On Zorich maps and other topics in quasiregular dynamics

Athanasios Tsantaris

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Abstract

The work in this thesis revolves around the study of dynamical systems arising from iterating quasiregular maps. Quasiregular maps are a natural generalization of holomorphic maps in higher (real) dimensions and their dynamics have only recently started being systematically studied.

We first study permutable quasiregular maps, i.e. maps that satisfy $f \circ g = g \circ f$, where we show that if the fast escaping sets of those functions are contained in their respective Julia sets then those two functions must have the same Julia set. We also obtain the same conclusion about commuting quasimeromorphic functions with infinite backward orbit of infinity. Furthermore we show that permutable quasiregular functions of the form f and $g = \phi \circ f$, where ϕ is a quasiconformal map, have the same Julia sets. Those results generalize well known theorems of Bergweiler, Hinkkanen and Baker on permutable entire functions.

Next we study the dynamics of Zorich maps which are among the most important examples of quasiregular maps and can be thought of as analogues of the exponential map on the plane. For the exponential family $E_\kappa : z \mapsto \kappa e^z$, $\kappa > 0$, it has been shown that when $\kappa > 1/e$ the Julia set of E_κ is the entire complex plane, essentially by Misiurewicz. Moreover, when $0 < \kappa \leq 1/e$ Devaney and Krych have shown that the Julia set of E_κ is an uncountable collection of disjoint curves. Bergweiler and Nicks have shown that a similar result is also true for Zorich maps.

First we construct a certain "symmetric" family of Zorich maps, and we show that the Julia set of a Zorich map in this family is the whole of $\mathbb{R}^3$ when the value of the parameter is large enough, thus generalizing Misiurewicz's result. Moreover, we show that the periodic points of those maps are dense in $\mathbb{R}^3$ and that their escaping set is connected, generalizing a result of Rempe. We also generalize a theorem of Ghys, Sullivan and Goldberg on the measurable dynamics of the exponential.

On a similar note, we study the set of endpoints of the Julia sets of Zorich maps in the case that the Julia set is a collection of curves. We show that ∞ is an explosion point for the set of endpoints by introducing a topological model for the Julia sets of certain Zorich maps, similar to the so called *straight brush* of Aarts

and Oversteegen. Moreover we introduce an object called a *hairy surface* which is a compactified version of the Julia set of Zorich maps and we show that those objects are not uniquely embedded in $\mathbb{R}^3$, unlike the corresponding two-dimensional objects which are all ambiently homeomorphic.

Finally, we study the question of how a connected component of the inverse image of a domain under a quasiregular map covers the domain. We prove that the subset of the domain that is not covered can be at most of conformal capacity zero. This partially generalizes a result due to Heins. We also show that all points in this omitted set are asymptotic values.

Publications

Most of the content of this thesis has appeared in the following publications:

[131] Athanasios Tsantaris, Permutable quasiregular maps, Math. Proc. Cambridge Philos. Soc., 1-17, 2021

[130] Athanasios Tsantaris, Julia sets of Zorich maps, Ergod. Theory Dyn. Syst., 1-37, 2021

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Chapter 1

Introduction

Complex dynamics is concerned with the study of dynamical systems arising from iterating holomorphic functions between Riemann surfaces. The two most prominent classes of such maps being rational and entire transcendental functions which have been studied extensively in the last 40 years. We refer to [9, 11, 30, 92] for thorough introductions to the subject and its history.

Complex dynamics has many deep connections with many classical areas in complex analysis such as value distribution theory, geometric function theory and Riemann surfaces to name a few.

This dissertation focuses on how we can generalize the theory of complex dynamics to the higher dimensional setting of $\mathbb{R}^d$, $d > 2$.

Our starting point is geometric function theory. Geometric function theory is the study of the geometric properties of analytic functions in the complex plane. Classical theorems that are part of this theory are for example the Riemann mapping theorem, Schwarz's lemma, the open mapping theorem and the maximum principle.

A natural question to ask now is: Is there something like the above theorems for a suitable class of maps in higher dimensions?

Since there is no natural notion of holomorphicity in higher dimensions we can only hope that we can find some appropriate class of maps for which the geometric properties of holomorphic functions still persist. Because holomorphic maps on $\mathbb{C}$ are conformal away from critical points (i.e. maps that locally preserve angles), the first class of maps that seems as a good candidate for such an endeavour is the class of sufficiently differentiable conformal maps in $\mathbb{R}^d$. We remind the reader here that a differentiable map $f : \mathbb{R}^d \to \mathbb{R}^d$ is conformal at $x \in \mathbb{R}^d$ when

$$Df(x) = \lambda U,$$

where $\lambda > 0$ and U is a map in the orthogonal group $O(d)$. Here $Df(x)$ denotes the derivative of f at x. Unfortunately, as Liouville's theorem on conformal maps informs us, this class of maps contains only higher dimensional Möbius transformations. In fact even if we relax the assumption on differentiability and consider only weakly differentiable conformal maps, Liouville's theorem still holds! We refer to the book [58] for the exact statement of Liouville's theorem and for a proof of the theorem for C^4 maps. The more general theorem for weakly differentiable maps in some Sobolev space can be found in [66] or [65].

Having said that, the only thing left to attempt is to relax the assumption on conformality. That proves to be a much more fruitful approach. Indeed there is a class of maps called *quasiregular* where the condition of angle preservation is relaxed to bounded distortion of angles and for which one can try to generalize the geometric aspects of function theory.

Intuitively quasiregular maps are a generalization of holomorphic maps in the sense that while holomorphic maps, thanks to the Cauchy-Riemann equations, send infinitesimally small circles to circles, quasiregular maps send infinitesimally small spheres to ellipsoids of bounded eccentricity. At this point one needs to be careful about the regularity assumptions imposed on the class that we are interested in studying. It is important not to restrict oneself to overly smooth maps since that restricts the class of quasiregular maps to homeomorphisms, see [118, p. 12] and the references mentioned there for more details. The natural amount of regularity in order to get a rich class of quasiregular maps is to assume that they belong to some suitable Sobolev space. To be more precise, if $d \geq 2$ and $G \subset \mathbb{R}^d$ is a domain, then for $1 \leq p < \infty$ the *Sobolev space* $W^1_{p,loc}(G)$ consists of functions $f = (f_1, f_2, \cdots, f_d) : G \to \mathbb{R}^d$ for which the first order weak partial derivatives $\partial_i f_j$ exist and are locally in L^p. A continuous map $f \in W^1_{d,loc}(G)$ is called quasiregular if there exists a constant $K_O \geq 1$ such that

$$|Df(x)|^d \leq K_O J_f(x) \ \ a.e., \tag{1.1}$$

where $Df(x)$ denotes again the (total) derivative,

$$|Df(x)| = \sup_{|h|=1} |Df(x)(h)|$$

denotes the operator norm of the derivative, and $J_f(x)$ denotes the Jacobian determinant.

Perhaps more well known are planar *quasiconformal* maps, which are quasiregular injective maps. Quasiconformal maps were first considered by Grötzsch in 1928. Soon after, their importance in complex analysis became clear and they were studied extensively by Ahlfors, Teichmuller, Lehto, Virtanen and others (see for example [76] and references therein).

The theory of quasiregular and quasiconformal maps in higher dimensions started being developed in the '60s by people such as Reshetnyak, Gehring, Väisälä and others. Pioneers in this entire area were the Finnish school of mathematics including Rickman, Martio, Väisälä, Vuorinen and others.

One of the most celebrated results that came out of this program was Rickman's proof [116, 117] of the big Picard theorem for quasiregular maps and the subsequent development of a value distribution theory for quasiregular maps. Today the theory of quasiregular maps is quite developed and a very active area of research with many deep connections with other areas of mathematics. We refer to the books [58, 60, 65, 118, 132] and the references therein for more information on quasiregular maps and their history.

Quasiregular and quasiconformal maps have been also used extensively in the study of the dynamics of holomorphic (and meromorphic) functions in the complex plane, ever since Sullivan's proof of the "no wandering domains conjecture" in [128]. Perhaps the most famous such use is in quasiconformal surgery (see for example [27]) where the theory of quasiconformal and quasiregular maps is used to construct holomorphic maps with specific dynamic behaviour.

However, the dynamics of quasiregular maps themselves had been studied only in the special case where the iterates of the map all have the same amount of local distortion. These maps are known in the literature as *uniformly quasiregular maps*. The dynamics of general quasiregular maps were much less well studied. That started to change around 2010 when Bergweiler, Fletcher, Nicks and others started to systematically develop an iteration theory for quasiregular maps in any dimension, see for example [16, 17, 20–23, 52, 53, 55, 57, 98, 99, 101].

In their work [17, 23] Bergweiler and Nicks defined a Julia set for quasiregular maps, by using something like the blow-up property that the classical Julia set has and showed, perhaps surprisingly, that this new Julia set behaves in many ways like the one for holomorphic maps on the plane. Their work opened up the way for many new research challenges and directions to follow.

Following this direction, in this thesis we work on three different projects on quasiregular dynamics. We will briefly describe below what each project is about. For more details we refer the reader to the introductions of the corresponding chapters.

Permutable quasiregular maps

Two maps f and g are called permutable if $f \circ g = g \circ f$. In Chapter 3 we study permutable quasiregular maps in $\mathbb{R}^d$. Inspired by the work of Fatou and Julia [48, 68] on permutable rational maps many people have worked on trying to classify all commuting maps, see for example [44, 106, 122, 123]. An important theorem in this endeavour is the fact that commuting rational maps have the same Julia sets. In the case of entire transcendental functions now we still do not know if any two commuting such maps have the same Julia set even though there has been a lot of research in this direction, see for example [10, 18, 97, 137, 138].

Most of the work in Chapter 3 first appeared in [131] and revolves around the question of whether or not permutable quasiregular maps have the same Julia set. Inspired by results in the complex plane we prove some theorems in that direction some of which are even new in the well studied case of holomorphic maps on the plane.

Dynamics of Zorich maps

In Chapter 4 we study the dynamics of Zorich maps. Zorich maps were first defined by Zorich in [141] and are one of the most important classes of quasiregular maps. They can be thought of as the higher dimensional counterpart to the exponential map in the complex plane.

The exponential family

$$E_\kappa(z) = \kappa e^z, \;\; \kappa \in \mathbb{C} \setminus \{0\},$$

is the simplest one-parameter family of transcendental entire functions since it only has one *singular value*. The singular values of a function are its asymptotic and critical values and they play a very important role in the dynamical behaviour of the function. For this reason the dynamics of exponential maps have been studied extensively and the literature on them is vast. Without wanting to be exhaustive we mention [6, 36, 39, 40, 59, 84, 109, 110, 112, 126]. We also refer the interested reader to the survey [38] on exponential dynamics.

Here we are mainly interested in two results about exponential maps. For $0 < \kappa \leq 1/e$, as was proven first by Devaney and Krych in [40], the Julia set $\mathcal{J}(E_\kappa)$ is a so called "*Cantor bouquet*" which consists of uncountably many disjoint curves each of which has a finite endpoint and goes off to infinity. On the other hand, when $\kappa > 1/e$ Misiurewicz in [94] proved that the Julia set $\mathcal{J}(E_\kappa)$ equals the entire complex plane $\mathbb{C}$.

In the higher dimensional setting of Zorich maps now, Bergweiler and Nicks in [16, 23], have managed to generalize the first of these results in $\mathbb{R}^3$. Our main goal in Chapter 4 is to obtain a generalization of Misiurewicz's result for Zorich maps in $\mathbb{R}^3$ and various other results along this theme, inspired by classical results for the exponential family in the complex plane. Most of this work first appeared in [130]. On a similar note, we study the topology of Julia sets of Zorich maps in the case where the Julia sets are collections of curves. We show that ∞ is an *explosion point* for the set of endpoints of those curves. This generalizes a result of Mayer [88] for exponential maps. To prove this we develop a topological model for the Julia sets of Zorich maps which is inspired by the straight brush model of Aarts and Oversteegen [1] and we show that there are some differences with the two-dimensional case. This work can be found in [129].

Mapping properties of domains under quasiregular maps

Finally, in Chapter 5 we study a problem on the mapping properties of domains in $\mathbb{R}^d$ under quasiregular maps. To be more precise, suppose that V is an open and connected set of $\mathbb{R}^d$. Let f be a quasimeromorphic map in $\mathbb{R}^d$. Consider now any connected component G of $f^{-1}(V)$. A very interesting and important problem would then be

Problem: How big can the set $V \setminus f(G)$ be?

That problem was solved by Heins in 1952 for the case of meromorphic maps in $\mathbb{C}$. Much later, using different methods, Herring and Bolsch improved Heins' theorem to a more general class of maps. The answer to the above problem for meromorphic maps is that $V \setminus f(G)$ can contain at most two points.

A particular case of interest is when G and V are quasi-Fatou components (the analogues of Fatou components for holomorphic maps).

In Chapter 5 we are going give some results towards a generalization in higher dimensions of Heins' theorem.

Chapter 2

Preliminaries

2.1 Complex dynamics

In this section we will give a brief overview of the iteration of holomorphic functions on the complex plane. Our focus will be the study of transcendental entire functions, which in large parts runs in parallel with the more developed theory of iteration of rational functions. For a more thorough introduction to these subjects we refer to [11, 30, 92].

The general theory of iteration of holomorphic/rational maps starts from the seminal work of Fatou [50] and Julia [67]. Fatou and Julia initially developed their theory for rational functions and later on Fatou [49] also considered iteration of entire transcendental functions. Both of them defined a partition of the complex plane in two sets. Those two sets today bear their name. They are the *Fatou set*, $\mathcal{F}$, and the *Julia set*, $\mathcal{J}$. In order to define them, let us consider a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ and denote by $\{f^n\}$ the family of iterates of f, namely the family

$$\{\underbrace{f \circ f \cdots \circ f}_{n \text{ times}} : n \in \mathbb{N}\}.$$

Then the *Fatou set* $\mathcal{F}$ is defined as the set of points in a neighbourhood of which this family is normal with respect to the spherical metric and the *Julia set* $\mathcal{J}$ is defined as its complement.

In what follows we are going to need the notions of *forward orbit* which is defined as

$$O^+(z) := \{f^n(z) : n \geq 0\}$$

and that of *backward orbit* which is defined as

$$O^-(z) := \{w \in \mathbb{C} : f^n(w) = z \text{ for some } n \geq 0\} = \bigcup_{n\geq 0} f^{-n}(z).$$

Another class of maps for which we are going to need the notion of a Julia set is the class of transcendental meromorphic functions for which the set $O^{-}(\infty)$ is infinite. For such maps the Julia set is simply defined as $\mathcal{J}(f) = \overline{O^{-}(\infty)}$.

We will say that a set $S \subset \mathbb{C}$ is *completely invariant* if $z \in S$ implies $f(z) \in S$ and conversely. The *exceptional set* $E(f)$ is defined as the set of all points whose backward orbit is finite. An easy consequence of Picard's Theorem is that the exceptional set can contain at most one point.

Also a point $z \in \mathbb{C}$ is called *periodic* for f if $f^n(z) = z$, for some $n \in \mathbb{N}$.

The following theorem contains some of the important properties of the Fatou and Julia sets. We also refer to [11] for the proofs and many more properties. In what follows we denote by $|S|$ the cardinality of a set S.

Theorem 2.1.1. *Let f be either a rational function of degree ≥ 2 or a transcendental entire function. Then*

a. $\mathcal{J}(f) \neq \emptyset$. *In fact,* $\mathrm{card}(\mathcal{J}(f)) = \infty$.

b. $\mathcal{F}(f)$ *is open and* $\mathcal{J}(f)$ *is closed.*

c. $\mathcal{F}(f) = \mathcal{F}(f^n)$ *and* $\mathcal{J}(f) = \mathcal{J}(f^n)$, $\forall n \geq 1$.

d. $\mathcal{F}(f)$ *and* $\mathcal{J}(f)$ *are completely invariant sets.*

e. If S is a closed and completely invariant set and $\mathrm{card}(S) \geq 3$, *then* $\mathcal{J}(f) \subset S$.

f. If U is open and $U \cap \mathcal{J}(f) \neq \emptyset$ then $\mathbb{C} \setminus E(f) \subset \bigcup_{n\geq 0} f^n(U)$. In fact the following stronger result is true.

g. If K is any compact subset of $\mathbb{C}$ that does not contain exceptional points and U is as above then $K \subset f^n(U)$ for all large enough n.

h. If $z \in \mathcal{J}(f) \setminus E(f)$ then $\mathcal{J}(f) = \overline{O^{-}(z)}$.

i. The periodic points of f are dense in $\mathcal{J}(f)$.

It is worth mentioning here that the property (f) is the so called blow-up property of the Julia set and it will be used later to define the Julia set for quasiregular maps.

An important thing to note here is that through the Arzela-Ascoli theorem the Julia set can be viewed as the set of points in the complex plane at which the dynamical system produced by the iterates of f has sensitive dependence on initial conditions. This means that nearby points in the Julia set will end up far apart after some number of iterations. In other words the family is not equicontinuous with respect to the spherical metric. Moreover, as was shown by Fatou and Julia in the case of rational maps and Baker [4] in the case of transcendental entire maps, the periodic points of f are dense in the Julia set. Combining those two facts with the blow up property we obtain that the Julia set is the set where the dynamics are chaotic, according to the definition of chaos by Devaney (see [37]).

Another important set in the theory of iteration of transcendental entire functions is the *escaping set*, which was defined first by Eremenko [43] and is the set

$$I(f) := \{z \in \mathbb{C} : \lim_{n\to\infty} |f^n(z)| = \infty\}.$$

Eremenko in [43] proved that for transcendental entire functions the escaping set is non-empty and furthermore

$$\partial I(f) = \mathcal{J}(f).$$

Eremenko also showed that all the connected components of $\overline{I(f)}$ are unbounded and he conjectured that the same is true for $I(f)$. That conjecture is one of the major open problems in complex dynamics.

Eremenko's Conjecture: Every component of $I(f)$ is unbounded.

His conjecture is motivated by examples such as κe^z for $0 < \kappa \leq 1/e$ where the escaping set is a collection of disjoint, unbounded curves and thus its connected components (those curves) are unbounded. In fact it has been shown in [126] that for all values of the parameter $\kappa \in \mathbb{C} \setminus \{0\}$ the path-connected components of $I(f)$ are unbounded.

Although Eremenko's conjecture has been shown to hold for large classes of entire transcendental maps, see for example [7, 111, 120, 124], the general case is still open.

Finally, let us discuss the *fast escaping set*. The fast escaping set is an important subset of the escaping set and was first defined by Bergweiler and Hinkkanen in [18]. For a transcendental entire function they defined it as

$$\mathcal{A}(f) := \{z \in \mathbb{C} : \text{ there exists } L \in \mathbb{N} \text{ such that } |f^n(z)| > M(R, f^{n-L}), \text{ for } n > L\} \tag{2.1}$$

where $M(r, f) = \max_{|z|=r} |f(z)|$, $r > 0$ and $R > 0$ is large. It is obvious that $\mathcal{A}(f) \subset I(f)$. Intuitively the fast escaping set is the set of points that escape to infinity as fast as possible. In [18] it is also proved that

$$\mathcal{J}(f) = \partial\mathcal{A}(f).$$

Rippon and Stallard in [121] prove that all components of $\mathcal{A}(f)$ are unbounded which implies that at least one component of $I(f)$ is unbounded. In the last years the fast escaping set has been studied extensively and it is now a central object in the study of the dynamics of transcendental entire functions. Besides the definition that we mentioned above Rippon and Stallard, in their papers [120, 121], gave two other equivalent definitions for the fast escaping set which are useful. They showed that

$$\mathcal{A}(f) = \{z \in \mathbb{C} : \text{ there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| > M^n(R, f), \text{ for } n > 0\}, \tag{2.2}$$

where $M^n(r, f)$ denotes the iteration of $M(r, f)$ with respect to the variable r, and $R > 0$ is any value such that $M(r, f) > r$ for $r \geq R$ or, equivalently, such that $M^n(R, f) \to \infty$ as $n \to \infty$. Also they proved that

$$\mathcal{A}(f) = \{z \in \mathbb{C} : \text{ there exists } L \in \mathbb{N} \text{ such that } f^{n+L}(z) \notin T(f^n(D)), \text{ for all } n \in \mathbb{N}\}, \tag{2.3}$$

where D is any open disc meeting $\mathcal{J}(f)$ and $T(X)$ is the *topological hull* of the set $X \subset \mathbb{C}$, in other words the union of X with its bounded complementary components.

For the proof that those definitions are equivalent, and many more interesting properties of the fast escaping set, we refer to [121].

2.2 Finite quotients of affine maps

In this section we will briefly describe a very important class of rational functions on the complex plane which appear in the study of permutable maps. Using the terminology introduced by Milnor in [91] we will call those maps *finite quotients of affine maps*. Since our study here will not be extensive we refer the interested reader to [91] for more details.

Let Λ be a lattice on the plane, meaning a discrete additive subgroup of $\mathbb{C}$. Any lattice, assuming that it is non-trivial, is generated by either one element or two elements. In what follows we assume that the lattice Λ is non-trivial.

