \tag{2.1.24}$$

This map is a connection on the left B-module $\mathcal{E}_B \in {}_B\mathsf{Mod}$ since both $\nu \in \Omega^1_A$ and $\nabla(\nu) \in \Omega^1_A \otimes_A \Omega^1_A$ are central (see (2.1.19) for the latter statement).

We need a last assumption to make sure that the data we just introduced defines a spinorial structure in the sense of Definition 1.1.9:

Assumption 2.1.13. The element $\nabla(\nu) \in \Omega^1_A \otimes_A \Omega^1_A$ satisfies

$$\big[\sigma_{23}\sigma_{12}\big(\Pi(\omega) \otimes_A \nabla(\nu)\big)\big] \;=\; \big[\nabla(\nu) \otimes_A \Pi(\omega)\big] \;\in\; \Omega^1_B \otimes_B q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \quad , \tag{2.1.25}$$

for all $\omega \in \Omega^1_A$, where $\sigma_{12} := \sigma \otimes_A \mathrm{id}$ and $\sigma_{23} := \mathrm{id} \otimes_A \sigma$.

Proposition 2.1.14. *The triple* $(\mathcal{E}_B, \nabla^{\mathrm{sp}}_B, \gamma_B)$ *defined in* (2.1.22), (2.1.24) *and* (2.1.23) *is a spinorial structure on the Riemannian structure* $(g_B, (\nabla_B, \sigma_B))$.

Proof. It remains to prove the Clifford relations and Clifford compatibility properties from Definition 1.1.9. In these calculations we frequently use the identities

$$\big[\gamma_{[2]}\big(\Pi(\omega) \otimes_A \nu \otimes_A s\big)\big] = -\big[\gamma_{[2]}\big(\nu \otimes_A \Pi(\omega) \otimes_A s\big)\big] \tag{2.1.26a}$$

and

$$\big[\gamma_{[2]}\big(\nu \otimes_A \nu \otimes_A s\big)\big] = -[s] \quad , \tag{2.1.26b}$$

which follow from the Clifford relations (1.1.14) for γ, Assumption 2.1.6, Lemma 2.1.7 (ii) and the normalization condition (2.1.6).

The Clifford relations (1.1.14) for γ_B follow from a direct calculation, for which we introduce the convenient short notation $\sigma(\omega \otimes_A \zeta) = \sum_\alpha \zeta^\alpha \otimes_A \omega_\alpha$. We compute

$$\begin{aligned}
\gamma_{B[2]}\Big([\omega] \otimes_B [\zeta] \otimes_B [s] + \sigma_{B12}\big([\omega] \otimes_B [\zeta] \otimes_B [s]\big)\Big)&\\
= \Big[\gamma_{[4]}\Big(&\Pi(\omega) \otimes_A \nu \otimes_A \Pi(\zeta) \otimes_A \nu \otimes_A s\\
&+ \sum_\alpha \Pi(\zeta^\alpha) \otimes_A \nu \otimes_A \Pi(\omega_\alpha) \otimes_A \nu \otimes_A s\Big)\Big]\\
= \Big[\gamma_{[2]}\Big(&\Pi(\omega) \otimes_A \Pi(\zeta) \otimes_A s + \sum_\alpha \Pi(\zeta^\alpha) \otimes_A \Pi(\omega_\alpha) \otimes_A s\Big)\Big]\\
= -2\, g_B^{-1}&\big([\omega] \otimes_B [\zeta]\big)\,[s] \quad ,
\end{aligned} \tag{2.1.27}$$

where in the second step we used (2.1.26). The last step follows from Assumption 2.1.11, the Clifford relations for γ and the definition of g_B^{-1} in (2.1.9b).

Proving the Clifford compatibility property (1.1.15) for ∇_B, ∇_B^{sp} and γ_B is a lengthier computation. Using as above (2.1.26), Assumption 2.1.11 and also Clifford compatibility for ∇, ∇^{sp} and γ, one finds that the desired equality $\nabla_B^{\mathrm{sp}}\, \gamma_B\big([\omega] \otimes_B [s]\big) = (\mathrm{id} \otimes_B \gamma_B)\big(\nabla_B^{\otimes}\big([\omega] \otimes_B [s]\big)\big)$ is equivalent to the statement that the two expressions

$$\Big[(\mathrm{id} \otimes_A \gamma_{[2]})\Big(\sigma_{12}\big(\Pi(\omega) \otimes_A \nabla(\nu) \otimes_A s\big) + \frac{1}{2}\nabla(\nu) \otimes_A \Pi(\omega) \otimes_A s\Big)\Big] \tag{2.1.28a}$$

and

$$\begin{aligned}
\Big[&(\mathrm{id} \otimes_A g^{-1})\big(\nabla\Pi(\omega) \otimes_A \nu\big) \otimes_A s\\
&+ \frac{1}{2}(\mathrm{id} \otimes_A \gamma_{[4]})\Big(\sigma_{12}\sigma_{23}\big(\Pi(\omega) \otimes_A \nu \otimes_A \nabla(\nu) \otimes_A \nu \otimes_A s\big)\Big)\Big]
\end{aligned} \tag{2.1.28b}$$

are equal. (The term with g^{-1} in (2.1.28b) arises from computing $(\mathrm{id} \otimes_A \Pi)\nabla\Pi(\omega) = \nabla\Pi(\omega) - (\mathrm{id} \otimes_A g^{-1})\big(\nabla\Pi(\omega) \otimes_A \nu\big) \otimes_A \nu$ via (2.1.7).) Using metric compatibility (1.1.12) for $(g, (\nabla, \sigma))$, Lemma 2.1.7 (ii) and the Clifford relations for γ, we can rewrite the first term of (2.1.28b) as

$$\begin{aligned}
(2.1.28\mathrm{b})^{1\mathrm{st}} &= \Big[-(\mathrm{id} \otimes_A g^{-1})\sigma_{12}\big(\Pi(\omega) \otimes_A \nabla(\nu)\big) \otimes_A s\Big]\\
&= \Big[\frac{1}{2}(\mathrm{id} \otimes_A \gamma_{[2]})\Big(\sigma_{12}\big(\Pi(\omega) \otimes_A \nabla(\nu) \otimes_A s\big)\\
&\qquad + \sigma_{23}\sigma_{12}\big(\Pi(\omega) \otimes_A \nabla(\nu) \otimes_A s\big)\Big)\Big] \quad .
\end{aligned} \tag{2.1.29}$$

Concerning the second term of (2.1.28b), we use the Clifford relations for γ to bring the left factor of ν to the right and observe that there is no g^{-1} contribution as a result of Lemma 2.1.7 (iii). Hence, we can rewrite the second term of (2.1.28b) as

$$(2.1.28\mathrm{b})^{2\mathrm{nd}} = \Big[\frac{1}{2}(\mathrm{id}\otimes_A \gamma_{[2]})\Big(\sigma_{12}\big(\Pi(\omega)\otimes_A \nabla(\nu)\otimes_A s\big)\Big)\Big] \quad . \tag{2.1.30}$$

From these simplifications and Assumption 2.1.13, it follows that the expressions in (2.1.28b) and (2.1.28a) are equal. This completes our proof of the Clifford compatibility property. □

To conclude this section, we provide an explicit expression for the induced Dirac operator on the hypersurface B

$$D_B \,:\, \mathcal{E}_B \xrightarrow{\nabla^{\mathrm{sp}}_B} \Omega^1_B \otimes_B \mathcal{E}_B \xrightarrow{\gamma_B} \mathcal{E}_B \quad . \tag{2.1.31}$$

Proposition 2.1.15. *The induced Dirac operator* (2.1.31) *reads explicitly as*

$$\begin{aligned} D_B([s]) = \Big[-\frac{1}{2}\,\big(\gamma_{[2]} - \gamma_{[2]}\,(\sigma\otimes_A \mathrm{id})\big)\big(\nu\otimes_A \nabla^{\mathrm{sp}}(s)\big) \\ + \frac{1}{2}\,\gamma_{[2]}\big((\Pi\otimes_A \mathrm{id})\nabla(\nu)\otimes_A s\big)\Big] \quad , \end{aligned} \tag{2.1.32}$$

for all $[s] \in \mathcal{E}_B$.

Proof. This is a straightforward calculation using (2.1.24), (2.1.23), the projector (2.1.7) and the Clifford relations (1.1.14) for γ, in particular (2.1.26). Since the relevant steps are similar to those in the proof of Proposition 2.1.14, we do not have to write out the details of this calculation. □

2.2 EXAMPLES

In this section, we will apply our framework to the sequence of noncommutative hypersurface embeddings $\mathbb{T}^2_\theta \hookrightarrow \mathbb{S}^3_\theta \hookrightarrow \mathbb{R}^4_\theta$ studied by Arnlind and Norkvist [AN19]. We have already in Section 1.1.1 described the embedding space $\mathbb{R}^4_\theta$ together with a differential, Riemannian and spinorial structure on it. In the following, we will use the framework developed in Section 2.1 to induce these structures first to the noncommutative hypersurface $\mathbb{S}^3_\theta \hookrightarrow \mathbb{R}^4_\theta$ and subsequently to the noncommutative hypersurface $\mathbb{T}^2_\theta \hookrightarrow \mathbb{S}^3_\theta$. As a result, we will also obtain explicit expressions for the Dirac operators (in the sense of Definition 1.1.9) on these noncommutative hypersurfaces.

2.2.1 *Noncommutative hypersurface* $\mathbb{S}^3_\theta \hookrightarrow \mathbb{R}^4_\theta$

The first part of the sequence of embeddings is the noncommutative 3-sphere $\mathbb{S}^3_\theta \hookrightarrow \mathbb{R}^4_\theta$, to which we will apply our framework from Section 2.1 to induce a differential, Riemannian and spinorial structure from the corresponding structures on $\mathbb{R}^4_\theta$. In practice, it means we need to confirm that this is an example of a noncommutative hypersurface as described in Definition 2.1.2 and that Assumptions 2.1.6, 2.1.11 and 2.1.13 are satisfied. We will from here on simplify the notation by omitting the square brackets for denoting equivalence classes as it will be evident from context and the general construction in Section 2.1 which expressions are considered in quotient spaces.

Consider the algebra $A = A_{\mathbb{R}^4_\theta}$ of $\mathbb{R}^4_\theta$ (see (1.1.18)). The algebra $B = B_{\mathbb{S}^3_\theta}$ of the noncommutative *Connes-Landi* 3-*sphere* [CL01, CD-V02] is given by the quotient algebra

$$B := A/(f) \tag{2.2.1}$$

where (f) is the ideal generated by the unit sphere relation (in complex coordinates)

$$f := \frac{1}{2}\Big(\sum_{i,j=1}^{4} g_{ij}\, z^i\, z^j \,-1\Big) \;=\; \frac{1}{2}\Big(z^1\,\overline{z^1} + z^2\,\overline{z^2} - 1\Big) \quad . \tag{2.2.2}$$

The $\frac{1}{2}$-prefactor is chosen for the generator of the ideal $(f) \subset A$ in order to agree with the conditions in Example 2.1.4 for $\mathbb{S}^3_\theta \hookrightarrow \mathbb{R}^4_\theta$ to be a noncommutative level set hypersurface. Using the commutation relations of the coordinate functions of $\mathbb{R}^4_\theta$ (1.1.18) and (1.1.19), one sees that $f \in \mathcal{Z}(A) \subseteq A$ is central as required.

Proposition 2.2.1. *The* 1*-form*

$$\nu := \mathrm{d}f = \sum_{i,j=1}^{4} g_{ij}\, z^i\, \mathrm{d}z^j \in \Omega^1_A \tag{2.2.3}$$

is central and normalized. Hence, by Example 2.1.4, $B = B_{\mathbb{S}^3_\theta}$ is a noncommutative hypersurface of $A = A_{\mathbb{R}^4_\theta}$ in the sense of Definition 2.1.2. The projector $\Pi : q_!(\Omega^1_A) \to q_!(\Omega^1_A)$ from Proposition 2.1.5 reads explicitly as

$$\Pi(\mathrm{d}z^i) = \mathrm{d}z^i - z^i\,\nu \quad . \tag{2.2.4}$$

Proof. Centrality of ν is a simple check using (1.1.21) and (1.1.19) and the normalization condition (2.1.6) is proven by

$$g^{-1}(\nu \otimes_A \nu) = \sum_{i,j,k,l=1}^{4} g_{ij}\, z^i\, g^{jl}\, g_{kl}\, z^k = \sum_{i,k=1}^{4} g_{ik}\, z^i\, z^k = 1 \quad . \tag{2.2.5}$$

The explicit expression for the projector is obtained from a short calculation

$$\begin{aligned}\Pi(\mathrm{d}z^i) &= \mathrm{d}z^i - g^{-1}\Big(\mathrm{d}z^i \otimes_A \sum_{j,k=1}^{4} g_{jk}\, z^j\, \mathrm{d}z^k\Big)\, \nu \\ &= \mathrm{d}z^i - \sum_{j,k=1}^{4} g^{ik}\, g_{jk}\, z^j\, \nu = \mathrm{d}z^i - z^i\, \nu \quad , \end{aligned} \tag{2.2.6}$$

where in the second step we used (1.1.21), (1.1.19) and (1.1.23) in order to write $\sum_{j,k=1}^{4} g_{jk}\, z^j\, \mathrm{d}z^k = \sum_{j,k=1}^{4} \mathrm{d}z^k\, g_{jk}\, z^j$. □

In the following, we will verify that Assumptions 2.1.6, 2.1.11 and 2.1.13 hold for the noncommutative hypersurface $\mathbb{S}^3_\theta \hookrightarrow \mathbb{R}^4_\theta$. We will also present explicit expressions for the Riemannian and spinorial structures.

Proposition 2.2.2. *Assumptions 2.1.6 and 2.1.11 hold true. The induced Riemannian structure from Proposition 2.1.12 reads explicitly as*

$$g_B \;=\; \sum_{i,j=1}^{4} g_{ij}\, \mathrm{d}z^i \otimes_B \mathrm{d}z^j \;\in\; \Omega^1_B \otimes_B \Omega^1_B \quad , \tag{2.2.7a}$$

$$g_B^{-1}\big(\mathrm{d}z^i \otimes_B \mathrm{d}z^j\big) \;=\; g^{ij} - z^i\, z^j \quad , \tag{2.2.7b}$$

$$\nabla_B(\mathrm{d}z^i) \;=\; -z^i \sum_{k,l=1}^{4} g_{kl}\, \mathrm{d}z^k \otimes_B \mathrm{d}z^l \quad , \tag{2.2.7c}$$

$$\sigma_B\big(\mathrm{d}z^i \otimes_B \mathrm{d}z^j\big) \;=\; R^{ji}\, \mathrm{d}z^j \otimes_B \mathrm{d}z^i \quad . \tag{2.2.7d}$$

Proof. Verifying Assumption 2.1.6 is a simple check using (1.1.21), (1.1.27) and (1.1.19). To prove commutativity of the top diagram in Assumption 2.1.11, we use (2.2.4) and compute

$$\begin{aligned}\sigma\big(\Pi(\mathrm{d}z^i) \otimes_A \mathrm{d}z^j\big) &= \sigma\big(\mathrm{d}z^i \otimes_A \mathrm{d}z^j\big) - \sigma\big(z^i\, \nu \otimes_A \mathrm{d}z^j\big) \\ &= R^{ji}\, \mathrm{d}z^j \otimes_A \mathrm{d}z^i - R^{ji}\, \mathrm{d}z^j \otimes_A z^i\, \nu \\ &= R^{ji}\, \mathrm{d}z^j \otimes_A \Pi(\mathrm{d}z^i) \\ &= (\mathrm{id} \otimes_A \Pi)\, \sigma\big(\mathrm{d}z^i \otimes_A \mathrm{d}z^j\big) \quad , \end{aligned} \tag{2.2.8}$$

where we in the second step also used (2.1.10) and (1.1.21). Commutativity of the bottom diagram in Assumption 2.1.11 is proven by a similar calculation.

Concerning the explicit expressions for the induced Riemannian structure, we observe that (2.2.7a) follows trivially from (2.1.9a). Equation (2.2.7b) follows from (2.1.9b), (2.2.4) and a straightforward calculation. Equation (2.2.7c) follows from (2.1.17) and (1.1.26) by a short calculation

$$\begin{aligned}\nabla_B(\mathrm{d}z^i) &= \nabla(\mathrm{d}z^i) - g^{-1}\big(\mathrm{d}z^i \otimes_A \nu\big)\, \nabla(\nu) = -z^i\, \nabla(\nu)\\ &= -z^i \sum_{k,l=1}^{4} g_{kl}\, \mathrm{d}z^k \otimes_B \mathrm{d}z^l \quad ,\end{aligned} \tag{2.2.9}$$

where in the last step we used that

$$\nabla(\nu) = \sum_{k,l=1}^{4} g_{kl}\, \nabla\big(z^k\, \mathrm{d}z^l\big) = \sum_{k,l=1}^{4} g_{kl}\, \mathrm{d}z^k \otimes_A \mathrm{d}z^l \tag{2.2.10}$$

via the left Leibniz rule and (1.1.26). Finally, (2.2.7d) follows trivially from (2.1.16) and (1.1.27). □

Proposition 2.2.3. *Assumption 2.1.13 holds true. The induced spinorial structure from Proposition 2.1.14 reads explicitly as*

$$\mathcal{E}_B = \frac{\mathcal{E}}{f\,\mathcal{E}} \quad , \tag{2.2.11a}$$

$$\gamma_B\big(\mathrm{d}z^i \otimes_B e_\alpha\big) = -\Big(\sum_{k,l=1}^{4} g_{kl}\, z^k\, \gamma_\theta^l\, \gamma_\theta^i + z^i\Big)\, e_\alpha \quad , \tag{2.2.11b}$$

$$\nabla_B^{\mathrm{sp}}(e_\alpha) = \frac{1}{2} \sum_{i,j,k,l=1}^{4} g_{ij}\, g_{kl}\, z^k\, \mathrm{d}z^i \otimes_B \gamma_\theta^j\, \gamma_\theta^l\, e_\alpha \quad . \tag{2.2.11c}$$

Proof. Recalling (2.2.10), Assumption 2.1.13 is verified by a similar calculation as the one that proves centrality of the metric g.

Concerning the explicit expressions for the induced spinorial structure, we observe that (2.2.11a) is just the definition in (2.1.22). Equation (2.2.11b) follows from (2.1.23) by a short calculation

$$\begin{aligned}\gamma_B\big(\mathrm{d}z^i \otimes_B e_\alpha\big) &= -\gamma_{[2]}\big(\nu \otimes_A \Pi(\mathrm{d}z^i) \otimes_A e_\alpha\big)\\ &= -\gamma_{[2]}\big(\nu \otimes_A \mathrm{d}z^i \otimes_A e_\alpha\big) + g^{-1}\big(\mathrm{d}z^i \otimes_A \nu\big)\, \gamma_{[2]}\big(\nu \otimes_A \nu \otimes_A e_\alpha\big)\\ &= -\Big(\sum_{k,l=1}^{4} g_{kl}\, z^k\, \gamma_\theta^l\, \gamma_\theta^i + z^i\Big)\, e_\alpha \quad ,\end{aligned} \tag{2.2.12}$$

where in the first step we used (2.1.26a) and in the third step we used (2.1.26b). Finally, equation (2.2.11c) follows from writing out (2.1.24) and using (1.1.31) and (2.2.10). □

We now have all the building blocks for computing the induced Dirac operator on $\mathbb{S}^3_\theta$.

Proposition 2.2.4. *The induced Dirac operator* (2.1.31) *on* $\mathbb{S}^3_\theta$ *is given by*

$$D_B(s) \;=\; -\frac{1}{2}\sum_{i,j=1}^{4}[\gamma^j_\theta,\gamma^i_\theta]_\theta\,\partial_i s\, z_j - \frac{3}{2}\, s \quad , \tag{2.2.13}$$

where $z_i := \sum_{k=1}^4 g_{ik}\, z^k$, $\partial_i s := \sum_{\alpha=1}^4 \partial_i s^\alpha\, e_\alpha$ *and* $[\gamma^j_\theta,\gamma^i_\theta]_\theta$ *is the* θ*-commutator from Lemma 1.1.16.*

Proof. We have to compute the induced Dirac operator from Proposition 2.1.15 for our example. Using (1.1.26) and (2.2.3), the first term of (2.1.32) is given by

$$\begin{aligned}(2.1.32)^{1\text{st}} &= -\frac{1}{2}\sum_{\alpha=1}^{4}\sum_{i,j,k=1}^{4}\partial_i s^\alpha\, g_{kj}\, z^k\,\big(\gamma^j_\theta\,\gamma^i_\theta - R^{ij}\,\gamma^i_\theta\,\gamma^j_\theta\big)\, e_\alpha\\ &= -\frac{1}{2}\sum_{i,j=1}^{4}[\gamma^j_\theta,\gamma^i_\theta]_\theta\,\partial_i s\, z_j \quad ,\end{aligned} \tag{2.2.14}$$

which yields the first term of (2.2.13). To compute the second term of (2.1.32), we first observe that

$$(\Pi\otimes_A \mathrm{id})\nabla(\nu) = \sum_{i,j=1}^{4} g_{ij}\,\Pi\big(\mathrm{d}z^i\big)\otimes_A \mathrm{d}z^j = \sum_{i,j=1}^{4}\big(g_{ij} - z_j\, z_i\big)\,\mathrm{d}z^i\otimes_A \mathrm{d}z^j \quad , \tag{2.2.15}$$

where in the first step we used (2.2.10) and in the second step (2.2.4). This element is invariant under applying σ, i.e. $\sigma(\Pi\otimes_A \mathrm{id})\nabla(\nu) = (\Pi\otimes_A \mathrm{id})\nabla(\nu)$, hence we can write

$$(\Pi\otimes_A \mathrm{id})\nabla(\nu) = \frac{1}{2}\Big((\Pi\otimes_A \mathrm{id})\nabla(\nu) + \sigma(\Pi\otimes_A \mathrm{id})\nabla(\nu)\Big) \tag{2.2.16}$$

in the second term of (2.1.32). Using the Clifford relations (1.1.14), we obtain

$$(2.1.32)^{2\text{nd}} = -\frac{1}{2}\sum_{i,j=1}^{4}\big(g_{ij} - z_j\, z_i\big)\, g^{ij}\, s = -\frac{1}{2}\big(4-1\big)\, s = -\frac{3}{2}\, s \quad , \tag{2.2.17}$$

where in the second step we used $\sum_{i,j=1}^4 g_{ij}\, g^{ij} = \sum_{i=1}^4 \delta^i_i = 4$ (see (1.1.24)) and the sphere relation $\sum_{i,j=1}^4 z_j\, z_i\, g^{ij} = \sum_{i,j=1}^4 g_{ij}\, z^i\, z^j = 1$ (see (2.2.2)). □

Remark 2.2.5. Observe that for vanishing deformation parameter $\theta = 0$, the Dirac operator (2.2.13) on $\mathbb{S}^3_\theta$ reduces to the usual Dirac operator on the commutative 3-sphere $\mathbb{S}^3 \subseteq \mathbb{R}^4$, see e.g. [Tra95, Section 7.1] or [Tra93]. △

Having obtained the Dirac operator on $\mathbb{S}^3_\theta$ with our framework, we now want to compare it with the Connes-Landi Dirac operator [CL01, CD-V02] which is obtained from an isospectral deformation [BLvS13]. The techniques used in the following are examples of *Drinfeld twist* (or 2-*cocycle*) *deformations*, see e.g. [BLvS13, AS14, BSS14]. First, we consider the coaction from Example 1.2.13 in Section 1.2, $\delta : A_{\mathbb{R}^4} \to H \otimes A_{\mathbb{R}^4}$ given by

$$\begin{aligned} \delta(z^1) &= t_{(2,0)} \otimes z^1 \ , \quad \delta(z^2) = t_{(0,2)} \otimes z^2 \\ \delta(z^3) &= t_{(-2,0)} \otimes z^3 \ , \quad \delta(z^4) = t_{(0,-2)} \otimes z^4 \quad . \end{aligned} \tag{2.2.18}$$

where $H = \mathcal{O}(\mathbb{T}^2)$ is the algebra of functions on the algebraic 2-torus and $A_{\mathbb{R}^4}$ is the *commutative* algebra of functions on $\mathbb{R}^4 \cong \mathbb{C}^2$. We may obtain the noncommutative algebra (1.1.18) as a deformation quantisation of the commutative algebra $A_{\mathbb{R}^4}$ by introducing the star-product

$$a \star_\theta a' := \sigma_\theta\big(a_{\underline{-1}} \otimes a'_{\underline{-1}}\big) \, a_{\underline{0}} \, a'_{\underline{0}} \tag{2.2.19}$$

where $\sigma_\theta : H \otimes H \to \mathbb{C}$ is defined by

$$\sigma_\theta\big(t_{(m_1,m_2)} \otimes t_{(n_1,n_2)}\big) := \exp\Big(\tfrac{i\theta}{8}\big(m_1\, n_2 - m_2\, n_1\big)\Big) \quad . \tag{2.2.20}$$

(The map σ_θ is an instance of a so called 2-*cocycle*.) In a similar fashion, one obtains the algebra $B = B_{\mathbb{S}^3_\theta}$ of the Connes-Landi sphere (2.2.1) as a deformation quantisation of the commutative 3-sphere algebra $B_{\mathbb{S}^3}$.

A coaction on the differential calculus $\Omega^1_{A_{\mathbb{R}^4}}$ is induced from the coaction on $A_{\mathbb{R}^4}$ by demanding the differential d to be an H-comodule morphism. Explicitly, on the basis $\{\mathrm{d}z^i\}_{i=1}^4$,

$$\begin{aligned} \delta(\mathrm{d}z^1) &= t_{(2,0)} \otimes \mathrm{d}z^1 \ , \quad \delta(\mathrm{d}z^2) = t_{(0,2)} \otimes \mathrm{d}z^2 \\ \delta(\mathrm{d}z^3) &= t_{(-2,0)} \otimes \mathrm{d}z^3 \ , \quad \delta(\mathrm{d}z^4) = t_{(0,-2)} \otimes \mathrm{d}z^4 \quad . \end{aligned} \tag{2.2.21}$$

This is done similarly in the case of the sphere $\mathbb{S}^3$.

To obtain the module of noncommutative spinors $\mathcal{E}$ on $\mathbb{R}^4_\theta$ in (1.1.30), we perform a deformation quantisation of the module of commutative spinors $\mathcal{E}_c$. This is realised by defining the left H-coaction

$$\begin{aligned} \delta(e_1) &= t_{(1,1)} \otimes e_1 \ , \quad \delta(e_2) = t_{(-1,-1)} \otimes e_2 \\ \delta(e_3) &= t_{(1,-1)} \otimes e_3 \ , \quad \delta(e_4) = t_{(-1,1)} \otimes e_4 \end{aligned} \tag{2.2.22}$$

and extension by the relation $\delta(a\,s) = a_{\underline{-1}}\,s_{\underline{-1}} \otimes a_{\underline{0}}\,s_{\underline{0}}$ for all $a \in A_{\mathbb{R}^4}$ and $s \in \mathcal{E}_c$, together with the associated star-module structure

$$a \star_\theta s \;:=\; \sigma_\theta\big(a_{\underline{-1}} \otimes s_{\underline{-1}}\big)\; a_{\underline{0}}\, s_{\underline{0}} \quad . \tag{2.2.23}$$

The spinor module $\mathcal{E}_B$ on $\mathbb{S}^3_\theta$ given in (2.2.11a) is obtained in the same vein. In terms of the star products, our Dirac operator (2.2.13) on $\mathbb{S}^3_\theta$ takes the form

$$D_B(s) \;=\; -\frac{1}{2}\sum_{i,j=1}^{4} [\gamma^j_\theta, \gamma^i_\theta]_\theta\, \partial^\theta_i s \star_\theta z_j - \frac{3}{2}\, s \quad , \tag{2.2.24}$$

where ∂^θ_i is defined by $\mathrm{d}a = \partial^\theta_i a \star_\theta \mathrm{d}z^i$ with respect to the deformed module structure.

The Connes-Landi Dirac operator D_{CL} on $\mathbb{S}^3_\theta$ is given by regarding the classical Dirac operator on $\mathbb{S}^3$ as an operator on the deformed spinor module, see [CL01, CD-V02, BLvS13] for details. Concretely, it is given by setting the deformation parameter $\theta = 0$ in (2.2.24), i.e.

$$D_{\mathrm{CL}}(s) \;=\; -\frac{1}{2}\sum_{i,j=1}^{4} [\gamma^j, \gamma^i]\, \partial_i s\, z_j - \frac{3}{2}\, s \quad , \tag{2.2.25}$$

where ∂_i is defined by $\mathrm{d}a = \partial_i a\, \mathrm{d}z^i$ with respect to the undeformed module structure. The Connes-Landi Dirac operator D_{CL} is equivariant under the torus action. In other words D_{CL} is an H-comodule morphism (see Definition 1.2.9). (One can show this explicitly through a direct calculation.) Hence, we may compute

$$\begin{aligned}
D_{\mathrm{CL}}(a \star_\theta s) &= \sigma_\theta\big(a_{\underline{-1}} \otimes s_{\underline{-1}}\big)\, D_{\mathrm{CL}}(a_{\underline{0}}\, s_{\underline{0}})\\
&= \sigma_\theta\big(a_{\underline{-1}} \otimes s_{\underline{-1}}\big)\, \Big(a_{\underline{0}}\, D_{\mathrm{CL}}(s_{\underline{0}}) + \gamma_{\mathbb{S}^3}\big(\mathrm{d}a_{\underline{0}} \otimes_{B_{\mathbb{S}^3}} s_{\underline{0}}\big)\Big)\\
&= \sigma_\theta\big(a_{\underline{-1}} \otimes D_{\mathrm{CL}}(s)_{\underline{-1}}\big)\, a_{\underline{0}}\, D_{\mathrm{CL}}(s)_{\underline{0}}\\
&\qquad + \sigma_\theta\big((\mathrm{d}a)_{\underline{-1}} \otimes s_{\underline{-1}}\big)\, \gamma_{\mathbb{S}^3}\big((\mathrm{d}a)_{\underline{0}} \otimes_{B_{\mathbb{S}^3}} s_{\underline{0}}\big)\Big)\\
&= a \star_\theta D_{\mathrm{CL}}(s) + \gamma_{\mathbb{S}^3}\big(\mathrm{d}a \otimes_{B_{\mathbb{S}^3_\theta}} s\big) \quad ,
\end{aligned} \tag{2.2.26}$$

where $\omega \otimes_{B_{\mathbb{S}^3_\theta}} s := \sigma_\theta(\omega_{\underline{-1}} \otimes s_{\underline{-1}})\; \omega_{\underline{0}} \otimes_{B_{\mathbb{S}^3}} s_{\underline{0}}$ denotes the deformed tensor product and $\gamma_{\mathbb{S}^3}$ the classical Clifford multiplication.[2] In the third equality, we used that D_{CL} and d are equivariant under the torus action. Now, we see that this relation (2.2.26) is also satisfied by our hypersurface Dirac operator by Proposition 1.1.10. The reason is that

2 In fact, this is how the deformed gamma matrices in (1.1.34) were constructed; the Clifford multiplication for $\mathbb{R}^4_\theta$ is given by $\gamma^i_\theta\, e_\alpha := \sigma\big((\mathrm{d}z^i)_{\underline{-1}} \otimes (e_\alpha)_{\underline{-1}}\big)\, \gamma_{\mathbb{R}^4}\big((\mathrm{d}z^i)_{\underline{0}} \otimes_{A_{\mathbb{R}^4}} (e_\alpha)_{\underline{0}}\big)$, where $\gamma_{\mathbb{R}^4}$ is the standard Clifford multiplication on $\mathbb{R}^4$ (in complex coordinates given by the gamma matrices (1.1.32)).

our noncommutative Clifford multiplication (1.1.35) by construction coincides with the classical Clifford multiplication regarded as a map on the deformed modules. This holds similarly for the induced Clifford multiplication (2.1.23) on the noncommutative hypersurface $\mathbb{S}^3_\theta$ since the normalised normal form ν in (2.2.3) is invariant under the torus action, which translates to $\delta(\nu) = \mathbb{1} \otimes \nu$. We may therefore finally provide the following comparison result:

Proposition 2.2.6. *The hypersurface Dirac operator* (2.2.24) *on* $\mathbb{S}^3_\theta$ *coincides with the Connes-Landi Dirac operator* D_{CL}.

Proof. Because both D_B and D_{CL} satisfy the same property (2.2.26), they coincide if and only if $D_B(e_\alpha) = D_{\mathrm{CL}}(e_\alpha)$, for all basis spinors e_α. The latter follows from (2.2.24) and (2.2.25) because $D_B(e_\alpha) = -\frac{3}{2} e_\alpha = D_{\mathrm{CL}}(e_\alpha)$. □

2.2.2 *Noncommutative hypersurface* $\mathbb{T}^2_\theta \hookrightarrow \mathbb{S}^3_\theta$

The next part of the example is to further apply the construction of Section 2.1 to induce the differential, Riemannian and spinorial structure on $\mathbb{S}^3_\theta$ (cf. Section 2.2.1) to the noncommutative 2-torus $\mathbb{T}^2_\theta \hookrightarrow \mathbb{S}^3_\theta$. Following the procedure in Section 2.2.1, we need to check that $\mathbb{T}^2_\theta \hookrightarrow \mathbb{S}^3_\theta$ is a noncommutative hypersurface (see Definition 2.1.2) and that Assumptions 2.1.6, 2.1.11 and 2.1.13 are satisfied. As before, we will also write down the explicit formulas for the induced structures.

The algebra $C = C_{\mathbb{T}^2_\theta}$ for the noncommutative 2-torus $\mathbb{T}^2_\theta$ is given by the quotient algebra

$$C := B/(\widetilde{f}) \tag{2.2.27}$$

of the sphere algebra $B = B_{\mathbb{S}^3_\theta}$ (see (2.2.1)) by the ideal generated by

$$\widetilde{f} := \frac{1}{2}\Big(\sum_{i,j=1}^{4} h_{ij}\, z^i\, z^j\Big) = \frac{1}{2}\Big(z^1\,\overline{z^1} - z^2\,\overline{z^2}\Big) \quad , \tag{2.2.28}$$

where h_{ij} are the entries of the matrix

$$(h_{ij}) := \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad . \tag{2.2.29}$$

Due to the commutation relations given by (1.1.18) and (1.1.19), the element $\widetilde{f} \in \mathcal{Z}(B) \subseteq B$ is central. The quotient map will be denoted by

$$\widetilde{q} \,:\, B \longrightarrow C \tag{2.2.30}$$

to avoid confusion with the quotient map $q : A \to B$ in Section 2.2.1. To see that $C = C_{\mathbb{T}^2_\theta}$ really is the algebra for the noncommutative 2-torus $\mathbb{T}^2_\theta$, recall that the noncommutative sphere algebra (2.2.1) is the quotient $B = A/(f)$. This means that $C = A/(f, \widetilde{f})$ is the quotient of the algebra $A = A_{\mathbb{R}^4_\theta}$ of $\mathbb{R}^4_\theta$ by the ideal generated by the two relations f and $\widetilde{f}$ in (2.2.2) and (2.2.28). By rescaling the coordinates, $u := \sqrt{2}\, z^1$ and $v := \sqrt{2}\, z^2$, we obtain the usual torus relations from the linear combinations

$$2\,(f+\widetilde{f}) \,=\, 2\,z^1\,\overline{z^1} - 1 = u\,\overline{u} - 1 \quad , \quad 2\,(f-\widetilde{f}) \,=\, 2\,z^2\,\overline{z^2} - 1 = v\,\overline{v} - 1 \quad . \tag{2.2.31}$$

Proposition 2.2.7. *The* 1*-form*

$$\widetilde{\nu} \,:=\, \mathrm{d}\widetilde{f} \,=\, \sum_{i,j=1}^{4} h_{ij}\, z^i\, \mathrm{d}z^j \,\in\, \Omega^1_B \tag{2.2.32}$$

is central and normalized. Hence, by Example 2.1.4, $C = C_{\mathbb{T}^2_\theta}$ is a noncommutative hypersurface of $B = B_{\mathbb{S}^3_\theta}$ in the sense of Definition 2.1.2. The projector $\widetilde{\Pi} : \widetilde{q}_!(\Omega^1_B) \to \widetilde{q}_!(\Omega^1_B)$ from Proposition 2.1.5 reads explicitly as

$$\widetilde{\Pi}(\mathrm{d}z^i) \,=\, \mathrm{d}z^i + (-1)^i\, z^i\, \widetilde{\nu} \quad . \tag{2.2.33}$$

Proof. Centrality of $\widetilde{\nu}$ is a simple check using (1.1.21) and (1.1.19). To prove the normalization condition, we use (2.2.7b) and compute

$$g_B^{-1}\big(\widetilde{\nu} \otimes_B \widetilde{\nu}\big) = \sum_{i,j,k,l=1}^{4} h_{ij}\, z^i\, \big(g^{jl} - z^j\, z^l\big)\, h_{kl}\, z^k = \sum_{i,k=1}^{4} g_{ik}\, z^i\, z^k = 1 \quad , \tag{2.2.34}$$

where in the second step we used (2.2.28) and the identity

$$\sum_{j,l=1}^{4} h_{ij}\, g^{jl}\, h_{kl} = g_{ik} \quad , \tag{2.2.35}$$

and in the last step we used (2.2.2). The explicit expression for the projector is obtained from a short calculation

$$\begin{aligned}\widetilde{\Pi}(\mathrm{d}z^i) &= \mathrm{d}z^i - g_B^{-1}\big(\mathrm{d}z^i \otimes_B \widetilde{\nu}\big)\, \widetilde{\nu} = \mathrm{d}z^i - \sum_{k,l=1}^{4} \big(g^{il} - z^i\, z^l\big)\, h_{kl}\, z^k\, \widetilde{\nu} \\ &= \mathrm{d}z^i - \sum_{k,l=1}^{4} g^{il}\, h_{kl}\, z^k\, \widetilde{\nu} = \mathrm{d}z^i + (-1)^i\, z^i\, \widetilde{\nu} \quad ,\end{aligned} \tag{2.2.36}$$

where in the second step we used (2.2.7b) and the third step (2.2.28). The last step follows from $\sum_{l=1}^4 g^{il}\, h_{kl} = -(-1)^i\, \delta^i_k$. □

Following the structure of Section 2.2.1, we now verify Assumptions 2.1.6, 2.1.11 and 2.1.13 while also providing explicit expressions for the Riemannian and spinorial structures.

Proposition 2.2.8. *Assumptions 2.1.6 and 2.1.11 hold true. The induced Riemannian structure from Proposition 2.1.12 reads explicitly as*

$$g_C \;=\; \sum_{i,j=1}^4 g_{ij}\, \mathrm{d}z^i \otimes_C \mathrm{d}z^j \;\in\; \Omega^1_C \otimes_C \Omega^1_C \quad , \tag{2.2.37a}$$

$$g_C^{-1}\big(\mathrm{d}z^i \otimes_C \mathrm{d}z^j\big) \;=\; g^{ij} - \big(1 + (-1)^i\,(-1)^j\big)\, z^i\, z^j \quad , \tag{2.2.37b}$$

$$\nabla_C(\mathrm{d}z^i) \;=\; -z^i \sum_{k,l=1}^4 \big(g_{kl} - (-1)^i\, h_{kl}\big)\, \mathrm{d}z^k \otimes_C \mathrm{d}z^l \quad , \tag{2.2.37c}$$

$$\sigma_C\big(\mathrm{d}z^i \otimes_C \mathrm{d}z^j\big) \;=\; R^{ji}\, \mathrm{d}z^j \otimes_C \mathrm{d}z^i \quad . \tag{2.2.37d}$$

Proof. Verifying Assumption 2.1.6 is a simple check using (1.1.21), (1.1.27) and (1.1.19). To prove commutativity of the top diagram in Assumption 2.1.11, we use (2.2.33) and compute

$$\begin{aligned}\sigma_B\big(\widetilde{\Pi}(\mathrm{d}z^i) \otimes_B \mathrm{d}z^j\big) &= R^{ji}\, \mathrm{d}z^j \otimes_B \mathrm{d}z^i + (-1)^i\, z^i\, \mathrm{d}z^j \otimes_B \widetilde{\nu} \\ &= R^{ji}\, \mathrm{d}z^j \otimes_B \mathrm{d}z^i + R^{ji}\, \mathrm{d}z^j \otimes_B (-1)^i\, z^i\, \widetilde{\nu} \\ &= (\mathrm{id} \otimes_B \widetilde{\Pi})\, \sigma_B\big(\mathrm{d}z^i \otimes_B \mathrm{d}z^j\big) \quad ,\end{aligned} \tag{2.2.38}$$

where in the second step we used (1.1.21). Commutativity of the bottom diagram in Assumption 2.1.11 is proven by a similar calculation.

We observe that (2.2.37a) follows trivially from (2.1.9a) and (2.2.37b) follows from (2.1.9b), (2.2.33) and a straightforward calculation. Equation (2.2.37c) follows from (2.1.17), (2.2.7c) and

$$\nabla_B(\widetilde{\nu}) = \sum_{k,l=1}^4 h_{kl}\, \mathrm{d}z^k \otimes_B \mathrm{d}z^l \tag{2.2.39}$$

by a short calculation. Finally, (2.2.37d) follows trivially from (2.1.16) and (2.2.7d). □

Proposition 2.2.9. *Assumption 2.1.13 holds true. The induced spinorial structure from Proposition 2.1.14 reads explicitly as*

$$\mathcal{E}_C = \frac{\mathcal{E}_B}{\widetilde{f}\,\mathcal{E}_B} = \frac{\mathcal{E}}{f\mathcal{E}\cup\widetilde{f}\mathcal{E}} \quad , \tag{2.2.40a}$$

$$\gamma_C\big(\mathrm{d}z^i\otimes_C e_\alpha\big) = \Big(z^i\sum_{k,l,m,n=1}^{4} g_{mn}\,z^m\,h_{kl}\,z^k\,\gamma_\theta^l\,\gamma_\theta^n - \sum_{k,l=1}^{4} h_{kl}\,z^k\,\gamma_\theta^l\,\gamma_\theta^i + (-1)^i\,z^i\Big)\,e_\alpha \quad , \tag{2.2.40b}$$

$$\nabla_C^{\mathrm{sp}}(e_\alpha) = \frac{1}{2}\sum_{i,j,k,l=1}^{4}\Big(g_{kl}\,z^k\,g_{ij}\,\mathrm{d}z^i + h_{kl}\,z^k\,h_{ij}\,\mathrm{d}z^i\Big)\otimes_C\gamma_\theta^j\,\gamma_\theta^l\,e_\alpha \quad . \tag{2.2.40c}$$

Proof. Recalling (2.2.39), Assumption 2.1.13 is verified by a similar calculation as the one that proves centrality of $\widetilde{f}$ given in (2.2.28). The explicit expressions in (2.2.40a), (2.2.40b) and (2.2.40c) follow easily from the definitions (cf. (2.1.22), (2.1.23) and (2.1.24)) by straightforward calculations. (To obtain (2.2.40c), one has to recall that $\widetilde{\nu} = \mathrm{d}\widetilde{f} = 0$ in Ω^1_C.) □

We can now finally give the explicit expression for the induced Dirac operator on $\mathbb{T}^2_\theta$.

