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Higher linear algebraic quantum field theory

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Abstract

In this thesis several homotopical aspects of linear algebraic quantum field theory are treated. These homotopical aspects are crucial when formalizing quantum gauge theories in a way that fully respects their gauge symmetries. After preliminaries introducing the relevant elements of category theory, Lorentz geometry, algebraic quantum field theory, chain complexes, model categories and operads, the definition of semi-strict homotopy algebraic quantum field theory is given, weakening only the time-slice axiom.

Building on the work done in [**BSW20**], an operadic definition of general algebraic field theories is obtained. This allows for adjunctions between different kinds of field theories: a descent adjunction, related to local-to-global features of a field theory; a localization adjunction, related to the existence of dynamics for a field theory; and a canonical quantization adjunction between linear field theories and quantum field theories. The latter adjunction is shown to generalize to homotopy algebraic field theories, and to preserve weak equivalences. This yields a general machinery to produce linear homotopy quantum field theories.

After this, the construction of examples of homotopy algebraic quantum field theories is studied. From the input data of a field complex and an equation of motion, the solution complex is formed as a derived critical locus. This retrieves several features of the BV formalism: ghost fields, antifields and an antibracket, i.e. a canonical shifted Poisson structure on the solution complex. Crucially, it is found that using features of the Lorentzian geometry of spacetime, this shifted Poisson structure can be trivialized in two ways, yielding an unshifted Poisson structure on the solution complex and thus a homotopy algebraic linear field theory.

The canonical quantization functor then produces a linear homotopy algebraic quantum field theory. This is illustrated by two examples: Klein-Gordon theory, which is shown to be equivalent to the usual treatment in algebraic quantum field theory; and linear Yang-Mills theory, which is a first nontrivial example of a linear homotopy algebraic quantum field theory, and is not equivalent to any ordinary algebraic quantum field theory.

Finally, the issue of relative Cauchy evolution for linear homotopy algebraic quantum field theory is treated. Using the localization adjunction an equivalent perspective on relative Cauchy evolution for ordinary algebraic field theories is proposed, which is found to be more suitable for homotopy algebraic field theories since the weakening of the time-slice axiom turns out to severely complicate the usual approach. A rectification theorem is proven for linear observables, and a suitable Poisson structure is found on the strictified model. Combined with the homotopical properties of the linear quantization functor this allows for a well-defined notion of relative Cauchy evolution for linear homotopy algebraic quantum field theories, and it is shown that for the linear observables in such a theory this notion agrees with the naive approach of quasi-inverting the maps involved. The relative Cauchy evolution for the linear Yang-Mills model is then computed, and it is shown that the associated stress-energy tensor agrees with the usual Maxwell stress-energy tensor.

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Contents

CHAPTER 1

Introduction

The theory of *quantum fields* is arguably one of the greatest achievements of modern physics. In particle physics, the Standard Model accounts for all known matter, describes all known interactions besides gravity and it has led to what is possibly the most precise prediction in physics, the anomaly of the magnetic moment of the electron. Beyond particle physics, quantum field theory has broad applications in condensed matter theory and cosmology, it has revolutionized certain areas of mathematics, and has led to the creation of others.

While quantum field theory has been extremely successful in both physics and mathematics, a complete mathematical formalization has not been achieved in full generality. Feynman's path integral infamously refuses to be rigorously defined for quantum fields, and even beyond that the field is rife with calculations leading to infinities in all but the simplest models. Methods of perturbative renormalization have been developed to remedy this, but a full description of the nonperturbative theory is still out of reach.

An important class of quantum fields is that of the so-called *gauge fields*. For example, the Yang-Mills fields are a crucial ingredient of the Standard Model, as they carry the electroweak and strong forces between elementary particles. Gauge fields are fields with a gauge symmetry: they are invariant under the action of a local gauge group, which is an action by a Lie group at every point in spacetime. Field configurations that are related by a gauge symmetry are considered to be equivalent. One can think of a choice of representative of a gauge equivalency class as a choice of coordinates in which the value of the field is expressed, at every point in spacetime.

The ingredient of gauge equivalence in a quantum field theory turns out to bring its own set of issues when constructing a mathematical theory of quantum gauge fields. Specifically, it leads us to a weaker notion of *equivalence* of theories: a pair of theories can be equivalent even when one has more fields than the other, for example when one of the theories is procured by gauge-fixing the other or by adding a redundant gauge symmetry. So in certain cases we can think of two non-isomorphic theories as equivalent, and we need to work in a framework that treats them as such. This leads us beyond the usual set- and category theory-based approaches, in turn leading to new conceptual and definitional challenges, which are of a different nature than the infinities arising in non-gauge quantum field theory; of course, the latter will in general still be present, and need to be addressed too.

This thesis is concerned with some of these former challenges. Specifically, we work in the framework of *homotopy algebraic quantum field theory*, which generalizes the framework of algebraic quantum field theory to the language of model categories. *Algebraic quantum field theory* is one of the main approaches toward rigorously defining quantum field theories, focusing on the algebras of observables which are defined on any subspacetime of the spacetime one works on. And *model categories* are categories defined explicitly to deal with weak equivalences such as the ones described above.

Algebraic quantum field theory and respecting weak equivalences are the two fundamental concepts underlying the work in this thesis. So let us informally introduce both, building on ideas and notation that are hopefully familiar to the reader.

1.1. From quantum fields to algebraic quantum field theory

In a first course on quantum field theory one usually starts with a scalar field $\phi(x)$ on Minkowski spacetime $\mathbb{M}$ with the Klein-Gordon equation

$$(\Box - m^2)\phi = 0 ,$$

as the equation of motion. On a spatial slice in $\mathbb{M}$ (say the $t = 0$ hyperplane with spatial coordinate $\mathbf{x}$) we then have the quantum field $\phi(\mathbf{x})$ and its conjugate momentum $\pi(\mathbf{x}) = \partial_t\phi(\mathbf{x})$. These fields are operator valued distributions acting on the Hilbert space of states and they satisfy the equal time canonical commutation relations

$$\big[\phi(\mathbf{x}), \pi(\mathbf{x}')\big] = i\delta^3(\mathbf{x} - \mathbf{x}') \quad ; \quad \big[\phi(\mathbf{x}), \phi(\mathbf{x}')\big] = \big[\pi(\mathbf{x}), \pi(\mathbf{x}')\big] = 0 .$$

We think of the operator $\phi(\mathbf{x})$ as observing the value of the field ϕ at the point $\mathbf{x}$ so we call ϕ and other quantum fields *observables*. This is the equal time approach, or Hamiltonian approach, to quantum fields.

The treatment then usually moves from the Schrödinger picture to the Heisenberg picture, making the observables time dependent instead of the states. Where in the Schrödinger picture the equation of motion for the states was given by the Schrödinger equation, in the Heisenberg picture we have the Heisenberg equation for the observables, a differential equation for ϕ involving the commutator of the field and the Hamiltonian. This means we now have a covariant quantum field operator $\phi(x)$ defined on all of spacetime, obtained by solving the Heisenberg equation with initial data $\phi(\mathbf{x})$ and $\pi(\mathbf{x}) = \partial_t\phi(\mathbf{x})$. In the usual treatment of quantum Klein-Gordon theory this is evidenced by the fact that the Fourier expansion of $\phi(x)$ is given by a three-dimensional integral over the momenta $\mathbf{k}$, with the zero component k_0 appearing in the time dependent part e^{ik_0t} given by $k_0 = \sqrt{\mathbf{k}^2 + m^2}$. So we work on-shell, with the *dynamics* now encoded in the quantum fields: in a sense, the quantum fields $\phi(x)$ only observe solutions to the equation of motion.

The commutator for the quantum fields is then

$$\big[\phi(x), \phi(x')\big] = iG(x, x')$$

where G is the (singular) causal Green function for the Klein-Gordon operator. Importantly $G(x, x')$ is zero when the difference $x - x'$ is a spacelike vector. This is the algebraic expression of *causality*: the fact that a measurement in a region U_1 of Minkowski space cannot influence a measurement in a region U_2 that is spacelike to U_1.

From here, a treatment of quantum field theory usually moves on to scattering and perturbation theory, adding interactions, defining in and out states, finding Feynman diagrams, calculating n-point functions and generally doing calculations within the framework. We move in the other direction, formalizing the quantum fields and (some of) their properties. We will still primarily think of the Klein-Gordon field $\phi(x)$ but the lessons we learn should be broadly applicable.

We start by collecting the observables $\phi(x)$, their products and a unit, into the algebra of observables $\mathfrak{A}$. If we have a Hilbert space of states $\mathcal{H}$ we can then think of $\mathfrak{A}$ as a subalgebra of the algebra of operators on $\mathcal{H}$. However, an algebra of quantum field observables will in general admit many different inequivalent Hilbert space representations (the Stone-von

Neumann theorem does not apply because the algebra is infinitely generated). So we decide to consider the algebra $\mathfrak{A}$ as the fundamental object of our field theory; hence the name algebraic quantum field theory. The Hilbert space of states is then a choice of representation of $\mathfrak{A}$.

We then recognize that the observables $\phi(x)$ are too sharp: they are operator valued distributions, which need to be integrated against a test function to give true operators. If we do not do this, this results in infinities, evidenced by the singular behaviour of the Green function appearing in the commutation relations, for example. So we define the observables $\phi(f)$ that are smeared with compactly supported functions $f \in C^\infty_{\mathrm{c}}(\mathbb{M})$,

$$\phi(f) := \int_{\mathbb{M}} f(x)\,\phi(x)\,d^4x \ .$$

The operator valued distributions $\phi(x)$ are not part of $\mathfrak{A}$ anymore, but they can be approximated by the smeared observables $\phi(f)$ by letting f approach $\delta^4(x)$. The observables $\phi(f)$ naturally carry an involution, making $\mathfrak{A}$ into a $*$-algebra[1]. The commutation relations between quantum fields then are

$$\big[\phi(f), \phi(f')\big] = i\int_{\mathbb{M}} f(x)\,G(f')(x)\,d^4x$$

where G is now the causal Green operator for Klein-Gordon theory, which will not lead to infinities since f is still compactly supported and $G(f')$ is smooth.

One of the fundamental features of (quantum) fields that we have not yet touched on is their *local* nature: a field located at a point x can only influence particles and other fields at that point or its neighbours. Implementing locality leads to an even more rigorous reinterpretation of $\mathfrak{A}$. This procedure has two steps. First, we recognize that on any subspacetime U of Minkowski space, we can define the algebra $\mathfrak{A}(U)$ of observables located on U. These are all smeared observables $\phi(f)$ where f is supported on U. The local nature of the fields then ensures that $\mathfrak{A}(U)$ is closed under products and the involution, so it is indeed still a $*$-algebra. So $\mathfrak{A}$ is now a function

$$\mathfrak{A}: \quad \mathbf{COpens}(\mathbb{M}) \quad\longrightarrow\quad *\mathbf{Alg}$$

assigning to any causal open of $\mathbb{M}$ (i.e. any subspacetime) its local algebra of observables.

The second step is to realize that some of these algebras are related: if $U_1 \subseteq U_2$ is an inclusion then any observable $\phi(f) \in \mathfrak{A}(U_1)$ is also local to U_2. So the inclusion $\iota : U_1 \hookrightarrow U_2$ leads to an inclusion of algebras $\mathfrak{A}(\iota) : \mathfrak{A}(U_1) \hookrightarrow \mathfrak{A}(U_2)$[2]. This leads us to recognize $\mathfrak{A}$ as not just a function, but a functor

$$\mathfrak{A}: \quad \mathbf{COpens}(\mathbb{M}) \quad\longrightarrow\quad *\mathbf{Alg} \ .$$

So $\mathfrak{A}$ is in fact a *net* of algebras, one for each subspacetime U of $\mathbb{M}$, with maps between algebras corresponding to inclusions in $\mathbb{M}$.

We can now give our two conditions of causality and dynamics in this new language. We saw that the causality of fields was algebraically expressed by the fact that if the difference between two points in $\mathbb{M}$ is spacelike, the commutator of fields located there is zero. In our

[1] To make the algebra of observables into a C*-algebra we need to form the Weyl algebra, which we will not pursue in this text.

[2] In general $\mathfrak{A}(\iota)$ might not be an inclusion, as we will see in Remark 2.3.3.

functorial discription this translates to the condition that if U_1 and U_2 are spacelike separated in a larger spacetime U',

$$\Big[\mathfrak{A}(\iota_1)\big(\phi(f_1)\big), \mathfrak{A}(\iota_2)\big(\phi(f_2)\big)\Big] = 0 \in \mathfrak{A}(U')$$

where $\iota_i : U_i \to U'$ are the inclusion maps and $\phi(f_i) \in \mathfrak{A}(U_i)$. We call this condition *Einstein causality.*

As for the dynamics, we saw that the quantum fields $\phi(x)$ only observe solutions in the theory. We formalize this as follows. A *Cauchy surface* of a spacetime U is a spacelike hypersurface Σ such that every inextendible causal curve in U intersects Σ exactly once. A *Cauchy morphism* $\iota : U \to U'$ is then an inclusion such that the image $\iota(U)$ contains a Cauchy surface of the bigger spacetime U'. This means that Σ is a good surface to define initial conditions on for the theory on U'; in other words, a solution on U will uniquely determine a solution on U'. In terms of the algebras this implies that the observables local to U already completely observe all possible solutions on U'. This is formalized by the *time-slice axiom*: the map of algebras

$$\mathfrak{A}(\iota) : \mathfrak{A}(U) \longrightarrow \mathfrak{A}(U')$$

is an isomorphism whenever $\iota : U \to U'$ is a Cauchy morphism.

This leads us to our first definition of an *algebraic quantum field theory*: it is a functor $\mathfrak{A} : \mathbf{COpens}(\mathbb{M}) \longrightarrow *\mathbf{Alg}$ that satisfies Einstein causality and the time-slice axiom. This is a quite abstract definition, and also quite general: it says nothing about the size of the algebras or which (if any) fields the theory describes. In this language, a field ϕ is upgraded to be a natural transformation

$$\phi : C^\infty_{\mathrm{c}} \longrightarrow \mathfrak{A}$$

of the underlying functors valued in vector spaces, with components

$$\begin{array}{rccc} \phi_U : & C^\infty_{\mathrm{c}}(U) & \longrightarrow & \mathfrak{A}(U) \\ & f & \longmapsto & \phi(f) \end{array} .$$

The algebraic approach to quantum field theory was first proposed in the seminal paper [**HK64**]. The essential step was letting of the representation Hilbert space that is central in the Wightman axioms [**WG65**]; it is considered to come second in the algebraic approach. Originally, the only regions considered were bounded opens U_{c} of $\mathbb{M}$, which is not a big ask: any $\phi(f)$ will live in a $\mathfrak{A}(U_{\mathrm{c}})$ since f is compactly supported. In [**BFV03**] *locally covariant quantum field theories* were introduced by defining theories on all suitable spacetimes M, elegantly incorporating symmetries of a spacetime (such as the Poincaré group) as automorphisms of spacetimes without relying on them. This will be our Definition 2.3.1 of algebraic quantum field theory.

A complete survey of the successes of algebraic quantum field theory is beyond the scope of this section, see e.g. [**Fre15**] and the other chapters in the book [**BDFY15**] for a recent overview. By moving to formal power series in $\hbar$, one can treat perturbative models in AQFT, see e.g. [**FR15**] and [**Rej16**]. It is still an open question how to move beyond perturbation theory, and include non-perturbative features; for recent progress on convergence for simple models see [**BR18**] and [**BFR21**].

We end by noting that other functorial formalizations of quantum field theory exist, though we will not cover them beyond this paragraph. Some, such as topological quantum field theory [**Ati88**], and one definition of conformal field theory [**Seg88**], focus on cobordisms as formalizing the time-evolution of states. A more recent approach is the use of factorization algebras of observables in quantum field theory [**CG16**], axiomatizing the operator product expansion and correlation functions.

1.2. Gauge theory and weak equivalences

We now turn to gauge theory. Mathematically, (classical) gauge fields on a spacetime M are connections on a corresponding principal G-bundle P, where G is a compact Lie group. These connections carry a natural action of the gauge group $\mathcal{G}(P)$, which locally looks like the group of G-valued functions on M. Two field configurations that are related by a gauge transformation $g \in \mathcal{G}(P)$ describe the same physical configuration, and are therefore considered to be physically equivalent.

For our purposes, the essential point is now that the classical space of fields $\mathfrak{F}(M)$, or the classical space of solutions to the equation of motion $\mathfrak{Sol}(M)$, is no longer simply a set (or a set with extra structure like a vector space or a manifold): it is a *groupoid* of fields (a category in which all maps are isomorphisms). The objects of $\mathfrak{F}(M)$ are field configurations, i.e. the connections on P, and the morphisms are the gauge transformations between field configurations. Field configurations that are different can then still be gauge equivalent when there exists a gauge transformation between them. And once we accept that we will work with fields up to equivalence, we have no choice but to also work with a broader notion of equivalence for theories than isomorphisms of categories.

To see this, consider the following simple example: let $\mathfrak{F}(M)$ be a set of classical field configurations without any symmetries, say the smooth functions on M for concreteness. In our broader framework we can then simply double this set and introduce a $\mathbb{Z}/2\mathbb{Z}$-symmetry between each field configuration and its copy: we define the disjoint union $\overline{\mathfrak{F}}(M) := \mathfrak{F}(M) \sqcup \mathfrak{F}(M)$, together with the $\mathbb{Z}/2\mathbb{Z}$-action

$$f_1 \xleftrightarrow{\;\bar{1}\;} f_2$$

where we write $f_1 \in \overline{\mathfrak{F}}(M)$ for elements of the first of the two copies of $\mathfrak{F}(M)$ in the disjoint union, and $f_2 \in \overline{\mathfrak{F}}(M)$ for elements of the second. The content of this new theory $\overline{\mathfrak{F}}(M)$ is manifestly the same: in essence we have simply given two different labels to each field configuration.

But we find that there is no way of defining a map $F : \mathfrak{F}(M) \to \overline{\mathfrak{F}}(M)$ that has a strict inverse $F^{-1} : \overline{\mathfrak{F}}(M) \to \mathfrak{F}(M)$. We see this directly: for any $f \in \mathfrak{F}(M)$, F^{-1} will map its copies f_1 and f_2 in $\overline{\mathfrak{F}}(M)$ to isomorphic, hence equal, elements in $\mathfrak{F}(M)$, since there are no (non-identity) symmetries in $\mathfrak{F}(M)$. So F^{-1} can never be injective. On the other hand, from the perspective of category theory, there is no problem: we can define the obvious maps

$$F: \begin{array}{ccc} \mathfrak{F}(M) & \longrightarrow & \overline{\mathfrak{F}}(M) \\ f & \longmapsto & f_1 \end{array} \quad ; \qquad F^{-1}: \begin{array}{ccc} \overline{\mathfrak{F}}(M) & \longrightarrow & \mathfrak{F}(M) \\ f_1, f_2 & \longmapsto & f \end{array}$$

and although the composition $F \circ F^{-1}$ is not the identity $\mathrm{id}_{\overline{\mathfrak{F}}(M)}$, it is isomorphic to $\mathrm{id}_{\overline{\mathfrak{F}}(M)}$.

So we have learned that the right concept of equivalence for groupoids (and more broadly for categories) is weaker than that of isomorphism. And that if we start working with gauge fields, there is no way around working with this weaker notion of equivalence for groupoids of gauge fields. A word of warning: in the above example we introduced a *redundant* gauge symmetry: the group $\mathbb{Z}/2\mathbb{Z}$ acts freely, and as such our groupoid of gauge fields $\overline{\mathfrak{F}}(M)$ is equivalent to the set of fields $\mathfrak{F}(M)$ that we started with, by design. In actual gauge theory, however, this is not the case: most groupoids of gauge fields are not equivalent to a set. As such, one will in general lose information when working with the set of gauge equivalence classes instead of its underlying groupoid of gauge fields.

We conclude that, if we are to take these symmetries seriously, we necessarily have to work in a mathematical framework that is capable of working up to *weak equivalence*. Concretely, for our purpose of working with algebraic quantum gauge field theories, this means two things. First, any definitions will only be *up to weak equivalence*. For example, the time-slice axiom will state that to any Cauchy morphism ι, the field theory functor $\mathfrak{A}$ will assign a weak equivalence rather than an isomorphism. And second, any construction we use to construct an algebraic quantum gauge field theory will need to *preserve weak equivalences*. As an example, for the linear quantization functor $\mathfrak{Q}_{\mathrm{lin}}$ that we define in Chapter 3 we will have to check that if $f : \mathfrak{L} \to \mathfrak{L}'$ is a weak equivalence of linear field theories, $\mathfrak{Q}_{\mathrm{lin}}(f) : \mathfrak{Q}_{\mathrm{lin}}(\mathfrak{L}) \to \mathfrak{Q}_{\mathrm{lin}}(\mathfrak{L}')$ is a weak equivalence of quantum field theories (it is). And if this is not the case, we need to *derive* the functor: adapt it in such a way that the new version does preserve equivalences, while being as close to the old functor as possible.

In several areas of mathematics, working up to weak equivalences is well-trodden ground. The above description of equivalence of categories has long been recognized as the correct one. In topology one often studies topological spaces up to homotopy, which is a much more flexible notion than that of homeomorphism. And in homological algebra, chain complexes of vastly different size (like the simplical complex and the singular complex of a space in algebraic topology) can still be thought of as equivalent if they have the same homology groups: we work up to quasi-isomorphism.

In all of these cases, we find that being a weak equivalence is not just a property, but that there exist extra structures that *witness* this fact: 2-morphisms of categories (called natural transformations) compare the 1-morphisms (functors) between categories. Homotopies can deform topological spaces, or continuous maps between them. And in homological algebra, chain homotopies are used to compare chain maps. So we are naturally led to *higher structures*, these homotopies or 2-morphisms, between maps. And of course, there is no reason to stop there: we could try to define homotopies of homotopies of continuous maps, 4-morphisms between 3-morphisms between 2-morphisms between functors between categories, and so on.

In the past half century of mathematics, several ways of handling weak equivalences in a category have been developed. One approach is introducing 2-morphisms between morphisms in a category, while letting all conditions for 1-morphisms (for example, associativity of composition) only hold up to these 2-morphisms. We can then repeat this, introducing 3-morphisms between 2-morphisms and weakening the conditions on 2-morphisms to hold up to 3-morphisms, and so on. This leads to the theory of *infinity categories* or $(\infty, 1)$-categories, which are famously extremely hard to define in this way directly. Probably the most common approach to define infinity categories is to use simplicial sets, resulting in the definition of quasi-categories [**Joy02, Lur09a**]. *Homotopical categories* [**Rie14**] yield a more flexible but less powerful framework, as categories with only a class of weak equivalences to be specified.

The main approach we will take is that of *model category theory*. A model category is a category $\mathbf{C}$ that comes with a class of weak equivalences, and two auxiliary classes morphisms, the fibrations and the cofibrations. These can be thought of as the "good" surjections and injections, respectively, which means that certain lifting problems have a solution: they are used to define two weak factorization systems. This turns out to be enough data to form the homotopy category of $\mathbf{C}$, which is a localization of $\mathbf{C}$: a category where we can simply treat the weak equivalences as isomorphisms. Importantly for our purposes, model category theory is also a good framework to derive functors, i.e. deform them such that they preserve weak equivalences.

Model category theory was initially developed by Quillen [**Qui06, Qui69**], to find a framework that encompasses the similar homotopical ideas across topology, homological algebra and simplicial sets. He called it *homotopical algebra*, seeing it as a generalization of homological algebra. We will regularly mix the use of the terms homotopy and weak equivalence in this text, using the term homotopical to mean "treating weak equivalences in an appropriate way". Specifically, we use the term homotopical functor for a functor that preserves weak equivalences.

1.3. Homotopy algebraic quantum field theory

So we arrive at the fundamental question driving this thesis: how do we define and construct algebraic quantum field theories in a manifestly homotopical way? In other words, how do we give definitions that naturally incorporate weak equivalences, and how do we ensure our constructions preserve weak equivalences? Answering these questions is the goal of the *homotopy algebraic quantum field theory project*.

Of course, noting that extra care should be taken when working with gauge theories is not an original or new insight. The Faddeev-Popov ghosts fields [**FP67**] were a first expression of this fact, which lead to the development of the BRST formalism [**BRS76, Tyu75**] and the Batalin-Vilkovisky formalism [**BV84, BV83**]. And in algebraic quantum field theory, [**Hol08**] constructed quantum Yang-Mills theory as a perturbative quantum field theory on any spacetime, while [**FR12, FR13**] developed the BV formalism for algebraic quantum field theories, arguably giving first examples of homotopy algebraic quantum field theories before this concept was precisely defined.

The first signs that the usual approach of algebraic quantum field theory was not adequate for the study of gauge theories were found in [**DL12**]. There, it was shown that the isotony axiom of locally covariant quantum field theory in [**BFV03**] is too stringent for Maxwell theory, because observables of topological charges might not survive moving to a larger spacetime. This was expanded upon in several works, [**SDH14, BDS14, BDHS14, BSS17, BBSS17**]. The universal algebra of the theory on a general spacetime was found to be deficient, since it was missing Dirac charge quantization and flat connections.

In [**BSS15**] this issue was resolved at the level of linear observables. The solution was to use homotopy colimits, providing evidence that gauge theories should be studied in an appropriate homotopical framework. First attempts to develop this homotopical framework of algebraic quantum field theory include work on groupoid actions in [**BS17**], obtaining first toy models of homotopy algebraic quantum field theory, and the study of the classical solution stack of non-abelian Yang-Mills fields in [**BSS18**].

At this moment it was still unclear what the definition of a homotopy algebraic quantum field theory was. In [**BSW20**] the crucial step of using operads to define algebraic quantum field theory was introduced. This led to several insights into the structure of algebraic quantum field theory, see also [**BDS18, BSW19b**]. Importantly, building on this operadic framework of algebraic quantum field theory, a definition of homotopy algebraic quantum field theory was obtained in [**BSW19a**].

Further developments inspired by homotopy algebraic quantum field theory include the categorification of algebraic quantum field theories [**BPSW21**] and the definition of smooth structures on algebraic quantum field theories [**BPS20**]. In [**Yau20**] further resolution techniques were developed for homotopy algebraic quantum field theory, and in [**Car21**] new model structures on the category of homotopy algebraic quantum field theories were developed via left and right Bousfield localization.

We see that a lot of formalism on homotopy algebraic quantum field theory has been developed. There has, however, been a lack of examples beyond toy models. The goal of this PhD project was to remedy this by developing linear homotopy quantum field theories, a class of more realistic examples.

1.4. Summary and contents

This thesis is called *Higher linear algebraic quantum field theory*. We have now hopefully shed light on all the words in the title, except for the term linear. Giving a full formalization of gauge theories in the framework of homotopy algebraic quantum field theory is far beyond the scope of this text or this author. Instead, we will mostly work with *linear* (quantum) field theories in this text. These are theories with a linear equation of motion and a linear gauge action, so they are in particular non-interacting, which excludes e.g. nonabelian Yang-Mills theory. Of course, these are not realistic models for Nature as a whole. But they have proven to be an ideal testing ground for the homotopical aspects of homotopy algebraic quantum field theory.

Practically, the focus on linear theories means that we are working with *chain complexes*, the homotopical analogues of vector spaces. Field theories are then valued in *differential graded algebras*, algebras in chain complexes, which is also true for general homotopy algebraic field theories. The quantum field theories are then obtained by linear quantization of the linear observables, which sidesteps the problems with the quantization of general Poisson algebras. Working solely with chain complexes, we will not encounter any nonlinear objects like stacks.

Let us now summarize the contents of this thesis. In Chapter 2 we give the necessary preliminaries for the text. We give a short introduction to category theory, and then move on to the Lorentzian geometry and results on Green operators that we will need. We then give a more formal introduction to algebraic quantum field theory, taking care to work out the example of Klein-Gordon theory first. Afterward, we turn to chain complexes and their model category theory, the language in which much of this text has been written. We introduce operads and their algebras, which are powerful tools for introducing homotopies to algebra, and we end the preliminaries by giving a definition of semi-strict homotopy algebraic field theories.

In Chapter 3 we use these operads for algebraic field theories: first to characterize algebraic field theories satisfying only the Einstein causality axiom as being valued in the algebras over an operad, and then to redefine these theories as algebras over a field theory operad. The machinery of operads and their algebras then allows us to obtain adjunctions between different types of field theories, leading to characterizations of field theories that satisfy the conditions of descent and the time-slice axiom. This also results in a linear quantization adjunction that allows us to construct linear quantum field theories from linear field theories. We then treat homotopical aspects of linear quantum field theories, finding that linear quantization is a homotopical functor and that it preserves both the homotopical descent condition and the homotopical time-slice axiom.

In Chapter 4 we use our linear quantization adjunction to give a full example of a homotopical linear algebraic quantum field theory, which we call linear Yang-Mills theory. We start by defining the solution complex of a linear theory as a derived critical locus, obtaining the ghost field and the antifields of the BV formalism in the process. This complex carries a natural shifted Poisson structure, and remarkably the Lorentzian structure of a spacetime M and the existence of retarded and advanced Green operators allows us to extract an unshifted Poisson structure on the solution complex. After treating functoriality and the algebraic field theory axioms, the full linear field theory is under control, and the quantization of the theory is straightforward with the results from Chapter 3. Throughout the chapter we use the chain complex valued Klein-Gordon model as a sanity check for our constructions, while developing the linear Yang-Mills model as a true homotopy AQFT.

In Chapter 5 we consider relative Cauchy evolution for homotopy algebraic quantum field theories, which tests how fields respond to compactly supported perturbations of a spacetime metric. We start by giving a slightly different approach to relative Cauchy evolution for regular algebraic field theories, using the time-slice adjunction from Chapter 3. We then use this approach to hypothesize several rectification theorems that would allow one to define relative Cauchy evolution for homotopy algebraic quantum field theories at various levels of generality. We then prove such a rectification theorem for linear field theories, first treating the complexes of linear observables and then their Poisson structures. This yields a well-defined notion of relative Cauchy evolution for linear homotopy quantum field theories, which agrees with the naive approach on the linear observables of a theory. We end the chapter by working out the relative Cauchy evolution for our linear Yang-Mills model, finding that its stress-energy tensor agrees with the usual Maxwell stress-energy tensor. In particular, the ghost field and antifields do not carry energy.

CHAPTER 2

Preliminaries

2.1. Category theory

Category theory is the language in which much of modern algebra, geometry and topology is written. It is also the language of algebraic quantum field theory, and other mathematical approaches to quantum field theory. The classic reference is MacLane [**ML13**]. The book by Leinster [**Lei14**] is a beautiful and gentle introduction. Riehl [**Rie17**] is another introduction, which covers more ground.

2.1.1. Basic definitions and examples. The three fundamental definitions in category theory are the following.

DEFINITION 2.1.1. A *category* $\mathbf{C}$ consists of:

- a collection of *objects* $\mathbf{Ob}\,\mathbf{C}$,
- for any two objects $c, c' \in \mathbf{Ob}\,\mathbf{C}$ a set of *morphisms* $\mathbf{Hom}_{\mathbf{C}}(c, c')$.

The category also comes equipped with:

- a *composition* of morphisms
$$\begin{array}{rccc} \circ: & \mathbf{Hom}_{\mathbf{C}}(c', c'') \times \mathbf{Hom}_{\mathbf{C}}(c, c') & \longrightarrow & \mathbf{Hom}_{\mathbf{C}}(c, c'') \\ & (g, f) & \longmapsto & g \circ f \end{array}$$
for all $c, c', c'' \in \mathbf{Ob}\,\mathbf{C}$,
- an *identity morphism*
$$\mathrm{id}_c \in \mathbf{Hom}_{\mathbf{C}}(c, c)$$
for every $c \in \mathbf{Ob}\,\mathbf{C}$.

The composition and identity morphisms satisfy the compatibility relations

- *associativity*:
$$h \circ (g \circ f) = (h \circ g) \circ f$$
for any triple of composable morphisms f, g, h,
- *identity*:
$$\mathrm{id}_{c'} \circ f = f = f \circ \mathrm{id}_c$$
for any $f \in \mathbf{Hom}_{\mathbf{C}}(c, c')$.

We will often write $c \in \mathbf{C}$ instead of $c \in \mathbf{Ob}\,\mathbf{C}$ for an object in $\mathbf{C}$ and write gf for the composition $g \circ f$. We will also use the notation $f : c \to c'$ or $c \xrightarrow{f} c'$ for $f \in \mathbf{Hom}_{\mathbf{C}}(c, c')$ and $\mathbf{C}(c, c')$ for the set of morphisms $\mathbf{Hom}_{\mathbf{C}}(c, c')$. Lastly, for any morphism $c \xrightarrow{f} c'$ we say that $\mathsf{s}(f) := c$ is the *source* of f, and $\mathsf{t}(f) := c'$ is its *target*.

DEFINITION 2.1.2. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $F : \mathbf{C} \to \mathbf{D}$ is an assignment
$$\begin{array}{ccc} \mathbf{Ob}\,\mathbf{C} & \longrightarrow & \mathbf{Ob}\,\mathbf{D} \\ c & \longmapsto & F(c) \end{array}$$

together with maps of morphisms

$$F : \mathbf{Hom_C}(c, c') \longrightarrow \mathbf{Hom_D}(F(c), F(c'))$$

for all $c, c' \in \mathbf{Ob}\,\mathbf{C}$. These maps are required to preserve the composition,

$$F(g \circ f) = F(g) \circ F(f) \ ,$$

and the identities,

$$F(\mathrm{id}_c) = \mathrm{id}_{F(c)} \ .$$

Functors can be composed in the obvious way, and for any category $\mathbf{C}$ we have the obvious identity functor $\mathbb{1}_\mathbf{C} : \mathbf{C} \to \mathbf{C}$. These two structures make the collection of categories and functors into a category itself. We will write $\mathbf{Fun}(\mathbf{C}, \mathbf{D})$ or $\mathbf{D}^\mathbf{C}$ for the set of functors from $\mathbf{C}$ to $\mathbf{D}$. In fact, these in turn also form a category themselves, the morphisms of which are the natural transformations.

Definition 2.1.3. Let $\mathbf{C}$ and $\mathbf{D}$ be categories, and F and G be two functors from $\mathbf{C}$ to $\mathbf{D}$. A *natural transformation* $\eta : F \Rightarrow G$ is a family of morphisms

$$\eta_c : F(c) \longrightarrow G(c)$$

for all $c \in \mathbf{Ob}\,\mathbf{C}$ such that for any $f : c \to c'$ in $\mathbf{C}$, the diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{\eta_c} & G(c) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow{\eta_{c'}} & G(c') \end{array} \tag{2.1.1}$$

commutes.

The notation $\eta : F \Rightarrow G$ is useful when talking about both functors and natural transformations; the notation $\eta : F \to G$ can also be used when there is no risk of confusion, and we will do so throughout this thesis.

Definition 2.1.4. A natural transformation $\eta : F \Rightarrow G$ is a *natural isomorphism* if each component $\eta_c : F(c) \xrightarrow{\cong} G(c)$ is an isomorphism.

Natural transformations can be equipped with two compositions. One is called *vertical composition*: for $\eta : F \Rightarrow G$ and $\zeta : G \Rightarrow H$ we have $\zeta \circ_v \eta : F \Rightarrow H$ by composition at each component:

$$(\zeta \circ_v \eta)_c := \zeta_c \eta_c : F(c) \to H(c) \ .$$

To define *horizontal composition* consider two pairs of composable functors $F, F' : \mathbf{C} \to \mathbf{D}$ and $G, G' : \mathbf{D} \to \mathbf{E}$, with natural transformations $\eta : F \Rightarrow F'$ and $\zeta : G \Rightarrow G'$ between them. Then we have the horizontal composition $\zeta \circ_h \eta : GF \Rightarrow G'F'$ through

$$(\zeta \circ_h \eta)_c := \zeta_{F'(c)} G(\eta_c) = G'(\eta_c) \zeta_{F(c)} : GF(c) \to G'F'(c) \ .$$

Note that for any functor F we can define the identity natural transformation $\mathrm{id}_F : F \Rightarrow F$ between the compositions of functors through

$$(\mathrm{id}_F)_c := \mathrm{id}_{F(c)} : F(c) \to F(c) \ .$$

For a diagram

$$\mathbf{C} \xrightarrow{F} \mathbf{D} \overset{G}{\underset{G'}{\;\Downarrow \eta\;}} \mathbf{E} \xrightarrow{H} \mathbf{F}$$

we can then form the horizontal composition $\mathrm{id}_H \circ_h \eta \circ_h \mathrm{id}_F$, which we will denote by $H\eta F : HGF \Rightarrow HG'F$ to simplify notation.

There are many examples of categories. We give a few below, and will encounter more in later chapters. The basic example of a category is the category of sets.

Example 2.1.5. **Set** is the category of sets. Objects of **Set** are sets and morphisms are maps. The composition is the standard composition of maps, and the identity map is the identity morphism.

Sets with extra structure give many other examples of categories. As with **Set**, composition is given by composition of maps and identity maps are identity morphisms in each of the below examples.

Example 2.1.6. $\mathbf{Vect}_{\mathbb{K}}$ is the category of vector spaces over a field $\mathbb{K}$. Objects of $\mathbf{Vect}_{\mathbb{K}}$ are vector spaces over $\mathbb{K}$ and morphisms are linear maps.

In a sense, vector spaces can be thought of as special types of chain complexes, which we introduce in Section 2.4.1.

Example 2.1.7. **Top** is the category of topological spaces and continuous maps. **Man** is the category of smooth manifolds and smooth (i.e. infinitely differentiable) maps.

A choice of extra structure on manifolds (like orientation, metric or symplectic structure) with corresponding morphisms gives a refinement of **Man**. We will encounter **Loc**, one such refinement, in Definition 2.2.8.

Example 2.1.8. $\mathsf{Alg}_{\mathsf{As}_{\mathbb{K}}}$ is the category of unital associative algebras over a field $\mathbb{K}$. Objects are associative algebras with unit (A, μ, η) and morphisms are linear maps that preserve the multiplication and unit.

We will often suppress the choice of ground field $\mathbb{K}$, writing $\mathsf{Alg}_{\mathsf{As}}$ instead of $\mathsf{Alg}_{\mathsf{As}_{\mathbb{K}}}$.

Example 2.1.9. If we work over the field $\mathbb{C}$ (or more generally over a field with an involution) we can define the category $*\mathsf{Alg}_{\mathsf{As}}$ of $*$-algebras. Objects are unital associative algebras with involution $(A, \mu, \eta, *)$ and morphisms are linear maps that preserve the multiplication, unit and involution.

Other choices of algebraic structure lead to other categories of algebras. These can often be characterized as algebras over operads, which we will do in Section 2.5.2. One structure that cannot be defined using operads is that of a Poisson vector space.

Example 2.1.10. $\mathbf{PoissVect}_{\mathbb{K}}$ is the category of Poisson vector spaces over a field $\mathbb{K}$. Objects are pairs of a vector space V together with a Poisson structure $\tau : V \wedge V \to \mathbb{K}$, where we note that the wedge product forces τ to be antisymmetric. A morphism of Poisson vector spaces $L : (V, \tau_V) \to (W, \tau_W)$ is a linear map $L : V \to W$ that preserves the Poisson structure: $L^*\tau_W = \tau_W \circ (L \wedge L) = \tau_V$.

Example 2.1.11. As already mentioned above, our definitions of categories, functors and natural transformations define two examples of categories.

Cat is the category of categories[1], with categories as objects and functors as morphisms. Functors can be composed in the obvious way, composing the maps of objects and the maps of morphisms, and the identity functor is the identity map on objects and morphisms.

Given two categories $\mathbf{C}$ and $\mathbf{D}$, we can also define the *functor category* $\mathbf{Fun}(\mathbf{C},\mathbf{D})$ or $\mathbf{D}^{\mathbf{D}}$. Objects are functors from $\mathbf{C}$ to $\mathbf{D}$ and morphisms are the natural transformations between them, with vertical composition of natural transformations as composition of morphisms, and the identity natural transformation as the identity.

In fact, we can combine these two structures and think of **Cat** as a 2-category, though we will not pursue 2-categories in this text. We just note here that categories carry a broader notion of equivalence than isomorphism, as alluded to in Section 1.2 of the introduction.

Definition 2.1.12. An *equivalence* of categories $\mathbf{C}$ and $\mathbf{D}$ is a pair of functors

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : G$$

together with a pair of natural isomorphisms

$$\eta : \mathbb{1}_{\mathbf{C}} \Rightarrow GF \quad ; \quad \epsilon : FG \Rightarrow \mathbb{1}_{\mathbf{D}} \ .$$

There is another characterization of functors that establish an equivalence, for which we need a few more definitions. We start by noting that functors act on both objects and morphisms, and we can consider their behaviour on either. For example, are they surjective or injective? It turns out that on objects, this is not quite the right question.

Definition 2.1.13. A functor $F : \mathbf{C} \to \mathbf{D}$ is *essentially surjective* if for all $d \in \mathbf{D}$, $d \cong F(c)$ for a $c \in \mathbf{C}$.

Asking the question of surjectivity and injectivity for the sets of morphisms leads to the definitions of full and faithful functors.

Definition 2.1.14. A functor $F : \mathbf{C} \to \mathbf{D}$ is *full* if for any two $c, c' \in \mathbf{C}$, the map

$$F : \mathbf{Hom}_{\mathbf{C}}(c, c') \longrightarrow \mathbf{Hom}_{\mathbf{D}}(F(c), F(c'))$$

is surjective, and F is *faithful* if this map is injective. A functor that is both full and faithful is called *fully faithful*.

Lemma 2.1.15. A functor $F : \mathbf{C} \to \mathbf{D}$ is part of an equivalence if and only if it is fully faithful and essentially surjective.

Categories and functors themselves allow for several constructions and definitions that give new categories.

Definition 2.1.16. Given a category $\mathbf{C}$, we define its *opposite category* $\mathbf{C}^{\mathrm{op}}$ by reversing the arrows of $\mathbf{C}$: objects of $\mathbf{C}^{\mathrm{op}}$ are the objects of $\mathbf{C}$, and $\mathbf{Hom}_{\mathbf{C}^{\mathrm{op}}}(c, c') := \mathbf{Hom}_{\mathbf{C}}(c', c)$. The identities of $\mathbf{C}^{\mathrm{op}}$ are the identities of $\mathbf{C}$, and composition similarly is the same, but with the arguments reversed: $f \circ_{\mathbf{C}^{\mathrm{op}}} g = g \circ_{\mathbf{C}} f$.

Remark 2.1.17. Definition 2.1.2 of a functor is really the definition of a *covariant functor*. A *contravariant functor* $F : \mathbf{C} \to \mathbf{D}$ is a functor that reverses the direction of the morphisms and the order of composition; so equivalently, it is a covariant functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{D}$ or $F : \mathbf{C} \to \mathbf{D}^{\mathrm{op}}$.

[1]Really, **Cat** is the (large) category of all small categories, see also Remark 2.1.24

Definition 2.1.18. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Their *product category* $\mathbf{C} \times \mathbf{D}$ is defined as follows. The objects of $\mathbf{C} \times \mathbf{D}$ are pairs $(c \in \mathbf{C}, d \in \mathbf{D})$. For $(c, d), (c', d') \in \mathbf{C} \times \mathbf{D}$, a morphism $(c, d) \to (c', d')$ is a pair (f, g) with $f : c \to c'$ and $g : d \to d'$.

Definition 2.1.19. Let $\mathbf{C}$ be a category. A *subcategory* $\mathbf{C}'$ of $\mathbf{C}$ is a category such that

$$\mathbf{Ob}\,\mathbf{C}' \subseteq \mathbf{Ob}\,\mathbf{C}$$

and for any $c, c' \in \mathbf{Ob}\,\mathbf{C}'$,

$$\mathbf{Hom}_{\mathbf{C}'}(c, c') \subseteq \mathbf{Hom}_{\mathbf{C}}(c, c') \ .$$

We will write $\mathbf{C}' \subseteq \mathbf{C}$.

If $\mathbf{C}'$ is a subcategory of $\mathbf{C}$, there is a natural inclusion functor $\iota : \mathbf{C}' \hookrightarrow \mathbf{C}$. This functor is clearly faithful, but not necessarily full.

Definition 2.1.20. A *full subcategory* $\mathbf{C}' \subseteq \mathbf{C}$ is a subcategory such that the inclusion functor is full. In other words, for any $c, c' \in \mathbf{C}'$,

$$\mathbf{Hom}_{\mathbf{C}'}(c, c') = \mathbf{Hom}_{\mathbf{C}}(c, c') \ .$$

An even more refined type of subcategory is that of a (co)reflective subcategory. We will return to these subcategories in Definition 2.1.27 because we first need to define adjoint functors.

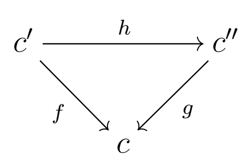
In the same way that surjectivity on objects is not the right way to think about surjectivity of functors, the naive definition of the image of a functor is also not the right one. In fact, it is not even always a category.

Definition 2.1.21. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. The *essential image* of F is the smallest subcategory of $\mathbf{D}$ that contains all objects $F(c)$ and all morphisms $F(f)$ for c and f in $\mathbf{C}$, and also all objects isomorphic to any $F(c)$ and all isomorphisms $f : F(c) \to d$ and $g : d \to F(c)$.

Note that this definition fits with Definition 2.1.13 of essential surjectivity: a functor is always essentially surjective onto its essential image.

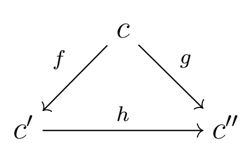
Given a category $\mathbf{C}$ and an object $c \in \mathbf{C}$, we can ask what $\mathbf{C}$ looks like from the perspective of c: which maps to c exist, and which maps out of c?

Definition 2.1.22. Let $\mathbf{C}$ be a category and $c \in \mathbf{C}$. The *overcategory* or *slice category* $\mathbf{C}/c$ or $\mathbf{C} \downarrow c$ has all morphisms $f : c' \to c$ into c as objects. Morphisms from $f : c' \to c$ to $g : c'' \to c$ are all $h : c' \to c''$ in $\mathbf{C}$ such that the diagram



commutes.

Dually, the *undercategory* or *coslice category* $c/\mathbf{C}$ or $c \downarrow \mathbf{C}$ has all morphisms $f : c \to c'$ out of c as objects, and morphisms from $f : c \to c'$ to $g : c \to c''$ are all $h : c' \to c''$ in $\mathbf{C}$ such that the diagram



commutes.

Note that there exists a forgetful functor (or a projection) from $\mathbf{C}/c$ to $\mathbf{C}$, forgetting the perspective of c:

$$\begin{array}{cccc} \Pi : & \mathbf{C}/c & \longrightarrow & \mathbf{C} \\ & (f : c' \to c) & \longmapsto & c' \\ & \left(\begin{array}{ccc} c' & \xrightarrow{h} & c'' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & c & \end{array} \right) & \longmapsto & (h : c' \to c'') \end{array} \tag{2.1.2}$$

and likewise we can define the projection $c/\mathbf{C} \to \mathbf{C}$.

The above definition of (co)slice categories can in fact be expanded to incorporate functors.

Definition 2.1.23. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and $d \in \mathbf{D}$. The *comma category* F/d or $F \downarrow d$ of F over d has as objects pairs $(c, f : F(c) \to d)$ where $c \in \mathbf{C}$ and f is a $\mathbf{D}$-morphism. A morphism from (c, f) to (c', f') is a $\mathbf{C}$-morphism $g : c \to c'$ such that the diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{F(g)} & F(c') \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & d & \end{array}$$

commutes.

Dually, the *mma category* d/F or $d \downarrow F$ of F under d has pairs $(c, f : d \to F(c))$ as objects, where $c \in \mathbf{C}$ and f is a $\mathbf{D}$-morphism. A morphism from (c, f) to (c', f') is a $\mathbf{C}$-morphism $g : c \to c'$ such that the diagram

$$\begin{array}{ccc} & d & \\ & {\scriptstyle f}\swarrow \quad \searrow{\scriptstyle f'} & \\ F(c) & \xrightarrow{F(g)} & F(c') \end{array}$$

commutes.

Similar to (2.1.2) there exist forgetful functors

$$\Pi : F/d \longrightarrow \mathbf{C} \quad ; \quad \Pi : d/F \longrightarrow \mathbf{C} \, . \tag{2.1.3}$$

Remark 2.1.24. Lastly, a note about the size of categories. A category $\mathbf{C}$ is *small* if the collection of objects $\mathbf{Ob}\,\mathbf{C}$ is a set and the collection of all morphisms $\mathbf{Mor}\,\mathbf{C}$ is a set, and *large* otherwise. Not every category we consider will be small, but every one will be equivalent to a small category. We will for the most part ignore issues of size in this text.

2.1.2. All concepts. Many concepts in algebra, geometry and topology can be captured by a universal property. One of the strengths of category theory is that it is an efficient language for these universal properties, expressing parallels between concepts in different fields. In this thesis we will encounter adjoint functors, (co)limits and Kan extensions. Miraculously, we will not use the Yoneda lemma anywhere (at least explicitly).

2.1.2.1. *Adjoint functors.*

Definition 2.1.25. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and

$$F : \mathbf{C} \rightleftarrows \mathbf{D} : G$$

be a pair of functors. F is *left adjoint* to G and G is *right adjoint* to F, if

$$\mathbf{D}(F(c), d) \cong \mathbf{C}(c, G(d))$$

naturally in $c \in \mathbf{C}$ and $d \in \mathbf{D}$.

Naturality here means that for any pair of composable morphisms

$$F(c) \xrightarrow{\;g\;} d \xrightarrow{\;q\;} d'$$

in $\mathbf{D}$, $\overline{qg} = G(q)\overline{g}$, and for any pair of composable morphisms

$$c' \xrightarrow{\;p\;} c \xrightarrow{\;f\;} G(d)$$

in $\mathbf{C}$, $\overline{fp} = \overline{f}F(p)$. Here we write $\overline{f}$ for the image under the above bijection of a morphism $f : c \to G(d)$, which we call the *transpose*. Likewise, we write $\overline{g}$ for the transpose of a $g : F(c) \to d$.

If F is left adjoint to G, we write $F \dashv G$ (or $G \vdash F$).

For an arbitrary category $\mathbf{C}$, the only morphisms that we know exist are the identities $\mathrm{id}_c : c \to c$. For an adjunction $F \dashv G$ this means we always have the transposes

$$\eta_c := \overline{\mathrm{id}_{F(c)}} : c \longrightarrow GF(c)$$

and

$$\epsilon_d := \overline{\mathrm{id}_{G(d)}} : FG(d) \longrightarrow d\ .$$

It turns out that these are the components of a natural transformation $\eta : \mathbb{1}_{\mathbf{C}} \Rightarrow GF$ called the *unit* and a natural transformation $\epsilon : FG \Rightarrow \mathbb{1}_{\mathbf{D}}$ called the *counit*. They satisfy the *triangle identities*: we have the commutative diagrams of natural transformations

$$\begin{array}{ccc} F & \overset{F\eta}{\Longrightarrow} & FGF \\ & {}_{\mathrm{id}_F}\searrow\!\!\searrow & \Downarrow \epsilon F \\ & & F \end{array} \quad ; \quad \begin{array}{ccc} G & \overset{\eta G}{\Longrightarrow} & GFG \\ & {}_{\mathrm{id}_G}\searrow\!\!\searrow & \Downarrow G\epsilon \\ & & G \end{array} \quad . \tag{2.1.4}$$

The unit and counit completely characterize an adjunction: by naturality, for any $g : F(c) \to d$ its transpose is

$$\overline{g} = G(g)\eta_c$$

and for any $f : c \to G(d)$ its transpose is

$$\overline{f} = \epsilon_d F(f)\ .$$

And in fact there exists a one-to-one correspondence between adjunctions between F and G, and pairs of natural transformations $\eta : \mathbb{1}_{\mathbf{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathbb{1}_{\mathbf{D}}$ that satisfy the triangle identities (2.1.4).

Example 2.1.26. An important class of adjunctions are so-called *free-forget* adjunctions. These are adjunctions where the right adjoint forgets part of the structure of an object, while the left adjoint freely generates it.

As an example, let $\mathbb{K}$ be a field and consider the category of vector spaces $\mathbf{Vect}_{\mathbb{K}}$ (see Example 2.1.6) and the category of associative algebras with unit $\mathsf{Alg}_{\mathsf{As}}$ over $\mathbb{K}$ (see Example 2.1.8). Any associative algebra with unit (A, μ, η) has an underlying vector space A, and this defines a forgetful functor

$$U : \mathsf{Alg}_{\mathsf{As}} \longrightarrow \mathbf{Vect}_{\mathbb{K}}.$$

This functor has a left adjoint which is constructed as follows. For a vector space $V \in \mathbf{Vect}_{\mathbb{K}}$, we construct the *tensor algebra* $(T^{\otimes}(V), \mu_{\otimes}, \eta_{\otimes})$,

$$T^{\otimes}(V) := \bigoplus_{n \geq 0} V^{\otimes n} \in \mathsf{Alg}_{\mathsf{As}}$$

with multiplication

$$\mu\big(v_1 \otimes \cdots \otimes v_m, v_1' \otimes \cdots \otimes v_n'\big) := v_1 \otimes \cdots \otimes v_m \otimes v_1' \otimes \cdots \otimes v_n'$$

and unit

$$\eta_{\otimes} = 1 \in \mathbb{K} =: V^{\otimes 0}.$$

This defines a functor

$$T^{\otimes} : \mathbf{Vect}_{\mathbb{K}} \to \mathsf{Alg}_{\mathsf{As}}$$

which is quickly seen to be left adjoint to U:

$$\mathsf{Alg}_{\mathsf{As}}(T^{\otimes}(V), A) \cong \mathbf{Vect}_{\mathbb{K}}(V, U(A))$$

naturally for any vector space V and algebra A. The components of the unit of this adjunction are linear maps

$$\iota_1 : V \to U(T^{\otimes}(V)) \tag{2.1.5}$$

which embed V into the direct summand $V^{\otimes 1} \cong V$ of $T^{\otimes}(V)$.

A somewhat less violent forgetful functor on associative algebras yields a unital Lie algebra; we will return to this in Section 3.3.4.

Adjoint functors are also used in defining (co)reflective subcategories, two refinements of full subcategories (see Definition 2.1.20).

Definition 2.1.27. A full subcategory $\iota : \mathbf{C}' \hookrightarrow \mathbf{C}$ is *reflective* if the inclusion functor ι has a left adjoint $T \dashv \iota$. Likewise, it is a *coreflective* subcategory if ι has a right adjoint $R \vdash \iota$.

Given an adjunction $F \dashv G$, we can ask if either of the functors is equivalent to a (co)reflective inclusion. These turn out to be characterized by their (co)units.

Definition 2.1.28. Let

$$F : \mathbf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{D} : G$$

be a pair of adjoint functors. We say that F *exhibits* $\mathbf{C}$ *as a coreflective subcategory* of $\mathbf{D}$ if F is fully faithful, or equivalently if the unit $\eta : \mathbb{1}_{\mathbf{C}} \Rightarrow GF$ is a natural isomorphism. In this case, $\mathbf{C}$ is equivalent to its essential image under F (see Definition 2.1.21 and Lemma 2.1.15 and recall that a functor is essentially surjective on its essential image).

Dually, G *exhibits* $\mathbf{D}$ *as a reflective subcategory* of $\mathbf{C}$ if G is fully faithful, or equivalently if the counit $\epsilon : FG \Rightarrow \mathbb{1}_{\mathbf{D}}$ is a natural isomorphism. Then $\mathbf{D}$ is equivalent to its essential image under G.

If an adjunction exhibits both reflective and coreflective subcategories, we find that it is an equivalence.

DEFINITION 2.1.29. An adjunction is an *adjoint equivalence* if its unit and counit are natural isomorphisms. Equivalently, see Lemma 2.1.15, it is an equivalence when either component is fully faithful and essentially surjective.

Not every equivalence of categories (see Definition 2.1.12) is an adjoint equivalence: this is only the case when the natural transformations η and ϵ are the unit and counit of an adjunction, or equivalently when they satisfy the triangle identities (2.1.4).

2.1.2.2. *Limits and colimits.*

DEFINITION 2.1.30. Let $\mathbf{C}$ be a category, and I be a small category. A *diagram of shape I in $\mathbf{C}$* is a functor $D : I \to \mathbf{C}$.

A *cone* (c, f_i) on a diagram D is then an object $c \in \mathbf{C}$ together with a family of morphisms $f_i : c \to D(i)$ for all $i \in I$, such that if $g : i \to i'$ is a morphism in I, the diagram

$$\begin{array}{ccc} c & \xrightarrow{f_i} & D(i) \\ & {\scriptstyle f_{i'}}\searrow & \downarrow {\scriptstyle D(g)} \\ & & D(i') \end{array}$$

commutes.

A *limit* $(\lim_I(D), p_i)$ of D is a universal cone on D: a cone on D such that for any cone (c, f_i) on D there exists a unique morphism

$$c \xrightarrow{\overline{f}} \lim_I(D)$$

such that $f_i = p_i \circ \overline{f}$ for all $i \in I$.

Dually, a *cocone* (c, f_i) on a diagram D is an object $c \in \mathbf{C}$ together with a family of morphisms $f_i : D(i) \to c$ for all $i \in I$, such that if $g : i \to i'$ is a morphism in I, the diagram

$$\begin{array}{ccc} D(i) & \xrightarrow{f_i} & c \\ {\scriptstyle D(g)} \downarrow & \nearrow {\scriptstyle f_{i'}} & \\ D(i') & & \end{array}$$

commutes.

A *colimit* $(\operatorname{colim}_I(D), \iota_i)$ of D is a universal cocone on D: a cocone on D such that for any cocone (c, f_i) on D there exists a unique morphism

$$\operatorname{colim}(D)_I \xrightarrow{\overline{f}} c$$

such that $f_i = \overline{f} \circ \iota_i$ for all $i \in I$.

We call a (co)limit *finite* if the category I has finitely many objects and morphisms. Note that postcomposing a morphism $c \to \lim_I(D)$ with the morphisms p_i yields a cone on D. So giving a morphism into $\lim_I(D)$ is equivalent to giving a cone on D. Dually, giving a morphism out of $\operatorname{colim}_I(D)$ is equivalent to giving a cocone on D. Finally, if they exist, limits and colimits are unique up to isomorphism; as such, we will often be speaking of *the* (co)limit of a diagram.

Example 2.1.31. Many well-known constructions are examples of limits or colimits.

- If I is a discrete category (i.e. a category with no morphisms that are not the identity) then its limit is called a *product*. We write $\prod_i D(i) := \lim_I(D)$. This is the usual product in the categories **Set**, $\mathbf{Vect}_{\mathbb{K}}$, **Man**, **Cat**, et cetera.
- Dually, the colimit of a diagram of a discrete category is called a *coproduct*. We write $\coprod_i D(i) := \operatorname{colim}_I(D)$; in **Set** this is the disjoint union. In $\mathbf{Vect}_{\mathbb{K}}$ (and $\mathbf{Ch}_{\mathbb{K}}$, see Section 2.4.1) we also write $\bigoplus_i D(i) := \operatorname{colim}_I(D)$; this is the direct sum.
- Consider the category
$$\bullet \rightrightarrows \bullet$$
of two objects and two nonidentity morphisms. A limit of a diagram on this category is called an *equalizer*; a colimit a *coequalizer*.

  As an example, if $A \in \mathsf{Alg}_{\mathsf{As}}$ is an associative algebra and $W \subseteq U(A)$ is a vector subspace of A with inclusion $i : W \hookrightarrow U(A)$ and zero map $0 : W \to U(A)$ (see Example 2.1.26), the colimit of the diagram
$$F(W) \underset{0}{\overset{i}{\rightrightarrows}} A$$
in $\mathsf{Alg}_{\mathsf{As}}$ is the quotient of A by the ideal generated by W.
- Consider the category
$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$
of three objects and two nonidentity morphisms. Limits of diagrams on this category are called *pullbacks*; they include several constructions known as such and they also include intersections.

  As an example, if $C^\infty(M)$ is the vector space of real-valued smooth functions on M and $P : C^\infty(M) \to C^\infty(M)$ is a linear differential operator, the pullback of
$$\begin{array}{ccc} & & C^\infty(M) \\ & & \big\downarrow{\scriptstyle (\mathrm{id},P)} \\ C^\infty(M) & \xrightarrow{(\mathrm{id},0)} & C^\infty(M) \times C^\infty(M) \end{array}$$
in $\mathbf{Vect}_{\mathbb{R}}$ will be the *solution space* of P: the vector space of functions $f \in C^\infty(M)$ such that $Pf = 0$. This vector space can be thought of as the intersection of the section (id, P) with the section $(\mathrm{id}, 0)$ in $C^\infty(M) \times C^\infty(M)$.
- If we take $I = \emptyset$ to be the empty category, we see that a limit $\lim_\emptyset(D)$ in $\mathbf{C}$ is an object $*$ such that for any object $c \in \mathbf{C}$, there exists a unique morphism $c \to *$. Such a $*$ is called a *terminal object* in $\mathbf{C}$.
- Dually, a colimit $\operatorname{colim}_\emptyset(D)$ is an object $\emptyset$ such that for any $c \in \mathbf{C}$, there exists a unique morphism $\emptyset \to c$. This $\emptyset$ is called an *initial object* in $\mathbf{C}$.

Limits do not exist in general for all categories.

Definition 2.1.32. Let $\mathbf{C}$ be a category. If I is a small category, we say that $\mathbf{C}$ *has limits of shape* I if a limit $\lim_I(D)$ exists in $\mathbf{C}$ for every diagram D of shape I. We call $\mathbf{C}$ *complete* if it has all small limits, i.e. if a limit $\lim_I(D)$ exists in $\mathbf{C}$ for every diagram D of every shape I.

Dually, $\mathbf{C}$ *has colimits of shape* I if a colimit $\mathrm{colim}_I(D)$ exists in $\mathbf{C}$ for every diagram D of shape I and we call $\mathbf{C}$ *cocomplete* if it has all small colimits.

$\mathbf{C}$ is called *bicomplete* if it is both complete and cocomplete.

Examples of bicomplete categories are the categories $\mathbf{Set}$ and $\mathbf{Vect}_{\mathbb{K}}$. If $\mathbf{C}$ has all limits of shape I, the operation of forming a limit defines a functor

$$\lim_I : \mathbf{C}^I \longrightarrow \mathbf{C}$$

from the category of diagrams of shape I in $\mathbf{C}$ to $\mathbf{C}$. This functor is right adjoint to the diagonal functor

$$\Delta : \mathbf{C} \longrightarrow \mathbf{C}^I$$

that assigns to any object $c \in \mathbf{C}$ the constant diagram $D(i) = c$. Dually, $\mathbf{C}$ has all colimits of shape I, this defines a functor

$$\mathrm{colim}_I : \mathbf{C}^I \longrightarrow \mathbf{C}$$

that is left adjoint to Δ.

If $F : \mathbf{C} \to \mathbf{D}$ is a functor, $F(\lim_I(D))$ will be a cone of the diagram $FD : I \to \mathbf{D}$, but it is not necessarily a limit of that diagram. In particular, the functors $\lim_I$ and colim_J do not commute in general, if they exist.

Definition 2.1.33. A functor $F : \mathbf{C} \to \mathbf{D}$ is said to *preserve limits* if, when $(L, p_i : L \to D(i))$ is a limit cone on D in $\mathbf{C}$, $(F(L), F(p_i) : F(L) \to FD(i))$ is a limit cone on FD in $\mathbf{D}$.

Dually, F *preserves colimits* if, when $(C, \iota_i : C \to D(i))$ is a colimit cone on D in $\mathbf{C}$, $(F(C), F(\iota_i) : FD(i) \to F(C)$ is a colimit cone on FD in $\mathbf{D}$.

Theorem 2.1.34. Right adjoints preserve limits, and left adjoints preserve colimits: if

$$F : \mathbf{C} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{D} : G$$

is a pair of adjoint functors, the left adjoint F preserves colimits, and the right adjoint G preserves limits.

Proof. This is shown in Section 6.3 in [**Lei14**]. □

2.1.2.3. *Kan extensions.* Let $\mathbf{C}, \mathbf{D}, \mathbf{E}$ be categories and $F : \mathbf{C} \to \mathbf{D}$ and $K : \mathbf{C} \to \mathbf{E}$ be functors:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\quad F \quad} & \mathbf{D} \\ & {\scriptstyle K}\searrow & \\ & & \mathbf{E} \end{array} \quad . \tag{2.1.6}$$

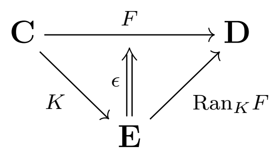
We can then ask if there exists a functor $\mathbf{E} \to \mathbf{D}$ that makes the resulting triangle commute as well as possible. This results in the definition of a Kan extension.

Definition 2.1.35. Let F and K be functors as above. A *left Kan extension* of F along K is a functor $\mathrm{Lan}_K F : \mathbf{E} \to \mathbf{D}$ and a natural transformation $\eta : F \Rightarrow \mathrm{Lan}_K F \circ K$,

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\quad F \quad} & \mathbf{D} \\ {\scriptstyle K}\searrow & \Downarrow \eta & \nearrow {\scriptstyle \mathrm{Lan}_K F} \\ & \mathbf{E} & \end{array} \quad ,$$

such that $(\mathrm{Lan}_K F, \eta)$ is universal for this property: for any other pair $(G : \mathbf{E} \to \mathbf{D}, \gamma : F \Rightarrow GK)$, γ factors uniquely as $\gamma = (\tilde{\gamma}K)\eta$ for a $\tilde{\gamma} : \mathrm{Lan}_K F \Rightarrow G$.

Dually, a *right Kan extension* of F along K is a functor $\mathrm{Ran}_K F : \mathbf{E} \to \mathbf{D}$ and a natural transformation $\epsilon : \mathrm{Ran}_K F \circ K \Rightarrow F$,



,

such that $(\mathrm{Ran}_K F, \epsilon)$ is universal for this property: for any other pair $(G : \mathbf{E} \to \mathbf{D}, \delta : GK \Rightarrow F)$, δ factors uniquely as $\delta = \epsilon(\tilde{\delta}K)$ for a $\tilde{\delta} : G \Rightarrow \mathrm{Ran}_K F$.

If they exist, Kan extensions are unique up to natural isomorphism, so it makes sense to speak of the left Kan extension. In turn, certain (co)limits give the Kan extensions explicitly, if the (co)limits exist. We give the formula for the left Kan extension, since we will use it later. Recall Definition 2.1.23 of comma categories and that there exists a canonical projection $\Pi : K/e \to \mathbf{C}$ for the comma category K/e of a functor $K : \mathbf{C} \to \mathbf{E}$.

Theorem 2.1.36. Let F and K be functors as above. If the colimits

$$\mathrm{colim}(K/e \xrightarrow{\Pi} \mathbf{C} \xrightarrow{F} \mathbf{D}) \tag{2.1.7}$$

exist for every $e \in \mathbf{E}$, then these define the left Kan extension $\mathrm{Lan}_K F : \mathbf{E} \to \mathbf{D}$ on the object $e \in \mathbf{E}$. For $g : e \to e'$ in $\mathbf{E}$, $\mathrm{Lan}_K F(g)$ is defined by the pushforward functor $f_* : K/e \to K/e'$. The natural transformation $\eta : F \Rightarrow \mathrm{Lan}_K F \circ K$ is given by $\eta_c := \iota^e_{(c,\mathrm{id}_{Kc})} : Fc \to \mathrm{Lan}_K F(Kc)$ where $\iota^e_{(c,f)} : F(\Pi(c, Kc \xrightarrow{f} e)) = Fc \to \mathrm{Lan}_K F(e)$ are the canonical maps into the colimit.

The above expression of Kan extensions as (co)limits is equivalent to the Kan extension being pointwise (which is a stronger condition on the definition of Kan extensions and argued to be the correct one in [**KK82**]), see Theorem 6.3.7 in [**Rie17**].

Remark 2.1.37. If the left and right Kan extensions can be extended to functors

$$\begin{array}{rccc} \mathrm{Lan}_K, \mathrm{Ran}_K : & \mathbf{Fun}(\mathbf{C}, \mathbf{D}) & \longrightarrow & \mathbf{Fun}(\mathbf{E}, \mathbf{D}) \\ & F & \longmapsto & \mathrm{Lan}_K F, \mathrm{Ran}_K F \end{array}$$

they are left and right adjoint to the pullback functor

$$\begin{array}{rccc} K^* : & \mathbf{Fun}(\mathbf{E}, \mathbf{D}) & \longrightarrow & \mathbf{Fun}(\mathbf{C}, \mathbf{D}) \\ & G & \longmapsto & GK \end{array}$$

respectively. In particular, we will use the notation

$$L_! : \mathbf{Fun}(\mathbf{C}, \mathbf{D}) \rightleftarrows \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{D}) : L^*$$

in Chapter 5, where $L : \mathbf{C} \to \mathbf{B}\mathbb{Z}$ is the localization of $\mathbf{C}$, writing $L_!$ for Lan_L. We will extend this notation in Section 2.5.2, finding that this left Kan extension is a special case of operadic left Kan extension.

2.1.3. Symmetric monoidal categories and additive categories. Monoidal categories formalize an extra structure that categories such as **Set** and $\mathbf{Vect}_{\mathbb{K}}$ have: the (tensor) product.

Definition 2.1.38. A *monoidal category* $(\mathbf{T}, \otimes, I)$ is a triple consisting of a category $\mathbf{T}$, a bifunctor

$$\otimes : \mathbf{T} \times \mathbf{T} \to \mathbf{T}$$

called a *monoidal product*, and an object $I \in \mathbf{T}$ called the *monoidal unit*, together with three natural isomorphisms, the *associator*

$$\alpha_{t,t',t''} : (t \otimes t') \otimes t'' \xrightarrow{\cong} t \otimes (t' \otimes t'')$$

and the left and right *unitors*,

$$\lambda_t : I \otimes t \xrightarrow{\cong} t \quad ; \quad \rho_t : t \otimes I \xrightarrow{\cong} t.$$

These natural isomorphisms are required to satisfy two coherence conditions: the pentagon identity,

$$\begin{array}{ccccc}
((t \otimes t') \otimes t'') \otimes t''' & \xrightarrow{\alpha\otimes\mathrm{id}} & (t \otimes (t' \otimes t'')) \otimes t''' & \xrightarrow{\alpha} & t \otimes ((t' \otimes t'') \otimes t''') \\
\downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\mathrm{id}\otimes\alpha} \\
(t \otimes t') \otimes (t'' \otimes t''') & & \xrightarrow{\qquad\qquad\alpha\qquad\qquad} & & t \otimes (t' \otimes (t'' \otimes t'''))
\end{array}$$

and the triangle identity,

$$\begin{array}{ccc}
(t \otimes I) \otimes t' & \xrightarrow{\qquad\alpha\qquad} & t \otimes (I \otimes t') \\
& {\scriptstyle\rho\otimes\mathrm{id}}\searrow \quad \swarrow{\scriptstyle\mathrm{id}\otimes\lambda} & \\
& t \otimes t' &
\end{array} .$$

To simplify notation and equations, we will suppress the associators and unitors when possible, writing $(\mathbf{T}, \otimes, I)$ where we should really write $(\mathbf{T}, \otimes, I, \alpha, \lambda, \rho)$ and writing $t \otimes t' \otimes t''$ without worrying about brackets.

Definition 2.1.39. A monoidal category $(\mathbf{T}, \otimes, I)$ is called *braided* if it is equipped with a *braiding* B, a natural isomorphism

$$B_{t,t'} : t \otimes t' \xrightarrow{\cong} t' \otimes t$$

that satisfies the two hexagon identities,

$$\begin{array}{ccccc}
(t \otimes t') \otimes t'' & \xrightarrow{\alpha} & t \otimes (t' \otimes t'') & \xrightarrow{B} & (t' \otimes t'') \otimes t \\
\downarrow{\scriptstyle B\otimes\mathrm{id}} & & & & \downarrow{\scriptstyle\alpha} \\
(t' \otimes t) \otimes t'' & \xrightarrow{\alpha} & t' \otimes (t \otimes t'') & \xrightarrow{\mathrm{id}\otimes B} & t' \otimes (t'' \otimes t)
\end{array}$$

and

$$\begin{array}{ccccc}
t \otimes (t' \otimes t'') & \xrightarrow{\alpha^{-1}} & (t \otimes t') \otimes t'' & \xrightarrow{B} & t'' \otimes (t \otimes t') \\
{\scriptstyle\mathrm{id}\otimes B}\downarrow & & & & \downarrow{\scriptstyle\alpha^{-1}} \\
t \otimes (t'' \otimes t') & \xrightarrow{\alpha^{-1}} & (t \otimes t'') \otimes t' & \xrightarrow{B\otimes\mathrm{id}} & (t'' \otimes t) \otimes t'
\end{array}$$

where α is the associator of $\mathbf{T}$.

If $B^2 = \mathrm{id}$ the braiding is called *symmetric*, and $\mathbf{T}$ is called a *symmetric monoidal category*.

DEFINITION 2.1.40. A symmetric monoidal category is called *closed* if for any $t \in \mathbf{T}$, the functor $- \otimes t : \mathbf{T} \to \mathbf{T}$ has a right adjoint $[t, -]$. This means that we have a bijection

$$\mathbf{Hom}_{\mathbf{T}}(t \otimes t', t'') \cong \mathbf{Hom}_{\mathbf{T}}(t, [t', t''])$$

that is natural in every argument. The functor

$$[\,,\,] : \mathbf{T}^{\mathrm{op}} \times \mathbf{T} \to \mathbf{T}$$

is called the *internal hom*, and is also denoted by **hom**.

EXAMPLE 2.1.41. **Set** is an example of a closed symmetric monoidal category, with the cartesian product $\times$ as the monoidal product, and the one point set $\{*\}$ as the monoidal unit. The symmetric braiding is the obvious map $X \times Y \xrightarrow{\cong} Y \times X$ that switches the factors, and the internal hom is the usual mapping set:

$$[X, Y] = \mathbf{Set}(X, Y).$$

which explains the name internal hom: it is an object in the category (so it is internal) that plays a role that is played by the set of morphisms in **Set**.

