Theoretical aspects of alternative theories of gravity

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Abstract

General Relativity (GR) is a very elegant theory with very powerful predictions. However, it's an incomplete theory since it cannot incorporate quantum phenomena and cosmological phenomena related to dark matter, dark energy and inflation. Since the theory's discovery by Einstein, the scientific community has been trying to modify and expand General Relativity to a more complete theory. Contemporary theoretical ideas, together with recent observational advances such as the LIGO experiment and the Event Horizon Telescope experiment are very promising, and the area of modified gravity is a very active and fruitful area of research today. The purpose of this report is to introduce modified gravity theories such as Lorentz violating theories and then generalised scalar tensor theories which are the most straightforward extensions of GR. Also, the spontaneous scalarization mechanism will be presented. This is a mechanism that is receiving a lot of attention lately since it is the most promising description of new physics at the strong gravity regime. More precisely, this mechanism endows relativistic stars and black holes with a nontrivial configuration which does not appear in the context of the non-modified GR theory.

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Chapter 1

Introduction

General Relativity is the gravitational theory of Albert Einstein which was published on 1915. Since this year of publication, a huge effort has been made by theoretical and experimental physicists to test, validate, improve this theory. After years of work, theoretical and observational insights have convinced us that General Relativity is an incomplete theory. It is a non renormalizable theory, it predicts singularities in space time and it does not describe the dark matter and dark energy. Although this theory describes the weak-field gravity phenomena (e.g. Solar system experiments) very well, it has not been fully tested in the strong gravity regime. This regime is now being tested with the hope that strong- field modifications may provide solutions to the before-mentioned problems of General Relativity. Strong gravitational interactions of black holes and neutrons stars provide us with the opportunity to test this strong gravity regime. Our main tool for these tests have been the recently discovered gravitational waves which reveal the behavior of strongly interacting systems and in this way, any possible deviations from General Relativity. By using this theory (GR), Einstein in 1916 predicted the existence of gravitational waves which are spacetime disturbances, generated by accelerated masses. After a century's quest, gravitational waves were detected by the Laser Interferometer Gravitational-Wave Observatory (LIGO), a large-scale physics experiment and observatory.The last few years LIGO has been used to test more intensively GR in the strong gravity regime.

In general, there are different ways to modify GR that can give different dynamics and different gravitational wave signals. The first way, a very common one, is the addition of extra dynamical fields at the Einstein-Hilbert action. And the simpler non trivial extra field is the scalar one, which leads to scalar-tensor which theories are the most well studied alternative theories of gravity since they are simple extensions of General Relativity. At these theories a new scalar field is introduced in the action which is non-minimally coupled to the matter. The field equations derived from the action are relatively simple. Scalar tensor theories are also the dimensionally reduced theories of higher dimensional ones and they also have important applications in cosmology. Nowadays, the validity of these theories is being tested through gravitational observations in the strong gravity regime. A phenomenon called spontaneous scalarization has received a lot of attention recently because it's a promising mechanism which describes solutions around relativistic and black holes different than the ones of General Relativity. This phenomenon is interesting, from an astrophysical perspective, since it can be experimentally tested through the gravitational radiation of systems that exhibit this mechanism. Spontaneous scalarization in the context of generalised scalar tensor theories will be later discussed in more detail.

A different way to modify GR is by giving up the assumption that Lorentz invariance is a necessary ingredient of gravitational theories at the strong gravity regime. Although Lorentz invariance has been tested very well at the Standard model sector and no violations of this symmetry have been observed, Lorentz symmetry violation in gravity is still under research. One way to violate Lorentz symmetry is by adding a preferred timelike vector in the action. There are two different theories that include this timelike vector field; the Einstein-AEther action and the khronometric theory. The difference at the latter one is that the vector field is hypersurface orthogonal. This is the low energy limit of Horava gravity which is a proposal for a quantum theory. Black holes can serve as theoretical laboratories of Lorentz violating theories. They can provide predictions of binary systems gravitational emission which could be different than the emission in General Relativity. However, the study of Black Holes in the context of Lorentz violating gravity is more complicated because of certain properties not appearing in General Relativity, namely the two extra gravitational propagation modes with speeds generally different from the speed of light.

Generaly, different variational procedures and different assumptions about the geometric structures such as a connection with a non zero torsion instead of the Levi-Civita connection, can lead to different field equations. Einstein Field Equations are considered to be the core of the theory of GR. Given a distribution of matter and energy in spacetime, these equations can describe the geometry of the spacetime. The usual way to obtain the equations is through the principle of least action and more precisely through the variation of the Einstein-Hilbert action with respect to the metric.When extra degrees of freedom are added to the action, variation with respect to these degrees must be performed. Therefore, the resulting equations of motion are different than the Einstein Field Equations and they are usually called Modified Einstein Field Equations. After careful derivation of the equations, simple settings in the context of the modified theories can be studied. Some common example cases that apppear in the literature are rotating black holes, slowly moving black holes. Many times even relatively simple systems of black holes or neutron stars require semi-analytical and numerical tools because of the complexity of the equations that describe the systems. However, after careful analysis insightul results are often being produced which reveal system behaviors different than in the context of GR theory. These theoretical analysis are used by experimental physicists who can compare theoretical predictions of different theories with gravitational signals by distant black hole or neutron star systems. The relevant branch of observational astronomy is called Gravitational Wave Astronomy. This scientific methodology allows the scientific community to test different alternative theories of gravity with the hope of extending GR to a more complete theory.