Definition 2.2.1. A rational map f of degree two or more will be called a finite quotient of an affine map if there is a lattice Λ on the complex plane, an affine map $A(z) = az + b$ from $\mathbb{C}/\Lambda$ to itself and a finite to one holomorphic map $\Theta : \mathbb{C}/\Lambda \to \overline{\mathbb{C}} \setminus E(f)$ which make the following diagram commute:

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{A} & \mathbb{C}/\Lambda \\ \Theta\big\downarrow & & \big\downarrow\Theta \\ \overline{\mathbb{C}} & \xrightarrow{f} & \overline{\mathbb{C}} \end{array} \tag{2.4}$$

Note that instead of having chosen A to be an affine map from $\mathbb{C}/\Lambda$ to itself we could have demanded A to be an affine map on the complex plane with $A\Lambda A^{-1} \subset \Lambda$.

The above diagram essentially tells us that f comes as a solution to the functional equation

$$f \circ \Theta = \Theta \circ A,$$

which when A is a linear map is the so called *Schröder functional equation.* That equation plays a fundamental role, through the Koenigs Linearization Theorem [92, Chapter 8], when studying the dynamics of a holomorphic map around attracting fixed points.

Let us now see some examples of finite quotients of affine maps.

Power Maps. The simplest example is that of power maps which are the maps $z \mapsto z^n$, $n \in \mathbb{N}$. To obtain the power from the construction we described above take $\Lambda = 2\pi i\mathbb{Z}$, $\Theta(z) = e^z$ and $A(z) = nz$. It is easy to see then that the diagram

$$\begin{array}{ccc} \mathbb{C}/2\pi i\mathbb{Z} & \xrightarrow{nz} & \mathbb{C}/2\pi i\mathbb{Z} \\ e^z\big\downarrow & & \big\downarrow e^z \\ \overline{\mathbb{C}} & \xrightarrow{z^n} & \overline{\mathbb{C}} \end{array}$$

commutes. So power maps are finite quotients of affine maps.

Tchebycheff polynomials. Tchebycheff polynomials, denoted by T_n, are the degree n polynomials that are defined as solutions of the equation

$$T_n(\cos z) = \cos(nz).$$

By their very definition we can see that Tchebycheff polynomials are indeed finite quotients of affine maps which make the following diagram commute

$$\begin{array}{ccc} \mathbb{C}/2\pi\mathbb{Z} & \xrightarrow{nz} & \mathbb{C}/2\pi\mathbb{Z} \\ \cos z\big\downarrow & & \big\downarrow\cos z \\ \overline{\mathbb{C}} & \xrightarrow{T_n} & \overline{\mathbb{C}}\,. \end{array}$$

In the above examples note that the lattice Λ has rank one. We will now split the class of finite quotients of affine maps in two classes. First we assume that the lattice we used in the definition has rank one. Then the following Theorem tells us, in a sense, that in this case the only finite quotients of affine maps are power maps and Tchebycheff polynomials.

Theorem 2.2.1 (Milnor, [91])**.** *If f is a finite quotient of an affine map and Λ has rank one, then f is conformally conjugate to a power map or to a Tchebycheff polynomial, T_n or to $-T_n$, for some $n \in \mathbb{N}$.*

Lattès maps. In the case where the lattice Λ has rank two then we call the map that we obtain from the commutative diagram 2.4 a *Lattès map.* Lattès maps were the first examples of rational functions whose Julia set is the entire Riemann sphere and they play an important role in the study of the dynamics of rational functions.

2.3 Background on quasiregular maps

2.3.1 Quasiregular maps

Here we will review some basic notions and definitions from the theory of quasiregular maps. For a more detailed treatment of quasiregular maps we refer to [60, 65, 118, 134].

First we recall the definition of a quasiregular map. Let G be a domain in $\mathbb{R}^d$. A continuous map $f \in W^1_{d,loc}(G)$ is called quasiregular (abbreviated qr) if there exists a constant $K_O \geq 1$ such that

$$|Df(x)|^d \leq K_O J_f(x) \ \ a.e., \tag{2.5}$$

where $Df(x)$ denotes the total derivative,

$$|Df(x)| = \sup_{|h|=1} |Df(x)(h)|$$

denotes the operator norm of the derivative, and $J_f(x)$ denotes the Jacobian determinant. Also let

$$\ell(Df(x)) = \inf_{|h|=1} |Df(x)(h)|.$$

The condition that (2.5) is satisfied for some $K_O \geq 1$ implies that

$$K_I \ell(Df(x))^d \geq J_f(x), \ \ a.e.,$$

for some $K_I \geq 1$. The smallest constants K_O and K_I for which those two conditions hold are called the *outer dilatation* and *inner dilatation* respectively. We call the maximum of those two numbers the dilatation of f and we denote it by $K(f)$. We say that f is K-quasiregular if $K(f) \leq K$, for some $K \geq 1$.

The notion of quasiregularity can also be defined more generally for maps $f : M \to N$, where M and N are oriented connected Riemannian n-manifolds. A particular case of interest here is when $N = \overline{\mathbb{R}^d} := \mathbb{R}^d \cup \{\infty\}$ equipped with the spherical metric (obtained via the stereographic projection from the unit sphere) and M is a domain in $\overline{\mathbb{R}^d}$. In that case, we call a quasiregular map $f : M \to N$ *quasimeromorphic* (abbreviated qm).

As it turns out if $G \subset \overline{\mathbb{R}^d}$, a non constant and continuous map $f : G \to \overline{\mathbb{R}^d}$ is quasimeromorphic if $f^{-1}(\infty)$ is discrete and f is quasiregular in $G \setminus (f^{-1}(\infty) \cup \{\infty\})$.

Next we note that if f and g are quasiregular maps, then the composition $f \circ g$ (assuming it is well defined) is also quasiregular. Moreover, it is true that $K(f \circ g) \leq K(f)K(g)$.

We say that f is *uniformly K-quasiregular* (abbreviated uqr) if all the iterates of f are K-quasiregular.

Quasiregular maps have many of the properties that holomorphic maps have. In particular, we will often use the fact that non-constant quasiregular maps are open and discrete.

Theorem 2.3.1 (Reshetnyak,[114, 115]). *Non-constant qr mappings are open and discrete. Moreover quasiregular maps are differentiable almost everywhere.*

We are also going to need the following theorem. Note that m denotes the Lebesgue measure and $N(y, f, E) = \operatorname{card}(f^{-1}(y) \cap E)$, $y \in \overline{\mathbb{R}^d}$.

Theorem 2.3.2 ([118] I. Proposition 4.14). *Let $f : G \to \mathbb{R}^d$ be a qr map. Then the following hold:*

1. *let $E \subset G$ be such that $m(E) = 0$ then $m(f(E)) = 0$ (Lusin's N property).*

2. *The change of variables formula holds*

$$\int_E (h \circ f) J_f dm = \int_{f(E)} h(y) N(y, f, E) dy,$$

   for any measurable $h : \mathbb{R}^d \to [0, \infty]$ and all measurable $E \subset G$.

A quasiregular map $f : \mathbb{R}^d \to \mathbb{R}^d$ is said to be of *transcendental type* if $\lim_{x\to\infty} f(x)$ does not exist and it is said to be of *polynomial type* if this limit is ∞.

For a quasiregular map f we define the degree of the map, and we denote it by $\deg(f)$, as

$$\deg(f) := \max_{x \in \overline{\mathbb{R}^d}} \operatorname{card}(f^{-1}(x)).$$

Note that f is of polynomial type if and only if $\deg(f) < \infty$ or equivalently f is of transcendental type if and only if $\deg(f) = \infty$.

Another important result about quasiregular maps is the quasiregular analogue of Picard's theorem. That was first proven by Rickman in [116] and takes the following form:

Theorem 2.3.3 (The big Rickman-Picard Theorem)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a non-constant K-quasiregular map, where $d \geq 2$ and $K \geq 1$. Then there exists a constant $q = q(d, K)$ such that there are at most $q(d, K)$ points that are taken only finitely often by f.*

Of particular interest to us here is the corresponding "small" version of the above result.

Theorem 2.3.4 (The small Rickman-Picard Theorem)**.** *For every $d \geq 3$ and $K \geq 1$ there exists a positive integer $q = q(d, K)$ such that if $a_1, \dots, a_q \in \mathbb{R}^d$ are distinct and f is a K-qr mapping $f : \mathbb{R}^d \to \mathbb{R}^d \setminus \{a_1, \dots, a_q\}$ then f is constant.*

Both of the above theorems hold for quasimeromorphic maps as well, we refer to [118, Chapter IV] for more details. Also the small Rickman-Picard theorem is shown to be sharp in [41, 117], meaning that for any number q there is a quasiregular map in $\mathbb{R}^d$ omitting q points.

Moreover, it is worth mentioning here that the Rickman-Picard theorem has been reproven many times and in different contexts by using different methods than those of Rickman, see for example [26, 45, 64, 65, 78, 108].

The Rickman-Picard theorem now implies that if we define the exceptional set $E(f)$ for a K-quasiregular map as the points with finite backward orbit, then $|E(f)| \leq q(d, K)$.

Finally, let us give some easy examples of quasiregular maps.

Examples.

1. All holomorphic maps are 1-quasiregular.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $f(x, y) = (kx, y)$, where $k > 1$. A simple computation can show that this map, while not holomorphic, is indeed k-quasiregular. Of course we can generalize this construction to higher dimensions.

3. Another easy example is provided by winding maps which are maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ given in polar coordinates by $(r, \theta) \mapsto (r, N\theta)$, where $N \in \mathbb{N}$ and $N \geq 2$.

Again a simple computation shows that those maps are N-quasiregular and $K_I(f) = K_O(f) = N$. This construction can also be generalized in higher dimensions.

4. Another interesting class of examples is in a sense a generalization of the exponential map in the complex plane called the Zorich maps. For simplicity let us describe its construction on $\mathbb{R}^3$. Consider the unit square $Q = [0,1]^2$, the hemisphere

$$S = \{(x,y,z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$$

and any L bi-Lipschitz map $h : Q \to S$. Define now the map $Z : Q \times \mathbb{R} \to \mathbb{R}^3$ as

$$Z(x,y,z) = e^z h(x,y).$$

This map sends the square beam $Q \times \mathbb{R}$ to the upper half-space. We can now extend this map to a map Z defined in the whole $\mathbb{R}^3$ by repeatedly reflecting across the sides of square beams and the xy-plane. We will show that this map is quasiregular. Moreover its dilatation is bounded in terms of the bi-Lipschitz constant L.

First notice that since bi-Lipschitz maps are a.e. differentiable we will have that when $(x,y,z) \in Q \times \mathbb{R}$ then

$$DZ(x,y,z) = e^z \begin{pmatrix} \frac{\partial h_1}{\partial x}(x,y) & \frac{\partial h_1}{\partial y}(x,y) & h_1(x,y) \\ \\ \frac{\partial h_2}{\partial x}(x,y) & \frac{\partial h_2}{\partial y}(x,y) & h_2(x,y) \\ \\ \frac{\partial h_3}{\partial x}(x,y) & \frac{\partial h_3}{\partial y}(x,y) & h_3(x,y) \end{pmatrix}.$$

We now have that

$$|DZ(x,y,z)|^3 = e^{3z} \sup_{u_1^2+u_2^2+u_3^2=1} |Au_1 + Bu_2 + Cu_3| \leq e^{3z}(2L+1)^3,$$

where A, B and C are the columns of DZ and we have used the triangle inequality and the fact that h is L-Lipschitz. We can now also show that $J_Z \geq \frac{e^{3z}}{L^2}$, see proof of Lemma 4.2.6. Also since the map is defined through reflections in the rest of $\mathbb{R}^3$ the bounds on J_Z and $|DZ|^3$ do not change when (x,y,z) is in other beams. Thus the Zorich map is quasiregular and its outer dilatation is bounded in terms of the bi-Lipschitz constant. Similarly we can estimate its inner dilatation.

Note also that this map is doubly periodic with periods $(4, 0, 0)$ and $(0, 4, 0)$ while it also omits 0. Of course this construction can be done in any dimensions. It is also worth noting that the definition of the Zorich maps is flexible. For instance we could have chosen a bigger square for our map h. Moreover, instead of mapping the initial square in a hemisphere we could have mapped it to a sufficiently "nice" surface (see section 4.6).

For more on Zorich maps see Chapter 4.

5. Another important family of quasiregular maps (in fact uniformly quasiregular) is the analogues of power, Tchebycheff and Lattés maps. Those were first constructed by Mayer in [89, 90] by using Schröder's equation.

   We start again with a discrete subgroup Λ of $\mathbb{R}^d$ which is isomorphic to either $\mathbb{Z}^d$ or $\mathbb{Z}^{d-1}$, a similarity $A(x) = \lambda O x$, where O is an orthogonal transformation and $\lambda > 0$, such that $A\Lambda A^{-1} \subset \Lambda$ and a quasiconformal map $\Theta : \mathbb{R}^d/\Lambda \to \overline{\mathbb{R}^d}$. The functions f that we are looking for come as solutions to the Schröder equation.

   Theorem 2.3.5 (Theorem 21.4.1, [65]). *Let Λ, A and Θ be as above then there is a solution f to the functional equation*

$$f \circ \Theta = \Theta \circ A$$

   and f is a uniformly quasiregular map of $\overline{\mathbb{R}^d}$.

   When the subgroup Λ is isomorphic to $\mathbb{Z}^{d-1}$ then depending on the quasiregular map Θ, which can be taken to be any of the quasiregular analogues of the exponential and trigonometric functions, the above theorem gives us the uqr analogues of power and Tchebycheff maps. When Λ is isomorphic to $\mathbb{Z}^d$ we obtain the uqr analogues of Lattés maps. We refer to [65] for more details on the construction.

   It is also worth mentioning here that the above construction can be done in a more general setting, see [54], but that is not needed for our purposes.

2.3.2 Dynamics of uniformly quasiregular maps

As we already mentioned in the previous section uniformly quasiregular maps are quasiregular maps whose iterates are all K-quasiregular for the same constant K. For those kinds of quasiregular maps there is a version of Montel's theorem available which one can prove using the analogue of Zalcman's lemma (see for example [13]) for quasiregular maps which was proven by Miniowitz [93].

Theorem 2.3.6. *Let $d \geq 2$, $K \geq 1$ and G be a domain in $\mathbb{R}^d$. If $a_1, \dots, a_q \in \mathbb{R}^d$ are distinct, where q is as in Theorem 2.3.4 then the family of all K-qr maps $f : G \to \mathbb{R}^d \setminus \{a_1, \dots, a_q\}$ is normal.*

As a result uniformly quasiregular maps usually have better dynamical properties than general quasiregular maps and their iterative theory resembles more closely that of holomorphic maps, see for example [63]. However uqr maps is a much more restricted class of maps. For example it is not known if there exists a transcendental type uqr map $f : \mathbb{R}^d \to \mathbb{R}^d$ for $d \geq 3$.

For more on the dynamics of uniformly quasiregular maps we refer to [15], [65, Chapter 21] and we also mention here the more recent papers [56, 69, 103, 104].

The Fatou set and the Julia set for a uniformly quasiregular map are defined exactly in the same way they were defined for holomorphic maps. Namely the Fatou set is the set where the iterates form a normal family and the Julia set is its complement. Theorem 2.1.1 holds almost unchanged for uniformly quasiregular maps.

Theorem 2.3.7. *Let f be a uniformly quasiregular function which is a self map, with degree ≥ 2, of $\overline{\mathbb{R}^d}$. Then*

a. $\mathcal{J}(f) \neq \emptyset$. *In fact,* $\operatorname{card}(\mathcal{J}(f)) = \infty$*;*

b. $\mathcal{F}(f)$ *is open and* $\mathcal{J}(f)$ *is closed;*

c. $\mathcal{F}(f) = \mathcal{F}(f^n)$ *and* $\mathcal{J}(f) = \mathcal{J}(f^n)$, $\forall n \geq 1$*;*

d. $\mathcal{F}(f)$ *and* $\mathcal{J}(f)$ *are completely invariant sets;*

e. If S is a closed and completely invariant set and $\operatorname{card}(S) \geq q(d, K) + 1$*, where q is Rickman's constant, then $\mathcal{J}(f) \subset S$;*

f. If U is open and $U \cap \mathcal{J}(f) \neq \emptyset$ then $\mathbb{R}^d \setminus E(f) \subset \bigcup_{n \geq 0} f^n(U)$;

g. If $z \in \mathcal{J}(f) \setminus E(f)$ then $\mathcal{J}(f) = \overline{O^-(z)}$;

Remark. We could also study the dynamics of uniformly quasiregular maps of transcendental type. However, for $d \geq 3$ the existence of such maps is still an open problem.

2.3.3 Quasiregular dynamics

In this section we drop the condition that all iterates of f are K-quasiregular. In [17] Bergweiler developed a Fatou-Julia theory for quasiregular self-maps of $\overline{\mathbb{R}^d}$, which include polynomial type quasiregular maps, and can be thought of as analogues of rational maps, while in [23] Bergweiler and Nicks did the same but for transcendental type quasiregular maps.

An important tool that we will need in order to define the Julia set of a quasiregular map is the capacity of a condenser. A condenser in $\mathbb{R}^d$ is a pair $E = (A, C)$, where A is an open set in $\mathbb{R}^d$ and C is a compact subset of A. The *conformal capacity* or just *capacity* of the condenser E is defined as

$$\operatorname{cap} E = \inf_u \int_A |\nabla u|^d dm,$$

where the infimum is taken over all non-negative functions $u \in C_0^\infty(A)$ which satisfy $u_{|C} \geq 1$ and m is the d-dimensional Lebesgue measure. We will call such functions *admissible.*

If $\operatorname{cap}(A, C) = 0$ for some bounded open set A containing C, then it is also true that $\operatorname{cap}(A', C) = 0$ for every other bounded set A' containing C;[118, Lemma III.2.2]. In this case we say that C has zero capacity and we write $\operatorname{cap} C = 0$; otherwise we say that C has positive capacity and we write $\operatorname{cap} C > 0$. Also for an arbitrary set $C \subset \mathbb{R}^d$, we write $\operatorname{cap} C = 0$ when $\operatorname{cap} F = 0$ for every compact subset F of C. If the capacity of a set is zero then this set has Hausdorff dimension zero [118, Theorem VII.1.15]. Thus a zero capacity set is small in this sense. It is also quite easy to see that for any two sets S, B with $S \subset B$ if $\operatorname{cap} B = 0$ then $\operatorname{cap} S = 0$.

A useful property of quasiregular maps is that they do not increase too much the capacity of condensers, namely the following theorem holds, which is known as the K_I inequality, [118, Theorem II.10.10].

Theorem 2.3.8. *Let $f : G \to \mathbb{R}^d$ be a nonconstant quasiregular map and $E = (A, C)$ a condenser in G, then*

$$\operatorname{cap} f(E) \leq K_I(f) \operatorname{cap} E.$$

Following [17, 23], we define the Julia set of $f : \mathbb{R}^d \to \mathbb{R}^d$, denoted $\mathcal{J}(f)$, to be the set of all those $x \in \mathbb{R}^d$ such that

$$\operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{k=1}^{\infty} f^k(U)\right) = 0$$

for every neighbourhood U of x. We call the complement of $\mathcal{J}(f)$ the *quasi-Fatou set*, and we denote it by $QF(f)$. We also want to define the Julia set for a quasimeromorphic map of transcendental type with at least one pole, $f : \mathbb{R}^d \to \overline{\mathbb{R}^d}$. This was done by Warren in [140] where he defined

$$\mathcal{J}(f) = \left\{ x \in \overline{\mathbb{R}^d} \setminus \overline{O_f^-(\infty)} : \operatorname{card}\left(\overline{\mathbb{R}^d} \setminus O_f^+(U_x) \right) < \infty \right\} \cup \overline{O_f^-(\infty)},$$

where U_x is any neighbourhood of x with $U_x \subset \overline{\mathbb{R}^d} \setminus \overline{O_f^-(\infty)}$ and $O_f^+(U_x) = \bigcup_{n=0}^{\infty} f^n(U_x)$. In particular if f has an infinite backward orbit of infinity then $\mathcal{J}(f) = \overline{O_f^-(\infty)}$.

Note here that we used something like the blow-up property, that the Julia set in complex dynamics has, in order to define our Julia set. Also note that we do not assume anything about the normality of the family of iterates of f in the quasi-Fatou set. For the motivation behind those definitions we refer to [15, 17]. Also let us mention that the definition of the Julia set as given in the section above also includes uniformly quasiregular maps and the two definitions are equivalent in this case. This is also true in the case of holomorphic maps in the complex plane.

The theorem below summarizes the basic properties of the Julia set. For the proofs of these facts we refer to [17, 23].