Proposition 2.2.10. *The induced Dirac operator* (2.1.31) *on* $\mathbb{T}^2_\theta$ *is given by*

$$D_C(s) = -\frac{1}{2}\sum_{i,j=1}^{4}[\gamma_\theta^j,\gamma_\theta^i]_\theta\Big(\partial_i s\,\widetilde{z}_j - \sum_{k=1}^{4}\partial_k s\,z^k\,z_i\,\widetilde{z}_j - s\,z_i\,\widetilde{z}_j\Big) \quad , \tag{2.2.41}$$

where $z_i := \sum_{k=1}^4 g_{ik}\,z^k$, $\widetilde{z}_i := \sum_{k=1}^4 h_{ik}\,z^k$, $\partial_i s := \sum_{\alpha=1}^4 \partial_i s^\alpha\,e_\alpha$ *and* $[\gamma_\theta^j,\gamma_\theta^i]_\theta$ *is the* θ*-commutator from Lemma 1.1.16.*

Proof. The proof is a straightforward but slightly lengthy calculation and hence will not be written out in detail. □

In its current form, it is not obvious how to interpret (2.2.41) as a Dirac operator on the flat noncommutative torus $\mathbb{T}^2_\theta$. We will therefore in the following rewrite (2.2.41) in a form which allows us to easier understand this. To this end, let us use the convenient standard generators

$$u := \sqrt{2}\,z^1 \quad , \quad v := \sqrt{2}\,z^2 \quad , \quad \overline{u} := \sqrt{2}\,\overline{z^1} \quad , \quad \overline{v} := \sqrt{2}\,\overline{z^2} \tag{2.2.42a}$$

of the algebra C of $\mathbb{T}^2_\theta$, which satisfy the relations

$$\overline{u}\,u \;=\; 1 \quad , \quad \overline{v}\,v \;=\; 1 \quad , \quad u\,v = e^{i\theta}\,v\,u \quad . \tag{2.2.42b}$$

We choose a central basis for the free 2-dimensional module of 1-forms Ω^1_C

$$\mathrm{d}\phi^1 \;:=\; \frac{1}{\mathrm{i}}\,\overline{u}\,\mathrm{d}u \quad , \quad \mathrm{d}\phi^2 \;:=\; \frac{1}{\mathrm{i}}\,\overline{v}\,\mathrm{d}v \quad . \tag{2.2.43}$$

(Our notation is inspired by thinking of $u = e^{i\,\phi^1}$ and $v = e^{i\,\phi^2}$ as exponential functions.) In this basis, the inverse metric (2.2.37b) is written as

$$g_C^{-1}\big(\mathrm{d}\phi^i \otimes \mathrm{d}\phi^j\big) \;=\; 2\,\delta^{ij} \quad . \tag{2.2.44}$$

The explanation for the factor 2 is that our embedded noncommutative torus $\mathbb{T}^2_\theta \hookrightarrow \mathbb{S}^3_\theta$ has radius $\frac{1}{\sqrt{2}}$, see (2.2.31). The differential of any $a \in C$ can be written in terms of the basis (2.2.43) as $\mathrm{d}a = \partial_{\phi^1} a\,\mathrm{d}\phi^1 + \partial_{\phi^2} a\,\mathrm{d}\phi^2$. By comparing this with $\mathrm{d}a = \sum_{i=1}^4 \partial_i a\,\mathrm{d}z^i \in \Omega^1_C$, we obtain

$$\partial_1 a \;=\; \frac{2}{i}\,\partial_{\phi^1} a\,\overline{z^1} \quad , \quad \partial_2 a \;=\; \frac{2}{i}\,\partial_{\phi^2} a\,\overline{z^2} \quad , \quad \partial_3 a \;=\; 0 \quad , \quad \partial_4 a \;=\; 0 \tag{2.2.45}$$

for the noncommutative partial derivatives along z^i.

Using the Clifford relations in the form of Lemma 1.1.16 (iii), we rewrite the Dirac operator (2.2.41) on $\mathbb{T}^2_\theta$ as

$$\begin{aligned} D_C(s) \;=\; & -\gamma\Big(\widetilde{\nu} \otimes_C \sum_{i=1}^4 \Big(\gamma^i_\theta\,\partial_i s - \gamma^i_\theta\,s\,z_i\Big)\Big) + \gamma_{[2]}\Big(\widetilde{\nu} \otimes_C \nu \otimes_C \sum_{i=1}^4 \partial_i s\,z^i\Big) \\ & + \sum_{i=1}^4 (-1)^i\,\partial_i s\,z^i \quad . \end{aligned} \tag{2.2.46}$$

Consider the map $\gamma(\widetilde{\nu} \otimes_C (-)) : \mathcal{E}_C \to \mathcal{E}_C$. From the Clifford relations (1.1.14) and because $\widetilde{\nu}$ is normalised, this map squares to $-\mathrm{id}$. Applying it to (2.2.46), we define

$$\begin{aligned} \widetilde{D}_C(s) \;:=\; & \gamma\big(\widetilde{\nu} \otimes_C D_C(s)\big) \\ =\; & \sum_{i=1}^4 \Big(\gamma^i_\theta\,\partial_i s - \gamma^i_\theta\,s\,z_i\Big) - \gamma\Big(\nu \otimes_C \sum_{i=1}^4 \partial_i s\,z^i\Big) + \gamma\Big(\widetilde{\nu} \otimes_C \sum_{i=1}^4 (-1)^i\,\partial_i s\,z^i\Big) \quad . \end{aligned} \tag{2.2.47}$$

Inserting (2.2.45) and the expressions for the normalised 1-forms (2.2.3) and (2.2.32) into (2.2.47), and carrying out all summations, we receive

$$\begin{aligned} \widetilde{D}_C(s) \;=\; & \frac{1}{\mathrm{i}}\Big(\gamma^1_\theta\,\partial_{\phi^1} s\,\overline{z^1} - \gamma^3_\theta\,\partial_{\phi^1} s\,z^1\Big) - \frac{1}{2}\,\Big(\gamma^1_\theta\,s\,\overline{z^1} + \gamma^3_\theta\,s\,z^1\Big) \\ & + \frac{1}{\mathrm{i}}\Big(\gamma^2_\theta\,\partial_{\phi^2} s\,\overline{z^2} - \gamma^4_\theta\,\partial_{\phi^2} s\,z^2\Big) - \frac{1}{2}\,\Big(\gamma^2_\theta\,s\,\overline{z^2} + \gamma^4_\theta\,s\,z^2\Big) \quad . \end{aligned} \tag{2.2.48}$$

We define the C-module map $\widetilde{\gamma} : \Omega^1_C \otimes_C \mathcal{E}_C \to \mathcal{E}_C$ by

$$\widetilde{\gamma}\big(\mathrm{d}\phi^1 \otimes_C s\big) \ := \ \frac{1}{\mathrm{i}}\Big(\gamma^1_\theta\, s\,\overline{z^1} - \gamma^3_\theta\, s\, z^1\Big) \quad , \quad \widetilde{\gamma}\big(\mathrm{d}\phi^2 \otimes_C s\big) \ := \ \frac{1}{\mathrm{i}}\Big(\gamma^2_\theta\, s\,\overline{z^2} - \gamma^4_\theta\, s\, z^2\Big) \quad , \tag{2.2.49}$$

for all $s \in \mathcal{E}_C$. It is straightforward to check that $\widetilde{\gamma}$ satisfies the Clifford relations

$$\begin{aligned}\widetilde{\gamma}_{[2]}\big(\mathrm{d}\phi^i \otimes_C \mathrm{d}\phi^j \otimes_C s\big) + \widetilde{\gamma}_{[2]}\big(\mathrm{d}\phi^j \otimes_C \mathrm{d}\phi^i \otimes_C s\big) &= -2\, g_C^{-1}\big(\mathrm{d}\phi^i \otimes \mathrm{d}\phi^j\big)\, s \\ &= -4\, \delta^{ij}\, s \end{aligned} \tag{2.2.50}$$

for the inverse metric (2.2.44). Note that there is no σ in this expression because $\sigma(\mathrm{d}\phi^i \otimes_C \mathrm{d}\phi^j) = \mathrm{d}\phi^j \otimes_C \mathrm{d}\phi^i$. Hence we may simplify (2.2.48) and we obtain

$$\begin{aligned}\widetilde{D}_C(s) \ &= \ \widetilde{\gamma}\Big(\mathrm{d}\phi^1 \otimes_C \Big(\partial_{\phi_1} s + \frac{1}{4}\widetilde{\gamma}\big(\mathrm{d}\phi^1 \otimes_C \big(\gamma^1_\theta\, s\,\overline{z^1} + \gamma^3_\theta\, s\, z^1\big)\big)\Big)\Big) \\ &\qquad + \widetilde{\gamma}\Big(\mathrm{d}\phi^2 \otimes_C \Big(\partial_{\phi_2} s + \frac{1}{4}\widetilde{\gamma}\big(\mathrm{d}\phi^2 \otimes_C \big(\gamma^2_\theta\, s\,\overline{z^2} + \gamma^4_\theta\, s\, z^2\big)\big)\Big)\Big) \\ &= \widetilde{\gamma}\Big(\mathrm{d}\phi^1 \otimes_C \Big(\partial_{\phi_1} s + \frac{1}{8i}[\gamma^1_\theta, \gamma^3_\theta]_\theta\, s\Big)\Big) + \widetilde{\gamma}\Big(\mathrm{d}\phi^2 \otimes_C \Big(\partial_{\phi_2} s + \frac{1}{8i}[\gamma^2_\theta, \gamma^4_\theta]_\theta\, s\Big)\Big) \quad . \end{aligned} \tag{2.2.51}$$

This is recognised as the Dirac operator on $\mathbb{T}^2_\theta$ corresponding to a rotating frame spin structure, see [BGa19]. The spectrum of $\widetilde{D}_C$, and hence the spectrum of the Dirac operator D_C in (2.2.41) on the noncommutative torus $\mathbb{T}^2_\theta$, can be found through a direct calculation and is given by

$$\Big\{ \pm \sqrt{2}\, \sqrt{\big(m + \tfrac{1}{2}\big)^2 + \big(n + \tfrac{1}{2}\big)^2} \ : \ m, n \in \mathbb{Z} \Big\} \quad . \tag{2.2.52}$$

It is observed that this spectrum coincides with the one of the Dirac operator corresponding to the $(1,1)$ spin structure on the commutative 2-torus $\mathbb{T}^2$, see e.g. [Fri84]. (The factor $\sqrt{2}$ in (2.2.52) is present because our noncommutative torus $\mathbb{T}^2_\theta \hookrightarrow \mathbb{S}^3_\theta$ has radius $\frac{1}{\sqrt{2}}$.)

The noncommutative hypersurface Dirac operator (2.2.51) on $\mathbb{T}^2_\theta$ coincides with the isospectral deformation [BLvS13] of the classical Dirac operator of type $(1,1)$ on the commutative 2-torus, acting as in [BGa19] on doubled, i.e. 4-dimensional, spinors. This is shown by utilising the same arguments as in Proposition 2.2.6, however with significantly longer calculations to compute $D_C(e_\alpha)$ on the basis spinors.

3

CONCLUSIONS AND OUTLOOK

In this part of the thesis, which is based on [NS20], we have devised a method for inducing differential, Riemannian and spinorial structures from ambient noncommutative spaces to an appropriate notion of noncommutative hypersurfaces. The induced structures are then used to build noncommutative hypersurface Dirac operators. Our effort is another attempt towards understanding the nature of noncommutative Riemannian spin geometry. Through our methods, we managed in Section 2.2 to construct Dirac operators on the Connes-Landi sphere and the noncommutative Clifford torus which match with ones obtained by other means in the literature.

Due to the systematic nature of our framework, a possible application would be the construction of novel examples of noncommutative Riemannian spaces (with spinorial structures) from old ones, and in particular obtain expressions for Dirac operators on them. It could also perhaps be of value to construct, as we did, Dirac operators on already known spaces. However, our considerations fall in the realm of almost commutative geometry and do not treat well strongly noncommutative algebras such as matrix algebras due to the various centrality assumptions. For instance, there is only one noncommutative level set hypersurface (in the sense of Example 2.1.4) associated to $A = \mathrm{Mat}_N(\mathbb{K})$, namely the trivial algebra $B = A/(f) = 0$, because the center of $\mathrm{Mat}_N(\mathbb{K})$ consists of scalar multiples of the identity matrix $\mathbb{1}$ so $(f) = A$. Regardless, the construction is still interesting because examples of noncommutative spaces are far and few between and often not very well understood. Another potential direction would be to link this to other approaches to noncommutative geometry, in particular to spectral triples, in which Dirac operators play a vital role.

Part II: Batalin-Vilkovisky quantisation of noncommutative field theories

4

PRELIMINARIES

The second part of the thesis is based on the papers [NSS21] and [GNS22], which adapts the modern formulation of the BV formalism of Costello and Gwilliam [Gwi12, CG16, CG21] to include finite noncommutative field theories. In [NSS21], this is applied to *fuzzy field theories*. Furthermore, the (finite) BV formalism is generalised to also account for fuzzy field theories with a symmetry encoded by a triangular Hopf algebra, also known as *braided fuzzy field theories*. In [GNS22], we apply the BV formalism on so called *dynamical fuzzy spectral triple models*. This chapter aims to cover the mathematical background, which includes a treatment of cochain complex in Section 4.1 and cyclic L_∞-algebras in Section 4.2, as well as the BV formalism for finite noncommutative field theories in Section 4.3.

4.1 COCHAIN COMPLEXES

In order to fix the notation and conventions, we recall some facts about cochain complexes. For more details, see e.g. [Wei94], which is one of many references on this subject. Most of the material covered here can be found in [NSS21]. We begin by providing the basic definitions. For a graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V^n$, we define the k-shifted vector space $V[k]$, $k \in \mathbb{Z}$, by $V[k]^n := V^{n+k}$ for all $n \in \mathbb{Z}$.

Definition 4.1.1. A *cochain complex* of $\mathbb{K}$-vector spaces consists a $\mathbb{Z}$-graded vector space $V = \bigoplus_{p\in\mathbb{Z}} V^p$ together with a $\mathbb{K}$-linear map $\mathrm{d} : V \to V[1]$ called *differential* such that $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0$.

Remark 4.1.2. The dual notion of a cochain complex is that of a *chain complex*, i.e. a $\mathbb{Z}$-graded vector space $V = \bigoplus_{p\in\mathbb{Z}} V_p$ together with a $\mathbb{K}$-linear map $\mathrm{d} : V \to V[-1]$ such that $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0$. △

We will sometimes refer to a cochain complex (V, d) only by its underlying graded vector space V. We will use the terminology *cohomological degree* to refer to the degree of an element in a cochain complex. We will also say that a map $f : V \to W[k]$ between graded vector spaces V and W is a map of (cohomological) degree k, i.e. f increases the degree by k.

We define the k-shifted cochain complex $V[k] \in \mathsf{Ch}_{\mathbb{K}}$ of any $V \in \mathsf{Ch}_{\mathbb{K}}$ and $k \in \mathbb{Z}$ as the cochain complex with underlying graded vector space $V[k]$ and differential $\mathrm{d}_{V[k]} := (-1)^k \,\mathrm{d}_V$. Note that $V[k] \cong \mathbb{K}[k] \otimes V$. The k-shifted cochain map $f[k] : V[k] \to W[k]$ of any cochain map $f : V \to W$ is defined by $f[k] := f$.

Note that the differential d consists of maps $\mathrm{d}_p : V^p \to V^{p+1}$ (also called differentials) such that $\mathrm{d}_{p+1} \circ \mathrm{d}_p = 0$. The fact that $\mathrm{d}^2 = 0$, is a crucial property as it implies that $\mathrm{im}(k) \subset \ker(\mathrm{d}_{k+1})$. This motivates the following definition.

Definition 4.1.3. The *kth cohomology* of a cochain complex (V, d) is the quotient space

$$H^k(V, \mathrm{d}) = \frac{\ker(\mathrm{d}_k)}{\mathrm{im}(\mathrm{d}_{k-1})} \quad . \tag{4.1.1}$$

We sometimes write only $H^k(V)$ when the differential d is understood.

Definition 4.1.4. A *cochain map* $f : V \to W$ between cochain complexes $(V, \mathrm{d}_V) = (\bigoplus_{p\in\mathbb{Z}} V^p, \mathrm{d}_V)$ and $(W, \mathrm{d}_W) = (\bigoplus_{p\in\mathbb{Z}} W^p, \mathrm{d}_W)$ is a collection of linear maps $f_p : V^p \to W^p$ such that for all $p \in \mathbb{Z}$, the squares

$$\begin{tikzcd} V^p \arrow[r, "\mathrm{d}_V"] \arrow[d, "f_p"'] & V^{p+1} \arrow[d, "f_{p+1}"] \\ W^p \arrow[r, "\mathrm{d}_W"'] & W^{p+1} \end{tikzcd} \tag{4.1.2}$$

commute.

Observe that a cochain map $f : V \to W$ induces linear maps

$$H^k(f) : H^k(V) \longrightarrow H^k(W) \;, \quad [v] \longmapsto [f(v)] \tag{4.1.3}$$

in cohomology due to (4.1.2). If $H^k(f)$ is a linear isomorphism for each $k \in \mathbb{Z}$, we say that $f : V \to W$ is a *quasi-isomorphism*.

The cochain complexes of $\mathbb{K}$-vector spaces and cochain maps form a category which we denote by $\mathsf{Ch}_{\mathbb{K}}$. The category $\mathsf{Ch}_{\mathbb{K}}$ is closed symmetric monoidal. The monoidal product of two cochain complexes $V, W \in \mathsf{Ch}_{\mathbb{K}}$ is the tensor product of cochain complexes, defined by the graded vector space

$$(V \otimes W)^n := \bigoplus_{m \in \mathbb{Z}} V^m \otimes W^{n-m} \quad , \tag{4.1.4a}$$

for all $n \in \mathbb{Z}$, together with the differential

$$\mathrm{d}_{V \otimes W}(v \otimes w) := (\mathrm{d}_V v) \otimes w + (-1)^{|v|}\, v \otimes (\mathrm{d}_W w) \quad , \tag{4.1.4b}$$

where $|v| \in \mathbb{Z}$ denotes the degree of a homogeneous element $v \in V$. The monoidal unit is given by the underlying field $\mathbb{K} \in \mathsf{Ch}_{\mathbb{K}}$, regarded as a cochain complex concentrated in degree 0. The symmetric braiding is given by the Koszul sign rule

$$\tau \,:\, V \otimes W \longrightarrow W \otimes V \,, \;\; v \otimes w \longmapsto (-1)^{|v|\,|w|}\, w \otimes v \quad . \tag{4.1.5}$$

The internal hom $\underline{\mathrm{hom}}(V, W) \in \mathsf{Ch}_{\mathbb{K}}$ between two cochain complexes $V, W \in \mathsf{Ch}_{\mathbb{K}}$ is the *mapping complex*, defined by the graded vector space

$$\underline{\mathrm{hom}}(V, W)^n := \prod_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}\big(V^m, W^{n+m}\big) \quad , \tag{4.1.6a}$$

for all $n \in \mathbb{Z}$, along with the “adjoint” differential

$$\partial(\zeta) := \mathrm{d}_W \circ \zeta - (-1)^{|\zeta|}\, \zeta \circ \mathrm{d}_V \quad . \tag{4.1.6b}$$

The vector space $\mathrm{Hom}_{\mathbb{K}}\big(V^m, W^{n+m}\big)$ is the vector space of all linear maps between V^m and W^{n+m}. The map ∂ is by construction of cohomological degree $+1$ and is nilpotent,

$$\begin{aligned} \partial(\mathrm{d}_W \circ \zeta - (-1)^{|\zeta|}\, \zeta \circ \mathrm{d}_V) &= \mathrm{d}_W \circ (\mathrm{d}_W \circ \zeta - (-1)^{|\zeta|}\, \zeta \circ \mathrm{d}_V) \\ &\qquad - (-1)^{|\zeta|+1}\, (\mathrm{d}_W \circ \zeta - (-1)^{|\zeta|}\, \zeta \circ \mathrm{d}_V) \circ \mathrm{d}_V \\ &= 0 \end{aligned} \tag{4.1.6c}$$

so it is indeed a differential. Observe that the 0-cocycles of the mapping complex, i.e. maps $f \in \underline{\mathrm{hom}}(V, W)^0 = \prod_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}\big(V^m, W^m\big)$ with $\partial(f) = \mathrm{d}_W \circ f - f \circ \mathrm{d}_V = 0$, are exactly the cochain maps $f : V \to W$. Furthermore, a *cochain homotopy* between two cochain maps $f, g : V \to W$ is a (-1)-cochain $h \in \underline{\mathrm{hom}}(V, W)^{-1} = \prod_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}\big(V^m, W^{m-1}\big)$ such that

$$f - g = \partial(h) = \mathrm{d}_W \circ h + h \circ \mathrm{d}_V \quad . \tag{4.1.7}$$

The cohomology of the mapping complex at degree 0 thus consists of the *homotopy classes* of cochain maps.

Let (V, d) be a cochain complex of finite dimensional vector spaces such that $V^n = 0$ for all $|n| > N$, for some $N \in \mathbb{N}$. We define its *dual* as the internal hom $V^\vee := \underline{\mathrm{hom}}(V, \mathbb{K})$. Explicitly, the direct summands of the underlying graded vector space is given by $\underline{\mathrm{hom}}(V, \mathbb{K})^n \cong \mathrm{Hom}_{\mathbb{K}}(V^{-n}, \mathbb{K})$. (Concretely, an element in $\underline{\mathrm{hom}}(V, \mathbb{K})^n$ is a map defined by a linear map in $\mathrm{Hom}_{\mathbb{K}}(V^{-n}, \mathbb{K})$ and the zero map in all other degrees.) The adjoint differential is given by $\partial(\zeta) = -(-1)^{|\zeta|}\, \zeta \circ \mathrm{d}$.

Definition 4.1.5. A *differential graded algebra* (dg-algebra) is a monoid in the monoidal category $\mathsf{Ch}_{\mathbb{K}}$, i.e. a cochain complex $A \in \mathsf{Ch}_{\mathbb{K}}$ together with cochain maps $\mu_A : A \otimes A \to A$ and $\eta_A : \mathbb{K} \to A$ such that the diagrams

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \mu_A"] \arrow[d, "\mu_A \otimes \mathrm{id}"'] & A \otimes A \arrow[d, "\mu_A"] \\ A \otimes A \arrow[r, "\mu_A"'] & A \end{tikzcd} \tag{4.1.8a}$$

$$\begin{tikzcd} \mathbb{K} \otimes A \arrow[r, "\eta_A \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu_A"'] & A \otimes \mathbb{K} \arrow[l, "\mathrm{id} \otimes \eta_A"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd} \tag{4.1.8b}$$

commute.

Explicitly, this means that A is a unital associative algebra such that the product is degree preserving with the unit element in degree zero and the graded Leibniz rule

$$\mathrm{d}(a\, a') = \mathrm{d}(a)\, a' + (-1)^{|a|}\, a\, \mathrm{d}(a') \tag{4.1.9}$$

is satisfied for all homogeneous elements $a, a' \in A$. In other words, $\mu_A : A \otimes A \to A$ is a cochain map (see (4.1.4)). A dg-algebra is commutative (cdg-algebra) if the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "\tau"] \arrow[dr, "\mu_A"'] & A \otimes A \arrow[d, "\mu_A"] \\ & A \end{tikzcd} \tag{4.1.10}$$

commutes, which translates to the Koszul sign rule $a\, a' = (-1)^{|a|\,|a'|}\, a'\, a$.

Example 4.1.6. Let M be a smooth d-dimensional manifold. One of the standard examples of a cochain complex is the de Rham complex of differential forms $\Omega^\bullet(M) = \bigoplus_{p=0}^{d} \Omega^p(M)$ and the exterior derivative $\mathrm{d}_{\mathrm{dR}} : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$. Furthermore, it is a commutative differential graded algebra via the wedge product. Examples of cochain maps are pullback maps $f^* : \Omega^\bullet(M') \to \Omega^\bullet(M)$ of forms along smooth maps $f : M \to M'$ between smooth manifolds M and M'. $\triangledown$

Example 4.1.7. An important example of a commutative dg-algebra is that of the *symmetric algebra* of a cochain complex (V, d). It is given by the quotient $\mathrm{Sym}\, V := T(V)/I$, where $T(V) = \bigoplus_{k\in\mathbb{N}} V^{\otimes k}$ is the tensor algebra and I the ideal generated by $v \otimes v' - \tau(v \otimes v')$, where τ is the symmetric braiding of $\mathsf{Ch}_{\mathbb{K}}$ (4.1.5). We will often denote the product in $\mathrm{Sym}\, V$ simply by concatenation. The differential is simply the one induced by (4.1.4b) and linear extension. Note that it is well defined with respect to the quotient.

Because our ground field $\mathbb{K}$ is of characteristic 0, the cohomology of the symmetric algebra $\mathrm{Sym}\, V$ can be written in terms of the cohomology of V, i.e.

$$H^\bullet(\mathrm{Sym}\, V) \cong \mathrm{Sym}\, H^\bullet(V) \quad . \tag{4.1.11}$$

This can be proven using standard arguments, which we sketch here for completeness sake. First, observe that one can identify the symmetric algebra with

$$\mathrm{Sym}\, V = \bigoplus_{k\in\mathbb{N}} (V^{\otimes k})_{S_k} \tag{4.1.12}$$

where $(V^{\otimes k})_{S_k}$ are the *coinvariants* of $V^{\otimes k}$ with respect to the action of the symmetric group S_k. A permutation $\sigma \in S_k$ acts on $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ by permuting the factors v_i according to σ using the symmetric braiding (4.1.5). We denote this action by $\tau^\sigma : V^{\otimes k} \to V^{\otimes k}$. The coinvariants are simply the quotient $(V^{\otimes k})_{S_k} := V^{\otimes k}/W_{S_k}$, where W_{S_k} is the smallest subspace containing all elements of the form $v_1 \otimes \cdots \otimes v_k - \tau^\sigma(v_1 \otimes \cdots \otimes v_k)$ with $v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$ and $\sigma \in S_k$. The fact that the underlying field $\mathbb{K}$ is of characteristic 0 is an integral part of the proof because it allows for a further identification

$$\mathrm{Sym}\, V \cong \bigoplus_{k\in\mathbb{N}} (V^{\otimes k})^{S_k} \tag{4.1.13}$$

with the *invariants* of the symmetric group action, i.e. $(V^{\otimes k})^{S_k} := \{v \in V^{\otimes k} : \tau^\sigma(v) = v \text{ for all } \sigma \in S_k\}$. Let us explain this fact for general group actions with finite groups G on some (graded) vector space W. There is a canonical linear map from the invariants to the coinvariants $W^G \to W_G$ defined by $w \mapsto [w]$.[1] The field $\mathbb{K}$ being of characteristic 0 allows for the formation of group averages. This leads to the linear map $W_G \to W^G$ given by $[w] \mapsto \frac{1}{|G|} \sum_{g\in G} g \triangleright w$, where $|G|$ denotes the order of the group (division by

1 To clarify, $W^G := \{w \in W : w = g \triangleright w \text{ for all } g \in G\}$ and $W_G := W/\langle w - g \triangleright w\rangle$, where $\langle w - g \triangleright w\rangle$ is the smallest subspace containing all elements of the form $w - g \triangleright w$ with $w \in W$ and $g \in G$.

$|G|$ is permitted because $\mathbb{K}$ is of characteristic 0). This map is well-defined because $w - g' \triangleright w$ is sent to 0, and is an inverse to the former map. Hence $W^G \cong W_G$ so (4.1.13) is justified. We now have

$$H^\bullet(\operatorname{Sym} V) \cong H^\bullet\Big(\bigoplus_{k\in\mathbb{N}}(V^{\otimes k})^{S_k}\Big) \cong \bigoplus_{k\in\mathbb{N}} H^\bullet((V^{\otimes k})^{S_k}) \cong \bigoplus_{k\in\mathbb{N}} \big(H^\bullet(V^{\otimes k})\big)^{S_k}$$
$$\cong \bigoplus_{k\in\mathbb{N}} \big((H^\bullet(V))^{\otimes k}\big)^{S_k} \cong \operatorname{Sym} H^\bullet(V) \quad . \tag{4.1.14}$$

In equality two, one uses that cohomology commutes with direct sums (see e.g. [Wei94, Exercise 1.2.1]), in the third equality that cohomology commutes with taking invariants in the case when the ground field is of characteristic 0 and the acting group is finite[2], and finally in equality four the Künneth theorem (see e.g. [Wei94, Theorem 3.6.3]). $\triangledown$

Example 4.1.8. The *exterior algebra* of V is related to the symmetric algebra and is simply given by $\bigwedge^\bullet V := \operatorname{Sym} V[-1]$. $\triangledown$

Example 4.1.9. There is an intrinsic notion of cohomology related to Lie algebras, which can be computed as the cohomology of the so called *Chevalley-Eilenberg cochain complex*, see e.g. [Wei94, Corollary 7.7.3]. Let $\mathfrak{g}$ be a Lie algebra and M be a left $\mathfrak{g}$-module[3]. Then the Chevalley-Eilenberg cochain complex with coefficients in M is given by

$$\mathrm{CE}^\bullet(\mathfrak{g}, M) := \operatorname{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes \textstyle\bigwedge^\bullet \mathfrak{g}, M) \cong \operatorname{Hom}_{\mathbb{K}}(\textstyle\bigwedge^\bullet \mathfrak{g}, M) \tag{4.1.15}$$

together with the differential

$$\begin{aligned}\mathrm{d}_{\mathrm{CE}}\phi(X_1,\dots,X_{n+1}) &= \sum_{i=1}^{n+1}(-1)^{i+1}\, X_i \triangleright \phi(X_1,\dots,\hat{X}_i,\dots,X_{n+1}) \\ &\quad + \sum_{1\le i<j\le n+1}(-1)^{i+j}\,\phi([X_i,X_j],X_1,\dots\hat{X}_i,\dots,\hat{X}_j,\dots,X_{n+1})\end{aligned} \tag{4.1.16}$$

2 Taking invariants $(\,\cdot\,)^G$ defines a right adjoint functor, which implies that it preserves limits. Analogously, taking coinvariants $(\,\cdot\,)_G$ is a left adjoint functor which implies that it preserves colimits. (See e.g. [Wei94, Exercise 6.1.1].) Hence, $(\,\cdot\,)^G \cong (\,\cdot\,)_G$ (see below (4.1.13)) preserves both limits and colimits, and therefore cohomologies; kernels, images and quotients are all special instances of limits and colimits.

3 A left $\mathfrak{g}$-module M is a $\mathbb{K}$-vector space M together with a left action $\triangleright : \mathfrak{g} \otimes M \to M$ such that $[x,y] \triangleright m = x \triangleright (x' \triangleright m) - x' \triangleright (x \triangleright m)$ for all $x, x' \in \mathfrak{g}$ and $m \in M$

for all $\phi \in \mathrm{CE}^\bullet(\mathfrak{g}, M)$ and $X_i \in \mathfrak{g}$. As usual, the hat $\hat{\ }$ means omission. Note that in the case when $\mathfrak{g}$ is finite dimensional, $\mathrm{CE}^\bullet(\mathfrak{g}, M) \cong M \otimes \bigwedge^\bullet \mathfrak{g}^\vee$. We will in the following assume finite dimensionality of $\mathfrak{g}$.

Let us consider the case when $M = A$ is an algebra and the left action acts as a derivation, i.e. $X \triangleright (a\, a') = (X \triangleright a)\, a' + a\, (X \triangleright a')$ for all $a, a' \in A$. Then, we may define an algebra structure on $\mathrm{CE}^\bullet(\mathfrak{g}, A) \cong A \otimes \bigwedge^\bullet \mathfrak{g}^\vee$ given by $(a \otimes \alpha) \wedge (a' \otimes \alpha') = a\, a' \otimes \alpha \wedge \alpha'$ for all $a, a' \in A$ and $\alpha, \alpha' \in \bigwedge^\bullet \mathfrak{g}^\vee$. If $\phi \in \mathrm{CE}^p(\mathfrak{g}, A)$ and $\phi' \in \mathrm{CE}^q(\mathfrak{g}, A)$, the action of $\phi \wedge \phi'$ on $X_1, \dots, X_{p+q} \in \mathfrak{g}$ is given by

$$(\phi \wedge \phi')(X_1, \dots, X_{p+q}) = \frac{1}{p!\, q!} \sum_{\sigma \in S_{p+q}} \mathrm{sign}(\sigma) \phi(X_{\sigma(1)}, \dots, X_{\sigma(p)}) \times \phi'(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}) \quad . \tag{4.1.17}$$

Since the left action acts as a derivation, one may check that the differential satisfies the graded Leibniz rule

$$\mathrm{d}_{\mathrm{CE}}(\phi \wedge \phi') \;=\; (\mathrm{d}_{\mathrm{CE}}\phi) \wedge \phi' + (-1)^p\, \phi \wedge (\mathrm{d}_{\mathrm{CE}}\phi') \quad , \tag{4.1.18}$$

where $\phi \in A \otimes \bigwedge^p \mathfrak{g}^\vee$ and $\phi' \in A \otimes \bigwedge^\bullet \mathfrak{g}^\vee$, making $\mathrm{CE}^\bullet(\mathfrak{g}, A)$ a dg-algebra. In such case, we call $\mathrm{CE}^\bullet(\mathfrak{g}, A)$ the *Chevalley-Eilenberg dg-algebra* of $\mathfrak{g}$ with coefficients in A. Given a basis $\{t_i\}_{i=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$ with dual basis $\{\theta^i\}_{i=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}^\vee$, the Chevalley-Eilenberg differential (4.1.16) is explicitly given by

$$\mathrm{d}_{\mathrm{CE}} a \;=\; (t_i \triangleright a) \otimes \theta^i \quad , \qquad \mathrm{d}_{\mathrm{CE}} \theta^i \;=\; -\frac{1}{2}\, f^i_{jk}\, \theta^j \wedge \theta^k \quad , \tag{4.1.19}$$

for all $a \in A$, where f^i_{jk} are the structure constants of $\mathfrak{g}$ (summation is understood over repeated indices). ▽

Definition 4.1.10. A *differential graded Lie algebra* (dg-Lie algebra) $(L, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ consists of a cochain complex (L, d) and a bilinear map $[\,\cdot\,,\,\cdot\,] : L \otimes L \to L$ of degree 0 (called *Lie bracket*) satisfying

(i) graded antisymmetry: $[v, v'] = (-1)^{|v|\,|v'|+1}\, [v', v]$

(ii) graded Jacobi identity:
$(-1)^{|v|\,|v''|}\, [v, [v', v'']] + (-1)^{|v'|\,|v|}\, [v', [v'', v]] + (-1)^{|v''|\,|v'|}\, [v'', [v, v']] = 0$

(iii) graded Leibniz rule: $\mathrm{d}[v, v'] = [\mathrm{d}v, v'] + (-1)^{|v|}\, [v, \mathrm{d}v']$

for all homogeneous $v, v', v'' \in L$.

The condition that the Lie bracket should satisfy the graded Leibniz rule simply translates to the requirement that the Lie bracket $[\,\cdot\,,\,\cdot\,] : L \otimes L \to L$ should be a cochain map (see (4.1.4)). Note that a differential graded Lie algebra is in general not a differential graded algebra because the associativity condition (4.1.8a) is generically not satisfied by the Lie bracket. A differential graded Lie algebra concentrated in degree 0 is nothing but an ordinary Lie algebra.

4.2 CYCLIC L_∞-ALGEBRAS

L_∞-algebras are higher generalisations of Lie algebras where the Jacobi identity is no longer satisfied on the nose. Instead, the Jacobi identity only holds up to homotopy, encoded by a tower of n-ary maps known as the higher Lie brackets. These in turn satisfy a higher generalisation of the Jacobi identity. Furthermore, if the L_∞-algebra is equipped with an appropriate notion of inner product, a so called cyclic structure, we obtain the concept of a cyclic L_∞-algebra. There are several approaches to give the definition of an L_∞-algebra. For instance, they emerge naturally through the concept of *homotopy transfer*, see e.g. [Val14] for a review in the context of A_∞-algebras (which translates to the case of L_∞-algebras). There is also a formulation in terms of coalgebras, see e.g. [JRSW19, Appendices A]. We open this section by providing the explicit version. For references, see e.g. [Sta92, LS93, LM94], see also [JRSW19, BKJMSW21, NSS21].

Definition 4.2.1. An *L_∞-algebra* (or *strong homotopy Lie algebra*) is a $\mathbb{Z}$-graded vector space L together with a collection $\{\ell_n : L^{\otimes n} \to L\}_{n\in\mathbb{Z}_{\geq 1}}$ of graded antisymmetric linear maps of degree $|\ell_n| = 2-n$ that satisfy the homotopy Jacobi identities

$$\sum_{k=0}^{n-1} (-1)^k \, \ell_{k+1} \circ \left(\ell_{n-k} \otimes \mathrm{id}_{L^{\otimes k}}\right) \circ \sum_{\sigma\in \mathrm{Sh}(n-k;k)} \mathrm{sgn}(\sigma)\, \tau^\sigma \;=\; 0 \quad , \tag{4.2.1}$$

for all $n \geq 1$, where $\mathrm{Sh}(n-k;k) \subset S_n$ denotes the set of $(n-k;k)$-shuffled permutations on n letters and $\tau^\sigma : L^{\otimes n} \to L^{\otimes n}$ denotes the action of the permutation σ via the symmetric braiding on the category of graded vector spaces. We refer to the maps ℓ_n as *higher* (*Lie*) *brackets*.

To clarify, a $(n-k;k)$-*shuffled permutation* $\sigma \in \mathrm{Sh}(n-k;k) \subset S_n$ is a permutation on n letters $(1,2,\dots,n)$ such that $\sigma(1) < \cdots < \sigma(n-k)$ and $\sigma(n-k+1) < \cdots < \sigma(n)$. (Note that the terminology *unshuffle* is used for this concept in other sources.)

To get a feeling of what the higher brackets encode, we write out (4.2.1) explicitly for the first three n. The identity for $n = 1$ simply states that the unary bracket $\ell_1 : L \to L[1]$ is nilpotent, $\ell_1 \circ \ell_1 = 0$. In other words, every L_∞-algebra has an underlying cochain complex $(L, \mathrm{d}_L := \ell_1)$. For $n = 2$, denoting the binary bracket by $[\,\cdot\,,\,\cdot\,] := \ell_2 : L \otimes L \to L$, the homotopy Jacobi identity takes the form

$$\mathrm{d}_L[v_1, v_2] = [\mathrm{d}_L(v_1), v_2] + (-1)^{|v_1|}\, [v_1, \mathrm{d}_L(v_2)] \tag{4.2.2}$$

for all homogeneous $v_1, v_2 \in L$. That is, $\mathrm{d}_L = \ell_1$ is a graded derivation with respect to the binary bracket $[\,\cdot\,,\,\cdot\,] = \ell_2$. Finally, for $n = 3$, the homotopy Jacobi identity will also include the ternary bracket $h := \ell_3 : L \otimes L \otimes L \to L[-1]$ and we obtain

$$\begin{aligned}
&[v_1, [v_2, v_3]] + (-1)^{|v_1|(|v_2|+|v_3|)}\, [v_2, [v_3, v_1]] + (-1)^{|v_3|(|v_1|+|v_2|)}\, [v_3, [v_1, v_2]] \\
&\quad = -\Big(\mathrm{d}_L h(v_1 \otimes v_2 \otimes v_3) + h(\mathrm{d}_L(v_1) \otimes v_2 \otimes v_3) \\
&\qquad\qquad + (-1)^{|v_1|}\, h(v_1 \otimes \mathrm{d}_L(v_2) \otimes v_3) + (-1)^{|v_1|+|v_2|} h(v_1 \otimes v_2 \otimes \mathrm{d}_L(v_3))\Big) \\
&\quad = -(\mathrm{d}_L \circ h + h \circ \mathrm{d}_{L\otimes L\otimes L})(v_1 \otimes v_2 \otimes v_3)
\end{aligned} \tag{4.2.3}$$

where $\mathrm{d}_{L^{\otimes k}} : L^{\otimes k} \to (L^{\otimes k})[1]$ is the induced differential on $L^{\otimes k}$ from $\mathrm{d}_L = \ell_1$ (see (4.1.4)). We thus see that the Jacobi identity for $[\,\cdot\,,\,\cdot\,] = \ell_2$ is satisfied up to the cochain homotopy $h = \ell_3$ (see around Equation (4.1.7)). It is now evident that a dg-Lie algebra is nothing but an L_∞-algebra where $\ell_n = 0$ for all $n \geq 3$. The pattern continues for higher n: the brackets ℓ_k with $k < n$ will satisfy some relation up to higher homotopy given by ℓ_n. In fact, for a fixed n, the terms in the homotopy Jacobi identity (4.2.1) containing ℓ_n can collectively be written as an exact element in the mapping complex $\underline{\mathrm{hom}}(L^{\otimes n}, L) \in \mathsf{Ch}_{\mathbb{K}}$ (4.1.6),

$$\begin{aligned}
&\mathrm{d}_L \ell_n(v_1 \otimes \cdots \otimes v_n) + (-1)^{n-1} \Big(\ell_n(\mathrm{d}_L(v_1) \otimes \cdots \otimes v_n) + \cdots \\
&\qquad\qquad\qquad + (-1)^{|v_1|+\cdots+|v_{n-1}|}\, \ell_n(v_1 \otimes \cdots \otimes \mathrm{d}_L(v_n))\Big) \\
&\quad = (\mathrm{d}_L \circ \ell_n - (-1)^{|\ell_n|}\, \ell_n \circ \mathrm{d}_{L^{\otimes n}}) \\
&\quad = \partial(\ell_n) \quad .
\end{aligned} \tag{4.2.4}$$

Definition 4.2.2. A *cyclic L_∞-algebra* is an L_∞-algebra $(L, \{\ell_n\})$ together with a nondegenerate symmetric cochain map $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle : L \otimes L \to \mathbb{K}[k]$ (called *cyclic structure*) that satisfies the cyclicity condition

$$\langle\!\langle v_0, \ell_n(v_1, \dots, v_n)\rangle\!\rangle \;=\; (-1)^{n+n\,(|v_0|+|v_n|)+|v_n|\,\sum_{i=0}^{n-1}|v_i|}\, \langle\!\langle v_n, \ell_n(v_0, \dots, v_{n-1})\rangle\!\rangle \quad , \tag{4.2.5}$$

for all $n \geq 1$ and all homogeneous elements $v_0, v_1, \dots, v_n \in L$.

Remark 4.2.3. By symmetric, we mean symmetric with respect to the braiding τ in $\mathsf{Ch}_{\mathbb{K}}$ (4.1.5), i.e. $\langle\!\langle \cdot , \cdot \rangle\!\rangle = \langle\!\langle \cdot , \cdot \rangle\!\rangle \circ \tau$.