The purpose of this thesis is firstly to give a technical introduction to some common non-quantum alternative theories of gravity. Then spontaneous scalarization in the context of generalised scalar tensor theories and rotating black holes in the context of Lorentz violating theories will be described. The description of the scalar tensor system will be based on the scientific article published by me and three other colleagues during our research. The description of the Lorentz violating example will be based on some authentic calculations I produced for the derivation of the Modified Field Equations that describe the slowly moving black hole, with the guidance of my supervisor.

Chapter 2

Scalar-Tensor theories

2.1 Einstein's Theory

General Relativity (GR) is a gravitational theory that treats gravity as a geometric property of a 4-dimensiona spacetimel manifold. It correlates the curvature of the spacetime with the energy and momentum of the masses in a given spacetime. Flat spacetime corresponds to the Newton's law of gravitation.

A basic tool that allows us to construct the GR theory is the notion of the metric tensor $g_{\mu\nu}$ which captures the notion of distance on a manifold. Along the curve γ this gives the measure of distance

$$s = \int_\gamma d\lambda \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \tag{2.1}$$

where λ is a parameter along the curve.

Given a metric tensor on a manifold, we can define the Levi-Civita connection which is a unique torsion-free connection such that the metric is covariantly constant. ($\nabla_\mu g_{\alpha\beta} = 0$). The components of this connection are called Christoffel symbols and are defined as

$$\Gamma^\mu{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu} \right). \tag{2.2}$$

Now, the Einstein equation is given by the expression:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{2.3}$$

Here $T_{\mu\nu}$ is the energy-momentum tensor of matter fields in the space-time.

Notice that in vacuum where $T_{\mu\nu} = 0$ the Einstein's equation gives $R = 0$ (after indices contraction). Hence the vacuum Einstein equation is written as

$$R_{\mu\nu} = 0 \tag{2.4}$$

Also notice that, Einstein equation is a set of 10 non-linear, second order, coupled partial differential equations for the components of the metric $g_{\mu\nu}$. Therefore, the solution of the equations is usually difficult and requires numerical methods.

The Einstein equation can be derived by varying the following Einstein-Hilbert action:

$$S = \frac{1}{16\pi G} \int \sqrt{-g} R d^4x + \int \mathcal{L}_m(g_{\mu\nu}, \psi) d^4x, \tag{2.5}$$

where $\mathcal{L}_m$ is the Lagrangian density of the matter fields,ψ and $g = det(g_\mu \nu)$ is the determinant of metric tensor matrix. The metric signature here is $(-,+,+,+)$. More specifically, variation with respect to the Riemann tensor, Ricci tensor, Ricci scalar and the determinant and a few calculations...gives the Einstein field equation (2.3).

Also variation of Eq. (2.5) with respect to the metric tensor give,

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}}. \tag{2.6}$$

GR has been well tested in our solar system and has given important insight into cosmological problems. But it has not been fully tested at the strong gravity regime. In addition, it has not yet been reconciled with quantum physics and has some other restrictions as well as mentioned before. So, theoretical physicists make efforts to modify this theory in a combatible with experiment way.

2.2 Scalar-Tensor Theories background

When trying to modify GR, there is an important theorem that should be kept in mind. According to Lovelock's theorem there are some restrictions in the way we modify the theory, if we want to construct a gravitational theory from an action principle involving the metric tensor and its derivatives only and the field equations are up to second order. The simplest way to modify GR, is by adding an extra scalar field in the action. (In general, vectors,tensors or higher rank fields can be added). Scalar-tensor theories include the usual rank-2 tensor field of GR and an extra scalar field. They are some of the most well studied alternative theories of gravity in the literature.

Up to a few scalar field redefinitions the most general action of the scalar tensor theories with an extra scalar field is give by,

$$S = \frac{1}{16\pi G}\int \left[\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla^\mu\phi\right] + \mathcal{L}_m(\Psi, g_{\mu\nu})\right] d^4x, \tag{2.7}$$

where $\omega(\phi)$ is an arbitrary function, often referred to as the 'coupling parameter'. This theory approaches General Relativity (2.5) in the limit $\omega \to \infty$, $\omega'/\omega^2 \to 0$.

Variation of the action (2.7) with respect to $g^{\mu\nu}$, gives the field equations

$$\phi G_{\mu\nu} + \left[\Box\phi + \frac{1}{2}\frac{\omega}{\phi}(\nabla\phi)^2\right] g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi = 8\pi T_{\mu\nu}. \tag{2.8}$$

and variation with respect to ϕ gives

$$(2\omega + 3)\Box\phi + \omega'(\nabla\phi)^2 = 8\pi T. \tag{2.9}$$

where primes here denote differentiation with respect to ϕ. These are the field equations of the scalar-tensor theories of gravity.