Theorem 2.3.9. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a quasiregular map with $\deg(f) > K_I(f)$. Then*

a. $\mathcal{J}(f) \neq \emptyset$. *In fact,* $\operatorname{card}(\mathcal{J}(f)) = \infty$;

b. The classical definition of $\mathcal{J}(f)$, for holomorphic $f : \mathbb{C} \to \mathbb{C}$, using non-normality agrees with the one we have given;

c. The set $\mathcal{J}(f)$ is closed and the set $QF(f)$ is open;

d. The set $\mathcal{J}(f)$ and the set $QF(f)$ are completely invariant sets;

If we further assume that $\operatorname{cap} \overline{O_f^-(x)} > 0$ *for all $x \in \mathbb{R}^d \setminus E(f)$ then*

e. $\mathcal{J}(f) \subset \overline{O_f^-(x)}$ *for all $x \in \mathbb{R}^d \setminus E(f)$;*

f. $\mathcal{J}(f) = \overline{O_f^-(x)}$ *for all $x \in \mathcal{J}(f) \setminus E(f)$;*

g. $\mathcal{J}(f^p) = \mathcal{J}(f)$, *for all $p \in \mathbb{N}$;*

h. $\mathbb{R}^d \setminus O_f^+(U) \subset E(f)$ *for every open set U, with $U \cap \mathcal{J}(f) \neq \emptyset$;*

i. $\mathcal{J}(f)$ *is perfect.*

Remark. The condition $\deg g > K(g)$ for g a polynomial type quasiregular map appears naturally in quasiregular dynamics, see for example [15, 17, 55]. It plays the same role as the condition $\deg g \geq 2$ in holomorphic dynamics, when g is a polynomial.

Remark. It has been conjectured in [17, 23] that $\operatorname{cap} \overline{O_f^-(x)} > 0$, for all $x \in \mathbb{R}^d \backslash E(f)$ always holds for quasiregular maps in $\mathbb{R}^d$ (assuming $\deg(f) > K_I$) and thus that assumption in the theorem above is not needed.

The escaping set of a quasiregular map can be defined in the same way as we did for entire maps. Namely, if $f : \mathbb{R}^d \to \mathbb{R}^d$ is a quasiregular map

$$I(f) = \{x \in \mathbb{R}^d : \lim_{n\to\infty} |f^n(x)| \to \infty\}.$$

It was first studied in [21, 55] where the authors showed that $I(f) \neq \emptyset$ and also provided an example of a quasiregular map with a bounded connected component of the escaping set. Thus Eremenko's conjecture does not hold in this setting. Moreover, in the quasiregular case it is only true that $\mathcal{J}(f) \subset \partial I(f)$.

Finally let us discuss the fast escaping set of a quasiregular map, which was first described by Bergweiler-Drasin-Fletcher in [20]. It is defined in a very similar way to the complex case, namely

$$\mathcal{A}(f) := \{x \in \mathbb{R}^d : \text{ there exists } L \in \mathbb{N} \text{ such that } f^{n+L}(x) \notin T(f^n(B(0, R))), \text{ for all } n \in \mathbb{N}\}, \tag{2.6}$$

where $R > 0$ is chosen so large that

$$T(f^n(B(0, R))) \supset B(0, r_n)$$

and $r_n > 0$ is a sequence that tends to ∞. Such an R is guaranteed to exist by [21, Lemma 4.1].

Also Bergweiler-Drasin-Fletcher in [20] gave two other equivalent definitions, in the same spirit as those for the complex case.

$$\mathcal{A}(f) = \{x \in \mathbb{R}^d : \text{ there exists } L \in \mathbb{N} \text{ such that } |f^n(x)| > M(R, f^{n-L}), \text{ for all } n > L\} \tag{2.7}$$

and

$$\mathcal{A}(f) = \{x \in \mathbb{R}^d : \text{ there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(x)| > M^n(R, f), \text{ for all } n > 0\}. \tag{2.8}$$

Furthermore they proved that

Theorem 2.3.10. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a quasiregular map of transcendental type. Then $\mathcal{A}(f)$ is non-empty and every connected component of $\mathcal{A}(f)$ is unbounded.*

For more details we refer to [20]. Unfortunately, in the quasiregular case it is still not known if $\mathcal{J}(f) = \partial\mathcal{A}(f)$. But let us mention that the above equality is known to be true if f does not grow too slowly and we always know that $\mathcal{J}(f) \subset \partial\mathcal{A}(f)$. We refer to [22] for more details on this.

Chapter 3

Permutable quasiregular maps

3.1 Introduction

Two functions f and g are called *permutable* or *commuting* if they satisfy the equation

$$f \circ g = g \circ f, \tag{3.1}$$

whenever both sides are defined. A very natural question to ask, once you confine your functions to a certain function space, is for which functions does this equation hold? Is there any sort of classification that can be achieved about permutable functions in different function spaces? Of course this problem at first seems extremely hard and probably not even solvable in certain function spaces. However if we confine ourselves to the class of rational functions in the complex plane there is a solution.

The first people who managed to make some progress in the problem of classifying commuting rational functions on the complex plane were Fatou and Julia. The way they approached the problem was through the then newly-found theory of complex dynamics. Both Fatou and Julia in [48] and [68] first show that commuting rational functions have the same Julia sets and then use this fact to find all commuting rational functions that do not share an iterate (i.e. $f^m \neq g^n$ for all $n, m \in \mathbb{N}$) and do not have as their common Julia set the entire complex plane. Much later Eremenko in [44] developed this method further and managed to classify all commuting rational functions that do not share an iterate. It is also worth mentioning here that Ritt had, long before Eremenko, solved the problem of finding all commuting rational functions, in [123], and all commuting polynomials in [122]. However Ritt used completely different methods than those of Fatou and Julia.

In the theorem that follows two rational maps f_1 and f_2 are called *conformally conjugate* if there is a conformal map ϕ of the Riemann sphere with $\phi \circ f_1 \circ \phi^{-1} = f_2$.

Theorem 3.1.1 (Ritt [123] and Eremenko [44])**.** *Let $f, g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be two commuting rational functions such that $f^n \neq g^m$, for all $m, n \in \mathbb{N}$. Then f and g are conformally conjugate to maps that are both either Láttes maps, power maps or Tchebycheff polynomials.*

For transcendental entire functions the problem is much harder and is still open to this day. It is not even known if permutable transcendental entire functions have the same Julia set or not. However Bergweiler and Hinkkanen [18] in 1999, by introducing the fast escaping set $\mathcal{A}(f)$, managed to prove the following.

Theorem 3.1.2 (Bergweiler and Hinkkanen [18])**.** *Let f and g be permutable, transcendental entire functions such that $\mathcal{A}(f) \subset \mathcal{J}(f)$ and $\mathcal{A}(g) \subset \mathcal{J}(g)$ then $\mathcal{J}(f)$=$\mathcal{J}(g)$.*

Recently, Benini, Rippon and Stallard in [10] managed to improve the above theorem and include some cases where $\mathcal{A}(f) \not\subset \mathcal{J}(f)$ and $\mathcal{A}(g) \not\subset \mathcal{J}(g)$. In particular they managed to show that two commuting functions will have the same Julia set if all their wandering Fatou components are multiply connected (such components are in the fast escaping set but not the Julia set). However the general case still remains open.

Passing to higher dimensions now, there are examples of permutable quasiregular maps. So the natural thing to ask is: Do permutable quasiregular maps have a similar dynamic behavior? Can we generalize Theorem 3.1.2 to quasiregular maps?

Our first result tells us that indeed Theorem 3.1.2 holds in this setting as well.

Theorem 3.1.3. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^d$ be permutable, quasiregular maps of transcendental type such that $\mathcal{A}(f) \subset \mathcal{J}(f)$ and $\mathcal{A}(g) \subset \mathcal{J}(g)$, then $\mathcal{J}(f) = \mathcal{J}(g)$.*

The more general version of this theorem would be the analogous result to that of Benini-Rippon-Stallard. Unfortunately their proof relies heavily on the properties of the hyperbolic metric under holomorphic maps. Such an approach does not work in higher dimensions.

Another interesting question would be whether or not permutable quasiregular maps of polynomial type must have the same Julia set. Thanks to the result of Fatou and Julia [48, 68] we would expect that the answer is yes. However this problem seems much harder in the quasiregular case. On the other hand we can prove the following:

Theorem 3.1.4. *Let $f : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ and $g : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ be permutable, uniformly quasiregular self maps of $\overline{\mathbb{R}^d}$ of degree ≥ 2. Then $\mathcal{J}(f) = \mathcal{J}(g)$.*

Moreover, we can generalize a result of Baker [5, Lemma 4.5] which deals with a special case and can be applied to quasiregular maps of polynomial or transcendental type.

Theorem 3.1.5. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^d$ be permutable quasiregular maps. Assume that $\operatorname{cap} \mathcal{J}(f) > 0$, $\operatorname{cap} \mathcal{J}(g) > 0$ and $g = \phi \circ f$, where $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a quasiconformal map. Then $\mathcal{J}(f) = \mathcal{J}(g)$.*

Note here that in the above theorem we assume that the capacity of the Julia sets of our functions is positive. It is conjectured that this always holds when the Julia set is infinite and thus we do not actually need this assumption. However, we can prove that this condition can be dropped if g has a very specific form. Namely the following holds.

Theorem 3.1.6. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^d$ be permutable quasiregular maps of transcendental type. Assume that $g = af + c$, where a is a positive real number and c is a constant in $\mathbb{R}^d$. Then $\mathcal{J}(f) = \mathcal{J}(g)$.*

We can also consider the case where f, g are quasimeromorphic (see Chapter 2 for the definition). It is interesting to ask whether something similar with Theorem 3.1.3 holds in this case. For quasimeromorphic maps we say that they are permutable if $f \circ g = g \circ f$ holds for points in $\mathbb{R}^d$ where both sides are defined.

When studying the dynamics of meromorphic functions we usually divide them in two classes. The first one, and the most general one, is

$$\mathcal{M} := \{f : f \text{ is transcendental meromorphic and } \operatorname{card}(O_f^-(\infty)) = \infty\},$$

while the other one is

$$\mathcal{P} := \{f : f \text{ is transcendental meromorphic and } \operatorname{card}(O_f^-(\infty)) < \infty\}.$$

A typical example of a map in class $\mathcal{P}$ is $\frac{e^z}{z}$. The iteration theory and the methods of proof in those two classes are often quite different with class $\mathcal{P}$ being often closer to the class of transcendental entire functions instead. The situation is similar for quasimeromorphic maps. For functions in the analogous class $\mathcal{M}$ in higher dimensions we prove the following.

Theorem 3.1.7. *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}^d}$ and $g : \mathbb{R}^d \to \overline{\mathbb{R}^d}$ be permutable, quasimeromorphic maps of transcendental type with $\operatorname{card}(O_f^-(\infty)) = \infty = \operatorname{card}(O_g^-(\infty))$. Then $\mathcal{J}(f) = \mathcal{J}(g)$.*

This theorem appears to be new even for meromorphic functions in $\mathbb{C}$. However, as is often the case, the method used to prove this theorem cannot be used in class $\mathcal{P}$. Let us also note here that it is highly non trivial to construct higher dimensional maps in class $\mathcal{P}$ and until recently (see [139]) this had not been done.

It is also worth mentioning that Baker in [3, Theorem 1 p. 244] proved that given an entire function f, which is either transcendental or polynomial of degree at least two, then there are only countably many entire functions g that are permutable with f. We will give examples which show that this theorem cannot hold in the quasiregular case. To be more specific, by modifying an example given in [18], we are able to prove the following result.

Theorem 3.1.8. *There exists an entire transcendental map f that is permutable with uncountably many quasiregular maps $g : \mathbb{C} \to \mathbb{C}$.*

3.2 The case of uniformly quasiregular maps and generalizing the theorem of Bergweiler and Hinkkanen

In this section we will prove Theorems 3.1.3 and 3.1.4. First we prove Theorem 3.1.4. The proof is similar to that for rational functions (see [5]). We split the proof in two lemmas.

Lemma 3.2.1. *Let $f : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ and $g : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ be permutable uniformly quasiregular self maps of $\overline{\mathbb{R}^d}$ of degree at least two. Then $g(\mathcal{F}(f)) \subset \mathcal{F}(f)$.*

Proof. Let $x_0 \in \mathcal{F}(f)$ be a point in the Fatou set of f. Then there is neighbourhood U of that point in which $\{f^n : n \in \mathbb{N}\}$ is a normal family. Thus for any sequence f^{n_j} there is a locally uniformly converging subsequence and without loss of generality assume that f^{n_j} itself is this subsequence. Now because g is a continuous map on a compact space it will also be uniformly continuous. Thus the sequence $g \circ f^{n_j}$ also converges locally uniformly in U. This implies that the family $\{g \circ f^n : n \in \mathbb{N}\}$ is normal in U. Because f and g commute now, we have that the family $\{f^n : n \in \mathbb{N}\}$ is normal in $g(U)$. Thus $g(x_0) \in \mathcal{F}(f)$ which means $g(\mathcal{F}(f)) \subset \mathcal{F}(f)$. $\square$

Lemma 3.2.2. *With the same assumptions as in Lemma 3.2.1 we have that $g(\mathcal{J}(f)) \subset \mathcal{J}(f)$.*

Proof. Take a point $x_1 \in \mathcal{J}(f)$ then from Theorem 2.3.7 we have that $\mathcal{J}(f) = \overline{O_f^-(z)}$, for any point $z \in \mathcal{J}(f) \setminus E(f)$. Take such a point which also satisfies

$$z \in g^{-1}(\mathcal{J}(f) \setminus E(f)). \tag{3.2}$$

Such points exist since $g(\mathcal{F}(f)) \subset \mathcal{F}(f)$ implies that $g^{-1}(\mathcal{J}(f)) \subset \mathcal{J}(f)$. Thus in any neighbourhood V of x_1 there is a point w_0 such that $f^n(w_0) = z$ for some $n \in \mathbb{N}$. Take $g(x_1)$ and its neighbourhood $g(V)$ then thanks to the fact that f and g commute $f^n(g(w_0)) = g(z)$. Hence $g(w_0) \in O_f^-(g(z))$ and thus $g(x_1) \in \overline{O_f^-(g(z))}$. But from (3.2) we know $g(z) \in \mathcal{J}(f) \setminus E(f)$ and thus $\overline{O_f^-(g(z))} = \mathcal{J}(f)$. Hence $g(x_1) \in \mathcal{J}(f)$. □

Proof of Theorem 3.1.4. It is easy to see that Lemma 3.2.1 implies that $g^{-1}(\mathcal{J}(f)) \subset \mathcal{J}(f)$. Thus by Lemma 3.2.2 the Julia set of f is completely invariant under g. Since by Theorem 2.3.7 the Julia set of f contains infinitely many points and since by the same Theorem the Julia set of g is the smallest such set, we will have that $\mathcal{J}(g) \subset \mathcal{J}(f)$. If we now reverse the roles of f and g we can prove the other inclusion too. □

In order to prove Theorem 3.1.3 we will need several lemmas. First comes a general lemma about capacities.

Lemma 3.2.3. *Let $V \subset \mathbb{R}^d$ be a set of zero capacity. If $x_0 \in \mathbb{R}^d$ then the set $V \cup \{x_0\}$ is also of zero capacity.*

Proof. Let B be an open set containing $V \cup \{x_0\}$. Take an admissible function $u \in C_0^\infty(B)$ for the condenser (B, V) and an admissible function $v \in C_0^\infty(B)$ for $(B, \{x_0\})$. Then $u + v$ will an admissible $C_0^\infty(B)$ function for $(B, V \cup \{x_0\})$. Since

$$\operatorname{cap} V = \operatorname{cap}\{x_0\} = 0,$$

for every $\varepsilon > 0$ we can choose u, v in such a way that

$$\int_B |\nabla u|^d dm < \varepsilon \text{ and } \int_B |\nabla v|^d dm < \varepsilon.$$

Then by using Hölder's inequality we will have

$$\begin{aligned}\int_B |\nabla(u+v)|^d dm &= \int_B |\nabla u + \nabla v|^d dm \\ &\leq \left(\left(\int_B |\nabla u|^d dm \right)^{1/d} + \left(\int_B |\nabla v|^d dm \right)^{1/d} \right)^d \leq 2^d \varepsilon.\end{aligned}$$

Since ε is arbitrary this means that $\operatorname{cap}(V \cup \{x_0\}) = 0$. □

The previous lemma implies that if we add a finite number of points to a set of zero capacity then the new set will also be of zero capacity.

Lemma 3.2.4. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^d$ be permutable quasiregular maps. Then*

$$g(\mathcal{J}(f)) \subset \mathcal{J}(f) \ \text{ and } \ f(\mathcal{J}(g)) \subset \mathcal{J}(g).$$

Proof. Take a $x_0 \in \mathcal{J}(f)$ and let U be a neighbourhood of $g(x_0)$. Name V the component of $g^{-1}(U)$ which contains x_0. We know, by the definition of the Julia set, that

$$\operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} f^n(V)\right) = 0. \tag{3.3}$$

But since f, g are permutable we have that $f^n(g(x)) = g(f^n(x))$, for all $x \in V$, which implies that

$$f^n(x) \in g^{-1}(f^n(U)), \text{ for all } x \in V.$$

Hence,

$$\mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} f^n(V) \supset \mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} g^{-1}(f^n(U)).$$

Thus, by (3.3) and the fact that subsets of zero capacity sets have zero capacity, we have that

$$\operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} g^{-1}(f^n(U))\right) = 0.$$

But since $\bigcup_{n=1}^{\infty} g^{-1}(f^n(U)) = g^{-1}\left(\bigcup_{n=1}^{\infty} f^n(U)\right)$ and $g^{-1}(\mathbb{R}^d) = \mathbb{R}^d$ this implies that

$$\operatorname{cap}\left(g^{-1}\left(\mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} f^n(U)\right)\right) = 0.$$

Hence, by the K_I-inequality (Theorem 2.3.8) we will have that

$$\operatorname{cap}\left(g\left(g^{-1}\left(\mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} f^n(U)\right)\right)\right) = 0.$$

Since g is a quasiregular self-map of $\mathbb{R}^d$ we know by Rickman's generalization of Picard's theorem that it omits at most a finite number of points. Thus

$$g\left(g^{-1}\left(\mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} f^n(U)\right)\right) = \mathbb{R}^d \setminus \left(\bigcup_{n=1}^{\infty} f^n(U) \cup \{a_1, a_2, \cdots, a_m\}\right),$$

where $a_1, a_2, \cdots, a_m$ are the omitted values of g. Hence, by using Lemma 3.2.3 we will have that

$$\operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} f^n(U)\right) = 0.$$

Since U was an arbitrary neighbourhood of $g(x_0)$, this implies that $g(x_0) \in \mathcal{J}(f)$.

For the other half of the lemma, the proof is completely analogous to this one with f and g changing roles. $\square$

Lemma 3.2.5. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^d$ be permutable quasiregular maps of transcendental type. Then*

$$g^{-1}\left(\mathcal{A}(f)\right) \subset \mathcal{A}(f) \text{ and } g^{-1}\left(\overline{\mathcal{A}(f)}\right) \subset \overline{\mathcal{A}(f)}.$$

Also

$$f^{-1}\left(\mathcal{A}(g)\right) \subset \mathcal{A}(g) \text{ and } f^{-1}\left(\overline{\mathcal{A}(g)}\right) \subset \overline{\mathcal{A}(g)}.$$

Proof. Recalling the definition of the fast escaping set in 2.6, we take $R_1 > 0$ so large that

$$T(f^n(B(0, R_1))) \supset B(0, r_n)$$

for some sequence r_n with $r_n \to \infty$. Also choose an $R > 0$ large enough so that $g(B(0, R_1)) \subset B(0, R)$ while at the same time $R > R_1$, which implies that

$$T(f^n(B(0, R))) \supset B(0, r_n).$$

Pick now an $x_0 \in \mathbb{R}^d$ such that $g(x_0) \in \mathcal{A}(f)$. We will then show that $x_0 \in \mathcal{A}(f)$. We know from (2.6), in other words the definition of the fast escaping set, that there exists an $L \in \mathbb{N}$ such that

$$f^{n+L}(g(x_0)) \notin T(f^n(B(0, R))), \text{ for all } n \in \mathbb{N}.$$

Since $f^{n+L}(g(x_0)) = g(f^{n+L}(x_0))$ we will have that

$$g(f^{n+L}(x_0)) \notin T(f^n(B(0, R))), \text{ for all } n \in \mathbb{N}.$$

This together with the fact that $g(B(0, R_1)) \subset B(0, R)$ implies that

$$\begin{aligned} & g(f^{n+L}(x_0)) \notin T(f^n(g(B(0, R_1))) \\ \Rightarrow\ & g(f^{n+L}(x_0)) \notin T(g(f^n(B(0, R_1))). \end{aligned} \tag{3.4}$$

Assume now that there is an $n \in \mathbb{N}$ such that

$$f^{n+L}(x_0) \in T(f^n(B(0, R_1)))$$

then

$$g(f^{n+L}(x_0)) \in g(T(f^n(B(0, R_1)))).$$

But it is true [20, Proposition 2.4] that $g(T(f^n(B(0,R_1)))) \subset T(g(f^n(B(0,R_1))))$. Thus we would have that $g(f^{n+L}(x_0)) \in T(g(f^n(B(0,R_1))))$ which contradicts (3.4). Hence it is true that

$$f^{n+L}(x_0) \notin T(f^n(B(0,R_1))), \text{ for all } n \in \mathbb{N}$$

and thus $x_0 \in \mathcal{A}(f)$. Hence $g^{-1}(\mathcal{A}(f)) \subset \mathcal{A}(f)$.