Note that the cyclicity condition for the cyclic structure $\langle\!\langle \cdot , \cdot \rangle\!\rangle : L \otimes L \to \mathbb{K}[k]$ in the case when k is odd simplifies to

$$\langle\!\langle v_0, \ell_n(v_1, \dots, v_n) \rangle\!\rangle \;=\; (-1)^{n\,(|v_0|+1)}\, \langle\!\langle v_n, \ell_n(v_0, \dots, v_{n-1}) \rangle\!\rangle \quad , \tag{4.2.6}$$

for all $n \geq 1$ and all homogeneous elements $v_0, v_1, \dots, v_n \in L$. This is the case for the BV formalism, where $k = -3$ (see Section 4.3.2). △

It turns out that there is a natural L_∞-algebra structure on the tensor product of an L_∞-algebra $(L, \{\ell_n\})$ and a commutative dg-algebra (4.1.10) (A, d). The underlying vector space is given by (see (4.1.4))

$$L_A := A \otimes L \quad . \tag{4.2.7}$$

The extended higher brackets $\ell_n^{\mathrm{ext}} : L_A^{\otimes n} \to L_A[2-n]$ are for $n = 1$ given by

$$\ell_1^{\mathrm{ext}}(a_1 \otimes v_1) := \mathrm{d}a_1 \otimes v_1 + (-1)^{|a_1|}\, a_1 \otimes \ell_1(v_1) \tag{4.2.8a}$$

and for $n \geq 2$ by

$$\ell_n^{\mathrm{ext}}(a_1 \otimes v_1, \dots, a_n \otimes v_n) := (-1)^{n\, \sum_{i=1}^n |a_i| + \sum_{j=2}^n \left(|a_j|\, \sum_{l=1}^{j-1} |v_l|\right)}\, (a_1 \cdots a_n) \otimes \ell_n(v_1, \dots, v_n) \quad , \tag{4.2.8b}$$

for all homogeneous $a_1, \dots, a_n \in A$ and homogeneous $v_1, \dots, v_n \in L$, and linear extension. As usual, $a_1 \cdots a_n$ denotes the (commutative) associative product of elements $a_1, \dots, a_n \in A$. Furthermore, for completeness sake, let us also mention that if L is equipped with a cyclic structure $\langle\!\langle \cdot , \cdot \rangle\!\rangle$ and A with an inner product $\langle \cdot , \cdot \rangle$ there is a natural cyclic structure on L_A given by

$$\langle\!\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle\!\rangle_{L_A} := (-1)^{|a_2|\,|v_1|} \langle a_1, a_2 \rangle\, \langle\!\langle v_1, v_2 \rangle\!\rangle \quad . \tag{4.2.9}$$

For more details, see e.g. [JRSW19].

4.3 BATALIN-VILKOVISKY FORMALISM IN FINITE DIMENSIONS

Quantum field theory, in the Lagrangian viewpoint, is described by a space of fields $\mathcal{M}$ together with an action functional $S : \mathcal{M} \to \mathbb{R}$. The expectation values of observables $\mathcal{O} : M \to \mathbb{C}$ are often calculated through the (typically ill-defined) Feynman path integral[4]

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{\mathcal{M}} \mathcal{O}(\phi)\, e^{-S[\phi]}\, \mathrm{d}\phi \tag{4.3.1}$$

with the partition function $Z = \int_{\mathcal{M}} e^{-S[\phi]}\, \mathrm{d}\phi$. The dominating contributions to the expectation values are dictated by the classical physics, i.e. the critical locus of S, which consists of the fields $\phi \in \mathcal{M}$ that are solutions to the Euler-Lagrange equations.

In many cases, the physical system comes with a gauge symmetry, i.e. the action functional S is invariant under the action of some Lie group G. This leads to redundancies in the path integral and one would like to calculate it only up to this gauge symmetry. However, one cannot simply take the naive quotient $\mathcal{M}/G$ and integrate over it. Indeed, the orbit space $\mathcal{M}/G$ of a manifold $\mathcal{M}$ with a non-free action of G is no longer a manifold as it contains singularities. The *Batalin-Vilkovisky* (*BV*) *formalism* [BV81, BV83, BV85, Sch93] offers a solution to this problem in the infinitesimal setting, i.e. for infinitesimal gauge transformations where the symmetry is encoded by the Lie algebra $\mathfrak{g}$ of G. A study of the case of gauge transformations from group actions (as opposed to Lie algebra actions) in the finite dimensional setting was conducted in [BSS21]. The BV formalism also sets up for a homological approach to integration.

We will adopt the modern approach to the BV formalism of Costello and Gwilliam [Gwi12, CG16, CG21]. The focus will be on the finite dimensional setting since the systems we apply these techniques to will be of such nature; our space of fields will generically comprise a finite dimensional noncommutative algebra. This allows us to work in a purely algebraic framework and hence avoid functional analytical subtleties otherwise found in continuum field theories. These issues are addressed in [Gwi12, CG16, CG21]. There is also a useful earlier exposition of BV quantisation in finite dimensions in [GJF18]. For an account of perturbative quantum field theories, the BV formalism and homotopy algebras, complete with an interpretation of various objects introduced in the following subsections, see [BKJMSW21, Section 4].

4 In this thesis, we consider Euclidean field theories.

4.3.1 *Finite-dimensional BV formalism*

We begin this subsection by recalling the definition of a classical free BV theory from [Gwi12, CG16, CG21], see also [NSS21, Section 2]. As no assumptions on commutativity on the space of fields are made (the algebra structure on the space of fields is not present in the construction), the BV formalism as presented here will apply also to finite dimensional noncommutative field theories, i.e matrix models or so called *fuzzy field theories*.

Definition 4.3.1. A *free BV theory* is a cochain complex $E \in \mathsf{Ch}_{\mathbb{K}}$, with differential denoted by $\mathrm{d}_E = -Q$,[5] together with a cochain map $\langle\,\cdot\,,\,\cdot\,\rangle : E \otimes E \to \mathbb{K}[-1]$ that is nondegenerate and antisymmetric, i.e. $\langle\,\cdot\,,\,\cdot\,\rangle = -\langle\,\cdot\,,\,\cdot\,\rangle \circ \tau$ where τ denotes the symmetric braiding (4.1.5) on $\mathsf{Ch}_{\mathbb{K}}$.

Remark 4.3.2. The cochain complex $E = (E, -Q)$ should be interpreted as a *derived* solution space (*derived critical locus*) of some action. In physics language, the elements in degree 0 are the *fields* of the theory, while the negative degrees account for the *ghost fields* and the elements in the positive degrees encode the *antifields*. The pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ plays the role of a (-1)-shifted symplectic structure.

Since we are working with finite dimensional systems, we will implicitly assume that each homogeneous component E^n of the cochain complex E is a finite dimensional vector space and that the complex itself is bounded both from above and below, i.e. there exists a positive integer $N \in \mathbb{Z}_{>0}$ such that $E^n = 0$ for both $n > N$ and $n < -N$. Hence, E is a perfect complex and therefore dualisable. △

The next step is to assign a commutative dg-algebra of polynomial observables to E, i.e. polynomial functions which take elements in E and return an element in $\mathbb{K}$. These will play the role of classical observables. This can be done for every free BV theory $(E, -Q, \langle\,\cdot\,,\,\cdot\,\rangle)$. Observe first that due to the nondegenerate pairing $\langle\,\cdot\,,\,\cdot\,\rangle : E \otimes E \to \mathbb{K}[-1]$, the dual (see Section 4.1) to the complex E is $E^\vee \cong E[1]$ with duality pairing given by

$$E[1] \otimes E \;\cong\; \mathbb{K}[1] \otimes E \otimes E \xrightarrow{\mathrm{id}\otimes\langle\,\cdot\,,\,\cdot\,\rangle} \mathbb{K}[1] \otimes \mathbb{K}[-1] \;\cong\; \mathbb{K} \quad . \tag{4.3.2}$$

5 We have included a minus sign in the definition of Q in order to avoid unpleasant sign factors in the dual differential on the observables, which we shall use more frequently.

(Note that the map $E[1] \to \underline{\mathrm{hom}}(V, \mathbb{K})$ given by $v \mapsto \langle v, \cdot \rangle$ really is an isomorphism of cochain complexes because the nondegenerate pairing $\langle \cdot, \cdot \rangle$ is a cochain map, which translates to the compatibility condition $\langle Qv, w\rangle + (-1)^{|v|} \langle v, Qw\rangle = 0$.) The differential on $E[1]$ acquires an extra sign

$$\mathrm{d}_{E[1]} \;=\; -\mathrm{d}_E \;=\; Q \tag{4.3.3}$$

due to the degree shift, as mentioned in Section 4.1. The polynomial observables are defined as $\mathrm{Sym}\, E^\vee \cong \mathrm{Sym}\, E[1] \in \mathsf{CAlg}(\mathsf{Ch}_\mathbb{K})$. Abusing the notation, the differential on the polynomial observables is denoted by the same symbol Q as the differential on $E[1]$ as in Example 4.1.7. Additionally, the pairing $\langle \cdot, \cdot \rangle$ induces a shifted Poisson bracket on $\mathrm{Sym}\, E[1]$, forming a so called *P_0-algebra*. We refer to [Saf17] for more details on shifted Poisson structures.

Definition 4.3.3. A *P_n-algebra* is a commutative dg-algebra $A \in \mathsf{CAlg}(\mathsf{Ch}_\mathbb{K})$ together with a Lie bracket $\{\cdot, \cdot\}_n : A[n-1] \otimes A[n-1] \to A[n-1]$ on the shifted cochain complex $A[n-1]$ such that $\{a, \cdot\}$ defines a derivation on A for each $a \in A$, i.e.

$$\{a, b\, c\} \;=\; \{a, b\}\, c + (-1)^{(|a|+1-n)\,|b|}\, b\, \{a, c\} \tag{4.3.4}$$

for all $a, b, c \in A$.

In this thesis, we will exclusively work with P_0-algebras. The Lie bracket $\{\cdot, \cdot\} := \{\cdot, \cdot\}_0$ is called *shifted Poisson bracket* or *antibracket*.

Remark 4.3.4. Let us explicitly spell out the properties of the antibracket.

(i) Graded antisymmetry: for all $a, b \in A$

$$\{a, b\} \;=\; -(-1)^{(|a|+1)\,(|b|+1)}\, \{b, a\} \quad . \tag{4.3.5}$$

(ii) Graded Jacobi identity: for all $a, b, c \in A$

$$\begin{aligned} 0 \;=\; &(-1)^{(|a|+1)\,(|c|+1)}\, \{a, \{b, c\}\} \\ &+ (-1)^{(|b|+1)\,(|a|+1)}\, \{b, \{c, a\}\} \\ &+ (-1)^{(|c|+1)\,(|b|+1)}\, \{c, \{a, b\}\} \quad . \end{aligned} \tag{4.3.6}$$

(iii) Derivation property: for all $a, b, c \in A$

$$\{a, b\, c\} \;=\; \{a, b\}\, c + (-1)^{(|a|+1)\,|b|}\, b\, \{a, c\} \quad . \tag{4.3.7}$$

(iv) Compatibility with the differential: for all $a, b \in A$

$$\mathrm{d}\{a,b\} \;=\; \{\mathrm{d}a, b\} + (-1)^{|a|+1}\,\{a, \mathrm{d}b\} \quad . \tag{4.3.8}$$

The compatibility condition with the differential is nothing but the statement that the antibracket is a cochain map $A[-1] \otimes A[-1] \to A[-1]$. Note that a different convention was used in [NSS21]. △

Remark 4.3.5. A concept related to P_0-algebras is that of *Gerstenhaber algebras*. In fact, Gerstenhaber algebras are also examples of shifted Poisson algebras, namely P_2-*algebras*, where the corresponding Lie bracket is $\{\,\cdot\,,\,\cdot\,\}_2 : A[1] \otimes A[1] \to A[1]$. △

In the following, we will describe how the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : E \otimes E \to \mathbb{K}[-1]$ gives rise to an antibracket on $\mathrm{Sym}\,E[1]$. First, observe the trivial fact that $E \cong (E[1])[-1]$. Then the pairing $\langle\,\cdot\,,\,\cdot\,\rangle : (E[1])[-1] \otimes (E[1])[-1] \to \mathbb{K}[-1]$ satisfies the graded anti-symmetry property $\{\varphi,\psi\} \;=\; -(-1)^{(|\varphi|+1)\,(|\psi|+1)}\,\{\psi,\varphi\}$ for all $\varphi,\psi \in E[1]$. Next, we define the map $\{\,\cdot\,,\,\cdot\,\} : (\mathrm{Sym}\,E[1])[-1] \otimes (\mathrm{Sym}\,E[1])[-1] \to (\mathrm{Sym}\,E[1])[-1]$. This is accomplished by setting for all $\varphi,\psi \in E[1]$

$$\{\varphi,\psi\} \;:=\; \langle\varphi,\psi\rangle\,\mathbb{1} \quad , \tag{4.3.9}$$

where $\mathbb{1} \in \mathrm{Sym}\,E[1]$ denotes the unit element, and extension to the entirety of $\mathrm{Sym}\,E[1]$ by the derivation property (4.3.7), together with the requirement that the antisymmetry property (4.3.5) should be satisfied. Due to the shifted Poisson structure being constant, the graded Jacobi identity (4.3.6) holds trivially. The compatibility condition (4.3.8) follows from the fact that the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ is a cochain map. Thus, we have a P_0-algebra

$$\mathrm{Obs}^{\mathrm{cl}} \;:=\; \big(\mathrm{Sym}\,E[1], Q, \{\,\cdot\,,\,\cdot\,\}\big) \quad , \tag{4.3.10}$$

interpreted as the classical observables of the free BV theory $(E, -Q, \langle\,\cdot\,,\,\cdot\,\rangle)$.

So far, we have only described free field theories. Interactions, as well as quantisation, are treated as certain deformations of the classical observables $\mathrm{Obs}^{\mathrm{cl}}$. We will begin by outlining how to incorporate interactions into the picture before moving on to quantisation. To this end, let λ be a formal parameter interpreted as a coupling constant. We consider the formal power series extension of $\mathrm{Obs}^{\mathrm{cl}}$, which by abuse of notation will be denoted with the same symbol. For any 0-cochain $I \in (\mathrm{Sym}\,E[1])^0$, interpreted as an interaction term in the classical BV action, the deformed differential on $\mathrm{Obs}^{\mathrm{cl}}$ is defined as

$$Q^{\mathrm{int}} \;:=\; Q + \{\lambda\, I, \,\cdot\,\} \tag{4.3.11}$$

such that the nilpotency condition $(Q^{\mathrm{int}})^2 = 0$ is satisfied. Using properties of the antibracket from Remark 4.3.4 together with nondegeneracy of the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$, the nilpotency condition can be shown to be equivalent to the *classical master equation*

$$Q(\lambda\, I) + \tfrac{1}{2}\,\{\lambda\, I, \lambda\, I\} \;=\; 0 \quad . \tag{4.3.12}$$

Thus, the classical observables for the interacting BV theory, with interaction term $I \in (\mathrm{Sym}\, E[1])^0$ satisfying the classical master equation (4.3.12), is the P_0-algebra

$$\mathrm{Obs}^{\mathrm{int,cl}} \;:=\; \big(\mathrm{Sym}\, E[1], Q^{\mathrm{int}}, \{\,\cdot\,,\,\cdot\,\}\big) \quad . \tag{4.3.13}$$

Next, we treat quantisation of free BV theories. Let $\hbar$ be another formal parameter, interpreted as Planck's constant. Similarly, we consider the formal power series extension of $\mathrm{Obs}^{\mathrm{cl}}$ in $\hbar$ and again denote it by the same symbol. The differential on $\mathrm{Obs}^{\mathrm{cl}}$ is deformed along the *BV Laplacian*

$$Q^{\hbar} \;:=\; Q + \hbar\, \Delta_{\mathrm{BV}} \quad . \tag{4.3.14}$$

The BV Laplacian $\Delta_{\mathrm{BV}} : \mathrm{Sym}\, E[1] \to (\mathrm{Sym}\, E[1])[1]$ is the cochain map defined on symmetric powers 0, 1 and 2 by

$$\Delta_{\mathrm{BV}}(\mathbb{1}) \;:=\; 0 \quad , \quad \Delta_{\mathrm{BV}}(\varphi) \;:=\; 0 \quad , \quad \Delta_{\mathrm{BV}}(\varphi\, \psi) \;:=\; (-1)^{|\varphi|}\, \{\varphi, \psi\} = (-1)^{|\varphi|}\, \langle \varphi, \psi\rangle \quad , \tag{4.3.15a}$$

for all generators $\varphi, \psi \in E[1]$ and extended to all of $\mathrm{Sym}\, E[1]$ by

$$\Delta_{\mathrm{BV}}(a\, b) \;=\; \Delta_{\mathrm{BV}}(a)\, b + (-1)^{|a|}\, a\, \Delta_{\mathrm{BV}}(b) + (-1)^{|a|}\, \{a, b\} \quad , \tag{4.3.15b}$$

for all $a, b \in \mathrm{Sym}\, E[1]$. Using the properties of the antibracket in Remark 4.3.4, we derive the explicit expression for the BV Laplacian

$$\begin{aligned} \Delta_{\mathrm{BV}}\big(\varphi_1 \cdots \varphi_n\big) \;&=\; \sum_{i<j} (-1)^{\sum_{k=1}^{i} |\varphi_k| + |\varphi_j|\, \sum_{k=i+1}^{j-1} |\varphi_k|}\, \{\varphi_i, \varphi_j\}\, \varphi_1 \cdots \widehat{\varphi}_i \cdots \widehat{\varphi}_j \cdots \varphi_n \\ &=\; \sum_{i<j} (-1)^{\sum_{k=1}^{i} |\varphi_k| + |\varphi_j|\, \sum_{k=i+1}^{j-1} |\varphi_k|}\, \langle\varphi_i, \varphi_j\rangle\, \varphi_1 \cdots \widehat{\varphi}_i \cdots \widehat{\varphi}_j \cdots \varphi_n \\ &\qquad \in \mathrm{Sym}\, E[1] \end{aligned} \tag{4.3.15c}$$

for all $\varphi_1, \dots, \varphi_n \in E[1]$ with $n \geq 2$, where the hat $\widehat{\cdot}$ signifies omission of the corresponding factor.[6] Using this explicit formula for the BV Laplacian, it is straightforward to show that

$$(\Delta_{\mathrm{BV}})^2 \,=\, 0 \quad , \quad Q\,\Delta_{\mathrm{BV}} + \Delta_{\mathrm{BV}}\,Q \,=\, 0 \quad . \tag{4.3.16}$$

This implies that the deformed differential $Q^\hbar$ (4.3.14) is indeed a differential, i.e. $(Q^\hbar)^2 = 0$. The resulting deformed cochain complex

$$\mathrm{Obs}^\hbar \,:=\, \big(\mathrm{Sym}\,E[1], Q^\hbar\big) \tag{4.3.17}$$

is interpreted as the quantum observables for the free BV theory. Note however that the quantum observables $\mathrm{Obs}^\hbar$ do *not* form a dg-algebra because the deformed differential $Q^\hbar$ does *not* respect the multiplication on $\mathrm{Sym}\,E[1]$. Instead, $\mathrm{Obs}^\hbar$ forms a so called E_0*-algebra*, i.e. a cochain complex with a distinguished 0-cocycle, which in this case is the unit element $\mathbb{1} \in \mathrm{Sym}\,E[1]$.

Finally, we shall treat interacting quantum BV theories. These are obtained by combining the two types of deformations. We define a deformed differential

$$Q^{\mathrm{int},\hbar} \,:=\, Q + \{\lambda\, I, \,\cdot\,\} + \hbar\,\Delta_{\mathrm{BV}} \quad . \tag{4.3.18}$$

In this case, the nilpotency condition $(Q^{\hbar,\mathrm{int}})^2 = 0$ is equivalent to the *quantum master equation*

$$Q(\lambda\, I) + \tfrac{1}{2}\,\{\lambda\, I, \lambda\, I\} + \hbar\,\Delta_{\mathrm{BV}}(\lambda\, I) \,=\, 0 \quad . \tag{4.3.19}$$

This can be shown by using (4.3.16) in conjunction with the identity

$$\Delta_{\mathrm{BV}}\big(\{a,b\}\big) \,=\, \{\Delta_{\mathrm{BV}}(a), b\} + (-1)^{|a|+1}\,\{a, \Delta_{\mathrm{BV}}(b)\} \quad , \tag{4.3.20}$$

for all $a, b \in \mathrm{Sym}\,E[1]$, which may be derived from (4.3.15b) by applying Δ_{BV} on both sides of the equality sign. The following E_0-algebra

$$\mathrm{Obs}^{\mathrm{int},\hbar} \,:=\, \big(\mathrm{Sym}\,E[1], Q^{\mathrm{int},\hbar}\big) \tag{4.3.21}$$

is interpreted as the quantum observables for the interacting BV theory corresponding to the interaction term $I \in (\mathrm{Sym}\,E[1])^0$ satisfying the quantum master equation (4.3.19).

6 The sign factors in (4.3.15c) can be understood as follows: Since the pairing $\langle\,\cdot\,,\,\cdot\,\rangle \,:\, (E[1])[-1] \otimes (E[1])[-1] \to \mathbb{K}[-1]$ is of degree 1 (with respect to elements in $E[1]$), the sign factor $(-1)^{\sum_{k=1}^{i-1}|\varphi_k|}$ appears when moving it across $\varphi_1 \cdots \varphi_{i-1}$. Permuting φ_j to the $(i+1)$th position yields the sign factor $(-1)^{|\varphi_j|\,\sum_{k=i+1}^{j-1}|\varphi_k|}$.

4.3.2 *Interaction terms and cyclic L_∞-algebras*

There is a powerful method for the construction of interaction terms $I \in (\operatorname{Sym} E[1])^0$ satisfying the classical (and also the quantum) master equation from cyclic L_∞-algebra structures (see Section 4.2). We will give a brief outline in this section. For more details surrounding this and more generally the relation between L_∞-algebras, classical field theories and the BV formalism, see [JRSW19]. There is a natural Abelian cyclic L_∞-algebra structure associated to every free BV theory $(E, -Q, \langle\,\cdot\,,\,\cdot\,\rangle)$. The cyclic L_∞-algebra structure is defined by the shifted cochain complex $E[-1]$ with $\ell_1 = \mathrm{d}_{E[-1]} = Q$ and $\ell_n = 0$ for all $n \geq 2$, with cyclic structure given by

$$\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle \,:\, E[-1]\otimes E[-1] \;\cong\; (E\otimes E)[-2] \xrightarrow{\langle\,\cdot\,,\,\cdot\,\rangle[-2]} \mathbb{K}[-1][-2] \;\cong\; \mathbb{K}[-3] \quad . \tag{4.3.22}$$

The isomorphism $E[-1]\otimes E[-1] \;\cong\; (E\otimes E)[-2]$ is given by $\varphi\otimes\psi \mapsto (-1)^{|\varphi|-1}\,\varphi\otimes\psi$, for all $\varphi,\psi \in E[-1]$. (To understand this sign factor, recall that $E[-1] \cong \mathbb{K}[-1]\otimes E$ and observe that for $\varphi \in E[-1]$ with $E[-1]$-degree $|\varphi|$, the E-degree is $|\varphi| - 1$.) One checks that the antisymmetry property of $\langle\,\cdot\,,\,\cdot\,\rangle$ implies the symmetry property of the cyclic structure $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$.

By endowing the cochain complex $(E[-1], \ell_1 = Q)$ with higher brackets $\{\ell_n\}_{n\geq 2}$ such that (4.3.22) defines a cyclic structure, one can obtain interaction terms $I \in (\operatorname{Sym} E[1])^0$ satisfying the classical master equation. Even better, the problem of finding an interaction term $I \in (\operatorname{Sym} E[1])^0$ that satisfies the classical master equation (4.3.12) is equivalent to endowing the cochain complex $(E[-1], \ell_1 = Q)$ with higher brackets $\{\ell_n\}_{n\geq 2}$ that result in a cyclic L_∞-algebra with respect to (4.3.22). The relationship between the interaction term $I \in (\operatorname{Sym} E[1])^0$ and the higher brackets $\{\ell_n\}_{n\geq 2}$ is described by the so called *homotopy Maurer-Cartan action* (see e.g. [JRSW19, Section 4.3]).

Before writing down the interaction term stemming from the higher brackets, we introduce the concept of "contracted coordinate functions" to simplify the presentation. We begin by choosing any basis $\{\varepsilon_\alpha \in E[-1]\}$ of the L_∞-algebra and denote by $\{\varrho^\alpha \in E[-1]^* \cong E[2]\}$ its dual with respect to the cyclic structure, i.e. $\langle\!\langle \varrho^\alpha, \varepsilon_\beta\rangle\!\rangle = \delta^\alpha_\beta$ for all α, β. The contracted coordinate functions are defined as the element

$$\mathsf{a} := \sum_\alpha \varrho^\alpha \otimes \varepsilon_\alpha \;\in\; \big((\operatorname{Sym} E[1])\otimes E[-1]\big)^1 \tag{4.3.23}$$

of degree 1 in the tensor product of the dg-algebra of polynomial observables and the L_∞-algebra $E[-1]$. The L_∞-algebra structure on $E[-1]$ induces a natural L_∞ structure on the tensor product $(\mathrm{Sym}\,E[1]) \otimes E[-1]$, with extended brackets

$$\ell_n^{\mathrm{ext}} \,:\, \big((\mathrm{Sym}\,E[1]) \otimes E[-1]\big)^{\otimes n} \longrightarrow (\mathrm{Sym}\,E[1]) \otimes E[-1] \quad , \tag{4.3.24}$$

for all $n \geq 2$. The explicit formulas are given by (4.2.8). Using the cyclic structure of $E[-1]$, we define the extended pairing

$$\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_{\mathrm{ext}} \,:\, \big((\mathrm{Sym}\,E[1]) \otimes E[-1]\big) \otimes \big((\mathrm{Sym}\,E[1]) \otimes E[-1]\big) \longrightarrow (\mathrm{Sym}\,E[1])[-3] \quad . \tag{4.3.25}$$

Explicitly, it is given by

$$\langle\!\langle a_1 \otimes \varphi_1, a_2 \otimes \varphi_2 \rangle\!\rangle_{\mathrm{ext}} := (-1)^{|a_1|+|a_2|+|\varphi_1|\,|a_2|}\, a_1\, a_2 \langle\!\langle \varphi_1, \varphi_2 \rangle\!\rangle \tag{4.3.26}$$

for all homogeneous $a_1, a_2 \in \mathrm{Sym}\,E[1]$ and homogeneous $\varphi_1, \varphi_2 \in E[-1]$.

Using the contracted coordinate functions and the extended higher brackets, we now write down the interaction term

$$\lambda\, I = \sum_{p\geq 3} \frac{\lambda^{p-2}}{p!}\, \langle\!\langle \mathsf{a}, \ell_{p-1}^{\mathrm{ext}}(\mathsf{a}, \dots, \mathsf{a}) \rangle\!\rangle_{\mathrm{ext}} \,\in\, (\mathrm{Sym}\,E[1])^0 \quad , \tag{4.3.27}$$

where we recall that λ is a formal parameter interpreted as a coupling constant.[7] It can be shown (see [JRSW19, Section 4.3]) that the interaction term (4.3.27) satisfies the classical master equation (4.3.12). Additionally, it can also be proven that it is annihilated by the BV Laplacian, i.e. $\Delta_{\mathrm{BV}}(\lambda\, I) = 0$, implying that it satisfies the quantum master equation (4.3.19) as well.

4.3.3 *Correlation functions and homological perturbation theory*

The goal is to compute the expectation values (4.3.1) of (polynomial) observables $\mathcal{O} \in \mathrm{Sym}\,E[1]$. (In our finite dimensional setting, the path integral can be rigorously defined.) The problem is broken down to describing a method for calculating the *n-point correlation functions* $\langle \varphi_1 \cdots \varphi_n \rangle$, with $\varphi_1, \dots, \varphi_n \in E[1]$. To this end, we employ a homological approach to integration using techniques from homological perturbation

7 It is easy to check that given any L_∞-algebra structure $\{\ell_n\}_{n\geq 1}$ and any (formal or non-formal) parameter λ, the rescaled brackets $\{\lambda^{n-1}\,\ell_n\}_{n\geq 1}$ also satisfy the homotopy Jacobi identities (4.2.1). This explains the powers of λ on the right-hand side of (4.3.27).

theory, see e.g. [CG16, Gwi12]. For some original references regarding this topic, see e.g. [Bro67, Gug72, GS86, LmS87, GLm89, Hueb89, GLmS90, HaTa90, GLmS91, HK91, Lm91]. The correlation functions are computed perturbatively and are determined by the cohomology of the cochain complex (4.3.21). We will provide a brief review of the relevant constructions following [NSS21]. In the following, we will consider the cohomology $H^\bullet(V)$ of a cochain complex $V \in \mathsf{Ch}_\mathbb{K}$ as a cochain complex with trivial differential.

Definition 4.3.6. A *strong deformation retract* of a cochain complex $V \in \mathsf{Ch}_\mathbb{K}$ onto its cohomology $H^\bullet(V)$ is given by the following data:

(i) A cochain map $\iota : H^\bullet(V) \to V$;

(ii) A cochain map $\pi : V \to H^\bullet(V)$;

(iii) A (-1)-cochain $\xi \in \underline{\mathrm{hom}}(V,V)^{-1}$.

These data are required to satisfy the following conditions:

(a) $\pi\,\iota = \mathrm{id}_{H^\bullet(V)}$;

(b) $\iota\,\pi - \mathrm{id}_V = \partial(\xi) = \mathrm{d}\,\xi + \xi\,\mathrm{d}$;[8]

(c) $\xi^2 = 0$, $\xi\,\iota = 0$ and $\pi\,\xi = 0$.

A strong deformation retract may be visualized by

$$\begin{tikzcd} (H^\bullet(V),0) \arrow[r, shift left, leftarrow, "\pi"] \arrow[r, shift right, "\iota"'] & (V,\mathrm{d}) \arrow[loop right, "\xi"] \end{tikzcd} \quad , \tag{4.3.28}$$

where we also explicitly display the differentials.

A key result is the homological perturbation lemma (see e.g. [Cra04]), which states that small perturbations $\mathrm{d}+\delta$ of the differential d on V lead to perturbations of strong deformation retracts. A perturbation is *small* when, in addition to $(\mathrm{d}+\delta)^2 = 0$, the map $\mathrm{id}_V - \delta\,\xi$ is invertible. In particular, the formal deformations of Section 4.3.1 are all small perturbations in this sense. We provide the precise statement of the homological perturbation lemma below.

8 I.e. ξ is a cochain homotopy between $\iota\,\pi$ and id_V, see (4.1.7)

Theorem 4.3.7. *Consider any strong deformation retract as in* (4.3.28) *and let* $\delta \in \underline{\mathrm{hom}}(V,V)^1$ *be a small perturbation. Then there exists a strong deformation retract*

$$\begin{tikzcd} (H^\bullet(V),\widetilde{\delta}) \arrow[r, shift right, "\widetilde{\iota}"'] & (V,\mathrm{d}+\delta) \arrow[l, shift right, "\widetilde{\pi}"'] \arrow[loop right, "\widetilde{\xi}"] \end{tikzcd} \tag{4.3.29a}$$

with

$$\widetilde{\delta} \,=\, \pi\,(\mathrm{id}_V - \delta\,\xi)^{-1}\,\delta\,\iota \quad , \tag{4.3.29b}$$

$$\widetilde{\iota} \,=\, \iota + \xi\,(\mathrm{id}_V - \delta\,\xi)^{-1}\,\delta\,\iota \quad , \tag{4.3.29c}$$

$$\widetilde{\pi} \,=\, \pi + \pi\,(\mathrm{id}_V - \delta\,\xi)^{-1}\,\delta\,\xi \quad , \tag{4.3.29d}$$

$$\widetilde{\xi} \,=\, \xi + \xi\,(\mathrm{id}_V - \delta\,\xi)^{-1}\,\delta\,\xi \quad . \tag{4.3.29e}$$

Let $(E,-Q,\langle\,\cdot\,,\,\cdot\,\rangle)$ be a free BV theory as in Definition 4.3.1 and choose a strong deformation retract[9] for its dual complex $E^* \cong E[1]$,

$$\begin{tikzcd} (H^\bullet(E[1]),0) \arrow[r, shift right, "\iota"'] & (E[1],Q) \arrow[l, shift right, "\pi"'] \arrow[loop right, "\xi"] \end{tikzcd} \quad . \tag{4.3.30}$$

In physical terms, different choices for a strong deformation retract can be seen as different gauge fixings. We will also see in Chapter 5 that such deformation retracts are related to Green operators through some concrete examples of matrix models. Next, we will use a result in [Gwi12, Proposition 2.5.5] which states that there exists an associated strong deformation retract

$$\begin{tikzcd} \big(\mathrm{Sym}\,H^\bullet(E[1]),0\big) \arrow[r, shift right, "J"'] & \big(\mathrm{Sym}\,E[1],Q\big) \arrow[l, shift right, "\Pi"'] \arrow[loop right, "\Xi"] \end{tikzcd} \tag{4.3.31}$$

at the level of symmetric algebras. Note that $\mathrm{Sym}\,H^\bullet(E[1]) \cong H^\bullet(\mathrm{Sym}\,E[1])$ (see Example 4.1.7). Here, the cochain maps $\Pi := \mathrm{Sym}\,\pi$ and $J := \mathrm{Sym}\,\iota$ are the usual extensions of π and ι to commutative dg-algebra morphisms, i.e.

$$J\big([\psi_1]\cdots[\psi_n]\big) \,:=\, \iota([\psi_1])\cdots\iota([\psi_n]) \quad , \quad \Pi\big(\varphi_1\cdots\varphi_n\big) \,:=\, \pi(\varphi_1)\cdots\pi(\varphi_n) \quad , \tag{4.3.32}$$

for all $[\psi_1],\dots,[\psi_n] \in H^\bullet(E[1])$ and $\varphi_1,\dots,\varphi_n \in E[1]$. The extended cochain homotopy $\Xi := \mathrm{Sym}\,\xi$ is a bit more complicated to describe. First, note that from the

9 A strong deformation retract in our setting always exists, see e.g. [JRSW19, Appendices B]

definition of a strong deformation retract, the composite $\iota\,\pi : E[1] \to E[1]$ defines a projector, i.e. $(\iota\,\pi)^2 = \iota\,\pi$. This allows for the decomposition

$$E[1] \;\cong\; E[1]^{\perp} \oplus H^{\bullet}(E[1]) \tag{4.3.33a}$$

and consequently

$$\operatorname{Sym} E[1] \;\cong\; \operatorname{Sym} E[1]^{\perp} \otimes \operatorname{Sym} H^{\bullet}(E[1]) \;\cong\; \bigoplus_{n\geq 0} \operatorname{Sym}^n E[1]^{\perp} \otimes \operatorname{Sym} H^{\bullet}(E[1]) \quad , \tag{4.3.33b}$$

where Sym^n denotes the n-th symmetric power. The cochain homotopy is then defined by setting

$$\Xi\big(\varphi_1^{\perp} \cdots \varphi_n^{\perp} \otimes a\big) \;:=\; \frac{1}{n} \sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i-1} |\varphi_j^{\perp}|}\, \varphi_1^{\perp} \cdots \varphi_{i-1}^{\perp}\, \zeta(\varphi_i^{\perp})\, \varphi_{i+1}^{\perp} \cdots \varphi_n^{\perp} \otimes a \quad , \tag{4.3.34}$$

for all homogeneous elements $\varphi_1^{\perp} \cdots \varphi_n^{\perp} \otimes a \;\in\; \operatorname{Sym}^n E[1]^{\perp} \otimes \operatorname{Sym} H^{\bullet}(E[1])$ in this decomposition. The case $n = 0$ should be read as $\Xi(a) = 0$, for all $a \in \operatorname{Sym} H^{\bullet}(E[1])$.

Recall from Section 4.3.1 that addition of interactions or quantisation of a free BV theory amounts to the deformation of the differential Q by adding $\{\lambda I, \cdot\}$ or $\hbar\,\Delta_{\mathrm{BV}}$ to it. Applying the homological perturbation lemma (Theorem 4.3.7) to the strong deformation retract (4.3.31) and the deformed differentials from Section 4.3.1, a strong deformation retract for the quantum observables is obtained,

$$\begin{tikzcd} \big(\operatorname{Sym} H^{\bullet}(E[1]), \widetilde{\delta}\,\big) \arrow[r, shift right, "\widetilde{J}"'] & \big(\operatorname{Sym} E[1], Q + \delta\big) \arrow[l, shift right, "\widetilde{\Pi}"'] \arrow[loop right, "\widetilde{\Xi}"] \end{tikzcd} \quad , \tag{4.3.35}$$

where $\delta := \{\lambda\, I, \cdot\} + \hbar\,\Delta_{\mathrm{BV}}$ (this of course also applies to the non-interacting case, i.e. $\lambda = 0$). The n-point correlation functions are then defined by the application of the map $\widetilde{\Pi}$ on a product of "test functions" $\varphi_1, \dots, \varphi_n \in E[1]$, i.e.

$$\langle \varphi_1 \cdots \varphi_n \rangle := \widetilde{\Pi}\big(\varphi_1 \cdots \varphi_n\big) \;\in\; \operatorname{Sym} H^{\bullet}(E[1]) \quad . \tag{4.3.36}$$

This can be computed perturbatively (as a formal power series in λ or $\hbar$, or both). Explicitly, from the formula for $\widetilde{\Pi}$ in Theorem 4.3.7

$$\widetilde{\Pi} \;=\; \Pi + \Pi\,\big(\mathrm{id} - \delta\,\Xi\big)^{-1}\,\delta\,\Xi \;=\; \Pi \circ \sum_{k=0}^{\infty} \big(\delta\,\Xi\big)^k \quad , \tag{4.3.37}$$

we obtain

$$\langle \varphi_1 \cdots \varphi_n \rangle \;=\; \sum_{k=0}^{\infty} \Pi\,\Big(\big(\delta\,\Xi\big)^k\,\big(\varphi_1 \cdots \varphi_n\big)\Big) \in\; \operatorname{Sym} H^{\bullet}(E[1]) \quad . \tag{4.3.38}$$

Observe that in general, the correlation functions are not numbers, but elements of $\mathrm{Sym}\, H^\bullet(E[1])$. The symmetric algebra $\mathrm{Sym}\, H^\bullet(E[1])$ should be interpreted as the algebra of polynomial functions on the space of vacua, which is the cohomology $H^\bullet(E)$ of the derived solution complex E (see Remark 4.3.2). The n-point correlation functions are therefore functions on the space of vacua, which when evaluated at a particular vacuum, yield the usual numerical correlations of the perturbative field theory around the said vacuum. This will be elucidated through concrete examples in Chapter 5. We shall also see that one can introduce graphical tools (in the form of Feynman diagrams) to facilitate the computation of correlation functions.

5

BATALIN-VILKOVISKY QUANTISATION OF FUZZY FIELD THEORIES

This chapter based on Section 3, 4 and 5 of [NSS21]. We will begin with applying the finite dimensional BV formalism as described in Section 4.3 on a couple of noncommutative field theories on the fuzzy 2-sphere in Section 5.1. First out in Section 5.1.1 are scalar field theories and we compute the 2-point function at 1-loop order for the particular case of Φ^4-theory. Then in Section 5.1.2, we treat Chern-Simons theory as an illustration of the formalism in the case when the gauge symmetry really is non-trivial. Having investigated these examples, we move on to Section 5.2 where we generalise the finite dimensional BV formalism to also include noncommutative field theories with a triangular Hopf algebra symmetry, so called *braided field theories*. This is then applied to scalar field theories on the fuzzy torus in Section 5.3. As for the scalar field on the fuzzy 2-sphere, we compute the 2-point function for Φ^4-theory to lowest non-trivial order in coupling constant. We also calculate the connected part of the 4-point function to lowest non-trivial order in coupling constant to check that the formalism really detects the triangular Hopf algebra symmetry.

5.1 FIELD THEORY ON THE FUZZY SPHERE

In this section, we will apply the techniques from Section 4.3 to both scalar field theory and Chern-Simons theory on the fuzzy 2-sphere in order to illustrate the formalism. In the following, we will always assume that the underlying field $\mathbb{K} = \mathbb{C}$ is the field of complex numbers.

5.1.1 *Scalar field theories*

The first case in consideration is the simplest one of scalar field theories. We will see that the formalism reproduces the known 1-loop 2-point function for Φ^4-theory on the fuzzy sphere, see e.g. [CMS01]. However, as opposed to the more traditional approach of [CMS01], our correlation functions are generally disconnected and 1-particle reducible, and contain unamputated external legs.

THE FUZZY 2-SPHERE. We first describe the fuzzy sphere following [CMS01], see also [NSS21]. For a positive integer $N \in \mathbb{Z}_{>0}$, let V denote the irreducible spin $N/2$-representation for the Lie algebra $\mathfrak{su}(2)$. The algebra of functions for the fuzzy sphere $\mathbb{S}^2_N$ is defined by

$$A := \underline{\mathrm{end}}(V) = V \otimes V^\vee \quad , \tag{5.1.1}$$

where $V^\vee$ denotes the dual representation. It follows that $A \cong \mathrm{Mat}_{N+1}(\mathbb{C})$ is isomorphic to the algebra of $(N+1) \times (N+1)$ matrices with complex entries since V is $(N+1)$-dimensional. The associated $\mathfrak{su}(2)$-action on V is the Lie algebra homomorphism

$$\rho : \mathfrak{su}(2) \longrightarrow A \quad , \tag{5.1.2}$$

where the Lie bracket on A is the matrix commutator. Let $\{e_i \in \mathfrak{su}(2)\}_{i=1,2,3}$ be a basis of $\mathfrak{su}(2)$ with the Lie bracket relations $[e_i, e_j] = \mathrm{i}\, \epsilon_{ijk}\, e_k$, where ϵ_{ijk} is the Levi-Civita symbol. By introducing the normalisation constant

$$\lambda_N := \frac{1}{\sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}} \in \mathbb{R} \quad , \tag{5.1.3}$$

the elements

$$X_i := \lambda_N\, \rho(e_i) \in A \tag{5.1.4a}$$

satisfy the fuzzy unit sphere relations

$$[X_i, X_j] = \mathrm{i}\, \lambda_N\, \epsilon_{ijk}\, X_k \quad , \quad \delta_{ij}\, X_i\, X_j = I_{N+1} \quad , \quad X_i^* = X_i \quad , \tag{5.1.4b}$$

where δ_{ij} is the Kronecker delta and I_{N+1} is the $(N+1) \times (N+1)$ identity matrix. Furthermore, by Burnside's theorem, see e.g. [Lam98, Theorem 2], the elements X_i generate $A \cong \mathrm{Mat}_{N+1}(\mathbb{C})$.

Integration on $\mathbb{S}^2_N$ is given by the normalised trace map

$$\int : A \longrightarrow \mathbb{C} \ , \ \ a \longmapsto \frac{4\pi}{N+1}\,\mathrm{Tr}(a) \quad , \tag{5.1.5}$$

and the scalar Laplacian is defined as

$$\Delta \, : \, A \longrightarrow A \ , \ \ a \longmapsto \Delta(a) \, := \, \frac{1}{\lambda_N^2}\,\delta_{ij}\,[X_i,[X_j,a]] \quad . \tag{5.1.6}$$

The eigenfunctions of the Laplacian, the fuzzy spherical harmonics $Y^J_j \in A$, for $J = 0,1,\dots,N$ and $-J \leq j \leq J$, satisfy the identities

$$\Delta(Y^J_j) \, = \, J\,(J+1)\,Y^J_j \tag{5.1.7a}$$

$$Y^{J\,*}_j \, = \, (-1)^J\,Y^J_{-j} \tag{5.1.7b}$$

$$\frac{4\pi}{N+1}\,\mathrm{Tr}\big(Y^{J\,*}_j\,Y^{J'}_{j'}\big) \, = \, \delta_{JJ'}\,\delta_{jj'} \quad . \tag{5.1.7c}$$

The fuzzy spherical harmonics constitute a basis for A. Let us also mention that there is a 'fusion formula' for the products $Y^I_i\,Y^J_j$ of fuzzy spherical harmonics in terms of Wigner's $3j$ and $6j$ symbols, see e.g. [CMS01]. We will however not present them here as they are not required in this thesis.

FREE BV THEORY. Having set up the required geometric data of the fuzzy sphere, we now move on to describe the non-interacting scalar field theory on the fuzzy sphere as a free BV theory in the sense of Definition 4.3.1.

