Self-tuning and the Cosmological Constant Problem

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Thesis submitted to the University of Nottingham
for the degree of Doctor of Philosophy

supervised by

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16th November 2022

Abstract

The cosmological constant problem is one of the biggest theoretical hurdles of the modern age. It comes out of two great theories in physics: General Relativity (GR) and Quantum Field Theory (QFT). QFT predicts the existence of vacuum energy and GR predicts that it will gravitate like a cosmological constant, Λ_{vac}. Through observations we find that the Universe is undergoing an accelerated expansion which can be sourced by a constant term, Λ_{obs}, whose value is set by observations. Therefore it would be tempting to compare Λ_{obs} with Λ_{vac}, but through this we find that the predicted value of the vacuum energy is far, far greater than that of observation. In fact, at a lower estimate it represents a fine-tuning of $\sim 10^{36}$ orders of magnitude. However, this is not the full extent of the problem. Even if we accept a fine-tuning in Λ_{vac}, its value is unstable to higher order perturbations, leading to repeated fine-tunings and re-tunings. This is known as *radiative instability* of the vacuum, and it is the true source of the cosmological constant problem.

This thesis chooses to focus on *self-tuning* as a method for alleviating this problem. Self-tuning refers to the practice of modifying GR by adding extra fields which act to force $\Lambda_{\text{vac}} \sim \Lambda_{\text{obs}}$, removing the need for fine-tuning. In this thesis we review a variety of self-tuning mechanisms to allow the reader to get a basic idea of the different approaches adopted.

The bulk of the thesis focuses on self-tuning with a massive scalar-tensor theory on an Anti-de Sitter (AdS) background, the idea for which originated by examining a range of allowed modifications to GR and placing some self-tuning conditions upon them. Here, we construct an explicit model and analyse the resultant field equations to check whether it can or cannot self-tune.

We then perform a numerical analysis on the resultant cosmological equations to understand the dynamics of the system; focusing specifically on whether our model can self-tune regardless of initial conditions. Finally, we conduct a rudimentary analysis on the stability of this model to further understand whether we can consistently self-tune without fine-tuning.

Overall this work serves as an initial point of exploration in self-tuning on an AdS background. As we later discuss, there are many exciting future directions this model can take beyond this thesis.

Acknowledgements

Firstly, I would like to thank my supervisors Ed and Tony, whose boundless support, knowledge, and patience have been the bedrock of my PhD journey. None of this would be possible without their guidance. A special thanks to Florian who has acted almost like a third supervisor. Our discussions have been immensely enjoyable, and I have thoroughly appreciated all your help. I would also like to thank Paul for his role as my undergraduate pastoral tutor and for being a friendly face around the office. A final (academic) thanks to Moustafa for being an all-round lovely person, who I have thoroughly enjoyed learning from as an undergraduate, and teaching with as a postgraduate.

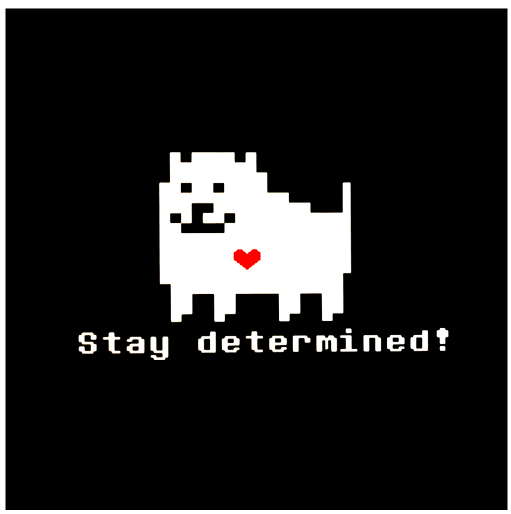
Turning my attention to CAPT, there are so many great people I would like to thank for their academic and non-academic contributions to my life. I could not have asked for a better department, and I have thoroughly enjoyed our lunch times, tea times, journal clubs, cake days, pub meets, DA visits, etc. There are far too many people to name specifically, so instead I will focus on some of my fellow fourth years who I have shared so many great memories with: Leo, Liza, Jacob, Mick, and Roan. You guys are the best!

Outside of CAPT, I feel that I have to thank the University of Nottingham dodgeball club. Despite giving me a blood clot which resulted in the removal of my rib, it has been a constant source of joy for me whether it playing, coaching, or just screaming at the ref.

Alongside dodgeball friends, I would like to thank Vesey friends and old uni-halls friends for putting up with me over the years. You've all been great!

Everyone in my family has been incredibly supportive, surprisingly interested in my work, and having the word “PHYSICS” yelled at me every time I walk into a room always makes me laugh: so I thank them for that. As before, there are far too many to name specifically so I will focus on those closest to me. Firstly, thank you to my parents who have had my back every step of the way, I wouldn't be the man I am today if it weren't for you. Likewise, I am grateful to my grandma and my granddad who have both had a major hand in raising me and I only hope that I have done them proud. To my brothers Mandeep, Sanjeevan, and Jasroop: my only hope is that you never get a doctorate, so I can hold this over you forever.

Finally, thank you to Lucy for all your love and support over the years. You have enriched my life in so many ways and I would not trade our time together for the world. See you on the other side!



Stay Determined - Undertale Dog, 2015

The above is an image from a video game that has remained on the whiteboard by my desk throughout almost my entire PhD. As silly as it is, it did make me smile whenever I hit a rough patch in my work so I think it's worth an inclusion in the acknowledgments. Hopefully it gives the next person at my desk some joy.

Conventions

Throughout this thesis we adopt the following conventions, notations, and definitions unless otherwise specified:

The spacetime metric uses the "mostly positive" signature, $(-,+,+,+)$, where Greek indices $\mu, \nu, ... = (0,1,2,3)$ represent the spacetime components. Here, 0 refers to the time coordinate and $1,2,3$ refers to the spatial coordinates. Alongside this, we sometimes refer to the time components as t (which is equivalent to 0) and the spatial components as i,j,k (which could refer to 1, 2, or 3 interchangeably). This is especially used when we are being agnostic about the specific spatial coordinate used, but we want to specify which components are spatial or temporal. For example, if we write A_{ij} and let $i \neq j$ this could refer to a set $A_{ij} = \{A_{12}, A_{13}, A_{21}, A_{23}, A_{31}, A_{32}\}$, whereas $A_{ti} = \{A_{01}, A_{02}, A_{03}\}$. We also work in natural units where $c = \hbar = 1$.

When computing derivatives we use ∂_μ, ∇_μ, and $\Box \equiv \nabla^\mu \nabla_\mu$ to represent partial derivatives, covariant derivatives, and the d'Alembert operator respectively. To simplify notation we often use dots to represent derivatives with respect to t, for example $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$. We also use primes to represent partial derivatives with respect to the argument of the function, for example $f'(x) = \frac{\partial f}{\partial x}$.

Finally to simplify our calculations we use the following notational devices:

- $A_{[ab]} = \frac{1}{2}\Big(A_{ab} - A_{ba}\Big)$,
- $A_{(ab)} = \frac{1}{2}\Big(A_{ab} + A_{ba}\Big)$,
- $L_{a,b} = \frac{\partial L_a}{\partial b}$.

Declaration

This thesis has been composed solely to be reviewed by the University of Nottingham and has not been previously submitted, either partially or wholly, to any other application or University. All works in this thesis are my own unless otherwise properly referenced and acknowledged.

Chapter 5 contains work on the published paper:

- Edmund J. Copeland, Sukhraj Ghataore, Florian Niedermann, Antonio Padilla, *Generalised scalar-tensor theories and self-tuning,* Published in the Journal of Cosmology and Astroparticle Physics (JCAP), DOI:10.1088/1475-7516/2022/03/004, arXiv:2111.11448, 2022

Chapter 4 also contains unpublished work.

Supervisors: Prof. E. J. Copeland and Prof. A. Padilla

Examiners: Prof. A. N. Taylor and Prof. P. M. Saffin

Submitted: 16/08/2022

Examined: 23/09/2022

Final Version: 16/11/2022

Abbreviations

AdS	Anti-de Sitter
BAO	Baryon Acoustic Oscillations
CMB	Cosmic Microwave Background
DE	Dark Energy
DOF	Degrees of Freedom
dRGT	de Rham-Gabadadze-Tolley
dS	de Sitter
EEP	Einstein's Equivalence Principle
FLRW	Friedmann–Lemaître–Robertson–Walker
GR	General Relativity
KG	Klein-Gordon
KL	Källén-Lehmann
QFT	Quantum Field Theory
RS	Randall-Sundrum
SLED	Supersymmetric Large Extra Dimension
SUSY	Supersymmetry

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Chapter 1

Introduction

The cosmological constant problem is one of the most elusive problems in modern theoretical physics. Annoyingly, the problem is born from two of the most prestigious and well-tested theories of the 20th century: General Relativity (GR) and Quantum Field Theory (QFT). This introductory chapter first focuses on the basics of GR in section 1.1, before discussing vacuum energy and how it arises from QFT in section 1.2. Finally, in section 1.3 we outline how the cosmological constant problem emerges from the existence of vacuum energy and GR.

1.1 General Relativity

Gravity can be simply thought of as the theory of a massless spin-2 propagator (a graviton). Although this is completely sufficient, it is through Einstein's theory of GR that we arrive at a far more intuitive and insightful approach to gravity. GR is based around Einstein's Equivalence Principle (EEP), which states that local non-gravitational experiments performed in a free falling laboratory are independent of the position and velocity of the laboratory. In other words, acceleration in a flat (or Minkowski) spacetime is locally indistinguishable from gravity. Note, that this is only true while in **infinitesimally** small regions of spacetime, as the gravitational field will be (approximately) homogeneous. As we expand to finite regions, inhomogeneities in the gravitational field will apply and so too will its effects. Returning to the local level, the acceleration that different forms of matter and energy experience will depend only on the gravitational field i.e. the acceleration will be independent of its internal properties. This idea suggests that gravity is an intrinsic feature of spacetime, hence we can build GR as a geometric theory. Furthermore, matter and energy have a non-trivial effect on the curvature of this spacetime. In fact, matter (and energy) distort the geometry of spacetime and objects within it simply follow geodesics (the shortest distance between two points). This phenomenon of following geodesics can be thought of as the force of gravity. Hence, gravity can be understood by analysing the geometry of the spacetime and the effect of matter and energy on it.

GR is built using objects called tensors which transform covariantly. By covariant transformation, we mean that tensors will transform in a specific way under different coordinate systems such that the underlying physics is unchanged. The metric tensor, $g_{\mu\nu}$, describes the distance, $\mathrm{d}s$, between two infinitesimally separated points using a line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. As such, the metric tensor is constructed as a 4×4 matrix with one time coordinate (denoted with t) and 3 spatial coordinates (denoted with i, j, k). Using these metric tensors we can construct the Riemann curvature tensor given by

$$R^\rho{}_{\sigma\mu\nu} = \Gamma^\rho_{\nu\sigma,\mu} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \tag{1.1}$$

with Christoffel symbols:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\big(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\big), \tag{1.2}$$

where a comma before indices refers to a partial derivative with respect to those indices i.e. $L_{a,b} = \frac{\partial L_a}{\partial b}$. $R^\rho{}_{\sigma\mu\nu}$ ascribes a tensor to each point in a pseudo-Riemannian manifold. These manifolds, $\mathcal{M}$, describe a Minkowski spacetime at a local level, but can admit to a non-trivial geometry as we extend beyond that. Based on our description of EEP we can see that this

Riemann curvature tensor would be an ideal candidate to model GR[1].

From this, we can construct the Einstein-Hilbert action which describes gravity:

$$S_{\text{EH}} = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_{\text{Pl}}^2}{2}R - \Lambda_{\text{bare}}\right), \tag{1.3}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ is the Ricci tensor, $g = \det(g_{\mu\nu})$, and $M_{\text{Pl}}^2 = \frac{1}{8\pi G}$ is the Planck mass. We can also add a bare cosmological constant term, Λ_{bare}, as there is nothing that disallows this kind of inclusion. Now, we can minimally couple the matter sector to gravity such that

$$S_{\text{GR}} = S_{\text{EH}} + S_m[g_{\mu\nu}, \psi], \tag{1.4}$$

where ψ denotes the matter fields. Note, that throughout this thesis $S_m[g_{\mu\nu}, \psi]$ represents the generic action for these matter fields. We can vary the above action with respect to $g_{\mu\nu}$ to obtain Einstein's field equation given by

$$M_{\text{Pl}}^2 G_{\mu\nu} = T_{\mu\nu} - \Lambda_{\text{bare}} g_{\mu\nu} \tag{1.5}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, and the energy-momentum tensor, $T_{\mu\nu}$, is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{1.6}$$

The 4×4 tensors in Einstein's field equation (eq. (1.5)) are symmetric so contain 10 independent components, but they can be simplified to 6 using the Bianchi Identity, $\nabla^{\mu}G_{\mu\nu} = 0$, and the conservation of the energy-momentum tensor, $\nabla^{\mu}T_{\mu\nu} = 0$, which provides 4 gauge-fixing conditions. Here, ∇ denotes a covariant derivative that we use to compute gradients of objects independent of the underlying coordinate system, which we define as: $\nabla_{\mu}\phi = \partial_{\mu}\phi$ for a scalar;

[1] Of course, the mathematical details are far more involved, whereas this description gives a broad overview of GR's relationship to geometry. Mathematical details can be found in [1, 2].

$\nabla_\mu \lambda_\nu = \partial_\mu \lambda_\nu - \Gamma^\sigma_{\mu\nu}\lambda_\sigma$ for a vector; and $\nabla_\sigma T_{\mu\nu} = \partial_\sigma T_{\mu\nu} - \Gamma^\rho_{\sigma\mu} T_{\rho\nu} - \Gamma^\rho_{\sigma\nu} T_{\mu\rho}$ for a tensor.

Through Einstein's field equation (eq. (1.5)) we can now model the effects of gravity and its interaction with the matter sector. Particularly relevant to this thesis is modelling the cosmology of the Universe through these equations. For this we use a Friedmann–Lemaître–Robertson–Walker (FLRW) metric, which is an exact solution to eq. (1.5) and describes a homogeneous, isotropic Universe given by

$$\begin{aligned} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= -\mathrm{d}t^2 + a^2(t)\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \\ &= -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right]. \end{aligned} \tag{1.7}$$

Here, $a(t)$ is the scale factor which is a function that ensures that distances are standardised for a dynamic Universe (with $a = 1$ today), and k is the spatial curvature of the Universe. The above describes the metric for a unit plane when $k = 0$; a sphere when $k = 1$; or a hyperboloid when $k = -1$. These different choices can be rewritten as

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}r^2 + \Omega(r)^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{1.8}$$

with

$$\Omega(r) = \begin{cases} \frac{1}{\sqrt{k}} \sin\left(r\sqrt{k}\right), & k > 0 \\ r, & k = 0 \\ \frac{1}{|\sqrt{k}|} \sinh\left(r\sqrt{|k|}\right), & k < 0. \end{cases} \tag{1.9}$$

But for now we will take the simpler example of eq. (1.7).

We also often take the stress-energy tensor to be that of a perfect fluid, such that

$$T^\mu_\nu = diag(-\rho, p, p, p), \tag{1.10}$$

where ρ and p are the energy density and pressure of a given fluid. Substituting eqs. (1.7) and (1.10) into eq. (1.5) yields two equations. The tt components give the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} + \frac{8\pi G\rho}{3} + \frac{\Lambda_{\text{bare}}}{3}, \tag{1.11}$$

whereas the ij components give the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda_{\text{bare}}}{3}. \tag{1.12}$$

Through these equations we can effectively model the cosmology of the Universe. In particular this thesis focuses on the interaction of gravity with the cosmological constant term. To simplify we can set $k, \rho, p = 0$ so we are only left with Λ_{bare} and the scale factor terms. These cosmological equations will have different solutions depending on the value of Λ_{bare}. Notice that if $\Lambda_{\text{bare}} = 0$ then $\ddot{a}, \dot{a} = 0$, therefore $a =$ constant. This corresponds to a Universe that is completely static with a flat spacetime curvature which is referred to as a Minkowski solution. This is not to be confused with the **spatial** curvature denoted by k, which simply refers to the way we can slice our cosmological solutions. Different spatial curvatures will come with different symmetries. Whereas the **spacetime** curvature refers to an intrinsic curvature which governs the dynamics of our Universe. Returning to our cases for Λ, if it is positive then $a = \exp\left(\sqrt{\Lambda_{\text{bare}}/3}\,t\right)$ which corresponds to a Universe undergoing an accelerated expansion, where the spacetime is positively curved. This is known as a de Sitter (dS) solution. Finally if $\Lambda_{\text{bare}} < 0$ then $a = \frac{\sin Qt}{Q}$ with $\Lambda_{\text{bare}} = -3Q^2$, where the parameter "$Q$" is simply another way of describing the spacetime curvature. This corresponds to a Universe that expands for $0 < t \leq \frac{\pi}{2Q}$, but contracts for $\frac{\pi}{2Q} < t \leq \frac{\pi}{Q}$. This is known as an Anti-de Sitter (AdS) solution where the spacetime is negatively curved.

GR has celebrated much experimental success over the years [3], and recently there have been experiments verifying GR's prediction of the existence of gravitational waves [4]. These experiments cement GR as the best current description of gravity.

Despite these experimental successes, GR faces some modern problems. Firstly, it is well known that GR is incomplete when considering smaller scale phenomena where quantum effects tend to dominate[2]. At this point a complete quantum theory of gravity is required to further understand these interactions, but even without this expected breakdown of the theory problems arise on a cosmological scale.

In [8, 9], they claimed that the Universe was accelerating through observations of type 1A supernovae. Type 1A supernovae come from exploding white dwarfs which were believed to be "standard candles" as they all have similar masses, hence similar intrinsic luminosities. In reality the intrinsic luminosity of these supernovae are varied, but they do exhibit a characteristic luminosity decline rate [10], which allows the type 1A supernovae to be somewhat standardised. Through measuring the spectral lines for many different supernovae we find that some of these lines will be red-shifted, implying that these supernovae are moving away from us. We can also independently measure the distances of galaxies that contain these supernovae through measurements of the parallax[3]. Through this, [8, 9] find that galaxies with a higher red-shift are further in distance. Therefore galaxies that are further away from us are also travelling faster (away from us). This implies that the Universe is undergoing an accelerated expansion. However, the notion that type 1A supernovae can be used as standard candles is contested [11, 12] (these papers are also challenged by [13]). Furthermore, a more recent study of type 1A supernovae [14] has determined that the evidence for an accelerated expansion is "marginal" [15]. However, these observations are still consistent with a uniform rate of expansion. Importantly, both studies into type 1A supernovae show that the Universe

[2] Note that both quantum and gravitational effects tend to dominate in black holes (see [5,6] and for more a recent review see [7]).

[3] In simple terms, this is achieved by observing the apparent position of a given galaxy at two separate observation points. Extending the line of sights of these apparent positions to the observation points and "drawing a line" between the two observation points produces a triangle, where the line of sight intersection is the actual position of the galaxy. By knowing the distance between the two observation points and the angle of the triangle, we can use simple geometry to determine the distance of the galaxy from the observer.

is expanding.

This has been verified through measurements of the Cosmic Microwave Background (CMB) [16, 17] and large scale structure [18, 19]. In the early Universe photons and baryons were coupled in a plasma. Over-densities within this plasma would cause contractions and subsequent expansions, creating vibrations within the plasma called Baryon Acoustic Oscillations (BAO). As the Universe cooled the photons decoupled from the baryons and were allowed to free-stream out. These free-streaming photons created the CMB, and the characteristic shape of it was due to these BAO. These BAO would have also left imprints within the decoupled baryons corresponding to over-densities, which would go on to form large scale structure. By comparing the shape of the CMB with the angular distance between these large scale structures, we can determine the expansion rate of the Universe that would allow such a distance. [16–19] show that the Universe must be undergoing an accelerated expansion. But this method also has its flaws. Firstly, the above observations are also consistent with a non-accelerating Universe [20] (but it once again confirms expansion). Secondly, the above method is model dependent as it is based on a FLRW cosmology [21]. This is an assumption that has been made, so it cannot be a true "objective" test of the expansion rate.

Despite its problems there are still a multitude of different observations that seem to suggest that the Universe is currently undergoing an accelerated expansion (see [22] and references therein for further discussions). So for the purposes of this thesis we will assume that is the case[4].

To describe the phenomenon of an accelerated expansion using our description of GR, we can exchange the Λ_{bare} term in eq. (1.5) for a cosmological constant term that is consistent with observation, Λ_{obs}. However, this does not explain where the source of the accelerated expansion comes from, which has since been given the generic term "Dark Energy". In this

[4]Even if this was not the case, this still does not change the nature of the cosmological constant problem which will discuss in section 1.3.

sense we can write $\Lambda_{\rm eff} = \Lambda_{\rm bare} + \Lambda_{\rm DE}$, where the effective cosmological constant, $\Lambda_{\rm eff}$, is the combination of an arbitrary bare cosmological constant, $\Lambda_{\rm bare}$, with that of DE, $\Lambda_{\rm DE}$. Through this construction we require $\Lambda_{\rm eff} \sim \Lambda_{\rm obs}$ to match with observations. In [23] there is an excellent review for both the observational evidence for DE and its possible candidates. However, this thesis chooses to specifically focus on the candidate that arises from the vacuum energy, $\Lambda_{\rm vac}$.

1.2 Vacuum Energy

Vacuum energy can arise from two different places. One is from the classical contribution, $\Lambda_{\rm class}$, which comes from the value of a particle's potential at its minimum. If this is non-zero then it will gravitate like a cosmological constant. The other is due to the quantum nature of the source. This contribution, $\Lambda_{\rm zp}$, comes from the consideration of zero-point fluctuations around the ground state of a source, which will also gravitate as a cosmological constant. Both of these contributions combine to give the cosmological constant due to the vacuum energy i.e. $\Lambda_{\rm vac} = \Lambda_{\rm class} + \Lambda_{\rm zp}$. In this section we show where these contributions come from, following the discussions in [24] (which provides an excellent review, both on the nature of vacuum energy and how its existence creates the cosmological constant problem).

To see where the classical contribution comes from we consider an action for a real scalar field, Φ:

$$S = -\int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + V(\Phi)\right], \tag{1.13}$$

where $V(\Phi)$ is the scalar potential. Then, we vary the action with respect to the metric to find the corresponding stress-energy tensor

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + V(\Phi)\right], \tag{1.14}$$

where the definition of $T_{\mu\nu}$ is given by eq. (1.6). The energy of this scalar is at its lowest when the kinetic terms vanish and the scalar sits at its minimum such that

$$T_{\mu\nu} = -V(\Phi_{\rm min})g_{\mu\nu}. \tag{1.15}$$

Notice that this form corresponds to that of a cosmological constant as in eq. (1.5), therefore as long as the $V(\Phi_{\rm min})$ is non-zero it will contribute to $\Lambda_{\rm vac}$. This problem is most apparent with phase transitions in the vacuum [25].

To demonstrate this we consider an explicit example with two interacting scalar fields, with a potential given by [24]

$$V(\Phi,\Psi) = V_0 + \frac{\lambda}{4}\left(\Phi^2 - v^2\right)^2 + \frac{g}{2}\Phi^2\Psi^2, \tag{1.16}$$

where λ and g are arbitrary coupling constants, V_0 is an arbitrary potential constant, and v is the value of the scalar needed to minimise the potential without Ψ. If Ψ is in thermal equilibrium we can identify the Ψ^2 term with $\langle\Psi^2\rangle_T$, which is the average value of the field at temperature, T. This average will be proportional to the temperature squared [26], which alters the form of the potential to be

$$V_{\rm eff}(\Phi) = V_0 + \frac{\lambda}{4}\left(\Phi^2 - v^2\right)^2 + \frac{\bar{g}}{2}T^2\Phi^2, \tag{1.17}$$

where the coupling constant $\bar{g}$ is used to accommodate any constants when we used $\langle\Psi^2\rangle_T \propto T^2$.

Now, we can express the potential as

$$V_{\rm eff}(\Phi) = V_0 + \frac{\lambda v^4}{4} + \frac{1}{2}m_{\rm eff}^2\Phi^2 + \frac{\lambda}{4}\Phi^4, \tag{1.18}$$

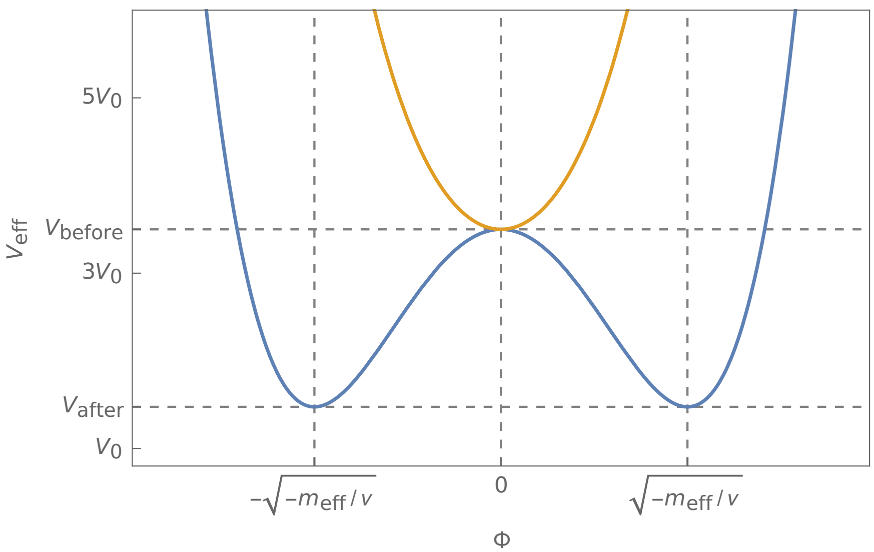


Figure 1.1: This plot showcases the behaviour of the effective potential given by eq. (1.18). The orange and blue lines denote the behaviour of the effective potential before and after the transition respectively. Here, we have defined $V_{\text{before}} \equiv V_0 + \frac{\lambda v^4}{4}$, which corresponds to the minimum of the potential before the transition. Likewise $V_{\text{after}} \equiv V_0 + \frac{\lambda \nu^4}{4} - \frac{m_{\text{eff}}^4}{4\lambda}$ corresponds to the minimum of the potential after.

where

$$m_{\text{eff}}^2(T) \equiv \lambda v^2 \left(\frac{T^2}{T_{\text{crit}}^2} - 1 \right),$$

and we have defined $T_{\text{crit}} \equiv v\sqrt{\frac{\lambda}{\bar{g}}}$. We can see that there is an emergent phase transition for this potential. For $T > T_{\text{crit}}$ (before the transition) the effective mass squared is positive, whereas for $T < T_{\text{crit}}$ (after the transition) it is negative. To minimise the potential before the transition (when m_{eff} is positive) we require $\Phi_{\text{min}} = 0$, hence $V_{\text{eff}}(\Phi_{\text{min}}) = V_0 + \frac{\lambda v^4}{4}$. However, if we want to minimise the potential after the transition (when m_{eff} is negative) we require $\Phi_{\text{min}} = \pm\sqrt{\frac{-m_{\text{eff}}^2}{\lambda}}$, hence $V_{\text{eff}}(\Phi_{\text{min}}) = V_0 + \frac{\lambda\nu^4}{4} - \frac{m_{\text{eff}}^4}{4\lambda}$. See fig. 1.1 where we have plotted the behaviour of the effective potential both before and after the transition. Notice that the minimising value of the scalar after the transition, $\Phi_{\text{min}} = \pm\sqrt{\frac{-m_{\text{eff}}^2}{\lambda}}$, is imaginary if we take $m_{\text{eff}}^2 > 0$ (i.e. the value of m_{eff}^2 before the transition). Interestingly, even if we can normalise $V_0 = \frac{-\lambda v^4}{4}$, such that the potential vanishes before the phase transition, it will reappear as $V_{\text{eff}}(\Phi_{\text{min}}) = -\frac{m_{\text{eff}}^4}{4\lambda}$ after. Likewise, we can also set $V_0 = -\frac{\lambda\nu^4}{4} + \frac{m_{\text{eff}}^4}{4\lambda}$, so that the potential vanishes after the phase transition but is present before. To reiterate, a non-trivial minimised potential acts as a cosmological constant term, which will contribute to the vacuum energy. Even if we can somehow construct a potential that automatically vanishes while at its minimum, phase transitions will reintroduce the Λ_{class} term.

The next contribution comes in the form of *zero-point energy* which arises from QFT. Broadly, QFT predicts that particles found in nature are actually excitations on a quantum field [5]. It combines special relativity with quantum mechanics to describe standard model particle physics. Using perturbative techniques, one finds that calculations of observable quantities tend to result in infinities. To avoid this, QFT uses so-called "renormalisation" procedures [29] to predict these observable quantities whilst removing infinities. Unfortunately, there is no such consistent renormalisation technique that can remove the infinities inside calculations involving gravity. Therefore we consider GR a *non-renormalisable* theory [30]. However, similar to GR we still uphold QFT as pillar of modern theoretical physics due to its predictability (for example its highly prediction of the anomalous magnetic dipole moment [31, 32]).

To reiterate, a key prediction of QFT is that particles can be thought of as excited states of a quantum field, where each state corresponds to a quanta. It also predicts that there are quantum fluctuations about a vacuum (or the ground state of a quantum field). These fluctuations are a result of Heisenberg's uncertainty principle, which states that there is a fundamental uncertainty between a given particle's position and momentum. As a consequence, there is a corresponding uncertainty in the energy and duration of a particle, $\Delta E \Delta t \geq \frac{\hbar}{2}$. This implies that even in a vacuum the energy cannot truly be zero for a finite time interval, as this would violate the uncertainty principle.

[5] Due to the breadth of the topic and the complexity of the calculations we cannot give QFT its due diligence nor is the goal of the thesis. This thesis is mostly concerned with calculations within GR, how they relate to cosmology, and how they relate to processes that arise from QFT. Instead we offer a very broad qualitative overview of QFT focusing particularly on how it relates to vacuum energy. For a more detailed review see [27, 28]

Qualitatively, this results in virtual particles that will "pop" into and out of existence. To be more specific, the uncertainty principle allows for the creation of virtual particle/anti-particle pairs (in accordance with conservation laws) from nothing. These pairs can exist as long they then annihilate back into nothing in a "short enough" time period (the size of which depends on the energy of the given virtual pair). These fluctuations will occur at all points in the vacuum randomly, such that the culmination of all possible virtual particle/anti-particle pair creations will produce a **constant** zero-point energy.

GR tells us that this zero-point energy must gravitate, where its energy density is given by $T^{\text{zp}}_{\mu\nu} = -\Lambda_{\text{zp}} g_{\mu\nu}$. After regularising away the infinities (which is standard procedure in QFT) the 1-loop energy density of the vacuum energy for a canonical scalar of mass m is given by [24]

$$\Lambda^{\text{1-loop}}_{\text{zp}} = \frac{m^4}{64\pi^2}\left[\ln\left(\frac{m^2}{M^2}\right) + \text{finite}\right], \tag{1.19}$$

where M is some arbitrary mass scale and the second term is used to denote some finite constants. We can write further higher order vacuum interactions, for example 2-loop ("figure of eight") terms, but for now we will restrict our analysis to 1-loop; we mention it here for reasons that will become apparent later [6]. Qualitatively this calculation corresponds to considering the ground state interaction of a simple scalar field. Using the stress-energy tensor of a simple scalar field, we can explicitly calculate an interaction with the field that starts and ends in the vacuum (or ground) state. This creates an integral which can be solved with a cut-off to the limits that respect Lorentz invariance. Finally, an explicit calculation of this integral yields an expression that contains infinities, which can be cancelled using a consistent *renormalisation* scheme.

To highlight the key points of zero-point fluctuations: it gravitates like a cosmological

[6] Also note that we have not included any concrete details of the above calculation, which we leave to [24]. To fully appreciate and understand these calculations requires an intimate knowledge of QFT. To reiterate, this thesis cannot offer a full description of QFT, so instead we surmise a qualitative explanation of the 1-loop calculation.

constant; and $\Lambda_{\rm zp}^{\text{1-loop}}$ scales with m^4. Recall however that the above calculation only applies for a single scalar field. To obtain an estimate for the value of all zero-point fluctuations for 1-loop we must sum over all particles, such that

$$\Lambda_{\rm zp}^{\text{1-loop}} \sim \sum_{\text{particles}} m_{\text{particles}}^4. \tag{1.20}$$

It should also be noted that bosons and fermions contribute opposite signs to their respective 1-loop calculations, therefore there will be some cancellation between them when we sum over all particles. However, since we do not live in a world with equal parts bosons and fermions the sum (eq. (1.20)) will not automatically vanish (other than in Supersymmetry (SUSY) theories which we discuss more in section 2.6).

Therefore, we see that zero-point fluctuations will gravitate as a cosmological constant if they exist. But there is one key question that we have not addressed yet: do they even exist? *A priori* there is no reason why we should exclude zero-point fluctuations, as they are completely allowed from QFT. Therefore, to verify the existence of zero-point fluctuations we must provide observable evidence. So far we have only presented the ongoing accelerated expansion as evidence of this. But as we have described previously, this can be accounted for by other forms of DE [23] or through $\Lambda_{\rm class}$. Therefore, we present two different experiments that provide strong evidence for the existence of zero-point fluctuations. Namely, the Lamb shift [33] and the Casimir effect [34]. Similar to before, we will only briefly explore these effects where detailed calculations can be found in [24].

The first clue to the existence of zero-point fluctuations came in 1947 from the Lamb-Retherford experiment [35]. This experiment involved measuring the energy difference between the $^2S_{1/2}$ and $^2P_{1/2}$ orbitals in a Hydrogen atom. Previously, it had been thought that these two levels will have the same energy. Instead they discovered that there is a very tiny shift in the energy, which can be explained by taking into account the interaction between the electron and the zero-point fluctuations. These interactions will cause the electron to "wiggle" about its usual orbit around a proton. These "wiggles" induce a "smearing" effect on the electron's wavefunction, which in turn will cause a shift in its energy. Since the ${}^2S_{1/2}$ orbital is closer to the proton, the "smearing" will be more pronounced than in ${}^2P_{1/2}$, therefore there will be a difference in their energies.

Another piece of evidence comes in the form of the Casimir effect [34], which (in the original 1948 paper [36]) theorised an experiment to detect zero-point fluctuations. Imagine a setup of two conducting plates very close together. Due to quantum mechanics, the vacuum energy states exist in discrete quanta rather than being continuous. The value of these discrete energy states are governed by the size of the "box" it resides in. In this sense, the space in-between the two plates acts as a "box", limiting the energy states that can form from zero-point fluctuations. Meanwhile outside of the plates, they are free to occupy all viable energy states. Both of these effects will induce a force on the plates, but since more energy states are available outside of the plates it will induce a greater force, pushing the plates together. This effect has since been tested and verified multiple times [37–40].

However, we will note that this experimental evidence does not necessarily explicitly link zero-point fluctuations to the Lamb shift or Casimir effect. In fact, it is entirely possible to derive the Casimir force from the two conducing plates without invoking zero-point fluctuations [41]. Despite this, we are not aware of any equivalent challenge to the Lamb shift, therefore we assume that it is highly likely that zero-point fluctuations exist.

Another aspect we have not yet discussed relates to whether zero-point fluctuations actually gravitate. Thus far, we have only shown that there is strong evidence for its existence, not its relationship to gravity. Again, *a priori* there is no reason why this should not be allowed, but it would be useful to verify that these zero-point fluctuations gravitate. To see this, consider the experiment that measured the ratio of the gravitational mass to the inertial mass for Aluminium and Platinum nuclei [42]. They found that these ratios agreed to the order of 10^{-12}, despite the fact that the "Lamb shift" energy effect in Platinum is 3 times that of Aluminium (see [24, 43–45] and references therein for further details). In other words, the inertial mass of Platinum is affected stronger than that of Aluminium due to this "Lamb shift" energy effect. If zero-point fluctuations did not gravitate, it would lead to a far larger difference in the ratios of the gravitational mass to the inertial mass for each nuclei. Instead, [42] shows that this is not the case to an extremely high degree of precision. Similar experiments agreed with the above result using Gold and Aluminium nuclei [46], as well as Potassium and Rubidium nuclei [47]. This evidence suggests that zero-point fluctuations gravitate.

However, there are still problems with these experiments. Firstly, despite the great deal of evidence for its existence, there is no reason to suggest that the corresponding energy effect for the Lamb shift must be that of a cosmological constant. Moreover, both the Lamb shift and the Casimir effect only measure the energy **differences** caused by zero-point fluctuations, so (for now) we can only look towards its cosmological effect to measure the absolute value.

Despite this, we have demonstrated that there is a great deal of evidence for the existence of zero-point fluctuations and *a priori* there is no reason to not include them. Therefore, absent a convincing mechanism to completely remove them, we proceed with the understanding that they exist and their gravitational response is that of a cosmological constant. This, alongside the classical contributions, combine to create what we refer to as vacuum energy such that $\Lambda_{\text{vac}} = \Lambda_{\text{class}} + \Lambda_{\text{zp}}$. Now, all that is left to do is to measure it.

1.3 The Cosmological Constant Problem

This section focuses on *the cosmological constant problem*, which comes from the difference between the observed cosmological constant and that of its theoretical prediction. Starting with the theoretical prediction, recall that eq. (1.19) gives us the value of a cosmological constant arising from 1-loop zero-point fluctuations for a given particle. Of course, to get a precise value for the cosmological constant we must take into account all particle contributions (eq. (1.20)) alongside the classical contributions (a detailed calculation of this can be found in [24]). However, to get a "ballpark" estimate we instead only consider the contribution from eq. (1.19) for an electron field. As long as this is somewhat similar to the observed value of the cosmological constant, we can accept deviations and hash out the details later. The mass of an electron is $m_e = 0.511\text{MeV}$, therefore our estimate becomes $\Lambda_{\text{vac}} \sim (\text{MeV})^4$. However, recall that the only way we can measure this is through its cosmological impact. Therefore, we must consider how $\Lambda_{\text{vac}} \sim (\text{MeV})^4$ affects the cosmology of the Universe.

To see this, we reconsider Einstein's field equation (eq. (1.5)) sourced only by the above vacuum energy ($T_{\mu\nu} = -\Lambda_{\text{vac}} g_{\mu\nu}$) and we set $\Lambda_{\text{bare}} = 0$ for now. We also use a Friedmann–Lemaître–Robertson–Walker (FLRW) metric: $g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}s^2 = \mathrm{d}t^2 + a(t)^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j$, where $a(t)^2$ is the scale factor. With these alterations eq. (1.5) becomes $H_{\text{vac}}^2 = \frac{\Lambda_{\text{vac}}}{3M_{\text{Pl}}^2}$, where the Hubble parameter, $H \equiv \frac{\dot{a}}{a}$, measures the rate of expansion of the Universe. This corresponds to the Friedmann equation sourced only by the vacuum energy, and if we only consider the 1-loop zero-point energy for an electron we find that $H_{\text{vac}}^2 \sim \frac{(\text{MeV})^4}{M_{\text{Pl}}^2}$. In other words $\Lambda_{\text{vac}} \sim (\text{MeV})^4$. Now we want to measure the observed value of the Hubble parameter today, and since this a rough estimate we would consider it a success if it is similar to that of the vacuum energy i.e. $H_0 \sim H_{\text{vac}}$ where H_0 is the current value of the Hubble parameter set by observations. Unfortunately we find through measurements of CMB anisotropies [48]

[7] that $H_0^2 \sim \frac{(\mathrm{meV})^4}{M_{\mathrm{Pl}}^2}$, or in other words the observed value of the cosmological constant is $\Lambda_{\mathrm{obs}} \sim (\mathrm{meV})^4$. This means that the observed and theoretical values of the vacuum energy differ by about 10^{36}. This gap is only worsened when we consider particles with higher masses, for example a proton has a mass that is close to the GeV scale. This means that the prediction of $\Lambda_{\mathrm{vac}} \sim (\mathrm{MeV})^4$ is actually a lower estimate, and the actual value is likely to be much higher. Or in other words

$$\Lambda_{\mathrm{vac}} > (\mathrm{MeV})^4 \sim 10^{36}\Lambda_{\mathrm{obs}}. \tag{1.21}$$

Often this discrepancy between observed and predicted values is cited as the source of the cosmological constant problem, but the problem is far deeper than that. Recall that we had previously set the bare cosmological constant to vanish. QFT in a Minkowski spacetime ordinarily does not have any problems with using counterterms to cancel off infinite divergences when calculating measurable quantities. Therefore, we should not have any problems fine-tuning the bare cosmological constant as a counterterm, such that

$$\Lambda_{\mathrm{bare}} + \Lambda_{\mathrm{vac}} = \Lambda_{\mathrm{obs}}. \tag{1.22}$$

After all, there is no reason why Λ_{bare} should vanish: at the moment it is a free parameter in our theory. The problem arises when we consider phase transitions, unknown particle species, and higher order loops.

To focus on the former, whenever a phase transition occurs we must accordingly re-tune Λ_{bare}. However, we could instead accept that $\Lambda_{\mathrm{bare}} + \Lambda_{\mathrm{vac}} = \Lambda_{\mathrm{eff}} \sim \Lambda_{\mathrm{obs}}$ today, but then the cosmological constant term would differ before any phase transitions. This may cause problems for early Universe cosmology as Λ_{eff} will no longer be similar to Λ_{obs}. If it is the

[7] Note, there is a discrepancy between this measurement and that of supernovae [49] which is known as the *Hubble tension* [50]. Despite this, the differing values of these measurements are still well within an order of magnitude.

dominant term it will contribute to early Universe dynamics in a non-trivial fashion. For example, if we want a period of radiation domination before a phase transition, we would require a different mechanism to remove the gravitational response of Λ_{eff}. This is undesirable but it is a sacrifice we could be willing to make to ensure that the cosmological constant problem is solved (for the Universe **after** phase transitions). Also the introduction of unknown particles species beyond the standard model will give their own contributions to the vacuum energy. This means we cannot get a truly accurate prediction of the vacuum energy density until we know of every particle that exists. However, another major component of this problem comes from the consideration of higher order loops of zero-point fluctuations.

Recall that we had only previously given an expression for 1-loop zero-point fluctuations (eq. (1.19)). To truly calculate the full value of the vacuum energy we must consider contributions from higher order loops. To start, we can consider 2-loop ("figure of eight") interactions which scale as λm^4 [44], where λ is some coupling constant. Without finely tuned couplings these perturbations are not sufficiently suppressed, therefore we need to go back and re-tune Λ_{bare} in order to account for this. Similarly, we will have to do the same for higher order loops, re-tuning the counterterm at each step. This process of fine-tuning and re-tuning is the "real" cosmological constant problem: one of *radiative instability* in the vacuum energy. In this sense, the true value of the vacuum energy is sensitive to higher order UV physics, which we are (so far) ignorant of when it comes to GR and QFT. Therefore simply "summing over all the loops" is neither feasible nor possible given our current understanding of physics. Beyond this, it shows that low energy physics is incredibly sensitive to the details of higher order perturbations. This goes against our "natural" understanding of physics, where we believe that low energy physics are emergent from some higher order theory. This has led some to seek out anthropic considerations [51], which proposes that there are many possible versions of this low energy physics and we just happen to live in the one with $\Lambda_{\text{eff}} = \Lambda_{\text{obs}} \sim (\text{meV})^4$. Despite this, our thesis focuses on solutions that are emergent due to modifications of our current theories. Even if anthropic principles prove to be true, we believe we must first exhaust all other natural options. One such method to alleviate the cosmological constant problem is by modifying GR in ways that naturally reduce the value of Λ_{eff} using *self-tuning* mechanisms. This will be the main focus of our thesis. Note, that from this point onward we use "vacuum energy" to refer to classical contributions, zero-point fluctuations, and their combination interchangeably unless we specify otherwise.