Now for the other part of the theorem, choose any point $x \in g^{-1}(\overline{\mathcal{A}(f)})$, then $g(x) \in \overline{\mathcal{A}(f)}$. Thus there is a sequence $y_n \in \mathcal{A}(f)$ with $y_n \to g(x)$. If we now take any open neighbourhood, U of x then $g(U)$ will be an open neighbourhood of $g(x)$, since g is open, and thus it will contain all y_n, for all $n > N$ and some $N \in \mathbb{N}$. Thus U will contain points x_n with $g(x_n) = y_n$, $\forall n > N$. And because U can be made arbitrarily small we will have that $x_n \to x$. Hence

$$x \in \overline{g^{-1}(\mathcal{A}(f))}.$$

This means that

$$g^{-1}(\overline{\mathcal{A}(f)}) \subset \overline{g^{-1}(\mathcal{A}(f))} \subset \overline{\mathcal{A}(f)}.$$

Lastly, for the other half of the theorem we just change the roles of f and g. $\square$

The next lemma tells us that the Julia set of a quasiregular map is the smallest closed set of positive capacity that is completely invariant under f, which is the analogous to a well known property of the Julia set in the complex plane.

Lemma 3.2.6. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a quasiregular map. If K is a closed set with $f(K) \subset K$ and $f^{-1}(K) \subset K$ and $\operatorname{cap} K > 0$ then $\mathcal{J}(f) \subset K$.*

Proof. Take any neighbourhood, U, of a point $x \in \mathcal{J}(f)$, then by the definition of the Julia set

$$\operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{n=1}^{\infty} f^n(U)\right) = 0.$$

Hence, $K \cap \bigcup_{n=1}^{\infty} f^n(U) \neq \emptyset$. This means that there is a $x_0 \in U$ with $f^n(x_0) \in K$ for some $n \in \mathbb{N}$, and because K is completely invariant under f we will have that $x_0 \in K$. Hence, every neighbourhood, U of a point in $\mathcal{J}(f)$ contains a point of K and because K is a closed set, this implies that $\mathcal{J}(f) \subset K$. $\square$

Proof of Theorem 3.1.3. First of all, since $\mathcal{A}(f) \subset \mathcal{J}(f)$ and since $\mathcal{J}(f)$ is closed we obtain that $\overline{\mathcal{A}(f)} \subset \mathcal{J}(f)$. As we have already mentioned, in the end of Chapter 2, by [22] we always know that

$$\mathcal{J}(f) \subset \partial\mathcal{A}(f) \subset \overline{\mathcal{A}(f)}.$$

Hence we will have that

$$\mathcal{J}(f) = \overline{\mathcal{A}(f)}.$$

Hence by Lemma 3.2.4 we will have that $g(\mathcal{J}(f)) \subset \mathcal{J}(f)$ while from Lemma 3.2.5 we will have that $g^{-1}(\mathcal{J}(f)) \subset \mathcal{J}(f)$. This means that $\mathcal{J}(f)$ is completely invariant under g. Also from Theorem 2.3.10 we know that $\overline{\mathcal{A}(f)}$ contains continua, since its components are unbounded, and thus it cannot have zero capacity because zero capacity sets are totally disconnected (see [118, Corollary III.2.5]) namely

$$\operatorname{cap} \overline{\mathcal{A}(f)} = \operatorname{cap} \mathcal{J}(f) > 0.$$

Hence, we can now apply Lemma 3.2.6 and conclude that $\mathcal{J}(g) \subset \mathcal{J}(f)$.

By a completely analogous argument we can also show that $\mathcal{J}(f) \subset \mathcal{J}(g)$ and thus $\mathcal{J}(g) = \mathcal{J}(f)$. □

3.3 Improving Baker's Lemma

In this section we will prove Theorems 3.1.5 and 3.1.6. Those theorems can be seen as generalizations of a theorem of Baker in [5]. We prove here a slightly more general theorem than that in [5].

Theorem 3.3.1. *Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be permutable transcendental entire functions. Suppose that $g = af + c$, where $a \in \mathbb{C} \setminus \{0\}$. Then $\mathcal{J}(f) = \mathcal{J}(g)$.*

Proof. First we prove that $\mathcal{J}(f)$ is completely invariant under g. From the analogue of Lemma 3.2.4 for entire functions we already know that $g(\mathcal{J}(f)) \subset \mathcal{J}(f)$ so we only need to prove that $g^{-1}(\mathcal{J}(f)) \subset \mathcal{J}(f)$. This is easily seen to be equivalent to $g(\mathcal{F}(f)) \subset \mathcal{F}(f)$.

Let $x_0 \in \mathcal{F}(f)$ and U a neighbourhood of x_0 with $\overline{U} \subset \mathcal{F}(f)$. Assume, towards a contradiction, that $g(x_0) \in \mathcal{J}(f)$. Then $g(U)$ contains points in $\mathcal{J}(f)$ and thus by the blow up property we know that

$$\bigcup_{n=0} f^n(g(U)) = \mathbb{C} \setminus E(f).$$

Hence by the fact that f and g commute we obtain

$$a \bigcup_{n=0} f^{n+1}(U) + c = \mathbb{C} \setminus E(f).$$

This implies that $\bigcup_{n=1} f^n(U)$ is the whole complex plane minus at most two points. This is impossible since U is inside the Fatou set of f and thus $\bigcup_{n=1} f^n(U)$ must miss the Julia set which is an infinite set. □

Proof of Theorem 3.1.5. We recall here that f and g are permutable quasiregular maps with $g = \phi \circ f$, where ϕ is a quasiconformal map. We will prove that $\mathcal{J}(f)$ is completely invariant under g. We already know from Lemma 3.2.4 that $g(\mathcal{J}(f)) \subset \mathcal{J}(f)$. Hence, it is enough to prove that $g^{-1}(\mathcal{J}(f)) \subset \mathcal{J}(f)$.

Take a point $x_0 \in \mathbb{R}^d$ such that $g(x_0) = \phi(f(x_0)) \in \mathcal{J}(f)$. Take V to be any neighbourhood of x_0, then $U = \phi(f(V)) = \{\phi(f(x)) : x \in V\}$ is a neighbourhood of $\phi(f(x_0))$. Hence,

$$\operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{k=0}^{\infty} f^k(U)\right) = 0. \tag{3.5}$$

But, it is true that $f\left(\phi^{-1}(U)\right) = \phi^{-1}\left(f(U)\right)$, which easily implies that

$$f^n\left(\phi^{-1}(U)\right) = \phi^{-1}(f^n(U)), \text{ for all } n \in \mathbb{N}. \tag{3.6}$$

Indeed, using the fact that f commutes with $\phi \circ f$,

$$\begin{aligned} f\left(\phi^{-1}(U)\right) &= f\left(\phi^{-1}(\phi(f(V)\right) = f(f(V)) \\ &= \phi^{-1}(\phi(f(f(V))) = \phi^{-1}(f(\phi(f(V))) \\ &= \phi^{-1}(f(U)). \end{aligned}$$

By using (3.5) and (3.6) now, we conclude that

$$\operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{k=0}^{\infty} \phi\left(f^k\left(\phi^{-1}(U)\right)\right)\right) = \operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{k=0}^{\infty} f^k(U)\right) = 0.$$

But it is true that

$$\mathbb{R}^d \setminus \bigcup_{k=0}^{\infty} \phi\left(f^k\left(\phi^{-1}(U)\right)\right) = \phi\left(\mathbb{R}^d \setminus \bigcup_{k=0}^{\infty} f^k\left(\phi^{-1}(U)\right)\right).$$

Hence, by using the K_I-inequality (Theorem 2.3.8), we conclude that

$$\operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{k=0}^{\infty} f^k\left(\phi^{-1}(U)\right)\right) = 0.$$

In other words,

$$\operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{k=0}^{\infty} f^k(f(V))\right) = 0,$$

which implies that

$$\operatorname{cap}\left(\mathbb{R}^d \setminus \bigcup_{k=0}^{\infty} f^k(V)\right) = 0.$$

Thus $x_0 \in \mathcal{J}(f)$. By a similar argument we can also prove that $\mathcal{J}(g)$ is invariant under f.

Now, since we know that $\operatorname{cap} \mathcal{J}(f) > 0$ and $\operatorname{cap} \mathcal{J}(g) > 0$, we can finish the proof in the same way we did in the proof of Theorem 3.1.3. $\square$

In our proof of Theorem 3.1.6 we will need the notion of a function having the *pits effect*. This concept was first introduced by Littlewood and Offord in [82] and a variant of it was used by Bergweiler and Nicks in [23] in their attempt to develop an iteration theory for quasiregular maps of transcendental type. This variant is what we will need here as well. In what follows with $|\cdot|$ we denote the usual Euclidean norm.

Definition 3.3.1. A quasiregular map $f : \mathbb{R}^d \to \mathbb{R}^d$ of transcendental type is said to have the *pits effect* if there exists $N \in \mathbb{N}$ such that, for all $\alpha > 1$, for all $\lambda > 1$ and all $\varepsilon > 0$ there exists R_0 such that if $R > R_0$, then

$$\{x \in \mathbb{R}^d : R \leq |x| \leq \lambda R,\ |f(x)| \leq R^\alpha\}$$

can be covered by N balls of radius εR.

We must also mention here that in [23] the authors first define the pits effect using the condition $|f(x)| \leq 1$ instead of $|f(x)| \leq R^\alpha$ and later prove that those two are actually the same [23, Theorem 8.1].

Lemma 3.3.2. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^d$ be permutable quasiregular maps. Assume that $g = f + c$, where $c \neq 0$ is a constant in $\mathbb{R}^d$. Then f does not have the pits effect.*

Proof. For any $N \in \mathbb{N}$, we will find a sequence $R_m \to \infty$ and $\lambda > 1$, $\varepsilon > 0$, $\alpha > 1$ such that

$$A = \{x \in \mathbb{R}^d : R_m \leq |x| \leq \lambda R_m,\ |f(x)| \leq R_m^\alpha\}$$

cannot be covered by N balls of radius εR_m.

First pick a $N \in \mathbb{N}$. Choose also a point $x_0 \in \mathbb{R}^d$ that lies in the half-line connecting 0 with c. Hence the sequence $|x_0 + nc|, n \in \mathbb{N}$ is an increasing sequence. Also since the number of omitted values of f is finite we can assume that this half line does not contain any omitted values from the point $x_0 - c$ onwards. We now set $R_m = |x_0 + mc|$ and we will show that a segment of the half line is contained in

A and that it is not possible to cover it with N balls. Choose $\varepsilon = 1/10$, then with N balls of radius $R_m/10$ we can cover distance at most $\frac{NR_m}{5}$. Hence, if we take

$$(\lambda - 1)R_m > \frac{NR_m}{5} \Leftrightarrow \lambda > \frac{N}{5} + 1,$$

then we cannot cover the part of the half line that lies between the circles with radius R_m and λR_m with those N balls. Now we only need to show that this part of the half line also satisfies the other condition; namely that $|f(x)| \leq R_m^\alpha$ for some $\alpha > 1$. Observe that all the points on the half line, after x_0, can be written as $y + nc$ for some y on the line segment from $x_0 - c$ to x_0 and some $n \in \mathbb{N}$. Then since those points are not omitted by f we have that

$$f(y + nc) = f(y + (n-1)c + c) = f(f(w_n) + c),$$

for some $w_n \in \mathbb{R}^d$ with $f(w_n) = y + (n-1)c$. Thus thanks to the fact that f is commuting with $f + c$ we obtain that

$$f(y + nc) = f(f(w_n)) + c = f(y + (n-1)c) + c.$$

By repeating this argument n times we obtain that

$$f(y + nc) = f(y) + nc, \text{ for all } n \in \mathbb{N}.$$

Hence for any point, $y + nc$, on the half line that lies between the circles with radius R_m and λR_m we have that

$$\begin{aligned} |f(y + nc)| = |f(y) + nc| &\leq |f(y) - y| + |y + nc| \\ &\leq |f(y) - y| + \lambda R_m. \end{aligned}$$

If we now take $\alpha = 2$ we will have $|f(y) - y| + \lambda R_m \leq R_m^\alpha$, for all big enough R_m and thus the second condition will hold for all points on this line segment. $\square$

Lemma 3.3.3. *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a quasiregular map of transcendental type and $L(x) = aUx$ a linear map where $a \in \mathbb{R} \setminus \{0\}$ and $U \in SO(d)$ ($SO(d)$ is the special orthogonal group). If $g = L \circ f + c$, where $c \in \mathbb{R}^d$, commutes with f then $|a| = 1$.*

Proof. Without loss of generality let us assume, towards a contradiction, that $|a| > 1$. Pick a large positive number $r' > 0$. Then there is a $y_{r'} \in \mathbb{R}^d$ with $|y_{r'}| = r'$ such that $M(r', f) = |f(y_{r'})|$, where $M(r', f) = \max_{|z|=r'}\{|f(z)|\}$. Now by Rickman's generalization of Picard's theorem, the fact that f is a transcendental quasiregular map and the fact that L is injective there is a point $x_{r'} \in \mathbb{R}^d$ such that $y_{r'} = (L \circ f)(x_{r'}) + c$ and thus

$$M(r', f) = |f(y_{r'})| = |f((L \circ f)(x_{r'}) + c)|.$$

We set $r = |f(x_{r'})|$. Note that

$$r = \left|L^{-1}(y_{r'} - c)\right| \geq |y_{r'}| \left|L^{-1}\left(\frac{y_{r'}}{|y_{r'}|}\right)\right| - |L^{-1}(c)| \geq \frac{r'}{|a|} - |L^{-1}(c)|.$$

Thus $r \to \infty$ as $r' \to \infty$. Also note that

$$r' = |y_{r'}| = |L(f(x_{r'})) + c| \geq |a|r - |c|.$$

Hence, if we take a $1 < \lambda < |a|$ then for all large enough r' we have that $r' \geq \lambda r$. From the fact that quasiregular maps are open, we can conclude now that they obey the maximum modulus principle and thus $M(r, f)$ is an increasing function of r. Hence

$$\begin{aligned}\frac{M(\lambda r, f)}{M(r, f)} \leq \frac{M(r', f)}{M(r, f)} &= \frac{|f(L(f(x_{r'})) + c)|}{M(r, f)} \\ &= \frac{|L(f(f(x_{r'}))) + c|}{M(r, f)} \\ &\leq \frac{|a|\,|f(f(x_{r'}))| + |c|}{M(r, f)} \\ &\leq \frac{M(r, f)|a| + |c|}{M(r, f)}.\end{aligned}$$

Hence $\frac{M(\lambda r, f)}{M(r, f)}$ stays bounded as $r \to \infty$, which is a contradiction since this ratio tends to infinity as $r \to \infty$ (see [14, Lemma 3.3]). Hence $|a| = 1$. □

Proof of Theorem 3.1.6. First, from Lemma 3.3.3 we will have that $a = 1$. From Lemma 3.3.2 we will have that f does not have the pits effect. Hence, from [23, Corollary 1.1] we have that $\operatorname{cap} \mathcal{J}(f) > 0$. Also note here that if f does not have the pits effect then, by the definition, $f + c$ also does not. This again implies that $\operatorname{cap} \mathcal{J}(g) > 0$. Now we can apply Theorem 3.1.5 and conclude that $\mathcal{J}(f) = \mathcal{J}(g)$. □

3.4 The quasimeromorphic case

In this section we prove Theorem 3.1.7.

Proof of Theorem 3.1.7. We will prove first that $O_f^-(\infty) \subset O_g^-(\infty)$. To that end, let $x_0 \in \mathbb{R}^d$ be a point in $O_f^-(\infty)$. This means that $f^n(x_0) = \infty$ for some $n \in \mathbb{N}$. This in turn implies that $f^{n-1}(x_0)$ is a pole of f. Now, note that f and g must have the same poles since if z_0 is a pole of f but not g then $g \circ f$ has an essential singularity in z_0 while $f \circ g$ does not, thus $f \circ g \neq g \circ f$ on a punctured neighbourhood of z_0. Hence, we will also have that $g(f^{n-1}(x_0)) = \infty$. By using the fact that f commutes with g

we have that $f(g(f^{n-2}(x_0))) = \infty$ and thus $g(f^{n-2}(x_0))$ is a pole of f which again implies it is also a pole of g. Using this argument n times yields that $g^n(x_0) = \infty$ and thus $x_0 \in O_g^-(\infty)$.

The other inclusion follows similarly by switching the roles of f and g. Hence we have that $O_f^-(\infty) = O_g^-(\infty)$ and thus, since $\overline{O_f^-(\infty)} = \mathcal{J}(f)$, we have $\mathcal{J}(f) = \mathcal{J}(g)$. $\square$

3.5 Examples

In this section we will first prove Theorem 3.1.8 and then discuss about examples of commuting transcendental and quasiregular maps.

Proof of Theorem 3.1.8. We want to construct uncountably many quasiregular maps that commute with a specific entire function. In order to do that we will follow the example given in [18, section 2] where the authors construct uncountably many continuous functions g that commute with the function $f(z) = c(e^{z^2} - 1)$, where c is a large positive number. Note that f has a superattracting fixed point at 0 and there is a conformal function ϕ, from the immediate basin of attraction A of f to the unit disk, that conjugates f with z^2. That map ϕ is in fact quasiconformal on an open set that contains A and that is due to the fact that f is a polynomial-like mapping (see [18] and references therein for more details). In order to construct the required map they first define a function G which commutes with z^2.

We define $G : \mathbb{C} \to \mathbb{C}$ by $G(z) = |z|^{m-1}z$ for some positive real number m. As we can easily see G commutes with $z \mapsto z^2$. In fact G is well known to be M-quasiconformal with $M = \max\{m, 1/m\}$. Also, note that $G(z) = z$, for $|z| = 1$. Because we have uncountably many choices for m we also have uncountably many such maps G.

Next, by defining $g(z) = \phi^{-1}(G(\phi(z)))$ for $z \in A$ we see that

$$g(f(z)) = \phi^{-1}(G(\phi(f(z)))) = \phi^{-1}(G(\phi(z)^2)) = f(\phi^{-1}(G(\phi(z)))) = f(g(z)), z \in A.$$

In order to extend g to the whole plane we argue as follows. If B is a component of the basin of attraction of the fixed point at 0 then there is a n with $f^n(B) = A$. Choose n to be the minimal with that property and define $g(z) = f^{-n}(g(f^n(z)))$, for all $z \in B$, with the appropriate branch of f^{-n}. Since G coincides with the identity map on the unit circle and ϕ extends continuously and bijectively on ∂A then we can extend g to $\partial A \cup \partial B$ by setting $g(z) = z$ there. We can now extend g to the rest of the plane, the complement of the basin of attraction, by setting $g(z) = z$ there. The functions $g : \mathbb{C} \to \mathbb{C}$ we obtain through this method will be quasiregular and because our choices for G are uncountably many, so are our functions g. $\square$

In the above construction it is not hard to see that the Julia set of the quasiconformal function g is empty. That is because if B_0 is the basin of attraction of 0 then $g(B_0) \subset B_0$ while outside of B_0, g is the identity. Thus there are no points having the blow-up property of the Julia set. It would be desirable to also have an example of uncountably many maps commuting with a given map f while also having the same Julia set.

3.5.1 Modifying the above construction

This next example was inspired by a question asked to me by Davoud Cheraghi in a seminar at the Open University.

In the above construction let us consider the map $h = f \circ g = g \circ f$ which is easy to see that also commutes with f. Note here that by following the natural extension process used in [18] in order to extend g gives us that inside the basin B_0, $g(z) = (f^{-n} \circ g \circ f^n)(z)$, for a suitable branch of f^{-n}. Hence, this new map h equals f outside of the basin of attraction B_0 of g and on the inside it is

$$f(f^{-n} \circ g \circ f^n(z)) = f^{-n+1} \circ g \circ f^n,$$

for suitable n which might also be 0.
This new map h has a non-empty Julia set, since it is a quasiregular map of transcendental type (see theorem 2.3.9). Also, by [23, Theorem 1.11] we know that $\operatorname{cap} \mathcal{J}(h) > 0$ and it is true that $\operatorname{cap} \mathcal{J}(f) > 0$. Hence we can apply Theorem 3.1.5 and conclude that h has the same Julia set with f. We also have uncountably many such maps since we have uncountably many choices for g.

3.5.2 Other examples

We give here some other examples of permutable quasiregular maps by generalizing holomorphic examples in the complex plane. First we give examples of holomorphic functions. We note here that the first three classes of examples are all finite quotients of affine maps and are essentially the only ones possible in the case of rational functions that do not share a common iterate as Theorem 3.1.1 informs us. The problem of classification is still open in the case where the functions share an iterate and a sensible solution does not seem to exist. See [44, 123] for more details.

Holomorphic examples.

1. Consider the family of power maps $f_n(z) = z^n, n \geq 2$. Obviously
$$f_n \circ f_m = f_{nm} = f_m \circ f_n.$$
We can also easily see that $\mathcal{J}(f_n) = S^1$, for all $n \geq 2$, where S^1 denotes the unit circle. Thus $\mathcal{J}(f_n) = \mathcal{J}(f_m)$.

2. Consider the family of Tchebycheff polynomials, that we introduced in Chapter 1 and satisfy $T_n(\cos z) = \cos(nz), n \geq 2$. It is easy to see that $T_n \circ T_m = T_m \circ T_n$. Also each of the Tchebycheff polynomials has as its Julia set the interval $[-2, 2]$ (see [30, p.30]), so clearly $\mathcal{J}(T_n) = \mathcal{J}(T_m)$.