EXAMPLE 2.1.42. The category $\mathbf{Vect}_{\mathbb{K}}$ is closed symmetric monoidal; the tensor product plays the role of the monoidal product, and the ground field $\mathbb{K}$ is the monoidal unit. As with **Set**, the braiding is the obvious map $V \otimes W \to W \otimes V$ that flips the factors. The internal hom is the mapping set, interpreted as a vector space over $\mathbb{K}$:

$$[V, W] = \mathbf{Hom}_{\mathbb{K}}(V, W).$$

Another closed symmetric monoidal category is the category $\mathbf{Ch}_{\mathbb{K}}$ of chain complexes over a field $\mathbb{K}$; we will treat these in Section 2.4.1.

REMARK 2.1.43. If $\mathbf{T}$ is closed, the functor $- \otimes t$ is a left adjoint. Because left adjoint functors preserve colimits (Theorem 2.1.34) this implies that

$$\mathrm{colim}_I(t_i \otimes t') \cong \mathrm{colim}_I(t_i) \otimes t'$$

for any diagram $t : I \to \mathbf{T}$

DEFINITION 2.1.44. If $\mathbf{T}$ is cocomplete, it allows for **Set**-*tensoring*: if $S \in \mathbf{Set}$ and $t \in \mathbf{T}$, we define

$$S \otimes t := \coprod_{s \in S} t.$$

Note that for any $s \in S$ we have a corresponding inclusion map $\iota_s : t \to S \otimes t$.

Additive categories formalize another structure on $\mathbf{Vect}_{\mathbb{K}}$: the addition of morphisms.

DEFINITION 2.1.45. An *additive category*[2] is a category $\mathbf{C}$ that is enriched over the abelian groups: for any $c, c' \in \mathbf{C}$, $\mathbf{Hom}_{\mathbf{C}}(c, c')$ is an abelian group. The composition of morphisms is bilinear: $(g_1 + g_2)(f_1 + f_2) = g_1 f_1 + g_1 f_2 + g_2 f_1 + g_2 f_2$.

[2] In other contexts this is called a *preadditive* category.

2.2. Lorentzian geometry and normally hyperbolic operators

All constructions of field theories in this thesis are on the category of globally hyperbolic spacetimes **Loc**, which we will introduce here. These are spacetimes (oriented and time-oriented Lorentzian manifolds) that one can equip with a global time coordinate (in a sense made precise below). This in turn ensures that our theories can exhibit a dynamical law. The reference used throughout this section is [**BGP07**]; other references are [**Bär15**] and [**BD15**].

Throughout this section we assume that our manifolds are oriented and connected.

2.2.1. Some features of Lorentzian manifolds. We start by considering some basic structures of Lorentzian manifolds.

Definition 2.2.1. A *Lorentzian manifold* (M, g) is a manifold M of fixed dimension n together with a metric g on M of Lorentzian signature $(- + + \cdots +)$.

Note that we use the "mostly plus" convention here: the signature of our metric has one minus sign, and all the others are plusses. A mostly minus convention is also often found in the literature. From here on out we will frequently abuse notation and write M for the pair (M, g).

Let us import some notions from special relativity to the setting of Lorentzian manifolds. First recall that on Minkowski space $\mathbb{M}$, and on a vector space V with Lorentzian inner product $(\ , \ )$ like T_pM for $p \in M$ more generally, vectors $v \in V \backslash \{0\}$ are classified as either

- *spacelike*: $(v, v) > 0$
- *lightlike*: $(v, v) = 0$
- *timelike*: $(v, v) < 0$.

The origin $0 \in V$ is considered to be spacelike, and a vector is called *causal* if it is lightlike or timelike.

These vectors are organized as follows. We define $I(0)$ to be the set of all timelike vectors; then $J(0) := \overline{I(0)}$ is the set of all causal vectors (and the origin), and $C(0) := \partial I(0)$ is the set of all lightlike vectors (and the origin), which is also called the *lightcone*. If the dimension of V is at least 1, $I(0)$ has two connected components, and a choice $I_+(0)$ of one of these is called a *time-orientation*. Vectors in $I_+(0)$ are called *future-directed*, and we define $I_-(0) := -I_+(0)$, the *past-directed* vectors, such that $I(0) = I_+(0) \sqcup I_-(0)$. Likewise, one can define $J_\pm(0)$ and $C_\pm(0)$.

Definition 2.2.2. A *time-orientation* $\mathfrak{t}$ on an oriented Lorentzian manifold M is a smooth choice of time-orientation $I_+(p)$ on T_pM for all $p \in M$. Equivalently, it is an equivalence class of smooth timelike (and therefore nonzero) vector fields $\mathfrak{t}$ on M.

A time-oriented and oriented Lorentzian manifold is called a *spacetime*.

A curve $\gamma(t)$ in a Lorentzian manifold M is called *timelike* if all its tangent vectors are timelike. Likewise, we define *lightlike*, *spacelike* and *causal* curves. On a spacetime M (which means M has a time-orientation) we also define *future-directed* and *past-directed* curves.

These different classes of curves allow us to define notions like the causal future of a point on a general spacetime. For a point $p \in M$, we define its *chronological future* $I_+(p)$ to be all points $q \in M$ such that there exists a future-directed timelike path from p to q in M, and its *chronological past* $I_-(p)$ all $q \in M$ such that a future-directed path from q to p exists (or equivalently, a past-oriented path from p to q). Similarly, the *causal future* $J_+(p)$ of p consists of p itself and all $q \in M$ such that there exists a future-directed causal curve from p to q, and the *causal past* $J_-(p)$ consists of p and all $q \in M$ such that there exists a future-directed causal curve from q to p.

These definitions extend to subsets of M in the obvious way: for a subset $U \subseteq M$ we define $I_\pm(U) = \bigcup_{p \in U} I_\pm(p)$ and $J_\pm(U) = \bigcup_{p \in U} J_\pm(p)$, and we have $J(U) = J_+(U) \cup J_-(U)$ and $I(U) = I_+(U) \cup I_-(U)$.

2.2.2. Globally hyperbolic spacetimes and Cauchy surfaces. There exists a whole hierarchy of causality conditions on spacetimes. One of the most basic demands is for M to not have closed causal curves; we will work with spacetimes that are globally hyperbolic, which is a somewhat stronger definition.

Definition 2.2.3. A spacetime M is said to satisfy the *strong causality condition* if it contains no almost closed causal curves: for any $p \in M$ with neighbourhood $U \ni p$, p has a neighbourhood $V \ni p$ contained in U such that any causal curve that starts and ends in V is completely contained in V.

A spacetime M is called *globally hyperbolic* if it satisfies the strong causality condition and for any two $p, q \in M$, $J_+(p) \cap J_-(q)$ is compact.

It turns out that there are two other characterizations of globally hyperbolic spacetimes, which are related to Cauchy surfaces. To understand these, we need a few preliminary definitions.

A piecewise C^1 curve $\gamma : I \to M$ is called *inextendible* if no piecewise C^1 reparametrization of γ can continuously be extended to the end points of the reparametrized interval. For a subset $U \subseteq M$, its *domain of dependence* $D(U)$ is the collection of points $p \in M$ such that any inextendible timelike curve through p intersects U. A subset $\Sigma \subseteq M$ is called *achronal* if every timelike curve in M intersects Σ at most once.

Definition 2.2.4. A *Cauchy hypersurface* or *Cauchy surface* in a spacetime M is a subset $\Sigma \subseteq M$ such that every inextendible timelike curve in M intersects Σ exactly once. In other words, Σ is achronal and $D(\Sigma) = M$.

Intuitively, a Cauchy surface Σ is a spatial slice of our spacetime, and timelike curves give a local time-coordinate. The following result shows that globally hyperbolic spacetimes look like this globally, and that the metric splits in a nice way relative to Σ.

Theorem 2.2.5. Let M be a connected spacetime. The following are equivalent:

(1) M is globally hyperbolic.
(2) M has a Cauchy surface Σ.
(3) M is isometric to $\mathbb{R} \times \Sigma$ with metric $-\beta \mathrm{d}t^2 + h(t)$, where t is the time coordinate on the first factor $\mathbb{R}$, β is a positive smooth function on $\mathbb{R} \times \Sigma$ and $h(t)$ is a smooth family of Riemannian metrics on Σ parametrized by t. Moreover, $\{t\} \times \Sigma$ is a Cauchy surface for any t.

Proof. This is Theorem 1.3.10 in [**BGP07**]. For a long time it was a folk theorem in Lorentzian geometry; it was finally completely proven in [**BS05**]. □

The metric and time-orientation on a spacetime M allow us to define several notions related to compactness, which we will need when considering Green operators.

Definition 2.2.6. A subset $U \subseteq M$ is called

- *past compact* if for any $p \in M$, the closure of $U \cap J_-(p)$ is compact,
- *future compact* if for any $p \in M$, the closure of $U \cap J_+(p)$ is compact,
- *spacelike compact* if it is contained in the causal cone $J(K)$ of a compact subset K,
- *timelike compact* if it is both past and future compact,
- *strong past compact* if it is both past and spacelike compact,
- *strong future compact* if it is both future and spacelike compact.

Note that the closure of U is compact if and only if it is past, future and spacelike compact.

If U is past (resp. future) compact, one can find a Cauchy surface $\Sigma \subset M$ in the past (resp. future) of U: a Σ such that $U \subseteq I_+(\Sigma)$ (resp. $U \subseteq I_-(\Sigma)$).

Two other notions that have more specific analogues in the context of Lorentzian geometry are convexity and disjointness.

Definition 2.2.7. On a spacetime M, two subsets $U, V \subset M$ are called *causally disjoint* if $J(U) \cap V = \emptyset$ (or equivalently, $U \cap J(V) = \emptyset$).

On a globally hyperbolic spacetime[3] M, a subset $U \subseteq M$ is called *causally convex* or *causal* if $J_+(U) \cap J_-(U) = U$. In other words, any causal curve that starts and ends in U is contained in U.

We organize the globally hyperbolic spacetimes into the category **Loc**.

Definition 2.2.8. The category **Loc** is

$$\mathbf{Loc} = \begin{cases} \text{obj}: & \text{globally hyperbolic spacetimes } M \text{ of fixed dimension} \\ \text{mor}: & \text{smooth isometric open embeddings } f: M \to N \text{ that preserve orientation} \\ & \text{and time-orientation such that } f(M) \subseteq N \text{ is causally convex in } N \end{cases}$$

The fact that $f(M) \subset N$ is causally convex in N means that through f we can consider M to be a subspacetime of N. Any physical phenomenon on $f(M)$ can then equivalently be described on M.

There are two extra structures on **Loc** that will be important for us.

Definition 2.2.9. A morphism $f : M \to N$ in **Loc** is called *Cauchy* if $f(M) \subseteq N$ contains a Cauchy surface of N.

Definition 2.2.10. A pair of maps in **Loc** with the same target, $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$, are called *causally disjoint* if $f_1(M_1)$ and $f_2(M_2)$ are causally disjoint, i.e. if $J(f_1(M_1)) \cap f_2(M_2) = \emptyset$.

Informally speaking, when a morphism $f : M \to N$ is Cauchy, a solution of field equations on M uniquely determines a solution on N. When two regions are causally disjoint, no physical (causal) signal can travel between the two regions. Our definitions of field theories will satisfy axioms that ensure these properties.

A choice of spacetime $\overline{M} \in \mathbf{Loc}$ gives a subcategory of **Loc**:

Definition 2.2.11. For $\overline{M} \in \mathbf{Loc}$, $\mathbf{COpens}(\overline{M})$ is the category of *causally convex opens* of $\overline{M}$. Objects of $\mathbf{COpens}(\overline{M})$ are causally convex open subsets U of $\overline{M}$, and the morphisms are inclusions, if they exist: for $U, V \subseteq \overline{M}$ causally open,

$$\mathbf{COpens}(\overline{M})(U,V) = \begin{cases} \{U \subseteq V\} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise.} \end{cases}$$

[3]The definition on more general Lorentzian manifolds is more involved, see [**BGP07**] Def. 1.3.3

Let us note a few facts about $\mathbf{COpens}(\overline{M})$. First, it has a terminal object, which is $\overline{M}$ itself. Furthermore, $\mathbf{COpens}(\overline{M})$ is not a full subcategory of $\mathbf{Loc}$: for example, if $\overline{M} = \mathbb{M}$ is Minkowski space, the set of endomorphisms $\mathbf{COpens}(\mathbb{M})(\mathbb{M}, \mathbb{M})$ contains just one element, while $\mathbf{Loc}(\mathbb{M}, \mathbb{M})$ is the connected component of the Lorentz group containing the identity. We also remark that $\mathbf{COpens}(\overline{M})$ can be characterized as the overcategory $\mathbf{Loc} \downarrow \overline{M}$ (see Definition 2.1.22). From this perspective, for all $\overline{M} \in \mathbf{Loc}$ there exists an obvious forgetful functor

$$\mathbf{COpens}(\overline{M}) \longrightarrow \mathbf{Loc} \tag{2.2.1}$$

that forgets the morphism to $\overline{M}$.

Another subcategory of $\mathbf{Loc}$ is $\mathbf{Loc}_\diamond$, which is central to the question of descent for field theories.

Definition 2.2.12. The category $\mathbf{Loc}_\diamond \subseteq \mathbf{Loc}$ is the full subcategory of $\mathbf{Loc}$ consisting of spacetimes M such that M is diffeomorphic to $\mathbb{R}^n$ (as a manifold).

Importantly, both $\mathbf{COpens}(\overline{M})$ and $\mathbf{Loc}_\diamond$ inherit the notions of Cauchy morphisms and causally disjoint pairs from $\mathbf{Loc}$.

2.2.3. Normally hyperbolic operators. We now turn to partial differential operators on a globally hyperbolic spacetime M.

Let V be a vector space over either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Recall that a *vector bundle* over M with fiber V is the data $E \xrightarrow{\pi} M$ such that

- π is a smooth surjective map,
- for any $p \in M$, $E_p := \pi^{-1}(p)$ is a vector space with $E_p \cong V$,
- any $p \in M$ has a neighbourhood $U \subset M$ such that $\pi^{-1}(U) \cong U \times V$ in a way that is compatible with π: the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\cong} & U \times V \\ & \overset{\pi}{\searrow} & \downarrow{\scriptstyle \pi_1} \\ & & U \end{array}$$

  commutes, where π_1 is the projection of $U \times V$ on the first factor U, and the restriction of this map to any fiber $\pi^{-1}(q)$ ($q \in U$) gives an isomorphism $\pi^{-1}(q) \cong V$ of vector spaces.

E is called the *total space*, π is the *projection* and $\pi^{-1}(U) \cong U \times V$ is a *local trivialization*. The dimension of V is called the *rank* of E.

For $E \xrightarrow{\pi} M$ a vector bundle, a smooth map $s : M \to E$ is called a *section* if $\pi \circ s = \mathrm{id}_M$. We write $\Gamma(E)$ for the $C^\infty(M)$-module of sections. We will also consider various subspaces of $\Gamma(E)$, characterized by their support properties: $\Gamma_{\mathrm{c}}(E)$ is the space of sections with compact support, and $\Gamma_{\mathrm{tc}}(E)$, $\Gamma_{\mathrm{pc}}(E)$, $\Gamma_{\mathrm{fc}}(E)$, $\Gamma_{\mathrm{sc}}(E)$ contain the sections with timelike compact, past compact, future compact and spacelike compact support, respectively.

If $E \xrightarrow{\pi} N$ is a vector bundle and $f : M \to N$ is a map, we can form the *pullback bundle*

$$\begin{array}{ccc} f^*E & \dashrightarrow & E \\ {\scriptstyle \pi'} \downarrow & & \downarrow {\scriptstyle \pi} \\ M & \xrightarrow{f} & N \end{array}$$

as a limit in **Man**. Explicitly,

$$f^*E = \{(x,e) \in X \times E | f(x) = \pi(e)\}$$

and $\pi'(x,e) = x$. In turn, sections of E can be pulled back to f^*E:

$$\begin{array}{rccc} f^*: & \Gamma(E) & \longrightarrow & \Gamma(f^*E) \\ & s & \longmapsto & f^*s(x) = (x, s(f(x)))\ . \end{array}$$

The section f^*s of f^*E is called the *pullback* of s along f.

If M and N are globally hyperbolic spacetimes with vector bundle $E \to N$, and $f : M \to N$ is a smooth open embedding (for example, if f is a **Loc**-morphism) then compactly supported sections of f^*E can be pushed forward along f: for $s \in \Gamma_c(f^*E)$ we can define the *pushforward* of s along f as

$$f_*s(x) = \begin{cases} s(f^{-1}(x)) & x \in f(M) \\ 0 & x \notin f(M) \end{cases} \tag{2.2.2}$$

for $x \in N$.

A *fiber metric* h on E is an inner product h_p on E_p for each $p \in M$ that depends smoothly on p. If E is equipped with a fiber metric h, this induces the integration pairing

$$\langle s, s' \rangle := \int_M h(s, s')\, \mathrm{vol}_M$$

on all sections $s, s' \in \Gamma(E)$ with compactly overlapping support, where vol_M is the volume form induced by the metric on M.

Given M of dimension n and two vector bundles $E \xrightarrow{\pi} M$ and $F \xrightarrow{\rho} M$ with fibers V and W, a *linear differential operator of order at most* k is a $\mathbb{K}$-linear map $P : \Gamma(E) \to \Gamma(F)$ that is locally of the form one would expect: for $p \in M$ there exists a neighbourhood U of p in M with coordinates x^i and local trivializations $\pi^{-1}(U) \cong U \times V$ and $\rho^{-1}(U) \cong U \times W$ such that

$$Ps = \sum_{\alpha : |\alpha| \le k} A_\alpha \Big(\frac{\partial}{\partial x^1}\Big)^{\alpha_1} \dots \Big(\frac{\partial}{\partial x^n}\Big)^{\alpha_n} s \tag{2.2.3}$$

in these coordinates. Here, $\alpha = (\alpha_1, \dots, \alpha_n)$ are all multi-indices valued in $\mathbb{N}_0^n$ and we sum over all α such that $|\alpha| := \sum_i \alpha_i \le k$, and $A_\alpha \in \mathbf{Hom}_{\mathbb{K}}(V, W)$. P is said to be of *order* k if it is of order at most k, but not of order at most $k-1$.

If E and F are equipped with fiber metrics h_E and h_F with induced integration pairings $\langle\, ,\, \rangle_E$ and $\langle\, ,\, \rangle_F$, the *formal adjoint* of a linear differential operator $P : \Gamma(E) \to \Gamma(F)$ is the differential operator $P^* : \Gamma(F) \to \Gamma(E)$ defined by

$$\langle s', Ps \rangle_F = \langle P^*s', s \rangle_E\ .$$

A linear differential operator $P : \Gamma(E) \to \Gamma(E)$ is *formally self-adjoint* if $P^* = P$.

For a differential $P : \Gamma(E) \to \Gamma(F)$ of order k with local expression (2.2.3), its *principal symbol* σ_P is the map

$$\sigma_P : T^*M \longrightarrow \mathbf{Hom}_{\mathbb{K}}(E, F)$$

that one gets by contracting a covector with the linear maps in the order k part of P: for $\zeta = \sum_i \zeta_i\, \mathrm{d}x^i \in T_p^*M$,

$$\sigma_P(\zeta) := \sum_{\alpha : |\alpha| = k} A_\alpha(p) \Big(\prod_i (\zeta_i)^{\alpha_i} \Big).$$

A linear differential operator $P : \Gamma(E) \to \Gamma(E)$ of order two is called *normally hyperbolic* if its principal symbol is

$$\sigma_P(\zeta) = g(\zeta, \zeta)\,\mathrm{id}_E$$

where we recall that g is the metric on M.

EXAMPLE 2.2.13. An important example of normally hyperbolic operators is the *d'Alembert operator* $\Box$, also known as the *Hodge Laplacian* or the *Laplace-de Rham* operator, which will play an central role in chapter 4. It acts on the k-forms $\Omega^k(M) = \Gamma(\bigwedge^k T^*M)$ of any oriented pseudo-Riemannian manifold (M, g) of dimension n.

To define $\Box$, we first note that g induces a fiberwise inner product

$$(\ ,\ ) : \bigwedge^k T_p^*M \otimes \bigwedge^k T_p^*M \to \mathbb{R}$$

such that (η, ζ) is a smooth function on M when $\eta, \zeta \in \Omega^k(M)$ are k-forms on M. The volume form vol_M induced by the metric g then allows us to define the *Hodge star operator*

$$* : \Omega^k(M) \to \Omega^{n-k}(M)$$

through

$$\eta \wedge *\zeta = (\eta, \zeta)\,\mathrm{vol}_M$$

for $\eta, \zeta \in \Omega^k(M)$. Integrating over M yields the integration pairing

$$\langle \eta, \zeta \rangle = \int_M \eta \wedge *\zeta$$

which is defined if the supports of η and ζ have compact overlap.

The Hodge star in turn allows us to define the *codifferential*

$$\delta := (-1)^k *^{-1} \mathrm{d} * : \Omega^k(M) \to \Omega^{k-1}(M). \tag{2.2.4}$$

Since d squares to zero, δ does too, and using Stokes' theorem, δ is seen to be formally adjoint to the de Rham differential d with respect to the integration pairing:

$$\langle \mathrm{d}\eta, \zeta \rangle = \langle \eta, \delta\zeta \rangle \tag{2.2.5}$$

where now $\eta \in \Omega^k(M)$ and $\zeta \in \Omega^{k+1}(M)$.

The d'Alembertian $\Box$ is then defined as

$$\Box := (\mathrm{d} + \delta)^2 = \mathrm{d}\delta + \delta\mathrm{d}.$$

Note that since $\mathrm{d}^2 = \delta^2 = 0$, both d and δ commute with $\Box$:

$$\mathrm{d}\,\Box = \mathrm{d}\delta\mathrm{d} = \Box\,\mathrm{d} \quad , \qquad \delta\,\Box = \delta\mathrm{d}\delta = \Box\,\delta.$$

Moreover, because d and δ are formally adjoint to each other with respect to the integration pairing, we immediately see that $\Box$ is self-adjoint:

$$\langle \Box\eta, \zeta \rangle = \langle \eta, \Box\zeta \rangle.$$

2.2.4. Green operators. Normally hyperbolic operators form a class of linear differential operators that can be solved using Green operators.

DEFINITION 2.2.14. For a linear differential operator $P : \Gamma(E) \to \Gamma(E)$ on a time-oriented Lorentzian manifold M, a *retarded Green operator* for P is a linear map

$$G^+ : \Gamma_{\mathrm{pc}}(E) \to \Gamma_{\mathrm{pc}}(E)$$

such that

(1) $P \circ G^+ = \mathrm{id}_{\Gamma_{\mathrm{pc}}(E)} = G^+ \circ P\big|_{\Gamma_{\mathrm{pc}}(E)}$
(2) $\mathrm{supp}(G^+ s) \subseteq J_+(\mathrm{supp}(s))$

Similarly, an *advanced Green operator* for P is a linear map

$$G^- : \Gamma_{\mathrm{fc}}(E) \to \Gamma_{\mathrm{fc}}(E)$$

such that

(1) $P \circ G^- = \mathrm{id}_{\Gamma_{\mathrm{fc}}(E)} = G^- \circ P\big|_{\Gamma_{\mathrm{fc}}(E)}$
(2) $\mathrm{supp}(G^- s) \subseteq J_-(\mathrm{supp}(s))$

An operator P for which both G^+ and G^- exist is called *Green hyperbolic.*

For a Green hyperbolic operator P with advanced and retarded Green operators $G^\pm$, its *retarded-minus-advanced* operator or *causal propagator* is the difference

$$G := G^+ - G^- : \Gamma_{\mathrm{tc}}(E) \to \Gamma(E).$$

Recall that timelike compact means both future and past compact, which is exactly where G is defined.

REMARK 2.2.15. Note that if $G^\pm$ exist, they are unique: for example, if both G^+ and $\tilde{G}^+$ are retarded Green operators for P, we immediately find

$$G^+ s = G^+ P \tilde{G}^+ s = \tilde{G}^+ s$$

for all $s \in \Gamma_{\mathrm{pc}}(E)$.

The existence of Green operators for the operator P implies several properties for P related to sections of various support: for example, by condition (1) in Definition 2.2.14 we immediately see that P is injective on $\Gamma_{\mathrm{pc}}(E)$. These properties are most compactly stated in the form of an exact sequence, as follows.

THEOREM 2.2.16. Let $P : \Gamma(E) \to \Gamma(E)$ be a Green hyperbolic operator with Green operators $G^\pm$. Then the following sequence is an exact complex:

$$0 \longrightarrow \Gamma_{\mathrm{tc}}(E) \xrightarrow{P} \Gamma_{\mathrm{tc}}(E) \xrightarrow{G} \Gamma(E) \xrightarrow{P} \Gamma(E) \longrightarrow 0\ .$$

Restricting to sections of compact support (at the start of the sequence) we find that

$$0 \longrightarrow \Gamma_{\mathrm{c}}(E) \xrightarrow{P} \Gamma_{\mathrm{c}}(E) \xrightarrow{G} \Gamma_{\mathrm{sc}}(E) \xrightarrow{P} \Gamma_{\mathrm{sc}}(E) \longrightarrow 0$$

is also exact.

PROOF. These statements are Lemma 2.14, Theorem 2.15 and Proposition 2.16 in [**BD15**]. □

Remark 2.2.17. If E comes equipped with a fiber metric and $P : \Gamma(E) \to \Gamma(E)$ is both Green hyperbolic and self-adjoint, the Green operators G^+ and G^- for P are also adjoint on compactly supported sections: if $s, s' \in \Gamma_c(E)$,

$$\langle s, G^+ s'\rangle = \langle PG^- s, G^+ s'\rangle = \langle G^- s, PG^+ s'\rangle = \langle G^- s, s'\rangle \ .$$

This immediately implies that the causal propagator $G = G^+ - G^-$ is skew-adjoint:

$$\langle s, Gs'\rangle = -\langle Gs, s'\rangle$$

for $s, s' \in \Gamma_c(E)$.

Remark 2.2.18. Let P be a Green hyperbolic operator with Green operators $G^\pm$, and let Q be another linear operator that commutes with P and does not increase the support of sections. Then Q will also commute with $G^\pm$: for example, for $s \in \Gamma_{pc}(E)$,

$$QG^+ s = G^+ PQG^+ s = G^+ QPG^+ s = G^+ Qs.$$

For our purposes, the main result on normally hyperbolic operators is the following theorem.

Theorem 2.2.19. *Normally hyperbolic operators on globally hyperbolic spacetimes are Green hyperbolic.*

Proof. For sections of compact support, this was proven in [**BGP07**], see Corollary 3.4.3. The extension to sections of future and past compact support is proven in [**Bär15**]. $\square$

Example 2.2.20. Continuing Example 2.2.13, we note that since the d'Alembert operator $\Box$ is normally hyperbolic, it has advanced and retarded Green operators $G^+_\Box$ and $G^-_\Box$ on globally hyperbolic spacetimes. Moreover, since d and δ commute with $\Box$, they also commute with its Green operators by Remark 2.2.18.

2.3. Algebraic quantum field theory

In this section we define algebraic quantum field theory (AQFT), the framework in which this thesis operates. The conceptual basis has already been laid in the introduction. We first treat the case of Klein-Gordon theory in some detail, and then give the necessary definition of algebraic quantum field theory. The book by Haag [**Haa12**] is probably the best introduction to the algebraic viewpoint on quantum field theory. Other references for this section are [**FV15**] and [**BD15**].

2.3.1. An example: Klein-Gordon observables. As an illuminating example of an algebra of observables, let us consider Klein-Gordon theory on a globally hyperbolic spacetime M, following [**BD15**].

To start with, scalar fields are simply (real-valued) smooth functions on M:

$$\phi \in C^\infty(M).$$

The equation of motion for a field of mass m is

$$P\phi := (\Box - m^2)\phi = 0$$

where we use the d'Alembertian $\Box$ defined in Example 2.2.13. Then the (classical) solution space is the kernel of the Klein-Gordon operator:

$$\mathfrak{Sol}^{KG} := \big\{\phi \in C^\infty(M) \mid P\phi = 0\big\} = \ker\big(P\big). \tag{2.3.1}$$

For the observables on $C^\infty(M)$ we choose to use the compactly supported smooth functions on M,

$$\psi \in C^\infty_{\mathrm{c}}(M).$$

Such a ψ defines a functional F_ψ on $C^\infty(M)$ through the integration pairing:

$$\begin{aligned} F_\psi : C^\infty(M) &\longrightarrow \quad \mathbb{R} \\ \phi \quad &\longmapsto \int_M \psi\, \phi \; \mathrm{vol}_M. \end{aligned}$$

These F_ψ can be interpreted as evaluation functionals, smeared by ψ: if

$$\begin{aligned} \mathrm{ev}_x : C^\infty(M) &\longrightarrow \; \mathbb{R} \\ \phi \quad &\longmapsto \phi(x) \end{aligned}$$

is the functional that evaluates ϕ at the point $x \in M$, then F_ψ can be interpreted as

$$F_\psi = \int_M \psi(x)\; \mathrm{ev}_x \; \mathrm{vol}_M$$

or more suggestively,

$$\phi(\psi) := F_\psi(\phi) = \int_M \psi(x)\phi(x)\; \mathrm{vol}_M \tag{2.3.2}$$

where only here we use notation $\phi(x) := \mathrm{ev}_x(\phi)$ for the operator evaluating the field ϕ at x.

The assignment $\psi \mapsto F_\psi$ is injective, so we choose the real vector space of (off-shell) linear observables to be

$$\mathfrak{L}^{KG}_{\text{off-shell}}(M) := \{F_\psi | \psi \in C^\infty_{\mathrm{c}}(M)\} \cong C^\infty_{\mathrm{c}}(M).$$

Note that we are indeed making a *choice* of observables here: for example, the evaluation functionals ev_x defined above are not included. So we have chosen a subspace of the algebraic dual of $C^\infty(M)$, which we call the *smooth dual space* to $C^\infty(M)$. Importantly, our choice of dual space is large enough, in that it separates $C^\infty(M)$: if $\phi_1 \neq \phi_2 \in C^\infty(M)$, there exists an $F_\psi \in \mathfrak{L}^{KG}_{\text{off-shell}}(M)$ such that $F_\psi(\phi_1) \neq F_\psi(\phi_2)$.

Moving on-shell (i.e. restricting our observables to $\mathfrak{Sol}^{KG} \subset C^\infty(M)$) we see that some F_ψ are identically zero on $\mathfrak{Sol}^{KG}$: if $\psi = P\alpha$ with $\alpha \in C^\infty_{\mathrm{c}}(M)$ and $\phi \in \mathfrak{Sol}^{KG}$,

$$F_\psi(\phi) = \int_M (P\alpha)\; \phi \; \mathrm{vol}_M = \int_M \alpha \; (P\phi)\; \mathrm{vol}_M = 0$$

where we have used the fact that $\Box$ (and therefore the Klein-Gordon operator P) is formally self-adjoint. Using the fact that P is Green hyperbolic and the properties expressed in the exact sequences of Theorem 2.2.16 we see that in fact,

$$\left\{\psi \in C^\infty_{\mathrm{c}}(M) | F_\psi\big|_{\mathfrak{Sol}^{KG}} = 0\right\} = PC^\infty_{\mathrm{c}}(M)$$

so we define the linear observables of our Klein-Gordon theory to be the equivalence classes of functions representing the functionals:

$$\mathfrak{L}^{KG}(M) := C^\infty_{\mathrm{c}}(M)\big/PC^\infty_{\mathrm{c}}(M) \cong \mathfrak{L}^{KG}_{\text{off-shell}}(M)\big/\left\{F_\psi \in \mathfrak{L}^{KG}_{\text{off-shell}}(M) | F_\psi\big|_{\mathfrak{Sol}^{KG}} = 0\right\} \tag{2.3.3}$$

Note that this is part of a more general pattern: restriciting to a subspace ($\mathfrak{Sol} = \ker(P) \subseteq V$) corresponds to taking a quotient on the (choice of) dual ($\mathbf{Obs} = V^*\big/P^*V^*$).

The next step for both the classical and the quantum theory is to specify a Poisson structure τ on the observables. This Poisson structure is constructed using the Green operators $G^{\pm}_{KG}$ of $P = \Box - m^2$, which exist because of Theorem 2.2.19:

$$\begin{array}{rcccl} \tau_M : \quad \mathfrak{L}^{KG}(M) & \otimes & \mathfrak{L}^{KG}(M) & \longrightarrow & \mathbb{R} \\ {[\psi_1]} & \otimes & [\psi_2] & \longmapsto & \int_M \psi_1 \; G_{KG} \; \psi_2 \; \mathrm{vol}_M \end{array} \tag{2.3.4}$$

which is well-defined because of the properties of $G^{\pm}_{KG}$ (recall that $G_{KG} = G^{+}_{KG} - G^{-}_{KG}$).

Restricting to a Cauchy surface, one can in fact show that this Poisson structure corresponds to the usual equal-time canonical commutation relations encountered when treating the Klein-Gordon field as a quantum field theory (see e.g. [**Wal94**]; note that there and in many other texts τ is called a symplectic structure). We also remark that while the introduction of the Poisson structure as presented here seems ad hoc, there are several constructions that lead to this specific Poisson form; for example, the Peierls formula (see [**Kha14**]) and Zuckerman's variational bicomplex ([**Zuc87**]). We will see this Poisson structure arise in yet another way in chapter 4 through certain canonical shifted Poisson structures.

The assignment of $(\mathfrak{L}^{KG}(M), \tau_M)$ to M can be extended to a functor on **Loc**: for any causal embedding $f : M \to N$ in **Loc**, the pushforward f_* (2.2.2) is defined by extending a compactly supported function ψ by zero outside of $f(M) \subseteq N$:

$$\begin{aligned} f_* : \mathfrak{L}^{KG}(M) &\longrightarrow \mathfrak{L}^{KG}(N) \\ [\psi] &\longmapsto [\psi_N] \end{aligned}$$

where

$$\psi_N(x) = \begin{cases} \psi(f^{-1}(x)) & x \in f(M) \\ 0 & x \notin f(M) \end{cases}$$

for $x \in N$. We also see that for $[\psi_1], [\psi_2] \in \mathfrak{L}(M)$

$$(f_*)^*(\tau_N)([\psi_1], [\psi_2]) = \tau_N(f_*[\psi_1], f_*[\psi_2]) = \tau_M([\psi_1], [\psi_2])$$

using the uniqueness of Green operators and the support of ψ_1. So we have a functor

$$(\mathfrak{L}^{KG}, \tau) : \mathbf{Loc} \longrightarrow \mathbf{PoissVect}_{\mathbb{R}}$$

where $\mathbf{PoissVect}_{\mathbb{R}}$ is the category of Poisson vector spaces, see Example 2.1.10.

To construct the classical theory we form the symmetric algebra

$$\mathrm{Sym}^{\otimes}\big(\mathfrak{L}^{KG}(M)\big) := \bigoplus_{n \geq 0} \mathfrak{L}^{KG}(M)^{\otimes n} / \Sigma_n$$

which is the symmetrization of the tensor algebra found in Example 2.1.26. The tensor product still gives the multiplication,

$$([\psi_1] \otimes \cdots \otimes [\psi_n]) \cdot ([\tilde{\psi}_1] \otimes \cdots \otimes [\tilde{\psi}_m]) := [\psi_1] \otimes \cdots \otimes [\psi_n] \otimes [\tilde{\psi}_1] \otimes \cdots \otimes [\tilde{\psi}_m]$$

and the unit is

$$1 \in \mathbb{R} =: \mathfrak{L}^{KG}(M)^{\otimes 0}.$$

We extend the Poisson structure τ_M to the tensor algebra as a derivation, which defines a Poisson bracket $\{,\}_M$ on $\mathrm{Sym}^{\otimes}\big(\mathfrak{L}^{KG}(M)\big)$. The resulting Poisson algebra

$$\mathfrak{A}^{KG}_{cl}(M) = \big(\mathrm{Sym}^{\otimes}(\mathfrak{L}^{KG}(M)), \{,\}_M\big)$$

is the *classical algebra of observables* for Klein-Gordon theory.

The quantum theory is obtained through canonical quantization of the linear observables: we form the tensor $*$-algebra of the complexified linear observables $\mathfrak{L}^{KG}_{\mathbb{C}}(M) := \mathfrak{L}^{KG}(M) \otimes_{\mathbb{R}} \mathbb{C}$,

$$T^{\otimes}\big(\mathfrak{L}^{KG}_{\mathbb{C}}(M)\big) := \bigoplus_{n\geq 0} \mathfrak{L}^{KG}_{\mathbb{C}}(M)^{\otimes n}$$

where multiplication and unit are defined as in Example 2.1.26. and the $*$-involution is given by

$$([\psi_1] \otimes \cdots \otimes [\psi_n])^* := [\psi_n] \otimes \cdots \otimes [\psi_1] \tag{2.3.5}$$

which is extended antilinearly (recall that all $\psi_i \in C^\infty_{\mathrm{c}}(M)$ are real-valued). The $*$-ideal that implements canonical commutation relations is

$$\mathcal{I}_{CCR}(M) := \big([\psi_1] \otimes [\psi_2] - [\psi_2] \otimes [\psi_1] - i\tau_M([\psi_1], [\psi_2]) \;\big|\; [\psi_1], [\psi_2] \in \mathfrak{L}^{KG}(M)\big)$$

and the algebra of quantum observables is then the quotient of the tensor algebra by this ideal:

$$\mathfrak{A}^{KG}_{qu}(M) := T^{\otimes}\big(\mathfrak{L}^{KG}_{\mathbb{C}}(M)\big)\big/\mathcal{I}_{CCR}(M) \in *\mathbf{Alg}.$$

For $[\psi] \in \mathfrak{L}(M) = C^\infty_{\mathrm{c}}(M)\big/PC^\infty_{\mathrm{c}}(M)$, we write its class in $\mathfrak{A}^{KG}_{qu}(M)$ as

$$\hat{\phi}(\psi) := [\psi] + \mathcal{I}_{CCR}(M) \in \mathfrak{A}^{KG}_{qu}(M). \tag{2.3.6}$$

Like $\mathfrak{L}^{KG}$, both $\mathfrak{A}^{KG}_{cl}$ and $\mathfrak{A}^{KG}_{qu}$ can be extended to functors on **Loc**. The pushforward immediately extends to a map of tensor algebras

$$f_* : T^{\otimes}(\mathfrak{L}^{KG}_{(\mathbb{C})}(M)) \longrightarrow T^{\otimes}(\mathfrak{L}^{KG}_{(\mathbb{C})}(N)).$$

which also defines a map of symmetric algebras. Since $(f_*)^*(\tau_N) = \tau_M$,

$$f_* : \mathfrak{A}^{KG}_{cl}(M) \longrightarrow \mathfrak{A}^{KG}_{cl}(N)$$

is a map of Poisson algebras; moreover, it implies that

$$f_*\big(\mathcal{I}_{CCR}(M)\big) \subseteq \mathcal{I}_{CCR}(N)$$

so we find that f_* descends to a map of algebras

$$f_* : \mathfrak{A}^{KG}_{qu}(M) \longrightarrow \mathfrak{A}^{KG}_{qu}(N).$$

This concludes our (very succinct) treatment of the classical and quantum observables of Klein-Gordon theory on an arbitrary globally hyperbolic spacetime M. Besides functoriality, $\mathfrak{A}^{KG}_{qu}$ exhibits two other features that make it into an algebraic quantum field theory, as we will see in the next section.

2.3.2. Algebraic quantum field theory. With the necessary preliminaries given, we now define algebraic quantum field theory.

DEFINITION 2.3.1. An *algebraic quantum field theory* on **Loc** is a functor

$$\mathfrak{A} : \mathbf{Loc} \longrightarrow *\mathbf{Alg}$$

that assigns to each globally hyperbolic spacetime M its *$*$-algebra of observables* $\mathfrak{A}(M)$. Moreover, it satisfies two axioms:

(1) *Einstein causality*: if $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are causally disjoint morphisms in **Loc** (see Definition 2.2.10), their observables commute in $\mathfrak{A}(N)$:

$$\Big[\mathfrak{A}(f_1)\Big(\mathfrak{A}(M_1)\Big), \mathfrak{A}(f_2)\Big(\mathfrak{A}(M_2)\Big)\Big] = \{0\} \subseteq \mathfrak{A}(N).$$

(2) *Time-slice*: if $f : M \to N$ is a Cauchy morphism in **Loc** (see Definition 2.2.9), then it induces an isomorphism of algebras of observables:

$$\mathfrak{A}(f) : \mathfrak{A}(M) \xrightarrow{\cong} \mathfrak{A}(N).$$

If $\overline{M} \in \mathbf{Loc}$, an *algebraic quantum field theory* on $\overline{M}$ is a functor

$$\mathfrak{A} : \mathbf{COpens}(\overline{M}) \longrightarrow *\mathbf{Alg}$$

satisfying the Einstein causality and time-slice axioms.

Remark 2.3.2. As mentioned in the introduction, this definition of an algebraic quantum field theory (AQFT) $\mathfrak{A}$ ensures that our theory exhibits some key features that one would demand from a quantum field theory:

- Since $\mathfrak{A}$ is a functor, the theory is *local* in the sense that for a subspacetime $U \subseteq M$ the inclusion $i : U \hookrightarrow M$ tells us which observables on M are located on U: $\mathfrak{A}(i)\big(\mathfrak{A}(U)\big) \subseteq \mathfrak{A}(M)$;
- As its name suggests, the Einstein causality axiom ensures that $\mathfrak{A}$ is *causal*: since the observables supported in spacelike separated regions in a spacetime N commute, the subsystems on the two regions are independent;
- Because of the time-slice axiom, $\mathfrak{A}$ has a *dynamical law*: a solution around a Cauchy surface can be propagated to the whole spacetime, so dually, the observables supported around that Cauchy surface are enough to probe the entire solution space.

Remark 2.3.3. One other axiom that is often included in the definition of an algebraic quantum field theory is isotony:

(3) *Isotony*: for any $f : M \to N$ in **Loc**, the map of algebras

$$\mathfrak{A}(f) : \mathfrak{A}(M) \longrightarrow \mathfrak{A}(N)$$

is injective.

Isotony tells us that non-zero observables on M will still be non-zero observables on N: none are lost when moving to the larger spacetime. However, this condition turns out to be too stringent for some models; for example, non-zero topological charges in Abelian Yang-Mills theory on M will be zero on N if N has trivial cohomology groups (this was first seen in [**DL12**] and expanded upon in [**SDH14, BDS14, BDHS14, BSS17, BBSS17**]. One proposal to resolve this is to replace the isotony axiom by a descent axiom; we will return to this in Section 3.3.1.

One direct advantage of the algebraic definition of quantum field theory is that it allows one to define and study quantum field theories on all globally hyperbolic spacetimes, curved or otherwise, with no reference to specifics of that spacetime (such as the Poincaré symmetries of Minkowski spacetime). As such it is an excellent framework to study QFTs on curved spacetimes. It is also a suitable setting to study general aspects of quantum field theories: one would expect that causality and dynamics are features of any QFT, and the prominence of observables in the theory allows locality to be baked in from the start. Note that while we are considering quantum fields on curved spacetimes, these are not theories of quantum gravity: the geometry of the spacetimes is kept fixed but arbitrary, so at best these are solutions to the classical Einstein equations.

EXAMPLE 2.3.4. The functor $\mathfrak{A}^{KG}_{qu}$ describing the observables of quantum Klein-Gordon theory discussed in the previous Section 2.3.1 defines an algebraic quantum field theory on **Loc**. We already saw that $\mathfrak{A}^{KG}_{qu} : \mathbf{Loc} \to *\mathbf{Alg}$ is a functor. Einstein causality for arbitrary elements follows from Einstein causality on generators: if $\mathrm{supp}(\psi_1)$ and $\mathrm{supp}(\psi_2)$ are spacelike separated,

$$[\hat{\phi}(\psi_1), \hat{\phi}(\psi_2)] = i\tau_M(\psi_1, \psi_2) = i \int_M \psi_1 \; G_{KG} \; \psi_2 \; \mathrm{vol}_M = 0$$

because of the support properties of G_{KG} and $\mathrm{supp}(\psi_1) \cap J(\mathrm{supp}(\psi_2)) = \emptyset$. Proving the time-slice axiom is somewhat more involved, requiring the use of Green operators and partitions of unity; this is done in the two theorems in section 3.1 of [**BD15**].

The algebras $\mathfrak{A}^{KG}_{qu}(M)$ are generated by the smearings $\hat{\phi}(\psi)$ of the linear field ϕ (2.3.2), where $\psi \in C^\infty_{\mathrm{c}}(M)$ is a test function. This assignment of observables clearly respects the morphisms in **Loc**: for $f : M \to N$ in **Loc** and $\psi \in C^\infty_{\mathrm{c}}(M)$,

$$\hat{\phi}(f_*\psi) = \mathfrak{A}(f)\hat{\phi}(\psi) \in \mathfrak{A}^{KG}_{qu}(N) \; .$$

Here, f_* is the pushforward of compactly supported functions along f. The spaces $C^\infty_{\mathrm{c}}(M)$ of compactly supported sections in fact form their own functor, the cosheaf

$$\begin{array}{rccc} C^\infty_{\mathrm{c}} : & \mathbf{Loc} & \longrightarrow & \mathbf{Vect}_{\mathbb{R}} \\ & M & \longmapsto & C^\infty_{\mathrm{c}}(M) \\ & (f : M \to N) & \longmapsto & (f_* : C^\infty_{\mathrm{c}}(M) \to C^\infty_{\mathrm{c}}(N)) \; . \end{array}$$

For general algebraic quantum field theories, note that a field on M with values in a vector bundle $E(M)$ should be smeared against a compactly supported section of the dual bundle $E^*(M)$. In these theories, quantum fields like $\hat{\phi}$ can then be defined as follows.

DEFINITION 2.3.5. Let $\mathfrak{A} : \mathbf{Loc} \to *\mathbf{Alg}$ be an algebraic quantum field theory and $E : \mathbf{Loc} \to \mathbf{VecBun}$ a contravariant functor that assigns to $M \in \mathbf{Loc}$ a vector bundle $E \to M$ of rank k over M. A *quantum field* $\hat{\Phi}$ of type E in $\mathfrak{A}$ is then a natural transformation

$$\hat{\Phi} : \Gamma_{\mathrm{c}}(E^*) \longrightarrow \mathfrak{A}$$

between the underlying functors of vector spaces (recall that we write $\Gamma_c(E^*)$ for the compactly supported sections of the dual bundle E^*).

We cast our definition of classical observables of a field theory into the same algebraic mold.

DEFINITION 2.3.6. An *algebraic classical field theory* on **Loc** is a functor

$$\mathfrak{A} : \mathbf{Loc} \longrightarrow \mathbf{PoissAlg}_{\mathbb{K}}$$

that assigns to each globally hyperbolic spacetime M its Poisson algebra of observables $\mathfrak{A}(M) = \big(\mathfrak{A}(M), \{,\}_M\big)$ and to each morphism $f : M \to N$ in **Loc** a Poisson algebra morphism

$$\mathfrak{A}(f) : \big(\mathfrak{A}(M), \{,\}_M\big) \to \big(\mathfrak{A}(N), \{,\}_N\big)$$

i.e. a linear map $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N)$ such that $\mathfrak{A}(f) \circ \{,\}_M = \{,\}_N \circ \mathfrak{A}(f) \otimes \mathfrak{A}(f)$. Moreover, it satisfies two axioms:

(1) *Einstein causality*: if $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are causally disjoint morphisms in **Loc** (see Definition 2.2.10), their observables have trivial Poisson bracket in $\mathfrak{A}(N)$:

$$\Big\{\mathfrak{A}(f_1)\big(\mathfrak{A}(M_1)\big), \mathfrak{A}(f_2)\big(\mathfrak{A}(M_2)\big)\Big\}_N = \{0\} \subseteq \mathfrak{A}(N)$$

(2) *Time-slice*: if $f : M \to N$ is a Cauchy morphism in **Loc** (see Definition 2.2.9), then it induces an isomorphism of Poisson algebras of observables:

$$\mathfrak{A}(f) : \mathfrak{A}(M) \xrightarrow{\cong} \mathfrak{A}(N) \ .$$

Remark 2.3.7. Both the classical and quantum Klein-Gordon theories from the previous Section 2.3.1 are free theories: they are constructed using the Poisson vector space of linear observables. Constructing an interacting theory is significantly more involved; specifically, canonical quantization is a lot less straightforward.

The theories constructed in this thesis will all be free; in fact, one of the main topics of chapter 3 is a generalization of the quantization of Klein-Gordon theory in Section 2.3.1. This leads us to the following algebraic definition of linear field theory.

Definition 2.3.8. An *algebraic linear field theory* on **Loc** is a functor

$$(\mathfrak{L}, \tau) : \mathbf{Loc} \longrightarrow \mathbf{PoissVect}_{\mathbb{R}}$$

that assigns to each globally hyperbolic spacetime M its *Poisson vector space of linear observables* $(\mathfrak{L}(M), \tau_M)$. Moreover, it satisfies two axioms:

(1) *Einstein causality*: if $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are causally disjoint morphisms in **Loc** (see Definition 2.2.10), their linear observables in $\mathfrak{L}(N)$ pair to zero with respect to τ_N:

$$\tau_N\Big(\mathfrak{L}(f_1)\Big(\mathfrak{L}(M_1)\Big), \mathfrak{L}(f_2)\Big(\mathfrak{L}(M_2)\Big)\Big) = \{0\} \subseteq \mathbb{R} \ .$$

(2) *Time-slice*: if $f : M \to N$ is a Cauchy morphism in **Loc** (see Definition 2.2.9), then it induces an isomorphism of Poisson vector spaces of linear observables:

$$\mathfrak{L}(f) : \big(\mathfrak{L}(M), \tau_M\big) \xrightarrow{\cong} \big(\mathfrak{L}(N), \tau_N\big) \ .$$

Definition 2.3.9. All three classes of algebraic field theories are subcategories of functor categories: the morphisms between two field theories are the natural transformations between the functors. As such, we have the full subcategories

$$\mathbf{QFT}(\mathbf{Loc}) \subseteq {*}\mathbf{Alg}^{\mathbf{Loc}} \ ; \quad \mathbf{ClFT}(\mathbf{Loc}) \subseteq {\mathbf{PoissAlg}_{\mathbb{K}}}^{\mathbf{Loc}} \ ; \quad \mathbf{LFT}(\mathbf{Loc}) \subseteq {\mathbf{PoissVect}_{\mathbb{R}}}^{\mathbf{Loc}} \ .$$

Definition 2.3.10. Given the definitions of algebraic quantum field theory and algebraic linear field theory, we can define a *linear canonical quantization functor*

$$\mathfrak{CCR} : \mathbf{LFT}(\mathbf{Loc}) \longrightarrow \mathbf{QFT}(\mathbf{Loc})$$

in the spirit of Section 2.3.1: for a Poisson vector space $(V, \tau) \in \mathbf{PoissVect}_{\mathbb{R}}$, we define

$$\mathfrak{ccr}(V, \tau) := T^{\otimes}\big(V_{\mathbb{C}}\big) / \mathcal{I}_{CCR}(V, \tau)$$

where $\mathcal{I}_{CCR}(V, \tau)$ is the $*$-ideal of $T^{\otimes}\big(V_{\mathbb{C}}\big)$ generated by the canonical commutation relation:

$$\mathcal{I}_{CCR}(V, \tau) = \big(x \otimes y - y \otimes x - i\tau(x, y)\big).$$

One immediately sees that this defines a functor $\mathfrak{ccr} : \mathbf{PoissVect}_{\mathbb{R}} \to *\mathbf{Alg}$ and postcomposition by this functor yields[4]

$$\mathfrak{CCR} := (\mathfrak{ccr})_* = \mathfrak{ccr} \circ (-) : \mathbf{LFT}(\mathbf{Loc}) \longrightarrow \mathbf{QFT}(\mathbf{Loc}).$$

If $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau) \in \mathbf{QFT}(\mathbf{Loc})$, we will say that $\mathfrak{A}$ is a *linear algebraic quantum field theory.* The inclusion unit $\iota_1 : V_{\mathbb{C}} \to T^{\otimes}(V_{\mathbb{C}})$ (2.1.5) of the free-forget adjunction $T^{\otimes} \dashv U$ yields the canonical natural transformation

$$\iota_1 : \mathfrak{L} \longrightarrow \mathfrak{CCR}(\mathfrak{L}, \tau) \tag{2.3.7}$$

of vector space valued functors, where we suppress the forgetful functor U.

DEFINITION 2.3.11. For linear algebraic quantum field theories $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau)$, there exists a special class of quantum fields in the sense of Definition 2.3.5: first, a *linear field* Φ in $(\mathfrak{L}, \tau)$ of type E is a natural transformation

$$\Phi : \Gamma_{\mathrm{c}}(E^*) \longrightarrow \mathfrak{L}$$

of the underlying functors into the category of vector spaces. A *linear quantum field* $\hat{\Phi}$ of type E in $\mathfrak{A}$ is then a quantum field

$$\hat{\Phi} = \iota_1 \circ \Phi : \Gamma_{\mathrm{c}}(E^*) \longrightarrow \mathfrak{A}$$

where Φ is a linear field in $(\mathfrak{L}, \tau)$ and ι_1 is the canonical natural transformation (2.3.7) from $\mathfrak{L}$ to $\mathfrak{A}$.

2.4. Chain complexes and model categories

The field theories in this text will regularly be homotopy field theories, in the sense that they are functors into a category of algebras valued in chain complexes. In turn, the category of chain complexes admits a structured study of its weak equivalences: it is a model category. Weibel [**Wei95**] will be our main source for chain complexes and we refer to [**MR19**] for our treatment of bicomplexes. We will use [**DS95, Rie14, Hov07**] for model categories.

2.4.1. Chain complexes and bicomplexes. We collect the facts on chain complexes we will need here and refer to [**Wei95**] for a proper introduction. Let $\mathbb{K}$ be a field of characteristic 0.

DEFINITION 2.4.1. A *chain complex* $(C_\bullet, \mathrm{d})$ over $\mathbb{K}$ is a sequence of vector spaces $\{C_n\}_{n\in\mathbb{Z}}$ over $\mathbb{K}$ with a *differential*, a sequence of linear maps $\{\mathrm{d} : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ such that $\mathrm{d}^2 = 0$. A *cochain complex* $(C^\bullet, \mathrm{d})$ over $\mathbb{K}$ is a sequence of vector spaces $\{C^n\}_{n\in\mathbb{Z}}$ over $\mathbb{K}$ with a *codifferential*, a sequence of linear maps $\{\mathrm{d} : C^n \to C^{n+1}\}_{n\in\mathbb{Z}}$ such that $\mathrm{d}^2 = 0$.

We will almost exclusively work with chain complexes in this text. We visualize a chain complex $(C_\bullet, \mathrm{d})$ as

$$\dots \xleftarrow{\mathrm{d}} C_{n-1} \xleftarrow{\mathrm{d}} C_n \xleftarrow{\mathrm{d}} C_{n+1} \xleftarrow{\mathrm{d}} \dots$$

and often abuse notation by denoting it by $C_\bullet$, (C, d) or C. The index $n \in \mathbb{Z}$ is called the *degree* of the component of the complex, so the differential reduces the degree of elements in the chain complex by 1. For an element $c \in C_n$ we write $|c| := n$ for its degree. We sometimes

[4]We should really prove that this construction preserves Einstein causality and the time-slice axiom; we will do this in Chapter 3. See Corollary 3.3.21.

use the notation $\overset{(n)}{A}$ to emphasize that A is the component at degree n of the complex in question.

REMARK 2.4.2. If V is a vector space, we will abuse notation and also write V for the chain complex

$$\dots \longleftarrow \overset{(-1)}{0} \overset{0}{\longleftarrow} \overset{(0)}{V} \overset{0}{\longleftarrow} \overset{(1)}{0} \longleftarrow \dots$$

that is only nonzero in degree 0, with $V_0 = V$. In particular we will write $\mathbb{K}$ for the chain complex that consists of the ground field $\mathbb{K}$ concentrated in degree 0.

DEFINITION 2.4.3. A morphism $f : (C_\bullet, \mathrm{d}_C) \to (D_\bullet, \mathrm{d}_D)$ of chain complexes is a sequence of linear maps $f_n : C_n \to D_n$ that commutes with the differentials:

$$f_n \mathrm{d}_C = \mathrm{d}_D f_{n+1}$$

for all $n \in \mathbb{Z}$; f is called a *chain map*. A *chain homotopy* λ between two chain maps $f, g : C \to D$ is a sequence of maps $\lambda_n : C_n \to D_{n+1}$ increasing the degree by 1 such that

$$f_n - g_n = \mathrm{d}_D \lambda_n + \lambda_{n-1} \mathrm{d}_C \ .$$

We write $\partial\lambda$ for the chain map $(\partial\lambda)_n := \mathrm{d}_D \lambda_n + \lambda_{n-1}\mathrm{d}_C$, a notation that will become more clear when discussing the internal hom below.

We think of the essential information of a chain complex as being its homology:

DEFINITION 2.4.4. Let C be a chain complex. For $n \in \mathbb{Z}$ we define its nth homology as

$$H_n(C) := \ker\big(C_{n-1} \overset{\mathrm{d}}{\longleftarrow} C_n \big) \big/ \operatorname{im}\big(C_n \overset{\mathrm{d}}{\longleftarrow} C_{n+1} \big)$$

which is well-defined because $\mathrm{d}^2 = 0$. A complex C is *exact* if $H_n(C) = 0$ for all n; in other words, when $\ker\big(C_{n-1} \overset{\mathrm{d}}{\longleftarrow} C_n \big) = \operatorname{im}\big(C_n \overset{\mathrm{d}}{\longleftarrow} C_{n+1} \big)$ for all n.

Because the components of a chain map $f : C \to D$ commute with the differentials, f defines a map on homology,

$$H_n(f) : H_n(C) \longrightarrow H_n(D)$$

which we will often simply denote by f. If f is an isomorphism on homology, i.e. if $H_n(f)$ is an isomorphism for all n, we call f a *quasi-isomorphism*. If $f : C \to D$ is a chain map, a *quasi-inverse* of f is a chain map $g : D \to C$ such that $gf - \mathrm{id}_C = \partial\lambda$ and $fg - \mathrm{id}_D = \partial\rho$ for chain homotopies λ and ρ.

We might write f^{-1} for a quasi-inverse of f, but we of course need to be careful with this notation: a quasi-inverse is not an actual inverse to f unless $\partial\lambda = 0$ and $\partial\rho = 0$, and a quasi-inverse will in general also not be unique. Since we consider the homology of a chain complex its relevant information, we think of quasi-isomorphisms as establishing an equivalence between two complexes, even when the two complexes are not isomorphic. We will consider this issue further in the next section.

The chain complexes over $\mathbb{K}$, together with the chain maps, form the category $\mathbf{Ch}_{\mathbb{K}}$. This is a category rich in additional structure: the quasi-isomorphisms make it into a homotopical category, and even a model category, as we will see in Example 2.4.19. It is also a closed symmetric monoidal category.

DEFINITION 2.4.5. The tensor product of two chain complexes (C, d_C) and (D, d_D) is

$$(C \otimes D)_n := \bigoplus_{k+l=n} C_k \otimes D_l$$

with differential $\mathrm{d}^{\otimes}$ defined by extending the differentials as derivations with a sign from the Koszul rule: for $c \in C_k$ and $d \in D_l$,

$$\mathrm{d}^{\otimes}(c \otimes d) := \mathrm{d}_C(c) \otimes d + (-1)^k c \otimes \mathrm{d}_D(d) \ .$$

The monoidal unit is then given by the complex $\mathbb{K}$, that is, the complex that is $\mathbb{K}$ in degree 0, and 0 elsewhere, with trivial differential. The braiding is given by a graded flip: for $c \otimes d \in C_k \otimes D_l$,

$$B(c \otimes d) = (-1)^{kl} d \otimes c \ . \tag{2.4.1}$$

The internal hom for $\mathbf{Ch}_{\mathbb{K}}$ is quite a bit bigger than the set of morphisms in $\mathbf{Ch}_{\mathbb{K}}$: for $(C, \mathrm{d}_V), (D, \mathrm{d}_W) \in \mathbf{Ch}_{\mathbb{K}}$, it is the mapping complex

$$[C, D]_n = \prod_{m \in \mathbb{Z}} \mathbf{Hom}_{\mathbb{K}}(C_m, D_{m+n})$$

where $\mathbf{Hom}_{\mathbb{K}}(C_n, D_{n+m})$ is the vector space of linear maps between vector spaces. For $L = \{L_m : C_m \to D_{m+n}\}_m \in [C, D]_n$ we have the differential

$$\partial L = \left\{ \mathrm{d}_D L_m - (-1)^n L_{m-1} \, \mathrm{d}_C : C_m \to D_{m+n-1} \right\}_m \in [C, D]_{n-1}.$$

The internal hom gives a characterization of the chain maps in $\mathbf{Ch}_{\mathbb{K}}$: for $(C_n, \mathrm{d}_C), (D_n, \mathrm{d}_D) \in \mathbf{Ch}_{\mathbb{K}}$,

$$\mathbf{Ch}_{\mathbb{K}}(C, D) = \ker \left(\partial : [C, D]_0 \to [C, D]_{-1} \right) \ .$$

It also characterizes the chain homotopies: a chain homotopy λ between f and g is a $\lambda \in [C, D]_1$ such that $f - g = \partial \lambda$.

DEFINITION 2.4.6. Given a chain complex (C, d_C) we define its *shifting* $C[k]$ by an integer k by $C[k]_n := C_{n-k}$ and $\mathrm{d}_{C[k]} := (-1)^k \mathrm{d}_C$.

It is clear that $C[k][l] = C[k + l]$ for any two $k, l \in \mathbb{Z}$ and that $C[0] = C$. Moreover, the shifting interacts with the closed monoidal structure: first, we see that $C[k] \cong \mathbb{K}[k] \otimes C$. And for two chain complexes C and D we have the isomorphism of internal homs $\mathbf{hom}(C, D[k]) \cong \mathbf{hom}(C, D)[k]$ through

$$\begin{array}{ccc} \mathbf{hom}(C, D[k])_n & \xrightarrow{\cong} & \mathbf{hom}(C, D)[k]_n \\ \{L_m : C_m \to D[k]_{m+n}\}_m & \longmapsto & \{L_m : C_m \to D_{m+n-k}\}_m \end{array} \ . \tag{2.4.2}$$

Importantly, this isomorphism preserves the differentials ∂ of the internal homs.

EXAMPLE 2.4.7. The differential forms $\omega \in \Omega^k(M)$ on a manifold M of dimension n form the well-known de Rham cochain complex

$$\Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(M)$$

and we will write $H^k_{\mathrm{dR}}(M)$ for its cohomology in degree k. If M is equipped with a metric, the codifferential δ (2.2.4) defines the complex

$$\Omega^0(M) \xleftarrow{\delta} \Omega^1(M) \xleftarrow{\delta} \cdots \xleftarrow{\delta} \Omega^n(M) \ .$$

We write $H^k_\delta(M)$ for its homology in degree k and $\Omega^k_\delta(M)$ for $\ker\big(\delta : \Omega^k(M) \to \Omega^{k-1}(M)\big)$. Of course, the Hodge star $*$ is an isomorphism, so we have $H^k_\delta(M) = *H^{n-k}_{\mathrm{dR}}(M)$.

Taking the compactly supported differential forms $\Omega^k_{\mathrm{c}}(M)$ gives subcomplexes of these two complexes, and we write $H^k_{\mathrm{c,dR}}(M)$ and $H^k_{\mathrm{c},\delta}$ for their (co)homology. Note that this reverses functoriality:

$$\Omega^\bullet : \mathbf{Man}^{\mathrm{op}} \longrightarrow \mathbf{coCh}_{\mathbb{R}}$$

is a contravariant functor using the pullback of forms, where $\mathbf{coCh}_{\mathbb{R}}$ is the category of cochain complexes. On the other hand, if we write $\mathbf{Man}_{emb}$ for the category of manifolds with smooth open embeddings as morphisms,

$$\Omega^\bullet_{\mathrm{c}} : \mathbf{Man}_{emb} \longrightarrow \mathbf{coCh}_{\mathbb{R}}$$

is a covariant functor using the pushforward of compactly supported forms.

Theorem 2.4.8. (Poincaré duality) The integration pairing

$$\begin{array}{rccc} (\;,\;): & \Omega^k(M) \otimes \Omega^{n-k}_{\mathrm{c}}(M) & \longrightarrow & \mathbb{R} \\ & \omega \otimes \zeta & \longmapsto & \int_M \omega \wedge \zeta \end{array}$$

induces a nondegenerate bilinear pairing

$$(\;,\;) : H^k_{\mathrm{dR}}(M) \otimes H^{n-k}_{\mathrm{c,dR}}(M) \longrightarrow \mathbb{R}\ .$$

So if the cohomologies are finite dimensional,

$$H^k_{\mathrm{dR}}(M) \cong (H^{n-k}_{\mathrm{c,dR}})^*\ .$$

We can define algebraic structures like a multiplication or a Poisson structure on chain complexes in the same way we define them on vector spaces; we just have to be careful that all maps involved are chain maps.

Definition 2.4.9. A *differential graded unital associative algebra* is a triple (A, μ, η) where $A = (A, \mathrm{d})$ is a chain complex, $\mu : A \otimes A \to A$ is an associative multiplication and $\eta : \mathbb{K} \to A$ is a unit. Both μ and η are required to be chain maps, which in particular implies that $\eta(1) \in A_0$, since $\mathbb{K}$ is concentrated in degree 0. These algebras form the category $\mathrm{dg}\mathsf{Alg}_{\mathsf{As}}$; we include the dg to emphasize that we are working with chain complexes. Morphisms in $\mathrm{dg}\mathsf{Alg}_{\mathsf{As}}$ are chain maps that preserve the multiplication and the unit.

If we work over the field $\mathbb{C}$ we can also define differential graded $*$-algebras. These are differential graded algebras that have an involution $* : A \to A$. We write $\mathrm{dg}^*\mathsf{Alg}_{\mathsf{As}}$ for the category of differential graded $*$-algebras.

Other algebraic structures (like that of a Lie algebra, or a Poisson algebra) can also be defined on chain complexes. As with vector spaces, we will define them using operads in Section 2.5.2.

Definition 2.4.10. A *Poisson chain complex* (V, τ) is a chain complex $V = (V, \mathrm{d})$ together with a Poisson structure, a chain map $\tau : V \otimes V \to \mathbb{K}$ that is graded antisymmetric: $\tau(v, w) = -(-1)^{|v||w|}\tau(w, v)$. Note that since τ is a chain map and $\mathbb{K}$ is concentrated in degree 0, τ will only pair an element of degree n with elements of degree $-n$. Morphisms of Poisson chain complexes $(V, \tau) \to (V', \tau')$ are chain maps $f : V \to V'$ that preserve the Poisson structure: $f^*(\tau') = \tau' \circ (f \wedge f) = \tau$. The Poisson chain complexes form the category $\mathbf{PoissCh}_{\mathbb{K}}$.

DEFINITION 2.4.11. We can also use the shifting defined above to shift the Poisson structure. A *k-shifted Poisson chain complex* (V, Υ) is a chain complex $V = (V, \mathrm{d})$ together with a shifted Poisson structure $\Upsilon : V \otimes V \to \mathbb{K}[k]$ that is shifted graded antisymmetric:

$$\Upsilon(v, w) = -(-1)^{|v||w|+k|v|+k|w|}\Upsilon(w, v) \ .$$

Since Υ pairs elements of degree n with elements of degree $-n + k$, we see that

- when k is odd, Υ is symmetric: $\Upsilon(v, w) = \Upsilon(w, v)$;
- when k is even, Υ is graded antisymmetric: $\Upsilon(v, w) = -(-1)^{|v||w|}\Upsilon(w, v)$.

This is a somewhat unintuitive expression of antisymmetry. Heuristically, it helps to think of Υ as an object of degree k sitting in between v and w, and then applying the Koszul sign rule. In this regard, $(v)\Upsilon(w)$ would perhaps be a more appropriate notation, though confusing in its own way.

To end this section, we also define bicomplexes, which we will use in chapter 5 as a tool to calculate the bar construction. The convention to let the two differentials anticommute is nonstandard but it is convenient for our purposes; the opposite convention where they commute is equivalent by adding a sign to the vertical differential, see Remark 2.2 in [**MR19**].

DEFINITION 2.4.12. A *bicomplex* $C = (C_{\bullet\bullet}, \delta, \mathrm{d})$ over $\mathbb{K}$ is a family of vector spaces $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$ over $\mathbb{K}$ together with a horizontal differential $\mathrm{d} : C_{p,q} \to C_{p-1,q}$ such that $\mathrm{d}^2 = 0$ and a vertical differential $\delta : C_{p,q} \to C_{p,q-1}$ such that $\delta^2 = 0$. The differentials are required to anticommute:

$$\mathrm{d}\delta + \delta\mathrm{d} = 0 \ .$$

A morphism of bicomplexes $f : C \to D$ is a family of linear maps $\{f_{p,q} : C_{p,q} \to D_{p,q}\}_{p,q\in\mathbb{Z}}$ that commute with both differentials. We write $\mathbf{bCh}_{\mathbb{K}}$ for the category of bicomplexes over $\mathbb{K}$.

For an element $c \in C_{p,q}$ we say that p is its *horizontal degree*, q is its *vertical degree*, (p, q) is its *bidegree* and $|c| := p+q$ is its *total degree*. For any fixed q, $C_{\bullet,q}$ will be a chain complex, as will $C_{p,\bullet}$ for a fixed p. And vice versa, we can specify a bicomplex C by describing for example all the complexes of fixed vertical degree $C_{\bullet,q}$ and the vertical differential δ.

Like $\mathbf{Ch}_{\mathbb{K}}$, the category $\mathbf{bCh}_{\mathbb{K}}$ is symmetric monoidal.

DEFINITION 2.4.13. The tensor product $C \otimes D$ of two bicomplexes C and D is the bicomplex

$$(C \otimes D)_{p,q} := \bigoplus_{i+k=p,\, j+l=q} C_{i,j} \otimes D_{k,l}$$

with differentials

$$\mathrm{d}(c \otimes d) = \mathrm{d}(c) \otimes d + (-1)^{|c|} c \otimes \mathrm{d}(d)$$
$$\delta(c \otimes d) = \delta(c) \otimes d + (-1)^{|c|} c \otimes \delta(d)$$

where we emphasize that $|c| = i + j$ is the total degree of $c \in C_{i,j}$. The monoidal unit is given by $\mathbb{K}$, the bicomplex that is $\mathbb{K}$ in bidegree $(0, 0)$ and 0 elsewhere, with trivial horizontal and vertical differential. The braiding is given by a graded flip

$$B(c \otimes d) = (-1)^{|c||d|} d \otimes c \ .$$

Definition 2.4.14. Bicomplexes can be made into chain complexes by $\bigoplus$-totalization:

$$\begin{array}{rccc} \mathrm{Tot}^{\oplus}: & \mathbf{bCh}_{\mathbb{K}} & \longrightarrow & \mathbf{Ch}_{\mathbb{K}} \\ & (C, \mathrm{d}, \delta) & \longmapsto & \big(\mathrm{Tot}^{\oplus}(C), \mathrm{d}^{\mathrm{tot}}\big) \end{array}$$

with $\mathrm{Tot}^{\oplus}(C)$ given by

$$\mathrm{Tot}^{\oplus}(C)_n := \bigoplus_{p+q=n} C_{p,q}$$

and the total differential

$$\mathrm{d}^{\mathrm{tot}} := \mathrm{d} + \delta \ .$$

This defines a strong monoidal functor.

We will routinely abuse notation and simply write totalization for $\bigoplus$-totalization; since we do not use $\prod$-totalization in this text this should not lead to confusion.

2.4.2. Weak equivalences and localization. In Section 2.4.1 we noted that the important information of a chain complex lies in its homology. From the perspective of category theory this means that we think of any quasi-isomorphism $f : C \to D$ (i.e. any chain map that induced an isomorphism in homology) as establishing an equivalence between the chain complexes C and D. Note however, that C and D do not need to be isomorphic.

Quasi-isomorphisms in $\mathbf{Ch}_{\mathbb{K}}$ are an example of *weak equivalences*: a class of morphisms in a category that we think of as establishing an equivalence, that includes the isomorphisms but will in general also contain non-isomorphisms. Homotopy equivalences between topological spaces give another example, and equivalences between categories (like the equivalences between groupoids of gauge fields discussed in Section 1.2 of the introduction) give a third.

In all cases mentioned here we find a concept of equivalence that is weaker than isomorphism, but we want to treat it as being a true equivalence. A first impulse would be to try to make them into isomorphisms, i.e. adapt the category in such a way that they can be inverted. This leads to the notion of localization.

A localization of a category $\mathbf{C}$ at a subset of its morphisms W is a new category $\mathbf{C}[W^{-1}]$ such that every $f \in W$ becomes an isomorphism in $\mathbf{C}[W^{-1}]$. Moreover, the localization is universal for this property: any functor $F : \mathbf{C} \to \mathbf{D}$ that sends all morphisms in W to isomorphisms must factor through $\mathbf{C}[W^{-1}]$. This results in the following definition, Definition 7.1.1 in [**KS06**].

Definition 2.4.15. Let $\mathbf{C}$ be a category and $W \subseteq \mathbf{Mor}\,\mathbf{C}$ a subset of its morphisms. Then a *localization* of $\mathbf{C}$ at W is a category $\mathbf{C}[W^{-1}]$ together with a functor

$$L : \mathbf{C} \longrightarrow \mathbf{C}[W^{-1}]$$

such that

- for any $f \in W$, $L(f)$ is an isomorphism;
- for any functor $F : \mathbf{C} \to \mathbf{D}$ to an arbitrary category $\mathbf{D}$ such that $F(f)$ is an isomorphism for all $f \in W$, there exists a functor $F_L : \mathbf{C}[W^{-1}] \to \mathbf{D}$ and a natural isomorphism $\eta : F \xrightarrow{\cong} F_L\, L$;
- the pullback functor

  $$L^* : \mathbf{Fun}(\mathbf{C}[W^{-1}], \mathbf{D}) \longrightarrow \mathbf{Fun}(\mathbf{C}, \mathbf{D})$$

  is fully faithful.