The action of (2.7) is the most general one can write that is quadratic in the first derivatives of the scalar. However, this action can be generalized further if we want to include more derivatives of the scalar field. The general scalar-tensor theory with a single scalar field and with at most second order derivatives in the field equations is given by the Horndeski action [7] which can be written as,

$$S = \frac{1}{2\kappa}\sum_{i=2}^{5}\int \mathrm{d}^4x\,\sqrt{-g}\mathcal{L}_i + S_{\mathrm{M}}, \tag{2.10}$$

where we have defined

$$\begin{aligned}
\mathcal{L}_2 =& G_2(\phi, X), && (2.11)\\
\mathcal{L}_3 =& -G_3(\phi, X)\,\Box\phi, && (2.12)\\
\mathcal{L}_4 =& G_4(\phi, X)R + G_{4X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2], && (2.13)\\
\mathcal{L}_5 =& G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi\\
&-\frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\,\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right], && (2.14)
\end{aligned}$$

and $X = -\nabla_\mu\phi\nabla^\mu\phi/2$, $(\nabla_\mu\nabla_\nu\phi)^2 = \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi$, $(\nabla_\mu\nabla_\nu\phi)^3 = \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\lambda\phi\nabla_\lambda\nabla^\mu\phi$ and $G_{iX} = \partial G_i/\partial X$. We have also defined $\kappa = 8\pi G/c^4$ and S_{M} is the matter action. Matter is assumed to couple minimally to the metric only and this means we are working in the so-called Jordan frame.

Varying the action with respect to the metric $g^{\mu\nu}$ and the scalar field ϕ yields respectively

$$\sum_{i=2}^{5}\mathcal{G}^i_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.15}$$

$$\sum_{i=2}^{5}\left(P^i_\phi - \nabla^\mu J^i_\nu\right) = 0, \tag{2.16}$$

where $T_{\mu\nu}$ is the matter stress-energy tensor.

At the following section we will describe an interesting case, in the context of the above theory, with phenomenology that deviates from GR in the strong-gravity regime.

2.3 Spontaneous scalarization in generalized scalar tensor theories

In 1993, Damour and Esposito-Farese [5] have studied spontaneous scalarization in the case of neutron stars and they showed that deviation from General Relativity can occur in these systems. More precisely,they showed that a linear tachyonic instability around the neutron star configuration can trigger the growth of the scalar field. The instability is eventually quenched due to the effect of non-linear terms. By mild tuning parameters, the threshold for the instability trigger can be set at typical densities of neutron stars and we can therefore evade weak-field constraints of our Solar System which is described by GR.

At this section, a study on spontaneous scalarization in the case of black holes in Horndeski theory which we recently conducted at [1], will be presented. Recently, it has been shown at [11] that in the context of scalar Gauss-Bonnet gravity there is a similar mechanism for systems of black holes, according to which scalar hair can be developed after a tachyonic instability is triggered. At [1] we extend, this study to a more general case, namely in the case of Horndeski theory, the general scalar-tensor theory with a single scalar field and with at most second order derivatives in the field equations. In addition, we consider the coupling with the Gauss-Bonnet invariant. More specifically, we find all the possible subclasses of the aforementioned theories for which the effective mass in the action becomes negative and therefore we are led to a tachyonic instability. Note that, we treat the scalar field as a small perturbation and therefore this study does not reveal if we necessarily obtain stable scalar hair solutions. However this study is sufficient to identify terms that can cause or contribute to the tachyonic instability that triggers scalarization. Although scalarization is triggered by a tachyonic instability at the linear level, nonlinear terms eventually take over and quench the instability, thereby determining the properties of the final configuration.

For our analysis we used the Horndeski action (2.10) and its field equations (2.15),(2.16). We started by requiring analycity around $X = 0$ since we were interested in theories in which the scalar exhibits a tachyonic instability around solutions of GR with $\phi = \phi_0 = const$. This can be achieved by expanding G_i in a power series in terms of X [1],

$$G_i = g_{i0}(\phi) + g_{i1}(\phi)X + \dots. \tag{2.17}$$

However, there is a caveat here which concerns our analycity assumption for the G_i functions. That's because we already know at least one theory for which non-analytic $\tilde{G}_i$ functions generate an analytic representation of the action. This theory is the scalar Gauss-Bonnet (sGB) gravity which contains a term of the form $\xi(\phi)\mathscr{G}$, where $\xi(\phi)$ is a generic function of ϕ and $\mathscr{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is the GB invariant. We therefore, consider functions which are the sum of analytic ones denoted by $\tilde{G}_i$ and the non-analytic sGB ones denoted by G_i^{GB},

$$G_i(\phi, X) = \tilde{G}_i(\phi, X) + G_i^{\text{GB}}(\phi, X), \tag{2.18}$$

$$\tilde{G}_i(\phi, X) = g_{i0}(\phi) + g_{i1}(\phi)X + \dots \tag{2.19}$$

where G_i^{GB} are given by,

$$\begin{aligned}
G_2^{\text{GB}} =& 8\xi^{(4)}X^2(3 - \ln X),\\
G_3^{\text{GB}} =& 4\xi^{(3)}X(7 - 3\ln X),\\
G_4^{\text{GB}} =& 4\xi^{(2)}X(2 - \ln X),\\
G_5^{\text{GB}} =& -4\xi^{(1)}\ln X,
\end{aligned} \tag{2.20}$$