3. The family of Lattès maps provides another example of commuting functions. As we have said, a rational map, of degree at least two, of the form
$$f = \Theta \circ L \circ \Theta^{-1}$$
is called Lattès. Here L is an affine self map of the torus $\mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice of rank two, and Θ is a holomorphic map from the torus to $\overline{\mathbb{C}}$. One possible option is to choose $L(z) = az$ for any $a \in \Lambda = \{x + yi : x, y \in \mathbb{Z}\}$, $|a| \geq 2$ and $\Theta = \wp^2(z)$, where $\wp(z)$ is the Weierstrass elliptic function with periods 1 and i. Then the Lattès maps that we take for the different values of a are commuting. Also it is well known that the Julia set of Lattès maps is the entire Riemann sphere.

4. Let f be an entire or a rational function. Consider the family $f_n = f^n$, for all $n \geq 1$. Then obviously $f_n \circ f_m = f_m \circ f_n$ and also it is well known that $\mathcal{J}(f_n) = \mathcal{J}(f)$, for all $n \geq 1$.

5. Consider an entire periodic function $P : \mathbb{C} \to \mathbb{C}$, with period $c \in \mathbb{C}$. Take $f(z) = P(z) + z$ and $g(z) = P(z) + z + kc$, where $k \in \mathbb{Z}$. Then f, g are permutable. Using now a result of Baker, Theorem 3.3.1, which we generalized in Theorems 3.1.5 and 3.1.6, we conclude that $\mathcal{J}(f) = \mathcal{J}(g)$.

6. Let
$$f(z) = 2ia \cos\left(\frac{(4k+3)\pi}{8a^2} iz^2\right) \text{ and } g(z) = 2ia \sin\left(\frac{(4k+3)\pi}{8a^2} iz^2\right),$$
where $a \in \mathbb{C} \setminus \{0\}$. A simple calculation shows that f and g commute. In [138] the authors generalize Baker's Theorem 3.3.1 and show that the functions f and g satisfy the conditions of their Theorem. Thus $\mathcal{J}(f) = \mathcal{J}(g)$.

7. The next example, which I learned from Gustavo Ferreira, is about permutable meromorphic maps. Suppose that $f_0 : \mathbb{C} \to \mathbb{C}$ is an entire function with $f(0) \neq 0$. Consider now the maps

$$f(z) = \frac{f_0(z^n)}{z^{n-1}} \text{ and } g(z) = \frac{\omega}{z^{n-1}} f_0(z^n),$$

where ω is an n-th root of unity.

It is easy to see now that f and g are meromorphic with a pole at 0 and commute. Moreover if 0 is not an omitted value of f_0 then it is easy to see that $\operatorname{card} O_f^-(\infty) = \operatorname{card} O_g^-(\infty) = \infty$ so that Theorem 3.1.7 applies and $\mathcal{J}(f) = \mathcal{J}(g)$.

8. Once we have a pair of commuting maps f and g it is quite easy to construct others. For example f and $f^n \circ g^m$, where $n, m \in \mathbb{N}$ will also commute.

Quasiregular examples.

1. As we have already mentioned in Chapter 2, Mayer in [89, 90] constructs uniformly quasiregular analogues of the power maps, of Tchebycheff polynomials and of Lattès type maps which can be easily seen, just like in the complex case, that are permutable. Also those families of maps have the same Julia sets: the unit sphere, the unit disc and $\overline{\mathbb{R}^d}$ respectively.

2. There is a quasiregular analogue of the exponential map in the complex plane called the Zorich map which was first defined by Zorich in [141] and which we introduced in Chapter 1. For simplicity assume we work on $\mathbb{R}^3$ and denote this map by $Z : \mathbb{R}^3 \to \mathbb{R}^3$. This map is periodic, with period 4, in its first two variables. In [98] Nicks and Sixsmith defined a quasiregular map $g : \mathbb{R}^3 \to \mathbb{R}^3$ of transcendental type such that

$$g = \begin{cases} Z + Id & x_3 > L \\ Id & x_3 < 0 \end{cases},$$

where Id the identity map and $L > 0$ is a constant. By its construction this map satisfies $g(x_1 + 4, x_2, x_3) = g(x_1, x_2, x_3) + (4, 0, 0)$ for $0 \leq x_3 \leq L$ and hence for all x_3 (see [98, section 6] for details). Now define the function $f(x_1, x_2, x_3) = g(x_1, x_2, x_3) + (4, 0, 0)$. It is quite easy to see that f commutes with g. Hence, by applying Theorem 3.1.6 we conclude that $\mathcal{J}(f) = \mathcal{J}(g)$.

3. Another example is provided by [98, section 7] where the authors define the map

$$h(x_1, x_2, x_3) = g(x_1, x_2, x_3) - (0, 0, L'),$$

where g is the function of the previous example and $L' > 0$ is a large constant. They also prove that $\mathcal{A}(h) \subset \mathcal{J}(h)$ and is quite easy to see that h commutes with $h + (4, 0, 0)$. Hence, in this example we can apply Theorem 3.1.3 and conclude that the two functions have the same Julia set.

4. Consider any two permutable holomorphic (or meromorphic) maps f and g on the complex plane. Then for any quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ consider the maps $f' = \phi^{-1} \circ f \circ \phi$ and $g' = \phi^{-1} \circ g \circ \phi$. The maps f' and g' are quasiregular (or quasimeromorphic), commute and, assuming f and g have the same Julia set, so do f' and g'.

5. Consider the maps $f(z) = |z|^{t_1} z^{k_1}$ and $g(z) = |z|^{t_2} z^{k_2}$, where $z \in \mathbb{C}$, $t_1, t_2 > 0$ and $k_1, k_2 \in \mathbb{N}$. It is easy to see that those two maps are quasiregular and that they commute for any values of the parameters t_1, t_2, k_1, k_2. Also since
$$K_I(f) = \frac{t_1 + k_1}{k_1} \text{ and } K_I(g) = \frac{t_2 + k_2}{k_2},$$
we have that when
$$k_i \geq \frac{t_i + k_i}{k_i} \Leftrightarrow k_i > \frac{1 + \sqrt{1 + 4t_i}}{2} \text{ or } k_i < \frac{1 - \sqrt{1 + 4t_i}}{2}, \; i = 1, 2$$
the Julia sets of f and g are non empty and in fact are the unit circle.

3.6 Open questions

In this section we will discuss some of the open problems in the study of permutable functions. The first and perhaps the most important is

Conjecture 3.6.1 (Fatou)**.** *Permutable entire transcendental maps have the same Julia set.*

Of course the same conjecture can be made about permutable quasiregular maps of transcendental type. However in the quasiregular case it is not even known if polynomial type permutable maps have the same Julia set.

Question 3.6.2. *Do permutable quasiregular maps of polynomial type have the same Julia set?*

Note that in the quasimeromorphic case, for functions in the corresponding class $\mathcal{M}$, we have already given a positive answer to the above question in Theorem 3.1.7.

Recently Ferreira in [51] also studied commuting meromorphic functions in class $\mathcal{P}$ and proved that there is a corresponding result to that of Bergweiler and Hinkkanen in that class.

It seems quite plausible that such a theorem can also be shown for quasimeromorphic maps in the corresponding class $\mathcal{P}$. For class $\mathcal{P}$ there is a sensible definition of the fast escaping set due to Rippon and Stallard [119]. Generalizing Ferreira's results would require the study of the fast escaping set for the quasiregular analogue of class $\mathcal{P}$ which has not been done before.

Question 3.6.3. *Is there an analogue of Theorem 3.1.3 for permutable maps in the quasiregular version of class $\mathcal{P}$?*

Question 3.6.4. *Do permutable maps in class $\mathcal{P}$ have the same Julia sets?*

It is also quite interesting to study permutable holomorphic maps with two essential singularities which are also omitted values for the map. Without loss of generality those two points can be taken to be 0 and ∞. We call those maps *transcendental self maps* of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Starting with Radström [107] the dynamics of those maps have been studied by many people, see for example [12, 24, 73, 75]. All such maps are of the form

$$z^k e^{g(z)+h(\frac{1}{z})}, \text{ where } g, h \text{ are entire functions and } k \in \mathbb{Z}.$$

Of course the first thing one should ask is whether or not there can be any permutable self maps of $\mathbb{C}^*$. Indeed for any entire maps f_0 and g_0 we can take

$$f(z) = ze^{g_0(z^n)+f_0(\frac{1}{z^n})} \text{ and } g(z) = \omega ze^{g_0(z^n)+f_0(\frac{1}{z^n})},$$

where ω is an n-th root of unity. It is easy to see that those maps commute so we can ask whether or not they have the same Julia set. Moreover Marti-Pete [85] has defined a fast escaping set of transcendental self maps of $\mathbb{C}^*$ and thus we can ask

Question 3.6.5. *Is there an analogue of Theorem 3.1.2 for transcendental self maps of $\mathbb{C}^*$? Do any two commuting such maps have the same Julia set?*

In the quasiregular setting now Nicks and Sixsmith [100] studied the dynamics of the quasiregular analogues of transcendental self maps of $\mathbb{C}^*$ and they also defined a fast escaping set for such maps. Thus the above question also makes sense in the higher dimensional setting.

Another interesting question is whether or not we can classify permutable uniformly quasiregular self maps of $\overline{\mathbb{R}^d}$ that do not share an iterate. In other words whether or not we can generalize Theorem 3.1.1 in the setting of uqr maps.

As we have already mentioned Eremenko gave a proof of Theorem 3.1.1 by further developing the method Fatou and Julia had used in their study of commuting rational maps in [48, 68]. The first result that this approach requires is that permutable maps have the same Julia set. As we have already shown, permutable uqr self maps of $\overline{\mathbb{R}^d}$ indeed have the same Julia sets. The next important piece of information that the proof of Theorem 3.1.1 by Eremenko requires is the fact that commuting rational functions have a common repelling periodic point. We remind the reader here that a point $z \in \mathbb{C}$ is called a periodic point of a rational map $f : \mathbb{C} \to \mathbb{C}$ if $f^n(z) = z$, for some $n \in \mathbb{N}$ and it is called repelling if $\left|(f^n)'(z)\right| > 1$. That definition of a repelling periodic point does not carry over to the higher dimensional case since uqr maps might not even be differentiable at a point. However there is a way to define the notion of a repelling periodic point, see [63].

Thus we can ask

Question 3.6.6. *Do permutable uniformly quasiregular self maps of $\overline{\mathbb{R}^d}$ have a common repelling periodic point?*

Chapter 4

Zorich maps

4.1 Introduction

In the study of the dynamics of complex analytic functions one of the most well studied and important families of functions is the exponential family $E_\kappa : z \mapsto \kappa e^z$, $\kappa \in \mathbb{C} \setminus \{0\}$. Perhaps the most fundamental fact about this family concerns its Julia set. For $0 < \kappa \leq 1/e$, as was proven first by Devaney and Krych in [40], the Julia set $\mathcal{J}(E_\kappa)$ is a so called "*Cantor bouquet*".

We say that a subset H of $\mathbb{C}$ (or $\mathbb{R}^d$) is a *hair* if there exists a homeomorphism $\gamma : [0, \infty) \to H$ such that $\gamma(t) \to \infty$ as $t \to \infty$. We call $\gamma(0)$ the endpoint of the hair H.

Theorem 4.1.1 (Devaney and Krych, [40])**.** *For $0 < \kappa \leq 1/e$ the Julia set $\mathcal{J}(E_\kappa)$ of the exponential map E_κ consists of uncountably many disjoint hairs each of which has a finite endpoint and tends to infinity by having their real parts go to infinity while their imaginary part remains bounded.*

On the other hand, when $\kappa > 1/e$ Misiurewicz in [94] proved that the Julia set $\mathcal{J}(E_\kappa)$ equals the entire complex plane $\mathbb{C}$ (actually Misiurewicz only proved this for $\kappa = 1$ but his proof can easily be adapted to cover the other cases as well, see [37]).

Theorem 4.1.2 (Misiurewicz, [94])**.** *For $\kappa > 1/e$ the Julia set $\mathcal{J}(E_\kappa)$ of the exponential map E_κ equals the entire complex plane $\mathbb{C}$.*

For a different proof of the same theorem see [127]. For all these facts and much more we refer to Devaney's survey [38] on exponential dynamics.

As we have already mentioned in the first chapter, in the higher dimensional setting of quasiregular maps there is a whole family of maps that can be considered analogues of the exponential map called the *Zorich maps* which were first constructed by Zorich in [141]. Before we proceed with the construction of Zorich maps

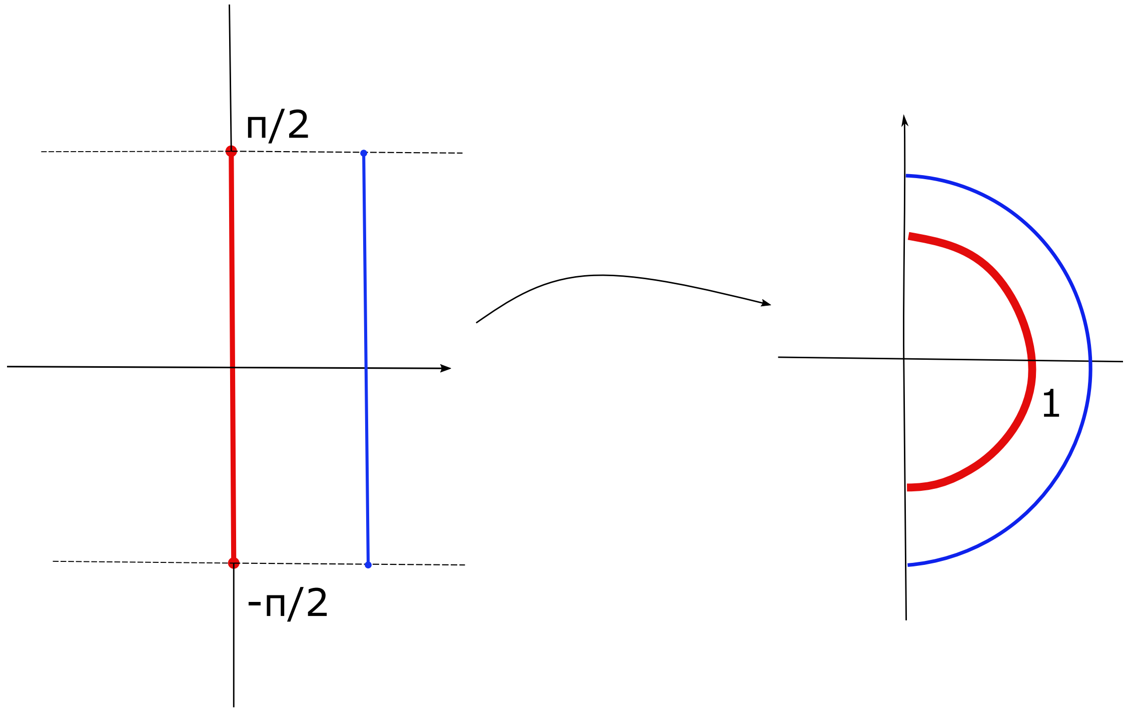

Figure 4.1: The construction of the exponential map.

let us recall the construction of the exponential map. First we start with the map $y \mapsto \phi(y) = (\cos y, \sin y)$ from $[\frac{-\pi}{2}, \frac{\pi}{2}]$ to the right half circle. Notice that this map is bi-Lipschitz. We now define the map $E(x, y) = e^x \phi(y)$ on the strip $[\frac{-\pi}{2}, \frac{\pi}{2}] \times \mathbb{R}$ (see figure 4.1) which is none other than the usual exponential map. To extend this map to the rest of $\mathbb{C}$ we repeatedly reflect across the boundary of the strip in the domain and across the imaginary axis in the codomain.

Following [65] we describe now the construction of the Zorich maps in three dimensions. Note that the construction can be done in arbitrary dimensions but we will confine ourselves in three dimensions for simplicity. First consider an L bi-Lipschitz, sense-preserving map $\mathfrak{h}$ that maps the square

$$Q := \Big\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\Big\}$$

to the upper hemisphere

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \geq 0\}.$$

This is in analogy with what we had in two dimensions, in the case of the exponential map, where line segments were getting mapped to right half circles. We then define $Z : Q \times \mathbb{R} \to \mathbb{R}^3$ as

$$Z(x_1, x_2, x_3) = e^{x_3}\mathfrak{h}(x_1, x_2).$$

The map Z maps the square beam $Q \times \mathbb{R}$ to the upper half-space. By repeatedly reflecting now, across the sides of the square beam in the domain and the x_1x_2 plane

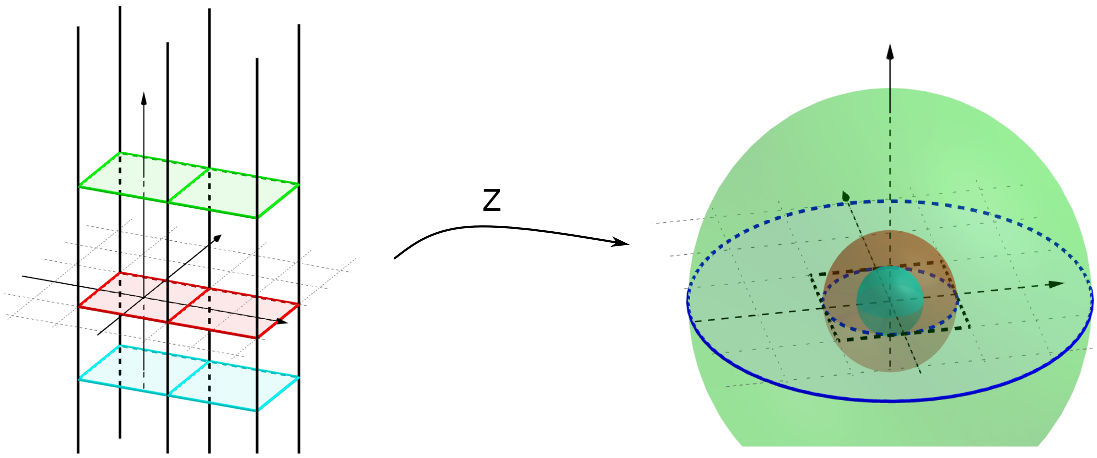

Figure 4.2: The construction of Zorich maps. The coloured squares are mapped to the corresponding coloured upper and lower hemispheres.

in the range, we obtain a map $Z : \mathbb{R}^3 \to \mathbb{R}^3$ (see also the Figure 4.1). We now make a few remarks about the map we just defined.

- First, note that this map is doubly periodic meaning that $Z(x_1 + 4, x_2, x_3) = Z(x_1, x_2 + 4, x_3) = Z(x_1, x_2, x_3)$.
- Moreover, this map is not locally injective everywhere. The lines $x_1 = 2n + 1, x_2 = 2m + 1$, $n, m \in \mathbb{Z}$ belong to the *branch set*, namely the set

$$\mathcal{B}_Z := \{x \in \mathbb{R}^3 : Z \text{ is not locally homeomorphic at } x\}.$$

- It can be shown that this map is quasiregular (see example in subsection 2.3.1) with the dilatation depending on the bi-Lipschitz constant. Also it has an essential singularity at infinity and omits 0, just like the exponential map on the plane does.

We can also introduce a parameter $\nu > 0$ and consider the family $Z_\nu = \nu Z$, where Z is a Zorich map. This family can be considered as an analogue of the exponential family E_κ in higher dimensions (at least in the case where $\kappa > 0$). Hence, it would be very interesting to know whether or not this family has a similar behaviour with the exponential in terms of dynamics.

Indeed, Bergweiler in [16] and Bergweiler and Nicks in [23, Section 7] have proven that for small values of ν this family has as its Julia set uncountably many, pairwise disjoint curves. For those curves, Bergweiler in [16] proved a counterpart to Karpińska's paradox (see [70, 71]) for the exponential map, namely the fact that the endpoints of those curves have Hausdorff dimension 3 while the curves minus the endpoints have Hausdorff dimension 1. Moreover, Comdühr in [33] proved that those curves are smooth generalizing a result of Viana in [133]. See also [19] for other results concerning the Hausdorff dimension of different subsets of the endpoints.

Having said all that it seems quite reasonable to expect that for large values of ν the Julia set of the Zorich family would be the whole $\mathbb{R}^3$ just like in the exponential family where the Julia set is the whole complex plane. One of the goals of this chapter is to study the dynamics of a slightly modified version of this family for big values of the parameter ν and show that they are chaotic.

Let us now define the modified bi-Lipschitz map we will use and state our main theorem. The first thing that we require is that our map

$$\mathfrak{h}(x_1, x_2) = (\mathfrak{h}_1(x_1, x_2), \mathfrak{h}_2(x_1, x_2), \mathfrak{h}_3(x_1, x_2))$$

must satisfy $\mathfrak{h}_1(x_1, x_1) = \mathfrak{h}_2(x_1, x_1)$ and $\mathfrak{h}_1(x_1, -x_1) = -\mathfrak{h}_2(x_1, -x_1)$. This way the planes $x_1 = x_2$ and $x_1 = -x_2$ are invariant under the Zorich map we obtain. Note that this implies that $\mathfrak{h}(0, 0) = (0, 0, 1)$. Second, we need to scale things by a factor $\lambda > 1$. To be more precise we define the function

$$h(x_1, x_2) = \lambda \mathfrak{h}\left(\frac{1}{\lambda}(x_1, x_2)\right), \ (x_1, x_2) \in \lambda Q.$$

We define now the Zorich maps we obtain by this h, which we denote by $\mathcal{Z}$

$$\mathcal{Z}_\nu(x_1, x_2, x_3) = \nu e^{x_3} h(x_1, x_2), \quad (x_1, x_2, x_3) \in \lambda Q \times \mathbb{R}, \ \ \nu > 0. \tag{4.1}$$

Again we extend this map to $\mathbb{R}^3$ by reflecting across the sides of the square beam and the plane $x_3 = 0$. Another important thing to note here is that during the extension process of our map $\mathcal{Z}$ from the initial square beam to the whole $\mathbb{R}^3$ we can also extend $\mathfrak{h}$ to a Lipschitz map from $\mathbb{R}^2$ to $\mathbb{R}^3$ with the same Lipschitz constant L. We will always assume that this extension has been done and when we talk about $\mathfrak{h}$ we will mean the extended one unless otherwise stated. Moreover, let us note here that this new Zorich map $\mathcal{Z}$ is conjugate to $x \mapsto Z(x_1, x_2, \lambda x_3)$, where Z is the classic Zorich map without the scaling.