Localizations are unique, up to categorical equivalence. From a formal perspective, localizations are very nice to work with, and in many cases one can prove that they exist. However, finding a model for a localization that is not too large to work with is in general a very hard problem, and several techniques have been developed to construct workable models. Perhaps the most famous one is model category theory.

2.4.3. Model categories. A model category is a category with a class of weak equivalences and two other classes of morphisms that we think of as the "good" surjections and injections. Again, we will give a short review of the facts that we will need here; [**DS95**] is a good introduction to the subject.

DEFINITION 2.4.16. A *model category* is a category $\mathbf{C}$ with three classes of morphisms in $\mathbf{Mor\,C}$:

- the *weak equivalences* $(c \xrightarrow{\sim} c') \in W$;
- the *fibrations* $(c \twoheadrightarrow c') \in \mathrm{Fib}$;
- the *cofibrations* $(c \hookrightarrow c') \in \mathrm{Cof}$

which are all closed under composition and include all identities. Fibrations that are also weak equivalences are called *trivial* or *acyclic fibrations*; likewise, cofibrations that are also weak equivalences are called *trivial* or *acyclic cofibrations*.

Moreover, the following axioms hold:

MC1 $\mathbf{C}$ is complete and cocomplete: it contains all small limits and colimits.

MC2 The *2-out-of-3 property* holds for the weak equivalences: if f and g are composable morphisms and two of the three maps f, g and gf are weak equivalences, then so is the third

MC3 All three classes of morphisms are closed under retracts: if we have the diagram

$$\begin{array}{ccccc} c & \xrightarrow{i} & d & \xrightarrow{r} & c \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ c' & \xrightarrow{i'} & d' & \xrightarrow{r'} & c' \end{array}$$

in $\mathbf{C}$ such that ri and $r'i'$ are the identities on c and c' respectively, then if g is a weak equivalence, fibration or cofibration, so is f.

MC4 In the diagram

$$\begin{array}{ccc} c & \xrightarrow{f} & d \\ {\scriptstyle i}\big\downarrow & \overset{h}{\nearrow} & \big\downarrow{\scriptstyle p} \\ c' & \xrightarrow[g]{} & d' \end{array}$$

where i is a cofibration and p is a fibration, the lift h exists whenever either i or p is also a weak equivalence. (A *lift* in this diagram is a map h that makes both resulting triangles commute.)

MC5 Any map $f : c \to c'$ in $\mathbf{C}$ factors in two ways:

$$\begin{array}{ccc} c & \xrightarrow{\quad f \quad} & c' \\ & {\scriptstyle i}\searrow^{\sim} \quad \nearrow{\scriptstyle p} & \\ & \tilde{c} & \end{array} \quad ; \quad \begin{array}{ccc} c & \xrightarrow{\quad f \quad} & c' \\ & {\scriptstyle i}\searrow \quad {}^{\sim}\nearrow{\scriptstyle p} & \\ & \tilde{c} & \end{array}$$

i.e. as an acyclic cofibration followed by a fibration, or as a cofibration followed by an acyclic fibration.

REMARK 2.4.17. In axiom MC5, we do not demand that the factorizations are functorial. However, in all model categories we will encounter, it will be possible to choose these factorizations to be functorial. This is true for most model structures one encounters: "it seems to be exceedingly difficult to find model categories that fail to satisfy this condition" [**Rie19**]. Specifically, it is true for categories that admit *Quillen's small object argument*. We refer to Section 2.1 in [**Hov07**], Chapter 12 in [**Rie14**] and Section 3.2 in [**Rie19**] for more on this. We note here that functorial factorizations in particular allow for functorial fibrant and cofibrant replacements, see Definition 2.4.28.

REMARK 2.4.18. In a sense these axioms are overdetermined: Proposition 3.13 in [**DS95**] tells us that if we have chosen the classes of weak equivalences and fibrations, the class of cofibrations is completely determined by the lifting diagram in axiom MC4. We say that the cofibrations are the morphisms that satisfy the *left lifting property* with respect to the acyclic fibrations. And dually, the fibrations are completely determined by a choice of classes of weak equivalences and cofibrations: they are the morphisms that satisfy the *right lifting property* with respect to the acyclic cofibrations.

There are several immediate implications to the model category axioms. For example, all three classes of morphisms contain all isomorphisms. This follows from the fact that all three classes of morphisms contain the identity, and axiom MC3 where f is an isomorphism, $g = i' = r' = \mathrm{id}_{c'}$, $i = f$ and $r = f^{-1}$. Furthermore, since we demand that a model category $\mathbf{C}$ is complete and cocomplete in axiom MC1, $\mathbf{C}$ always contains an initial object $\emptyset$ and a terminal object $*$. For any object $c \in \mathbf{C}$ we then say that c is *fibrant* if $c \to *$ is a fibration, and *cofibrant* if $\emptyset \to c$ is a cofibration.

It is in general a hard exercise to prove that a given class of weak equivalences and (co)fibrations in a category defines a model category structure. We will solely work with the model category of chain complexes and categories related to it, so for the purposes of this thesis, the most important result of this section is the following, which is shown in Section 2.3 of [**Hov07**].

EXAMPLE 2.4.19. The category $\mathbf{Ch}_{\mathbb{K}}$ over chain complexes over a field of characteristic 0 forms a model category, where

- the weak equivalences are the quasi-isomorphisms;
- the fibrations are the degreewise surjections;
- the cofibrations are the maps that satisfy the left lifting property (see Remark 2.4.18) with respect to the acyclic fibrations.

It is cofibrantly generated and admits Quillen's small object argument, so it allows for functorial factorization, see Remark 2.4.17. Every object in $\mathbf{Ch}_{\mathbb{K}}$ is fibrant and cofibrant. Note that we consider all (unbounded) chain complexes; there is no need for them to be bounded from below as in [**DS95**].

REMARK 2.4.20. Model category theory is a powerful framework for forming the localization of a category (see Definition 2.4.15): it allows for the construction of the *homotopy category* $\mathbf{C} \xrightarrow{\gamma} \mathrm{Ho}\,\mathbf{C}$ of a model category, which is a localization of the category at its weak equivalences. It is also a strong tool for deriving functors, i.e. making sure they preserve weak equivalences (see Section 2.4.5). In terms of the homotopy category, this condition of preserving weak equivalences is that the functor descends to a well-defined functor on the homotopy category.

2.4.4. Model structures on functor categories. As noted in the previous section, it is in general a hard problem to prove that a choice of weak equivalences and (co)fibrations defines a model structure. But a given model structure on a category may be used to define other model structures. One such case that will be relevant to us is the model structure on functor categories.

If $\mathbf{C}$ is a model category, and $\mathbf{D}$ is a small category, the functor category $\mathbf{Fun}(\mathbf{D},\mathbf{C})$ has a natural notion of weak equivalence: a natural transformation η between functors is a weak equivalence if each component η_d is a weak equivalence. In this case we say that the weak equivalences are defined *objectwise* or *componentwise*. We can likewise define the fibrations or the cofibrations objectwise, but recall Remark 2.4.18: a choice of weak equivalences and (say) fibrations determines the cofibrations, so we can in general only hope to define one of the two auxiliary classes of maps this way. This leads to the following definition.

Definition 2.4.21. Let $\mathbf{C}$ be a model category and $\mathbf{D}$ be a small category. Then we define the *projective weak equivalences* in $\mathbf{Fun}(\mathbf{D},\mathbf{C})$ to be the objectwise weak equivalences, and the *projective fibrations* to be the objectwise fibrations. The *projective cofibrations* are the natrual transformations that satisfy the left lifting property with respect to the projective acyclic fibrations. If these choices define a model category structure we call this the *projective model structure* on $\mathbf{Fun}(\mathbf{D},\mathbf{C})$.

Dually, we define the *injective weak equivalences* in $\mathbf{Fun}(\mathbf{D},\mathbf{C})$ to be the objectwise weak equivalences, and the *injective cofibrations* to be the objectwise cofibrations. The *injective fibrations* are the natrual transformations that satisfy the right lifting property with respect to the injective acyclic cofibrations. If these choices define a model category structure we call this the *injective model structure* on $\mathbf{Fun}(\mathbf{D},\mathbf{C})$.

As is implied in the definition, neither projective nor injective model structures on functor categories exist in general. But the former do exist in quite broad cases by the following result, Theorem 12.3.2 in [**Rie14**].

Proposition 2.4.22. If a model category $\mathbf{C}$ is cofibrantly generated and admits the small object argument, the functor category $\mathbf{C}^{\mathbf{D}}$ admits the projective model structure for any small category $\mathbf{D}$.

Example 2.4.23. When $\mathbb{K}$ is of characteristic 0, the functor category ${\mathbf{Ch}_{\mathbb{K}}}^{\mathbf{D}}$ admits the projective model structure for any small category $\mathbf{D}$ by Example 2.4.19. Note that in the projective model structure, every object is fibrant.

2.4.5. Derived functors. Say that $\mathbf{C}$ and $\mathbf{D}$ are two model categories and $F:\mathbf{C}\to\mathbf{D}$ is a functor. If $f:c\to c'$ is a weak equivalence, we would like $F(f):F(c)\to F(c')$ to be a weak equivalence, too. Otherwise, F is deficient in the sense that equivalent inputs have inequivalent outcomes. Unfortunately, many functors one encounters in the wild do not preserve weak equivalences; the process of approximating these functors by functors that do so is called *deriving* the functors.

Definition 2.4.24. Let $F:\mathbf{C}\to\mathbf{D}$ be a functor of model categories that have homotopy categories $\mathbf{C}\xrightarrow{\gamma}\operatorname{Ho}\mathbf{C}$ and $\mathbf{D}\xrightarrow{\delta}\operatorname{Ho}\mathbf{D}$. A *left derived* functor for F is a functor $\mathbb{L}F:\mathbf{C}\to\mathbf{D}$ that preserves weak equivalences, together with a natural transformation $q:\mathbb{L}F\Rightarrow F$, such that $\big((\delta\mathbb{L}F)_\gamma, (\delta q) \circ_h \eta^{-1}\big)$ is a right Kan extension of δF along γ. Here we use the natural isomorphism $\eta : \delta\mathbb{L}F \Rightarrow (\delta\mathbb{L}F)_\gamma\gamma$ from Definition 2.4.15.

Dually, a *right derived* functor for F is a functor $\mathbb{R}F : \mathbf{C} \to \mathbf{D}$ that preserves weak equivalences and a natural transformation $r : F \Rightarrow \mathbb{R}F$, such that $\big((\delta\mathbb{R}F)_\gamma, \eta \circ_h (\delta r)\big)$ is a left Kan extension of δF along γ.

Derived functors are in general not unique up to natural isomorphism, and to be precise we should really say that we are finding *a* derived functor of F, or a *model* for the derived functor of F.

There is a general theory of deriving functors via left and right deformations, see for example Chapter 2 in [**Rie14**]. We will specialize this method to the particularly nice case where our functor is part of a Quillen adjunction.

Definition 2.4.25. A *Quillen adjunction* is an adjunction

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$$

between model categories, such that F preserves cofibrations and G preserves fibrations.

Remark 2.4.26. By Remark 9.8 in [**DS95**] the definition of a Quillen adjunction is equivalent to G preserving fibrations and acyclic fibrations, or also to F preserving cofibrations and acyclic cofibrations.

Happily, in the situation where one of the components of a Quillen adjunction preserves weak equivalences, we do not need to derive that component.

Lemma 2.4.27. If the left or the right adjoint of a Quillen adjunction preserves weak equivalences, it is a model for its derived functor.

The crucial ingredient for our derivation of functors are (co)fibrant replacement functors, which are the deformations in [**Rie14**].

Definition 2.4.28. Let $\mathbf{C}$ be a model category. A *cofibrant replacement* on $\mathbf{C}$ is an endofunctor $Q : \mathbf{C} \to \mathbf{C}$ together with a natural acyclic fibration $q : Q \Rightarrow \mathbb{1}_\mathbf{C}$ such that $Q(c)$ is cofibrant for all $c \in \mathbf{C}$.

Dually, a *fibrant replacement* on $\mathbf{C}$ is an endofunctor $R : \mathbf{C} \to \mathbf{C}$ together with a natural acyclic cofibration $r : \mathbb{1}_\mathbf{C} \Rightarrow R$ such that $R(c)$ is fibrant for all $c \in \mathbf{C}$.

The (co)fibrant replacements can be used to derive both components of a Quillen adjunction as follows, see Theorem 9.7 in [**DS95**].

Proposition 2.4.29. If $\mathbf{C}$ allows for functorial factorization (see Remark 2.4.17) fibrant and cofibrant replacements on it exist.

If $F \dashv G$ is a Quillen adjunction, (Q, q) is a cofibrant replacement on $\mathbf{C}$ and (R, r) is a fibrant replacement on $\mathbf{D}$, then $(\mathbb{L}F := FQ, Fq)$ and $(\mathbb{R}G := GR, Gr)$ are models for the left derived functor for F and the right derived functor for G, respectively.

One technical result that is used in proving the existence of derived functors is Ken Brown's lemma, which we will also independently use in this text.

Lemma 2.4.30. (Ken Brown) Let F be a functor between model categories that carries acyclic cofibrations between cofibrant objects to weak equivalences (for example, if F is the left adjoint in a Quillen adjunction). Then F preserves all weak equivalences between cofibrant objects.

Dually, let G be a functor between model categories that carries acyclic fibrations between fibrant objects to weak equivalences (for example, when G is the right adjoint in a Quillen adjunction). Then G preserves all weak equivalences between fibrant objects.

Definition 2.4.31. If $F \dashv G$ is a Quillen adjunction with unit $\eta : \mathbb{1}_{\mathbf{C}} \to GF$ and counit $\epsilon : FG \to \mathbb{1}_{\mathbf{D}}$, and it is derived in the manner of Proposition 2.4.29, the derived unit $\tilde{\eta} : Q \to \mathbb{R}G\mathbb{L}F$ and the derived counit $\tilde{\epsilon} : \mathbb{L}F\mathbb{R}G \to R$ are given by

$$\begin{array}{ll} \tilde{\eta}_c : & Qc \xrightarrow{\eta_{Qc}} GFQc \xrightarrow{Gr_{FQc}} GRFQc = \mathbb{R}G\mathbb{L}Fc \\ \tilde{\epsilon}_d : & \mathbb{L}F\mathbb{R}Gd = FQGRd \xrightarrow{FqGRd} FGRd \xrightarrow{\epsilon_{Rd}} Rd \end{array} . \tag{2.4.3}$$

Remark 2.4.32. The functors $\mathbb{L}F = FQ$ and $\mathbb{R}G = GR$ are in general not strictly adjoint, and the triangle identities (2.1.4) do not in general hold strictly for the derived unit and counit. This improves on the homotopy categories (see Remark 2.4.20): there, the descended derived functors again form an adjunction, and the triangle identities hold for the unit and counit.

Remark 2.4.33. Similar to the case for localizations of categories, the (co)fibrant replacements Q and R that go into the definitions of the derived functors $\mathbb{L}F$ and $\mathbb{R}G$ always exist, but it is in general quite hard to find (co)fibrant replacements that are small enough to manage. Finding these workable models for Q and R (or for the derived functors) is a lot of the work that goes into computations involving underived functors.

Example 2.4.34. Limit and colimit functors are an important class of functors that often need to be derived. For example, consider the following two diagrams in the category of topological spaces for any $n \geq 1$:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \\ D^n & & \end{array} \quad ; \quad \begin{array}{ccc} S^{n-1} & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$$

Here, D^n is the n-disc, $S^{n-1} = \partial(D^n)$ is its boundary, the $n-1$-sphere, and $*$ is the singleton topological space, the terminal object in **Top**. The maps in the left diagram are the inclusions of the boundary of D^n into D^n. Weak equivalences in the diagram category $\mathbf{Top}^{\bullet\leftarrow\bullet\rightarrow\bullet}$ are defined componentwise, and **Top** has a model structure where the weak equivalences are the weak homotopy equivalences, so these diagrams are weakly equivalent in the diagram category.

Forming the colimit of these diagrams in **Top**, the pushout, means gluing the outer two spaces along the images of the middle one. We see that the pushout of the left diagram is S^n, but the pushout of the right one is $*$. So the pushout does not preserve weak equivalences. It turns out that in the right diagram, we need to cofibrantly replace. One such replacement is by the left diagram, so S^n is the homotopy pushout of the diagram.

An example of a limit that needs to be derived is the pullback of chain complexes; we will carry out an explicit computation of such a homotopy pullback in in Theorem 4.1.7.

(Co)fibrantly replacing is not the only way of deriving a functor, as we will see in the next section.

2.4.6. The bar construction. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between any two categories and consider the Kan extension diagram (2.1.6)

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{Ch}_{\mathbb{K}} \\ & \searrow^{F} & \\ & & \mathbf{D} \end{array}$$

where $\mathbf{Ch}_{\mathbb{K}}$ is the model category of chain complexes over a field $\mathbb{K}$. Recall Remark 2.1.37: if it exists (as is the case for $\mathbf{Ch}_{\mathbb{K}}$), the left Kan extension functor is part of the adjunction

$$F_! : \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}}) \rightleftarrows \mathbf{Fun}(\mathbf{D}, \mathbf{Ch}_{\mathbb{K}}) : F^* \ .$$

The following is a construction to derive the left Kan extension $F_!$ in the projective model structures on $\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$ and $\mathbf{Fun}(\mathbf{D}, \mathbf{Ch}_{\mathbb{K}})$, by fattening up the complex to a bicomplex using all possible maps in $\mathbf{C}$ and $\mathbf{D}$ and then totalizing the resulting bicomplex. It works in much more generality than just for $\mathbf{Ch}_{\mathbb{K}}$ and the left Kan extension (see Chapter 4 in [**Rie14**]), but we restrict ourselves to complexes and bicomplexes here to avoid treating simplicial sets in these preliminaries. Effectively, we have already used the Dold-Kan correspondence to move from simplicial chain complexes to bicomplexes that are nonnegatively graded in the degree corresponding to the simplicial degree. This is the bar we put on the functor B_Δ below (no relation to the name of the construction).

The bar construction associated to F is a functor

$$\overline{B_\Delta}(F, \mathbf{C}, -) : \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}}) \longrightarrow \mathbf{Fun}(\mathbf{D}, \mathbf{bCh}_{\mathbb{K}}) \ .$$

To a functor $X : \mathbf{C} \to \mathbf{Ch}_{\mathbb{K}}$ it assigns the functor $\overline{B_\Delta}(F, \mathbf{C}, X)$ which in turn assigns to an object $d \in \mathbf{D}$ the bicomplex $\overline{B_\Delta}(F, \mathbf{C}, X)(d)_{\bullet,\bullet}$ which is concentrated in nonnegative vertical degrees $q \geq 0$. In vertical degree $q = 0$ the bicomplex is given by the complex

$$\overline{B_\Delta}(F, \mathbf{C}, X)(d)_{\bullet,0} = \bigoplus_{c \in \mathbf{C}} \ \bigoplus_{(Fc \xrightarrow{g} d) \in \mathbf{D}} X(c)_\bullet \tag{2.4.4}$$

collecting a copy of $X(c)$ for every map of the shape $g : Fc \to d$ in $\mathbf{D}$. In positive vertical degrees $q \geq 1$ the bicomplex is the complex

$$\overline{B_\Delta}(F, \mathbf{C}, X)(d)_{\bullet,q} = \bigoplus_{c \in \mathbf{C}} \ \bigoplus_{(Fc \xrightarrow{g} d) \in \mathbf{D}} \ \bigoplus_{\substack{(c \xleftarrow{f_1} \cdots \xleftarrow{f_q} \mathsf{s} f_q) \in \mathbf{C} \\ f_i \neq \mathrm{id}}} X(\mathsf{s} f_q)_\bullet$$

collecting objects $X(\mathsf{s} f_q)$ for all $g : Fc \to d$ in $\mathbf{D}$ as above and all chains of composable morphisms

$$c \xleftarrow{f_1} \cdots \xleftarrow{f_q} \mathsf{s} f_q$$

in $\mathbf{C}$ that do not include any identity morphisms. The horizontal differential on $\overline{B_\Delta}(F, \mathbf{C}, X)(d)$ is simply the differential on the complexes $X(c)$ while the vertical differential is given by the alternating sum

$$\delta(c,g,f_1,\cdots,f_q,x) = (-1)^{|x|}\Big(\ (\mathsf{s}f_1, g\circ Ff_1, f_2,\dots,f_q,x) \\ \sum_{j=1}^{q-1}(-1)^j(c,g,f_1,\dots,f_j\circ f_{j+1},\dots,f_q,x) \\ (-1)^q(c,g,f_1,\dots,f_{q-1},f_q(x)) \ \Big)$$

for an element $(c,g,f_1,\dots,f_q,x) \in \overline{B_\Delta}(F,\mathbf{C},X)(d)_{\bullet,q}$ of degree $q\geq 1$ (since the bicomplex is nonnegatively graded, δ is 0 on $\overline{B_\Delta}(F,\mathbf{C},X)(d)_{\bullet,0}$). Here, $|x|$ is the (chain complex) degree of $x\in X(\mathsf{s}f_q)$.

To morphisms $k: d\to d'$ in $\mathbf{D}$, the functor $\overline{B_\Delta}(F,\mathbf{C},X)$ assigns the bicomplex morphism

$$\begin{array}{cccc} \overline{B_\Delta}(F,\mathbf{C},X)(k): & \overline{B_\Delta}(F,\mathbf{C},X)(d) & \longrightarrow & \overline{B_\Delta}(F,\mathbf{C},X)(d) \\ & (c,g,f_1,\cdots,f_q,x) & \longmapsto & (c,k\circ g,f_1,\cdots,f_q,x) \end{array}$$

which composes the index morphism g with k.

We recognize the sum in the definition of $\overline{B_\Delta}(F,\mathbf{C},X)(d)_{\bullet,0}$ (2.4.4) as a sum over the objects in the overcategory $F\downarrow d$. And in fact, if we calculate the zeroth vertical homology of the bicomplex, we find the model (2.1.7) of the left Kan extension. So we have indeed fattened up the left Kan extension in some sense. In Section 13.3 and Theorem 17.2.7 in [**Fre09**] it is proven that after totalizing back to $\mathbf{Ch}_\mathbb{K}$,

$$\mathbb{L}F_! := \mathrm{Tot}^\oplus\big(\overline{B_\Delta}(F,\mathbf{C},-)\big) : \mathbf{Fun}(\mathbf{C},\mathbf{Ch}_\mathbb{K}) \longrightarrow \mathbf{Fun}(\mathbf{D},\mathbf{Ch}_\mathbb{K})$$

is a model for the left derived left Kan extension functor $F_!$.

Let us also give the derived unit and counit of the derived adjunction

$$\mathbb{L}F_! : \mathbf{Fun}(\mathbf{C},\mathbf{Ch}_\mathbb{K}) \rightleftarrows \mathbf{Fun}(\mathbf{D},\mathbf{Ch}_\mathbb{K}) : \mathbb{R}F^* \quad .$$

First, note that we do not need to derive the pullback F^*; we see this either by the fact that every object is fibrant in the projective model structure on $\mathbf{Fun}(\mathbf{D},\mathbf{Ch}_\mathbb{K})$ or directly by the fact that if a natural transformation $\gamma: G\to G'$ is a weak equivalence, so is $F^*\gamma = \gamma F : GF\to G'F$ because the weak equivalences are defined objectwise. So we have $\mathbb{R}F^* = F^*$ and the cofibrant replacement (R,r) is chosen to be the identity. The derived counit $\tilde\epsilon : \mathbb{L}F_!F^*\to \mathrm{id}_{\mathbf{Fun}(\mathbf{D},\mathbf{Ch}_\mathbb{K})}$ at the component $Y\in\mathbf{Fun}(\mathbf{D},\mathbf{Ch}_\mathbb{K})$ then is the natural transformation

$$\tilde\epsilon_Y : \mathbb{L}F_!F^*(Y) \longrightarrow Y \tag{2.4.5a}$$

that is in turn at the component $d\in\mathbf{D}$

$$\begin{array}{cccc} \tilde\epsilon_{Y,d}: & \mathrm{Tot}^\oplus\big(\overline{B_\Delta}(F,\mathbf{C},F^*Y)\big)(d) & \longrightarrow & Y(d) \\ & (c,g,y) & \longmapsto & Y(g)(y) \ . \\ & (c,g,f_1,\dots,f_q,y) & \longmapsto & 0 \end{array} \tag{2.4.5b}$$

for any positive vertical degree $q\geq 1$.

For the derived unit we have the cofibrant replacement

$$Q = \mathrm{Tot}^\oplus\big(\overline{B_\Delta}(\mathrm{id}_\mathbf{C},\mathbf{C},-)\big) : \mathbf{Fun}(\mathbf{C},\mathbf{Ch}_\mathbb{K}) \longrightarrow \mathbf{Fun}(\mathbf{C},\mathbf{Ch}_\mathbb{K}) \tag{2.4.6}$$

that is obtained by applying the bar construction to the identity functor $\mathrm{id}:\mathbf{C}\to\mathbf{C}$. The derived unit $\tilde\eta : Q\to F^*\mathbb{L}F_!$ at the component $X\in\mathbf{Fun}(\mathbf{C},\mathbf{Ch}_\mathbb{K})$ then is the natural transformation

$$\tilde\eta_X : Q(X) \longrightarrow F^*\mathbb{L}F_!(X) \tag{2.4.7a}$$

which in turn at the component $\tilde{c} \in \mathbf{C}$ is given by

$$\begin{array}{rccc} \tilde{\eta}_{X,\tilde{c}}: & \mathrm{Tot}^{\oplus}\big(\overline{B_{\Delta}}(\mathrm{id}_{\mathbf{C}}, \mathbf{C}, X)(\tilde{c})\big) & \longrightarrow & \mathrm{Tot}^{\oplus}\big(\overline{B_{\Delta}}(F, \mathbf{C}, Y)(F\tilde{c})\big) \\ & (c, f_0, f_1, \ldots, f_q, x) & \longmapsto & (c, Ff_0, f_1, \ldots, f_q, x) \end{array} . \tag{2.4.7b}$$

Note that this is a different way of deriving functors than the one described in the previous section: indeed, $\mathbb{L}F_!$ is not equal to $F_!Q$ with the cofibrant replacement Q given above. However, the two functors are equivalent.

2.4.7. Homotopical categories. Not all categories with a class of weak equivalences can be equipped with the structure of a model category. In particular, not all categories are complete or cocomplete. A more flexible definition is that of a homotopical category. We collect the definitions we need here from [**Rie14**].

Definition 2.4.35. A *homotopical category* is a category $\mathbf{C}$ with a class of weak equivalences $W \subseteq \mathbf{Mor}\,\mathbf{C}$ which is closed under composition and contains all identities. Moreover, W satisfies the *2-out-of-6 property*: if f, g and h are composable morphisms, and hg and gf are weak equivalences, then f, g, h and hgf are weak equivalences too.

As is the case with model categories, all isomorphisms are weak equivalences (this follows from the 2-out-of-6 property and the fact that W contains all identities).

Example 2.4.36. The category of Poisson chain complexes $\mathbf{PoissCh}_{\mathbb{K}}$ (see Definition 2.4.10) is a homotopical category; the weak equivalences are the maps for which the underlying chain map is a quasi-isomorphism. It is not a model category, since it does not contain all coequalizers, so it is not cocomplete.

Note that besides the underlying chain complex V, we can also weakly vary the Poisson structure τ by adding a homotopy $\partial\rho$ for $\rho \in \mathbf{hom}(\bigwedge^2 V, \mathbb{K})_1$. There is in general no direct Poisson chain map $(V, \tau) \to (V, \tau + \partial\rho)$, so we will have to take care of this issue separately; we will do so in Proposition 3.4.6 for example.

Remark 2.4.37. The 2-out-of-6 property is stronger than the 2-out-of-3 property. However, it can be shown that the weak equivalences of a model category also satisfy the 2-out-of-6 property. Therefore, any model category also forms a homotopical category with its class of weak equivalences.

Example 2.4.38. We will find in Section 2.5.3 that all differential graded algebras that we consider form model categories where the weak equivalences are the maps of algebras such that the underlying chain maps are quasi-isomorphisms. By the above remark we see that this choice of weak equivalences also makes these categories into homotopical categories.

Definition 2.4.39. A *homotopical functor* between homotopical categories is a functor that preserves weak equivalences, i.e. it maps weak equivalences to weak equivalences.

2.5. Operads

Operads are structures that encode algebra. For example, an algebra over the associative operad is an associative algebra. They become essential when trying to do algebra up to homotopy [**Val14, LV12**], though we will not pursue these ideas in this text. Here we introduce operads and their algebras, following [**Yau16**], and we will then introduce model structures for algebras over operads valued in chain complexes, following [**Hin97, Hin13**].

2.5.1. Colored operads. Fix a closed symmetric monoidal category $\mathbf{T}$ that is complete and cocomplete. We first introduce some notation. For a set $\mathfrak{C}$, we write $\underline{c} := (c_1, \dots, c_n) \in \mathfrak{C}^n$ for any finite sequence in $\mathfrak{C}$ with $n \geq 0$. We call $\underline{c}$ a $\mathfrak{C}$-*profile*. The *length* of $\underline{c}$ is $|\underline{c}| := n$. We can compose two profiles $\underline{a} = (a_1, \dots, a_m)$ and $\underline{b} = (b_1, \dots, b_n)$ by concatenation:

$$(\underline{a}, \underline{b}) := (a_1, \dots, a_m, b_1, \dots, b_n).$$

Definition 2.5.1. Let $\mathfrak{C}$ be a set. A $\mathfrak{C}$-*colored sequence* X in $\mathbf{T}$ is an assignment

$$X\tbinom{t}{\underline{c}} \in \mathbf{T}$$

for all $(\underline{c}, t) \in \mathfrak{C}^n \times \mathfrak{C}$ and all $n \geq 0$. The length n of $\underline{c}$ is called the *arity* of $X\binom{t}{\underline{c}}$ and its elements. A morphism of $\mathfrak{C}$-colored sequences $f : X \to Y$ is a family of $\mathbf{T}$-morphisms

$$f : X\tbinom{t}{\underline{c}} \longrightarrow Y\tbinom{t}{\underline{c}}$$

for all $(\underline{c}, t) \in \mathfrak{C}^n \times \mathfrak{C}$ and all $n \geq 0$. We collect the $\mathfrak{C}$-colored sequences in $\mathbf{T}$ and their morphisms into the category $\mathbf{Seq}_{\mathfrak{C}}(\mathbf{T})$.

For $n \geq 0$ we denote the symmetric group on n elements by Σ_n. We define a right-Σ_n-action on the profiles of length n by

$$\underline{c}\sigma := (c_{\sigma(1)}, \dots, c_{\sigma(n)}).$$

Definition 2.5.2. A $\mathfrak{C}$-*colored symmetric sequence* X in $\mathbf{T}$ is a $\mathfrak{C}$-colored sequence together with a right-action of the symmetric group: if $\sigma \in \Sigma_n$ then we have a $\mathbf{T}$-morphism

$$X(\sigma) : X\tbinom{t}{\underline{c}} \longrightarrow X\tbinom{t}{\underline{c}\sigma}$$

such that $X(\tau)X(\sigma) = X(\sigma\tau)$ for $\sigma, \tau \in \Sigma_n$ and $X(e) = \mathrm{id}$ for the unit $e \in \Sigma_n$. A morphism of symmetric sequences is a $\mathbf{Seq}_{\mathfrak{C}}(\mathbf{T})$-morphism that commutes with the symmetric action: $Y(\sigma)f = fX(\sigma)$. The $\mathfrak{C}$-colored symmetric sequences in $\mathbf{T}$ and their morphisms are collected in the category $\mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{T})$.

Operads are symmetric sequences, equipped with a composition that matches the colors and a corresponding unit, as follows.

Definition 2.5.3. Let $\mathfrak{C}$ be a set. A $\mathfrak{C}$-*colored operad* $(\mathcal{O}, \gamma, \mathbb{1})$ with values in $\mathbf{T}$ consists of the following data:

- a $\mathfrak{C}$-colored symmetric sequence $\mathcal{O}$ in $\mathbf{T}$;
- the *operadic composition*: $\mathbf{T}$-morphisms

$$\gamma : \mathcal{O}\tbinom{t}{\underline{a}} \otimes \bigotimes_i \mathcal{O}\tbinom{a_i}{\underline{b}_i} \longrightarrow \mathcal{O}\tbinom{t}{\underline{b}}$$

  for all all $n > 0$, $k_1, \dots, k_n \geq 0$, $t \in \mathfrak{C}$, $\underline{a} \in \mathfrak{C}^n$ and $\underline{b}_i \in \mathfrak{C}^{k_i}$ for $i = 1, \dots, n$, where $\underline{b} = (\underline{b}_1, \dots, \underline{b}_n)$;
- The *operadic unit*: $\mathbf{T}$-morphisms

$$\mathbb{1} : I \longrightarrow \mathcal{O}\tbinom{c}{c}$$

  for all $c \in \mathfrak{C}$, where I is the monoidal unit in $\mathbf{T}$.

These operations are required to satisfy associativity, unitality and equivariance axioms:

- Associativity: the diagram

$$\begin{array}{ccc}
\Big[\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_i \mathcal{O}\binom{a_i}{\underline{b}_i}\Big] \otimes \bigotimes_i \Big[\bigotimes_j \mathcal{O}\binom{b_{ij}}{\underline{c}_{ij}}\Big] & \xrightarrow{\gamma\otimes\mathrm{id}} & \mathcal{O}\binom{t}{\underline{b}} \otimes \bigotimes_i \Big[\bigotimes_j \mathcal{O}\binom{b_{ij}}{\underline{c}_{ij}}\Big] \\
\Big\downarrow{\scriptstyle \text{permute}} & & \\
\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_i \Big[\mathcal{O}\binom{a_i}{\underline{b}_i} \otimes \bigotimes_j \mathcal{O}\binom{b_{ij}}{\underline{c}_{ij}}\Big] & & \Big\downarrow{\scriptstyle \gamma} \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes\bigotimes_i \gamma} & & \\
\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_i \mathcal{O}\binom{a_i}{\underline{c}_i} & \xrightarrow{\quad\gamma\quad} & \mathcal{O}\binom{t}{\underline{c}}
\end{array}$$

  commutes for all possible t, $\underline{a}$, $\underline{b}_i$ and $\underline{c}_{ij}$, where $\underline{b}$, $\underline{c}_i$ and $\underline{c}$ are defined by concatenation. "Permute" means the relevant actions of the associator and the braiding of $\mathbf{T}$ to permute the tensor products.
- Right unitality: the diagram

$$\begin{array}{ccc}
\mathcal{O}\binom{t}{\underline{c}} \otimes I^{\otimes n} & \xrightarrow[\rho^{\otimes n}]{\cong} & \mathcal{O}\binom{t}{\underline{c}} \\
{\scriptstyle \mathrm{id}\otimes\bigotimes_i \mathbb{1}} \searrow & & \nearrow {\scriptstyle \gamma} \\
& \mathcal{O}\binom{t}{\underline{c}} \otimes \bigotimes_i \mathcal{O}\binom{c_i}{c_i} &
\end{array}$$

  commutes for all $(\underline{c}, t) \in \mathfrak{C}^n \times \mathfrak{C}$, where ρ is the right unitor of $\mathbf{T}$.
- Left unitality: the diagram

$$\begin{array}{ccc}
I \otimes \mathcal{O}\binom{t}{\underline{c}} & \xrightarrow[\lambda]{\cong} & \mathcal{O}\binom{t}{\underline{c}} \\
{\scriptstyle \mathbb{1}\otimes\mathrm{id}} \searrow & & \nearrow {\scriptstyle \gamma} \\
& \mathcal{O}\binom{t}{t} \otimes \mathcal{O}\binom{t}{\underline{c}} &
\end{array}$$

  commutes for all $(\underline{c}, t) \in \mathfrak{C}^n \times \mathfrak{C}$, where λ is the left unitor of $\mathbf{T}$.
- Top equivariance: the diagram

$$\begin{array}{ccc}
\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_i \mathcal{O}\binom{a_i}{\underline{b}_i} & \xrightarrow{\mathcal{O}(\sigma)\otimes\text{permute}} & \mathcal{O}\binom{t}{\underline{a}\sigma} \otimes \bigotimes_i \mathcal{O}\binom{a_{\sigma(i)}}{\underline{b}_{\sigma(i)}} \\
\Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \gamma} \\
\mathcal{O}\binom{t}{\underline{b}_1,\ldots,\underline{b}_n} & \xrightarrow{\mathcal{O}(\sigma\langle k_1,\ldots,k_n\rangle)} & \mathcal{O}\binom{t}{\underline{b}_{\sigma(1)},\ldots,\underline{b}_{\sigma(n)}}
\end{array}$$

  commutes for all $\underline{a} \in \mathfrak{C}^n$, $\sigma \in \Sigma_n$ and all possible t and $\underline{b}_i$. Here $\sigma\langle k_1, \ldots, k_n\rangle \in \Sigma_{\sum_i k_i}$ is the block permutation induced by σ, and "permute" is the permutation of the factors in $\bigotimes_i \mathcal{O}\binom{a_i}{\underline{b}_i}$ induced by σ, using the braiding of $\mathbf{T}$.
- Bottom equivariance: the diagram

$$\begin{array}{ccc}
\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_i \mathcal{O}\binom{a_i}{\underline{b}_i} & \xrightarrow{\mathrm{id}\otimes\bigotimes_i \mathcal{O}(\sigma_i)} & \mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_i \mathcal{O}\binom{a_i}{\underline{b}_i\sigma_i} \\
\Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \gamma} \\
\mathcal{O}\binom{t}{\underline{b}_1,\ldots,\underline{b}_n} & \xrightarrow{\mathcal{O}(\sigma_1\oplus\cdots\oplus\sigma_n)} & \mathcal{O}\binom{t}{\underline{b}_1\sigma_1,\ldots,\underline{b}_n\sigma_n}
\end{array}$$

commutes for all $\underline{b}_i \in \mathfrak{C}^{k_i}$, $\sigma_i \in \Sigma_{k_i}$ and all possible t and $\underline{a}$. Here $\sigma_1 \oplus \cdots \oplus \sigma_n \in \Sigma_{\sum_i k_i}$ is the block sum induced by the σ_i.

Definition 2.5.4. If $\mathbf{C} = \{*\}$ has only one element, we call $\mathfrak{C}$-colored operads *uncolored operads*. For an uncolored operad $\mathcal{O}$, we write $\mathcal{O}(n)$ for the object $\mathcal{O}\binom{*}{*\cdots*}$ of arity n.

Definition 2.5.5. A *morphism of $\mathfrak{C}$-colored operads* $\phi : \mathcal{O} \to \mathcal{P}$ is a morphism of the underlying symmetric sequences that preserves the composition,

$$\begin{array}{ccc}
\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_i \mathcal{O}\binom{a_i}{\underline{b}_i} & \xrightarrow{\gamma_{\mathcal{O}}} & \mathcal{O}\binom{t}{\underline{b}} \\
{\scriptstyle \phi \otimes \bigotimes_i \phi} \downarrow & & \downarrow {\scriptstyle \phi} \\
\mathcal{P}\binom{t}{\underline{a}} \otimes \bigotimes_i \mathcal{P}\binom{a_i}{\underline{b}_i} & \xrightarrow{\gamma_{\mathcal{P}}} & \mathcal{P}\binom{t}{\underline{b}}
\end{array}$$

for all n, t, $\underline{a}$ and $\underline{b}_i$, and the unit,

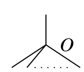
$$\begin{array}{ccc}
I & \xrightarrow{\mathbb{1}_{\mathcal{O}}} & \mathcal{O}\binom{c}{c} \\
 & {\scriptstyle \mathbb{1}_{\mathcal{P}}} \searrow & \downarrow {\scriptstyle \phi} \\
 & & \mathcal{P}\binom{c}{c}
\end{array}$$

for all c.

We write $\mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$ for the category of $\mathfrak{C}$-colored operads valued in $\mathbf{T}$, and $\mathbf{Op}(\mathbf{T}) := \mathbf{Op}_{\{*\}}(\mathbf{T})$ for the category of uncolored operads. For $\mathcal{O} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$, we will say that $\mathcal{O}$ is $\mathbf{T}$-*valued*. Since $\mathbf{T}$ is complete and cocomplete, so is $\mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$, see Theorem 3.8 in [**PS18**].

Example 2.5.6. Let $\mathbf{C}$ be a small category. Its *diagram operad* $\mathsf{Diag}_{\mathbf{C}}$ is the $\mathbf{Ob}\,\mathbf{C}$-colored operad which is concentrated in arity 1: for $c, c' \in \mathbf{C}$, $\mathsf{Diag}_{\mathbf{C}}\binom{c'}{c} := \mathbf{C}(c, c')$. Operadic composition is defined by composition of morphisms, and the operadic unit $\mathbb{1}_c$ is simply the identity.

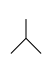
In general, it is useful to visualize the elements $o \in \mathcal{O}(n)$ in arity n as (directed and rooted) trees with n inputs (or *leaves*), and one output,



.

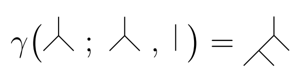
In this visualization, grafting the outputs of trees to the leaves of another and then contracting the internal edges represents composition, the identity is a one-input tree, and permutation of the leaves is the symmetric action. This heuristic can in fact be used to construct operads; let us illustrate this by giving a presentation of the (uncolored) associative operad in $\mathbf{T} = \mathbf{Set}$.

Start with a single tree with two leaves,

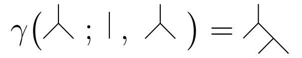


to represent the multiplication μ in an algebra, and a tree with one leaf,

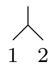
which is the operadic unit $\mathbb{1}$ (representing the identity morphism of an algebra; the object representing the unit element of the algebra will be defined below). Composition (in the operad) is now achieved by grafting these trees: for example,

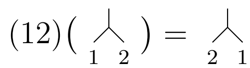


and

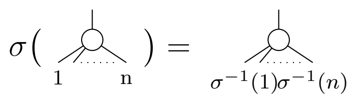


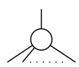
Of course, we can keep grafting larger and larger trees like this, which results in binary trees of every arity (i.e. every number of leaves).

The symmetric action will now permute the leaves of trees, which we label to keep track of this. So we write  for the generating tree with two leaves; the symmetric action then yields

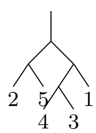


and in general we find



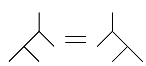
for any $\sigma \in \Sigma_n$, where  represents any binary tree we can construct through grafting.

The elements we can construct this way are flat binary trees (trees with two inputs and one output at every node) with the n leaves of the tree labeled by $\sigma(1)$ to $\sigma(n)$ for any $\sigma \in \Sigma$. For example, we have the element

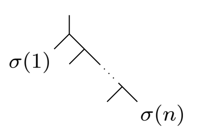


in arity 5.

We then implement the associativity relation

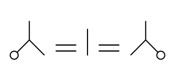


for all possible permutations of the inputs. We can then use this relation to reshape every one of our trees into a standard shape, say



Lastly, we introduce one more generator, the tree

with zero leaves, which represents the unit η of the algebra. The relations



then encode unitality.

One can now check that this construction indeed defines an operad, taking care with the permutations in the top equivariance axiom.

Example 2.5.7. The *associative operad* As is the operad constructed above.

There exists an obvious forgetful functor

$$U : \mathbf{Op}_{\mathfrak{C}}(\mathbf{T}) \longrightarrow \mathbf{Seq}_{\mathfrak{C}}(\mathbf{T}) \tag{2.5.1}$$

that forgets the symmetric action, the operadic composition and the operadic unit. If fact, if we define $\mathbf{NOp}_{\mathfrak{C}}(\mathbf{T})$ to be the category of $\mathfrak{C}$-colored nonsymmetric operads by removing the symmetric action and the equivariance axioms from Definition 2.5.3, we have the commuting square of forgetful functors

$$\begin{array}{ccc} \mathbf{Seq}_{\mathfrak{C}}(\mathbf{T}) & \longleftarrow & \mathbf{SymSeq}_{\mathfrak{C}}(\mathbf{T}) \\ {\scriptstyle U^{\Omega}}\uparrow & \nwarrow^{U} & \uparrow \\ \mathbf{NOp}_{\mathfrak{C}}(\mathbf{T}) & \xleftarrow[U^{\Sigma}]{} & \mathbf{Op}_{\mathfrak{C}}(\mathbf{T}) \end{array} . \tag{2.5.2}$$

Like in Example 2.1.26, the functor U has a left adjoint.

Theorem 2.5.8. The forgetful functor U (2.5.1) is part of the adjunction

$$F : \mathbf{Seq}_{\mathfrak{C}}(\mathbf{T}) \rightleftarrows \mathbf{Op}_{\mathfrak{C}}(\mathbf{T}) : U \quad . \tag{2.5.3}$$

The functor F is the *free $\mathfrak{C}$-colored operad functor* and for $G \in \mathbf{Seq}_{\mathfrak{C}}(\mathbf{T})$ we say that $F(G)$ is the *free operad* generated by G.

Proof. This is proven in Chapter 20 in [**Yau16**]. The functor F is built by finding left adjoints to the forgetful functors

$$\mathbf{Seq}_{\mathfrak{C}}(\mathbf{T}) \xleftarrow[U^{\Omega}]{} \mathbf{NOp}_{\mathfrak{C}}(\mathbf{T}) \xleftarrow[U^{\Sigma}]{} \mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$$

where U^{Σ} forgets the symmetric action and U^{Ω} forgets the operadic composition and unit. □

Given a free operad $F(G)$ generated by $G \in \mathbf{Seq}_{\mathfrak{C}}(\mathbf{T})$, we can implement relations on it as follows. Let $R \in \mathbf{Seq}_{\mathfrak{C}}(\mathbf{T})$ be another sequence, and

$$r_1, r_2 : \ R \rightrightarrows UF(G)$$

be two parallel $\mathbf{Seq}_{\mathfrak{C}}(\mathbf{T})$-morphisms. By the adjunction (2.5.3) this is equivalent to two parallel $\mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$-morphisms

$$r_1, r_2 : \ F(R) \rightrightarrows F(G)$$

which we denote by the same symbols. Since $\mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$ is cocomplete, we can then form the coequalizer.

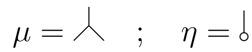
Definition 2.5.9. For $G \in \mathbf{Seq}_{\mathfrak{C}}(\mathbf{T})$ and relations $r_1, r_2 : R \to UF(G)$, the $\mathfrak{C}$-colored operad presented by G and the relations r_i is the coequalizer

$$F(R) \underset{r_2}{\overset{r_1}{\rightrightarrows}} F(G) \dashrightarrow F(G)/(r_1 = r_2)$$

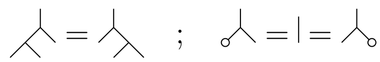
in $\mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$.

As in the construction of the associative operad As above it is helpful to denote the generators of a free operad by trees, and the relations as relations of tree diagrams.

EXAMPLE 2.5.10. The associative operad As constructed above is the operad generated by



representing multiplication and the unit, respectively, with the relations



(2.5.4)

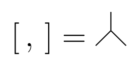
implementing associativity and unitality.

Note that this construction was done in **Set**; for any bicomplete closed symmetric monoidal category **T** we can use the **Set**-tensoring of Definition 2.1.44 and the monoidal unit $I \in \mathbf{T}$ to define generators $G \otimes I$ and relations $r_i : R \otimes I \to UF(G) \otimes I \cong UF(G \otimes I)$ and form the operad

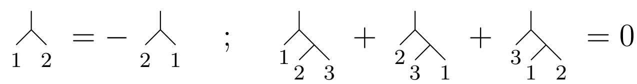
$$\mathsf{As} := F(G \otimes I)/(r_1 = r_2)$$

in $\mathbf{Op}(\mathbf{T})$.

EXAMPLE 2.5.11. Let $\mathbf{T} = \mathbf{Vect}_{\mathbb{K}}$. The *Lie operad* Lie is the operad generated by



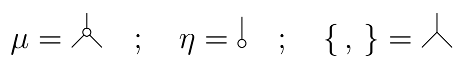
representing the Lie bracket, with the relations

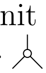
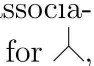


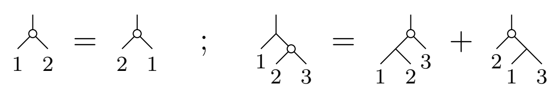
(2.5.5)

implementing antisymmetry and the Jacobi identity.

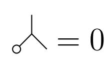
EXAMPLE 2.5.12. Again let $\mathbf{T} = \mathbf{Vect}_{\mathbb{K}}$. The *Poisson operad* Pois is the operad generated by



representing multiplication, unit and Poisson bracket, respectively, and as relations associativity and unitality (2.5.4) for  and , antisymmetry and the Jacobi identity (2.5.5) for , together with the relations

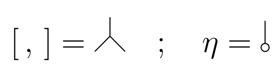


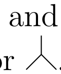
implementing commutativity of the multiplication and the fact that the Poisson bracket is a derivation in the second entry (antisymmetry of  then implies that it is also a derivation in the first entry). The relation

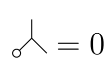


then follows from the derivation relation.

EXAMPLE 2.5.13. Let $\mathbf{T} = \mathbf{Vect}_{\mathbb{K}}$. The *unital Lie operad* uLie is the operad generated by



representing the Lie bracket and the unit, respectively, with as relations antisymmetry and the Jacobi identity (2.5.5) for , and



implementing infinitesimal unitality for the Lie bracket.

Since in the above three examples of Lie, Pois and uLie we are using addition in our relations, the three operads cannot be defined in any monoidal category $\mathbf{T}$. They can, however, be defined for any additive $\mathbf{T}$. Specifically, interpreting vector spaces as chain complexes concentrated in degree zero defines $\mathsf{Lie}, \mathsf{Pois}, \mathsf{uLie} \in \mathbf{Op}(\mathbf{Ch}_{\mathbb{K}})$. When more than one base field is involved, we will use the notation $\mathcal{O}_{\mathbb{K}}$ to specify that $\mathcal{O} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{Vect}_{\mathbb{K}})$ or $\mathbf{Op}_{\mathfrak{C}}(\mathbf{Ch}_{\mathbb{K}})$.

Definition 2.5.14. If $\mathcal{Q} \in \mathbf{Op}_{\mathfrak{D}}(\mathbf{T})$ is a $\mathfrak{D}$-colored operad and $f : \mathfrak{C} \to \mathfrak{D}$ is a map of sets of colors, we can define the *pullback* $\mathfrak{C}$*-colored operad* $f^*\mathcal{Q} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$ by

$$f^*\mathcal{P}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) := \mathcal{P}\big(\begin{smallmatrix} f(t) \\ f(\underline{c}) \end{smallmatrix}\big)\big)$$

and restricting the symmetric action, operadic composition and operadic unit in the obvious way.

This allows us to define the *category of operads with varying colors* $\mathbf{COp}(\mathbf{T})$: an object is a pair $(\mathfrak{C}, \mathcal{P})$ where $\mathfrak{C}$ is a set of colors and $\mathcal{P} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$ is a $\mathfrak{C}$-colored operad. A morphism

$$(f, \phi) : (\mathfrak{C}, \mathcal{P}) \longrightarrow (\mathfrak{D}, \mathcal{Q})$$

in $\mathbf{COp}(\mathbf{T})$ is then a map of sets of colors $f : \mathfrak{C} \to \mathfrak{D}$ together with an $\mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$-morphism $\phi : \mathcal{P} \to f^*\mathcal{Q}$.

2.5.2. Algebras over operads. In the previous sections we saw several operads that encode algebraic structures, like the associative operad As or the Lie operad Lie. The objects that exhibit these algebraic structures are the algebras over the operad. Recall that $\mathbf{T}$ is a bicomplete closed symmetric monoidal category.

Definition 2.5.15. Let $\mathfrak{C}$ be a set of colors (which is in particular a category without nonidentity morphisms). A $\mathfrak{C}$*-colored object* is a functor $X : \mathfrak{C} \to \mathbf{T}$, i.e. an assignment X_c to every color $c \in \mathfrak{C}$.

A map of $\mathfrak{C}$-colored objects $f : X \to Y$ is a natural transformation, i.e. a map $f_c : X_c \to Y_c$ for every $c \in \mathfrak{C}$.

Any map $f : \mathfrak{C} \to \mathfrak{D}$ induces the pullback functor of $\mathfrak{D}$-colored objects

$$f^* : \mathbf{T}^{\mathfrak{D}} \longrightarrow \mathbf{T}^{\mathfrak{C}} \tag{2.5.6}$$

via $f^*(Y)_c = Y_{f(c)}$. This functor has a left adjoint

$$f_! : \mathbf{T}^{\mathfrak{C}} \longrightarrow \mathbf{T}^{\mathfrak{D}} \tag{2.5.7}$$

with $f_!(X)_d = \coprod_{c \in f^{-1}(d)} X_c$.

Definition 2.5.16. Let $(\mathcal{O}, \gamma, \mathbb{1}) \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$ be a $\mathfrak{C}$-colored operad valued in $\mathbf{T}$. An *algebra* A over $\mathcal{O}$ is a $\mathfrak{C}$-colored object, i.e. an assignment $A_c \in \mathbf{T}$ for each $c \in \mathfrak{C}$, together with an $\mathcal{O}$*-action*: for all $n \geq 0$ and all $(\underline{c}, t) \in \mathfrak{C}^n \times \mathfrak{C}$, a $\mathbf{T}$-morphism

$$\alpha : \mathcal{O}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \otimes A_{\underline{c}} \longrightarrow A_t$$

where we write $A_{\underline{c}} := A_{c_1} \otimes \cdots \otimes A_{c_n}$. The $\mathcal{O}$-action is required to satisfy the following conditions:

- Associativity: the diagram

$$\begin{array}{ccc}
\Big[\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_i \mathcal{O}\binom{a_i}{\underline{b}_i}\Big] \otimes A_{\underline{b}} & \xrightarrow{\gamma\otimes\mathrm{id}} & \mathcal{O}\binom{t}{\underline{b}} \otimes A_{\underline{b}} \\
\downarrow{\scriptstyle \text{permute}} & & \\
\mathcal{O}\binom{t}{\underline{a}} \otimes \bigotimes_i \Big[\mathcal{O}\binom{a_i}{\underline{b}_i} \otimes A_{\underline{b}_i}\Big] & & \downarrow{\scriptstyle \alpha} \\
\downarrow{\scriptstyle \mathrm{id}\otimes\bigotimes_i \alpha} & & \\
\mathcal{O}\binom{t}{\underline{a}} \otimes A_{\underline{a}} & \xrightarrow{\alpha} & A_t
\end{array}$$

  commutes for all possible t, $\underline{a}$ and $\underline{b}_i$.
- Unity: the diagram

$$\begin{array}{ccc}
I \otimes A_c & \xrightarrow[\lambda]{\cong} & A_c \\
{\scriptstyle \mathbb{1}\otimes\mathrm{id}} \searrow & & \nearrow {\scriptstyle \alpha} \\
& \mathcal{O}\binom{c}{c} \otimes A_c &
\end{array}$$

  commutes for all c.
- Equivariance: the diagram

$$\begin{array}{ccc}
\mathcal{O}\binom{t}{\underline{c}} \otimes A_{\underline{c}} & \xrightarrow{\mathcal{O}(\sigma)\otimes\text{permute}} & \mathcal{O}\binom{t}{\underline{a}\sigma} \otimes A_{\underline{a}\sigma} \\
{\scriptstyle \alpha} \searrow & & \swarrow {\scriptstyle \alpha} \\
& A_t &
\end{array}$$

  commutes for all t, $\underline{c} \in \mathfrak{C}^n$ and $\sigma \in \Sigma_n$.

DEFINITION 2.5.17. A *morphism of $\mathcal{O}$-algebras* $\kappa : (A, \alpha_A) \to (B, \alpha_B)$ is a map of $\mathfrak{C}$-colored objects $\kappa : A \to B$ that commutes with the $\mathcal{O}$-action:

$$\alpha_B \circ (\mathrm{id} \otimes \bigotimes_i \kappa_{c_i}) = \kappa_t \circ \alpha_A : \mathcal{O}\binom{t}{\underline{c}} \otimes A_{\underline{c}} \to B_t$$

for all $n \geq 0$ and all $(\underline{c}, t) \in \mathfrak{C}^n \times \mathfrak{C}$.

We write $\mathsf{Alg}_{\mathcal{O}}$ for the category of algebras over the operad $\mathcal{O}$.

EXAMPLE 2.5.18. An algebra (A, α) over the diagram operad $\mathsf{Diag}_{\mathbf{C}}$ of a category $\mathbf{C}$ of Example 2.5.6 assigns an object $A(c) := A_c \in \mathbf{T}$ to all objects $c \in \mathbf{C}$. The operation $A(f) := \alpha(f; -) : A(c) \to A(c')$ for $f : c \to c'$ then defines maps in $\mathbf{T}$. One can check that the algebra axioms now exactly ensure that $A : \mathbf{C} \to \mathbf{T}$ is a functor. So $\mathsf{Alg}_{\mathsf{Diag}_{\mathbf{C}}} \cong \mathbf{T}^{\mathbf{C}}$ is the functor category.

EXAMPLE 2.5.19. As the notation suggests, algebras over the operads $\mathsf{As}, \mathsf{Lie}, \mathsf{Pois}, \mathsf{uLie} \in \mathbf{Op}(\mathbf{Vect}_{\mathbb{K}})$ are associative, Lie, Poisson and unital Lie algebras in $\mathbf{Vect}_{\mathbb{K}}$, respectively. In particular, we see that our earlier notation of $\mathsf{Alg}_{\mathsf{As}}$ for the category of associative algebras agrees with our current notation. Similarly, interpreting these vector space-valued operads as chain complex-valued operads concentrated in degree zero, their algebras are differential graded associative, Lie, Poisson and unital Lie algebras, respectively. We will keep using the notation $\mathrm{dg}\mathsf{Alg}_{\mathcal{O}}$ for differential graded algebras, to make clear that we are considering algebras valued in chain complexes.

Working over the field $\mathbb{C}$, adding an involution $*$ and implementing the obvious relations turns the above algebras into $*$-algebras and differential graded $*$-algebras, though we must note that we are not using operads to formalize this structure (this last note will be relevant when discussing the model structure on differential graded algebras). In [**BSW19b**] such $*$-operads are studied, as well as their applications to algebraic quantum field theory.

For an operad $\mathcal{O} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{T})$, there exists an obvious forgetful functor from $\mathcal{O}$-algebras to $\mathfrak{C}$-colored objects

$$U_{\mathcal{O}} : \mathsf{Alg}_{\mathcal{O}} \longrightarrow \mathbf{T}^{\mathfrak{C}} \tag{2.5.8}$$

that forgets the $\mathcal{O}$-action on an algebra. As was true for operads (Theorem 2.5.8) and associative algebras (Example 2.1.26), this forgetful functor has a left adjoint.

THEOREM 2.5.20. The forgetful functor $U_{\mathcal{O}}$ (2.5.8) is part of the adjunction

$$\mathcal{O} \circ (-) : \mathbf{T}^{\mathfrak{C}} \rightleftarrows \mathsf{Alg}_{\mathcal{O}} : U_{\mathcal{O}} \quad . \tag{2.5.9}$$

The functor $\mathcal{O} \circ (-)$ is the *free $\mathcal{O}$-algebra functor* and for $X \in \mathbf{T}^{\mathfrak{C}}$ we say that $\mathcal{O} \circ X$ is the *free $\mathcal{O}$-algebra* generated by X.

The $\mathcal{O}$-algebra $\mathcal{O} \circ X$ can be computed very explicitly using coends: see Theorem 2.8 in [**BSW20**].

DEFINITION 2.5.21. Let $(f, \phi) : (\mathfrak{C}, \mathcal{P}) \to (\mathfrak{D}, \mathcal{Q})$ be an $\mathbf{Op}(\mathbf{T})$-morphism and $(A, \alpha) \in \mathsf{Alg}_{\mathcal{Q}}$ be a $\mathcal{Q}$-algebra. Then the *pullback* $(f, \phi)^* A$ of (A, α) along (f, ϕ) is defined by $(f, \phi)^* A := f^* A$ (2.5.6) and the $\mathcal{P}$-action

$$\alpha_{(f,\phi)^* A} : \mathcal{P}\big(\begin{smallmatrix} t \\ \underline{c} \end{smallmatrix}\big) \otimes ((f, \phi)^* A)_{\underline{c}} \xrightarrow{\ \phi \otimes \mathrm{id}\ } \mathcal{Q}\big(\begin{smallmatrix} f(t) \\ f(\underline{c}) \end{smallmatrix}\big) \otimes A_{f(\underline{c})} \xrightarrow{\ \alpha\ } A_{f(t)} = ((f, \phi)^* A)_t \quad .$$

This defines a functor

$$(f, \phi)^* : \mathsf{Alg}_{\mathcal{Q}} \longrightarrow \mathsf{Alg}_{\mathcal{P}} \ .$$

As is true for functor categories (see Remark 2.1.37), i.e. algebras over the diagram operad (see Example 2.5.18), the pullback of algebras over operads has a left adjoint.

THEOREM 2.5.22. Let $(f, \phi) : (\mathfrak{C}, \mathcal{P}) \to (\mathfrak{D}, \mathcal{Q})$ be a $\mathbf{COp}(\mathbf{T})$-morphism. Then the pullback of algebras defined in Definition 2.5.21 is part of an adjunction on the algebra categories

$$(f, \phi)_! : \mathsf{Alg}_{\mathcal{P}} \rightleftarrows \mathsf{Alg}_{\mathcal{Q}} : (f, \phi)^* \quad .$$

We call the functor $(f, \phi)_!$ the *operadic left Kan extension.*

PROOF. This follows from the adjoint lifting theorem for the square of functors and adjoints

$$\begin{array}{ccc} \mathsf{Alg}_{\mathcal{P}} & \xleftarrow[\ (f,\phi)^*\ ]{} & \mathsf{Alg}_{\mathcal{Q}} \\ {\scriptstyle \mathcal{P}\circ(-)}\uparrow\downarrow{\scriptstyle U_{\mathcal{P}}} & & {\scriptstyle \mathcal{Q}\circ(-)}\uparrow\downarrow{\scriptstyle U_{\mathcal{Q}}} \\ \mathbf{T}^{\mathfrak{C}} & \underset{f^*}{\overset{f_!}{\rightleftarrows}} & \mathbf{T}^{\mathfrak{D}} \end{array} \ .$$

because the square of pullbacks and forgetful functors commutes and the category $\mathbf{T}$ is cocomplete, see Section 4.5 in [**Bor94**]. One finds that for a $\mathcal{P}$-algebra (A, α),

$$(f, \phi)_!(A, \alpha) = \operatorname{colim}\Big(\mathcal{Q} \circ f_!(\mathcal{P} \circ A) \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \mathcal{Q} \circ f_!(A) \Big)$$

where we suppress the forgetful functors $U_{\mathcal{P}}$ and $U_{\mathcal{Q}}$, $\partial_0 = \mathcal{Q} \circ f_!(\alpha)$ and ∂_1 is constructed using the units and counits of the other three adjunctions, which involves the operadic composition of $\mathcal{Q}$. □

EXAMPLE 2.5.23. If $F : \mathbf{C} \to \mathbf{D}$ is a functor, it induces a morphism $F : \mathsf{Diag}_{\mathbf{C}} \to \mathsf{Diag}_{\mathbf{D}}$ of diagram operads. Using the identification of Example 2.5.18 the pullback $F^* : \mathsf{Alg}_{\mathsf{Diag}_{\mathbf{D}}} \to \mathsf{Alg}_{\mathsf{Diag}_{\mathbf{C}}}$ is then seen to be the ordinary pullback of functor categories. Therefore, the operadic left Kan extension is the ordinary left Kan extension of functors in this case.

2.5.3. Model structures for differential graded algebras over operads. Now let $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$ be the model category of chain complexes over a field of characteristic 0, see Example 2.4.19. We saw that the functor category ${\mathbf{Ch}_{\mathbb{K}}}^{\mathbf{D}} = \mathrm{dg}\mathsf{Alg}_{\mathsf{Diag}_{\mathbf{D}}}$ admits the projective model structure in Example 2.4.23, for which the weak equivalences and fibrations are defined objectwise. A natural question to now ask is if this is also true for algebras over other $\mathbf{Ch}_{\mathbb{K}}$-valued operads.

Let $\mathcal{O} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{Ch}_{\mathbb{K}})$ be an operad. Recall the free-forget adjunction (2.5.9)

$$\mathcal{O} \circ (-) : {\mathbf{Ch}_{\mathbb{K}}}^{\mathfrak{C}} \rightleftarrows \mathrm{dg}\mathsf{Alg}_{\mathcal{O}} : U_{\mathcal{O}} \quad .$$

By Proposition 2.4.22, ${\mathbf{Ch}_{\mathbb{K}}}^{\mathfrak{C}}$ carries the projective model structure. We can define the classes of weak equivalences (fibrations) on $\mathrm{dg}\mathsf{Alg}_{\mathcal{O}}$ to be the algebra morphisms f such that $U_{\mathcal{O}}(f)$ is a weak equivalence (fibration) in ${\mathbf{Ch}_{\mathbb{K}}}^{\mathfrak{C}}$, i.e. an objectwise weak equivalence (fibration) in $\mathbf{Ch}_{\mathbb{K}}$. Note that, if this defines a model structure on $\mathrm{dg}\mathsf{Alg}_{\mathcal{O}}$, this makes $U_{\mathcal{O}}$ right Quillen.

DEFINITION 2.5.24. An operad $\mathcal{O} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{Ch}_{\mathbb{K}})$ is called *admissible* if the above choice of objectwise weak equivalences and fibrations defines a model structure on $\mathrm{dg}\mathsf{Alg}_{\mathcal{O}}$, with the cofibrations as the algebra morphisms that satisfy the left lifting property with respect to the acyclic fibrations.

THEOREM 2.5.25. If $\mathbb{K}$ is of characteristic zero, all operads $\mathcal{O} \in \mathbf{Op}_{\mathfrak{C}}(\mathbf{Ch}_{\mathbb{K}})$ are admissible: they admit a model structure with objectwise weak equivalences (quasi-isomorphisms) and objectwise fibrations (degreewise surjections).

PROOF. This is shown in Theorem 2.6.1 in [**Hin13**], see also [**Hin97**][5]. The crucial point is that because the characteristic of $\mathbb{K}$ is 0, the operad $\mathcal{O}$ is Σ-*split*: since every $n!$ is invertible, the natural sum map

$$\pi : \mathcal{O}^{\Sigma} = F^{\Sigma} U^{\Sigma}(\mathcal{O}) \longrightarrow \mathcal{O}$$

has a Σ-splitting. Here, $F^{\Sigma} \dashv U^{\Sigma}$ is the part of the free-forget adjunction of Theorem 2.5.8 that forgets the symmetric action and freely generates it, and π is the counit of this adjunction. □

[5]Its erratum [**Hin03**] is related to a counterexample when the characteristic of $\mathbb{K}$ is nonzero, so it does not apply to our case.

Example 2.5.26. We find that the category of differential graded associative algebras $\mathrm{dg}\mathsf{Alg}_{\mathsf{As}}$ is a model category, with quasi-isomorphisms of the underlying chain complex as the weak equivalences and degreewise surjections as the fibrations. The same is true for the categories $\mathrm{dg}\mathsf{Alg}_{\mathsf{Lie}}$, $\mathrm{dg}\mathsf{Alg}_{\mathsf{Pois}}$ and $\mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}}$. Working over $\mathbb{C}$ and adding an involution as in Example 2.5.19 we find that by Remark 2.4.37, $\mathrm{dg}^*\mathsf{Alg}_{\mathsf{As}}$ is a homotopical category (see Definition 2.4.35), as are $\mathrm{dg}^*\mathsf{Alg}_{\mathsf{Lie}}$, $\mathrm{dg}^*\mathsf{Alg}_{\mathsf{Pois}}$ and $\mathrm{dg}^*\mathsf{Alg}_{\mathsf{uLie}}$. To the best of our knowledge, the question of whether these are model categories has not yet been studied.

2.6. Homotopy algebraic quantum field theory

With the model structure on the categories of operad algebras from Section 2.5.3 in hand, we can give definitions of algebraic field theories as in Section 2.3.2 with the axioms only holding up to homotopy.

Recall (Definition 2.3.9) that algebraic field theories are subcategories of functor categories. From Example 2.5.26 we see that $\mathrm{dg}\mathsf{Alg}_{\mathcal{O}}$ is a model category for the operads As, Pois and uLie. We also see that the corresponding category $\mathrm{dg}^*\mathsf{Alg}_{\mathcal{O}_{\mathbb{C}}}$ is a homotopical category (see Definition 2.4.35), as is $\mathbf{PoissCh}_{\mathbb{K}}$ by Example 2.4.36. In all cases, the weak equivalences are the maps such that the underlying chain map is a quasi-isomorphism. This also makes the functor categories

$$\mathrm{dg}\mathsf{Alg}_{\mathcal{O}}{}^{\mathbf{C}} \quad ; \quad \mathrm{dg}^*\mathsf{Alg}_{\mathcal{O}_{\mathbb{C}}}{}^{\mathbf{C}} \quad ; \quad \mathbf{PoissCh}_{\mathbb{K}}{}^{\mathbf{C}}$$

into homotopical categories for any category $\mathbf{C}$ and uncolored operad $\mathcal{O}$, with the weak equivalences being the natural weak equivalences, i.e. objectwise quasi-isomorphisms.

Picking $\mathcal{O} = \mathsf{As}_{\mathbb{C}}$ and $\mathbf{C} = \mathbf{Loc}$ we can try to define homotopy algebraic quantum field theory. Generalizing the time-slice axiom is straightforward: we replace the condition that $\mathfrak{A}(f)$ is an isomorphism with the condition that it is a weak equivalence. Generalizing Einstein causality is more complicated: this is done in [**BSW19a**] and involves finding a Σ-cofibrant resolution of the corresponding field theory operad (we will introduce the field theory operad in the next chapter, though not the resolution). Effectively, the commutator map now is only homotopic to the zero map, and there is a whole tower of homotopies governing this.

Since all field theories in this text satisfy the strict Einstein causality axiom, we choose to only weaken the time-slice axiom here, arriving at the definition of semi-strict homotopy algebraic quantum field theories. Note that by the strictification theorem 4.3 in [**BSW19a**] there is no loss of generality when doing this in characteristic 0. Compare Definition 2.3.1 and recall Definition 2.2.10 of causally disjoint morphisms and Definition 2.2.9 of Cauchy morphisms.

Definition 2.6.1. A *semi-strict homotopy algebraic quantum field theory* on $\mathbf{Loc}$ is a functor

$$\mathfrak{A} : \mathbf{Loc} \longrightarrow \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$$

that satisfies the following two axioms:

(1) *Strict Einstein causality*: if $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are causally disjoint morphisms in $\mathbf{Loc}$, their observables commute in $\mathfrak{A}(N)$: the commutator map

$$\big[\mathfrak{A}(f_1)(-), \mathfrak{A}(f_2)(-)\big] : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \longrightarrow \mathfrak{A}(N)$$

is the zero map.

(2) *Homotopy time-slice*: if $f : M \to N$ is a Cauchy morphism in **Loc**, then it induces an weak equivalence
$$\mathfrak{A}(f) : \mathfrak{A}(M) \xrightarrow{\sim} \mathfrak{A}(N)$$
in $\mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$.

The category of semi-strict homotopy algebraic quantum field theories on **Loc** is the full subcategory
$$\mathbf{hQFT}(\mathbf{Loc}) \subseteq \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}{}^{\mathbf{Loc}}$$
and $\mathbf{hQFT}(\mathbf{Loc})$ inherits the structure of a homotopical category since the weak equivalences in $\mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}{}^{\mathbf{Loc}}$ are defined objectwise.

REMARK 2.6.2. Involutions are an extra structure on the category of differential graded As-algebras that are not governed by the operad (see Example 2.5.19 and also [**BSW19b**]). So in order to streamline the following chapters, we have chosen to suppress the datum of a $*$-involution in the above definition and in the text that follows. An added advantage of this definition is that the category of homotopy algebraic quantum field theories is a model category, as will be shown in Proposition 3.4.1.

In all examples constructed in this text, we find that we can add the involution if needed. In particular, the linear homotopy algebraic quantum field theories obtained by the canonical commutation relations functor $\mathfrak{CCR}$ (3.4.2) admit an extension of the involution (2.3.5), see Appendix A in [**BBS20**]. As such, the linear Yang-Mills model obtained in Theorem 4.4.12 does, too.

We will propose another characterization of the homotopy time-slice axiom in Definition 3.4.12. We will sometimes drop the prefix *semi-strict* in this text; however, every field theory considered will satisfy the strict Einstein causality axiom.

Picking different algebraic categories and adapting the axioms accordingly we find semi-strict homotopy versions of Definitions 2.3.6 and 2.3.8 of algebraic classical and linear field theories.

DEFINITION 2.6.3. A *semi-strict homotopy algebraic classical field theory* on **Loc** is a functor
$$\mathfrak{A} : \mathbf{Loc} \longrightarrow \mathrm{dg}\mathsf{Alg}_{\mathsf{Pois}}$$
that satisfies two axioms:

(1) *Strict Einstein causality*: if $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are causally disjoint morphisms in **Loc**, the induced Poisson bracket $\mathfrak{A}(N)$
$$\Big\{\mathfrak{A}(f_1)\big(-\big), \mathfrak{A}(f_2)\big(-\big)\Big\}_N : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \longrightarrow \mathfrak{A}(N)$$
is the zero map.