Chapter 2

Self-tuning Theories

Previously in section 1.3 we saw that the cosmological constant problem arises due to the large difference in the values of Λ_{vac} and Λ_{obs}. Whilst we could proceed with an initial fine-tuning of a Λ_{bare} counterterm, such that $\Lambda_{\mathrm{bare}} + \Lambda_{\mathrm{vac}} = \Lambda_{\mathrm{obs}}$, this ignores the underlying problem. Vacuum energy is incredibly sensitive to unknown UV physics, so its value drastically changes when considering higher order loops. Changes in the value of Λ_{vac} leads to repeated fine-tunings in Λ_{bare}, owing to the *radiative instability* of the vacuum energy. In this chapter we discuss a range of modifications to GR that seek to solve or alleviate this problem through *self-tuning* mechanisms.

Broadly, self-tuning (or *self-adjusting*) mechanisms act to shield or cancel off the gravitational effects of a large vacuum energy from the spacetime curvature using additional fields. This suppression enables the cosmological constant to be naturally small, or in other words it forces $\Lambda_{\mathrm{vac}} \sim \Lambda_{\mathrm{obs}}$, eliminating the need for repeated fine-tunings in Λ_{bare}. Note, that the self-tuning fields should adjust their value in response to the value of the vacuum energy. If instead the value of a self-tuning field is predetermined it would be no different from a counterterm, leading us back to the issue of radiative instability.

Depending on the theory, a self-tuning mechanism can act to either partially or wholly cancel the gravitational effects of the vacuum energy. If a theory predicts $\Lambda_{\mathrm{vac}} = \Lambda_{\mathrm{obs}}$, it is trivial to see that this completely solves the cosmological constant problem. However, if a theory predicts that $\Lambda_{\text{vac}} \neq \Lambda_{\text{obs}}$, it requires other forms of DE [23] to explain the discrepancy. Regardless, any good self-tuning mechanism should seek to heavily reduce the value of Λ_{vac}.

However, before constructing a self-tuning theory we must first contend with Weinberg's no-go theorem [51]. In his review on the cosmological constant problem Weinberg argues that, under general assumptions, it is not possible to achieve a phenomenologically viable self-tuning theory in Minkowski space without fine-tuning (the very thing we want to avoid!). Weinberg was able to demonstrate this for an arbitrary (but finite) number of self-tuning fields, alongside a general metric theory that is not necessarily minimally coupled. For a clear review on this see [44]. Here, we will briefly demonstrate the essence of his no-go theorem using a simplified example with arguments presented in [52].

Consider a modified GR theory with a single scalar[8], ϕ, whose vacuum configuration is a constant, ϕ_0. In Minkowski space we expect that the curvature of this system will vanish on the background. In other words the trace of Einstein's field equation (eq. (1.5)) becomes

$$R = 4(\Lambda_{\text{bare}} + \Lambda_{\text{vac}}) - \tau^{\gamma}_{\gamma}(\phi_0) = 4\Lambda_{\text{eff}} - \tau^{\gamma}_{\gamma}(\phi_0) = 0, \tag{2.1}$$

where we have set $M_{\text{Pl}} = 1$ for simplicity and we have written the energy-momentum tensor as a linear combination of the vacuum contribution and the self-tuning field: $T^{\mu}_{\nu} = -\Lambda_{\text{vac}}\delta^{\mu}_{\nu} + \tau^{\mu}_{\nu}(\phi)$. For this to truly be a self-tuning field, the scalar must adjust dynamically to changes in the value of Λ_{eff}. So we require that

$$\frac{\partial V(\phi)}{\partial \phi} = 4\Lambda_{\text{eff}} - \tau^{\gamma}_{\gamma}(\phi), \tag{2.2}$$

where the potential reaches its minimum at ϕ_0 such that $\frac{\partial V}{\partial \phi}|_{\phi_0} = 0$. For a constant solution there are no kinetic terms, so we can identify $\tau_{\mu\nu}(\phi_0) = -V(\phi_0)g_{\mu\nu}$ such that $\tau^{\gamma}_{\gamma}(\phi_0) = -4V(\phi_0)$. At first this seems promising, we have an adjustment mechanism with ϕ that will

[8] For a more in-depth description on scalar-tensor theories in general see section 2.2.

move the system to a Minkowski background in response to changes in Λ_{eff}. Now let us extend this to $\phi \neq \phi_0$ where we only consider small changes in the scalar, hence we can discount kinetic terms (i.e. $\partial_\mu \phi \, \partial_\nu \phi = 0$). This creates a differential equation for $V(\phi)$ which can be solved with

$$V(\phi) = V_0 e^{4\phi} - \Lambda_{\text{eff}}, \tag{2.3}$$

where V_0 is an arbitrary constant. Unfortunately this potential is a runaway in ϕ and will not generically have a minimum, therefore it cannot reproduce a vanishing curvature solution. We can artificially enforce a minimum in two ways. Firstly, we can set $V_0 = 0$ which allows the potential to be minimised, but corresponds to a fine-tuning in the scalar. Any changes in the value of the vacuum energy, will change the combination of the cosmological constants i.e. $\Lambda_{\text{bare}} + \delta\Lambda_{\text{vac}} = \delta\Lambda_{\text{eff}}$. Now, there is no dynamical mechanism that can change this, which means that it cannot self-tune. We could instead set $e^{4\phi} \to 0$, however this choice is not phenomenologically sound. In this setup the exponent will act as a conformal factor with the metric tensor in the Lagrangian [44], therefore all masses should scale with $e^{4\phi}$. This setup should be avoided because all masses within this theory would go to zero, which clearly does not correspond to the Universe we live in.

As shown, a self-tuning mechanism is not possible under this regime without fine-tuning. This is because the value of the potential is determined automatically by the theory, rather than responding to the cosmological constant. However, it is important to recognise that this is derived under some very general assumptions. Weinberg assumes that: the theory admits to local kinetic structure in four dimensions; Poincaré invariance holds across all fields; there are a finite number of fields; and general covariance holds. To address the first assumption, this means that physics is completely causal and interacts entirely at a local level. However, breaking this is not necessarily out of the realm of possibilities (see [53, 54] for further discussions). Poincaré invariance involves full symmetry in translations, rotations, and boosts. In other words, Weinberg demands that the vacuum solutions involve a constant scalar field alongside a Minkowski spacetime metric, and we explicitly break this assumption in chapters 4 and 5. These chapters contain original work, thus are the main focus of this thesis. Further discussions are contained within the above chapters, alongside chapter 3 and section 2.2. There are also possible avenues to explore when breaking the finite fields assumption. In fact some Kaluza-Klein theories contain infinite towers of fields, whose effects are hidden due to compactifications [55] (we will discuss compactification more in section 2.6). Finally, general covariance or diffeomorphism invariance involves the laws of physics being invariant under different coordinate transformations. As with the other assumptions there are ways we can break this which we discuss further in section 2.7. To conclude, evading at least one of these assumptions will be pivotal to any self-tuning theory that wishes to avoid fine-tuning.

To reiterate, a viable self-tuning theory must modify GR such that it produces a naturally small cosmological constant, all whilst evading Weinberg's no-go theorem. Also note that precautions should be taken if one wishes to make such modifications. As discussed in section 1.1, a multitude of experimental and observational results have cemented GR as our best current description of gravity. Any theory that wishes to be phenomenologically viable must either recover GR in some limit, or be prepared to explain away the discrepancies.

The rest of this chapter focuses on different approaches to obtain viable self-tuning theories. This is by no means an exhaustive list nor will we provide full derivations of these models. Instead, this chapter offers the reader a chance to understand some of the approaches adopted, their mechanisms, and some of the pitfalls of self-tuning.

2.1 Unimodular gravity

We begin with *unimodular gravity* which involves modifying GR such that the object $\det(g_{\mu\nu})$ is immediately restricted. Restricting the allowed values of the metric will restrict the gravitational effect of the vacuum energy. To see this we construct a modified version of the Einstein-Hilbert action (eq. (1.3)) such that [56, 57]

$$S_{\mathrm{UMG}} = \int \mathrm{d}^4 x \left(\sqrt{-g}\frac{M_{\mathrm{Pl}}^2}{2} R - \lambda(x)\left(\sqrt{-g} - \epsilon_0\right)\right) + S_m[g_{\mu\nu}, \psi], \tag{2.4}$$

where λ is a Lagrange multiplier (essentially an auxiliary field), ϵ_0 is a constant, and we have also removed the bare cosmological constant term for now. Varying with respect to λ sets $\sqrt{-g} = \epsilon_0$, such that $\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g} = 0$. This represents a gauge choice in GR, but it is one that breaks full diffeomorphism invariance. Note, that the "addition" of extra fields (in this case λ) only fixes the gauge condition. In reality, there are no physical extra fields that will act to directly cancel off a cosmological constant. Instead, it changes the value of the cosmological constant through an alternate mechanism.

To see this, we vary the action (eq. (2.4)) with respect to $g_{\mu\nu}$ to obtain

$$M_{\mathrm{Pl}}^2 G_{\mu\nu} + \lambda g_{\mu\nu} = T_{\mu\nu}, \tag{2.5}$$

where $T_{\mu\nu}$ is given by eq. (1.6). Taking the trace of the above leads to $\lambda = \frac{1}{4}\left(M_{\mathrm{Pl}}^2 R + T\right)$ which we then substitute back into eq. (2.5):

$$M_{\mathrm{Pl}}^2 \left[G_{\mu\nu} + \frac{1}{4} R g_{\mu\nu}\right] = T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu}. \tag{2.6}$$

Notice that if we expect there to be a pure vacuum source (i.e. $T_{\mu\nu} = -\Lambda_{\mathrm{vac}} g_{\mu\nu}$) then the right hand side of eq. (2.5) automatically vanishes. At first this seems great! We seem to have completely removed vacuum energy from affecting the curvature. However the problem comes when take the divergence of above field equation. Einstein's field equation (eq. (1.5)) automatically vanishes if we act on it with ∇^μ, but doing the same to eq. (2.6) creates a constraint:

$$\nabla^\mu(T + M_{\rm Pl}^2 R) = 0 \Longrightarrow T + M_{\rm Pl}^2 R = 4\Lambda_{\rm bare}, \tag{2.7}$$

where $\Lambda_{\rm bare}$ is an arbitrary integration constant and we have used the Bianchi identity ($\nabla^\mu G_{\mu\nu} = 0$) alongside conservation of the energy-momentum tensor ($\nabla^\mu T_{\mu\nu} = 0$). Now we can see that substituting the constraint (eq. (2.7)) back into the above field equation (eq. (2.6)) yields Einstein's field equation (eq. (1.5)). In this sense, the above version of unimodular gravity is no different from GR.

To illustrate this phenomenon again, we consider a generalisation of unimodular gravity by reintroducing diffeomorphism invariance [58]. As demonstrated by [59], this generalisation is just a special case of the Henneaux-Teitelboim action [60]:

$$S_{\rm HT} = \int {\rm d}^4x \left[\sqrt{-g}\frac{M_{\rm Pl}^2}{2}R - \lambda\left(\sqrt{-g} - \partial_\mu w^\mu\right)\right] + S_m[g_{\mu\nu}, \psi], \tag{2.8}$$

where w^μ is a vector density and λ is the Lagrange multiplier. Similar to before, we can vary eq. (2.8) with respect to λ and w^μ, which yields $\sqrt{-g} = \partial_\mu w^\mu$ and $\partial_\mu \lambda = 0$ respectively. Once again, this restricts $\det(g_{\mu\nu})$. But the variation with respect to $g_{\mu\nu}$ yields eq. (2.5) leading to the same problem as before: it will be indistinguishable from GR[9]. There is some "difference" to GR in that the $\Lambda_{\rm bare}$ term is an integration constant that we can set, rather than having it be arbitrary. However, this just hearkens back to the problem of radiative instability. Even if we have more power to change $\Lambda_{\rm bare}$, this will not stop the repeated fine-tunings and re-tunings that we need to avoid. Note, that there is some debate as to whether **quantum** unimodular differs from **quantum** GR (see [59, 61] and references therein). However, this

[9] A full generalisation of this action is analysed in [59] and is also shown to be indistinguishable from GR.

thesis is only concerned with GR's relationship to the cosmological constant problem. So for our purposes we can say that unimodular gravity is indistinguishable from GR, thus we cannot create a viable self-tuning theory from it.

2.2 Fab Four

A simple way to modify gravity is through the addition of scalar fields. These alternatives to GR have a rich history ranging from the seminal Brans-Dicke gravity [62] to more recent examples [63,64][10]. To briefly introduce scalar-tensor theories we will explore how two different examples modify GR. Namely quintessence [68] and Brans-Dicke [62]. To focus on the former consider the modified Einstein-Hilbert action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}g^{\mu\nu}\partial_\nu\phi\partial_\nu\phi - V(\phi)\right] + S_m[g_{\mu\nu}, \psi]. \tag{2.9}$$

This modification simply corresponds to adding a scalar field which has a kinetic and a potential term. We have also set $M_{\mathrm{Pl}}^2 = 1$ for simplicity. The field equations are then given by

$$G^\mu_\nu - \frac{1}{2}\partial_\sigma\phi\partial_\nu\phi + \frac{1}{4}\delta^\mu_\nu g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{4}\delta^\mu_\nu V(\phi) = T^\mu_\nu, \tag{2.10}$$

$$\Box\phi - \frac{\partial V(\phi)}{\partial\phi} = T, \tag{2.11}$$

where we have varied with respect to the $g_{\mu\nu}$ and ϕ respectively. Note that $\Box = \nabla_\mu\nabla^\mu$. Here, we see that if $\phi =$ constant then the potential term in eq. (2.10) acts as a cosmological constant, which can induce a cosmic acceleration. Furthermore, we can decompose the stress-energy tensor into vacuum and local matter parts, such that $T_{\mu\nu} = \tau_{\mu\nu} - \Lambda_{\mathrm{vac}}g_{\mu\nu}$. In doing so, we can see that the potential term can also be used to explicitly cancel a large vacuum

[10] See [65–67] for reviews on scalar-tensor theories.

term. However, as we have discussed in section 1.3 this reintroduces the problem of radiative instability. Therefore these kind of simple modifications are often used in inflationary physics (see [68,69] and references therein). Either way, clearly the addition of scalar fields can contain non-trivial dynamics that alter the cosmology of our system. In fact, care should be taken to ensure the so-called "fifth force" produced by these scalar field do not drastically alter shorter distance physics, unless we admit to a departure from GR. As we discussed in section 1.1 there is a very close agreement between GR and observation [3]. Therefore (to recover GR in some limit) we must either work within the allowed bounds of experimentation [3] or introduce non-trivial mechanisms that can screen these fifth forces [70].

Turning our attention Brans-Dicke gravity, we can get a better insight into how scalar fields can affect gravity. The explicit action is given by

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\left(\nabla\phi\right)^2\right] + S_m[g_{\mu\nu},\psi], \tag{2.12}$$

where ω is a coupling constant and we have once again set $M_{\mathrm{Pl}}^2 = 1$. The field equations for this are given by

$$\phi G_{\mu\nu} - \nabla_\nu\nabla_\mu\phi - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi + g_{\mu\nu}\left(\Box\phi + \frac{\omega}{2\phi}\left(\nabla\phi\right)^2\right) = T_{\mu\nu}, \tag{2.13}$$

$$(2\omega+3)\,\Box\phi = T. \tag{2.14}$$

Notice that the scalar is coupled to $G_{\mu\nu}$. In a sense, it allows the scalar field to act as effective gravitational constant where $G_{\mathrm{eff}} \propto \frac{1}{\phi}$, which modifies the strength of gravity. This makes it even clearer that one must be careful when using scalar fields to avoid a departure from GR. Similarly to quintessence, often Brans-Dicke models are confined to inducing an accelerated expansion [71]. To demonstrate how Brans-Dicke can achieve this we can write the modified Friedmann equation from eq. (2.13). For simplicity, we do so using a flat FLRW metric (eq. (1.7) with $k = 0$), taking $\omega = 0$, and a Universe that is dominated by matter

($T^{\mu}_{\nu} = diag(-\rho_m, 0, 0, 0)$) to obtain $3(H^2 + H\dot{\phi}) = \rho_m$. Using $' = \frac{\mathrm{d}}{\mathrm{d}\ln a}$ this can then be rewritten as

$$3H^2 = \frac{\rho_m}{\phi + \phi'}. \tag{2.15}$$

Recall that matter scales with the scale factor as $\rho_m \sim a^{-3}$[11]. To induce an accelerated expansion we must have $\phi + \phi' \sim a^{-3}$ at least (where $H =$ constant in this example). If this is the case, then the value of the scalar field is decreasing with time. Recall that the effective gravitational constant is $G_{\text{eff}} \propto \frac{1}{\phi}$, therefore G_{eff} is actually increasing. In a sense, we are increasing effect of gravity in order to produce an accelerated expansion. To reaffirm this point, we can demonstrate what happens if we decrease the effective gravitational constant. We can do this by taking the limit where $G_{\text{eff}} \propto \frac{1}{\phi} \to 0$. Re-adding spatial curvature, k, causes the modified Friedmann equation to be given by

$$H^2 = -\frac{k^2}{a^2}. \tag{2.16}$$

The only real solution is when $k \leq 0$. If $k = 0$ this corresponds to a completely static Universe. Whereas if $k < 0$ the solution to the scale factor becomes $a(t) \propto t$. This is an expansive solution, but not one of an **accelerated** expansion. Instead it describes a Universe that has some initial 'kick' and then continues to expand linearly. To summarise we cannot achieve an accelerated expansion by decreasing the strength of gravity, but we can by increasing it[12]. Returning our attention to self-tuning with scalar-tensor theories, we will use the rest of this section to explore Fab Four, which is derived from Horndeski's action.

By avoiding theories with higher than second order field equations Horndeski was able to write the most general four dimensional scalar-tensor theory in 1974 [72] (which was later rediscovered in 2011 [73]). Due to this, any other scalar-tensor theory, with no more than

[11] This can be derived by considering canonical GR without a scalar field and with $\Lambda_{\text{eff}} = 0 = k$. Then, the solution to the Friedmann equation for matter becomes $\rho_m \sim a^{-3}$.

[12] A special thank you to Andy Taylor for their helpful discussions on this topic.

second order field equations, must necessarily be a special case of Horndeski's theory. Note, that we avoid higher order theories in order to evade the *Ostrogradsky instability* [74,75]. The theory states that (generically) Lagrangians with higher than second order field equations contain instabilities because the corresponding Hamiltonian will be unbounded both above and below[13]. These unbounded Hamiltonians lead to states with a negative kinetic energy, which are known as *ghosts* [67] (specifically the Ostrogradsky ghost in this case). In general, ghosts are states with a negative norms or negative energy densities (depending on how they are quantised). The former violates unitarity as it allows for the existence of states with a negative probability, whereas the former insinuates the existence of negative energy eigenvalues. If they are unphysical states we can accept them within the theory. However, if they correspond to physical excitations they are unacceptable,

Starting from the Horndeski action, the model (first proposed in [79] and whose full derivation is contained within [80]) was able to obtain a class of solutions that give rise to a self-tuning mechanism on FLRW backgrounds. To achieve this without fine-tuning, the theory must first overcome Weinberg's no-go theorem [51] by breaking a key assumption. Inspired by the approach of a bigalileon theory [81, 82], this model maintains Poincaré invariance on the metric but breaks it for the self-tuning scalar field. In other words, it maintains a flat spacetime geometry, but the scalar field does not remain constant on the background.

Now that the no-go theorem has been evaded, Fab Four places some very general self-tuning conditions on Horndeski's action. Namely, they require that:

1. the theory produces a (patch of)[14] Minkowski spacetime for any value of the cosmological constant;
2. the above condition continues to work through phase transitions, where the vacuum

[13]There are examples to the contrary, but these require Lagrangians that are carefully constructed to avoid an unbounded Hamiltonian [76–78].

[14]Note, that the reason for the inclusion of "patch of" will become apparent later.

energy instantaneously changes;

3. the theory allows for a non-trivial cosmology, such that the system is allowed to evolve dynamically to an attractive fixed point.

The final assumption is that matter is minimally coupled to the metric, satisfying EEP.

Remarkably, these conditions reduce the complicated Horndeski action to just four base Lagrangians, each containing an arbitrary function of the scalar field coupled to a curvature term. These Lagrangians combine linearly in the action given by

$$S_{FabFour} = \int \mathrm{d}^4x\sqrt{-g}\,[\mathcal{L}_j + \mathcal{L}_p + \mathcal{L}_g + \mathcal{L}_r - \Lambda_{\text{bare}}] + S_m[g_{\mu\nu}, \psi]. \tag{2.17}$$

Here, $\mathcal{L}_j$, $\mathcal{L}_p$, $\mathcal{L}_g$, and $\mathcal{L}_r$ are known as the *Fab Four*, where the indices refer to *John*, *Paul*, *George*, and *Ringo*. The form of these Lagrangians are given explicitly by

$$\mathcal{L}_j = V_j(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi, \tag{2.18}$$

$$\mathcal{L}_p = V_p(\phi)P^{\mu\nu\alpha\beta}\nabla_\mu\phi\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi, \tag{2.19}$$

$$\mathcal{L}_g = V_g(\phi)R, \tag{2.20}$$

$$\mathcal{L}_r = V_r(\phi)G_{GB}, \tag{2.21}$$

where R is the Ricci Scalar, $G_{\mu\nu}$ is the Einstein tensor, $P^{\mu\nu}{}_{\alpha\beta} \equiv -\frac{1}{4}\delta^{\mu\nu\rho\delta}_{\sigma\lambda\alpha\beta}R^{\sigma\lambda}{}_{\rho\delta} = -R^{\mu\nu}{}_{\alpha\beta} + 2R^{\mu}_{[\alpha}\delta^{\nu}_{\beta]} - 2R^{\nu}_{[\alpha}\delta^{\mu}_{\beta]} - R\delta^{\mu}_{[\alpha}\delta^{\nu}_{\beta]}$ [15] is the double dual of the Riemann tensor [83], and $G_{GB} \equiv R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - 4R^{\mu\nu}R_{\mu\nu} + R^2$ is the Gauss-Bonnet combination [80]. The action (eq. (2.17)) also includes an arbitrary bare cosmological constant term, Λ_{bare}. Requiring Λ_{bare} to vanish in order to self-tune corresponds to an effective fine-tuning between it and the vacuum energy, Λ_{vac}. In this sense, allowing for an arbitrary bare cosmological constant acts as a consistency check to achieve a successful self-tuning mechanism. In other words, although we have previously

[15] Reminder that $A_{[ab]} = \frac{1}{2}\left(A_{ab} - A_{ba}\right)$ as defined in Conventions.

stated that $\Lambda_{\text{vac}} \sim \Lambda_{\text{obs}}$, in reality we require the combination $\Lambda_{\text{bare}} + \Lambda_{\text{vac}} = \Lambda_{\text{eff}} \sim \Lambda_{\text{obs}}$. This ensures that there are no hidden fine-tunings present, where Λ_{bare} can effectively be absorbed into Λ_{vac}.

First, [80] begins by writing Horndeski's action and field equations, albeit in a slightly altered but still equivalent form to [72]. To study the cosmology of this system they consider an FLRW metric as in eq. (1.7). Using this cosmological setup they are able to find some useful identities, the full details of which are contained within [80]. They further recognise that two Lagrangians will be equivalent in their equations of motion if they differ only by a total derivative. Using these identities and performing several integration by parts to the original Horndeski Lagrangian [72], $\mathcal{L}_H$, they arrive at simplified cosmological minisuperspace Lagrangian, L, which is given by

$$L = \frac{\int \mathrm{d}^3x\sqrt{-g}\,\mathcal{L}_H}{\int \mathrm{d}^3x\sqrt{\gamma}}, \tag{2.22}$$

where $\gamma = \det(\gamma_{ij})$. To clarify, this Lagrangian, L, is essentially a simplified form of the Horndeski Lagrangian, $\mathcal{L}_H$, which [80] uses to fully capture the most general scalar-tensor theory (in four dimensions with equations of motion up to second order). To further simplify this, they demand the matter sector takes the form of a homogeneous cosmological fluid with energy density, ρ, and pressure, p.

From this, they write down the field equations

$$\mathcal{H} = -\rho, \qquad E_\phi = 0, \tag{2.23}$$

where

$$\mathcal{H} = \frac{1}{a^3}\left[\frac{\partial L}{\partial \dot{a}}\dot{a} + \frac{\partial L}{\partial \dot{\phi}}\dot{\phi} - L\right], \tag{2.24}$$

and

$$E_\phi = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{\phi}}\right] + \frac{\partial L}{\partial \phi}, \tag{2.25}$$

are the Hamiltonian density and the scalar equation of motion respectively. Here, they find that E_ϕ is linear in second derivatives, therefore it can be written in the form

$$E_\phi = \ddot{\phi} f(\phi, \dot{\phi}, a, \dot{a}) + g(\phi, \dot{\phi}, a, \dot{a}, \ddot{a}), \tag{2.26}$$

where the functions of f and g are determined eq. (2.25). Currently, everything we have written is still following Horndeski's full theory: the next step is to apply the three self-tuning conditions outlined above.

The first self-tuning condition requires the vacuum energy density within the matter sector to be in the form of a cosmological constant, $\rho_{\text{vac}} = \Lambda_{\text{vac}}$, which will combine linearly with Λ_{bare} such that $\Lambda_{\text{vac}} + \Lambda_{\text{bare}} = \Lambda_{\text{eff}}$. Regardless of the value of Λ_{eff} the system should always return to a solution with a portion of flat spacetime. For this to hold the solutions must be Ricci-flat which correspond to

$$H^2 = -\frac{k}{a^2}, \tag{2.27}$$

for eq. (1.7), where $k = 0$ is flat, $k = -1$ is a flat Milne slicing, and there is no slicing possible for $k = 1$. Next, we go "*on-shell-in-a*" for the field equations by inserting $a = a_k(t) \equiv a_0 + \sqrt{-k}\,t$, satisfying the Ricci-flat solution (eq. (2.27)). This means that the equations are solved for the scale factor, but ϕ remains as a variable. This changes the functional forms of $\mathcal{H}(\phi, \dot{\phi}, a, \dot{a}) \to \mathcal{H}_k(\phi, \dot{\phi}, a_k)$, $f(\phi, \dot{\phi}, a, \dot{a}) \to f_k(\phi, \dot{\phi}, a_k)$, and $g(\phi, \dot{\phi}, a, \dot{a}, \ddot{a}) \to g_k(\phi, \dot{\phi}, a_k)$. Therefore, from eq. (2.23) the *on-shell-in-a* field equations take the functional form

$$\mathcal{H}_k(\phi, \dot{\phi}, a_k) = -\Lambda_{\text{eff}}, \qquad \ddot{\phi} f_k(\phi, \dot{\phi}, a_k) + g_k(\phi, \dot{\phi}, a_k) = 0, \tag{2.28}$$

where $\mathcal{H}_k$, f_k, and g_k do not contain any explicit time dependence apart from those contained within a_k.

The second self-tuning condition demands that the first condition holds in the presence of (effectively) instantaneous, finite changes in the value of the vacuum energy. At the phase transition the scalar field should act to maintain a Ricci-flat solution, or in other words the scalar field should tune itself in response to changes in the vacuum energy. If we assume this change is instantaneous, then Λ_{eff} in eq. (2.28) must be discontinuous at the time of transition, $t = t_*$. For the *on-shell-in-a* Hamiltonian equation to hold, there must be a discontinuity in $\mathcal{H}_k(\phi, \dot{\phi}, a_k)$. We can achieve this by assuming that $\phi(t)$ is continuous, whereas $\dot{\phi}$ can be discontinuous. This leads us to our first self-tuning constraint: the *on-shell-in-a* Hamiltonian, $\mathcal{H}_k$, must have a non-trivial dependence on $\dot{\phi}$ to account for the discontinuity in Λ_{eff} at the phase transition.

To find the second constraint, we (again) use the fact that the value of the vacuum energy changes instantaneously at the time of transition. Therefore, the rate of change of Λ_{eff} will be proportional to a Dirac delta function, $\delta(t - t_*)$, centered about $t = t_*$. To study this phenomenon we can differentiate the *on-shell-in-a* Hamiltonian equation in eq. (2.28) to obtain

$$\sqrt{-k}\frac{\partial \mathcal{H}_k}{\partial a_k} + \dot{\phi}\frac{\partial \mathcal{H}_k}{\partial \phi} + \ddot{\phi}\frac{\partial \mathcal{H}_k}{\partial \dot{\phi}} \propto \delta(t - t_*). \tag{2.29}$$

Notice that the $\delta(t - t_*)$ on the right-hand side of the equation must be supported by one of the terms on the left. Neither $\phi(t)$ nor $\dot{\phi}$ can do this, so $\ddot{\phi}$ must be proportional to a Dirac delta function. This tracks with the fact that $\dot{\phi}$ is discontinuous at the phase transition, so its rate of change, $\ddot{\phi}$, must be $\propto \delta(t - t_*)$.

Returning to eq. (2.28), the *on-shell-in-a* scalar equation of motion contains a $\ddot{\phi}$ term without a $\delta(t - t_*)$ to support it. This implies that both $f_k(\phi, \dot{\phi}, a_k)$ and $g_k(\phi, \dot{\phi}, a_k)$ must vanish to satisfy this equation. First, we consider

$$f_k(\phi, \dot{\phi}, a_k) = 0. \tag{2.30}$$

If f_k contains a non-trivial $\dot{\phi}$ dependence, the left-hand side of this equation must also be discontinuous at $t = t_*$. However, this discontinuity is not supported by the right-hand side of the equation, so we conclude that f_k is independent of $\dot{\phi}$, or in other words

$$f_k(\phi, \dot{\phi}, a_k) \to f_k(\phi, a_k). \tag{2.31}$$

This can be constrained further by taking the time derivative of $f_k(\phi, a_k) = 0$ around the time of the transition:

$$\sqrt{-k}\,\frac{\partial f_k}{\partial a_k} + \dot{\phi}\frac{\partial f_k}{\partial \phi} = 0. \tag{2.32}$$

As before, there is no discontinuity on the right-hand side of the equation, therefore $\frac{\partial f_k}{\partial \phi}$ must vanish which implies

$$f_k(\phi, a_k) \to f_k(a_k). \tag{2.33}$$

Following the same process it is trivial to see that

$$g_k(\phi, \dot{\phi}, a_k) \to g_k(a_k). \tag{2.34}$$

Note, this is technically only valid around the region of $t = t_*$. However to allow a phase transition to occur at any t, we extend this result to all times. Recall that $a_k(t) \equiv a_0 + \sqrt{-k}\,t$ is fixed, so $f_k = 0$ and $g_k = 0$ do not contain any dynamics. Therefore f_k, g_k, and by extension E_ϕ must vanish identically *on-shell-in-a*.

This impacts the *on-shell-in-a* Lagrangian, $L_k = L_k(\phi, \dot{\phi}, a_k)$, as the scalar equations of motion become

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_k}{\partial\dot{\phi}}\right)+\frac{\partial L_k}{\partial\phi}=0, \tag{2.35}$$

$$\Rightarrow\left[-L_{k,\dot{\phi}\dot{\phi}}\right]\ddot{\phi}+\left[-\sqrt{-k}\,L_{k,\dot{\phi}a_k}-\dot{\phi}L_{k,\dot{\phi}\phi}+L_{k,\phi}\right]=0, \tag{2.36}$$

$$\Rightarrow f_k\ddot{\phi}+g_k=0, \tag{2.37}$$

where we recognise that $f_k=-L_{k,\dot{\phi}\dot{\phi}}$ and $g_k=-\sqrt{-k}\,L_{k,\dot{\phi}a_k}-\dot{\phi}L_{k,\dot{\phi}\phi}+L_{k,\phi}$. As a reminder we use the notation: $A_{,a}=\frac{\partial A}{\partial a}$. By applying $f_k=0$, the Lagrangian becomes

$$L_k=\zeta_{k,\phi}(\phi,a_k)\dot{\phi}+\xi_k(\phi,a_k), \tag{2.38}$$

where $\xi_k(\phi,a_k)$ is a general function of ϕ and a_k. Notice that $\zeta_{k,\phi}(\phi,a_k)$ is also general, but we have written it in the form of a partial derivative for later convenience. Then, by applying $g_k=0$, $\xi_k(\phi,a_k)$ must take the form

$$\xi_k(\phi,a_k)=\sqrt{-k}\,\zeta_{k,a_k}(\phi,a_k)+\nu_k(a_k), \tag{2.39}$$

where $\nu(a_k)$ is a generic function of a_k. This can be directly inserted into eq. (2.38) to yield the final form of the *on-shell-in-a* Lagrangian:

$$L_k=\zeta_{k,\phi}(\phi,a_k)\dot{\phi}+\sqrt{-k}\,\zeta_{k,a_k}(\phi,a_k)+\nu(a_k)=\dot{\zeta}+\nu_k(a_k)\cong\nu_k(a_k), \tag{2.40}$$

where we have used the form of $\zeta_{k,\phi}(\phi,a_k)$ to write $\dot{\zeta}$. We have then discarded $\dot{\zeta}$ because it is a total derivative, therefore it will not contribute to the equations of motion. This form of L_k is the second self-tuning constraint: the *on-shell-in-a* Lagrangian must be independent of ϕ and $\dot{\phi}$.

The third constraint appears when we apply the last self-tuning condition: the full theory must have a non-trivial cosmology. Before going *on-shell-in-a*, recall that the scalar equation of motion (eq. (2.23)) vanishes when imposing the Ricci-flat solution (eq. (2.27)). This can be achieved if $E_\phi = 0$ is either algebraic or dynamic in $H - \frac{\sqrt{-k}}{a}$. If it is algebraic, then the scalar equation of motion will demand a Minkowski solution at all times. This immediately forces a trivial cosmology, which goes against the above self-tuning condition. To avoid this $E_\phi = 0$ must contain derivatives of $H - \frac{\sqrt{-k}}{a}$ i.e. it must not be independent of $\ddot{a}$.

To conclude these conditions imply that a self-tuning scalar-tensor theory must:

1. **not be** independent of $\dot{\phi}$ in the *on-shell-in-a* Hamiltonian density,
2. **be** independent of ϕ and $\dot{\phi}$ in the *on-shell-in-a* minisuperspace Lagrangian,
3. **not be** independent of $\ddot{a}$ in the full scalar equation of motion.

Now these self-tuning constraints can be applied to Horndeski's action to reveal the final form of Fab Four. After some manipulation [80] finds that the action becomes eq. (2.17). They also discover that self-tuning with a homogeneous scalar for $k = 0$ is forbidden, however an identical setup is allowed for $k = -1$. Whilst this regime is not purely flat, by changing the self-tuning ansatz and coordinate choice, the geometry of the system can correspond to (a patch of) Minkowski.

Another interesting property can be found by explicitly calculating the Hamiltonian for eq. (2.17):

$$\mathcal{H} = \mathcal{H}_j + \mathcal{H}_p + \mathcal{H}_g + \mathcal{H}_r + \Lambda_{\text{bare}}, \tag{2.41}$$

where

$$\mathcal{H}_j = 3V_j(\phi)\dot{\phi}^2\left(3H^2 + \frac{k}{a^2}\right), \tag{2.42}$$

$$\mathcal{H}_p = -3V_p(\phi)\dot{\phi}^3 H\left(5H^2 + 3\frac{k}{a^2}\right), \tag{2.43}$$

$$\mathcal{H}_g = -6V_g(\phi)\left[\left(H^2 + \frac{k}{a^2}\right) + H\dot{\phi}\frac{V_g'}{V_g}\right], \tag{2.44}$$

$$\mathcal{H}_r = -24V_r'(\phi)\dot{\phi}H\left(H^2 + \frac{k}{a^2}\right), \tag{2.45}$$

with primes denoting a derivative with respect to ϕ. Recall, that our first self-tuning constraint required the *on-shell-in-a* Hamiltonian to **not** be independent of $\dot{\phi}$. By substituting the Ricci-flat solution (eq. (2.27)) into eqs. (2.42) to (2.45), we find that $\mathcal{H}_j$, $\mathcal{H}_p$, and $\mathcal{H}_g$ contain non-trivial $\dot{\phi}$ dependencies, whereas $\mathcal{H}_r$ does not. Therefore, to achieve self-tuning we must have

$$\{V_j, V_p, V_g\} \neq \{0, 0, \text{constant}\}. \tag{2.46}$$

This clearly rules out GR (given by eq. (1.3)) which is not too surprising. But it does mean that some care should be taken if Fab Four hopes to be a phenomenologically viable cosmological theory (such that GR is recovered in some limit). This constraint also implies that the Ringo term cannot achieve self-tuning alone despite its non-trivial effects on the cosmological dynamics.

Next, we seek cosmological solutions for the Fab Four model that admit to self-tuning. Further studies of the model with a large vacuum energy that dominates over other sources can be found in [84]. Using specific scalar potentials, they show that the fixed point solutions can correspond to inflation, radiation, and matter dominated epochs. In other words, the scalar gives the appearance of a cosmology that is dominated by other sources, despite the energy density being dominated by the vacuum. They find that there exists a class of scaling solutions with a net cosmological constant term. However, these include a "matter dominated" solution that upon perturbing contains a gradient instability that grows too fast to be phenomenologically sound. By setting $\Lambda_{\text{eff}} = 0$, a "matter dominated solution" can be found that is stable against cosmological perturbations. Whilst allowing the cosmological constant to vanish goes against the principle self-tuning, the solutions represent a good approximation to a viable cosmology through numerical simulations when a large Λ_{eff} is reintroduced. Although this is not ideal for a self-tuning theory, it is remarkable that Fab Four can find such solutions.

To conclude, Fab Four is derived from Horndeski's Lagrangian [72], making it a highly generalised self-tuning, scalar-tensor theory in four dimensions. In this respect, it is similar to Weinberg's no-go theorem [51]: providing constraints on the possibilities of viable self-tuning models using scalars. Despite its cosmological shortcomings, it is able to retrieve solutions that mimic well-known cosmological epochs. However, it should be noted that the discovery of gravitational waves [4] led to observational constraints on Horndeski's action (and by extension Fab Four). [85–87] were among the first papers to focus specifically on constraining Horndeski theories through these observations. This has since been addressed by [88] (see references therein for further details) which identify a class of "rescued" Horndeski (and beyond Horndeski) theories that reside within these observational bounds. Future explorations into Horndeski have often been through the lens of DE (for a review on the subject see [89]). For example, through an effective field theory for DE, [90,91] have been able to reconstruct a class of Horndeski theories that reproduce the same background dynamics and linear perturbations, which lies within the observational constraints from gravitational waves. However to refocus on self-tuning, even absent of gravitational wave considerations, it was later found that issues arise when one tries to reconstruct viable cosmological histories with Fab Four [92]. But we will save this discussion for the end of section 2.3, where we explore an extension to Fab Four that suffers from the same issue.

2.3 Fab Five

In section 2.2 we showed that Fab Four is the most general scalar-tensor theory that can self-tune through the linear combination of four Lagrangians, eqs. (2.18) to (2.21). To extend this, the idea of combining these Lagrangians in a **non-linear** fashion was first proposed in [93], which extends beyond Horndeski's action [16]. Naively, these non-linear theories do not necessarily reproduce equations of motion that only go up to second order, but in certain situations they can. One example is a theory involving a non-linear combination of the Ricci Scalar and the Gauss-Bonnet term [96] (which can be identified in Fab Four as mixing the George and Ringo terms). The lack of higher order terms within the equations of motion are a consequence of symmetries within these particular theories. Likewise, the equations of motion for the Lagrangian in [93] remain up to second order. In this paper they analyse a non-linear version of a "purely kinetic gravity" model as a proof of concept. The linear version of this model involves the addition of a standard kinetic term alongside derivatives of the scalar field coupled to the Einstein tensor (which can be identified as the John term in Fab Four) to GR [97]. Whilst this was shown to give rise to an accelerated expansion, it was later ruled out due to various instabilities in the model [98].

We start with the non-linear action [93]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}R + c_1 X + f_{f5}\left(\zeta\right)\right] + S_m[g_{\mu\nu}, \psi], \tag{2.47}$$

where $f_{f5}(\zeta)$ is a generic non-linear function with a variable $\zeta \equiv c_2 X + \frac{c_G}{M^2}G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$; $g_{\mu\nu}$ is the metric; c_1, c_2, and c_G are constants; $X \equiv -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$; and M is a mass scale that renders c_G dimensionless. From the equations of motion, we can study the cosmology of the system using a FLRW metric, assuming spatial flatness, alongside a fluid with energy density, ρ, and pressure, p. To study the dynamics we can use dimensionless parameters $\bar{H} \equiv \frac{H}{H_0}$

[16] For more information on scalar-tensor theories beyond Horndeski see [94, 95] and references therein.

and $x = \frac{\phi'}{M_{\rm Pl}}$, where $H_0 = \frac{\dot{a}_0}{a_0}$ is the Hubble parameter at the present day and primes denote derivatives with respect to τ.

First, consider the cosmological evolution of the system **without** an explicit cosmological constant term, $\Lambda_{\rm eff}$. By analytically and numerically studying the late-time evolution, [93] finds that there are dS attractor solutions (with $f(\zeta) = \zeta^{1.5}$). These fixed point solutions occur when $\bar{H}' = 0$, or in other words, $\bar{H} \to$ constant. In a particular case ($c_1 = 0$) there is a solution that corresponds to $\bar{H}^2_{dS} = -\frac{c_2}{6c_G}$, where c_2 and c_G must have opposite signs. This is encouraging, but it is not self-tuning. To show self-tuning, the model must demonstrate that dS solutions can be found upon the reintroduction of a large $\Lambda_{\rm eff}$ term.

For simplicity, [93] neglects any matter/radiation terms and considers the dynamics when $c_1 = 0$ (alongside the reintroduction of $\Lambda_{\rm eff}$). It can be analytically shown that there are two fixed point dS attractors for this. The first point corresponds to the explicit cosmological constant, $\bar{H}^2_1 = \frac{8\pi G\Lambda_{\rm eff}}{3H_0} = \Omega_\Lambda$, where the scalar field contribution decays away, i.e. $\rho_\phi \to 0$. Whilst this is a dS solution, clearly this is not one that self-tunes. The second point corresponds to $\bar{H}^2_2 = \bar{H}^2_{dS} = -\frac{c_2}{6c_G}$ where $\bar{H}_1 \gg \bar{H}_2$. This point **is** a self-tuning solution: not only does the solution severely reduce the effect of the cosmological constant, but the scalar field also dynamically adjusts to cancel it off. This can be verified numerically as the system can evolve to either point (depending on the initial conditions). To further cement this idea, it can be shown that the negative of the scalar field energy density asymptotically approaches the effective cosmological constant at late times i.e. $\rho_\phi \to -\Lambda_{\rm eff}$ with $\rho_\phi < 0$. Remarkably, a numerical analysis of the system shows that this result holds even in the presence of phase transitions. Finally, by considering linear perturbations the theory is shown to be free of ghosts. Unfortunately, there is a possibility of an early time instability and predictions that the gravitational couplings of the theory could differ from GR for a given form of f_{f5}. Whilst it renders this specific model untenable, the goal of [93] was to explore the possibilities of non-linear extensions to Fab Four. In that sense it was successful in showing that eq. (2.47) can self-tune.

Upon further research, it was shown that the above cannot reproduce phenomenologically viable cosmologies. By placing constraints on non-linear purely kinetic gravity, [92] showed that these theories must admit to an early-time instability if it hopes to generically produce periods of radiation/matter domination. Likewise, for the theory to be stable it must also admit to fine-tuning in the initial conditions to produce a viable early Universe cosmology. Whilst this is a bit of a dampener, one can still proceed with a different non-linear theory in the hopes it can ameliorate these problems.

However, [92] also showed that all non-linear (and by extension linear) Fab Four theories self-tune **too** well. These scalar fields act to cancel off any energy density within the theory. In other words, the value of the scalar field energy density dynamically changes to counteract any form of matter, vacuum energy or otherwise. Whilst this self-tunes away the cosmological constant, it also "self-tunes" away any radiation or matter contributions. Therefore, even if the theory initially starts in an era of radiation or matter domination, the scalar field quickly acts to push the system into the given background solution. The speed at which this occurs is far too swift to allow for radiation/matter to dominate for long enough to be consistent with our Universe's cosmological history. This unfortunately renders Fab Four (and Fab Five) theories in general to not be phenomenologically viable.