Remark. Here it is worth elaborating on that last sentence. Instead of studying the family $\mathcal{Z}_\nu$, defined in (4.1), we could have studied the family $\alpha \circ Z$, where $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear map induced by the matrix

$$\begin{pmatrix} \nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu\lambda \end{pmatrix}$$

and Z is the Zorich map that leaves the planes $x_1 = \pm x_2$ invariant and comes from using $\mathfrak{h}$. It is easy to see that the map $\alpha \circ Z$ is conjugate with $\nu Z(x_1, x_2, \lambda x_3)$ and thus with $\mathcal{Z}_\nu$. The advantage of this viewpoint is that the Zorich maps we consider here and the maps considered by Bergweiler in [16] can all be seen as maps in the space $\{\mathcal{A} \circ Z : \mathcal{A} \in GL_3(\mathbb{R})\}$, where $GL_3(\mathbb{R})$ is the general linear group of degree 3. Thus $GL_3(\mathbb{R}) \setminus \{0\}$ becomes the parameter space for Zorich maps in analogy with $\mathbb{C} \setminus \{0\}$ being the parameter space for the exponential map.

Due to the conjugacy all of the theorems we are going to prove here are also true for the family $\alpha \circ Z$. We have chosen to use a different presentation of Zorich maps than the one described here since that way the definition seems more natural and the connection with the exponential family is more apparent.

For the type of Zorich maps defined in (4.1) we will prove

Theorem 4.1.3. *Let $\lambda > L^5$. Then for all $\nu > \sqrt{\frac{2L}{\lambda}}$ the Zorich map $\mathcal{Z}_\nu$ we obtain using this scale factor λ has as its Julia set the whole $\mathbb{R}^3$.*

Remark. We will actually prove a slightly stronger result. Namely, that if the assumptions of the above theorem are satisfied and V is any open set of $\mathbb{R}^3$ then $\bigcup_{n\geq 0} \mathcal{Z}_\nu^n(V)$ covers $\mathbb{R}^3 \setminus \{0\}$.

The above Theorem implies that the behaviour of the iterates of the Zorich maps, for those particular choices of the parameters, are chaotic in the whole $\mathbb{R}^3$. Along the way of proving Theorem 4.1.3 we will also prove a Theorem on the measurable dynamics of Zorich maps which can be seen as the analogous result of a theorem for exponential maps due to Ghys, Sullivan and Goldberg, see Theorem 4.5.1 below.

Another fact usually associated with chaotic behaviour in a set is the density of periodic points on that set. In the complex plane it is well known and was first proven by Baker in [4], that periodic points of an entire transcendental map (in fact even repelling periodic points) are dense in its Julia set. However, it still unknown whether or not the periodic points of a quasiregular map on $\mathbb{R}^3$ are dense in its Julia set. We are able to prove that this is indeed the case for Zorich maps.

Theorem 4.1.4. *Let ν and λ be as in Theorem 4.1.3. The periodic points of $\mathcal{Z}_\nu$ are dense in $\mathbb{R}^3$.*

Another object of study in transcendental complex dynamics, which we have already mentioned, and is intimately connected with the Julia set is the escaping set. As we have mentioned in the introduction it is known that $I(f) \neq \emptyset$ and that $\partial I(f) = \mathcal{J}(f)$.

Moreover, for the exponential family from [42] it is true that $I(f) \subset \mathcal{J}(f)$ and thus $I(f)$ is dense in the Julia set. When the Julia set is a Cantor bouquet, the escaping set consists of the disjoint curves that make up the Julia set together with some of their endpoints. In this case $I(f)$ is disconnected while $I(f) \cup \{\infty\}$ is connected (see [38]). On the other hand, when the Julia set of a map in the exponential family is the entire complex plane, the escaping set is dense in the complex plane and Rempe in [112] has proven that it is also connected (see also [113] for a more general version of the same result).

The situation is similar for the Zorich maps as well. As we already mentioned, in [16, 23] it is proven that for some values of the parameter ν the Julia set consists of disjoint curves together with their endpoints and $I(\mathcal{Z}_\nu)$ is again a disconnected subset of the Julia set. On the other hand we are able to show that

Theorem 4.1.5. *For the same choice of ν and λ as in Theorem 4.1.3, we have that the escaping set $I(\mathcal{Z}_\nu)$ is a connected subset of $\mathbb{R}^3$.*

It is also worth mentioning here that there are other methods of constructing Zorich-like maps where instead of mapping squares to hemispheres through bi-Lipschitz functions we map squares to surfaces whose boundary lies on the plane $x_3 = 0$ and the half ray connecting the origin with a point on the surface intersects the surface only once. If we further impose some bound on the angle between that ray and the tangent plane to the surface (see section 4.6 for more details) we can use our methods and prove a Theorem similar to Theorem 4.1.3.

To state the theorem let us denote those Zorich maps with $\mathcal{Z}_{gen}$. In the construction of those maps we will use a bi-Lipschitz map h_{gen} which will be the rescaled version, by a factor of λ, of another L bi-Lipschitz map $\mathfrak{h}$. Note that we do not introduce the parameter ν in this case for simplicity.

For those maps we are able to prove the corresponding result to Theorem 4.1.3.

Theorem 4.1.6. *For $\lambda > C_{h_{gen}}$ the Julia set $\mathcal{J}(\mathcal{Z}_{gen})$ is the entire $\mathbb{R}^3$, where $C_{h_{gen}}$ a constant depending on h_{gen}.*

The proof of this theorem is essentially the same as the one we give for Theorem 4.1.3. We will give a sketch of the proof in section 4.6 where we also find an explicit value for the constant $C_{h_{gen}}$.

Theorems 4.1.4 and 4.1.5 possibly also hold for those kind of Zorich maps with very similar proofs although we forgo the effort of proving them here.

Let us now return to the case where the values of the parameter ν are small and without imposing any further restrictions to the bi-Lipschitz map $\mathfrak{h}$. We remind that in this case we denoted Zorich maps by Z and also that the scaling factor λ is

1 in this case. As we have already mentioned in this case the Julia set is a collection of curves.

Another amazing fact about exponential maps, proven by Mayer in [88], concerning the endpoints of the curves comprising the Julia set (for small values of κ) is that the point at ∞ is what is called a *dispersion point* for the set of endpoints. In other words if we denote the set of endpoints by $\mathcal{E}\left(\mathcal{J}(E_\kappa)\right)$ then this set is totally disconnected while $\mathcal{E}\left(\mathcal{J}(E_\kappa)\right) \cup \{\infty\}$ is connected. That this amazing property is even possible for a planar set is usually exhibited through a set known as Cantor's leaky tent or Knaster-Kuratowski fan (see for example [83]). In fact something even stronger is true here, the work of Aarts and Oversteegen in [1] shows that the point at ∞ is an *explosion point*. This means that the set $\mathcal{E}\left(\mathcal{J}(E_\kappa)\right)$ is in fact totally separated, meaning that for any two of its points, x, y there is a clopen subset $U \subset \mathcal{E}\left(\mathcal{J}(E_\kappa)\right)$ such that $x \in U$ but $y \notin U$. Note that this property implies that $\mathcal{E}\left(\mathcal{J}(E_\kappa)\right)$ is totally disconnected and thus any explosion point is also a dispersion point. We also warn the reader that dispersion points have been sometimes called explosion points in the literature.

Moreover, it is worth mentioning that explosion and dispersion points for exponential maps have been studied for subsets of all the endpoints in [2, 46] and they have been studied for more general classes of maps than the exponential family in [2, 47]. Also, in [34] the presence of Cantor bouquets is proved for maps on the complex plane that are more general than Zorich maps (not necessarily quasiregular) and in fact those bouquets also have ∞ as an explosion point for their set of endpoints.

In the higher dimensional setting of Zorich maps now, Bergweiler in [16] pointed out that ∞ might also be an explosion point for the set of endpoints of the hairs of $\mathcal{J}(Z_\nu)$. In this chapter we will show that this is indeed the case.

Theorem 4.1.7 (Bergweiler and Nicks, [16, 23])**.**
For small enough $\nu > 0$ the Julia set, $\mathcal{J}(Z_\nu)$ of the Zorich family consists of uncountably many disjoint hairs. Each of the hairs tends to ∞ by having their third coordinate go to ∞ while the other two coordinates remain bounded. Moreover, for each point $x \notin \mathcal{J}(Z_\nu)$, $Z_\nu^n(x)$ converges to a fixed point.

Remark. In [16, 23] Theorem 4.1.7 is shown to be true for small enough values of ν without stating an explicit estimate for suitable values of ν. In section 4.7 we make this more precise by proving that Theorem 4.1.7 holds for $0 < \nu < e^{-(\log L + L)}$.

We now state our result on explosion points.

Theorem 4.1.8. *Let $0 < \nu < e^{-(\log L + L)}$. Then ∞ is an explosion point for the set of endpoints of hairs in the Julia set, $\mathcal{E}(\mathcal{J}(Z_\nu))$.*

In the process of proving the above theorem we will need to introduce a topological model for $\mathcal{J}(Z_\nu)$. That model is a generalization, in three dimensions, of a model that Aarts and Oversteegen introduced in [1] in order to study the topology of Julia sets. They called their model *straight brush* and proved that the Julia set of the exponential map E_κ, $0 < \kappa \leq 1/e$ is homeomorphic to a straight brush. We call our new three dimensional model a *3-d straight brush* (see section 4.7 for details) and we will prove the following.

Theorem 4.1.9. *Let $0 < \nu < e^{-(\log L + L)}$. Then there is a 3-d straight brush B and a homeomorphism of B onto the Julia set $\mathcal{J}(Z_\nu)$ of the Zorich map. Moreover, this homeomorphism extends to a homeomorphism between $B \cup \{\infty\}$ and $\mathcal{J}(Z_\nu) \cup \{\infty\}$.*

Another object that Aarts and Oversteegen introduced in the same paper was the *straight one-sided hairy arc* which is compact object living in $[0, 1]^2$. It turns out that if we suitably embed a straight brush in the square $[0, 1]^2$ and then compactify we obtain such an object. It also turns out that any two such objects are ambiently homeomorphic in the complex plane so that there is homeomorphism between them that also extends to the entire complex plane. Moreover, if we compactify $\mathcal{J}(E_\kappa)$, $0 < \kappa \leq 1/e$ in a certain way then we obtain a space which is homeomorphic to a straight one-sided hairy arc. Aarts and Oversteegen call the homeomorphic images of straight one-sided hairy arcs just *hairy arcs*. Moreover, they show that if we embed a hairy arc to the plane in a way that it has the extra property of one-sidedness then it is ambiently homeomorphic, this time the ambient space being the Riemann sphere, with a straight one-sided hairy arc. As a result the compactified version of $\mathcal{J}(E_\kappa)$ is a hairy arc which when suitably embedded in the Riemann sphere is ambiently homeomorphic to a straight one-sided hairy arc.

Such considerations make sense in higher dimensions too. Thus in section 4.10 we define a *straight one-sided hairy square* and a *hairy surface* which are the analogous objects to straight one-sided hairy arcs and hairy arcs respectively (see section 4.10 for details).

Similarly with the case of exponential maps on the plane we can show

Theorem 4.1.10. *Let $0 < \nu < e^{-(\log L + L)}$. Then there is a compactification $\widetilde{\mathcal{J}(Z_\nu)}$ of $\mathcal{J}(Z_\nu)$ which is a hairy surface.*

Two other natural questions to ask in this higher dimensional setting would be whether or not any two straight one-sided hairy squares are ambiently homeomorphic (the ambient space being $\mathbb{R}^3$) and whether all one-sided hairy surfaces are ambiently homeomorphic to a straight one-sided hairy square. Both of these are true in the two-dimensional setting as we have already mentioned.

Although we will not give an answer to the first question here, we can show that in higher dimensions not all one-sided hairy surfaces are ambiently homeomorphic to a straight one-sided hairy square as the next theorem shows.

Theorem 4.1.11. *There exists a straight one-sided hairy square S and a homemorphism*

$$H : S \to H(S) \subset \mathbb{R}^3$$

such that $H(S)$ is one sided and $H(S)$ and S are not ambiently homeomorphic, that is there is no homeomorphism of $\mathbb{R}^3$ that maps S to $H(S)$.

4.2 Julia sets of Zorich maps

4.2.1 Studying the modified Zorich family $\nu\mathcal{Z}$ in the planes $x_1 = \pm x_2$

In this subsection we are going to study the modified Zorich maps $\nu\mathcal{Z}$ defined in(4.1) (recall that we introduced a scaling parameter λ in this case) on the planes $x_1 = \pm x_2$ that are forward invariant under our Zorich maps by construction. Our goal is to show that the dynamics there are chaotic and the planes belong to the Julia set.

We start with:

Proposition 4.2.1. *The x_3-axis belongs to $\mathcal{J}(\mathcal{Z}_\nu)$ for all $\lambda \geq 1$ and ν with $\lambda\nu > 1/e$.*

Before we prove this proposition let us name a few things first. Using the same notation as in [16], for $r = (r_1, r_2) \in \mathbb{Z}^2$ we define

$$P(r) = P(r_1, r_2) := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - 2\lambda r_1| < \lambda, |x_2 - 2\lambda r_2| < \lambda\}.$$

For $c \in \mathbb{R}$ we also define $H_{>c}$ to be the half-space $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > c\}$ and we define $H_{\geq c}$, $H_{<c}$ similarly. We observe here that $\mathcal{Z}_\nu$ maps $P(r_1, r_2) \times \mathbb{R}$ bijectively to $H_{>0}$, when $r_1 + r_2$ is even and to $H_{<0}$ when $r_1 + r_2$ is odd. Thus there is an inverse branch $\Lambda_{(0,0)} : H_{>0} \to P(0,0) \times \mathbb{R}$. We can now, as in [16], find constants $M_0 \in \mathbb{R}$ and $\alpha \in (0,1)$ such that

$$|\Lambda_{(0,0)}(x) - \Lambda_{(0,0)}(y)| \leq \alpha|x - y|, \text{ for all } x, y \in H_{>\nu\lambda e^{M_0}}. \tag{4.2}$$

The next lemma is similar to [23, Lemma 7.1].

Lemma 4.2.2. *Let $M > M_0 > 0$ be a large positive number and $x \in \Lambda_{(0,0)}(H_{>\nu\lambda e^M})$. Then*

$$\mathcal{Z}_\nu(B(x,R) \cap H_{\geq M}) \supset B(\mathcal{Z}_\nu(x), \alpha^{-1}R) \cap H_{>\nu\lambda e^M}, \tag{4.3}$$

where $R > 0$ and $B(x,R)$ denotes the ball of centre x and radius R.

Proof. Note that $x \in \Lambda_{(0,0)}(H_{>\nu\lambda e^M})$ implies that $x \in P(0,0) \times (M, \infty)$. Let

$$y \in B(\mathcal{Z}_\nu(x), \alpha^{-1}R) \cap H_{>\nu\lambda e^M}.$$

Then by (4.2) we have that

$$|x - \Lambda_{(0,0)}(y)| = |\Lambda_{(0,0)}(\mathcal{Z}_\nu(x)) - \Lambda_{(0,0)}(y)| \leq \alpha|\mathcal{Z}_\nu(x) - y| < R.$$

Hence, $\Lambda_{(0,0)}(y) \in B(x,R) \cap P(0,0)$ and thus $y = \mathcal{Z}_\nu(\Lambda_{(0,0)}(y)) \in \mathcal{Z}_\nu(B(x,R) \cap H_{\geq M})$. □

Proof of Proposition 4.2.1. Let us fix a point $x = (0,0,x_0)$ on the x_3-axis and consider a neighbourhood U of that point. It is easy to see now that $\mathcal{Z}_\nu^k(x) = (0,0,E_{\nu\lambda}^k(x_0))$, where $E_{\nu\lambda}^k$ denotes the k-th iterate of the map $E_{\nu\lambda}(t) = \nu\lambda e^t$. Since the x_3-axis is invariant under our Zorich map and since $\nu\lambda > 1/e$ we have that $E_{\nu\lambda}^k(x) \to \infty$ and thus we may assume that $x \in H_{\geq M}$, for some $M > M_0$. By repeatedly applying (4.3) we may now obtain a sequence $R_k \to \infty$ with

$$\mathcal{Z}_\nu^k(U) \supset B(\mathcal{Z}_\nu^k(x), R_k) \cap H_{E_{\nu\lambda}^k(M)}$$

and we note that the intersection on the right hand side always contains the upper half of the ball $B(\mathcal{Z}_\nu^k(x), R_k)$. Hence, for large enough k, the set

$$V_k = \{(x_1,x_2) \in \mathbb{R}^2 : |x_1| \leq 2\lambda, |x_2| \leq 2\lambda\} \times \left[E_{\nu\lambda}^k(x_0), E_{\nu\lambda}^k(x_0) + R_k/2\right],$$

is contained in $B(\mathcal{Z}_\nu^k(x), R_k) \cap H_{E_{\nu\lambda}^k(M)}$. Observe now that $\mathcal{Z}_\nu$ maps V_k onto the shell

$$A_k = \{x \in \mathbb{R}^3 : \nu\lambda \exp\left(E_{\nu\lambda}^k(x_0)\right) \leq |x| \leq \nu\lambda \exp\left(E_{\nu\lambda}^k(x_0) + R_k/2\right)\}.$$

It is easy to see now that this shell, for large enough k and since $R_k \to \infty$, contains a set of the form

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3 : |x_1 - 2\lambda q_{k,1}| \leq 2\lambda, |x_2 - 2\lambda q_{k,2}| \leq 2\lambda, |x_3| \leq t_k\},$$

with $q_{k,1}, q_{k,2} \in \mathbb{Z}$ and $t_k \to \infty$. This implies that

$$\{x \in \mathbb{R}^3 : \nu\lambda e^{-t_k} \leq |x| \leq \nu\lambda e^{t_k}\} \subset \mathcal{Z}_\nu(A_k) \subset \mathcal{Z}_\nu^{k+2}(U).$$

Hence $\bigcup_{k=1}^\infty \mathcal{Z}_\nu^k(U) = \mathbb{R}^3 \setminus \{(0,0,0)\}$ which means that $x \in \mathcal{J}(\mathcal{Z}_\nu)$. □

Here we will prove some basic facts about the Zorich family we have constructed. As we already have mentioned in section 4.1 our Zorich maps send the planes $x_1 = x_2$ and $x_1 = -x_2$ to themselves. We would like to know the behaviour of $\mathcal{Z}_\nu$ restricted to those planes. With that in mind, we observe that restricted to the plane $x_1 = x_2$ our Zorich map is conjugate through $\phi(x_1, x_1, x_3) = \frac{1}{\lambda}(x_3 + i\sqrt{2}x_1)$ to the map $g : \mathbb{C} \to \mathbb{C}$ given by

$$g(z) := \begin{cases} \psi(\bar{z} + 2\sqrt{2}i), & \operatorname{Im}(z) \in \left[(4k+1)\sqrt{2}, (4k+3)\sqrt{2}\right] \\ \\ \psi(z), & \operatorname{Im}(z) \in \left[(4k-1)\sqrt{2}, (4k+1)\sqrt{2}\right], \end{cases}$$

where $k \in \mathbb{Z}$ and $\psi(x+iy) = \nu e^{\lambda x}\left(\mathfrak{h}_3\left(\frac{y}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right) + i\sqrt{2}\mathfrak{h}_1\left(\frac{y}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right)\right)$. Similarly the Zorich map is conjugate to a similar map to g on the plane $x_2 = -x_1$ and everything that follows works in that case as well. For simplicity let us write $a(y)$ and $b(y)$ instead of $\mathfrak{h}_3(y/\sqrt{2}, y/\sqrt{2})$ and $\sqrt{2}\mathfrak{h}_1(y/\sqrt{2}, y/\sqrt{2})$. Note that $a^2(y) + b^2(y) = 1$. Also let us note here that the function ψ is quasiregular and that $g(\mathbb{C}) = \{\operatorname{Re} z > 0\}$. Furthermore, g is not a quasiregular map, since it is not sense preserving, and is a two to one function in the strip $\{z \in \mathbb{C} : (4k-1)\sqrt{2} \leq \operatorname{Im}(z) \leq (4k+3)\sqrt{2}\}$.