(2) *Homotopy time-slice*: if $f : M \to N$ is a Cauchy morphism in **Loc**, then it induces a weak equivalence
$$\mathfrak{A}(f) : \mathfrak{A}(M) \xrightarrow{\sim} \mathfrak{A}(N)$$
in $\mathrm{dg}\mathsf{Alg}_{\mathsf{Pois}}$.

The semi-strict homotopy algebraic classical field theories form the full homotopical subcategory
$$\mathbf{hClFT}(\mathbf{Loc}) \subseteq \mathrm{dg}\mathsf{Alg}_{\mathsf{Pois}}{}^{\mathbf{Loc}}\ .$$

DEFINITION 2.6.4. A *semi-strict homotopy algebraic linear field theory* on **Loc** is a functor

$$(\mathfrak{L}, \tau) : \mathbf{Loc} \longrightarrow \mathbf{PoissCh}_{\mathbb{R}}$$

that satisfies two axioms:

(1) *Strict Einstein causality*: if $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are causally disjoint morphisms in **Loc**, the induced Poisson structure

$$\tau_N\big(\mathfrak{L}(f_1)(-), \mathfrak{L}(f_2)(-)\big)\mathfrak{L}(M_1) \otimes \mathfrak{L}(M_2) \longrightarrow \mathbb{R}$$

is equal to zero.

(2) *Homotopy time-slice*: if $f : M \to N$ is a Cauchy morphism in **Loc**, then it induces a weak equivalence

$$\mathfrak{L}(f) : \mathfrak{L}(M) \xrightarrow{\sim} \mathfrak{L}(N)$$

in $\mathbf{PoissCh}_{\mathbb{R}}$.

The semi-strict homotopy algebraic linear field theories form the full homotopical subcategory

$$\mathbf{hLFT}(\mathbf{Loc}) \subseteq \mathbf{PoissCh}_{\mathbb{R}}{}^{\mathbf{Loc}} .$$

We see that all of our categories of homotopy field theories are homotopical categories, with objectwise quasi-isomorphisms as weak equivalences. In Proposition 3.4.1 we will see that the categories of field theories valued in operad algebras (i.e. the classical field theories and the quantum field theories) are in fact model categories with these classes of weak equivalences.

We end by extending Definitions 2.3.5 and 2.3.11 of (linear) quantum fields to the context of differential graded algebras.

DEFINITION 2.6.5. Let $\mathfrak{A} \in \mathbf{hQFT}(\mathbf{Loc})$ be a homotopy algebraic quantum field theory, and let E be a graded vector space with a differential on $\Gamma_{\mathrm{c}}(E^*)$. A *quantum field of type E* is a natural transformation

$$\hat{\Phi} : \Gamma_{\mathrm{c}}(E^*) \longrightarrow \mathfrak{A}$$

between the underlying functors of chain complexes.

If $(\mathfrak{L}, \tau)$ is a homotopical linear field theory, a *linear field of type E* is a natural transformation

$$\Phi : \Gamma_{\mathrm{c}}(E^*) \longrightarrow \mathfrak{L}$$

between the underlying functors of chain complexes. The corresponding *linear quantum field of type E* of $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau)$ (see Section 3.4.2 for a description of the canonical quantization scheme for chain complex-valued theories) is the natural transformation

$$\hat{\Phi} = \iota_1 \circ \Phi : \Gamma_{\mathrm{c}}(E^*) \longrightarrow \mathfrak{A}$$

where $\iota_1 : \mathfrak{L} \to \mathfrak{A}$ is the canonical natural transformation.

CHAPTER 3

Operadic quantization

In this chapter we build on the work done in [**BSW20**] on operads for algebraic quantum field theory, expanding it to any uncolored operad. We recast the Einstein causality axiom in terms of operads, and then define algebraic field theories as algebras over a field theory operad. Varying spacetime categories leads to adjunctions related to descent and the time-slice axiom, while the canonical operad map $\mathsf{uLie} \to \mathsf{As}$ yields a linear quantization adjunction. We end by treating linear quantization for chain complex valued theories, which will be used throughout the rest of this thesis. The results in this chapter have earlier appeared in [**BS19b**] and were summarized in [**Bru19**].

3.1. Formalizing the definition of algebraic field theories

In this section we recast our definitions of algebraic quantum field theory (Definition 2.3.1) and algebraic classical field theory (Definition 2.3.6) into an operadic form. Broadly speaking, our definitions of field theory[1] incorporated two data: a spacetime category $\mathbf{C}$ (such as **Loc** or $\mathbf{COpens}(\overline{M})$) and a choice of algebraic structure (such as an associative algebra or a Poisson algebra). We now recognize the latter as a choice of operad $\mathcal{P}$ governing the algebraic structure of the field theory, and we note that this choice comes with a third one, that of the closed symmetric monoidal category $\mathbf{T}$. For now, we fix $\mathbf{T}$ and assume it is complete and cocomplete; we will start investigating what happens if it is a model category in Section 3.4.

Field theories are then functors $\mathbf{C} \to \mathsf{Alg}_{\mathcal{P}}$ satisfying two properties, Einstein causality and time-slice. We would like to formalize these two properties, turning them into structure built into the functors. For the time-slice axiom, this entails localizing at the Cauchy morphisms. This is the topic of Section 3.3.2. Formalizing Einstein causality is more involved, and it is one of the reasons for developing the operadic definition of field theory.

To start with we formalize spacetime categories with a notion of pairs of causally disjoint maps (Definition 2.2.10).

DEFINITION 3.1.1. An *orthogonal category* $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ is a small category $\mathbf{C}$ with an *orthogonality relation* $\perp$: a set of *orthogonal maps* $\perp \subseteq \mathbf{Mor}\,\mathbf{C}\,{}_t\!\times_t \mathbf{Mor}\,\mathbf{C}$. This notation means that $\perp$ consists of pairs of morphisms in $\mathbf{C}$ with a common target, $(f_1, f_2) = (c_1 \xrightarrow{f_1} c \xleftarrow{f_2} c_2)$; f_1 and f_2 are called *orthogonal*. We write $f_1 \perp f_2$ for $(f_1, f_2) \in \perp$ and we demand that $\perp$ is

- symmetric: if $f_1 \perp f_2$ then $f_2 \perp f_1$;
- stable under composition: if $(f_1, f_2) = (c_1 \xrightarrow{f_1} c \xleftarrow{f_2} c_2) \in \perp$ then for $c \xrightarrow{g} c'$ in $\mathbf{C}$ and $c_i' \xrightarrow{h_i} c_i$, $g f_1 h_1 \perp g f_2 h_2$.

[1]We will mostly drop the prefix “algebraic” from now on

A functor $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ between orthogonal categories is called *orthogonal* if it preserves orthogonality: if $f_1 \perp_{\mathbf{C}} f_2$ then $F(f_1) \perp_{\mathbf{D}} F(f_2)$. We write **OrthCat** for the category of orthogonal categories and orthogonal functors.

Example 3.1.2. The category of globally hyperbolic spacetimes **Loc**, with the pairs of causally disjoint maps as pairs of orthogonal maps, would be an orthogonal category if it were small. Abusing notation, we will from here on out write **Loc** for any small category equivalent to **Loc**, and $\overline{\mathbf{Loc}} = (\mathbf{Loc}, \perp_{\mathbf{Loc}})$ is the orthogonal category for which $\perp_{\mathbf{Loc}}$ is the set of pairs of causally disjoint maps.

Similarly, $\overline{\mathbf{Loc}_\diamond}$ and $\overline{\mathbf{COpens}(\overline{M})}$ are orthogonal categories choosing causally disjoint pairs as orthogonal maps. Another example is the category $\overline{\mathbf{Man}}$ of smooth manifolds with open embeddings as morphisms, where the orthogonal pairs are the pairs of maps with disjoint images.

Lemma 3.1.3. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between small categories.

- If $\mathbf{D}$ is an orthogonal category with orthogonality relation $\perp_{\mathbf{D}}$, then
$$F^*(\perp_{\mathbf{D}}) := \big\{(f_1, f_2) | F(f_1) \perp_{\mathbf{D}} F(f_2)\big\}$$
defines an orthogonality relation on $\mathbf{C}$. We call $F^*(\perp_{\mathbf{D}})$ the *pullback* of $\perp_{\mathbf{D}}$ along F. $F : (\mathbf{C}, F^*(\perp_{\mathbf{D}})) \to (\mathbf{D}, \perp_{\mathbf{D}})$ is an orthogonal functor.
- If $\perp_{\mathbf{C}}$ is an orthogonality relation on $\mathbf{C}$, then
$$F_*(\perp_{\mathbf{C}}) := \big\{(gF(f_1)h_1, gF(f_2)h_2) | f_1 \perp_{\mathbf{C}} f_2; g, h_1, h_2 \in \mathbf{Mor}\,\mathbf{D}\big\}$$
defines an orthogonality relation on $\mathbf{D}$. We call $F_*(\perp_{\mathbf{C}})$ the *pushforward* of $\perp_{\mathbf{C}}$ along F. $F : (\mathbf{C}, \perp_{\mathbf{C}}) \to (\mathbf{D}, F_*(\perp_{\mathbf{C}}))$ is an orthogonal functor.

Proof. This follows immediately from Definition 3.1.1. □

Einstein causality says that these orthogonal pairs of morphisms should lead to commuting observables at the level of algebras, so the next step is to formalize what "commuting" means in this framework. Define the uncolored sequence $I[2] \in \mathbf{Seq}(\mathbf{T})$ as

$$I[2](n) = \begin{cases} I & n = 2 \\ \emptyset & n \neq 2 \end{cases}$$

where $I \in \mathbf{T}$ is the monoidal unit.

Definition 3.1.4. A *bipointed (of arity 2) uncolored operad* $\mathcal{P}^r = (\mathcal{P}, r_1, r_2)$ is an uncolored operad $\mathcal{P} \in \mathbf{Op}(\mathbf{T})$ together with two $\mathbf{Seq}(\mathbf{T})$-morphisms

$$r_1, r_2 : I[2] \rightrightarrows U(\mathcal{P}).$$

Recall that $U : \mathbf{Op}(\mathbf{T}) \to \mathbf{Seq}(\mathbf{T})$ is the forgetful functor; we will suppress the U when able, and just write $r_i : I[2] \to \mathcal{P}$ for the $\mathbf{Seq}(\mathbf{T})$-morphisms.

We say that an $\mathbf{Op}(\mathbf{T})$-morphism $\phi : \mathcal{P}^r \to \mathcal{Q}^s$ of bipointed operads *preserves the points* if the diagram

$$\begin{array}{ccc} I[2] & \xrightarrow{r_i} & \mathcal{P} \\ \| & & \downarrow{\scriptstyle \phi} \\ I[2] & \xrightarrow[s_i]{} & \mathcal{Q} \end{array}$$

in $\mathbf{Seq}(\mathbf{T})$ commutes for both i. We then define $\mathbf{Op}^{2\mathrm{pt}}(\mathbf{T})$ to be the category of bipointed uncolored operads with $\mathbf{Op}(\mathbf{T})$-morphisms that preserve the points as morphisms.

The maps r_1 and r_2 pick out two arity 2 operations in $\mathcal{P}$, and we will interpret "commuting" to mean that these two operations are equal. Note that this definition could easily be extended to operations of higher arity, encoding more general conditions than commutation, and that likewise the definition of orthogonal category can include n-tuples of orthogonal morphisms; since we will not use any other operations than binary ones we have restricted to the arity 2 case.

We can now give a definition of a field theory satisfying Einstein causality in this language (recall that the time-slice axiom will be treated later).

Definition 3.1.5. Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ be an orthogonal category, and $\mathcal{P}^r$ be a bipointed uncolored operad. A *field theory* of type $\mathcal{P}^r$ on $\overline{\mathbf{C}}$ is a functor

$$\mathfrak{A} : \mathbf{C} \longrightarrow \mathsf{Alg}_{\mathcal{P}}$$

such that for all $(c_1 \xrightarrow{f_1} c \xleftarrow{f_2} c_2) \in \perp$, the diagram

$$\begin{array}{ccc} I \otimes \mathfrak{A}(c_1) \otimes \mathfrak{A}(c_2) & \xrightarrow{r_1 \otimes \mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)} & \mathcal{P}(2) \otimes \mathfrak{A}(c)^{\otimes 2} \\ {\scriptstyle r_2 \otimes \mathfrak{A}(f_1) \otimes \mathfrak{A}(f_2)} \downarrow & & \downarrow {\scriptstyle \alpha_c^{\mathcal{P}}} \\ \mathcal{P}(2) \otimes \mathfrak{A}(c)^{\otimes 2} & \xrightarrow[\alpha_c^{\mathcal{P}}]{} & \mathfrak{A}(c) \end{array} \tag{3.1.1}$$

commutes, where $\alpha_c^{\mathcal{P}}$ is the action of $\mathcal{P}$ on the algebra $\mathfrak{A}(c)$.

The category of field theories of type $\mathcal{P}^r$ on $\overline{\mathbf{C}}$ is a full subcategory of $\mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$, which we call $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$.

Note that we define $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) \subseteq \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$ to be a full subcategory, so we do not impose any conditions on the morphisms. Since morphisms in $\mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$ are natural transformations in $\mathsf{Alg}_{\mathcal{P}}$ they respect both the algebraic structure and the functoriality in $\mathbf{C}$, so they automatically preserve the structures that encode Einstein causality and any such natural transformation is a satisfactory morphism of field theories.

As expected, our previous definitions of (algebraic) field theories (without the time-slice axiom) fit into the framework of Definition 3.1.5. Note that in Section 2.3.2 we implicitly used the closed symmetric monoidal category $\mathbf{T} = \mathbf{Vect}_{\mathbb{K}}$; we will see that the below definitions work in larger generality.

Example 3.1.6. Let $\overline{\mathbf{C}}$ an orthogonal category. An (algebraic) quantum field theory (see Definition 2.3.1) on $\overline{\mathbf{C}}$ is a field theory of type $\mathsf{As}^{\mu,\mu^{\mathrm{op}}}$ (see Example 2.5.10), where μ and μ^{op} pick out the multiplication and opposite multiplication, respectively:

$$\begin{array}{cccc} \mu : & I[2] & \longrightarrow & \mathsf{As} \\ & 1 & \longmapsto & \curlywedge = \underset{1\ 2}{\curlywedge} \end{array} \quad ; \quad \begin{array}{cccc} \mu^{\mathrm{op}} : & I[2] & \longrightarrow & \mathsf{As} \\ & 1 & \longmapsto & \underset{2\ 1}{\curlywedge} \end{array} .$$

If $\mathbf{T}$ is additive, we can also present quantum field theories using the commutator $[\ ,\ ] := \mu - \mu^{\mathrm{op}}$ and 0, which turns out to be more convenient for the purpose of linear quantization. These two definitions are equivalent, so we have

$$\mathbf{QFT}(\overline{\mathbf{C}}) := \mathbf{FT}(\overline{\mathbf{C}}, \mathsf{As}^{\mu,\mu^{\mathrm{op}}}) \cong \mathbf{FT}(\overline{\mathbf{C}}, \mathsf{As}^{[\ ,\ ],0}).$$

EXAMPLE 3.1.7. Let $\mathbf{T}$ be additive and let $\overline{\mathbf{C}}$ be an orthogonal category. A classical field theory (see Definition 2.3.6) on $\overline{\mathbf{C}}$ is a field theory of type $\mathsf{Pois}^{\{\,,\,\},0}$ (see Example 2.5.12), where $\{\,,\,\}$ picks out the Poisson bracket operation in Pois:

$$\begin{array}{rccc} \{\,,\,\}: & I[2] & \longrightarrow & \mathsf{Pois} \\ & 1 & \longmapsto & \curlywedge \end{array}$$

(recall that $\curlywedge$ corresponds to the Poisson bracket, while $\dot{\curlywedge}$ corresponds to the commutative multiplication in the notation of Example 2.5.12). So we have

$$\mathbf{ClFT}(\overline{\mathbf{C}}) := \mathbf{FT}(\overline{\mathbf{C}}, \mathsf{Pois}^{\{\,,\,\},0}).$$

As observed in Remark 2.3.7, the Klein-Gordon theory constructed in Section 2.3.1 is a *free* theory. This means that both the classical and the quantum theories can be constructed from the linear observables $\mathfrak{L}^{KG}$ in a straightforward way. We will treat the process of linear quantization in our operadic language in Section 3.3.4; for now we will give an operadic definition of linear field theories.

EXAMPLE 3.1.8. Trying to define linear field theories (see Definition 2.3.8) we run into an issue: for a Poisson vector space (V, τ) the Poisson structure $\tau : V \otimes V \to \mathbb{R}$ lands in $\mathbb{R}$, not V. As such $\mathbf{PoissVect}_{\mathbb{R}}$ is not a category of algebras over an operad, and Definition 2.3.8 of linear field theory is not a field theory in the sense of Definition 3.1.5. This is resolved by taking the Heisenberg Lie algebra of (V, τ), in effect complexifying V and adding the ground field to it. We define

$$\begin{array}{rccc} \mathfrak{heis}: & \mathbf{PoissVect}_{\mathbb{R}} & \longrightarrow & \mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}} \\ & (V,\tau) & \longmapsto & \big(V_{\mathbb{C}} \oplus \mathbb{C}, [\,,\,]\big) \end{array}$$

where the bracket is defined by

$$[v_1 \oplus \lambda_1, v_2 \oplus \lambda_2] := 0 \oplus i\tau(v_1, v_2)$$

and $0 \oplus 1$ is the unit.

So we adapt Definition 2.3.8 and we say that if $\mathbf{T}$ is additive, a linear field theory on $\overline{\mathbf{C}}$ is a field theory of type $\mathsf{uLie}^{[\,,\,],0}$ (see Example 2.5.13) where $[\,,\,]$ picks out the Lie bracket operation in uLie:

$$\begin{array}{rccc} [\,,\,]: & I[2] & \longrightarrow & \mathsf{uLie} \\ & 1 & \longmapsto & \curlywedge\,. \end{array}$$

and we have

$$\widetilde{\mathbf{LFT}}(\overline{\mathbf{C}}) := \mathbf{FT}(\overline{\mathbf{C}}, \mathsf{uLie}^{[\,,\,],0}).$$

REMARK 3.1.9. The pushforward along $\mathfrak{heis}$ defines a functor of functor categories,

$$\mathfrak{heis}_* : \mathbf{PoissVect}_{\mathbb{R}}{}^{\mathbf{C}} \longrightarrow \mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}}{}^{\mathbf{C}}\,.$$

Recall Definitions 2.3.8 and 2.3.9. From the definition of the Lie bracket on $\mathfrak{heis}(V, \tau)$ it is clear that if $\mathfrak{L} \in \mathbf{LFT}(\overline{\mathbf{C}})$ satisfies Einstein causality, so does $\mathfrak{heis}_*(\mathfrak{L})$. Moreover, looking ahead, if $\mathfrak{L}$ satisfies the time-slice axiom, so does $\mathfrak{heis}_*(\mathfrak{L})$. So $\mathfrak{heis}_*$ restricts to a functor of field theories,

$$\mathfrak{heis}_* : \mathbf{LFT}(\overline{\mathbf{C}}) \longrightarrow \widetilde{\mathbf{LFT}}(\overline{\mathbf{C}})$$

Note that while the above definition of $\widetilde{\mathbf{LFT}}(\overline{\mathbf{C}})$ works for any additive closed symmetric monoidal category $\mathbf{T}$, for a linear field theory to be of the form $\mathfrak{heis}_*(\mathfrak{L})$ for $\mathfrak{L} \in \mathbf{LFT}(\overline{\mathbf{C}})$, we require that $\mathbf{T} = \mathbf{Vect}_{\mathbb{C}}$ (and we will see in Section 3.4 that $\mathbf{T} = \mathbf{Ch}_{\mathbb{C}}$ also works).

3.2. Field theories as algebras over field theory operads

We will now translate our definition of field theories to yet another form, realizing them as an algebra over a colored operad (as opposed to a functor with values in algebras over an uncolored operad, as in last section). First, we reformulate $\mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$ as the algebras over an $\mathrm{obj}(\mathbf{C})$-colored operad $\mathcal{P}_{\mathbf{C}}$. Then, we implement our generalized version of Einstein causality from the previous section, building it into the definition of the field theory operad $\mathcal{P}^{r}_{\overline{\mathbf{C}}}$.

Throughout this section we will use the multi-morphism sets

$$\mathbf{C}(\underline{c}, t) := \prod_i \mathbf{C}(c_i, t)$$

which have elements $\underline{f} = (f_1, \ldots, f_n) \in \mathbf{C}(\underline{c}, t)$. The multi-morphisms carry a permutation group action

$$\underline{f}\sigma = (f_{\sigma(1)}, \ldots, f_{\sigma(n)}) \in \mathbf{C}(\underline{c}\sigma, t)$$

for $\sigma \in \Sigma_n$ and $\underline{f} \in \mathbf{C}(\underline{c}, t)$, and a composition defined by composition in $\mathbf{C}$,

$$\underline{f}(\underline{g}_1, \ldots, \underline{g}_n) = (f_1 g_{11}, \ldots, f_n g_{nk_n}) \in \mathbf{C}(\underline{b}, t)$$

for $\underline{f} \in \mathbf{C}(\underline{a}, t)$ and $\underline{g}_i \in \mathbf{C}(\underline{b}_i, t)$. With the unit $\mathrm{id} \in \mathbf{C}(t, t)$ it becomes an operad.

Recall the definition of a colored operad, Definition 2.5.3, and recall that we have fixed a closed symmetric monoidal category $\mathbf{T}$ that is complete and cocomplete. In particular, this means that we can **Set**-tensor (see Definition 2.1.44).

Definition 3.2.1. Let $\mathbf{C}$ be a small category with set of objects $\mathbf{C}_0 := \mathrm{obj}(\mathbf{C})$ and let $\mathcal{P}$ be an uncolored operad valued in $\mathbf{T}$. We define $\mathcal{P}_{\mathbf{C}} \in \mathbf{Op}_{\mathbf{C}_0}(\mathbf{T})$, the $\mathbf{C}$-*coloring of* $\mathcal{P}$, as the following $\mathbf{C}_0$-colored operad.

- For $(\underline{c}, t) \in {\mathbf{C}_0}^n \times \mathbf{C}_0$,

  $$\mathcal{P}_{\mathbf{C}}\tbinom{t}{\underline{c}} := \mathbf{C}(\underline{c}, t) \otimes \mathcal{P}(n)$$

  with inclusion maps $\iota_{\underline{f}} : \mathcal{P}(n) \to \mathcal{P}_{\mathbf{C}}\tbinom{t}{\underline{c}}$ for $\underline{f} = (f_1, \ldots, f_n) \in \mathbf{C}(\underline{c}, t)$.
- For $(\underline{c}, t) \in {\mathbf{C}_0}^n \times \mathbf{C}_0$ and $\sigma \in \Sigma_n$, the permutation action $\mathcal{P}_{\mathbf{C}}(\sigma)$ is defined by

  $$\begin{array}{ccc} \mathcal{P}(n) & \xrightarrow{\mathcal{P}(\sigma)} & \mathcal{P}(n) \\ \downarrow{\scriptstyle \iota_{\underline{f}}} & & \downarrow{\scriptstyle \iota_{\underline{f}\sigma}} \\ \mathcal{P}_{\mathbf{C}}\tbinom{t}{\underline{c}} & \overset{\mathcal{P}_{\mathbf{C}}(\sigma)}{\dashrightarrow} & \mathcal{P}_{\mathbf{C}}\tbinom{t}{\underline{c}\sigma} \end{array}$$

  for all $\underline{f} = (f_1, \ldots, f_n) \in \mathbf{C}(\underline{c}, t)$.
- For $(\underline{a}, t) \in {\mathbf{C}_0}^n \times \mathbf{C}_0$ and $(\underline{b}_i, a_i) \in {\mathbf{C}_0}^{k_i} \times \mathbf{C}_0$, the operadic composition $\gamma^{\mathcal{P}_{\mathbf{C}}}$ is defined by

  $$\begin{array}{ccc} \mathcal{P}(n) \otimes \bigotimes_i \mathcal{P}(k_i) & \xrightarrow{\gamma^{\mathcal{P}}} & \mathcal{P}(\sum_i k_i) \\ \downarrow{\scriptstyle \iota_{\underline{f}} \otimes \iota_{\underline{g}_1} \otimes \cdots \otimes \iota_{\underline{g}_n}} & & \downarrow{\scriptstyle \iota_{\underline{f}(\underline{g}_1, \ldots, \underline{g}_n)}} \\ \mathcal{P}_{\mathbf{C}}\tbinom{t}{\underline{a}} \otimes \bigotimes_i \mathcal{P}_{\mathbf{C}}\tbinom{a_i}{\underline{b}_i} & \overset{\gamma^{\mathcal{P}_{\mathbf{C}}}}{\dashrightarrow} & \mathcal{P}_{\mathbf{C}}\tbinom{t}{\underline{b}} \end{array}$$

  for all $\underline{f} = (f_1, \ldots, f_n) \in \mathbf{C}(\underline{a}, t)$ and $\underline{g}_i = (g_{i1}, \ldots, g_{ik_i}) \in \mathbf{C}(\underline{b}_i, a_i)$.

- The operadic unit $\mathbb{1}^{\mathcal{P}_{\mathbf{C}}}$ is defined by

$$\begin{array}{ccc} I & \xrightarrow{\mathbb{1}^{\mathcal{P}}} & \mathcal{P}(1) \\ & \overset{\mathbb{1}^{\mathcal{P}_{\mathbf{C}}}}{\searrow} & \downarrow{\scriptstyle \iota_{\mathrm{id}_c}} \\ & & \mathcal{P}_{\mathbf{C}}\binom{c}{c} \end{array}$$

for all $c \in \mathbf{C}_0$.

Checking that this defines a colored operad is straightforward but lengthy; essentially, this follows because $\mathcal{P}$ is an (uncolored) operad, and because composition in $\mathbf{C}$ works in an appropriate way.

The central result about $\mathcal{P}_{\mathbf{C}}$ is the following, which tells us that $\mathcal{P}_{\mathbf{C}}$ encodes the functors on $\mathbf{C}$ valued in $\mathsf{Alg}_{\mathcal{P}}$.

LEMMA 3.2.2. Let $\mathbf{C}$ be a small category and $\mathcal{P}$ be an uncolored operad. Then there is a canonical isomorphism

$$\mathsf{Alg}_{\mathcal{P}_{\mathbf{C}}} \cong \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}.$$

PROOF. By Definition 2.5.16 an algebra over $\mathcal{P}_{\mathbf{C}}$ is a family of objects $A_c \in \mathbf{T}$ for each $c \in \mathbf{C}_0$, with a $\mathcal{P}_{\mathbf{C}}$-action

$$\alpha : \mathcal{P}_{\mathbf{C}}\binom{t}{\underline{c}} \otimes A_{\underline{c}} \to A_t.$$

Because $\mathcal{P}_{\mathbf{C}}$ is a coproduct and $\mathbf{T}$ is closed (see Remark 2.1.43), the definition of the operad action is equivalent to a famliy of $\mathbf{T}$-morphisms

$$\alpha_{\underline{f}} : \mathcal{P}(n) \otimes A_{\underline{c}} \to A_t$$

for all $n \geq 0$, $(\underline{c}, t) \in \mathbf{C}_0{}^n \times \mathbf{C}_0$ and $\underline{f} \in \mathbf{C}(\underline{c}, t)$, satisfying the three axioms of Definition 2.5.16: the diagrams

$$\begin{array}{ccc} \Big[\mathcal{P}(n) \otimes \bigotimes_i \mathcal{P}(k_i)\Big] \otimes A_{\underline{b}} & \xrightarrow{\gamma^{\mathcal{P}} \otimes \mathrm{id}} & \mathcal{P}(\sum_i k_i) \otimes A_{\underline{b}} \\ \downarrow{\scriptstyle \text{permute}} & & \\ \mathcal{P}(n) \otimes \bigotimes_i \Big[\mathcal{P}(k_i) \otimes A_{\underline{b}_i}\Big] & & \downarrow{\scriptstyle \alpha_{\underline{f}(\underline{g}_1, \ldots, \underline{g}_n)}} \\ \downarrow{\scriptstyle \mathrm{id} \otimes \bigotimes_i \alpha_{\underline{g}_i}} & & \\ \mathcal{P}(n) \otimes A_{\underline{a}} & \xrightarrow{\alpha_{\underline{f}}} & A_t \end{array} \tag{3.2.1a}$$

$$\begin{array}{ccc} I \otimes A_c & \xrightarrow[\lambda]{\cong} & A_c \\ & {\scriptstyle \mathbb{1}^{\mathcal{P}} \otimes \mathrm{id}} \searrow \quad \nearrow {\scriptstyle \alpha_{\mathrm{id}_c}} & \\ & \mathcal{P}(1) \otimes A_c & \end{array} \tag{3.2.1b}$$

$$\begin{array}{ccc} \mathcal{P}(n) \otimes A_{\underline{c}} & \xrightarrow{\mathcal{O}(\sigma) \otimes \text{permute}} & \mathcal{P}(n) \otimes A_{\underline{a}\sigma} \\ & {\scriptstyle \alpha_{\underline{f}}} \searrow \quad \swarrow {\scriptstyle \alpha_{\underline{f}\sigma}} & \\ & A_t & \end{array} \tag{3.2.1c}$$

commute.

The first diagram and the right unitality axiom for $\mathcal{P}$ imply that for any $\underline{f} \in \mathbf{C}(\underline{c}, t)$, $\alpha_{\underline{f}}$ factorizes as

$$
\begin{array}{ccc}
\mathcal{P}(n) \otimes \left[\bigotimes_i I \otimes A_{c_i}\right] & \xrightarrow[\lambda]{\cong} & \mathcal{P}(n) \otimes A_{\underline{c}} \\
\Big\downarrow{\scriptstyle \mathrm{id} \otimes \bigotimes_i \mathbb{1}^{\mathcal{P}} \otimes \mathrm{id}} & & \\
\mathcal{P}(n) \otimes \bigotimes_i \left[\mathcal{P}(1) \otimes A_{c_i}\right] & & \Big\downarrow{\scriptstyle \alpha_{\underline{f}}} \\
\Big\downarrow{\scriptstyle \mathrm{id} \otimes \bigotimes_i \alpha_{f_i}} & & \\
\mathcal{P}(n) \otimes {A_t}^{\otimes n} & \xrightarrow{\alpha_{\underline{\mathrm{id}_t}}} & A_t
\end{array}
\tag{3.2.2}
$$

since $\underline{f} = (f_1, \ldots, f_n) = (\mathrm{id}_t, \ldots, \mathrm{id}_t)(f_1, \ldots, f_n)$, where the sequence of identities $\underline{\mathrm{id}_t} = (\mathrm{id}_t, \ldots, \mathrm{id}_t)$ is also of length n. So the $\mathcal{P}_{\mathbf{C}}$ action is uniquely specified by the $\mathbf{T}$-morphisms

$$\tilde{\alpha}_t := \alpha_{\underline{\mathrm{id}_t}} : \mathcal{P}(n) \otimes {A_t}^{\otimes n} \to A_t$$

for all $n \geq 0$ and all $t \in \mathbf{C}_0$, and

$$A(f) := \alpha_f(\mathbb{1}^{\mathcal{P}} \otimes \mathrm{id})\lambda^{-1} : A_c \to A_t$$

for all $f : c \to t$ in $\mathbf{C}$. Using $\underline{f} = \underline{\mathrm{id}_t}$ in the diagrams (3.2.1) we see that $(A_t, \tilde{\alpha}_t)$ is a $\mathcal{P}$-algebra for every $t \in \mathbf{C}_0$. And using $n = 1$ we see that $A : \mathbf{C} \to \mathsf{Alg}_{\mathcal{P}}(\mathbf{T})$ defines a functor.

Reversing this argument, equation (3.2.2) defines a $\mathcal{P}_{\mathbf{C}}$-algebra structure on a functor $A : \mathbf{C} \to \mathsf{Alg}_{\mathcal{P}}$, $A(c) = (A_c, \tilde{\alpha}_c)$. Using this correspondence between $\mathcal{P}_{\mathbf{C}}$-algebras and functors $\mathbf{C} \to \mathsf{Alg}_{\mathcal{P}}$ we see that an $\mathsf{Alg}_{\mathcal{P}_{\mathbf{C}}}$-morphism exactly translates to a natural transformation of functors valued in $\mathsf{Alg}_{\mathcal{P}}$. □

Now we turn our attention to the generalized Einstein causality axiom as implemented in Definition 3.1.5. As such, fix an orthogonal category $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ and a bipointed operad $\mathcal{P}^r = (\mathcal{P}; r_1, r_2 : I[2] \rightrightarrows \mathcal{P})$. For $(\underline{c}, t) \in {\mathbf{C}_0}^n \times \mathbf{C}_0$ we define

$$\perp(\underline{c}, t) := \perp \cap\, \mathbf{C}(\underline{c}, t);$$

note that $\perp(\underline{c}, t)$ can only be nonempty if $|\underline{c}| = n = 2$ because $\perp$ was defined as a set of pairs of maps. The $\mathbf{C}_0$-colored sequence of relations $R_\perp \in \mathbf{Seq}_{\mathbf{C}_0}(\mathbf{T})$ is then defined as

$$R_\perp\tbinom{t}{\underline{c}} := \perp(\underline{c}, t) \otimes I$$

which is also only nontrivial if $|\underline{c}| = n = 2$. The two morphisms $r_i : I[2] \to \mathcal{P}$ are extended to $\mathbf{Seq}_{\mathbf{C}_0}(\mathbf{T})$-morphisms

$$r_{i,\mathbf{C}} : R_\perp\tbinom{t}{\underline{c}} \to \mathcal{P}_{\mathbf{C}}\tbinom{t}{\underline{c}}$$

through

$$
\begin{array}{ccc}
I & \xrightarrow{r_i} & \mathcal{P}(2) \\
\Big\downarrow{\scriptstyle \iota_{\underline{f}}} & & \Big\downarrow{\scriptstyle \iota_{\underline{f}}} \\
R_\perp\tbinom{t}{(c_1,c_2)} & \overset{r_{i,\mathbf{C}}}{\dashrightarrow} & \mathcal{P}_{\mathbf{C}}\tbinom{t}{(c_1,c_2)}
\end{array}
$$

for all $\underline{f} = (f_1, f_2) \in \perp((c_1, c_2), t)$.

Definition 3.2.3. Let $\overline{\mathbf{C}}$ be an orthogonal category and $\mathcal{P}^r$ be a bipointed operad. We define $\mathcal{P}^r_{\overline{\mathbf{C}}}$, the *operad of field theories of type* $\mathcal{P}^r$ *on* $\overline{\mathbf{C}}$, as the coequalizer

$$F(R_\perp) \overset{r_{1,\mathbf{C}}}{\underset{r_{2,\mathbf{C}}}{\rightrightarrows}} \mathcal{P}_{\mathbf{C}} \dashrightarrow \mathcal{P}^r_{\overline{\mathbf{C}}} \tag{3.2.3}$$

in $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{T})$, where $F : \mathbf{Seq}_{\mathbf{C}_0}(\mathbf{T}) \to \mathbf{Op}_{\mathbf{C}_0}(\mathbf{T})$ is the free operad functor of Theorem 2.5.8.

This, it turns out, is the correct definition of an operad encoding Einstein causality, as evidenced by the following result.

Theorem 3.2.4. Let $\overline{\mathbf{C}}$ be an orthogonal category and $\mathcal{P}^r$ be a bipointed operad. Then there exists a canonical isomorphism

$$\mathsf{Alg}_{\mathcal{P}^r_{\overline{\mathbf{C}}}} \cong \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$$

between the category of algebras over the field theory operad $\mathcal{P}^r_{\overline{\mathbf{C}}}$ of Definition 3.2.3 and the category of field theories on $\overline{\mathbf{C}}$ of type $\mathcal{P}^r$ from Definition 3.1.5.

Proof. The operad $\mathcal{P}^r_{\overline{\mathbf{C}}}$ is defined as a coequalizer in Definition 3.2.3, so the $\mathcal{P}^r_{\overline{\mathbf{C}}}$-action on an algebra A is equivalently defined by a morphism

$$\mathcal{P}_{\mathbf{C}}\tbinom{t}{\underline{c}} \otimes A_{\underline{c}} \longrightarrow A_t$$

for all $(\underline{c}, t) \in {\mathbf{C}_0}^n \times \mathbf{C}_0$, i.e. a $\mathcal{P}_{\mathbf{C}}$-action, such that the operations picked out by $r_{1,\mathbf{C}}, r_{2,\mathbf{C}} : R_\perp\tbinom{t}{\underline{c}} \to \mathcal{P}_{\mathbf{C}}\tbinom{t}{\underline{c}}$ result in the same action on A. This means that for $(f_1, f_2) \in \perp\big((c_1, c_2), t\big)$, the diagram

$$\begin{array}{ccc} I \otimes A_{c_1} \otimes A_{c_2} & \xrightarrow{r_{2,\mathbf{C}}\otimes \mathrm{id}\otimes \mathrm{id}} & \mathcal{P}(2) \otimes A_{c_1} \otimes A_{c_2} \\ {\scriptstyle r_{1,\mathbf{C}}\otimes\mathrm{id}\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \alpha_{f_1,f_2}} \\ \mathcal{P}(2) \otimes A_{c_1} \otimes A_{c_2} & \xrightarrow{\alpha_{f_1,f_2}} & A_t \end{array}$$

commutes, where we use the notation $\alpha_{\underline{f}}$ from the proof of Lemma 3.2.2. Using the isomorphism constructed in Lemma 3.2.2, we see that this exactly translates to a functor on $\mathbf{C}$ valued in $\mathsf{Alg}_{\mathcal{P}}$ satisfying the diagram given in Definition 3.1.5, i.e. a field theory of type $\mathcal{P}^r$ on $\overline{\mathbf{C}}$. □

Remark 3.2.5. This construction gives yet another way of characterizing (algebraic) field theories. As such, the quantum field theories of Example 3.1.6, the classical field theories of Example 3.1.7 and the linear field theories of Example 3.1.6 (formalized as Heisenberg Lie algebras) are all algebras over the corresponding colored operad.

Having formalized our definition of field theory as algebras over the operad $\mathcal{P}^r_{\overline{\mathbf{C}}}$ we will now examine the relations between different kinds of field theory by varying $\overline{\mathbf{C}}$ and $\mathcal{P}^r$. Before we do this, we must first check that the assignment $(\overline{\mathbf{C}}, \mathcal{P}^r) \mapsto \mathcal{P}^r_{\overline{\mathbf{C}}}$ is appropriately functorial. Recall that the orthogonal categories were collected in the category **OrthCat**, see Definition 3.1.1, and the uncolored bipointed operads were collected in the category $\mathbf{Op}^{\mathrm{2pt}}(\mathbf{T})$, see Definition 3.1.4.

Proposition 3.2.6. The construction of $\mathcal{P}^r_{\overline{\mathbf{C}}}$ in Definition 3.2.3 naturally extends to a functor

$$\mathbf{OrthCat} \times \mathbf{Op}^{\mathrm{2pt}}(\mathbf{T}) \longrightarrow \mathbf{COp}(\mathbf{T})$$

into the category $\mathbf{COp}(\mathbf{T})$ of operads with varying colors.

PROOF. A morphism in $\mathbf{OrthCat} \times \mathbf{Op}^{2\mathrm{pt}}(\mathbf{T})$ is of the form

$$(G, \phi) : (\overline{\mathbf{C}}, \mathcal{P}^r) \longrightarrow (\overline{\mathbf{D}}, \mathcal{Q}^s)$$

where $G : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ is an orthogonal functor and $\phi : \mathcal{P}^r \to \mathcal{Q}^s$ is a morphism of bipointed uncolored operads. This defines an $\mathbf{COp}(\mathbf{T})$-morphism of the colored operads

$$\phi_G : \mathcal{P}_{\mathbf{C}} \longrightarrow \mathcal{Q}_{\mathbf{D}}$$

through the components

$$\begin{array}{ccccc}
\mathcal{P}(n) & \xrightarrow{\ \phi\ } & \mathcal{Q}(n) & & \\
\Big\downarrow{\scriptstyle \iota_{\underline{f}}} & & \Big\downarrow{\scriptstyle \iota_{G(\underline{f})}} & & \\
\mathcal{P}_{\mathbf{C}}\binom{t}{\underline{c}} & \overset{\phi_G}{\dashrightarrow} & \mathcal{Q}_{\mathbf{D}}\binom{G(t)}{G(\underline{c})} & = & G^*\mathcal{Q}_{\mathbf{D}}\binom{t}{\underline{c}}
\end{array} \ .$$

Similarly, this defines a morphism of $\mathbf{C}_0$-colored sequences

$$R_G : R_{\perp_{\mathbf{C}}} \longrightarrow U_{\mathbf{C}_0} G^* F_{\mathbf{D}_0} R_{\perp_{\mathbf{D}}}$$

through the components

$$\begin{array}{ccccc}
I & = & I & & \\
\Big\downarrow{\scriptstyle \iota_{\underline{f}}} & & \Big\downarrow{\scriptstyle \eta_{R_{\perp_{\mathbf{D}}}, Gt, G\underline{c}}\, \iota_{G(\underline{f})}} & & \\
R_{\perp_{\mathbf{C}}}\binom{t}{\underline{c}} & \overset{R_G}{\dashrightarrow} & U_{\mathbf{D}_0} F_{\mathbf{D}_0} R_{\perp_{\mathbf{D}}}\binom{G(t)}{G(\underline{c})} & = & U_{\mathbf{C}_0} G^* F_{\mathbf{D}_0} R_{\perp_{\mathbf{D}}}\binom{t}{\underline{c}}
\end{array} \ .$$

Recall Theorem 2.5.8: here we use both the free-forget adjunction $F_{\mathbf{C}_0} \dashv U_{\mathbf{C}_0}$ for $\mathbf{C}_0$-colored operads and sequences, and the free-forget adjunction $F_{\mathbf{D}_0} \dashv U_{\mathbf{D}_0}$ for $\mathbf{D}_0$-colored operads and sequences. The morphism $\eta_{R_{\perp_{\mathbf{D}}}, Gt, G\underline{c}}$ is the unit for the adjunction $F_{\mathbf{D}_0} \dashv U_{\mathbf{D}_0}$ at the component $R_{\perp_{\mathbf{D}}}\binom{G(t)}{G(\underline{c})}$.

Because G is an orthogonal functor and ϕ preserves the points, this yields a morphism of parallel pairs: the diagram

$$\begin{array}{ccccc}
F_{\mathbf{C}_0}(R_{\perp_{\mathbf{C}}}) & \overset{r_{1,\mathbf{C}}}{\underset{r_{2,\mathbf{C}}}{\rightrightarrows}} & \mathcal{P}_{\mathbf{C}} & \dashrightarrow & \mathcal{P}^r_{\overline{\mathbf{C}}} \\
\Big\downarrow{\scriptstyle R_G} & & \Big\downarrow{\scriptstyle \phi_G} & & \\
G^* F_{\mathbf{D}_0}(R_{\perp_{\mathbf{D}}}) & \overset{s_{1,\mathbf{C}}}{\underset{s_{2,\mathbf{C}}}{\rightrightarrows}} & G^*\mathcal{Q}_{\mathbf{D}} & \dashrightarrow & \operatorname{colim}\Big(G^* F_{\mathbf{D}_0}(R_{\perp_{\mathbf{D}}}) \overset{s_{1,\mathbf{C}}}{\underset{s_{2,\mathbf{C}}}{\rightrightarrows}} G^*\mathcal{Q}_{\mathbf{D}} \Big)
\end{array}$$

commutes in $\mathbf{Op}_{\mathbf{C}_0}$ when either taking r_1 and s_1, or r_2 and s_2. Because forming colimits is functorial, this gives an $\mathbf{Op}_{\mathbf{C}_0}$-morphism

$$\mathcal{P}^r_{\overline{\mathbf{C}}} \longrightarrow \operatorname{colim}\Big(G^* F_{\mathbf{D}_0}(R_{\perp_{\mathbf{D}}}) \overset{s_{1,\mathbf{C}}}{\underset{s_{2,\mathbf{C}}}{\rightrightarrows}} G^*\mathcal{Q}_{\mathbf{D}} \Big) \ . \tag{3.2.4}$$

Pulling back the defining diagram (3.2.3) of $\mathcal{Q}_{\mathbf{D}}$ along G we get

$$G^* F_{\mathbf{D}_0}(R_{\perp_{\mathbf{D}}}) \overset{s_{1,\mathbf{C}}}{\underset{s_{2,\mathbf{C}}}{\rightrightarrows}} G^*\mathcal{Q}_{\mathbf{D}} \longrightarrow G^*\mathcal{Q}^s_{\overline{\mathbf{D}}}$$

noting that $G^*(s_{i,\mathbf{D}}) = s_{i,\mathbf{C}}$. The universal property of the colimit then yields the morphism

$$\operatorname{colim}\Big(G^* F_{\mathbf{D}_0}(R_{\perp_{\mathbf{D}}}) \underset{s_{2,\mathbf{C}}}{\overset{s_{1,\mathbf{C}}}{\rightrightarrows}} G^*\mathcal{Q}_{\mathbf{D}}\Big) \longrightarrow G^*\mathcal{Q}^s_{\overline{\mathbf{D}}}\ . \tag{3.2.5}$$

Composing the morphisms (3.2.4) and (3.2.5) results in the desired operad morphism

$$\phi_G : \mathcal{P}^r_{\overline{\mathbf{C}}} \longrightarrow G^*\mathcal{Q}^s_{\overline{\mathbf{D}}}$$

which is the operadic part of the $\mathbf{COp}(\mathbf{T})$-morphism

$$(G, \phi_G) : (\mathbf{C}_0, \mathcal{P}^r_{\overline{\mathbf{C}}}) \longrightarrow (\mathbf{D}_0, \mathcal{Q}^s_{\overline{\mathbf{D}}})$$

which we also denote by ϕ_G. □

3.3. Field theory adjunctions: universal constructions

With the operadic definition of field theory and the functoriality of $\mathcal{P}^r_{\overline{\mathbf{C}}}$ we can now reap the rewards, constructing adjunctions between field theories of different types. Proposition 3.2.6 tells us that for any morphism $(F, \phi) : (\overline{\mathbf{C}}, \mathcal{P}^r) \to (\overline{\mathbf{D}}, \mathcal{Q}^s)$ in $\mathbf{OrthCat} \times \mathbf{Op}^{2\mathrm{pt}}(\mathbf{T})$ we get a morphism of colored operads

$$\phi_F : \mathcal{P}^r_{\overline{\mathbf{C}}} \longrightarrow \mathcal{Q}^s_{\overline{\mathbf{D}}}$$

in $\mathbf{COp}(\mathbf{T})$. Using Theorems 2.5.22 and 3.2.4 this yields an adjunction

$$(\phi_F)_! : \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) \rightleftarrows \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{Q}^s) : (\phi_F)^* \tag{3.3.1}$$

of categories of field theories.

The morphism ϕ_F has two ingredients, F and ϕ, and we have

$$\phi_F = \mathrm{id}_F \circ \phi_{\mathrm{id}} = \phi_{\mathrm{id}} \circ \mathrm{id}_F.$$

As such, a pair (F, ϕ) really determines a square of adjunctions

$$\begin{array}{ccc} \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s) & \underset{(\mathrm{id}_F)^*}{\overset{(\mathrm{id}_F)_!}{\rightleftarrows}} & \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{Q}^s) \\ {\scriptstyle (\phi_{\mathrm{id}})_!}\uparrow\downarrow{\scriptstyle (\phi_{\mathrm{id}})^*} & & {\scriptstyle (\phi_{\mathrm{id}})_!}\uparrow\downarrow{\scriptstyle (\phi_{\mathrm{id}})^*} \\ \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) & \underset{(\mathrm{id}_F)^*}{\overset{(\mathrm{id}_F)_!}{\rightleftarrows}} & \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r). \end{array}$$

where the square formed by the right adjoints (the pullbacks $(\mathrm{id}_F)^*$ and $(\phi_{\mathrm{id}})^*$) commutes and, because left adjoints are unique up to unique natural isomorphism, the square formed by the left adjoints $(\mathrm{id}_F)_!$ and $(\phi_{\mathrm{id}})_!$ commutes up to natural isomorphism. By considering the maps separately we can find a more explicit description of the pullbacks in this square of adjunctions.

An orthogonal functor $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ induces a pullback on the categories of functors,

$$F^* = (-) \circ F : \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{D}} \longrightarrow \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}.$$

Considering the definition of id_F in Proposition 3.2.6 (for $\mathrm{id} : \mathcal{P}^r \to \mathcal{P}^r$) and the identification of $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ with algebras over $\mathcal{P}^r_{\overline{\mathbf{C}}}$ in Theorem 3.2.4 we see that $(\mathrm{id}_F)^*$ is the restriction of this pullback F^* to the full subcategory $\mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r)$, so we write

$$F^* := (\mathrm{id}_F)^* : \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r) \longrightarrow \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r). \tag{3.3.2}$$

As a special case, if for $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$ we consider the orthogonal functor

$$F := \mathrm{id} : (\mathbf{C}, \emptyset) \longrightarrow (\mathbf{C}, \perp),$$

we see that

$$F^* : \mathbf{FT}\big((\mathbf{C}, \perp), \mathcal{P}^r\big) \longrightarrow \mathbf{FT}\big((\mathbf{C}, \emptyset), \mathcal{P}^r\big) = \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}} \tag{3.3.3}$$

is the inclusion functor of the full subcategory $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ into the functor category $\mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$, see Definition 3.1.5. The left adjoint $F_!$ then establishes $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ as a reflective full subcategory of $\mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$.

A morphism $\phi : \mathcal{P}^r \to \mathcal{Q}^s$ in $\mathbf{Op}^{2\mathrm{pt}}(\mathbf{T})$ induces a pullback of algebras over uncolored operads

$$\phi^* : \mathsf{Alg}_{\mathcal{Q}} \longrightarrow \mathsf{Alg}_{\mathcal{P}}$$

as in Definition 2.5.21, which can be used to push forward functors into algebra categories:

$$(\phi^*)_* = \phi^* \circ (-) : \mathsf{Alg}_{\mathcal{Q}}{}^{\mathbf{C}} \longrightarrow \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}.$$

And again, with the definition of ϕ_{id} (now for $\mathrm{id} : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$) and the identification of field theory categories and algebras over field theory operads, we see that $(\phi_{\mathrm{id}})^*$ is the restriction of $(\phi^*)_*$ to the full subcategory $\mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r)$, so we write

$$(\phi^*)_* := (\phi_{\mathrm{id}})^* : \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s) \longrightarrow \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r).$$

Combining these two findings, we find the following.

LEMMA 3.3.1. The right adjoint $(\phi_F)^*$ in the adjunction (3.3.1) has an explicit description as either of the two compositions in the commutative diagram

$$\begin{array}{ccc} \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s) & \xleftarrow{\;F^*\;} & \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{Q}^s) \\ {\scriptstyle (\phi^*)_*}\downarrow & \swarrow{\scriptstyle (\phi_F)^*} & \downarrow{\scriptstyle (\phi^*)_*} \\ \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) & \xleftarrow{\;F^*\;} & \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r). \end{array}$$

So we will find two different types of adjunctions: adjunctions that arise from an orthogonal functor $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ and adjunctions arising from a morphism of bipointed operads $\phi : \mathcal{P}^r \to \mathcal{Q}^s$, and we will study adjunctions of both types. F and ϕ determine a square of adjunctions

$$\begin{array}{ccc} \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s) & \underset{F^*}{\overset{F_!}{\rightleftarrows}} & \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{Q}^s) \\ {\scriptstyle (\phi^*)^!}\uparrow\downarrow{\scriptstyle (\phi^*)_*} & & {\scriptstyle (\phi^*)^!}\uparrow\downarrow{\scriptstyle (\phi^*)_*} \\ \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) & \underset{F^*}{\overset{F_!}{\rightleftarrows}} & \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r). \end{array} \tag{3.3.4}$$

where we denote the left adjoints of F^* and $(\phi^*)_*$ by $F_!$ and $(\phi^*)^!$, respectively. Note that these are the left adjoints for the field theory categories: both $F^* : \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{D}} \to \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$ and $(\phi^*)_* : \mathsf{Alg}_{\mathcal{Q}}{}^{\mathbf{C}} \longrightarrow \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$ have left adjoints, but the field theory left adjoints are not necessarily the restictions of these left adjoints. As mentioned above, the square formed by the right adjoints F^* and $(\phi^*)_*$ commutes and the square of the left adjoints $F_!$ and $(\phi^*)^!$ commutes up to unique natural isomorphism, but the other two squares do not necessarily commute. So we will also study the interplay of left and right adjoints in the above diagram.

3.3.1. Full orthogonal subcategories: descent. The first type of adjunction we encounter is one arising from full orthogonal subcategories $\overline{\mathbf{C}} \subseteq \overline{\mathbf{D}}$. Recall Definition 2.1.20 of a full subcategory $\mathbf{C} \subseteq \mathbf{D}$: this is a category that contains only part of the objects of $\mathbf{D}$, but all of the morphisms in $\mathbf{D}$ between objects in $\mathbf{C}$ are also morphisms in $\mathbf{C}$. Our guiding example is the category $\overline{\mathbf{Loc}_\diamond} \subseteq \overline{\mathbf{Loc}}$. A natural question to ask about a field theory $\mathfrak{A} \in \mathbf{FT}(\overline{\mathbf{Loc}}, \mathcal{P}^r)$ is whether or not it is determined by its restriction to $\overline{\mathbf{Loc}_\diamond}$. This is the issue of *descent* mentioned in Remark 2.3.3.

First we define full orthogonal subcategories.

Definition 3.3.2. Let $\overline{\mathbf{D}} = (\mathbf{D}, \perp_{\mathbf{D}})$ be an orthogonal category. A full orthogonal subcategory $\overline{\mathbf{C}} \subseteq \overline{\mathbf{D}}$ is a full subcategory $\mathbf{C} \subseteq \mathbf{D}$ with the pullback orthogonality relation: $f_1 \perp_{\mathbf{C}} f_2$ if and only if $f_1 \perp_{\mathbf{D}} f_2$.

For a full orthogonal subcategory $\overline{\mathbf{C}} \subseteq \overline{\mathbf{D}}$, the embedding functor $j : \mathbf{C} \to \mathbf{D}$ defines an orthogonal functor $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$.

Proposition 3.3.3. Let $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ be a full orthogonal subcategory and $\mathcal{P}^r$ be a bipointed uncolored operad. Then the field theory adjunction

$$j_! : \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) \rightleftarrows \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r) : j^* \tag{3.3.5}$$

exhibits $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ as a full coreflective subcategory of $\mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r)$: the unit $\eta : \mathrm{id} \to j^* j_!$ is a natural isomorphism.

Proof. This is proven in Proposition 4.6 in [**BSW20**] for the case $\mathcal{P} = \mathsf{As}$. The proof for the general case is completely similar: the image of a $\mathcal{P}^r_{\overline{\mathbf{C}}}$-algebra $(\mathfrak{A}, \alpha)$ under the left adjoint $j_!$ is given by the colimit

$$j_!(\mathfrak{A}) = \mathrm{colim}\Big(\ \mathcal{P}^r_{\overline{\mathbf{D}}} \circ j_!(\mathcal{P}^r_{\overline{\mathbf{C}}} \circ \mathfrak{A}) \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \mathcal{P}^r_{\overline{\mathbf{D}}} \circ j_!(\mathfrak{A})\ \Big) \tag{3.3.6}$$

where $\mathcal{P}^r_{\overline{\mathbf{C}}} \circ \mathfrak{A}$ denotes the free algebra (see Theorem 2.5.20) and the $j_!$ used in the colimit is the left adjoint (2.5.7) to the pullback $j^* : \mathbf{T}^{\mathbf{D}_0} \to \mathbf{T}^{\mathbf{C}_0}$ of categories of colored objects (Definition 2.5.15). The map ∂_0 is defined using the $\mathcal{P}^r_{\overline{\mathbf{C}}}$-action on $\mathfrak{A}$, while ∂_1 is constructed from the units and counits of both the adjunction $j_! \dashv j^*$ of categories of colored objects, and of the adjunction $F \dashv U$ of Theorem 2.5.8, which involves the operadic composition of the factor $\mathcal{P}^r_{\overline{\mathbf{D}}}$ with $j(\mathcal{P}^r_{\overline{\mathbf{C}}})$.

At this point, one uses the fact that j is a full orthogonal inclusion, which implies that $j^*(\mathcal{P}^r_{\overline{\mathbf{D}}}) = \mathcal{P}^r_{\overline{\mathbf{C}}}$ and the fact that the left adjoint $j_!$ (2.5.7) has the particularly easy form

$$j_! X_d = \begin{cases} X_d & \text{if } d \in \mathbf{C} \\ \emptyset & \text{if } d \notin \mathbf{C}\ . \end{cases}$$

In turn, this is used to show that after applying the pullback j^*, the expression (3.3.6) simplifies to

$$j^* j_!(\mathfrak{A}) = \mathrm{colim}\Big(\ \mathcal{P}^r_{\overline{\mathbf{C}}} \circ (\mathcal{P}^r_{\overline{\mathbf{C}}} \circ \mathfrak{A}) \underset{\gamma \circ \mathfrak{A}}{\overset{\mathcal{P}^r_{\overline{\mathbf{C}}} \circ \alpha}{\rightrightarrows}} \mathcal{P}^r_{\overline{\mathbf{C}}} \circ (\mathfrak{A})\ \Big)\ .$$

This coequalizer diagram is part of the diagram expressing associativity of the $\mathcal{P}^r_{\overline{\mathbf{C}}}$-action on $\mathfrak{A}$, and by Lemma 4.3.3 in [**Bor94**] the colimit of the diagram is naturally isomorphic to $\mathfrak{A}$. By Lemma 1.3 in [**JM89**] this proves the result. ☐

EXAMPLE 3.3.4. As mentioned in the introduction, one example of a full orthogonal subcategory is the subcategory $\overline{\mathbf{Loc}_\diamond} \subseteq \overline{\mathbf{Loc}}$ of globally hyperbolic spacetimes that are diffeomorphic to $\mathbb{R}^n$, see Definition 2.2.12. The pullback orthogonality relation on $\overline{\mathbf{Loc}_\diamond}$ is the same as the one defined in Definition 3.1.2, and the functor $j : \overline{\mathbf{Loc}_\diamond} \to \overline{\mathbf{Loc}}$ gives the adjuntion

$$j_! : \mathbf{FT}(\overline{\mathbf{Loc}_\diamond}, \mathcal{P}^r) \rightleftarrows \mathbf{FT}(\overline{\mathbf{Loc}}, \mathcal{P}^r) : j^*$$

between field theories on all of $\overline{\mathbf{Loc}}$ on the right and field theories only defined on $\overline{\mathbf{Loc}_\diamond}$. We saw in Lemma 3.3.1 that the right adjoint j^* is the restriction of the pullback $j^* : \mathsf{Alg}_{\mathcal{P}}{}^{\overline{\mathbf{Loc}}} \to \mathsf{Alg}_{\mathcal{P}}{}^{\overline{\mathbf{Loc}_\diamond}}$, which takes a field theory defined on all of $\overline{\mathbf{Loc}}$ and restricts it to $\overline{\mathbf{Loc}_\diamond}$. The left adjoint $j_!$ is more interesting: it takes a theory defined on $\overline{\mathbf{Loc}_\diamond}$ and extends it to all globally hyperbolic spacetimes of $\overline{\mathbf{Loc}}$. In [**BSW20**] it was shown that $j_!$ is a generalization and refinement of Fredenhagen's universal algebra construction, see [**Fre90**], [**Fre93**], [**FRS92**] and [**Lan14**].

REMARK 3.3.5. Let $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ be a full orthogonal subcategory. The right adjoint j^* in the adjunction (3.3.5) is a pullback, so it is the functor restricting field theories on $\overline{\mathbf{D}}$ to $\overline{\mathbf{C}}$, while as per the example, we interpret the left adjoint $j_!$ as a universal extension functor. Proposition 3.3.3 then tells us that for a theory $\mathfrak{B} \in \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ on $\overline{\mathbf{C}}$, the restriction of the extension of $\mathfrak{B}$, $j^* j_!(\mathfrak{B})$, is isomorphic to $\mathfrak{B}$ by the unit $\eta_{\mathfrak{B}}$ of the adjunction. In other words, $j_!$ extends field theories on $\overline{\mathbf{C}}$ to $\overline{\mathbf{D}}$ without changing their values on the category $\overline{\mathbf{C}}$ where they are defined.

This also functions as a sanity check for $j_!$: in general, there is no guarantee that $j_!(\mathfrak{B})$ is non-trivial on any $d \in \overline{\mathbf{D}}$. For example, field theories that have local solutions but not always global solutions on topologically non-trivial spacetimes might provide examples of theories that are non-trivial on $\overline{\mathbf{Loc}_\diamond}$ but trivial on other parts of $\overline{\mathbf{Loc}}$. And in fact, Fredenhagen's universal algebra may indeed be trivial in some cases, see e.g. [**RV12**] and [**Lan14**]. However, Proposition 3.3.3 ensures that if $\mathfrak{B}$ is non-trivial, so is $j_!(\mathfrak{B})$, since its restriction $j^* j_!(\mathfrak{B})$ is.

The adjunctions given by full orthogonal subcategories $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ allow us to formalize a local-to-global condition, which we will call a *descent* condition. A theory $\mathfrak{B} \in \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ on $\overline{\mathbf{C}}$ will always yield a theory $j_!(\mathfrak{B})$ on $\overline{\mathbf{D}}$. This is a theory that is completely determined by its values on $\overline{\mathbf{C}}$ and through the restriction functor j^*, $j_!(\mathfrak{B})$ can again be made into a theory on $\overline{\mathbf{C}}$. This naturally leads one to pose the reverse question: given a theory $\mathfrak{A} \in \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r)$, is it determined by its values on $\overline{\mathbf{C}}$? In other words, is $\mathfrak{A} \cong j_! j^*(\mathfrak{A})$?

DEFINITION 3.3.6. Let $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ be a full orthogonal subcategory. A field theory $\mathfrak{A} \in \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r)$ on $\overline{\mathbf{D}}$ is called *j-local* if the component $\epsilon_{\mathfrak{A}} : j_! j^*(\mathfrak{A}) \to \mathfrak{A}$ of the counit is an isomorphism. We write $\mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r)^{j-\mathrm{loc}} \subseteq \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r)$ for the full subcategory of j-local field theories.

COROLLARY 3.3.7. If we restrict to $\mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r)^{j-\mathrm{loc}} \subseteq \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r)$ on the right side of the adjunction (3.3.5), we get an adjoint equivalence

$$j_! : \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) \underset{\sim}{\rightleftarrows} \mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r)^{j-\mathrm{loc}} : j^* \ .$$

PROOF. This is immediate: Proposition 3.3.3 tells us that the unit of the adjunction is an isomorphism, while the counit is an isomorphism by Definition 3.3.6. $\square$

REMARK 3.3.8. In [**Lan14**] a similar condition was investigated, restricting to connected spacetimes and using the categorical left Kan extension along j rather than the operadic left Kan extension $j_!$. It was shown there that Klein-Gordon theory is j-local in this sense.

3.3.2. Orthogonal localization: time-slice. We will now consider another adjunction arising from varying the orthogonal category $\overline{\mathbf{C}}$, focusing on localizations $L : \overline{\mathbf{C}} \to \overline{\mathbf{C}[W^{-1}]}$. In a loose sense this discussion is dual to the previous one: where in Section 3.3.1 we considered subcategories with a subset of objects but all possible morphisms, we now keep the set of objects equal, while adding morphisms. The example we keep in mind is $L : \overline{\mathbf{Loc}} \to \overline{\mathbf{Loc}[\mathrm{Cauchy}^{-1}]}$ where Cauchy is the set of Cauchy morphisms in **Loc**, see Definition 2.2.9. This is of course related to the time-slice property of field theories. Recall Definition 2.4.15.

DEFINITION 3.3.9. Let $\overline{\mathbf{C}} = (\mathbf{C}, \perp_{\mathbf{C}})$ be an orthogonal category with a subset $W \subseteq \mathbf{Mor}\,\mathbf{C}$ of morphisms. An *orthogonal localization* $\overline{\mathbf{C}[W^{-1}]}$ is given by a localization $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$ of $\mathbf{C}$ at W, together with the pushforward orthogonality relation $\perp_{\mathbf{C}[W^{-1}]} := L_*(\perp_{\mathbf{C}})$ of $\perp_{\mathbf{C}}$ along L, see Lemma 3.1.3.

If $L : \mathbf{C} \to \mathbf{C}[W^{-1}]$ is a localization of $\mathbf{C}$ at W, $L : \overline{\mathbf{C}} \to \overline{\mathbf{C}[W^{-1}]}$ is an orthogonal functor, and we can study the corresponding field theory adjunction.

PROPOSITION 3.3.10. Let $L : \overline{\mathbf{C}} \to \overline{\mathbf{C}[W^{-1}]}$ be an orthogonal localization, and $\mathcal{P}^r$ be a bipointed uncolored operad. Then the field theory adjunction

$$L_! : \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) \rightleftarrows \mathbf{FT}(\overline{\mathbf{C}[W^{-1}]}, \mathcal{P}^r) : L^* \tag{3.3.7}$$

exhibits $\mathbf{FT}(\overline{\mathbf{C}[W^{-1}]}, \mathcal{P}^r)$ as a full reflective subcategory of $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$: the counit $\epsilon : L_! L^* \to \mathrm{id}$ is a natural isomorphism.

PROOF. We saw in (3.3.2) that the pullback $L^* : \mathbf{FT}(\overline{\mathbf{C}[W^{-1}]}, \mathcal{P}^r) \to \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ of field theories is given by the restriction of the pullback $L^* : \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}[W^{-1}]} \to \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$ of functors. By Definition 2.4.15 of localization, this pullback is fully faithful. Because $\mathbf{FT}(\overline{\mathbf{D}}, \mathcal{P}^r) \subseteq \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{D}}$ is a reflective full subcategory inclusion for both $\overline{\mathbf{D}} = \overline{\mathbf{C}}$ and $\overline{\mathbf{C}[W^{-1}]}$, see (3.3.3), the restriction to field theory categories is also fully faithful. ☐

Orthogonal localization is directly related to the time-slice axiom. We will use the following generalization.

DEFINITION 3.3.11. A field theory $\mathfrak{A} \in \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ is called *W-constant* if for any $f : c \to c'$ in W, $\mathfrak{A}(f) : \mathfrak{A}(c) \to \mathfrak{A}(c')$ is an isomorphism. We write $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)^{W-\mathrm{const}} \subseteq \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ for the full subcategory of W-constant field theories.

PROPOSITION 3.3.12. The adjunction (3.3.7) restricts to an adjoint equivalence

$$L_! : \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)^{W-\mathrm{const}} \underset{\sim}{\rightleftarrows} \mathbf{FT}(\overline{\mathbf{C}[W^{-1}]}, \mathcal{P}^r) : L^* \ .$$

PROOF. Recall from Lemma 2.1.15 that an adjunction is an adjoint equivalence when either functor is fully faithful and essentially surjective. Proposition 3.3.10 proves that L^* is fully faithful: because the counit ϵ is an isomorphism, L^* is a bijection on **Hom**-sets. So it remains to prove that L^* is essentially surjective on $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)^{W-\mathrm{const}}$.

First, note that the image of L^* lies in $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)^{W-\mathrm{const}}$: if $\mathfrak{B} \in \mathbf{FT}(\overline{\mathbf{C}[W^{-1}]}, \mathcal{P}^r)$, then for any $f \in W$, Lf is an isomorphism in $\mathbf{C}[W^{-1}]$ by the definition of localization. So $\mathfrak{B}(Lf)$ is an isomorphism, and therefore $L^*(\mathfrak{B})(f) = \mathfrak{B}(Lf)$ is, too, so $L^*(\mathfrak{B})$ is W-constant.

For essential surjectivity, let $\mathfrak{A} \in \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)^{W-\text{const}}$. The underlying functor $\mathfrak{A} : \mathbf{C} \to \mathsf{Alg}_{\mathcal{P}}$ will send all $f \in W$ to isomorphisms, so by definition of localization, it factors through $\mathbf{C}[W^{-1}]$: there exists a $\mathfrak{B} : \mathbf{C}[W^{-1}] \to \mathsf{Alg}_{\mathcal{P}}$ such that $\mathfrak{A}$ is naturally isomorphic to $\mathfrak{B}L = L^*(\mathfrak{B})$. By Definition 3.3.9 of orthogonal localization (and the definition of the pushforward orthogonality relation, see Lemma 3.1.3), we see that in fact $\mathfrak{B} \in \mathbf{FT}(\overline{\mathbf{C}[W^{-1}]}, \mathcal{P}^r)$: the diagram corresponding to diagram (3.1.1) that $\mathfrak{B}$ has to satisfy to be a field theory commutes because $\mathfrak{A}$ is a field theory and $\mathfrak{A}$ and $L^*(\mathfrak{B})$ are naturally isomorphic in $\mathsf{Alg}_{\mathcal{P}}$. □

Example 3.3.13. For our purposes, the central example of an othogonal localization is $L : \overline{\mathbf{Loc}} \to \overline{\mathbf{Loc}[\text{Cauchy}^{-1}]}$, the localization of $\overline{\mathbf{Loc}}$ at all Cauchy morphisms. As noted in Section 2.4.2, one can prove that this localization exists, though it is very hard to find a workable model for it. Sidestepping this practical issue for the moment, such a localization L gives the adjunction

$$L_! : \mathbf{FT}(\overline{\mathbf{Loc}}, \mathcal{P}^r) \rightleftarrows \mathbf{FT}(\overline{\mathbf{Loc}[\text{Cauchy}^{-1}]}, \mathcal{P}^r) : L^*$$

between the category of field theories without any condition for Cauchy morphisms on the left and the category $\mathbf{FT}(\overline{\mathbf{Loc}[\text{Cauchy}^{-1}]}, \mathcal{P}^r)$ on the right, which Proposition 3.3.12 tells us is equivalent to the category of field theories satisfying the time-slice axiom. A field theory $\mathfrak{A} \in \mathbf{FT}(\overline{\mathbf{Loc}}, \mathcal{P}^r)$ then satisfies the time-slice axiom if and only if the corresponding component of the unit $\eta_{\mathfrak{A}} : \mathfrak{A} \to L^* L_!(\mathfrak{A})$ is an isomorphism.

The right adjoint L^* is a pullback functor: it takes a theory $\mathfrak{B} \in \mathbf{FT}(\overline{\mathbf{Loc}[\text{Cauchy}^{-1}]}, \mathcal{P}^r)$ for which the time-slice axiom holds, and interprets it as a more general field theory $L^*(\mathfrak{B}) \in \mathbf{FT}(\overline{\mathbf{Loc}}, \mathcal{P}^r)$, forgetting that the time slice axiom holds for $L^*(\mathfrak{B})$. The left adjoint can then be viewed as a *time-slicification* functor: it makes a theory $\mathfrak{A} \in \mathbf{FT}(\overline{\mathbf{Loc}}, \mathcal{P}^r)$ that not necessarily satisfies the time-slice axiom into a theory $L_!(\mathfrak{A}) \in \mathbf{FT}(\overline{\mathbf{Loc}[\text{Cauchy}^{-1}]}, \mathcal{P}^r)$ that does. Proposition 3.3.10 then tells us that this process of time-slicification is not too violent: if $\mathfrak{B} \in \mathbf{FT}(\overline{\mathbf{Loc}[\text{Cauchy}^{-1}]}, \mathcal{P}^r)$ is a theory for which the time-slice axiom holds, forgetting this condition and then applying the time-slicification functor $L_!$ results in a theory $L_! L^*(\mathfrak{B})$ that is isomorphic with $\mathfrak{B}$ through the counit.

To give a concrete example of time-slicification and show that it does not in general lead to trivial field theories, first recall the treatment of Klein-Gordon theory from Section 2.3.1. There, we defined

$$\mathfrak{L}^{KG}(M) := \mathfrak{L}^{KG}_{\text{off-shell}}(M) \big/ \big\{ F_\psi \in \mathfrak{L}^{KG}_{\text{off-shell}}(M) | F_\psi \big|_{\mathfrak{Sol}^{KG}} = 0 \big\}$$

implementing the time-slice axiom for the linear observables by forming a quotient. Similarly, say that $\mathfrak{B} = \mathfrak{A}/\mathcal{I} \in \mathbf{FT}(\overline{\mathbf{Loc}[\text{Cauchy}^{-1}]}, \mathcal{P}^r)$ is a field theory satisfying the time-slice axiom, which is obtained from an off-shell theory $\mathfrak{A}$ by dividing out an ideal $\mathcal{I}$ that implements the equation of motion. What we mean by this is that the theory $L^*(\mathfrak{B}) \in \mathbf{FT}(\overline{\mathbf{Loc}}, \mathcal{P}^r)$ is given by the coequalizer

$$F(\mathcal{I}) \rightrightarrows \mathfrak{A} \dashrightarrow L^*(\mathfrak{B})$$

in $\mathbf{FT}(\overline{\mathbf{Loc}}, \mathcal{P}^r)$ where $F(\mathcal{I})$ is the free field theory generated by $\mathcal{I}$ and the two parallel maps are the adjuncts of the inclusion $\mathcal{I} \hookrightarrow U(\mathfrak{A})$ and the zero map $\mathcal{I} \to U(\mathfrak{A})$ obtained through the adjunction $F \dashv U$ (Theorem 2.5.8). Now apply $L_!$ to the above diagram. $L_!$ is a left adjoint, so it preserves colimits and we get

$$L_!F(\mathcal{I}) \rightrightarrows L_!(\mathfrak{A}) \dashrightarrow L_!L^*(\mathfrak{B}) \xrightarrow[\cong]{\epsilon_{\mathfrak{B}}} \mathfrak{B}$$

as a coequalizer in $\mathbf{FT}(\overline{\mathbf{Loc}[\mathrm{Cauchy}^{-1}]}, \mathcal{P}^r)$. Proposition 3.3.10 provides the last arrow in the diagram. We see that the field theory $\mathfrak{B} = \mathfrak{A}/\mathcal{I}$ equivalently may be presented as a quotient of the time-slicification $L_!(\mathfrak{A})$. Of course, this is true for any such $\mathfrak{B}$, so we find that any on-shell quotient of the off-shell theory $\mathfrak{A}$ can be also be obtained as a quotient of $L_!\mathfrak{A}$. So assuming the theory $\mathfrak{A}$ allows for on-shell quotients, $\mathfrak{L}_!(\mathfrak{A})$ is a nontrivial theory that in particular contains the information for any such possible quotient.