Then by substituting the functions of Eq. (2.18) into the scalar field equation (2.16) and keeping only the terms that contribute to the linearization of the equation around a constant value ϕ_0, we get a simplified scalar field equation which is given as

$$\tilde{g}^{\mu\nu}\nabla_\mu\nabla_\nu\phi + \frac{g_{20\phi} + g_{40\phi}R + \xi^{(1)}\mathscr{G}}{A(\phi)} = 0, \tag{2.21}$$

where

$$A(\phi) = g_{21} + g_{41} R, \tag{2.22}$$

and the effective metric reads

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - \frac{2 g_{41} R^{\mu\nu}}{A(\phi)}. \tag{2.23}$$

We now impose that $\phi = \phi_0$ is a solution of Eq. (2.21). There are two distinct cases for which this happens,

$$\begin{aligned} \text{case I:} \quad & g^0_{20\phi} + g^0_{40\phi} R + \xi_0^{(1)} \mathscr{G} = 0, \\ & A_0 \text{ finite;} && (2.24) \\ \text{case II:} \quad & g^0_{20\phi} + g^0_{40\phi} R + \xi_0^{(1)} \mathscr{G} \neq 0, \\ & A_0 \to \infty, && (2.25) \end{aligned}$$

where

$$A_0 \equiv A(\phi_0) = g^0_{21} + g^0_{41} R. \tag{2.26}$$

In the next few steps we will impose these conditions to our linearized scalar equation and in this way we will restrict the Horndeski theory, and consider only classes which admit GR solutions. By linearizing Eq. (2.21) for small for small $\delta\phi = \phi - \phi_0$ yields

$$\tilde{g}^{\mu\nu} \nabla_\mu \nabla_\nu \delta\phi - m^2_{\mathrm{I}} \delta\phi - m^2_{\mathrm{II}} \delta\phi = 0, \tag{2.27}$$

where

$$m^2_{\mathrm{I}} = -\frac{g^0_{20\phi\phi} + g^0_{40\phi\phi} R + \xi_0^{(2)} \mathscr{G}}{A_0}, \tag{2.28}$$

$$m^2_{\mathrm{II}} = \frac{g^0_{20\phi} + g^0_{40\phi} R + \xi_0^{(1)} \mathscr{G}}{A_0^2} \left.\frac{\partial A}{\partial \phi}\right|_{\phi_0} \tag{2.29}$$

are the effective masses obtained in the two separate cases.

We can now write down the two minimal actions (one for each case) which contain all the terms that contribute to the linearized equation and admit GR solutions.
The minimal action for case I is

$$S_{\mathrm{I}} = \int \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa} \Big[R - 2\Lambda + (a_{21} + a_{41}R)X \\ + a_{41}R_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi - \frac{m_\phi^2\phi^2 - \alpha\phi^2 R - \beta\,\phi^2\,\mathscr{G}}{2} \Big] + S_{\mathrm{M}}, \quad (2.30)$$

whereas for case II we have

$$S_{\mathrm{II}} = \int \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa} \Big[R - 2\Lambda + \frac{b_{21} + b_{41}R}{\phi} X \\ + \frac{b_{41}}{\phi} R_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi + \tau\phi + \eta\,\phi R + \lambda\,\phi\,\mathscr{G} \Big] + S_{\mathrm{M}}. \quad (2.31)$$

where all the new terms that where introduced for simplicity are functions of the terms that contribute to the linearized equation.

However, it can be shown that these actions can be mapped to each other after a scalar field redefinition and suitable change of the parameters α_{21} ,α_{41}, α, β,$m^2{}_\phi$. Therefore, for our analysis we just consider every term of the action (2.30) that contributes to the mass. In this way we can map the general minimal action we constructed, with already known models that can exhibit scalarization and in this way we can check if there are any new models.
Now, if the bare mass of the scalar field $m^2{}_\phi$ is negative it can lead to a tachyonic instability that would persist in flat space. Therefore, this term cannot lead to scalarization and we fix it to zero for the following analysis. In addition, α_{21} can be fixed to 1 through scalar field redefinition and also we set $\alpha_{41} = 0$ for now, for the sake of simplicity. The case for which α_{41} is not zero will be discussed later.
For the choice $m_\phi = \alpha = 0$ (and $a_{21} = 1$, $a_{41} = 0$) one has the action

$$S = \int \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa} \left[R - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi + \beta\phi^2\mathscr{G} \right] + S_{\mathrm{M}}, \quad (2.32)$$

However this model is the known scalar-Gauss Bonnet scalarization model. Then, if we set $m_\phi = \beta = 0$ and after some field redefinitions we get

$$S = \int \mathrm{d}^4x \frac{\sqrt{-g}}{2\kappa} \left[\Phi R - \frac{\omega(\Phi)}{\Phi} \nabla^\mu \Phi \nabla_\mu \Phi \right] + S_{\mathrm{M}} \,, \tag{2.33}$$

where Φ is a function of ϕ. After some further scalar field redefinitions, it can be shown that this model is the already known DEF model which exhibits spontaneous scalarization. Finally, the only term we have not yet considered is α_{41}. To do that, we disformally transform the action (2.30) and after some tedious calculations we prove that this transformation leaves the action formally invariant. Also we prove that the condition $\alpha_{41} = 0$ is equivalent to a specific type of disformal coupling. Therefore, when α_{41} term is nonzero, it's producing already known scalarization models which are disformally coupled to matter. These models are interesting from an astrophysical perspective since they can affect the structure of relativistic stars according to [8].