Definition 5.1.1. The free BV theory associated to a scalar field with mass parameter $m^2 \geq 0$ on the fuzzy sphere is given by the cochain complex

$$E \, = \, \Big(\overset{(0)}{A} \xrightarrow{\ -Q\ } \overset{(1)}{A} \Big) \qquad \text{with} \qquad Q \, := \, \Delta + m^2 \tag{5.1.8}$$

concentrated in degrees 0 and 1, together with the pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle \, : \, E\otimes E \longrightarrow \mathbb{C}[-1] \ , \ \ \varphi\otimes\psi \longmapsto \langle\varphi,\psi\rangle \, := \, (-1)^{|\varphi|}\,\frac{4\pi}{N+1}\,\mathrm{Tr}\big(\varphi\,\psi\big) \quad . \tag{5.1.9}$$

Following the construction of Section 4.3.1, we obtain from this free BV theory a P_0-algebra of classical observables

$$\mathrm{Obs}^{\mathrm{cl}} \, := \, \big(\mathrm{Sym}\,E[1], Q, \{\,\cdot\,,\,\cdot\,\}\big) \quad . \tag{5.1.10}$$

Observe that the complex

$$E[1] \, = \, \Big(\overset{(-1)}{A} \xrightarrow{\ \ Q\ \ } \overset{(0)}{A} \Big) \tag{5.1.11}$$

is concentrated in degrees -1 and 0.

INTERACTIONS. Since the generators of $E[1]$ is concentrated in degrees -1 and 0, the dg-algebra $\mathrm{Sym}\,E[1]$ is concentrated in non-positive degrees. Therefore, every 0-cochain $I \in (\mathrm{Sym}\,E[1])^0$ satisfies

$$Q(I) \;=\; \{I,I\} \;=\; \Delta_{\mathrm{BV}}(I) \;=\; 0 \quad , \tag{5.1.12}$$

implying that the classical master equation (4.3.12) as well as the quantum master equation (4.3.19) are satisfied. Hence, every 0-cochain $I \in (\mathrm{Sym}\,E[1])^0$ is a well-defined interaction term for a scalar field in both the classical and quantum cases.

We may still apply the cyclic L_∞-algebra formalism from Section 4.3.2 to build interaction terms. We will focus on the usual p-point interactions. From the free scalar field theory of Definition 5.1.1, we obtain the associated Abelian cyclic L_∞-algebra given by the cochain complex

$$E[-1] \;=\; \Big(\overset{(1)}{A} \xrightarrow{\;Q\;} \overset{(2)}{A} \Big) \tag{5.1.13}$$

with the cyclic structure

$$\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle \,:\, E[-1]\otimes E[-1] \;\longrightarrow\; \mathbb{C}[-3] \;,\;\; \varphi\otimes\psi \;\longmapsto\; \langle\!\langle \varphi,\psi \rangle\!\rangle \;=\; \frac{4\pi}{N+1}\,\mathrm{Tr}\big(\varphi\,\psi\big) \quad . \tag{5.1.14}$$

For any $p \geq 3$, we may augment the above Abelian L_∞-algebra with the compatible $(p-1)$-bracket

$$\ell_{p-1} \,:\, E[-1]^{\otimes p-1} \;\longrightarrow\; E[-1] \;,\;\; \varphi_1\otimes\cdots\otimes\varphi_p \;\longmapsto\; \frac{1}{(p-1)!}\sum_{\sigma\in S_{p-1}} \varphi_{\sigma(1)}\cdots\varphi_{\sigma(p-1)} \quad . \tag{5.1.15}$$

For degree reasons, the higher brackets (5.1.15) is only non-vanishing if each $\varphi_i \in E[-1]$ is of degree 1 in $E[-1]$. As required, the higher brackets ℓ_{p-1} are graded antisymmetric due to the symmetrisation of the matrix multiplications in the definition of ℓ_{p-1}.

Next, we write out the contracted coordinate functions (4.3.23). In this case, they may be described in terms of the fuzzy spherical harmonics Y^J_j. We will use the notation $Y^J_j \in E[-1]^1 = A$ when we refer to the fuzzy spherical harmonics as elements of degree 1 in $E[-1]$ and $\widetilde{Y}^J_j \in E[-1]^2 = A$ when we regard them as elements of degree 2 in $E[-1]$. With this notation, the contracted coordinate functions take the form

$$\mathsf{a} \;=\; \sum_{J,j} Y^{J\,*}_j \otimes Y^J_j + \sum_{J,j} \widetilde{Y}^{J\,*}_j \otimes \widetilde{Y}^J_j \;\in\; \big((\mathrm{Sym}\,E[1])\otimes E[-1]\big)^1 \quad , \tag{5.1.16}$$

where $Y_j^{J\,*} \in E[1]^0 = A$ denotes elements of degree 0 in $E[1]$ and $\widetilde{Y}_j^{J\,*} \in E[1]^{-1} = A$ stands for elements of degree -1 in $E[1]$.

In terms of the contracted coordinate functions (5.1.16), the interaction term (4.3.27) corresponding to a p-point interaction takes the explicit form

$$\begin{aligned}\lambda\, I &= \frac{\lambda^{p-2}}{p!}\, \langle\!\langle \mathsf{a}, \ell^{\rm ext}_{p-1}(\mathsf{a},\dots,\mathsf{a})\rangle\!\rangle_{\rm ext} \\ &= \frac{\lambda^{p-2}}{p!} \sum_{J_0,j_0,\dots,J_{p-1},j_{p-1}} Y_{j_0}^{J_0\,*}\, Y_{j_1}^{J_1\,*} \cdots Y_{j_{p-1}}^{J_{p-1}\,*}\ \langle\!\langle Y_{j_0}^{J_0}, \ell_{p-1}(Y_{j_1}^{J_1},\dots,Y_{j_{p-1}}^{J_{p-1}})\rangle\!\rangle \\ &\in\ (\mathrm{Sym}\, E[1])^0 \quad . \end{aligned} \tag{5.1.17a}$$

We would like to stress that the products of the $Y_{j_i}^{J_i\,*} \in E[1]^0$ in the second line are *not* given by matrix multiplication but rather by the product in the symmetric algebra $\mathrm{Sym}\, E[1]$. The constants

$$I^{J_0 J_1 \cdots J_{p-1}}_{j_0 j_1 \cdots j_{p-1}} := \langle\!\langle Y_{j_0}^{J_0}, \ell_{p-1}(Y_{j_1}^{J_1},\dots,Y_{j_{p-1}}^{J_{p-1}})\rangle\!\rangle \ \in\ \mathbb{C} \tag{5.1.17b}$$

can be explicitly written out in terms of the Wigner $3j$ and $6j$ symbols as in [CMS01]. However, this level of detail is not necessary in the context of this thesis. From the cyclicity property (4.2.6) of the cyclic structure, the constants $I^{J_0 J_1 \cdots J_{p-1}}_{j_0 j_1 \cdots j_{p-1}}$ are symmetric under the exchange of any neighbouring pairs of indices, i.e.

$$I^{J_0 J_1 \cdots J_i J_{i+1} \cdots J_{p-1}}_{j_0 j_1 \cdots j_i j_{i+1} \cdots j_{p-1}} = I^{J_0 J_1 \cdots J_{i+1} J_i \cdots J_{p-1}}_{j_0 j_1 \cdots j_{i+1} j_i \cdots j_{p-1}} \quad . \tag{5.1.18}$$

STRONG DEFORMATION RETRACT. The cohomology of the complex (5.1.11) is different depending on whether the scalar field is massless or not. Knowing that the spectrum of the Laplacian (5.1.6) is $\{J\,(J+1)\ :\ J = 0,1,\dots,N\}$, we compute

$$H^\bullet(E[1]) \ \cong\ \begin{cases} 0 & \text{for } m^2 > 0 \quad , \\ \mathbb{C}[1] \oplus \mathbb{C} & \text{for } m^2 = 0 \quad . \end{cases} \tag{5.1.19}$$

Therefore, we need to consider the two cases separately. In the massive case $m^2 > 0$, we have the strong deformation retract

$$(0,0) \ \underset{\iota=0}{\overset{\pi=0}{\rightleftarrows}}\ (E[1],Q) \circlearrowleft \xi = -G \qquad \text{for } m^2 > 0 \quad , \tag{5.1.20}$$

where G is the inverse of $Q = \Delta + m^2$, i.e. the Green operator, and $\xi = -G$ is defined to act as a degree -1 map on $E[1]$ (see Definition 4.3.6 (iii)).

The massless case $m^2 = 0$ needs to be considered more carefully since the scalar Laplacian $Q = \Delta$ has a non-trivial kernel given by complex multiples of the unit $\mathbb{1} \in A$. The linear map $\eta \, \frac{1}{N+1} \operatorname{Tr} : A \to A \,,\ a \mapsto \frac{1}{N+1} \operatorname{Tr}(a) \, \mathbb{1}$, which is obtained by composing the normalized trace and the unit map $\eta : \mathbb{C} \to A$, defines a projector onto the kernel of Δ. This leads to the orthogonal decomposition[1]

$$A \,\cong\, A^{\perp} \oplus \mathbb{C} \quad . \tag{5.1.21}$$

By the rank-nullity theorem of linear algebra, the scalar Laplacian therefore restricts to an isomorphism $\Delta^{\perp} : A^{\perp} \to A^{\perp}$, whose inverse we denote by $G^{\perp} : A^{\perp} \to A^{\perp}$. Extending $G^{\perp}$ by 0 to all of A, we obtain the linear map

$$G_0 \,:=\, G^{\perp} \left(\mathrm{id}_A - \eta \, \tfrac{1}{N+1} \operatorname{Tr}\right) \,:\, A \,\longrightarrow\, A \quad . \tag{5.1.22}$$

From these data, we obtain a strong deformation retract

$$\begin{tikzcd} (\mathbb{C}[1] \oplus \mathbb{C}, 0) \arrow[r, shift right, "\iota = \eta"'] & (E[1], Q) \arrow[l, shift right, "\pi = \frac{1}{N+1} \operatorname{Tr}"'] \arrow[loop right, "\xi = -G_0"] \end{tikzcd} \qquad \text{for } m^2 = 0 \tag{5.1.23}$$

for the massless case. The strong deformation retracts for both the massive (5.1.20) and massless (5.1.23) cases extend to the symmetric algebras via the construction outlined in Section 4.3.3, Equation (4.3.31) and below.

Remark 5.1.2. Both G and G_0 are anti-self-adjoint with respect to the pairing (5.1.9). We show this for G_0 as the arguments in the case of G are a simplified version of the arguments for the former case. First, one can verify through straightforward calculations using the cyclicity of the trace that the Laplacian $\Delta^{\perp} : A^{\perp} \to A^{\perp}$ is anti-self-adjoint. Then, since the decomposition (5.1.21) is orthogonal,

$$\langle \varphi, G_0 \psi \rangle = \langle \Delta G_0 \varphi, G_0 \psi \rangle = -\langle G_0 \varphi, \Delta G_0 \psi \rangle = -\langle G_0 \varphi, \psi \rangle \quad . \tag{5.1.24}$$

$\triangle$

CORRELATION FUNCTIONS FOR $m^2 > 0$. Having described the strong deformation retract, we now would like to compute correlation functions for the scalar field theory. We will explain the process and provide some explicit examples. The focus

1 Note that the projection onto $A^{\perp}$ is given by $\mathrm{id}_A - \eta \, \frac{1}{N+1} \operatorname{Tr}$, meaning $A^{\perp}$ consists of the traceless $(N+1) \times (N+1)$-matrices in $\mathrm{Mat}_{N+1}(\mathbb{C})$, so the decomposition is indeed orthogonal with respect to the pairing (5.1.9).

will be mostly on the massive case $m^2 > 0$ and later we briefly comment on the massless case.

Recall again that the strong deformation retract (5.1.20) extends to the symmetric algebras. Let δ be a small deformation of the differential Q on $\mathrm{Sym}\, E[1]$. From the homological perturbation lemma (Theorem 4.3.7), we obtain the deformed strong deformation retract

$$\begin{tikzcd} (\mathrm{Sym}\, 0 \cong \mathbb{C}, 0) \arrow[r, shift right, "\widetilde{\jmath}"'] & (\mathrm{Sym}\, E[1], Q + \delta) \arrow[l, shift right, "\widetilde{\Pi}"'] \arrow[loop right, "\widetilde{\Xi}"] \end{tikzcd} \quad . \tag{5.1.25}$$

In particular, the n-point correlation functions of $\varphi_1, \dots, \varphi_n \in E[1]$ are computed through the map $\widetilde{\Pi}$ (4.3.38), which we repeat here

$$\langle \varphi_1 \cdots \varphi_n \rangle \,=\, \sum_{k=0}^{\infty} \Pi\Big(\big(\delta\, \Xi\big)^k \,(\varphi_1 \cdots \varphi_n)\Big) \,\in\, \mathrm{Sym}\, 0 \cong \mathbb{C} \quad , \tag{4.3.38}$$

where we recall that Π and Ξ are the respective extensions of π and ξ to the symmetric algebras (see (4.3.32) and around (4.3.34)). Recall also that the perturbations δ in our case are of the form

$$\delta \,=\, \hbar\, \Delta_{\mathrm{BV}} + \{\lambda\, I, \,\cdot\,\} \quad , \tag{5.1.26}$$

where Δ_{BV} is the BV Laplacian (4.3.15c) and $\lambda\, I \in (\mathrm{Sym}\, E[1])^0$ denotes the p-point interaction term (5.1.17) for some $p \geq 3$.

We are mainly interested in correlation functions $\widetilde{\Pi}(\varphi_1 \cdots \varphi_n)$ for test functions $\varphi_1, \dots, \varphi_n \in E[1]^0$ of degree zero, which describe the correlators of physical fields, in contrast to correlators involving antifields. In order to understand the perturbative expansion (4.3.37), we need to pin down how the maps Π and $\delta\, \Xi$ act on elements $\varphi_1 \cdots \varphi_n \in \mathrm{Sym}\, E[1]$ with all $\varphi_i \in E[1]^0$ of degree zero. Since $\pi = 0$ in the present case (see (5.1.20)), we have

$$\Pi(\mathbb{1}) \,=\, 1 \quad , \quad \Pi(\varphi_1 \cdots \varphi_n) \,=\, 0 \quad , \tag{5.1.27}$$

for all $n \geq 1$. For $\delta\, \Xi = \hbar\, \Delta_{\mathrm{BV}}\, \Xi + \{\lambda\, I, \,\cdot\,\}\, \Xi$ let us consider the two terms separately. In the case of the first term, using the definition of (4.3.34) of $\Xi = \mathrm{Sym}\, \xi$, the explicit formula for the BV Laplacian (4.3.15c), and anti-self-adjointness of G (Remark 5.1.2), we compute

$$\hbar\, \Delta_{\mathrm{BV}}\, \Xi\big(\varphi_1 \cdots \varphi_n\big) \,=\, -\frac{2\,\hbar}{n} \sum_{i<j} \big\langle \varphi_i, G(\varphi_j) \big\rangle\; \varphi_1 \cdots \widehat{\varphi}_i \cdots \widehat{\varphi}_j \cdots \varphi_n \quad , \tag{5.1.28}$$

where we recall that $G = -\xi$ is the Green operator for $Q = \Delta + m^2$. The second term is written out using the axioms of P_0-algebras (see Remark 4.3.4), the explicit expression (5.1.17) for the p-point interaction term (together with its symmetry property (5.1.18)), and also recalling how the antibracket was defined on generators (4.3.9), resulting in

$$\begin{aligned}\big\{\lambda\, I, \Xi(\varphi_1 \cdots \varphi_n)\big\} \;&=\; -\frac{\lambda^{p-2}}{(p-1)!\,n} \sum_{i=1}^{n} \sum_{J_0, j_0, \dots, J_{p-1}, j_{p-1}} \Big(\varphi_1 \cdots \varphi_{i-1} \\ &\qquad \times I^{J_0 J_1 \cdots J_{p-1}}_{j_0 j_1 \cdots j_{p-1}} \, \big\langle Y^{J_0\,*}_{j_0}, G(\varphi_i)\big\rangle \; Y^{J_1\,*}_{j_1} \, \cdots \, Y^{J_{p-1}\,*}_{j_{p-1}} \; \varphi_{i+1} \cdots \varphi_n\Big) \quad . \end{aligned} \tag{5.1.29}$$

From the two expressions (5.1.28) and (5.1.29), we derive a graphical calculus. By depicting the product $\varphi_1 \cdots \varphi_n$ by n vertical lines, the map in (5.1.28) may be depicted as

$$\hbar\, \Delta_{\mathrm{BV}}\, \Xi\Big(|\,|\,|\cdots|\,|\Big) \;=\; -\frac{2\,\hbar}{n}\,\Big(\bigcap\,|\cdots|\,|\, + \,|\bigcap\cdots|\,|\, + \,\cdots\, + \,|\,|\,|\cdots\bigcap\Big) \quad , \tag{5.1.30}$$

where the cap expresses a contraction of two elements with respect to $\{\,\cdot\,, G(\cdot)\}$. The map in (5.1.29) may be pictured as

$$\Big\{\lambda\, I\,,\, \Xi\Big(|\,|\,|\cdots|\,|\Big)\Big\} \;=\; -\frac{\lambda^{p-2}}{(p-1)!\,n}\,\Big(\overset{p-1 \text{ legs}}{\Psi}\,|\,|\cdots|\,|\, + \,\cdots\, + \,|\,|\,|\cdots|\,\overset{p-1 \text{ legs}}{\Psi}\Big) \quad , \tag{5.1.31}$$

where the vertex acts on an element as $\sum_{J_0, j_0, \dots, J_{p-1}, j_{p-1}} I^{J_0 J_1 \cdots J_{p-1}}_{j_0 j_1 \cdots j_{p-1}} \, \big\langle Y^{J_0\,*}_{j_0}, G(\cdot)\big\rangle$ $\times\, Y^{J_1\,*}_{j_1} \, \cdots \, Y^{J_{p-1}\,*}_{j_{p-1}}$, i.e. it turns a single vertical line into $p-1$ legs.

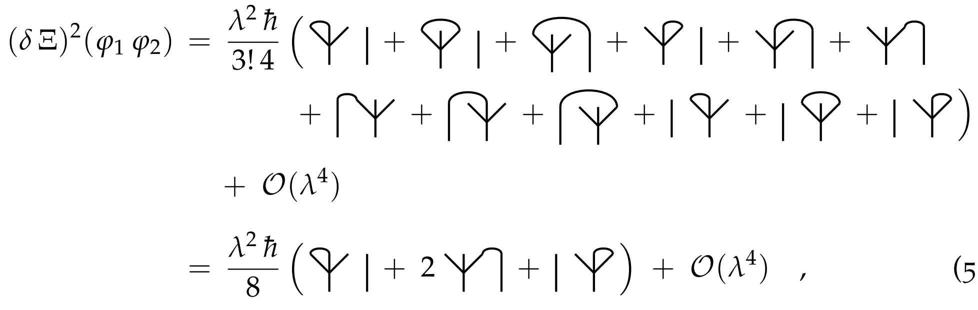
Example 5.1.3. As an example, we compute the 2-point function

$$\widetilde{\Pi}(\varphi_1\, \varphi_2) \;=\; \sum_{k=0}^{\infty} \Pi\big((\delta\, \Xi)^k(\varphi_1\, \varphi_2)\big) \tag{5.1.32}$$

for Φ^4-theory, i.e. we set $p = 4$. (Note that by using our conventions in (5.1.17), the 4-point interaction vertex has coupling constant λ^2.) Using the graphical calculus, we first compute

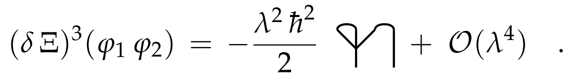
$$\delta\, \Xi(\varphi_1\, \varphi_2) \;=\; -\hbar\, \bigcap - \frac{\lambda^2}{3!\,2}\,\big(\Psi\,| + |\,\Psi\big) \quad . \tag{5.1.33}$$

Applying $\delta\,\Xi$ twice, we get



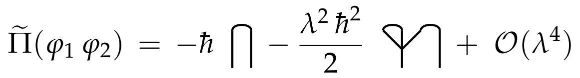
(5.1.34)

where we have used the symmetry property of the interaction term (5.1.18) for the simplification in the second equality. The 3-fold application of $\delta\,\Xi$ is given by



(5.1.35)

From the above iterated applications of $\delta\,\Xi$, we compute the 2-point function (5.1.32) to leading order in the coupling constant as



(5.1.36)

$$
\begin{aligned}
&= -\hbar\,\big\langle \varphi_1, G(\varphi_2)\big\rangle \\
&\quad - \frac{\lambda^2\,\hbar^2}{2} \sum_{J_0, j_0, \dots, J_3, j_3} I^{J_0 J_1 J_2 J_3}_{j_0 j_1 j_2 j_3}\,\big\langle Y^{J_0}_{j_0}{}^{*}, G(\varphi_1)\big\rangle\,\big\langle Y^{J_1}_{j_1}{}^{*}, G(Y^{J_2}_{j_2}{}^{*})\big\rangle\,\big\langle Y^{J_3}_{j_3}{}^{*}, G(\varphi_2)\big\rangle \\
&\quad + \mathcal{O}(\lambda^4) \quad .
\end{aligned}
$$

Note that the 2-point function at order λ^2 (and higher), even though it is not directly apparent in our graphical presentation, receives contributions from both planar and non-planar diagrams. This is analogous to the computation of [CMS01] by traditional perturbative techniques. These two kinds of contributions stem from the (graded anti-)symmetrisation of the higher L_∞-bracket (5.1.15), which enters the definition of the constants $I^{J_0 J_1 J_2 J_3}_{j_0 j_1 j_2 j_3}$ in (5.1.17). ▽

CORRELATION FUNCTIONS FOR $\boldsymbol{m^2 = 0}$. We end this subsection by briefly treating the massless $m^2 = 0$ case. In contrast to the massive case, the cochain map $\pi = \frac{1}{N+1}\operatorname{Tr}$ in the massless strong deformation retract (5.1.23) is not the zero map. Hence, the extension of π to the symmetric algebra in this case is given by

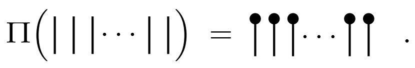
$$
\Pi(\mathbb{1}) = 1 \in \operatorname{Sym}\mathbb{C} \quad , \quad \Pi(\varphi_1 \cdots \varphi_n) = \pi(\varphi_1) \odot \cdots \odot \pi(\varphi_n) \in \operatorname{Sym}\mathbb{C} \quad , \tag{5.1.37}
$$

for all $\varphi_1, \dots, \varphi_n \in E[1]^0$ in degree 0, where we use the symbol $\odot$ to denote the product of the symmetric algebra $\operatorname{Sym}\mathbb{C}$ in order to distinguish it from the multiplication of complex numbers. The factors $\pi(\varphi_i) \in \operatorname{Sym}\mathbb{C}$ are regarded as linear functions on the space of vacua $H^{-1}(E[1]) \cong \ker(\Delta : A \to A) \cong \mathbb{C}$ via

$$\pi(\varphi_i) \;:\; \ker(\Delta : A \to A) \;\longrightarrow\; \mathbb{C} \;,\;\; \underline{\Phi} \;=\; \Phi_0\, \mathbb{1} \;\longmapsto\; \frac{1}{N+1}\operatorname{Tr}\big(\varphi_i\, \underline{\Phi}\big) \;=\; \pi(\varphi_i)\, \Phi_0 \quad , \tag{5.1.38}$$

where the product $\pi(\varphi_i)\, \Phi_0$ in the last step is the usual multiplication of complex numbers (denoted by juxtaposition).

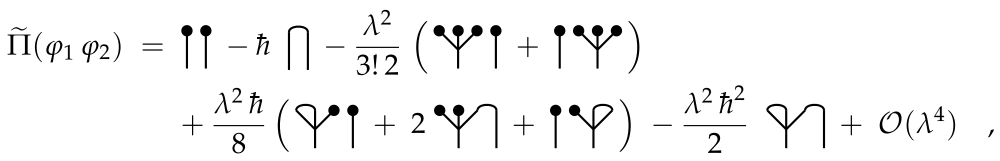
We denote (5.1.37) graphically by attaching vertices on top of the vertical lines



(5.1.39)

The vertices above should be interpreted as empty slots which can be evaluated against classical vacua $\underline{\Phi} \in \ker(\Delta : A \to A) \cong \mathbb{C}$. These purely classical contributions to the correlation functions are completely analogous to those one would obtain in traditional approaches to quantum field theory by expanding the field operator $\widehat{\Phi} + \underline{\Phi}$ around a generic classical solution $\underline{\Phi}$.

Example 5.1.4. For the 2-point function of massless Φ^4-theory, using (5.1.33)–(5.1.35), we obtain



(5.1.40)

as an element in $\operatorname{Sym}\mathbb{C}$. $\triangledown$

5.1.2 *Chern-Simons theory*

It is possible to define Chern-Simons theory on the fuzzy 2-sphere $\mathbb{S}^2_N$ because it admits a well-known 3-dimensional calculus, see e.g. [ARS00, GMS01]. We will focus on the Abelian Chern-Simons theory on $\mathbb{S}^2_N$ as in [GMS01] which, due to the noncommutativity of the differential calculus on $\mathbb{S}^2_N$, includes a ternary interaction term. Extending to the non-Abelian case with matrix gauge algebras such as $\mathfrak{gl}(n)$ or $\mathfrak{u}(n)$ is a straightforward process but will not present any essential novelties. This is the simplest example of a noncommutative field theory with gauge symmetries on which we will apply the BV formalism.

DIFFERENTIAL CALCULUS ON THE FUZZY 2-SPHERE. We begin by setting up the basic theory of differential forms on $\mathbb{S}^2_N$ which is required for the definition of Chern-Simons theory. The standard $\mathfrak{su}(2)$-equivariant differential calculus on the fuzzy sphere algebra (5.1.1) is given by the Chevalley-Eilenberg dg-algebra (see Example 4.1.9)

$$\Omega^\bullet(A) \,:=\, \mathrm{CE}^\bullet(\mathfrak{su}(2),A) \,=\, A\otimes \textstyle\bigwedge^\bullet \mathfrak{su}(2)^* \quad . \tag{5.1.41}$$

The dual of the Lie algebra basis $\{e_i\in\mathfrak{su}(2)\}_{i=1,2,3}$ defines a basis $\{\theta^i\in\Omega^1(A)\}_{i=1,2,3}$ for the A-module of 1-forms. This in turn generates the entire differential calculus $\Omega^\bullet(A)$. The basis for $\Omega^1(A)$ is central, i.e. $a\,\theta^i=\theta^i\,a$ for all $a\in A=\Omega^0(A)$, and $\theta^i\wedge\theta^j=-\theta^j\wedge\theta^i$, for all $i,j=1,2,3$. The de Rham differential is given by the Chevalley-Eilenberg differential

$$\mathrm{d}a \,=\, \frac{1}{\lambda_N}\,[X_i,a]\,\theta^i \quad , \qquad \mathrm{d}\theta^i \,=\, -\frac{\mathrm{i}}{2}\,\epsilon^{ijk}\,\theta^j\wedge\theta^k \quad , \tag{5.1.42}$$

for all $a\in A=\Omega^0(A)$ and $i=1,2,3$, together with the graded Leibniz rule

$$\mathrm{d}(\omega\wedge\zeta) \,=\, (\mathrm{d}\omega)\wedge\zeta+(-1)^p\,\omega\wedge(\mathrm{d}\zeta) \quad , \tag{5.1.43}$$

for all $\omega\in\Omega^p(A)$ and $\zeta\in\Omega^\bullet(A)$.[2] A significant difference is that the differential calculus on the fuzzy 2-sphere is 3-dimensional, while the differential calculus on the regular commutative 2-sphere is 2-dimensional. Higher-dimensional (covariant) calculi are in fact a common feature in noncommutative geometry and arise in a myriad of examples, from the fuzzy sphere to quantum groups.

As usual, one can define integration on top degree forms $\Omega^3(A)$, in this case by

$$\int \,:\, \Omega^3(A) \,\longrightarrow\, \mathbb{C}\ ,\ \ \omega \,=\, a\,\theta^1\wedge\theta^2\wedge\theta^3 \,\longmapsto\, \int\omega \,:=\, \frac{4\pi}{N+1}\,\mathrm{Tr}(a) \quad . \tag{5.1.44}$$

Using (5.1.42) together with the graded Leibniz rule (5.1.43), one checks that the integration map satisfies the Stokes theorem

$$\int \mathrm{d}\zeta \,=\, 0 \quad , \tag{5.1.45}$$

for all 2-forms $\zeta=\frac{1}{2}\,\zeta_{ij}\,\theta^i\wedge\theta^j\in\Omega^2(A)$.

2 The differential calculus $(\Omega^\bullet(A),\mathrm{d})$ we just introduced on the fuzzy 2-sphere is a so called *derivation based calculus* introduced in [D-V88], see also [D-V01, Mas08, Sch14] for reviews of this concept.

We end the discussion surrounding the differential calculus with some Hodge-theoretical constructions. The Hodge operator $\ast : \Omega^\bullet(A) \to \Omega^{3-\bullet}(A)$ on the fuzzy sphere is defined on the A-module basis of $\Omega^\bullet(A)$ by

$$\begin{aligned} \ast(\mathbb{1}) \,:=\, \tfrac{1}{3!}\,\epsilon^{ijk}\,\theta^i\wedge\theta^j\wedge\theta^k \quad , \quad \ast\,(\theta^i) \,:=\, \tfrac{1}{2!}\,\epsilon^{ijk}\,\theta^j\wedge\theta^k \quad , \\ \ast(\theta^i\wedge\theta^j) \,:=\, \epsilon^{ijk}\,\theta^k \quad , \quad \ast\,(\theta^i\wedge\theta^j\wedge\theta^k) \,:=\, \epsilon^{ijk}\,\mathbb{1} \quad . \end{aligned} \tag{5.1.46}$$

Note that the Hodge operator is idempotent, i.e. $\ast\ast(\omega) = \omega$, for all $\omega \in \Omega^p(A)$. From here we can define the codifferential

$$\delta \,:=\, (-1)^p \,\ast\mathrm{d}\ast \,:\, \Omega^p(A) \;\longrightarrow\; \Omega^{p-1}(A) \tag{5.1.47}$$

and the Hodge-de Rham Laplacian

$$\Delta \,:=\, -\big(\delta\,\mathrm{d} + \mathrm{d}\,\delta\big) \,:\, \Omega^p(A) \;\longrightarrow\; \Omega^p(A) \quad , \tag{5.1.48}$$

for all $p = 0, 1, 2, 3$. Acting with the Hodge-de Rham Laplacian on 0-forms, one shows that it coincides with the scalar Laplacian (5.1.6).

FREE BV THEORY. Having outlined the necessary geometric structures, we can now proceed to describe the non-interacting part of Abelian Chern-Simons theory on the fuzzy sphere as a free BV theory as in Definition 4.3.1.

Definition 5.1.5. The free BV theory associated to Abelian Chern-Simons theory is given by the cochain complex

$$E \,=\, \Omega^\bullet(A)[1] \,=\, \Big(\overset{(-1)}{\Omega^0(A)} \xrightarrow{-\mathrm{d}} \overset{(0)}{\Omega^1(A)} \xrightarrow{-\mathrm{d}} \overset{(1)}{\Omega^2(A)} \xrightarrow{-\mathrm{d}} \overset{(2)}{\Omega^3(A)} \Big) \quad , \tag{5.1.49}$$

i.e. $Q := \mathrm{d}$ is the de Rham differential, together with the pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle \,:\, E\otimes E \;\longrightarrow\; \mathbb{C}[-1]\,,\;\; \alpha\otimes\beta \;\longmapsto\; (-1)^{|\alpha|}\int \alpha\wedge\beta \quad , \tag{5.1.50}$$

where $|\alpha|$ denotes the cohomological degree of $\alpha \in E$. (Note that the latter differs from the de Rham degree as $|\alpha|_{\mathrm{dR}} = |\alpha| + 1$.)

Following the procedure in Section 4.3.1, we construct from the above input data the corresponding P_0-algebra

$$\mathrm{Obs}^{\mathrm{cl}} \,:=\, \big(\mathrm{Sym}\,E[1], Q, \{\,\cdot\,,\,\cdot\,\}\big) \tag{5.1.51}$$

of classical observables of the non-interacting theory. Note that the complex

$$E[1] \;=\; \Omega^\bullet(A)[2] \;=\; \Big(\overset{(-2)}{\Omega^0(A)} \xrightarrow{\ \mathrm{d}\ } \overset{(-1)}{\Omega^1(A)} \xrightarrow{\ \mathrm{d}\ } \overset{(0)}{\Omega^2(A)} \xrightarrow{\ \mathrm{d}\ } \overset{(1)}{\Omega^3(A)} \Big) \tag{5.1.52}$$

is concentrated in degrees -2, -1, 0 and 1.

INTERACTIONS. The next step is to augment the free Chern-Simons theory in Definition 5.1.5 with interaction terms using the formalism in Section 4.3.2. The associated Abelian cyclic L_∞-algebra is given by the cochain complex

$$E[-1] \;=\; \Omega^\bullet(A) \;=\; \Big(\overset{(0)}{\Omega^0(A)} \xrightarrow{\ \mathrm{d}\ } \overset{(1)}{\Omega^1(A)} \xrightarrow{\ \mathrm{d}\ } \overset{(2)}{\Omega^2(A)} \xrightarrow{\ \mathrm{d}\ } \overset{(3)}{\Omega^3(A)} \Big) \tag{5.1.53}$$

together with the cyclic structure

$$\langle\!\langle \,\cdot\,,\,\cdot\,\rangle\!\rangle \,:\, E[-1]\otimes E[-1] \;\longrightarrow\; \mathbb{C}[-3]\,,\;\; \alpha\otimes\beta \;\longmapsto\; \langle\!\langle \alpha,\beta\rangle\!\rangle \;=\; \int \alpha\wedge\beta \quad . \tag{5.1.54}$$

The above structure can be endowed with a compatible 2-bracket

$$\ell_2 \,:\, \Omega^\bullet(A)\otimes\Omega^\bullet(A) \;\longrightarrow\; \Omega^\bullet(A)\,,\;\; \alpha\otimes\beta \;\longmapsto\; [\alpha,\beta] \;:=\; \alpha\wedge\beta - (-1)^{|\alpha|\,|\beta|}\,\beta\wedge\alpha \tag{5.1.55}$$

given by the graded commutator in the differential calculus $\Omega^\bullet(A)$. Observe that, in opposition to commutative Chern-Simons theory, the bracket ℓ_2 is *not* zero due to the noncommutativity of the differential calculus on the fuzzy sphere $\mathbb{S}^2_N$.

We now would like to write down the contracted coordinate functions for this non-Abelian cyclic dg-Lie algebra. To this end, we choose a basis of $E[-1]$ which we denote by $c_a \in E[-1]^0 = \Omega^0(A)$, $A_b \in E[-1]^1 = \Omega^1(A)$, $A^+_c \in E[-1]^2 = \Omega^2(A)$ and $c^+_d \in E[-1]^3 = \Omega^3(A)$. In terms of this basis, the contracted coordinate functions take the form

$$\begin{aligned} \mathsf{a} \;=\; &\sum_a c^*_a\otimes c_a + \sum_b A^*_b\otimes A_b + \sum_c A^{+*}_c\otimes A^+_c + \sum_d c^{+*}_d\otimes c^+_d \\ &\in\; \big((\mathrm{Sym}\,E[1])\otimes E[-1]\big)^1 \quad , \end{aligned} \tag{5.1.56}$$

where the dual basis with respect to the cyclic structure $\langle\langle\,\cdot\,,\,\cdot\,\rangle\rangle$ is denoted by $c_a^* \in E[1]^1 = \Omega^3(A)$, $A_b^* \in E[1]^0 = \Omega^2(A)$, $A_c^{+*} \in E[1]^{-1} = \Omega^1(A)$ and $c_d^{+*} \in E[1]^{-2} = \Omega^0(A)$. The Chern-Simons interaction term thus reads as

$$\begin{aligned}\lambda\, I &= \frac{\lambda}{3!}\, \langle\langle \mathsf{a}, \ell_2^{\mathrm{ext}}(\mathsf{a},\mathsf{a})\rangle\rangle_{\mathrm{ext}} \\ &= \frac{\lambda}{3!} \sum_{b,b',b''} A_b^*\, A_{b'}^*\, A_{b''}^*\ \langle\langle A_b, [A_{b'}, A_{b''}]\rangle\rangle - \lambda \sum_{a,b,c} c_a^*\, A_b^*\, A_c^{+*}\ \langle\langle c_a, [A_b, A_c^+]\rangle\rangle \\ &\quad + \frac{\lambda}{2} \sum_{d,a,a'} c_d^{+*}\, c_a^*\, c_{a'}^*\ \langle\langle c_d^+, [c_a, c_{a'}]\rangle\rangle \ \in\ (\mathrm{Sym}\, E[1])^0 \quad . \end{aligned} \tag{5.1.57}$$

We would again like to emphasise that the products of the dual basis elements in (5.1.57) are given by the product of the symmetric algebra $\mathrm{Sym}\, E[1]$.

STRONG DEFORMATION RETRACT. In order to compute the cohomology of the complex (5.1.52) we will make use of the Whitehead lemma, see e.g. [Wei94, Theorem 7.8.9]. To do this, we recall that the differential calculus $\Omega^\bullet(A) = \mathrm{CE}^\bullet(\mathfrak{su}(2), A)$ is by definition the Chevalley-Eilenberg cochain complex of $\mathfrak{su}(2)$ with coefficients in the fuzzy sphere algebra (5.1.1). Regarding the latter as a $\mathfrak{su}(2)$-representation, we decompose it as $A \cong \bigoplus_{J=0}^N (J)$, where (J) denotes the irreducible spin J representation. Therefore, we have

$$\Omega^\bullet(A) \cong \bigoplus_{J=0}^{N} \mathrm{CE}^\bullet(\mathfrak{su}(2), (J)) \quad . \tag{5.1.58}$$

The Whitehead lemma tells us that the cohomology of $\mathrm{CE}^\bullet(\mathfrak{su}(2),(J))$ is trivial for all $J > 0$. From [Wei94, Corollary 7.8.10,Corollary 7.8.12] (also known as Whitehead's first and second lemmas), we know that $H^1\big(\mathrm{CE}^\bullet(\mathfrak{su}(2),\mathbb{C}\big) = H^2\big(\mathrm{CE}^\bullet(\mathfrak{su}(2),\mathbb{C}\big) = 0$. The cohomology in degree 0 can be read off from (5.1.42) to be $H^0\big(\mathrm{CE}^\bullet(\mathfrak{su}(2),\mathbb{C}\big) \cong \mathbb{C}$. For the cohomology in top degree, observe that $\mathrm{d}(\theta^i \wedge \theta^j) = 0$ from (5.1.42) and (5.1.43), which can be seen after rewriting $\theta^i \wedge \theta^j \wedge \theta^k = \epsilon^{ijk}\, \theta^1 \wedge \theta^2 \wedge \theta^3$. Therefore, the image of d in the top degree is trivial. Because $\mathrm{CE}^3(\mathfrak{su}(2),\mathbb{C}) \cong \mathbb{C}$ is 1-dimensional, $H^3\big(\mathrm{CE}^\bullet(\mathfrak{su}(2),\mathbb{C}\big) \cong \mathbb{C}$. It follows that

$$H^\bullet\big(\Omega^\bullet(A)\big) \cong H^\bullet\big(\mathrm{CE}^\bullet(\mathfrak{su}(2),(0))\big) \cong \mathbb{C} \oplus \mathbb{C}[-3] \tag{5.1.59}$$

is concentrated in differential form degrees 0 and 3. Therefore, the cohomology of the complex (5.1.52) is given by

$$H^\bullet(E[1]) = H^\bullet\big(\Omega^\bullet(A)[2]\big) \cong \mathbb{C}[2] \oplus \mathbb{C}[-1] \quad . \tag{5.1.60}$$

Following the procedure, we now construct a strong deformation retract as in (4.3.30). We begin by defining the maps ι and π. Using the unit $\eta(1) = \mathbb{1} \in \Omega^0(A) = A$ and its Hodge dual $*\eta(1) = *(\mathbb{1}) \in \Omega^3(A)$ (see (5.1.46)), we define the cochain map

$$\begin{array}{c} H^\bullet(E[1]) \\ {\scriptstyle\iota}\big\downarrow \\ E[1] \end{array} \quad = \quad \left(\begin{tikzcd} \mathbb{C} \arrow[r, "0"] \arrow[d, "\eta"'] & 0 \arrow[r, "0"] \arrow[d, "0"'] & 0 \arrow[r, "0"] \arrow[d, "0"'] & \mathbb{C} \arrow[d, "*\eta"'] \\ \Omega^0(A) \arrow[r, "\mathrm{d}"'] & \Omega^1(A) \arrow[r, "\mathrm{d}"'] & \Omega^2(A) \arrow[r, "\mathrm{d}"'] & \Omega^3(A) \end{tikzcd} \right) \quad . \tag{5.1.61}$$

The cochain map

$$\begin{array}{c} E[1] \\ {\scriptstyle\pi}\big\downarrow \\ H^\bullet(E[1]) \end{array} \quad = \quad \left(\begin{tikzcd} \Omega^0(A) \arrow[r, "\mathrm{d}"] \arrow[d, "\frac{1}{4\pi}\int *"'] & \Omega^1(A) \arrow[r, "\mathrm{d}"] \arrow[d, "0"'] & \Omega^2(A) \arrow[r, "\mathrm{d}"] \arrow[d, "0"'] & \Omega^3(A) \arrow[d, "\frac{1}{4\pi}\int"'] \\ \mathbb{C} \arrow[r, "0"'] & 0 \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \mathbb{C} \end{tikzcd} \right) \tag{5.1.62}$$

is defined using the Hodge operator (5.1.46) and the integration map (5.1.44), where the normalization factor $\frac{1}{4\pi}$ is chosen so that $\pi\,\iota = \mathrm{id}_{H^\bullet(E[1])}$ as required by the axioms of a strong deformation retract (see Definition 4.3.6).

To obtain a strong deformation retract, we also need a cochain homotopy. We begin by observing that the composite cochain map $\iota\,\pi : E[1] \to E[1]$ defines a projector onto the harmonic forms[3], leading to the decomposition

$$E[1] \;\cong\; E[1]^\perp \oplus H^\bullet(E[1]) \quad . \tag{5.1.63}$$

In virtue of the rank-nullity theorem of linear algebra, the Hodge-de Rham Laplacian restricts to an isomorphism $\Delta^\perp : E[1]^\perp \to E[1]^\perp$, whose inverse (i.e. the Green operator) we denote by $G^\perp : E[1]^\perp \to E[1]^\perp$. Together with the codifferential δ, we define the cochain homotopy

$$\xi \;:=\; \delta\, G^\perp \left(\mathrm{id}_{E[1]} - \iota\,\pi\right) \;\in\; \underline{\mathrm{hom}}(E[1], E[1])^{-1} \quad , \tag{5.1.64}$$

which results in the sought after strong deformation retract

$$\big(\mathbb{C}[2] \oplus \mathbb{C}[-1], 0\big) \underset{\iota}{\overset{\pi}{\leftrightarrows}} \big(\Omega^\bullet(A)[2], \mathrm{d}\big) \circlearrowleft \xi \tag{5.1.65}$$

3 A differential form $\alpha \in \Omega^\bullet(A)$ is *harmonic* if it is annihilated by the Hodge-de Rham Laplacian $\Delta\alpha = 0$. In this case, one can prove that the harmonic forms coincide with the de Rham cohomology (5.1.60) by adapting the usual Hodge theory [War83, Chapter 6] to this setting (some proofs are vastly simplified due to finite dimensionality).

for Chern-Simons theory. The relevant properties of Definition 4.3.6 are straightforward to check. For instance, one may check that

$$\begin{aligned}\partial(\xi) &= \mathrm{d}\,\xi + \xi\,\mathrm{d} = \mathrm{d}\,\delta\,G^{\perp}\left(\mathrm{id}_{E[1]} - \iota\,\pi\right) + \delta\,G^{\perp}\left(\mathrm{id}_{E[1]} - \iota\,\pi\right)\mathrm{d}\\ &= \mathrm{d}\,\delta\,G^{\perp}\left(\mathrm{id}_{E[1]} - \iota\,\pi\right) + \delta\,\mathrm{d}\,G^{\perp}\left(\mathrm{id}_{E[1]} - \iota\,\pi\right)\\ &= -\Delta^{\perp}\,G^{\perp}\left(\mathrm{id}_{E[1]} - \iota\,\pi\right) = \iota\,\pi - \mathrm{id}_{E[1]} \quad .\end{aligned} \tag{5.1.66}$$

The second line follows from the fact that the projector $\iota\,\pi$ commutes with d since it is a cochain map, together with the property that the Green operator commutes with d because $\mathrm{d}\,\Delta^{\perp} = \Delta^{\perp}\,\mathrm{d}$. In the final line, we used the definition of the Hodge-de Rham Laplacian (5.1.48) and that $G^{\perp}$ is the inverse of $\Delta^{\perp}$.