3.3.3. Change of operad. We will now study the field theory adjunction

$$(\phi^*)^! : \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) \rightleftarrows \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s) : (\phi^*)_* \tag{3.3.8}$$

resulting from a change of operad $\phi : \mathcal{P}^r \to \mathcal{Q}^s$ while keeping the orthogonal category $\overline{\mathbf{C}}$ fixed. Specifically, we will see what happens to j-locality (see Definition 3.3.6) and W-constancy (see Definition 3.3.11) under these kinds of adjunctions. In the next Section, we will consider a specific case, that of linear quantization.

The general result on preserving j-locality and W-constancy is the following.

Proposition 3.3.14. Let $\phi : \mathcal{P}^r \to \mathcal{Q}^s$ be a morphism of bipointed uncolored operads and $\overline{\mathbf{C}}$ an orthogonal category, with corresponding adjunction of field theories (3.3.8).

(1) If $j : \overline{\mathbf{D}} \to \overline{\mathbf{C}}$ is a full orthogonal subcategory, the left adjoint $(\phi^*)^!$ in the adjunction (3.3.8) preserves j-local field theories. (Note the reversal of $\overline{\mathbf{C}}$ and $\overline{\mathbf{D}}$ as compared with Section 3.3.1.)
(2) If $W \subseteq \mathbf{Mor}\ \mathbf{C}$ is a subset of morphisms in $\mathbf{C}$, the right adjoint $(\phi^*)_*$ in the adjunction (3.3.8) preserves W-constant field theories.

Proof. Item (1): Let $\mathfrak{A} \in \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ be j-local, so the counit $\epsilon_{\mathfrak{A}} : j_!j^*(\mathfrak{A}) \to \mathfrak{A}$ is an isomorphism. Then we have the commutative diagram

$$\begin{array}{ccc}
j_!j^*(\phi^*)^!(\mathfrak{A}) & \xrightarrow{\epsilon_{(\phi^*)^!(\mathfrak{A})}} & (\phi^*)^!(\mathfrak{A}) \\
{\scriptstyle j_!j^*(\phi^*)^!\epsilon_{\mathfrak{A}}}\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle(\phi^*)^!\epsilon_{\mathfrak{A}}} \\
j_!j^*(\phi^*)^!j_!j^*(\mathfrak{A}) & \xrightarrow{\epsilon_{(\phi^*)^!j_!j^*(\mathfrak{A})}} & (\phi^*)^!j_!j^*(\mathfrak{A}) \\
\uparrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow \\
j_!j^*j_!(\phi^*)^!j^*(\mathfrak{A}) & \xrightarrow{\epsilon_{j_!(\phi^*)^!j^*(\mathfrak{A})}} & j_!(\phi^*)^!j^*(\mathfrak{A}) \\
{\scriptstyle j_!\eta_{(\phi^*)^!j^*(\mathfrak{A})}}\uparrow{\scriptstyle\cong} & ⫽ & \\
j_!(\phi^*)^!j^*(\mathfrak{A}) & &
\end{array}$$

for the counit $\epsilon_{(\phi^*)^!(\mathfrak{A})}$ of $(\phi^*)^!(\mathfrak{A})$. The top diagram is the diagram (2.1.1) for the natural transformation $\epsilon : j_!j^* \to \mathrm{id}$ at the morphism $(\phi^*)^!\epsilon_{\mathfrak{A}}$, so it commutes, and $j_!j^*(\phi^*)^!\epsilon_{\mathfrak{A}}$ and $(\phi^*)^!\epsilon_{\mathfrak{A}}$ are isomorphisms because $\epsilon_{\mathfrak{A}}$ is. The middle diagram expresses the fact that the left adjoints $j_!$ and $(\phi^*)^!$ commute up to unique natural isomorphism, which was mentioned after equation (3.3.4). The bottom diagram is the triangle identity (2.1.4) for the adjunction $j_! \dashv j^*$

at $(\phi^*)^! j^* \mathfrak{A}$, and because of Proposition 3.3.3 we know that the unit η is an isomorphism. The diagram proves that $\epsilon_{(\phi^*)^!(\mathfrak{A})}$ is an isomorphism, so $(\phi^*)^!(\mathfrak{A})$ is j-local.

Item (2): Let $\mathfrak{A} \in \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s)$ be W-constant: for any $f : c \to c'$ in W, $\mathfrak{A}(f) : \mathfrak{A}(c) \to \mathfrak{A}(c')$ is an isomorphism. Then $(\phi^*)_*(\mathfrak{A})(f) = \phi^*(\mathfrak{A}(f))$ is also an isomorphism, since ϕ^* is a functor, so $(\phi^*)_*(\mathfrak{A})$ is W-constant. $\square$

This is half of the answer to the question "does the adjunction (3.3.8) preserve j-locality and W-constancy?" Note the symmetry of this result: the left adjoint $(\phi^*)^!$ preserves j-local theories, which form a coreflective subcategory, while the right adjoint $(\phi^*)_*$ preserves W-constant theories, which form a reflective subcategory. In fact, we could draw a diagram dual to the one on j-locality to prove preservation of W-constancy: the essential ingredient is the fact that the right adjoints in the square (3.3.4) commute and the left adjoints commute up to unique natural isomorphism. The converse statements are not always true, but we do find that the left adjoint preserves W-locality in a special case, as follows.

Recall that $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^r)$ is defined as a full subcategory of the functor category $\mathsf{Alg}_{\mathcal{Q}}{}^{\mathbf{C}}$ and that the right adjoint $(\phi^*)_*$ on $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^r)$ is the restriction of $(\phi^*)_*$ from the functor category $\mathsf{Alg}_{\mathcal{Q}}{}^{\mathbf{C}}$ to the field theory category $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^r)$. In turn, the right adjoint $(\phi^*)_*$ on $\mathsf{Alg}_{\mathcal{Q}}{}^{\mathbf{C}}$ is the pushforward along the right adjoint ϕ^* in the adjunction

$$\phi_! : \mathsf{Alg}_{\mathcal{P}} \rightleftarrows \mathsf{Alg}_{\mathcal{Q}} : \phi^*$$

of algebras over uncolored operads. In contrast, the left adjoint $(\phi^*)^!$ on $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ is in general not related to the functor category. But we do have a candidate coming from this adjunction of algebras over uncolored operads: the pushforward along $\phi_!$, i.e. the left adjoint in the adjunction

$$(\phi_!)_* : \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}} \rightleftarrows \mathsf{Alg}_{\mathcal{Q}}{}^{\mathbf{C}} : (\phi^*)_* \ . \tag{3.3.9}$$

Proposition 3.3.15. Let $\phi : \mathcal{P}^r \to \mathcal{Q}^s$ be a morphism of bipointed uncolored operads, $\overline{\mathbf{C}}$ be an orthogonal category and $W \subseteq \mathbf{Mor}\ \mathbf{C}$ be a subset. If the left adjoint $(\phi^*)^!$ in the adjunction (3.3.8) is naturally isomorphic to the restriction of $(\phi_!)_*$ in (3.3.9) from $\mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$ to $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$, $(\phi^*)^!$ preserves W-constant field theories.

Proof. As in the proof of item (2) in Proposition 3.3.14, this is immediate: since $(\phi_!)_*$ is the pushforward along the functor $\phi_!$, if $\mathfrak{A}(f)$ is an isomorphism for $f \in W$, so is $(\phi_!)_*(\mathfrak{A})(f) = \phi_!(\mathfrak{A}(f))$. $\square$

Let us investigate the condition of $(\phi^*)^!$ being naturally isomorphic to the restriction of $(\phi_!)_*$. The field theory operad $\mathcal{P}^r_{\overline{\mathbf{C}}}$ was defined as a coequalizer in Definition 3.2.3. Say that $\pi : \mathcal{P}_{\mathbf{C}} \to \mathcal{P}^r_{\overline{\mathbf{C}}}$ is the corresponding morphism in $\mathbf{Op}_{\mathbf{C}_0}(\mathbf{T})$. Then π defines an adjunction of algebra categories

$$\pi_! : \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}} \rightleftarrows \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) : \pi^*$$

where we use Lemma 3.2.2 and Theorem 3.2.4 to rewrite both categories of algebras. One immediately sees that π^* is the inclusion functor of $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ into $\mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$. Moreover, this adjunction establishes $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ as a full reflective subcategory of $\mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}}$, since π^* is fully faithful.

For a morphism of bipointed uncolored operads $\phi : \mathcal{P}^r \to \mathcal{Q}^s$ we now get a square of adjunctions

$$\begin{array}{ccc} \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) & \underset{(\phi^*)_*}{\overset{(\phi^*)^!}{\rightleftarrows}} & \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s) \\ {\scriptstyle \pi_!}\uparrow\downarrow{\scriptstyle \pi^*} & & {\scriptstyle \pi_!}\uparrow\downarrow{\scriptstyle \pi^*} \\ \mathsf{Alg}_{\mathcal{P}}{}^{\mathbf{C}} & \underset{(\phi^*)_*}{\overset{(\phi_!)_*}{\rightleftarrows}} & \mathsf{Alg}_{\mathcal{Q}}{}^{\mathbf{C}}. \end{array}$$

As before, the square of right adjoints $(\phi^*)_*$ and π^* commutes because both maps are pullbacks, which implies that the square of left adjoints commutes up to unique natural isomorphism. In these terms, saying that $(\phi^*)^!$ is naturally isomorphic to the restriction of $(\phi_!)_*$ is saying that a third square commutes: that $\pi^*(\phi^*)^!$ is naturally isomorphic to $(\phi_!)_*\pi^*$.

Lemma 3.3.16. If $(\phi_!)_*\pi^*$ factors through $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s)$, i.e. if for $\mathfrak{A} \in \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$,

$$(\phi_!)_*\pi^*(\mathfrak{A}) \in \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s) \subseteq \mathsf{Alg}_{\mathcal{Q}}{}^{\mathbf{C}},$$

then $(\phi^*)^!$ is naturally isomorphic to the restriction of $(\phi_!)_*$.

Proof. Since $(\phi_!)_*\pi^*(\mathfrak{A}) \in \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s)$, $(\phi_!)_*\pi^*(\mathfrak{A}) \cong \pi^*(\mathfrak{B})$ for a $\mathfrak{B} \in \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s)$, so $\pi^*\pi_!(\phi_!)_*\pi^* \cong (\phi_!)_*\pi^*$ since $\pi_!\pi^*$ is naturally isomorphic to id (because $\pi^* : \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s) \to \mathsf{Alg}_{\mathcal{Q}}{}^{\mathbf{C}}$ is a reflective subcategory). So we find

$$(\phi_!)_*\pi^* \cong \pi^*\pi_!(\phi_!)_*\pi^* \cong \pi^*(\phi^*)^!\pi_!\pi^* \cong \pi^*(\phi^*)^!$$

where the second equivalence holds because the left adjoints commute up to natural isomorphism, and the third one holds again because $\pi^* : \mathbf{FT}(\overline{\mathbf{C}}, \mathcal{Q}^s) \to \mathsf{Alg}_{\mathcal{Q}}{}^{\mathbf{C}}$ is a reflective subcategory. □

3.3.4. Linear quantization. We will now consider a specific field theory adjunction resulting from a change of operad: the quantization of linear field theories. Because we will work with the unital Lie operad, we fix an additive closed symmetric monoidal category $\mathbf{T}$.

Recall that we defined quantum field theories on an orthogonal category $\overline{\mathbf{C}}$ as

$$\mathbf{QFT}(\overline{\mathbf{C}}) := \mathbf{FT}(\overline{\mathbf{C}}, \mathsf{As}^{[\,,\,],0})$$

in Example 3.1.6, while we defined linear field theories on $\overline{\mathbf{C}}$ as

$$\widetilde{\mathbf{LFT}}(\overline{\mathbf{C}}) := \mathbf{FT}(\overline{\mathbf{C}}, \mathsf{uLie}^{[\,,\,],0})$$

in Example 3.1.8. Consider the canonical uncolored operad morphism

$$\phi : \mathsf{uLie} \to \mathsf{As}$$

which sends the unit of uLie to the unit of As and sends the generator $[\,,\,] \in \mathsf{uLie}$ representing the Lie bracket to the commutator $[\,,\,] = \mu - \mu^{\mathrm{op}} \in \mathsf{As}$. Of course, this clearly also defines a morphism of bipointed uncolored operads

$$\phi : \mathsf{uLie}^{[\,,\,],0} \to \mathsf{As}^{[\,,\,],0}$$

so as before this yields a field theory adjunction

$$\mathfrak{Q}_{\mathrm{lin}} := (\phi^*)^! : \widetilde{\mathbf{LFT}}(\overline{\mathbf{C}}) \rightleftarrows \mathbf{QFT}(\overline{\mathbf{C}}) : (\phi^*)_* =: \mathfrak{U}_{\mathrm{lin}}. \tag{3.3.10}$$

It turns out that this is a linear quantization adjunction; in particular, $\mathfrak{Q}_{\mathrm{lin}}$ makes a linear field theory into a quantum field theory in a way that can rightfully be interpreted as quantization.

First, consider $\mathfrak{U}_{\mathrm{lin}}$. Writing out the functor $\phi^* : \mathsf{Alg}_{\mathsf{As}} \to \mathsf{Alg}_{\mathsf{uLie}}$ we find that it assigns to an associative algebra with unit (A, μ, η) the Lie algebra with unit $(A, [\,,\,] := \mu - \mu^{\mathrm{op}}, \eta)$. So this is the forgetful functor, that forgets the multiplication on an algebra, but remembers the commutator. The corresponding pushforward $\mathfrak{U}_{\mathrm{lin}} = (\phi^*)_*$ simply carries out this operation objectwise: for $\mathfrak{A} \in \mathbf{QFT}(\overline{\mathbf{C}})$, $\mathfrak{U}_{\mathrm{lin}}(\mathfrak{A})$ assigns $\phi^*(\mathfrak{A}(c)) \in \mathsf{Alg}_{\mathsf{uLie}}$ to each $c \in \mathbf{C}$.

The left adjoint $\mathfrak{Q}_{\mathrm{lin}}$ in the adjunction (3.3.10) can also be explicitly described, by constructing a model for the left adjoint $\phi_!$ in the adjunction

$$\phi_! : \mathsf{Alg}_{\mathsf{uLie}} \rightleftarrows \mathsf{Alg}_{\mathsf{As}} : \phi^*$$

in the spirit of Lemma 3.3.16: we will see that $(\phi^*)^!$ is isomorphic to the restriction of $(\phi_!)_*$. The construction is a variation of the well-known universal enveloping algebra $U^{\otimes}(L)$ of a Lie algebra L, which we call the *unital universal enveloping algebra*.

Start with a unital Lie algebra $(V, [\,,\,], \eta) \in \mathsf{Alg}_{\mathsf{uLie}}$ and form the usual tensor algebra

$$T^{\otimes}(V) := \bigoplus_{n \geq 0} V^{\otimes n} \in \mathsf{Alg}_{\mathsf{As}}$$

as in Example 2.1.26, with multiplication $\mu_{\otimes}$, unit $\eta_{\otimes}$ and inclusion ι_1 in the direct summand $V^{\otimes 1}$. The tensor algebra $T^{\otimes}(V)$ remembers nothing about the Lie bracket $[\,,\,]$ or the unit η. To remedy this, first consider the $\mathbf{T}$-morphisms

$$V \otimes V \underset{q_2 := \iota_1 [\,,\,]}{\overset{q_1 := (\mu_{\otimes} - \mu_{\otimes}^{\mathrm{op}})(\iota_1 \otimes \iota_1)}{\rightrightarrows}} T^{\otimes} V$$

picking out the commutator in $T^{\otimes}V$ and the Lie bracket on V (we suppress the forgetful functor $U : \mathsf{Alg}_{\mathsf{As}} \to \mathbf{T}$ here: to be precise we should write $U(T^{\otimes}V)$). By the adjunction $T^{\otimes} \dashv U$ this diagram in $\mathbf{T}$ defines an $\mathsf{Alg}_{\mathsf{As}}$-diagram of which we form the coequalizer

$$T^{\otimes}(V \otimes V) \underset{q_2}{\overset{q_1}{\rightrightarrows}} T^{\otimes}V \overset{\pi}{\dashrightarrow} U^{\otimes}V$$

in $\mathsf{Alg}_{\mathsf{As}}$, which is the usual universal enveloping algebra of the Lie algebra $(V, [\,,\,])$. For the unit η we consider the two $\mathbf{T}$-morphisms

$$I \underset{s_2 := \pi \eta_{\otimes}}{\overset{s_1 := \pi \iota_1 \eta}{\rightrightarrows}} U^{\otimes}V$$

which compare η, the Lie unit of $V \in \mathsf{Alg}_{\mathsf{uLie}}$, and $\eta_{\otimes}$, the multiplicative unit of $T^{\otimes}V \in \mathsf{Alg}_{\mathsf{As}}$. This again defines a diagram in $\mathsf{Alg}_{\mathsf{As}}$ of which we take the coequalizer

$$T^{\otimes}(I) \underset{s_2}{\overset{s_1}{\rightrightarrows}} U^{\otimes}V \overset{\pi'}{\dashrightarrow} \phi_!(V).$$

Both of the above diagrams in $\mathbf{T}$ implementing the bracket and the unit are clearly functorial in $\mathsf{Alg}_{\mathsf{uLie}}$-morphisms. Because taking colimits is functorial, we find that we have a functor

$$\phi_! : \mathsf{Alg}_{\mathsf{uLie}} \longrightarrow \mathsf{Alg}_{\mathsf{As}}$$

LEMMA 3.3.17. The functor $\phi_!$ is left adjoint to the forgetful functor

$$\phi^* : \mathsf{Alg}_{\mathsf{As}} \longrightarrow \mathsf{Alg}_{\mathsf{uLie}}$$

described above.

PROOF. For $V \in \mathsf{Alg}_{\mathsf{uLie}}$ and $A \in \mathsf{Alg}_{\mathsf{As}}$, we want a natural bijection

$$\mathsf{Alg}_{\mathsf{As}}(\phi_!(V), A) \cong \mathsf{Alg}_{\mathsf{uLie}}(V, \phi^*(A)).$$

An $\mathsf{Alg}_{\mathsf{As}}$-morphism $\kappa : \phi_!(V) \to A$ defines a unital Lie morphism $\kappa\pi'\pi\iota_1 : V \to \phi^*(A)$ (recall that $\phi^*(A)$ is the same $\mathbf{T}$-object as A). Conversely, if $\rho : V \to \phi^*(A)$ is an $\mathsf{Alg}_{\mathsf{uLie}}$-morphism, the underlying $\mathbf{T}$-morphism defines an $\mathsf{Alg}_{\mathsf{As}}$-morphism $\rho : T^{\otimes}V \to A$ which descends to the quotients $U^{\otimes}V$ and $\phi_!(V)$ because ρ is a unital Lie morphism. □

We now check the condition for Lemma 3.3.16 for our operad morphism $\phi : \mathsf{uLie} \to \mathsf{As}$.

PROPOSITION 3.3.18. If $\mathfrak{B} \in \widetilde{\mathbf{LFT}}(\overline{\mathbf{C}})$ is a linear field theory on $\overline{\mathbf{C}}$, the pushforward along the unital universal enveloping functor $(\phi_!)_*(\mathfrak{B}) = \phi_!\mathfrak{B} \in \mathsf{Alg}_{\mathsf{As}}{}^{\mathbf{C}}$ is a quantum field theory: $(\phi_!)_*(\mathfrak{B}) \in \mathbf{QFT}(\overline{\mathbf{C}})$.

PROOF. Let $(c_1 \xrightarrow{f_1} c \xleftarrow{f_2} c_2) \in \perp$ be a pair of orthogonal maps in $\overline{\mathbf{C}} = (\mathbf{C}, \perp)$. Because $\mathfrak{B} \in \widetilde{\mathbf{LFT}}(\overline{\mathbf{C}})$ is a field theory in the sense of Definition 3.1.5, the induced bracket

$$\big[\mathfrak{B}(f_1)(-), \mathfrak{B}(f_2)(-)\big]_c : \mathfrak{B}(c_1) \otimes \mathfrak{B}(c_2) \longrightarrow \mathfrak{B}(c)$$

is the zero map (see Example 3.1.8). For $(\phi_!)_*(\mathfrak{B}) \in \mathsf{Alg}_{\mathsf{As}}{}^{\mathbf{C}}$ to be a quantum field theory, we need the induced commutator

$$\big[\phi_!\mathfrak{B}(f_1)(-), \phi_!\mathfrak{B}(f_2)(-)\big]_c : \phi_!\mathfrak{B}(c_1) \otimes \phi_!\mathfrak{B}(c_2) \longrightarrow \phi_!\mathfrak{B}(c)$$

to also be the zero map (see Example 3.1.6). On linear observables (the image of $\pi'\pi\iota_1$ in $\phi_!\mathfrak{B}(c_i)$) this immediately follows from the definition of the universal enveloping algebra $U^{\otimes}(\mathfrak{B}(c))$ above and the fact that $\mathfrak{B}$ is a field theory. The fact that the Leibniz rule allows us to expand the commutator of polynomials of linear observables then proves the general case. □

The upshot of this proposition and Lemma 3.3.16 is now

COROLLARY 3.3.19. The restriction of the pushforward $(\phi_!)_* : \mathsf{Alg}_{\mathsf{uLie}}{}^{\mathbf{C}} \to \mathsf{Alg}_{\mathsf{As}}{}^{\mathbf{C}}$ to the field theory category $\widetilde{\mathbf{LFT}}(\overline{\mathbf{C}})$ is a model for the left adjoint $\mathfrak{Q}_{\mathrm{lin}} : \widetilde{\mathbf{LFT}}(\overline{\mathbf{C}}) \to \mathbf{QFT}(\overline{\mathbf{C}})$ in the linear quantization adjunction (3.3.10).

REMARK 3.3.20. With the explicit description of $\mathfrak{Q}_{\mathrm{lin}}$, we can now explain the name "linear quantization functor". Let $\mathbf{T} = \mathbf{Vect}_{\mathbb{K}}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Recall that in Definition 2.3.10 we defined the linear canonical quantization functor $\mathfrak{CCR} : \mathbf{LFT}(\overline{\mathbf{C}}) \to \mathbf{QFT}(\overline{\mathbf{C}})$. Using $\mathfrak{heis}$ defined in Example 3.1.8 and the discussion there, we recognize that

$$\mathfrak{Q}_{\mathrm{lin}} \circ \mathfrak{heis}_* \cong \mathfrak{ccr}_* \cong \mathfrak{CCR} : \mathbf{LFT}(\overline{\mathbf{C}}) \longrightarrow \mathbf{QFT}(\overline{\mathbf{C}})$$

so $\mathfrak{Q}_{\mathrm{lin}}$ is one half of the canonical quantization functor $\mathfrak{CCR}$.

Let us summarize the results of this section and add a proof of the claim in Definition 2.3.10, that the canonical quantization functor $\mathfrak{CCR}$ preserves Einstein causality, the time-slice axiom and the descent condition.

COROLLARY 3.3.21. Let $\overline{\mathbf{C}}$ be an orthogonal category. The canonical operad morphism $\phi : \mathsf{uLie} \to \mathsf{As}$ induces the linear quantization adjunction

$$\mathfrak{Q}_{\mathrm{lin}} := (\phi_!)_* : \widetilde{\mathbf{LFT}}(\overline{\mathbf{C}}) \rightleftarrows \mathbf{QFT}(\overline{\mathbf{C}}) : (\phi^*)_* =: \mathfrak{U}_{\mathrm{lin}}. \tag{3.3.11}$$

Moreover,

(1) If $j : \overline{\mathbf{D}} \to \overline{\mathbf{C}}$ is a full orthogonal subcategory, the linear quantization functor $\mathfrak{Q}_{\text{lin}} : \widetilde{\mathbf{LFT}}(\overline{\mathbf{C}}) \to \mathbf{QFT}(\overline{\mathbf{C}})$ maps j-local linear field theories to j-local quantum field theories.
(2) If $W \subseteq \mathbf{Mor}\ \mathbf{C}$ is a subset of morphisms, the linear quantization functor $\mathfrak{Q}_{\text{lin}}$: $\widetilde{\mathbf{LFT}}(\overline{\mathbf{C}}) \to \mathbf{QFT}(\overline{\mathbf{C}})$ maps W-constant linear field theories to W-constant quantum field theories.

Proof. Item (1) follows from item (1) in Proposition 3.3.14. Item (2) follows from Proposition 3.3.15 and Corollary 3.3.19. □

3.4. Quantization of linear gauge theories

We now choose the closed symmetric monoidal category $\mathbf{T}$ to be the category of chain complexes $\mathbf{Ch}_{\mathbb{K}}$ over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Of course, all constructions and results on field theories from the previous sections still hold, but recall from Example 2.4.19 that $\mathbf{Ch}_{\mathbb{K}}$ is a model category. This in particular means that $\mathbf{Ch}_{\mathbb{K}}$ has a class of weak equivalences, the quasi-isomorphisms, which is a bigger class of morphisms than just the isomorphisms. We will see that the model structure on $\mathbf{Ch}_{\mathbb{K}}$ induces a model structure on the categories of field theories. This raises the question that we will answer in the rest of the section: are our constructions, definitions and results preserved under weak equivalences?

3.4.1. Model structures for field theory categories. Write $\mathbf{hFT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ for the category of field theories $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ when $\mathcal{P}^r$ is valued in $\mathbf{Ch}_{\mathbb{K}}$. We start by defining a model structure on $\mathbf{hFT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ that is induced by the model structure on $\mathbf{Ch}_{\mathbb{K}}$. This means that the weak equivalences (and the fibrations) are defined objectwise, as one would expect. Recall Definition 3.1.5.

Proposition 3.4.1. Let $\overline{\mathbf{C}}$ be an orthogonal category and $\mathcal{P}^r \in \mathbf{Op}^{2\text{pt}}(\mathbf{Ch}_{\mathbb{K}})$ be an uncolored bipointed operad valued in chain complexes. If we define a field theory morphism $\zeta : \mathfrak{A} \to \mathfrak{B}$ in $\mathbf{hFT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ (which is a natural transformation between the underlying functors $\mathfrak{A}, \mathfrak{B} : \mathbf{C} \to \text{dg}\mathsf{Alg}_{\mathcal{P}}$) to be

(1) a weak equivalence if all the components $\zeta_c : \mathfrak{A}(c) \to \mathfrak{B}(c)$ are weak equivalences (i.e. quasi-isomorphisms) in $\mathbf{Ch}_{\mathbb{K}}$;
(2) a fibration if all the components $\zeta_c : \mathfrak{A}(c) \to \mathfrak{B}(c)$ are degreewise surjective in $\mathbf{Ch}_{\mathbb{K}}$;
(3) a cofibration if ζ has the left-lifting property with respect to acyclic fibrations;

then $\mathbf{hFT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ is a model category with these choices.

Proof. Recall Theorem 3.2.4: $\mathbf{hFT}(\overline{\mathbf{C}}, \mathcal{P}^r) \cong \mathsf{Alg}_{\mathcal{P}^r_{\overline{\mathbf{C}}}}$. The result then immediately follows from Theorem 2.5.25. □

This proposition improves on the results from Section 2.6 that the objectwise quasi-isomorphisms make $\text{dg}\mathsf{Alg}_{\mathcal{O}}{}^{\mathbf{C}}$ and the subcategory of field theories into a homotopical category, by showing that they make these categories into a model category. Note that here we are still working with the category $\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ of field theories, which means these theories satisfy the strict Einstein causality axiom. So we are considering semi-strict field theories, as mentioned in Section 2.6. We will cover the homotopy time-slice axiom in Definition 3.4.12.

Remark 3.4.2. With this model structure, every object $\mathfrak{A} \in \mathbf{hFT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ is fibrant: the unique chain map from $\mathfrak{A}(c)$ to the terminal object 0 (the chain complex that is $\{0\}$ in every entry) is always surjective for any c.

The other result we will need for a model categorical treatment of our linear quantization functor is the fact that our algebra adjunctions are Quillen adjunctions, see Definition 2.4.25.

PROPOSITION 3.4.3. Let $F : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ be an orthogonal functor, and $\phi : \mathcal{P}^r \to \mathcal{Q}^s$ be a morphism in $\mathbf{Op}^{2\mathrm{pt}}(\mathbf{Ch}_{\mathbb{K}})$. Then the corresponding adjunction (3.3.1) of algebras is a Quillen adjunction with respect to the model structures on $\mathbf{hFT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ and $\mathbf{hFT}(\overline{\mathbf{D}}, \mathcal{Q}^s)$ defined in Proposition 3.4.1.

PROOF. This is immediate with Remark 2.4.26, since the pullback $(\phi_F)^*$ obviously preserves weak equivalences and fibrations, which are both defined componentwise. □

3.4.2. Homotopical linear quantization. We now turn to the linear quantization of chain complex-valued field theories. In Remark 3.3.20 we saw that linear quantization for $\mathbf{T} = \mathbf{Vect}_{\mathbb{K}}$ consists of two parts:

$$\mathfrak{CCR} = \mathfrak{ccr}_* \cong \mathfrak{Q}_{\mathrm{lin}} \circ \mathfrak{heis}_*$$

and both of these functors are pushforwards: we have $\mathfrak{Q}_{\mathrm{lin}} = (\phi_!)_*$ and

$$\mathbf{PoissVect}_{\mathbb{R}} \xrightarrow{\mathfrak{heis}} \mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}} \xrightarrow{\phi_!} \mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}\ , \qquad \mathfrak{ccr} = \phi_! \circ \mathfrak{heis}\ .$$

The morphism $(\phi_!)_*$ is of course defined for any additive $\mathbf{T}$. The left morphism $\mathfrak{heis}$ was originally defined for $\mathbf{T} = \mathbf{Vect}_{\mathbb{R}}$ on the left and $\mathbf{T} = \mathbf{Vect}_{\mathbb{C}}$ on the right (since the tensor product $\otimes_{\mathbb{R}}\mathbb{C}$ is involved) but it can easily be extended to Poisson chain complexes: for $(V, \tau) \in \mathbf{PoissCh}_{\mathbb{R}}$, $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ is obtained by tensoring with $\mathbb{C}$, the chain complex with $\mathbb{C}$ at degree 0 as the only non-trivial entry (or equivalently, $V_n \otimes_{\mathbb{R}} \mathbb{C}$ at each degree, with the differentials extended linearly). An extra summand of $\mathbb{C}$ at degree 0 provides the Lie unit, and the Lie bracket on $\mathfrak{heis}(V, \tau) := \big(V_{\mathbb{C}} \oplus \mathbb{C}, [\ ,\ ]\big)$ is defined by

$$[v_1 \oplus \lambda_1, v_2 \oplus \lambda_2] := 0 \oplus i\tau(v_1, v_2) \tag{3.4.1}$$

similarly as for vector spaces. Lastly, as in Remark 3.1.9 we immediately see that $\mathfrak{heis}_*$ preserves the Einstein causality axiom.

So we end up with the analogous quantization scheme for chain complex valued field theories,

$$\mathfrak{CCR} = \mathfrak{ccr}_* \cong \mathfrak{Q}_{\mathrm{lin}} \circ \mathfrak{heis}_* \tag{3.4.2}$$

with $\mathfrak{Q}_{\mathrm{lin}} = (\phi_!)_*$ and

$$\mathbf{PoissCh}_{\mathbb{R}} \xrightarrow{\mathfrak{heis}} \mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}} \xrightarrow{\phi_!} \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}\ , \qquad \mathfrak{ccr} = \phi_! \circ \mathfrak{heis} \tag{3.4.3}$$

where we now use the notation $\mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}}$ and $\mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$ to emphasize that we are working with differential graded (i.e. chain complex-valued) algebras. This means that we can separately investigate the homotopical properties of the two components $\mathfrak{heis}_*$ and $\mathfrak{Q}_{\mathrm{lin}}$. Since both functors are pushforwards, and the weak equivalences in $\mathbf{PoissCh}_{\mathbb{R}}$ and $\mathbf{hFT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ are defined componentwise, it suffices to consider the functors $\mathfrak{heis}$ and $\phi_!$.

For the functor $\mathfrak{heis} : \mathbf{PoissCh}_{\mathbb{R}} \to \mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}}$, recall Example 2.4.36: $\mathbf{PoissCh}_{\mathbb{R}}$ is a homotopical category, not a model category. The weak equivalences in $\mathbf{PoissCh}_{\mathbb{R}}$ are the Poisson morphisms $f : (V, \tau) \to (V', \tau')$ such that the underlying chain map $f : V \to V'$ is a quasi-isomorphism in $\mathbf{Ch}_{\mathbb{R}}$. Since $\mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}}$ is a model category by Theorem 2.5.25 it is a homotopical category with the componentwise quasi-isomorphisms as weak equivalences.

PROPOSITION 3.4.4. The functor $\mathfrak{heis}$ is a homotopical functor: it preserves the weak equivalences on $\mathbf{PoissCh}_{\mathbb{R}}$ and $\mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}}$.

PROOF. This is immediate by the definition of $\mathfrak{heis}$ and the definitions of weak equivalences: $f : V \to V'$ is a quasi-isomorphism, so its extension $\mathfrak{heis}(f) : V_{\mathbb{C}} \oplus \mathbb{C} \to V'_{\mathbb{C}} \oplus \mathbb{C}$ is too. □

Since weak equivalences in $\widetilde{\mathbf{hLFT}}(\overline{\mathbf{C}})$ are defined objectwise in Proposition 3.4.1 and we saw above that $\mathfrak{heis}_*$ preserves the Einstein causality axiom, we immediately find

COROLLARY 3.4.5. The functor

$$\mathfrak{heis}_* : \mathbf{hLFT}(\overline{\mathbf{C}}) \longrightarrow \widetilde{\mathbf{hLFT}}(\overline{\mathbf{C}})$$

is a homotopical functor.

Another datum that can be weakly varied is the Poisson structure $\tau \in \underline{\mathrm{hom}}(V \wedge V, \mathbb{R})_0$: we can consider the chain homotopic Poisson structure $\tau + \partial\rho$ for a homotopy $\rho \in \underline{\mathrm{hom}}(V \wedge V, \mathbb{R})_1$.

PROPOSITION 3.4.6. For a Poisson complex $(V, \tau) \in \mathbf{PoissCh}_{\mathbb{R}}$ and a 1-chain $\rho \in \underline{\mathrm{hom}}(V \wedge V, \mathbb{R})_1$ there exists a zig-zag

$$\mathfrak{heis}(V,\tau) \xleftarrow{\ \sim\ } H_{V,\tau,\rho} \xrightarrow{\ \sim\ } \mathfrak{heis}(V, \tau + \partial\rho)$$

of weak equivalences in $\mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}}$.

PROOF. Consider the acyclic chain complex

$$D := \Big(\overset{(-1)}{\mathbb{C}} \xleftarrow{\ \mathrm{id}\ } \overset{(0)}{\mathbb{C}} \Big) \tag{3.4.4}$$

and write $x := 1 \in D_0$ and $y := \mathrm{d}x = 1 \in D_{-1}$ for the two vectors spanning D. We construct the interpolating object $H_{V,\tau,\rho} \in \mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}}$ as

$$H_{V,\tau,\rho} := V_{\mathbb{C}} \oplus D \oplus \mathbb{C}$$

(recall that the chain complex $\mathbb{C}$ is $\mathbb{C}$ concentrated in degree 0) with Lie bracket

$$\big[v_1 \oplus \alpha_1 \oplus \lambda_1, v_2 \oplus \alpha_2 \oplus \lambda_2\big] := 0 \oplus \big(i(\partial\rho)(v_1,v_2)x + i\rho(v_1,v_2)y\big) \oplus i\tau(v_1,v_2)$$

and unit $0 \oplus 0 \oplus 1$. For $s \in \mathbb{R}$, define the differential graded unital Lie ideal $\mathcal{I}_s \subseteq H_{V,\tau,\rho}$ to be generated by the relations

$$0 \oplus x \oplus 0 = 0 \oplus 0 \oplus s \quad ; \quad 0 \oplus y \oplus 0 = 0.$$

Forming the quotient we see that

$$H_{V,\tau,\rho}/\mathcal{I}_s \cong \mathfrak{heis}(V, \tau + s\ \partial\rho)$$

with quotient map

$$\pi_s = \mathrm{id}_{V_{\mathbb{C}}} + q_s : H_{V,\tau,\rho} \longrightarrow \mathfrak{heis}(V, \tau + s\ \partial\rho)$$

which is the identity on $V_{\mathbb{C}}$ and

$$\begin{array}{cccc} q_s : & D \oplus \mathbb{C} & \longrightarrow & \mathbb{C} \\ & (c_1 x + c_2 y) \oplus \lambda & \longmapsto & s c_1 + \lambda \end{array}$$

on $D \oplus \mathbb{C}$. Since q_s is a quasi-isomorphism, all π_s are weak equivalences, and filling in $s = 0, 1$ now gives the desired result. □

We now turn to the other ingredient of linear quantization. Recall the model structures on $\widetilde{\mathbf{hLFT}}(\overline{\mathbf{C}})$ and $\mathbf{hQFT}(\overline{\mathbf{C}})$ from Proposition 3.4.1 and the linear quantization adjunction (3.3.11),

$$\mathfrak{Q}_{\mathrm{lin}} := (\phi_!)_* : \widetilde{\mathbf{hLFT}}(\overline{\mathbf{C}}) \rightleftarrows \mathbf{hQFT}(\overline{\mathbf{C}}) : (\phi^*)_* =: \mathfrak{U}_{\mathrm{lin}} \ .$$

By Proposition 3.4.3 this is a Quillen adjunction, so we can pursue the strategy of deriving it as in Proposition 2.4.29, (co)fibrantly replacing on either side of the adjunction.

For $\mathfrak{U}_{\mathrm{lin}}$ we are in luck: since every object in $\mathbf{hQFT}(\overline{\mathbf{C}})$ is fibrant (recall that the fibrations are defined as objectwise surjections), $R := \mathrm{id}$ suffices as a fibrant replacement and we find

Proposition 3.4.7. The underived functor $\mathfrak{U}_{\mathrm{lin}}$ in the adjunction (3.3.11) is a model for the derived functor $\mathbb{R}\mathfrak{U}_{\mathrm{lin}}$.

For the left adjoint $\mathfrak{Q}_{\mathrm{lin}}$ we are not done quite so quickly, since not every object in $\widetilde{\mathbf{hLFT}}(\overline{\mathbf{C}})$ is cofibrant. However, we do have the following.

Proposition 3.4.8. The unital universal enveloping algebra functor $\phi_! : \mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}} \to \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$ constructed in Section 3.3.4 preserves weak equivalences, i.e. morphisms of differential graded algebras such that the underlying chain maps are quasi-isomorphisms.

Proof. Let $\rho : V \to V'$ be a weak equivalence in $\mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}}$, i.e. a quasi-isomorphism. Recall the construction of $\phi_!$: now working with $\mathbf{T} = \mathbf{Ch}_{\mathbb{C}}$, we see that if $(V, [\ ,\ ], \eta) \in \mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}}$ is a differential graded unital Lie algebra, we have

$$\phi_!(V) = T^{\otimes}V \big/ \mathcal{I}$$

where $T^{\otimes}V$ is the tensor algebra

$$T^{\otimes}V = \bigoplus_{n\geq 0} V^{\otimes n}$$

and $\mathcal{I}$ is the differential graded ideal generated by

$$v_1 \otimes v_2 - (-1)^{|v_1||v_2|} v_2 \otimes v_1 = [v_1, v_2] \quad ; \quad \mathbb{1}_{\otimes} = \mathbb{1}$$

for all homogeneous elements $v_i \in V$ and the units $\mathbb{1}_{\otimes} = \eta_{\otimes}(1)$ of $T^{\otimes}V$ and $\mathbb{1} = \eta(1)$ of V. The tensor algebra $T^{\otimes}V$ has a filtration

$$T^{\leq m}V := \bigoplus_{n=0}^{m} V^{\otimes n}$$

which allows us to define

$$\phi_!(V)^m := T^{\leq m}V \big/ (T^{\leq m}V \cap \mathcal{I})$$

for all $n \geq 0$. This in turn defines a sequential diagram

$$\phi_!(V)^0 \hookrightarrow \phi_!(V)^1 \hookrightarrow \phi_!(V)^2 \hookrightarrow \cdots$$

in $\mathbf{Ch}_{\mathbb{C}}$ with colimit $\phi_!(V)$. Because filtered colimits are exact (Theorem 2.6.15 in [**Wei95**]), they preserve quasi-isomorphisms, and it suffices to prove that the induced chain map

$$\phi_!(\rho) : \phi_!(V)^m \longrightarrow \phi_!(V')^m$$

is a quasi-isomorphism for all m.

Using the relations defining $\mathcal{I}$, we see that the quotient of two neighbours in the sequence is

$$\phi_!(V)^{m+1} \big/ \phi_!(V)^m \cong \tilde{V}^{\otimes m+1} \big/ \Sigma_{m+1}$$

where $\tilde{V} := V/\mathbb{C}\mathbb{1}$ is the quotient of V by the unit and Σ_{m+1} is the symmetric group acting on the tensor product by permuting the entries. So we have a short exact sequence of complexes

$$0 \longrightarrow \phi_!(V)^m \longrightarrow \phi_!(V)^{m+1} \longrightarrow \tilde{V}^{\otimes m+1}/\Sigma_{m+1} \longrightarrow 0 \tag{3.4.5}$$

for all $m \geq 0$. Form the long exact sequance of homology from the short exact sequence (3.4.5),

$$\cdots \to H_{l+1}\big(\overline{V}^{m+1}\big) \to H_l\big(\phi_!(V)^m\big) \to H_l\big(\phi_!(V)^{m+1}\big) \to H_l\big(\overline{V}^{m+1}\big) \to H_{l-1}\big(\phi_!(V)^m\big) \to \cdots$$

writing $\overline{V}^{m+1}$ for $V^{\otimes m+1}/\Sigma_{m+1}$. Using the five lemma, we see that we can proceed by induction on m, provided that ρ induces isomorphisms on the homology of $\tilde{V}^{\otimes m}/\Sigma_m$. The first steps of induction are clear: the chain maps

$$\big(\phi_!(\rho) : \phi_!(V)^0 \longrightarrow \phi_!(V')^0\big) = \big(\mathrm{id} : \mathbb{C} \longrightarrow \mathbb{C}\big)$$

and

$$\big(\phi_!(\rho) : \phi_!(V)^1 \longrightarrow \phi_!(V')^1\big) = \big(\rho : \tilde{V} \longrightarrow \tilde{V}'\big)$$

are clearly quasi-isomorphims.

For the chain complexes $\tilde{V}^{\otimes m}/\Sigma_m$ we have

$$\begin{aligned} H_\bullet(\tilde{V}^{\otimes m}/\Sigma_m) &\cong H_\bullet((\tilde{V}^{\otimes m})^{\Sigma_m}) \\ &\cong (H_\bullet(\tilde{V}^{\otimes m}))^{\Sigma_m} \\ &\cong (H_\bullet(\tilde{V})^{\otimes m})^{\Sigma_m} \ . \end{aligned}$$

Here we use the fact that for a finite group action G on a chain complex V over a field of characteristic zero, we can use the projection $\frac{1}{|G|}\sum_{g\in G} g$ to show that the coinvariants V/G are isomorphic to the invariants V^G and taking the homology commutes with taking the invariants. In the last step we use the Künneth formula for fields. So the induced map

$$\tilde{\rho} : \tilde{V}^{\otimes m}/\Sigma_m \longrightarrow \tilde{V}'^{\otimes m}/\Sigma_m$$

is indeed a quasi-isomorphism, and we have proven the result. □

COROLLARY 3.4.9. The linear quantization functor $\mathfrak{Q}_{\mathrm{lin}}$ in the adjunction (3.3.11) preserves weak equivalences, and is therefore a model for the left derived functor $\mathbb{L}\mathfrak{Q}_{\mathrm{lin}}$.

PROOF. The first claim follows from Proposition 3.4.8 and the fact that the weak equivalences in $\widetilde{\mathbf{hLFT}}(\overline{\mathbf{C}})$ are defined objectwise in Proposition 3.4.1. The second claim follows from Lemma 2.4.27. □

Combining our results on $\mathfrak{heis}$ and $\mathfrak{Q}_{\mathrm{lin}}$ we find

PROPOSITION 3.4.10. The linear canonical quantization functor

$$\mathfrak{CCR} = \mathfrak{Q}_{\mathrm{lin}} \circ \mathfrak{heis}_* : \mathbf{hLFT}(\overline{\mathbf{C}}) \longrightarrow \mathbf{hQFT}(\overline{\mathbf{C}})$$

is a homotopical functor: it preserves weak equivalences. Moreover, for a linear field theory $(V,\tau) \in \mathbf{hLFT}(\overline{\mathbf{C}})$ and a natural 1-chain $\rho \in \underline{\mathrm{hom}}(V \wedge V, \mathbb{R})_1$ there exists a zig-zag

$$\mathfrak{CCR}(V,\tau) \xleftarrow{\ \sim\ } A_{V,\tau,\rho} \xrightarrow{\ \sim\ } \mathfrak{CCR}(V,\tau+\partial\rho)$$

of weak equivalences in $\mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$.

PROOF. The first claim follows from Corollary 3.4.5 and Corollary 3.4.9. The second follows from Proposition 3.4.6 and Corollary 3.4.9, with $A_{V,\tau,\rho} = \mathfrak{Q}_{\mathrm{lin}}(H_{V,\tau,\rho})$. □

3.4.3. Homotopical descent and time-slice. We end this chapter by considering homotopical generalizations of j-locality (see Definition 3.3.6) and W-constancy (see Definition 3.3.11) in the context of the model category $\mathbf{T} = \mathbf{Ch}_{\mathbb{K}}$, and investigating their behaviour under the linear quantization functor $\mathfrak{Q}_{\mathrm{lin}}$.

First, consider a full orthogonal subcategory $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ as in Section 3.3.1. Proposition 3.4.3 tells us that the field theory adjunction (3.3.5)

$$j_! : \mathbf{hFT}(\overline{\mathbf{C}}, \mathcal{P}^r) \rightleftarrows \mathbf{hFT}(\overline{\mathbf{D}}, \mathcal{P}^r) : j^*$$

is a Quillen adjunction. Since every object in $\mathbf{hFT}(\overline{\mathbf{D}}, \mathcal{P}^r)$ is fibrant, $R = \mathrm{id}$ suffices as a fibrant replacement, so we can choose $\mathbb{R}j^* = j^*$ as a model for the right derived functor. Unfortunately, the left adjoint $j_!$ will in general not preserve weak equivalences (see Appendix A in [**BSW19a**]) so it has to be derived: $\mathbb{L}j_! = j_! Q$ for a cofibrant replacement Q with natural weak equivalence $q : Q \xrightarrow{\sim} \mathrm{id}$. This also alters our Definition 3.3.6 of j-locality.

DEFINITION 3.4.11. A field theory $\mathfrak{A} \in \mathbf{hFT}(\overline{\mathbf{D}}, \mathcal{P}^r)$ on $\overline{\mathbf{D}}$ is called *homotopy j-local* if the component

$$\tilde{\epsilon}_{\mathfrak{A}} : j_! Q j^*(\mathfrak{A}) \xrightarrow{j_! q_{j^*(\mathfrak{A})}} j_! j^*(\mathfrak{A}) \xrightarrow{\epsilon_{\mathfrak{A}}} \mathfrak{A}$$

of the derived counit is a weak equivalence.

Now let $\overline{\mathbf{C}}$ be an orthogonal category, and $W \subseteq \mathbf{Mor}\ \mathbf{C}$ a subset of morphisms. The natural homotopical generalization of Definition 3.3.11 is

DEFINITION 3.4.12. A field theory $\mathfrak{A} \in \mathbf{hFT}(\overline{\mathbf{C}}, \mathcal{P}^r)$ is called *homotopy W-constant* if for any $f : c \to c'$ in W, $\mathfrak{A}(f) : \mathfrak{A}(c) \to \mathfrak{A}(c')$ is a quasi-isomorphism in $\mathbf{Ch}_{\mathbb{K}}$.

Compare Definition 2.6.1 of semi-strict homotopy algebraic field theories: we see that such a theory on $\overline{\mathbf{C}}$ is exactly a Cauchy-constant field theory $\mathfrak{A} \in \mathbf{hFT}(\overline{\mathbf{Loc}}, \mathsf{As}^{[\,,\,],0})$.

PROPOSITION 3.4.13. Let $j : \overline{\mathbf{D}} \to \overline{\mathbf{C}}$ be a full orthogonal subcategory, and $W \subseteq \mathbf{Mor}\ \mathbf{C}$ be a subset of morphisms. The linear quantization functor $\mathfrak{Q}_{\mathrm{lin}} : \widetilde{\mathbf{hLFT}}(\overline{\mathbf{C}}) \to \mathbf{hQFT}(\overline{\mathbf{C}})$ (see (3.3.11) and Corollary 3.4.9) preserves homotopy j-local field theories and homotopy W-constant field theories.

PROOF. For homotopy j-locality, let $\mathfrak{B} \in \widetilde{\mathbf{hLFT}}(\overline{\mathbf{C}})$ be a homotopy j-local linear field theory, so the derived counit $\tilde{\epsilon}_{\mathfrak{B}} : j_! Q j^*(\mathfrak{B}) \to \mathfrak{B}$ is a weak equivalence. We want to show that the derived counit $\tilde{\epsilon}_{\mathfrak{Q}_{\mathrm{lin}}(\mathfrak{B})} : j_! Q j^* \mathfrak{Q}_{\mathrm{lin}}(\mathfrak{B}) \to \mathfrak{Q}_{\mathrm{lin}}(\mathfrak{B})$ of the quantized theory $\mathfrak{Q}_{\mathrm{lin}}(\mathfrak{B}) \in \mathbf{hQFT}(\overline{\mathbf{C}})$ is also a weak equivalence.

First, consider the following diagram:

$$\begin{array}{ccc}
j_!Qj^*\mathfrak{Q}_{\mathrm{lin}}(\mathfrak{B}) & \xrightarrow{\tilde{\epsilon}_{\mathfrak{Q}_{\mathrm{lin}}(\mathfrak{B})}} & \mathfrak{Q}_{\mathrm{lin}}(\mathfrak{B}) \\
{\scriptstyle j_!Qj^*\mathfrak{Q}_{\mathrm{lin}}\tilde{\epsilon}_{\mathfrak{B}}}\uparrow\wr & & \wr\uparrow{\scriptstyle \mathfrak{Q}_{\mathrm{lin}}\tilde{\epsilon}_{\mathfrak{B}}} \\
j_!Qj^*\underbrace{\mathfrak{Q}_{\mathrm{lin}}j_!}_{\cong j_!\mathfrak{Q}_{\mathrm{lin}}}Qj^*(\mathfrak{B}) & \xrightarrow{\tilde{\epsilon}_{\mathfrak{Q}_{\mathrm{lin}}j_!Qj^*(\mathfrak{B})}} & \underbrace{\mathfrak{Q}_{\mathrm{lin}}j_!}_{\cong j_!\mathfrak{Q}_{\mathrm{lin}}}Qj^*(\mathfrak{B}) \\
{\scriptstyle j_!Qj^*j_!q_{\mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B})}}\uparrow\wr & & \wr\uparrow{\scriptstyle j_!q_{\mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B})}} \\
j_!Qj^*j_!Q\mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B}) & \xrightarrow{\tilde{\epsilon}_{j_!Q\mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B})}} & j_!Q\mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B})\ .
\end{array}$$

The top vertical arrows are weak equivalences because $\tilde{\epsilon}_{\mathfrak{B}}$ is, and because $\mathfrak{Q}_{\mathrm{lin}}$, j^* and $j_!Q = \mathbb{L}j_!$ preserve weak equivalences. We have $j_!\mathfrak{Q}_{\mathrm{lin}} \cong \mathfrak{Q}_{\mathrm{lin}}j_!$ because the left adjoints in the diagram (3.3.4) commute up to natural isomorphism. Left Quillen functors preserve cofibrant objects, so $\mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B})$ is cofibrant, and of course $Q\mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B})$ is cofibrant too. By Ken Brown's lemma (Lemma 2.4.30) the left Quillen functor $j_!$ preserves weak equivalences between cofibrant objects, like $q_{\mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B})} : Q\mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B}) \to \mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B})$. Since j^* and $j_!Q = \mathbb{L}j_!$ preserve weak equivalences we see that bottom vertical arrows in the diagram are also weak equivalences. By the 2-out-of-3 property of weak equivalences, we see that $\tilde{\epsilon}_{\mathfrak{Q}_{\mathrm{lin}}(\mathfrak{B})}$ is a weak equivalence if and only if $\tilde{\epsilon}_{j_!Q\mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B})}$ is.

So write $\mathfrak{A} := \mathfrak{Q}_{\mathrm{lin}}Qj^*(\mathfrak{B})$; we then want to prove that $\tilde{\epsilon}_{j_!Q\mathfrak{A}}$ is a weak equivalence. This follows from the 2-out-of-3 property and the diagram

$$\begin{array}{ccccc}
& & \overset{\tilde{\epsilon}_{j_!Q(\mathfrak{A})}}{\frown\!\!\!\longrightarrow} & & \\
j_!Qj^*j_!Q(\mathfrak{A}) & \xrightarrow{j_!q_{j^*j_!Q(\mathfrak{A})}} & j_!j^*j_!Q(\mathfrak{A}) & \xrightarrow{\epsilon_{j_!Q(\mathfrak{A})}} & j_!Q(\mathfrak{A})\ . \\
{\scriptstyle j_!Q\eta_{Q(\mathfrak{A})}}\uparrow\cong & & {\scriptstyle j_!\eta_{Q(\mathfrak{A})}}\uparrow\cong & \nearrow\!\!\!= & \\
j_!QQ(\mathfrak{A}) & \underset{\sim}{\xrightarrow{j_!q_{Q(\mathfrak{A})}}} & j_!Q(\mathfrak{A}) & &
\end{array}$$

The two top arrows give the definition of the derived counit, while the right triangle is the triangle for the (underived) unit and counit. The vertical arrows are isomorphisms by Proposition 3.3.3. Lastly, the bottom arrow is a weak equivalence because $j_!$ is left Quillen and $q_{Q(\mathfrak{A})}$ is a weak equivalence between cofibrant objects.

For homotopy W-constancy, the argument is a lot simpler: recall that $\mathfrak{Q}_{\mathrm{lin}} = (\phi_!)_*$ is the pushforward along $\phi_!$. We saw in Proposition 3.4.8 that $\phi_!$ preserves weak equivalences, so since weak equivalences in $\widetilde{\mathbf{hLFT}}(\overline{\mathbf{C}})$ and $\mathbf{hQFT}(\overline{\mathbf{C}})$ are defined objectwise, $\mathfrak{Q}_{\mathrm{lin}}(\mathfrak{A})$ will be homotopy W-constant if $\mathfrak{A}$ is. $\square$

CHAPTER 4

Linear Yang-Mills theory

In the previous chapter the canonical quantization functor $\mathfrak{CCR} : \mathbf{hLFT}(\overline{\mathbf{C}}) \to \mathbf{hQFT}(\overline{\mathbf{C}})$ was constructed. In this chapter we will use this functor to develop the linear Yang-Mills model, one of the first examples of a homotopy algebraic quantum field theory. For the underlying homotopy linear field theory the solution complex is obtained as a derived critical locus. The Poisson structure on the solution complex is then constructed using the crucial insight that the Green operators of the d'Alembertian $\square$ allow us to trivialize the canonical shifted Poisson structure in two different ways. The results in this chapter were previously published in [**BBS20**]. A note on notation: the functor $\mathfrak{CCR}$ in that article is denoted by $\mathfrak{ccr}$ in this text.

4.1. Field and solution complexes

In Section 2.3.1 we encountered the data used to define a linear field theory: the space of fields and the equation of motion, with which we defined the solution space. In this section we encode these data into chain complexes. We will consider observables and the Poisson structure in the next section. For now we fix a globally hyperbolic spacetime M; we will consider functoriality in Section 4.4. Note that all complexes in this section are over $\mathbb{R}$; complex numbers will show up when discussing quantization.

We start with the chain complex of fields.

Definition 4.1.1. A *field theory complex* $\mathfrak{F}$ on M is a chain complex

$$\mathfrak{F}(M) = \Big(\mathfrak{F}_0(M) \xleftarrow{\;Q\;} \mathfrak{F}_1(M) \Big) \tag{4.1.1}$$

concentrated in degrees 0 and 1, such that both $\mathfrak{F}_i = \Gamma(E_i)$ are vector spaces of sections of vector bundles $E_i \to M$ of finite rank with fiber metric h_i and $Q : \mathfrak{F}_1(M) \to \mathfrak{F}_0(M)$ is a linear differential operator.

We interpret the sections in $\mathfrak{F}_0(M)$ as the physical fields; the sections in $\mathfrak{F}_1(M)$ then are the gauge transformations (if there are no gauge transformations, E_1 is the zero bundle). This is illustrated by the following two examples, which will be guiding us throughout this chapter.

Example 4.1.2. The *scalar field complex* on M is the chain complex

$$\Big(\Omega^0(M) \xleftarrow{\;0\;} 0 \Big)$$

concentrated in degree 0. The fiber metric is given by scalar multiplication. This is exactly what is expected when considering the treatment of Klein-Gordon theory in Section 2.3.1: there are scalar fields $\phi \in \Omega^0(M) = C^\infty(M)$ and no nontrivial gauge transformations.

EXAMPLE 4.1.3. The *linear gauge theory complex* with gauge group $\mathbb{R}$ on M is the chain complex

$$\Big(\; \Omega^1(M) \xleftarrow{\;\mathrm{d}\;} \Omega^0(M) \; \Big)$$

where d is the de Rham differential. The fiber metric given by the metric (or equivalently by the Hodge star operator) as in Example 2.2.13. Sections $A \in \Omega^1(M)$ in degree 0 are interpreted as gauge fields and sections $\epsilon \in \Omega^0(M)$ in degree 1 are gauge transformations. The gauge transformations ϵ act on gauge fields as $A \to A + \mathrm{d}\epsilon$.

REMARK 4.1.4. It is possible to encode higher linear gauge field theories (with gauge transformations of gauge transformations and so on) into this framework by allowing for longer chain complexes,

$$\mathfrak{F}(M) = \Big(\; \mathfrak{F}_0(M) \xleftarrow{\;Q_1\;} \mathfrak{F}_1(M) \xleftarrow{\;Q_2\;} \mathfrak{F}_2(M) \xleftarrow{\;Q_3\;} \dots \; \Big),$$

where as in Definition 4.1.1 every $\mathfrak{F}_i(M)$ is the space of sections of a vector bundle over M equipped with a fiber metric. For example we could consider the complex

$$\Big(\; \Omega^p(M) \xleftarrow{\;\mathrm{d}\;} \Omega^{p-1}(M) \xleftarrow{\;\mathrm{d}\;} \dots \xleftarrow{\;\mathrm{d}\;} \Omega^0(M) \; \Big)$$

of p-form gauge fields and gauge transformations, 2-gauge transformations and so on. The results in this chapter (specifically, the shape of the solution complex as a homotopy pullback in Theorem 4.1.7) extend to these kinds of field theories in an obvious way.

The next step is to encode the equation of motion, which determines the dynamics of the theory. We take the equation of motion operator to be a formally self-adjoint linear differential operator

$$P : \mathfrak{F}_0(M) \longrightarrow \mathfrak{F}_0(M) \tag{4.1.2}$$

that acts on the gauge fields in degree 0. The corresponding action functional is then

$$S(s) := \frac{1}{2}\langle s, Ps\rangle = \frac{1}{2}\int_M h(s, Ps)\mathrm{vol}_M \ . \tag{4.1.3}$$

The action is gauge-invariant if and only if the equation of motion operator P satisfies

$$PQ = 0 \tag{4.1.4}$$

so we will assume this from now on. Note that since P is formally self-adjoint, this implies that

$$(PQ)^* = Q^*P^* = Q^*P = 0 \ .$$

To encode the equation of motion operator in our chain complex language, we define the contangent complex $T^*\mathfrak{F}(M)$ to $\mathfrak{F}(M)$ as

$$T^*\mathfrak{F}(M) = \mathfrak{F}(M) \times \mathfrak{F}_{\mathrm{c}}(M)^*$$

where

$$\mathfrak{F}_{\mathrm{c}}(M)^* := \Big(\; \overset{(-1)}{\mathfrak{F}_1(M)} \xleftarrow{\;-Q^*\;} \overset{(0)}{\mathfrak{F}_0(M)} \; \Big)$$

with Q^* being the linear differential operator that is formally adjoint to Q. $\mathfrak{F}_{\mathrm{c}}(M)^*$ is the complex to which the complex of compactly supported sections

$$\mathfrak{F}_{\mathrm{c}}(M) := \Big(\; \overset{(0)}{\mathfrak{F}_{0,c}(M)} \xleftarrow{\;Q\;} \overset{(1)}{\mathfrak{F}_{1,c}(M)} \; \Big)$$

is the smooth dual. Note that there are two choices made here: we choose the tangent space to a point in $\mathfrak{F}(M)$ to be $\mathfrak{F}_c(M)$, and we choose the cotangent space to that point to be the complex $\mathfrak{F}_c(M)^*$ that the tangent space is smoothly dual to. Since $\mathfrak{F}(M)$ models a linear space, the total cotangent space is then $T^*\mathfrak{F}(M) = \mathfrak{F}(M) \times \mathfrak{F}_c(M)^*$. Explicitly, we have

$$T^*\mathfrak{F}(M) = \Big(\overset{(-1)}{\mathfrak{F}_1(M)} \xleftarrow{-Q^*\pi_2} \mathfrak{F}_0(M) \overset{(0)}{\times} \mathfrak{F}_0(M) \xleftarrow{\iota_1 Q} \overset{(1)}{\mathfrak{F}_1(M)} \Big)$$

where the first copy of $\mathfrak{F}_0(M)$ in $\mathfrak{F}_0(M) \times \mathfrak{F}_0(M)$ is the zero degree term in $\mathfrak{F}(M)$ and the second one is the zero degree term in $\mathfrak{F}_c(M)^*$. The map $\iota_1 : \mathfrak{F}_0(M) \to \mathfrak{F}_0(M) \times \mathfrak{F}_0(M)$ includes $\mathfrak{F}_0(M)$ in the first factor, while $\pi_2 : \mathfrak{F}_0(M) \times \mathfrak{F}_0(M) \to \mathfrak{F}_0(M)$ is the projection on the second factor.

The section $\delta^v S : \mathfrak{F}(M) \to T^*\mathfrak{F}(M)$ obtained from varying the action is then

$$\begin{array}{c} \mathfrak{F}(M) \\ {\scriptstyle \delta^v S}\big\downarrow \\ T^*\mathfrak{F}(M) \end{array} = \left(\begin{array}{ccccc} 0 & \xleftarrow{\;\;0\;\;} & \mathfrak{F}_0(M) & \xleftarrow{\;\;Q\;\;} & \mathfrak{F}_1(M) \\ {\scriptstyle 0}\big\downarrow & & {\scriptstyle (\mathrm{id},P)}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ \mathfrak{F}_1(M) & \xleftarrow{-Q^*\pi_2} & \mathfrak{F}_0(M) \times \mathfrak{F}_0(M) & \xleftarrow{\iota_1 Q} & \mathfrak{F}_1(M) \end{array} \right). \tag{4.1.5}$$

To find the space of solutions $\mathfrak{Sol}(M)$, we need to enforce the equation of motion: $P\phi = 0$, and we see that this is done by intersecting $\delta^v S$ with the zero-section

$$\begin{array}{c} \mathfrak{F}(M) \\ {\scriptstyle 0}\big\downarrow \\ T^*\mathfrak{F}(M) \end{array} = \left(\begin{array}{ccccc} 0 & \xleftarrow{\;\;0\;\;} & \mathfrak{F}_0(M) & \xleftarrow{\;\;Q\;\;} & \mathfrak{F}_1(M) \\ {\scriptstyle 0}\big\downarrow & & {\scriptstyle (\mathrm{id},0)}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ \mathfrak{F}_1(M) & \xleftarrow{-Q^*\pi_2} & \mathfrak{F}_0(M) \times \mathfrak{F}_0(M) & \xleftarrow{\iota_1 Q} & \mathfrak{F}_1(M) \end{array} \right) \tag{4.1.6}$$

which is the content of the following definition.

Definition 4.1.5. Let $\mathfrak{F}(M)$ be a field complex with equation of motion operator $P : \mathfrak{F}_0(M) \to \mathfrak{F}_0(M)$ that satisfies (4.1.4). Then the *solution complex* $\mathfrak{Sol}(M)$ is the derived critical locus of the associated action functional S (4.1.3): it is the homotopy pullback

$$\begin{array}{ccc} \mathfrak{Sol}(M) & \dashrightarrow & \mathfrak{F}(M) \\ \big\downarrow & h & \big\downarrow {\scriptstyle \delta^v S} \\ \mathfrak{F}(M) & \xrightarrow{\;\;0\;\;} & T^*\mathfrak{F}(M) \end{array} \tag{4.1.7}$$

in the model category $\mathbf{Ch}_{\mathbb{R}}$.

Remark 4.1.6. As in the fourth item of Example 2.1.31 we can think of this pullback as solving an intersection problem, i.e. forming the solution space to our equation of motion. The fact that it is a *homotopy* pullback then means that we are implementing the equations of motion only up to weak equivalence (i.e. up to homology).

Theorem 4.1.7. The complex

$$\mathfrak{Sol}(M) = \Big(\overset{(-2)}{\mathfrak{F}_1(M)} \xleftarrow{\;Q^*\;} \overset{(-1)}{\mathfrak{F}_0(M)} \xleftarrow{\;P\;} \overset{(0)}{\mathfrak{F}_0(M)} \xleftarrow{\;Q\;} \overset{(1)}{\mathfrak{F}_1(M)} \Big) \tag{4.1.8}$$

is a model for the solution complex of Definition 4.1.5.

PROOF. This is Proposition 3.21 in [**BS19a**]. The proof uses the technology of *proper model categories*, see Chapter 13 in [**Hir09**]: since all objects in $\mathbf{Ch}_{\mathbb{K}}$ are fibrant, it is a *right proper* model category by its Corollary 13.1.3. By Proposition 13.3.4 we see that we can equivalently calculate the homotopy pullback of a weakly equivalent diagram, and by Corollary 13.3.8 we see that when either of the maps g or h in a pullback diagram

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow{\scriptstyle h} \\ X & \xrightarrow{\ g\ } & Z \end{array}$$

in $\mathbf{Ch}_{\mathbb{K}}$ is a fibration (i.e. a degreewise surjection), the homotopy pullback of the diagram is weakly equivalent to the ordinary pullback $X \times_Z Y$. So we can calculate the homotopy pullback by replacing one of the two maps 0 or $\delta^v S$ in the diagram (4.1.7) by an equivalent fibration.

We proceed with the zero map (4.1.6). Note that this map is already surjective on the part $\mathfrak{F}(M)$ of the complex $T^*\mathfrak{F}(M) = \mathfrak{F}(M) \times \mathfrak{F}_c(M)^*$. So we introduce the term $\mathfrak{F}_c(M)^* \otimes D$ where D is the acyclic chain complex (3.4.4) concentrated in degrees -1 and 0, now with the field $\mathbb{R}$ instead of $\mathbb{C}$. Define $\tilde{\mathfrak{F}}(M) := \mathfrak{F}(M) \times (\mathfrak{F}_c(M)^* \otimes D)$. The maps $\mathrm{id} : \mathfrak{F}_c(M)^* \to \mathfrak{F}_c(M)^*$ and

$$\begin{array}{c} D \\ \big\downarrow \\ \mathbb{R} \end{array} = \left(\begin{array}{ccc} \overset{(-1)}{\mathbb{R}} & \xleftarrow{\ \mathrm{id}\ } & \overset{(0)}{\mathbb{R}} \\ {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ 0 & \xleftarrow{\ 0\ } & \mathbb{R} \end{array} \right)$$

then define the fibration $\tilde{0} : \tilde{\mathfrak{F}}(M) \to T^*\mathfrak{F}(M)$ by

$$\begin{array}{c} \tilde{\mathfrak{F}}(M) \\ \big\downarrow{\scriptstyle \tilde{0}} \\ T^*\mathfrak{F}(M) \end{array} = \left(\begin{array}{ccccccc} \overset{(-2)}{\mathfrak{F}_1(M)} & \xleftarrow{\mathrm{id}\pi_1 + Q^*\pi_2} & \overset{(-1)}{\mathfrak{F}_1(M) \times \mathfrak{F}_0(M)} & \xleftarrow{(-Q^*,\mathrm{id})\pi_2} & \overset{(0)}{\mathfrak{F}_0(M) \times \mathfrak{F}_0(M)} & \xleftarrow{\ \iota_1 Q\ } & \overset{(1)}{\mathfrak{F}_1(M)} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi_1 \mathrm{id}} & & \big\downarrow{\scriptstyle (\mathrm{id}\pi_1, \mathrm{id}\pi_2)} & & \big\downarrow{\scriptstyle \mathrm{id}} \\ 0 & \longleftarrow & \mathfrak{F}_1(M) & \xleftarrow{\ -Q^*\pi_2\ } & \mathfrak{F}_0(M) \times \mathfrak{F}_0(M) & \xleftarrow{\ \iota_1 Q\ } & \mathfrak{F}_1(M) \end{array} \right)$$

which is weakly equivalent to the zero map $0 : \mathfrak{F}(M) \to T^*\mathfrak{F}(M)$ because D is acyclic.

So we see that a model for $\mathfrak{Sol}(M)$ can be computed by calculating the ordinary pullback in the diagram

$$\begin{array}{ccc} \mathfrak{Sol}(M) & \dashrightarrow & \mathfrak{F}(M) \\ \big\downarrow & & \big\downarrow{\scriptstyle \delta^v S} \\ \tilde{\mathfrak{F}}(M) & \xrightarrow{\ \tilde{0}\ } & T^*\mathfrak{F}(M)\ . \end{array}$$

The chain complex (4.1.8) is then seen to indeed be the pullback in this diagram in $\mathbf{Ch}_{\mathbb{K}}$. $\square$

To illustrate this, consider the examples of Klein-Gordon theory and linear Yang-Mills theory defined above.

EXAMPLE 4.1.8. For the scalar field complex of Example 4.1.2 we have the formally self-adjoint Klein-Gordon operator

$$P = \square - m^2 : \Omega^0(M) \longrightarrow \Omega^0(M)$$

as the equation of motion operator. This yields the usual Klein-Gordon action

$$S(\phi) = \frac{1}{2}\langle \phi, (\Box - m^2)\phi \rangle = \frac{1}{2}\int_M \big(\mathrm{d}\phi \wedge *\mathrm{d}\phi - m^2\phi^2 \mathrm{vol}_M\big)$$

and the solution complex from Theorem 4.1.7 is

$$\mathfrak{Sol}^{KG}(M) = \Big(0 \xleftarrow{\;0\;} \overset{(-1)}{\Omega^0(M)} \xleftarrow{\;\Box - m^2\;} \overset{(0)}{\Omega^0(M)} \xleftarrow{\;0\;} 0 \Big) . \tag{4.1.9}$$

In terms of the BRST/BV formalism, we interpret the two terms in the complex as follows:

- the fields $\phi \in \Omega^0(M)$ in degree 0 are the scalar fields;
- the fields $\phi^\ddagger \in \Omega^0(M)$ in degree -1 are the antifields.

The homology of $\mathfrak{Sol}^{KG}(M)$ is zero in degree -1: this follows from the fact that the first complex in Theorem 2.2.16 is exact in the last term. In degree 0, the homology

$$H_0(\mathfrak{Sol}^{KG}(M)) \cong \big\{\phi \in \Omega^0(M) \mid P\phi = 0\big\} = \ker\big(P\big)$$

is isomorphic to the ordinary Klein-Gordon solution space (2.3.1) we found earlier. So the solution complex of Klein-Gordon theory is quasi-isomorphic to the usual solution space, interpreted as a chain complex concentrated in degree 0. Recall the general tenet of chain complexes: the essential information of a chain complex lies in its homology, so we only consider chain complexes up to quasi-isomorphism. This means that for Klein-Gordon theory we can equivalently work with either the solution complex or the solution space, and the solution complex contains no new information.

Example 4.1.9. For the linear gauge theory complex of Example 4.1.3 we choose the linear Yang-Mills operator

$$P = \delta\mathrm{d} : \Omega^1(M) \longrightarrow \Omega^1(M)$$

as the equation of motion operator. Since d and δ are mutually adjoint with respect to the fiber metric (see Example 2.2.13), P is formally self-adjoint, and it is clear that $PQ = \delta\mathrm{d}\mathrm{d} = 0$. The resulting action is then the usual linear Yang-Mills action

$$S(A) = \frac{1}{2}\langle A, \delta\mathrm{d}A\rangle = \frac{1}{2}\langle \mathrm{d}A, \mathrm{d}A\rangle = \frac{1}{2}\int_M F \wedge *F$$

where $F = \mathrm{d}A \in \Omega^2(M)$ is the usual field strength 2-form. The solution complex from Theorem 4.1.7 is

$$\mathfrak{Sol}^{LYM}(M) = \Big(\overset{(-2)}{\Omega^0(M)} \xleftarrow{\;\delta\;} \overset{(-1)}{\Omega^1(M)} \xleftarrow{\;\delta\mathrm{d}\;} \overset{(0)}{\Omega^1(M)} \xleftarrow{\;\mathrm{d}\;} \overset{(1)}{\Omega^0(M)} \Big) . \tag{4.1.10}$$

We can again interpret the terms in this complex in the language of the BRST/BV formalism:

- the fields $A \in \Omega^1(M)$ in degree 0 are the gauge fields;
- the fields $c \in \Omega^0(M)$ in degree 1 are the ghost fields corresponding to the gauge transformations;
- the fields $A^\ddagger \in \Omega^1(M)$ in degree -1 are the antifields for the gauge field;
- the fields $c^\ddagger \in \Omega^0(M)$ in degree -2 are the antifields for the ghost field.