2.4 Well-tempered Cosmology

As shown in sections 2.2 and 2.3, we can construct scalar-tensor theories that admit to self-tuning by using the additional scalar fields to dynamically cancel off a cosmological constant. However, these theories also screen any radiation or matter fields, as such they cannot produce a viable cosmological history. To alleviate this problem, a scalar-tensor model would have to self-tune the vacuum energy density, whilst ignoring anything else from the matter sector. A potential solution to this problem comes in the form of *well-tempered cosmology*. The idea of a well-tempered cosmology, first introduced in [99], is to create a model whereby the scalar field has a bias towards self-tuning a cosmological constant. We consider the specific action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}R + K(\phi, X) - G(\phi, X)\Box\phi + \Lambda_{\mathrm{bare}}\right] + S_m[g_{\mu\nu}, \psi], \tag{2.48}$$

where K and G are arbitrary functions of ϕ and X. To study the cosmology of this system they use: a perfect fluid; a FLRW metric $g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$; and a time varying scalar field $\phi = \phi(t)$. From this, they find the field equations where the Hamiltonian constraint is given by

$$3M_{\mathrm{Pl}}^2H^2 = \rho + \Lambda_{\mathrm{bare}} + 2K_{,X}X - K + 3G_{,X}H\dot{\phi}^3 - 2G_{,\phi}X, \tag{2.49}$$

the acceleration equation is given by

$$-M_{\mathrm{Pl}}^2\left(3H^2 + 2\dot{H}\right) = p - \Lambda_{\mathrm{bare}} + K - 2\left(G_{,\phi} + G_{,X}\ddot{\phi}\right)X, \tag{2.50}$$

and the scalar field equation is given by

$$\begin{aligned} K_{,X}\left(\ddot{\phi} + 3H\dot{\phi}\right) + 2K_{,XX}X\ddot{\phi} + 2K_{,X\phi}X - K_{,\phi} - 2\left(G_{,\phi} - G_{,X\phi}X\right)\left(\ddot{\phi} + 3H\dot{\phi}\right) \\ + 6G_{,X}\left[(\dot{HX}) + 3H^2X\right] - 4G_{,X\phi}X\ddot{\phi} - 2G_{,\phi\phi}X + 6HG_{,XX}X\dot{X} = 0. \end{aligned} \tag{2.51}$$

If the scalar has shift symmetry, $\phi \to \phi + \mathrm{constant}$, these arbitrary functions can be written as $K(\phi, X) = c_3M^3\phi + M^4A(X)$[17] and $G(\phi, X) = MB(X)$. $A(X)$ and $B(X)$ are arbitrary dimensionless functions, M is some mass scale, and c_3 is a constant. Next, we can

[17] The $c_3M^3\phi$ term is shift symmetric as any constant shift can be absorbed by the cosmological constant.

write the field equations in terms of dimensionless parameters

$$\psi = \frac{\phi}{M}, \qquad h = \frac{H}{M}, \qquad \tau = Mt, \qquad \Lambda_{\rm bare} = M_{\rm Pl}^2 M_\Lambda^2. \tag{2.52}$$

Using these definitions, the Hamiltonian constraint (eq. (2.49)) becomes

$$3\frac{M_{\rm Pl}^2}{M^2}h^2 = \frac{\rho}{M^4} + \frac{M_{\rm Pl}^2 M_\Lambda^2}{M^4} + A_{,\psi'}\psi' - c_3\psi - A + 3hB_{,\psi'}\psi'^2, \tag{2.53}$$

the acceleration equation (eq. (2.50)) becomes

$$2\frac{M_{\rm Pl}^2}{M^2}h' + 3\frac{M_{\rm Pl}^2}{M^2}h^2 = -\frac{p}{M^4} + \frac{M_{\rm Pl}^2 M_\Lambda^2}{M^4} - c_3\psi - A + B_{,\psi'}\psi'\psi'', \tag{2.54}$$

and the scalar field equation (eq. (2.51)) becomes

$$0 = 3hA_{,\psi'} + A_{,\psi'\psi'}\psi'' - c_3 + 3B_{,\psi'}(h'\psi' + h\psi'' + 3h^2\psi') + 3hB_{,\psi'\psi'}\psi'\psi'', \tag{2.55}$$

where primes denote derivatives with respect to τ[18]. Note, that eq. (2.54) can be simplified through a linear combination with eq. (2.53), such that the $\Lambda_{\rm bare}$ terms vanish:

$$2\frac{M_{\rm Pl}^2}{M^2}h' = -\frac{\rho}{M^4} - \frac{p}{M^4} + A_{,\psi'}\psi' + B_{,\psi'}\psi'(\psi'' - 3h\psi'). \tag{2.56}$$

First, we examine whether the theory can self-tune in any capacity by considering vacuum energy contributions only, $\rho = -p = \Lambda_{\rm vac}$ (or in other words, when the equation of state is $w \equiv \frac{p}{\rho} = -1$). For purposes that will become clear later, eq. (2.56) is multiplied by an arbitrary non-trivial function $f_{WT}(\psi')$. To impose a self-tuning condition we go *on-shell* for a dS solution, $H = H_{\rm dS}$, such that $h = h_{\rm ds} \equiv \frac{H_{\rm dS}}{M} =$ constant. With this choice, eqs. (2.55)

[18]Reminder that $L_{a,b} = \frac{\partial L_a}{\partial b}$, as defined in Conventions.

and (2.56) become

$$0 = -A_{,\psi'} F_{WT}(\psi') + B_{,\psi'}(\psi'' - 3h_{\rm ds}\psi') F_{WT}(\psi'), \tag{2.57}$$

$$0 = 3h_{\rm ds} A_{,\psi'} + A_{,\psi'\psi'}\psi'' - c_3 + 3B_{,\psi'}(h_{\rm ds}\psi'' + 3h_{\rm ds}^2\psi') + 3h_{\rm ds} B_{,\psi'\psi'}\psi'\psi'', \tag{2.58}$$

where $F_{WT}(\psi') \equiv \psi' f_{WT}(\psi')$. Recall that Fab Four (section 2.2) and Fab Five (section 2.3) would solve this system by demanding that the scalar field equation (eq. (2.57)) vanishes identically *on-shell-in-a*. The Hamiltonian constraint (eq. (2.53)) would then determine the evolution of the scalar that cancels off a large cosmological constant *on-shell-in-a*. Using the above method the model can self-tune, but it will still be plagued by the problems discussed in section 2.3 i.e. self-tuning **too** well.

To avoid this [99] takes an alternative approach: through the requirement that eqs. (2.57) and (2.58) are equivalent. This equivalency can be written by equating the ψ'' term coefficients, and then equating everything else such that

$$F_{WT}(\psi') B_{,\psi'} = A_{,\psi'\psi'} + 3h_{\rm ds}(B_{,\psi'\psi'}\psi' + B_{,\psi'}), \tag{2.59}$$

$$-F_{WT}(\psi')(A_{,\psi'} + 3h_{\rm ds} B_{,\psi'}\psi') = 3h_{\rm ds}(A_{,\psi'} + 3h_{\rm ds} B_{,\psi'}\psi') - c_3. \tag{2.60}$$

From this, we can write $A_{,\psi'}$ and $B_{,\psi'}$ in terms of $F_{WT}(\psi')$:

$$A_{,\psi'} = \frac{c_3}{3h_{\rm ds} + F_{WT}} + 3c_3 h_{\rm ds} \frac{(F_{WT})_{,\psi'}\psi'}{F_{WT}(3h_{\rm ds} + F_{WT})^2}, \qquad B_{,\psi'} = -c_3 \frac{(F_{WT})_{,\psi'}}{F_{WT}(3h_{\rm ds} + F_{WT})^2}. \tag{2.61}$$

Plugging these forms back into eqs. (2.57) and (2.58) yields

$$\psi'' = -\frac{(3h_{\rm ds} + F_{WT})F_{WT}}{(F_{WT})_{,\psi'}}. \tag{2.62}$$

Clearly this shows that the *on-shell* field equations (eqs. (2.57) and (2.58)) are equivalent for this specific choice of ψ, which depends on the form of $F_{WT}(\psi')$. In [99] they showcase various solutions, one of which is $A(\psi') = \text{constant}$ (this sets the form of $F_{WT}(\psi')$ and $B(\psi')$). By substituting these forms into eqs. (2.53), (2.55), and (2.56) they were able to find that solutions exist for $h = h_{ds}$, where eqs. (2.55) and (2.56) both reduce to $\psi'' = 3h_{\rm ds}\psi'$. This corresponds to an evolving field such that $\psi' \sim e^{3h_{\rm ds}\tau}$. This is a form of self-tuning because the solution to h is that of a dS solution. Also, the form of the evolving field is set by eqs. (2.55) and (2.56) which do not contain any vacuum energy terms. Furthermore, the scalar field within this dS solution is time varying, hence we avoid Weinberg's no-go theorem [51]. To further cement this idea, the numerical solutions for $A(\psi') = \text{constant}$ show that these conditions do indeed correspond to a dS attractor. It can also be shown that a large vacuum energy can be screened both before and after a phase transition. There is a brief period during the transition where this is not the case, which could lead to observational traces[19]. However, for this specific theory the duration and amplitude of these deviations from $w \equiv \frac{p}{\rho} = -1$ are small. In short, this well-tempered model also seems to be resistant to the impact of phase transitions.

Now that we have established that the model (eq. (2.48)) can self-tune a large cosmological constant, we turn our attention to the problems presented in [92]. To evade this, the scalars must not screen other forms of matter that we introduce to the system. To see this return to eqs. (2.55) and (2.56) but this time keep the matter components generic. These field equations are now no longer equivalent unless $w = -1$, i.e. during vacuum energy domination. Whilst $w \neq -1$, the system will no longer be satisfied by a dS solution. Instead, the scale factor, h, and the scalar field, ψ, will react to these other matter components forming a different solution. This shows that a well-tempered cosmology can self-tune by forcing a dS solution during vacuum energy domination, but the scalars that screen this large cosmological constant can allow other forms of matter to dominate. To reiterate, this contrasts with previous "Fab" models (sections 2.2 and 2.3), where the scalar field equation (eq. (2.55)) would be trivially

[19]For example [100] examines the effect of phase transitions on the energy spectrum of primordial gravitational waves.

satisfied *on-shell* to enforce a self-tuning solution irrespective of the matter component. In the well-tempered approach, the background solution is satisfied **only** for constant energy densities. This has been verified numerically, where [99] shows that the model can admit to a conventional period of matter domination before settling into a late-time dS state.

To conclude, in this section we have established that the model (eq. (2.48)) can use scalar fields that dynamically cancel off a large cosmological constant. Moreover, unlike previous models, the theory does not screen other forms of matter, thus allowing for a viable cosmological history. However, despite its successes at demonstrating a well-tempered cosmology, [99] is a highly simplified model. [101] provides a generalisation by applying a well-tempered approach to Horndeski's original model using a series expansion of its arbitrary functions. [102] takes a different approach by assuming various ansätze for functions within the model. Both papers go on to present example solutions that verify the results of [99]. Further explorations into this method of self-tuning are contained within [103–106]. The future of well-tempering seems positive, prompting further investigations into its properties to constrain the theory and verify its cosmological viability.

We will also very briefly mention a minimal self-tuning model that uses scalar fields to self-tune differently to well-tempering, whilst avoiding the "Fab" problem [92]. Recently, [107] have devised a "Kinetic Gravity Braiding" model [20] (also studied using well-tempering in [102,104]) given by the action eq. (2.48). To study the self-tuning properties, they consider: a perfect fluid where the energy-momentum tensor is given by $T^{\nu}_{\mu} = diag(-\rho, p, p, p)$; with a FLRW metric $g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$; and a time-dependent scalar field $\phi = \phi(t)$. They then take the specific potentials $K(\phi, X) = -3c_0\frac{H_{ds}}{M}X + c_1 M^3\phi$ and $G(X) = \frac{c_0}{M}\sqrt{2X}$ where c_0 and c_1 are dimensionless parameters; M is an arbitrary mass scale; H_{ds} is the late-time attractor solution for the Hubble parameter; and recall $X \equiv -\frac{1}{2}g^{\mu\nu}\nabla_{\mu}\phi\nabla_{\nu}\phi = \frac{\dot{\phi}^2}{2}$ such that

[20] So-called due to the "braiding" of the scalar and metric terms in the equations of motion. For further background reading on this topic we refer the reader to [108–110].

the action becomes

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_{\rm Pl}^2}{2}R - 3c_0\frac{H_{ds}}{M}X + c_1M^3\phi - \frac{c_0}{M}\sqrt{2X}\,\Box\phi + \Lambda_{\rm bare}\right) + S_m[g_{\mu\nu},\psi], \quad (2.63)$$

where they consistently choose the positive branch of $\sqrt{2X}$. From the above (eq. (2.63)), they derive the Friedmann equation:

$$3M_{\rm Pl}^2H^2 = \rho + \Lambda_{\rm bare} + \frac{3c_0}{2M}(2H - H_{ds})\dot{\phi}^2 - c_1M^3\phi, \quad (2.64)$$

the acceleration equation:

$$3M_{\rm Pl}^2H^2 + 2M_{\rm Pl}^2\dot{H} = -p + \Lambda_{\rm bare} + \frac{3c_0H_{ds}}{2M}\dot{\phi}^2 + \frac{c_0}{M}\ddot{\phi}\dot{\phi} - c_1M^3\phi, \quad (2.65)$$

and the scalar equation:

$$(\ddot{\phi} + 3H\dot{\phi})(H - H_{ds}) + \dot{H}\dot{\phi} - \frac{c_1M^4}{3c_0} = 0, \quad (2.66)$$

where the Hubble parameter is $H \equiv \frac{\dot{a}}{a}$ and dots refer to derivatives with respect to t. Also, the energy conservation equation is given by

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (2.67)$$

First, [107] combines eqs. (2.64) and (2.65) to obtain the "Hubble evolution equation":

$$2M_{\rm Pl}^2\dot{H} = -(\rho + p) - \frac{3c_0}{M}\dot{\phi}^2(H - H_{ds}) + \frac{c_0}{M}\dot{\phi}\ddot{\phi}. \quad (2.68)$$

This is similar to eq. (2.56) in well-tempering, in that eq. (2.68) is an expression that automatically removes its dependency on the vacuum term (i.e. when $\rho = -p = \Lambda_{\rm vac}$). Where this model differs is how they use the scalar equation. Well-tempering would enforce an equivalency between the scalar and Hubble evolution equation (eqs. (2.66) and (2.68)), then solve the resultant expressions. Instead, [107] recognises that eq. (2.66) is not independent as it can be derived through a combination of eqs. (2.64), (2.65), and (2.67). This means that eq. (2.66) can now be solved explicitly for $\dot{\phi}$, and in this sense it is different to well-tempering. But similar to well-tempering it seeks to find self-tuning solutions that consistently remove cosmological constant terms, whilst allowing for other forms of matter. To solve these equations they first consider an explicit energy density that contains matter and vacuum energy parts only (i.e. $\rho = \rho_m + \Lambda_{\rm vac}$ and $p = -\Lambda_{\rm vac}$). Now, they make the following dimensionless substitutions: $h \equiv \frac{H}{M}$, $\psi \equiv \frac{\phi}{M}$, $\tau \equiv Mt$, $\mu \equiv \frac{M_{\rm Pl}^2}{M^2}$, $\rho_{m,0} \equiv \frac{\rho_m}{M^4}$, $\Lambda_{\rm bare,0} \equiv \frac{\Lambda_{\rm bare}}{M^4}$, and $\Lambda_{\rm vac,0} \equiv \frac{\Lambda_{\rm vac}}{M^4}$. With these substitutions they use eqs. (2.64), (2.67), and (2.68) to find two dimensionless evolution equations given by

$$h' = \frac{-3(h-\alpha)(3c_0\psi'^2(2h-\alpha)+\rho_{m,0})+c_1\psi'}{6\mu(h-\alpha)+3c_0\psi'^2}, \tag{2.69}$$

$$\psi'' = \frac{9c_0\psi'(h-\alpha)(c_0\psi'^2-2\mu h)+3c_0\psi'\rho_{m,0}+2\mu c_1}{6\mu c_0(h-\alpha)+3c_0^2\psi'^2}, \tag{2.70}$$

where the attractor is denoted by the dimensionless parameter $\alpha = \frac{H_{ds}}{M}$ and primes denote a derivative with respect to τ. For completeness the dimensionless fluid conservation equation becomes

$$\rho'_{m,0} + 3h\rho_{m,0} = 0. \tag{2.71}$$

Now, eqs. (2.69) to (2.71) represent a full numerical system to be solved. Notice that none of the above equations contain explicit $\Lambda_{\rm vac,0}$ or $\Lambda_{\rm bare,0}$ dependencies. Instead, the value of the vacuum energy is used to choose the initial value of ψ using eq. (2.64) (so that they have the freedom to choose the initial conditions of h and ψ'). The evolution of the system itself does not depend on the value of $\Lambda_{\rm vac,0}$ (or $\Lambda_{\rm bare,0}$), but it does depend on $\rho_{m,0}$. Hence, this mechanism has (similar to well-tempering) enforced a removal of the cosmological constant terms from the dynamics of the system, whilst allowing for matter to be included. Finally, the scalar equation (eq. (2.66)) is merely used to measure the behaviour of ψ' (which is represents $\dot{\phi}$). Full details of these calculations can be found within [107]. But we have provided this short description here to demonstrate to the reader that well-tempering is not necessarily the only way to self-tune whilst allowing for other forms of matter domination.

They go on to show that the above model can cancel off a large vacuum energy, whilst self-tuning to a dS attractor solution. They further demonstrate that the model can undergo a period of matter domination, and that these solutions are stable under phase transitions in the value of the vacuum energy. Overall, [107] exists as an initial exploration into a self-tuning mechanism, that is similar but separate to well-tempering. To our knowledge the above is the only exploration into this mechanism, therefore exploring it using another model or through generalising the above model would be interesting.

2.5 Vacuum Energy Sequestering

Vacuum Energy Sequestering is a unique method of self-tuning that recovers GR and requires a simple modification to the usual gravitational equations. The idea is to promote a bare cosmological constant term, $\Lambda_{\rm bare}$, to a variable to provide constraints on the action. In this section we will showcase the sequestering mechanism by considering a global vacuum energy sequestering model [111–114]. This theory also introduces another variable, λ, where both $\Lambda_{\rm bare}$ and λ are global variables i.e. variations occur on the level of the action, but not in spacetime. The explicit action considered is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \Lambda_{\rm bare} - \lambda^4\mathcal{L}_{\rm m}(\lambda^{-2}g^{\mu\nu},\psi)\right] + \sigma(z), \tag{2.72}$$

where the matter sector $\mathcal{L}_{\rm m}$ is minimally coupled to the "Jordan" frame metric, $\tilde{g}_{\mu\nu}$, and $z \equiv \frac{\Lambda_{\rm bare}}{\lambda^4\mu^4}$. Note that a Jordan frame usually describes a scalar-tensor theory, whereby the

Ricci scalar is coupled directly to a scalar field. Our action (eq. (2.72)) is not written in this way, therefore it is in the "Einstein" frame. But it can be changed into an equivalent "Jordan" frame using a redefinition of the metric, $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$ [115]. However, λ is not related to a scalar field in this model, hence the quotation marks on "Jordan" and "Einstein". Finally, σ is a smooth function that lies outside of the integral that is dimensionless due to a mass scale, μ.

Varying the action with respect to Λ_{bare} and λ yields

$$\frac{\sigma'}{\lambda^4 \mu^4} = \int \mathrm{d}^4 x \sqrt{-g}, \tag{2.73}$$

and

$$4\Lambda_{\text{bare}} \frac{\sigma'}{\lambda^4 \mu^4} = \int \mathrm{d}^4 x \sqrt{-g}\, \lambda^4 \tilde{T}^{\alpha}_{\alpha}, \tag{2.74}$$

respectively, where $\sigma' = \frac{\partial \sigma(z)}{\partial z}$ and $\tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta S_m}{\delta \tilde{g}^{\mu\nu}}$ is the energy-momentum tensor defined in the "Jordan" frame, which is $T^{\mu}_{\nu} = \lambda^4 \tilde{T}^{\mu}_{\nu}$ in the "Einstein" frame. By combining these two equations the constraint equation becomes

$$4\Lambda_{\text{bare}} = \langle T^{\alpha}_{\alpha} \rangle, \tag{2.75}$$

where the 4-volume average quantity is defined by $\langle Q \rangle = \frac{\int \mathrm{d}^4 x \sqrt{-g}\, Q}{\int \mathrm{d}^4 x \sqrt{-g}}$. Then, varying the action with respect to $g_{\mu\nu}$ yields

$$M_{\text{Pl}}^2 G^{\mu}_{\nu} = \lambda^4 \tilde{T}^{\mu}_{\nu} - \delta^{\mu}_{\nu} \Lambda_{\text{bare}}, \tag{2.76}$$

which becomes

$$M_{\text{Pl}}^2 G^{\mu}_{\nu} = T^{\mu}_{\nu} - \frac{1}{4} \langle T^{\alpha}_{\alpha} \rangle \delta^{\mu}_{\nu}, \tag{2.77}$$

using eq. (2.75). Now, we can split the energy-momentum tensor such that $T^{\mu}_{\nu} = \tau^{\mu}_{\nu} - \Lambda_{\text{vac}} \delta^{\mu}_{\nu}$, where τ and Λ_{vac} represent the contributions from local matter sources and vacuum energy respectively. By writing T^{μ}_{ν} in this way, eq. (2.77) becomes

$$M_{\rm Pl}^2 G^{\mu}_{\nu} = \tau^{\mu}_{\nu} - \frac{1}{4}\langle \tau^{\alpha}_{\alpha} \rangle \delta^{\mu}_{\nu}, \tag{2.78}$$

where $\langle \Lambda_{\rm vac} \rangle \equiv \Lambda_{\rm vac}$. Notice that both the $\Lambda_{\rm bare}$ and $\Lambda_{\rm vac}$ terms have completely dropped out of the equations, but the local matter sector has survived. This shows that the cosmological constant terms no longer have a direct gravitational response, whereas the τ^{μ}_{ν} term still contributes. In other words, the explicit self-tuning mechanism here uses eq. (2.75) to set a constraint that requires $\Lambda_{\rm bare}$ to cancel off $\Lambda_{\rm vac}$ irrespective of its overall value. Here, eq. (2.78) has left behind a residual cosmological constant term, $\Lambda_{\rm eff} = \frac{1}{4}\langle \tau^{\alpha}_{\alpha} \rangle$, where $\langle \tau^{\alpha}_{\alpha} \rangle$ is the 4-volume average of the trace of the local matter sector. However, since $\langle \tau^{\alpha}_{\alpha} \rangle$ is unrelated to vacuum loops, $\Lambda_{\rm eff}$ does not suffer from the same radiative instability that plagues the cosmological constant problem. Instead, the value of $\Lambda_{\rm eff}$ is determined by a measurement of the historic (and future) matter density. In this sense, the Universe must have a finite spacetime to ensure that the value of $\Lambda_{\rm eff}$ is bounded. In other words, the Universe must be spatially finite and collapse in the future. Similarly, due to the infinite volume in their interiors, black holes would ordinarily provide significant contributions to this historic matter density measurement. However, as [112] demonstrated, this is no longer the case in finite spacetimes. They were further able to show that this $\Lambda_{\rm eff}$ is automatically small in large, old Universes [111], with [112] providing a mechanism to achieve this.

This action (eq. (2.72)) was later modified into a local sequestering theory [116, 117], by promoting λ and $\Lambda_{\rm bare}$ to scalar fields. Performing a similar analysis grants a global constraint, evoking a cancellation that sets the value of $\Lambda_{\rm eff}$, as shown previously. Note, these models avoid Weinberg's no-go theorem [51] as they contain non-gravitating 4-forms which are used to break translational invariance in the corresponding 3-forms [116]. Most recently, the "Omnia Sequestria" model [118, 119] proposes a method that gravitationally decouples matter loops (along with its predecessors) **and** graviton loops.

To conclude, vacuum energy sequestering theories modify GR to achieve self-tuning. Promoting a bare cosmological constant to a variable alongside introducing a separate variable allows us to vary the action with respect to them, which provides constraints on the model. These constraints set the bare cosmological constant to automatically cancel vacuum energy contributions without altering local matter sources. This cancellation leaves behind a residual cosmological constant that is not related to vacuum loops and is naturally small in large, old Universes for finite spacetimes (with [120] setting $|\langle\tau^{\alpha}_{\alpha}\rangle| \lesssim M_{\rm Pl}^2 H_0^2$, where H_0 is the Hubble parameter today). In general, both global and local sequestering theories yield similar cosmological responses through self-tuning the value of the vacuum energy. The future direction of these sequestering theories would be to understand these mechanisms as a UV complete theory to further test its viability as a gravitational alternative to GR. We also briefly mention a similar theory, which involves promoting the Planck mass to a variable [121–123]. As in vacuum energy sequestering, it acts to self-tune an arbitrary cosmological constant in exchange for an effective cosmological constant that is set by historic matter excitations (i.e. $\Lambda_{\rm eff} \propto \langle\tau\rangle$).

2.6 Braneworlds and supersymmetry in cosmology

Despite the fact that we appear to live in a 4D Universe, models such as string theory [124,125] exist in 10D. The structure of these extra dimensions affect the physics within our 4D observable Universe, but their effect is (usually) suppressed as a consequence of Kaluza-Klein compactifications [126]. If these compactifications are sufficiently small they do not spoil standard model particle physics.

Braneworld gravity [127] uses the idea that our 4D observable Universe exists in an object called a "brane". This brane is a hyperspace inside a larger (4+N)D spacetime called the "bulk". Analogous to string theories, the structure of this bulk affects physics within the brane. The difference is that all matter fields are restricted to the brane, whilst gravity and other non-standard model physics are not. This means that the usual constraints on the size of these extra dimensions no longer apply, but instead they can be as large as a few micrometres [128–130]. This non-trivial embedding of the brane in the bulk can offer mechanisms that "hide" the gravitational response of vacuum energy in the bulk, whilst recovering GR on the brane.

The 5D Randall-Sundrum (RS) braneworld model [131][21] was the first to include the self-gravity of branes. This model is described by a 5D action, where the matter Lagrangian is confined to a 4D spacetime that recovers GR in a low energy limit [133]. These models can be oriented to produce a self-tuning mechanism [133–135]. Consider the relatively simply 5D action

$$S = \int \mathrm{d}^5X\sqrt{-{}^{(5)}G}\left[{}^{(5)}R - \frac{4}{3}(\nabla\phi)^2 - \Lambda_5\right] - \int \mathrm{d}^4x\sqrt{-g}\,\mathcal{L}_m, \tag{2.79}$$

where ${}^{(5)}G$ and ${}^{(5)}R$ are the determinant of the metric, ${}^{(5)}G_{MN}$, and trace of the Ricci tensor, ${}^{(5)}R_{MN}$, respectively. Furthermore ϕ is a scalar, Λ_5 is the cosmological constant on the bulk, and X are the coordinates. All of the objects we have just described are 5-dimensional, such that the indices run as $M, N = 0, ..., n-1$, where n is the dimension of the model (in this case $n = 4$ with $M, N = 0$ as the time dimension and $M, N = 1, ..., 4$ as the 4 spatial dimensions). The 4D integral in eq. (2.79) corresponds to the brane where $g = \det(g_{\mu\nu})$ and $g_{\mu\nu}$ is the 4D metric (such that $\mu, \nu = 0, ..., 3$). For simplicity we consider $\Lambda_5 = 0$ and $\mathcal{L}_m = \sigma e^{b\phi}$. Here, σ is the *brane tension* which corresponds to the value of the vacuum energy as calculated by a 4D observer and b is an arbitrary constant. Then, we demand a Minkowski solution on the

[21]This is an example of a RSII model which contains two branes, where one is placed infinitely far away, thereby effectively removing it (at least physically). The other brane contains our standard model physics. The RSI model has the same geometrical setup, but instead both branes are a finite distance away and our standard model physics is contained within one of them. This model was first introduced as a method of solving the hierarchy problem [132].

brane such that the 5D metric ansatz is given by

$$^{(5)}G_{MN}\mathrm{d}X^M\mathrm{d}X^N = \mathrm{d}y^2 + e^{2A(y)}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu, \tag{2.80}$$

where $A(y)$ is the warp factor, $y = X^5$ denotes the extra dimension, and for simplicity we have chosen that the brane lies at $y = 0$. To enter into a self-tuning regime we must find solutions that recover a flat spacetime for any value of σ. Substituting eq. (2.80) into the $(M, N) = (4, 4)$ components of Einstein's field equation yields $A' = \alpha\phi'$, where $\alpha = \pm\frac{1}{3}$ and primes denote derivatives with respect to y. Substituting this solution into a combination of the scalar equation, alongside the $(4, 4)$ and (μ, ν) components of Einstein's field equation gives $\phi'' \pm \frac{4}{3}(\phi')^2 = 0$. By assuming that $\alpha = \frac{1}{3}$ for $y > 0$ and $\alpha = -\frac{1}{3}$ for $y < 0$ the solution to the above relationship becomes

$$\phi(y) = \begin{cases} \frac{3}{4}\log\left(\frac{4}{3}y + c_1\right) + d_1, & \text{for } y < 0 \\ -\frac{3}{4}\log\left(\frac{4}{3}y + c_2\right) + d_2, & \text{for } y > 0 \end{cases} \tag{2.81}$$

where c_1, c_2, d_1, and d_2 are integration constants. First, d_2 can be determined by demanding continuity in $\phi(y)$ across $y = 0$. Then, substituting eq. (2.81) back into the field equations yields two expressions for the integration constants c_1 and c_2, in terms of d_1, b and σ for $b \neq \pm\frac{4}{3}$[22]. Therefore for any given b, there exists a Minkowski solution for the brane, regardless of the value of σ. In other words, the system exhibits self-tuning, without fine-tuning the vacuum energy density. We can "see" this at the level of the action (eq. (2.79)) where the scalar field in the 4D integral can absorb shifts in σ. Note, a similar result can also be shown for a general $\mathcal{L}_m$ [135]. Unfortunately these models have several issues. Firstly, notice that eq. (2.81) contains a singularity at $y = -\frac{3}{4}c_1$ and $y = -\frac{3}{4}c_2$. Any attempt to regularise these singularities re-introduced fine-tuning [136–138]. Secondly, there are instabilities related to external expansion or singular collapse of the braneworld [139].

[22]There are solutions that exist for $b = \pm\frac{4}{3}$ which requires $\alpha = \frac{1}{3}$ for all y as detailed in [135]. But for simplicity we will only consider solutions for $b \neq \pm\frac{4}{3}$.

A different approach increases the dimensions from 5 to 6. [140] considers a 6D bulk action

$$S = \int \mathrm{d}^6 X \sqrt{-{}^{(6)}G} \left[\frac{1}{2} {}^{(6)}R - \Lambda_6 - \frac{1}{4} F_{MN} F^{MN} \right], \tag{2.82}$$

where ${}^{(6)}G$ and ${}^{(6)}R$ are the 6 dimensional determinant of the metric and Ricci scalar respectively. Furthermore F_{MN} is the Maxwell field strength (which provides the pressure needed to stabilise the extra dimensions [140]), Λ_6 is a cosmological constant on the bulk and X are the six-dimensional coordinates. The notation is consistent with the previous example such that indices run as $M, N = 0, ..., 5$, and $\mu, \nu = 0, ..., 3$ refer to indices on the brane. As before we consider a Minkowski metric ansatz given by

$$\mathrm{d}s^2 = G_{MN} \mathrm{d}X^M \mathrm{d}X^N = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \gamma_{ab}(y) \mathrm{d}y^a \mathrm{d}y^b, \tag{2.83}$$

but this time $\gamma_{ab}(y)$ is the metric of the two extra dimensions y^a, y^b, where the indices refer to $(a, b) = (4, 5)$[23]. Now, we can choose F_{MN} to be a magnetic flux threading the extra-dimensional space such that

$$F_{ab} = \sqrt{\gamma} B_0 \epsilon_{ab}, \tag{2.84}$$

where B_0 is a constant, $\gamma = \det(\gamma_{ab})$, ϵ_{ab} is anti-symmetric such that $\epsilon_{12} = 1$, and all other components of F_{ab} vanish. Next we make the choice of a two-sphere geometry for γ_{ab},

$$\gamma_{ab}(y) \mathrm{d}y^a \mathrm{d}y^b = a_0^2 (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{2.85}$$

alongside fixing the magnetic field strength B_0 and the radius a_0 in terms of Λ_6:

$$B_0^2 = 2\Lambda_6, \qquad a_0^2 = \frac{1}{2\Lambda_6}, \tag{2.86}$$

[23]To be clear this does not necessarily mean that $a = 4$ and $b = 5$. Similar to how (i, j, k) refer **only** to the spatial coordinates on a 4D spacetime, (a, b) are used to refer **only** to the extra dimensions $(4, 5)$ and can do so interchangeably.

which is consistent with a static, stable solution [140, 141].

Now, branes can be added using the action

$$S_4 = -\sigma \int \mathrm{d}^4x\sqrt{-g}, \tag{2.87}$$

where σ is the brane tension. This explicit addition is done by placing the two branes atop the bulk on opposite ends of the 2-sphere. The metric now adjusts such that [140]

$$\gamma_{ab}(y)\mathrm{d}y^i\mathrm{d}y^j = a_0^2(\mathrm{d}\theta^2 + \alpha^2\sin^2\theta\mathrm{d}\phi^2), \tag{2.88}$$

where

$$\alpha = 1 - \frac{\sigma}{2\pi}, \qquad a_0^2 = \frac{1}{2\Lambda_6}, \qquad B_0^2 = 2\Lambda_6. \tag{2.89}$$

This metric corresponds to removing a wedge that stretches from the north pole to the south pole from the 2-sphere bulk, where σ is proportional to the size of the wedge (see figure 1 from [140]). To picture this, imagine the geometry of the bulk is a rugby ball[24] with branes that sit atop opposite points of the ball. Notice that the only object that changes due to the brane tension, σ, is α. a_0 and B_0, which are necessary to ensure that the 4D spacetime is flat, remain the same. In other words, the brane tension affects the bulk geometry but leaves the 4D geometry untouched. Unfortunately, this does not solve the cosmological constant problem. In [142] they show that effective 4D cosmological constant corresponds to $\Lambda_4 \propto \frac{1}{2}\Lambda_6 - \frac{1}{4}B_0^2$. For Λ_4 to vanish it requires $B_0^2 = 2\Lambda_6$ as in eq. (2.86), which is a choice that we made. So for Λ_4 to vanish it requires a fine-tuning between Λ_6 and B_0. In a sense, this moves the fine-tuning of the cosmological constant onto the magnetic field.

This problem can be avoided by considering a Supersymmetric Large Extra Dimension

[24] Note, the authors in [140] refer to the geometry as an “American football”.

(SLED) model [143–145], which is a modified version of [140]. The action is given by

$$S = \int \mathrm{d}^6 x \sqrt{-{}^{(6)}G} \left[\frac{1}{2}({}^{(6)}R - \partial_M \phi \partial^M \phi) - \frac{1}{4} e^{-\phi} F_{MN} F^{MN} - e^{\phi} \Lambda_6 \right], \tag{2.90}$$

where there is a constant scalar solution $\phi = \phi_0 =$ constant, and the effective 4D cosmological constant can be shown to be $\Lambda_4 \propto \frac{1}{2}\Lambda_6 e^{\phi_0} - \frac{1}{4} B_0^2 e^{-\phi_0}$ [142]. Now, notice we can write the effective scalar potential from eq. (2.90) as $V(\phi) \equiv -\frac{1}{2} B_0^2 e^{-\phi} - \Lambda_6 e^{\phi}$ (where the B_0^2 picks up a factor of 2 by summing over MN). For there to be a constant scalar solution, the potential must be minimised at ϕ_0, i.e. $V'(\phi_0) \equiv -\frac{1}{2} B_0^2 e^{-\phi_0} + \Lambda_6 e^{\phi_0} = 0$. Notice that this is the exact cancellation needed for Λ_4 to vanish. By introducing a scalar which has a constant solution, the theory automatically chooses the value of the magnetic flux to counter the cosmological constant in the bulk, such that the effective 4D cosmological constant vanishes. This fixes the problem we have outlined above without introducing Supersymmetry (SUSY) on the brane. Instead SLED models use SUSY to improve the model in other ways.

First a short note on SUSY in relation to the vacuum energy (for a review on SUSY in general see [146]). The basic idea is that each boson has a corresponding SUSY partner that is a fermion of the same mass, and vice versa. As we have previously discussed, bosons and fermions contribute opposite signs to the value of the vacuum energy. Therefore the vacuum energy will automatically vanish if we live in a fully SUSY Universe [24]. Unfortunately this is not the case, so how does this help us? If any form of SUSY exists, it must exist above a certain scale due to the lack of experimental evidence for superpartners below this scale [147]. In accordance with experimental observations, [143] places this lower bound of SUSY breaking at the level of the electroweak scale, M_w. This sets the coefficient of cosmological constant in the bulk to be $\Lambda_6 \lesssim M_w^4$, as the cancellation between boson and fermion fields will stop at the SUSY breaking scale. Firstly, this allows one to introduce SUSY particles in the bulk, further generalising the theory. But without fully exploring these different particles (which is beyond the scope of this thesis, see [143–145] and references therein for details), introducing SUSY generates another interesting property. As shown in [133], the SUSY breaking scale can be linked to the compactification scale, which in turn will be linked to the value of Λ_4. This provides further constraints on the size of the extra dimensions, which in turn could leave experimental and observational traces [144, 145]. However, we find that these SLED models cannot return to a Minkowski spacetime when considering phase transitions [142, 148]. In fact, it was later shown that there is still a hidden fine-tuning condition in the form of the flux quantisation condition, which arises from the Maxwell sector in eq. (2.90) (see [149, 150] and references therein). Without this flux sector, the extra dimensions will either decompactify or will collapse; either way this destroys our self-tuning mechanism. To be clear, the flux sector can still stabilise our extra dimensions without fine-tuning, but without the fine-tuning a large cosmological constant will reappear on the brane. Despite its lack of success, the SLED proposal has some interesting properties. To reiterate, the idea is that the gravitational response of the 4D cosmological constant can be "pushed" into the bulk to recover a flat spacetime. Here, SUSY acts to introduce new particles in the bulk and restrict the size of the cosmological constant in the bulk, which in turn restricts the size of the compactifications.

To conclude this section we briefly point towards the future directions of self-tuning with branes and SUSY. First, to focus on the former, recent advancements come in the form of holographic self-tuning (see [151] and references therein). This alters the original 5D RS branes [131] and casts solutions to be AdS on the bulk, rather than flat. But it still invokes a self-tuning mechanism, whereby there is an interplay between the bulk and the brane. [152–154] verify the stability of this proposal and explore its dynamics further. In general there are still many open questions to this model as [151, 154] points out, which must be explored further. As for SUSY, most recently [155] has showcased a class of 4D supergravity models, which can be used to impose a scalar potential that can dynamically act to reduce the gravitational response of the vacuum energy. However, due to the complexity of this model there are several unaddressed issues[25] which demand further research.

[25] For example the theory introduces a dilaton that couples to ordinary matter, which could result in

2.7 Massive gravity

Massive gravity is a modification of GR that propagates with a massive spin-2 particle of mass m (see [156, 157] for a review). It started with the original Fierz-Pauli action in 1939 [158], which describes a linear free massive graviton. Interestingly, a 4D massive theory propagates with 5 Degrees of Freedom (DOF) instead of the usual 2 of GR. In the 1970's van Dam, Veltman, and Zakharov studied this linear theory coupled to a source [159, 160]. They discovered that in the limit where $m \to 0$ GR is not fully recovered, which is known as the *vDVZ discontinuity*. This arises due to the extra DOF present in a massive gravity theory. Even when a massless limit is taken, these extra DOF do not fully decouple from the source. The number of DOF instead become 2 from a massless graviton, 2 from a massless vector, and 1 from a massless scalar. Specifically, the scalar remains coupled to the source, leading to the discontinuity between the predictions of GR and massive gravity in the massless limit. Vainshtein later studied non-linearities within massive theories [161], utilising the *Vainshtein mechanism* in order to evade the vDVZ discontinuity (see [157, 162] and references therein). However, these non-linear massive gravity theories were still plagued by Boulware-Deser ghosts [163]. These theories propagate with 6 DOF, where the extra mode presents itself as an instability. To remove this instability one can tune the coefficients in the expansion of the metric perturbation, to write a massive gravity theory that is ghost free [164–167]. Since then there have been successful demonstrations of non-linear massive gravity theories that are absent of ghosts [168–170].

Degravitation [53, 54, 171, 172] is the idea of modifying GR to give it a "high pass filter" that does not allow higher energy (longer wavelength) sources to pass through, but leaves that of lower energy (shorter wavelength) untouched. In other words, the gravitational response of a large cosmological constant (with an effectively infinite wavelength) is screened, but GR

inconsistencies with gravitational tests.

(which involves shorter wavelength interactions) are kept intact.

To demonstrate this phenomenon we first consider the case of a massive vector, reiterating the arguments made in [157]. We start with the action

$$S = \int \mathrm{d}^D x \bigg\{ -\frac{1}{4} H_{\mu\nu} H^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu - m A_\mu \partial^\mu \phi - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + A_\mu J^\mu \bigg\}, \tag{2.91}$$

where $H_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, ϕ is a scalar, A_μ is a vector, J is a source, m is the mass of the vector and D are the dimensions of the model. Here, ϕ is a scalar that is introduced as part of a *Stückelberg trick*, to ensure no DOF are lost when $m \to 0$ [173–175][26]. The scalar equation of motion becomes

$$\Box \phi + m\, \partial_\mu A^\mu = 0, \tag{2.92}$$

which can then be plugged back into eq. (2.91) to eliminate ϕ, such that

$$S = \int \mathrm{d}^D x \bigg\{ -\frac{1}{4} H_{\mu\nu} \left(1 - \frac{m^2}{\Box}\right) H^{\mu\nu} + A_\mu J^\mu \bigg\}, \tag{2.93}$$

where we have used $H_{\mu\nu} \frac{1}{\Box} H^{\mu\nu} = -2 A_\mu A^\mu - 2 \partial_\mu A^\mu \frac{1}{\Box} \partial_\nu A^\nu$. The resulting equation of motion becomes

$$\left(1 - \frac{m^2}{\Box}\right) \partial_\mu H^{\mu\nu} = -J^\nu. \tag{2.94}$$

If we allow A_μ to be a massive photon we see that eq. (2.94) is equivalent to Maxwell's electromagnetism seen through a high pass filter, $\left(1 - \frac{m^2}{\Box}\right)^{-1}$. For higher momenta (shorter wavelength) sources the filter ~ 1, so ordinary electromagnetism is preserved. Whereas for low momenta (longer wavelength) sources the filter is small, weakening the response of J.