CORRELATION FUNCTIONS. By applying the homological perturbation lemma (Theorem 4.3.7) on the extension of the strong deformation retract (5.1.65) to symmetric algebras, together with the small perturbation

$$\delta = \hbar\,\Delta_{\mathrm{BV}} + \{\lambda\,I,\,\cdot\,\} \quad , \tag{5.1.67}$$

where $\lambda\,I$ is the Chern-Simons interaction term (5.1.57), we obtain the deformed strong deformation retract

$$\begin{tikzcd} \left(\mathrm{Sym}\left(\mathbb{C}[2]\oplus\mathbb{C}[-1]\right),\widetilde{\delta}\,\right) \arrow[r, shift right, "\widetilde{J}"'] & \left(\mathrm{Sym}\,E[1], Q+\delta\right) \arrow[l, shift right, "\widetilde{\Pi}"'] \arrow[loop right, "\widetilde{\Xi}"] \end{tikzcd} \quad . \tag{5.1.68}$$

The perturbative expansion

$$\widetilde{\Pi}(\varphi_1\cdots\varphi_n) = \sum_{k=0}^{\infty} \Pi\big((\delta\,\Xi)^k(\varphi_1\cdots\varphi_n)\big) \tag{5.1.69}$$

of the n-point correlation functions can then be computed in orders of λ and/or $\hbar$ by using the algebraic properties of $\Xi = \mathrm{Sym}\,\xi$ (4.3.34), the BV Laplacian Δ_{BV} (4.3.15c) and the graded derivation $\{\lambda\,I,\,\cdot\,\}$ (see the P_0-algebra axioms in Remark 4.3.4). We will however not write out any explicit examples of correlation functions in this case; the computations are vastly more cumbersome than for the case of scalar field theories due to the additional field species.

5.2 BV QUANTISATION OF BRAIDED FIELD THEORIES

The BV formalism outlined in Section 4.3.1 can be generalised to also include theories with a *triangular* Hopf algebra symmetry, i.e. field theories defined in the representation category of a triangular Hopf algebra. The goal of this section is to make this explicit and provide the details. We will use the adjective "braided" (instead of the categorically more accurate "symmetric braided") as in [DCGRS20, DCGRS21] in order to indicate that the theories might contain symmetries encoded by a non-identity triangular R-matrix, together with equivariance. We will unfortunately not treat the more general quasi-triangular case as it is considerably more complicated since it obstructs the formulation of symmetry properties and the Jacobi identity. We refer to Section 1.2 for a review of the relevant Hopf algebra constructions.

5.2.1 *Finite-dimensional braided BV formalism*

Let H be a triangular Hopf algebra with triangular structure R. Recall from Section 1.2 that the category ${}_H\mathsf{Mod}$ of left H-modules is a closed symmetric monoidal category. Since ${}_H\mathsf{Mod}$ is an Abelian category, we may study cochain complexes in ${}_H\mathsf{Mod}$ using standard techniques form homological algebra. We denote the category of cochain complexes of left H-modules by

$$
{}_H\mathsf{Ch} := \mathsf{Ch}\big({}_H\mathsf{Mod}\big) \quad . \tag{5.2.1}
$$

An object in ${}_H\mathsf{Ch}$ is a $\mathbb{Z}$-graded left H-module V (in particular, each graded component is itself an H-module) together with an H-equivariant differential $d : V \to V[1]$, i.e. $\mathrm{d}(h \triangleright v) = h \triangleright (\mathrm{d}v)$, for all $h \in H$ and $v \in V$. The morphisms consist of H-equivariant cochain maps. Just as for $\mathsf{Ch}_{\mathbb{K}}$ in Section 4.1, ${}_H\mathsf{Ch}$ is a closed symmetric monoidal category. The monoidal product is given by endowing (4.1.4) with the left tensor product H-action (1.2.5), the monoidal unit is $\mathbb{K}$ regarded as a left H-module with the trivial left H-action ($h \triangleright c = \epsilon(h)\, c$, for all $h \in H$ and $c \in \mathbb{K}$) and the internal hom is given by equipping (4.1.6) with the left adjoint H-action (1.2.7). The symmetric braiding is the combination of (4.1.5) with (1.2.17),

$$
\tau_R \,:\, V \otimes W \longrightarrow W \otimes V \ , \quad v \otimes w \longmapsto (-1)^{|v|\,|w|}\, (R_\alpha \triangleright w) \otimes (R^\alpha \triangleright v) \quad , \tag{5.2.2}
$$

for all $V, W \in {}_H\mathsf{Ch}$, involving both the Koszul signs and the R-matrix $R = R^\alpha \otimes R_\alpha \in H \otimes H$.

We now give the generalisation of Definition 4.3.1 to the present case.

Definition 5.2.1. A *free braided BV theory* is an object $E = (E, -Q) \in {}_H\mathsf{Ch}$ together with an ${}_H\mathsf{Ch}$-morphism $\langle\,\cdot\,,\,\cdot\,\rangle : E \otimes E \to \mathbb{K}[-1]$ that is nondegenerate and antisymmetric, i.e. $\langle\,\cdot\,,\,\cdot\,\rangle = -\langle\,\cdot\,,\,\cdot\,\rangle \circ \tau_R$, or explicitly

$$\langle \varphi, \psi \rangle \ = \ -(-1)^{|\varphi|\,|\psi|} \ \langle R_\alpha \triangleright \psi, R^\alpha \triangleright \varphi \rangle \quad , \tag{5.2.3}$$

for all $\varphi, \psi \in E$.

Before introducing the (symmetric) braided version of P_0-algebras (generalising Definition 4.3.3), we need the concept of *braided commutative dg-algebras* (generalising commutative differential graded algebras, see Definition 4.1.5 and below). The fact that ${}_H\mathsf{Ch}$ is symmetric monoidal means that there is an associated category $\mathsf{CAlg}({}_H\mathsf{Ch})$ of commutative algebras in ${}_H\mathsf{Ch}$. That is, the objects of $\mathsf{CAlg}({}_H\mathsf{Ch})$ are the monoids in ${}_H\mathsf{Ch}$ such that the associated product is (braided) commutative. Let us spell this out explicitly. An object in $\mathsf{CAlg}({}_H\mathsf{Ch})$ (i.e. a *braided commutative dg-algebra*) is a triple (A, μ, η), consisting of an object $A \in {}_H\mathsf{Ch}$ and two ${}_H\mathsf{Ch}$-morphisms $\mu : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ satisfying the associativity and unitality axioms (4.1.8), together with commutativity $\mu \circ \tau_R = \mu$ (with braiding (5.2.2)) which in this case takes the explicit form

$$a\, b \ = \ (-1)^{|a|\,|b|} \ (R_\alpha \triangleright b)\, (R^\alpha \triangleright a) \quad , \tag{5.2.4}$$

for all $a, b \in A$. Being morphisms in ${}_H\mathsf{Ch}$, the product μ and unit η are by definition H-equivariant, which translates to

$$h \triangleright (a\, b) \ = \ (h_{\underline{1}} \triangleright a)\ (h_{\underline{2}} \triangleright b) \quad , \qquad h \triangleright \mathbb{1} \ = \ \epsilon(h)\, \mathbb{1} \quad , \tag{5.2.5}$$

for all $h \in H$ and $a, b \in A$. We will mainly be concerned with a specific example of a braided commutative dg-algebra, namely that of the *braided symmetric algebra* $\mathrm{Sym}_R V \in \mathsf{CAlg}({}_H\mathsf{Ch})$ associated with an object $V \in {}_H\mathsf{Ch}$. In analogy with the usual case, the H-equivariant dg-algebra $\mathrm{Sym}_R V \in \mathsf{CAlg}({}_H\mathsf{Ch})$ is generated by all $v \in V$, modulo the commutation relations involving the R-matrix

$$v\, v' \ = \ (-1)^{|v|\,|v'|} \ (R_\alpha \triangleright v')\, (R^\alpha \triangleright v) \quad , \tag{5.2.6}$$

for all homogeneous $v, v' \in V$. As in the non-braided case, we have that

$$\mathrm{Sym}_R\, H^\bullet(V) \cong H^\bullet(\mathrm{Sym}_R\, V) \tag{5.2.7}$$

since we are working in characteristic 0 (the proof from Example 4.1.7 can be generalised to accommodate the braided case in a straightforward manner).

Having introduced braided commutative dg-algebras, we may now provide the generalisation of Definition 4.3.3, given here in the explicit format of Remark 4.3.4.

Definition 5.2.2. A *braided P_0-algebra* is a braided commutative dg-algebra $A \in \mathsf{CAlg}({}_H\mathsf{Ch})$ together with an ${}_H\mathsf{Ch}$-morphism $\{\,\cdot\,,\,\cdot\,\} : A[-1] \otimes A[-1] \to A[-1]$ satisfying the following axioms:

(i) Braided antisymmetry: For all $a, b \in A$,

$$\{a, b\} \;=\; -(-1)^{(|a|+1)\,(|b|+1)}\,\{R_\alpha \triangleright b, R^\alpha \triangleright a\} \quad . \tag{5.2.8}$$

(ii) Braided Jacobi identity: For all $a, b, c \in A$,

$$\begin{aligned} 0 \;=\; &(-1)^{(|a|+1)\,(|c|+1)}\;\{a, \{b, c\}\} \\ &+ (-1)^{(|b|+1)\,(|a|+1)}\;\{R_\alpha \triangleright b, \{R_\beta \triangleright c, R^\beta\, R^\alpha \triangleright a\}\} \\ &+ (-1)^{|c|+1)\,(|b|+1)}\;\{R_\beta\, R_\alpha \triangleright c, \{R^\beta \triangleright a, R^\alpha \triangleright b\}\} \quad . \end{aligned} \tag{5.2.9}$$

(iii) Braided derivation property: For all $a, b, c \in A$,

$$\{a, b\,c\} \;=\; \{a, b\}\,c + (-1)^{(|a|+1)\,|b|}\,(R_\alpha \triangleright b)\;\{R^\alpha \triangleright a, c\} \quad . \tag{5.2.10}$$

Remark 5.2.3. The fact that $\{\,\cdot\,,\,\cdot\,\} : A[-1] \otimes A[-1] \to A[-1]$ is a ${}_H\mathsf{Ch}$-morphism, translates to the compatibility with the differential condition

$$\mathrm{d}\{a, b\} \;=\; \{\mathrm{d}a, b\} + (-1)^{|a|+1}\,\{a, \mathrm{d}b\} \tag{5.2.11}$$

for all $a, b \in A$, just as before. △

The classical observables of a free BV theory $(E, -Q, \langle\,\cdot\,,\,\cdot\,\rangle)$ are given by the braided P_0-algebra

$$\mathrm{Obs}^{\mathrm{cl}} \;:=\; \big(\mathrm{Sym}_R E[1], Q, \{\,\cdot\,,\,\cdot\,\}\big) \tag{5.2.12}$$

consisting of the symmetric braided algebra of the dual $E^* \cong E[1]$ together with the bracket defined by

$$\{\varphi, \psi\} \;=\; \langle \varphi, \psi \rangle \, \mathbb{1} \quad , \tag{5.2.13}$$

for all $\varphi, \psi \in E[1]$ such that the braided antisymmetry (i) and braided derivation (iii) properties of Definition 5.2.2 are satisfied.

As before, interactions and quantisation are given by certain deformations of the differential Q in (5.2.12). To obtain a deformed cochain complex of left H-modules, one should consider *H-invariant* deformations. We treat both cases one by one.

The interaction terms consist of 0-cochains $I \in (\mathrm{Sym}_R E[1])^0$ which are H-invariant, i.e. $h \triangleright I = \epsilon(h)\, I$ for all $h \in H$, and satisfy the classical master equation

$$Q(\lambda\, I) + \tfrac{1}{2}\, \{\lambda\, I, \lambda\, I\} \;=\; 0 \quad , \tag{5.2.14}$$

where λ is an H-invariant formal parameter (coupling constant). The H-invariance of $\lambda\, I$ guarantees the H-equivariance of the map $\{\lambda\, I, \,\cdot\,\}$ as a consequence of the fact that $\{\,\cdot\,,\,\cdot\,\} : A[-1] \otimes A[-1] \to A[-1]$ is a ${}_H\mathsf{Ch}$-morphism (recall how H acts on tensor products (1.2.5)) and counitality of the coproduct (1.2.3a). The classical observables for the interacting braided BV theory with interaction term I is then the braided P_0-algebra

$$\mathrm{Obs}^{\mathrm{cl,int}} \;:=\; \big(\mathrm{Sym}_R E[1], Q^{\mathrm{int}}, \{\,\cdot\,,\,\cdot\,\}\big) \quad \text{with} \quad Q^{\mathrm{int}} \;:=\; Q + \{\lambda\, I, \,\cdot\,\} \quad . \tag{5.2.15}$$

In the case of quantisation, observe that the definition of the BV Laplacian in (4.3.15) is compatible with our braided case without modification. Therefore, it defines an ${}_H\mathsf{Ch}$-morphism $\Delta_{\mathrm{BV}} : \mathrm{Sym}_R E[1] \to (\mathrm{Sym}_R E[1])[1]$ that is nilpotent $\Delta_{\mathrm{BV}}^2 = 0$. The explicit formula (4.3.15c) is however slightly different in the braided case, modified with suitable R-matrix actions. One finds that

$$\begin{aligned} \Delta_{\mathrm{BV}}\big(\varphi_1 \cdots \varphi_n\big) \;=\; \sum_{i<j} (-1)^{\sum_{k=1}^{i} |\varphi_k| + |\varphi_j| \sum_{k=i+1}^{j-1} |\varphi_k|} \, \big\langle \varphi_i, R_{\alpha_{i+1}} \cdots R_{\alpha_{j-1}} \triangleright \varphi_j \big\rangle \\ \times\; \varphi_1 \cdots \varphi_{i-1}\, \widehat{\varphi}_i \, (R^{\alpha_{i+1}} \triangleright \varphi_{i+1}) \cdots (R^{\alpha_{j-1}} \triangleright \varphi_{j-1})\, \widehat{\varphi}_j \, \varphi_{j+1} \cdots \varphi_n \quad , \end{aligned}$$

for all $\varphi_1, \dots, \varphi_n \in E[1]$ with $n \geq 2$. The quantum observables for the non-interacting braided BV theory is then given by the braided E_0-algebra

$$\mathrm{Obs}^{\hbar} \;:=\; \big(\mathrm{Sym}_R E[1], Q^{\hbar}\big) \quad \text{with} \quad Q^{\hbar} \;:=\; Q + \hbar\, \Delta_{\mathrm{BV}} \quad . \tag{5.2.16}$$

Lastly, in order to describe the combined case, we consider interaction terms in the form of H-invariant 0-cochains $I \in (\mathrm{Sym}_R E[1])^0$ that satisfy the quantum master equation

$$Q(\lambda\, I) + \hbar\, \Delta_{\mathrm{BV}}(\lambda\, I) + \tfrac{1}{2}\, \{\lambda\, I, \lambda\, I\} \;=\; 0 \quad . \tag{5.2.17}$$

Then the quantum observables for the interacting braided BV theory with interaction term I constitute the braided E_0-algebra

$$\mathrm{Obs}^{\mathrm{int},\hbar} \;:=\; \big(\mathrm{Sym}_R E[1], Q^{\mathrm{int},\hbar}\big) \tag{5.2.18}$$

with

$$Q^{\mathrm{int},\hbar} \;:=\; Q + \hbar\, \Delta_{\mathrm{BV}} + \{\lambda\, I, \,\cdot\,\} \quad . \tag{5.2.19}$$

5.2.2 *Braided L_∞-algebras, their cyclic versions and interaction terms*

The construction in Section 4.3.2 of interaction terms satisfying the classical (and quantum) master equation using cyclic L_∞-algebra structures generalises to braided BV theories by using *braided L_∞-algebras*, which were introduced in [DCGRS20, DCGRS21]. We review the relevant constructions here.

Definition 5.2.4. A *braided L_∞-algebra* is a $\mathbb{Z}$-graded left H-module L together with a collection $\{\ell_n : L^{\otimes n} \to L\}_{n\in\mathbb{Z}_{\geq 1}}$ of H-equivariant graded braided antisymmetric linear maps of degree $|\ell_n| = 2-n$ that satisfy the braided homotopy Jacobi identities

$$\sum_{k=0}^{n-1} (-1)^k\, \ell_{k+1} \circ \big(\ell_{n-k} \otimes \mathrm{id}_{L^{\otimes k}}\big) \;\circ \sum_{\sigma\in \mathrm{Sh}(n-k;k)} \mathrm{sgn}(\sigma)\, \tau_R^\sigma \;=\; 0 \quad , \tag{5.2.20}$$

for all $n \geq 1$, where $\tau_R^\sigma : L^{\otimes n} \to L^{\otimes n}$ denotes the action of the permutation σ via the symmetric braiding τ_R on the category of graded left H-modules.

Remark 5.2.5. Let us unpack some of the details of the definition above. The graded braided antisymmetry of $\ell_n : L^{\otimes n} \to L$ means that

$$\ell_n\big(v_1,\dots,v_n\big) \;=\; -(-1)^{|v_i|\,|v_{i+1}|}\; \ell_n\big(v_1,\dots,v_{i-1}, R_\alpha \triangleright v_{i+1}, R^\alpha \triangleright v_i, v_{i+2},\dots,v_n\big) \tag{5.2.21}$$

for all $i = 1,\dots,n-1$ and all homogeneous elements $v_1,\dots,v_n \in L$. The permutation action $\tau_R^\sigma : L^{\otimes n} \to L^{\otimes n}$ in (5.2.20) also includes, apart from the usual Koszul signs, appropriate actions of the R-matrix as in (5.2.2). As discussed in Section 4.2, below Definition 4.2.1, every braided L_∞-algebra has an underlying cochain complex $(L, \mathrm{d}_L := \ell_1) \in {}_H\mathsf{Ch}$, and the binary bracket $\ell_2 : L \otimes L \to L$ is an ${}_H\mathsf{Ch}$-morphism. If ℓ_2 is the only non-vanishing bracket, we recover a braided Lie algebra in the sense of [Maj94]. △

The definition of a cyclic braided L_∞-algebra given below is specialised to our situation, see Remark 4.2.3.

Definition 5.2.6. A *cyclic braided L_∞-algebra* is a braided L_∞-algebra $(L, \{\ell_n\})$ together with a nondegenerate braided symmetric ${}_H\mathsf{Ch}$-morphism $\langle\!\langle \cdot , \cdot \rangle\!\rangle : L \otimes L \to \mathbb{K}[-3]$ that satisfies the cyclicity condition

$$\langle\!\langle v_0, \ell_n(v_1, \dots, v_n) \rangle\!\rangle = (-1)^{n\,(|v_0|+1)} \, \langle\!\langle R_{\alpha_0} \cdots R_{\alpha_{n-1}} \triangleright v_n, \ell_n(R^{\alpha_0} \triangleright v_0, \dots, R^{\alpha_{n-1}} \triangleright v_{n-1}) \rangle\!\rangle \quad , \tag{5.2.22}$$

for all $n \geq 1$ and all homogeneous elements $v_0, v_1, \dots, v_n \in L$.

As in the ordinary case in Section 4.3.2, any free braided BV theory $(E, -Q, \langle \cdot , \cdot \rangle)$ in the sense of Definition 5.2.1 defines an Abelian cyclic braided L_∞-algebra given by $E[-1]$, $\ell_1 := \mathrm{d}_{E[-1]} = Q$ and cyclic structure

$$\langle\!\langle \cdot , \cdot \rangle\!\rangle \, : \, E[-1] \otimes E[-1] \, \cong \, (E \otimes E)[-2] \xrightarrow{\langle \cdot , \cdot \rangle[-2]} \mathbb{K}[-1][-2] \, \cong \, \mathbb{K}[-3] \quad . \tag{5.2.23}$$

Then, as before, the introduction of interaction terms $I \in (\mathrm{Sym}_R E[1])^0$ that satisfy the classical master equation (5.2.14) is equivalent to endowing the Abelian cyclic braided L_∞-algebra $(E[-1], \ell_1, \langle\!\langle \cdot , \cdot \rangle\!\rangle)$ with compatible higher brackets $\{\ell_n\}_{n\geq 2}$. Again, this is obtained from the homotopy Maurer-Cartan action and is given by

$$\lambda \, I \, = \, \sum_{p \geq 3} \frac{\lambda^{p-2}}{p!} \, \langle\!\langle \mathsf{a}, \ell^{\mathrm{ext}}_{p-1}(\mathsf{a}, \dots, \mathsf{a}) \rangle\!\rangle_{\mathrm{ext}} \, \in \, (\mathrm{Sym}_R E[1])^0 \quad , \tag{5.2.24}$$

where the contracted coordinate functions

$$\mathsf{a} \, := \, \sum_\alpha \varrho^\alpha \otimes \varepsilon_\alpha \, \in \, \big((\mathrm{Sym}_R E[1]) \otimes E[-1] \big)^1 \tag{5.2.25}$$

are defined by making a choice of basis $\{\varepsilon_\alpha \in E[-1]\}$ with dual basis $\{\varrho^\alpha \in E[-1]^* \cong E[2]\}$. We would like to stress that since this is the braided case, the extended brackets

$$\ell^{\mathrm{ext}}_n \, : \, \big((\mathrm{Sym}_R E[1]) \otimes E[-1] \big)^{\otimes n} \, \longrightarrow \, (\mathrm{Sym}_R E[1]) \otimes E[-1] \tag{5.2.26}$$

and the extended pairing

$$\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_{\mathrm{ext}}\,:\,\big((\mathrm{Sym}_R E[1])\otimes E[-1]\big)\otimes\big((\mathrm{Sym}_R E[1])\otimes E[-1]\big)\;\longrightarrow\;(\mathrm{Sym}_R E[1])[-3] \tag{5.2.27}$$

receive appropriate actions of the R-matrix (in addition to the obvious Koszul signs, see [JRSW19, Section 2.3]). For the extended pairing (cf. (4.3.26)) we have

$$\langle\!\langle a\otimes v, a'\otimes v'\rangle\!\rangle_{\mathrm{ext}} \;=\; (-1)^{|a|+|a'|+|v|\,|a'|}\;a\,(R_\alpha\triangleright a')\;\langle\!\langle R^\alpha\triangleright v, v'\rangle\!\rangle \quad , \tag{5.2.28}$$

for all homogeneous $a, a'\in\mathrm{Sym}_R E[1]$ and $v, v'\in E[-1]$, and similarly for the extended brackets ℓ_n^{ext} with $n\geq 2$ (cf. (4.2.8b)).

Remark 5.2.7. The claim that (5.2.24) satisfies the classical master equation (5.2.14) can be proven by the exact same calculation as in the ordinary case , see e.g. [JRSW19, Section 4.3]. The reason is that the contracted coordinate functions (5.2.25) are H-invariant elements of $(\mathrm{Sym}_R E[1])\otimes E[-1]$. This implies that there are no occurrences of R-matrices in the properties of the extended brackets ℓ_n^{ext} and the extended pairing $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle_{\mathrm{ext}}$ in the evaluation on tensor products of the H-invariant element a. In the same vein, the proofs from the ordinary case [JRSW19, Section 4.3] can be used to show that the interaction term (5.2.24) is annihilated by the BV Laplacian, i.e. $\Delta_{\mathrm{BV}}(\lambda\,I)=0$, and consequently that it also satisfies the quantum master equation (5.2.17). △

5.2.3 *Correlation functions and braided homological perturbation theory*

The homological perturbation lemma (Theorem 4.3.7) extends to the braided setting if we assume that the perturbations $\delta\in\underline{\mathrm{hom}}(V,V)^1$ are H-invariant. This is the situation in our case: recall that the perturbations corresponding to interactions and quantisation from Section 5.2.1 are of such nature. We will spell out the details in the following.

Definition 5.2.8. A *braided strong deformation retract* of an object $V\in{}_H\mathsf{Ch}$ onto its cohomology $H^\bullet(V)$ is a strong deformation retract

$$\begin{tikzcd} (H^\bullet(V),0) \arrow[r, shift right, "\iota"'] & (V,\mathrm{d}) \arrow[l, shift right, "\pi"'] \arrow[loop right, "\xi"] \end{tikzcd} \quad , \tag{5.2.29}$$

in the sense of Definition 4.3.6, such that π and ι are H-equivariant, i.e. morphisms in ${}_H\mathsf{Ch}$, and the homotopy $\xi\in\underline{\mathrm{hom}}(V,V)^{-1}$ is H-invariant, i.e. $h\triangleright\xi=\epsilon(h)\,\xi$ for all $h\in H$.

Corollary 5.2.9. *Consider a braided strong deformation retract as in* (5.2.29). *Let* $\delta \in \underline{\mathrm{hom}}(V,V)^1$ *be a small* H*-invariant perturbation, i.e.* $h \triangleright \delta = \epsilon(h)\,\delta$ *for all* $h \in H$. *Then the expressions* (4.3.29) *define a braided strong deformation retract.*

Proof. By direct inspection, one observes that the explicit formulas in (4.3.29) satisfy the necessary H-equivariance or H-invariance properties. $\square$

Let $(E, -Q, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a free BV theory in the sense of Definition 5.2.1 and choose any strong deformation retract for its dual complex

$$\begin{tikzcd} (H^\bullet(E[1]),0) \arrow[r, shift right, "\iota"'] & (E[1],Q) \arrow[l, shift right, "\pi"'] \arrow[loop right, "\xi"] \end{tikzcd} \quad . \tag{5.2.30}$$

A similar construction as in Section 4.3.3 gives rise to a strong deformation retract

$$\begin{tikzcd} \big(\mathrm{Sym}_R\, H^\bullet(E[1]),0\big) \arrow[r, shift right, "\mathrm{Sym}_R\,\iota"'] & \big(\mathrm{Sym}_R\, E[1],Q\big) \arrow[l, shift right, "\mathrm{Sym}_R\,\pi"'] \arrow[loop right, "\mathrm{Sym}_R\,\xi"] \end{tikzcd} \quad . \tag{5.2.31}$$

The cochain homotopy $\mathrm{Sym}_R\xi$ is defined analogously to before, see (4.3.34) and below. This works because the homotopy $\xi \in \underline{\mathrm{hom}}(V,V)^{-1}$ is H-invariant. Note that $\mathrm{Sym}_R\, H^\bullet(E[1]) \cong H^\bullet(\mathrm{Sym}_R\, E[1]$ (see (5.2.7)). The correlation functions of both braided non-interacting and interacting quantum BV theories can be computed by applying this version of the homological perturbation lemma (Corollary 5.2.9) to (5.2.31) and the deformed differentials of Section 5.2.1. This works in complete analogy to the ordinary case reviewed at the end of Section 4.3.3.

5.3 BRAIDED FIELD THEORIES ON THE FUZZY TORUS

To illustrate the construction from Section 5.2, we consider the example of scalar field theories on the fuzzy 2-torus. We shall see that this reproduces many aspects of Oeckel's braided quantum field theory for symmetric braidings [Oec00]. In this section we will fix the underlying field to be the field of complex numbers $\mathbb{K} = \mathbb{C}$.

THE FUZZY 2-TORUS. We begin by introducing the fuzzy torus and its Hopf algebra symmetry, see e.g. [BGa19, BSS17] for further details. Let $N \in \mathbb{Z}_{>0}$ be a positive integer and set

$$q := \mathrm{e}^{2\pi\,\mathrm{i}/N} \in \mathbb{C} \quad . \tag{5.3.1}$$

The algebra of functions of the fuzzy torus $\mathbb{T}^2_N$ is defined as the noncommutative $*$-algebra

$$A := \mathbb{C}[U,V] \,/\, \big(U\,U^* - \mathbb{1}\,,\; V\,V^* - \mathbb{1}\,,\; U\,V - q\,V\,U\,,\; U^N - \mathbb{1}\,,\; V^N - \mathbb{1}\big) \tag{5.3.2}$$

freely generated by two elements U and V, modulo the $*$-ideal generated by the displayed relations. The generators U and V should be thought of as the two exponential functions corresponding to the two 1-cycles of $\mathbb{T}^2_N$. An element $a \in A$ can be uniquely written as $a = \sum_{i,j\in\mathbb{Z}_N} a_{ij}\,U^i\,V^j$, where the constants $a_{ij} \in \mathbb{C}$ should be interpreted as Fourier coefficients.[4]

The fuzzy 2-torus has a (discrete) translation symmetry encoded by the left action $\triangleright : H \otimes A \to A$ of the group Hopf algebra $H := \mathbb{C}[\mathbb{Z}^2_N]$, see Example 1.2.8. Recall that we denoted the basis vectors by $\underline{k} = (k_1, k_2) \in H$, with $k_1, k_2 \in \mathbb{Z}_N$ integers modulo N. Concretely these act on the generators of A as

$$\underline{k} \triangleright U \;:=\; q^{k_1}\,U \quad , \qquad \underline{k} \triangleright V \;:=\; q^{k_2}\,V \quad . \tag{5.3.3}$$

To extend this action to the entirety of A, we impose that A is a left H-module algebra, i.e. $\underline{k} \triangleright (a\,b) = (\underline{k}_{\underline{1}} \triangleright a)\,(\underline{k}_{\underline{2}} \triangleright b) = (\underline{k} \triangleright a)\,(\underline{k} \triangleright b)$, for all $a, b \in A$, where we have used the coproduct of $H = \mathbb{C}[\mathbb{Z}^2_N]$.

We equip H with the triangular structure as in Example 1.2.8

$$R \;:=\; \frac{1}{N^2} \sum_{\underline{s},\underline{t}\in\mathbb{Z}^2_N} q^{\underline{s}\Theta\underline{t}}\;\underline{s} \otimes \underline{t} \;\in\; H \otimes H \qquad \text{with} \quad \Theta \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad . \tag{5.3.4}$$

With this choice of triangular structure, A becomes a *braided commutative* left H-module algebra, i.e. $a\,b = (R_\alpha \triangleright b)\,(R^\alpha \triangleright a)$ for all $a, b \in A$. This can without loss of generality be shown through a straightforward calculation on the basis elements $a = U^i\,V^j$ and $b = U^k\,V^l$, for some $i, j, k, l \in \mathbb{Z}_N$. By the commutation relations in (5.3.2), we calculate

$$(U^i\,V^j)\,(U^k\,V^l) \;=\; q^{\,i\,l - j\,k}\;(U^k\,V^l)\,(U^i\,V^j) \quad . \tag{5.3.5a}$$

4 The torus algebra A can be identified as $A \cong \mathrm{Mat}_N(\mathbb{C})$ by realizing the elements U and V as $N \times N$ clock and shift matrices, see e.g. [LLS01]. This level of concreteness is however not required in our case.

On the other hand, from the definition of the R-matrix (5.3.4) and of the left action (5.3.3), one computes

$$\begin{aligned}\big(R_\alpha \triangleright (U^k\, V^l)\big)\, \big(R^\alpha \triangleright (U^i\, V^j)\big) \;&=\; \frac{1}{N^2} \sum_{\underline{s},\underline{t}\in\mathbb{Z}_N^2} q^{\underline{s}\Theta\underline{t}}\, \big(\underline{t} \triangleright (U^k\, V^l)\big)\, \big(\underline{s} \triangleright (U^i\, V^j)\big) \\ &=\; \frac{1}{N^2} \sum_{\underline{s},\underline{t}\in\mathbb{Z}_N^2} q^{\underline{s}\Theta\underline{t}}\, q^{t_1\, k + t_2\, l}\, q^{s_1\, i + s_2\, j}\, (U^k\, V^l)\, (U^i\, V^j) \\ &=\; q^{i\,l - j\,k}\, (U^k\, V^l)\, (U^i\, V^j) \quad , \end{aligned} \tag{5.3.5b}$$

where the last step follows from (1.2.27). Since the two expressions coincide, we have showed that A is braided commutative.

Integration on the fuzzy torus is defined via the linear map

$$\int \,:\, A \longrightarrow \mathbb{C}\;,\quad a \,=\, \sum_{i,j\in\mathbb{Z}_N} a_{ij}\, U^i\, V^j \;\longmapsto\; \int a \,:=\, a_{00} \quad . \tag{5.3.6}$$

This map is both H-equivariant and cyclic, i.e. $\int a\, b = \int b\, a$ for all $a, b \in A$. The scalar Laplacian $\Delta \,:\, A \longrightarrow A$ is defined by

$$\Delta(a) \,:=\, -\frac{1}{\big(q^{1/2} - q^{-1/2}\big)^2}\, \Big([U, [U^*, a]] + [V, [V^*, a]]\Big) \quad , \tag{5.3.7}$$

for all $a \in A$, where we have chosen the square root $q^{1/2} := \mathrm{e}^{\pi\,\mathrm{i}/N} \in \mathbb{C}$ of q. One observes that the scalar Laplacian is H-equivariant under the action (5.3.3) because the powers of q emerging from the action on U and on U^* compensate each other, and similarly for those from V and V^*. There is a basis of eigenfunctions for the Laplacian given by

$$e_{\underline{k}} \,:=\, U^{k_1}\, V^{k_2} \,\in\, A \quad , \tag{5.3.8}$$

for all $\underline{k} = (k_1, k_2) \in \mathbb{Z}_N^2$, with the corresponding eigenvalues given by

$$\Delta(e_{\underline{k}}) \,=\, \big([k_1]_q^2 + [k_2]_q^2\big)\, e_{\underline{k}} \quad , \tag{5.3.9a}$$

where the q-numbers are defined as

$$[n]_q \,:=\, \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \quad . \tag{5.3.9b}$$

For later use, we list the properties

$$e_{\underline{k}}^* \,=\, q^{-k_1\, k_2}\, e_{-\underline{k}} \quad , \quad e_{\underline{k}}\, e_{\underline{l}} \,=\, q^{-l_1\, k_2}\, e_{\underline{k}+\underline{l}} \quad , \quad \int e_{\underline{k}}^*\, e_{\underline{l}} \,=\, \delta_{\underline{k},\underline{l}} \quad , \tag{5.3.10a}$$

and

$$\tau_R(e_{\underline{k}} \otimes e_{\underline{l}}) \;=\; q^{-\underline{k}\Theta\underline{l}}\; e_{\underline{l}} \otimes e_{\underline{k}} \;=\; q^{\underline{l}\Theta\underline{k}}\; e_{\underline{l}} \otimes e_{\underline{k}} \quad , \tag{5.3.10b}$$

for all $\underline{k}, \underline{l} \in \mathbb{Z}_N^2$. In particular, this implies that $\{e^*_{\underline{k}}\}$ is the dual basis $\{e_{\underline{k}}\}$ under the integration pairing.

FREE BRAIDED BV THEORY. We have now set up the underlying geometry. The next step is to describe a non-interacting scalar field theory on the fuzzy torus as a free braided BV theory in the sense of Definition 5.2.1.

Definition 5.3.1. The free braided BV theory associated to a scalar field with mass parameter $m^2 \geq 0$ on the fuzzy torus is given by the ${}_H\mathsf{Ch}$-object

$$E \;=\; \Big(\overset{(0)}{A} \xrightarrow{\;-Q\;} \overset{(1)}{A} \Big) \qquad \text{with} \qquad Q \;:=\; \Delta + m^2 \tag{5.3.11}$$

concentrated in degrees 0 and 1, together with the ${}_H\mathsf{Ch}$-pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle \;:\; E \otimes E \;\longrightarrow\; \mathbb{C}[-1]\;,\;\; \varphi \otimes \psi \;\longmapsto\; \langle \varphi, \psi\rangle \;:=\; (-1)^{|\varphi|} \int \varphi\,\psi \quad . \tag{5.3.12}$$

Remark 5.3.2. Observe that the H-equivariance of the pairing is a direct consequence of the fact that both the product on A and the integration map (5.3.6) are H-equivariant. To prove the antisymmetry property, observe first that

$$\begin{aligned} \langle \varphi, \psi\rangle \;&=\; (-1)^{|\varphi|} \int \varphi\,\psi \;=\; (-1)^{|\varphi|} \int \psi\,\varphi \;=\; (-1)^{|\varphi|+|\psi|}\, \langle \psi, \varphi\rangle \\ &=\; -(-1)^{|\varphi|\,|\psi|}\, \langle \psi, \varphi\rangle \end{aligned} \tag{5.3.13}$$

is *strictly* antisymmetric, i.e. antisymmetric *without* R-matrix actions. Then, using the explicit form of the R-matrix (5.3.4), together with the Hopf algebra structure on $H = \mathbb{C}[\mathbb{Z}_N^2]$ from Example 1.2.8, we check that

$$\begin{aligned} \langle R_\alpha \triangleright \psi, R^\alpha \triangleright \varphi\rangle \;&=\; \frac{1}{N^2} \sum_{\underline{s},\underline{t} \in \mathbb{Z}_N^2} q^{\underline{s}\Theta\underline{t}}\; \langle \underline{t} \triangleright \psi, \underline{s} \triangleright \varphi\rangle \\ &=\; \frac{1}{N^2} \sum_{\underline{s},\underline{t} \in \mathbb{Z}_N^2} q^{\underline{s}\Theta\underline{t}}\; \langle \psi, (\underline{s} - \underline{t}) \triangleright \varphi\rangle \\ &=\; \frac{1}{N^2} \sum_{\underline{s}',\underline{t} \in \mathbb{Z}_N^2} q^{\underline{s}'\Theta\underline{t}}\; \langle \psi, \underline{s}' \triangleright \varphi\rangle \\ &=\; \sum_{\underline{s}' \in \mathbb{Z}_N^2} \delta_{\underline{s}',\underline{0}}\, \langle \psi, \underline{s}' \triangleright \varphi\rangle \;=\; \langle \psi, \varphi\rangle \quad . \end{aligned} \tag{5.3.14}$$

In the second equality we used H-equivariance of the pairing to write $\langle\,\cdot\,,\,\cdot\,\rangle = (-\underline{t})\triangleright\langle\,\cdot\,,\,\cdot\,\rangle = \langle(-\underline{t})\triangleright\,\cdot\,,(-\underline{t})\triangleright\,\cdot\,\rangle$ and in the fourth step we used (1.2.27). This, together with the strict antisymmetry (5.3.13), implies that the pairing 5.3.12 is *both* strictly antisymmetric and braided antisymmetric. The latter is the property required by Definition 5.2.1. △

From the general construction in Section 5.2.1, we obtain from the free braided BV theory in Definition 5.3.1 a braided P_0-algebra

$$\mathrm{Obs}^{\mathrm{cl}} := \big(\mathrm{Sym}_R E[1], Q, \{\,\cdot\,,\,\cdot\,\}\big) \tag{5.3.15}$$

of classical observables of the non-interacting theory. Recall that Sym_R denotes the braided symmetric algebra (defined via (5.2.6)). Note that the complex

$$E[1] = \big(\overset{(-1)}{A} \xrightarrow{\;Q\;} \overset{(0)}{A}\big) \in {}_H\mathsf{Ch} \tag{5.3.16}$$

is concentrated in degrees -1 and 0.

INTERACTIONS. The braided symmetric algebra $\mathrm{Sym}_R E[1]$ in this case is concentrated in non-positive degrees (see (5.3.16)), just as for the scalar field on the fuzzy sphere in Section 5.1.1. Hence, for degree reasons, every H-invariant 0-cochain $I \in (\mathrm{Sym}_R E[1])^0$ satisfies

$$Q(I) = \{I, I\} = \Delta_{\mathrm{BV}}(I) = 0 \tag{5.3.17}$$

and therefore both the classical and quantum master equations (5.2.14) and (5.2.17).

Regardless, we consider the p-point interactions obtained from the cyclic L_∞-algebra formalism from Section 5.2.2 as a concrete example. From the free BV theory of Definition 5.3.1, there is the associated Abelian cyclic braided L_∞-algebra defined by the complex

$$E[-1] = \big(\overset{(1)}{A} \xrightarrow{\;Q\;} \overset{(2)}{A}\big) \in {}_H\mathsf{Ch} \tag{5.3.18}$$

and the cyclic structure

$$\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle : E[-1]\otimes E[-1] \longrightarrow \mathbb{C}[-3]\,,\;\; \varphi\otimes\psi \longmapsto \langle\!\langle\varphi,\psi\rangle\!\rangle = \int \varphi\,\psi \quad . \tag{5.3.19}$$

For any $p \geq 3$, we may augment the above with the compatible $(p-1)$-bracket

$$\ell_{p-1} : E[-1]^{\otimes p-1} \longrightarrow E[-1]\,,\;\; \varphi_1\otimes\cdots\otimes\varphi_{p-1} \longmapsto \varphi_1\cdots\varphi_{p-1} \tag{5.3.20}$$

given by the multiplication in the left H-module algebra A. In comparison with (5.1.15) in the case of the fuzzy sphere, note that the symmetrisation simplifies to the expression in (5.3.20) due to the braided commutativity of the fuzzy torus algebra A. Similarly to before, for degree reasons, ℓ_{p-1} is only non-vanishing if each $\varphi_i \in E[-1]$ is of degree 1 in $E[-1]$. Using this, together with the fact that A is braided commutative, we verify the braided graded antisymmetry property of ℓ_{p-1},

$$\begin{aligned} \ell_{p-1}(\varphi_1, \dots, \varphi_{p-1}) &= \varphi_1 \cdots \varphi_i \, \varphi_{i+1} \cdots \varphi_{p-1} \\ &= \varphi_1 \cdots (R_\alpha \triangleright \varphi_{i+1}) \, (R^\alpha \triangleright \varphi_i) \cdots \varphi_{p-1} \\ &= -(-1)^{|\varphi_i| \, |\varphi_{i+1}|} \, \ell_{p-1}(\varphi_1, \dots, R_\alpha \triangleright \varphi_{i+1}, R^\alpha \triangleright \varphi_i, \dots, \varphi_{p-1}) \quad , \end{aligned} \tag{5.3.21}$$

for all $i = 1, \dots, p-2$.