We would now like to show that the planes $x_1 = \pm x_2$ belong to the Julia set of $\mathcal{Z}_\nu$. We already know, from Proposition 4.2.1, that the x_3-axis belongs to the Julia set. With that in mind we will prove that any open set in $\mathbb{R}^3$ that intersects those planes also intersects the x_3-axis under iteration by $\mathcal{Z}_\nu$. Now since we know that $\mathcal{Z}_\nu$ is conjugate to g on those planes it is enough to prove that any open set in the complex plane intersects the real axis under iteration by g.

Theorem 4.2.3. *Let $\nu^2\lambda > 2L$ and $V \subset \mathbb{C}$ be a connected set with $m(V) > 0$, where m is the 2 dimensional Lebesgue measure. Then $g^n(V)$ intersects the real axis for some $n \in \mathbb{N}$.*

For the proof of this Theorem we will need several lemmas. Note here that since $\mathfrak{h}$ is a Lipschitz function it will also be differentiable almost everywhere. This implies that g is differentiable almost everywhere.

Lemma 4.2.4. *Let g be the function we defined above. Then*

$$|\det(Dg(z))| \geq \frac{\nu^2\lambda e^{2\lambda \operatorname{Re}(z)}}{L} \; a.e.$$

Proof. It is enough to find a lower bound for $\operatorname{Im} z \in \left[(4k-1)\sqrt{2}, (4k+1)\sqrt{2}\right]$. This is true because for other z we have that $T(z) = \bar{z} + 2\sqrt{2}i$ has imaginary part in $\left[(4k'-1)\sqrt{2}, (4k'+1)\sqrt{2}\right]$ for some $k' \in \mathbb{Z}$ and thus for those z we have $g(z) =$

$g(T(z)) = \psi(T(z))$. Then by the chain rule we have that $Dg(z) = Dg(T(z))DT(z)$. Since $DT(z) = -1$ this implies that $|\det Dg(z)| = |\det Dg(T(z))|$.

With that in mind, if $z = x + iy$ we have that $Dg(z)$ is the linear transformation induced by the matrix

$$\begin{pmatrix} \nu\lambda e^{\lambda x} a(y) & \nu e^{\lambda x} \frac{da}{dy}(y) \\ \\ \nu\lambda e^{\lambda x} b(y) & \nu e^{\lambda x} \frac{db}{dy}(y) \end{pmatrix} = \begin{pmatrix} \nu\lambda e^{\lambda x} \mathfrak{h}_3(\frac{y}{\sqrt{2}}) & \frac{\nu}{\sqrt{2}} e^{\lambda x} \frac{d\mathfrak{h}_3}{dy}(\frac{y}{\sqrt{2}}) \\ \\ \nu\lambda\sqrt{2} e^{\lambda x} \mathfrak{h}_1(\frac{y}{\sqrt{2}}) & \nu e^{\lambda x} \frac{d\mathfrak{h}_1}{dy}(\frac{y}{\sqrt{2}}) \end{pmatrix},$$

if $y \in \big[(4k-1)\sqrt{2}, (4k+1)\sqrt{2}\big]$.

Thus

$$\begin{aligned} |\det Dg(z)| &= \nu^2 \lambda e^{2\lambda x} \left| \mathfrak{h}_3(y) \frac{d\mathfrak{h}_1}{dy}(y) - \mathfrak{h}_1(y) \frac{d\mathfrak{h}_3}{dy}(y) \right| \\ &= \nu^2 \lambda e^{2\lambda x} \left| \det \begin{pmatrix} a(y) & \frac{da}{dy}(y) \\ \\ b(y) & \frac{db}{dy}(y) \end{pmatrix} \right|, \text{ for } y \in \Big[(4k-1)\sqrt{2}, (4k+1)\sqrt{2}\Big]. \end{aligned}$$

We now claim that

$$\left| \det \begin{pmatrix} a(y) & \frac{da}{dy}(y) \\ \\ b(y) & \frac{db}{dy}(y) \end{pmatrix} \right| > \frac{1}{L}, \text{ a.e.}$$

Hence

$$|\det Dg(z)| \geq \frac{\nu^2 \lambda e^{2\lambda x}}{L} \text{ a.e.}$$

Indeed, because $a^2(y) + b^2(y) = 1$ we have that the vectors $(a(y), b(y))$ and $(\frac{da}{dy}, \frac{db}{dy})$ are orthogonal and thus so is their matrix. This implies that

$$\left| \det \begin{pmatrix} a(y) & \frac{da}{dy}(y) \\ \\ b(y) & \frac{db}{dy}(y) \end{pmatrix} \right| = |(a(y), b(y))| \left| \left(\frac{da}{dy}(y), \frac{db}{dy}(y) \right) \right| = \left| \left(\frac{da}{dy}(y), \frac{db}{dy}(y) \right) \right|.$$

Now because $\mathfrak{h}$ is a locally bi-Lipschitz map almost everywhere we have that

$$\left| \frac{d\mathfrak{h}}{dy} \left(\frac{y}{\sqrt{2}}, \frac{y}{\sqrt{2}} \right) \right| \geq \frac{1}{L} \text{ a.e.}$$

Since $\left| \left(\frac{da}{dy}(y), \frac{db}{dy}(y) \right) \right| = \left| \frac{d\mathfrak{h}}{dy} \left(\frac{y}{\sqrt{2}}, \frac{y}{\sqrt{2}} \right) \right|$ we obtain what we wanted.

□

Lemma 4.2.5. *Let $\nu^2\lambda > 2L$ where $\lambda \geq 1$, $\nu > 0$ and $V \subset \mathbb{C}$ be a connected subset of the complex plane with $m(V) > 0$ and such that its iterates under g do not intersect the real axis. Then $m(g^n(V)) \to \infty$ as $n \to \infty$, where m is the 2-dimensional Lebesgue measure.*

Proof. We can assume that V lies on the right half plane $\{z : \mathrm{Re}(z) > 0\}$ otherwise just consider $g(V)$ since g maps $\mathbb{C}$ to the right half plane. We know that $m(g(V)) = \int_{g(V)} dm$. Since none of the iterates of V under g intersects the real axis we have that those iterates also do not intersect any of the pre-images of the real axis, meaning the lines $y = 2\sqrt{2}k$, $k \in \mathbb{Z}$. Thus $g^n(V)$ is always inside strips of the form $\{z \in \mathbb{C} : \mathrm{Im}\, z \in (2\sqrt{2}k, 2\sqrt{2}(k+1))\}$. In each of those strips it is easy to see, by the definition of g, that it is a two-to-one map. By using Lemma 4.2.4 and since $\mathrm{Re}\, z > 0$ for all $z \in V$ we now have

$$\int_{g(V)} dm \geq \frac{1}{2}\int_V |\det(Dg)| dm \geq \frac{\nu^2\lambda}{2L}\int_V e^{2\lambda \mathrm{Re}(z)} dm \geq \frac{\nu^2\lambda}{2L} m(V).$$

This means that

$$m(g(V)) \geq \frac{\nu^2\lambda}{2L} m(V).$$

Hence, since $\nu^2\lambda > 2L$ if we iterate that inequality we have that $m(g^n(V)) \to \infty$.

$\square$

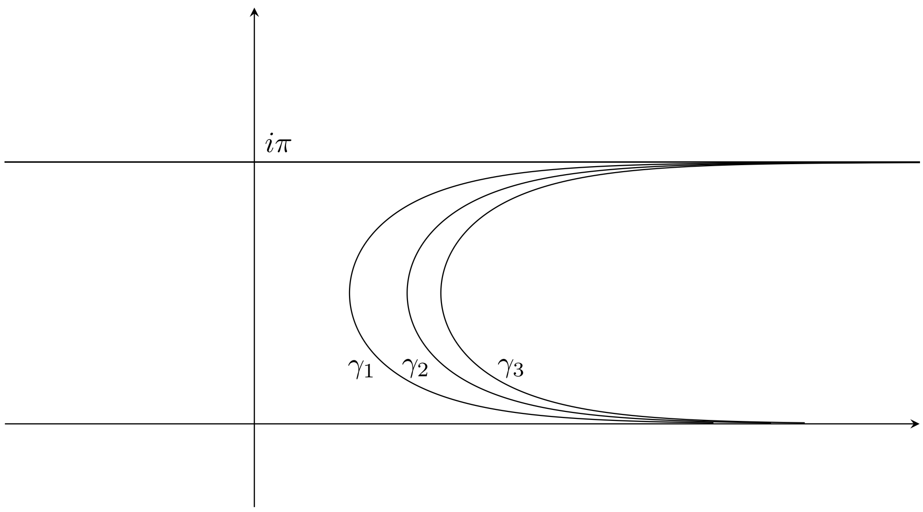

Figure 4.3: The pre-images of the lines $\mathrm{Im}\, z = 2k\pi i$, $k \in \mathbb{Z} \setminus \{0\}$ under the exponential map. The curves γ_m in the proof of Theorem 4.2.3 have a similar structure.

Proof of Theorem 4.2.3. Suppose, towards a contradiction, that there is a connected set V with $m(V) > 0$ whose iterates never intersect the real axis. Then by Lemma 4.2.5 we have that $m(g^n(V)) \to \infty$ as $n \to \infty$. Since $g^n(V)$ never intersects the real axis it also does not intersect its pre-images meaning the lines $\mathrm{Im}(z) = 2\sqrt{2}k$, $k \in \mathbb{Z}$. This means that $g^n(V)$ stays always inside strips of the form $\{z \in \mathbb{C} : \mathrm{Im}(z) \in (2\sqrt{2}k, 2\sqrt{2}(k+1)), k \in \mathbb{Z}\}$. If we now take the pre-image of the lines $\mathrm{Im}(z) = 2\sqrt{2}m$, $m \in \mathbb{Z} \setminus \{0\}$, that lie inside all those strips we obtain curves γ_m

which the iterates $g^n(V)$ of our set must not cross. By symmetry we can confine ourselves in the strip

$$S := \left\{ z \in \mathbb{C} : \mathrm{Im}(z) \in \left(0, 2\sqrt{2}\right) \right\}.$$

From now on let us write $\mathfrak{h}_i(y)$ instead of $\mathfrak{h}_i(y,y)$, $i = 1,2,3$ for simplicity. We have two cases to consider now. Either S contains the pre-images of the lines $\mathrm{Im}(z) = 2\sqrt{2}m$, $m > 0$, in which case $\mathfrak{h}_1(y) > 0$ for $y > 0$ or S contains the pre-images of the lines for $m < 0$ in which case $\mathfrak{h}_1(y) < 0$ for $y > 0$. We will only consider the first case here. The second one can be dealt similarly. So suppose $m > 0$. We claim that the curves γ_m partition the strip S in sets of uniformly bounded area. In fact if we name A_m the area of the set defined by γ_m and γ_{m+1} and we name A_0 the area inside the strip between the imaginary axis and γ_1, then we claim that A_m is an eventually decreasing sequence. Clearly, since $m(g^n(V)) \to \infty$ and $g^n(V)$ must stay inside those sets A_m this is impossible.

In order to prove those claims note that the curves γ_m are given by the equations $\nu e^{\lambda x}\mathfrak{h}_1\left(\frac{y}{\sqrt{2}}\right) = 2m$, when $y \in \left(0, \sqrt{2}\right]$ and $\nu e^{\lambda x}\mathfrak{h}_1\left(\frac{2\sqrt{2}-y}{\sqrt{2}}\right) = 2m$, when $y \in \left[\sqrt{2}, 2\sqrt{2}\right)$. It is also easy to see that those curves do not have self intersections and do not intersect with each other. The area we are looking for will be given by

$$\begin{aligned} A_m = &\int_0^{\sqrt{2}} \frac{1}{\lambda}\left(\log\frac{2(m+1)}{\nu\mathfrak{h}_1\left(\frac{y}{\sqrt{2}}\right)} - \log\frac{2m}{\nu\mathfrak{h}_1\left(\frac{y}{\sqrt{2}}\right)}\right) dy \\ &+ \int_{\sqrt{2}}^{2\sqrt{2}} \frac{1}{\lambda}\left(\log\frac{2(m+1)}{\nu\mathfrak{h}_1\left(\frac{2\sqrt{2}-y}{\sqrt{2}}\right)} - \log\frac{2m}{\nu\mathfrak{h}_1\left(\frac{2\sqrt{2}-y}{\sqrt{2}}\right)}\right) dy. \end{aligned}$$

Thus $A_m = \frac{2\sqrt{2}}{\lambda}\log\frac{m+1}{m}$ which proves what we wanted. We also need to find A_0 for which it is true that

$$A_0 = \int_0^{\sqrt{2}} \frac{1}{\lambda}\log\frac{2}{\nu\mathfrak{h}_1\left(\frac{y}{\sqrt{2}}\right)}dy + \int_{\sqrt{2}}^{2\sqrt{2}} \frac{1}{\lambda}\log\frac{2}{\nu\mathfrak{h}_1\left(\frac{2\sqrt{2}-y}{\sqrt{2}}\right)}dy = 2\int_0^{\sqrt{2}} \frac{1}{\lambda}\log\frac{2}{\nu\mathfrak{h}_1\left(\frac{y}{\sqrt{2}}\right)}dy,$$

if $\nu\mathfrak{h}_1\left(\frac{y}{\sqrt{2}}\right) = 2$ has no solution. If this equation has solutions then, although we can find the area again we do not need to since A_0 will be even smaller in this case (see figure 4.4).

Notice that because $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$ is always a point on the unit sphere we have that

$$\mathfrak{h}_1(x,y)^2 + \mathfrak{h}_2(x,y)^2 = \sin^2\theta|\mathfrak{h}(x,y) - \mathfrak{h}(0,0)|^2 = \sin^2\theta\left(\mathfrak{h}_1(x,y)^2 + \mathfrak{h}_2(x,y)^2 + (\mathfrak{h}_3(x,y)-1)^2\right),$$

where θ is the angle between the x_3-axis and the segment that connects $(0,0,1)$ with $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$. Taking $x = y$, the fact that $\mathfrak{h}$ is bi-Lipschitz on $[-1,1]^2$ and noticing

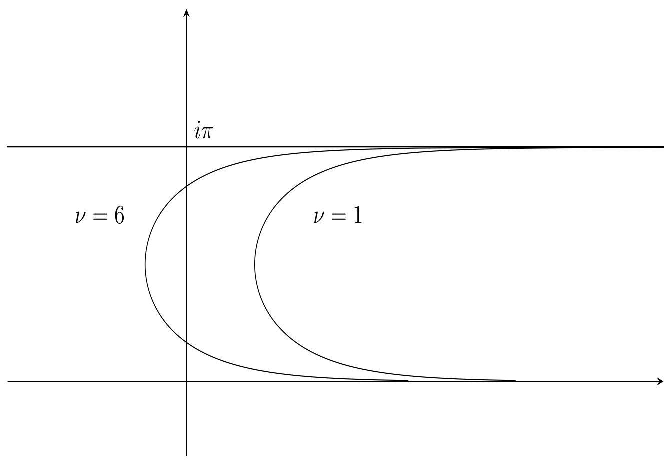

Figure 4.4: The curve γ_0 for the exponential map for two different values of ν. The situation is similar with our maps as well.

that $\theta \geq \pi/4$ gives us $|\mathfrak{h}_1(y)| \geq \frac{|y|}{\sqrt{2}L}$. Hence because $\mathfrak{h}_1(y) > 0$ for $y \in (0,1)$ we have

$$A_0 \leq \frac{2}{\lambda} \int_0^{\sqrt{2}} \log\left(\frac{4L}{\nu y}\right) dy,$$

which is finite. $\square$

4.2.2 Proof of Theorem 4.1.3

Having proved Theorem 4.2.3 we are now ready to proceed to our main theorem.

First we prove some lemmas that we will later need. Note here that since we proved that the planes $x_1 = \pm x_2$ are in the Julia set of $\mathcal{Z}_\nu$ we will also know that all their inverse images are in the Julia set. Those inverse images are again planes of the form $x_1 = \pm x_2 + 2\lambda k$, where $k \in \mathbb{Z}$. Those planes partition $\mathbb{R}^3$ in square beams. Name $B_{(0,0)}$ the open rectangular beam that is the union of the two square beams that touch the x_3-axis and are in the half-space $x_2 \leq x_1$. We can partition the space now in rectangular beams that are translates of this $B_{(0,0)}$. Let us name them

$$B_{(i,j)} = B_{(0,0)} + i(2\lambda, 2\lambda, 0) + j(\lambda, -\lambda, 0), \quad i, j \in \mathbb{Z}.$$

Note that the map $\mathcal{Z}_\nu$ is a homeomorphism in those rectangular beams. The next lemma is inspired by the one Misiurewicz used in his proof (compare [94, Lemma 1]) and is the main reason that we need the scale factor λ in our definition of the Zorich map. It will be convenient to introduce some notation. Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map defined by $p(x_1, x_2, x_3) = (x_1, x_2)$. Also let $p_3(x)$, $x \in \mathbb{R}^3$ denote the third coordinate of x, in other words $p_3(x_1, x_2, x_3) = x_3$.

Lemma 4.2.6. *If λ and L are the numbers we used in the construction of the Zorich map and $\nu > 0$ then*

$$\det\left(D\mathcal{Z}_\nu^n(x)\right) \geq \left(\frac{\lambda}{L^5}\right)^n \frac{1}{\lambda^3} \left|\left(p \circ \mathcal{Z}_\nu^n\right)(x)\right|^3 \quad a.e.$$

Proof. First note that $|p(\mathcal{Z}_\nu^n(x))| = \nu e^{p_3(\mathcal{Z}_\nu^{n-1})} \cdot |(p \circ h \circ p \circ \mathcal{Z}_\nu^{n-1})(x)|$. Also, using the fact that $h(0) = (0, 0, \lambda)$ and $h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2), h_3(x_1, x_2))$ is a Lipschitz map we have that

$$|p(h(x))| = |p\left(h(x) - h(0)\right)| \leq |h(x) - h(0)| \leq L|x|.$$

Hence

$$\begin{aligned} |p(\mathcal{Z}_\nu^n(x))| &= \nu e^{p_3(\mathcal{Z}_\nu^{n-1})} \left|\left(p \circ h \circ p \circ \mathcal{Z}_\nu^{n-1}\right)(x)\right| \\ &\leq L\nu e^{p_3(\mathcal{Z}_\nu^{n-1})} \left|\left(p \circ \mathcal{Z}_\nu^{n-1}\right)(x)\right| \leq \cdots \\ &\leq (\nu L)^n e^{p_3(\mathcal{Z}_\nu^{n-1})} e^{p_3(\mathcal{Z}_\nu^{n-2})} \cdots e^{x_3} \sqrt{h_1^2\Big(x_1, x_2\Big) + h_2^2\Big(x_1, x_2\Big)} \\ &\leq \lambda(\nu L)^n e^{p_3(\mathcal{Z}_\nu^{n-1})} e^{p_3(\mathcal{Z}_\nu^{n-2})} \cdots e^{x_3}. \end{aligned} \tag{4.4}$$

From the chain rule we know that

$$\det\left(D\mathcal{Z}_\nu^n(x)\right) = \prod_{k=0}^{n-1} \det\left(D\mathcal{Z}_\nu\left(\mathcal{Z}_\nu^k(x)\right)\right). \tag{4.5}$$

Now from the definition of $\mathcal{Z}_\nu$ we have

$$\det(D\mathcal{Z}_\nu(x)) = \nu^3 e^{3x_3} \det H, \tag{4.6}$$

where

$$H = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(p(x)) & \frac{\partial h_1}{\partial x_2}(p(x)) & h_1(p(x)) \\ \frac{\partial h_2}{\partial x_1}(p(x)) & \frac{\partial h_2}{\partial x_2}(p(x)) & h_2(p(x)) \\ \frac{\partial h_3}{\partial x_1}(p(x)) & \frac{\partial h_3}{\partial x_2}(p(x)) & h_3(p(x)) \end{pmatrix}.$$

We now set

$$A = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}\left(p(x)\right) \\ \frac{\partial h_2}{\partial x_1}\left(p(x)\right) \\ \frac{\partial h_3}{\partial x_1}\left(p(x)\right) \end{pmatrix}, \quad B = \begin{pmatrix} \frac{\partial h_1}{\partial x_2}\left(p(x)\right) \\ \frac{\partial h_2}{\partial x_2}\left(p(x)\right) \\ \frac{\partial h_3}{\partial x_2}\left(p(x)\right) \end{pmatrix}, \quad C = \begin{pmatrix} h_1\left(p(x)\right) \\ h_2\left(p(x)\right) \\ h_3\left(p(x)\right) \end{pmatrix}.$$

Recall now that from linear algebra, the determinant of a matrix equals the scalar triple product. This means that

$$\det H = \langle A \times B, C \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. Since $\mathcal{Z}_\nu$ is sense preserving we will have that $\det H > 0$ and since A and B are orthogonal to C we will have that $A \times B$ is parallel to C. Remember that $|C| = \lambda$ so

$$\det H = \lambda |A \times B| \left\langle \frac{A \times B}{|A \times B|}, \frac{C}{|C|} \right\rangle = \lambda |A \times B|. \tag{4.7}$$

Now because $\mathfrak{h}$ is a locally bi-Lipschitz map we have that

$$|h\left(p(x) + tv\right) - h\left(p(x)\right)| \geq \frac{|tv|}{L},$$

for all small $t > 0$ where $v = (v_1, v_2) \in \mathbb{R}^2$. This implies that

$$|Dh\left(p(x)\right)(v)| \geq \frac{|v|}{L} \tag{4.8}$$

and if we set $v = \left(|B|, \frac{-\langle A,B \rangle}{|B|}\right)$ and square both sides we have

$$\left| |B|A - \frac{\langle A, B \rangle}{|B|} B \right|^2 \geq \frac{1}{L^2} \left(|B|^2 + \frac{\langle A, B \rangle^2}{|B|^2} \right) \geq \frac{|B|^2}{L^2}.$$

Simplifying we have

$$|A|^2 |B|^2 - \langle A, B \rangle^2 \geq \frac{|B|^2}{L^2}.$$

Now note that by elementary properties of the cross product $|A \times B|^2 = |A|^2|B|^2 - \langle A, B \rangle^2$ and thus

$$|A \times B|^2 \geq \frac{|B|^2}{L^2} \geq \frac{1}{L^4},$$

where the last inequality comes from (4.8) for $v = (0, 1)$. Hence, (4.7) becomes $\det H \geq \frac{\lambda}{L^2}$. Putting everything together in (4.6) we have

$$\det\left(D\mathcal{Z}_\nu(x)\right) \geq \nu^3 e^{3x_3} \frac{\lambda}{L^2}, \text{ a.e.} \tag{4.9}$$

Hence, by (4.4) and (4.9) we have that

$$\begin{aligned} |p(\mathcal{Z}_\nu^n(x))|^3 &\leq \lambda^3 (L\nu)^{3n} e^{3p_3(\mathcal{Z}_\nu^{n-1})} e^{3p_3(\mathcal{Z}_\nu^{n-2})} \cdots e^{3x_3} \\ &\leq \frac{\lambda^3 L^{5n}}{\lambda^n} \det\left(D\mathcal{Z}_\nu\left(\mathcal{Z}_\nu^{n-1}(x)\right)\right) \cdots \det(D\mathcal{Z}_\nu(x)) \text{ a.e.} \end{aligned}$$

By rearranging and (4.5) now we obtain the desired inequality. □

The next lemma describes the behaviour of points near the x_3-axis under iteration.