In conclusion, in this general study we identified all the possible classes of theories within Horndeski theory which exhibit spontaneous scalarization. We proved that these classes of theories are the already known models, namely the scalar-Gauss-Bonnet and the DEF model. We also proved, that when these models are disformally coupled to matter can also exhibit this mechanism. It is important to note that, other non-linear terms will affect the end state of the tachyonic instability but not the onset of the scalarization. This study is important because it can narrow down the possible theories with interesting strong field phenomenology. Future studies will enable the scientific community to study this phenomenology with the aim to measure any possible scalarized binary systems and therefore any deviations from GR.

Chapter 3

Lorentz violating theories

3.1 Lorentz violating theories background

A different way to modify GR is by giving up the assumption that Lorentz invariance is a necessary ingredient of gravitational theories at the strong gravity regime. Although Lorentz invariance has been tested very well at the Standard model sector and no violations of this symmetry have been observed, Lorentz symmetry violation in gravity is still under research. One way to violate Lorentz symmetry is by adding a preferred timelike vector in the action. There are two different theories that include this timelike vector field; the Einstein-AEther action and the khronometric theory. The difference at the latter one is that the vector field is hypersurface orthogonal. This this is the low energy limit of Horava gravity which is a proposal for a quantum theory.

In Hořava gravity [9], Lorentz symmetry is violated by introducing a dynamical scalar field T, the "khronon", a timelike vector ($\nabla_\mu T\, \nabla^\mu T > 0$), i.e. hypersurfaces of constant khronon are spacelike, which defines a preferred time foliation. Using T as the time coordinate, the action for Hořava gravity can be written as

$$
\begin{aligned}
S =& \frac{1-\beta}{16\pi G}\int \mathrm{d}T\mathrm{d}^3x\, N\sqrt{\gamma}\Big(K_{ij}\,K^{ij} - \frac{1+\lambda}{1-\beta}K^2 \\
&+ \frac{1}{1-\beta}{}^{(3)}R + \frac{\alpha}{1-\beta}a_i\,a^i + \frac{1}{M_\star^2}L_4 + \frac{1}{M_\star^4}L_6\Big) \\
&+ S_{\mathrm{matter}}[g_{\mu\nu},\Psi]\,,
\end{aligned}
\tag{3.1}
$$

where K^{ij}, ${}^{(3)}R$, and γ_{ij} are respectively the extrinsic curvature, 3-dimensional Ricci scalar and 3-metric of the $T=$ const hypersurfaces; $K=K^{ij}\gamma_{ij}$; N is the lapse; $a_i\equiv\partial_i\ln N$; α, β and λ are dimensionless coupling constants; and Latin (spatial) indices are raised/lowered with the 3-metric γ_{ij}.

The effect of the higher-order terms L_6 and L_4 appearing in the action (3.1) is typically small for astrophysical objects. If we want to focus on the low-energy limit of Hořava gravity we can neglect them. The resulting theory is often referred to as khronometric theory. This is a Lorentz violating due to the presence of the timelike æther vector u^μ, which defines a preferred time direction. By defining the an æther timelike vector u^μ of unit norm in terms of the khronon field T as,

$$
u_\mu = \frac{\nabla_\mu T}{\sqrt{\nabla^\alpha T\nabla_\alpha T}}\,. \tag{3.2}
$$

the action (3.1) becomes

$$
\begin{aligned}
S_{kh} =& -\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\Big(R+\alpha(u^\mu\nabla_\mu u_\nu)^2 \\
&+\beta\nabla_\mu u^\nu\nabla_\nu u^\mu+\lambda(\nabla_\mu u^\mu)^2\Big)+S_{\mathrm{matter}}[g_{\mu\nu},\Psi]
\end{aligned}
\tag{3.3}
$$

where g, R and ∇ are the metric determinant, Ricci scalar and Levi-Civita connection respectively. α,β,λ are dimensionless coupling constants. Note that we set the speed of light c=1 and we use the metric convention $(+,-,-,-)$.

By varying the khronometric action with respect to the metric we get the generalised

Einstein equation

$$G_{\mu\nu} - T^{kh}_{\mu\nu} = 8\pi G\, T^{matter}_{\mu\nu}\,, \tag{3.4}$$

where $G_{\mu\nu} = R_{\mu\nu} - R\, g_{\mu\nu}/2$ is the Einstein tensor and $T^{matter}_{\mu\nu}$ is the matter stress-energy tensor which is defined as

$$T^{\mu\nu}_{matter} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g_{\mu\nu}}\,, \tag{3.5}$$