For the contracted coordinate functions, we consider the basis $\{e_{\underline{k}} \in A\}_{\underline{k} \in \mathbb{Z}_N^2}$ introduced in (5.3.8). Using its properties listed in (5.3.10), the contracted coordinate functions take the form

$$\mathsf{a} = \sum_{\underline{k} \in \mathbb{Z}_N^2} e^*_{\underline{k}} \otimes e_{\underline{k}} + \sum_{\underline{k} \in \mathbb{Z}_N^2} \widetilde{e}^*_{\underline{k}} \otimes \widetilde{e}_{\underline{k}} \ \in \ \big((\mathrm{Sym}_R E[1]) \otimes E[-1]\big)^1 \quad , \tag{5.3.22}$$

where, as in the fuzzy sphere example from Section 5.1.1, the individual elements live in the vector spaces $e_{\underline{k}} \in E[-1]^1$, $e^*_{\underline{k}} \in E[1]^0$, $\widetilde{e}_{\underline{k}} \in E[-1]^2$ and $\widetilde{e}^*_{\underline{k}} \in E[1]^{-1}$. The interaction term (5.2.24) corresponding to a p-point interaction is then written out as

$$\begin{aligned} \lambda \, I &= \frac{\lambda^{p-2}}{p!} \, \langle\!\langle \mathsf{a}, \ell^{\mathrm{ext}}_{p-1}(\mathsf{a}, \dots, \mathsf{a}) \rangle\!\rangle_{\mathrm{ext}} && (5.3.23\mathrm{a}) \\ &= \frac{\lambda^{p-2}}{p!} \sum_{\underline{k}_0, \dots, \underline{k}_{p-1} \in \mathbb{Z}_N^2} q^{\sum_{i<j} \underline{k}_i \Theta \underline{k}_j} \, e^*_{\underline{k}_0} \cdots e^*_{\underline{k}_{p-1}} \, \langle\!\langle e_{\underline{k}_0}, \ell_{p-1}(e_{\underline{k}_1}, \dots, e_{\underline{k}_{p-1}}) \rangle\!\rangle \\ &\quad \in \ (\mathrm{Sym}_R E[1])^0 \quad , \end{aligned}$$

with the factors of q emerging from the braiding identity in (5.3.10). Again, as before, the products of the elements $e^*_{\underline{k}_i}$ in the second line are of the braided symmetric algebra $\mathrm{Sym}_R E[1]$ and *not* A.

The constants

$$I_{\underline{k}_0 \underline{k}_1 \cdots \underline{k}_{p-1}} := q^{\sum_{i<j} \underline{k}_i \Theta \underline{k}_j} \, \langle\!\langle e_{\underline{k}_0}, \ell_{p-1}(e_{\underline{k}_1}, \dots, e_{\underline{k}_{p-1}}) \rangle\!\rangle \ \in \ \mathbb{C} \tag{5.3.23b}$$

can be written out explicitly by using (5.3.19), (5.3.20), (5.3.6) and (5.3.10), from which one obtains

$$\begin{aligned} I_{\underline{k}_0\underline{k}_1\cdots\underline{k}_{p-1}} &= q^{\sum_{i<j}\underline{k}_i\Theta\underline{k}_j}\int e_{\underline{k}_0}\,e_{\underline{k}_1}\cdots e_{\underline{k}_{p-1}} \\ &= q^{\sum_{i<j}\underline{k}_i\Theta\underline{k}_j}\,q^{-\sum_{i<j}k_{i2}\,k_{j1}}\int e_{\underline{k}_0+\underline{k}_1+\cdots+\underline{k}_{p-1}} \\ &= q^{-\sum_{i<j}k_{i1}\,k_{j2}}\,\delta_{\underline{k}_0+\underline{k}_1+\cdots+\underline{k}_{p-1},\underline{0}} \quad , \end{aligned} \tag{5.3.23c}$$

where the double subscript notation denotes the components $\underline{k}_i = (k_{i1}, k_{i2}) \in \mathbb{Z}_N^2$ for $i = 0, 1, \ldots, p-1$. The constants $I_{\underline{k}_0\underline{k}_1\cdots\underline{k}_{p-1}}$ satisfy the q-deformed symmetry property

$$I_{\underline{k}_0\underline{k}_1\cdots\underline{k}_i\underline{k}_{i+1}\cdots\underline{k}_{p-1}} = q^{\underline{k}_i\Theta\underline{k}_{i+1}}\,I_{\underline{k}_0\underline{k}_1\cdots\underline{k}_{i+1}\underline{k}_i\cdots\underline{k}_{p-1}} \tag{5.3.24a}$$

for any exchange of neighbouring indices. In particular, this implies the *strict* cyclicity property

$$I_{\underline{k}_0\underline{k}_1\cdots\underline{k}_{p-1}} = I_{\underline{k}_1\cdots\underline{k}_{p-1}\underline{k}_0} \tag{5.3.24b}$$

by further using momentum conservation imposed by the Kronecker delta-symbol $\delta_{\underline{k}_0+\underline{k}_1+\cdots+\underline{k}_{p-1},\underline{0}}$.

braided strong deformation retract. For the remaining part of this section, we will assume that $m^2 > 0$. (The massless case $m^2 = 0$ can be treated analogously to the example on the fuzzy sphere in Section 5.1.1.) Since the spectrum of the operator $Q = \Delta + m^2$ is positive (see (5.3.9) for the spectrum of the Laplacian Δ), the cohomology for the complex $E[1]$ in (5.3.16) is trivial, i.e. $H^\bullet(E[1]) \cong 0$. There is thus a strong deformation retract

$$\begin{tikzcd} (0,0) \arrow[r, shift left, "\iota=0"'] & (E[1],Q) \arrow[l, shift left, "\pi=0"'] \arrow[loop right, "\xi=-G"] \end{tikzcd} \quad , \tag{5.3.25}$$

where G is the inverse of $Q = \Delta + m^2$, i.e. the Green operator, and $\xi = -G$ is defined to act as a degree -1 map on $E[1]$. The homotopy ξ acts on the (dual) basis $e^*_{\underline{k}} \in E[1]^0$ as

$$\xi(e^*_{\underline{k}}) = -G(e^*_{\underline{k}}) = -\frac{1}{[k_1]_q^2 + [k_2]_q^2 + m^2}\,\widetilde{e}^*_{\underline{k}} \in E[1]^{-1} \quad , \tag{5.3.26}$$

where we have used the same notation as in (5.3.22) for the basis vectors. The strong deformation retract (5.3.25) satisfies the H-equivariance and H-invariance conditions of a braided strong deformation retract in the sense of Definition 5.2.8.

CORRELATION FUNCTIONS. We have now arrived at the point where we can compute correlation functions. The braided strong deformation retract (5.3.25) extends to the braided symmetric algebras. For any small H-invariant perturbation δ of the differential Q on $\mathrm{Sym}_R E[1]$, we obtain from the homological perturbation lemma for braided strong deformation retracts (Corollary 5.2.9) a deformed braided strong deformation retract

$$\begin{tikzcd} (\mathrm{Sym}_R 0 \cong \mathbb{C}, 0) \arrow[r, shift right, "\widetilde{J}"'] & (\mathrm{Sym}_R E[1], Q+\delta) \arrow[l, shift right, "\widetilde{\Pi}"'] \arrow[loop right, "\widetilde{\Xi}"] \end{tikzcd} \quad , \qquad (5.3.27)$$

where the tilded quantities are computed through the homological perturbation lemma, see Theorem 4.3.7. Following the previous notation, we have denoted the extension of the maps ι, π and ξ to the symmetric algebras by $J := \mathrm{Sym}_R\, \iota$, $\Pi := \mathrm{Sym}_R\, \pi$ and $\Xi := \mathrm{Sym}_R\, \xi$, respectively. The correlation functions can be computed via the ${}_H\mathsf{Ch}$-morphism

$$\widetilde{\Pi} \,=\, \Pi \circ \sum_{k=0}^{\infty} (\delta\, \Xi)^k \quad . \qquad (5.3.28)$$

The relevant perturbations δ are of the form

$$\delta \,=\, \hbar\, \Delta_{\mathrm{BV}} + \{\lambda\, I, \,\cdot\,\} \quad , \qquad (5.3.29)$$

where Δ_{BV} is the BV Laplacian (5.2.16) and $\lambda\, I \in (\mathrm{Sym}_R E[1])^0$ denotes the p-point interaction term (5.3.23) for some $p \geq 3$.

We wish to compute the n-point correlation functions $\langle \varphi_1 \cdots \varphi_n \rangle = \widetilde{\Pi}(\varphi_1 \cdots \varphi_n)$ for test functions $\varphi_1, \dots, \varphi_n \in E[1]^0$ of degree zero. Therefore, we need to understand how the maps Π and $\delta\, \Xi$ act on elements $\varphi_1 \cdots \varphi_n \in \mathrm{Sym}_R E[1]$ with all $\varphi_1, \dots, \varphi_n \in E[1]^0$ of degree zero. The case of Π is simple,

$$\Pi(\mathbb{1}) \,=\, 1 \quad , \quad \Pi(\varphi_1 \cdots \varphi_n) \,=\, 0 \quad , \qquad (5.3.30)$$

for all $n \geq 1$. In the case of $\delta\, \Xi = \hbar\, \Delta_{\mathrm{BV}}\, \Xi + \{\lambda\, I, \,\cdot\,\}\, \Xi$, it is useful to individually investigate the summands. For the first term, from the formula (5.2.16) for the BV Laplacian, one finds

$$\begin{aligned} \hbar\, \Delta_{\mathrm{BV}}\, \Xi(\varphi_1 \cdots \varphi_n) \,=\, & -\frac{2\hbar}{n} \sum_{i<j} \big\langle \varphi_i, R_{\alpha_{i+1}} \cdots R_{\alpha_{j-1}} \triangleright G(\varphi_j) \big\rangle \\ & \times\; \varphi_1 \cdots \widehat{\varphi}_i\, (R^{\alpha_{i+1}} \triangleright \varphi_{i+1}) \cdots (R^{\alpha_{j-1}} \triangleright \varphi_{j-1})\, \widehat{\varphi}_j \cdots \varphi_n \quad . \end{aligned} \qquad (5.3.31)$$

The second term can explicitly be expressed as

$$\big\{\lambda\, I, \Xi(\varphi_1\cdots\varphi_n)\big\} \;=\; -\frac{\lambda^{p-2}}{(p-1)!\,n}\sum_{i=1}^{n}\ \sum_{\underline{k}_0,\dots,\underline{k}_{p-1}\in\mathbb{Z}_N^2}\Big(\varphi_1\cdots\varphi_{i-1} \\ \times I_{\underline{k}_0\underline{k}_1\cdots\underline{k}_{p-1}}\,\big\langle e^*_{\underline{k}_0}, G(\varphi_i)\big\rangle\; e^*_{\underline{k}_1}\cdots e^*_{\underline{k}_{p-1}}\,\varphi_{i+1}\cdots\varphi_n\Big)\quad, \tag{5.3.32}$$

using the expression (5.3.23) for the p-point interaction term, together with its properties (5.3.24).

As in Section 5.1.1, the two expressions (5.3.31) and (5.3.32) can be visualised graphically. Using n vertical lines to represent the element $\varphi_1\cdots\varphi_n$, the map in (5.3.31) may be depicted as

$$\hbar\,\Delta_{\rm BV}\,\Xi\big(|\,|\,|\cdots|\,|\big) \;=\; -\frac{2\,\hbar}{n}\Big(\bigcap|\cdots|\,| + \bigcap\cdots|\,| + \cdots + |\,|\,|\cdots\bigcap\Big)\quad, \tag{5.3.33}$$

where the cap indicates a contraction of two elements with respect to $\big\langle\,\cdot\,, G(\cdot)\big\rangle$. Note that in the pictures that the right leg of the contraction is permuted using the symmetric braiding τ_R across intermediate vertical lines, leading to the correct R-matrix insertions in (5.3.31). The map (5.3.32) may be drawn as

$$\Big\{\lambda\, I\,,\,\Xi\big(|\,|\,|\cdots|\,|\big)\Big\} \;=\; -\frac{\lambda^{p-2}}{(p-1)!\,n}\Big(\overset{p-1\text{ legs}}{\Upsilon}\,|\,|\cdots|\,| + \cdots + |\,|\,|\cdots|\,\overset{p-1\text{ legs}}{\Upsilon}\Big)\quad, \tag{5.3.34}$$

where the vertex acts on an element as $\sum_{\underline{k}_0,\dots,\underline{k}_{p-1}\in\mathbb{Z}_N^2} I_{\underline{k}_0\underline{k}_1\cdots\underline{k}_{p-1}}\,\big\langle e^*_{\underline{k}_0}, G(\cdot)\big\rangle\; e^*_{\underline{k}_1}\cdots e^*_{\underline{k}_{p-1}}$, attaching $p-1$ legs to a vertical line.

Example 5.3.3. We begin by computing the 4-point function

$$\widetilde{\Pi}(\varphi_1\,\varphi_2\,\varphi_3\,\varphi_4) \;=\; \Pi\big((\hbar\,\Delta_{\rm BV}\,\Xi)^2(\varphi_1\,\varphi_2\,\varphi_3\,\varphi_4)\big) \tag{5.3.35}$$

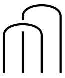
of a non-interacting scalar field, i.e. $I=0$. In our graphical notation, we compute

$$\hbar\,\Delta_{\rm BV}\,\Xi\big(|\,|\,|\,|\big) \;=\; -\frac{\hbar}{2}\Big(\bigcap|\,| + \bigcap| + \bigcap + |\bigcap| + |\bigcap + |\,|\bigcap\Big) \tag{5.3.36}$$

and

$$(\hbar\,\Delta_{\rm BV}\,\Xi)^2\big(|\,|\,|\,|\big) \;=\; \frac{\hbar^2}{2}\Big(2\bigcap\bigcap + \bigcap + \bigcap + \bigcap + \bigcap\Big) \\ =\; \hbar^2\Big(\bigcap\bigcap + \bigcap + \bigcap\Big)\quad. \tag{5.3.37}$$

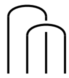
The simplification in the last step works as follows. Using H-equivariance of the paring $\{\,\cdot\,,\,\cdot\,\}$ and the standard identity $(S \otimes \mathrm{id}_H)R = R^{-1} = R_{21}$ (see (1.2.16b)) for a triangular R-matrix, it follows that



$$= \big\langle \varphi_1, R_\alpha \triangleright G(\varphi_3)\big\rangle \, \big\langle R^\alpha \triangleright \varphi_2, G(\varphi_4)\big\rangle$$

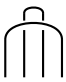
$$= \big\langle \varphi_1, R_\alpha \triangleright G(\varphi_3)\big\rangle \, \big\langle \varphi_2, S(R^\alpha) \triangleright G(\varphi_4)\big\rangle$$

$$= \big\langle \varphi_1, R^\alpha \triangleright G(\varphi_3)\big\rangle \, \big\langle \varphi_2, R_\alpha \triangleright G(\varphi_4)\big\rangle \; = $$



$$, \qquad (5.3.38)$$

and so the second and the fifth term in the first line of (5.3.37) coincide, yielding the second term in the second line. To show that the third and the fourth term in the first line of (5.3.37) agree and hence yield the third term in the second line, we again use H-equivariance of the pairing $\{\,\cdot\,,\,\cdot\,\}$, together with the third R-matrix property in (1.2.15) and the normalization condition $\epsilon(R^\alpha)\,R_\alpha = 1$ (see (1.2.16a)) to obtain

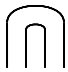


$$= \big\langle \varphi_1, R_\alpha\, R_\beta \triangleright G(\varphi_4)\big\rangle \, \big\langle R^\alpha \triangleright \varphi_2, R^\beta \triangleright G(\varphi_3)\big\rangle$$

$$= \big\langle \varphi_1, R_\alpha \triangleright G(\varphi_4)\big\rangle \, \big\langle R^\alpha_{\underline{1}} \triangleright \varphi_2, R^\alpha_{\underline{2}} \triangleright G(\varphi_3)\big\rangle$$

$$= \big\langle \varphi_1, R_\alpha \triangleright G(\varphi_4)\big\rangle \, \epsilon(R^\alpha)\, \big\langle \varphi_2, G(\varphi_3)\big\rangle$$

$$= \big\langle \varphi_1, G(\varphi_4)\big\rangle \, \big\langle \varphi_2, G(\varphi_3)\big\rangle \; = $$

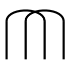


$$. \qquad (5.3.39)$$

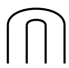
The free 4-point function is hence given by

$$\widetilde{\Pi}(\varphi_1\,\varphi_2\,\varphi_3\,\varphi_4) \;=\; \hbar^2 \Big($$

$$+$$



$$+$$



$$\Big)$$

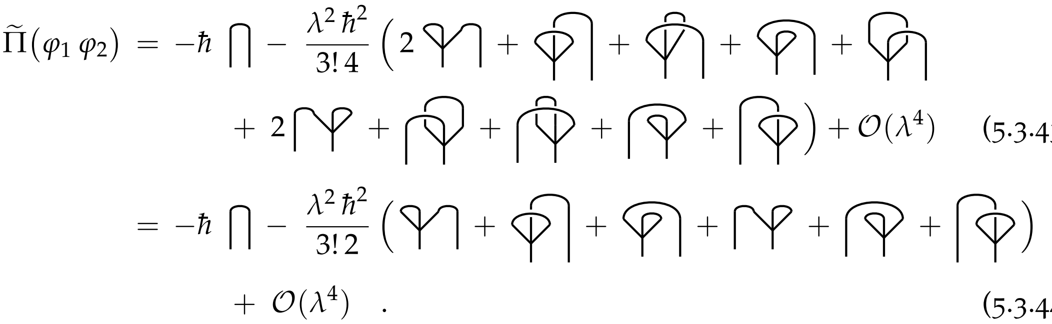
$$\begin{aligned}
&= \hbar^2 \Big(\big\langle \varphi_1, G(\varphi_2)\big\rangle \, \big\langle \varphi_3, G(\varphi_4)\big\rangle \\
&\qquad + \big\langle \varphi_1, R_\alpha \triangleright G(\varphi_3)\big\rangle \, \big\langle R^\alpha \triangleright \varphi_2, G(\varphi_4)\big\rangle && (5.3.40) \\
&\qquad + \big\langle \varphi_1, G(\varphi_4)\big\rangle \, \big\langle \varphi_2, G(\varphi_3)\big\rangle \Big) \quad . && (5.3.41)
\end{aligned}$$

Observe that there is a single appearance of an R-matrix, corresponding to the line crossing in the second term. Because our R-matrix is triangular, we do not have to distinguish between over and under crossings, just as in the case of Oeckl's *symmetric* braided quantum field theory [Oec00]. $\triangledown$

Example 5.3.4. The next example we would like to compute is the 2-point function

$$\widetilde{\Pi}(\varphi_1\,\varphi_2) \;=\; \sum_{k=0}^{\infty} \Pi\big((\delta\,\Xi)^k(\varphi_1\,\varphi_2)\big) \qquad (5.3.42)$$

of Φ^4-theory (i.e. we set $p = 4$) to the lowest non-trivial order in the coupling constant λ. The graphical expansion (without simplifications) is completely analogous to the computation on the fuzzy sphere Example 5.1.3,

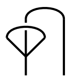


(5.3.43)

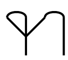
(5.3.44)

The simplifications in the last step use the same arguments as in Example 5.3.3, see in particular (5.3.38) and (5.3.39).

There are further simplifications since one can show that all six loop contributions coincide. To illustrate the arguments, we show how the second and the third loop diagrams agree with the first one. The other terms are rewritten using similar arguments. For the second term, we have that

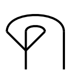


$$
\begin{aligned}
&= \sum_{\underline{k}_0,\underline{k}_1,\underline{k}_2,\underline{k}_3 \in \mathbb{Z}_N^2} I_{\underline{k}_0\underline{k}_1\underline{k}_2\underline{k}_3} \, \big\langle e^*_{\underline{k}_0}, G(\varphi_1)\big\rangle \, \big\langle e^*_{\underline{k}_1}, G(R_\alpha \triangleright e^*_{\underline{k}_3})\big\rangle \, \big\langle R^\alpha \triangleright e^*_{\underline{k}_2}, G(\varphi_2)\big\rangle \\
&= \sum_{\underline{k}_0,\underline{k}_1,\underline{k}_2,\underline{k}_3 \in \mathbb{Z}_N^2} q^{-\underline{k}_2 \Theta \underline{k}_3} \, I_{\underline{k}_0\underline{k}_1\underline{k}_2\underline{k}_3} \, \big\langle e^*_{\underline{k}_0}, G(\varphi_1)\big\rangle \, \big\langle e^*_{\underline{k}_1}, G(e^*_{\underline{k}_3})\big\rangle \, \big\langle e^*_{\underline{k}_2}, G(\varphi_2)\big\rangle \\
&= \sum_{\underline{k}_0,\underline{k}_1,\underline{k}_2,\underline{k}_3 \in \mathbb{Z}_N^2} I_{\underline{k}_0\underline{k}_1\underline{k}_2\underline{k}_3} \, \big\langle e^*_{\underline{k}_0}, G(\varphi_1)\big\rangle \, \big\langle e^*_{\underline{k}_1}, G(e^*_{\underline{k}_2})\big\rangle \, \big\langle e^*_{\underline{k}_3}, G(\varphi_2)\big\rangle \; =
\end{aligned}
$$

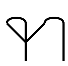


$$. \qquad (5.3.45)$$

In the first step we used the properties (5.3.10) of the (dual) basis $e^*_{\underline{k}}$ and in the second step the q-deformed symmetry property (5.3.24) of the interaction term. In the case of the third term, we find



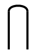
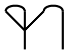
$$
\begin{aligned}
&= \sum_{\underline{k}_0,\underline{k}_1,\underline{k}_2,\underline{k}_3 \in \mathbb{Z}_N^2} I_{\underline{k}_0\underline{k}_1\underline{k}_2\underline{k}_3} \, \big\langle e^*_{\underline{k}_0}, G(\varphi_1)\big\rangle \, \big\langle e^*_{\underline{k}_2}, G(e^*_{\underline{k}_3})\big\rangle \, \big\langle e^*_{\underline{k}_1}, G(\varphi_2)\big\rangle \\
&= \sum_{\underline{k}_0,\underline{k}_1,\underline{k}_2,\underline{k}_3 \in \mathbb{Z}_N^2} I_{\underline{k}_0\underline{k}_1\underline{k}_2\underline{k}_3} \, \big\langle e^*_{\underline{k}_0}, G(\varphi_1)\big\rangle \, \big\langle R_{\alpha\underline{1}} \triangleright e^*_{\underline{k}_2}, G(R_{\alpha\underline{2}} \triangleright e^*_{\underline{k}_3})\big\rangle \, \big\langle R^\alpha \triangleright e^*_{\underline{k}_1}, G(\varphi_2)\big\rangle \\
&= \sum_{\underline{k}_0,\underline{k}_1,\underline{k}_2,\underline{k}_3 \in \mathbb{Z}_N^2} I_{\underline{k}_0\underline{k}_1\underline{k}_2\underline{k}_3} \, \big\langle e^*_{\underline{k}_0}, G(\varphi_1)\big\rangle \, \big\langle R_\alpha \triangleright e^*_{\underline{k}_2}, G(R_\beta \triangleright e^*_{\underline{k}_3})\big\rangle \, \big\langle R^\beta R^\alpha \triangleright e^*_{\underline{k}_1}, G(\varphi_2)\big\rangle \\
&= \sum_{\underline{k}_0,\underline{k}_1,\underline{k}_2,\underline{k}_3 \in \mathbb{Z}_N^2} I_{\underline{k}_0\underline{k}_1\underline{k}_2\underline{k}_3} \, \big\langle e^*_{\underline{k}_0}, G(\varphi_1)\big\rangle \, \big\langle e^*_{\underline{k}_1}, G(e^*_{\underline{k}_2})\big\rangle \, \big\langle e^*_{\underline{k}_3}, G(\varphi_2)\big\rangle \; =
\end{aligned}
$$



$$. \qquad (5.3.46)$$

In the first step we used H-equivariance of $\langle \,\cdot\, , G(\cdot) \rangle$ together with the normalization condition $R^{\alpha}\, \epsilon(R_{\alpha}) = 1$ (see (1.2.16a)). The second step follows from the second identity in (1.2.15), and the third step from applying the q-deformed symmetry property (5.3.24) of the interaction term twice.

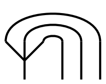
In summary, the 2-point function of Φ^4-theory on the fuzzy torus to leading order in the coupling constant reads as

$$\widetilde{\Pi}(\varphi_1\, \varphi_2) \;=\; -\hbar$$  $$-\; \frac{\lambda^2\, \hbar^2}{2}$$  $$+\; \mathcal{O}(\lambda^4)$$

$$= -\hbar \, \langle \varphi_1, G(\varphi_2) \rangle - \frac{\lambda^2\, \hbar^2}{2} \sum_{\underline{k},\underline{l} \in \mathbb{Z}^2_N} \frac{\langle e^*_{\underline{k}}, G(\varphi_1) \rangle \, \langle e_{\underline{k}}, G(\varphi_2) \rangle}{[l_1]^2_q + [l_2]^2_q + m^2} + \mathcal{O}(\lambda^4) \quad , \tag{5.3.47}$$

where we also used the explicit expression (5.3.26) for the Green operator to write out the $\langle e^*_{\underline{k}_1}, G(e^*_{\underline{k}_2}) \rangle$ factor in the loop diagram. Note that in contrast to the traditional (unbraided) approaches to noncommutative quantum field theory [IIKK00, MVRS00], there is no distinction between planar and non-planar loop corrections. This is closely tied with the braided commutativity property of the fuzzy torus algebra (5.3.2). In particular, this automatically implies the braided (graded) antisymmetry of the higher L_∞-algebra brackets (5.3.20), without the need for graded antisymmetrization, just as in the fuzzy sphere case (5.1.15). The absence of non-planar features in loop corrections has similarly been observed by Oeckl in his framework of (symmetric) braided quantum field theory [Oec00]. $\triangledown$

Example 5.3.5. We would like to emphasize that the disappearance of the q-factors from the interaction term (5.3.23) in the loop diagram in (5.3.47) is only due to the shape of that diagram and not a general feature of our formalism. To demonstrate this, we again consider Φ^4-theory and compute the *connected part* of the 4-point function to the first non-trivial order in the coupling constant. Using the same arguments as in the previous examples, in particular the q-deformed symmetry property (5.3.24) of the interaction term, one finds

$$\widetilde{\Pi}(\varphi_1\, \varphi_2\, \varphi_3\, \varphi_4)_{\text{connected}} \;=\; \lambda^2\, \hbar^3$$  $$+\; \mathcal{O}(\lambda^4) \tag{5.3.48}$$

$$= \lambda^2\, \hbar^3 \sum_{\underline{k}_0,\underline{k}_1,\underline{k}_2,\underline{k}_3 \in \mathbb{Z}^2_N} I_{\underline{k}_0 \underline{k}_1 \underline{k}_2 \underline{k}_3} \, \langle e^*_{\underline{k}_0}, G(\varphi_1) \rangle \, \langle e^*_{\underline{k}_1}, G(\varphi_4) \rangle \, \langle e^*_{\underline{k}_2}, G(\varphi_3) \rangle \, \langle e^*_{\underline{k}_3}, G(\varphi_2) \rangle$$

$$+\; \mathcal{O}(\lambda^4) \quad .$$

From the explicit expression (5.3.23c) for the constants $I_{\underline{k}_0\underline{k}_1\underline{k}_2\underline{k}_3}$, we indeed see that this correlation function includes the expected q-factors from the interaction term. $\triangledown$

6

BV QUANTISATION OF DYNAMICAL FUZZY SPECTRAL TRIPLES

This chapter is based on [GNS22] and is dedicated to the study of so called *dynamical fuzzy spectral triple models* through the BV formalism which was outlined in Section 4.3. This calls for a review of the theory of fuzzy spectral triples, which can be found in Section 6.1 below. Having set up the underlying geometric concepts, we proceed in Section 6.2 by describing the natural notion of gauge symmetries intrinsic to the such geometries, and subsequently also describing the constituents of dynamical fuzzy spectral triples. The nature of these allows for the use of the *classical* BV formalism to obtain a P_0-algebra of classical observables. This is covered in Section 6.3. An adaptation of the BV quantisation and computation of correlators to this setting is briefly outlined in Section 6.4. One of the goals of [GNS22] was to see if the gauge symmetry contributes to correlation functions for dynamical fuzzy spectral triples. This is split up and investigated in two parts: perturbations around a trivial background $D_0 = 0$ in Section 6.5 and a non-zero background $D_0 \neq 0$ in Section 6.6.

6.1 FUZZY SPECTRAL TRIPLES

In this section, we will give a brief review of the concept of real spectral triples, which is another approach to noncommutative Riemannian spin geometry. For more details, we refer to e.g. [Con94, CM08, vanS15]. Associated to every commutative Riemannian spin manifold M with metric g is a triple of spectral data $(C^\infty(M), \Gamma_{L^2}(S), D_M)$. As usual, sweeping many details under the carpet, $C^\infty(M)$ are the smooth functions on M and $\Gamma_{L^2}(S)$ is the Hilbert space of L^2-spinors, i.e. the completion (under a certain norm) of the square integrable sections of the spinor bundle S . The space of spinors furthermore carries a $C^\infty(M)$-action, making it a $C^\infty(M)$-module. Lastly, $D_M : \Gamma_{L^2}(S) \to \Gamma_{L^2}(S)$ is the Dirac operator built from the spinorial data. A result which motivates the definition of spectral triples is that one can recover the geodesic distance from the Dirac operator. Expressed in this language, it takes the form

$$d(x,y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : ||[D_M, f]|| \leq 1\}. \tag{6.1.1}$$

The spectral triple formalism generalises this to also include noncommutative function algebras; a spectral triple $(A, \mathcal{H}, D)$ consists of a (possibly noncommutative) $*$-algebra[1] A represented on a Hilbert space $\mathcal{H}$ and a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$.

A spin manifold is further defined by a certain antiunitary operator $J_M : \Gamma(S) \to \Gamma(S)$ called the *charge conjugation operator* (the J_M operators can be seen as the operators which selects the spin structures among the spin^c structures). Additionally, when M is even dimensional, $\Gamma(S)$ can be assigned a $\mathbb{Z}_2$-grading defined by an operator γ_M known as the *chirality operator*. These two operators satisfy certain sign rules (related to the theory of Clifford algebras), which will be outlined in the definition of a real spectral triple. The given definitions will focus on the algebraic side of spectral triples as in e.g. [BM17, BM20] and not touch upon the functional analytical aspects. This has no impact in our considerations because we are mainly concerned with the finite dimensional case.

Definition 6.1.1. A *spectral triple* $(A, \mathcal{H}, D)$ consists of a $*$-algebra A together with a faithful $*$-representation[2] $\rho : A \to \mathrm{End}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ with Hermitian inner product $\langle\langle \cdot, \cdot \rangle\rangle$ and a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$.

A *real spectral triple* $(A, \mathcal{H}, D; \Gamma, J)$ of KO-dimension $s \in \mathbb{Z}/8\mathbb{Z}$ is a spectral triple $(A, \mathcal{H}, D)$ equipped with a $\mathbb{C}$-linear operator $\Gamma : \mathcal{H} \to \mathcal{H}$ called the *chirality operator* and a $\mathbb{C}$-antilinear operator $J : \mathcal{H} \to \mathcal{H}$ called the *real structure* such that

(i) $\Gamma^* = \Gamma$ and $\Gamma^2 = \mathrm{id}$

(ii) $\Gamma\, \rho(a) = \rho(a)\, \Gamma$ for all $a \in A$

(iii) J is antiunitary, i.e. $\langle\langle J\, u, J\, v \rangle\rangle = \langle\langle v, u \rangle\rangle$ for all $u, v \in \mathcal{H}$

(iv) $J^2 = \epsilon$ and $J\, \Gamma = \epsilon''\, \Gamma\, J$

1 All algebras in this section will be over $\mathbb{C}$ unless stated otherwise.
2 I.e. $\rho(a^*) = \rho(a)^*$ for all $a \in A$, where $\rho(a)^*$ is the adjoint of $\rho(a)$ in $\mathrm{End}(\mathcal{H})$.

(v) $[\rho(a), J\,\rho(a)\,J^{-1}] = 0$ for all $a, b \in A$

(vi) $D\,\Gamma = -\Gamma\,D$ for even s and $D\,\Gamma = \Gamma\,D$ for odd s

(vii) $J\,D = \epsilon'\,D\,J$

(viii) $[[D, \rho(a)], J\,\rho(b)J^{-1}] = 0$ for all $a, b \in A$.

The signs ϵ, ϵ' and ϵ'' depend on s and are given in the table below:

s	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1	1	-1	1	1	1	-1	1

The operator D satisfying axioms (vi)-(viii) is called the *Dirac operator*.

The archetypical example is of course the canonical real spectral triple extracted from a commutative Riemannian spin manifold. One might also ask if the converse also works, i.e. if it is possible to obtain a Riemannian spin manifold from a given real spectral triple $(A, \mathcal{H}, D; \Gamma, J)$ with A commutative. It turns out that under suitable conditions, this is possible. This is known as *Connes' Reconstruction Theorem* [Con13].

We will adopt the terminology used in [Bar15]. As the Dirac operators are the objects containing the metric data in the case of commutative spectral triples, it is motivated to consider them separately. Thus, a real spectral triple without the data of a Dirac operator (and consequently, without axioms (vi)-(viii)), is called a *fermion space* $(A, \mathcal{H}; \Gamma, J)$. Looking back at the definition of a Dirac operator in Definition 6.1.1, the $\mathbb{R}$-vector space of Dirac operators

$$\mathcal{D} := \left\{ D \in \mathrm{End}(\mathcal{H}) \,:\, D = D^* \text{ and (vi)-(viii) are satisfied} \right\} \subseteq \mathrm{End}(\mathcal{H}) \qquad (6.1.2)$$

is called the *space of geometries* or the *Dirac ensemble* corresponding to the fermion space $(A, \mathcal{H}; \Gamma, J)$.

Remark 6.1.2. The left action of A on H via $\rho : A \to \mathrm{End}(\mathcal{H})$ gives $\mathcal{H}$ a left A-module structure. The real structure $J : \mathcal{H} \to \mathcal{H}$ of a real spectral triple gives $\mathcal{H}$ a right A-action which commutes with the left action: for $\psi \in \mathcal{H}$ and $a \in A$,

$$\psi \cdot a := J\,\rho(a)^*\,J^{-1}\,\psi \quad . \qquad (6.1.3)$$

By axiom (v) in Definition 6.1.1, the left and right actions commute. The right action is well-defined precisely because of the involution $*$. △

Remark 6.1.3. Though we will not use it in this thesis, it is worth mentioning that there is a notion of differential calculus associated to every spectral triple. Given a spectral triple $(A, \mathcal{H}, D)$, the *Connes' differential one-forms* is the A-bimodule

$$\Omega^1_D(A) := \Big\{ \sum_k a_k\, [D, b_k] : a_k, b_k \in A \Big\} \tag{6.1.4}$$

with differential given by $[D, \cdot] : A \to \Omega^1_D$. △

We will now focus on the finite dimensional incarnations of real spectral triples. A *finite (real) spectral triple* is a (real) spectral triple with finite dimensional Hilbert space $\mathcal{H}$. A standard result, see e.g. [vanS15] states that unital $*$-algebras A acting faithfully on finite dimensional Hilbert spaces are automatically matrix algebras, i.e.

$$A \cong \bigoplus_{i=1}^{N} \mathrm{Mat}_{n_i}(\mathbb{C}) \tag{6.1.5}$$

for some $N \in \mathbb{Z}_{\geq 0}$. Finite spectral triples were classified by Krajewski in [Kra98]. In this thesis, we are in particular interested in a special case of finite spectral triples developed in [Bar15] called *fuzzy spaces*. The definition written below is a slightly adapted version.

Before writing down the definition, let us briefly recall some concepts surrounding real Clifford algebras, see e.g. [BT88, LM89], or [Bar15, vanS15] for a discussion in the context of spectral triples. Consider $\mathbb{R}^n$ together with a (symmetric and nondegenerate) bilinear form $\eta : \mathbb{R}^n \otimes \mathbb{R}^n \to \mathbb{R}$ defined (after a choice of orthonormal basis $\{e_i\}_{i=1}^n$) by the diagonal matrix (η_{ij}) with p occurrences of $+1$ and q occurrences of -1 with $n = p + q$. The corresponding *Clifford algebra* $\mathrm{Cl}_{p,q}(\mathbb{R})$ is given by

$$\mathrm{Cl}_{p,q}(\mathbb{R}) = T(\mathbb{R}^n) / \langle v\, w + w\, v - 2\, \eta(v \otimes w)\, 1 \rangle \quad , \tag{6.1.6}$$

where $T(\mathbb{R}^n) = \bigoplus_{k \geq 0} (\mathbb{R}^n)^{\otimes k}$ denotes the tensor algebra of $\mathbb{R}^n$. (Note that there is a difference between the sign conventions here and in the Clifford relations (1.1.14) in Definition 1.1.9. We choose this sign here in order to match with the discussions in [Bar15].) These Clifford algebras depend to great extent on $s = q - p \mod 8$ and are completely classified in terms of matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. A *Clifford module* is a vector space $V = \mathbb{C}^k$ on which $\mathrm{Cl}_{p,q}(\mathbb{R}^n)$ acts via a representation ρ_V :

$\mathrm{Cl}_{p,q}(\mathbb{R}^n) \to \mathrm{End}(V) \cong \mathrm{Mat}_k(\mathbb{C})$. We further assume that V is equipped with the standard Hermitian inner product which we denote by $\langle \cdot, \cdot \rangle$. The matrices $\widehat{\gamma}_i := \rho(e_i)$ are called *gamma matrices* and the $\mathbb{C}$-linear operator

$$\widehat{\gamma} := i^{\frac{s(s+1)}{2}}\, \widehat{\gamma}_1 \cdots \widehat{\gamma}_n : V \to V \tag{6.1.7}$$

is called the *chirality operator*. Since

$$\widehat{\gamma}_i\, \widehat{\gamma}_j + \widehat{\gamma}_j\, \widehat{\gamma}_i = 2\, \eta_{ij} \quad , \tag{6.1.8}$$

one can show that $\widehat{\gamma}^2 = 1$ and $\widehat{\gamma}^* = \widehat{\gamma}$, where $*$ denotes the adjoint with respect to $\langle \cdot, \cdot \rangle$, i.e. Hermitian conjugation. In the case when $n = p + q$ is even, there is one unique (up to equivalence) irreducible representation of $\mathrm{Cl}_{p,q}(\mathbb{R})$. When n is odd, there are two inequivalent irreducible representations V_+ and V_-, respectively characterised by $\widehat{\gamma} = 1$ and $\widehat{\gamma} = -1$. Lastly, a *real structure* for the Clifford module V is an $\mathbb{C}$-antilinear operator $C : V \to V$ satisfying

(i) $C^2 = \epsilon$,

(ii) $\langle Cv, Cw \rangle = \langle w, v \rangle$,

(iii) $C\, \widehat{\gamma}_i = \epsilon'\, \widehat{\gamma}_i\, C$.

From this, the sign ϵ'' can be derived,

$$C\, \widehat{\gamma} = \epsilon''\, \widehat{\gamma}\, C \quad . \tag{6.1.9}$$

The signs ϵ, ϵ' and ϵ'' are given in the table in Definition 6.1.1.

Definition 6.1.4. Let $N \in \mathbb{Z}_{>0}$ and $V = \mathbb{C}^k$ be a (p, q)-Clifford module with chirality operator $\widehat{\gamma} : V \to V$ and real structure $C : V \to V$. For $p + q$ even, assume V is irreducible and for $p + q$ odd, assume $V = V_+ \oplus V_-$. A *fuzzy space* (or *fuzzy spectral triple*) *of type* (p, q) is a finite real spectral triple $(A, \mathcal{H}, D; \Gamma, J)$ of KO dimension $s = q - p \mod 8$ with

(i) $A = \mathrm{Mat}_N(\mathbb{C})$ with the $*$-involution given by Hermitian conjugation,

(ii) $\mathcal{H} = \mathrm{Mat}_N(\mathbb{C}) \otimes V = A \otimes V$,

(iii) $\langle\langle m \otimes v, m' \otimes v' \rangle\rangle = \mathrm{Tr}(m^*\, m)\langle v, v' \rangle$, where Tr is the trace on $A = \mathrm{Mat}_N(\mathbb{C})$,

(iv) $\rho(a)(m \otimes v) = a\, m \otimes v$, i.e. matrix multiplication of a and m on the first tensor factor,

(v) $\Gamma(m \otimes v) = m \otimes \widehat{\gamma}\, v$,

(vi) $J(m \otimes v) = m^* \otimes C\, v$,

for all $m \otimes v, m' \otimes v' \in \mathcal{H}$ and $a \in A$.

As before, a fuzzy spectral triple of type (p, q) without the data of a Dirac operator will be called a (p, q)-fermion space $(A, \mathcal{H}; \Gamma, J)$.

Remark 6.1.5. Viewing Remark 6.1.2 in the context of fuzzy spaces, $\mathcal{H} = \mathrm{Mat}_N(\mathbb{C}) \otimes V$ is an A-bimodule with the right action of $A = \mathrm{Mat}_N(\mathbb{C})$ given simply by right matrix multiplication,

$$J\, \rho(a)^*\, J^{-1}(m \otimes v) = m\, a \otimes v \quad . \tag{6.1.10}$$

△

The Dirac operators of fuzzy spaces can be concretely expressed in terms of products of gamma matrices and Hermitian and anti-Hermitian matrices. Let us denote Hermitian matrices and anti-Hermitian matrices in $\mathrm{Mat}_N(\mathbb{C})$ by H_i and L_j, respectively. Furthermore, we use the notation τ^i to represent Hermitian products of gamma matrices and α^j for anti-Hermitian products. There are two separate cases depending on ϵ'. For $\epsilon' = 1$, the Dirac operator can be written as

$$D(m \otimes v) = \sum_i [L_i, m] \otimes \alpha^i\, v + \sum_j \{H_j, m\} \otimes \tau^j v \quad . \tag{6.1.11a}$$

For $\epsilon' = -1$, the Dirac operator takes the form

$$\begin{aligned} D(m \otimes v) = & \sum_i [L_i, m] \otimes \alpha^i_-\, v + \sum_j \{H_j, m\} \otimes \tau^j_- v \\ & + \sum_k \{L_k, m\} \otimes \alpha^k_+\, v + \sum_l [H_l, m] \otimes \tau^l_+ v \quad , \end{aligned} \tag{6.1.11b}$$

where α^i_+, τ^j_+ are products of even numbers of gamma matrices and α^k_-, τ^l_- odd numbers of gamma matrices (apart from being Hermitian/anti-Hermitian as prescribed above). In both cases of ϵ', whenever s is even, one only has products of odd numbers of gamma matrices.

Example 6.1.6. Let us begin with some simple examples [Bar15] for fuzzy spaces with $N \geq 2$. In the following, we will use $\overline{c}$ to denote complex conjugation of $c \in \mathbb{C}$.