We can demonstrate a parallel phenomenon in linearised massive gravity in flat spacetime

[26] A similar "trick" can be used within massive gravity theories [176].

by taking the action [157]

$$S = \int \mathrm{d}^D x \Big\{ \mathcal{L}_{m=0} - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{2} m^2 H_{\mu\nu} H^{\mu\nu} - 2m^2 (h_{\mu\nu} \partial^\mu A^\nu - h \partial_\mu A^\mu) \\ + g_m h_{\mu\nu} T^{\mu\nu} \Big\}, \tag{2.95}$$

where $h_{\mu\nu}$ is the perturbation of the metric such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, g_m is the coupling strength to the source, and m now refers to the mass of the graviton. Similar to the role of ϕ in eq. (2.91), A_μ is a Stückelberg field. Finally, $\mathcal{L}_{m=0}$ corresponds to the Lagrangian of a linearised massless graviton given by

$$\mathcal{L}_{m=0} = \frac{1}{2} h_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} h_{\alpha\beta}, \tag{2.96}$$

where

$$\mathcal{E}^{\mu\nu}{}_{\alpha\beta} = \left(\eta_\alpha^{(\mu} \eta_\beta^{\nu)} - \eta^{\mu\nu} \eta_{\alpha\beta} \right) \Box - 2\partial^{(\mu} \partial_{(\alpha} \eta_{\beta)}^{\nu)} + \partial^\mu \partial^\nu \eta_{\alpha\beta} + \partial_\alpha \partial_\beta \eta^{\mu\nu}. \tag{2.97}$$

To further simplify we add an auxiliary field, $M = \frac{1}{2} h + \partial_\mu A^\mu$, so that the action becomes

$$S = \int \mathrm{d}^D x \Big\{ \mathcal{L}_{m=0} + m^2 \Big[-\frac{1}{2} h_{\mu\nu} h^{\mu\nu} + \frac{1}{4} h^2 + A_\mu \Box A^\mu + M(h - M) \\ - A^\mu (\partial_\mu h - 2\partial^\nu h_{\mu\nu} + 2\partial_\mu M) \Big] + g_m h_{\mu\nu} T^{\mu\nu} \Big\}. \tag{2.98}$$

We can then eliminate the vector field A_μ using the equation of motion

$$A_\mu = \frac{1}{\Box} \left(\frac{1}{2} \partial_\mu h - \partial^\nu h_{\mu\nu} + \partial_\mu M \right). \tag{2.99}$$

This elimination yields

$$S = \int \mathrm{d}^D x \Big\{ \frac{1}{2} h_{\mu\nu} \left(1 - \frac{m^2}{\Box} \right) \mathcal{E}^{\mu\nu\alpha\beta} h_{\alpha\beta} - 2M \frac{1}{\Box} (\partial_\mu \partial_\nu h^{\mu\nu} - \Box h) + g_m h_{\mu\nu} T^{\mu\nu} \Big\}. \tag{2.100}$$

Notice the incredibly similar structure between this and eq. (2.93). Therefore, computing the field equations of eq. (2.100) will yield a similar structure to eq. (2.94), such that the massive graviton acts as a filter to the source.

Although degravitation can be shown to work at a linear level, an exploration into a non-linear extension has so far been unfruitful (see [177] and references therein). Despite this, a study into its cosmology [178] has shown that massive gravity can produce an accelerated expansion [179, 180]. It also seems that a related theory, bi-metric gravity [181], can also reproduce phenomenologically accurate cosmologies [182]. Finally, there remains a small possibility that the non-linear form of another related theory, partially massless gravity, could provide a solution to the cosmological constant problem [183]. However, constructing such a theory has thus far proved unsuccessful [183–187].

Chapter 3

Obstructions to self-tuning

In this chapter we review the work from F. Niedermann and A. Padilla in [188], which explored obstructions to self-tuning and possible ways around these. The reason for this review is due to the fact that our work comes as a direct reaction to the results of this paper. Much like Fab Four (section 2.2) and Weinberg's no-go theorem [51], they construct a generalised model and place self-tuning conditions upon it to examine what types of theories can and cannot self-tune. To explore this, they first construct an exchange amplitude, $\mathcal{A}$, which represents the interaction between a "probe" source, $\tau'_{\mu\nu}$, and a given field. For example, to see how a scalar, ϕ, interacts linearly with a source we can examine the object $\tau'\phi$. Then, we can understand its gravitational effect by coupling it to the metric and examining the resultant integral (similar to the form of an action):

$$\mathcal{A}_0 = \int \mathrm{d}^4x\sqrt{-\bar{g}}\,\tau'\phi, \tag{3.1}$$

where $\mathcal{A}_0$ represents the exchange amplitude for a single massive spin-0 state, $\tau' = \tau'_{\mu\nu}\bar{g}^{\nu\nu}$, and $\bar{g}_{\mu\nu}$ is the background metric where $\bar{g} = \det(\bar{g}_{\mu\nu})$. Note, the use of the background metric is due to the fact that we are exploring linearised interactions. These amplitudes are used here to represent the strength of a given interaction. From these amplitudes, one can study how they behave under certain limits. This method is simply a different way of studying the gravitational interaction of a given theory [27]. It offers a different perspective

[27] Alongside this it offers a way to consider a range of fields with different masses and coupling strengths, using the Källén-Lehmann (KL) spectral representation. We will discuss this in more detail later in the

from the "usual method" of solving the field equations from an explicit action, to provide a new insight into a given model's interaction with gravity. Then from this linear study of the exchange amplitudes, one can construct an explicit non-linear action to study the model's field equations and cosmology based on some desired characteristics. Specifically with self-tuning, [188] focuses on what conditions require the exchange amplitude to vanish when interacting with a vacuum energy source in a long wavelength limit (where a constant source has an infinite wavelength). This corresponds to the gravitational effect from a vacuum source vanishing, which is the desired result for self-tuning to be realised. Alongside this, they also demand a close agreement with the exchange amplitude of GR for short wavelength sources. This ensures that the theory to be constructed from this analysis can fit within observational bounds. Note that the exchange amplitude for GR can be found by computing the interaction between a probe source and a massless spin-2 particle.

To write eq. (3.1) into a more functional form, we can think of this scalar as being generated by a separate source, τ. Therefore, we can rewrite eq. (3.1) by considering the inhomogeneous KG equation for a scalar interacting with a separate source via $(-\Box+m^2)\phi = \tau$, where m^2 is effective mass squared of the scalar and $\tau = \tau_{\mu\nu}\bar{g}^{\mu\nu}$. Using this expression we can now write $\phi = (-\Box + m^2)^{-1}\,\tau$ and substitute this into eq. (3.1) to obtain

$$\mathcal{A}_0 = -\int \mathrm{d}^4x\sqrt{-\bar{g}}\,\tau'\frac{1}{\Box - m^2}\tau.$$

This now represents a source $\tau'_{\mu\nu}$ interacting with another source $\tau_{\mu\nu}$ via a propagator. In this case the propagator takes the explicit form $(-\Box + m^2)^{-1}$, which is that of a massive scalar field.

Furthermore, we want an exchange amplitude that is mediated by both single and multi-particle states using a Källén-Lehmann (KL) spectral representation [189, 190] to ensure that

chapter.

we fully capture all possible configurations. This idea of a KL spectral representation is to create an expression for the sum of all the possible propagators (which will differ based on the value of the mass term) for a given spin state. Returning to our scalar field example, we want a way to write the sum of all propagators for spin-0 (from a lower mass limit, s_0, all the way up to ∞) such that

$$\Delta = -\int_{s_0}^{\infty} \mathrm{d}s \rho_0(s) \frac{1}{\Box - s},$$

where Δ represents the sum of the propagators, $\rho_0(s)$ is the "spectral density function" for a massive scalar, and s refers to the effective mass squared. The key point here is that the above integral effectively creates a sum of all possible massive spin-0 configurations. Then the spectral density function, $\rho_0(s)$, simply describes the coupling of that propagator at a given mass s. This construction allows us to represent the full range of propagators when we build our exchange amplitude.

In [188] they also consider the exchange amplitude for all states up to spin-2. Therefore, we can write the entire exchange amplitude, assuming linear couplings to the source, by summing the massless and massive components of each given spin state such that

$$\mathcal{A} = \bar{\rho}_2 \bar{\mathcal{A}}_2 + \bar{\rho}_0 \bar{\mathcal{A}}_0 + \int_{s_2}^{\infty} \mathrm{d}s \, \rho_2(s) \mathcal{A}_2 \,(s) + \int_{s_0}^{\infty} \mathrm{d}s \, \rho_0(s) \, \mathcal{A}_0(s). \tag{3.2}$$

The model includes the exchange amplitude of: a single massless spin-2 state, $\bar{\mathcal{A}}_2$, with coupling, $\bar{\rho}_2$; a single massless spin-0 state, $\bar{\mathcal{A}}_0$, with coupling, $\bar{\rho}_0$; massive spin-2 states, $\mathcal{A}_2$, of mass s with spectral density, $\rho_2(s)$; and massive spin-0 states, $\mathcal{A}_0$, of mass s with spectral density, $\rho_0(s)$. Notice, we have not included any spin-1 states since we only consider linear couplings to conserved sources. A linear spin-1 state can only contribute via a derivative, therefore if we assume conservation of the source, $\bar{\nabla}_\mu \tau^{\mu\nu} = 0$, a spin-1 term will never contribute as we can consistently remove it by integrating by parts. Finally, s_2 and s_0 are the lower limits of the mass of the spin-2 and spin-0 states respectively. Note, in order to evade Weinberg they also allow for breaks in translational invariance and non-local kinetic operators. It is further assumed that free field propagators are in a canonical form such that they obey unitarity and Lorentz invariance, and the spectral densities are positive to preserve unitarity (i.e. $\bar{\rho}_2, \bar{\rho}_0, \rho_2, \rho_0 \geq 0$).

We can keep the couplings and spectral densities entirely arbitrary for now, as these objects only infer the strength of the given state. But we must define the exchange amplitude for each state. First, we reiterate our definition for the exchange amplitude of a massive scalar, $\mathcal{A}_0$, and state that of a massless scalar, $\bar{\mathcal{A}}_0$, by taking $s = 0$ to obtain

$$\bar{\mathcal{A}}_0 = -\int \mathrm{d}^4x\sqrt{-\bar{g}}\,\tau' \frac{1}{\bar{\Box}}\tau, \tag{3.3}$$

$$\mathcal{A}_0 = -\int \mathrm{d}^4x\sqrt{-\bar{g}}\,\tau' \frac{1}{\bar{\Box} - s}\tau, \tag{3.4}$$

where $\bar{\Box} \equiv \bar{\nabla}_\mu \bar{\nabla}^\mu$, which refers to the derivatives when computed on the background since we are dealing with *linearised* sources. For spin-2 we start by examining the object

$$\mathcal{A} = \int \mathrm{d}^4x\sqrt{-\bar{g}}\,\tau'^{\mu\nu} h_{\mu\nu}, \tag{3.5}$$

where $h_{\mu\nu}$ is the graviton fluctuation due to a separate source $\tau_{\mu\nu}$. Note, that eq. (3.5) is the most general way of writing the exchange amplitude for models up to spin-2. However, in order to write down $\mathcal{A}_2$ separately we must first write down an expression for the $h_{\mu\nu}$ that is generated by $\tau_{\mu\nu}$ for a massive spin-2 state. By considering this explicit expression on a curved background, [188] were able to write the exchange amplitude as two sources interacting via a massive spin-2 propagator. Unlike before, we cannot generically find the massless counterpart by setting the mass to 0 as this will give rise to the well-known vDVZ discontinuity [159,160] (as discussed in section 2.7). Although the vDVZ discontinuity can be evaded on a curved background, the same cannot be said for that of Minkowski. Therefore, we must reconstruct an expression for the graviton fluctuation generated by a source for a massless spin-2 state to find $\bar{\mathcal{A}}_2$. The full derivation is fairly involved, but can be found in the Appendix of [188], so we will instead state the form here[28]:

$$\bar{\mathcal{A}}_2 = -\int \mathrm{d}^4x\sqrt{-\bar{g}}\left\{\tau'^{\mu\nu}_{(TT)}\frac{1}{\bar{\Box}-2\kappa}\tau^{(TT)}_{\mu\nu} - \frac{1}{6}\tau'\frac{1}{\bar{\Box}+4\kappa}\tau\right\}, \tag{3.6}$$

$$\mathcal{A}_2 = -\int \mathrm{d}^4x\sqrt{-\bar{g}}\left\{\tau'^{\mu\nu}_{(TT)}\frac{1}{\bar{\Box}-2\kappa-s}\tau^{(TT)}_{\mu\nu} - \frac{\kappa}{3(2\kappa-s)}\tau'\frac{1}{\bar{\Box}+4\kappa}\tau\right\}, \tag{3.7}$$

where κ is the spacetime curvature ($\kappa = 0$ is a Minkowski background, $\kappa > 0$ is a dS background, and $\kappa < 0$ is an AdS background) with

$$\tau^{(TT)}_{\mu\nu} = \tau_{\mu\nu} + \frac{1}{3}\left[\bar{\nabla}_\mu\bar{\nabla}_\nu - \bar{g}_{\mu\nu}(\bar{\Box}+3\kappa)\right]\left(\frac{1}{\bar{\Box}+4\kappa}\right)\tau. \tag{3.8}$$

Note κ, which refers to the intrinsic curvature of the system, should not be confused with k in eq. (1.7), which refers to the spatial curvature. Usually $3\kappa = (-\Lambda_{\text{eff}}, 0, \Lambda_{\text{eff}})$ for AdS, Minkowski, and dS solutions respectively without a self-tuning mechanism. However with an apt mechanism, we can force 3κ (the intrinsic spacetime curvature) to be small (or 0 if we want to achieve a Minkowski solution for $\Lambda_{\text{eff}} \neq 0$) for an arbitrarily large Λ_{eff}. This ensures the gravitational effect of Λ_{eff}, which would usually produce a large intrinsic curvature, can be cancelled off. In other words, the value of the effective cosmological constant will not affect the value of the spacetime curvature. Using eq. (3.8) they also derive the spin-2 amplitudes for *localised* sources:

$$\bar{\mathcal{A}}_2 = -\int \mathrm{d}^4x\sqrt{-\bar{g}}\left\{\tau'^{\mu\nu}\frac{1}{\bar{\Box}-2\kappa}\tau_{\mu\nu} - \frac{1}{2}\tau'\left[\frac{1/2}{\bar{\Box}-2\kappa} + \frac{1/2}{\bar{\Box}+6\kappa}\right]\tau\right\}, \tag{3.9}$$

$$\begin{aligned}\mathcal{A}_2 = -\int \mathrm{d}^4x\sqrt{-\bar{g}}\Bigg\{&\tau'^{\mu\nu}\frac{1}{\bar{\Box}-2\kappa-s}\tau_{\mu\nu} \\ &- \frac{1}{2}\tau'\left[\frac{1/2}{\bar{\Box}-2\kappa-s} + \frac{\kappa-s/6}{\kappa-s/2}\frac{1/2}{\bar{\Box}+6\kappa-s}\right]\tau\Bigg\}.\end{aligned} \tag{3.10}$$

[28] In section 4.2 we calculate the exchange amplitude of a theory which involves a single massive graviton (alongside a single massive scalar) on a curved background. So a slightly altered derivation is contained within this thesis.

Conservation of localised sources ensures that it vanishes in response to $\bar{\nabla}_\mu \bar{\nabla}_\nu$ (contained within our definition for eq. (3.8)). However, since vacuum sources are constant they do not generically vanish in response to $\bar{\nabla}_\mu \bar{\nabla}_\nu$ due to the presence of non-local operators. Therefore we cannot generically use eqs. (3.9) and (3.10), instead we must use the general expressions when considering vacuum interactions (eqs. (3.6) and (3.7)). Note, that eqs. (3.3) and (3.4) do not contain any $\bar{\nabla}_\mu \bar{\nabla}_\nu$ terms, so these expressions can be used for both localised and vacuum sources.

Now that we have constructed our general exchange amplitude we place two self-tuning conditions upon it, in order to constrain the parameter space of the theory. Self-tuning here requires the gravitational response of an infinite wavelength source to be screened, whilst leaving short distance gravity sources mostly untouched. In other words our conditions are defined such that:

(1) the exchange amplitude, $\mathcal{A}$, should vanish in response to a vacuum energy source, which corresponds to a low energy (long wavelength) limit,

(2) the exchange amplitude, $\mathcal{A}$, should closely agree with that of GR (due to its experimental success) in response to localised sources, which corresponds to a high energy (short wavelength) limit.

We start by considering this theory on a Minkowski background ($\kappa = 0$). In [188] they take very general lower mass limits such that $s_2 = 0^+ = s_0$ to ensure the positivity of each effective mass squared (where the above refers to $s_2, s_0 > 0$, but we have written it as above for notational convenience). We first examine condition (1) by allowing the vacuum energy source, $\tau_{\mu\nu} = -\Lambda_{\text{vac}} \eta_{\mu\nu}$, to interact with a localised probe, $\tau'_{\mu\nu}$. Substituting this into eq. (3.2) with our exchange amplitudes for *generalised* sources (eqs. (3.3), (3.4), (3.6), and (3.7)) yields the "vacuum" constraint

$$-\frac{2}{3}\bar{\rho}_2\frac{1}{\bar{\Box}}1 + 4\bar{\rho}_0\frac{1}{\bar{\Box}}1 - 4\int_{0^+}^{\infty} \mathrm{d}s\rho_0(s)\frac{1}{s}1 = 0, \tag{3.11}$$

where we have used $\frac{1}{\bar{\Box}-s}1 = -\frac{1}{s}$ (as $|\bar{\Box}| \ll |s|$) and the fact that the transverse-tracefree part of the energy-momentum tensor $\tau_{\mu\nu}^{(TT)}$ vanishes for a constant vacuum source. If we operate on the above (eq. (3.11)) with $\bar{\Box}$ we notice that the third term is much smaller than the first two terms since $|\bar{\Box}| \ll |s|$. Therefore, we obtain our first vacuum constraint for a Minkowski background: $\bar{\rho}_2 = 6\bar{\rho}_0$[29].

Turning our attention to condition (2), shorter wavelength sources must closely match the exchange amplitude of GR such that

$$\mathcal{A} \to -\frac{1}{M_{\mathrm{Pl}}^2}\int \mathrm{d}^4x\sqrt{-\bar{g}}\left[\tau'^{\mu\nu}\frac{1}{\bar{\Box}}\tau_{\mu\nu} - \frac{1}{2}(1-\epsilon)\tau'\frac{1}{\bar{\Box}}\tau\right], \tag{3.12}$$

as $\bar{\Box} \to -\infty$. The exchange amplitude here is constructed by considering a massless spin-2 state on a generic background with a coupling constant $\bar{\rho}_2 = \frac{1}{M_{\mathrm{Pl}}^2}$. Here, we can set $\kappa = 0$ as the kinetic sources will dominate over κ for a high energy (short wavelength) limit, $|\bar{\Box}| \gg |\kappa|$. Finally, ϵ represents a small deviation from GR in accordance with solar system tests of gravity where $|\epsilon| \lesssim 10^{-5}$ [3]. The ϵ can be written coupled to the tensor parts, but instead we have chosen to normalise it to the scalar parts. It is used to ensure that we can encode a deviation from GR into the above description, whilst still residing within observational bounds. To satisfy condition (2) we demand that the general exchange amplitude (eq. (3.2)) agrees with that of eq. (3.12) (hence it closely agrees with GR) for *localised* sources (using the "*localised*" amplitudes eqs. (3.3), (3.4), (3.9), and (3.10)). From this we can write two more constraints by separating the tensor ($\propto \tau^{\mu\nu}$) and the scalar ($\propto \tau$) components, and

[29] Note that this also implies that the third term must vanish separately, which is a property that we will exploit when $\kappa \neq 0$.

expressing them in momentum space such that

$$\frac{\bar{\rho}_2}{x} + \int_{0^+}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{x+s} \to \frac{1}{M_{\rm Pl}^2}\frac{1}{x}, \tag{3.13}$$

$$-\frac{1}{2}\frac{\bar{\rho}_2}{x} - \frac{1}{3}\int_{0^+}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{x+s} + \frac{\bar{\rho}_0}{x} + \int_{0^+}^{\infty} \mathrm{d}s \frac{\rho_0(s)}{x+s} \to -\frac{1}{2}\left(1-\epsilon\right)\frac{1}{M_{\rm Pl}^2}\frac{1}{x}. \tag{3.14}$$

where $\bar{\Box} \propto x \equiv p_\mu p^\mu \to \infty$. We can also modify the "scalar" constraint (eq. (3.14)) by removing the $\bar{\rho}_2$ term using the "tensor" constraint (eq. (3.13)):

$$\frac{1}{3}\int_{0^+}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{x+s} + 2\frac{\bar{\rho}_0}{x} + 2\int_{0^+}^{\infty} \mathrm{d}s \frac{\rho_0(s)}{x+s} \to \frac{\epsilon}{M_{\rm Pl}^2}\frac{1}{x}, \tag{3.15}$$

which we will now refer to as the scalar constraint. We can immediately recognise that the size of each term in the scalar constraint (eq. (3.15)) is limited due to the positivity of x and the spectral densities. In other words, $x\int_{0^+}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{x+s}$, $\bar{\rho}_0$, and $x\int_{0^+}^{\infty} \mathrm{d}s \frac{\rho_0(s)}{x+s} \lesssim \frac{|\epsilon|}{M_{\rm Pl}^2}$. Since $x\int_{0^+}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{x+s}$ is small, the tensor constraint (eq. (3.13)) implies that $\bar{\rho}_2 \sim \frac{1}{M_{\rm Pl}^2}$. Notice there is a contradiction: the scalar constraint implies that $\bar{\rho}_0 \lesssim \frac{|\epsilon|}{M_{\rm Pl}^2}$, whereas the tensor constraint implies that $\bar{\rho}_2 \sim \frac{1}{M_{\rm Pl}^2}$. This directly contradicts the vacuum constraint: $\bar{\rho}_2 = 6\bar{\rho}_0$. Therefore self-tuning, as we have defined it, is not allowed on a Minkowski background.

We can repeat the same process on a curved background ($\kappa \neq 0$). Through condition (1) we, once again, demand that the exchange amplitude (eq. (3.2)) vanishes for $\tau = -\Lambda_{\rm vac}\bar{g}_{\mu\nu}$:

$$\frac{\bar{\rho}_2}{6\kappa} - 4\bar{\rho}_0\frac{1}{\bar{\Box}}1 - \frac{1}{3}\int_{s_2}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{s-2\kappa} + 4\int_{s_0}^{\infty} \mathrm{d}s \frac{\rho_0(s)}{s} = 0. \tag{3.16}$$

This is an analogous expression to the vacuum constraint on a Minkowski background (eq. (3.11)), where this time we have used eqs. (3.3), (3.4), (3.6), and (3.7) with $\kappa \neq 0$. Operating on the above (eq. (3.16)) with $\bar{\Box}$ renders the first, third, and fourth terms as small since $|\bar{\Box}| \ll |s|, |\kappa|$. Therefore the liner combination of the three terms will vanish according to eq. (3.16). This also implies that the second term must vanish separately, such that $\bar{\rho}_0 = 0$.

This is our vacuum constraint.

To satisfy condition (2) we, once again, split the tensor and scalar parts for our exchange amplitude eq. (3.2) (using the "localised" amplitudes eqs. (3.3), (3.4), (3.9), and (3.10) with $\kappa \neq 0$) and compare with that of eq. (3.12) to obtain

$$\frac{\bar{\rho}_2}{x} + \int_{s_2}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{x+s} \to \frac{1}{M_{\mathrm{Pl}}^2} \frac{1}{x}, \tag{3.17}$$

$$-\frac{1}{2}\frac{\bar{\rho}_2}{x} - \frac{1}{3}\int_{0^+}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{x+s}\left[1 - \frac{\kappa}{s-2\kappa}\right] + \frac{\bar{\rho}_0}{x} + \int_{0^+}^{\infty} \mathrm{d}s \frac{\rho_0(s)}{x+s} \to -\frac{1}{2}(1-\epsilon)\frac{1}{M_{\mathrm{Pl}}^2}\frac{1}{x}, \tag{3.18}$$

where we have used the fact that $|x| \gg |\kappa|$ for the high energy (short wavelength) limit. Similar to deriving eq. (3.15), we can remove $\bar{\rho}_2$ from the "scalar" constraint (eq. (3.18)) using the "tensor" constraint (eq. (3.17)) to obtain our new scalar constraint:

$$\frac{1}{3}\int_{s_2}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{x+s} + \frac{2\kappa}{3}\int_{s_2}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{x+s}\frac{1}{s-2\kappa} + 2\frac{\bar{\rho}_0}{x} + 2\int_{s_0}^{\infty} \mathrm{d}s \frac{\rho_0(s)}{x+s} \to \frac{\epsilon}{M_{\mathrm{Pl}}^2}\frac{1}{x}. \tag{3.19}$$

Note, the lower limit in dS space for s_2 is set by perturbative unitarity on spin-2 states such that $s > 2\kappa$ [183, 184, 191–193], which is often called the "Higuchi bound". However, for AdS space we must take the lower limit as $s_2 > 0$ to avoid helicity-1 ghosts [194]. Also in [188] they set a very general lower mass limit for the scalar, $s_0 > 0$, in dS space ensuring that the effective mass squared of the scalar is positive. However, for AdS space they use the Breitenlohner-Freedman bound [195] such that $s_0 = 4\kappa$.

First, we specifically focus on the implications of these constraints for a dS background ($\kappa > 0$). Once again, the positivity of the spectral densities requires each term in the scalar constraint (eq. (3.19)) to sum to be small, such that $x\int_{s_2}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{x+s} \lesssim \frac{|\epsilon|}{M_{\mathrm{Pl}}^2}$. This then sets $\bar{\rho}_2 \sim \frac{1}{M_{\mathrm{Pl}}^2}$ from the tensor constraint (eq. (3.17)). We then narrow our focus to the second term in scalar constraint to obtain

$$\begin{aligned}\frac{|\epsilon|}{M_{\rm Pl}^2} &\gtrsim x\kappa \int_{2\kappa^+}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{s-2\kappa}\frac{1}{x+s}, \\ &\gtrsim \kappa \int_{2\kappa^+}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{s-2\kappa}\frac{1}{1+s/x}.\end{aligned} \tag{3.20}$$

Then, we use the fact that $x \to \infty$ in the high energy (short wavelength) limit such that

$$\frac{|\epsilon|}{M_{\rm Pl}^2} \gtrsim \int_{2\kappa^+}^{s_*} \mathrm{d}s \frac{\rho_2(s)}{s-2\kappa} > 0, \tag{3.21}$$

where s_* is some arbitrarily large but finite constant. Note, there is some loss of generality now that we have assumed that the masses for spin-2 states do not extend to ∞, rather the ceiling is s_*. Nevertheless, we can still proceed with the knowledge that we can capture spin-2 states with large masses in our self-tuning analysis. Now, we substitute $\bar{\rho}_2 \sim \frac{1}{M_{\rm Pl}^2}$ and $\bar{\rho}_0 = 0$ into the vacuum constraint (eq. (3.16)), alongside the fact that $\int_{2\kappa^+}^{s_*} \mathrm{d}s \frac{\rho_2(s)}{s-2\kappa}$ is small to obtain

$$24\kappa M_{\rm Pl}^2 \int_{s_0}^{\infty} \mathrm{d}s \frac{\rho_0(s)}{s} \sim -1 + \mathcal{O}(\epsilon), \tag{3.22}$$

which directly contradicts the positivity of ρ_0. This suggests self-tuning is not allowed in dS space either. Note, that these findings appear to rule out Fab Four (constructed in Minkowski space) and well-tempering (constructed in dS space) as viable self-tuning models. However to reiterate, the above self-tuning conditions only apply to *linearised* theories. Indeed, the non-linearities within Fab Four and well-tempering ensure that both models can self-tune.

Turning our attention to an AdS background, we find that there exists an "AdS loophole" that can satisfy all three constraints (eqs. (3.16), (3.17), and (3.19)) simultaneously. Now that $\kappa < 0$, the $\frac{2\kappa}{3}\int_{s_2}^{\infty} \mathrm{d}s \frac{\rho_2(s)}{s-2\kappa}$ term in the scalar constraint (eq. (3.19)) must be negative. Consequently, in contrast to Minkowski and dS backgrounds, each term in the scalar constraint (eq. (3.19)) does not necessarily have to be small in order to sum to $\frac{|\epsilon|}{M_{\rm Pl}^2}\frac{1}{x}$. Instead, there are theoretically a great number of loopholes we can exploit, so we demonstrate that the constraints are satisfied through an explicit example. In [188], they achieved this by writing a choice of spectral densities that described a theory with a massless graviton, a single massive graviton, and a single massive scalar. Instead, we consider a choice of a spectral densities that describes a single massive graviton alongside a single massive scalar:

$$\bar{\rho}_2 = 0, \quad \bar{\rho}_0 = 0, \quad \rho_2(s) = \frac{1}{M_{\rm Pl}^2}\delta(s - m_g^2), \quad \rho_0(s) = \frac{1}{M_{\rm Pl}^2}\frac{m_\phi^2}{12\Gamma}\delta(s - m_\phi^2), \tag{3.23}$$

where $\Gamma \equiv 2q^2 + m_g^2$ and we have made a change in variables such that $\kappa \to -q^2$ with $q^2 > 0$ for convenience. Furthermore, m_g^2 and m_ϕ^2 are the effective mass squared of the graviton and the scalar respectively, where both masses are finite. Substituting this explicit example into the vacuum constraint (eq. (3.16)) yields

$$\frac{1}{M_{\rm Pl}^2}\left[-\frac{1}{3}\frac{1}{2q^2 + m_g^2} + \frac{4}{12}\frac{m_\phi^2}{m_\phi^2\Gamma}\right] = 0,$$

which is clearly automatically satisfied. Likewise, the tensor constraint (eq. (3.17)) is immediately satisfied when we recognise that $x \to \infty$. However, the scalar constraint (eq. (3.19)) yields

$$\frac{1}{M_{\rm Pl}^2}\left\{\left(\frac{1}{3} - \frac{2q^2}{3\Gamma}\right)\frac{1}{x + m_g^2} + 2\frac{m_\phi^2}{12\Gamma}\frac{1}{x + m_\phi^2}\right\} \to \frac{\epsilon}{M_{\rm Pl}^2}\frac{1}{x},$$

which produces the constraint $\epsilon = \frac{m_\phi^2 + 2m_g^2}{6\Gamma}$ when we take $x \to \infty$. This suggests that it is viable to construct a linear self-tuning theory on an AdS background, and it is through the specific example in eq. (3.23) that we write an explicit action.

Chapter 4

Generalised Fab Four in anti-de Sitter space

This chapter is based on the work published in [196], but before we consider the specific action we will briefly discuss the efficacy of working in AdS space. We start by mentioning the AdS/CFT correspondence (see [197] for a review), which posits the duality between quantum field theories and gravitational theories on an AdS background (first hypothesised by Maldacena in 1997 [198]). These principles are interesting, having been used in other areas of modern day physics (such as condensed matter [199]) and they could provide key insights into a complete picture of quantum gravity. However, despite its success this thesis does not focus on this aspect of AdS space, but instead approaches it with a purely gravitational lens. Observations seem to suggest that the Universe is undergoing an accelerated expansion (as discussed in chapter 1), which implies that the curvature of the Universe corresponds to a small dS space. In contrast to this, our theory seeks to create a Universe with a small AdS background regardless of the value of the vacuum energy. However, we still consider it a success provided that the value of the cosmological constant is heavily reduced. As discussed in chapter 2, even if $\Lambda_{\rm vac} \neq \Lambda_{\rm obs}$, as long as $\Lambda_{\rm vac}$ is on the same scale as $\Lambda_{\rm obs}$ this removes the need for repeated fine-tunings in $\Lambda_{\rm bare}$. Hence radiative instability is no longer a problem. After this, it is on the onus of a DE candidate to make up the difference. Even so, AdS spacetime makes for a great testing ground to explore the underlying mechanisms of a model. As highlighted in [188] it is not immediately possible to construct a linear self-tuning theory on a Minkowski or dS background[30]. AdS **is** allowed, serving as an exploration point that could point us towards possible loopholes for self-tuning in dS or Minkowski space (as we will later show).

Returning to chapter 3, our choice of spectral densities in eq. (3.23) corresponds to a single massive scalar field, alongside a single massive graviton. To fully capture the scalar component **and** the massive graviton we consider a generalisation of the Fab Four action (eq. (2.17) in section 2.2) on an AdS background [196]:

$$S = \int \mathrm{d}^4x\sqrt{-g}\,[\mathcal{L}_j + \mathcal{L}_p + \mathcal{L}_g + \mathcal{L}_r - \mathcal{K}(\phi)g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \mathcal{U}_{mg} - \Lambda_{\text{bare}}] + S_m[g_{\mu\nu}, \psi], \quad (4.1)$$

where

$$\mathcal{L}_j = V_j(\phi)(G^{\mu\nu} - 3q^2 g^{\mu\nu})\nabla_\nu\phi\nabla_\mu\phi, \quad (4.2)$$

$$\mathcal{L}_p = V_p(\phi)(P^{\mu\nu\alpha\beta} + 2q^2 g^{\mu[\alpha}g^{\beta]\nu})\nabla_\mu\phi\nabla_\alpha\phi\nabla_\mu\nabla_\beta\phi, \quad (4.3)$$

$$\mathcal{L}_g = V_g(\phi)(R + 12q^2), \quad (4.4)$$

$$\mathcal{L}_r = V_r(\phi)(G_{GB} - 24q^4), \quad (4.5)$$

where $g_{\mu\nu}$ is the Jordan frame metric. The usual Fab Four terms (eqs. (2.18) to (2.21)) have been modified to include an AdS curvature term, $-q^2$ (where $q^2 > 0$), by shifting $R_{\mu\nu\alpha\beta} \to R_{\mu\nu\alpha\beta} + q^2(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$. Here, we have extracted the latter term to ensure that the background curvature from R is negative. This choice ensures that each term vanishes on an AdS background solution. Recall, that we allow for an arbitrary Λ_{bare} term to ensure that there are no hidden fine-tunings between it and Λ_{vac}. We have also included two additional terms. The first is an extra kinetic scalar term coupled to the arbitrary function $\mathcal{K}(\phi)$, with the reasons for its inclusion becoming apparent later.

[30]Note the use of the word **linear**. Non-linearities within other scalar-tensor models (as in sections 2.2 to 2.4) ensure that self-tuning can still be achieved.

The second additional term is that of a de Rham-Gabadadze-Tolley (dRGT) massive gravity term that takes the form [168]

$$\mathcal{U}_{mg} = \frac{1}{4} m^2 V_g^2 \Big[\mathcal{U}(\Sigma)\Big]_{\bar{F}_E}^{F_E}, \tag{4.6}$$

where we use $\Big[\mathcal{U}_{mg}(\Sigma)\Big]_{\bar{F}_E}^{F_E} = \mathcal{U}_{mg}(F_E) - \mathcal{U}_{mg}(\bar{F}_E)$ as shorthand. The explicit arguments take the form $(F_E^2)^\mu_\nu = \tilde{g}^{\mu\alpha}\bar{g}_{\nu\alpha} = \frac{1}{V_g} g^{\mu\alpha}\bar{g}_{\nu\alpha}$ and $(\bar{F}_E^2)^\mu_\nu = \frac{1}{\bar{V}_g}\delta^\mu_\nu$ through the Einstein frame metric, $\tilde{g}_{\mu\nu} = V_g g_{\mu\nu}$, where the background AdS metric is $\bar{g}_{\mu\nu}$ and $\bar{V}_g$ is the George potential evaluated on the background (when $\phi = \bar{\phi}$). The explicit form of the $\mathcal{U}(\Sigma)$ is given by

$$\mathcal{U}(\Sigma) = -4\big(12 - 6\,\Sigma^\alpha_\alpha + \Sigma^\alpha_\alpha \Sigma^\beta_\beta - \Sigma^\alpha_\beta \Sigma^\beta_\alpha\big). \tag{4.7}$$

Here, we will briefly describe the derivation for our form of the potential. In [168], the form of the metric is given by $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\alpha\beta}\partial_\mu\Phi^\alpha\partial_\nu\Phi^\beta + h'_{\mu\nu}$, where $h_{\mu\nu}$ is the metric perturbation, $h'_{\mu\nu}$ is the "covariantised" metric perturbation, $\eta_{\mu\nu}$ is the Minkowski background metric, and Φ are the Stückelberg fields. This simply refers to the metric perturbation when Stückelberg fields are added to the theory. We set the Stückelberg fields to vanish so $h'_{\mu\nu} \to h_{\mu\nu}$ and we set $\eta_{\mu\nu} \to \bar{g}_{\mu\nu}$ to cast the background into AdS space. Now, the metric takes the form $\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. The explicit form of $\mathcal{U}$ (eq. (4.7)) in [168] is given by

$$\mathcal{U}(\tilde{g}, h) = -4\big(H^\mu_\mu H^\nu_\nu - H^\mu_\nu H^\nu_\mu\big), \tag{4.8}$$

where $H^\mu_\nu \equiv \delta^\mu_\nu - \sqrt{\delta^\mu_\nu - h^\mu_\nu}$. Through a substitution of $h^\mu_\nu = \delta^\mu_\nu - \tilde{g}^{\mu\alpha}\bar{g}_{\nu\alpha}$, we can write $H^\mu_\nu = \delta^\mu_\nu - \sqrt{\tilde{g}^{\mu\alpha}\bar{g}_{\nu\alpha}}$. Plugging this into eq. (4.8) yields

$$\mathcal{U}(\tilde{g}, h) = -4\Big(12 - 6\sqrt{\tilde{g}_{\mu\nu}\bar{g}^{\mu\nu}} + \sqrt{\tilde{g}_{\mu\nu}\bar{g}^{\mu\nu}\tilde{g}_{\alpha\beta}\bar{g}^{\alpha\beta}} - \sqrt{\tilde{g}_{\nu\alpha}\bar{g}^{\mu\alpha}\tilde{g}_{\mu\beta}\bar{g}^{\nu\beta}}\Big). \tag{4.9}$$

Since this only contains terms that are summed over we can simplify our notation such that

$(F_E^2)^\mu_\nu = \tilde{g}^{\mu\alpha}\bar{g}_{\nu\alpha}$, so eq. (4.9) becomes eq. (4.7) if we identify Σ with F_E. Finally, the explicit form of U_{mg} is given in [168] as

$$\mathcal{U}_{mg} = -\frac{M_{\rm Pl}^2}{2}\frac{m^2}{4}\mathcal{U}(\tilde{g}, h). \tag{4.10}$$

Recall, that $\tilde{g}_{\mu\nu}$ is the Einstein frame metric, which can be cast into the Jordan frame using $\tilde{g}_{\mu\nu} = V_g g_{\mu\nu}$. Then, we can identify $\frac{M_{\rm Pl}^2}{2}$ with V_g by comparing the Lagrangian in [168] with eq. (4.1). Lastly, in a similar fashion to our reconstruction of the Fab Four terms (eqs. (4.2) to (4.5)), we demand that $\mathcal{U}_{mg}$ vanishes on the background yielding eq. (4.6). Notice $\mathcal{U}_{mg}$ contains explicit forms of the background metric, hence breaks diffeomorphism invariance. Usually this can be restored with the addition of Stückelberg fields [156], which we have set to vanish. However, to simplify our analysis we proceed with this break in diffeomorphism invariance, knowing that it could be restored in an extended analysis.

To achieve self-tuning we explore whether the field equations at the level of the background can accommodate for a cancellation between $\Lambda_{\rm bare}$ and $\Lambda_{\rm vac}$, whilst the linearised perturbations remain free of instabilities and closely align with GR. Unlike the original Fab Four (section 2.2), we evade Weinberg's no-go theorem by breaking translational invariance on the level of the metric (as we are no longer in Minkowski spacetime) rather than the scalars. This leaves us free to find a background solution that can accommodate for constant scalars, which also allows us to consistently remove the John and Paul terms ($\mathcal{L}_j$ and $\mathcal{L}_p$), as they will not contribute to our analysis. Therefore the addition of the $\mathcal{K}(\phi)$ term within our action is to provide the model with a kinetic dependence for ϕ. This will ensure that the scalar field can be dynamical; without it we run the risk of a static scalar field. If ϕ is static then it will not be able to evolve towards its background solution when perturbed. Instead, the scalar field will remain at its perturbed position which could spoil our self-tuning vacuum solution (as shown in section 4.1). Also notice that the addition of the $\mathcal{K}(\phi)$ term will not alter our vacuum solution (section 4.1) but it will contribute to the linearised perturbation (section 4.2).

These truncations reduce the action to

$$S = \int d^4x\sqrt{-g}\Bigg[\mathcal{G}(\phi)^2(R+12q^2) + V_r(\phi)(G_{GB} - 24q^4) \\ - \mathcal{K}(\phi)g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{m^2}{4}\mathcal{G}(\phi)^4\Big[\mathcal{U}(\Sigma)\Big]^{F_E}_{\frac{1}{\mathcal{G}}\mathbb{1}} - \Lambda_{\text{bare}}\Bigg] + S_m[g_{\mu\nu},\Psi], \tag{4.11}$$

where we have written the George potential as $V_g(\phi) = \mathcal{G}(\phi)^2$ to ensure that V_g remains positive irrespective of the value of ϕ and the form of $\mathcal{G}$. As per our discussion of Brans-Dicke in section 2.2, V_g acts as an effective gravitational "constant", therefore we require it be positive. Now we calculate Einstein's field equation ($\mathcal{E}^\mu_\nu = 0$) and the scalar field equation ($\mathcal{E}_\phi = 0$) which are varied with respect to the metric and the scalar respectively such that

$$\mathcal{E}^\mu_\nu = 2\big(\nabla^\mu\nabla_\nu - \delta^\mu_\nu\Box - G^\mu_\nu\big)\mathcal{G}^2 + 12q^2\mathcal{G}^2\delta^\mu_\nu + 2\mathcal{K}\Big\{\nabla^\mu\phi\nabla_\nu\phi - \frac{1}{2}\big(\nabla\phi\big)^2\delta^\mu_\nu\Big\} - \delta^\mu_\nu\Lambda_{\text{bare}} \\ + \frac{m^2}{4}\Big\{\mathcal{W}^\mu_\nu - \delta^\mu_\nu\Big[\mathcal{U}(\Sigma)\Big]^{F_E}_{\frac{1}{\mathcal{G}}\mathbb{1}}\Big\}\mathcal{G}^4 + \big(8P^{\mu\alpha}{}_{\nu\beta}\nabla_\alpha\nabla^\beta - 24q^4\delta^\mu_\nu\big)V_r + T^\mu_\nu, \tag{4.12}$$

$$\mathcal{E}_\phi = 2\mathcal{G}\mathcal{G}'\big(R+12q^2\big) + \mathcal{K}'\big(\nabla\phi\big)^2 + 2\mathcal{K}\Box\phi - \frac{m^2}{4}\mathcal{G}^3\mathcal{G}'\Bigg[4\mathcal{U}(\Sigma) - \text{Tr}\Big(\Sigma\frac{\partial\mathcal{U}}{\partial\Sigma}\Big)\Bigg]^{F_E}_{\frac{1}{\mathcal{G}}\mathbb{1}} \\ + V_r'\big[G_{GB} - 24q^4\big], \tag{4.13}$$

where $\mathcal{W}$ is given by

$$\mathcal{W}^\mu_\nu = \bigg(\frac{\partial\mathcal{U}}{\partial F_E}\bigg)^\alpha_\nu (F_E)^\mu_\alpha = -4\big(-6(F_E)^\mu_\nu + 2(F_E)^\alpha_\alpha(F_E)^\mu_\nu - 2(F_E)^\mu_\alpha(F_E)^\alpha_\nu\big), \tag{4.14}$$

$T_{\mu\nu} = -\frac{2}{\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta\bar{g}^{\mu\nu}}$ is the energy-momentum tensor for matter minimally coupled to the metric, and primes denote a derivative with respect to ϕ. Finally, recall that the form of both $P^{\mu\alpha}{}_{\nu\beta}$ and G_{GB} are given in section 2.2.

Here, we will address the non-trivial parts of these calculations. We first focus on

Einstein's field equation (eq. (4.12)), where the first and third terms are given by Brans-Dicke gravity [62], and the Ringo terms (coupled to V_r) are given by Fab Four [80]. Turning our attention to the massive terms, it is not immediately obvious where the $\mathcal{W}^{\mu}_{\nu}$ term arises. Varying $\mathcal{U}(F_E)$ yields

$$\left(\frac{\partial \mathcal{U}(F_E)}{\partial F_E}\delta_g F_E\right)^{\alpha}_{\alpha} \ , \tag{4.15}$$

where δ_g denotes a variation with respect to the metric. Recall that $(F_E^2)^{\mu}_{\nu} = \tilde{g}^{\mu\alpha}\bar{g}_{\nu\alpha}$, so we can write $(F_E)^{\mu}_{\alpha}(\delta_g F_E)^{\alpha}_{\nu} + (\delta_g F_E)^{\mu}_{\beta}(F_E)^{\beta}_{\nu} = \delta\tilde{g}^{\mu\alpha}\bar{g}_{\nu\alpha} = \frac{1}{\mathcal{G}^2}\delta g^{\mu\alpha}\bar{g}_{\nu\alpha}$, which reduces to $2(\delta_g F)^{\mu}_{\nu} = \delta\tilde{g}^{\beta\alpha}\bar{g}_{\nu\alpha}((F_E)^{-1})^{\mu}_{\beta}$ if we assume that $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ only contain diagonal components. Now we can write

$$\left(\frac{\partial \mathcal{U}(F_E)}{\partial F_E}\delta_g F_E\right)^{\alpha}_{\alpha} = \frac{1}{2}\left(\frac{\partial \mathcal{U}(F_E)}{\partial F_E}(F_E)^{-1}\delta_g\tilde{g}^{-1}\bar{g}\right)^{\alpha}_{\alpha}. \tag{4.16}$$

Then, by raising the indices with the Einstein frame metric and once again recognising the definition of F_E we obtain $\mathcal{W}^{\mu}_{\nu}$ as in eq. (4.14).