and the khronon stress-energy tensor is given by

$$\begin{aligned} T^{kh}_{\mu\nu} &\equiv \nabla_\rho \left[J_{(\mu}{}^{\rho} u_{\nu)} - J^{\rho}{}_{(\mu} u_{\nu)} - J_{(\mu\nu)} u^\rho \right] + \alpha\, a_\mu\, a_\nu \\ &+ \left(u_\sigma\, \nabla_\rho J^{\rho\sigma} - \alpha\, a_\rho a^\rho \right) u_\mu\, u_\nu + \frac{1}{2} L_{kh}\, g_{\mu\nu} + 2\text{Æ}_{(\mu} u_{\nu)}\,, \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} & J^{\rho}{}_{\mu} \equiv \lambda \left(\nabla_\sigma u^\sigma \right) \delta^\rho_\mu + \beta\, \nabla_\mu u^\rho + \alpha\, a_\mu u^\rho\,, \\ & \text{Æ}_\mu \equiv \gamma_{\mu\nu} (\nabla_\rho J^{\rho\nu} - \alpha\, a_\rho \nabla^\nu u^\rho)\,, \\ & \gamma_{\mu\nu} = g_{\mu\nu} - u_\mu\, u_\nu\,, \\ & L_{kh} = \lambda\, (\nabla_\mu u^\mu)^2 + \beta \nabla_\mu u^\nu \nabla_\nu u^\mu + \alpha\, a_\mu a^\mu\,, \\ & a_\mu = u^\rho \nabla_\rho u_\mu\,. \end{aligned} \tag{3.7}$$

Variation of the khronometric action with respect to T gives the scalar khronon equation

$$\nabla_\mu \left(\frac{\text{Æ}^\mu}{\sqrt{\nabla^\alpha T \nabla_\alpha T}} \right) = 0\,. \tag{3.8}$$

3.2 Slowly translated black holes in Lorentz violating theories

Black holes can serve as theoretical laboratories of Lorentz violating theories. They can provide predictions of binary systems gravitational emission which could be different than the emission in General Relativity. However, the study of Black Holes in the context of

Lorentz violating gravity is complicated because of certain properties not appearing in General Relativity, namely the two extra gravitational propagation modes with speeds generally different from the speed of light.

Black holes in Horava gravity have multiple distinct horizons. There is one horizon for each gravitational mode of the theory. Note that, these modes have speeds which depend on coupling parameters that appear in the action and in principle can differ from the speed of light. At Hořava gravity there is a matter horizon for light and other matter fields, a spin-2 horizon for gravitational waves and a spin-0 horizon for scalar excitations. Because of the requirement that there is no gravitational Cherenkov radiation, since it has not been observed experimentally, the propagation speeds of the modes must be larger than the speed of light. Therefore, the corresponding horizons must be enclosed by the matter horizon. An open problem in Lorentz violating theories is the causal structure of black hole solutions and the question that must be answered is whether there a stable universal horizon which traps modes of any speed.

It has been shown at [3],[2],[4] that there is a universal horizon at spherical and slowly rotating black hole solutions but it's not clear if they generally exist. This is something which requires further investigation since without a universal horizon the concept of a black hole in these theories is meaningless. A recent study [10] has investigated slowly moving, relatively to the preferred foliation, black hole solutions in Hořava gravity. One of the main conclusions of this study is that for generic values of the three dimensionless coupling constants of the khronometric theory the black holes present finite area curvature singularities. They also reached the conclusion that in a one-dimensional subset of the parameter space of the theory's constants (i.e 2 of the constants are fixed to zero and this can be justified when the speed of spin-2 modes exactly matches that of light and predictions of the theory in the solar system exactly match those of GR) black holes are regular everywhere and the solution coincides with the Schwarzchild solution. In this case, dipolar gravitational emission is absent.

In the following calculations, our purpose was to study slowly translated black in a slightly different setting than in [10]. We extracted the equation of the moving black hole solutions in khronometric theory by just pertubinq the static symmetric solution at rest relative to the khronon. For our analysis we treated theæther field as a small perturbation around a fixed GR background.

In khronometric theory and in (t, r, θ, ϕ) coordinates, the metric and æther vector that describe regular, spherically symmetric, static and asymptoticaly flat black hole solutions and the æther vector take the the form [2]

$$
\begin{aligned}
d\bar{s}^2 =& f(r)dt^2 - \frac{B(r)^2}{f(r)}dr^2 - r^2 d\Omega^2 \\
\bar{u}_\mu dx^\mu =& \left(\frac{1+f(r)A(r)^2}{2A(r)}\right)dt + \frac{B(r)}{2A(r)}\left(\frac{1}{f(r)} - A(r)^2\right)dr\,,
\end{aligned}
\tag{3.9}
$$

where $f(r)$, $B(r)$ and $A(r)$ depend on the coupling constants α, β and λ. We use the bar notation because the above solution will be our background solution which we will perturb later. The asymptotic solution of A(r), B(r), f(r) is given by [6, 2, 10]

$$
f(r) = 1 - \frac{2G_N\tilde{m}}{r} - \frac{\alpha(G_N\tilde{m})^3}{6r^3} + \cdots \tag{3.10}
$$

$$
B(r) = 1 + \frac{\alpha(G_N\tilde{m})^2}{4r^2} + \frac{2\alpha(G_N\tilde{m})^3}{3r^3} + \cdots \tag{3.11}
$$

$$
\begin{aligned}
A(r) =& 1 + \frac{G_N\tilde{m}}{r} + \frac{a_2(G_N\tilde{m})^2}{r^2} + \\
& (24a_2 + \alpha - 6)\,\frac{(G_N\tilde{m})^3}{12r^3} + \cdots\,,
\end{aligned}
\tag{3.12}
$$

where the parameter a_2 is determined numerically once the mass $\tilde{m}$ is fixed.