We can also calculate the homology in each degree:

- In degree 1, we find the zeroth de Rham cohomology of M,
$$H_1\big(\mathfrak{Sol}^{LYM}(M)\big) \cong \ker\big(\mathrm{d} : \Omega^0(M) \to \Omega^1(M)\big) \cong H^0_{\mathrm{dR}}(M) \ .$$
These are the locally constant ghost fields, and as such they describe the gauge transformations that act trivially on the gauge fields, i.e. the degree to which the gauge group acts non-freely. Since $H^0_{\mathrm{dR}}(M) \cong \mathbb{R}^{\pi_0(M)}$ is a vector space of dimension equal to the number of connected components of M, this homology is never zero.
- In degree 0 we find the usual space of gauge equivalence classes of solutions to the linear Yang-Mills equation,
$$H_0\big(\mathfrak{Sol}^{LYM}(M)\big) = \big\{A \in \Omega^1(M) | \delta \mathrm{d} A = 0\big\}\big/\, \mathrm{d}\Omega^0(M) \ .$$
This is the object that would traditionally be considered the solution space of the theory.
- In degree -1 we find
$$\begin{aligned} H_{-1}\big(\mathfrak{Sol}^{LYM}(M)\big) &= \ker\big(\delta : \Omega^1(M) \to \Omega^0(M)\big)\big/\, \delta\mathrm{d}\Omega^1(M) \\ &\cong \ker\big(\delta : \Omega^1(M) \to \Omega^0(M)\big)\big/\, \delta\Omega^2(M) \\ &= H^1_\delta(M) \cong H^{m-1}_{\mathrm{dR}}(M) \end{aligned}$$
where $m = \dim(M)$. The second step follows by a standard argument using partitions of unity and Green operators for the d'Alembert operator $\Box$ that will also be employed in Section 5.6. The last step uses the isomorphism given by the Hodge star between the de Rham cohomology and the δ-homology. This homology captures obstructions to solving the inhomogeneous linear Yang-Mills equation $\delta \mathrm{d} A = j$, where $j \in \Omega^1_\delta(M)$ is a 1-form such that $\delta j = 0$.
- In degree -2 we find the top de Rham cohomology of M,
$$H_{-2}\big(\mathfrak{Sol}^{LYM}(M)\big) = \Omega^0(M)\big/\delta\Omega^1(M) = H^0_\delta(M) \cong H^m_{\mathrm{dR}}(M)$$
by the same arguments as above. Since $M \cong \mathbb{R} \times \Sigma$ by Theorem 2.2.5, $H^m_{\mathrm{dR}}(M) \cong 0$ is trivial.

Note that in contrast to the Klein-Gordon complex, the linear Yang-Mills complex contains more information than the classical gauge orbit space of solutions $H_0\big(\mathfrak{Sol}^{LYM}(M)\big)$. In particular it is not quasi-isomorphic to a complex concentrated in degree 0.

4.2. Observables and Poisson structure, shifted and unshifted

We now turn to the linear observables on a solution complex. A general result from derived algebraic geometry (see [**PTVV13, CPT⁺17, Pri18**]) is that a derived critical locus comes equipped with a shifted Poisson structure. As we will see below, this is true in particular for our solution complex. The crucial feature of both of our examples (Klein-Gordon theory and linear Yang-Mills theory) is that because of the existence of retarded and advanced Green operators for the relevant differential operators, this Poisson structure is homologically trivial and there exist retarded and advanced chain homotopies that trivialize it. Taking the difference, we will see that this allows us to define an unshifted Poisson structure on the solution complex, which is the crucial ingredient for quantizing the theory as in Section 3.4.2.

We first define the chain complex of linear observables on $\mathfrak{Sol}(M)$ in a similar manner to how we defined the linear observables $\mathfrak{L}^{KG}(M)$ in Section 2.3.1.

Definition 4.2.1. For a solution complex $\mathfrak{Sol}(M)$ as in (4.1.8), the *complex of linear observables* $\mathfrak{L}(M)$ on $\mathfrak{Sol}(M)$ is defined as

$$\mathfrak{L}(M) := \Big(\overset{(-1)}{\mathfrak{F}_{1,c}(M)} \xleftarrow{-Q^*} \overset{(0)}{\mathfrak{F}_{0,c}(M)} \xleftarrow{P} \overset{(1)}{\mathfrak{F}_{0,c}(M)} \xleftarrow{-Q} \overset{(2)}{\mathfrak{F}_{1,c}(M)} \Big) \tag{4.2.1}$$

where the subscript c means we are considering sections with compact support. The integration pairing defined by the fiber metrics h_i on the bundles E_i defines evaluation chain maps

$$\begin{array}{rrcl} \langle\,,\,\rangle: & \mathfrak{L}(M)\otimes\mathfrak{Sol}(M) & \longrightarrow & \mathbb{R} \\ \langle\,,\,\rangle: & \mathfrak{Sol}(M)\otimes\mathfrak{L}(M) & \longrightarrow & \mathbb{R} \end{array} \tag{4.2.2}$$

between the linear observables and the solutions.

Note that the choice of signs for the differentials in (4.2.1) ensures that the evaluation pairings (4.2.2) are chain maps.

Clearly, $\mathfrak{Sol}(M)$ and $\mathfrak{L}(M)$ look quite similar: we get the latter by shifting the former, restricting to sections of complex support and adding minus signs to some of the boundary maps. This is the feature that allows us to define the shifted Poisson structure. First, consider the [1]-shifted solution complex,

$$\mathfrak{Sol}(M)[1] = \Big(\overset{(-1)}{\mathfrak{F}_{1}(M)} \xleftarrow{-Q^*} \overset{(0)}{\mathfrak{F}_{0}(M)} \xleftarrow{-P} \overset{(1)}{\mathfrak{F}_{0}(M)} \xleftarrow{-Q} \overset{(2)}{\mathfrak{F}_{1}(M)} \Big)$$

(recall the definition of the shifting of a chain complex from Definition 2.4.6). Then the natural inclusions $\iota : \mathfrak{F}_{i,c}(M) \to \mathfrak{F}_i(M)$ define the chain map

$$\begin{array}{c}\mathfrak{L}(M)\\ \big\downarrow j \\ \mathfrak{Sol}(M)[1]\end{array} := \left(\begin{array}{ccccccc} \overset{(-1)}{\mathfrak{F}_{1,c}(M)} & \xleftarrow{-Q^*} & \overset{(0)}{\mathfrak{F}_{0,c}(M)} & \xleftarrow{P} & \overset{(1)}{\mathfrak{F}_{0,c}(M)} & \xleftarrow{-Q} & \overset{(2)}{\mathfrak{F}_{1,c}(M)} \\ \big\downarrow -\iota & & \big\downarrow -\iota & & \big\downarrow \iota & & \big\downarrow \iota \\ \mathfrak{F}_1(M) & \xleftarrow{-Q^*} & \mathfrak{F}_0(M) & \xleftarrow{-P} & \mathfrak{F}_0(M) & \xleftarrow{-Q} & \mathfrak{F}_1(M) \end{array} \right). \tag{4.2.3}$$

In turn, j can be used to define the shifted Poisson structure as follows.

Definition 4.2.2. The 1-*shifted Poisson structure* Υ on a solution complex $\mathfrak{Sol}(M)$ (4.1.8) with linear observables $\mathfrak{L}(M)$ (4.2.1) is the chain map $\Upsilon : \mathfrak{L}(M)\otimes\mathfrak{L}(M) \to \mathbb{R}[1]$ defined by the composition

$$\begin{array}{ccc} \mathfrak{L}(M)\otimes\mathfrak{L}(M) & \xrightarrow{\qquad\Upsilon\qquad} & \mathbb{R}[1] \\ \big\downarrow \mathrm{id}\otimes j & & \big\uparrow \mathrm{id}\otimes\langle\,,\,\rangle \\ \mathfrak{L}(M)\otimes\mathbb{R}[1]\otimes\mathfrak{Sol}(M) & \xrightarrow{B\otimes\mathrm{id}} & \mathbb{R}[1]\otimes\mathfrak{L}(M)\otimes\mathfrak{Sol}(M) \end{array} \tag{4.2.4}$$

where B is the braiding in $\mathbf{Ch}_{\mathbb{K}}$ (see (2.4.1)) and we have used the isomorphism $\mathfrak{Sol}(M)[1] \cong \mathbb{R}[1]\otimes\mathfrak{Sol}(M)$.

Remark 4.2.3. In BRST/BV terminology, Υ is called the *antibracket*. It is shifted graded antisymmetric. In Section 2.4.1 we saw that this means that for homogeneous elements $\alpha \in \mathfrak{L}_k(M)$, $\beta \in \mathfrak{L}_{-k+1}(M)$,

$$\Upsilon(\alpha,\beta) = \Upsilon(\beta,\alpha)\ .$$

This follows from the fact that the fiber metrics h_i defining $\langle\,,\,\rangle$ are symmetric, the sign in the definition of j (4.2.3) and the symmetry of the solution and observable complexes.

In the terminology of [**CPT⁺17**], Υ is a (-1)-*shifted Poisson structure.*

It turns out that in our two guiding examples of Klein-Gordon theory and linear Yang-Mills theory, the homology class of the chain map j is trivial. We will find two different trivializations of j, which will allow us to define an unshifted Poisson structure on $\mathfrak{Sol}(M)$.

First, we define the past compact and future compact versions of the complex of linear observables $\mathfrak{L}(M)$:

$$\mathfrak{L}_{\mathrm{pc/fc}}(M) := \Big(\overset{(-1)}{\mathfrak{F}_{1,\mathrm{pc/fc}}(M)} \xleftarrow{-Q^*} \overset{(0)}{\mathfrak{F}_{0,\mathrm{pc/fc}}(M)} \xleftarrow{P} \overset{(1)}{\mathfrak{F}_{0,\mathrm{pc/fc}}(M)} \xleftarrow{-Q} \overset{(2)}{\mathfrak{F}_{1,\mathrm{pc/fc}}(M)} \Big) \ . \tag{4.2.5}$$

The sections in these complexes are not observables; we will only use $\mathfrak{L}_{\mathrm{pc/fc}}(M)$ as auxiliary complexes. Note that $\mathfrak{L}(M)$ includes in both $\mathfrak{L}_{\mathrm{pc}}(M)$ and $\mathfrak{L}_{\mathrm{fc}}(M)$ and that the map j factors through both inclusions:

$$\begin{array}{ccc} \mathfrak{L}(M) & \xrightarrow{\quad j \quad} & \mathfrak{Sol}(M)[1] \\ & {\scriptstyle \iota} \searrow \quad \nearrow {\scriptstyle j_{\mathrm{pc/fc}}} & \\ & \mathfrak{L}_{\mathrm{pc/fc}}(M) & \end{array} \tag{4.2.6}$$

where ι is the obvious inclusion chain map (without any signs) and $j_{\mathrm{pc/fc}}$ is the obvious extension of j (4.2.3).

These complexes of past and future compact observables allow for the following definition.

Definition 4.2.4. If $\mathfrak{Sol}(M)$ is a solution complex with linear observable complex $\mathfrak{L}(M)$, a *retarded trivialization* Λ^+ is a contracting homotopy of $\mathfrak{L}_{\mathrm{pc}}(M)$: a homotopy

$$\Lambda^+ \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc}}(M), \mathfrak{L}_{\mathrm{pc}}(M)\big)_1$$

such that $\mathrm{id}_{\mathfrak{L}_{\mathrm{pc}}(M)} = \partial\Lambda^+$. Likewise, an *advanced trivialization* Λ^- is a contracting homotopy of $\mathfrak{L}_{\mathrm{fc}}(M)$: a homotopy

$$\Lambda^- \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{fc}}(M), \mathfrak{L}_{\mathrm{fc}}(M)\big)_1$$

such that $\mathrm{id}_{\mathfrak{L}_{\mathrm{fc}}(M)} = \partial\Lambda^-$.

Lemma 4.2.5. Retarded and advanced trivializations $\Lambda^\pm$ have the following properties:

(1) If $\Lambda^\pm \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_1$ is a retarded or advanced trivialization, then

$$j = \partial(j_{\mathrm{pc/fc}}\, \Lambda^\pm \iota) \tag{4.2.7}$$

and

$$\Upsilon = \partial\big((\mathrm{id} \otimes \langle\, ,\, \rangle)(B \otimes \mathrm{id})(\mathrm{id} \otimes j_{\mathrm{pc/fc}}\, \Lambda^\pm \iota)\big) \ .$$

So the homology classes $[j] = 0$ and $[\Upsilon] = 0$ are trivial.

(2) If $\Lambda^\pm, \tilde{\Lambda}^\pm \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_1$ are two retarded or two advanced trivializations,

$$\tilde{\Lambda}^\pm - \Lambda^\pm = \partial\lambda^\pm$$

where $\lambda^\pm \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_2$ is a 2-chain.

(3) If $\Lambda^+, \Lambda^- \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_1$ is a pair of retarded and advanced trivializations, then

$$\mathcal{G} := j_{\mathrm{pc}}\Lambda^+\iota - j_{\mathrm{fc}}\Lambda^-\iota \in \mathbf{hom}\big(\mathfrak{L}(M), \mathfrak{Sol}(M)[1]\big)_1 \tag{4.2.8}$$

is a 1-cycle: $\partial\mathcal{G} = 0$. So $\mathcal{G}$ defines a chain map

$$\mathcal{G} : \mathfrak{L}(M) \longrightarrow \mathfrak{Sol}(M) \ .$$

PROOF. (1) This follows immediately from the factorization of j (4.2.6) and the definition of Υ (4.2.4) and the fact that all maps involved are chain maps.

(2) Because $\mathrm{id}_{\mathfrak{L}_{\mathrm{pc/fc}}(M)} = \partial\Lambda^{\pm}$, $\mathfrak{L}_{\mathrm{pc/fc}}(M)$ has trivial homology. This implies that the internal hom $\mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)$ also has trivial homology: for any $\rho \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_m$ with $\partial\rho = 0$,

$$\rho = \mathrm{id} \circ \rho = \partial\Lambda^{\pm} \circ \rho = \partial(\Lambda^{\pm} \circ \rho) + \Lambda^{\pm} \circ \partial\rho = \partial(\Lambda^{\pm} \circ \rho)$$

so $[\rho] = 0$. This immediately implies that since

$$\partial(\tilde{\Lambda}^{\pm} - \Lambda^{\pm}) = \mathrm{id}_{\mathfrak{L}_{\mathrm{pc/fc}}(M)} - \mathrm{id}_{\mathfrak{L}_{\mathrm{pc/fc}}(M)} = 0$$

we have $\tilde{\Lambda}^{\pm} - \Lambda^{\pm} = \partial\lambda^{\pm}$ for a 2-chain $\lambda^{\pm} \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_2$.

(3) The first claim follows from item (1):

$$\partial\mathcal{G} = j - j = 0 \ .$$

The second claim is immediate with the bookkeeping of shiftings of chain complexes (Definition 2.4.6) and the isomorphism (2.4.2):

$$\begin{aligned} \mathbf{hom}\big(\mathfrak{L}(M), \mathfrak{Sol}(M)[1]\big)_1 &\cong \mathbf{hom}\big(\mathfrak{L}(M), \mathfrak{Sol}(M)\big)[1]_1 \\ &\cong \mathbf{hom}\big(\mathfrak{L}(M), \mathfrak{Sol}(M)\big)_0 \ . \end{aligned}$$

□

REMARK 4.2.6. Note that the argument in the proof of item (2) of Lemma 4.2.5 extends to higher homotopies: if both $\lambda^{\pm}$ and $\tilde{\lambda}^{\pm}$ are 2-chains that trivialize $\tilde{\Lambda}^{\pm} - \Lambda^{\pm}$ then $\tilde{\lambda}^{\pm} - \lambda^{\pm}$ is a 2-cycle, and since the internal hom has trivial homology, they are related by the boundary of a 3-chain. This argument applies to all higher homotopies, so we can say that retarded/advanced trivializations are unique up to contractible choice.

A pair of retarded and advanced trivializations $\Lambda^{\pm}$ thus defines a chain map (4.2.8)

$$\begin{array}{c} \mathfrak{L}(M) \\ {\scriptstyle \mathcal{G}}\big\downarrow \\ \mathfrak{Sol}(M) \end{array} = \left(\begin{array}{ccccccccc} 0 & \longleftarrow & \mathfrak{F}_{1,c}(M) & \xleftarrow{-Q^*} & \mathfrak{F}_{0,c}(M) & \xleftarrow{P} & \mathfrak{F}_{0,c}(M) & \xleftarrow{-Q} & \mathfrak{F}_{1,c}(M) \\ \big\downarrow & & \big\downarrow {\scriptstyle \mathcal{G}_{-1}} & & \big\downarrow {\scriptstyle \mathcal{G}_0} & & \big\downarrow {\scriptstyle \mathcal{G}_1} & & \big\downarrow \\ \mathfrak{F}_1(M) & \xleftarrow{Q^*} & \mathfrak{F}_0(M) & \xleftarrow{P} & \mathfrak{F}_0(M) & \xleftarrow{Q} & \mathfrak{F}_1(M) & \longleftarrow & 0 \end{array} \right) \tag{4.2.9}$$

by item (3) in Lemma 4.2.5. With this map, we can try and construct an unshifted Poisson structure. However, not every pair of trivializations will do.

DEFINITION 4.2.7. A pair $\Lambda^+, \Lambda^- \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_1$ of retarded and advanced trivializations is called *compatible* if the chain map $\mathcal{G}$ defined in (4.2.8) is formally skew-adjoint with respect to the evaluation maps (4.2.2) $\langle\, , \rangle$: the diagram

$$\begin{array}{ccc} \mathfrak{L}(M) \otimes \mathfrak{L}(M) & \xrightarrow{\mathrm{id}\otimes\mathcal{G}} & \mathfrak{L}(M) \otimes \mathfrak{Sol}(M) \\ {\scriptstyle -\mathcal{G}\otimes\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \langle\, , \rangle} \\ \mathfrak{Sol}(M) \otimes \mathfrak{L}(M) & \xrightarrow{\langle\, , \rangle} & \mathbb{R} \end{array} \tag{4.2.10}$$

commutes.

DEFINITION 4.2.8. Let $\mathfrak{Sol}(M)$ be a solution complex with linear observable complex $\mathfrak{L}(M)$ and $\Lambda^+, \Lambda^- \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_1$ be a pair of compatible retarded and advanced trivializations. Then the corresponding *unshifted Poisson structure* τ on the solution complex $\mathfrak{Sol}(M)$ is the chain map defined by

$$\begin{array}{ccc} \mathfrak{L}(M) \otimes \mathfrak{L}(M) & \xrightarrow{\quad\tau\quad} & \mathbb{R} \\ & {\scriptstyle \mathrm{id}\otimes\mathcal{G}} \searrow \quad \nearrow {\scriptstyle \langle\,,\,\rangle} & \\ & \mathfrak{L}(M) \otimes \mathfrak{Sol}(M) & \end{array} \qquad (4.2.11)$$

REMARK 4.2.9. Note that since $\Lambda^\pm$ are by assumption compatible, the unshifted Poisson structure τ can be defined by either composition in the diagram (4.2.10). This ensures that τ is graded antisymmetric, $\tau B = -\tau$ where B is the braiding in $\mathbf{Ch}_\mathbb{R}$. So τ also defines a chain map

$$\tau : \mathfrak{L}(M) \wedge \mathfrak{L}(M) \longrightarrow \mathbb{R}$$

on the graded exterior product that we again denote by τ, i.e. a 0-cycle $\tau \in \mathbf{hom}\big(\bigwedge^2 \mathfrak{L}(M), \mathbb{R}\big)_0$.

REMARK 4.2.10. As will be more clear from the examples in Sections 4.2.1 and 4.2.2, the chain map $\mathcal{G}$ plays a similar role to that of the causal propagator in ordinary quantum field theory, so we will sometimes refer to it as such. We will call its ingredients

$$\mathcal{G}^\pm := j_{\mathrm{pc/fc}} \Lambda^\pm \iota \in \mathbf{hom}\big(\mathfrak{L}(M), \mathfrak{Sol}(M)[1]\big)_1 \qquad (4.2.12)$$

retarded and advanced Green homotopies because they play a role similar to that of Green operators, and in our examples Green operators will be the crucial ingredient in their definition. Note that they are in general not chain maps $\mathfrak{L}(M) \to \mathfrak{Sol}(M)[1]$ even though they are of the right degree. We will use these Green homotopies extensively in Section 5.6.

COROLLARY 4.2.11. If $\Lambda^\pm, \tilde{\Lambda}^\pm \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_1$ are two pairs of compatible retarded and advanced trivializations with corresponding unshifted Poisson structures $\tau, \tilde{\tau} \in \mathbf{hom}\big(\bigwedge^2 \mathfrak{L}(M), \mathbb{R}\big)_0$, there exists a 1-chain $\rho \in \mathbf{hom}\big(\bigwedge^2 \mathfrak{L}(M), \mathbb{R}\big)_1$ such that $\tilde{\tau} - \tau = \partial\rho$. So $[\tau] = [\tilde{\tau}] \in H_0\big(\mathbf{hom}(\bigwedge^2 \mathfrak{L}(M), \mathbb{R})\big)$.

PROOF. From item (2) in Lemma 4.2.5 we know that $\tilde{\Lambda}^\pm - \Lambda^\pm = \partial\lambda^\pm$ where $\lambda^\pm \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_2$ are 2-chains. So if we define

$$\tilde{\rho} := \langle\,,\,\rangle\big(\mathrm{id} \otimes (j_{\mathrm{pc}}\lambda^+\iota - j_{\mathrm{fc}}\lambda^-\iota)\big)$$

we have

$$\tilde{\tau} - \tau = \partial\tilde{\rho}$$

by the definitions of τ (4.2.11) and $\mathcal{G}$ (4.2.8). The chain homotopy $\tilde{\rho}$ is not necessarily graded antisymmetric, so consider its decomposition

$$\tilde{\rho} = \tilde{\rho}_s + \tilde{\rho}_a = \frac{1}{2}\,\tilde{\rho}\,(\mathrm{id} + B) + \frac{1}{2}\,\tilde{\rho}\,(\mathrm{id} - B)$$

into its symmetric and antisymmetric parts (recall that B is the braiding on $\mathbf{Ch}_\mathbb{R}$). Since both $\tilde{\tau}$ and τ are graded antisymmetric, we find that after antisymmetrizing $\tilde{\tau} - \tau = \partial\tilde{\rho}$,

$$\tilde{\tau} - \tau = \partial\tilde{\rho}_a$$

where $\rho := \tilde{\rho}_a \in \mathbf{hom}\big(\bigwedge^2 \mathfrak{L}(M), \mathbb{R}\big)_1$. □

This ends the general treatment of shifted and unshifted Poisson structures on solution complexes. Note that while the shifted Poisson structure always exists, the crucial new ingredient necessary for the existence of the unshifted Poisson structure is the pair of compatible retarded and advanced trivializations. It is not at all a given that these trivializations exist; our next task now is to investigate whether or not they exist for our two examples. We will see that they do, and that these retarded and advanced trivializations are closely related to retarded and advanced Green operators.

4.2.1. Klein-Gordon theory. Recall that the solution complex $\mathfrak{Sol}^{KG}(M)$ for Klein-Gordon theory is (4.1.9)

$$\mathfrak{Sol}^{KG}(M) = \Big(0 \xleftarrow{\;0\;} \overset{(-1)}{\Omega^0(M)} \xleftarrow{\;\square - m^2\;} \overset{(0)}{\Omega^0(M)} \xleftarrow{\;0\;} 0 \Big)$$

with scalar fields ϕ in degree 0 and antifields $\phi^\ddagger$ in degree -1. The complex of linear observables (see Definition 4.2.1) is then

$$\mathfrak{L}^{KG}(M) = \Big(0 \xleftarrow{\;0\;} \overset{(0)}{\Omega^0_{\mathrm{c}}(M)} \xleftarrow{\;\square - m^2\;} \overset{(1)}{\Omega^0_{\mathrm{c}}(M)} \xleftarrow{\;0\;} 0 \Big)~. \tag{4.2.13}$$

The observables $\psi \in \mathfrak{L}^{KG}_0(M) = \Omega^0_{\mathrm{c}}(M)$ in degree 0 observe the scalar field ϕ,

$$\langle \psi, \phi \rangle = \int_M \psi\, \phi \operatorname{vol}_M ~,$$

and the observables $\alpha \in \mathfrak{L}^{KG}_1(M) = \Omega^0_{\mathrm{c}}(M)$ in degree 1 observe the antifield $\phi^\ddagger$,

$$\langle \alpha, \phi^\ddagger \rangle = \int_M \alpha\, \phi^\ddagger \operatorname{vol}_M ~.$$

Note that as with the solution complex, the homology of $\mathfrak{L}^{KG}(M)$ is only nonzero in degree 0, where it is isomorphic to the vector space $\mathfrak{L}^{KG}(M)$ (2.3.3) we found in Section 2.3.1. So our study of Klein-Gordon theory so far is still quasi-isomorphic to the study there.

The shifted Poisson structure (see Definition 4.2.2) on $\mathfrak{Sol}^{KG}(M)$ pairs the scalar field observables and antifield observables:

$$\Upsilon^{KG} : \mathfrak{L}^{KG}(M) \otimes \mathfrak{L}^{KG}(M) \longrightarrow \mathbb{R}[1] \tag{4.2.14a}$$

with

$$\Upsilon^{KG}(\alpha, \psi) = -\int_M \alpha\, \psi \operatorname{vol}_M = \Upsilon^{KG}(\psi, \alpha) \tag{4.2.14b}$$

for $\alpha \in \mathfrak{L}^{KG}_1(M) = \Omega^0_{\mathrm{c}}(M)$ and $\psi \in \mathfrak{L}^{KG}_0(M) = \Omega^0_{\mathrm{c}}(M)$.

The retarded and advanced trivializations (see Definition 4.2.4) $\Lambda^\pm$ for Klein-Gordon theory have to be trivializations for the past compact and future compact versions of the observable complex $\mathfrak{L}^{KG}(M)$ (4.2.13):

$$\begin{array}{ccccccc}
0 & \xleftarrow{\;0\;} & \Omega^0_{\mathrm{pc/fc}}(M) & \xleftarrow{\;\square - m^2\;} & \Omega^0_{\mathrm{pc/fc}}(M) & \xleftarrow{\;0\;} & 0 \\
{\scriptstyle 0}\big\downarrow & \searrow^{0} & \big\downarrow{\scriptstyle \mathrm{id}} & \searrow^{\Lambda^\pm_0} & \big\downarrow{\scriptstyle \mathrm{id}} & \searrow^{0} & \big\downarrow{\scriptstyle 0} \\
0 & \xleftarrow{\;0\;} & \Omega^0_{\mathrm{pc/fc}}(M) & \xleftarrow{\;\square - m^2\;} & \Omega^0_{\mathrm{pc/fc}}(M) & \xleftarrow{\;0\;} & 0
\end{array}~.$$

We see that in the case of a theory without gauge symmetries, a retarded/advanced trivialization is given by a single map. For Klein-Gordon theory this is

$$\Lambda_0^{\pm} : \Omega^0_{\mathrm{pc/fc}}(M) \longrightarrow \Omega^0_{\mathrm{pc/fc}}(M)$$

and the defining relation $\partial\Lambda^{\pm} = \mathrm{id}_{\mathfrak{L}_{\mathrm{pc/fc}}(M)}$ is equivalent to the two equalities

$$(\Box - m^2)\,\Lambda_0^{\pm} = \mathrm{id} \quad ; \quad \Lambda_0^{\pm}\,(\Box - m^2) = \mathrm{id}\ . \tag{4.2.15}$$

PROPOSITION 4.2.12. Klein-Gordon theory has unique retarded and advanced trivializations $\Lambda^{\pm} \in \mathbf{hom}\big(\mathfrak{L}^{KG}_{\mathrm{pc/fc}}(M), \mathfrak{L}^{KG}_{\mathrm{pc/fc}}(M)\big)_1$:

$$\Lambda_0^{\pm} = G^{\pm} : \Omega^0_{\mathrm{pc/fc}}(M) \longrightarrow \Omega^0_{\mathrm{pc/fc}}(M)$$

where $G^{\pm}$ are the retarded and advanced Green operators for the Klein-Gordon operator $P = \Box - m^2$.

PROOF. The Klein-Gordon operator is normally hyperbolic, so by Theorem 2.2.19 its Green operators exist. We immediately see that the definition of retarded and advanced Green operators (Definition 2.2.14) is equivalent to the defining relations (4.2.15) of retarded and advanced trivializations.

Uniqueness of the trivializations follows either by the uniqueness of Green operators (Remark 2.2.15) or item (2) in Lemma 4.2.5 and the fact that $\mathbf{hom}\big(\mathfrak{L}^{KG}_{\mathrm{pc/fc}}(M), \mathfrak{L}^{KG}_{\mathrm{pc/fc}}(M)\big)_2 = 0$. □

REMARK 4.2.13. Note that the above proof immediately generalizes to any Green hyperbolic differential operator P defining a non-gauge complex $\mathfrak{Sol}^P(M)$: the retarded and advanced trivializations of this theory will always be given by the unique Green operators $G^{\pm}_P$. So one can think of these trivializations as linear gauge theory generalizations of Green operators.

Since the the Klein-Gordon operator is formally self-adjoint, the causal propagator $G = G^+ - G^-$ is skew-adjoint by Remark 2.2.17, so the pair of trivializations $\Lambda^{\pm}$ is compatible in the sense of Definition 4.2.7. So by Definition 4.2.8, $\Lambda^{\pm}$ define the unshifted Poisson structure

$$\tau(\psi_1, \psi_2) = \int_M \psi_1\, G\, \psi_2\, \mathrm{vol}_M \tag{4.2.16}$$

which is the same Poisson structure (2.3.4) that we found in Section 2.3.1.

In conclusion, we find that the chain complex treatment of Klein-Gordon theory as a linear field theory is equivalent to the treatment in Section 2.3.1: in Example 4.1.8 we saw that the solution chain complex $\mathfrak{Sol}^{KG}(M)$ is quasi-isomorphic to the Klein-Gordon solution space (2.3.1) we found; at the start of this section we saw that the complex of linear observables (4.2.13) is quasi-isomorphic to the space of linear observables (2.3.3); and we just found that the (unshifted) Poisson structure (4.2.16) is the same as the Poisson structure (2.3.4). This serves as a good sanity check for the theory constructed so far: it produces a treatment of Klein-Gordon theory that is equivalent to the usual one.

REMARK 4.2.14. If we compare our results with those found in Section 6.4 of [**GR20**], we see that the shifted Poisson structure (4.2.14) and unshifted Poisson structure (4.2.16) agree, up to a minus sign. The operators $\beta_{\pm}$, which are used in Section 6.4.2 in [**GR20**] to relate the Poisson bracket and the factorization product, are analogous to our retarded and advanced trivializations $\Lambda^{\pm}$, though our use of sections with past compact and future compact support means that we do not need to make use of partitions of unity.

4.2.2. Linear Yang-Mills theory. The solution complex for linear Yang-Mills theory is (4.1.10)

$$\mathfrak{Sol}^{LYM}(M) = \Big(\overset{(-2)}{\Omega^0(M)} \xleftarrow{\;\delta\;} \overset{(-1)}{\Omega^1(M)} \xleftarrow{\;\delta\mathrm{d}\;} \overset{(0)}{\Omega^1(M)} \xleftarrow{\;\mathrm{d}\;} \overset{(1)}{\Omega^0(M)} \Big)$$

and recall the names we gave the fields in $\mathfrak{Sol}^{LYM}(M)$ in Example 4.1.9. By Definition 4.2.1 the complex of linear observables is then

$$\mathfrak{L}^{LYM}(M) = \Big(\overset{(-1)}{\Omega^0_{\mathrm{c}}(M)} \xleftarrow{\;-\delta\;} \overset{(0)}{\Omega^1_{\mathrm{c}}(M)} \xleftarrow{\;\delta\mathrm{d}\;} \overset{(1)}{\Omega^1_{\mathrm{c}}(M)} \xleftarrow{\;-\mathrm{d}\;} \overset{(2)}{\Omega^0_{\mathrm{c}}(M)} \Big) \ . \tag{4.2.17}$$

We interpret the elements of the linear observable complex as follows:

- observables $\psi \in \mathfrak{L}^{LYM}_0(M) = \Omega^1_{\mathrm{c}}(M)$ in degree 0 observe the gauge field A,
$$\langle \psi, A\rangle = \int_M \psi \wedge *A$$
- observables $\chi \in \mathfrak{L}^{LYM}_{-1}(M) = \Omega^0_{\mathrm{c}}(M)$ in degree -1 observe the ghost field c,
$$\langle \chi, c\rangle = \int_M \chi\, c\, \mathrm{vol}_M$$
- observables $\alpha \in \mathfrak{L}^{LYM}_1(M) = \Omega^1_{\mathrm{c}}(M)$ in degree 1 observe the antifield $A^\ddagger$,
$$\langle \alpha, A^\ddagger\rangle = \int_M \alpha \wedge *A^\ddagger$$
- observables $\beta \in \mathfrak{L}^{LYM}_2(M) = \Omega^0_{\mathrm{c}}(M)$ in degree 2 observe the antifield $c^\ddagger$,
$$\langle \chi, c\rangle = \int_M \beta\, c^\ddagger\, \mathrm{vol}_M \ .$$

As in Example 4.1.9 the homology of the complex can be computed explicitly:

- In degree -1 we find
$$H_{-1}\big(\mathfrak{L}^{LYM}(M)\big) = H^0_{\mathrm{c},\delta}(M) \cong H^m_{\mathrm{c,dR}}(M)$$
where $m = \dim(M)$, which is the linear dual to $H_1\big(\mathfrak{Sol}^{LYM}(M)\big) \cong H^0_{\mathrm{dR}}(M)$ by Poincaré duality (Theorem 2.4.8). These are the linear observables that probe the gauge transformations that act trivially on the gauge fields.
- In degree 0 we find
$$H_0\big(\mathfrak{L}^{LYM}(M)\big) = \Omega^1_{\mathrm{c},\delta}(M)\big/\delta\mathrm{d}\Omega^1_{\mathrm{c}}(M)$$
which is the usual vector space of gauge-invariant on-shell linear observables, see e.g. [**SDH14, BDS14, BDHS14, FL16, Ben16, BSS17**].
- In degree 1 we find
$$H_1\big(\mathfrak{L}^{LYM}(M)\big) = \Omega^1_{\mathrm{c},\delta\mathrm{d}}(M)\big/\mathrm{d}\Omega^0_{\mathrm{c}}(M) \cong H^1_{\mathrm{c,dR}}(M)$$
again using properties of the Green operators for the d'Alembertian $\Box$. This is the linear dual to $H_{-1}\big(\mathfrak{Sol}^{LYM}(M)\big) \cong H^{m-1}_{\mathrm{dR}}(M)$ by Poincaré duality. As such, these are the linear observables that measure obstructions to solving the inhomogeneous linear Yang-Mills equation.

- In degree 2 we find
$$H_2\big(\mathfrak{L}^{LYM}(M)\big) = H^0_{\mathrm{c,dR}}(M) \cong 0$$
because $M \cong \mathbb{R} \times \Sigma$.

So we find that in homology, the observables in $\mathfrak{L}^{LYM}(M)$ exactly test the homology classes of the corresponding degree of $\mathfrak{Sol}^{LYM}(M)$.

The shifted Poisson structure (see Definition 4.2.2) on $\mathfrak{Sol}^{LYM}(M)$ pairs observables of a field with observables of its antifield:
$$\Upsilon^{LYM} : \mathfrak{L}^{LYM}(M) \otimes \mathfrak{L}^{LYM}(M) \longrightarrow \mathbb{R}[1]$$
is defined as
$$\Upsilon^{LYM}(\alpha, \psi) = -\int_M \alpha \wedge *\psi = \Upsilon^{LYM}(\psi, \alpha)$$
$$\Upsilon^{LYM}(\beta, \chi) = \int_M \beta\,\chi\,\mathrm{vol}_M = \Upsilon^{LYM}(\chi, \beta)$$
for $\psi \in \mathfrak{L}_0^{LYM}(M) = \Omega^1_{\mathrm{c}}(M)$, $\alpha \in \mathfrak{L}_1^{LYM}(M) = \Omega^1_{\mathrm{c}}(M)$, $\chi \in \mathfrak{L}_{-1}^{LYM}(M) = \Omega^0_{\mathrm{c}}(M)$ and $\beta \in \mathfrak{L}_2^{LYM}(M) = \Omega^0_{\mathrm{c}}(M)$.

The retarded and advanced trivializations $\Lambda^\pm$ for linear Yang-Mills theory (see Definition 4.2.4) will now be trivializations of the past and future compact versions of the linear observable complex $\mathfrak{L}^{LYM}(M)$ (4.2.17):

$$\begin{array}{ccccccccccc}
0 & \xleftarrow{\;0\;} & \Omega^0_{\mathrm{pc/fc}}(M) & \xleftarrow{\;-\delta\;} & \Omega^1_{\mathrm{pc/fc}}(M) & \xleftarrow{\;\delta\mathrm{d}\;} & \Omega^1_{\mathrm{pc/fc}}(M) & \xleftarrow{\;-\mathrm{d}\;} & \Omega^0_{\mathrm{pc/fc}}(M) & \xleftarrow{\;0\;} & 0 \\
{\scriptstyle 0}\downarrow & \searrow^{0} & \downarrow{\scriptstyle \mathrm{id}} & \searrow^{\Lambda^\pm_{-1}} & \downarrow{\scriptstyle \mathrm{id}} & \searrow^{\Lambda^\pm_{0}} & \downarrow{\scriptstyle \mathrm{id}} & \searrow^{\Lambda^\pm_{1}} & \downarrow{\scriptstyle \mathrm{id}} & \searrow^{0} & \downarrow{\scriptstyle 0} \\
0 & \xleftarrow{\;0\;} & \Omega^0_{\mathrm{pc/fc}}(M) & \xleftarrow{\;-\delta\;} & \Omega^1_{\mathrm{pc/fc}}(M) & \xleftarrow{\;\delta\mathrm{d}\;} & \Omega^1_{\mathrm{pc/fc}}(M) & \xleftarrow{\;-\mathrm{d}\;} & \Omega^0_{\mathrm{pc/fc}}(M) & \xleftarrow{\;0\;} & 0
\end{array} .$$

So a retarded or advanced trivialization is a triple of maps
$$\Lambda^\pm_{-1} : \Omega^0_{\mathrm{pc/fc}}(M) \longrightarrow \Omega^1_{\mathrm{pc/fc}}(M), \tag{4.2.18a}$$
$$\Lambda^\pm_{0} : \Omega^1_{\mathrm{pc/fc}}(M) \longrightarrow \Omega^1_{\mathrm{pc/fc}}(M), \tag{4.2.18b}$$
$$\Lambda^\pm_{1} : \Omega^1_{\mathrm{pc/fc}}(M) \longrightarrow \Omega^0_{\mathrm{pc/fc}}(M) \tag{4.2.18c}$$
such that
$$-\delta\Lambda^\pm_{-1} = \mathrm{id} \quad ; \quad \delta\mathrm{d}\Lambda^\pm_0 - \Lambda^\pm_{-1}\delta = \mathrm{id} \tag{4.2.18d}$$
$$-\mathrm{d}\Lambda^\pm_1 + \Lambda^\pm_0\delta\mathrm{d} = \mathrm{id} \quad ; \quad -\Lambda^\pm_1\mathrm{d} = \mathrm{id} \quad . \tag{4.2.18e}$$

PROPOSITION 4.2.15. Write $G^\pm : \Omega^1_{\mathrm{pc/fc}}(M) \to \Omega^1_{\mathrm{pc/fc}}(M)$ for the advanced and retarded Green operators of the d'Alembert operator $\Box : \Omega^1(M) \to \Omega^1(M)$ on 1-forms (see Example 2.2.13). Then the maps
$$\Lambda^\pm_{-1} = -G^\pm\mathrm{d}\ , \quad \Lambda^\pm_0 = G^\pm\ , \quad \Lambda^\pm_1 = -\delta G^\pm \tag{4.2.19}$$
define a compatible pair of retarded and advanced trivializations for linear Yang-Mills theory.

PROOF. From the definition of Green operators (see Definition 2.2.14) we immediately see that since $\Box = \mathrm{d}\delta + \delta\mathrm{d}$, the maps (4.2.19) satisfy the equations (4.2.18). Compatibility follows from the fact that since $\Box$ is formally self-adjoint, $G = G^+ - G^-$ is formally skew-adjoint (see Remark 2.2.17), the fact that d and δ are mutually adjoint (2.2.5) and the fact that since d and δ commute with $\Box$, they commute with $G^\pm$ (see Example 2.2.20). $\Box$

Remark 4.2.16. Note that in contrast to the situation for the Klein-Gordon field, the retarded and advanced trivializations for linear Yang-Mills theory will in general not be unique, though they are unique up to contractible choice. See item (2) in Lemma 4.2.5: for any 2-chain $\lambda^\pm \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_2$, $\tilde{\Lambda}^\pm = \Lambda^\pm + \partial\lambda^\pm$ will also be a retarded or advanced trivialization and all such trivializations are of this form.

The retarded and advanced trivializations define the chain map

$$\begin{array}{c}\mathfrak{L}^{LYM}(M)\\ \mathcal{G}\big\downarrow\\ \mathfrak{Sol}^{LYM}(M)\end{array} = \left(\begin{array}{ccccccccc} 0 & \xleftarrow{\quad} & \Omega^0_{\mathrm{c}}(M) & \xleftarrow{-\delta} & \Omega^1_{\mathrm{c}}(M) & \xleftarrow{\delta \mathrm{d}} & \Omega^1_{\mathrm{c}}(M) & \xleftarrow{-\mathrm{d}} & \Omega^0_{\mathrm{c}}(M)\\ \big\downarrow & & G\mathrm{d}\big\downarrow & & G\big\downarrow & & -\delta G\big\downarrow & & \big\downarrow\\ \Omega^0(M) & \xleftarrow{\delta} & \Omega^1(M) & \xleftarrow{\delta\mathrm{d}} & \Omega^1(M) & \xleftarrow{\mathrm{d}} & \Omega^0(M) & \xleftarrow{\quad} & 0 \end{array}\right) \tag{4.2.20}$$

where the relative minus sign compared to (4.2.19) is due to postcomposition with the maps $j_{\mathrm{pc/fc}}$. Moreover, the pair $\Lambda^\pm$ is compatible by Proposition 4.2.15, so $\mathcal{G}$ defines an unshifted Poisson structure τ^{LYM} by Definition 4.2.8. This is

$$\tau^{LYM}(\psi_1, \psi_2) = \int_M \psi_1 \wedge *G\psi_2 = -\tau^{LYM}(\psi_2, \psi_1) \tag{4.2.21a}$$

$$\tau^{LYM}(\alpha, \chi) = \int_M \alpha \wedge *Gd\chi = \tau^{LYM}(\chi, \alpha) \tag{4.2.21b}$$

for $\psi_1, \psi_2 \in \mathfrak{L}^{LYM}_0(M) = \Omega^1_{\mathrm{c}}(M)$, $\alpha \in \mathfrak{L}^{LYM}_1(M) = \Omega^1_{\mathrm{c}}(M)$ and $\chi \in \mathfrak{L}^{LYM}_{-1}(M) = \Omega^0_{\mathrm{c}}(M)$. Here $G = G^+ - G^-$ is the causal propagator for the d'Alembert operator $\Box$. This Poisson structure pairs the gauge field observables with themselves, and pairs the ghost field observables with the observables for the antifield of the gauge field. Note that a different choice of compatible $\tilde{\Lambda}^\pm$ (see Remark 4.2.16) would result in a different but homotopic unshifted Poisson structure $\tilde{\tau}^{LYM}$, see Corollary 4.2.11.

4.3. Linear quantum field theories

Using the results from Section 3.4 we can construct the quantum theories of our chain complex-valued presentations of Klein-Gordon theory and linear Yang-Mills theory. Recall that in Section 3.3.4 we defined the canonical quantization functor

$$\mathfrak{ccr} = \phi_! \circ \mathfrak{heis} : \mathbf{PoissCh}_{\mathbb{R}} \longrightarrow \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$$

which is a homotopical functor: $\mathfrak{heis}$ is homotopical by Proposition 3.4.4 while $\phi_!$ is homotopical by Proposition 3.4.8. Moreover, we saw in Proposition 3.4.6 that for a Poisson complex $(V, \tau) \in \mathbf{PoissCh}_{\mathbb{R}}$ and a 1-chain $\rho \in \underline{\mathrm{hom}}(V \wedge V, \mathbb{R})_1$ (which is the case when we have two pairs of compatible retarded and advanced trivializations, see Corollary 4.2.11), there exists a zig-zag

$$\mathfrak{heis}(V, \tau) \xleftarrow{\sim} H_{V,\tau,\rho} \xrightarrow{\sim} \mathfrak{heis}(V, \tau + \partial\rho)$$

of weak equivalences in $\mathrm{dg}\mathsf{Alg}_{\mathsf{uLie}_{\mathbb{C}}}$. The upshot of all this is the following.

Corollary 4.3.1. Let $\mathfrak{Sol}(M)$ be a solution complex of a field complex $\mathfrak{F}(M)$ with equation of motion operator P, and let $\mathfrak{L}(M)$ be its complex of linear observables. If $\Lambda^\pm$ is a compatible pair of retarded and advanced trivializations and τ is the corresponding unshifted Poisson structure, then

$$\mathfrak{ccr}\big(\mathfrak{L}(M), \tau\big) \in \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$$

is a quantum algebra of observables that only depends on the quasi-isomorphism class of $\mathfrak{L}(M)$. Moreover, if $\tilde{\Lambda}^{\pm}$ is another compatible pair of retarded and advanced trivializations with corresponding Poisson structure $\tilde{\tau}$, there exists a zig-zag of equivalences

$$\mathfrak{ccr}(\mathfrak{L}(M),\tau) \xleftarrow{\sim} A_{\mathfrak{L}(M),\tau,\tilde{\tau}} \xrightarrow{\sim} \mathfrak{ccr}(\mathfrak{L}(M),\tilde{\tau})$$

of corresponding quantum field theories.

This means that both of our examples give well-defined quantum field theories on M.

Example 4.3.2. For Klein-Gordon theory, we found in Section 4.2.1 that the complex $\mathfrak{L}^{KG}(M)$ is quasi-isomorphic to its zero degree homology, which were the on-shell linear observables we found in Section 2.3.1. In fact, we have the quotient map

$$\mathfrak{L}^{KG}(M) \xrightarrow{\sim} H_0\big(\mathfrak{L}^{KG}(M)\big) = \Omega^0_{\mathrm{c}}(M)/(\Box - m^2)\Omega^0_{\mathrm{c}}(M)$$

which is a quasi-isomorphism. The unshifted Poisson structure τ^{KG} (4.2.16) descends to the quotient because $G(\Box - m^2) = 0$, so the quotient map defines a weak equivalence

$$\big(\mathfrak{L}^{KG}(M),\tau^{KG}\big) \xrightarrow{\sim} \big(H_0\big(\mathfrak{L}^{KG}(M)\big),\tau^{KG}\big)$$

in $\mathbf{PoissCh}_{\mathbb{R}}$. By Corollary 4.3.1 we find that the quantum theory $\mathfrak{ccr}\big(\mathfrak{L}^{KG}(M),\tau^{KG}\big)$ is weakly equivalent to $\mathfrak{ccr}\big(H_0\big(\mathfrak{L}^{KG}(M)\big),\tau^{KG}\big)$ in $\mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$, which is the theory $\mathfrak{A}^{KG}_{qu}$ we constructed in Section 2.3.1 viewed as a differential graded algebra concentrated in degree 0.

Example 4.3.3. For linear Yang-Mills theory, we found the solution complex $\mathfrak{Sol}^{LYM}(M)$ in Example 4.1.9 and the complex of linear observables $\mathfrak{L}^{LYM}(M)$ in Section 4.2.2. By Corollary 4.3.1 this results in a consistent quantum theory $\mathfrak{ccr}\big(\mathfrak{L}^{LYM}(M),\tau^{LYM}\big)$ that (up to quasi-isomorphism) does not depend on the expression of $\mathfrak{Sol}^{LYM}(M)$ or our choice of retarded and advanced trivializations (4.2.19).

Explicitly, using the same notation as (2.3.6) and Example 2.3.4 we can write $\hat{A}(\psi)$ with $\psi \in \mathfrak{L}^{LYM}_0(M) = \Omega^1_{\mathrm{c}}(M)$ for the linear quantum observables for the gauge field A. Similarly, we write $\hat{c}(\chi)$ with $\chi \in \mathfrak{L}^{LYM}_{-1}(M) = \Omega^0_{\mathrm{c}}(M)$ for the linear quantum observable for the ghost field c and $\hat{A}^{\ddagger}(\alpha)$ and $\hat{c}^{\ddagger}(\beta)$ with $\alpha \in \mathfrak{L}^{LYM}_1(M) = \Omega^1_{\mathrm{c}}(M)$ and $\beta \in \mathfrak{L}^{LYM}_2(M) = \Omega^0_{\mathrm{c}}(M)$ for the linear quantum observables for the antifields $A^{\ddagger}$ and $c^{\ddagger}$. The nonzero commutation relations in the quantum algebra $\mathfrak{ccr}\big(\mathfrak{L}^{LYM}(M),\tau^{LYM}\big)$ are then

$$\big[\hat{A}(\psi_1),\hat{A}(\psi_2)\big] = i\int_M \psi_1 \wedge *G\psi_2\,\mathbb{1} \tag{4.3.1}$$

$$\big[\hat{A}^{\ddagger}(\alpha),\hat{c}(\chi)\big] = i\int_M \alpha \wedge *Gd\chi\,\mathbb{1} = \big[\hat{c}(\chi),\hat{A}^{\ddagger}(\alpha)\big] \tag{4.3.2}$$

by (4.2.21), (3.4.1) and the defining relations of $\phi_!$ in Section 3.3.4.

4.4. Linear Yang-Mills theory as a homotopy AQFT

So far we have worked on a fixed globally hyperbolic spacetime M. For our field theories to qualify as algebraic field theories, we will have to consider functoriality and the Einstein causality and time-slice axioms.

We will consider both the spacetime category $\mathbf{Loc}$ (see Definition 2.2.8) and $\mathbf{COpens}(\overline{M}) \cong \mathbf{Loc} \downarrow \overline{M}$ for a fixed $\overline{M}$ (see Defintion 2.2.11) here. Recall (2.2.1) that there exists a forgetful functor $\mathbf{COpens}(\overline{M}) \to \mathbf{Loc}$; this will be useful to pull back constructions on all of $\mathbf{Loc}$ to $\mathbf{COpens}(\overline{M})$.

To start with, we need our input data to be sufficiently functorial. So from now on we will assume that the vector bundles F_i, their fiber metrics h_i and the linear differential operator Q of Defintion 4.1.1 are natural on **Loc**. Given this, our assignment of field complexes in Definition 4.1.1 defines a functor

$$\mathfrak{F} : \mathbf{Loc}^{\mathrm{op}} \longrightarrow \mathbf{Ch}_{\mathbb{R}}$$

that assigns to a **Loc**-morphism $f : M \to N$ the chain map $\mathfrak{F}(f) : \mathfrak{F}(N) \to \mathfrak{F}(M)$ that is the pullback of sections $f^* : \mathfrak{F}_i(N) \to \mathfrak{F}_i(M)$ for each i.

Next, we also assume the action (4.1.3) or equivalently the equation of motion operator (4.1.2) is natural. This in turn implies that the assingment of solution complexes (4.1.8) defines a functor

$$\mathfrak{Sol} : \mathbf{Loc}^{\mathrm{op}} \longrightarrow \mathbf{Ch}_{\mathbb{R}}$$

also using the pullback of **Loc**-morphisms.

For the complex of linear observables (see Definition 4.2.1) we note that the pushforward of compactly supported sections $f_* : \mathfrak{F}_{i,c}(M) \to \mathfrak{F}_{i,c}(N)$ corresponding to a **Loc**-morphism $f : M \to N$ also ensures that

$$\mathfrak{L} : \mathbf{Loc} \longrightarrow \mathbf{Ch}_{\mathbb{R}}$$

is a functor. Moreover, the integration pairings (4.2.2) are natural: the diagram

$$\begin{array}{ccc} \mathfrak{L}(M) \otimes \mathfrak{Sol}(N) & \xrightarrow{f_* \otimes \mathrm{id}} & \mathfrak{L}(N) \otimes \mathfrak{Sol}(N) \\ {\scriptstyle \mathrm{id} \otimes f^*} \downarrow & & \downarrow {\scriptstyle \langle\, , \rangle_N} \\ \mathfrak{L}(M) \otimes \mathfrak{Sol}(M) & \xrightarrow{\langle\, , \rangle_M} & \mathbb{R} \end{array}$$

(and the diagram where $\mathfrak{L}(M)$ and $\mathfrak{Sol}(N)$ are swapped) commutes for all **Loc**-morphisms $f : M \to N$.

Given the above assumptions, one immediately sees that the inclusion maps $j : \mathfrak{L} \to \mathfrak{Sol}[1]$ (4.2.3) are natural, in the sense that

$$\begin{array}{ccc} \mathfrak{L}(M) & \xrightarrow{f_*} & \mathfrak{L}(N) \\ {\scriptstyle j_M} \downarrow & & \downarrow {\scriptstyle j_N} \\ \mathfrak{Sol}(M)[1] & \xleftarrow{f^*} & \mathfrak{Sol}(N)[1] \end{array}$$

commutes for all **Loc**-morphisms $f : M \to N$. The shifted Poisson structures $\Upsilon : \mathfrak{L} \otimes \mathfrak{L} \to \mathbb{R}[1]$ are then also natural: the diagram

$$\begin{array}{ccc} \mathfrak{L}(M) \otimes \mathfrak{L}(M) & \xrightarrow{\Upsilon_M} & \mathbb{R}[1] \\ {\scriptstyle f_* \otimes f_*} \downarrow & & \| \\ \mathfrak{L}(N) \otimes \mathfrak{L}(N) & \xrightarrow{\Upsilon_N} & \mathbb{R}[1] \end{array}$$

commutes for all **Loc**-morphisms $f : M \to N$.

REMARK 4.4.1. Note that all the above assumptions on naturality are satisfied in both of our examples of Klein-Gordon theory (see Examples 4.1.2 and 4.1.8) and linear Yang-Mills theory (see Examples 4.1.3 and 4.1.9), where in both cases the pullback is given by the pullback of differential forms and the pushforward is given by the pushforward of compactly supported differential forms.

REMARK 4.4.2. On the category of causal opens of $\overline{M}$, $\mathbf{COpens}(\overline{M})$, we can use the forgetful functor $\mathbf{COpens}(\overline{M}) \to \mathbf{Loc}$ to restrict any functors and natural transformations on **Loc**, as mentioned above. So naturality on **Loc** implies naturality on $\mathbf{COpens}(\overline{M})$, and conversely it is in general harder to establish naturality on **Loc** than on $\mathbf{COpens}(\overline{M})$. This will become apparent later.

For the unshifted Poisson structures, we need the retarded and advanced trivializations $\Lambda^{\pm}$ in Definition 4.2.4 to be appropriately natural. Our results on **Loc** will be weaker than the ones on $\mathbf{COpens}(\overline{M})$ so we will describe both cases separately. Recall Definition 4.2.4 and diagram (4.2.6) for the definitions of ι and $j_{\mathrm{pc/fc}}$.

DEFINITION 4.4.3. Let $\mathbf{C}$ be **Loc** or $\mathbf{COpens}(\overline{M})$ for a $\overline{M} \in \mathbf{Loc}$. Then a $\mathbf{C}$-*natural retarded or advanced trivialization* $\Lambda^{\pm}$ is a family

$$\Lambda^{\pm} = \big\{\Lambda^{\pm}_M \in \mathbf{hom}\big(\mathfrak{L}_{\mathrm{pc/fc}}(M), \mathfrak{L}_{\mathrm{pc/fc}}(M)\big)_1\big\}_{M\in\mathbf{C}}$$

of retarded or advanced trivializations for $M \in \mathbf{C}$ that is $\mathbf{C}$-natural in the sense that

$$f^*(j_{\mathrm{pc/fc}}\Lambda^{\pm}_N \iota) f_* = j_{\mathrm{pc/fc}}\Lambda^{\pm}_M \iota$$

for any morphism $f : M \to N$ in $\mathbf{C}$.

Note that as in Remark 4.4.2, a **Loc**-natural retarded or advanced trivialization can be restricted to a $\mathbf{COpens}(\overline{M})$-natural retarded or advanced trivialization.

PROPOSITION 4.4.4. The unique retarded and advanced trivializations for Klein-Gordon theory defined in Proposition 4.2.12 and the retarded and advanced trivializations for linear Yang-Mills theory defined in Proposition 4.2.15 both define **Loc**-natural retarded and advanced trivializations.

PROOF. This follows from the naturality of Green operators, which is Lemma 3.2 in [**BG12**]. In turn, this lemma hinges on the uniqueness of Green operators (Remark 2.2.15) and their (support) properties (Definition 2.2.14), and the fact that our vector bundles and equation of motion operator are assumed to be suitably natural at the start of this section. □

DEFINITION 4.4.5. Let $\mathbf{C}$ be **Loc** or $\mathbf{COpens}(\overline{M})$ for a $\overline{M} \in \mathbf{Loc}$. Then a $\mathbf{C}$-*natural unshifted Poisson structure* τ on a solution complex $\mathfrak{Sol} : \mathbf{C}^{\mathrm{op}} \to \mathbf{Ch}_{\mathbb{R}}$ is a 0-cycle $\tau \in \mathbf{hom}\big(\bigwedge^2 \mathfrak{L}, \mathbb{R}\big)_0$ in the chain complex

$$\mathbf{hom}\big(\textstyle\bigwedge^2\mathfrak{L}, \mathbb{R}\big) := \lim_{M\in\mathbf{C}^{\mathrm{op}}} \mathbf{hom}\big(\textstyle\bigwedge^2\mathfrak{L}(M), \mathbb{R}\big) \in \mathbf{Ch}_{\mathbb{R}} \tag{4.4.1}$$

where $\mathfrak{L} : \mathbf{C} \to \mathbf{Ch}_{\mathbb{R}}$ is the complex of linear observables of $\mathfrak{Sol}$ described above.

If $\tau, \tilde{\tau} \in \mathbf{hom}\big(\bigwedge^2 \mathfrak{L}, \mathbb{R}\big)_0$ are two $\mathbf{C}$-natural unshifted Poisson structures, a $\mathbf{C}$-*natural homotopy* ρ between τ and $\tilde{\tau}$ is a 1-chain $\rho \in \mathbf{hom}\big(\bigwedge^2 \mathfrak{L}, \mathbb{R}\big)_1$ such that $\tilde{\tau} - \tau = \partial\rho$.

REMARK 4.4.6. Using Definition 2.1.30 of limits, the somewhat abstract definition of τ as an element of a limit complex can be translated to be more concrete. We find a choice of τ to be equivalent to a family of unshifted Poisson structures, i.e. chain maps

$$\big\{\tau_M : \mathfrak{L}(M) \wedge \mathfrak{L}(M) \longrightarrow \mathbb{R}\big\}_{M\in\mathbf{C}}$$

for all $M \in \mathbf{C}$, which are $\mathbf{C}$-natural: the diagram

$$\begin{array}{ccc} \mathfrak{L}(M) \wedge \mathfrak{L}(M) & \xrightarrow{\tau_M} & \mathbb{R} \\ {\scriptstyle f_* \wedge f_*}\downarrow & & \Vert \\ \mathfrak{L}(N) \wedge \mathfrak{L}(N) & \xrightarrow{\tau_N} & \mathbb{R} \end{array}$$

commutes for all $f : M \to N$ in $\mathbf{C}$.

In this language, a $\mathbf{C}$-natural homotopy between τ and $\tilde{\tau}$ is a family of 1-chains

$$\big\{\rho_M \in \mathbf{hom}\big(\textstyle\bigwedge^2 \mathfrak{L}, \mathbb{R}\big)_1\big\}_{M \in \mathbf{C}}$$

such that $\tilde{\tau}_M - \tau_M = \partial \rho_M$ for all $M \in \mathbf{C}$ and $\rho_N(f_* \wedge f_*) = \rho_M$ for all $f : M \to N$ in $\mathbf{C}$.

As one would expect, $\mathbf{C}$-natural retarded and advanced trivializations define $\mathbf{C}$-natural unshifted Poisson structures. However, when it comes to the $\mathbf{C}$-naturality of the homotopies we run into complications when $\mathbf{C} = \mathbf{Loc}$.

Lemma 4.4.7. (1) Let $\mathbf{C}$ be $\mathbf{Loc}$ or $\mathbf{COpens}(\overline{M})$ for a $\overline{M} \in \mathbf{Loc}$. If $\Lambda^{\pm}$ is a compatible pair of $\mathbf{C}$-natural retarded and advanced trivializations, then the componentwise construction of unshifted Poisson structures in Definition 4.2.8 defines a $\mathbf{C}$-natural unshifted Poisson structure on $\mathfrak{Sol}$.

(2) Now let $\mathbf{C}$ be $\mathbf{COpens}(\overline{M})$ for a $\overline{M} \in \mathbf{Loc}$. Then the chain complex $\mathbf{hom}\big(\bigwedge^2 \mathfrak{L}, \mathbb{R}\big)$ defined in (4.4.1) is isomorphic to the mapping complex $\mathbf{hom}\big(\bigwedge^2 \mathfrak{L}(\overline{M}), \mathbb{R}\big)$. Therefore, a $\mathbf{COpens}(\overline{M})$-natural unshifted Poisson structure τ is uniquely determined by its value $\tau_{\overline{M}}$ on $\overline{M}$. Likewise, a $\mathbf{COpens}(\overline{M})$-natural homotopy ρ is uniquely determined by its value $\rho_{\overline{M}}$ on $\overline{M}$.

(3) Again let $\mathbf{C}$ be $\mathbf{COpens}(\overline{M})$ for a $\overline{M} \in \mathbf{Loc}$. If $\Lambda^{\pm}$ and $\tilde{\Lambda}^{\pm}$ are two $\mathbf{COpens}(\overline{M})$-natural compatible pairs of retarded and advanced trivializations, then the corresponding unshifted Poisson structures τ and $\tilde{\tau}$ from item (1) are $\mathbf{COpens}(\overline{M})$-naturally homotopic: $\tilde{\tau} - \tau = \partial \rho$ for a $\mathbf{COpens}(\overline{M})$-natural homotopy ρ.

Proof. (1) This follows directly from the definition of the unshifted Poisson structure (4.2.11), since its components $\mathcal{G}$ and $\langle\,,\,\rangle$ are by assumption $\mathbf{C}$-natural.

(2) Since $\mathbf{COpens}(\overline{M})$ has a terminal object $\overline{M}$, $\mathbf{COpens}(\overline{M})^{\mathrm{op}}$ has an initial object. And one immediately sees that the initial object of a diagram is isomorphic to the limit of that diagram. So we indeed find

$$\mathbf{hom}\big(\textstyle\bigwedge^2 \mathfrak{L}, \mathbb{R}\big) \cong \mathbf{hom}\big(\textstyle\bigwedge^2 \mathfrak{L}(\overline{M}), \mathbb{R}\big)$$

using (4.4.1).

(3) From item (2) we know that a $\mathbf{COpens}(\overline{M})$-natural homotopy ρ between τ and $\tilde{\tau}$ is equivalent to a homotopy $\rho_{\overline{M}}$ between $\tau_{\overline{M}}$ and $\tilde{\tau}_{\overline{M}}$. And by Corollary 4.2.11 we know that such a $\rho_{\overline{M}}$ always exists.

□

Remark 4.4.8. Here we find that the treatments on $\mathbf{Loc}$ and $\mathbf{COpens}(\overline{M})$ diverge: while item (1) of the above lemma holds for both spacetime categories, it is not obvious if item (3) holds for $\mathbf{Loc}$, because (2) does not. Let us consider this. Let $\Lambda^{\pm}$ and $\tilde{\Lambda}^{\pm}$ be two $\mathbf{Loc}$-natural retarded and advanced trivializations with corresponding unshifted Poisson structures τ and $\tilde{\tau}$. By Corollary 4.2.11 we know that for any $M \in \mathbf{Loc}$, we can find a 1-chain

$\rho_M \in \mathbf{hom}\big(\bigwedge^2 \mathfrak{L}(M), \mathbb{R}\big)_1$ such that $\tilde{\tau}_M - \tau_M = \partial \rho_M$. But there is no guarantee that these homotopies ρ_M can be chosen to be **Loc**-natural. The argument for $\mathbf{COpens}(\overline{M})$ hinges on the fact that it has a terminal object, which is not the case for **Loc**.

The upshot of this fact is that with our current results, we do not know if the model for linear Yang-Mills theory as an AQFT on **Loc** constructed below is unique up to natural weak equivalence. In particular, there might exist another compatible pair of **Loc**-natural retarded and advanced trivializations that differs from the ones found in Proposition 4.4.4. In turn, this would lead to a non-homotopic **Loc**-natural unshifted Poisson structure and potentially to a non-equivalent quantization. However, note that this difference would be quite subtle: on any object M, we saw above that τ_M and $\tilde{\tau}_M$ are homotopic. We just do not know if this homotopy is natural.

Let us now investigate the functoriality of our two examples.

Example 4.4.9. For Klein-Gordon theory, see Examples 4.1.2 and 4.1.8, and Section 4.2.1. In Remark 4.4.1 we noted that the naturality assumptions on the vector bundles and the equation of motion are satisfied for Klein-Gordon theory, so that the solution complex $\mathfrak{Sol}^{KG}$, the complex of linear observables $\mathfrak{L}^{KG}$ and the shifted Poisson structure Υ^{KG} are natural on **Loc**. In Proposition 4.4.4 we saw that the unique retarded and advanced trivializations for Klein-Gordon theory are **Loc**-natural, so by item (1) of Lemma 4.4.7 they define a **Loc**-natural unshifted Poisson structure τ^{KG} on $\mathfrak{Sol}^{KG}$, which is given by (4.2.16).

For Klein-Gordon theory, we in fact obtain stronger results than those found in Lemma 4.4.7: in Proposition 4.2.12 we saw that the retarded and advanced trivializations are unique, so τ^{KG} is unique, too. We find that our chain complex-valued treatment of Klein-Gordon theory results in a linear field theory that is quasi-isomorphic to the one constructed in Section 2.3.1 with no other possible choice for the unshifted Poisson structure τ^{KG}.

Considering Remark 4.2.13 we see that this result on uniqueness of τ immediately generalizes to any non-gauge theory with a Green hyperbolic equation of motion operator P: the Green operators will be unique, so the trivializations and unshifted Poisson structure will be, too.

Example 4.4.10. For linear Yang-Mills theory, see Examples 4.1.3 and 4.1.9, and Section 4.2.2. As in the case of Klein-Gordon theory, the naturality of the input data noted in Remark 4.4.1 implies that $\mathfrak{Sol}^{LYM}$, $\mathfrak{L}^{LYM}$ and Υ^{LYM} are **Loc**-natural. And the **Loc**-naturality of the retarded and advanced trivializations found in Proposition 4.4.4 yields a **Loc**-natural unshifted Poisson structure τ^{LYM} on $\mathfrak{Sol}^{LYM}$ by item (1) of Lemma 4.4.7, which is given by (4.2.21).

As noted in Remark 4.4.8, we cannot exclude the possibility of other, non-homotopic, **Loc**-natural compatible pairs of retarded and advanced trivializations existing. In turn, these might define **Loc**-natural unshifted Poisson structures that are not homotopic to our found τ^{LYM}. Of course, as shown in items (2) and (3) of Lemma 4.4.7, when restricting to $\mathbf{COpens}(\overline{M})$ for a $\overline{M} \in \mathbf{Loc}$ the situation improves. In that case, another choice of $\mathbf{COpens}(\overline{M})$-natural compatible pairs of retarded and advanced trivializations defines a homotopic $\mathbf{COpens}(\overline{M})$-natural unshifted Poisson structure $\tilde{\tau}^{LYM}$ because it is determined by $\tilde{\tau}^{LYM}_{\overline{M}}$. So on $\mathbf{COpens}(\overline{M})$, our constructions define a unique homology class

$$[\tau^{LYM}] \in H_0\big(\mathbf{hom}\big(\bigwedge^2 \mathfrak{L}^{LYM}, \mathbb{R}\big)\big) \ .$$

So we find that both $(\mathfrak{L}^{KG}, \tau^{KG})$ and $(\mathfrak{L}^{LYM}, \tau^{LYM})$ define functors

$$\mathbf{C} \longrightarrow \mathbf{PoissCh}_{\mathbb{R}}$$

for both $\mathbf{C} = \mathbf{Loc}$ and $\mathbf{C} = \mathbf{COpens}(\overline{M})$ for $\overline{M} \in \mathbf{Loc}$. To be able to use the results from Section 3.4 to make our linear field theories into quantum field theories, we still need to make sure that these functors satisfy the two field theory axioms, Einstein causality and the time-slice axiom. Recall Definition 2.6.4 of a semi-strict homotopy algebraic linear field theory.

PROPOSITION 4.4.11. Let $(\mathfrak{L}^{KG}, \tau^{KG}) : \mathbf{Loc} \to \mathbf{PoissCh}_{\mathbb{R}}$ be the functor found in Example 4.4.9 and let $(\mathfrak{L}^{LYM}, \tau^{LYM}) : \mathbf{Loc} \to \mathbf{PoissCh}_{\mathbb{R}}$ be the functor found in Example 4.4.10. These functors both define semi-strict homotopy linear field theories on $\overline{\mathbf{Loc}}$: they satisfy strict Einstein causality and the homotopy time-slice axiom.

PROOF. For Klein-Gordon theory, we found the unshifted Poisson structure (4.2.16)

$$\tau(\psi_1, \psi_2) = \int_M \psi_1 \, G \, \psi_2 \, \mathrm{vol}_M$$

and we immediately see that this theory satisfies (strict) Einstein causality by the same argument as in Example 2.3.4.

As for the time-slice axiom, we use the fact noted in Section 4.2.1: the complex of linear observables $\mathfrak{L}^{KG}(M)$ is (through the natural quotient map) quasi-isomorphic to its zeroth homology $H_0\big(\mathfrak{L}^{KG}(M)\big)$, which is the usual vector space of linear observables defined in Section 2.3.1. The map on the homology induced by a Cauchy morphism $f : M \to N$ is then exactly the usual pushforward f_* of equivalence classes of functions on M, which is proven to be an isomorphism in [**BD15**], as mentioned in Example 2.3.4. Because weak equivalences in $\mathbf{PoissCh}_{\mathbb{R}}{}^{\mathbf{C}}$ are defined objectwise (see Section 2.6) the theory $(\mathfrak{L}^{KG}, \tau^{KG})$ is equivalent to $(H_0(\mathfrak{L}^{KG}), \tau^{KG})$ and we see that $(\mathfrak{L}^{KG}, \tau^{KG})$ indeed satisfies the homotopy time-slice axiom.

Next we turn to linear Yang-Mills theory. As in the case of Klein-Gordon theory, the support properties of Green operators and the form (4.2.21) of the unshifted Poisson structure τ^{LYM} immediately imply that the theory satisfies strict Einstein causality.

For the time-slice axiom, let $f : M \to N$ be a Cauchy morphism in $\mathbf{Loc}$. This induces the pushforward $f_* : \Omega^i_{\mathrm{c}}(M) \to \Omega^i_{\mathrm{c}}(N)$ of compactly supported 0- and 1-forms, which are the components of the chain map $\mathfrak{L}(f) : \mathfrak{L}^{LYM}(M) \to \mathfrak{L}^{LYM}(N)$ (see Remark 4.4.1). This chain map in turn induces maps $H_n(f_*)$ on the homology; for the homotopy time-slice axiom to hold, we need these induced maps to be isomorphisms.

Recall from Section 4.2.2 that $H_*\big(\mathfrak{L}^{LYM}(M)\big)$ is nonzero in degrees $n = -1$, 0 and 1. The fact that $H_n(f_*)$ is an isomorphism is known in the literature: in degrees $n = \pm 1$ this follows from homotopy invariance of de Rham cohomology while in degree 0 the proof is more involved, see e.g. the discussion after Definition 4.13 in [**SDH14**]. But we note here that, similarly to Klein-Gordon theory, one can find an explicit quasi-inverse to $\mathfrak{L}(f)$ using Green operators and partitions of unity; we will carry out this calculation in the next chapter, see (5.6.7) and Proposition 5.6.3. □

We find that $(\mathfrak{L}^{KG}, \tau^{KG}) \in \mathbf{hLFT}(\overline{\mathbf{Loc}})$ and $(\mathfrak{L}^{LYM}, \tau^{LYM}) \in \mathbf{hLFT}(\overline{\mathbf{Loc}})$. With our results on the linear quantization functor $\mathfrak{CCR}$ from Section 3.4.2 we can finally state our main result: linear Yang-Mills theory $\mathfrak{A}^{LYM} = \mathfrak{CCR}(\mathfrak{L}^{LYM}, \tau^{LYM})$ is a homotopy AQFT (as is our presentation of Klein-Gordon theory).

Theorem 4.4.12. (1) Let $(\mathfrak{L}^{KG}, \tau^{KG}) : \overline{\mathbf{Loc}} \to \mathbf{PoissCh}_{\mathbb{R}}$ be the functor found in Example 4.4.9. Then the functor

$$\mathfrak{A}^{KG} := \mathfrak{CCR}(\mathfrak{L}^{KG}, \tau^{KG}) : \overline{\mathbf{Loc}} \longrightarrow \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$$

defines a homotopy AQFT on $\overline{\mathbf{Loc}}$.