(i) $(p, q) = (0, 0)$:
  - Clifford module: $V = \mathbb{C}$
  - Real structure: $C(c) = \overline{c}$
  - Gamma matrices: none
  - Chirality operator: $\widehat{\gamma} = 1$
  - Dirac operator: $D = 0$

(ii) $(p, q) = (1, 0)$
  - Clifford module: $V = \mathbb{C}$
  - Real structure: $C(c) = \overline{c}$
  - Gamma matrices: $\widehat{\gamma}^1 = 1$
  - Chirality operator: $\widehat{\gamma} = 1$
  - Dirac operator: $D = \{H, \cdot\}$

(iii) $(p, q) = (0, 1)$
  - Clifford module: $V = \mathbb{C}$
  - Real structure: $C(c) = \overline{c}$
  - Gamma matrices: $\widehat{\gamma}^1 = -\mathrm{i}$
  - Chirality operator: $\widehat{\gamma} = 1$
  - Dirac operator: $D = -\mathrm{i}\,[L, \cdot]$

(iv) $(p, q) = (2, 0)$
  - Clifford module: $V = \mathbb{C}^2$
  - Real structure: $C \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \overline{c_1} \\ \overline{c_2} \end{pmatrix}$
  - Gamma matrices: $\widehat{\gamma}^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\widehat{\gamma}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
  - Chirality operator: $\widehat{\gamma} = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}$

- Dirac operator: $D = \{H_1, \cdot\} \otimes \widehat{\gamma}^1 + \{H_2, \cdot\} \otimes \widehat{\gamma}^2$

(v) $(p,q) = (1,1)$

- Clifford module: $V = \mathbb{C}^2$
- Real structure: $C \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \overline{c_1} \\ \overline{c_2} \end{pmatrix}$
- Gamma matrices: $\widehat{\gamma}^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\widehat{\gamma}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- Chirality operator: $\widehat{\gamma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Dirac operator: $D = \{H, \cdot\} \otimes \widehat{\gamma}^1 + [L, \cdot] \otimes \widehat{\gamma}^2$

(vi) $(p,q) = (0,2)$

- Clifford module: $V = \mathbb{C}^2$
- Real structure: $C \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \overline{c_2} \\ -\overline{c_1} \end{pmatrix}$
- Gamma matrices: $\widehat{\gamma}^1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$, $\widehat{\gamma}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- Chirality operator: $\widehat{\gamma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Dirac operator: $D = [L_1, \cdot] \otimes \widehat{\gamma}^1 + [L_2, \cdot] \otimes \widehat{\gamma}^2$

$\triangledown$

Example 6.1.7. One of the standard examples of noncommutative geometry is the fuzzy 2-sphere [Mad92, GP95]. The fuzzy sphere can be realised as a fuzzy spectral triple, as we shall outline here.

Recall from Section 5.1 that the function algebra for the fuzzy sphere is $A \cong \mathrm{Mat}_{N+1}(\mathbb{C})$. We will consider the fuzzy sphere as a fuzzy space of type $(1,3)$. For more details, we refer to [Bar15]. To match with the notation used there, we make a change of basis for $\mathfrak{su}(2)$. Define $e_{ij} := [e_i, e_j] = \mathrm{i}\, \epsilon_{ijk}\, e_k$. The Lie bracket in the new basis is $[e_{ij}, e_{kl}] = \delta_{jk}\, e_{il} - \delta_{jl}\, e_{ik} - \delta_{ik}\, e_{jl} + \delta_{il}\, e_{jk}$. Let $L_{ij} := \rho(e_{ij})$. We now list the remaining data (without providing details). There is a procedure for obtaining Clifford modules for all (p,q) using the ones in Example 6.1.6, see e.g. [Bar15, Section II.B].

The $(1,3)$ case is obtained here by first forming a $(0,3)$ Clifford module from the $(0,2)$ and $(0,1)$ Clifford modules and then augmenting the $(1,0)$ instance with it. The $(1,3)$ Clifford module is $V = \mathbb{C}^4$ with gamma matrices

$$\widehat{\gamma}^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \quad , \quad , \widehat{\gamma}^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad , \quad , i = 1, 2, 3, \tag{6.1.12}$$

where σ^i are the Pauli matrices (1.1.33).[3] The corresponding chirality operator is given by

$$\widehat{\gamma} = -\,\mathrm{i}\; \widehat{\gamma}^0\, \widehat{\gamma}^1\, \widehat{\gamma}^2\, \widehat{\gamma}^3 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix} \quad . \tag{6.1.13}$$

The real structure is

$$C \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -\bar{c}_4 \\ \bar{c}_3 \\ -\bar{c}_2 \\ \bar{c}_1 \end{pmatrix} \quad . \tag{6.1.14}$$

Finally, the Dirac operator for the fuzzy sphere is defined to be

$$D_{\mathrm{S}^2_N} := I_{N+1} \otimes \widehat{\gamma}^0 + \frac{1}{2} \sum_{i,j=1}^{3} [L_{ij}, \,\cdot\,] \otimes \widehat{\gamma}^0\, \widehat{\gamma}^i\, \widehat{\gamma}^j \quad . \tag{6.1.15}$$

$\triangledown$

Another example of a fuzzy space is the fuzzy torus, see Section 5.3. We refer to [BGa19] for an extensive exposition of the subject in the context of fuzzy spectral triples.

6.2 AUTOMORPHISMS AND DYNAMICAL FUZZY SPECTRAL TRIPLES

There is a natural notion of automorphisms of (p,q)-fermion spaces. An *automorphism* of a (p,q)-fermion space $(A, \mathcal{H}, \langle\!\langle \,\cdot\,,\,\cdot\, \rangle\!\rangle, \pi, \Gamma, J)$ is a pair (φ, Φ) consisting of

3 It should be noted that by using the method mentioned above for constructing the $(1,3)$ Clifford module, one obtains a slight difference in signs and ordering of the gamma matrices, namely $\gamma^1 = -\widehat{\gamma^3}$, $\gamma^2 = \widehat{\gamma}^2$ and $\gamma^3 = \widehat{\gamma}^1$ (here, γ^i denote the gamma matrices before any change of signs or reordering). This does not influence any of the other formulas because the gamma matrices $\widehat{\gamma}^1$, $\widehat{\gamma}^2$ and $\widehat{\gamma}^3$ anticommute. We present them in this way in order to have a uniform notation and to follow the ordering of the Pauli matrices in (1.1.33). This also matches with how the $(1,3)$ fuzzy space is presented in [Bar15, Section VI].

- a $*$-algebra automorphism $\varphi : A \to A$
- a left A-module automorphism $\Phi : \mathcal{H} \to \mathcal{H}$ relative to φ, i.e. $\Phi(a\,h) = \varphi(a)\,\Phi(h)$ for all $a \in A$ and $h \in \mathcal{H}$ such that it preserves the inner product $\langle\!\langle \Phi(\,\cdot\,), \Phi(\,\cdot\,)\rangle\!\rangle = \langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$, the chirality operator $\Gamma\,\Phi = \Phi\,\Gamma$ and the real structure $J\,\Phi = \Phi\,J$.

In particular, the morphism Φ can be written out more explicitly by using the fact that $\mathcal{H} = A \otimes V$ is a free module and that the center of $A = \mathrm{Mat}_N(\mathbb{C})$ consists of complex multiples of the unit $\mathbb{1}$. Then one can conclude that Φ is of the form

$$\Phi(a \otimes v) \;=\; \varphi(a) \otimes T(v) \quad , \tag{6.2.1}$$

for all $a \otimes v \in \mathcal{H}$, where $T \in \mathrm{Aut}(V)$ is an automorphism of the Clifford module V which preserves the inner product $\langle T(\,\cdot\,), T(\,\cdot\,)\rangle = \langle\,\cdot\,,\,\cdot\,\rangle$, the chirality $\gamma\,T = T\,\gamma$ and the real structure $C\,T = T\,C$. Denoting the group resulting from the automorphisms T by $K \subseteq \mathrm{Aut}(V)$, we conclude that the automorphism group of the (p,q)-fermion space is isomorphic to the product group $\mathrm{Aut}(A) \times K$. The two factors carry the following interpretations: The group $\mathrm{Aut}(A)$ acts on the underlying $*$-algebra and thus should be thought of as the analogue to the diffeomorphism group in commutative differential geometry, whereas the group $K \subseteq \mathrm{Aut}(V)$ acts only on the Clifford module V and therefore should be viewed as as global, i.e. A-independent, transformations of spinors.

Definition 6.2.1. We call $\mathcal{G} := \mathrm{Aut}(A) \times K$ the *gauge group* of the (p,q)-fermion space over $A = \mathrm{Mat}_N(\mathbb{C})$. This group acts from the left as automorphisms of the (p,q)-fermion space

$$\rho_A \,:\, \mathcal{G} \times A \;\longrightarrow\; A\;,\;\; (\varphi, T, a) \;\longmapsto\; \rho_A(\varphi, T)(a) \;=\; \varphi(a) \quad , \tag{6.2.2a}$$

$$\rho_{\mathcal{H}} \,:\, \mathcal{G} \times \mathcal{H} \;\longrightarrow\; \mathcal{H}\;,\;\; (\varphi, T, a \otimes v) \;\longmapsto\; \rho_{\mathcal{H}}(\varphi, T)(a \otimes v) \;=\; \varphi(a) \otimes T(v) \quad . \tag{6.2.2b}$$

The induced left adjoint action on the space of Dirac operators is given by

$$\rho_{\mathcal{D}} \,:\, \mathcal{G} \times \mathcal{D} \;\longrightarrow \mathcal{D}\;,\;\; (\varphi, T, D) \;\longmapsto\; \rho_{\mathcal{H}}(\varphi, T) \circ D \circ \rho_{\mathcal{H}}(\varphi^{-1}, T^{-1}) \quad , \tag{6.2.3}$$

where $\circ$ denotes composition of maps.

Remark 6.2.2. A standard fact is that $\mathrm{Aut}(A) \cong PU(N) := U(N)/U(1)$ is isomorphic to the projective unitary group, see e.g. [vanS15, Example 6.3]. This isomorphism is given by assigning to an element $[u] \in PU(N)$ the automorphism $\varphi_{[u]} \in \mathrm{Aut}(A)$

defined by $\varphi_{[u]}(a) = u\,a\,u^*$, for all $a \in A$. Hence, the $\mathcal{G}$-actions from Definition 6.2.1 can be presented more explicitly as $\rho_A(\varphi_{[u]}, T)(a) = u\,a\,u^*$ and $\rho_{\mathcal{H}}(\varphi_{[u]}, T)(a \otimes v) = (u\,a\,u^*) \otimes T(v)$.

This isomorphic perspective facilitates the description of the infinitesimal gauge transformations. Since the Lie algebra of the projective unitary group is $\mathfrak{su}(N) \cong \mathfrak{pu}(N)$, the Lie algebra $\mathfrak{g}$ of the gauge group $\mathcal{G}$ is given by a direct sum

$$\mathfrak{g} = \mathfrak{su}(N) \oplus \mathfrak{k} \quad , \tag{6.2.4}$$

where $\mathfrak{k}$ denotes the Lie algebra of $K \subseteq \mathrm{Aut}(V)$.[4] The Lie algebra actions induced by the $\mathcal{G}$-actions from Definition 6.2.1 can then explicitly be expressed as

$$\rho_A(\epsilon \oplus k)(a) = [\epsilon, a]_A \quad , \tag{6.2.5a}$$

$$\rho_{\mathcal{H}}(\epsilon \oplus k)(a \otimes v) = [\epsilon, a]_A \otimes v + a \otimes k(v) \quad , \tag{6.2.5b}$$

$$\rho_{\mathcal{D}}(\epsilon \oplus k)(D) = [\rho_{\mathcal{H}}(\epsilon \oplus k), D]_{\mathrm{End}(\mathcal{H})} \quad , \tag{6.2.5c}$$

for all $\epsilon \oplus k \in \mathfrak{g}$, where $[\,\cdot\,,\,\cdot\,]_A$ denotes the commutator on A and $[\,\cdot\,,\,\cdot\,]_{\mathrm{End}(\mathcal{H})}$ the commutator on $\mathrm{End}(\mathcal{H})$. △

The next objective is to choose an appropriate action $S : \mathcal{D} \to \mathbb{R}$ on the Dirac ensemble $\mathcal{D}$ for a fixed (p,q)-fermion space $(A, \mathcal{H}, \langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle, \pi, \Gamma, J)$. A typical class of such actions are the *spectral actions*, introduced first in [Con96] and later examined in more depth and generality in [CC97]. One well studied choice is the action $S(D) = \mathrm{Tr}_{\mathrm{End}(\mathcal{H})}(\frac{g_2}{2}\,D^2 + \frac{g_4}{4!}\,D^4)$, where $g_2, g_4 \in \mathbb{R}$ are constants and $\mathrm{Tr}_{\mathrm{End}(\mathcal{H})}$ is the trace on the endomorphisms of the Hilbert space $\mathcal{H}$. This action was investigated in [BG16] due to its simplicity, as well as in [BDG19, KP21a]. There have also been later studies of more general actions of the form $S(D) = \mathrm{Tr}_{\mathrm{End}(\mathcal{H})}(f(D))$, where f is a real-valued polynomial in [PS19, HKP21, KP21b] and even more general multi-trace actions appeared in [AK19]. In our case, for the majority of the time, we can work in the more general setting given in the below definition, without making any specific choices of actions. Let $\mathcal{D}^\vee$ denote the dual to the real vector space $\mathcal{D}$ and $\mathrm{Sym}\,\mathcal{D}^\vee$ its symmetric algebra. Observe that $\mathrm{Sym}\,\mathcal{D}^\vee$ is (isomorphic to) the algebra of polynomial functions on $\mathcal{D}$.

Definition 6.2.3. An *action* $S : \mathcal{D} \to \mathbb{R}$ is a gauge-invariant and real-valued polynomial function on the space of Dirac operators, or equivalently a $\mathcal{G}$-invariant element $S \in \mathrm{Sym}\,\mathcal{D}^\vee$.

4 Recall that elements $\epsilon \in \mathfrak{su}(N) \subseteq \mathrm{Mat}_N(\mathbb{C})$ are anti-Hermitian and trace-free $N \times N$-matrices.

Recall that the BV formalism treats field theories perturbatively. We will end this section by briefly discussing the perturbative approach to dynamical fuzzy spectral triples. First, we choose an action $S \in \operatorname{Sym}\mathcal{D}^\vee$ and an exact solution $D_0 \in \mathcal{D}$ of its Euler-Lagrange equations. Then we may consider formal perturbations

$$D = D_0 + \lambda\,\tilde{D} \tag{6.2.6}$$

of the Dirac operator, where λ is a formal parameter and the perturbation $\tilde{D} \in \mathcal{D}$ is an element of the same vector space $\mathcal{D}$. The perturbation $\tilde{D}$ is considered as the dynamical field in this perturbative treatment. The infinitesimal gauge transformations (6.2.5) of the perturbation then take the form

$$\tilde{\rho}_{\mathcal{D}}(\epsilon \oplus k)(\tilde{D}) \;=\; [\rho_{\mathcal{H}}(\epsilon \oplus k), D_0]_{\mathrm{End}(\mathcal{H})} + \lambda\,[\rho_{\mathcal{H}}(\epsilon \oplus k), \tilde{D}]_{\mathrm{End}(\mathcal{H})} \quad . \tag{6.2.7}$$

Note that they act through a combination of a linear transformation $[\rho_{\mathcal{H}}(\epsilon \oplus k), \tilde{D}]_{\mathrm{End}(\mathcal{H})}$ and an inhomogeneous one $[\rho_{\mathcal{H}}(\epsilon \oplus k), D_0]_{\mathrm{End}(\mathcal{H})}$ that depends on the background solution D_0.[5] We then define the induced action for the perturbation $\tilde{D}$ as

$$\tilde{S}(\tilde{D}) \;:=\; \frac{1}{\lambda^2}\big(S(D_0 + \lambda\,\tilde{D}) - S(D_0)\big) \quad . \tag{6.2.8}$$

The subtraction of the constant term $S(D_0)$ is a convenient choice since it ensures that $\tilde{S}$ is a sum of monomials of degree ≥ 2; the degree 1 term vanishes because D_0 is a solution of the Euler-Lagrange equations for the original action S. The role of the normalisation $\frac{1}{\lambda^2}$ is to make sure that the quadratic term in $\tilde{S}$ is of order λ^0. By construction, the action $\tilde{S} \in \operatorname{Sym}\mathcal{D}^\vee$ is invariant under the infinitesimal gauge transformations (6.2.7).

6.3 CLASSICAL BV FORMALISM OF DYNAMICAL FUZZY SPECTRAL TRIPLE MODELS

In contrast to the description of the BV formalism in Section 4.3, the *classical* BV formalism starts from (in some sense) the more primitive data of a space of fields, infinitesimal gauge symmetries and a gauge invariant action, and associates to that the corresponding P_0-algebra of classical observables. Subsequently, we will utilise the

5 Note that this is the same form as in e.g. Yang-Mills theory, where the infinitesimal gauge transformations $\chi \in C^\infty(M, \mathfrak{g})$ act on connection one-forms $A \in \Omega^1(M, \mathfrak{g})$ via $\delta_\chi(A) = \mathrm{d}\chi + [\chi, A]$, where d is the de Rham differential.

formalism in Section 4.3 to quantise the theory and compute correlation functions. In this section, we will predominantly be concerned with constructing the classical observables for the perturbative dynamical fuzzy spectral triple models from Section 6.2. We will use the systematic techniques from [BSS21] without delving to far into the details as that would require substantial coverage of derived geometry, which is beyond the scope of this thesis. For a fixed (p,q)-fermion space $(A,\mathcal{H},\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle,\pi,\Gamma,J)$, the classical observables can be constructed by specialising the general construction from there to the following input data:

- The real vector space $\mathcal{D}$ (i.e. the Dirac ensemble of the fixed (p,q)-fermion space) of perturbations $\tilde{D}$ of a given background Dirac operator D_0 which is an exact solution to the Euler-Lagrange equations of an action S. The space $\mathcal{D}$ serves as the space of fields.
- The infinitesimal gauge symmetries given by the Lie algebra $\mathfrak{g} = \mathfrak{su}(N) \oplus \mathfrak{k}$ of the gauge group from Definition 6.2.1 acting on the fields according to (6.2.7).
- The $\mathfrak{g}$-invariant action $\tilde{S}$ defined in (6.2.8), which determines the perturbative dynamics.

From the above data, we will construct the P_0-algebra of interacting classical observables (4.3.13)

$$\mathrm{Obs}^{\mathrm{cl,int}} = (\mathrm{Sym}\,\mathcal{L}, \mathrm{d}, \{\,\cdot\,,\,\cdot\,\}) \tag{6.3.1}$$

where have used the notation $\mathcal{L} = E[1]$. We have also denoted the differential by $\mathrm{d} := Q^{\mathrm{int}}$ to emphasise the slight shift of perspective; instead of starting from the free theory, we immediately obtain the interacting theory. Using the general results in [BSS21, Section 7], we obtain the underlying graded commutative algebra $\mathrm{Sym}\,\mathcal{L}$ of the BV formalism for this model with

$$\mathcal{L} := \mathfrak{g}[2] \oplus \mathcal{D}[1] \oplus \mathcal{D}^\vee \oplus \mathfrak{g}^\vee[-1] \quad . \tag{6.3.2}$$

In order to obtain a more concrete description, we choose a dual pair of vector space bases

$$\big\{e_a \in \mathcal{D}\big\}_{a=1}^{\dim\mathcal{D}} \quad , \quad \big\{f^a \in \mathcal{D}^\vee\big\}_{a=1}^{\dim\mathcal{D}} \quad , \quad \big\{t_i \in \mathfrak{g}\big\}_{i=1}^{\dim\mathfrak{g}} \quad , \quad \big\{\theta^i \in \mathfrak{g}^\vee\big\}_{i=1}^{\dim\mathfrak{g}} \tag{6.3.3}$$

for $\mathcal{D}$ and $\mathcal{D}^\vee$ and for $\mathfrak{g}$ and $\mathfrak{g}^\vee$. Then $\mathrm{Sym}\,\mathcal{L}$ is the graded commutative algebra generated by the generators t_i in degree -2, e_a in degree -1, f^a in degree 0 and θ^i

in degree 1. The physical interpretation of the generators is as follows: θ^i are linear observables for the ghost field $\mathsf{c} \in \mathfrak{g}$, f^a are linear observables for the field $\tilde{D} \in \mathcal{D}$, e_a are linear observables for the antifield $\tilde{D}^+ \in \mathcal{D}^\vee$, and t_i are linear observables for the antifield for the ghost $\mathsf{c}^+ \in \mathfrak{g}^\vee$.

The differential d in (6.3.1) is abstractly given as the totalisation of an internal differential and the Chevalley-Eilenberg differential (see [BSS21, Section 7]). It is convenient to describe it in terms of the generators. With respect to our choice of bases, the Lie bracket on $\mathfrak{g}$ and the Lie algebra action (6.2.7) are characterised by structure constants and we write

$$[t_i, t_j] \;=\; \lambda\, f_{ij}^k\, t_k \quad , \quad \tilde{\rho}_{\mathcal{D}}(t_i)(e_a) = \beta_i^b\, e_b + \lambda\, g_{ia}^b\, e_b \quad , \tag{6.3.4}$$

where β_i^b describes the inhomogeneous term (depending on D_0) in (6.2.7) and g_{ia}^b the linear term. Here, and for the rest of the chapter, we employ the standard summation convention, i.e summation is understood over repeated indices. The action $\tilde{S}$ given in (6.2.8) can also be expanded in terms of the basis

$$\tilde{S} = \sum_{n\geq 2} \frac{\lambda^{n-2}}{n!}\, S_{a_1\cdots a_n}\, f^{a_1}\cdots f^{a_n} \;\in\; \mathrm{Sym}\,\mathcal{D}^\vee \quad , \tag{6.3.5}$$

which starts at $n = 2$ because the background Dirac operator D_0 is assumed to be an exact solution of the Euler-Lagrange equations of the given action S. The coefficients $S_{a_1\cdots a_n}$ vanish for n greater than the polynomial degree of $\tilde{S}$. The differential d in (6.3.1) is then determined by the graded Leibniz rule and the following action on the generators

$$\mathrm{d}t_i \;=\; \beta_i^a\, e_a + \lambda\, g_{ib}^a\, e_a\, f^b - \lambda\, f_{ij}^k\, t_k\, \theta^j \quad , \tag{6.3.6a}$$

$$\mathrm{d}e_a \;=\; \sum_{n\geq 2} \frac{\lambda^{n-2}}{(n-1)!} S_{a a_2\cdots a_n} f^{a_2}\cdots f^{a_n} - \lambda\, g_{ja}^b\, e_b\, \theta^j \quad , \tag{6.3.6b}$$

$$\mathrm{d}f^a \;=\; -\beta_j^a\, \theta^j - \lambda\, g_{jb}^a\, f^b\, \theta^j \quad , \tag{6.3.6c}$$

$$\mathrm{d}\theta^i \;=\; -\frac{\lambda}{2} f_{jk}^i\, \theta^j\, \theta^k \quad . \tag{6.3.6d}$$

Note that the differential encodes both the equation of motion and the gauge symmetries, which is a feature of the BV formalism. The nilpotency $\mathrm{d}^2 = 0$ follows from Lie algebra representation identities for the structure constants f_{ij}^k, g_{ia}^b and β_i^a and gauge invariance of the action $\tilde{S}$.

In order to describe the antibracket $\{\,\cdot\,,\,\cdot\,\}$ of (6.3.1), we use the canonical (-1)-shifted symplectic structure of the dg-algebra $(\mathrm{Sym}\,\mathcal{L},\mathrm{d})$ which in our case takes the form

$$\omega \;=\; \mathrm{d}^{\mathrm{dR}} e_a \wedge \mathrm{d}^{\mathrm{dR}} f^a - \mathrm{d}^{\mathrm{dR}} t_i \wedge \mathrm{d}^{\mathrm{dR}} \theta^i \quad . \tag{6.3.7}$$

Here, d^{dR} denotes the de Rham differential. The (-1)-shifted symplectic structure is nondegenerate and both d-closed and d^{dR}-closed. The antibracket is the shifted Poisson structure dual to ω. It is concretely defined by

$$\{a,b\} := \iota_{{}_aH}\iota_{{}_bH}\omega \quad , \tag{6.3.8}$$

for all $a,b \in \mathrm{Obs}^{\mathrm{cl}}$, where ι denotes the contraction between vector fields and forms, and ${}_aH$ is the shifted Hamiltonian vector field defined by $\mathrm{d}^{\mathrm{dR}}a = \iota_{{}_aH}\omega$. One can show that the antibracket satisfies the required graded antisymmetry property, the graded Jacobi identity, the derivation property and compatibility with the differential, see Remark 4.3.4 for the explicit formulas. As a result of these properties, the antibracket is completely determined by its value on the generators,

$$\{t_i,\theta^j\} \;=\; \delta_i^j \;=\; -\{\theta^j,t_i\} \quad , \quad \{e_a,f^b\} \;=\; -\delta_a^b \;=\; -\{f^b,e_a\} \quad , \tag{6.3.9}$$

and zero otherwise. We have thus constructed the P_0-algebra of (interacting) classical observables (6.3.1) for dynamical fuzzy spectral triples models.

Remark 6.3.1. For the purpose of putting the above considerations in a context, we make a few brief comments surrounding differential forms in this setting. The $\mathrm{Sym}\,\mathcal{L}$-module of 1-forms $\Omega^1_{\mathrm{Sym}\,\mathcal{L}} \cong \big(\mathrm{Sym}\,\mathcal{L}\big)\otimes\mathcal{L}$ (the underlying concept is that of *Kähler differentials*) has a basis consisting of $\mathrm{d}^{\mathrm{dR}}t_i$, $\mathrm{d}^{\mathrm{dR}}e_a$, $\mathrm{d}^{\mathrm{dR}}f^a$ and $\mathrm{d}^{\mathrm{dR}}\theta^i$ which we for now collectively denote by $\mathrm{d}x^\mu$. Higher differential forms $\Omega^\bullet_{\mathrm{Sym}\,\mathcal{L}}$ are as usual built using the wedge product, which satisfies $\alpha\wedge\alpha' = (-1)^{|\alpha|\,|\alpha'|+p\,p'}\alpha'\wedge\alpha$, where α and α' are differential forms of form degree p and p' respectively. As before, $|\cdot|$ denotes the cohomological degree. In our conventions, the differential d and the de Rham differential d^{dR} commute, $\mathrm{d}\,\mathrm{d}^{\mathrm{dR}} = \mathrm{d}^{\mathrm{dR}}\,\mathrm{d}$ (hence $\Omega^\bullet_{\mathrm{Sym}\,\mathcal{L}}$ forms a so called *double complex*).

In order to fix the signs, let us also swiftly write down the definition of the contraction operation. Dual to the 1-forms are the tangent vector fields $T_{\mathrm{Sym}\,\mathcal{L}} \cong \big(\mathrm{Sym}\,\mathcal{L}\big)\otimes\mathcal{L}^\vee$. Let us denote the dual basis by ∂_μ. Then the contraction is a $\mathrm{Sym}\,\mathcal{L}$-linear map

$$\iota : T_{\mathrm{Sym}\,\mathcal{L}}\otimes\Omega^\bullet_{\mathrm{Sym}\,\mathcal{L}} \longrightarrow \Omega^\bullet_{\mathrm{Sym}\,\mathcal{L}} \ , \ \ X\otimes\alpha \longmapsto \iota(X\otimes\alpha) := \iota_X\alpha \tag{6.3.10}$$

defined by $\iota_{\partial_\mu} \mathrm{d}x^\nu = \delta_\mu{}^\nu$ and $\iota_{\partial_\mu} 1 = 0$ such that the Leibniz rule is satisfied:

$$\iota_X(\alpha \wedge \alpha') = \big(\iota_X \alpha\big) \wedge \alpha' + (-1)^{|X|\,|\alpha| - p} \alpha \wedge \big(\iota_X \alpha'\big) \tag{6.3.11}$$

for homogeneous $X \in T_{\mathrm{Sym}\,\mathcal{L}}$, homogeneous $\alpha \in \Omega^p_{\mathrm{Sym}\,\mathcal{L}}$, and $\alpha' \in \Omega^\bullet_{\mathrm{Sym}\,\mathcal{L}}$. △

Remark 6.3.2. Let us rewrite what we have developed so far in this section in an equivalent description commonly found in the literature, namely the *BV action*. The BV extension of the action in (6.3.5) reads explicitly as

$$S_{\mathrm{BV}} \,=\, \sum_{n \geq 2} \frac{\lambda^{n-2}}{n!}\, S_{a_1 \cdots a_n} f^{a_1} \cdots f^{a_n} - \lambda\, g^a_{i\,b}\, e_a\, f^b\, \theta^i - \frac{\lambda}{2}\, f^k_{ij}\, t_k\, \theta^i\, \theta^j - \beta^a_i\, e_a\, \theta^i \quad . \tag{6.3.12}$$

From the properties of the antibracket (Remark 4.3.4), one shows that the differential (6.3.6) is given by

$$\mathrm{d} \,=\, \{S_{\mathrm{BV}}, \,\cdot\,\} \quad . \tag{6.3.13}$$

In fact, from here we see that the interacting part of d (i.e. the terms of order $\lambda^{\geq 1}$) indeed is given by $\{\lambda, I, \,\cdot\,\}$ with $I \in (\mathrm{Sym}\,\mathcal{L})^0$, matching with how interactions are treated in Section 4.3.1. The square-zero condition $\mathrm{d}^2 = 0$ for the differential is equivalent to the classical master equation (4.3.19)

$$\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} \,=\, 0 \tag{6.3.14}$$

for the BV action. △

In order to implement the formalism in Section 4.3, we extract the free part of the classical observables (6.3.1) and treat interactions and subsequent quantisation as perturbations. This means that we split the differential (6.3.6) into

$$\mathrm{d} = \mathrm{d}^{\mathrm{free}} + \lambda\, \mathrm{d}^{\mathrm{int}} \tag{6.3.15}$$

where $\mathrm{d}^{\mathrm{free}}$ is the part of the differential obtained by setting $\lambda = 0$ in (6.3.6). By definition, $\mathrm{d}^{\mathrm{free}}$ is linear in the generators. Hence the classical free observables (see (4.3.10))

$$\mathrm{Obs}^{\mathrm{free}} \,:=\, \big(\mathrm{Sym}\,\mathcal{L}, \mathrm{d}^{\mathrm{free}}, \{\,\cdot\,,\,\cdot\,\}\big) \,=\, \big(\mathrm{Sym}(\mathcal{L}, \mathrm{d}^{\mathrm{free}}), \{\,\cdot\,,\,\cdot\,\}\big) \tag{6.3.16a}$$

are given by the symmetric algebra of the cochain complex

$$(\mathcal{L}, \mathrm{d}^{\mathrm{free}}) \,=\, \Big(\, \mathfrak{g}[2] \xrightarrow{\mathrm{d}^{\mathrm{free}}} \mathcal{D}[1] \xrightarrow{\mathrm{d}^{\mathrm{free}}} \mathcal{D}^\vee \xrightarrow{\mathrm{d}^{\mathrm{free}}} \mathfrak{g}^\vee[-1] \,\Big) \quad . \tag{6.3.16b}$$

(The cochain complex $(\mathcal{L}, \mathrm{d}^{\mathrm{free}})$ corresponds to the cochain complex $(E[1], Q)$ in (4.3.10).) Explicitly, the action of the differential on the vector space bases (6.3.3) reads as

$$\mathrm{d}^{\mathrm{free}} t_i \,=\, \beta^a_i\, e_a \quad , \quad \mathrm{d}^{\mathrm{free}} e_a \,=\, S_{ab}\, f^b \quad , \quad \mathrm{d}^{\mathrm{free}} f^a \,=\, -\beta^a_j\, \theta^j \quad , \quad \mathrm{d}^{\mathrm{free}} \theta^i \,=\, 0 \quad . \tag{6.3.16c}$$

6.4 BV QUANTISATION AND CORRELATORS

We now have everything set up for the procedure outlined in Section 4.3 for quantisation and computation of correlation functions. We recall some of the parts here in order to set up the notation. For BV quantisation, recall that we deform the differential along the antibracket via the BV Laplacian (4.3.15). The quantum observables for the interacting theory (4.3.21) are thus given by

$$\mathrm{Obs}^{\mathrm{int},\hbar} \ := \ \big(\mathrm{Sym}\,\mathcal{L}, \mathrm{d}^{\hbar} := \mathrm{d}^{\mathrm{free}} + \lambda\,\mathrm{d}^{\mathrm{int}} + \hbar\,\Delta_{\mathrm{BV}}\big) \quad . \tag{6.4.1}$$

The correlation functions can be computed by choosing a strong deformation retract (see Definition 4.3.6)

$$\big(\mathrm{H}^\bullet(\mathcal{L},\mathrm{d}^{\mathrm{free}}),0\big) \ \underset{\iota}{\overset{\pi}{\rightleftarrows}} \ (\mathcal{L},\mathrm{d}^{\mathrm{free}}) \circlearrowleft \xi \tag{6.4.2}$$

of the cochain complex $(\mathcal{L},\mathrm{d}^{\mathrm{free}})$ onto its cohomology and running the machinery in Section 4.3.3. The n-point correlation function is computed perturbatively in λ and $\hbar$ from the formula (4.3.38), repeated here

$$\langle\varphi_1\cdots\varphi_n\rangle \ = \ \sum_{k=0}^{\infty} \Pi\Big(\big(\delta\ \Xi\big)^k\,\big(\varphi_1\cdots\varphi_n\big)\Big) \in \ \mathrm{Sym}\,H^\bullet(\mathcal{L},\mathrm{d}^{\mathrm{free}}) \quad , \tag{4.3.38}$$

where $\delta = \lambda\,\mathrm{d}^{\mathrm{int}} + \hbar\,\Delta_{\mathrm{BV}}$. The maps Π and Ξ are the extensions of π and ξ to the symmetric algebras, described in (4.3.32) and around (4.3.34) respectively. As before, one can use graphical tools to compute the correlation functions, which we will see in the upcoming sections.

6.5 PERTURBATIONS AROUND $D_0 = 0$

In this section we will investigate perturbations around the zero Dirac operator $D_0 = 0$ and provide some general results for this case. We fix an arbitrary (p,q)-fermion space over $A = \mathrm{Mat}_N(\mathbb{C})$ and consider any gauge-invariant polynomial action of the form

$$S(D) \ = \ \frac{g_2}{2}\,\mathrm{Tr}_{\mathrm{End}(\mathcal{H})}\big(D^2\big) + S^{\mathrm{int}}(D) \quad , \tag{6.5.1}$$

where $g_2 \neq 0$ is a non-zero constant and the interaction term S^{int} is a sum of monomials of degree ≥ 3. Observe that the zero Dirac operator $D_0 = 0$ is an exact solution to the Euler-Lagrange equations of this action and hence an admissible choice for background Dirac operator. Furthermore, the quadratic term in the action (6.5.1) is nondegenerate: the complex vector space $\mathrm{End}(\mathcal{H})$ can be equipped with the Hermitian inner product $\langle B, B' \rangle := \mathrm{Tr}_{\mathrm{End}(\mathcal{H})}(B^* \, B')$, for all $B, B' \in \mathrm{End}(\mathcal{H})$, which restricts to a real inner product on the real subspace $\mathcal{D} \subseteq \mathrm{End}(\mathcal{H})$ of Dirac operators (6.1.2). The nondegeneracy of the quadratic term will lead to a particularly simple propagator.

Having fixed the (p, q)-fermion space means we have also fixed the Dirac ensemble of perturbations and the gauge Lie algebra (6.2.4). Having also chosen the action, we now have the ingredients for running the classical BV formalism of Section 6.3, setting up for quantisation and computation of correlation functions following Section 6.4. We first describe the free theory. From the definition of the structure constants 6.3.4, we see that for $D_0 = 0$ the constants $\beta^a_i = 0$ vanish, hence the cochain complex of linear observables simplifies to

$$(\mathcal{L}, \mathrm{d}^{\mathrm{free}}) \;=\; \Big(\mathfrak{g}[2] \xrightarrow{\;\;0\;\;} \mathcal{D}[1] \xrightarrow{\;\mathrm{d}^{\mathrm{free}}\;} \mathcal{D}^\vee \xrightarrow{\;\;0\;\;} \mathfrak{g}^\vee[-1] \Big) \quad , \tag{6.5.2}$$

where we recall that $\mathrm{d}^{\mathrm{free}} e_a = S_{ab}\, f^b$ is controlled by the quadratic term of the action. The nondegeneracy of the quadratic term in the action (6.5.1) implies that S_{ab} is invertible. Hence the cohomology of the complex is

$$\mathrm{H}^\bullet(\mathcal{L}, \mathrm{d}^{\mathrm{free}}) \;=\; \mathfrak{g}[2] \oplus \mathfrak{g}^\vee[-1] \quad . \tag{6.5.3}$$

The other piece of data we need to feed into the formalism is a choice of strong deformation retract (6.4.2). For this, we choose

$$\pi \,:\, \begin{cases} t_i \mapsto t_i \\ e_a \mapsto 0 \\ f^a \mapsto 0 \\ \theta^i \mapsto \theta^i \end{cases} \quad , \qquad \iota \,:\, \begin{cases} t_i \mapsto t_i \\ \theta^i \mapsto \theta^i \end{cases} \quad , \qquad h \,:\, \begin{cases} t_i \mapsto 0 \\ e_a \mapsto 0 \\ f^a \mapsto -S^{ab}\, e_b \\ \theta^i \mapsto 0 \end{cases} \quad , \tag{6.5.4}$$

where S^{ab} denotes the inverse of S_{ab}, i.e. $S^{ab} S_{bc} = \delta^a_c = S_{cb} S^{ba}$. (The relevant properties of Definition 4.3.6 are straightforward to confirm.)

We now have all the building blocks to compute the quantum correlation functions for our model. Let us collect the relevant information about the maps appearing in (4.3.38):

(i) The cochain homotopy Ξ is defined by (4.3.34) together with its action on generators (6.5.4). The relevant direct sum decomposition (4.3.33) is in this case given by

$$\mathcal{L} \,=\, \mathcal{L}^{\perp} \oplus \mathrm{H}^{\bullet}(\mathcal{L}, \mathrm{d}^{\mathrm{free}}) \,=\, \Big(\mathcal{D}[1] \xrightarrow{\mathrm{d}^{\mathrm{free}}} \mathcal{D}^{\vee} \Big) \oplus \Big(\mathfrak{g}[2] \oplus \mathfrak{g}^{\vee}[-1] \Big) \quad . \tag{6.5.5}$$

(ii) The small perturbation of the differential is given by $\delta = \lambda\, \mathrm{d}^{\mathrm{int}} + \hbar\, \Delta_{\mathrm{BV}}$. The interaction part of the differential, $\mathrm{d}^{\mathrm{int}}$, is defined by the graded Leibniz rule and the order $\lambda^{\geq 1}$ terms in (6.3.6). The BV Laplacian Δ_{BV} is defined by (4.3.15) and (6.3.9).

(iii) The dg-algebra map Π is given by (4.3.32) and its action on the generators (6.5.4).

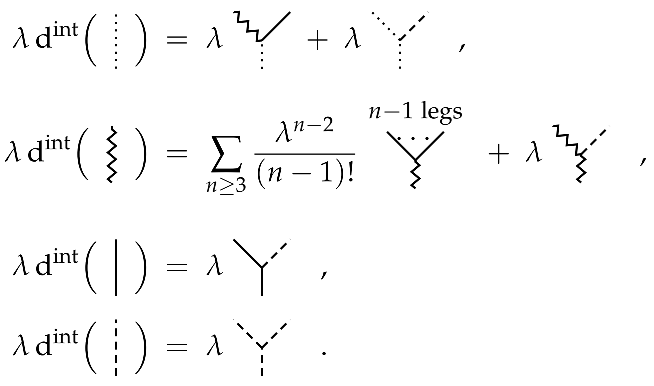
To make the computations of the correlation functions (4.3.38) more tractable, we introduce a graphical notation. We denote elements $\varphi_1 \cdots \varphi_n \in \mathrm{Sym}(\mathcal{L})$ by n vertical lines, where each $\varphi_i \in \mathcal{L}$. Since we have four different field species (fields, ghosts, antifields and antifields for ghosts) distinguished by their cohomological degree in $\mathcal{L}$, we need four different types of lines

$$t_i = \quad , \qquad e_a = \quad , \qquad f^a = \quad , \qquad \theta^i = \quad . \tag{6.5.6}$$

The action of the cochain homotopy Ξ on n lines can be expressed via (4.3.34) as a sum of actions on the individual lines. We would like to emphasize that, because of the direct sum decomposition (6.5.5), the number n in (4.3.34) counts only the number of wiggly and straight lines, i.e. there are *no* contributions to n from dashed and dotted lines. From (6.5.4), we observe that the homotopy Ξ is only non-zero when acting on f^a, which we depict as

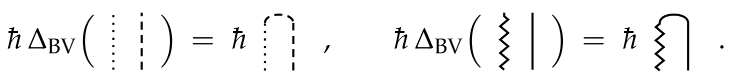
$$\Xi\Big(\quad \Big) = \quad . \tag{6.5.7}$$

The interaction part of the differential acting on n lines can be written as a sum of actions on the individual lines due to the (graded) Leibniz rule. The action on a single line may be depicted as interaction vertices



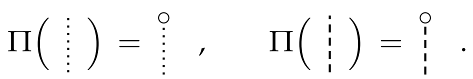
(6.5.8a) (6.5.8b) (6.5.8c) (6.5.8d)

The diagrams should be read from bottom to top and the numerical values are given in (6.3.6). Because the lines represent elements in the graded symmetric algebra $\operatorname{Sym}\mathcal{L}$, any two neighbouring lines can be permuted up to a Koszul sign determined by their cohomological degree. For the BV Laplacian, its action on n lines can be reduced to a sum of pairings between two lines by using the algebraic properties in (4.3.15). Taking into account (6.3.9), we see that the only non-vanishing terms are the ones with pairing between t_i and θ^j and when pairing between e_a and f^b. We will depict such pairings as



(6.5.9)

Similarly to before, the ingoing lines can be permuted (in this case, the Koszul signs are trivial). Finally, according to (4.3.32) the dg-algebra map Π acts on n lines as a product of actions on individual lines, evaluating them to their cohomology classes (6.5.4). This is only non-zero for t_i and θ^i, which we will depict by



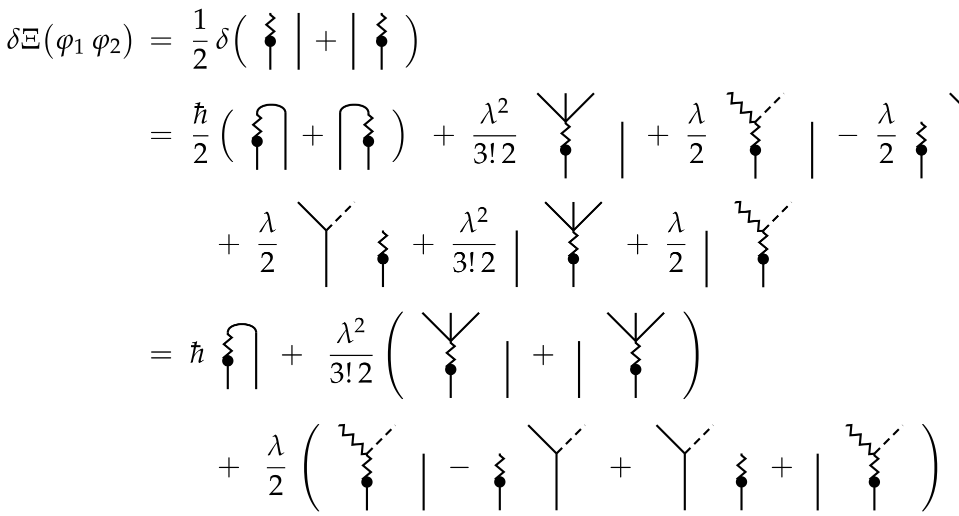
(6.5.10)

Remark 6.5.1. In comparison with [NSS21] and Chapter 5, note that there is a difference to how we approach the graphical calculus. In that case, there were fewer field species and the homotopy was only non-trivial on $\varphi \in E[1]^0 = \mathcal{L}^0$ (which will not be the case in the next section), allowing for a significantly simpler graphical presentation. $\triangle$

Example 6.5.2. Let us study the case of an action with quartic interaction term (i.e. the $n = 4$ term is the only non-trivial summand in (6.5.8b)) and the 2-point correlation function

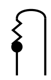
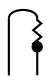
$$\langle \varphi_1\, \varphi_2 \rangle \;=\; \sum_{k=0}^{\infty} \Pi\big((\delta\Xi)^k(\varphi_1\, \varphi_2)\big) \quad , \qquad \varphi_1, \varphi_2 \in \mathcal{L}^0 = \mathcal{D}^\vee \quad , \tag{6.5.11}$$

on two generators in degree 0 to the lowest non-trivial order in the formal parameter λ. This will be computed by iteratively applying $\delta\Xi$. The first iteration is given by



(6.5.12)

The negative sign in the second equality is a Koszul sign due to $\delta = \lambda\, \mathrm{d}^{\mathrm{int}} + \hbar\, \Delta_{\mathrm{BV}}$ being a map of cohomological degree 1. In the third equality, we have sorted the interaction terms according to their power in λ. The term of order $\hbar$ is simplified as a consequence of the identity

 $= \Delta_{\mathrm{BV}}\big(\xi(\varphi_1)\, \varphi_2\big) \;=\; -\{\xi(\varphi_1), \varphi_2\} \;=\; \{\varphi_1, \xi(\varphi_2)\} \;=\; \Delta_{\mathrm{BV}}\big(\varphi_1\, \xi(\varphi_2)\big) \;=$  ,

(6.5.13a)

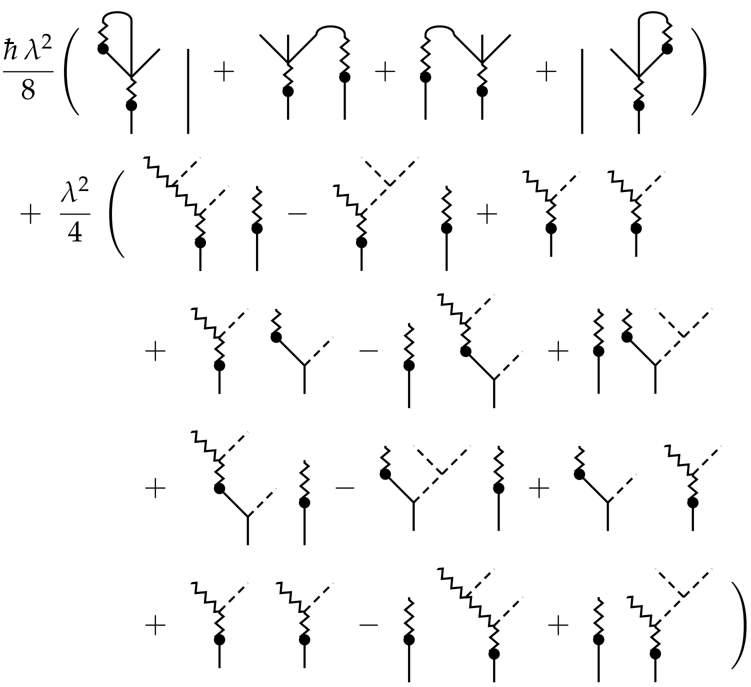
where the middle step is checked on basis elements

$$-\{\xi(f^a), f^b\} \;=\; S^{ac}\,\{e_c, f^b\} \;=\; -S^{ab} \;=\; -S^{ba} \;=\; -S^{bc}\{f^a, e_c\} \;=\; \{f^a, \xi(f^b)\} \tag{6.5.13b}$$

by using (6.3.9), (6.5.4) and symmetry of S^{ab}.