Now, we consider the decoupling limit of the khronometric theory, *i.e.* treating the æther field as a small perturbation around a fixed GR background. In this case, we have,

$$\begin{aligned} d\bar{s}^2 =& E(r)dt^2 - E(r)^{-1}dr^2 - r^2 d\Omega^2 \\ \bar{u}_\mu dx^\mu =& \left(\frac{1+E(r)A(r)^2}{2A(r)}\right)dt + \frac{1}{2A(r)}\left(\frac{1}{E(r)} - A(r)^2\right)dr \end{aligned} \tag{3.13}$$

where we introduced $E(r) \equiv \left(1 - \frac{2M}{r}\right)$. More precisely, in the decoupling case we set $f(r) = E(r)$, $B(r) = 1$ and we retrieve the Schwarzschild metric.

Now, we can perturb the decoupled spherical black hole solution, to get a slowly translated solution with velocity v in the z direction relative to the æther. We follow the idea which can be found at [12, 2]. However, we choose to construct the most general metric and æther ansatz in the reference frame in which the black hole is not moving (Schwarzschild black hole). In this way, we use all the gauge symmetry that arises from the Bianchi identity. It's also obvious that the usual Einstein equation at the vacuum (not the generalised Einstein equation since we are considering the decoupling limit) is automatically satisfied. So, the khronon equation (3.8) will be sufficient for fully exploring slowly moving black holes relative to the æther vector.

For the ansatz construction we find convenient to use cylindrical coordinates. We can also use the vectors $\boldsymbol{n} = (\rho, z)/r$ and $\boldsymbol{v} = (0, v)$ and we therefore have the most general perturbations,

$$\delta u_t(r) = E(r)C(r)(\boldsymbol{v}\cdot\boldsymbol{n}) = v\,E(r)C(r)\frac{z}{r} \tag{3.14}$$

$$\begin{pmatrix}\delta u_\rho(r)\\ \delta u_z(r)\end{pmatrix} = E(r)\left[Q(r)\boldsymbol{v} + W(r)(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}\right] \tag{3.15}$$

$$= v\,E(r)\left[Q(r)\begin{pmatrix}0\\1\end{pmatrix} + W(r)\frac{z}{r^2}\begin{pmatrix}\rho\\z\end{pmatrix}\right] \tag{3.16}$$

$$\tag{3.17}$$

Therefore, the slowly translated metric and æther vector can be written in (t, r, θ, ϕ) coordinates as,

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = E(r)dt^2 - E(r)^{-1}dr^2 - r^2 d\Omega^2 \tag{3.18}$$

and

$$u_{\mu}dx^{\mu} = \bar{u}_t(r)dt + \bar{u}_r(r)dr + v\Big\{C(r)\cos\theta dt + W^*(r)\cos\theta dr - rQ(r)\sin\theta d\theta\Big\}, \tag{3.19}$$

where $\bar{u}_t(r) = (1+E(r)A(r)^2)/(2A(r))$ and $\bar{u}_r(r) = (1-E(r)A(r)^2)/(2A(r)E(r))$. We used the coordinate transformations, $z = r\cos\theta, \rho = r\sin\theta$ and we introduced $W^*(r) \equiv W(r) + Q(r)$. $W^*(r)$ can be relabelled as $W(r)$ without loss of generality. We also want to impose the boundary conditions of our ansatz; $g_{\mu\nu} = \eta_{\mu\nu}$ and $u^{\mu}\partial_{\mu} = \partial_t - v\partial_z$ when $r \to \infty$. Notice that, when black hole moves with velocity v in the z direction relative to the æther we can equivalently say that the æther moves with velocity $-v$ in the z direction with respect to the black hole. We can impose this behavior through the boundary conditions $C(\infty) = 0, W(\infty) = -1, Q(\infty) = -1$.

Before solving the khronon equation (3.8) we should impose the hypersurface orthogonality and normalization condition. These conditions will reduce the three free parameters C(r), W(r), Q(r) to one.
The hypersurface orthogonality implies that $\epsilon^{\alpha\beta\gamma\delta}u_{\beta}\nabla_{\gamma}u_{\delta} = 0$ which be simplified as $\bar{u}_t(r)\nabla_{\theta}\delta u_r(r) - \bar{u}_r(r)\nabla_{\theta}\delta u_t(r) - \bar{u}_t(r)\nabla_r\delta u_{\theta}(r) + \delta u_{\theta}(r)\nabla_r\bar{u}_t(r) = 0$ since all the other contributions are zero. In this way we get,

$$\bar{u}_t(r)[W(r) - Q(r) - rQ'(r)] = \bar{u}_r(r)C(r) - r\bar{u}_t'(r)Q(r) \tag{3.20}$$