(2) Let $(\mathfrak{L}^{LYM}, \tau^{LYM}) : \overline{\mathbf{Loc}} \to \mathbf{PoissCh}_{\mathbb{R}}$ be the functor found in Example 4.4.10. Then the functor

$$\mathfrak{A}^{LYM} := \mathfrak{CCR}(\mathfrak{L}^{LYM}, \tau^{LYM}) : \overline{\mathbf{Loc}} \longrightarrow \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$$

defines a homotopy AQFT on $\overline{\mathbf{Loc}}$. Restricting to $\overline{\mathbf{COpens}(\overline{M})}$ for any $\overline{M} \in \overline{\mathbf{Loc}}$ defines a homotopy AQFT on $\overline{M}$. This homotopy AQFT on $\overline{M}$ does not depend, up to natural weak equivalence, on the choice of $\overline{\mathbf{Loc}}$-natural compatible pair of retarded and advanced trivializations.

Proof. Both functors are homotopy AQFTs by Proposition 4.4.11 and the first part of Proposition 3.4.10. Uniqueness (up to natural weak equivalence) of $\mathfrak{A}^{LYM}$ on $\overline{\mathbf{COpens}(\overline{M})}$ follows from item (3) in Lemma 4.4.7 and the second part of Proposition 3.4.10.

Note that the homotopy AQFT $\mathfrak{A}^{KG}$ on $\overline{\mathbf{Loc}}$ does not depend on the choice of $\overline{\mathbf{Loc}}$-natural compatible pair of retarded and advanced trivializations, since these are unique by Proposition 4.2.12. □

We end with some final remarks on the two theories.

Remark 4.4.13. The differential graded algebra $\mathfrak{A}^{KG}(M)$ will in general be much bigger than the algebra $\mathfrak{A}^{KG}(M)$ encountered in Section 2.3.1: while the latter algebra was valued in $\mathbf{Vect}_{\mathbb{C}}$ (so it is concentrated in degree 0 when interpreted as a differential graded algebra) the $\mathfrak{A}^{KG}(M)$ constructed above will in general be nonzero in positive degrees. Nevertheless we find that our chain complex-treatment of Klein-Gordon theory is equivalent (i.e. quasi-isomorphic) to the usual treatment of Section 2.3.1. Indeed, we saw that $\mathfrak{L}^{KG}(M)$ is quasi-isomorphic to its zeroth homology $H_0\big(\mathfrak{L}^{KG}(M)\big)$ in Section 4.2.1, so by Proposition 3.4.10, $\mathfrak{A}^{KG}(M)$ is quasi-isomorphic to $\mathfrak{CCR}\big(H_0\big(\mathfrak{L}^{KG}(M)\big), \tau^{KG}\big)$, which is the algebra concentrated in degree 0 found in Section 2.3.1. Since all these constructions are functorial, we find that the two field theories are equivalent.

Remark 4.4.14. As for linear Yang-Mills theory, we emphasize as in Remark 4.4.8 that while our constructions above define a homotopy AQFT on $\overline{\mathbf{Loc}}$, we cannot exclude the possibility of another $\overline{\mathbf{Loc}}$-natural compatible pair of retarded and advanced trivializations existing. This could yield another $\overline{\mathbf{Loc}}$-natural unshifted Poisson structure $\tilde{\tau}^{LYM}$ that is not homotopic to the one we found, in turn leading to a potentially non-equivalent quantization. The difference would of course have to be quite subtle: on any $\overline{M} \in \mathbf{Loc}$, we did find that $\tau^{LYM}_{\overline{M}}$ and $\tilde{\tau}^{LYM}_{\overline{M}}$ are homotopic.

Lastly, consider the theory $\mathfrak{A}^{LYM}_0 = \mathfrak{CCR}\big(H_0(\mathfrak{L}^{LYM}), \tau^{LYM}\big)$ that assigns the quantum algebra generated by the linear gauge-invariant on-shell observables to a spacetime $M \in \overline{\mathbf{Loc}}$. This is the usual model in the literature, see e.g. [**SDH14, BDS14, BDHS14, FL16, Ben16, BSS17**]. Unlike for Klein-Gordon theory in the above remark, we find that our model $\mathfrak{A}^{LYM}$ is not weakly equivalent to $\mathfrak{A}^{LYM}_0$: on generic spacetimes M, $H_n(\mathfrak{L}^{LYM}(M))$ will be nonzero in degrees $n = -1, 0, 1$, so $H_n(\mathfrak{A}^{LYM}(M))$ will be nonzero in both positive and negative degrees. In particular, $\mathfrak{A}^{LYM}(M)$ and $\mathfrak{A}^{LYM}_0(M)$ will already differ in zeroth homology: while the latter will only be generated by the linear gauge-invariant on-shell observables, $\mathfrak{A}^{LYM}(M)$ will also contain classes obtained by formal products of equal amounts of linear ghost field observables and linear antifield observables.

CHAPTER 5

Relative Cauchy evolution for homotopy AQFTs

Relative Cauchy evolution (RCE) [**BFV03, FV15, FL16**] probes the degree to which the observables of a quantum field theory respond to a perturbation in the spacetime metric. It is fundamentally built on the time-slice axiom, which encodes the dynamics of a theory; we can think of relative Cauchy evolution as testing what happens if we evolve a field over the perturbation. We will deal with some of the issues that arise when trying to treat relative Cauchy evolution in a homotopy quantum field theory, in particular when the time-slice axiom is weakened to the homotopy time-slice axiom, as is the case for the linear homotopy quantum field theories constructed in previous chapters, like the linear Yang-Mills model. The work done here was previously published in the preprint [**BFS21**].

5.1. Relative Cauchy evolution

In this section we review relative Cauchy evolution for regular (vector space-valued) field theories from several perspectives. This will turn out to be useful in the next sections, where we will move on to homotopy (i.e. chain complex-valued) field theories.

Let us start by reviewing the time-slice axiom. Recall Definition 2.2.8 of the category **Loc** of globally hyperbolic spacetimes and Definition 2.2.9 of Cauchy morphisms. A field theory $\mathfrak{A}$ then satisfies time-slice if, when $f : M \to N$ is a Cauchy morphism, $\mathfrak{A}(f)$ is an isomorphism (we consider the strong time-slice axiom here; in the next section we turn to the weaker case where $\mathfrak{A}(f)$ is a quasi-isomorphism). We first encountered it in our various Definitions 2.3.1, 2.3.6, 2.3.8 of algebraic field theories as the axiom that ensures the theory has a dynamical law.

In Section 3.3.2 we saw that the time-slice axiom can be characterized in two different ways: now writing W for the set of Cauchy morphisms in **Loc** we had the orthogonal localization functor

$$L : \overline{\mathbf{Loc}} \longrightarrow \overline{\mathbf{Loc}[W^{-1}]}$$

(recall Definition 3.3.9 of the orthogonal localization). In Proposition 3.3.10 we found the adjunction

$$L_! : \mathbf{QFT}(\overline{\mathbf{Loc}}) \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathbf{QFT}(\overline{\mathbf{Loc}[W^{-1}]}) : L^* \tag{5.1.1}$$

and in Proposition 3.3.12 we saw that it reduces to the adjoint equivalence

$$L_! : \mathbf{QFT}(\overline{\mathbf{Loc}})^{W-\mathrm{const}} \underset{\longleftarrow}{\overset{\longrightarrow}{\sim}} \mathbf{QFT}(\overline{\mathbf{Loc}[W^{-1}]}) : L^* \tag{5.1.2}$$

between quantum field theories satisfying W-constancy (i.e. the time-slice axiom) on the left and quantum field theories on the localized category $\overline{\mathbf{Loc}[W^{-1}]}$ on the right.

Given a field theory $\mathfrak{A}$ that satisfies the time-slice axiom, we define the *relative Cauchy evolution* as follows. Consider a spacetime $(M, g) \in \mathbf{Loc}$ with metric g, and a compactly supported metric perturbation $h \in \Gamma_c(\mathrm{Sym}^2 T^*M)$ that is small enough: $M_h := (M, g+h)$ is also a globally hyperbolic spacetime. We then can define the spacetimes

$$M_\pm := M \setminus J_\mp(\operatorname{supp} h)$$

with the causal past and future of $\operatorname{supp} h$ removed. Both $M_\pm$ naturally embed into both M and M_h, so we have the diagram

$$\begin{array}{ccc} & M_+ & \\ {\scriptstyle i_+}\swarrow & & \searrow{\scriptstyle j_+} \\ M & & M_h \\ {\scriptstyle i_-}\nwarrow & & \nearrow{\scriptstyle j_-} \\ & M_- & \end{array} \tag{5.1.3}$$

in **Loc**. Note that all four morphisms are Cauchy morphisms, though they are not isomorphisms (unless $h = 0$). Applying $\mathfrak{A}$ we then obtain the diagram

$$\begin{array}{ccc} & \mathfrak{A}(M)_+ & \\ {\scriptstyle \mathfrak{A}(i_+)}\swarrow & & \searrow{\scriptstyle \mathfrak{A}(j_+)} \\ \mathfrak{A}(M) & & \mathfrak{A}(M_h) \\ {\scriptstyle \mathfrak{A}(i_-)}\nwarrow & & \nearrow{\scriptstyle \mathfrak{A}(j_-)} \\ & \mathfrak{A}(M_-) & \end{array}$$

of isomorphisms in $\mathsf{Alg}_{\mathsf{As}}(\mathbf{Vect}_{\mathbb{C}})$. The relative Cauchy evolution associated to the pair (M, h) is then the automorphism

$$\mathrm{RCE}_{M,h} := \mathfrak{A}(i_-)\mathfrak{A}(j_-)^{-1}\mathfrak{A}(j_+)\mathfrak{A}(i_+)^{-1} : \mathfrak{A}(M) \longrightarrow \mathfrak{A}(M) \tag{5.1.4}$$

of the algebra of observables $\mathfrak{A}(M)$ that cycles clockwise through the diagram once. The map $\mathrm{RCE}_{M,h}$ is compatible with **Loc**-morphisms and morphisms of field theories, see Propositions 3.7 and 3.8 in [**FV12**].

Let us now consider relative Cauchy evolution from the other perspective offered by Proposition 3.3.12. Using the localization functor $L : \overline{\mathbf{Loc}} \to \overline{\mathbf{Loc}[W^{-1}]}$ we see that the images under L of all four Cauchy morphisms in (5.1.3) are isomorphisms in the localized category $\overline{\mathbf{Loc}[W^{-1}]}$. So we can define the RCE composition

$$r_{M,h} := L(i_-)L(j_-)^{-1}L(j_+)L(i_+)^{-1} : L(M) \longrightarrow L(M)$$

in $\overline{\mathbf{Loc}[W^{-1}]}$. For a given quantum field theory $\mathfrak{B} \in \mathbf{QFT}(\overline{\mathbf{Loc}[W^{-1}]})$ the relative Cauchy evolution associated to (M, h) is then defined as the image under $\mathfrak{B}$ of this automorphism:

$$\mathrm{RCE}_{M,h} := \mathfrak{B}(r_{M,h}) : \mathfrak{B}(L(M)) \longrightarrow \mathfrak{B}(L(M)) \,. \tag{5.1.5}$$

The two definitions (5.1.4) and (5.1.5) of the RCE morphism are of course equivalent when the theories $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent under the adjoint equivalence (5.1.2), i.e. when $\mathfrak{A} = L^*\mathfrak{B}$.

In order to simplify the problem of localizing $\overline{\mathbf{Loc}}$ we now restrict the adjoint equivalence (5.1.2) to the small category consisting only of the four spacetimes in question. As such we define the subcategory

$$\mathbf{C} := \left(\begin{array}{ccc} & M_+ & \\ {}^{i_+}\swarrow & & \searrow^{j_+} \\ M & & M_h \\ {}_{i_-}\nwarrow & & \nearrow_{j_-} \\ & M_- & \end{array} \right) \tag{5.1.6}$$

of **Loc** consisting of four objects and four non-identity morphisms. Note that no pair of morphisms in **C** is orthogonal (recall Definition 3.1.1) so the pullback orthogonality relation (recall Lemma 3.1.3) on **C** along the inclusion functor into $\overline{\mathbf{Loc}}$ is the empty one. So a quantum field theory on $\overline{\mathbf{C}}$ in the sense of Definition 3.1.5 and Example 3.1.6 (which satisfies Einstein causality but not necessarily the time-slice axiom) is simply a functor from **C** to $\mathsf{Alg}_{\mathsf{As}}(\mathbf{Vect}_{\mathbb{C}})$. From now on we suppress the orthogonality data on $\overline{\mathbf{C}}$ and simply write **C**.

We also note that every morphism in **C** is a Cauchy morphism. So the localization of **C** at the Cauchy morphisms (on which the pushforward orthogonality relation of Lemma 3.1.3 along the localization functor is also empty) is the localization $\mathbf{C}[\mathrm{All}^{-1}]$ of **C** at all morphisms. We can explicitly describe a such a localization as follows.

Write $\mathbf{B}\mathbb{Z}$ for the category with one object $*$ and $\mathbb{Z}$ as the set of morphisms from $*$ to itself:

$$\mathbf{B}\mathbb{Z} := \left(* \circlearrowleft \mathbb{Z} \right) . \tag{5.1.7}$$

We will routinely write $n + m$ for the composition of morphism n and m in $\mathbf{B}\mathbb{Z}$ for obvious reasons. Note that the data of a functor $\mathfrak{B}$ from $\mathbf{B}\mathbb{Z}$ to an arbitrary category $\mathbf{D}$ consists of the object $\mathfrak{B} := \mathfrak{B}(*)$ together with a $\mathbb{Z}$ action in $\mathbf{D}$, i.e. a group morphism $\mathbb{Z} \to \mathrm{Aut}_{\mathbf{D}}(\mathfrak{B})$ to the automorphism group of $\mathfrak{B}$ in $\mathbf{D}$, which is characterized by the automorphism $\mathfrak{B}(1)$ (or equivalently $\mathfrak{B}(-1)$). We then claim that the functor

$$\begin{array}{rccc} L : & \mathbf{C} & \longrightarrow & \mathbf{B}\mathbb{Z} \\ & M, M_h, M_+, M_- & \longmapsto & * \\ & i+, j_+, j_- & \longmapsto & 0 \\ & i_- & \longmapsto & 1 \end{array} \tag{5.1.8}$$

is a localization of **C** at all morphisms.

LEMMA 5.1.1. The above functor $L : \mathbf{C} \to \mathbf{B}\mathbb{Z}$ (5.1.8) is a localization of the category **C** (5.1.6) at all morphisms.

PROOF. Recall Definition 2.4.15 of the localization of a category. It is clear that every $L(f)$ is an isomorphism, since every morphism in $\mathbf{B}\mathbb{Z}$ is an isomorphism. Given a functor $F : \mathbf{C} \to \mathbf{D}$ such that every $F(f)$ is an isomorphism for f in $\mathbf{Mor}\,\mathbf{C}$, we define

$$\begin{array}{rccc} F_L : & \mathbf{B}\mathbb{Z} & \longrightarrow & \mathbf{D} \\ & * & \longmapsto & F(M) \\ & 1 & \longmapsto & F(i_-)F(j_-)^{-1}F(j_+)F(i_+)^{-1} \end{array}$$

such that $F_L(n) := F_L(1)^n$. Then F and $F_L L$ are naturally isomorphic by

$$\begin{array}{lccc} \eta_M = \mathrm{id}_{F(M)} : & F(M) & \longrightarrow & F(M) \\ \eta_{M_+} = F(i_+) : & F(M_+) & \longrightarrow & F(M) \\ \eta_{M_h} = F(i_+)F(j_+)^{-1} : & F(M_h) & \longrightarrow & F(M) \\ \eta_{M_-} = F(i_+)F(j_+)^{-1}F(j_-) : & F(M_-) & \longrightarrow & F(M) \end{array} .$$

Lastly, the pullback functor

$$L^* : \mathbf{Fun}(\mathbf{C}[W^{-1}], \mathbf{D}) \longrightarrow \mathbf{Fun}(\mathbf{C}, \mathbf{D})$$

is clearly faithful, since L is surjective on objects. To prove fullness, let $G, H : \mathbf{B}\mathbb{Z} \to \mathbf{D}$ be functors and $\kappa : GL \to HL$ be a natural transformation. Since L sends the three morphisms i_+, j_+ and j_- to 0, we find that $\kappa_M = \kappa_{M_+} = \kappa_{M_h} = \kappa_{M_-}$ by naturality of κ, so all components of κ coincide. Naturality with respect to the last morphism i_- then shows that $\kappa_M G(1) = H(1)\kappa_{M_-} = H(1)\kappa_M$ so $\zeta_* := \kappa_M$ defines a natural transformation ζ between G and H such that $\kappa = \zeta L$. □

Reducing to $\mathbf{C}$ and its localization $\mathbf{B}\mathbb{Z}$ the adjunction (5.1.1) simplifies to

$$L_! : \mathbf{Fun}(\mathbf{C}, \mathsf{Alg}_{\mathsf{As}}) \rightleftarrows \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathsf{Alg}_{\mathsf{As}}) : L^*$$

which becomes an adjoint equivalence when restricting the left category to $\mathbf{Fun}(\mathbf{C}, \mathsf{Alg}_{\mathsf{As}})^{\mathrm{All}}$ (recall that since $\mathbf{C}$ and $\mathbf{B}\mathbb{Z}$ have an empty orthogonality relation, quantum field theories on the categories are simply functors to $\mathsf{Alg}_{\mathsf{As}}$). In this adjunction, the right adjoint L^* is simply the pullback functor along L, while $L_!$ is the left Kan extension along L. In Theorem 2.1.36 we saw that $L_!$ can be computed as a colimit:

$$L_!(\mathfrak{A}) = \mathrm{Lan}_L \mathfrak{A} = \operatorname{colim}\left(L/{*} \xrightarrow{\ \Pi\ } \mathbf{C} \xrightarrow{\ \mathfrak{A}\ } \mathsf{Alg}_{\mathsf{As}} \right) . \tag{5.1.9}$$

Here, $L/{*}$ is the comma category of L over the unique object $* \in \mathbf{B}\mathbb{Z}$ (see Definition 2.1.23) and $\Pi : L/{*} \to \mathbf{C}$ is the canonical forgetful functor (2.1.3).

Objects in the comma category $L/{*}$ are pairs

$$\big(N \in \mathbf{C}, (f : Lc = * \longrightarrow *)\big) = (N, n) \in \mathbf{Ob}\,\mathbf{C} \times \mathbb{Z}$$

and morphisms between (N, n) and (N', n') are morphisms $f : N \to N'$ in $\mathbf{C}$ such that

$$\begin{array}{ccc} L(N) = * & \xrightarrow{\quad L(f)\quad} & L(N') = * \\ & {\scriptstyle n}\searrow \quad \swarrow{\scriptstyle n'} & \\ & * & \end{array}$$

commutes, i.e. such that $n = n' + L(f)$. These conditions imply that between two objects in $L/{*}$ there exists at most one morphism. In fact, recalling the definition of L we find that we can visualize the comma category as

$$\cdots \to (M, n-1) \xleftarrow{i_-} (M_-, n) \xrightarrow{j_-} (M_h, n) \xleftarrow{j_+} (M_+, n) \xrightarrow{i_+} (M, n) \xleftarrow{i_-} (M_-, n+1) \leftarrow \cdots \tag{5.1.10}$$

not writing the identity morphisms. Note that for any of composable morphisms

$$(N, n) \xrightarrow{f} (N', n') \xrightarrow{g} (N'', n'')$$

in $L/{*}$ we have either $f = \mathrm{id}$ or $g = \mathrm{id}$ (or both).

The projection $\Pi : L/{*} \to \mathbf{C}$ forgets the integer n: $\Pi(N, n) = N \in \mathbf{C}$ and $\Pi(f : (N, n) \to (N', n')) = (f : N \to N')$. Recalling the definitions of $\mathbf{C}$ (5.1.6) and the comma category (5.1.10) we see that Π exhibits $L/{*}$ as a kind of universal cover of $\mathbf{C}$. Pictorially, we think of $L/{*}$ as a spiral that is infinitely long in both directions, with four nodes at every level n corresponding to the four objects in $\mathbf{C}$. Adjacent nodes are then connected by a unique morphism. This spiral is projected by Π onto $\mathbf{C}$, which is a loop with four nodes where adjacent nodes are also connected by unique morphisms. We will use this intuitive picture throughout this chapter, hopefully eludicating the large formulas we will encounter. Specifically, we will refer to the index n as the level in the spiral.

Moving on to $L_!(\mathfrak{A})$ we have the canonical maps

$$\iota_{(N,n)} : \mathfrak{A}(N) \longrightarrow L_!(\mathfrak{A})$$

into the colimit (5.1.9). The $\mathbb{Z}$-action on $L_!(\mathfrak{A})$ is then defined through the universal property of the colimit, by

$$\begin{array}{ccc} & \mathfrak{A}(N) & \\ {}^{\iota_{(N,n)}}\swarrow & & \searrow^{\iota_{(N,n+k)}} \\ L_!(\mathfrak{A}) & \xrightarrow{\;L_!(\mathfrak{A})(k)\;} & L_!(\mathfrak{A}) \end{array}$$

for all $(N,n) \in L/*$ and all $k \in \mathbb{Z}$, which we visualize as moving up k levels in the spiral. If we let $\zeta : \mathfrak{A} \to \mathfrak{A}'$ be a natural transformation (i.e. a morphism in $\mathbf{Fun}(\mathbf{C}, \mathsf{Alg}_{\mathsf{As}})$) then $L_!(\zeta) : L_!(\mathfrak{A}) \to L_!(\mathfrak{A}')$ is determined by the functoriality of the colimit:

$$\begin{array}{ccc} \mathfrak{A}(N) & \xrightarrow{\;\zeta_N\;} & \mathfrak{A}'(N) \\ {\scriptstyle \iota_{(N,n)}}\downarrow & & \downarrow{\scriptstyle \iota'_{(N,n)}} \\ L_!(\mathfrak{A}) & \overset{L_!(\zeta)}{\dashrightarrow} & L_!(\mathfrak{A}') \end{array}$$

where $\iota_{(N,n)} : \mathfrak{A}(N) \to L_!(\mathfrak{A})$ and $\iota'_{(N,n)} : \mathfrak{A}'(N) \to L_!(\mathfrak{A}')$ are the canonical maps of the colimits.

Now let $\mathfrak{A} \in \mathbf{Fun}(\mathbf{C}, \mathsf{Alg}_{\mathsf{As}})^{\mathrm{All}}$ be a QFT that satisfies the time-slice axiom. We know that the adjunction $L_! \dashv L^*$ is an adjoint equivalence when restricting to $\mathbf{Fun}(\mathbf{C}, \mathsf{Alg}_{\mathsf{As}})^{\mathrm{All}}$, so $\mathfrak{A}$ is equivalent to $L_!(\mathfrak{A})$. Since we have

$$L(i_-)L(j_-)^{-1}L(j_+)L(i_+)^{-1} = 1$$

in $\mathbf{B}\mathbb{Z}$ by (5.1.8) we find that the relative Cauchy evolution on $L_!(\mathfrak{A})$ is given by

$$L_!(\mathfrak{A})(1) : L_!(\mathfrak{A}) \longrightarrow L_!(\mathfrak{A}) \tag{5.1.11}$$

so by moving up one level in the spiral.

In fact, we can directly relate the two approaches to relative Cauchy evolution (using $\mathfrak{A}$ or using $L_!(\mathfrak{A})$) by noting that $\iota_{(N,n)} : \mathfrak{A}(N) \to L_!(\mathfrak{A})$ is an isomorphism. We then have the commutative diagram

$$\begin{array}{ccc} \mathfrak{A}(M) & \xrightarrow{\;\mathrm{RCE}_{M,h}\;} & \mathfrak{A}(M) \\ {\scriptstyle \iota_{(M,n)}}\downarrow & & \downarrow{\scriptstyle \iota_{(M,n)}} \\ L_!(\mathfrak{A}) & \xrightarrow{\;L_!(\mathfrak{A})(1)\;} & L_!(\mathfrak{A}) \end{array}$$

intertwining our two notions of relative Cauchy evolution.

5.2. Relative Cauchy evolution for homotopy AQFTs

We now turn to homotopy algebraic quantum field theories[1], and the question of how to define relative Cauchy evolution for them. Recall the model structure on (quantum) field categories of Proposition 3.4.1: a weak equivalence of homotopy field theories is a natural weak equivalence (i.e. quasi-isomorphism) of the underlying functors. Crucially, a homotopy quantum field theory $\mathfrak{A}$ satisfies the homotopy time slice axiom (see Definition 2.6.1 and also Definition 3.4.12): if $f : M \to N$ is a Cauchy morphism, $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N)$ is a quasi-isomorphism.

The issue with defining relative Cauchy evolution as in (5.1.4) on general homotopy quantum field theories $\mathfrak{A} : \mathbf{Loc} \to \mathrm{dg}\mathsf{Alg}_{\mathsf{As}}$ is then that quasi-isomorphisms do not in general have inverses. For chain complexes, quasi-inverses exist (and we in fact will find these later in the chapter), but the quasi-inverse to a quasi-isomorphism of associative algebras will in general be an A_∞-morphism, see [**LV12**]. Moreover, we cannot simply invert objectwise: we have to take care of **Loc**-functoriality. Any choices made along the way would lead to a potentially infinite tower of coherences when considering compatibility with **Loc**-morphisms, morphisms of field theories or between RCE-morphisms themselves.

So we follow the strategy of the previous section. Recall that this involved two steps: using the adjunction $L_! \dashv L^*$ (which is Quillen, see Proposition 3.4.3), and considering only one single metric perturbation (M, h), in effect restricting to the category $\mathbf{C} \subseteq \mathbf{Loc}$. Let us consider both steps individually first.

On the one hand, if we restrict our theory from **Loc** to **C**, we do not need to worry about other RCE-morphisms anymore, and **C**-functoriality is a lot easier to handle than **Loc**-functoriality. We would still need to find the A_∞-quasi-inverse, which seems to be a hard but solvable problem, but it is not clear if a good physical interpretation could be extracted.

Conversely, we could keep working on **Loc** and use the Quillen adjunction

$$L_! : \mathbf{hQFT}(\overline{\mathbf{Loc}}) \rightleftarrows \mathbf{hQFT}(\overline{\mathbf{Loc}[W^{-1}]}) : L^* \quad .$$

The goal now would be to prove that the restricted derived unit of this adjunction is a weak equivalence when restricting the left side to the category $\mathbf{hQFT}(\overline{\mathbf{Loc}})^{\mathrm{ho}W}$ of quantum theories satisfying the time-slice axiom. In that case, a theory $\mathfrak{A} \in \mathbf{hQFT}(\overline{\mathbf{Loc}})^{\mathrm{ho}W}$ satisfying homotopy time-slice could equivalently be described by the theory $\mathbb{L}L_!(\mathfrak{A}) \in \mathbf{hQFT}(\overline{\mathbf{Loc}[W^{-1}]})$, where $\mathbb{L}L_!$ is the left derivation of $L_!$. The theory $\mathbb{L}L_!(\mathfrak{A})$ assigns to any Cauchy morphism an isomorphism, not just a quasi-isomorphism, so all RCE-morphisms (5.1.5) are defined in a coherent way: since all inverses are strict there would be no need to work with A_∞-quasi-inverses or with higher coherences. But finding a workable model for the localization $\mathbf{Loc}[W]^{-1}$ is beyond the scope of this thesis.

Employing both strategies we consider the adjunction

$$L_! : \mathbf{Fun}(\mathbf{C}, \mathrm{dg}\mathsf{Alg}_{\mathsf{As}}) \rightleftarrows \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathrm{dg}\mathsf{Alg}_{\mathsf{As}}) : L^* \tag{5.2.1}$$

remembering that since both **C** and $\mathbf{B}\mathbb{Z}$ have empty orthogonality relations, quantum field theories on them are simply functors to $\mathrm{dg}\mathsf{Alg}_{\mathsf{As}}$. L^* is the usual pullback functor, and $L_!$ is the differential graded version of the functor $L_!$ (5.1.9). As above, the goal now would

[1] Recall from Section 2.6 that in this text we use homotopy AQFT to mean semi-strict homotopy AQFT. Since we restrict to the category **C** with empty orthogonality data in this chapter, this has no bearing on the results presented here.

be to prove that the derived unit of this Quillen adjunction is a weak equivalence when restricting the left category in the adjunction to the category $\mathbf{Fun}(\mathbf{C}, \mathrm{dg}\mathsf{Alg}_{\mathsf{As}})^{\mathrm{hoAll}}$ of quantum theories satisfying the time-slice axiom. This would again mean that a field theory $\mathfrak{A} \in \mathbf{Fun}(\mathbf{C}, \mathrm{dg}\mathsf{Alg}_{\mathsf{As}})^{\mathrm{hoAll}}$ satisfying homotopy time-slice could equivalently be described by the theory $\mathbb{L}L_!(\mathfrak{A}) \in \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathrm{dg}\mathsf{Alg}_{\mathsf{As}})$, i.e. the differential graded algebra $\mathbb{L}L_!(\mathfrak{A})(*)$ with strict $\mathbb{Z}$-action $\mathbb{L}L_!(\mathfrak{A})(1)$ describing the relative Cauchy evolution as in (5.1.11).

Proving that the restriction of the derived unit of the above adjunction is a weak equivalence using simplicial methods does not seem impossible, but we did not find a complete proof before this thesis was due. So we reduce our scope once more, now restricting to quantizations of linear field theories $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau)$ where $\mathfrak{CCR}$ is the linear quantization functor (3.4.2) and $(\mathfrak{L}, \tau)$ is a homotopy linear field theory in the sense of Definition 2.6.4. The plan for the rest of this chapter is as follows: in Section 5.3 we will prove a rectification theorem as described above for the adjunction

$$L_! : \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}}) \rightleftarrows \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{Ch}_{\mathbb{K}}) : L^*$$

on linear observables $\mathfrak{L}$, finding a notion of relative Cauchy evolution for $\mathfrak{L}$. In Section 5.4 we will find a Poisson structure $\tau_{\mathbb{L}}$ on the strictified linear observables $\mathbb{L}L_!(\mathfrak{L})$. In Section 5.5 we use these results to prove a rectification theorem for linear quantum field theories and show that for the linear observables, the strict relative Cauchy evolution is homotopic to the naive version. Finally, in Section 5.6 we will treat relative Cauchy evolution for the example of linear Yang-Mills theory of Chapter 4 and calculate the stress-energy tensor.

5.3. A rectification theorem for the linear observables

The first step is to prove a rectification theorem for the linear observables $\mathfrak{L}$ such as the complexes from Definition 4.2.1. We are working on the category $\mathbf{C}$ (5.1.6), so the linear observables are functors

$$\mathfrak{L} : \mathbf{C} \longrightarrow \mathbf{Ch}_{\mathbb{K}}.$$

In turn, we saw that $\mathbf{C}$ localizes to the category $\mathbf{B}\mathbb{Z}$ through the functor L (5.1.8), which yields the Quillen adjunction

$$L_! : \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}}) \rightleftarrows \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{Ch}_{\mathbb{K}}) : L^* \tag{5.3.1}$$

where we endow both sides with the projective model structure. L^* is the pullback functor along L, and $L_!$ is the left Kan extension along L.

The time-slice axiom for a theory of linear observables $\mathfrak{L}$ says that $\mathfrak{L}$ assigns weak equivalences in $\mathbf{Ch}_{\mathbb{K}}$ (i.e. quasi-isomorphisms) to all Cauchy morphisms in $\mathbf{C}$, that is, to all morphisms in $\mathbf{C}$. We write $\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})^{\mathrm{hoAll}}$ for the category of functors satisfying homotopy time-slice; as mentioned in Section 5.2 our goal of proving a rectification theorem is then proving that the derived unit of the above adjunction is a weak equivalence when restricting the left side to $\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})^{\mathrm{hoAll}}$.

Let us give all the ingredients for this theorem: the underived and the derived functors, and the derived unit and counit. Recall the expression of $L_!$ (5.1.9), the projective model structure on functors valued in $\mathbf{Ch}_{\mathbb{K}}$ (Example 2.4.23), the process of deriving functors (Section 2.4.5) and the bar construction (Section 2.4.6).

First, the right adjoint L^*. Recall that a functor $Y : \mathbf{B}\mathbb{Z} \to \mathbf{Ch}_{\mathbb{K}}$ is characterized by the object $Y := Y(*)$ and the automorphism $Y(1) : Y \to Y$. The pullback of such a Y under L^*

is then the functor $L^*Y = YL$ which we visualize as

$$\begin{array}{ccccc} & & Y & & \\ & \overset{\text{id}}{\swarrow} & & \overset{\text{id}}{\searrow} & \\ Y & & & & Y \\ & \underset{Y(1)}{\nwarrow} & & \underset{\text{id}}{\nearrow} & \\ & & Y & & \end{array} \tag{5.3.2}$$

recalling the shape (5.1.6) of the category $\mathbf{C}$. As noted in Section 2.4.6, since each object is fibrant in the projective model structure on $\mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{Ch}_{\mathbb{K}})$, we do not need to fibrantly replace and the right adjoint L^* is the right derived functor $\mathbb{R}L^*$.

Then, the left adjoint. The explicit expression of the underived functor $L_!$ is (5.1.9); to a functor $X \in \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$ it assigns the object

$$L_! X = \bigoplus_{(N,n)\in L/*} X(N)_n \Big/ \big(X(\mathsf{s}f)_n - X(f)(X(\mathsf{s}f))_{n+L(f)}\big)_{f\in \mathbf{Mor}\, L/*} \,. \tag{5.3.3}$$

The sum $\bigoplus_{(N,n)\in L/*} X(N)_n$ consists of a copy of $X(N)$ for each object $(N,n) \in L/*$. The chain complex that we quotient out by then is generated by all elements

$$x_n - X(f)(x)_{n'}$$

where $x_n \in X(N)_n$ and $X(f)(x)_{n'} \in X(N')_{n'}$ for $f : (N,n) \to (N',n')$ in $L/*$, so $n' = n + L(f)$. We can visualize this quotient as an appropriate chain complex $X(N)$ at each node in the spiral, linked through the quotient to each of its neighbours. The $\mathbb{Z}$-action is then given by moving up and down the spiral:

$$L_!(X)(k) : \begin{array}{ccc} L_! X & \longrightarrow & L_! X \\ [x_n] & \longmapsto & [x_{n+k}] \end{array} .$$

To derive $L_!$ we use the bar construction, effectively unfolding this quotient into a bicomplex such that the quotient is retrieved in zeroth vertical homology. From Section 2.4.6 we see that

$$\mathbb{L}L_!(X) = \mathrm{Tot}^{\oplus}\big(\overline{B_\Delta}(L, \mathbf{C}, X)\big) \in \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{Ch}_{\mathbb{K}}) \tag{5.3.4}$$

is a model for the left derived functor $\mathbb{L}L_!$, where $\overline{B_\Delta}$ is the bar construction and $\mathrm{Tot}^{\oplus}$ is the functor totalizing the resulting bicomplex into a chain complex. To simplify notation we write

$$\tilde{X} := \overline{B_\Delta}(L, \mathbf{C}, X) \in \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{bCh}_{\mathbb{K}}) \tag{5.3.5a}$$

for the bicomplex valued functor obtained by the bar construction. This functor can be explicitly described as follows. The underlying bicomplex $\tilde{X} := \tilde{X}(*)$ is

$$\tilde{X} = \big(\, \tilde{X}_{\bullet,0} \xleftarrow{\;\delta\;} \tilde{X}_{\bullet,1} \,\big) \,. \tag{5.3.5b}$$

There are no terms in higher vertical degree, since there are no pairs of composable morphisms $(N,n) \xrightarrow{f} (N',n') \xrightarrow{g} (N'',n'')$ in $L/*$ for which f and g are not the identity. In turn,

$$\tilde{X}_{\bullet,0} = \bigoplus_{(N,n)\in L/*} X(N)_n = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{N\in\mathbf{C}} X(N)_\bullet \tag{5.3.5c}$$

is the sum we found in the expression of $L_!X$ (5.3.3) over all nodes in the spiral, and

$$\tilde{X}_{\bullet,1} = \bigoplus_{(N,n)\in L/*} \bigoplus_{\substack{(N \xleftarrow{f} N')\in \mathbf{C} \\ f\neq \mathrm{id}}} X(N')_\bullet = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{\substack{f\in \mathbf{Mor}\,\mathbf{C} \\ f\neq \mathrm{id}}} X(\mathsf{s}f)_\bullet \tag{5.3.5d}$$

is precisely the complex that implements the quotient in (5.3.3). So we have in a sense unfolded the quotient $L_!X$: the complex $\tilde{X}_{\bullet,0}$ still yields a complex at every node in our spiral, while $\tilde{X}_{\bullet,1}$ implements the quotient linking to neighbouring nodes.

For $\tilde{X}_{\bullet,0}$ we write (N,n,x) for the element $x\in X(N)$ in the summand indexed by (N,n), and similarly for $\tilde{X}_{\bullet,1}$ we write (f,n,x) for the element $x\in X(N)$ in the summand indexed by (f,n). We will also use this notation for the elements of the totalization $\mathrm{Tot}^\oplus(\tilde{X}) = \mathbb{L}L_!(X)$. The vertical differential $\delta : \tilde{X}_{\bullet,1}\to\tilde{X}_{\bullet,0}$ is then given by

$$\delta(f,n,x) = (-1)^{|x|}\big((\mathsf{s}f, n+L(f), x) - (\mathsf{t}f, n, X(f)x)\big) \tag{5.3.6}$$

where $|x|$ is the degree of x in $X(\mathsf{s}f)$. The $\mathbb{Z}$-action is then very explicitly given by addition:

$$\begin{array}{rccc} \tilde{X}(k): & \tilde{X} & \longrightarrow & \tilde{X} \\ & (N,n,x) & \longmapsto & (N,n+k,x) \\ & (f,n,x) & \longmapsto & (f,n+k,x) \end{array} .$$

The derived counit $\tilde{\epsilon} : \mathbb{L}L_!L^* \to \mathrm{id}$ at the component $Y\in \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{Ch}_\mathbb{K})$ is then given by (2.4.5):

$$\begin{array}{rccc} \tilde{\epsilon}_Y: & \mathbb{L}L_!L^*Y & \longrightarrow & Y \\ & (N,n,y) & \longmapsto & Y(n)(y) \\ & (f,n,y) & \longmapsto & 0 \end{array} \tag{5.3.7}$$

which is clearly equivariant with respect to both $\mathbb{Z}$-actions, so it is a natural transformation of functors from $\mathbf{B}\mathbb{Z}$ to $\mathbf{Ch}_\mathbb{K}$. Note that every component in the sums $\widetilde{L^*Y}_{\bullet,0}$ and $\widetilde{L^*Y}_{\bullet,1}$ (5.3.5) is Y.

For the derived unit, we have the cofibrant replacement (2.4.6)

$$Q(X) = \mathrm{Tot}^\oplus\big(\overline{B_\Delta}(\mathrm{id},\mathbf{C},X)\big) \in \mathbf{Fun}(\mathbf{C},\mathbf{Ch}_\mathbb{K})$$

through applying the bar construction to the identity functor $\mathrm{id}:\mathbf{C}\to\mathbf{C}$. Like $\mathbb{L}L_!(X)$ above it can be explicitly described. To simplify notation, write $X^\Delta := \overline{B_\Delta}(\mathrm{id},\mathbf{C},X) \in \mathbf{Fun}(\mathbf{C},\mathbf{bCh}_\mathbb{K})$. For $N\in\mathbf{C}$ we then have the bicomplex

$$X^\Delta(N) = \big(\, X^\Delta(N)_{\bullet,0} \xleftarrow{\ \delta\ } X^\Delta(N)_{\bullet,1} \,\big) \in \mathbf{bCh}_\mathbb{K} \tag{5.3.8a}$$

with again no terms in higher vertical degrees because composable morphisms in $\mathbf{C}$ are very restricted. In turn, the terms appearing in the bicomplex are

$$X^\Delta(N)_{\bullet,0} = \bigoplus_{\substack{g\in\mathbf{Mor}\,\mathbf{C} \\ \mathsf{t}g=N}} X(\mathsf{s}g)_\bullet \tag{5.3.8b}$$

and

$$X^\Delta(N)_{\bullet,1} = \bigoplus_{\substack{(g,f)\in\mathbf{Mor}_2\,\mathbf{C} \\ \mathsf{t}g=N, f\neq\mathrm{id}}} X(\mathsf{s}f)_\bullet\ , \tag{5.3.8c}$$

where we use the notation $\mathbf{Mor}_2\,\mathbf{C}$ for the composable pairs of morphisms in $\mathbf{C}$. We write $(g,x)\in X^\Delta(N)_{\bullet,0}$ for elements of vertical degree 0, and $(g,f,x)\in X^\Delta(N)_{\bullet,1}$ for elements of vertical degree 1. Note that the condition $f \neq \mathrm{id}$ implies that $g = \mathrm{id}$ in this sum defining $X^\Delta(N)_{\bullet,1}$, so any element (g, f, x) of vertical degree 1 will always have $g = \mathrm{id}$. The vertical differential δ is given by

$$\delta(g, f, x) = (-1)^{|x|}\big((gf, x) - (g, X(f)x)\big)$$

for any $(g, f, x) \in X^\Delta(N)_{\bullet,1}$. **C**-functoriality is then given by given by postcomposition: for $h : N \to N'$ in **C**,

$$\begin{array}{rccc} X^\Delta(h): & X^\Delta(N) & \longrightarrow & X^\Delta(N') \\ & (g,x) & \longmapsto & (hg,x) \\ & (g,f,x) & \longmapsto & (hg,f,x) \end{array} .$$

It is a general result (Lemma 13.3.3 in [**Fre09**]) that $Q(X)$ is a naturally weakly equivalent to X through the natural weak equivalence $q_X : Q(X) \to X$ which has components

$$\begin{array}{rccc} q_{X,N}: & Q(X)(N) & \longrightarrow & X(N) \\ & (g,x) & \longmapsto & X(g)x \\ & (g,f,x) & \longmapsto & 0 \end{array} . \tag{5.3.9}$$

In our simple case this is explicit with the quasi-inverse $s_X : X \to Q(X)$ given by

$$\begin{array}{rccc} s_{X,N}: & X(N) & \longrightarrow & Q(X)(N) \\ & x & \longmapsto & (\mathrm{id}_N, x) \end{array} \tag{5.3.10}$$

so q (and s) are indeed quasi-isomorphisms.

With the resolution $Q(X)$ under control, the derived unit $\tilde\eta : Q \to L^*\mathbb{L}L_!$ is then given at components X and $N \in \mathbf{C}$ by (2.4.7)

$$\begin{array}{rccc} \tilde\eta_{X,N}: & Q(X)(N) & \longrightarrow & L^*\mathbb{L}L_!(X)(N) = \mathbb{L}L_!(X) \\ & (g,x) & \longmapsto & (L(g), \mathsf{s}g, x) \\ & (g,f,x) & \longmapsto & (L(g), f, x) \end{array} . \tag{5.3.11}$$

With all the ingredients in place, we can now prove the rectification theorem.

THEOREM 5.3.1. (1) The derived counit $\tilde\epsilon$ (5.3.7) of the Quillen adjunction $L_! \dashv L^*$ (5.3.1) is a weak equivalence.

(2) When we restrict to the full subcategory $\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_\mathbb{K})^{\mathrm{hoAll}} \subseteq \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_\mathbb{K})$ of functors such that any $X(f)$ for $f \in \mathbf{Mor}\,\mathbf{C}$ is a quasi-isomorphism, the derived unit (5.3.11) is also a weak equivalence.

PROOF. Item (1): For the derived counit, let $Y \in \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{Ch}_\mathbb{K})$ and recall the shape of L^*Y (5.3.2). To prove that the derived counit is a weak equivalence, we can construct an explicit quasi-inverse to the chain map underlying the component $\tilde\epsilon_Y$ (5.3.7).

So define

$$\begin{array}{rccc} \kappa: & Y & \longrightarrow & \mathbb{L}L_!(L^*Y) \\ & y & \longmapsto & (M, 0, y) \end{array}$$

that embeds Y at the node $(M, 0)$ in the spiral. Clearly κ does not preserve the $\mathbb{Z}$-action, but it does not need to: we just need it to prove that ϵ_Y is a quasi-isomorphism. It is immediate that $\tilde\epsilon_Y\kappa = \mathrm{id}$.

For the other composition we will find a homotopy $\rho \in \mathbf{hom}(\mathbb{L}L_!(L^*Y), \mathbb{L}L_!(L^*Y))_1$ such that

$$\kappa\tilde\epsilon_Y - \mathrm{id} = \partial\rho \; . \tag{5.3.12}$$

For any node (N,n) in the spiral $L/*$ (5.1.10), write

$$(M,0) =: (N_{-1},n_{-1}) \xleftarrow{f_0} (N_0,n_0) \xrightarrow{f_1} (N_1,n_1) \xleftarrow{f_2} \cdots \overset{f_m}{\leftrightarrow} (N_m,n_m) := (N,n) \tag{5.3.13}$$

for the shortest zig-zag between $(M,0)$ and (N,n) in the spiral, and write

$$f_i : (N_i^{\mathsf{s}}, n_i^{\mathsf{s}}) \longrightarrow (N_i^{\mathsf{t}}, n_i^{\mathsf{t}})$$

for the morphisms in this zig-zag. For an element of vertical degree 0 $(N,n,y) \in \mathbb{L}L_!(L^*Y) = \mathrm{Tot}^{\oplus}(\widetilde{L^*Y})$ we then define the homotopy

$$\rho(N,n,y) = -(-1)^{|y|}\sum_{i=0}^{m}(-1)^i(f_i, n_i^{\mathsf{t}}, Y(n-n_i^{\mathsf{s}})y)$$

and we let $\rho(N,f,y) = 0$ for elements of vertical degree 1.

We now find that (5.3.12) holds: for (N,n,y) of vertical degree 0,

$$\begin{aligned}
\partial\rho(N,n,y) &= (\delta + \mathrm{d})\rho(N,n,y) + \rho(\delta+\mathrm{d})(N,n,y)\\
&= \delta\rho(N,n,y)\\
&= -\sum_{i=0}^{m}(-1)^i\big((N_i^{\mathsf{s}}, n_i^{\mathsf{s}}, Y(n-n_i^{\mathsf{s}})y) - (N_i^{\mathsf{t}}, n_i^{\mathsf{t}}, Y(n-n_i^{\mathsf{t}})y)\big)\\
&= (M,0,Y(n)y) - (N,n,y)\\
&= (\kappa\tilde{\epsilon}_Y - \mathrm{id})(N,n,y)
\end{aligned}$$

where the horizontal differential d drops out because $Y(k)$ is a chain map and we use $\delta(N,n,y) = 0$. Furthermore, all neighbouring terms from the zig-zag (5.3.13) cancel except the terms from the boundary, yielding the desired result. Similarly, we find that

$$\partial\rho(f,n,y) = \rho\delta(f,n,y) = -(f,n,y)$$

for any element (f,n,y) of vertical degree 1. So we indeed have (5.3.12), and ϵ_Y indeed is a quasi-isomorphism.

Item (2): For the derived unit, let $X \in \mathbf{Fun}(\mathbf{C},\mathbf{Ch}_{\mathbb{K}})^{\mathrm{hoAll}}$ now be a functor such that for any $f \in \mathbf{Mor}\,\mathbf{C}$, $X(f)$ is a quasi-isomorphism. We need to prove that the component

$$\tilde{\eta}_{X,N} : Q(X)(N) \longrightarrow \mathbb{L}L_!(X)$$

(5.3.11) is a quasi-isomorphism for each $N \in \mathbf{C}$. Since the category $\mathbf{C}$ is connected, every $Q(X)(f)$ is a quasi-isomorphism and every $\mathbb{L}L_!(X)(f)$ is an isomorphism, it suffices to prove that the component $\tilde{\eta}_{X,M}$ at M is a quasi-isomorphism by the 2-out-of-3 property and naturality of $\tilde{\eta}_X$. In turn, $\tilde{\eta}_{X,M}$ is a quasi-isomorphism if and only

$$\phi := s_{X,M}\tilde{\eta}_{X,M} : \begin{array}{ccc} X(M) & \longrightarrow & \mathbb{L}L_!(X)\\ x & \longmapsto & (M,0,x)\end{array}$$

is a quasi-isomorphism, where $s_{X,M}$ is the quasi-inverse to $q_{X,M}$ (5.3.10).

The strategy now is to consider the short exact sequence of complexes

$$0 \longrightarrow X(M) \xrightarrow{\ \phi\ } \mathbb{L}L_!(X) \longrightarrow \mathrm{coker}(\phi) \longrightarrow 0$$

which is exact because ϕ is injective. If we then prove that the homology of $\mathrm{coker}(\phi)$ is trivial, the long exact sequence of homologies associated to this sequence then implies that

ϕ is an isomorphism on homology, i.e. a quasi-isomorphism. To do this, we will use spectral sequences, filtering $\mathrm{coker}(\phi)$.

First, recall the structure (5.3.5) of the complex $\mathbb{L}L_!(X)$ and note that by dividing out the copy of $X(M)$ at the node $(M,0)$ in the spiral, the cokernel admits the direct sum decomposition $\mathrm{coker}(\phi) = F^L \oplus F^R$. Here $F^L \in \mathbf{Ch}_{\mathbb{K}}$ is the chain complex associated with the objects and morphisms to the left of the node $(M,0)$ in the spiral (5.1.10), i.e. of negative and zero level n, and $F^R \in \mathbf{Ch}_{\mathbb{K}}$ is the chain complex associated with the objects and morphisms to the right of the node $(M,0)$ in the spiral, i.e. of positive level n. We can prove that $\mathrm{coker}(\phi)$ has trivial homology by proving this for F^L and F^R independently.

For notational convenience we write

$$\Big((M,0) \xleftarrow{f_0^R} N_0^R \xrightarrow{f_1^R} N_1^R \xleftarrow{f_2^R} \cdots\Big) := \Big((M,0) \xleftarrow{i_-} (M_-,1) \xrightarrow{j_-} (M_h,1) \xleftarrow{j_+} \cdots\Big)$$

for the part of the spiral associated to F^R. Then F^R is

$$F^R = \mathrm{Tot}^{\oplus}\Big(\bigoplus_{k\geq 0} X(N_k^R) \xleftarrow{\ \delta^R\ } \bigoplus_{k\geq 0} X(\mathsf{s}f_k^R) \Big)$$

with vertical differential on $(f_k,k,x) \in \bigoplus\limits_{k\geq 0} X(\mathsf{s}f_k^R)$ given by

$$\delta^R(f_k,k,x) = \begin{cases} (-1)^{|x|}(0,x) & k=0 \\ (-1)^{|x|}((k,x)-(k-1,X(f_k^R)x)) & k\geq 2 \text{ even} \\ (-1)^{|x|}((k-1,x)-(k,X(f_k^R)x)) & k \text{ odd}\ . \end{cases}$$

The difference with the vertical differential δ (5.3.6) at $k=0$ is a result of the fact that we divided out $X(M)$ at level 0 in the cokernel.

We then define the filtration $0 \subseteq F_0^R \subseteq F_1^R \subseteq \cdots \subseteq F_p^R \subseteq \cdots \subseteq F^R$ by restricting the direct sums defining F^R to $k \leq p$,

$$F_p^R := \mathrm{Tot}^{\oplus}\Big(\bigoplus_{0\leq k\leq p} X(N_k^R) \xleftarrow{\ \delta^R\ } \bigoplus_{0\leq k\leq p} X(\mathsf{s}f_k^R) \Big)\ .$$

This filtration is exhaustive and bounded below. The quotients

$$F_p^R/F_{p-1}^R = \begin{cases} \mathrm{cone}(\mathrm{id} : X(N_p^R) \to X(N_p^R)) & p\geq 0 \text{ even} \\ \mathrm{cone}(-X(f_p^R) : X(N_{p-1}^R \to X(N_p^R)) & p\geq 0 \text{ odd} \end{cases}$$

are then all mapping cone complexes (see Definition 1.5.1 in [**Wei95**]), each associated to a quasi-isomorphism because $X \in \mathbf{Fun}(\mathbf{C},\mathbf{Ch}_{\mathbb{K}})^{\mathrm{hoAll}}$ assigns quasi-isomorphisms to all maps in $\mathbf{C}$. By Corollary 1.5.4 in [**Wei95**] we have $H_\bullet(F_p^R/F_{p-1}^R) = 0$ for all p, and by the convergence theorem for spectral sequences (Theorem 5.5.1 in [**Wei95**]) we have $H_\bullet(F^R) = 0$.

A completely analogous argument also shows that $H_\bullet(F^L) = 0$, so we find $H_\bullet(\mathrm{coker}(\phi)) = 0$ as required. □

Let us summarize the results of this section for the next one. If $\mathfrak{L} : \mathbf{C} \to \mathbf{PoissCh}_{\mathbb{K}}$ is linear field theory on $\mathbf{C}$ satisfying the homotopy time-slice axiom, we also write $\mathfrak{L}$ for the underlying functor $\mathfrak{L} \in \mathbf{Fun}(\mathbf{C},\mathbf{Ch}_{\mathbb{K}})^{\mathrm{hoAll}}$ (we will treat the Poisson structure in the next section). Theorem 5.3.1 and $q_{\mathfrak{L}} : Q(\mathfrak{L}) \to \mathfrak{L}$ (5.3.9) then yield the zig-zag of weak equivalences

$$\begin{array}{ccccc} & & Q(\mathfrak{L}) & & \\ & {}^{q_{\mathfrak{L}}}\swarrow\sim & & \sim\searrow^{\tilde{\eta}_{\mathfrak{L}}} & \\ \mathfrak{L} & & & & L^*\mathbb{L}L_!(\mathfrak{L}) \end{array} \tag{5.3.14}$$

in $\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$ (and even in $\mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})^{\mathrm{hoAll}}$ since $Q(\mathfrak{L}) \in \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})^{\mathrm{hoAll}}$ also). This zig-zag shows that the original theory $\mathfrak{L}$ that satisfies the homotopy time-slice axiom is equivalent to the theory $L^*\mathbb{L}L_!(\mathfrak{L})$, which satisfies the strict time-slice axiom. The price to pay is that $L^*\mathbb{L}L_!(\mathfrak{L})$ will be a theory that assigns the same, larger complex $\mathbb{L}L_!(\mathfrak{L})$ to every object in $\mathbf{C}$.

The relative Cauchy evolution on these complexes $\mathbb{L}L_!(\mathfrak{L})$ is then very simple: we have

$$\begin{array}{rccc} \mathrm{rce}_{M,h} := \mathbb{L}L_!(\mathfrak{L})(1) : & \mathbb{L}L_!(\mathfrak{L}) & \longmapsto & \mathbb{L}L_!(\mathfrak{L}) \\ & (N, n, x) & \longmapsto & (N, n+1, x) \\ & (f, n, x) & \longmapsto & (f, n+1, x) \end{array} \tag{5.3.15}$$

where we write rce for the relative Cauchy evolution on linear observables, in contrast to the notation RCE for quantum field theories. So similar to the reformulation (5.1.11) this version of RCE is simply addition by 1 to the level in the spiral (5.1.10).

5.4. A Poisson structure on the strictified linear observables

We now turn to the Poisson structure. To form a quantum theory $\mathfrak{A}^{\mathrm{st}}$ from the strictified theory $L^*\mathbb{L}L_!(\mathfrak{L})$ we need a Poisson structure on the complex $\mathbb{L}L_!(\mathfrak{L})$. We then also need to make sure that this theory is equivalent to the quantum theory $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau) \in \mathbf{hQFT}(\mathbf{C})$ of the original model. Our strategy has two steps. First, we define the Poisson structure

$$\tau_{\mathbb{L}} : \mathbb{L}L_!(\mathfrak{L}) \wedge \mathbb{L}L_!(\mathfrak{L}) \longrightarrow \mathbb{K} \tag{5.4.1}$$

on the theory $\mathbb{L}L_!(\mathfrak{L}) : \mathbf{B}\mathbb{Z} \to \mathbf{Ch}_{\mathbb{K}}$. Then, we prove that this Poisson structure is compatible with the original τ on $\mathfrak{L}$ by showing that they are homotopic when pulled back to $Q(\mathfrak{L})$ in the diagram (5.3.14).

To define the Poisson structure $\tau_{\mathbb{L}}$ on $\mathbb{L}L_!(\mathfrak{L})$ we will use zig-zagging maps related to the spiral (5.1.10). First, choose a quasi inverse

$$\mathfrak{L}(f)^{-1} : \mathfrak{L}(N') \to \mathfrak{L}(N) \tag{5.4.2}$$

to the quasi-isomorphism $\mathfrak{L}(f) : \mathfrak{L}(N) \to \mathfrak{L}(N')$ for every morphism $f : N \to N'$ in $\mathbf{C}$. For any two nodes $(N, n), (N', n') \in L/*$ in the spiral we then note that there exists a unique chain of zig-zags of minimal length between them. Applying $\mathfrak{L}$ to the spiral and using $\mathfrak{L}(f)^{-1}$ for the arrows in the opposite direction this yields the *zig-zagging chain maps*

$$Z^{N,n}_{N',n'} : \mathfrak{L}(N') \longrightarrow \mathfrak{L}(N) \tag{5.4.3}$$

through these shortest zig-zags in $L/*$. For example, the zig-zag

$$(M_-, n) \xrightarrow{j_-} (M_h, n) \xleftarrow{j_+} (M_+, n) \xrightarrow{i_+} (M, n) \tag{5.4.4a}$$

results in the map

$$Z^{M,n}_{M_-,n} = \mathfrak{L}(i_+)\mathfrak{L}(j_+)^{-1}\mathfrak{L}(j_-) : \mathfrak{L}(M_-) \longrightarrow \mathfrak{L}(M) \tag{5.4.4b}$$

while the zig-zag

$$(M_-, n+1) \xrightarrow{i_-} (M, n) \xleftarrow{i_+} (M_+, n) \xrightarrow{j_+} (M_h, n) \tag{5.4.4c}$$

yields the map

$$Z^{M_h,n}_{M_-,n+1} = \mathfrak{L}(j_+)\mathfrak{L}(i_+)^{-1}\mathfrak{L}(i_-) : \mathfrak{L}(M_-) \longrightarrow \mathfrak{L}(M_h) \ . \tag{5.4.4d}$$

Note that for the second map we have reversed the direction of the spiral compared to (5.1.10) to keep the source object on the left; of course, how we write a zig-zag down does not influence the zig-zagging chain map, and we will in fact use this freedom below when defining the zig-zagging homotopies. We further note that the zig-zagging chain maps (5.4.3) are manifestly invariant under the $\mathbb{Z}$-action:

$$Z^{N,n+k}_{N',n'+k} = Z^{N,n}_{N',n'} \tag{5.4.5}$$

for all $k \in \mathbb{Z}$.

We now need to introduce homotopies that compare the zig-zagging chain map $Z^{N,n}_{N',n'}$ with its neighbours. First, choose chain homotopies $\lambda_f \in \mathbf{hom}(\mathfrak{L}(N'), \mathfrak{L}(N'))_1$ and $\gamma_f \in \mathbf{hom}(\mathfrak{L}(N), \mathfrak{L}(N))_1$ that witness that $\mathfrak{L}(f)^{-1}$ is a quasi-inverse to $\mathfrak{L}(f)$,

$$\mathfrak{L}(f)\mathfrak{L}(f)^{-1} - \mathrm{id} = \partial\lambda_f \tag{5.4.6a}$$

$$\mathfrak{L}(f)^{-1}\mathfrak{L}(f) - \mathrm{id} = \partial\gamma_f \tag{5.4.6b}$$

and a chain 2-homotopy $\xi_f \in \mathbf{hom}(\mathfrak{L}(N'), \mathfrak{L}(N'))_2$ that governs compatibility between $\mathfrak{L}(f)$ and the 1-homotopies,

$$\mathfrak{L}(f)\gamma_f - \lambda_f\mathfrak{L}(f) = \partial\xi_f \ . \tag{5.4.7}$$

For the identity morphisms id_N, we choose $\mathfrak{L}(\mathrm{id}_N)^{-1} = \mathrm{id}_{\mathfrak{L}(N)}$ to be the identity and the homotopies $\lambda_{\mathrm{id}_N} = 0$, $\gamma_{\mathrm{id}_N} = 0$ and $\xi_{\mathrm{id}_N} = 0$ to be zero.

Remark 5.4.1. The quasi-inverse and the corresponding homotopy data always exist. One can show this directly, by choosing projections from $\mathfrak{L}(N)$ onto $\ker \mathrm{d}$ and from $\ker \mathrm{d}$ onto $H_\bullet(\mathfrak{L}(N))$. There is also a nice 2-categorical argument, hinging on the fact that $\mathbf{Ch}_\mathbb{K}$ can be enriched to a 2-category where the 2-morphisms are homotopy classes of chain homotopies and the equivalences are the quasi-isomorphisms. The existence of our homotopy data then follows from the fact that every equivalence can be upgraded to an adjoint equivalence (see Exercise 2.2 in [**Lac10**]): $\mathfrak{L}(f)^{-1}$ is the right adjoint to $\mathfrak{L}(f)$, γ_f and $-\lambda_f$ are the unit and counit of this adjunction, respectively, and the equation for ξ_f is one of the two triangle identities. We will not need the homotopy $\tilde{\xi}_f$ resulting from the other triangle identity

$$\gamma_f\mathfrak{L}(f)^{-1} - \mathfrak{L}(f)^{-1}\lambda_f = \partial\tilde{\xi}_f \ .$$

In the linear Yang-Mills example of Section 5.6 we will explicitly calculate these homotopies using the familiar tools of Green operators and partitions of unity.

Now we use the basic homotopies λ_f, γ_f and ξ_f to define the homotopies that govern the behaviour of the zig-zagging chain maps $Z^{N,n}_{N',n'}$ when composing them with maps $\mathfrak{L}(f)$ and $\mathfrak{L}(f)^{-1}$.

Let us first consider composition from the left. Let (N', n') be an object in $L/\!*$ and $(f, n) : (\mathsf{s}f, n + L(f)) \to (\mathsf{t}f, n)$ be a morphism in $L/\!*$. A *zig-zagging chain homotopy for the left composition* is then a chain homotopy

$$\Lambda^{f,n}_{N',n'} \in \mathbf{hom}(\mathfrak{L}(N'), \mathfrak{L}(\mathsf{t}f))_1 \tag{5.4.8a}$$

such that

$$\mathfrak{L}(f) Z^{\mathsf{s}f, n+L(f)}_{N',n'} - Z^{\mathsf{t}f,n}_{N',n'} = \partial \Lambda^{f,n}_{N',n'} \ . \tag{5.4.8b}$$

We construct $\Lambda^{f,n}_{N',n'}$ as follows. First, note that there are two possibilities for the direction of f compared to the zig-zag: it either points *along* the zig-zag,

$$(N', n') \xleftrightarrow{\text{zig-zag}} (\mathsf{s}f, n + L(f)) \xrightarrow{\quad f \quad} (\mathsf{t}f, n) \quad , \tag{5.4.9a}$$

or it points *against* the zig-zag,

$$(N', n') \xleftrightarrow{\text{zig-zag}} (\mathsf{t}f, n) \xleftarrow{\quad f \quad} (\mathsf{s}f, n + L(f)) \quad . \tag{5.4.9b}$$

As with the examples of zig-zagging chain maps (5.4.4) we note that the orientation of the zig-zag as written down here might be opposite to the one of the spiral (5.1.10). We then define $\Lambda^{f,n}_{N',n'}$ to be

$$\Lambda^{f,n}_{N',n'} := \begin{cases} 0 & \text{for direction (5.4.9}a) \\ \lambda_f Z^{\mathsf{t}f,n}_{N',n'} & \text{for direction (5.4.9}b) \ . \end{cases} \tag{5.4.10}$$

To check that (5.4.8) holds we note that when f points along the zig-zag (5.4.9a) we have

$$\mathfrak{L}(f) Z^{\mathsf{s}f,n+L(f)}_{N',n'} = Z^{\mathsf{t}f,n}_{N',n'}$$

so $\Lambda^{f,n}_{N',n'} = 0$ indeed suffices. On the other hand, if f points against the zig-zag (5.4.9b),

$$\mathfrak{L}(f) Z^{\mathsf{s}f,n+L(f)}_{N',n'} - Z^{\mathsf{t}f,n}_{N',n'} = (\mathfrak{L}(f)\mathfrak{L}(f)^{-1} - \mathrm{id}) Z^{\mathsf{t}f,n}_{N',n'} \tag{5.4.11}$$

$$= \partial \lambda_f Z^{\mathsf{t}f,n}_{N',n'} \tag{5.4.12}$$

$$= \partial \Lambda^{f,n}_{N',n'} \tag{5.4.13}$$

because $Z^{\mathsf{t}f,n}_{N',n'}$ is a chain map. So $\Lambda^{f,n}_{N',n'}$ is indeed a zig-zagging chain homotopy for the left composition.

Similarly, let (N, n) be an object in $L/*$ and $(f', n') : (\mathsf{s}f', n' + L(f')) \to (\mathsf{t}f', n')$ be a morphism in $L/*$. A *zig-zagging chain homotopy for the right composition* is then a chain homotopy

$$\Gamma^{N,n}_{f',n'} \in \mathbf{hom}(\mathfrak{L}(\mathsf{s}f'), \mathfrak{L}(N))_1 \tag{5.4.14a}$$

such that

$$Z^{N,n}_{\mathsf{t}f',n'} \mathfrak{L}(f') - Z^{N,n}_{\mathsf{s}f',n'+L(f')} = \partial \Gamma^{N,n}_{f',n'} \ . \tag{5.4.14b}$$

We again distinguish the directions of f: along the zig-zag,

$$(\mathsf{s}f', n' + L(f')) \xrightarrow{\quad f' \quad} (\mathsf{t}f', n') \xleftrightarrow{\text{zig-zag}} (N, n) \quad , \tag{5.4.15a}$$

or against the zig-zag,

$$(\mathsf{t}f', n') \xleftarrow{\quad f' \quad} (\mathsf{s}f', n' + L(f')) \xleftrightarrow{\text{zig-zag}} (N, n) \quad . \tag{5.4.15b}$$

Then

$$\Gamma^{N,n}_{f',n'} := \begin{cases} 0 & \text{for direction (5.4.15}a) \\ Z^{N,n}_{\mathsf{s}f',n'+L(f')} \gamma_{f'} & \text{for direction (5.4.15}b) \ . \end{cases} \tag{5.4.16}$$

defines a zig-zagging chain homotopy for the right composition; proving this is analogous to the case of left composition.

Lastly, let $(f,n) : (\mathsf{s}f, n+L(f)) \to (\mathsf{t}f,n)$ and $(f',n') : (\mathsf{s}f', n'+L(f')) \to (\mathsf{t}f',n')$ be morphisms in $L/*$. A *zig-zagging 2-homotopy* is a 2-homotopy

$$\Xi^{f,n}_{f',n'} \in \mathbf{hom}(\mathfrak{L}(\mathsf{s}f'), \mathfrak{L}(\mathsf{t}f))_2 \tag{5.4.17a}$$

such that

$$\mathfrak{L}(f)\Gamma^{\mathsf{s}f,n+L(f)}_{f',n'} - \Gamma^{\mathsf{t}f,n}_{f',n'} + \Lambda^{f,n}_{\mathsf{s}f',n'+L(f')} - \Lambda^{f,n}_{\mathsf{t}f',n'}\mathfrak{L}(f') = \partial \Xi^{f,n}_{f',n'} \ . \tag{5.4.17b}$$

We now distinguish five cases for the directions of f and f' relative to the zig-zag:

$$(\mathsf{s}f', n'+L(f')) \xrightarrow{\ f'\ } (\mathsf{t}f',n') \xleftrightarrow{\text{zig-zag}} (\mathsf{s}f, n+L(f)) \xrightarrow{\ f\ } (\mathsf{t}f,n) \quad , \tag{5.4.18a}$$

$$(\mathsf{s}f', n'+L(f')) \xrightarrow{\ f'\ } (\mathsf{t}f',n') \xleftrightarrow{\text{zig-zag}} (\mathsf{t}f,n) \xleftarrow{\ f\ } (\mathsf{s}f, n+L(f)) \quad , \tag{5.4.18b}$$

$$(\mathsf{t}f',n') \xleftarrow{\ f'\ } (\mathsf{s}f', n'+L(f')) \xleftrightarrow{\text{zig-zag}} (\mathsf{s}f, n+L(f)) \xrightarrow{\ f\ } (\mathsf{t}f,n) \quad , \tag{5.4.18c}$$

$$(\mathsf{t}f',n') \xleftarrow{\ f'\ } (\mathsf{s}f', n'+L(f')) \xleftrightarrow{\text{zig-zag}} (\mathsf{t}f,n) \xleftarrow{\ f\ } (\mathsf{s}f, n+L(f)) \quad , \tag{5.4.18d}$$

$$(f,n) = (f',n') \ . \tag{5.4.18e}$$

The 2-homotopy

$$\Xi^{f,n}_{f',n'} := \begin{cases} 0 & \text{for cases (5.4.18a), (5.4.18b) and (5.4.18c)} \\ \lambda_f Z^{\mathsf{t}f,n}_{\mathsf{s}f',n'+L(f')}\gamma_{f'} & \text{for case (5.4.18d)} \\ \xi_f & \text{for case (5.4.18e)} \end{cases} \tag{5.4.19}$$

then defines a zig-zagging 2-homotopy, again by a similar proof to the previous two cases.

We note three things. First, in the case that $f = \mathrm{id}$ or $f' = \mathrm{id}$, the above case distinctions are nonsensical. This is not a problem however, since we had chosen $\lambda_{\mathrm{id}} = 0$, $\gamma_{\mathrm{id}} = 0$ and $\xi_{\mathrm{id}} = 0$ so in this case our definitions for Λ, Γ and Ξ are also zero in all cases. Second, like the zig-zagging chain map $Z^{N,n}_{N',n'}$ (5.4.5), the zig-zagging homotopies are all $\mathbb{Z}$-invariant. We immediately see that this follows from the $\mathbb{Z}$-invariance of $Z^{N,n}_{N',n'}$. So for example, $\Gamma^{N,n+k}_{f',n'+k} = \Gamma^{N,n}_{f',n'}$ for any $k \in \mathbb{Z}$. And third, we note that the purpose of these homotopy coherence data is to relate the shortest zig-zag maps $Z^{N,n}_{N',n'}$ to longer zig-zags that are obtained by composing $Z^{N,n}_{N',n'}$ with a map $\mathfrak{L}(f)$ on either side. In general we could continue this procedure, building a tower of homotopies to relate any zig-zag between two objects to the shortest one, but we will not need this for the Poisson structure $\tau_{\mathbb{L}}$.

We now have the tools to define the Poisson structure

$$\tau_{\mathbb{L}} : \mathbb{L}L_!(\mathfrak{L}) \wedge \mathbb{L}L_!(\mathfrak{L}) \longrightarrow \mathbb{K} \tag{5.4.20}$$

on the strictified functor $\mathbb{L}L_!(\mathfrak{L}) = \mathrm{Tot}^{\oplus}(\tilde{\mathfrak{L}}) : \mathbf{B}\mathbb{Z} \to \mathbf{Ch}_{\mathbb{K}}$ (recall the definition of $\tilde{\mathfrak{L}}$ (5.3.5)). To unpack the condition

$$\tau_{\mathbb{L}} \circ \mathrm{d}^{\mathbb{L}L_!(\mathfrak{L}) \wedge \mathbb{L}L_!(\mathfrak{L})} = \mathrm{d}^{\mathbb{K}} \circ \tau_{\mathbb{L}} = 0 \tag{5.4.21}$$

that $\tau_{\mathbb{L}}$ is a chain map, we use the underlying bicomplexes: note that $\mathbb{L}L_!(\mathfrak{L}) \wedge \mathbb{L}L_!(\mathfrak{L}) = \mathrm{Tot}^{\oplus}(\tilde{\mathfrak{L}}) \wedge \mathrm{Tot}^{\oplus}(\tilde{\mathfrak{L}}) \cong \mathrm{Tot}^{\oplus}(\tilde{\mathfrak{L}} \wedge \tilde{\mathfrak{L}})$. Decomposing the bicomplex $\tilde{\mathfrak{L}} \wedge \tilde{\mathfrak{L}}$ into chain complexes $(\tilde{\mathfrak{L}} \wedge \tilde{\mathfrak{L}})_{\bullet,n}$ of constant vertical degree, we first note that since $\tilde{\mathfrak{L}}$ is concentrated in vertical degrees 0 and 1, $\tilde{\mathfrak{L}} \wedge \tilde{\mathfrak{L}}$ is concentrated in vertical degrees 0, 1 and 2. A Poisson structure (5.4.20) is then given by a family of n-chains

$$\Big\{\tau_{\mathbb{L},n} \in \mathbf{hom}\big((\tilde{\mathfrak{L}} \wedge \tilde{\mathfrak{L}})_{\bullet,n}, \mathbb{K}\big)_n\Big\}_{n=0,1,2} \tag{5.4.22}$$

invariant under the $\mathbb{Z}$-action on $\tilde{\mathfrak{L}}$ and such that

$$\tau_{\mathbb{L},n} \circ \delta^{\tilde{\mathfrak{L}}\wedge\tilde{\mathfrak{L}}} + \tau_{\mathbb{L},n+1} \circ \mathrm{d}^{\tilde{\mathfrak{L}}\wedge\tilde{\mathfrak{L}}} = 0 \tag{5.4.23}$$

for all n, since $\mathrm{d}^{\mathrm{Tot}^{\oplus}(\tilde{\mathfrak{L}}\wedge\tilde{\mathfrak{L}})} = \delta^{\tilde{\mathfrak{L}}\wedge\tilde{\mathfrak{L}}} + \mathrm{d}^{\tilde{\mathfrak{L}}\wedge\tilde{\mathfrak{L}}}$.