Similarly with the scalar field equation (eq. (4.13)) the second and third terms are given by Brans-Dicke [62], whilst the Ringo terms are given by Fab Four [80]. The massive term contains $\left(\Sigma\frac{\partial\mathcal{U}}{\partial\Sigma}\right)^{\alpha}_{\alpha}$ which arises by varying $\mathcal{U}(\Sigma)$:

$$\delta_\phi\mathcal{U}(\Sigma) = \frac{\partial\mathcal{U}(\Sigma)}{\partial\phi}\delta\phi = \frac{\partial\mathcal{U}}{\partial\Sigma^{\mu}_{\nu}}\frac{\partial\Sigma^{\mu}_{\nu}}{\partial\phi}\delta\phi, \tag{4.17}$$

where δ_ϕ denotes a variation with respect to ϕ. From this, we automatically recognise $\left(\frac{\partial\Sigma}{\partial\phi}\right)^{\mu}_{\nu} = -\frac{\mathcal{G}'}{\mathcal{G}}\Sigma^{\mu}_{\nu}$ and substitute this result directly back into eq. (4.13).

4.1 Self-tuning constraints

First, we consider the field equations on the background using $\phi = \bar{\phi} =$ constant and $g_{\mu\nu} = \bar{g}_{\mu\nu}$, alongside an arbitrarily large vacuum energy source, $T_{\mu\nu} = -\Lambda_{\text{vac}}g_{\mu\nu}$. This will allow us to establish whether the theory can accommodate for an arbitrary cosmological constant term, to recover a small background AdS solution with curvature $-q^2$. It is clear to see that the scalar field equation is automatically satisfied for the background solution (which can be shown by substituting $\phi = \bar{\phi}$ and $g_{\mu\nu} = \bar{g}_{\mu\nu}$ into eq. (4.13)), whereas Einstein's field equation (eq. (4.12)) yields our "self-tuning equation":

$$\frac{\Lambda_{\text{vac}} + \Lambda_{\text{bare}}}{6\bar{\mathcal{G}}^2} = \frac{\Lambda_{\text{eff}}}{6\bar{\mathcal{G}}^2} = m^2\big(\bar{\mathcal{G}} - 1\big) + q^2\left(1 - \frac{4q^2}{\bar{\mathcal{G}}^2}\bar{V}_r\right), \tag{4.18}$$

where a bar denotes the corresponding potential evaluated at $\phi = \bar{\phi}$ (i.e. evaluated on the background). To understand the role of the massive graviton and the Ringo term we can remove them from eq. (4.18) and analyse the resulting self-tuning equation, which is now given by

$$\frac{\Lambda_{\text{eff}}}{6\bar{\mathcal{G}}^2} = q^2.$$

For self-tuning to occur the scale of the background curvature must be similar to or smaller than that of the curvature of the Universe today ($q^2 \lesssim H_0^2$), which sets $|\Lambda_{\text{eff}}| \lesssim M_{\text{Pl}}^2 H_0^2$. This combination fine-tunes Λ_{bare} to cancel Λ_{vac} to an extreme degree of precision, leading to the familiar problem of radiative instability. A reintroduction of the Ringo term can provide this cancellation between the cosmological constant term and the scalar within $\bar{V}_r$. Naively, the massive term can be used in a similar way, but upon closer inspection this is not allowed. The massive term's scalar dependence is contained within $\bar{\mathcal{G}}$, which we can identify with the Planck mass scale, $\mathcal{G} \sim M_{\text{Pl}} \sim 10^{18}$GeV, as it sets the scale of the gravitational interactions. This greatly restricts the allowed scalar dependence for the mass term. Whereas, $\bar{V}_r$ can (so far) be totally arbitrary. However, there is some potential use of the massive term still. If it is large enough to cancel Λ_{vac}, this is still a step in the right direction, but it effectively moves the fine-tuning from Λ_{bare} to the massive term. Unfortunately, through a detailed analysis on fluctuations about the vacuum (in section 4.2), we demonstrate that $m^2\left(\bar{\mathcal{G}} - 1\right)$ must be small. This reintroduces the fine-tuning onto the bare cosmological constant, highlighting the importance of the Ringo term.

4.2 Exchange amplitude between two conserved sources

To further understand this model, we write the exchange amplitude between two conserved sources (similar to the analysis in [188]). This will allow us to determine whether the two self-tuning conditions as defined in chapter 3 are met. That is to say: (1) the amplitude vanishes for a vacuum source; but (2) the amplitude closely matches that of GR for localised sources. Examining the resultant expression will further help us to identify potential instabilities that we wish to avoid. The most general exchange amplitude for this action is given by

$$\mathcal{A} = \int \mathrm{d}^4x\sqrt{-\bar{g}}\,\tau'^{\mu\nu}h_{\mu\nu}.$$

where the form matches that of eq. (3.5) in chapter 3 [188]. However, this time we consider a metric perturbation, $h_{\mu\nu}$, that can be sourced by both scalar and graviton fluctuations. Therefore, it can describe the exchange amplitude for our entire action, not just the spin-2 components (in contrast to eq. (3.5) which considered a metric perturbation that is sourced **only** by massive graviton fluctuations). In order to separate the tensor, vector, and scalar parts of the metric perturbation we decompose $h_{\mu\nu}$ such that

$$h_{\mu\nu} = h^{(TT)}_{\mu\nu} + 2\bar{\nabla}_{(\mu}A^{(T)}_{\nu)} + 2\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\chi + 2\bar{g}_{\mu\nu}\psi, \tag{4.19}$$

where $h^{(TT)}_{\mu\nu}$ is transverse-tracefree ($\bar{g}^{\mu\nu}h^{(TT)}_{\mu\nu} = 0 = \bar{\nabla}^{\mu}h^{(TT)}_{\mu\nu}$), $A^{(T)}_{\mu}$ is transverse ($\bar{\nabla}^{\mu}A^{(T)}_{\mu} = 0$), and indices are raised and lowered with respect to the background metric, $\bar{g}_{\mu\nu}$. Here, we assume that the couplings to the energy-momentum tensor are linear. In that case, $A^{(T)}_{\mu}$ will consistently enter into equations with the form $\bar{\nabla}^{\mu}A^{(T)}_{\mu}$, which vanishes due to the transverse condition. Therefore, we are free to write

$$h_{\mu\nu} = h_{\mu\nu}^{(TT)} + 2\bar{\nabla}_\mu\bar{\nabla}_\nu\chi + 2\bar{g}_{\mu\nu}\psi. \tag{4.20}$$

Now, substituting the decomposition of $h_{\mu\nu}$ (eq. (4.20)) into the general equation for the amplitude (eq. (3.5)) we obtain

$$\mathcal{A} = \int \mathrm{d}^4x\sqrt{-\bar{g}}\big(\tau'^{\mu\nu}h_{\mu\nu}^{(TT)} + 2\tau'\psi\big), \tag{4.21}$$

where we have removed the $\bar{\nabla}_\mu\bar{\nabla}_\nu\chi$ term: if we assume conservation of the source, $\bar{\nabla}_\mu\tau'^{\mu\nu} = 0$, these terms will vanish when we integrate by parts, hence we can remove them.

Now the goal is clear, to understand the interaction our theory has in relation to another source, we must find expressions for $h_{\mu\nu}^{(TT)}$ and ψ that are proportional **only** to the source, $\tau_{\mu\nu}$. To find this, we start by perturbing eqs. (4.12) and (4.13) about the background solution using

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \qquad \phi = \bar{\phi} + \delta\phi, \qquad T^\mu_\nu = -\Lambda_{\text{vac}}\delta^\mu_\nu + \tau^\mu_\nu. \tag{4.22}$$

To reiterate the metric is perturbed around an AdS background, the scalar is perturbed around a constant background scalar, and we separate the vacuum components from the local matter components in the energy-momentum tensor (where the vacuum components are essentially the background). The perturbed field equations, $\delta\mathcal{E}^\mu_\nu = 0$ and $\delta\mathcal{E}_\phi = 0$, are given by

$$\delta\mathcal{E}^\mu_\nu = -2\bar{\mathcal{G}}^2\delta G^\mu_\nu + (4\bar{\mathcal{G}}\bar{\mathcal{G}}' - 8q^2\bar{V}_r')D^\mu_\nu\delta\phi + m_g^2\big(6\bar{\mathcal{G}}\bar{\mathcal{G}}'\delta\phi\delta^\mu_\nu + \bar{\mathcal{G}}^2\big(h\delta^\mu_\nu - h^\mu_\nu\big)\big) + \tau^\mu_\nu, \tag{4.23}$$

$$\delta\mathcal{E}_\phi = 2\bar{\mathcal{G}}\bar{\mathcal{G}}'\delta R + 3m_g^2\bar{\mathcal{G}}\bar{\mathcal{G}}'h + 2\bar{\mathcal{K}}\bar{\Box}\delta\phi + \bar{V}_r'\big(4\bar{P}_\mu{}^\alpha{}_{\nu\beta}\bar{\nabla}_\alpha\bar{\nabla}^\beta h^{\mu\nu} - 12q^4h\big), \tag{4.24}$$

where

$$D^\mu_\nu \equiv \bar{\nabla}^\mu\bar{\nabla}_\nu - \bar{\Box}\delta^\mu_\nu + 3q^2\delta^\mu_\nu, \qquad m_g^2 \equiv m^2(3\bar{\mathcal{G}} - 2), \tag{4.25}$$

and primes are used such that $\bar{A}' = \frac{\partial A}{\partial \phi}\Big|_{\bar{\phi}}$. As before, with deriving the field equations we will address the non-trivial parts of this calculation. The massive terms can be simply derived by substituting the perturbations (eq. (4.22)) into the field equations (eqs. (4.12) and (4.13)), where we have used $\delta g^{\mu\alpha}\bar{g}_{\alpha\nu} = -h^{\mu\alpha}\bar{g}_{\alpha\nu} = -h^{\mu}_{\nu}$.

For the Ringo terms we first consider the perturbed Einstein field equation (eq. (4.23)), where only the $\delta\phi$ terms have survived since $\delta P^{\mu\alpha}{}_{\nu\beta}\bar{\nabla}_\alpha\bar{\nabla}^\beta\bar{V}_r = 0$, leaving $\big(8\bar{P}^{\mu\alpha}{}_{\nu\beta}\bar{\nabla}_\alpha\bar{\nabla}^\beta - 24q^4\delta^\mu_\nu\big)\bar{V}_r\delta\phi$. Evaluating this expression then yields $-8q^2\bar{V}_r' D^\mu_\nu\delta\phi$ as in eq. (4.23). However, the Ringo term derivation in the perturbed scalar equation (eq. (4.24)) is a little bit more involved. Of course, we could derive this expression by explicitly substituting the perturbations (eq. (4.22)) into the scalar equation (eq. (4.13)), but it is easier to see it emerge through a little "trick". We first only consider the Ringo parts of $\mathcal{E}_\phi$ (eq. (4.13)) where $(\mathcal{E}_\phi)_r = V_r'\big[G_{GB} - 24q^4\big]$ and by perturbing this we obtain

$$\begin{aligned}\delta\left(\mathcal{E}_\phi\right)_r &= \int_y \frac{\delta V_r'(x)}{\delta\phi(y)}\delta\phi(y)\left(\bar{G}_{GB} - 24q^2\right)\mathrm{d}y + \bar{V}_r'\int_y \frac{\delta G_{GB}(x)}{\delta g^{\mu\nu}(y)}\delta g^{\mu\nu}(y)\mathrm{d}y \\ &= \bar{V}_r'\int_y \frac{\delta G_{GB}(x)}{\delta g^{\mu\nu}(y)}\delta g^{\mu\nu}(y)\mathrm{d}y,\end{aligned} \tag{4.26}$$

where $\left(\mathcal{E}_\phi\right)_r$ is a variable in x and y is a dummy variable that we integrate over with $\int_y$. Note we have not included every instance where a function is dependent on x which we keep implicit. Also, notice that we have removed the term $\propto \delta\phi$ within eq. (4.26) as $\bar{G}_{GB} = 24q^2$. Next, we must find an expression for the term on the second line. In order to do this we first define $S_r = \int \mathrm{d}^4x\sqrt{-g}V_r\big[G_{GB} - 24q^2\big]$, which is simply the "Ringo action" given by eq. (4.5). An explicit variation with respect to the metric yields

$$\delta_g S_r = \int \mathrm{d}^4x\sqrt{-g}V_r\bigg[-\frac{1}{2}\left(G_{GB} - 24q^2\right)g_{\mu\nu}\delta g^{\mu\nu} + \int_y \frac{\delta G_{GB}(x)}{\delta g^{\mu\nu}(y)}\delta g^{\mu\nu}(y)\mathrm{d}y\bigg]. \tag{4.27}$$

The above is true, however the explicit form of the field equations for the Ringo term (given by eq. (4.12) in section 2.2 and [80]) is $\frac{-2}{\sqrt{-g}}\frac{\delta_g S_r}{\delta g^{\mu\nu}} = 8P^{\mu\alpha}{}_{\nu\beta}\nabla^\beta\nabla_\alpha V_r - 24q^2\delta^\mu_\nu V_r$. We can write this in the same form as eq. (4.27) such that

$$\delta_g S_r = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\Big[8P^{\mu\alpha}{}_{\nu\beta}\nabla^\beta\nabla_\alpha V_r - 24q^2\delta^\mu_\nu V_r\Big]\delta g^{\mu\nu}. \tag{4.28}$$

We can then perform several integration by parts on eq. (4.28) to obtain

$$\delta_g S_r = -\frac{1}{2}\int \mathrm{d}^4x V_r\sqrt{-g}\Big[8\nabla_\alpha\nabla^\beta\left(P_\mu{}^\alpha{}_{\nu\beta}\delta g^{\mu\nu}\right) - 24q^2 g_{\mu\nu}\delta g^{\mu\nu}\Big]. \tag{4.29}$$

This will allow us to compare eq. (4.27) with eq. (4.29) to find that

$$\int_y \frac{\delta G_{GB}(x)}{\delta g^{\mu\nu}(y)}\delta g^{\mu\nu}(y)\mathrm{d}y = -4\nabla_\alpha\nabla^\beta\left(P_\mu{}^\alpha{}_{\nu\beta}\delta g^{\mu\nu}\right) + \frac{1}{2}g_{\mu\nu}G_{GB}\delta g^{\mu\nu}. \tag{4.30}$$

We then substitute the above (eq. (4.30)) into eq. (4.26):

$$\delta\left(\mathcal{E}_\phi\right)_r = \bar{V}_r'\bigg[-4\bar{\nabla}_\alpha\bar{\nabla}^\beta\left(\bar{P}_\mu{}^\alpha{}_{\nu\beta}\delta g^{\mu\nu}\right) + \frac{1}{2}\bar{g}_{\mu\nu}\bar{G}_{GB}\delta g^{\mu\nu}\bigg], \tag{4.31}$$

where we have set everything apart from $\delta g^{\mu\nu}$ to its background value, as we are only considering first order perturbations. Finally, recognising that $\delta g^{\mu\nu} = -h^{\mu\nu}$ yields the Ringo term in the perturbed scalar equation (eq. (4.24)).

Next, we use $\delta R_{\mu\nu} = -\frac{1}{2}\Delta_L h_{\mu\nu}$ to derive $\delta G_{\mu\nu}$, where the Lichnerowicz operator is given by $\Delta_L h_{\mu\nu} = \bar{\Box}h_{\mu\nu} - 2\bar{\nabla}_{(\mu}\bar{\nabla}^\alpha h_{\nu)\alpha} - 2\bar{R}_{\alpha(\mu}h^\alpha_{\nu)} + 2\bar{R}_{\mu\alpha\nu\beta}h^{\alpha\beta}$. Now we can substitute this, alongside the metric decomposition (eq. (4.20)), into the perturbed Einstein field equation (eq. (4.23)) to obtain

$$\tau^{(TT)\mu}{}_\nu = -\bar{\mathcal{G}}^2\big(\bar{\Box} + 2q^2 - m_g^2\big)h^{(TT)\mu}{}_\nu, \tag{4.32}$$

where

$$\begin{aligned}\tau^{(TT)\mu}{}_\nu = \tau^\mu_\nu + \bar{\mathcal{G}}^2 m_g^2\big(2\bar{\Box}\delta^\mu_\nu - 2\bar{\nabla}^\mu\bar{\nabla}_\nu\big)\chi + 2\big(2D^\mu_\nu + 3m_g^2\delta^\mu_\nu\big)\big(\bar{\mathcal{G}}^2\psi + \bar{\mathcal{G}}\bar{\mathcal{G}}'\delta\phi\big)\\ - 8q^2\bar{V}_r'D^\mu_\nu\delta\phi.\end{aligned} \tag{4.33}$$

Note, we have kept the $\bar{\nabla}^\mu\bar{\nabla}_\nu\chi$ terms in eq. (4.33) as they are not directly interacting with the source (in contrast to eq. (4.21)). Therefore, they may yet contribute non-trivially to our exchange amplitude. Next, we find an equivalent expression for the perturbed scalar field equation (eq. (4.24)) by using the Lichnerowicz operator to derive δR, alongside substituting in the metric decomposition (eq. (4.20)) to obtain

$$0 = 12\bar{\mathcal{G}}\bar{\mathcal{G}}'\big(-\bar{\Box}+4q^2\big)\psi + 3m_g^2\bar{\mathcal{G}}\bar{\mathcal{G}}'\big(2\bar{\Box}\chi+8\psi\big) + 2\bar{\mathcal{K}}\bar{\Box}\delta\phi + 24q^2\bar{V}_r'\big(\bar{\Box}-4q^2\big)\psi. \tag{4.34}$$

Our first step is to simplify eq. (4.33) by introducing new variables:

$$\mathcal{S} \equiv \bar{\mathcal{G}}^2\big(m_g^2\chi - 2\psi - 2\bar{\gamma}\delta\phi\big) + 4q^2\bar{V}_r'\delta\phi, \tag{4.35}$$

and

$$\mathcal{C} \equiv 6\bar{\mathcal{G}}^2 m_g^2\big(\psi + q^2\chi + \bar{\gamma}\delta\phi\big) \tag{4.36}$$

where $\bar{\gamma} \equiv \frac{\bar{\mathcal{G}}'}{\bar{\mathcal{G}}}$. This trivial substitution alters the form of eq. (4.33) to be

$$\tau^{(TT)\mu}{}_\nu = \tau^\mu_\nu - 2D^\mu_\nu\mathcal{S} + \mathcal{C}\delta^\mu_\nu. \tag{4.37}$$

To further simplify, recall that we have assumed conservation of the source. Operating on eq. (4.37) with $\bar{\nabla}$ yields $\mathcal{C}=0$, or in other words from eq. (4.36):

$$6\bar{\mathcal{G}}^2 m_g^2(\psi + q^2\chi + \bar{\gamma}\delta\phi) = 0. \tag{4.38}$$

Notice that this is only non-trivial due to the break in diffeomorphism invariance caused by the massive graviton. If we have a massless graviton $m_g^2 = 0$, $\mathcal{C}$ will automatically vanish. Now we can use eq. (4.38) to simplify eq. (4.37) such that

$$\tau^{(TT)\mu}{}_\nu = \tau^\mu_\nu - 2D^\mu_\nu\mathcal{S}. \tag{4.39}$$

The constraint (eq. (4.38)) can also be used to simplify the form of $\mathcal{S}$ (eq. (4.35)) by removing χ such that

$$\mathcal{S} \equiv -\frac{1}{q^2}\left(\bar{\mathcal{G}}^2\Gamma\psi + \bar{\mathcal{R}}\delta\phi\right), \tag{4.40}$$

where $\bar{\mathcal{R}} \equiv \bar{\gamma}\Gamma\bar{\mathcal{G}}^2 - 4q^4\bar{V}_r'$ and recall that $\Gamma \equiv 2q^2 + m_g^2$. Returning to the scalar field perturbation (eq. (4.34)) we once again remove χ using the constraint equation (eq. (4.38)) yielding

$$-6\frac{\bar{\mathcal{R}}}{q^2}(\bar{\Box} - 4q^2)\psi + 2\bar{\mathcal{K}}_{\text{eff}}\bar{\Box}\delta\phi = 0, \tag{4.41}$$

where $\bar{\mathcal{K}}_{\text{eff}} = \bar{\mathcal{K}} - \frac{3\bar{\gamma}^2 m_g^2\bar{\mathcal{G}}^2}{q^2}$.

For completeness we reiterate our key equations. In eq. (4.32) we have an expression for $h_{\mu\nu}^{(TT)}$, where the definition of $\tau_{\mu\nu}^{(TT)}$ is given in eq. (4.39). Here, $\mathcal{S}$ is defined by the combination in eq. (4.40), which contains ψ and $\delta\phi$ terms. We also have eq. (4.41), which contains a relationship between ψ and $\delta\phi$. We now focus on finding an expression for $h_{\mu\nu}^{(TT)}$ that is proportional only to the source.

To explicitly state the expression we substitute eq. (4.39) into eq. (4.32) such that

$$\tau^\mu_\nu - 2D^\mu_\nu\mathcal{S} = -\bar{\mathcal{G}}^2\left(\bar{\Box} + 2q^2 - m_g^2\right)h^{(TT)\mu}{}_\nu. \tag{4.42}$$

Taking the trace of the above expression (eq. (4.42)) yields

$$\mathcal{S} = -\frac{1}{6}\frac{1}{\bar{\Box} - 4q^2}\tau, \tag{4.43}$$

where we have used the traceless properties of $h_{\mu\nu}^{(TT)}$ and the trace of the source is $\tau = g^{\mu\nu}\tau_{\mu\nu}$, likewise $\tau' = g^{\mu\nu}\tau'_{\mu\nu}$. To be clear, the form of

$$\mathcal{S} \equiv -\frac{1}{q^2}\left(\bar{\mathcal{G}}^2\Gamma\psi + \bar{\mathcal{R}}\delta\phi\right),$$

has been defined in eq. (4.40). Now, eq. (4.43) is merely a statement that

$$-\frac{1}{q^2}\left(\bar{\mathcal{G}}^2\Gamma\psi+\bar{\mathcal{R}}\delta\phi\right)=-\frac{1}{6}\frac{1}{\bar{\Box}-4q^2}\tau. \tag{4.44}$$

With this we can write an expression for $h^{(TT)}_{\mu\nu}$ that is proportional only to the source. For notational convenience we explicitly write this expression in terms of $\mathcal{S}$, where in eq. (4.43) we have shown that it is proportional only to the source:

$$h^{(TT)\mu}{}_{\nu}=-\frac{1}{\bar{\mathcal{G}}}\frac{1}{\bar{\Box}+2q^2-m_g^2}\left[\tau^{\mu}_{\nu}-2D^{\mu}_{\nu}\mathcal{S}\right]. \tag{4.45}$$

Similarly, we want to write an expression for ψ in terms $\mathcal{S}$ by using eq. (4.40) to eliminate $\delta\phi$ in eq. (4.41) which yields

$$\psi=\frac{m_\phi^2-4q^2}{4\bar{\mathcal{G}}^2\Gamma}\frac{\bar{\Box}}{\bar{\Box}-m_\phi^2}\mathcal{S}, \tag{4.46}$$

where

$$m_\phi^2\equiv\frac{12(\bar{\gamma}\bar{\mathcal{G}}^2\Gamma-4q^4\bar{V}_r')^2}{\bar{\mathcal{G}}^2\Gamma(\bar{\mathcal{K}}+6\bar{\gamma}^2\bar{\mathcal{G}}^2-24q^2\bar{\gamma}\bar{V}_r')+48q^6\bar{V}_r'^2}. \tag{4.47}$$

Now, we are in a position to explicitly calculate the exchange amplitude as we have the required expressions. In particular, substituting our expressions for $h^{(TT)}_{\mu\nu}$ (eq. (4.45)) and ψ (eq. (4.46)) into eq. (4.21) yields

$$\begin{aligned}\mathcal{A}=-\frac{1}{\bar{\mathcal{G}}^2}\int \mathrm{d}^4x\sqrt{-\bar{g}}\Bigg\{&\tau'^{\mu\nu}\frac{1}{\bar{\Box}+2q^2-m_g^2}\tau_{\mu\nu}-2\tau'^{\mu\nu}\frac{1}{\bar{\Box}+2q^2-m_g^2}D_{\mu\nu}\mathcal{S}\\ &-\tau'\frac{m_\phi^2-4q^2}{2\Gamma}\frac{\bar{\Box}}{\bar{\Box}-m_\phi^2}\mathcal{S}\Bigg\}.\end{aligned} \tag{4.48}$$

]

Recall, we can consistently work with $\mathcal{S}$ as it is effectively an expression for the source, τ, given by eq. (4.43). Then, we make use of the following formula in the appendix of [188]

$$\frac{1}{\bar{\Box}-M^2}\bar{\nabla}_\mu\bar{\nabla}_\nu\mathcal{S}=\bar{\nabla}_\mu\bar{\nabla}_\nu\frac{1}{\bar{\Box}-M^2-8q^2}\mathcal{S}+\frac{1}{4}\bar{g}_{\mu\nu}\left[\frac{M^2}{\bar{\Box}-M^2}-\frac{(M^2+8q^2)}{\bar{\Box}-M^2-8q^2}\right]\mathcal{S}, \tag{4.49}$$

alongside the conservation of the source such that the $\bar{\nabla}_\mu\bar{\nabla}_\nu\mathcal{S}$ terms vanish. The amplitude now becomes

$$\begin{aligned}\mathcal{A} = -\frac{1}{\bar{\mathcal{G}}^2}\int \mathrm{d}^4x\sqrt{-\bar{g}}\Bigg\{\tau'^{\mu\nu}\frac{1}{\bar{\Box}+2q^2-m_g^2}\tau_{\mu\nu} + 2\tau'\frac{\bar{\Box}-3q^2}{\bar{\Box}+2q^2-m_g^2}\mathcal{S} \\ -\frac{1}{2}\tau'\left[\frac{-2q^2+m_g^2}{\bar{\Box}+2q^2-m_g^2} - \frac{(6q^2+m_g^2)}{\bar{\Box}-6q^2-m_g^2}\right]\mathcal{S} - \tau'\frac{m_\phi^2-4q^2}{2\Gamma}\frac{\bar{\Box}}{\bar{\Box}-m_\phi^2}\mathcal{S}\Bigg\}. \quad (4.50)\end{aligned}$$

Finally, we use eq. (4.43) to write our final expression for the amplitude:

$$\begin{aligned}\mathcal{A} = & -\frac{1}{\bar{\mathcal{G}}^2}\int \mathrm{d}^4x\sqrt{-\bar{g}}\Bigg\{\tau'^{\mu\nu}\frac{1}{\bar{\Box}+2q^2-m_g^2}\tau_{\mu\nu} \\ & -\frac{\tau'}{2}\left[\frac{1/2}{\bar{\Box}+2q^2-m_g^2} + \left(\frac{q^2+m_g^2/6}{q^2+m_g^2/2}\right)\frac{1/2}{\bar{\Box}-6q^2-m_g^2}\right]\tau + g_\phi\tau'\frac{1}{\bar{\Box}-m_\phi^2}\tau\Bigg\},\end{aligned} \quad (4.51)$$

where we have defined $g_\phi \equiv \frac{m_\phi^2}{12\Gamma}$ as the scalar coupling (where the graviton coupling is normalised as $g_g = 1$). This closely matches our explicit example in eq. (3.23), where the first and second line of the amplitude describe a massive graviton with an effective mass $m_g^2 \equiv m^2(3\bar{\mathcal{G}}-2)$, as defined in eq. (3.10). Likewise, the third line describes a scalar with an effective mass given by eq. (4.47), as defined in eq. (3.4). Also, recall that $\Gamma \equiv 2q^2 + m_g^2$ and $\bar{\gamma} \equiv \frac{\bar{\mathcal{G}}'}{\bar{\mathcal{G}}}$. If we identify $\bar{\mathcal{G}} = M_{\rm Pl}$ this amplitude exactly matches our "AdS loophole" example in eq. (3.23). Firstly, we must check that our amplitude vanishes for a vacuum energy source (long wavelength sources) in correspondence with condition (1). Here, we can use the "low energy limit", $|\bar{\Box}| \ll |q^2|, |m_g^2|, |m_\phi^2|$, where vacuum sources will have a greater effect on physics over local sources. The amplitude will indeed vanish if we substitute this alongside a pure vacuum source, $\tau_{\mu\nu} = -\delta\Lambda_{\rm vac}\bar{g}_{\mu\nu}$ into eq. (4.51), provided that $\bar{\mathcal{G}}, m_\phi^2, q^2 + \frac{m_g^2}{6} \neq 0$.

Condition (2) requires the amplitude to closely agree with GR for short wavelength sources. For this we can take the "high energy limit", $|\bar{\Box}| \gg |q^2|, |m_g^2|, |m_\phi^2|$, where local sources will have a greater effect on physics over vacuum sources. Now, the amplitude becomes

$$\mathcal{A} \sim -\frac{1}{\bar{\mathcal{G}}^2}\int \mathrm{d}^4x \left\{ \tau'^{\mu\nu}\frac{1}{\bar{\Box}}\tau_{\mu\nu} - \frac{1}{2}\tau'\frac{1}{\bar{\Box}}\tau + \frac{2m_g^2+m_\phi^2}{12\Gamma}\tau'\frac{1}{\bar{\Box}}\tau \right\}. \tag{4.52}$$

If we allow $\bar{\mathcal{G}} = M_{\mathrm{Pl}}$, we can identify the third term in eq. (4.52) with ϵ in the "GR Amplitude" (eq. (3.12)). This is the exact constraint we derived to ensure that the scalar constraint for a curved background (eq. (3.19)) is satisfied for our explicit AdS loophole (eq. (3.23)), namely that

$$\epsilon = \frac{m_\phi^2 + 2m_g^2}{6\Gamma}, \tag{4.53}$$

where $|\epsilon| \lesssim 10^{-5}$ in accordance with solar system constraints [3] and recall that $\Gamma \equiv 2q^2 + m_g^2$. These checks ensure that there is an agreement between the above analysis and the results in [188]. We now proceed to analyse the stability of the resulting amplitude, to further understand the phenomenology of this model.

Scenario	m_g^2	m_ϕ^2	dS/AdS?
1.	$0 < m_g^2 \lesssim \mathcal{O}(10^{-5})q^2$	$0 < m_\phi^2 \lesssim \mathcal{O}(10^{-5})q^2$	AdS ($q^2 > 0$)
2.	$m_g^2 = 0$	$0 < m_\phi^2 \lesssim \mathcal{O}(10^{-5})q^2$	AdS ($q^2 > 0$)
3.	$m_g^2 = 0$	$0 > m_\phi^2 \gtrsim -\mathcal{O}(10^{-5})\lvert q^2\rvert$	dS ($q^2 < 0$)

Table 4.1: This table showcases the distinct phenomena of the three allowed scenarios, where other scenarios are forbidden due to observational constraints on the system. We display the allowed range of the graviton mass, m_g^2, the allowed range of the scalar mass, m_ϕ^2, alongside whether the system is in dS space or AdS space (i.e. whether $q^2 < 0$ or $q^2 > 0$).

Consider the massive graviton term on lines one and two of the exchange amplitude (eq. (4.51)). To prevent a helicity-2 ghost we must have $\bar{\mathcal{G}} > 0$, ensuring the positivity of the coupling. We must also have $m_g^2 \geq 0$, as it is shown in [194] that this limit avoids a helicity-1 ghost for AdS space. Similarly, if we choose to work in dS space (i.e. $q^2 < 0$), then we must have $m_g^2 \geq 2|q^2|$ due to the Higuchi bound [191]. Note, that setting $m_g^2 = 0$ changes the massive graviton to a massless graviton, which we are free to do in a curved spacetime. Turning our attention to the scalar on line three, this is ghost free when $g_\phi \geq 0$. However unlike the massive term, removing the scalar (i.e. setting $g_\phi \equiv \frac{m_\phi^2}{12\Gamma} = 0$) means that the amplitude no longer vanishes for a vacuum source, hence the model will not self-tune. To reiterate, these constraints imply $\bar{\mathcal{G}}$, $g_\phi > 0$, and $m_g^2 \geq 0$ (alongside $m_g^2 \geq 2|q^2|$ or $m_g^2 = 0$ in dS space) in order for the theory to be ghost free and to self-tune.

From this analysis we identify three distinct scenarios whose constraints are showcased in table 4.1. We summarise the phenomenologies of these scenarios here (with the assumption that the scale of the background curvature is smaller than that of the curvature of the Universe today, $|q^2| \lesssim H_0^2$):

1. **Massive graviton, no tachyons, AdS vacuum:** The no-ghost condition on the graviton, $m_g^2 > 0$, ensures that m_g^2 is positive. Also, the no-ghost condition on the scalar, $g_\phi \equiv \frac{m_\phi^2}{12\Gamma} > 0$, ensures that m_ϕ^2 is positive since $q^2 > 0$ in AdS space. The constraint in eq. (4.53) ensures that m_g^2 and m_ϕ^2 are both small (m_g^2, $m_\phi^2 \lesssim \mathcal{O}(10^{-5})q^2$). Reminder, that we can read off the coupling of each particle to a source from the exchange amplitude (eq. (4.51)). This shows that the coupling for the scalar field is g_ϕ, where the coupling for the graviton is $g_g = 1$. Therefore, this scenario corresponds to an ultralight graviton and an ultralight scalar, where the latter is very weakly coupled to matter ($0 < g_\phi \lesssim \mathcal{O}(10^{-5})g_g$).

2. **Massless graviton, no tachyons, AdS vacuum:** We have set $m_g^2 = 0$, changing the graviton from massive to massless. Also, the no-ghost condition on the scalar, $g_\phi \equiv \frac{m_\phi^2}{12\Gamma} > 0$, ensures that m_ϕ^2 is positive since $q^2 > 0$ in AdS space. The constraint in eq. (4.53) ensures that m_ϕ^2 is small ($m_\phi^2 \lesssim \mathcal{O}(10^{-5})q^2$). As before we can understand the strength of the coupling through the exchange amplitude (eq. (4.51)). This corresponds to an ultralight scalar that is very weakly coupled to matter ($0 < g_\phi \lesssim \mathcal{O}(10^{-5})g_g$).

3. **Massless graviton, tachyonic scalar, dS vacuum:** We have set $m_g^2 = 0$, once again changing the graviton from massive to massless. Also, the no-ghost condition

on the scalar, $g_\phi \equiv \frac{m_\phi^2}{12\Gamma} > 0$, ensures that m_ϕ^2 is negative since $q^2 < 0$ in dS space. The constraint in eq. (4.53) ensures that m_ϕ^2 is small ($|m_\phi^2| \lesssim \mathcal{O}(10^{-5})|q^2|$). Allowing $m_\phi^2 < 0$ could present an instability, which is perhaps the cause of dS solutions being ruled out in [188] (as discussed in chapter 3). But since we have assumed $|q^2| \lesssim H_0^2$ this ultralight tachyon has a lifetime that exceeds the current age of the Universe. Therefore we can potentially accept this instability, since it evolves very slowly. As before we can understand the strength of the coupling through the exchange amplitude (eq. (4.51)). This corresponds to an ultralight scalar that is very weakly coupled to matter ($0 < g_\phi \lesssim \mathcal{O}(10^{-5})g_g$).

These scenarios all exhibit self-tuning, are ghost-free, and can fit within solar system constraints if the coupling of the light scalar field remains weak. Note, that through examining each scenario $0 \leq m_g^2 \equiv m^2(3\bar{\mathcal{G}} - 2) \lesssim \mathcal{O}(10^{-5})|q^2|$. As alluded to in section 4.1, this also implies that the term $m^2(\bar{\mathcal{G}} - 1)$ cannot be much greater than $\sim \mathcal{O}(10^{-5})|q^2|$. This means that the massive term in self-tuning equation (eq. (4.18)) cannot counter an arbitrarily large Λ_{eff}. This, once again, highlights the importance of the Ringo term, implying that $\bar{V}_r$ must be the "self-tuning" term (hence the scalar acts as the self-tuning field).

As a demonstration of this theory we consider an explicit canonical example: $\bar{\mathcal{G}}^2 = M_{\text{Pl}}^2$, $\bar{\mathcal{K}} = \frac{1}{2}$, $\bar{V}_r = \frac{\bar{\phi}}{\mu}$, and $m^2 = 0$, which covers scenario 2 if $q^2 > 0$ and scenario 3 if $q^2 < 0$. The exchange amplitude (eq. (4.51)) is now given by

$$\mathcal{A} = -\frac{1}{M_{\text{Pl}}^2}\int \mathrm{d}^4x\sqrt{-\bar{g}}\left\{\tau'^{\mu\nu}\frac{1}{\bar{\Box}+2q^2}\tau_{\mu\nu} - \frac{\tau'}{2}\left[\frac{1/2}{\bar{\Box}+2q^2}+\frac{1/2}{\bar{\Box}-6q^2}\right]\tau + g_\phi\tau'\frac{1}{\bar{\Box}-m_\phi^2}\tau\right\}, \quad (4.54)$$

where

$$m_\phi^2 = \frac{384q^6}{M_{pl}^2\mu^2+96q^4}, \qquad g_\phi = \frac{16q^4}{M_{pl}^2\mu^2+96q^4}. \quad (4.55)$$

At large wavelengths, $|\bar{\Box}| \ll |q^2|, |m_\phi^2|$, the amplitude becomes

$$\mathcal{A} \sim -\frac{1}{M_{\rm Pl}^2} \int \mathrm{d}^4x \sqrt{-\bar{g}}\, \frac{1}{2q^2} \left\{ \tau'^{\mu\nu}\tau_{\mu\nu} - \frac{1}{4}\tau'\tau \right\}, \tag{4.56}$$

which clearly vanishes for a vacuum source, $\tau_{\mu\nu} = -\delta\Lambda_{\rm vac}\bar{g}_{\mu\nu}$. Note the importance of the background curvature, as we require $q^2 \neq 0$ for this to be well defined. At short wavelengths, $|\bar{\Box}| \gg |q^2|, |m_\phi^2|$, the amplitude becomes

$$\mathcal{A} \sim -\frac{1}{M_{\rm Pl}^2} \int \mathrm{d}^4x \sqrt{-\bar{g}} \left\{ \tau'^{\mu\nu} \frac{1}{\bar{\Box}} \tau_{\mu\nu} - \frac{1}{2}\tau' \frac{1}{\bar{\Box}} \tau + g_\phi \tau' \frac{1}{\bar{\Box}} \tau \right\}, \tag{4.57}$$

which exactly matches eq. (3.12) if we identify ϵ with g_ϕ. This requires $|\mu| \gtrsim 1000|q^2|/M_{pl}$ such that $g_\phi \lesssim 10^{-5}$, ensuring the scalar is very weakly coupled.

To conclude this section we have shown that the exchange amplitude for our theory will vanish for a vacuum source (condition (1)). At the same time, for localised sources it will closely match the exchange amplitude for GR (condition (2)). This can be achieved with three distinct scenarios, whose constraints are summarised in table 4.1. This clearly matches the goal of self-tuning as outlined in chapter 3, but for our model we demand a greater level of scrutiny. More specifically we now require a model that will **dynamically** tune to the given background solution for an arbitrary cosmological constant term. In other words, if we place the system away from the vacuum solution, the self-tuning fields will act to move the system accordingly. For this we study the cosmology of the model.

Chapter 5

Cosmology of the Fab Four in anti-de Sitter space

This chapter is based off unpublished works from the authors of [196]. In this chapter we derive the cosmological equations of this model, which can be used to explore how the self-tuning fields are used to produce a small AdS background curvature. They also provide a framework to solve the equations numerically. As alluded to previously, this will allow us to verify whether the system moves to a vacuum solution regardless of the initial conditions. This will ensure that we truly have a self-tuning theory i.e. the self-tuning fields will dynamically act to move the system to a small AdS background from an arbitrary cosmological constant. To find these cosmological solutions we use the ansatz metric $g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -N(t)^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\mathcal{H}_3^2$, where $N(t)$ is the lapse function and $a(t)$ is the scale factor. Here, $\mathrm{d}\mathcal{H}_3^2 = r^2\big(\mathrm{d}\theta_1^2 + \sinh^2\theta_1\big(\mathrm{d}\theta_2^2 + \sin^2\theta_2\,\mathrm{d}\theta_3^2\big)\big)$ corresponds to a 3-hyperbole metric, where the angular coordinates are $\theta_1,\, \theta_2 \in [0,\, \pi]$ and $\theta_3 \in [0,\, 2\pi]$. This choice corresponds to the third line in eq. (1.9), so we are considering a negative spatial curvature slicing. Alongside this, the background AdS metric is $\bar{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}t^2 + a_0^2\mathrm{d}\mathcal{H}_3^2$, where $a_0 \equiv \frac{\sin qt}{q}$ which corresponds to a small AdS background if $|q^2| \lesssim |H_0^2|$. We also assume the form of the energy-momentum tensor to be that of a perfect fluid: $T^\mu_\nu = diag(-\rho, p, p, p)$.