By varying the normalization condition $u^{\mu}u_{\mu} = 1$ we get $\bar{u}^{\nu}(r)\delta u_{\nu}(r) = 0$ (note that, $\delta g_{\mu\nu} = 0$). This condition gives,

$$C(r) = E(r)^2 W(r)\frac{\bar{u}_r(r)}{\bar{u}_t(r)} \tag{3.21}$$

By combining the equations (3.20), (3.21) we get,

$$
\begin{aligned}
W(r) =& \frac{\bar{u}_t(r)D(r)}{\bar{u}_t^2 - \bar{u}_r^2 E(r)^2} \\
C(r) =& \frac{\bar{u}_r(r)E(r)^2 D(r)}{\bar{u}_t^2 - \bar{u}_r^2 E(r)^2}
\end{aligned}
\tag{3.22}
$$

where $D(r) \equiv (\bar{u}_t(r) - r\bar{u}_t{}'(r))Q(r) + r\bar{u}_t(r)Q'(r)$. It is also easy to check that,

$$
\bar{u}_t^2 - \bar{u}_r^2 E(r)^2 = E(r) \tag{3.23}
$$

and due to this equation get,

$$
\begin{aligned}
W(r) =& \frac{\bar{u}_t(r)D(r)}{E(r)} \\
C(r) =& \bar{u}_r(r)E(r)D(r)
\end{aligned}
\tag{3.24}
$$

In conclusion, the æther ansatz that we should use to solve the khronon equation is the following,

$$
u_\mu dx^\mu = \bar{u}_t(r)dt + \bar{u}_r(r)dr + v\left\{\bar{u}_r(r)E(r)D(r)\cos\theta dt + \frac{\bar{u}_t(r)D(r)}{E(r)}\cos\theta dr - rQ(r)\sin\theta d\theta\right\} \tag{3.25}
$$

The khronon equation (3.8) contains the derivatives of the khronon T. Staticity of the background solution implies that the components of the aether vector u_μ must be constant in time. If we also consider spherical symmetry we conclude that $T = t + f(r)$ is a reasonable choice for the background khronon. Furthermore, in order to avoid time dependent contributions to the perturbed aether vector,the perturbation of the khronon should be linearly dependent on t. Therefore, by considering the symmetries of the problem and by generalizing the idea presented at [4] we construct the following ansatz of the khronon

$$T = t + f(r) + v\left[t + f(r) + \chi(r)\cos\theta\right] \tag{3.26}$$

where f(r) is a function of the radius and $\chi(r)$ is the function that should define the slow translation pertubation and should therefore be related to the function Q(r).

By using the above definition of the khronon and the definition of the æther vector (3.2) we get,

$$u_\mu dx^\mu = \frac{1}{\sqrt{N}}\Big[\Big(1+v\Big)dt + \Big(\frac{df}{dr} + v\Big[\frac{df}{dr} + \frac{d\chi}{dr}\cos\theta\Big]\Big)dr - v\chi(r)\sin\theta d\theta\Big] \tag{3.27}$$

where,

$$N \equiv g^{\mu\nu}\nabla_\mu T\nabla_\nu T = \frac{1}{E(r)} - E(r)\Big(\frac{df}{dr}\Big)^2 + 2v\Big(\frac{1}{E(r)} - E(r)\frac{df}{dr}\Big[\frac{df}{dr} + \frac{d\chi}{dr}\cos\theta\Big]\Big). \tag{3.28}$$

By Taylor expanding $1/\sqrt{N}$ and by comparing (order by order in v) the aether vectors (3.25) and (3.27) we conclude that,

$$\begin{aligned} \frac{df}{dr} &= \frac{\bar{u}_r(r)}{\bar{u}_t(r)} \\ \frac{d\chi}{dr} &= \frac{D(r)}{\bar{u}_t{}^2(r)} \\ \chi(r) &= \frac{rQ(r)}{\bar{u}_t} \end{aligned} \tag{3.29}$$

and

$$T = t + \int\frac{\bar{u}_r(r)}{\bar{u}_t(r)}dr + v\left[t + \int\frac{\bar{u}_r(r)}{\bar{u}_t(r)}dr + \frac{rQ(r)}{\bar{u}_t}\cos\theta\right] \tag{3.30}$$

By combining equations (3.28) and (3.29) we get,

$$\frac{1}{\sqrt{N}} = \bar{u}_t(r) - v\left(\bar{u}_t - \bar{u}_r(r)E(r)D(r)\cos\theta\right) \tag{3.31}$$

At this stage, we have done all the necessary calculations that will allow us to solve numerically the khronon equation $\nabla_\mu(\text{Æ}^\mu/\sqrt{N}) = 0$.

In future work, our final khronon equation can be solved numerically with the aim of checking whether black holes solutions are regular in a bigger parameter space than the one given by the work at [1].

Chapter 4

Discussion

Gravity, a century after the devolopment of GR by Einstein, remains the least well understood fundamental force of nature. At this work we presented work related to certain classes of alternative theories of gravity which are crucial for open problems in gravitational physics and for testing GR. Minimal deviations and mechanisms that hide modifications on scales tested by experiment allow us to test the limits of GR theory. Additional fields and broken symmetries are some of the ways we use to construct new models.

Black holes in the strong-gravity regime, our main theoretical laboratory for testing these theories provide us with many opportunities for unravelling nature's mysteries. Gravitational-wave astronomy is rapidly expanding and will allow us to test our black holes models in the context of modified gravity theories, with the hope of revealing new fundamental physics. New experimental tools such as Laser Interferometer Space Antenna (LISA), a space-based gravitational wave detector will directly measure of gravitational waves to study astrophysical systems with unprecedented precision. Tests of Lorentz invariance and the weak equivalence principle are being proposed as space missions away from earth's seismic noise. The gravitational physics future seems exciting...

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