So $\tau_{\mathbb{L}}$ has three nonzero components. To define them, recall that we write $(N, n, x) \in \tilde{\mathfrak{L}}_{\bullet,0}$ for elements in $\tilde{\mathfrak{L}}$ of vertical degree 0 and $(f, n, x) \in \tilde{\mathfrak{L}}_{\bullet,1}$ for elements of vertical degree 1, and write τ_N for the component at N of the Poisson structure τ on $\mathfrak{L} \in \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$. We define $\tau_{\mathbb{L}}$ by

$$\tau_{\mathbb{L},0}\big((N,n,x)\otimes(N',n',x')\big) := \frac{1}{2}\Big(\tau_N\big(x\otimes Z^{N,n}_{N',n'}x'\big) + \tau_{N'}\big(Z^{N',n'}_{N,n}x\otimes x'\big)\Big)\ , \tag{5.4.24a}$$

$$\tau_{\mathbb{L},1}\big((f,n,x)\otimes(N',n',x')\big) := \frac{1}{2}\Big(\tau_{\mathsf{t}f}\big(\mathfrak{L}(f)x\otimes \Lambda^{f,n}_{N',n'}x'\big) + (-1)^{|x'|}\tau_{N'}\big(\Gamma^{N',n'}_{f,n}x\otimes x'\big)\Big)\ , \tag{5.4.24b}$$

$$\tau_{\mathbb{L},1}\big((N,n,x)\otimes(f',n',x')\big) := (-1)^{|x|}\frac{1}{2}\Big(\tau_N\big(x\otimes \Gamma^{N,n}_{f',n'}x'\big) + (-1)^{|x'|}\tau_{\mathsf{t}f'}\big(\Lambda^{f',n'}_{N,n}x\otimes \mathfrak{L}(f')x'\big)\Big)\ , \tag{5.4.24c}$$

$$\tau_{\mathbb{L},2}\big((f,n,x)\otimes(f',n',x')\big) := -(-1)^{|x|}\frac{1}{2}\Big(\tau_{\mathsf{t}f}\big(\mathfrak{L}(f)x\otimes \Xi^{f,n}_{f',n'}x'\big) - \tau_{\mathsf{t}f'}\big(\Xi^{f',n'}_{f,n}x\otimes \mathfrak{L}(f')x'\big)\Big)\ . \tag{5.4.24d}$$

Proposition 5.4.2. The $\tau_{\mathbb{L}}$ defined in (5.4.24) defines a Poisson structure on $\mathbb{L}L_!(\mathfrak{L}) = \mathrm{Tot}^{\oplus}(\tilde{\mathfrak{L}})$.

Proof. Invariance under the $\mathbb{Z}$-action follows from the fact that Z, Λ, Γ and Ξ are all invariant.

For checking that $\tau_{\mathbb{L}}$ is a chain map, we see that the conditions (5.4.23) that are not zero are

$$\tau_{\mathbb{L},0}\circ \mathrm{d}^{\tilde{\mathfrak{L}}\wedge\tilde{\mathfrak{L}}} = 0 \quad ; \quad \tau_{\mathbb{L},0}\circ\delta^{\tilde{\mathfrak{L}}\wedge\tilde{\mathfrak{L}}} + \tau_{\mathbb{L},1}\circ \mathrm{d}^{\tilde{\mathfrak{L}}\wedge\tilde{\mathfrak{L}}} = 0 \quad ; \quad \tau_{\mathbb{L},1}\circ\delta^{\tilde{\mathfrak{L}}\wedge\tilde{\mathfrak{L}}} + \tau_{\mathbb{L},2}\circ \mathrm{d}^{\tilde{\mathfrak{L}}\wedge\tilde{\mathfrak{L}}} = 0\ .$$

The first condition is immediate since all ingredients of $\tau_{\mathbb{L},0}$ are chain maps, so $\tau_{\mathbb{L},0}$ is too. The second condition follows from a moderately lengthy computation using the definition of $\delta^{\tilde{\mathfrak{L}}}$ in (5.3.6) and the definitions of Λ (5.4.10) and Γ (5.4.16). The third similarly follows from the definition of $\delta^{\tilde{\mathfrak{L}}}$ and the definition of Ξ (5.4.19). □

This concludes the first of the two tasks we set for this section. The second one was to prove that this new Poisson structure $\tau_{\mathbb{L}}$ defines a Poisson structure on $Q(\mathfrak{L})$ that is homotopic to the one defined by τ. For this, we first note that the functor $L^* : \mathbf{Fun}(\mathbf{B}\mathbb{Z}, \mathbf{Ch}_{\mathbb{K}}) \to \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$ defines the Poisson structure $L^*(\tau_{\mathbb{L}})$ on $L^*\mathbb{L}L_!(\mathfrak{L})$. With pullbacks along the zig-zag of weak equivalences (5.3.14) we thus get the two Poisson structures

$$\begin{array}{rlcl} q^*_{\mathfrak{L}}(\tau) = \tau(q_{\mathfrak{L}}\wedge q_{\mathfrak{L}}) : & Q(\mathfrak{L})\wedge Q(\mathfrak{L}) & \longrightarrow & \mathbb{K} \\ \tilde{\eta}^*_{\mathfrak{L}}(L^*\tau_{\mathbb{L}}) = (L^*\tau_{\mathbb{L}})(\tilde{\eta}_{\mathfrak{L}}\wedge\tilde{\eta}_{\mathfrak{L}}) : & Q(\mathfrak{L})\wedge Q(\mathfrak{L}) & \longrightarrow & \mathbb{K} \end{array} \tag{5.4.25}$$

on $Q(\mathfrak{L}) \in \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_{\mathbb{K}})$. Here we can then compare the two Poisson structures, and our compatibility condition is to demand that they are homotopic:

$$\tilde{\eta}^*_{\mathfrak{L}}(L^*\tau_{\mathbb{L}}) - q^*_{\mathfrak{L}}(\tau) = \partial\rho = \rho \circ \mathrm{d}^{Q(\mathfrak{L})\wedge Q(\mathfrak{L})} \tag{5.4.26}$$

for a $\rho \in \mathbf{hom}(Q(\mathfrak{L}) \wedge Q(\mathfrak{L}), \mathbb{K})_1$.

Let us now construct the components $\rho_N \in \mathbf{hom}(Q(\mathfrak{L})(N) \wedge Q(\mathfrak{L})(N), \mathbb{K})_1$ for a $N \in \mathbf{C}$. We again use the underlying bicomplexes: $Q(\mathfrak{L})(N) = \mathrm{Tot}^{\oplus}(\mathfrak{L}^{\Delta}(N))$ (recall Definition (5.3.8) of $\mathfrak{L}^{\Delta}$) so we have

$$Q(\mathfrak{L})(N) \wedge Q(\mathfrak{L})(N) = \mathrm{Tot}^{\oplus}(\mathfrak{L}^{\Delta}(N)) \wedge \mathrm{Tot}^{\oplus}(\mathfrak{L}^{\Delta}(N)) \cong \mathrm{Tot}^{\oplus}(\mathfrak{L}^{\Delta}(N) \wedge \mathfrak{L}^{\Delta}(N)) \ .$$

Decomposing the bicomplex $\mathfrak{L}^{\Delta}(N) \wedge \mathfrak{L}^{\Delta}(N)$ into chain complexes $(\mathfrak{L}^{\Delta}(N) \wedge \mathfrak{L}^{\Delta}(N))_{\bullet,n}$, the homotopy ρ_N can now be defined by the family of $n+1$-chains

$$\Big\{\rho_{N,n} \in \mathbf{hom}\big((\mathfrak{L}^{\Delta}(N) \wedge \mathfrak{L}^{\Delta}(N))_{\bullet,n}, \mathbb{K}\big)_{n+1}\Big\}_{n\in\mathbb{Z}}$$

so that condition (5.4.26) becomes

$$\tau_{\mathbb{L},n} \circ (\tilde{\eta}_{\mathfrak{L}} \wedge \tilde{\eta}_{\mathfrak{L}}) - \tau_{N,n} \circ (q_{\mathfrak{L}} \wedge q_{\mathfrak{L}}) = \rho_{N,n-1} \circ \delta^{\mathfrak{L}^{\Delta}(N)\wedge\mathfrak{L}^{\Delta}(N)} + \rho_{N,n} \circ \mathrm{d}^{\mathfrak{L}^{\Delta}(N)\wedge\mathfrak{L}^{\Delta}(N)}$$

where we now consider the original Poisson complex $(\mathfrak{L}(N), \tau_N)$ as a bicomplex concentrated in vertical degree 0, so $\tau_{N,0} = \tau_N$ and $\tau_{N,n} = 0$ for $n \neq 0$.

As with $\tau_{\mathbb{L}}$, $\rho_{N,n}$ can only be nonzero for $n = 0, 1, 2$ because of the vertical degrees of the bicomplex $\mathfrak{L}^{\Delta}(N)$. In fact, we only need the $n = 0$ and $n = 1$ components. Recall that we write $(g, x) \in \mathfrak{L}^{\Delta}(N)_{\bullet,0}$ for elements of vertical degree 0, and $(g, f, x) \in \mathfrak{L}^{\Delta}(N)_{\bullet,1}$ for elements of vertical degree 1. We then define

$$\rho_{N,0}\big((g,x)\otimes(g',x')\big) := \frac{1}{2}\Big((-1)^{|x|}\tau_N\big(\mathfrak{L}(g)x \otimes \Lambda^{g,0}_{\mathsf{s}g',L(g')}x'\big) + \tau_N\big(\Lambda^{g',0}_{\mathsf{s}g,L(g)}x \otimes \mathfrak{L}(g')x'\big)\Big) \tag{5.4.27a}$$

$$\rho_{N,1}\big((g,f,x)\otimes(g',x')\big) := -(-1)^{|x'|}\frac{1}{2}\tau_N\big(\Xi^{g',0}_{f,0}x \otimes \mathfrak{L}(g')x'\big) \tag{5.4.27b}$$

$$\rho_{N,1}\big((g,x)\otimes(g',f',x')\big) := -\frac{1}{2}\tau_N\big(\mathfrak{L}(g)x \otimes \Xi^{g,0}_{f',0}x'\big) \tag{5.4.27c}$$

where we have used that for any (g, f, x) of vertical degree 1, $g = \mathrm{id}$.

Proposition 5.4.3. The components $\rho_{N,n}$ defined in (5.4.27) define a natural chain homotopy $\rho \in \mathbf{hom}(Q(\mathfrak{L}) \wedge Q(\mathfrak{L}), \mathbb{K})_1$ that satisfies (5.4.26).

Proof. As in the case of Proposition 5.4.2, this is a straightforward calculation using the properties of the zig-zagging homotopies Λ (5.4.10), Γ (5.4.16) and Ξ (5.4.19), and the naturality of the original Poisson structure τ. □

Propositions 5.4.2 and 5.4.3 allow us to upgrade the diagram (5.3.14) of chain complex-valued functors on $\mathbf{C}$ to the diagram

$$\begin{array}{ccc} & (Q(\mathfrak{L}), q^*_{\mathfrak{L}}(\tau)) \overset{\rho}{\simeq} (Q(\mathfrak{L}), \tilde{\eta}^*_{\mathfrak{L}}(L^*\tau_{\mathbb{L}})) & \\ \swarrow^{q_{\mathfrak{L}}}_{\sim} & & \searrow^{\tilde{\eta}_{\mathfrak{L}}}_{\sim} \\ (\mathfrak{L}, \tau) & & (L^*\mathbb{L}L_!(\mathfrak{L}), L^*\tau_{\mathbb{L}}) \end{array} \tag{5.4.28}$$

of linear field theories on $\mathbf{C}$.

5.5. Relative Cauchy evolution for linear quantum field theories

With the results of the previous sections in hand, we can now use the machinery of Chapter 3 to define a notion of relative Cauchy evolution for linear quantum field theories. Recall (3.4.3) that we defined the quantization functor

$$\mathfrak{ccr} : \mathbf{PoissCh}_{\mathbb{R}} \longrightarrow \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$$

of Poisson chain complexes, which in turn (3.4.2) defined the quantization functor

$$\mathfrak{CCR} = \mathfrak{ccr}_* : \mathbf{hLFT}(\mathbf{C}) \longrightarrow \mathbf{hQFT}(\mathbf{C})$$

of linear field theories (recall that since we are working on the small category $\mathbf{C}$, there is no orthogonality data, so we write $\mathbf{C}$ for $\overline{\mathbf{C}} = (\mathbf{C}, \emptyset)$). Given a linear field theory $(\mathfrak{L}, \tau) \in \mathbf{hLFT}(\mathbf{C})$, this yields the homotopy quantum field theory

$$\mathfrak{A} := \mathfrak{CCR}(\mathfrak{L}, \tau) \in \mathbf{hQFT}(\mathbf{C}) \ .$$

The strictified model $(L^*\mathbb{L}L_!(\mathfrak{L}), L^*\tau_{\mathbb{L}})$ then defines another homotopy quantum field theory,

$$\mathfrak{A}^{\mathrm{st}} := \mathfrak{CCR}(L^*\mathbb{L}L_!(\mathfrak{L}), L^*\tau_{\mathbb{L}}) = L^*(\mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_{\mathbb{L}})) \in \mathbf{hQFT}(\mathbf{C}) \ ,$$

which is constant on the objects of $\mathbf{C}$. This field theory satisfies the strict time-slice axiom by construction, since it is the pullback of $\mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_{\mathbb{L}}) : \mathbf{B}\mathbb{Z} \to \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}}$ along the localization functor $L : \mathbf{C} \to \mathbf{B}\mathbb{Z}$. So $\mathfrak{A}^{\mathrm{st}}$ is in fact a strict quantum field theory on $\mathbf{C}$, since the Einstein causality axiom is an empty condition when the orthogonality relation is empty. Explicitly, recalling (5.3.15) we see that on $\mathfrak{A}^{\mathrm{st}}$ we have the explicit RCE automorphism

$$\mathrm{RCE}_{M,h} := \mathfrak{CCR}(\mathrm{rce}_{M,h}) = \mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L})(1)) : \mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_{\mathbb{L}}) \xrightarrow{\cong} \mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_{\mathbb{L}}) \tag{5.5.1}$$

on the differential graded algebra $\mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_{\mathbb{L}})$ which $\mathfrak{A}^{\mathrm{st}}$ assigns to each object $N \in \mathbf{C}$. So for linear quantum field theories, relative Cauchy evolution for the strictified model is again addition by 1 to the level in the spiral (5.1.10).

We now use both parts of Proposition 3.4.10 with the diagram (5.4.28). Denote the canonical quantizations of the other two objects in the diagram by

$$\mathfrak{A}_Q := \mathfrak{CCR}(Q(\mathfrak{L}), q_{\mathfrak{L}}^*(\tau)) \in \mathbf{hQFT}(\mathbf{C})$$

and

$$\mathfrak{A}_Q^{\mathrm{st}} := \mathfrak{CCR}(Q(\mathfrak{L}), \tilde{\eta}_{\mathfrak{L}}^*(L^*\tau_{\mathbb{L}})) \in \mathbf{hQFT}(\mathbf{C}) \ .$$

The weak equivalences $q_{\mathfrak{L}}$ and $\tilde{\eta}_{\mathfrak{L}}$ in the diagram then define weak equivalences

$$\mathfrak{CCR}(q_{\mathfrak{L}}) : \mathfrak{A}_Q \xrightarrow{\sim} \mathfrak{A}$$

and

$$\mathfrak{CCR}(\tilde{\eta}_{\mathfrak{L}}) : \mathfrak{A}_Q^{\mathrm{st}} \xrightarrow{\sim} \mathfrak{A}^{\mathrm{st}}$$

in $\mathbf{hQFT}(\mathbf{C})$ (and therefore in $\mathbf{Fun}(\mathbf{C}, \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}})$) by the first part of Proposition 3.4.10. The second part of the proposition then tells us that since $\tilde{\eta}_{\mathfrak{L}}^*(L^*\tau_{\mathbb{L}}) = q_{\mathfrak{L}}^*(\tau) + \partial\rho$, there exists an object $A_{Q(\mathfrak{L}), q_{\mathfrak{L}}^*(\tau), \rho} \in \mathbf{Fun}(\mathbf{C}, \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_{\mathbb{C}}})$ together with a zig-zag of weak equivalences

$$\mathfrak{A}_Q \xleftarrow{\sim} A_{Q(\mathfrak{L}), q_{\mathfrak{L}}^*(\tau), \rho} \xrightarrow{\sim} \mathfrak{A}_Q^{\mathrm{st}} \ .$$

Combining these two results we find that our constructions lead to the diagram of weak equivalences

$$\begin{array}{ccccc} \mathfrak{A}_Q & \xleftarrow{\sim} & A_{Q(\mathfrak{L}),q^*_{\mathfrak{L}}(\tau),\rho} & \xrightarrow{\sim} & \mathfrak{A}_Q^{\mathrm{st}} \\ {\scriptstyle \mathfrak{CCR}(q_{\mathfrak{L}})}\downarrow{\scriptstyle\sim} & & & & {\scriptstyle\sim}\downarrow{\scriptstyle \mathfrak{CCR}(\tilde{\eta}_{\mathfrak{L}})} \\ \mathfrak{A} & & & & \mathfrak{A}^{\mathrm{st}} \end{array}$$

in $\mathbf{Fun}(\mathbf{C}, \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_\mathbb{C}})$. We have proven the following.

THEOREM 5.5.1. Let $(\mathfrak{L}, \tau) \in \mathbf{hLFT}(\mathbf{C})$ be a homotopy linear field theory on $\mathbf{C}$ that satisfies the homotopy time-slice axiom, so $\mathfrak{L} \in \mathbf{Fun}(\mathbf{C}, \mathbf{Ch}_\mathbb{R})^{\mathrm{hoAll}}$. Its homotopy quantum field theory $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau)$, which satisfies the weak time-slice axiom, is then equivalent via a zig-zag of weak equivalences in the category $\mathbf{Fun}(\mathbf{C}, \mathrm{dg}\mathsf{Alg}_{\mathsf{As}_\mathbb{C}})$ to the theory $\mathfrak{A}^{\mathrm{st}} = \mathfrak{CCR}(L^*\mathbb{L}L_!(\mathfrak{L}), L^*\tau_\mathbb{L})$ which satisfies the strict time-slice axiom.

This is the main result of this chapter: a linear homotopy quantum field theory $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau)$ which satisfies the homotopy time-slice axiom, for which the relative Cauchy evolution would in general be an A_∞-quasi-automorphism, is equivalent to its strictification, a linear homotopy quantum field theory $\mathfrak{A}^{\mathrm{st}} = \mathfrak{CCR}(L^*\mathbb{L}L_!(\mathfrak{L}), L^*\tau_\mathbb{L})$ that satisfies the strict time-slice axiom. From an abstract point of view, we are done: we have proven that a well-defined concept of relative Cauchy evolution exists for linear homotopy quantum field theories, using the adjunction (5.1.2). The field theory $\mathfrak{A}^{\mathrm{st}}$ is very unwieldy, however: the complexes $\mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_\mathbb{L})$ are very large. And moreover, the concept of relative Cauchy evolution as addition by 1 to the level in the spiral has no clear physical interpretation. In particular, it is not at all clear how to divine a stress-energy tensor from this point of view.

To remedy this, let us move forward by comparing linear observables in the theories $\mathfrak{A}$ and $\mathfrak{A}^{\mathrm{st}}$, in the spirit of Definitions 2.3.5, 2.3.11 and 2.6.5 of quantum fields as natural transformations. Our plan is to interpret some of the linear observables in the theory $\mathfrak{A}$ as linear observables in the theory $\mathfrak{A}^{\mathrm{st}}$, and to show that the strict relative Cauchy evolution $\mathrm{RCE}_{M,h} = \mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L})(1))$ on $\mathfrak{A}^{\mathrm{st}}$ (5.5.1) for these observables is equivalent to the naive approach obtained by quasi-inverting the maps involved.

For the original theory $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau)$, we have the canonical chain map

$$\mathfrak{L}(M) \longrightarrow \mathfrak{A}(M) = \mathfrak{CCR}(\mathfrak{L}, \tau)(M)$$

that assigns to any linear observable its corresponding generator in the CCR-algebra. Similarly, there is the canonical map

$$\mathbb{L}L_!(\mathfrak{L}) \longrightarrow \mathfrak{A}^{\mathrm{st}}(*) = \mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_\mathbb{L})$$

for the strictified algebra. Choosing the original spacetime M in $\mathbf{C}$ and the level $n = 0$, we then define the chain map

$$\begin{array}{rccc} \iota : & \mathfrak{L}(M) & \longrightarrow & \mathbb{L}L_!(\mathfrak{L}) \\ & \omega & \longmapsto & (M, 0, \omega) \ . \end{array}$$

So we embed the original complex $\mathfrak{L}(M)$ of linear observables on M in the complex of linear observables $\mathbb{L}L_!(\mathfrak{L})$ of the strictified theory, at the level $(M, 0)$ in the spiral. Composing ι with the canonical map,

$$\mathfrak{L}(M) \xrightarrow{\ \iota\ } \mathbb{L}L_!(\mathfrak{L}) \longrightarrow \mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_\mathbb{L})$$

we can then interpret the linear observables of the original theory $\mathfrak{L}$ on M as linear quantum observables in the strictified theory $\mathfrak{A}^{\mathrm{st}}$.

On the complex $\mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_{\mathbb{L}})$ we have constructed a concept of relative Cauchy evolution. On the complex $\mathfrak{L}(M)$ we have not, but there is the naive guess

$$\mathrm{rce}^{\mathrm{lin}}_{M,h} := Z^{M,0}_{M,1} = \mathfrak{L}(i_-)\mathfrak{L}(j_-)^{-1}\mathfrak{L}(j_+)\mathfrak{L}(i_+)^{-1} : \mathfrak{L}(M) \longrightarrow \mathfrak{L}(M) \tag{5.5.2}$$

of the zig-zagging chain map (5.4.3) that corresponds to moving down the spiral by one level that is obtained by using the quasi-inverses (5.4.2). These two concepts of relative Cauchy evolution agree per the following result.

PROPOSITION 5.5.2. The diagram

$$\begin{array}{ccccc} \mathfrak{L}(M) & \xrightarrow{\ \iota\ } & \mathbb{L}L_!(\mathfrak{L}) & \longrightarrow & \mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_{\mathbb{L}}) \\ {\scriptstyle Z^{M,0}_{M,1}}\big\downarrow & \overset{\theta}{\nearrow\!\!\!\nearrow} & \big\downarrow{\scriptstyle \mathrm{rce}_{M,h}} & & \big\downarrow{\scriptstyle \mathrm{RCE}_{M,h}} \\ \mathfrak{L}(M) & \xrightarrow{\ \iota\ } & \mathbb{L}L_!(\mathfrak{L}) & \longrightarrow & \mathfrak{CCR}(\mathbb{L}L_!(\mathfrak{L}), \tau_{\mathbb{L}}) \end{array} \tag{5.5.3}$$

is homotopy commutative: the right diagram commutes, and the left diagram commutes up to a homotopy,

$$\mathrm{rce}_{M,h}\iota - \iota Z^{M,0}_{M,1} = \partial\theta$$

where $\theta \in \mathbf{hom}(\mathfrak{L}(M), \mathbb{L}L_!(\mathfrak{L}))_1$ is constructed in the proof.

PROOF. The right square in the diagram commutes strictly by the definition of $\mathrm{RCE}_{M,h}$ on $\mathfrak{A}^{\mathrm{st}}$ (5.5.1). For the left diagram, first recall that $\mathbb{L}L_!(\mathfrak{L}) = \mathrm{Tot}^{\oplus}(\tilde{\mathfrak{L}})$ is the totalization of the bicomplex $\tilde{\mathfrak{L}}$ (5.3.5), which has elements (N, n, x) of vertical degree 0 and elements (f, n, x) of vertical degree 1. $\tilde{\mathfrak{L}}$ has a vertical differential $\delta^{\tilde{\mathfrak{L}}}$ (5.3.6) and a horizontal differential $\mathrm{d}^{\tilde{\mathfrak{L}}}$ obtained from the original complexes $\mathfrak{L}(N)$. So we have

$$\partial\theta = (\delta^{\tilde{\mathfrak{L}}} + \mathrm{d}^{\tilde{\mathfrak{L}}})\theta + \theta\mathrm{d}^{\mathfrak{L}(M)}$$

and after evaluating on an element $x \in \mathfrak{L}(M)$ we find that we need θ to satisfy

$$(1, M, x) - (0, M, Z^{M,0}_{M,1}x) = (\delta^{\tilde{\mathfrak{L}}} + \mathrm{d}^{\tilde{\mathfrak{L}}})\theta(x) + \theta(\mathrm{d}^{\mathfrak{L}(M)}(x)) \ .$$

We will construct θ by transporting $(1, M, x)$ along

$$(M, 0) \xleftarrow{\ i_-\ } (M_-, 1) \xrightarrow{\ j_-\ } (M_h, 1) \xleftarrow{\ j_+\ } (M_+, 1) \xrightarrow{\ i_+\ } (M, 1) \tag{5.5.4}$$

which is the part of the spiral (5.1.10) that corresponds to the zig-zagging chain map $Z^{M,0}_{M,1}$.

To do this, we again distinguish between the two directions of arrows in the zig-zag. Let $f : (\mathsf{s}f, n + L(f)) \to (\mathsf{t}f, n)$ be any of the maps in the zig-zag (5.5.4). If f points from right to left, i.e. if there is a term $\mathfrak{L}(f)$ in the map $Z^{M,0}_{M,1}$ (5.5.2), we define the homotopy $\theta_{\overleftarrow{f}} \in \mathbf{hom}(\mathfrak{L}(\mathsf{s}f), \mathbb{L}L_!(\mathfrak{L}))_1$ as

$$\theta_{\overleftarrow{f}}(x) := (-1)^{|x|}(f, n, x) \ .$$

Then

$$\begin{aligned} \partial\theta_{\overleftarrow{f}}(x) &= \big((\mathsf{s}f, n + L(f), x) - (\mathsf{t}f, n, \mathfrak{L}(f)x)\big) + (-1)^{|x|}(f, n, \mathrm{d}x) + (-1)^{|\mathrm{d}x|}(f, n, \mathrm{d}x) \\ &= (\mathsf{s}f, n + L(f), x) - (\mathsf{t}f, n, \mathfrak{L}(f)x) \end{aligned}$$

for all $x \in \mathfrak{L}(\mathsf{s}f)$.

Conversely, if f points from left in the zig-zag (5.5.4), i.e. if there is a term $\mathfrak{L}(f)^{-1}$ in the map $Z^{M,0}_{M,1}$ (5.5.2), we define the homotopy $\theta_{\overrightarrow{f}} \in \mathbf{hom}(\mathfrak{L}(\mathsf{t}f), \mathbb{L}L_!(\mathfrak{L}))_1$ as

$$\theta_{\overrightarrow{f}}(x) := -(-1)^{|x|}(f, n, \mathfrak{L}(f)^{-1}x) - (\mathsf{t}f, n, \lambda_f x) \ .$$

where λ_f is one of the two homotopies (5.4.6) witnessing that $\mathfrak{L}(f)^{-1}$ is a quasi-inverse to $\mathfrak{L}(f)$. We then have

$$\begin{aligned}\partial\theta_{\overrightarrow{f}}(x) &= -\big((\mathsf{s}f, n + L(f), \mathfrak{L}(f)^{-1}x) - (\mathsf{t}f, n, \mathfrak{L}(f)\mathfrak{L}(f)^{-1}x)\big) \\ &\quad - (-1)^{|x|}(f, n, \mathrm{d}\mathfrak{L}(f)^{-1}x) - (-1)^{|\mathrm{d}x|}(f, n, \mathfrak{L}(f)^{-1}\mathrm{d}x) \\ &\quad - (\mathsf{t}f, n, (\mathrm{d}\lambda_f + \lambda_f \mathrm{d})x) \\ &= (\mathsf{t}f, n, x) - (\mathsf{s}f, n + L(f), \mathfrak{L}(f)^{-1}x)\end{aligned}$$

for all $x \in \mathfrak{L}(\mathsf{t}f)$.

In both cases, the homotopies $\theta_{\overleftarrow{f}}$ and $\theta_{\overrightarrow{f}}$ move an element x from right to left in the zig-zag (5.5.4). Composing these homotopies, we find the desired homotopy θ,

$$\begin{aligned}\theta = \mathfrak{L}(i_-)\mathfrak{L}(j_-)^{-1}\mathfrak{L}(j_+)\,\theta_{\overrightarrow{i_+}} + \mathfrak{L}(i_-)\mathfrak{L}(j_-)^{-1}\,\theta_{\overleftarrow{j_+}}\,\mathfrak{L}(i_+)^{-1} \\ + \mathfrak{L}(i_-)\,\theta_{\overrightarrow{j_-}}\,\mathfrak{L}(j_+)\mathfrak{L}(i_+)^{-1} + \theta_{\overleftarrow{i_-}}\,\mathfrak{L}(j_-)^{-1}\mathfrak{L}(j_+)\mathfrak{L}(i_+)^{-1}\end{aligned}$$

□

We have found the second main result of this section: for the linear observables in $\mathfrak{A}$ we can use the naive definition (5.5.2) of relative Cauchy evolution, which up to homotopy agrees with the strict relative Cauchy evolution (5.5.1) for the theory $\mathfrak{A}^{\mathrm{st}}$. Crucially, this is only true for the linear observables in $\mathfrak{A}$, i.e. the image under the canonical map $\mathfrak{L}(M) \to \mathfrak{A}(M)$. Indeed, since the components do not preserve the Poisson structure τ, the naive RCE map $Z^{M,0}_{M,1}$ (5.5.2) will in general *not* lift to a map of algebras $\mathfrak{A} \to \mathfrak{A}$.

Summing up the results of this section, we see that Theorem 5.5.1 proves that a well-defined concept of relative Cauchy evolution exists for linear homotopy quantum field theories. Proposition 5.5.2 then shows how to extract a computable RCE for the linear observables in these theories that is compatible with the strict RCE for $\mathfrak{A}^{\mathrm{st}}$.

5.6. Relative Cauchy evolution for linear Yang-Mills theory

We end this chapter by calculating the relative Cauchy evolution (5.5.2) for the linear Yang-Mills model of chapter 4. We first construct an explicit quasi-inverse $\mathfrak{L}(f)^{-1}$ to any map $\mathfrak{L}(f)$ corresponding to a Cauchy morphism f in **Loc**. We then give explicit formulas for the relative Cauchy evolution of the four linear fields that generate $\mathfrak{A}^{LYM}$ and we close by computing the stress-energy tensor of the theory.

Recall the solution complex $\mathfrak{Sol}^{LYM}$ (4.1.10) of linear Yang-Mills theory, its complex of linear observables $\mathfrak{L}^{LYM}$ (4.2.17), the inclusion map $j : \mathfrak{L} \to \mathfrak{Sol}[1]$ (4.2.3), the retarded and advanced trivializations $\Lambda^{\pm}$ (4.2.19) yielding the Green homotopies $\mathcal{G}^{\pm} = j_{\mathrm{pc/fc}}\Lambda^{\pm}\iota$ (4.2.12) and the causal propagator $\mathcal{G}$ (4.2.20). To discern between maps, homotopies and other constructions on different spacetimes, we will add a subscript of the spacetime where necessary.

Let $f : N \to N'$ be a Cauchy morphism in **Loc**. We will now construct the quasi-inverse to $\mathfrak{L}(f) : \mathfrak{L}(N) \to \mathfrak{L}(N')$, making two choices:

(i) First, choose two Cauchy surfaces $\Sigma_{\pm} \subseteq N$ such that Σ_+ lies in the chronological future of Σ_-: $\Sigma_+ \subseteq I_{+,N}(\Sigma_-)$.

Note that the images $\Sigma'_\pm := f(\Sigma_\pm) \subseteq N'$ of $\Sigma_\pm$ in N' are also Cauchy surfaces, because f is a Cauchy morphism.

(ii) Second, choose a partition of unity $\rho'_\pm$ on N' such that ρ'_+ is zero in the past of Σ'_- and ρ'_- is zero in the future of Σ'_+; i.e. a partition of unity subordinate to the open cover $\{I_{+,N'}(\Sigma'_-), I_{-,N'}(\Sigma'_+)\}$ of N'.

Write $\rho_\pm := f^*(\rho'_\pm)$ for the pullback of $\rho'_\pm$ along f; this defines a partition of unity on N subordinate to the open cover $\{I_{+,N}(\Sigma_-), I_{-,N}(\Sigma_+)\}$. We then define the chain homotopies

$$\tilde{\lambda}_f := -\rho'_+\mathcal{G}^-_{N'} - \rho'_-\mathcal{G}^+_{N'} \in \mathbf{hom}\big(\mathfrak{L}^{LYM}(N'), \mathfrak{Sol}^{LYM}(N')[1]\big)_1 \tag{5.6.1}$$

$$\tilde{\gamma}_f := -\rho_+\mathcal{G}^-_{N} - \rho_-\mathcal{G}^+_{N} \in \mathbf{hom}\big(\mathfrak{L}^{LYM}(N), \mathfrak{Sol}^{LYM}(N)[1]\big)_1 \ . \tag{5.6.2}$$

In turn, these homotopies define homotopies λ_f on $\mathfrak{L}(N')$ and γ_f on $\mathfrak{L}(N)$ as follows.

Lemma 5.6.1. The chain homotopy $\tilde{\lambda}_f$ factors uniquely through the inclusion chain map $j_{N'} : \mathfrak{L}^{LYM}(N') \to \mathfrak{Sol}^{LYM}(N')[1]$ (4.2.3): there exists a unique chain homotopy

$$\lambda_f \in \mathbf{hom}\big(\mathfrak{L}^{LYM}(N'), \mathfrak{L}^{LYM}(N')\big)_1$$

such that

$$\tilde{\lambda}_f = j_{N'}\lambda_f \ .$$

Analogously, the chain homotopy $\tilde{\gamma}_f$ factors uniquely through the inclusion chain map $j_N : \mathfrak{L}^{LYM}(N) \to \mathfrak{Sol}^{LYM}(N)[1]$, so there exists a unique homotopy

$$\gamma_f \in \mathbf{hom}\big(\mathfrak{L}^{LYM}(N), \mathfrak{L}^{LYM}(N)\big)_1$$

such that

$$\tilde{\gamma}_f = j_N\gamma_f \ .$$

Proof. The proofs for both claims are identical; we will give the proof for $\tilde{\lambda}_f$. First, notice that uniqueness is immediate, since the inclusion chain map $j : \mathfrak{L} \to \mathfrak{Sol}[1]$ is injective. For existence, we have to show that for any $\omega \in \mathfrak{L}^{LYM}(N')$, $\tilde{\lambda}_f(\omega) \in \mathfrak{L}(N')$, i.e. that

$$\tilde{\lambda}_f(\omega) = -\rho'_+\mathcal{G}^-_{N'}(\omega) - \rho'_-\mathcal{G}^+_{N'}(\omega)$$

has compact support. Since ω has compact support, $\mathcal{G}^-_{N'}(\omega)$ has strong future compact support. And since ρ'_+ has past compact support, $-\rho'_+\mathcal{G}^-_{N'}(\omega)$ is compactly supported by Definition 2.2.6. Similarly, the other term $-\rho'_-\mathcal{G}^+_{N'}(\omega)$ has compact support, so $\tilde{\lambda}_f(\omega)$ has, too. □

Remark 5.6.2. Since we will use them for computing the relative Cauchy evolution, let us give explicit formulas for λ_f and γ_f. For any $\tilde{N} \in \mathbf{Loc}$, define the operator

$$Q_{\tilde{N}}(\chi) := -\mathrm{d}_{\tilde{N}}(\chi) \quad ; \quad Q_{\tilde{N}}(\psi) := \psi \quad ; \quad Q_{\tilde{N}}(\alpha) := -\delta_{\tilde{N}}\alpha \quad ; \quad Q_{\tilde{N}}(\beta) := 0 \tag{5.6.3}$$

for $\chi \in \mathfrak{L}^{LYM}(\tilde{N})_{-1}$, $\psi \in \mathfrak{L}^{LYM}(\tilde{N})_0$, $\alpha \in \mathfrak{L}^{LYM}(\tilde{N})_1$ and $\beta \in \mathfrak{L}^{LYM}(\tilde{N})_2$, Note that $\mathcal{G}^\pm_{\tilde{N}} = j_{\mathrm{pc/fc}}G^\pm_{\tilde{N}}Q_{\tilde{N}}$ where $G^\pm_{\tilde{N}}$ are the retarded and advanced Green operators for the d'Alembert operator $\Box_{\tilde{N}}$, and recall that the map $j_{\mathrm{pc/fc}}$ carries a minus sign in degrees -1 and 0, so the signs of Q are the same as those of the retarded and advanced trivializations $\Lambda^\pm$ (4.2.19). The chain homotopy λ_f is then given by

$$\lambda_f = -\rho'_+G^-_{N'}Q_{N'} - \rho'_-G^+_{N'}Q_{N'} \tag{5.6.4}$$

and the chain homotopy γ_f is

$$\gamma_f = -\rho_+G^-_{N}Q_N - \rho_-G^+_{N}Q_N \ . \tag{5.6.5}$$

Now consider the chain map

$$j_{N'} + \partial_{N'}\tilde{\lambda}_f : \mathfrak{L}(N') \longrightarrow \mathfrak{Sol}(N')[1].$$

Let $\omega \in \mathfrak{L}(N')$ be any differential form. We investigate the support of

$$\tilde{\omega} := j_{N'}(\omega) + \partial_{N'}\tilde{\lambda}_f(\omega) \in \mathfrak{Sol}(N')[1] \ .$$

On $U_+ := I_{+,N'}(\Sigma'_+) \subseteq N'$, the chronological future of Σ'_+, we have

$$\tilde{\omega}\big|_{U_+} = j_{N'}(\omega)\big|_{U_+} - \partial_{N'}\mathcal{G}^-_{N'}(\omega)\big|_{U_+} = 0$$

since $\rho'_+ = 1$ and $\rho'_- = 0$ on U_+, and $j_{N'} = \partial\mathcal{G}^\pm_{N'}$ (4.2.7). Likewise, on $U_- := I_{-,N'}(\Sigma'_-) \subseteq N'$, the chronological future of Σ'_-, $\tilde{\omega}$ will be zero. So the support of $\tilde{\omega} \in \mathfrak{Sol}(N')[1]$ lies in the closed time-slab

$$J_{-,N'}(\Sigma'_+) \cap J_{+,N'}(\Sigma'_-) \subseteq f(N) \subseteq N' \ .$$

It is also clear from its definition that $\tilde{\omega}$ has spacelike compact support, so as in Lemma 5.6.1 we see that the chain map $j_{N'} + \partial_{N'}\tilde{\lambda}_f$ factors as

$$j_{N'} + \partial_{N'}\tilde{\lambda}_f = j_{N'}(\mathrm{id} + \partial_{N'}\lambda_f)$$

where

$$\mathrm{id} + \partial_{N'}\lambda_f : \mathfrak{L}(N') \longrightarrow \mathfrak{L}(f(N)) \tag{5.6.6}$$

is a chain map that takes values in the subcomplex $\mathfrak{L}(f(N)) \subseteq \mathfrak{L}(N')$ and λ_f is the homotopy found in Lemma 5.6.1. On the subcomplex $\mathfrak{L}(f(N)) \subseteq \mathfrak{L}(N')$ the pullback $f^* : \mathfrak{L}(f(N)) \to \mathfrak{L}(N)$ of differential forms along $f : N \to N'$ is defined, so we obtain the chain map

$$\mathfrak{L}(f)^{-1} := f^*\left(\mathrm{id} + \partial_{N'}\lambda_f\right) : \mathfrak{L}(N') \longrightarrow \mathfrak{L}(N) \ . \tag{5.6.7}$$

This is indeed a quasi-inverse of $\mathfrak{L}(f) : \mathfrak{L}(N) \to \mathfrak{L}(N')$.

Proposition 5.6.3. Let $f : N \to N'$ be a Cauchy morphism in **Loc**, with the corresponding chain map $\mathfrak{L}(f)^{-1}$ (5.6.7) and homotopies λ_f and γ_f defined in Lemma 5.6.1. Then we have

$$\mathfrak{L}(f)\mathfrak{L}(f)^{-1} - \mathrm{id} = \partial_{N'}\lambda_f \tag{5.6.8a}$$

$$\mathfrak{L}(f)^{-1}\mathfrak{L}(f) - \mathrm{id} = \partial_N\gamma_f \tag{5.6.8b}$$

$$\mathfrak{L}(f)\gamma_f - \lambda_f\mathfrak{L}(f) = 0 \ . \tag{5.6.8c}$$

In particular, $\mathfrak{L}(f)^{-1}$ is a quasi-inverse to $\mathfrak{L}(f)$.

Proof. Recall that $\mathfrak{L}(f) = f_*$ is the pushforward of compactly supported forms. For the first identity we then have

$$\mathfrak{L}(f)\mathfrak{L}(f)^{-1} - \mathrm{id} = f_*f^*\left(\mathrm{id} + \partial_{N'}\lambda_f\right) - \mathrm{id} = \partial_{N'}\lambda_f$$

since f_*f^* is the identity on the subcomplex $\mathfrak{L}(f(N)) \subseteq \mathfrak{L}(N')$.

Since the chain map j_N is injective in every degree, the second identity is equivalent to

$$j_N\mathfrak{L}(f)^{-1}\mathfrak{L}(f) - j_N = j_N(\mathfrak{L}(f)^{-1}\mathfrak{L}(f) - \mathrm{id}) = j_N\partial_N\gamma_f = \partial_N\tilde{\gamma}_f \ .$$

We see that

$$j_N\mathfrak{L}(f)^{-1}\mathfrak{L}(f) - j_N = j_Nf^*\left(\mathrm{id} + \partial_{N'}\lambda_f\right)f_* - j_N = \partial_N(f^*\tilde{\lambda}_f f_*)$$

where for the second equality we have used the naturality of j and ∂ and the fact that $f^* f_* = \mathrm{id}$ on $\mathfrak{L}(N)$. We then find

$$f^*\tilde{\lambda}_f f_* = f^*(-\rho'_+\mathcal{G}^-_{N'} - \rho'_-\mathcal{G}^+_{N'})f_* = -\rho_+ f^*\mathcal{G}^-_{N'}f^* - \rho_- f^*\mathcal{G}^+_{N'}f_* = \tilde{\gamma}_f$$

because of the naturality of the Green homotopies and the definition of $\rho_\pm$, proving the second identity.

For the third identity, note that for any $\omega \in \mathfrak{L}(N)$, both $\mathfrak{L}(f)\gamma_f(\omega)$ and $\lambda_f\mathfrak{L}(f)(\omega)$ are supported on $f(N) \subseteq N'$. So these forms lie in $\mathfrak{L}(f(N))$ and we see that the third identity is equivalent to

$$f^*\lambda_f\mathfrak{L}(f) = f^*\mathfrak{L}(f)\gamma_f \ .$$

We find that

$$\begin{aligned} f^*\lambda_f\mathfrak{L}(f) &= -f^*\rho'_+ G^-_{N'} Q_{N'} f_* - f^*\rho'_- G^+_{N'} Q_{N'} f_* \\ &= -\rho_+ G^-_N Q_N - \rho_- G^+_N Q_N = \gamma_f = f^*\mathfrak{L}(f)\gamma_f \end{aligned}$$

where we use the explicit formulas for λ_f (5.6.4) and γ_f (5.6.5). For the second equality we use the naturality of $G^\pm$ and Q and the definition of $\rho^\pm$, and in the last step we use the fact that $f^* f_* = \mathrm{id}$ on $\mathfrak{L}(N)$. □

Equipped with the explicit formula for the quasi-inverse $\mathfrak{L}(f)$ (5.6.7) and its ingredient λ_f (5.6.4) we can now write down an explicit formula for the RCE chain map (5.5.2) of linear quantum fields for the linear Yang-Mills model. To keep notation manageable, we will suppress the pullbacks f^* and pushforwards f_* of differential forms: since all of the morphisms f in $\mathbf{C}$ are subset inclusions, f^* and f_* are simply restrictions and extensions by zero. We then have

$$\begin{aligned} Z^{M,0}_{M,1} &= \mathfrak{L}(i_-)\mathfrak{L}(j_-)^{-1}\mathfrak{L}(j_+)\mathfrak{L}(i_+)^{-1} \\ &= \big(\mathrm{id} + \partial_{M_h}\lambda_{j_-}\big)\big(\mathrm{id} + \partial_M\lambda_{i_+}\big) \\ &= \mathrm{id} + \big(\partial_{M_h}\lambda_{j_-}\big)\big(\mathrm{id} + \partial_M\lambda_{i_+}\big) + \partial_M\lambda_{i_+} \\ &= \mathrm{id} + \big((\partial_{M_h} - \partial_M)\lambda_{j_-}\big)\big(\mathrm{id} + \partial_M\lambda_{i_+}\big) + \partial_M\big(\lambda_{i_+} + \lambda_{j_-}(\mathrm{id} + \partial_M\lambda_{i_+})\big) \ . \end{aligned}$$

Since in diagram (5.5.3) of Proposition 5.5.2 the RCE map on $\mathfrak{L}M)$ is only determined up to homotopy, so we drop the last term and redefine the relative Cauchy evolution for the linear quantum fields to be

$$\mathrm{rce}^{\mathrm{lin}}_{M,h} := \mathrm{id} + \big((\partial_{M_h} - \partial_M)\lambda_{j_-}\big)\big(\mathrm{id} + \partial_M\lambda_{i_+}\big) : \mathfrak{L}(M) \longrightarrow \mathfrak{L}(M) \ . \tag{5.6.9}$$

Note that it is essential here that we are considering chain maps $\mathfrak{L}(M) \to \mathfrak{L}(M)$, which allows us to drop terms $\partial_M\rho$ for any homotopy $\rho \in \mathbf{hom}(\mathfrak{L}(M), \mathfrak{L}(M))_1$ when working up to homotopy. We cannot do the same for terms $\partial_{M_h}\rho$ (and in fact these are the terms that determine the relative Cauchy evolution).

Investigating the support properties of the different ingredients of the RCE map allows us to simplify it further.

PROPOSITION 5.6.4. The chain map $\mathrm{rce}^{\mathrm{lin}}_{M,h}$ (5.6.9) is equal to

$$\mathrm{rce}^{\mathrm{lin}}_{M,h} = \mathrm{id} + \big(\mathrm{d}^{\mathfrak{L}(M_h)} - \mathrm{d}^{\mathfrak{L}(M)}\big)G_{M_h}Q_{M_h}(\mathrm{id} + \partial_M\lambda_{i_+}) \tag{5.6.10}$$

where G_{M_h} is the causal propagator for the d'Alembert operator $\Box_{M_h}$ and Q_{M_h} is the operator defined in (5.6.3).

PROOF. We saw (5.6.6) that the chain map

$$\mathrm{id} + \partial_M \lambda_{i_+} : \mathfrak{L}(M) \longrightarrow \mathfrak{L}(M_+)$$

takes values in the sub-chain complex of differential forms supported on $M_+ \subseteq M$. We thus compute

$$\begin{aligned}
(\partial_{M_h} - \partial_M)\lambda_{j_-}\big|_{\mathfrak{L}(M_+)} &= \big(\mathrm{d}^{\mathfrak{L}(M_h)} - \mathrm{d}^{\mathfrak{L}(M)}\big)\lambda_{j_-}\big|_{\mathfrak{L}(M_+)} + \lambda_{j_-}\big(\mathrm{d}^{\mathfrak{L}(M_h)} - \mathrm{d}^{\mathfrak{L}(M)}\big)\big|_{\mathfrak{L}(M_+)} \\
&= \big(\mathrm{d}^{\mathfrak{L}(M_h)} - \mathrm{d}^{\mathfrak{L}(M)}\big)\big(- \rho_+ G^-_{M_h} Q_{M_h} - \rho_- G^+_{M_h} Q_{M_h}\big)\big|_{\mathfrak{L}(M_+)} \\
&= \big(\mathrm{d}^{\mathfrak{L}(M_h)} - \mathrm{d}^{\mathfrak{L}(M)}\big)\big(- G^-_{M_h} Q_{M_h}\big)\big|_{\mathfrak{L}(M_+)} \\
&= \big(\mathrm{d}^{\mathfrak{L}(M_h)} - \mathrm{d}^{\mathfrak{L}(M)}\big)\big(G^+_{M_h} - G^-_{M_h}\big) Q_{M_h}\big|_{\mathfrak{L}(M_+)} \\
&= \big(\mathrm{d}^{\mathfrak{L}(M_h)} - \mathrm{d}^{\mathfrak{L}(M)}\big) G_{M_h} Q_{M_h}\big|_{\mathfrak{L}(M_+)} \ .
\end{aligned}$$

The crucial ingredient here is that since the differentials $\mathrm{d}^{\mathfrak{L}(M)}$ and $\mathrm{d}^{\mathfrak{L}(M_h)}$ are differential operators, they are local, so they agree outside the support of h. In the second step we use this, since this shows that $\big(\mathrm{d}^{\mathfrak{L}(M_h)} - \mathrm{d}^{\mathfrak{L}(M)}\big)\big|_{\mathfrak{L}(M_+)} = 0$. Here we also used the formula (5.6.4) for λ_{j_-}, so the $\rho_\pm$ are the partitions of unity arising from a choice of Cauchy surfaces in M_-. In the third step we use that $\rho_+ = 1$ and $\rho_- = 0$ on $\mathrm{supp}(h)$. And in the fourth step we use that $J_{+,M_h}(M_+) \cap \mathrm{supp}(h) = \emptyset$. □

REMARK 5.6.5. Let us explicitly calculate the components of the chain map $\mathrm{rce}^{\mathrm{lin}}_{M,h}$ (5.6.10) for the four linear fields of linear Yang-Mills theory. Recall the complex $\mathfrak{L}(M)$ (4.2.17) and the operator Q (5.6.3).

- For the linear ghost field observable $\chi \in \mathfrak{L}(M)_{-1} = \Omega^0_{\mathrm{c}}(M)$, we find

$$\begin{aligned}
\mathrm{rce}^{\mathrm{lin}}_{M,h}(\chi) &= \chi + (\delta_{M_h} - \delta_M)\, G_{M_h} \mathrm{d}_{M_h} (\mathrm{id} + \partial_M \lambda_{i_+})(\chi) \\
&= \chi + (\Box_{M_h} - \Box_M)\, G_{M_h} (\mathrm{id} + \partial_M \lambda_{i_+})(\chi) \\
&= \chi - \Box_M\, G_{M_h} (\mathrm{id} + \partial_M \lambda_{i_+})(\chi)
\end{aligned} \tag{5.6.11}$$

  where we have used the fact that d_{M_h} commutes with G_{M_h}, that $\mathrm{d}_M = \mathrm{d}_{M_h}$, that $\Box_{M_h} G_{M_h} = 0$ and that $\delta = 0$ on 0-forms, so $\Box = \delta\mathrm{d}$ on 0-forms.

  For homology, we see that on 0-forms,

$$\begin{aligned}
\Box_M\, G_{M_h} &= \delta_M \mathrm{d}_M\, G_{M_h} \\
&= *_M^{-1} \mathrm{d}_M *_M \mathrm{d}_M G_{M_h} \\
&= *_M^{-1} \mathrm{d}_M (*_M - *_{M_h}) \mathrm{d}_M G_{M_h} \\
&= \delta_M (\mathrm{id} - *_M^{-1} *_{M_h}) \mathrm{d}_M G_{M_h}
\end{aligned}$$

  where in the third step we use that

$$*_M^{-1} \mathrm{d}_M *_{M_h} \mathrm{d}_M G_{M_h} = *_M^{-1} *_{M_h} \delta_{M_h} \mathrm{d}_{M_h} G_{M_h} = *_M^{-1} *_{M_h} \Box_{M_h} G_{M_h} = 0 \ .$$

  Since $(\mathrm{id} - *_M^{-1} *_{M_h})$ is only supported on $\mathrm{supp}(h)$, the resulting form is compactly supported, so we see that $\Box_M\, G_{M_h}(\mathrm{id} + \partial_M \lambda_{i_+})(\chi)$ will be exact in the chain complex $\mathfrak{L}(M)$, so the relative Cauchy evolution is trivial in homology,

$$[\mathrm{rce}^{\mathrm{lin}}_{M,h}(\chi)] = [\chi] \ .$$

- For the linear gauge field observable $\psi \in \mathfrak{L}(M)_0 = \Omega^1_{\mathrm{c}}(M)$, we find
$$\mathrm{rce}^{\mathrm{lin}}_{M,h}(\psi) = \psi + (\delta_{M_h}\mathrm{d}_{M_h} - \delta_M \mathrm{d}_M) G_{M_h}(\mathrm{id} + \partial_M \lambda_{i_+})(\psi) \ . \tag{5.6.12}$$
In homology this agrees with the relative Cauchy evolution for the field strength tensor of Maxwell theory computed in Section 6.3 of [**FL16**], if we take $\psi = \delta_M \omega$ for $\omega \in \Omega^2_{\mathrm{c}}(M_+)$.
- For the linear antifield observable for the gauge field $\alpha \in \mathfrak{L}(M)_1 = \Omega^1_{\mathrm{c}}(M)$, we find
$$\mathrm{rce}^{\mathrm{lin}}_{M,h}(\alpha) = \alpha + (\mathrm{d}_{M_h} - \mathrm{d}_M) G_{M_h} \delta_{M_h}(\mathrm{id} + \partial_M \lambda_{i_+})(\alpha) = \alpha \tag{5.6.13}$$
because $\mathrm{d}_{M_h} = \mathrm{d}_M$.
- Finally, for the linear antifield observable for the ghost field $\beta \in \mathfrak{L}(M)_2 = \Omega^0_{\mathrm{c}}(M)$, we find
$$\mathrm{rce}^{\mathrm{lin}}_{M,h}(\beta) = \beta \tag{5.6.14}$$
because $Q_{M_h}(\beta) = 0$.

We find that the relative Cauchy evolution is trivial on the linear antifield observables α and β and it is trivial in homology on the linear ghost field observable χ, so it is only nontrivial on the linear gauge field observables ψ, where it agrees with earlier results from [**FL16**].

We end this section by computing the stress-energy tensor of the linear Yang-Mills model. In ordinary algebraic quantum field theory, this is done by varying the perturbation h of the metric by multiplying it with ϵ^2 and calculating the first derivative
$$t_{M,h} := \frac{d}{d\epsilon} \mathrm{rce}^{\mathrm{lin}}_{M,\epsilon h}\big|_{\epsilon=0} : \mathfrak{L}(M) \longrightarrow \mathfrak{L}(M) \ .$$
We want to interpret $t_{M,h}$ as the commutator $[T_M(h), -]$ with the smeared stress-energy tensor, which in particular means that it needs to be a derivation on the algebra of observables. The stress-energy tensor T_M is then given by
$$\tau_M\big(t_{M,h}(\omega_1) \otimes \omega_2\big) = \int_M h_{ab}\, T^{ab}_M(\omega_1, \omega_2) \mathrm{vol}_M$$
where ω_i are fields in the theory and a and b are spacetime indices.

To simplify the calculation, we once more perturb the formula for linear RCE, by choosing observables that are supported on M_+. This means we can precompose the RCE map (5.6.10) with $\mathfrak{L}(i_+) : \mathfrak{L}(M_+) \to \mathfrak{L}(M)$, which is in turn homotopic to the chain map
$$\mathrm{rce}^{\mathrm{lin},+}_{M,h} := \mathrm{id} + \big(\mathrm{d}^{\mathfrak{L}(M_h)} - \mathrm{d}^{\mathfrak{L}(M)}\big) G_{M_h} Q_{M_h} : \mathfrak{L}(M_+) \longrightarrow \mathfrak{L}(M) \ .$$
Considering the formula for $\mathrm{rce}^{\mathrm{lin}}_{M,h}$ (5.6.10) and the explicit formulas of Remark 5.6.5 we see that this comes down to removing the map $\mathrm{id} + \partial_M \lambda_{i_+}$ that restricts from M to M_+ from the formulas.

We then compute the chain map
$$t_{M,h} := \frac{d}{d\epsilon} \mathrm{rce}^{\mathrm{lin},+}_{M,\epsilon h}\big|_{\epsilon=0} : \mathfrak{L}(M_+) \longrightarrow \mathfrak{L}(M) \ .$$
To do this, we need to use index notation $\omega = \omega_{a_1 \cdots a_n} \mathrm{d}x^{a_1} \wedge \cdots \wedge \mathrm{d}x^{a_n}$ for differential n-forms. In these terms, the action of both d_M and δ_M is expressed using the Levi-Civita connection ∇_M, see e.g. Section 3 in [**FL16**]. For both of the nontrivial RCE maps on χ (5.6.11) and

[2] For any perturbation h there always exists an $r > 0$ such that $(M, g + \epsilon h) \in \mathbf{Loc}$ for all $\epsilon \in [-r, r]$, see Theorem 7.2 in [**BEE96**].

ψ (5.6.12) we recognize that $\delta_{M_{\epsilon h}} - \delta_M$ is the crucial ingredient. So we calculate for any $k+1$-form $\omega = \omega_{a_0\cdots a_n} \mathrm{d}x^{a_0} \wedge \cdots \wedge \mathrm{d}x^{a_n}$,

$$(\delta_{M_{\epsilon h}}\omega - \delta_M \omega)_{a_1\cdots a_n} = \epsilon\Big(\nabla_{M\,a}(h^{ab}\omega_{ba_1\cdots a_n}) - \frac{1}{2}(\nabla_{M\,b}h^a{}_a)\omega^b{}_{a_1\cdots a_n} + \sum_{j=1}^{k}(-1)^{j-1}(\nabla_{M\,a}h_{ba_j})\omega^{ab}{}_{a_1\cdots\hat{a}_j\cdots a_k}\Big) + \mathcal{O}(\epsilon^2)$$

to first order in ϵ. Here we use the usual notation $\hat{a}_j$ for an index that has been removed, and since we are working to first order in ϵ we can raise and lower indices using either g or $g + \epsilon h$. We find that

$$\begin{aligned} t_{M,h}(\chi) &= \nabla_{M\,a}(h^{ab}(\mathrm{d}_M G_M \chi)_b) - \frac{1}{2}(\nabla_{M\,b}h^a{}_a)(\mathrm{d}_M G_M\chi)^b \\ t_{M,h}(\psi)_c &= \nabla_{M\,a}(h^{ab}(\mathrm{d}_M G_M \psi)_{bc}) - \frac{1}{2}(\nabla_{M\,b}h^a{}_a)(\mathrm{d}_M G_M\psi)^b{}_c \\ &\quad + (\nabla_{M\,a}h_{bc})(\mathrm{d}_M G_M \psi)^{ab} \\ t_{M,h}(\alpha)_c &= 0 \\ t_{M,h}(\beta) &= 0 \end{aligned} \tag{5.6.15}$$

for the observables $\chi \in \mathfrak{L}(M_+)_{-1}$, $\psi \in \mathfrak{L}(M_+)_0$, $\alpha \in \mathfrak{L}(M_+)_1$ and $\beta \in \mathfrak{L}(M_+)_2$.

One last issue now is that the chain map $t_{M,h} : \mathfrak{L}(M_+) \to \mathfrak{L}(M)$ does not extend to a derivation of algebras $\mathfrak{A}(M_+) \to \mathfrak{A}(M)$ relative to the algebra map $\mathfrak{A}(i_+) : \mathfrak{A}(M_+) \to \mathfrak{A}(M)$. This is an artifact of $\mathrm{rce}^{\mathrm{lin},+}_{M,h}$ not preserving the Poisson structures, which to first order in ϵ (at infinitesimal level) is expressed as

$$\tau \circ (t_{M,h} \wedge \mathfrak{L}(i_+) + \mathfrak{L}(i_+) \wedge t_{M,h}) \neq 0 : \mathfrak{L}(M_+) \wedge \mathfrak{L}(M_+) \longrightarrow \mathbb{R} \ .$$

Luckily, this can be remedied by adding a homotopy: if we define the map

$$\tilde{t}_{M,h} := t_{M,h} + \partial\nu : \mathfrak{L}(M_+) \longrightarrow \mathfrak{L}(M)$$

where $\nu \in \mathbf{hom}(\mathfrak{L}(M_+), \mathfrak{L}(M))_1$ is the homotopy defined by

$$\begin{aligned} \nu(\chi)_b &= \frac{1}{2}h^a{}_a(\mathrm{d}_M G_M\chi)_b - h_{ab}(\mathrm{d}_M G_M \chi)^a \\ \nu(\psi)_b &= 0 \qquad \nu(\alpha) = 0 \qquad \nu(\beta) = 0 \ . \end{aligned}$$

then

$$\tau \circ (\tilde{t}_{M,h} \wedge \mathfrak{L}(i_+) + \mathfrak{L}(i_+) \wedge \tilde{t}_{M,h}) = 0$$

by integration by parts.

Following the arguments in Section 6.2 of [**FL16**], we find that this defines the stress-energy tensor

$$T^{ab}_M(\omega_1, \omega_2) = \frac{1}{4}g^{ab}(F_{\psi_1})^{cd}(F_{\psi_2})_{cd} - (F_{\psi_1})^{ac}(F_{\psi_2})^b{}_c \tag{5.6.16}$$

through

$$\tau_M\big(\tilde{t}_{M,h}(\omega_1) \otimes \mathfrak{L}(i_+)(\omega_2)\big) = \int_M h_{ab}\, T^{ab}_M(\omega_1,\omega_2)\mathrm{vol}_M$$

for all $\omega_i \in \mathfrak{L}(M_+)$. Here, $F_\psi := \mathrm{d}_M G_M \psi \in \Omega^2(M)$ is the field strength 2-form of the gauge field, which is only nonzero when the $\omega_i = \psi_i$ are gauge field observables. So, as one would suspect, we find that the ghost fields and antifields do not contribute to the stress-energy tensor, and that for the gauge fields A it agrees with the usual Maxwell stress-energy tensor.

CHAPTER 6

Conclusion and outlook

In this thesis we covered several aspects of higher linear algebraic quantum field theory. Recall from the introduction that homotopy algebraic quantum field theory is a natural framework to study quantum gauge theories, since it allows for an appropriate handling of weak equivalences. The formalism of homotopy algebraic quantum field theory has undergone rapid development in the past few years, but there were not many nontrivial examples available beyond toy models.

The work done in this thesis has sought to remedy this. A general quantization functor for homotopy linear field theories was developed, which was shown to be appropriately homotopical. Using this functor the linear Yang-Mills model was constructed, a nontrivial example of a homotopy algebraic quantum gauge theory that is not equivalent to an ordinary algebraic quantum field theory. The existence of a well-defined concept of relative Cauchy evolution for linear homotopy algebraic quantum field theories was proven, yielding a stress-energy tensor for linear Yang-Mills theory.

In chapter 3 we built on the work in [**BSW20**] to characterize algebraic field theories in terms of operad algebras and in particular define a canonical quantization functor. We gave a general definition of algebraic field theories of type $\mathcal{P}^r$ on an orthogonal category $\overline{\mathbf{C}}$ satisfying the Einstein causality axiom (Definition 3.1.5), and found (Theorem 3.2.4) that these field theories are the algebras over a field theory operad $\mathcal{P}^r_{\overline{\mathbf{C}}}$:

$$\mathbf{FT}(\overline{\mathbf{C}}, \mathcal{P}^r) \cong \mathsf{Alg}_{\mathcal{P}^r_{\overline{\mathbf{C}}}} .$$

In turn, this yields several field theory adjunctions. For a full orthogonal subcategory $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$, the adjunction $j_! \dashv j^*$ (3.3.5) establishes the category of field theories on $\overline{\mathbf{C}}$ as a full coreflective subcategory of the category of field theories on $\overline{\mathbf{D}}$ (Proposition 3.3.3), characterizing j-local theories, i.e. theories that satisfy descent along j (Corollary 3.3.7). On the other hand, for an orthogonal localization $L : \overline{\mathbf{C}} \to \overline{\mathbf{C}[W^{-1}]}$, the adjunction $L_! \dashv L^*$ (3.3.7) establishes the field theories on $\overline{\mathbf{C}[W^{-1}]}$ as a full reflective subcategory of the field theories on $\overline{\mathbf{C}}$ (Proposition 3.3.10), characterizing W-constant theories, i.e. theories that satisfy the analogue of the time-slice axiom (Proposition 3.3.12).

The canonical operad map $\phi : \mathsf{uLie} \to \mathsf{As}$ results in the linear quantization adjunction $\mathfrak{Q}_{\mathrm{lin}} \dashv \mathfrak{U}_{\mathrm{lin}}$ (3.3.10), and $\mathfrak{Q}_{\mathrm{lin}}$ is the essential part of the canonical commutation relations functor

$$\mathfrak{CCR} : \mathfrak{Q}_{\mathrm{lin}} \circ \mathfrak{heis}_* : \mathbf{LFT}(\overline{\mathbf{C}}) \longrightarrow \mathbf{QFT}(\overline{\mathbf{C}})$$

from linear field theories to quantum field theories. This construction applies to both vector space-valued and chain complex-valued field theories. The category of chain complex-valued field theories carries a model structure with objectwise quasi-isomorphisms as weak equivalences (Proposition 3.4.1). The canonical commutation relations functor preserves these weak equivalences and results in weakly equivalent theories when varying the Poisson structure by a homotopy (Proposition 3.4.10), and it also preserves homotopy j-local field theories and W-constant field theories (Proposition 3.4.13).

In chapter 4 we constructed one of the first nontrivial examples of a linear homotopy quantum field theory, the linear Yang-Mills model. The general construction is that from the input data of a field complex $\mathfrak{F}(M)$ that includes gauge transformations and an equation of motion P, one forms the solution complex $\mathfrak{Sol}(M)$ as a derived critical locus (Theorem 4.1.7). The linear observables $\mathfrak{L}(M)$ (4.2.1) of the theory naturally embed into this solution complex, which defines a 1-shifted Poisson structure Υ on $\mathfrak{Sol}(M)$ (Definition 4.2.2), the antibracket of the BV formalism. Crucially, when working on a globally hyperbolic spacetime M, retarded and advanced Green operators related to P allow us in special cases to trivialize this shifted Poisson structure in two ways, and their difference defines an unshifted Poisson structure τ on $\mathfrak{Sol}(M)$ (Definition 4.2.8). This is a crucial result, since the Poisson structure allows for the canonical quantization of the theory as found in chapter 3 (Corollary 4.3.1), and as such yields examples of linear homotopy algebraic quantum field theories, provided all ingredients are appropriately functorial (Section 4.4).

The examples of Klein-Gordon theory and linear Yang-Mills theory both define linear homotopy algebraic quantum field theories on $\overline{\mathbf{Loc}}$ (Theorem 4.4.12). The chain complex model of Klein-Gordon theory was found to be equivalent to the usual vector space-valued treatment of Klein-Gordon theory in algebraic quantum field theory (Remark 4.4.13), after moving to the homology in degree 0 (the only degree where homology is nontrivial). The linear Yang-Mills model is *not* equivalent to a vector-space valued field theory (Remark 4.4.14), and as such provides a nontrivial example of a homotopy algebraic quantum field theory that is not equivalent to an algebraic quantum field theory in the usual sense.

In chapter 5 we treated relative Cauchy evolution for homotopy algebraic quantum field theories. Relative Cauchy evolution for a homotopy algebraic quantum field theory $\mathfrak{A}$ is not easily defined, because they only satisfy the homotopy time-slice axiom: Cauchy morphisms f in $\mathbf{Loc}$ only lead to weak equivalences $\mathfrak{A}(f)$ in $\mathrm{dg}\mathsf{Alg}_{\mathsf{As}}$, which in general only have A_∞-quasi-inverses. Using the adjunction $L_! \dashv L^*$ (3.3.7) we reformulated the notion of relative Cauchy evolution for ordinary algebraic quantum field theories, calculating the localization

$$L : \mathbf{C} \longrightarrow \mathbf{B}\mathbb{Z}$$

of the category $\mathbf{C}$ of objects and morphisms relevant to relative Cauchy evolution (Lemma 5.1.1).

The strategy was to strictify the homotopy quantum field theory using the adjunction $L_! \dashv L^*$. We restricted from $\mathbf{Loc}$ to $\mathbf{C}$, and restricted to linear homotopy quantum field theories $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau)$ constructed in the previous chapters. We then strictified a homotopy linear field theory $(\mathfrak{L}, \tau)$. First we proved a rectification theorem for the linear observables $\mathfrak{L}$, deriving the functor $L_!$ (Theorem 5.3.1). Then we defined an appropriate Poisson structure $\tau_{\mathbb{L}}$ on the resultant object $\mathbb{L}L_!(\mathfrak{L})$ (Proposition 5.4.2) that is compatible with the original Poisson structure τ (Proposition 5.4.3) through the diagram of equivalences

$$\begin{array}{ccc} & Q(\mathfrak{L}) & \\ {}^{q_{\mathfrak{L}}}\swarrow_{\sim} & & {}_{\sim}\searrow^{\tilde{\eta}_{\mathfrak{L}}} \\ \mathfrak{L} & & L^*\mathbb{L}L_!(\mathfrak{L}) \end{array} .$$

Using the results from Chapter 3 on the homotopical properties of the linear quantization functor $\mathfrak{CCR}$ we found that the above construction yields a strictification $\mathfrak{A}^{\mathrm{st}}$ of the linear homotopy quantum field theory $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau)$ (Theorem 5.5.1). This theory satisfies the strict time-slice axiom and as such carries a strict relative Cauchy evolution. The linear quantum observables in $\mathfrak{A}$ then carry a notion of relative Cauchy evolution obtained by the naive quasi-inversion of the maps involved (Proposition 5.5.2). We obtained formulas for relative Cauchy evolution for the linear Yang-Mills model, finding that only for the gauge field it is nontrivial in homology (Remark 5.6.5), and as such obtained a stress-energy tensor that the ghost and antifields do not contribute to (5.6.16).

Several avenues for further research present themselves. For one, while the existence of a well-defined concept of relative Cauchy evolution was proven for linear homotopy algebraic quantum field theories, it is still an open question if this can be done for a general homotopy algebraic quantum field theory. This question can be attacked on various levels. At the level of homotopy quantum field theories on the RCE category $\mathbf{C}$ (5.1.6), one can follow the same strategy as in Chapter 5 and derive the left adjoint in the adjunction (5.2.1) using a bar resolution. The simplicial bookkeeping this involves is heavier than that used in this text, but it does not seem impossible.

Moving beyond the RCE category $\mathbf{C}$ and its localization $\mathbf{B}\mathbb{Z}$, it does not seem feasible at the moment to find a workable model for the localization of $\mathbf{Loc}$ at all Cauchy morphisms. But perhaps one could localize other interesting subcategories larger than $\mathbf{C}$, such as the category of all spacetimes M_h obtained by a compact perturbation h to the metric of M and all possible $M_\pm$. If such a model were constructed, one could proceed in a similar way as in this thesis, deriving the localization functor $L_!$ for either the linear observables or generic homotopy algebraic quantum field theories.

Finally, one could try to directly compute the A_∞-quasi-inverses, ignoring localization. For the case of linear quantum field theories $\mathfrak{A} = \mathfrak{CCR}(\mathfrak{L}, \tau)$ this is an exercise involving corrections to the multiplication in $\mathfrak{A}$ governed by τ. In both this case and the generic one, one challenge is to ensure that these approaches lead to a result with a good physical interpretation.

Another topic that demands further research is the question of descent for homotopy algebraic quantum field theories. Recall that this is a local-to-global property: it tells us if a theory on a general spacetime can be reconstructed using local data. In several approaches to functorial field theory this plays a crucial role: for example, extended topological quantum field theories are determined by their value on a point by the cobordism hypothesis [**Lur09b**]. And in the context of factorization algebras [**CG16**], factorization homology constructs a theory on general manifolds from its values on disks [**AF15**]. The relevant example for algebraic quantum field theory is $j : \overline{\mathbf{Loc}_\diamond} \to \overline{\mathbf{Loc}}$. We found that a theory $\mathfrak{B}$ on $\overline{\mathbf{D}}$ satisfies descent with respect to the full subcategory inclusion $j : \overline{\mathbf{C}} \to \overline{\mathbf{D}}$ if the component $\mathbb{L}j_! j^* \mathfrak{B} \to \mathfrak{B}$ of the derived counit is a weak equivalence (Definition 3.4.11). The behaviour of $\mathfrak{B}$ on a spacetime M is then completely determined by its behaviour on spacetimes diffeomorphic to $\mathbb{R}^n$.

As noted in Remark 3.3.8, a descent condition for Klein-Gordon theory was found in [**Lan14**]. So far, however, descent for algebraic quantum field theory has not been systematically studied. The natural question to ask is now whether linear homotopy algebraic quantum field theories such as the linear Yang-Mills model satisfy descent for $j : \overline{\mathbf{Loc}_\diamond} \to \overline{\mathbf{Loc}}$. As in Chapter 5, this involves deriving the left adjoint $j_!$ and now determining the derived counit of the adjunction. Such a construction would allow us to answer if linear Yang-Mills theory satisfies descent; it would also shed a broader light on the concept and whether or not it is natural to ask field theories to satisfy this powerful condition.

Finally, all quantum gauge theories that occur in Nature have nonlinear gauge groups, so one should eventually go beyond the linear field theories presented in this thesis. The first step would be to move from gauge group $\mathbb{R}$ to $U(1)$, considering nonlinear (but still non-interacting) gauge theories. This immediately asks for much heavier technical machinery, moving from linear Poisson chain complexes to higher (derived and stacky) differential geometry.

The constructions of chapters 3 and 4 are clearly rooted in chain complexes, but perhaps analogous constructions for derived stacks of nonlinear non-interacting theories could be found. When trying to pursue this, at every step of the way one would have to provide constructions and make choices analogous to those made for linear theories. For example, the solution space $\mathfrak{Sol}(M)$ now is the derived critical locus of a function of a quotient stack rather than the derived critical locus of a function on a chain complex; in the context of derived algebraic geometry this was achieved in [**BSS21**]. Beyond this, a good analogue of the linear observables $\mathfrak{L}(M)$ would be crucial; Pontryagin duals might provide the solution here.

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