The subsequent calculations are performed using similar arguments as well as the fact that the interaction term $S_{a_1a_2a_3a_4}$ is symmetric under the exchange of indices. As a result, the second iteration is given by

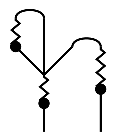
$$(\delta\Xi)^2(\varphi_1\,\varphi_2) =$$



$$+\ \mathcal{O}(\lambda^3) \qquad (6.5.14)$$

and the third by

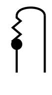
$$(\delta\Xi)^3(\varphi_1\,\varphi_2) = \frac{\hbar^2\lambda^2}{2}$$



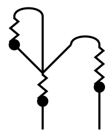
$$+\ \mathcal{O}(\lambda^3) \quad . \qquad (6.5.15)$$

By applying Π to these expressions we obtain the 2-point function to order λ^2,

$$\langle\varphi_1\,\varphi_2\rangle = \hbar$$



$$+\ \frac{\hbar^2\lambda^2}{2}$$



$$+\ \mathcal{O}(\lambda^3) \quad . \qquad (6.5.16)$$

Observe that neither the ghosts nor the antifields for ghosts contribute to the 2-point function at order λ^2. It turns out that in the case of perturbations around the zero Dirac operator $D_0 = 0$, this is true for *all* n-point functions of degree zero observables, *all* interaction terms and to *all* orders of the perturbation series. $\triangledown$

Proposition 6.5.3. *Consider an arbitrary (p, q)-fermion space over $A = \mathrm{Mat}_N(\mathbb{C})$ and any gauge-invariant action of the form* (6.5.1). *Then, for perturbations around the zero Dirac operator $D_0 = 0$, all n-point quantum correlation functions $\langle\varphi_1 \cdots \varphi_n\rangle$ for degree* 0 *observables $\varphi_1, \ldots, \varphi_n \in \mathcal{L}^0 = \mathcal{D}^\vee$ do* not *receive contributions from ghosts and antifields for ghosts.*

Proof. The proof is a simple argument using our graphical calculus. Starting from n straight lines, representing the element $\varphi_1 \cdots \varphi_n \in \mathrm{Sym}(\mathcal{L})$, one observes by direct inspection that iterated applications of $\delta\Xi$ do *not* include any dotted (antifield for ghost) lines due to the explicit form of the interaction vertices (6.5.8) and of the cochain homotopy (6.5.7). Because $\delta\Xi$ is of cohomological degree 0, the element $(\delta\Xi)^k(\varphi_1 \cdots \varphi_n) \in \mathrm{Sym}(\mathcal{L})$ is of cohomological degree 0 too, hence together with the previous observation it must contain an equal number of dashed (ghost) lines and wiggly (antifield) lines. Applying the dg-algebra homomorphism Π and using that it gives zero on every wiggly line, it follows that only those terms with no ghost lines contribute to the correlation function $\langle \varphi_1 \cdots \varphi_n \rangle = \sum_{k=0}^{\infty} \Pi\big((\delta\Xi)^k(\varphi_1 \cdots \varphi_n)\big)$. This completes the proof. □

6.6 EXAMPLE FOR $D_0 \neq 0$: THE QUARTIC $(0,1)$-MODEL

Having studied perturbations around the zero Dirac operator $D_0 = 0$, we now would like to consider the case when the background Dirac operator $D_0 \neq 0$ is non-zero. This will be performed in the simplest dynamical spectral triple model, the so called $(0,1)$-model from [Bar15], and we will choose to work with a specific action. We shall see that the quantum correlation functions are even in this primitive case dependent on the ghosts and antifield for ghosts. In fact, as opposed to the case when $D_0 = 0$ as in Proposition 6.5.3, it will become evident as one works through the computational details that this is a general feature of any model for perturbations around non-zero backgrounds $D_0 \neq 0$ that breaks some of the gauge symmetries. (Note that the zero Dirac operator $D_0 = 0$ is gauge invariant under (6.2.5).)

THE $(0,1)$-MODEL. We begin by recalling and collecting some relevant details of the $(0,1)$-model from [Bar15, BG16], of which some can be found in Example 6.1.6 (iii). The $(0,1)$-fermion space (see Section 6.1) over $A = \mathrm{Mat}_N(\mathbb{C})$ consists of the Hilbert space $\mathcal{H} = A$, on which A acts via left multiplication, with inner product $\langle\langle a, a' \rangle\rangle = \mathrm{Tr}_A(a^* a')$, chirality operator $\Gamma(a) = a$ and real structure $\Gamma(a) = a^*$. The Dirac operators (in the sense of Definition 6.1.1) are of the form

$$D = -\mathrm{i}\,[L, \cdot\,] \quad , \tag{6.6.1}$$

where $L \in \mathrm{Mat}_N(\mathbb{C})$ is a trace-free anti-Hermitian $N \times N$-matrix. The Dirac ensemble of the $(0,1)$-fermion space can therefore be identified as (the underlying real vector space of) the Lie algebra $\mathfrak{su}(N)$ of anti-Hermitian and trace-free $N \times N$-matrices.

$$\mathcal{D} \cong \mathfrak{su}(N) \quad . \tag{6.6.2}$$

Taking Remark 6.2.2 into consideration, we see that the Lie algebra of infinitesimal gauge transformations of this model is given by

$$\mathfrak{g} = \mathfrak{su}(N) \quad , \tag{6.6.3}$$

whose action on the Dirac ensemble is given by the Lie bracket of $\mathfrak{su}(N)$, i.e.

$$\rho_{\mathcal{D}} : \mathfrak{g} \times \mathcal{D} \longrightarrow \mathcal{D} \,, \;\; (\epsilon, L) \longmapsto \rho_{\mathcal{D}}(\epsilon)(L) = [\epsilon, L] \quad . \tag{6.6.4}$$

Our choice of action is the gauge invariant quartic action $S(D) = \mathrm{Tr}_{\mathrm{End}(\mathcal{H})}\big(\frac{g_2}{2}D^2 + \frac{g_4}{4!}D^4\big)$ under the assumption that $g_2 < 0$ is negative and $g_4 > 0$ is positive in order to obtain a "symmetry-breaking potential". We may without loss of generality assume that $g_2 = -1$ by rescaling the Dirac operator D. By inserting (6.6.1) into this action and also using that L is trace-free, we may rewrite the action S as a function of L (see also e.g. [BG16, Appendix A.2] for some useful identities),

$$S(L) = N\,\mathrm{Tr}_A(L^2) + \frac{g_4}{4!}\Big(2\,N\,\mathrm{Tr}_A(L^4) + 6\,\big(\mathrm{Tr}_A(L^2)\big)^2\Big) \quad . \tag{6.6.5}$$

By varying this action with respect to L,

$$\delta S(L) = \mathrm{Tr}_A\Big[\delta L\,\Big(2\,N\,L + \frac{g_4}{4!}\big(8\,N\,L^3 + 24\,L\,\mathrm{Tr}_A(L^2)\big)\Big)\Big] \quad , \tag{6.6.6a}$$

we obtain the Euler-Lagrange equation

$$L + \frac{g_4}{3!}\Big(L^3 - \frac{1}{N}\,\mathrm{Tr}_A(L^3) + \frac{3}{N}\,L\,\mathrm{Tr}_A(L^2)\Big) = 0 \quad . \tag{6.6.6b}$$

Note that the term $\mathrm{Tr}_A(L^3)$ in (6.6.6b) arises due to the fact that the variation δL is trace-free, hence the big round bracket in (6.6.6a) must be projected onto the space of trace-free and anti-Hermitian matrices. Indeed, this becomes evident by decomposing $L^3 = \big(L^3 - \frac{1}{N}\,\mathrm{Tr}_A(L^3)\big) + \frac{1}{N}\,\mathrm{Tr}_A(L^3)$ into its trace and trace-free parts in (6.6.6a).

Next, we need to find a non-trivial solution $D_0 \neq 0$ which we will perturb around. For this, we consider the case for even N and the simple exact solution

$$L_0 = \mathrm{i}\,\kappa \begin{pmatrix} \mathbb{1}_{N/2} & 0 \\ 0 & -\mathbb{1}_{N/2} \end{pmatrix} \quad , \qquad \kappa := \sqrt{\frac{3}{2\,g_4}} \tag{6.6.7}$$

of (6.6.6b), which we have written in block matrix notation. This background solution is enough to illustrate the main features; we shall see that it breaks the $\mathfrak{g} = \mathfrak{su}(N)$ gauge symmetry down to a Lie subalgebra. To this end, writing the Lie algebra elements in block matrix notation

$$\epsilon = \begin{pmatrix} \epsilon_1 & \epsilon_3 \\ -\epsilon_3^* & \epsilon_2 \end{pmatrix} \in \mathfrak{g} = \mathfrak{su}(N) \quad , \tag{6.6.8}$$

we check that the Lie algebra action (6.6.4) on L_0 results in

$$\rho_{\mathcal{D}}(\epsilon)(L_0) = [\epsilon, L_0] = -2\,\mathrm{i}\,\kappa \begin{pmatrix} 0 & \epsilon_3 \\ \epsilon_3^* & 0 \end{pmatrix} \quad . \tag{6.6.9}$$

From here, we see that the $\mathfrak{su}(N)$ gauge symmetry is indeed broken down to the Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g} = \mathfrak{su}(N)$ stabilising the background (6.6.7), whose elements are of the form

$$\epsilon = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \in \mathfrak{g}_0 \subset \mathfrak{g} = \mathfrak{su}(N) \quad . \tag{6.6.10}$$

We will later make use of the linearisation around L_0 of the Euler-Lagrange equation (6.6.6b)

$$P(\tilde{L}) := -\frac{1}{2}\tilde{L} + \frac{g_4}{3!}\,L_0\,[\tilde{L}, L_0] + \frac{g_4}{N}\,L_0\,\mathrm{Tr}_A(\tilde{L}\,L_0) = 0 \quad , \tag{6.6.11a}$$

which in block matrix notation explicitly reads as

$$P(\tilde{L}) = -\frac{1}{2}\begin{pmatrix} \tilde{L}_1 & 0 \\ 0 & \tilde{L}_2 \end{pmatrix} - \frac{3}{2N}\,\mathrm{Tr}(\tilde{L}_1 - \tilde{L}_2)\begin{pmatrix} \mathbb{1}_{N/2} & 0 \\ 0 & -\mathbb{1}_{N/2} \end{pmatrix} \quad , \tag{6.6.11b}$$

where the unadorned trace Tr is over $N/2 \times N/2$-matrices. Furthermore, the perturbed action (6.2.8) around L_0 can be written as

$$\begin{aligned} \tilde{S}(\tilde{L}) &= -\frac{N}{4}\,\mathrm{Tr}_A(\tilde{L}^2) + \frac{g_4}{4!}\Big(4\,N\,\mathrm{Tr}_A(L_0\,\tilde{L}\,L_0\,\tilde{L}) + 24\,\big(\mathrm{Tr}_A(L_0\,\tilde{L})\big)^2\Big) \\ &\quad + \frac{\lambda\,g_4}{4!}\Big(8\,N\,\mathrm{Tr}_A(L_0\,\tilde{L}^3) + 24\,\mathrm{Tr}_A(L_0\,\tilde{L})\,\mathrm{Tr}_A(\tilde{L}^2)\Big) \\ &\quad + \frac{\lambda^2\,g_4}{4!}\Big(2\,N\,\mathrm{Tr}_A(\tilde{L}^4) + 6\,\big(\mathrm{Tr}_A(\tilde{L}^2)\big)^2\Big) \quad . \end{aligned} \tag{6.6.12}$$

THE COMPLEX OF LINEAR OBSERVABLES. In what follows, we will explicitly describe the complex of linear free observables (6.3.16). First, recall that $\mathcal{D} \cong \mathfrak{su}(N)$ and $\mathfrak{g} = \mathfrak{su}(N)$. Hence, utilising the Killing form $2N\,\mathrm{Tr}_A(X\,Y)$, for $X, Y \in \mathfrak{su}(N)$, we may also identify $\mathcal{D}^\vee \cong \mathfrak{su}(N)$ and $\mathfrak{g}^\vee \cong \mathfrak{su}(N)$. The cochain complex (6.3.16) of linear observables for the $(0,1)$-model is then isomorphic to the complex

$$\mathcal{L} = \Big(\mathfrak{su}(N)[2] \xrightarrow{\mathrm{d}^{\mathrm{free}}} \mathfrak{su}(N)[1] \xrightarrow{\mathrm{d}^{\mathrm{free}}} \mathfrak{su}(N) \xrightarrow{\mathrm{d}^{\mathrm{free}}} \mathfrak{su}(N)[-1] \Big) \quad , \tag{6.6.13a}$$

where we used the same notation for the differential $\mathrm{d}^{\mathrm{free}}$. In order to distinguish between different $\mathfrak{su}(N)$-components in this complex, we introduce the notation

$$\beta \in \mathfrak{su}(N)[2] \ , \quad \alpha \in \mathfrak{su}(N)[1] \ , \quad \varphi \in \mathfrak{su}(N) \ , \quad \chi \in \mathfrak{su}(N)[-1] \tag{6.6.13b}$$

where we recall that they, in the presented order, are interpreted as linear observables for the antifield for the ghost c^+, the antifield $\tilde{D}^+$, the field $\tilde{D}$ and the ghost c. Continuing with using our block matrix notation (see (6.6.9) and (6.6.11)), the free differential (6.3.16) takes the explicit form

$$\mathrm{d}^{\mathrm{free}}\beta = [\beta, L_0] = -2\,\mathrm{i}\,\kappa \begin{pmatrix} 0 & \beta_3 \\ \beta_3^* & 0 \end{pmatrix} \quad , \tag{6.6.14a}$$

$$\mathrm{d}^{\mathrm{free}}\alpha = P(\alpha) = -\frac{1}{2}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} - \frac{3}{2N}\,\mathrm{Tr}(\alpha_1 - \alpha_2)\begin{pmatrix} \mathbb{1}_{N/2} & 0 \\ 0 & -\mathbb{1}_{N/2} \end{pmatrix} \quad , \tag{6.6.14b}$$

$$\mathrm{d}^{\mathrm{free}}\varphi = [\varphi, L_0] = -2\,\mathrm{i}\,\kappa \begin{pmatrix} 0 & \varphi_3 \\ \varphi_3^* & 0 \end{pmatrix} \quad , \tag{6.6.14c}$$

$$\mathrm{d}^{\mathrm{free}}\chi = 0 \quad . \tag{6.6.14d}$$

Remark 6.6.1. Let us explain the absence of the relative sign between $\mathrm{d}^{\mathrm{free}}\beta$ and $\mathrm{d}^{\mathrm{free}}\varphi$ in (6.3.16). Essentially, it is compensated by the minus sign that comes from forming adjoints with respect to the Killing form. (Explicitly, $2N\,\mathrm{Tr}_A(X\,[Y,\epsilon]) = -2N\,\mathrm{Tr}_A([X,\epsilon]\,Y)$.) Observe that the antibracket can be built from the Killing form (recall that the Killing form on $\mathfrak{su}(N)$ is negative definite) by setting (cf. (6.3.9))

$$\begin{aligned} \{\varphi,\alpha\} &:= -2N\,\mathrm{Tr}_A(\varphi\,\alpha)\ , \ \ \{\alpha,\varphi\} := 2N\,\mathrm{Tr}_A(\varphi\,\alpha) \quad , \\ \{\beta,\chi\} &:= -2N\,\mathrm{Tr}_A(\beta\,\chi)\ , \ \ \{\chi,\beta\} := 2N\,\mathrm{Tr}_A(\beta\,\chi) \quad . \end{aligned} \tag{6.6.15}$$

(One can reconstruct the corresponding symplectic form (6.3.7) from this by an appropriate choice of basis.) In particular, the antibracket is compatible with the differential (see (4.3.8) in Remark 4.3.4) which specifically implies that

$$\{\varphi, \mathrm{d}^{\mathrm{free}}\beta\} = \{\mathrm{d}^{\mathrm{free}}\varphi, \beta\} \quad . \tag{6.6.16}$$

For $\mathrm{d}^{\mathrm{free}}\beta = [\beta, L_0]$ (see (6.6.14)), the left hand side can be written out as

$$\{\varphi, \mathrm{d}^{\mathrm{free}}\beta\} = -2N\,\mathrm{Tr}_A(\varphi\,[\beta, L_0]) = 2N\,\mathrm{Tr}_A([\varphi, L_0]\,\beta) \tag{6.6.17}$$

and the right hand side is given by

$$\{\mathrm{d}^{\mathrm{free}}\varphi, \beta\} = 2N\,\mathrm{Tr}_A(\mathrm{d}^{\mathrm{free}}\varphi\,\beta) \quad . \tag{6.6.18}$$

Altogether, nondegeneracy of the Killing form implies that

$$\mathrm{d}^{\mathrm{free}}\varphi = [\varphi, L_0] \quad . \tag{6.6.19}$$

△

Let us write out in more detail what the complex of linear observables given by (6.6.13) and (6.6.14) looks like. From the Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g} = \mathfrak{su}(N)$ in (6.6.10), we introduce its orthogonal complement $\mathfrak{g}_0^\perp \subset \mathfrak{g} = \mathfrak{su}(N)$ with respect to the Killing form. The elements of $\mathfrak{g}_0^\perp$ are of the form

$$\epsilon = \begin{pmatrix} 0 & \epsilon_3 \\ -\epsilon_3^* & 0 \end{pmatrix} \in \mathfrak{g}_0^\perp \subset \mathfrak{g} = \mathfrak{su}(N) \quad . \tag{6.6.20}$$

By decomposing all components in (6.6.13) according to $\mathfrak{su}(N) = \mathfrak{g}_0 \oplus \mathfrak{g}_0^\perp$, we see directly from (6.6.14) that the differential $\mathrm{d}^{\mathrm{free}}$ is only non-zero on one of the two summands. This leads to the decomposition

$$\mathcal{L} \;=\; \left(\begin{array}{cccc} \mathfrak{g}_0[2] & \mathfrak{g}_0[1] \xrightarrow{\;\mathrm{d}^{\mathrm{free}}\;} & \mathfrak{g}_0 & \mathfrak{g}_0[-1] \\ \oplus & \oplus & \oplus & \oplus \\ \mathfrak{g}_0^\perp[2] \xrightarrow[\;\mathrm{d}^{\mathrm{free}}\;]{} & \mathfrak{g}_0^\perp[1] & \mathfrak{g}_0^\perp \xrightarrow[\;\mathrm{d}^{\mathrm{free}}\;]{} & \mathfrak{g}_0^\perp[-1] \end{array}\right) \quad . \tag{6.6.21}$$

of the complex (6.6.13). The cohomology of this complex is apparent from the observation that the displayed non-trivial differentials are all injective (which can be quickly concluded from simply comparing (6.6.14) with (6.6.10) and (6.6.20)) and therefore, by the rank-nullity theorem, also surjective on the corresponding summands. Hence, we obtain that

$$\mathrm{H}^\bullet\big(\mathcal{L}, \mathrm{d}^{\mathrm{free}}\big) \;=\; \mathfrak{g}_0[2] \oplus \mathfrak{g}_0[-1] \quad . \tag{6.6.22}$$

THE STRONG DEFORMATION RETRACT. The next step is to choose a strong deformation retract (6.4.2). Observe that our complex of linear observables (6.6.21) decomposes into the direct sum

$$\mathcal{L} \;=\; \mathcal{L}^\perp \oplus \mathrm{H}^\bullet\big(\mathcal{L}, \mathrm{d}^{\mathrm{free}}\big) \tag{6.6.23a}$$

of the acyclic complex

$$\mathcal{L}^{\perp} = \left(\begin{array}{ccccc} & & \mathfrak{g}_0[1] & \xrightarrow{\mathrm{d}^{\text{free}}} & \mathfrak{g}_0 \\ & & \oplus & & \oplus \\ \mathfrak{g}_0^{\perp}[2] & \xrightarrow[\mathrm{d}^{\text{free}}]{} & \mathfrak{g}_0^{\perp}[1] & \qquad \mathfrak{g}_0^{\perp} & \xrightarrow[\mathrm{d}^{\text{free}}]{} \mathfrak{g}_0^{\perp}[-1] \end{array} \right) \tag{6.6.23b}$$

and the cohomology (6.6.22). From this consideration, we see that we may take for the strong deformation retract (6.4.2) the π-map simply to be the projection onto $\mathrm{H}^{\bullet}(\mathcal{L}, \mathrm{d}^{\text{free}})$ and the ι-map to be the inclusion map. Then, a viable choice for a cochain homotopy ξ can be found by inverting (the non-trivial parts of) d^{free}, yielding

$$\xi(\beta) = 0 \quad , \tag{6.6.24a}$$

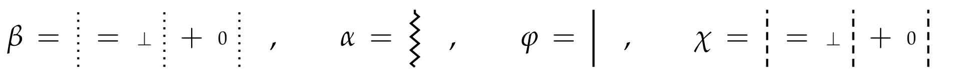
$$\xi(\alpha) = -\frac{\mathrm{i}}{2\kappa} \begin{pmatrix} 0 & \alpha_3 \\ \alpha_3^* & 0 \end{pmatrix} \quad , \tag{6.6.24b}$$

$$\xi(\varphi) = 2 \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} - \frac{3}{2N} \operatorname{Tr}(\varphi_1 - \varphi_2) \begin{pmatrix} \mathbb{1}_{N/2} & 0 \\ 0 & -\mathbb{1}_{N/2} \end{pmatrix} \quad , \tag{6.6.24c}$$

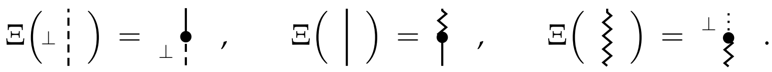
$$\xi(\chi) = -\frac{\mathrm{i}}{2\kappa} \begin{pmatrix} 0 & \chi_3 \\ \chi_3^* & 0 \end{pmatrix} \quad . \tag{6.6.24d}$$

The verification of the relevant properties of Definition 4.3.6 is straightforward.

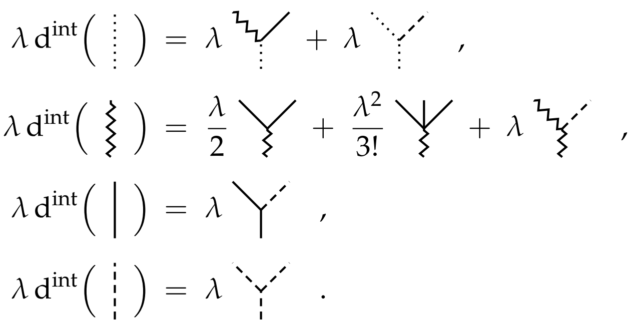
QUANTUM CORRELATION FUNCTIONS. We now have all the building blocks for the computation of the correlation functions. To achieve this, we introduce a slightly modified version of the diagrammatic calculus from Section 6.5. As in (6.5.6), there are four types of lines corresponding to each field species. In this case however, we further decompose them according to (6.6.23) into their $\mathcal{L}^{\perp}$ and cohomology components, which we graphically will depict as


(6.6.25)

As before, from (4.3.34), the action of the cochain homotopy Ξ on several lines can be expressed as a sum of action on individual lines. We further would like to emphasise that, by definition, the number n *not* count the cohomology components of the lines, i.e it only counts the wiggly, straight, $\perp$-dashed and $\perp$-dotted lines. The action of the homotopy on individual lines is given by (6.6.24) and is non-zero only on the $\mathcal{L}^{\perp}$-components. We will draw the non-vanishing components of the homotopy as

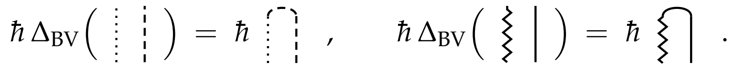

(6.6.26)

To obtain the graphical representation of the interaction part of the differential $\mathrm{d}^{\mathrm{int}}$ for our model, we specialise (6.5.8) to the perturbative action (6.6.12). Observe that this gives rise to cubic and quartic interaction vertices. Altogether, we visualise the action of $\mathrm{d}^{\mathrm{int}}$ as



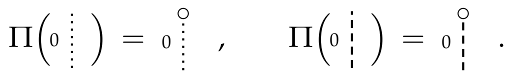
(6.6.27a) (6.6.27b) (6.6.27c) (6.6.27d)

The numerical values of the interaction vertices with respect to a choice of bases can be read off from (6.3.6). For the BV Laplacian there is no need for modification and we again represent it as capping off two lines,



(6.6.28)

Finally, the map Π acts only non-trivially on the cohomology components of the lines,



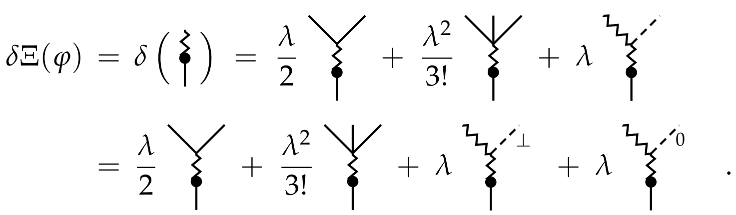
(6.6.29)

Altogether, this describes the diagrammatic calculus for the computation of correlation functions of the $(0,1)$ dynamical spectral triples model with action $S(D) = \mathrm{Tr}_{\mathrm{End}(\mathcal{H})}\big(-\frac{1}{2}D^2 + \frac{g_4}{4!}D^4\big)$, where $g_4 > 0$, perturbed around the non-trivial background Dirac operator D_0 given by (6.6.7).

Example 6.6.2. As a first example to illustrate the diagrammatic calculus we have set up, we compute the 1-point function

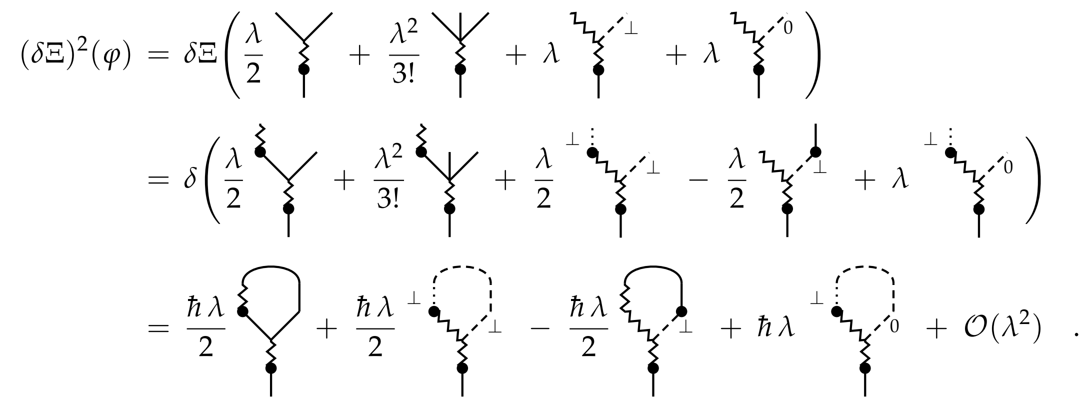
$$\langle\varphi\rangle = \sum_{k=0}^{\infty} \Pi\big((\delta\Xi)^k(\varphi)\big) \quad , \qquad \varphi \in \mathcal{L}^0 \quad , \tag{6.6.30}$$

for a generator in degree 0 to the lowest non-trivial order in the formal parameter λ. As before, this is performed by iteratively applying $\delta\Xi$. The first iteration is computed as



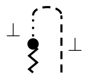
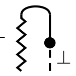
(6.6.31)

In the second line, we have decomposed the dashed line according to (6.6.23). Note that we need to go to the next perturbative order to see any quantum corrections because the application of Π to this expression yields 0. For the second iteration, we use the algebraic expression of the homotopy Ξ (4.3.34) to first write out

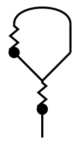
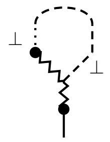


(6.6.32)

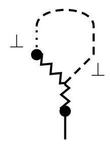
Observe that the fourth term vanishes due to the fact that the decomposition $\mathfrak{g} = \mathfrak{g}_0^\perp \oplus \mathfrak{g}_0$ is orthogonal with respect to the Killing form. Furthermore, there is an identity analogous to (6.5.13),

 $= \Delta_{\mathrm{BV}}\big(\xi(\alpha)\,\chi\big) = \{\xi(\alpha), \chi\} = \{\alpha, \xi(\chi)\} = -\Delta_{\mathrm{BV}}\big(\alpha\,\xi(\chi)\big) = -$  ,   (6.6.33)

which in particular implies that the second and the third term coincide. Hence, the 1-point function is given by

$\langle\varphi\rangle = \frac{\hbar\,\lambda}{2}$  $+ \hbar\,\lambda$  $+ \mathcal{O}(\lambda^2)$.   (6.6.34)

Let us write out the algebraic expression for the ghost contribution using explicit form of the interaction part of the differential (see (6.3.6)). One finds that

$\hbar\,\lambda$  $= \hbar\,\lambda \sum_{i=1}^{\dim(\mathfrak{g}_0^\perp)} 2\,N\,\mathrm{Tr}_A\Big(t_i\,\xi\big([t_i, \xi(\varphi)]\big)\Big)$,   (6.6.35)

where ξ are the components of the homotopy in (6.6.24), $[\,\cdot\,,\,\cdot\,]$ is the Lie bracket on $\mathfrak{g} = \mathfrak{su}(N)$, and $\{t_i \in \mathfrak{g}_0^\perp\}$ is an orthogonal basis with respect to the Killing form, i.e. $2N\,\mathrm{Tr}_A(t_i\,t_j) = -\delta_{ij}$ (the minus sign is a consequence of the Killing form being negative definite).

The above ghost field contribution (6.6.35) is in general non-zero. Let us check that for the $N = 2$ case. An orthogonal basis for $\mathfrak{g} = \mathfrak{su}(2)$ is given by the (appropriately normalized and anti-Hermitian) Pauli matrices (see (1.1.33))

$$t_i = \frac{\mathrm{i}}{\sqrt{8}}\,\sigma_i \quad , \quad \text{for } i = 1,2,3 \quad , \tag{6.6.36}$$

which satisfy the Lie bracket relations $[t_i, t_j] = -\frac{1}{\sqrt{2}}\,\epsilon_{ijk}\,t_k$. By comparing (1.1.33) with (6.6.10) and (6.6.20), we see that the Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is spanned by t_3 and its orthogonal complement $\mathfrak{g}_0^\perp$ is spanned by t_1 and t_2. Since the homotopy $\xi(\varphi)$ in (6.6.35) maps surjectively onto $\mathfrak{g}_0$, we may without loss of generality set $\xi(\varphi) = t_3$ by an appropriate choice of φ. For the other homotopy in (6.6.35), which is of the type $\xi(\alpha)$ from (6.6.24), we compute its action on the basis of $\mathfrak{g}_0^\perp$ to be

$$\xi(t_1) = \frac{1}{2\kappa}\,t_2 \quad , \quad \xi(t_2) = -\frac{1}{2\kappa}\,t_1 \quad . \tag{6.6.37}$$

We may therefore compute

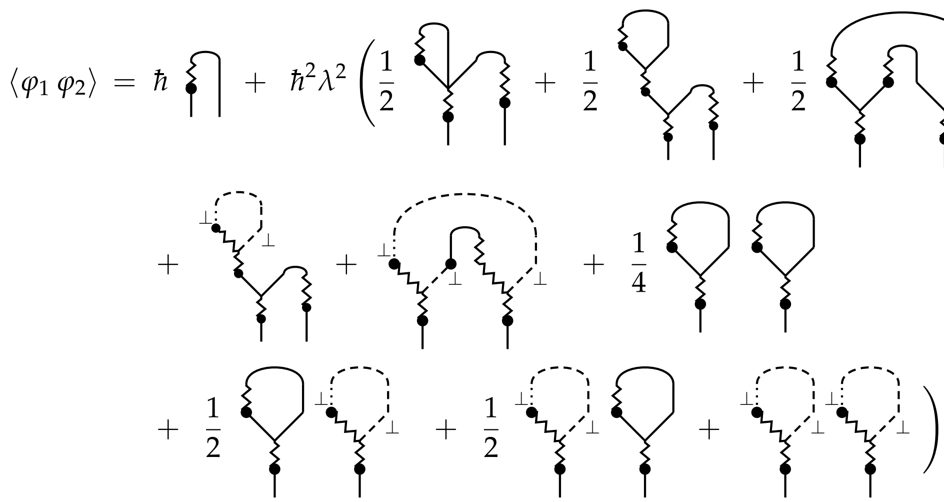
$$(6.6.35) \overset{N=2}{=} \hbar\,\lambda\,4\,\mathrm{Tr}_A\Big(t_1\,\xi\big([t_1,t_3]\big) + t_2\,\xi\big([t_2,t_3]\big)\Big) = \frac{\hbar\,\lambda}{\sqrt{2}\,\kappa} = \hbar\,\lambda\,\sqrt{\frac{g_4}{3}} \quad , \tag{6.6.38}$$

showing that the ghost field contribution does not vanish. $\triangledown$

Example 6.6.3. With considerably more effort, one can also compute the 2-point correlation function

$$\langle \varphi_1\,\varphi_2\rangle = \sum_{k=0}^{\infty} \Pi\big((\delta\Xi)^k(\varphi_1\,\varphi_2)\big) \quad , \qquad \varphi_1, \varphi_2 \in \mathcal{L}^0 = \mathcal{D}^\vee \quad , \tag{6.6.39}$$

for two generators in degree 0 to the lowest non-trivial order in the formal parameter λ. The final result is



$$+\ \mathcal{O}(\lambda^3) \quad . \tag{6.6.40}$$

Just as in the previous Example 6.6.2 of the the 1-point function, there are non-trivial contributions from the ghost field, which are absent for perturbations around the trivial Dirac operator $D_0 = 0$ (see Example 6.5.2). $\triangledown$

7

CONCLUSIONS AND OUTLOOK

In this part of the thesis, we have adapted and applied the modern formulation of the BV formalism by Costello and Gwilliam [Gwi12, CG16, CG21] to finite dimensional noncommutative quantum field theories. This provides powerful and systematic methods for dealing with gauge theoretic aspects in the noncommutative setting. This was covered in Chapter 4.

In Chapter 5, the main content of [NSS21] was presented. In Section 5.1, the BV formalism as described in Section 4.3 was applied to scalar field theory and Chern-Simons theory on the fuzzy 2-sphere. In particular, we observe for scalar field theories that the quantum correlation functions exhibit well-known noncommutative features such as non-planar contributions to the loop diagrams, see Example 5.1.3.

We then extended in Section 5.2 the BV formalism to include also noncommutative field theories defined in the representation category ${}_H\mathsf{Mod}$ of a Hopf algebra H equipped with a non-identity *triangular* R-matrix, i.e. the so called braided noncommutative field theories of [DCGRS20, DCGRS21]. (Again, "braided" here actually means "symmetric braided".) Because the representation category of a triangular Hopf algebra is symmetric monoidal, the procedure was fairly straightforward. The techniques were then applied in Section 5.3 to scalar field theories on the fuzzy 2-torus. The results coincide with the (symmetric) braided quantum field theory of Oeckl [Oec00]. In particular, non-planar contributions to the loop diagrams are absent in the braided framework, see Example 5.3.4.

There are two interesting directions one could pursue in future projects. The first idea would be to generalise the BV formalism as treated in this thesis to also include noncommutative quantum field theories on infinite-dimensional algebras (e.g. the Moyal plane) by adapting the analytical aspects of Costello and Gwilliam's formulation of the BV formalism [Gwi12, CG16, CG21]. Together with a better understanding of noncommutative phenomena such as UV/IR mixing, this would allow for the study of regularisation and renormalisation of (infinite dimensional) noncommutative quantum field theories. The other route would be to generalise the braided BV formalism to also include non-symmetric braidings, i.e. to symmetries encoded by quasitriangular Hopf algebras which are not triangular. In contrast to the symmetric braided case presented here, the truly braided case is expected to be substantially more difficult. This is in part due to the obstructions stemming from quasitriangularity one faces when trying to encode certain properties such as symmetry or the Jacobi identity for P_0-algebras. There are examples such as braided analogues of Lie algebras [Maj94] which, however, are substantially more involved. This level of complication is therefore to be expected when trying to define a truly braided version of P_0-algebras. One could, as a minimalistic approach, skip the P_0-algebra step and directly try to generalise the explicit formula of the BV Laplacian (5.2.16) to the case of a quasitriangular R-matrix. The drawback of this method is that it is unclear how to define the BV Laplacian Δ_{BV} such that the crucial nilpotency condition $\Delta_{\mathrm{BV}}^2 = 0$ holds. However, it does agree with Oeckl's braided Wick Theorem [Oec01]. We are currently unaware of any resolutions to these issues and why they are absent from Oeckl's truly braided approach.

In Chapter 6, which is based on [GNS22], we used the BV formalism to quantise and compute correlation functions of dynamical spectral triple models. These are field theories whose space of fields consists of a vector space of Dirac operators (the Dirac ensemble), which parametrise fuzzy spectral triples with a fixed fermion space. Since the dynamical variables are Dirac operators, which encode the (geo)metric structure of noncommutative spaces, the dynamical spectral triple models can be seen as describing (toy-)models of quantum gravity. The particular gauge symmetries we consider are analogues to the diffeomorphism symmetries of Riemannian (spin) manifolds, described in Section 6.2. We then in Section 6.3 describe the classical BV formalism based on the systematic techniques from [BSS21] to construct the classical observables. Using then the procedure outlined in Section 4.3 we then quantise the classical observables and compute the quantum correlation functions. It turns out that whether we have gauge theoretical contributions or not to the correlation functions (i.e. contributions from ghost fields and their antifields) is strongly impacted by the amount of gauge symmetry that is broken by the background Dirac operator D_0 (which is a solution to the classical equations of motion) we perturb around. The analysis is carried out for two cases, perturbations around the trivial Dirac operator $D_0 = 0$ and perturbations around a non-zero Dirac operator $D_0 \neq 0$. The zero Dirac operator $D_0 = 0$ is gauge invariant and thus breaks no symmetry. It turns out that in this case, there are no contributions from ghost fields and their antifields to the correlation functions, see Proposition 6.5.3, i.e. gauge theoretical modifications to the path integral are absent for perturbations around $D_0 = 0$. However, the situation is different for the case of non-zero Dirac operators $D_0 \neq 0$ breaking some of the gauge symmetries. This was investigated for the simplest dynamical spectral triple model, the quartic $(0,1)$-model of [BG16]. The model carries a "symmetry breaking potential" which leads to a Higgs-like mechanism. This is identified as the source of the non-trivial gauge theoretic modifications to the quantum correlation functions. This leads to the conclusion that when considering quantum fluctuations localised around a non-trivial classical solution $D_0 \neq 0$ which breaks the gauge symmetry, the quantum correlation functions receive gauge-theoretic contributions.

In the future, it would be interesting to investigate and gain a deeper understanding of the physical effects linked to the gauge-theoretic modifications to the correlation functions and interpretation in the context of quantum gravity. In addition, it would be useful to apply these techniques to dynamical spectral triple models with more involved solutions to the Euler-Lagrange equations as the $(0,1)$-model considered here is somewhat too simplistic and thus of limited physical relevance (though it did serve its purpose as a toy-model to demonstrate that there indeed are gauge-theoretic effects in the path integrals to be taken into consideration). Such solutions, e.g. the fuzzy sphere, have recently been investigated in [D'Ar22].

A further intriguing aspect to explore would be the large N behaviour of fuzzy spectral triple models, see e.g. [HKPV22] for an overview. With this in mind, it is natural to pose the question whether gauge symmetries and BV quantisation plays any role in large N phenomena such as phase transitions. A study related to this can be found in [GGHZ22, GHZ22], where homological methods are used to examine the large N limit of the Gaussian Unitary Ensemble from random matrix theory. A route for future research would be to adapt these techniques to the setting of fuzzy spectral triple models in order to obtain a homological perspective of the results by the Western University group [HKPV22]. Following this approach, it might be feasible to determine the effect of gauge symmetries in the large N limit on phenomena such as phase transitions.

Another research direction of physical relevance would be to incorporate and treat gauge symmetries coming from external gauge groups. These are not related to the gauge symmetries investigated here, which are intrinsic to the noncommutative spaces themselves. Further into the future, it would also be of interest to move away from the perturbative realm to the non-perturbative one. However, up to date, there are few such techniques available and they typically require more sophisticated machinery based on derived geometry. Such an account can be found in e.g. [BSS21] (from which we already extracted our perturbative techniques for the classical BV formalism).

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