Considering the tt parts of Einstein's field equation (eq. (4.12)) we obtain a modified

Friedman equation:

$$
\begin{aligned}
H^2 = m^2\left[\left(2-\frac{a_0^2}{a^2}\right)+\left(3\frac{a_0}{a}-4\right)\mathcal{G}\right]N^2 - 2\gamma H\dot{\phi} - 2q^2N^2 + \frac{\mathcal{K}\dot{\phi}^2}{6\mathcal{G}^2} + \frac{\rho N^2}{6\mathcal{G}^2} + \frac{N^2}{a^2r^2} \\
+ \frac{\Lambda_{\text{bare}}N^2}{6\mathcal{G}^2} - \frac{4}{\mathcal{G}^2N^2}\left[\left(H^2-\frac{N^2}{a^2r^2}\right)HV_r'\dot{\phi} - q^4N^4V_r\right)\right],
\end{aligned}
\tag{5.1}
$$

where $\gamma \equiv \frac{\mathcal{G}'}{\mathcal{G}}$, $H \equiv \frac{\dot{a}}{a}$, dots represent a differential with respect to t, and primes represent a differential with respect to the scalar, ϕ. Similarly, considering the ij parts of eq. (4.12) we obtain an acceleration equation:

$$
\begin{aligned}
2\frac{\ddot{a}}{a} + H^2 = m^2\left[\left(6-\frac{a_0^2}{a^2}-2\frac{a_0}{aN}\right)+\left(\frac{3}{N}+6\frac{a_0}{a}-12\right)\mathcal{G}\right]N^2 \\
-2\gamma\left[\ddot{\phi}+\left(\gamma+\frac{\mathcal{G}''}{\mathcal{G}'}\right)\dot{\phi}^2+\left(2H-\frac{\dot{N}}{N}\right)\dot{\phi}\right] - 6q^2N^2 - \frac{\mathcal{K}\dot{\phi}^2}{2\mathcal{G}^2} - \frac{pN^2}{2\mathcal{G}^2} + 2\frac{\dot{N}}{N}H + \frac{N^2}{a^2r^2} + \frac{\Lambda_{\text{bare}}N^2}{2\mathcal{G}^2} \\
-\frac{4}{\mathcal{G}^2N^2}\left[\left(H^2-\frac{N^2}{a^2r^2}\right)\left(V_r'\ddot{\phi}+V_r''\dot{\phi}^2-\frac{\dot{N}}{N}V_r'\dot{\phi}\right)+2\left(\frac{\ddot{a}}{a}-\frac{\dot{N}}{N}H\right)HV_r'\dot{\phi}-3q^4N^4V_r\right].
\end{aligned}
\tag{5.2}
$$

Also, the scalar field equation (eq. (4.13)) becomes

$$
\begin{aligned}
0 = 12\gamma\mathcal{G}^2\left\{\frac{\ddot{a}}{a}+H^2-\frac{\dot{N}}{N}H-\frac{N^2}{a^2r^2}+2q^2N^2+m^2\left[-2+\frac{a_0}{aN}+\frac{a_0^2}{a^2}+\left(6-\frac{3}{2N}-\frac{9a_0}{2a}\right)\mathcal{G}\right]N^2\right\} \\
-\mathcal{K}'\dot{\phi}^2 - 2\mathcal{K}\left(\ddot{\phi}-\frac{\dot{N}}{N}\dot{\phi}+3H\dot{\phi}\right)+\frac{24V_r'}{N^2}\left[\left(H^2-\frac{N^2}{a^2r^2}\right)\left(\frac{\ddot{a}}{a}-\frac{\dot{N}}{N}H\right)-q^4N^4\right].
\end{aligned}
\tag{5.3}
$$

Finally, to model the fluid we use the energy conservation equation

$$
\dot{\rho} = -3H(\rho+p). \tag{5.4}
$$

Through these equations we have been able to find consistent vacuum solutions where $N = 1$, $a = a_0 \equiv \frac{\sin qt}{q}$, $\rho = -p = \Lambda_{\text{vac}}$, and $\phi = \bar{\phi} =$ constant. As in section 4.1, this choice automatically satisfies the scalar equation (eq. (5.3)) as long as we set $r = 1$ (which we consistently do so from here). Similarly eq. (5.4) is automatically satisfied, whereas the modified Friedmann and acceleration equations (eqs. (5.1) and (5.2)) both yield the self-tuning equation (eq. (4.18)). In other words, if we choose an arbitrary value of Λ_{eff} it sets the background solution of the scalar, $\bar{\phi}$. This $\bar{\phi}$ counteracts the cosmological constant term such that we reproduce an AdS spacetime evolution. We can also form a further constraint due to the massive graviton, which explicitly breaks diffeomorphism invariance. Recall that we could have restored it in chapter 4 using Stückelberg fields. However we set the Stückelberg fields to vanish, hence the break in diffeomorphism invariance. To find this constraint we want to find an expression for eq. (5.4) using our definitions for ρ and p. To see this we first differentiate eq. (5.1) with respect to t, such that we have an expression for $\dot{\rho}$. We can then substitute this into eq. (5.4), alongside our definitions of ρ (eq. (5.1)) and p (eq. (5.2)). Finally, we can eliminate the $\ddot{a}$ and $\ddot{\phi}$ terms from this expression using eq. (5.3) to obtain

$$\frac{m^2\mathcal{G}^2}{aN}\left(3a\mathcal{G}-2a_0\right)\left(-\dot{a}_0N+\dot{a}+a\gamma\dot{\phi}\right)=0. \tag{5.5}$$

Notice that our background solution immediately satisfies the above expression due to the second bracket in this constraint. This expression can also be trivially satisfied by $m^2=0$, which changes the graviton from being massive to massless. It can also be satisfied by $\mathcal{G}=0$, but we reject this solution as we have previously found that we require $\mathcal{G}\neq 0$ to avoid ghosts in section 4.2. Interestingly, this constraint can also be satisfied through the first bracket, i.e. $3a\mathcal{G}-2a_0=0$. Since our background solution is satisfied for the second bracket, we do not have to satisfy this constraint. However, if we were to write different background solutions, then we must satisfy $3a\mathcal{G}-2a_0=0$. Writing a different background solution is not something we consider in this work, but we discuss the possibility further in chapter 6.

To expand on this cosmological analysis we explore numerical solutions to the model. To this end, we consider something similar to our canonical example in section 4.2 ($\mathcal{K}=\frac{1}{2}$, $\mathcal{G}^2=\frac{M_{pl}^2}{2}$, $V_r=\frac{\phi}{\mu}$), but we also allow $m^2\neq 0$, and set $M_{\text{Pl}}=1$ for simplicity. Recall, through our choice of the background metric, this set of cosmological equations can only be understood in AdS space i.e. for $q^2 > 0$. We also invoke a change in variables such that

$$\phi(t) \equiv \frac{\mu}{8q^4} x(t), \qquad \dot{\phi}(t) \equiv \sqrt{6}\, y(t), \qquad \alpha \equiv \frac{a_0}{a}. \tag{5.6}$$

This change in variables is used to rewrite the cosmological equations in terms of $H(t)$, $x(t)$, $y(t)$, and $\alpha(t)$. These parameters will have "*scaling solutions*", which correspond to the metric undergoing an AdS evolution and the scalar reaching a constant that satisfies the self-tuning equation (eq. (4.18)) for an arbitrary Λ_{eff}. In other words, we want the above variables to approach the background solutions (as outlined in section 4.1) in our numerical analysis such that

$$H(t) \to \frac{\dot{a}_0}{a_0} \equiv q \cot qt, \qquad x(t) \to x_0, \qquad y(t) \to 0, \qquad \alpha(t) \to 1, \tag{5.7}$$

where x_0 is a constant and is determined by substituting this scaling solution into eqs. (5.1) and (5.2):

$$x_0 = \frac{1}{6}\left(m^2\left(\pm 3\sqrt{2} - 6\right) + 6q^2 - 2\Lambda_{\text{eff}}\right). \tag{5.8}$$

This essentially sets the background value of the scalar in terms of m, q and Λ_{eff}.

Returning to our cosmological equations, alongside setting $N = 1$ and $\rho = -p = \Lambda_{\text{vac}}$ for simplicity, we substitute the change of variables (eq. (5.6)) into eqs. (5.1) to (5.3)[31] to obtain

$$\begin{aligned} H^2 = m^2\left[\left(2-\alpha^2\right) \pm \left(3\alpha - 4\right)\frac{1}{\sqrt{2}}\right] - 2q^2 + y^2 + \frac{\Lambda_{\text{eff}}}{3} + \frac{\alpha^2}{a_0^2} \\ - \frac{8}{\mu}\left[\sqrt{6}\left(H^2 - \frac{N^2}{a^2}\right)Hy - \frac{\mu}{8}x\right], \end{aligned} \tag{5.9}$$

$$\begin{aligned} 2\frac{\ddot{a}}{a} + H^2 = m^2\left[\left(6-\alpha^2-2\alpha\right) \pm \left(6\alpha - 9\right)\frac{1}{\sqrt{2}}\right] - 6q^2 - 3y^2 + \Lambda_{\text{eff}} + \frac{\alpha^2}{a_0^2} \\ - \frac{8}{\mu}\left[\left(H^2 - \frac{\alpha^2}{a_0^2}\right)\ddot{\phi} + 2\sqrt{6}\frac{\ddot{a}}{a}Hy - \frac{3\mu}{8}x\right], \end{aligned} \tag{5.10}$$

$$\ddot{\phi} + 3\sqrt{6}Hy = \frac{24}{\mu}\left[\left(H^2 - \frac{\alpha^2}{a_0^2}\right)\frac{\ddot{a}}{a} - q^4\right], \tag{5.11}$$

respectively. From this point on we consistently take the positive solution whenever there is a $\pm$ sign, however we obtain similar results upon consideration of the negative solution. Notice that we can find an explicit expression for α which effectively appears as a quadratic in eq. (5.9):

$$\alpha(t) = \frac{\alpha_{\text{nom}}}{\alpha_{\text{dom}}}, \tag{5.12}$$

where

$$\begin{aligned} \alpha_{\text{nom}} = -\frac{3m^2}{\sqrt{2}} + \Bigg[\frac{9m^4}{2} - 4\left(-\frac{8\sqrt{6}Hy}{\mu a_0^2} - \frac{1}{a_0^2} + m^2\right) \\ \left(\frac{8\sqrt{6}H^3y}{\mu} + H^2 + 2\sqrt{2}m^2 - 2m^2 + 2q^2 - \frac{\Lambda_{\text{eff}}}{3} - y^2 - x\right)\Bigg]^{\frac{1}{2}}, \end{aligned} \tag{5.13}$$

and

$$\alpha_{\text{dom}} = 2\left(\frac{8\sqrt{6}Hy}{\mu a_0^2} + \frac{1}{a_0^2} - m^2\right). \tag{5.14}$$

Since eq. (5.9) is quadratic in α it will have two solutions, but we (once again) consistently choose to take the positive solution (although we see a similar phenomenology when we use the negative solution).

[31]Equation (5.4) is trivially satisfied as $\rho = -p = \Lambda_{\text{vac}}$, hence $\dot{\rho} = 0$ in accordance with a cosmological constant.

Now that eq. (5.9) is used to determine α, we effectively have two equations that determine the cosmology: eqs. (5.10) and (5.11). Since they both contain $\frac{\ddot{a}}{a}$ and $\ddot{\phi}$ terms we can use eqs. (5.10) and (5.11) as two simultaneous equations. Explicitly solving these simultaneous equations yields two separate expressions for $\frac{\ddot{a}}{a}$ and $\ddot{\phi}$. To start with our expression for $\frac{\ddot{a}}{a}$ becomes

$$\frac{\ddot{a}}{a} = \frac{A}{B}, \tag{5.15}$$

where

$$A = \frac{8\left(H^2 - \frac{\alpha^2}{a_0^2}\right)\left(3\sqrt{6}Hy + \frac{24q^4}{\mu}\right)}{\mu} + \frac{\alpha^2}{a_0^2} - H^2 + m^2\left(\frac{6\alpha - 9}{\sqrt{2}} - \alpha^2 - 2\alpha + 6\right) + \Lambda_{\text{eff}} - 6q^2 - 3y^2 + 3x, \tag{5.16}$$

and

$$B = \frac{192\left(H^2 - \frac{\alpha^2}{a_0^2}\right)^2}{\mu^2} + \frac{16\sqrt{6}Hy}{\mu} + 2. \tag{5.17}$$

Then our expression for $\ddot{\phi}$ becomes

$$\ddot{\phi} = \frac{C}{D}, \tag{5.18}$$

where

$$\begin{aligned} C = & -12\mu\alpha^4 \\ & + 6\mu a_0^2\alpha^2\left[4H^2 + 2m^2\alpha\left(\alpha - 3\sqrt{2} + 2\right) + 6y^2 - 6x + 3m^2\left(3\sqrt{2} - 4\right) + 12q^2 - 2\Lambda_{\text{eff}}\right] \\ & + 3a_0^4\Big[2\mu H^2\Big(- 3m^2\left(3\sqrt{2} - 4\right) - 12q^2 - 2m^2\alpha\left(\alpha - 3\sqrt{2} + 2\right) - 30y^2 + 6x + 2\Lambda_{\text{eff}}\Big) \\ & \qquad - \sqrt{6}H\left(\mu^2 + 64q^4\right)y - 4\mu H^4 - 8\mu q^4\Big], \end{aligned} \tag{5.19}$$

and

$$D = -192a_0^2H^2\alpha^2 + a_0^4\left(8\sqrt{6}\mu Hy + 96H^4 + \mu^2\right) + 96\alpha^4, \tag{5.20}$$

where the form of α in both equations is given by eq. (5.12). In this sense we can rewrite these equations as

$$\frac{\ddot{a}}{a} = \frac{A}{B} = f(t, H, x, y), \tag{5.21}$$
$$\ddot{\phi} = \frac{C}{D} = g(t, H, x, y), \tag{5.22}$$

where f and g are given functions of t, H, x, y, whose forms can be found by examining eqs. (5.15) and (5.18) and using eq. (5.12) to eliminate α. Notice that we now have expressions for $\frac{\ddot{a}}{a}$ and $\ddot{\phi}$ solely in terms of our "scaling variables" and t. From this, we can form three rate equations to fully capture the dynamics of the system. Here, we differentiate our scaling variables with respect to t and then substitute in a given solution such that the rate equations are first order. First, consider H differentiated with respect to t. Through an explicit calculation $\dot{H} = \frac{\ddot{a}}{a} - H^2$, and now we can use eq. (5.21) to replace $\frac{\ddot{a}}{a}$ with $f(t, H, x, y)$. Likewise for y: $\dot{y} = \frac{\ddot{\phi}}{\sqrt{6}}$, and now we can use eq. (5.22) to replace the $\ddot{\phi}$ with $g(t, H, x, y)$. Finally, for x: $\dot{x} = \frac{8q^4}{\mu}\dot{\phi}$, and we recognise that $\dot{\phi} \equiv \sqrt{6}\,y$. These form our three rate equations:

$$\dot{H} = \frac{\ddot{a}}{a} - H^2 \equiv f - H^2, \tag{5.23}$$
$$\dot{x} = \frac{8q^4}{\mu}\dot{\phi} \equiv \frac{8q^4}{\mu}\sqrt{6}\,y, \tag{5.24}$$
$$\dot{y} = \frac{\ddot{\phi}}{\sqrt{6}} \equiv \frac{g}{\sqrt{6}}. \tag{5.25}$$

Now we can proceed to solve these specific rate equations numerically.

To address our values chosen for the initial and final times of our numerical system we can examine the background scale factor $a_0 \equiv \frac{\sin qt}{q}$ with fig. 5.1. We firstly restrict the initial time to be $t_i = 10^{-5}$. Ideally we would like to have $t_i = 0$, but a_0 will vanish when $t = 0$, therefore we set $t_i > 0$ to avoid singularities appearing in $\bar{H} \equiv \frac{\dot{a}_0}{a_0} \equiv q\frac{\cos qt}{\sin qt}$, where $\bar{H}$ is the value of the Hubble parameter on the scaling solution. To understand our choice for the final time, t_f, notice that in fig. 5.1 the scale factor increases for $0 \leq qt \leq \frac{\pi}{2}$ but decreases for $\frac{\pi}{2} < qt \leq \pi$. This corresponds to a cosmological expansion and subsequent cosmological crunch (inherent to an AdS evolution). It should be noted that this cosmological crunch does not happen in our own Universe. In a sense, $0 \leq qt \leq \frac{\pi}{2}$ is the most important era to study, as it corresponds to a cosmological expansion (similar to that of our own Universe). However, it would still be interesting to study the evolution for an entire cosmological cycle, $0 \leq qt \leq \pi$[32]. For this cosmological analysis we use the Wolfram Mathematica inbuilt numerical solver, NDSolve [200]. Unfortunately, we find that evolving past $qt = \frac{\pi}{2}$ pushes our numerical solver beyond its capability as numerical instabilities start to appear[33]. Despite this, setting $qt_f = \frac{\pi}{2}$ still allows us to study the cosmology for a full expansion cycle. To reiterate we set $t_i = 10^{-5}$ and $t_f = \frac{\pi}{2q}$ unless otherwise specified.

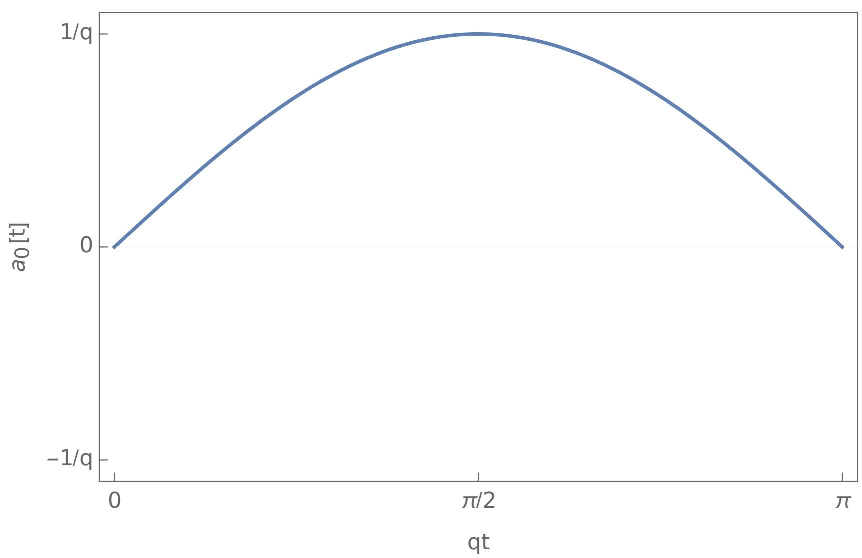


Figure 5.1: A graph showcasing the evolution for the background scale factor in AdS space, $a_0 \equiv \frac{\sin qt}{q}$. Notice that it undergoes an expansion for $0 \leq qt \leq \frac{\pi}{2}$, but contracts for $\frac{\pi}{2} < qt \leq \pi$. The above represents a full cosmological cycle for AdS space.

We then ensure we are consistent with the constraints discussed in section 4.2. Namely

[32]Where $qt > \pi$ corresponds to an entirely separate cosmological cycle, therefore we do not need to study this.

[33]Note, this is only true for a weak scalar coupling i.e. when $g_\phi \lesssim \mathcal{O}(10^{-5})$ in accordance with the constraint in eq. (5.28). In section 5.1 we later find that for the "strongly coupled regime" ($g_\phi > \mathcal{O}(10^{-5})$) we can evolve the system beyond $qt = \frac{\pi}{2}$.

we satisfy

$$0 < m_\phi^2 \lesssim \mathcal{O}(10^{-5})q^2, \qquad 0 \leq m_g^2 \lesssim \mathcal{O}(10^{-5})q^2, \tag{5.26}$$

to fit within solar system constraints. Here, our cosmological framework sets

$$m_\phi^2 = \frac{768q^8}{\Gamma\mu^2 + 192q^6}, \qquad m_g^2 = m^2\left(\frac{3}{\sqrt{2}} - 2\right), \tag{5.27}$$

where we have chosen the positive branch of $\mathcal{G} = \pm\frac{1}{\sqrt{2}}$ and recall that $\Gamma \equiv 2q^2 + m_g^2$. Alongside this, we must ensure that the scalar coupling $g_\phi \equiv \frac{m_\phi^2}{12\Gamma} > 0$. Recall, that g_ϕ is related to m_ϕ^2, so to further satisfy the solar system constraints we must have

$$0 < g_\phi \lesssim \mathcal{O}(10^{-5}), \tag{5.28}$$

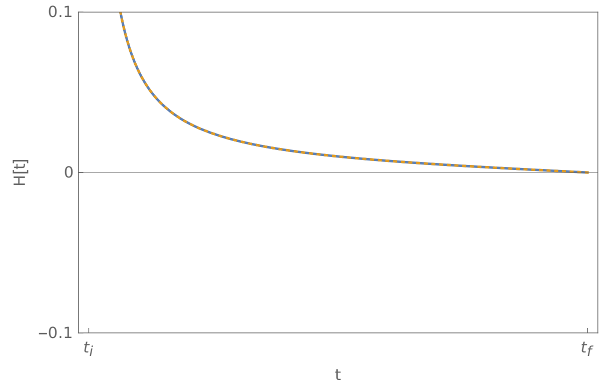
where

$$g_\phi = \frac{64q^8}{\Gamma\left(\Gamma\mu^2 + 192q^6\right)}. \tag{5.29}$$

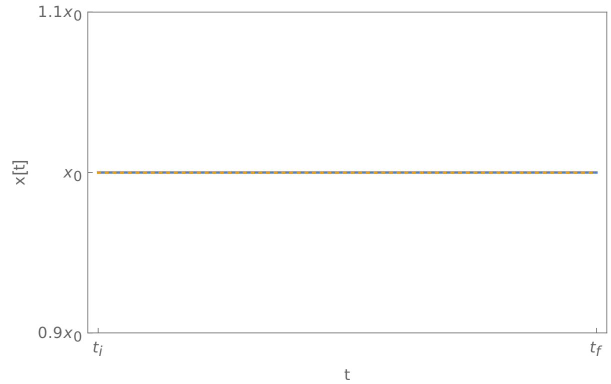
Alongside $H(t)$, $x(t)$, and $y(t)$ we measure the function $z(t) \equiv H(t)^2 - \frac{1}{a^2} = H(t)^2 - \frac{\alpha(t)^2}{a_0(t)^2}$ as a consistency check because its scaling solution, $z(t) \to -q^2$, is a constant. This is simply another way of measuring the scale factor, $a(t)$, but the value of the self-tuning solution is physically important. Recall that q^2 measures the intrinsic curvature, and if q^2 is small then the intrinsic curvature of the system is also small. Therefore, if $z(t) \to -q^2$ at late times this implies that the system is undergoing an AdS evolution with a small intrinsic curvature (if $|q^2| \lesssim |H_0^2|$), and that the gravitational effect of the vacuum has been cancelled out. Importantly (as we discussed in section 4.1 with eq. (4.18)) this will hopefully be achieved via the scalar field, which will dynamically adjust to cancel an arbitrarily large effective cosmological constant term on the background.

We first discuss the plots in fig. 5.2 where we have chosen the parameters such that $q^2 = 10^{-4}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = -10^{-2}$, $\mu = 1$, which sets $x_0 \sim 3.43 \times 10^{-3}$. In this example we place the initial conditions of $H(t)$, $x(t)$, and $y(t)$ on their respective scaling solutions i.e. $H_i = q\cot qt_i$, $x_i = x_0$, $y_i = 0$ where the i denotes the initial value of the variable (and recall that x_0 is set by eq. (5.8)). This will act as a consistency check to ensure that our system remains on the scaling solution if initially placed there. Firstly, $H(t)$ (fig. 5.2a), $x(t)$ (fig. 5.2b), and $z(t)$ (fig. 5.2d) all remain on their respective scaling solution ($q\cot qt$, x_0, and $-q^2$). Notice that $y(t)$ (fig. 5.2c) deviates from the scaling solution despite being initially placed there. But this is a tiny deviation where $\delta y \sim \mathcal{O}(10^{-19})$, which is much smaller than the scale of $x_0 \sim \mathcal{O}(10^{-3})$, so we can treat it practically as 0. Note that the machine precision of our chosen numerical solver is $\sim \mathcal{O}(10^{-16})$, so we can be sure that the deviation in δy is simply a product of the numerical analysis. This verifies that our system remains on the scaling solution when it initially starts there.

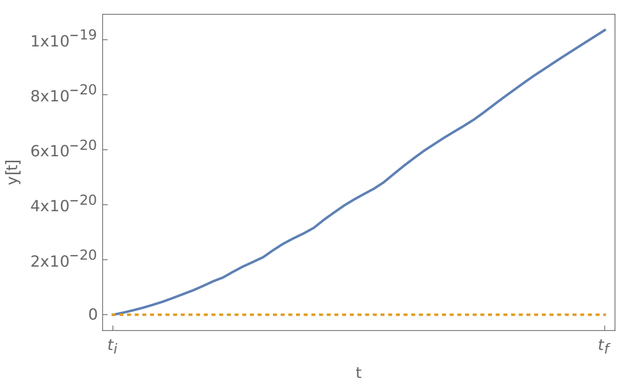
To further test this claim we numerically analyse the system for a variety of parameters with differing initial conditions for $H(t)$. In fig. 5.3 we set $H_i = 10q\frac{\cos qt_i}{\sin qt_i}$. Here, $H(t)$ rapidly approaches its scaling solution, but this is not displayed in fig. 5.3a as $H(t) \to q\cot qt$ too quickly to easily see on a plot. However, we can essentially track the evolution of the metric using $z(t) \equiv H(t)^2 - \frac{\alpha(t)^2}{a_0(t)^2}$. Fig. 5.3d shows that $z(t)$ starts away from its scaling solution but quickly approaches it and remains there. Similar to the above, in several other upcoming plots we also do not show $H(t)$ explicitly approach its scaling solution from a different initial value, with the understanding that $z(t)$ can track the metric evolution. Alongside this, $x(t)$ (fig. 5.3b) remains on its scaling solution. As before $y(t)$ (fig. 5.3c) deviates from its scaling solution, but it is still extremely small ($\delta y \sim \mathcal{O}(10^{-22})$) compared to the value of $x_0 \sim \mathcal{O}(10^{-2})$. These results are similar to the plots in fig. 5.4 where set $H_i = 10^{-4}q\frac{\cos qt_i}{\sin qt_i}$. Once again fig. 5.4a appears to always follows $H(t) \to q\cot qt$ as we have restricted the axis of $H(t)$. $z(t)$ in fig. 5.4d also appears to start at the self-tuning solution and remain there. This is likely due to the deviation from the self-tuning solution being initially very small (and in fact not detectable in our numerical solver). Finally, the $x(t)$ (fig. 5.4b) remains on its scaling solution, whereas $y(t)$ (fig. 5.4c) deviation is beyond the computation precision of our numerical solver.



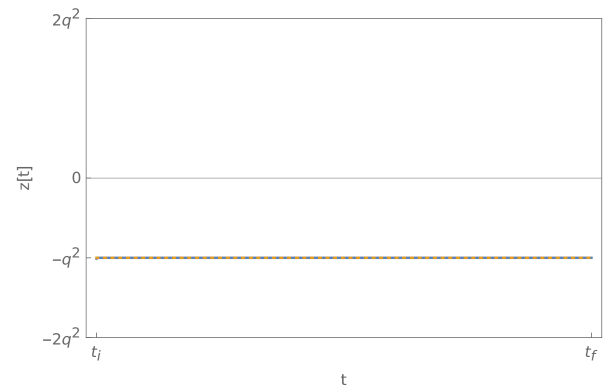
(a) $H(t)$



(b) $x(t)$

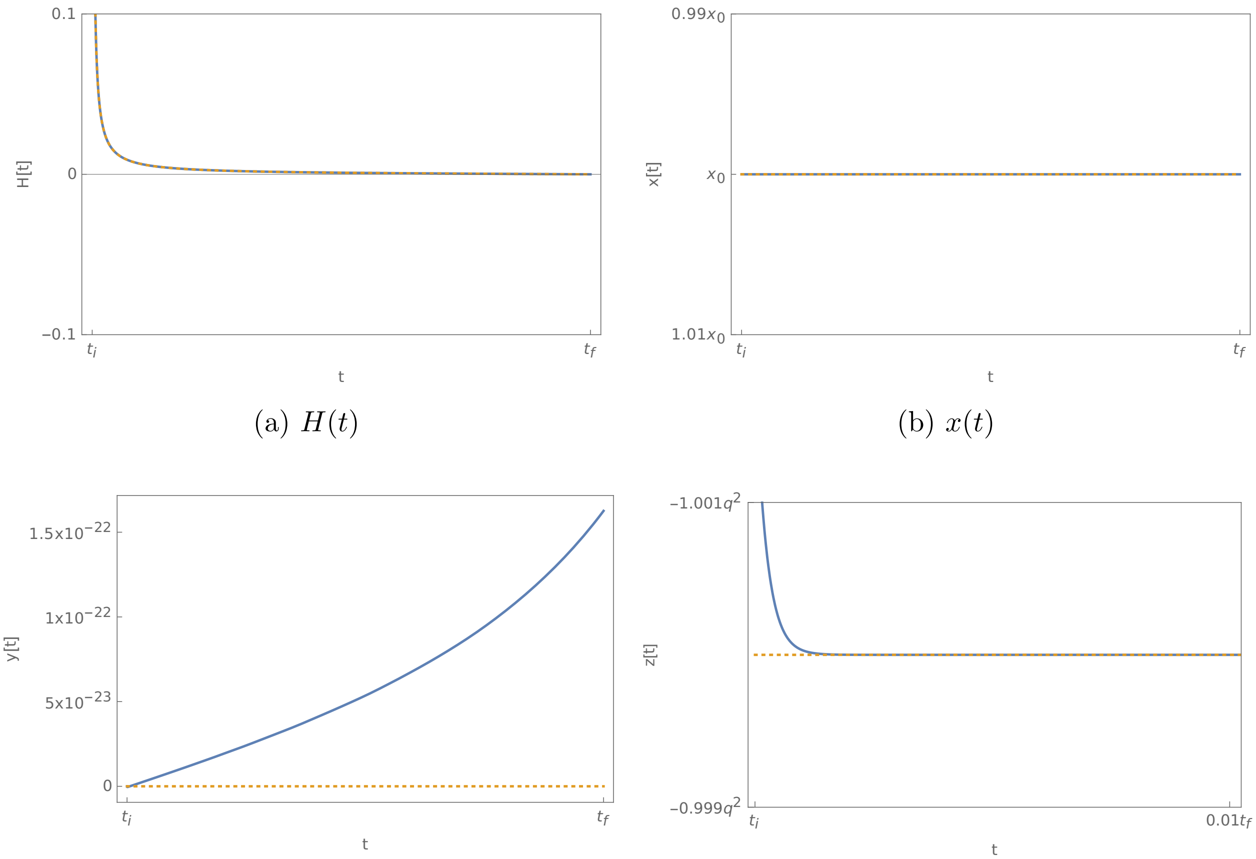


(c) $y(t)$



(d) $z(t)$

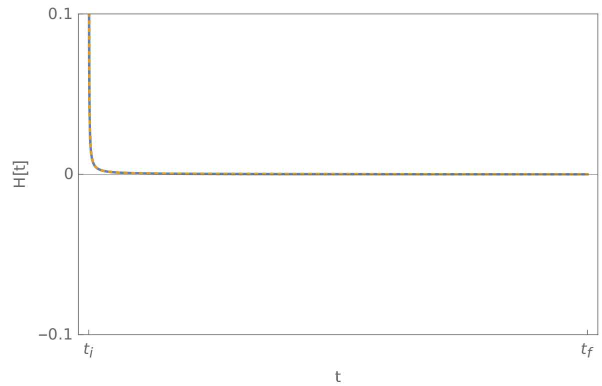
Figure 5.2: These figures showcase the numerical evolution of the system when $x_i = x_0$ (i.e. the initial condition corresponds to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q \cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-5} q^2$, $\Lambda_{\text{eff}} = -10^{-2}$, $\mu = 1$, which sets $x_0 \sim 3.43 \times 10^{-3}$ and is consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = q \cot qt_i$, $x_i = x_0$, $y_i = 0$. Note that $H(t)$ rapidly decreases from its initial value. Therefore, we have chosen to restrict the vertical axis of fig. 5.2a to better display its evolution.



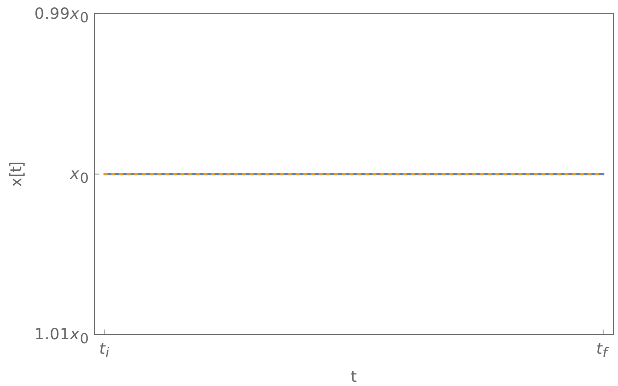
(a) $H(t)$ (b) $x(t)$

(c) $y(t)$ (d) $z(t)$

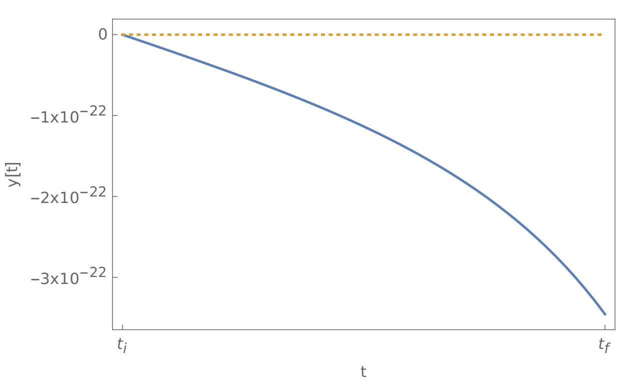
Figure 5.3: These figures showcase the numerical evolution of the system when $x_i = x_0$ (i.e. the initial condition corresponds to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q \cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-6}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = 10^{-1}$, $\mu = 100$, which sets $x_0 \sim -3.33 \times 10^{-2}$ and is consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 10q \cot qt_i$, $x_i = x_0$, $y_i = 0$. Note that both $H(t)$ and $z(t)$ rapidly decrease from their initial values. Therefore, we have chosen to restrict the vertical axis of figs. 5.3a and 5.3d and the horizontal axis fig. 5.3d to better display the evolution of each parameter.



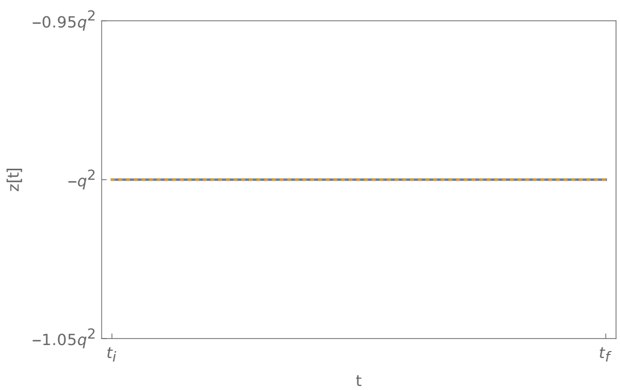
(a) $H(t)$



(b) $x(t)$



(c) $y(t)$



(d) $z(t)$

Figure 5.4: These figures showcase the numerical evolution of the system when $x_i = x_0$ (i.e. the initial condition corresponds to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q \cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-8}$, $m^2 = 10^{-8}q^2$, $\Lambda_{\text{eff}} = 10$, $\mu = 1000$, which sets $x_0 \sim -3.33$ and is consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 10^{-4} q \cot qt_i$, $x_i = x_0$, $y_i = 0$. Note that $H(t)$ rapidly decreases from its initial value. Therefore, we have chosen to restrict the vertical axis of fig. 5.4a to better display its evolution.

However, we should note that $H(t)$ is not generically robust to changes in H_i. As $H_i \to 0$, numerical instabilities arise within the solver (as we have previously discussed). Similarly, setting $H_i < 0$ pushes our numerical analysis beyond its validity. But this should be expected as the initial rate of expansion should never be negative for a Universe that starts at $a_0 \equiv \frac{\sin qt}{q} = 0$ for $t = 0$. Finally, making H_i too large also pushes our solver beyond its validity, but the same can be said for most parameters (dividing by very large numbers will result in instabilities within the numerical analysis).

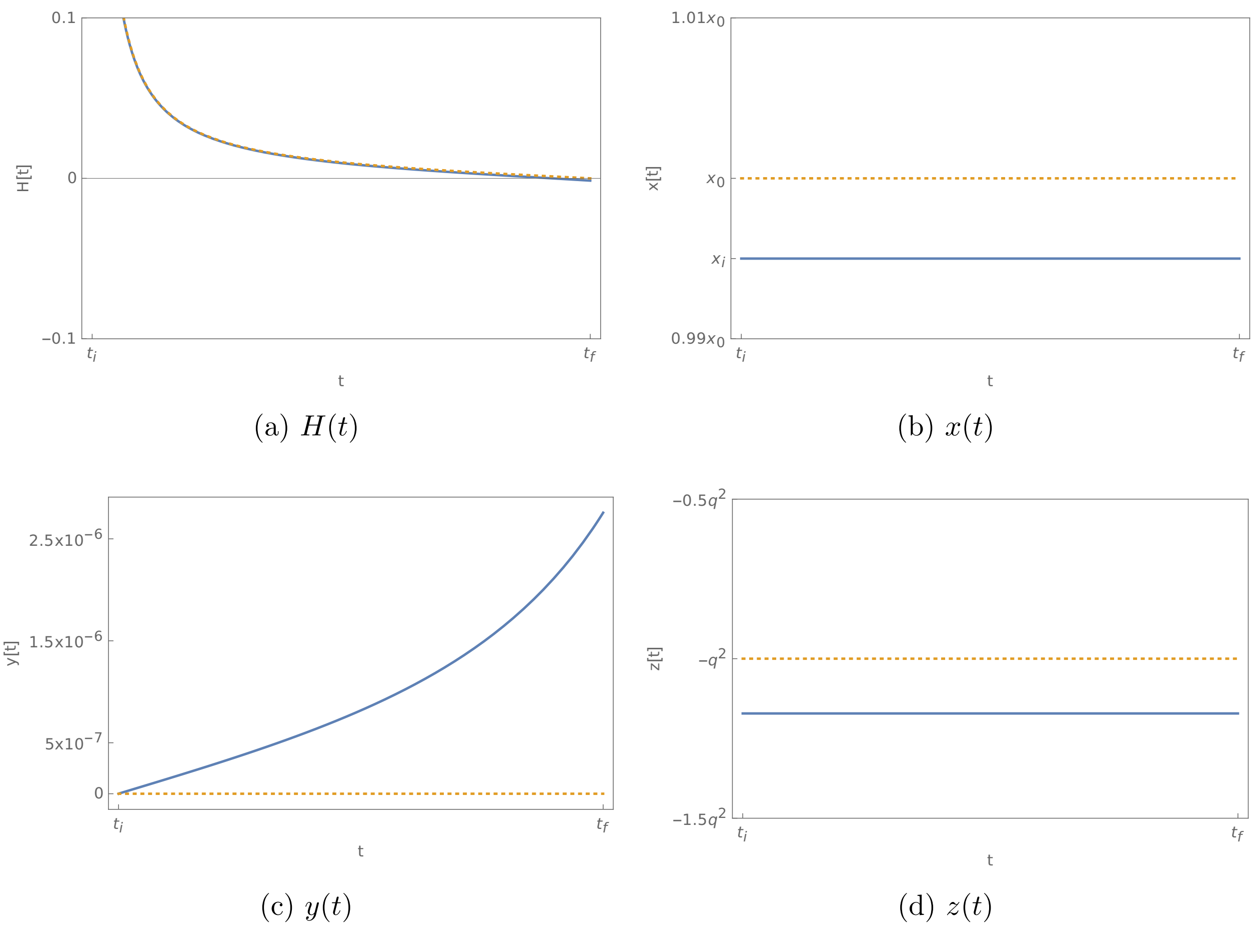
Altogether the above implies that we can achieve self-tuning if the initial conditions correspond to the self-tuning scaling solutions. In fact, despite changes in H_i the system still approaches the self-tuning solutions. Similarly the scalar field remains on the self-tuning solution when placed there. This shows that the gravitational effects of an arbitrary $\Lambda_{\rm eff}$ have been cancelled off to return to a small intrinsic negative curvature, as demonstrated by $z(t)->q^2$ at late times. Once again (as shown through eq. (4.18)), we believe this is due to an explicit cancellation from the scalar field inside of the Ringo term.

However, it would be interesting to see whether the scalar field can dynamically achieve this. In other words we want to test whether the scalar field, when initially placed away from the self-tuning solution, will dynamically evolve towards it to cancel off an arbitrary $\Lambda_{\rm eff}$ and recover small intrinsic curvature. To observe this we follow the plots in fig. 5.5 where we have shifted the initial conditions of $x(t)$ (where $x_i = 0.995x_0$). Notice how in fig. 5.5a $H(t)$ seems to reside on its scaling solution, but at late times deviates from it. However, its deviation from the scaling solution appears to be incredibly small (< 0.01). Similarly, from fig. 5.5d $z(t)$ seems to remain at a constant that deviates from its scaling solution ($-q^2$) by approximately $0.2q^2$. Note that although $z(t)$ appears to be constant, it could be varying very slowly. Despite this we choose to focus our analysis on the dynamics of the scalar, since we have already shown previously (through figs. 5.2a, 5.2d, 5.3a, 5.3d, 5.4a, and 5.4d) that both $H(t)$ and $z(t)$ approach their scaling solutions irrespective of H_i when $x_i = x_0$. Ideally we would want $x(t)$ to evolve towards the self-tuning solution. Once it reaches the self-tuning solution we would expect the scalar to remain there, such that it acts as a counterterm to an arbitrarily large cosmological constant (as in eq. (4.18)). Then, as long as the scale factor follows an AdS evolution (where $a(t) = \frac{\sin qt}{q}$) this corresponds to self-tuning. In other words, regardless of the value of $\Lambda_{\rm eff}$ the system undergoes an AdS evolution with small intrinsic curvature ($|q^2| \lesssim H_0^2$), where the scalar acts to cancel the gravitational effect of $\Lambda_{\rm eff}$. Interestingly, from fig. 5.5b, $x(t)$ appears to not be dynamically changing at all. This is in contrast to $y(t)$ (fig. 5.5c) which is non-zero and positive. Therefore $x(t)$ should be dynamically evolving towards x_0, albeit very slowly. In fact, it is so small that our numerical solver cannot detect this change in $x(t)$ (at least when $y_i = 0$).

We can better demonstrate this by reconsidering the same system, but instead we set $\mu = 0.01$. This violates our solar system constraints (eqs. (5.26) and (5.28)) and in particular it sets $g_\phi \sim \mathcal{O}(10^{-3})$ (from eq. (5.29)). Hence, $\mu = 0.01$ corresponds to an increase in the scalar coupling, which in turn increases the interaction strength of the scalar. We do this to demonstrate that $x(t)$ is indeed dynamical (see fig. 5.6a), but is very slow when the scalar coupling is weak. To verify this claim we can measure the form of the effective potential against the value of the scalar. To see this, consider the KG equation for a scalar with an effective potential, which is given by

$$\Box\phi = \frac{\partial V_{\text{eff}}}{\partial \phi}. \tag{5.30}$$

We further study the same example but instead we set $x_i = 0.99x_0$. Once again, we find that $H(t)$ and $z(t)$ (figs. 5.14a and 5.14d) quickly approach their respective scaling solutions. This shows that the intrinsic curvature is once again small for an arbitrarily large cosmological constant. Therefore the scalar field has been effectively able to self-tune Λ_{eff} As before, there is a slight deviation in $z(t)$ at late times (fig. 5.14e), which could be traced back to numerical instabilities. Similarly, $x(t)$ (fig. 5.14b) approaches and reaches its scaling solution at $t = t_f$, but moves beyond x_0 for $t > t_f$. Similar to before there is a turning point in $y(t)$ (fig. 5.14c) at $t = t_f$ which could imply some sort of oscillatory nature in $x(t)$. However, the behaviour of the effective potential is slightly altered in fig. 5.14f. Similar to fig. 5.13f, $\frac{\partial V_{\text{eff}}}{\partial \phi}$ crosses the horizontal axis at x_0. Unlike fig. 5.13f, $\frac{\partial V_{\text{eff}}}{\partial \phi}$ (in fig. 5.14f) is initially negative and becomes positive after crossing the axis. This implies that the scalar is rolling down the effective potential to x_0, which corresponds to its minimum. After this it then rolls back up to the potential away from x_0, but this is with the expectation that it will follow the slope back to the minimum. This is exactly the type of behaviour needed in order to evade fine-tuning.



(a) $H(t)$ (b) $x(t)$

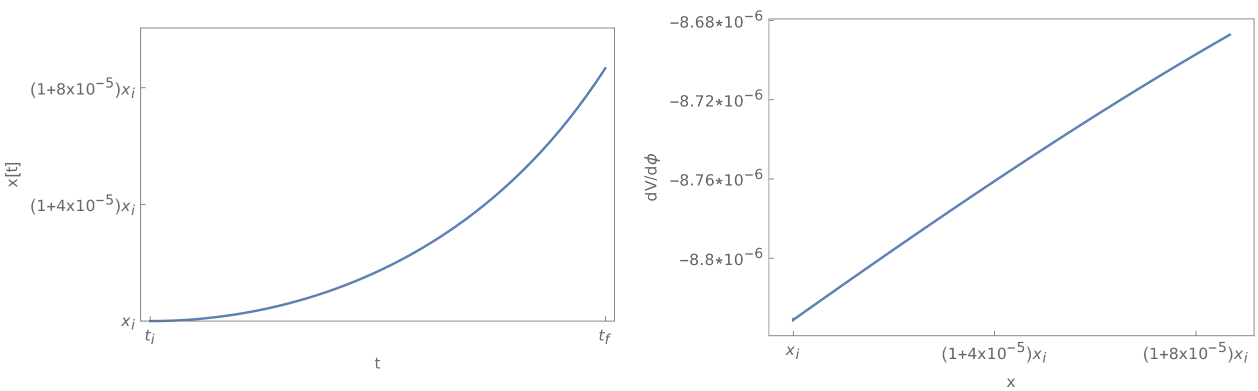
(c) $y(t)$ (d) $z(t)$

Figure 5.5: These figures showcase the numerical evolution of the system when $x_i = 0.995x_0$ (i.e. the initial condition does not correspond to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q \cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = -10^{-2}$, $\mu = 1$, which sets $x_0 \sim 3.43 \times 10^{-3}$ and is consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 10$, $x_i = 0.995x_0$, $y_i = 0$. Note that $H(t)$, rapidly decreases from its initial value. Therefore, we have chosen to restrict the vertical axis of fig. 5.5a to better display its evolution.

Comparing eqs. (4.13) and (5.3) we can see that $\Box\phi = -\frac{1}{N^2}\left(\ddot{\phi} - \frac{\dot{N}}{N}\dot{\phi} + 3H\dot{\phi}\right)$, which becomes $\Box\phi = -\left(g + 3H\sqrt{6}y\right)$ when we take $N = 1$, perform a change in variables using eq. (5.6), and replace $\ddot{\phi}$ with g which is given by eq. (5.22). According to the KG equation, the above is equal to $\frac{\partial V_{\text{eff}}}{\partial \phi}$, therefore

$$\frac{\partial V_{\text{eff}}}{\partial \phi} = -\left(g + 3H\sqrt{6}\,y\right) . \tag{5.31}$$

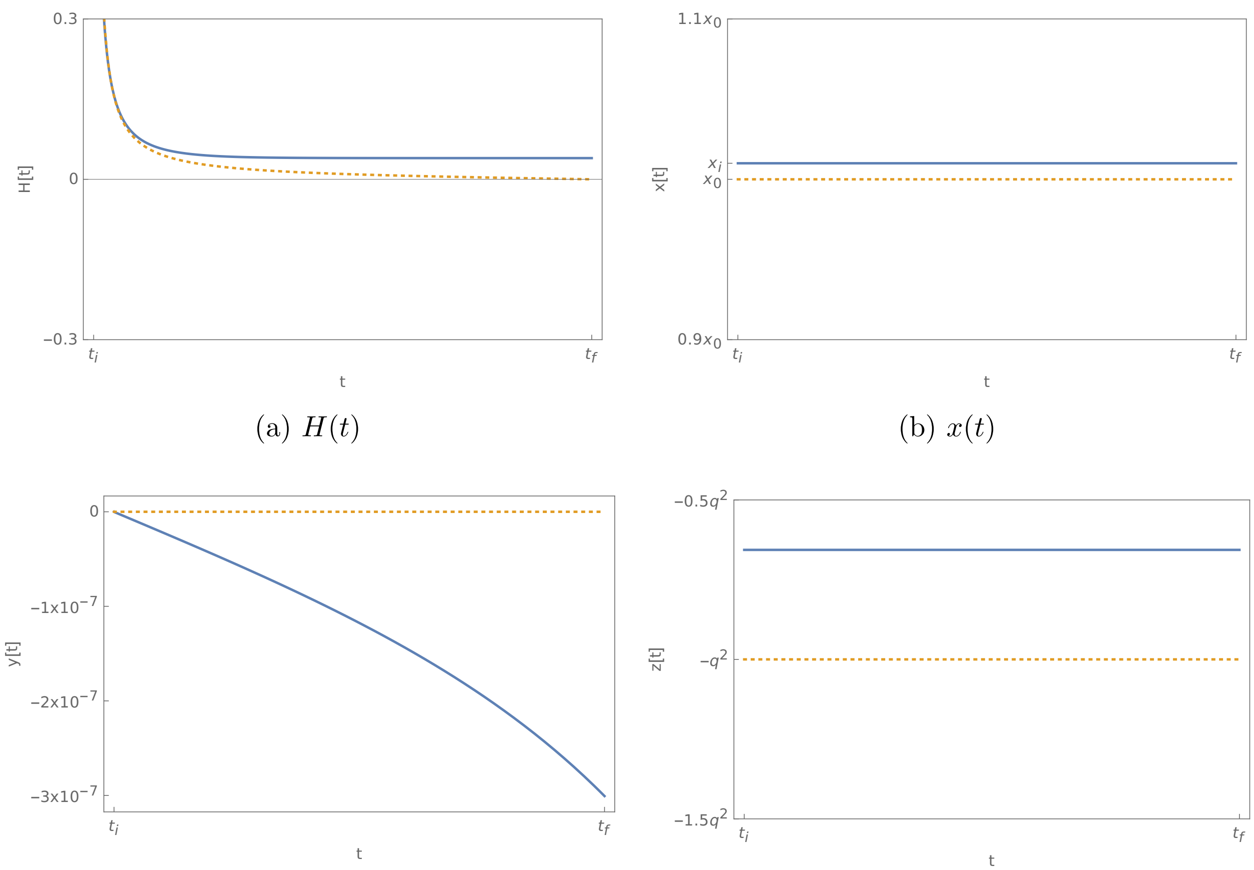
Now we can plot the right hand side of eq. (5.31) against x to verify that the scalar is moving as demonstrated by fig. 5.6b. Here, we see that the scalar is indeed rolling along the potential. Since $\frac{\partial V_{\text{eff}}}{\partial \phi} < 0$ it implies that this scalar is rolling down a slope, hopefully towards a minimum at x_0. Altogether, fig. 5.6 implies that our scalar is moving very slowly towards its scaling solution. As before the scalar does not reach the self-tuning solution as it evolves too slowly. However fig. 5.6a does imply that $x(t)$ is approaching x_0.



(a) $x(t)$

(b) $\partial V_{\text{eff}}/\partial \phi$ against x

Figure 5.6: These figures showcase the numerical evolution of the system when $x_i = 0.995x_0$ (i.e. the initial condition does not correspond to the scaling solution). The blue line represents the evolution of the system and the scaling solutions are given by $H(t) \to q \cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = -10^{-2}$, $\mu = 0.01$, which sets $x_0 \sim 3.43 \times 10^{-3}$. Our choice of μ now sets $g_\phi \sim \mathcal{O}(10^{-3})$, which is no longer consistent with eqs. (5.26) and (5.28). Instead these parameters are chosen to increase the scalar coupling, which demonstrates the dynamical nature of the scalar to compare with fig. 5.5. The initial conditions chosen are $H_i = 10$, $x_i = 0.995x_0$, $y_i = 0$.



(a) $H(t)$ (b) $x(t)$

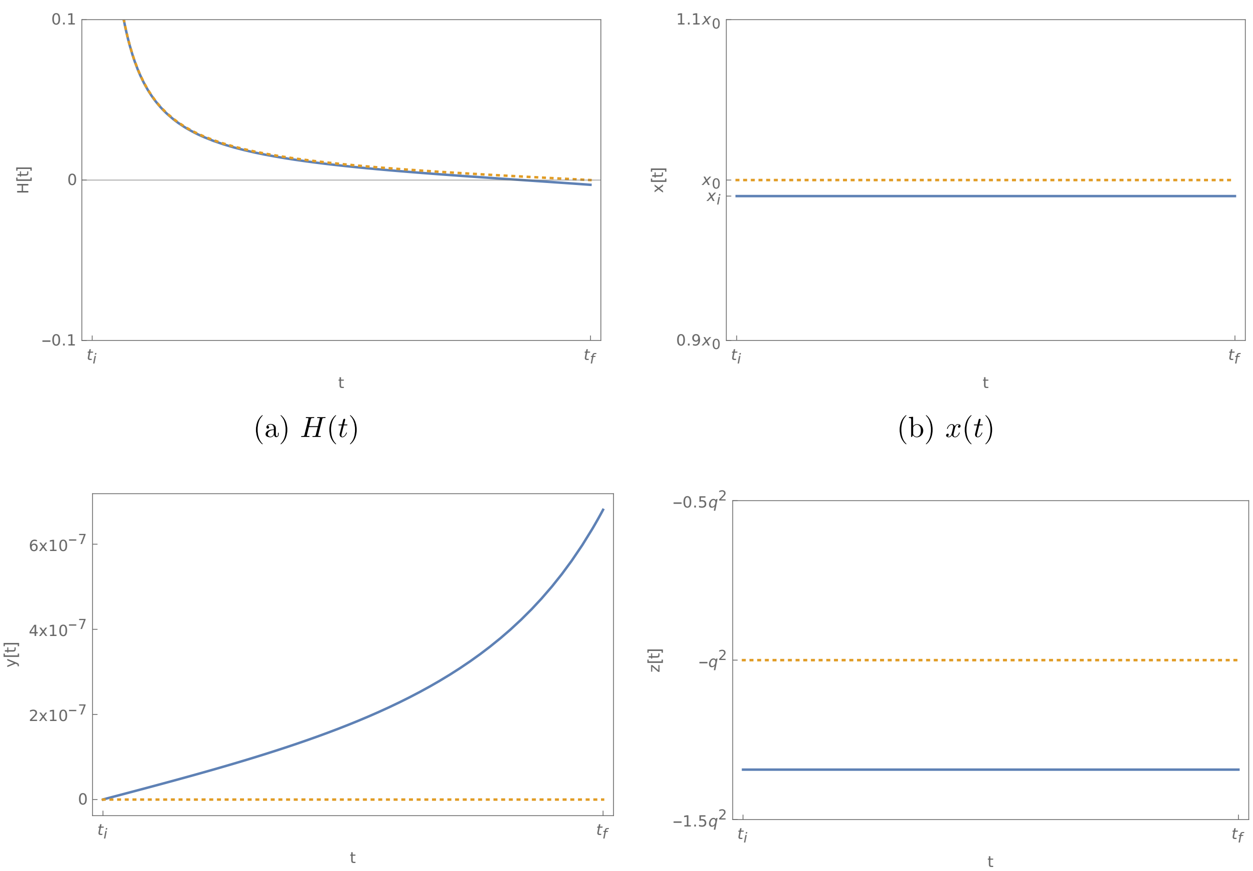
(c) $y(t)$ (d) $z(t)$

Figure 5.7: These figures showcase the numerical evolution of the system when $x_i = 1.05x_0$ (i.e. the initial condition does not correspond to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q \cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-8}q^2$, $\Lambda_{\text{eff}} = -0.1$, $\mu = 10$, which sets $x_0 \sim 3.34 \times 10^{-2}$ and is consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 10$, $x_i = 1.05x_0$, $y_i = 0$. Note that $H(t)$ rapidly decreases from its initial value. Therefore, we have chosen to restrict the vertical axis of fig. 5.7a to better display its evolution.

We retest this numerical analysis using a variety of parameters and initial conditions as showcased by fig. 5.7. In this example $H(t)$ (fig. 5.7a) initially appears to follow its scaling solution, but deviates from it at late times. Whilst $z(t)$ (fig. 5.7d) appears to remain at a constant that deviates from $-q^2$ by approximately $0.35q^2$ (again, it may be very slowly moving, but it certainly **appears** to be constant. Note that upcoming plots may also be slowly evolving even if it appears to be constant, but we will not state it explicitly to avoid repetition). Here, $x(t)$ (fig. 5.7b) appears to remain on $x_i = 1.05x_0$. However this is only because it evolves incredibly slowly as demonstrated by $y(t)$ (fig. 5.7c), which is negative as $x_i > x_0$. This further evidence that the scalar is trying to move towards the the self-tuning solution, but is far too slow to reach it.

Similarly, in fig. 5.8 $H(t)$ (fig. 5.8a) initially resides on the scaling solution, but deviates slightly from it at late times. Similarly $z(t)$ (fig. 5.8d) seems to remain at a constant that deviates from $-q^2$ by about $0.35q^2$. As before, $x(t)$ (fig. 5.8b) appears to remain on its initial condition, $x_i = 0.99x_0$. However, this is only because it evolves slowly as shown by $y(t)$ (fig. 5.8c), which is positive as $x_i < x_0$. To reiterate the point, the scalar is seemingly trying to evolve towards the self-tuning solution, but is too slow to reach it.

To recap we have shown how our system evolves for a wide range of parameters and initial conditions in $H(t)$. Furthermore, $H(t)$ seems to always approach its scaling solution for $x = x_0$, which can be more easily tracked via $z(t)$. For small deviations in $x_i = x_0$, we see that $H(t)$ and $z(t)$ seem to deviate slightly from their scaling solutions at late times. This is echoed by the behaviour of $x(t)$, which appears to be a constant that deviates from x_0. However, through the evolution of $y(t)$ we see that $x(t)$ does want to evolve to x_0, albeit very slowly. This can be demonstrated explicitly by analysing the system with a large scalar coupling, which resides outside of the solar system constraints. However, the above unfortunately does not correspond to dynamical self-tuning. Recall that we want Λ_{eff} to be arbitrary (and large), and the scalar should act to approach the self-tuning solution of x_0. These scalar fields will modify the value of the Ringo term within eq. (4.18), which is used to counter an arbitrary Λ_{eff}. In turn the system will undergo an AdS evolution with a small intrinsic curvature, showing that the gravitational effect of the cosmological constant has been cancelled. Instead, since $x(t)$ does not reach the x_0 at late times and the system deviates from an AdS evolution, as evidenced by the deviation of $z(t)$ (which effectively measures the evolution of $a(t)$) from its constant self-tuning solution, $-q^2$. Since this self-tuning solution represents the intrinsic curvature of the system, the fact that it deviates from $-q^2$ shows that the intrinsic curvature can be arbitrarily large depending on the deviation of x_i from x_0. In this sense, the scalar field has not been able to effectively self-tune away an arbitrary Λ_{eff} to recover a small intrinsic curvature.



(a) $H(t)$ (b) $x(t)$

(c) $y(t)$ (d) $z(t)$

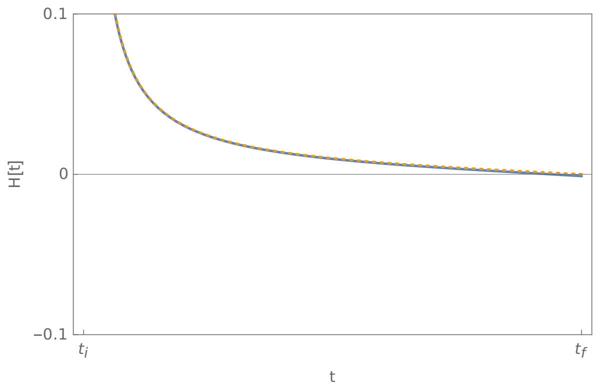
Figure 5.8: These figures showcase the numerical evolution of the system when $x_i = 0.99x_0$ (i.e. the initial condition does not correspond to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q \cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = 10^{-2}$, $\mu = 10$, which sets $x_0 \sim 3.43 \times 10^{-3}$ and is consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 100$, $x_i = 0.99x_0$, $y_i = 0$. Note that $H(t)$ rapidly decreases from its initial value. Therefore, we have chosen to restrict the vertical axis of fig. 5.8a to better display its evolution.

Now we want to force $x(t)$ to be dynamical in an attempt to force $x(t) \to x_0$ at late times, whilst keeping within the constraints set by eqs. (5.26) and (5.28). In doing so we hope that $z(t) \to -q^2$ such that the system is evolving with a small intrinsic curvature, having cancelled off an arbitrary Λ_{eff}. To do this we deviate y_i from its scaling solution, 0, which is equivalent to giving $x(t)$ a "kick". In fig. 5.9 we use similar parameters to fig. 5.2, but we set $x_i = 0.995x_0$ and $y_i = 400$. Notice that $H(t)$ (fig. 5.9a) seems to reside on its scaling solution (but slightly deviates from it at late times). Similarly, $z(t)$ (fig. 5.9e) seems to approach a constant which deviates from $-q^2$ (by about $0.1q^2$). Turning our attention to $x(t)$ (fig. 5.9b), it appears to have a large increase in a short amount of time before settling into a constant. This is evidenced by examining $y(t)$ at early times, which swiftly approaches its scaling solution from a large initial value (fig. 5.9c). Interestingly, by examining $y(t)$ at late times (fig. 5.9d), we see that it does not seem to approach its scaling solution. Instead, it changes from its early time trajectory and starts increasing rather than decreasing. This makes sense because at this time $x(t) > x_0$, therefore we must have a positive $y(t)$ in order for $x(t) \to x_0$. Unfortunately, the increase in $x(t)$ at late times is undetectable in our numerical analysis (even if it can be demonstrated in the behaviour of fig. 5.9d). Furthermore, it is strange that even if $y(t)$ is given a high initial value, it still rapidly decreases before increasing. For this system we also plot $\partial V_{\text{eff}}/\partial\phi$ against x to further understand how the effective potential moves the scalar. Fig. 5.9f implies that the scalar is rolling up the slope (as $\partial V_{\text{eff}}/\partial\phi > 0$) and it appears to be close to 0 at $t = t_f$. This hopefully implies that it then has the intention of rolling down the slope to a minimum at x_0. However, since it does not cross 0 we cannot make too many inferences about the form of V_{eff}.

We then perform the same analysis but this time we set $y_i = 2000$. This ensures that the $x(t)$ reaches and surpasses x_0 (fig. 5.10b). Firstly we will address the similarities. $H(t)$ (fig. 5.10a) seems to reside on its scaling solution. But we see that $z(t)$ (fig. 5.10e) seems to approach a constant at late times that deviates from $-q^2$ (by about $0.05q^2$). This implies there is a small deviation in $H(t)$ from its scaling solution at late times. At early times $y(t)$

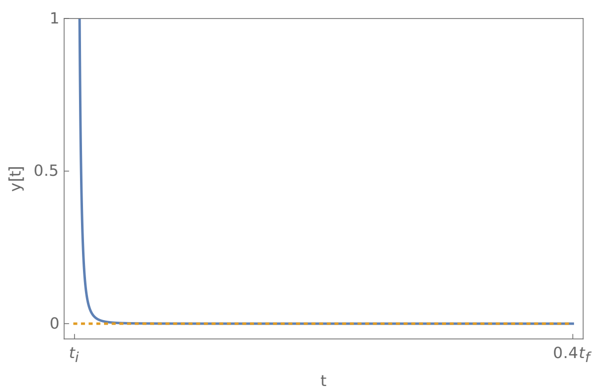
(fig. 5.10c) once again decreases rapidly, approaching the scaling solution. Examining $y(t)$ at late times (fig. 5.10d), we find that it does not increase after decreasing (in contrast to fig. 5.9d). This makes sense as the high initial value of y_i raised $x(t) > x_0$. However, what we would expect is that $y(t)$ becomes negative in an attempt to decrease $x(t) \to x_0$. Instead, we find that $y(t)$ at late times seems to approach a constant. Considering $\partial V_{\text{eff}}/\partial\phi$ against x (fig. 5.10f) we find that the scalar seems to roll up the slope of the potential (as $\partial V_{\text{eff}}/\partial\phi > 0$). This is despite moving past our scaling solution, x_0. However, once again, it does not cross the horizontal axis, therefore we cannot make any strong inferences about the form of V_{eff}.

Next, we examine a similar scenario, but this time we set $x_i = 1.005x_0$. Unfortunately, setting $y_i < 0$ (which we would like to do since $x_i > x_0$) seems to push the numerical solver beyond its validity. The reasons for this are unknown, and $y_i < 0$ remains as an unexplored region. Despite this, setting $y_i = 1$ and observing its evolution still has interesting results. Once again, $H(t)$ (fig. 5.11a) resides on its scaling solution but slightly deviates from it late times. Also, $z(t)$ (fig. 5.11e) seems to approach a constant that deviates from $-q^2$ (by about $0.1q^2$). $x(t)$ (fig. 5.11b) appears to remain at its initial value, but as we have seen before, it is far more instructive to track the evolution of $y(t)$. At early times $y(t)$ (fig. 5.11c) quickly decreases to its scaling solution. However, at late times $y(t)$ (fig. 5.11d) evolves past axis, and becomes negative. This is in contrast to (fig. 5.10d, which stays positive despite $x(t) > x_0$. Turning our attention to $\partial V_{\text{eff}}/\partial\phi$ against x (fig. 5.11f), the scalar seems to once again roll up the slope of a potential. As in figs. 5.10f and 5.11f it does not cross the axis, so we cannot make too many inferences about the form of V_{eff}.

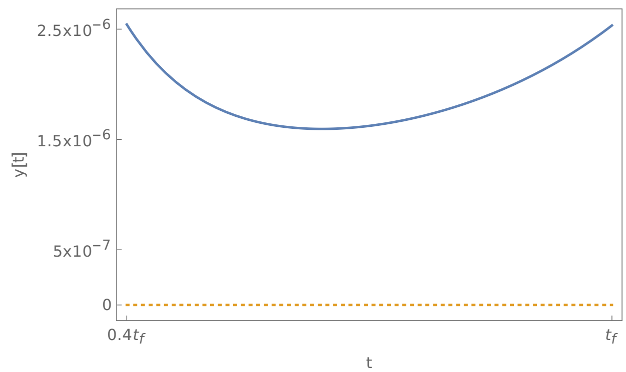


(a) $H(t)$

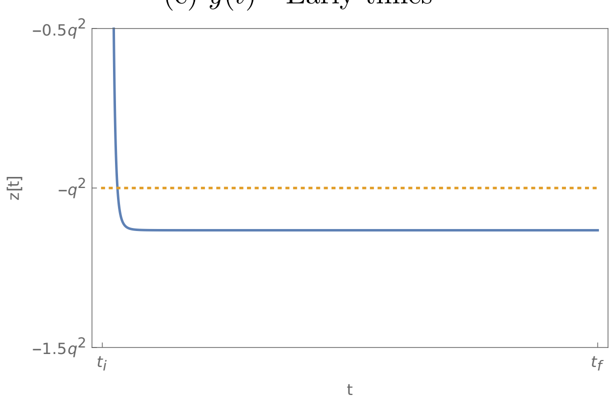
(b) $x(t)$



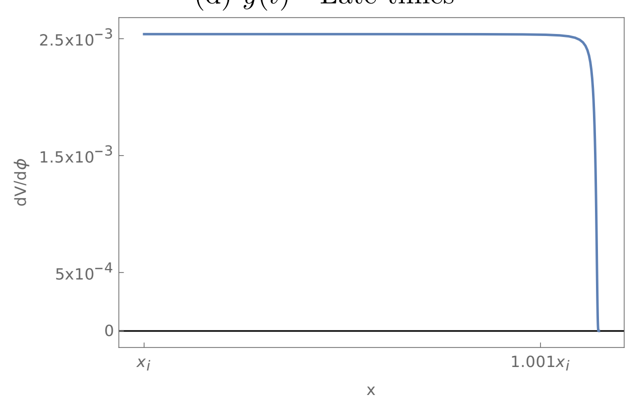
(c) $y(t)$ - Early times



(d) $y(t)$ - Late times

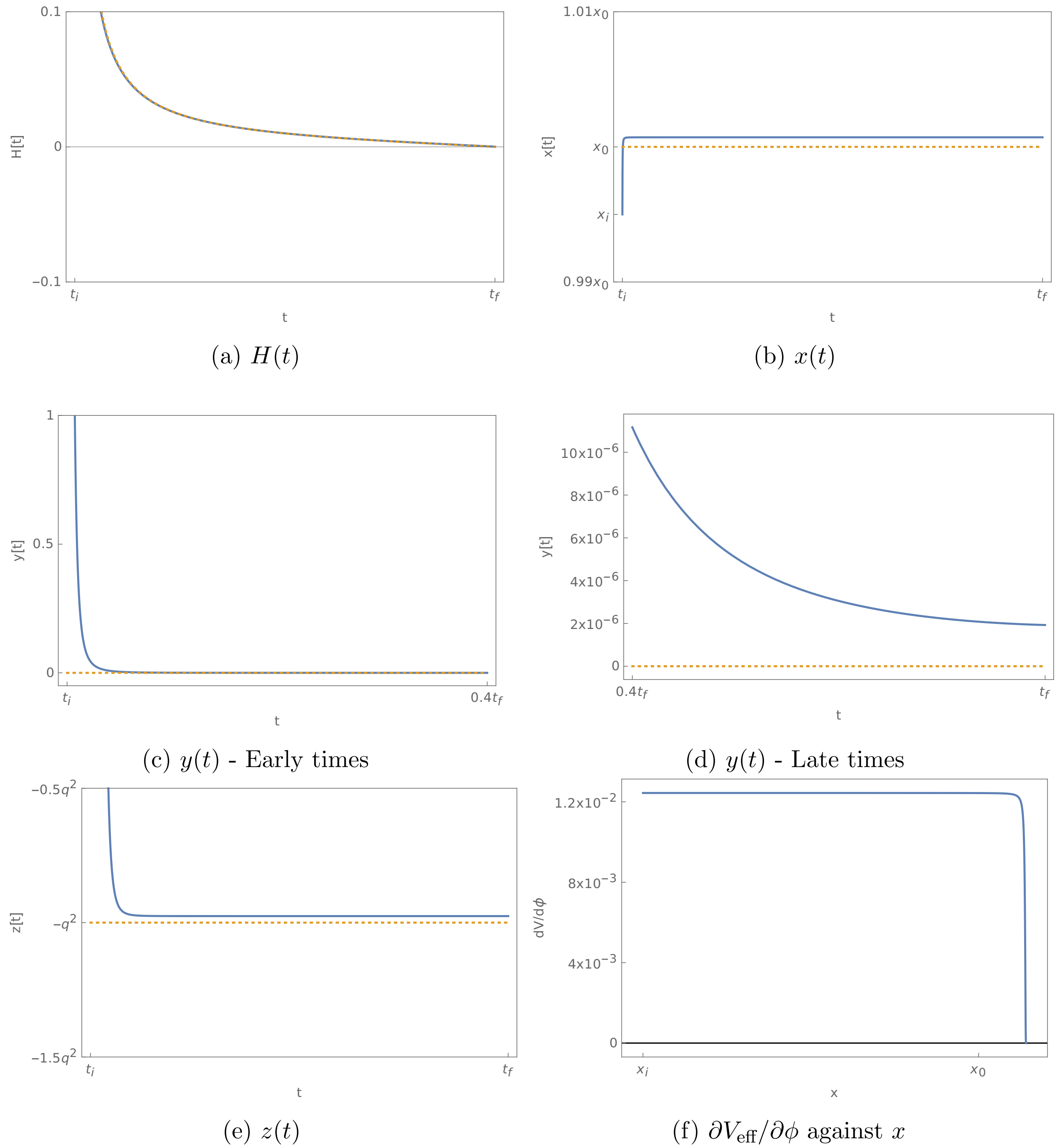


(e) $z(t)$



(f) $\partial V_{\text{eff}}/\partial\phi$ against x

Figure 5.9: These figures showcase the numerical evolution of the system when $x_i = 0.995x_0$ (i.e. the initial condition does not correspond to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q\cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = -10^{-2}$, $\mu = 1$, which sets $x_0 \sim 3.43 \times 10^{-3}$ and is consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 10$, $x_i = 0.995x_0$, $y_i = 400$. Note that $H(t)$, $y(t)$, and $z(t)$ rapidly decreases from their initial values. Therefore, we have chosen restrict the vertical axis of figs. 5.9a, 5.9c, and 5.9e to better display the evolution of each variable.



(a) $H(t)$

(b) $x(t)$

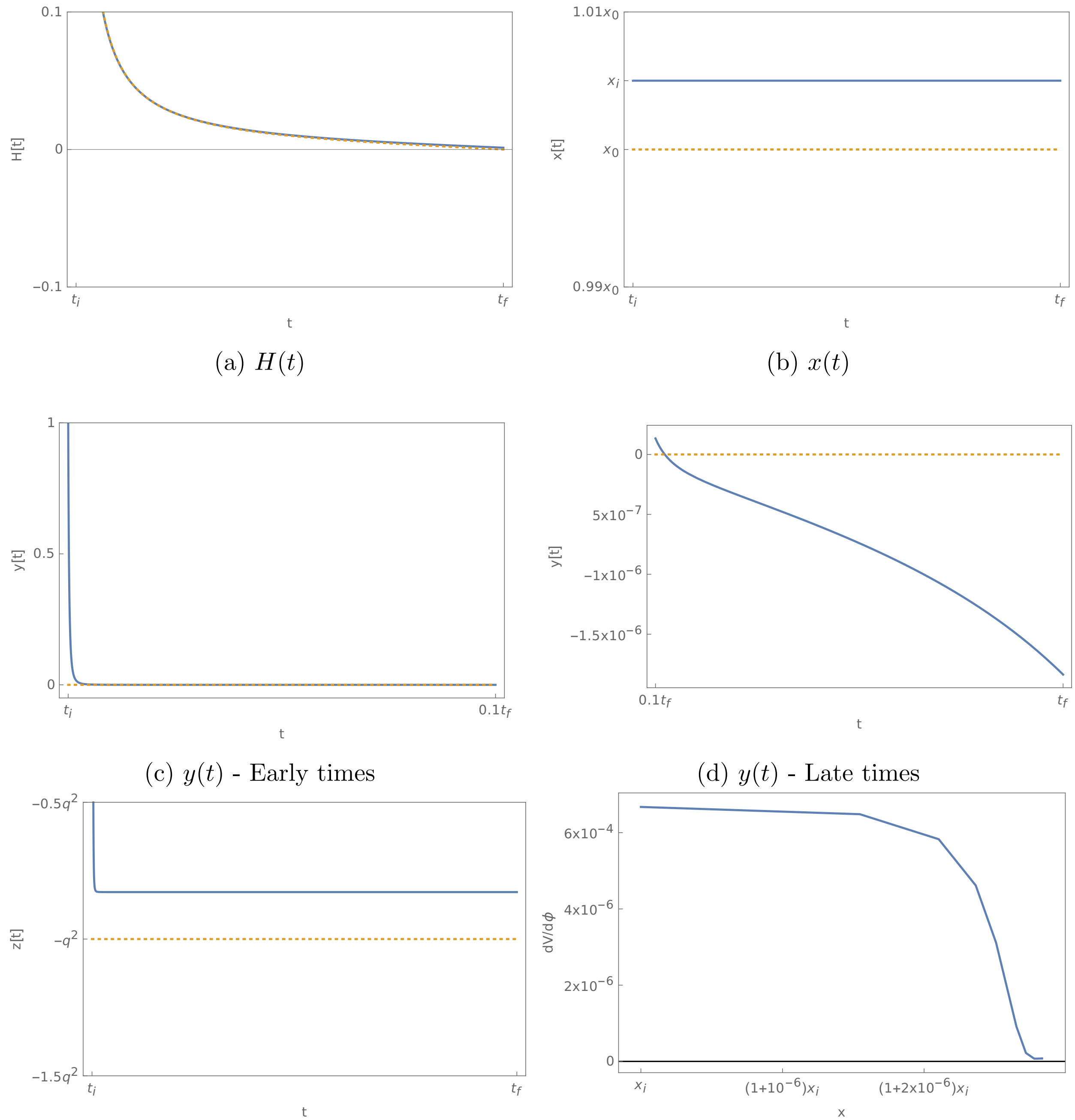
(c) $y(t)$ - Early times

(d) $y(t)$ - Late times

(e) $z(t)$

(f) $\partial V_{\text{eff}}/\partial\phi$ against x

Figure 5.10: These figures showcase the numerical evolution of the system when $x_i = 0.995x_0$ (i.e. the initial condition does not correspond to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q\cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = -10^{-2}$, $\mu = 1$, which sets $x_0 \sim 3.43 \times 10^{-3}$ and is consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 10$, $x_i = 0.995x_0$, $y_i = 2000$. Note that $H(t)$, $y(t)$, and $z(t)$ rapidly decreases from their initial values. Therefore, we have chosen to restrict the vertical axis of figs. 5.10a, 5.10c, and 5.10e to better display the evolution of each variable.



(a) $H(t)$

(b) $x(t)$

(c) $y(t)$ - Early times

(d) $y(t)$ - Late times

(e) $z(t)$

(f) $\partial V_{\text{eff}}/\partial\phi$ against x

Figure 5.11: These figures showcase the numerical evolution of the system when $x_i = 1.005x_0$ (i.e. the initial condition does not correspond to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q\cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = -10^{-2}$, $\mu = 1$, which sets $x_0 \sim 3.43 \times 10^{-3}$ and is consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 10$, $x_i = 1.005x_0$, $y_i = 1$. Note that $H(t)$, $y(t)$, and $z(t)$ rapidly decreases from their initial values. Therefore, we have chosen to restrict the vertical axis of figs. 5.11a and 5.11e to better display the evolution of each variable.

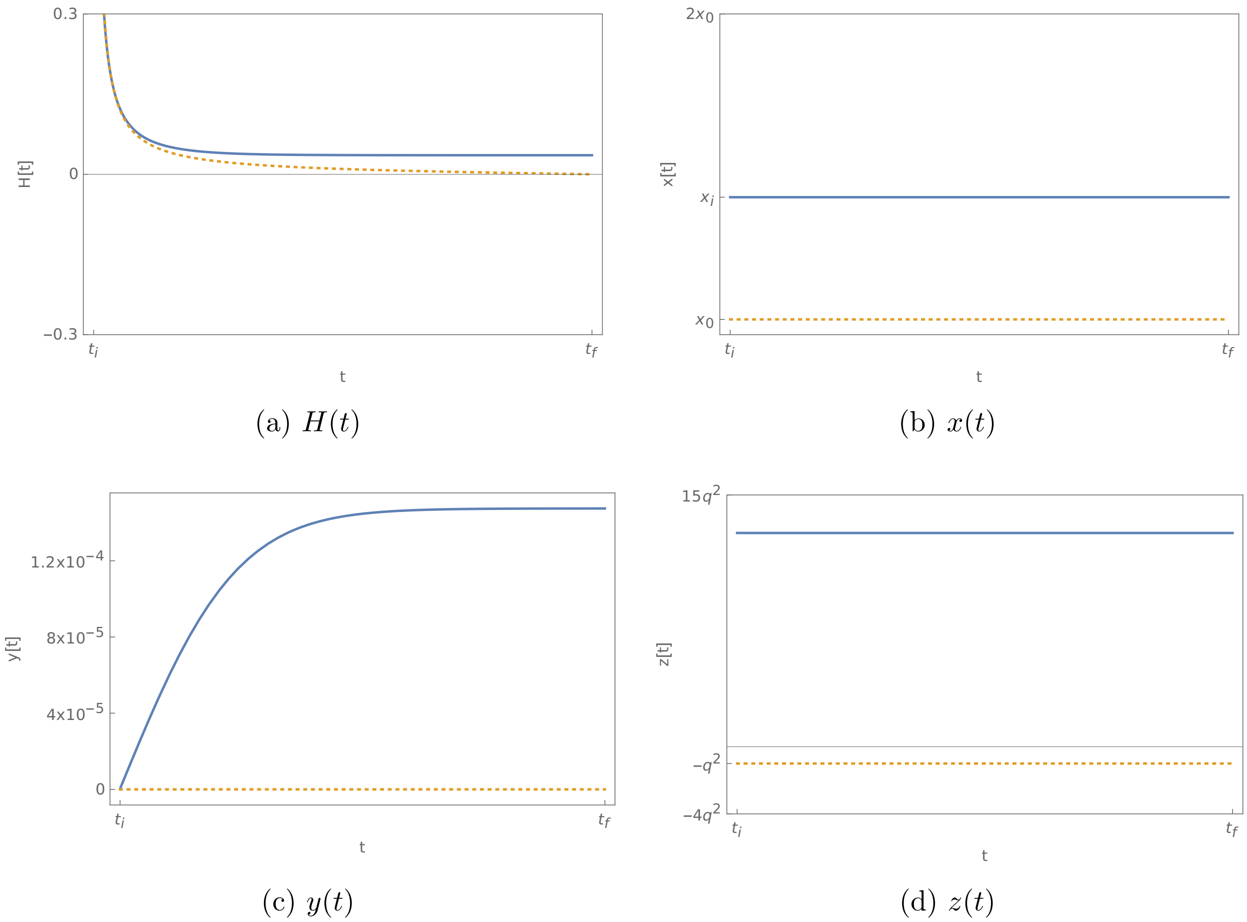


Figure 5.12: These figures showcase the numerical evolution of the system when $x_i = 1.4x_0$ (i.e. the initial condition does not correspond to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q \cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = -10^{-2}$, $\mu = 1$, which sets $x_0 \sim 3.43 \times 10^{-3}$ and is consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 10$, $x_i = 1.4x_0$, $y_i = 0$. Note that $H(t)$ rapidly decreases from its initial value. Therefore, we have chosen to restrict the vertical axis of fig. 5.12a to better display its evolution.

Finally, for completeness we consider what happens when we "push" the initial value of x_i too much. Here, we find that "pushing" too far in the direction of $x_i < x_0$ moves our system beyond the validity of the numerical solver. However, we can "push" x_i further if $x_i > x_0$. In fig. 5.12 we have set $x_i = 1.4x_0$ and $y_i = 0$. Here, $H(t)$ (fig. 5.12a) seems to reside on the scaling solution before deviating from it at late times. Likewise, $z(t)$ (fig. 5.12d) remains at a constant that deviates by a large amount away from $-q^2$ (by about $14q^2$). This is not too surprising as $x(t)$ (fig. 5.12b) seems to remain at x_i, therefore we are currently deviating from our scaling solutions. However despite setting $y_i = 0$, $y(t)$ (fig. 5.12c) appears to evolve in the positive direction (when $x(t) > x_0$). This implies that the scalar is trying to move away from the scaling solution, not towards it.

The above shows that even when we have given the scalar a “kick” (by setting $y_i \neq 0$) the scalar still does not find its scaling solution. Likewise, $z(t)$ approaches a constant that differs from its scaling solution. In this sense, we cannot generically recover a small AdS because $z(t)$ does not seem to select $-q^2$ consistently. This means that the curvature (represented by $-q^2$) can be large, showing that the scalar has been unable to effectively counter an arbitrarily large cosmological constant unless x_i and y_i have been fine-tuned. Therefore we have been unable to self-tune without fine-tuning the initial value of the scalar field.

To conclude, for this specific set of scaling equations, the system can recover the scaling solutions when initially placed there (fig. 5.2). In fact, we find that $H(t)$ approaches its scaling solution for a variety of initial conditions and parameters (figs. 5.3 and 5.4). But, we find that the scaling solutions are not recovered when we perturb x_i from x_0 (figs. 5.5, 5.7, and 5.8). However, (as fig. 5.6 suggests) $x(t)$ still dynamically evolves slowly towards its scaling solution. To further test this we give $x(t)$ a “kick” using $y_i \neq 0$. In figs. 5.9 to 5.12 we find that $H(t)$ seems to find its scaling solution at early times, even if it deviates from it at late times. Similarly, $z(t)$ very quickly settles into a constant that deviates from its scaling solution (as a reminder $z(t)$ could be moving so slowly that its evolution cannot be detected by our numerical solver). We also observe that $y(t)$ seems to approach its scaling solution at early times after it has given a “kick” to $x(t)$. Then at late times, it takes on a non-zero value dependent on the value of $x(t)$ at the time. In general, it appears that $x(t)$ can dynamically change its value to try and recover the scaling solutions (at least for small deviations from x_i), but it does so incredibly slowly. In this sense, we have only been able to exactly recover the scaling solutions through fine-tuning the initial values of the scalar (x_i and y_i).

Note, that this could be an artefact of the method we have chosen rather than it being intrinsic to the model. Firstly, we have used a couple of assumptions by consistently setting $N(t) = 1$, $\mathcal{G} = \frac{M_{\rm Pl}^2}{2}$, $\mathcal{K} = \frac{1}{2}$, and $V_r = \frac{\phi}{\mu}$, due to the complexity of the numerical analysis. Changing the form of these variables (in particular V_r) may result in alleviating the fine-tuning presented. Solving these equations via a different method could also eliminate this fine-tuning.

As we have previously alluded to, understanding the nature of this fine-tuning requires a stability analysis of the cosmological equations. That is to say, we want to perturb the cosmological variables about their background solutions to understand why the scalar seems to require such a great deal of fine-tuning. To this end, we perform a linear stability analysis [201] on our dynamical system. Our system (eqs. (5.23) to (5.25)) can be written in matrix form:

$$\begin{pmatrix} \dot{H} \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f - H^2 \\ \frac{8q^4}{\mu}\sqrt{6}\, y \\ \frac{g}{\sqrt{6}} \end{pmatrix}, \tag{5.32}$$

which we can write as $\dot{\boldsymbol{m}} = \boldsymbol{M}(\boldsymbol{m})$. Here, $\boldsymbol{m} = (H, x, y)$ and $\boldsymbol{M}(\boldsymbol{m}) = (M_1(\boldsymbol{m}), M_2(\boldsymbol{m}), M_3(\boldsymbol{m}))$ as defined by eq. (5.32). Now, we can perturb this system with $\boldsymbol{m} \to \bar{\boldsymbol{m}} + \delta\boldsymbol{m}$, where $\bar{\boldsymbol{m}}$ and $\delta\boldsymbol{m}$ are the background values and the perturbed values of $\boldsymbol{m}$ respectively. From this, we can write a perturbed matrix system such that

$$\begin{pmatrix} \delta\dot{H} \\ \delta\dot{x} \\ \delta\dot{y} \end{pmatrix} = \boldsymbol{M_p} \begin{pmatrix} \delta H \\ \delta x \\ \delta y \end{pmatrix}, \tag{5.33}$$

where

$$\boldsymbol{M_p} = \begin{pmatrix} \frac{\partial f}{\partial H} - 2\bar{H} & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ 0 & 0 & \frac{8q^4}{\mu}\sqrt{6} \\ \frac{1}{\sqrt{6}}\frac{\partial g}{\partial H} & \frac{1}{\sqrt{6}}\frac{\partial g}{\partial x} & \frac{1}{\sqrt{6}}\frac{\partial g}{\partial y} \end{pmatrix}, \tag{5.34}$$

where each partial derivative is evaluated on the scaling solution, $\bar{\boldsymbol{m}}$. Recall, that the definitions of f and g are given by eqs. (5.21) and (5.22) respectively. Now, by computing the eigenvalues, λ_i, we can study the linear stability of our system. According to linear stability theory, a solution to our system constitutes a linear combination of $e^{\lambda_i t}$. Therefore, if all the real parts of λ_i are negative ($\mathrm{Re}(\lambda_i) < 0$), the system is stable for linear perturbations since the exponent converges. Instead, if the system has eigenvalues that are positive ($\mathrm{Re}(\lambda_i) > 0$), it will be **unstable** as the exponent will diverge. Computing the eigenvalues of eq. (5.34) yields:

$$\left(0, -\frac{8\sqrt{6}}{\sqrt{\mu^2+96}}i, \frac{8\sqrt{6}}{\sqrt{\mu^2+96}}i\right), \tag{5.35}$$

or in other words $\mathrm{Re}(\lambda_i) = 0$. Our linear stability analysis does not provide us with any new information about the stability of the system. To overcome this we must use other techniques, for example the centre manifold theory studies the dynamics of a linear system, but this would be challenging given the complexity of f and g. Another example is using Lyapunov's method (see [201] and references therein), which we can use to explore the dynamics of a system through the construction of a Lyapunov function, $V(\boldsymbol{m})$. It states that a system is stable about $\bar{\boldsymbol{m}}$ if we can find a $V(\boldsymbol{m})$ that obeys a certain set of rules, one of which involves interactions with $\boldsymbol{m}$ and $\boldsymbol{M}(\boldsymbol{m})$. Unfortunately, the only way to find this function is through trial and error which, once again, would be incredibly difficult to find due to the complexity of the system.

5.1 Evading fine-tuning?

In this final section we return to the key question: does our theory truly evade "Weinberg's no go theorem"? Weinberg's no-go theorem only accounts for constant scalar fields on a Minkowski background, so there is no *a priori* reason to reject an AdS background. After all, the translational invariance inherent to Weinberg's construction of the no-go theorem **is** broken by the explicit coordinate dependence of the metric. So we do evade the no-go theorem, but what we are truly asking with this question is can we evade fine-tuning?

Of course, we have already shown that self-tuning without fine-tuning is possible. This can be shown through our exchange amplitude (eq. (4.51) in section 4.2), which satisfies both self-tuning conditions as defined by chapter 3. This is further demonstrated by our numerical analysis in chapter 5. Although the scalar has been unable to reach its scaling solution at late times for $x_i \neq x_0$, there is evidence that that it tries to evolve there (albeit very slowly). Instead, this section is used to **verify** that our model can indeed self-tune without fine-tuning. If this is the case, then it would appear that the difficulties faced in our numerical analysis are not due to fine-tuning. We later offer an alternative explanation as to why our system does not **appear** to self-tune.

To show this, we return to our analysis at the start of chapter 2 where we explored Weinberg's no-go theorem. Recall that we were able to show that self-tuning is not possible for a constant background scalar field in a Minkowski spacetime. Here, we demonstrated that the effective potential, V_{eff}, of the self-tuning scalar field evaluated on the background is used to cancel an arbitrary cosmological constant. However, for small perturbations in the scalar this potential cannot be minimised without fine-tuning. In other words, if a scalar is placed away from its background solution, it cannot return to it to cancel off an arbitrary cosmological constant. This can only be achieved through fine-tuning, which reintroduces the radiative instability inherent to the cosmological constant problem. Therefore, the goal of this section is to provide evidence that the effective potential within **our** model can be minimised for small changes in the scalar.

We start by extending the analysis in chapter 2 to our model by taking the trace of eq. (4.12) which yields

$$\begin{aligned} R = & \frac{3}{\mathcal{G}^2}\Box\mathcal{G}^2 - 24q^2 + \frac{\mathcal{K}}{\mathcal{G}^2}(\nabla\phi)^2 + \frac{2\Lambda_{\text{eff}}}{\mathcal{G}^2} \\ & - m^2\left[- 9(F_E)^\alpha_\alpha + (F_E)^\alpha_\alpha(F_E)^\beta_\beta - (F_E)^\alpha_\beta(F_E)^\beta_\alpha + \frac{48}{\mathcal{G}} - \frac{24}{\mathcal{G}^2}\right]\mathcal{G}^2 \\ & - \frac{4}{\bar{\mathcal{G}}^2}(G^\alpha_\beta\nabla_\alpha\nabla^\beta - 12q^4)V_r. \end{aligned} \tag{5.36}$$

Here we have written the equations in terms of R which represents the curvature and we have used $T_{\mu\nu} = -\Lambda_{\text{eff}}\, g_{\mu\nu}$. Note, we exclude other matter sources as we only want to analyse the self-tuning field, ϕ, alongside the cosmological constant and their effects on the curvature of the system. Now as a check the curvature should be negative ($R < 0$) on the level of the background. For this, we use $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\phi = \bar{\phi} =$ constant, such that eq. (5.36) becomes

$$R = -24q^2 + \frac{2\Lambda_{\text{eff}}}{\bar{\mathcal{G}}^2} - 12m^2\big[\bar{\mathcal{G}} - 1\big] + \frac{48q^4}{\bar{\mathcal{G}}^2}\bar{V}_r. \tag{5.37}$$

Using the self-tuning equation (eq. (4.18)) we find that $R = -12q^2$, which means the curvature is negative and can be small as long as $|q^2| \lesssim |H_0^2|$. This is not surprising as we were able to show, both analytically and numerically, that the model admits to this solution. We cannot immediately identify R with $\frac{\partial V_{\text{eff}}}{\partial\phi}$ (as we did in our analysis for a constant scalar in Minkowski space) as the curvature is non-zero. Instead, we recognise that the right hand side of eq. (5.36) must dynamically adjust to achieve a curvature of $-12q^2$. Therefore, the form of the effective potential must be such that it uses the right hand side of eq. (5.36) to dynamically evolve the scalar to the background solution, but when $\frac{\partial V_{\text{eff}}}{\partial\phi}$ is evaluated at $\phi = \bar{\phi}$ it vanishes. So we can say that

$$\begin{aligned}\frac{\partial V_{\text{eff}}}{\partial \phi} &= \frac{3}{\mathcal{G}^2}\Box\mathcal{G}^2 - 24q^2 + \frac{\mathcal{K}}{\mathcal{G}^2}(\nabla\phi)^2 + \frac{2\Lambda_{\text{eff}}}{\mathcal{G}^2} \\ &- m^2\left[-9(F_E)^\alpha_\alpha + (F_E)^\alpha_\alpha(F_E)^\beta_\beta - (F_E)^\alpha_\beta(F_E)^\beta_\alpha + \frac{48}{\mathcal{G}} - \frac{24}{\mathcal{G}^2}\right]\mathcal{G}^2 \\ &- \frac{4}{\bar{\mathcal{G}}^2}(G^\alpha_\beta\nabla_\alpha\nabla^\beta - 12q^4)V_r + 12q^2,\end{aligned} \tag{5.38}$$

where the right hand side of the above is identical to that of eq. (5.36), aside from a $+12q^2$ which ensures that eq. (5.38) vanishes on the background. Recall that V_{eff} refers to the effective potential of the scalar field.

To study how the scalar self-tunes we consider a metric that lies on the background, $g_{\mu\nu} = \bar{g}_{\mu\nu}$, alongside small perturbations around a constant background scalar, $\bar{\phi}$. Similar to our previous analysis (in chapter 2), this removes the explicit kinetic term coupled to $\mathcal{K}$. Furthermore, recall that $\mathcal{G}$ can be identified with M_{Pl} as discussed in sections 4.1 and 4.2, which means we do not want $\mathcal{G}$ to vary much (if at all). Therefore we set $\mathcal{G} = 1$ such that it cannot vary, which subsequently removes the massive terms[34]. We can also justify the massive term vanishing, as it is truly the Ringo term that accommodates for a large vacuum energy, as shown through the self-tuning equation (eq. (4.18)). With these truncations eq. (5.38) reduces to

$$\frac{\partial V_{\text{eff}}}{\partial \phi} = -12q^2 + 2\Lambda_{\text{eff}} - 4\left(G^\alpha_\beta\nabla_\alpha\nabla^\beta - 12q^4\right)V_r, \tag{5.39}$$

where ϕ represents small perturbations from $\bar{\phi}$ and we have kept the form of $G^\alpha_\beta\nabla_\alpha\nabla^\beta V_r$ for clarity. To address this we use the fact that[35]

$$G^\alpha_\beta\nabla_\alpha\nabla^\beta V_r = \frac{3}{N^4}\left(H^2 - \frac{N^2}{a^2r^2}\right)\left(\partial_t^2 - \frac{\dot{N}}{N}\partial_t\right)V_r, \tag{5.40}$$

[34]Note, we can make other constant substitutions for the $\mathcal{G}$. The key point is that $\mathcal{G}$ "loses" its ϕ dependence.

[35]We have derived this using our ansatz metric $g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -N(t)^2 + a(t)^2\mathrm{d}\mathcal{H}_3^2$, where $N(t)$ is the lapse function, $a(t)$ is the scale factor, and $\mathrm{d}\mathcal{H}_3^2 = r^2\left(\mathrm{d}\theta_1^2 + \sinh^2\theta_1\left(\mathrm{d}\theta_2^2 + \sin^2\theta_2\,\mathrm{d}\theta_3^2\right)\right)$.

and we set the metric on the background (such that $a = a_0 \equiv \frac{\sin qt}{q}$, $N = 1$ and $r = 1$) to obtain

$$G^\alpha_\beta \nabla_\alpha \nabla^\beta V_r = -3q^2 \left(\frac{\partial^2 V_r}{\partial \phi^2} \dot{\phi}^2 + \frac{\partial V_r}{\partial \phi} \ddot{\phi} \right). \tag{5.41}$$

However, recall that we are only considering small changes around a constant background scalar. This means that the $\dot{\phi}^2$ term will vanish, hence we can substitute eq. (5.41) back into eq. (5.39) to obtain

$$\frac{\partial V_{\text{eff}}}{\partial \phi} = -12q^2 + 2\Lambda_{\text{eff}} + 12q^2 \left(\left. \frac{\partial V_r}{\partial \phi} \right|_{\bar{\phi}} \ddot{\phi} + 4q^2 V_r \right). \tag{5.42}$$

First, notice that on a Minkowski background the above potential cannot be minimised unless $\Lambda_{\text{eff}} = 0$, which corresponds to fine-tuning. Whereas on an AdS background, the $\ddot{\phi}$ term opens up a pathway for V_{eff} to be minimised. This is a demonstration of how our model can accommodate for an effective potential that can be minimised without fine-tuning. Note, there are still some assumptions that we could avoid to further generalise the above analysis. In eq. (5.36) there is a G^α_β term coupled to V_r. Similar to R, this will have a non-trivial effect on the curvature, but we have set G^α_β on the background to simplify our analysis. Another key assumption we made is to set $\mathcal{G} =$ constant. Whilst it is still true that we do not want this value to vary much, a non-constant $\mathcal{G}$ could still provide some non-trivial effects. Even with these assumptions we have demonstrated that there is now a possibility of minimising the effective potential with the Ringo term on an AdS background.

With this in mind we search for other pieces of evidence that suggest that the effective potential can be minimised. For this we return to our expression for the KG equation with an effective potential (eq. (5.30)). Applying this to the scalar equation (eq. (4.13)) yields

$$\frac{\partial V_{\text{eff}}}{\partial\phi} = -\frac{1}{2\mathcal{K}}\bigg\{2\mathcal{G}\mathcal{G}'\big(R+12q^2\big)+\mathcal{K}'\big(\nabla\phi\big)^2 \\ -\frac{m^2}{4}\mathcal{G}^3\mathcal{G}'\bigg[4\mathcal{U}(\Sigma)-\text{Tr}\bigg(\Sigma\frac{\partial\mathcal{U}}{\partial\Sigma}\bigg)\bigg]_{\frac{1}{\mathcal{G}}\mathbb{1}}^{F_E}+V_r'\big[G_{GB}-24q^4\big]\bigg\}. \tag{5.43}$$

As before, we set the metric on the background and consider small changes in the scalar, which causes the right hand side of the equation to vanish[36]. Therefore, for small perturbations in the scalar, the effective potential (when constructed in this fashion) can also be generically minimised.

To numerically verify this claim we reconsider eq. (5.31), which is equivalent to eq. (5.43) using our previous cosmological setup in eqs. (5.23) to (5.25)[37]. Within figs. 5.13 and 5.14 we plot the derivative of the effective potential against x (the scalar field). Previously we have displayed these plots in figs. 5.6b, 5.9f, 5.10f, and 5.11f. But we found that x evolves too slowly to reach its scaling solution at late times, therefore $\frac{\partial V_{\text{eff}}}{\partial\phi}$ does not cross the horizontal axis. This is due to fact that the scalar in the "weakly coupled regime" (where $g_\phi \lesssim \mathcal{O}(10^{-5})$). To fix this we set $\mu = 10^{-6}$, which sets $g_\phi \sim \mathcal{O}(10^{-1})$ such that the scalar field is in the "strongly coupled regime", allowing x to be more dynamical. Now, our choice for μ is no longer consistent with eqs. (5.26) and (5.28), but this value is chosen to better display how the effective potential evolves with x. To be clear, this is far beyond the allowed bounds for the scalar coupling to be consistent with solar system constraints of GR. But its inclusion here is to demonstrate the behaviour of the scalar by increasing the strength of the coupling. In a way, this is not a representation of what our model can do, rather it demonstrates what our model is **trying** to do. Note, that by increasing the scalar coupling and only allowing for small perturbations from $x_i = x_0$, we find that we can let $qt > \frac{\pi}{2}$ without encountering numerical

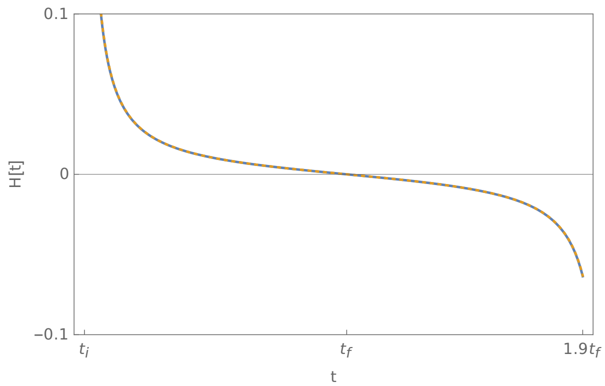
[36]Recall, that we have purposely constructed the scalar equation (eq. (4.13)) to vanish for the background solution. The only non-trivial part is $(\nabla\phi)^2$, which clearly vanishes for small perturbations in ϕ.

[37]Recall, this involves making the explicit choice: $\mathcal{K} = \frac{1}{2}$, $\mathcal{G}^2 = \frac{M_{\text{Pl}}^2}{2}$, $V_r = \frac{\phi}{\mu}$ with $M_{\text{Pl}} = 1$. Alongside a change in coordinates: $\phi(t) \equiv \frac{\mu}{8q^4}x(t)$, $\dot{\phi}(t) \equiv \sqrt{6}\,y(t)$, $\alpha \equiv \frac{a_0}{a}$.

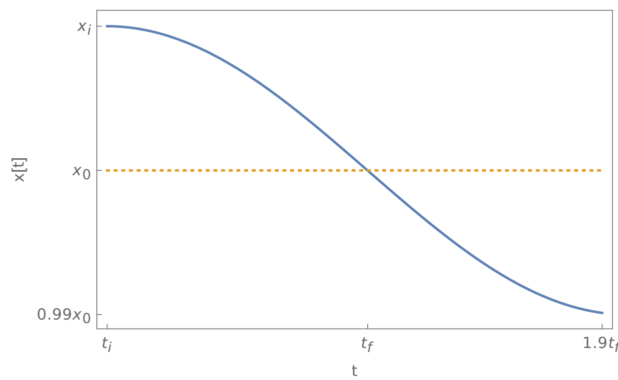
instabilities. But for consistency we define $t_f = \frac{\pi}{2q}$ whilst allowing for the system to evolve beyond that. Ideally we would like the system to evolve til $t = 2t_f = \frac{\pi}{q}$ as this represents a full cosmological cycle (see fig. 5.1). However we do encounter numerical instabilities at this point, possibly due to $a_0(\frac{\pi}{q}) = \frac{\sin \pi t}{q} = 0$ which will create instabilities in $H(t) \equiv \frac{\dot{a}}{a}$.

In fig. 5.13 we set $x_i = 1.01x_0$ to find that $H(t)$ and $z(t)$ (figs. 5.13a and 5.13d) quickly approach their respective scaling solutions. There is some deviation in $z(t)$ at late times (fig. 5.13e), however this could be due to the system approaching the aforementioned numerical instabilities. Interestingly, with an increased scalar coupling, $x(t)$ (fig. 5.13b) approaches and reaches its scaling solution at $t = t_f$ contrary to our previous numerical analysis. Note that this all happens during the "expansion era" (see fig. 5.1). However, for $t > t_f$ it seems to move away from its scaling solution, which corresponds to the "crunch era". To understand this, it is instructive to see how $y(t)$ (fig. 5.13c) behaves. At $t = t_f$ it reaches its maximum, but beyond this $y(t)$ decreases. This could imply some sort of oscillatory behaviour in the scalar. However, we cannot extend t beyond $2t_f$ to study this. Firstly due to the numerical instabilities, but more importantly evolving beyond this implies that the Universe is in a different cosmological cycle. But it is still interesting that with an increased scalar coupling, the system seems to remain on an AdS evolution where $z(t) \to -q^2$ with $x_i \neq x_0$. This is despite the fact that $x(t)$ does not settle into its scaling solution, x_0, for late times as we previously expected. Therefore the scalar field has been able to effectively counter a large cosmological constant, whilst deviating from its self-tuning scaling solution, to recover a Universe with a small intrinsic curvature. This is in contrast to previous attempts within the weak scalar coupling regime. When in the strong scalar coupling regime, the scalar can now freely evolve in order to cancel off an arbitrary Λ_{eff}. This shows that the increase in the scalar coupling strength has played a pivotal role in retrieving a small AdS solution. To reiterate, this regime is outside the allowed bounds of solar system tests, but it demonstrates what our system is trying to do. We can further examine the derivative of the effective potential against $x(t)$ in fig. 5.13f. This plot starts positive, crosses the horizontal axis at x_0, and then becomes negative. This could imply one of two things. It could imply that the scalar has rolled up the slope of the potential onto an unstable equilibrium point (at x_0), rather than settling into a global minimum. Or it could imply that the scalar is rolling up the slope and reaches its peak at x_0. After this it will roll down to a minimum that is $\neq x_0$. Both scenarios are not ideal, but they both demonstrate that the effective potential can be minimised.

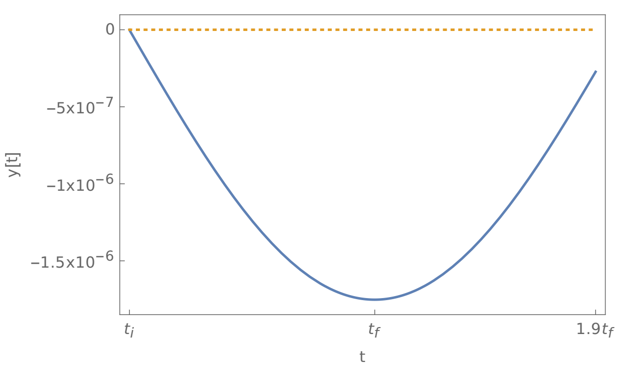
We further study the same example but instead we set $x_i = 0.99x_0$. Once again, we find that $H(t)$ and $z(t)$ (figs. 5.14a and 5.14d) quickly approach their respective scaling solutions. As before, there is a slight deviation in $z(t)$ at late times, which could be traced back to numerical instabilities. Similarly, $x(t)$ (fig. 5.14b) approaches and reaches its scaling solution at $t = t_f$, but moves beyond x_0 for $t > t_f$. Similar to before there is a turning point in $y(t)$ (fig. 5.14c) at $t = t_f f$ which could imply some sort of oscillatory nature in $x(t)$. Once again, despite deviating from its scaling solution, the scalar field has been effectively able to recover a small AdS solution (as shown by $z(t)- > q^2$) for an arbitrary Λ_{eff}. However, the behaviour of the effective potential is slightly altered in fig. 5.14f. Similar to fig. 5.13f, $\frac{\partial V_{\text{eff}}}{\partial \phi}$ crosses the horizontal axis at x_0. Unlike fig. 5.13f, $\frac{\partial V_{\text{eff}}}{\partial \phi}$ (in fig. 5.14f) is initially negative and becomes positive after crossing the axis. This implies that the scalar is rolling down the effective potential to x_0, which corresponds to its minimum. After this it then rolls back up to the potential away from x_0, but this is with the expectation that it will follow the slope back to the minimum. This is exactly the type of behaviour needed in order to evade fine-tuning.



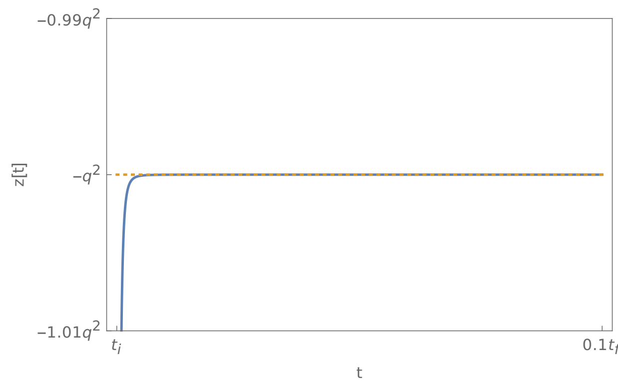
(a) $H(t)$



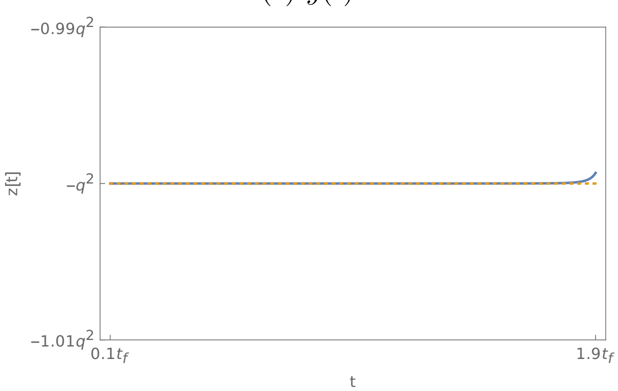
(b) $x(t)$



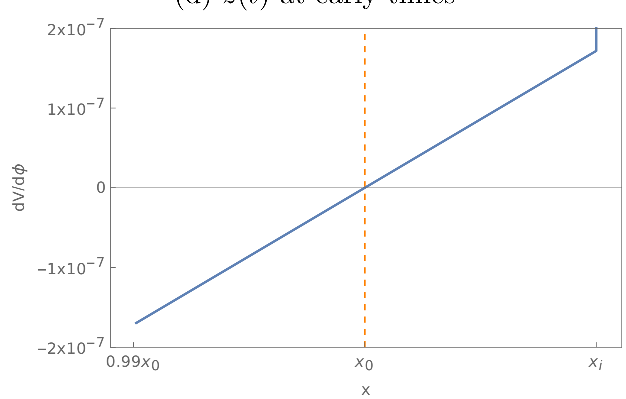
(c) $y(t)$



(d) $z(t)$ at early times

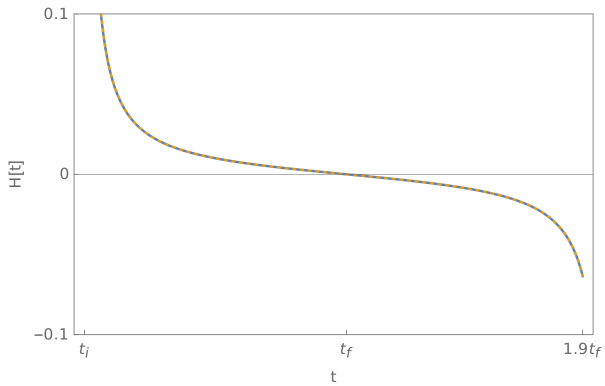


(e) $z(t)$ at late times

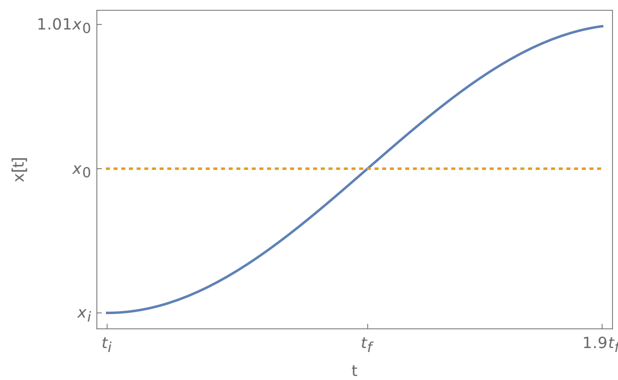


(f) $\partial V_{\text{eff}}/\partial\phi$ against x

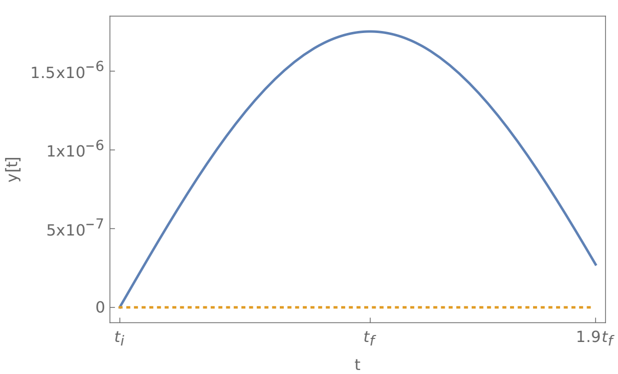
Figure 5.13: These figures showcase the numerical evolution of the system when $x_i = 1.01x_0$ (i.e. the initial condition does not correspond to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q\cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = -10^{-2}$, $\mu = 10^{-6}$, which sets $x_0 \sim 3.43 \times 10^{-3}$. But $g_\phi \sim \mathcal{O}(10^{-1})$, therefore it is no longer consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 10$, $x_i = 1.01x_0$, $y_i = 0$. Note that $H(t)$ and $z(t)$ rapidly change from their initial values. Therefore, we have chosen to restrict the vertical axis of figs. 5.13a and 5.13d to better display the evolution of each variable.



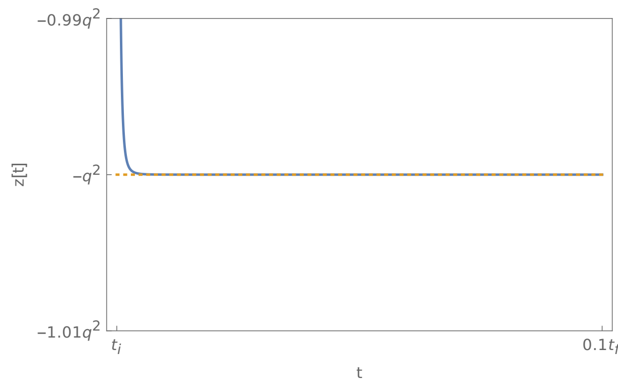
(a) $H(t)$



(b) $x(t)$

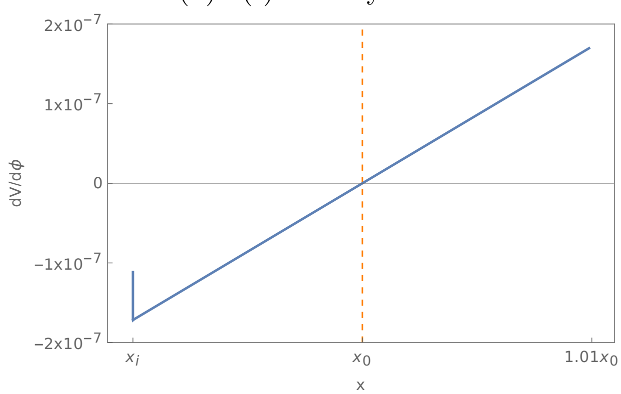


(c) $y(t)$



(d) $z(t)$ at early times

(e) $z(t)$ at late times



(f) $\partial V_{\text{eff}}/\partial\phi$ against x

Figure 5.14: These figures showcase the numerical evolution of the system when $x_i = 0.99x_0$ (i.e. the initial condition does not correspond to the scaling solution). The blue line represents the evolution of the system, whereas the orange dashed line represents the scaling solution of the given variable, where $H(t) \to q\cot qt$, $x(t) \to x_0$, $y(t) \to 0$, and $z(t) \to -q^2$. The parameters are chosen such that $q^2 = 10^{-4}$, $m^2 = 10^{-5}q^2$, $\Lambda_{\text{eff}} = -10^{-2}$, $\mu = 10^{-6}$, which sets $x_0 \sim 3.43 \times 10^{-3}$. But $g_\phi \sim \mathcal{O}(10^{-1})$, therefore it is no longer consistent with eqs. (5.26) and (5.28). The initial conditions chosen are $H_i = 10$, $x_i = 0.99x_0$, $y_i = 0$. Note that $H(t)$ and $z(t)$ rapidly change from their initial values. Therefore, we have chosen to restrict the vertical axis of figs. 5.14a and 5.14d to better display the evolution of each variable.

We conclude by reiterating that the purpose of this section is to show that our model truly evades fine-tuning by attempting to minimise its effective potential. In this section we have offered several pieces of evidence that demonstrate this. Firstly, we have shown that the effective potential (in a setup similar to our analysis of Weinberg's no-go theorem in chapter 2) has the possibility of being minimised. Next, we constructed a separate effective potential using our KG equation, and analytically showed that this can be generically minimised for small perturbations in the scalar. To verify this numerically we returned to our cosmological analysis whilst greatly increasing the scalar coupling and studying the corresponding dynamics for $x_i \neq x_0$. Here, we showed that an increase in the scalar coupling allows $x(t)$ to reach x_0 at $t = t_f = \frac{\pi}{2q}$ (in contrast to our numerical analysis in chapter 5). Strangely, the scalar evolves beyond its scaling solution for $t > t_f$, but the system remains on an AdS evolution until late times. By specifically examining $\frac{\partial V_{\text{eff}}}{\partial \phi}$ in fig. 5.13 we found that it crosses the horizontal axis at x_0, but is initially positive. We can infer two different behaviours from this plot, but the key point is that the effective potential can be minimised. We also found that the plot of $\frac{\partial V_{\text{eff}}}{\partial \phi}$ in fig. 5.14 crosses the horizontal axis at x_0. Since it is initially negative, this implies that the scalar is rolling down the effective potential to its minimum: x_0. These pieces of evidence together, alongside the analysis contained within section 4.2 and chapter 5, heavily imply that our model can self-tune without fine-tuning. It also implies that the previous difficulties with self-tuning (without fine-tuning) are due to the coupling of the scalar: specifically that it is too weak.

Chapter 6

Discussions and future avenues

This thesis has explored modifications to GR, such that the observational value of the cosmological constant better matches our theoretical understanding. Specifically we have focused on self-tuning, which requires adding extra fields to GR to achieve this.

In chapter 1 we discussed how GR and predictions within QFT combine to create the cosmological constant problem. QFT predicts the existence of vacuum energy, and GR predicts that it will gravitate as a cosmological constant, Λ_{vac}. Even a lower estimate of its theoretical value far surpasses the observational value, Λ_{obs}. However as we emphasised, the real problem is the radiative instability of vacuum energy. Naively, we can add a finely-tuned counterterm, Λ_{bare}, to "eat up" Λ_{vac}. But vacuum energy is incredibly sensitive to higher order perturbations, which will lead to repeated fine-tunings in the counterterm. This process of fine-tuning and re-tuning is something we want to avoid.

In chapter 2 we reviewed a range of self-tuning theories. By self-tuning we mean that additional fields act to naturally reduce the effective value of the cosmological constant. As explained previously, to expose any hidden fine-tunings a self-tuning theory should also admit to an arbitrary bare cosmological constant term. Setting $\Lambda_{\text{bare}} = 0$ corresponds to a fine-tuning between it and Λ_{vac}. Therefore, the real goal is to construct a model where $\Lambda_{\text{bare}} + \Lambda_{\text{vac}} = \Lambda_{\text{eff}} \sim \Lambda_{\text{obs}}$. First, we considered a modified version of GR with a scalar field that is constant on the background to show that it cannot self-tune without fine-tuning. In doing so we demonstrated the essence of Weinberg's no-go theorem, which places some very general constraints on self-tuning theories. Then, we reviewed a range of self-tuning theories which seek to avoid this. Here, we will briefly outline each section.

We began with **unimodular gravity** (section 2.1) which involves restricting the allowed values of the determinant of the metric. In a sense, this will prevent the vacuum terms from gravitating. At first glance it appeared to work well. But on closer inspection we found that the cosmological constant terms are just reintroduced via a constraint, and that the field equations become indistinguishable from GR (at least classically).

In section 2.2 we showed that **Fab Four** was able to write the most general linear scalar-tensor action (with field equations up to second order) that can self-tune using Horndeski's action. Using various self-tuning conditions, this complex action can be reduced to a linear combination of four Lagrangians. **Fab Five** (section 2.3) constructs a more general action by combining the Lagrangians non-linearly. We later discovered that both "Fabs" self-tune away a large cosmological constant, but they also self-tune away all other matter contributions. This removes the model's ability to enter into a radiation/matter dominated era, driven by radiation/matter. **Well-tempered** (section 2.4) theories can fix this by solving the field equations in a different way. The "Fab" theories use a solution that trivially satisfies the scalar field equation, regardless of the matter component. Whereas well-tempering demands that the scalar field equation is only satisfied for a cosmological constant. This ensures that non-constant matter components are not self-tuned away. We also briefly mentioned a recently discovered mechanism separate from well-tempering, that can both self-tune and allow for other forms of matter domination.

We then explored a different approach with **vacuum energy sequestering** (section 2.5), which involves promoting the bare cosmological constant term to a global variable, alongside the addition of another global variable. Varying the action with respect to these variables creates constraints, such that the Λ_{bare} automatically cancels the Λ_{vac}. But, this leaves behind a separate effective cosmological constant that is proportional to the 4-volume average of the trace of the local matter sector, $\langle\tau\rangle$. This object is determined by measurement, which removes the fine-tuning problem, and can be shown to be automatically small in large, old Universes.

In section 2.6 we discussed **braneworlds**, which is based on the idea that our observable Universe is a 4D brane that lies atop a bulk that exists in extra dimensions. We can effectively "hide" the gravitational response of the vacuum energy into these extra dimensions. If we existed in a full **SUSY** Universe, the vacuum energy would automatically vanish since fermions and bosons contribute opposite signs to its energy density. However we know that a full SUSY sector cannot exist on our 4D brane. Instead, SUSY can be used to introduce novel particles in the bulk (that can induce a non-trivial effect on the curvature), alongside constraining the compactification scale (to fit with observations).

Finally, we explored self-tuning with **massive gravity** in section 2.7. The massive graviton acts as a high-pass filter that allows shorter wavelength sources to pass through unhindered, but screens longer wavelength sources. This screens the effect of a cosmological constant, whilst allowing for an agreement with GR physics.

In chapter 3 we explored another self-tuning model by reiterating the arguments presented in [188]. Here, they analyse the linear exchange amplitude between two conserved sources, mediated by both single and multi-particle states up to spin-2. In order to self-tune, they demand that the amplitude vanishes for vacuum energy sources but is in close agreement with GR for short wavelength sources. Under some general assumptions, they argue that we cannot create a self-tuning theory for Minkowski or dS backgrounds since these types of theories violate the self-tuning constraints. However, they discover that there are models that can satisfy these constraints through the so-called "AdS loophole". They later construct an explicit example to demonstrate that there are indeed models that can self-tune in AdS space. Slightly departing from this example, we created a model that can satisfy the constraints using a single massive graviton and a single massive scalar. It is through this example that we constructed a full realisation of the model.

In chapter 4, we started by considering a generalisation of the Fab Four action, which we modified to include an AdS curvature, $-q^2$. Alongside this, we included a dRGT massive gravity term and an extra kinetic term. Unlike the original Fab Four we broke translational invariance at the level of the metric, rather than the scalar. This allowed us to evade Weinberg's no-go theorem, with a constant scalar field on the background. This also allowed us to remove the John and Paul terms, which do not contribute to the exchange amplitude or background solutions, whilst justifying the addition of the extra kinetic term (which will contribute).

Next, in section 4.1, we wrote down the field equations and examined them on the background solution ($g_{\mu\nu} = \bar{g}_{\mu\nu}$, $\phi = \bar{\phi} =$ constant, and $\rho = -p = \Lambda_{\text{vac}}$). The scalar field equation is automatically satisfied, whereas Einstein's field equation reduces to an expression for Λ_{vac}, which we call the "self-tuning equation". This expression tells us that we can indeed accommodate for a large cosmological constant, to recover a small AdS background solution. In other words, our combination of the cosmological constant terms, $\Lambda_{\text{bare}} + \Lambda_{\text{vac}} = \Lambda_{\text{eff}}$, can be countered by the Ringo term to create a background solution that is small and negative. By looking at the self-tuning equation one can naively conclude that the massive terms can also counter an arbitrarily large cosmological constant. Instead, we noticed that the massive terms are related to $\mathcal{G}$, which we later identified with M_{Pl} (as it sets the scale of gravitational interaction). Therefore we do not want to vary $\mathcal{G}$ too much. Even worse, we later found that this massive term must be small to avoid ghosts. This almost completely removes its effectiveness at countering the large cosmological constant, which highlights the importance of Ringo.

Then in section 4.2, we compared our analysis with [188] by writing the exchange amplitude between two conserved sources for the model. To do this we returned to the field equations, perturbed about the background solutions, and decomposed the metric perturbations. Using these expressions we were able to write an exchange amplitude for our model. Then, we showed that it both recovers GR in the high energy limit and vanishes for a cosmological constant source. We also used this expression to check the stability and the ghost-free conditions of our model. From this, we found that there are three distinct scenarios, each representing distinct models. We finally demonstrated this model through an explicit example, verifying that it vanishes for a vacuum source and recovers GR in the high energy limit.

To further understand the dynamics of our model we then computed its cosmological equations, studying them both analytically and numerically in chapter 5. For this we considered an AdS background metric, and we assumed the matter components to be that of a conserved perfect fluid. Applying these requirements yields four equations: the modified Friedmann equation; the modified acceleration equation; the scalar equation; and the energy conservation equation. From this, we first verified that the vacuum solutions satisfy these equations. The scalar equation and the energy conservation equation are immediately satisfied, whilst the modified Friedmann and acceleration equation yield the "self-tuning equation". Also, due to the break in diffeomorphism from the massive gravity term we were able to derive a constraint equation. By differentiating the modified Friedmann equation with respect to t, and combining again with the four "base" equations, we obtained a constraint which has two general solutions. The first is satisfied automatically by our vacuum solution, so it acts as a verification tool. The second creates an entirely new constraint: $3a\mathcal{G} - 2a_0 = 0$. This could provide a different set of solutions, should we choose to study a different solution in the future.

Expanding on this we used a numerical analysis to study the dynamical nature of the system. For this, we considered a canonical example (similar to section 4.2) and a change in variables such that $\alpha \equiv \frac{a_0}{a}$, $\phi(t) \equiv \frac{\mu}{8q^4}x(t)$, and $\dot{\phi}(t) \equiv \sqrt{6}\,y(t)$. Note that these variables will have constant vacuum scaling solutions of the form $x(t) \to x_0$, $y(t) \to 0$, and $\alpha(t) \to 1$, alongside $H(t) \to \frac{\dot{a}_0}{a_0} \equiv q \cot qt$. These scaling solutions correspond to the system undergoing an AdS evolution with a constant scalar. This is consistent with the background solution we derived in section 4.1, where $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $\phi = \bar{\phi} =$ constant. From the change of variables we rewrote the modified Friedmann equation in terms of α, which we used to simplify the modified acceleration equation and the scalar equation. By manipulating these equations we were able to derive two second order equations in terms of $\frac{\ddot{a}}{a}$ and $\ddot{\phi}$ separately, sourced by the t, $H(t)$, $x(t)$ and $y(t)$. Finally, using the above we derived three rate equations for $H(t)$, $x(t)$, and $y(t)$, such that $\dot{H} = \frac{\ddot{a}}{a} - H^2 \equiv f - H^2$, $\dot{x} = \frac{8q^4}{\mu}\dot{\phi} \equiv \frac{8q^4}{\mu}\sqrt{6}\,y$, and $\dot{y} = \frac{\ddot{\phi}}{\sqrt{6}} \equiv \frac{g}{\sqrt{6}}$ where f and g are given by eqs. (5.21) and (5.22) respectively.

Then, we proceeded with our numerical analysis, where we verified that the model recovers the scaling solutions given the initial conditions $H_i = q\frac{\cos qt_i}{\sin qt_i}$, $x_i = x_0$, and $y_i = 0$. We also tracked the variable $z(t) = H(t) - \frac{1}{a^2}$, where $z(t) \to -q^2$ on the scaling solution, to support this verification. We showed that the scaling solutions can be recovered for a wide range of different parameters and initial conditions in H_i. Next, we perturbed x_i from x_0, but we found that the system does not recover its scaling solution. However, upon examining the behaviour of $y(t)$ we found that $x(t)$ can dynamically evolve towards x_0, although it does so very slowly. To further understand this we gave $x(t)$ a "kick" with $y_i \neq 0$. These plots showed that $x(t)$ can dynamically change when given a "kick". However, we were not able to generically recover the scaling solutions without fine-tuning x_i and y_i.

To understand the nature of this fine-tuning we initially conducted a linear stability analysis by perturbing the cosmological equations about the scaling solutions. Unfortunately, our method produced results that are inconclusive. To overcome this requires analysing the stability of this model through a different method, beyond the scope of this thesis.

Finally, we verified that our model can self-tune without fine-tuning in section 5.1. Here, we provided several pieces of evidence showing that the effective potential for our model can be generically minimised. As explained towards the beginning of chapter 2 this is a requirement of self-tuning with a constant background scalar field without fine-tuning. This implies that the difficulties experienced in our numerical analysis (in chapter 5) is not actually due to the system needing to be fine-tuned. Through figs. 5.13 and 5.14 we have also shown that the system remains on an AdS evolution for small changes in $x_i = x_0$ (in contrast to our results in chapter 5). Similarly, $x(t)$ reaches its scaling solution at $t = t_f = \frac{\pi}{2q}$ (at the end of the cosmological expansion era), but moves beyond it at $t > t_f$ (during the cosmological crunch). We also examined the derivative of the effective potential, to show that V_{eff} can be minimised in both of these scenarios. However, this can only be achieved by working in the "strong scalar coupling regime", which is well beyond the allowed solar system constraints that we derived in eqs. (5.26) and (5.28). Therefore these results cannot be taken as a direct representation of the evolution of our system. Instead, it represents what the system is **trying** to do. However, it does imply that "fine-tuning" in our numerical analysis is really an artefact of the scalar being too weakly coupled.

An interesting aspect of our work is that there are many potential directions that it can take. In [188] they demonstrated that an explicit example (that differs from our model) satisfies all of their self-tuning constraints. Namely, one that includes a single massless graviton, a single massive graviton, and a single massive scalar. It would be interesting to conduct an analysis similar to ours on this aforementioned example. Furthermore, there are theoretically a great number of separate examples that can satisfy the constraints in [188]. These will create distinct models that require separate analyses to understand their phenomenologies. A separate project could further explore the parameter space that satisfies these constraints, rather than constructing singular examples.

We have been able to show that our model, both analytically and numerically, can self-tune. In fact, the scalar seems to evolve towards its scaling solution in our numerical analysis for small perturbations in its initial value (i.e. $x_i \neq x_0$). However, despite evidence that $x(t)$ can be dynamical, it evolves far too slowly to recover its scaling solution in the "weak scalar coupling regime". Therefore, there are still a number of potential avenues to explore through changes in our numerical analysis. This could take the form of numerically solving through an alternate method, through a different choice of arbitrary functions (corresponding to different choices of V_r, $\mathcal{K}$, $\mathcal{G}$, and $N(t)$), or by generalising these arbitrary functions entirely.

We could also seek to understand our system better through a stability analysis. Our linear stability analysis proved to be inconclusive, but we can still study it via other methods (that we discussed further in chapter 5).

An interesting approach to further extend our model would be to seek background solutions that involve a non-constant scalar. This represents a full AdS generalisation of Fab Four. Of course, reintroducing a non-constant scalar on the background will also reintroduce the John and Paul terms[38]. This requires a separate analysis (similar to our work) in order to explore whether these non-constant scalar background solutions exist. If they do, it would be interesting to study the phenomenology of such a system.

Finally, we briefly mention that our work resolves the field equations of our model in a similar way to Fab Four and Fab Five (sections 2.2 and 2.3). But as we pointed out, this method may result in the self-tuning technique working "too" well. By this we mean that it removes the gravitational effect for all matter components, not just that of a cosmological constant. Therefore, it would be useful to solve these equations in a similar way to the well-tempered analysis to ameliorate this problem. If this is the case, we must also satisfy the constraint equation due to the massive graviton (eq. (5.5) as derived in chapter 5), as it is no longer automatically satisfied by the background solution. In other words we must

[38]This will prompt the removal of $\mathcal{K}$ as the John and Paul terms will now be able to provide the system with a kinetic dependency i.e. terms containing $\nabla_\mu\phi\nabla^\nu\phi$.

satisfy $3a\mathcal{G} - 2a_0 = 0$, which could prove to be a useful testing ground.

Regardless, we must first find a consistent generalisation of Fab Four in AdS space that can **dynamically** self-tune away an arbitrary cosmological constant term. We have shown that self-tuning without fine-tuning is indeed analytically possible with our model. Our numerical analysis has also shown that it can self-tune when the initial conditions of the scalar ($x(t)$) match that of its self-tuning scaling solution (x_0), whilst allowing for the initial conditions of the Hubble parameter ($H(t) \equiv \frac{\dot{a}}{a}$) to be (mostly) arbitrary. Furthermore, when we perturbed $x(t)$ away from x_0, there is evidence that it tries to slowly evolve to it. However, it never quite reaches x_0 at late times without fine-tuning the initial conditions of $x(t)$ and $\dot{x} \propto y(t)$. Therefore, even though there is evidence that the scalar dynamically evolves towards the scaling solution, we have been unable to **numerically** show that our system can effectively counter an arbitrary cosmological constant at late times without some fine-tuning. Despite this, our work stands as a novel exploration into self-tuning in an AdS space. As demonstrated, there are many potential avenues that this project could take in order to find a model that can consistently solve the cosmological constant problem.

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