Lemma 4.2.7. *Let $\nu\lambda > \frac{1}{e}$.*

(a) There are $\delta > 0$ and $c > 0$ such that if $x \in C_\delta$, where C_δ is the cylinder around the x_3-axis with δ radius, then $p_3(\mathcal{Z}_\nu(x)) > p_3(x) + c$,

(b) For δ as in (a) and for every $x \in C_\delta$, with $p(x) \neq (0,0)$, there is an $n \in \mathbb{N}$ such that $\mathcal{Z}_\nu^n(x) \notin C_\delta$.

Proof. (a) We have if $h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2), h_3(x_1, x_2))$ then $p_3(\mathcal{Z}_\nu(x)) = \nu e^{x_3} h_3(x_1, x_2)$. Now since $h(0,0) = (0,0,\lambda)$ and h is continuous, for all $\varepsilon > 0$ there is a disk $D = D(0,\delta)$ of radius $\delta > 0$ on which we have $h_3(x_1, x_2) > \lambda - \varepsilon$. Hence if $x = (x_1, x_2, x_3) \in C_\delta = D \times \mathbb{R}$, then

$$p_3(\mathcal{Z}_\nu(x)) = \nu e^{x_3} h_3(x_1, x_2) > \nu e^{p_3(x)}(\lambda - \varepsilon) \geq p_3(x) + 1 + \log\left(\nu(\lambda - \varepsilon)\right),$$

where the last inequality follows by minimizing $\nu e^t(\lambda - \varepsilon) - t$. Now notice that since $\nu\lambda > \frac{1}{e}$ we can find a small enough $\varepsilon > 0$ such that $\nu(\lambda - \varepsilon) > \frac{1}{e}$, which implies $1 + \log(\nu(\lambda - \varepsilon)) > 0$. Hence, $p_3(\mathcal{Z}_\nu(x)) > p_3(x) + c$ with $c = 1 + \log(\nu(\lambda - \varepsilon))$.

(b) For a δ as in (a) and $\delta < \lambda$ now assume that there is a point $x \in C_\delta$ such that $p(x) \neq 0$ and $\mathcal{Z}_\nu^n(x) \in C_\delta$ for all $n \in \mathbb{N}$. Then according to (a) we would have that $p_3(\mathcal{Z}_\nu^n(x)) \to \infty$ when $n \to \infty$. We know that

$$|(p \circ \mathcal{Z}_\nu^{n+1})(x)| = e^{p_3(\mathcal{Z}_\nu^n)} |\left(p \circ h \circ p \circ \mathcal{Z}_\nu^n\right)(x)|. \tag{4.10}$$

Now its a simple geometric fact that for each $y \in [-\lambda, \lambda]^2$

$$|\left(p \circ h\right)(y)| = \sin\theta \left|h(y) - h(0)\right|, \tag{4.11}$$

where θ is the angle between the line segment joining the point $h(y)$, on the sphere, with the point $(0,0,\lambda)$ and the x_3-axis. Moreover, $\theta \geq \pi/4$ for all such y. Also by the fact that h is a bi-Lipschitz function we have that $|h(y) - h(0)| \geq \frac{|y|}{L}$.

Now taking $y = p\left(\mathcal{Z}_\nu^n(x)\right)$ and combining this with (4.10) and (4.11) implies that

$$|(p \circ \mathcal{Z}_\nu^{n+1})(x)| \geq \frac{e^{p_3(\mathcal{Z}^n(x))}}{\sqrt{2}L} |(p \circ \mathcal{Z}_\nu^n)(x)|.$$

Thus, since $p_3(\mathcal{Z}_\nu^n(x)) \to \infty$, for all large enough n we can say that

$$|(p \circ \mathcal{Z}_\nu^{n+1})(x)| \geq 2|(p \circ \mathcal{Z}_\nu^n)(x)|.$$

This of course contradicts the fact that $\mathcal{Z}^n(x) \in C_\delta$ for all $n \in \mathbb{N}$. $\square$

The next lemmas describe how sets of positive measure behave under iteration by the Zorich map assuming that their iterates never cross the planes that we already know belong to the Julia set.

Lemma 4.2.8. *Assume that $\lambda > L^5$. Let $V \subset \mathbb{R}^3$ be a connected set with $m(V) > 0$ and whose iterates under $\mathcal{Z}_\nu$ do not intersect any of the planes $x_1 = \pm x_2 + 2\lambda k$, where $k \in \mathbb{Z}$. Suppose also that there is sequence of integers $n_j > 0$ with $\mathcal{Z}_\nu^{n_j}(V) \cap C_a = \emptyset$, where C_a is a cylinder around the x_3-axis of any radius $a > 0$. Then $m(\mathcal{Z}_\nu^{n_j}(V)) \to \infty$ as $n_j \to \infty$, where m is the 3 dimensional Lebesgue measure.*

Proof. Since $\mathcal{Z}_\nu^{n_j}(V)$ stays out of the cylinder C_a we have that, for $x \in V$

$$|(p \circ \mathcal{Z}_\nu^{n_j})(x)| > a.$$

By using Lemma 4.2.6 we will now have that

$$\det\left(D\mathcal{Z}_\nu^{n_j}\right) \geq \left(\frac{\lambda}{L^5}\right)^{n_j} \cdot \frac{a^3}{\lambda^3} \text{ a.e. on } V.$$

Since all of the iterates of V do not intersect any of the planes $x_1 = \pm x_2 + 2\lambda k$, where $k \in \mathbb{Z}$ and since the Zorich map is a homeomorphism in the square beams that remain if we remove those planes we will have that $\mathcal{Z}_\nu^n$ is a homeomorphism in V. Hence, for all n_j we will have that

$$m(\mathcal{Z}_\nu^{n_j}(V)) = \int_V \left|\det\left(D\mathcal{Z}_\nu^{n_j}\right)\right| dm \geq \left(\frac{\lambda}{L^5}\right)^{n_j}\left(\frac{a^3}{\lambda^3}\right) m(V),$$

which tends to infinity as $n_j \to \infty$ since $\lambda > L^5$. □

For our next lemma let us assume that our Zorich map sends the beam $B_{(0,0)}$ to the half space $x_2 \leq x_1$. The other alternative is mapping it to the half space $x_1 \leq x_2$ but the methods work in a very similar way with minor modifications.

Consider the inverse image under $\mathcal{Z}_\nu$ of the boundary of $B_{(0,0)}$ that lies in the interior of $B_{(0,0)}$. This inverse image will be some surface which we will call S_0. In Figure 4.5 we have drawn the x_1x_2 plane and the rectangular beams $B_{(0,0)}$ and $B_{(0,-1)}$. Take now the planes $P_1 : x_2 = x_1 - 4\lambda$, $P_2 : x_2 = -x_1 + 4\lambda$, $P_3 : x_2 = -x_1 - 4\lambda$ and consider the rectangular beam they define together with the plane $x_1 = x_2$. Let us now take the boundary of this beam, without the part that belongs to $x_1 = x_2$, and name it L_1. Consider now the inverse image of L_1 that lies inside $B_{(0,0)}$. This image is a surface, let us call it S_1. We can now do the same with the planes $P_4 : x_2 = x_1 - 6\lambda$, $P_5 : x_2 = -x_1 + 6\lambda$, $P_6 : x_2 = -x_1 - 6\lambda$ and get the boundary of the beam they define, which we call L_2 and then the surface we obtain by taking the inverse image which we call S_2. If we continue with this construction we obtain a sequence of surfaces, S_0, S_1, S_2, S_3, $\cdots$ inside $B_{(0,0)}$. Each of those surfaces lies above its previous starting with S_0. We can also construct similar surfaces $K_0, K_1, K_2, \cdots$ inside the beam $B_{(0,-1)}$ by taking inverse images of the corresponding boundaries $\partial B_{(0,-1)}$, R_1, R_2, $\cdots$ (see Figure 4.5). Moreover, we construct similar surfaces in all the other rectangular beams $B_{(i,j)}$, that partition the space, depending on which half-space the beam is mapped to under the Zorich map. Let us denote by $\mathcal{S}$ the union of all those surfaces.

We will show that the space between S_n and S_{n+1} (similarly between K_n and K_{n+1}) is of finite volume and is decreasing as n increases.

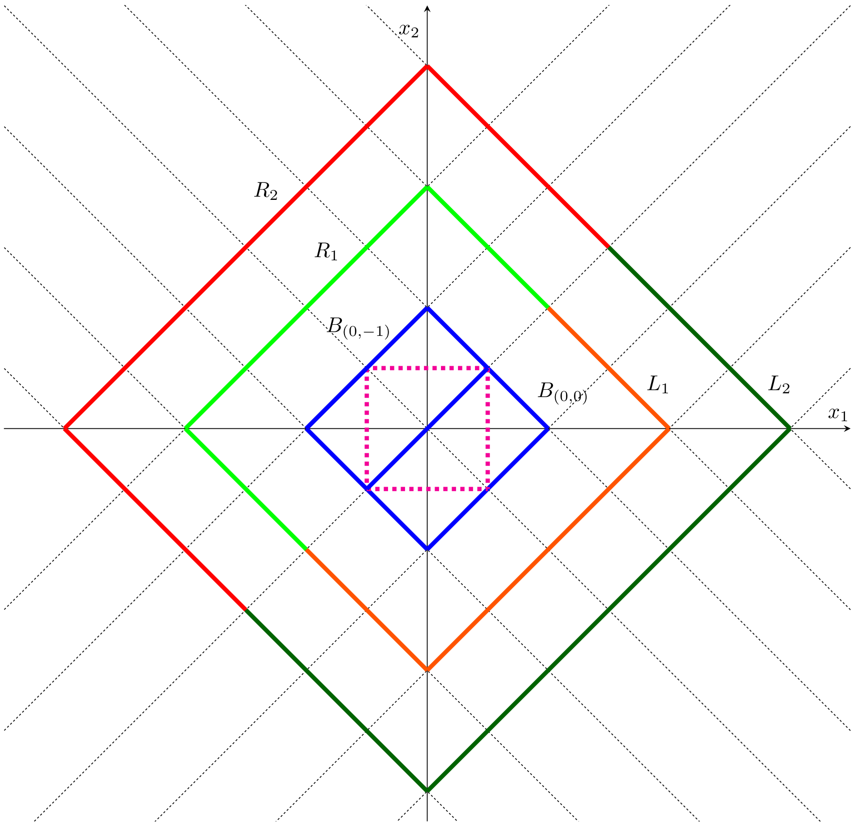

Figure 4.5: The x_1x_2 plane. In pink is the initial square we used to define our Zorich map. In blue the rectangular beams $B_{(0,0)}$ and $B_{(0,-1)}$ while in orange, dark green, green and red are the sets L_1, L_2, R_1 and R_2 respectively.

Lemma 4.2.9. *Let I_n be the volume that the surface S_n encloses together with the plane $x_3 = 0$ and inside the beam $B_{(0,0)}$. Then I_n is finite for all $n \in \mathbb{N}$. Furthermore, if $T_n := I_{n+1} - I_n$ is the volume between S_n and S_{n+1} then T_n is a decreasing sequence.*

Proof. Let us first find an implicit equation that describes each of these surfaces. We work on $B_{(0,0)}$ but the same can be done on all other rectangular beams. Let us split $B_{(0,0)}$ in three different beams whose cross-sections with the x_1x_2 plane are the sets

$$Q_1 := h^{-1}\left(\{(x_1,x_2,x_3)\in S(0,\lambda) : 0\le x_2\le x_1\}\right),$$

$$Q_2 := h^{-1}\left(\{(x_1,x_2,x_3)\in S(0,\lambda) : x_2\le 0, x_1\ge 0\}\right)$$

and

$$Q_3 := h^{-1}\left(\{(x_1,x_2,x_3)\in S(0,\lambda) : x_2\le x_1\le 0\}\right),$$

where $S(0,\lambda)$ the sphere of centre 0 and radius λ. In the beam corresponding to the first cross section, meaning $Q_1\times\mathbb{R}$, the points on the surface S_n satisfy

$$\nu e^{x_3}h_1(x_1,x_2) = -\nu e^{x_3}h_2(x_1,x_2) + 2(n+1)\lambda.$$

On the beam $Q_2\times\mathbb{R}$ the surface points satisfy

$$\nu e^{x_3}h_1(x_1,x_2) = \nu e^{x_3}h_2(x_1,x_2) + 2(n+1)\lambda,$$

while on the beam $Q_3\times\mathbb{R}$ that corresponds to the last cross section the points satisfy

$$\nu e^{x_3}h_1(x_1,x_2) = -\nu e^{x_3}h_2(x_1,x_2) - 2(n+1)\lambda.$$

Suppose now that the surfaces S_n do not intersect the plane $x_3=0$. Hence the volume that the surface S_n encloses together with the plane $x_3=0$ and inside the beam $B_{(0,0)}$ is given by the integrals

$$\begin{aligned} I_n = \int\int_{Q_1} \log\frac{2(n+1)\lambda}{\nu\left(h_2(x_1,x_2)+h_1(x_1,x_2)\right)}dx_2dx_1 \\ + \int\int_{Q_2} \log\frac{-2(n+1)\lambda}{\nu\left(h_1(x_1,x_2)-h_2(x_1,x_2)\right)}dx_2dx_1 \\ + \int\int_{Q_3} \log\frac{-2(n+1)\lambda}{\nu\left(h_2(x_1,x_2)+h_1(x_1,x_2)\right)}dx_2dx_1. \end{aligned}$$

It is not so hard to prove now that each of these integrals is finite since each one of them is an integral of a bounded function except on a neighbourhood of $(0,0)$ and the points where $h_1\pm h_2=0$ which are $(0,-2\lambda)$, $(2\lambda,0)$. So in order to show that this sum is finite, it is enough to consider the integrals only in neighbourhoods of those points. On the other hand, when those surfaces S_n intersect the plane $x_3=0$ the volumes are no longer given by the above integrals (the set where we integrate will change) but again we only have to consider them at a neighbourhood of $(0,0)$ as well as $(0,-2\lambda)$, $(2\lambda,0)$ so that is what we do next. In fact, those volumes in that case are even smaller.

We will only treat here the second integral around an ε-neighbourhood of $(0,0)$ and the rest follows similarly. So we are looking at the integral

$$\int\int_{Q_2\cap B(0,\varepsilon)} \log\frac{-2(n+1)\lambda}{\nu\left(h_2(x_1,x_2)-h_1(x_1,x_2)\right)}dx_2dx_1,$$

where $B(0,\varepsilon)$ is a ball centred at 0 with radius ε. Equivalently, we want to show that the integral

$$\int\int_{Q_2\cap B(0,\varepsilon)} -\log\left(h_1(x_1,x_2)-h_2(x_1,x_2)\right)dx_2dx_1 \tag{4.12}$$

is finite. Now because h is a bi-Lipschitz map and because $h_2(x_1,x_2)\leq 0$ and $h_1(x_1,x_2)\geq 0$ in the set we are integrating we have that

$$(h_1-h_2)^2 = h_1^2+h_2^2+2h_1(-h_2) \geq h_1^2+h_2^2 = \sin^2\theta(h_1^2+h_2^2+(h_3-\lambda)^2) \geq \frac{c_\varepsilon}{L^2}(x_1^2+x_2^2),$$

where θ is the angle between the x_3 axis and the segment that connects $(0,0,\lambda)$ with (h_1,h_2,h_3) and $c_\varepsilon>0$ a constant that depends only on ε. Hence

$$|h_1(x_1,x_2)-h_2(x_1,x_2)| \geq \frac{\sqrt{c_\varepsilon}}{L}\sqrt{x_1^2+x_2^2}.$$

Now since $h_1-h_2\geq 0$ in the set we are integrating, we will have that

$$\int\int_{Q_2\cap B(0,\varepsilon)} \log\left(h_1(x_1,x_2)-h_2(x_1,x_2)\right)dx_2dx_1 \geq \int\int_{Q_2\cap B(0,\varepsilon)} \log\left(\frac{\sqrt{c_\varepsilon}}{L}\sqrt{x_1^2+x_2^2}\right)dx_2dx_1.$$

Now since the last integral is finite we will have that the integral (4.12) is also finite.

Finally, let us show that the sequence T_n is a decreasing one. Indeed,

$$T_n = I_{n+1}-I_n = \lambda^2\log\frac{n+2}{n+1}+\lambda^2\log\frac{n+2}{n+1}+2\lambda^2\log\frac{n+2}{n+1} = 4\lambda^2\log\frac{n+2}{n+1},$$

which can be easily seen to be a decreasing sequence. □

Lemma 4.2.10. *Assume $\lambda > L^5$ and let V be a connected subset of $\mathbb{R}^3$ with $m(V) > 0$ and such that $\mathcal{Z}_\nu^n(V)$ does not intersect any of the planes $x_1=\pm x_2+2\lambda k$, where $k\in\mathbb{Z}$ for all $n\in\mathbb{N}$. Then $\mathcal{Z}_\nu^n(V)$ visits infinitely often one of the two rectangular beams $B_{(0,0)}$, $B_{(0,-1)}$, that have the x_3-axis in their boundary.*

Proof. Consider the iterates $V_i=\mathcal{Z}_\nu^i(V)$ of the set V. The sets V_i stay always inside one of the rectangular beams by assumption and also they cannot intersect any of the surfaces in $\mathcal{S}$ that are in those beams since if they did on the next iterate they would intersect the boundary of one of the beams. Suppose now that we can find a $N\in\mathbb{N}$ such that $V_i\notin B_{(0,0)}\cup B_{(0,-1)}$ for all $i>N$. Then by Lemma 4.2.8 we have that $m(V_i)\to\infty$. This implies that our sets V_i cannot lie between any two of the surfaces in $\mathcal{S}$, for all large i since there is finite volume between them. Thus V_i stays below the lowest surface in the relevant rectangular beam for all $i>N_1>N$, where $N_1\in\mathbb{N}$. This is a contradiction since being below that surface implies that V_{i+1} is in either $B_{(0,0)}$ or $B_{(0,-1)}$. □

The next lemma tells us that when a set remains in $B_{(0,0)}$ under iteration by $\mathcal{Z}_\nu$ then we can find points with large x_3 coordinate in its iterates.

Lemma 4.2.11. *Assume $\lambda > L^5$, $\nu > \frac{1}{\lambda e}$ and let V be a connected set of $\mathbb{R}^3$ with $m(V) > 0$ and such that $\mathcal{Z}_\nu^n(V)$ does not intersect any of the planes $x_1 = \pm x_2 + 2\lambda k$, where $k \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Suppose that there is an $N_0 \in \mathbb{N}$ such that $\mathcal{Z}_\nu^n(V) \subset B_{(0,0)}$, for all $n > N_0$. Then for all $M > 0$ and $\varepsilon > 0$ there is some $n_0 > N_0$ and a point $x \in \mathcal{Z}_\nu^{n_0}(V)$ such that $p_3(x) > M$ and $d(x, x_3\text{-axis}) < \varepsilon$, where d is the Euclidean distance.*

Proof. Either an $\varepsilon_0 > 0$ exists such that $d(\mathcal{Z}_\nu^n(V), x_3\text{-axis}) > \varepsilon_0$ for all $n > N_0$ or such an ε_0 does not exist. In the first case we know from Lemma 4.2.8 that $m(\mathcal{Z}_\nu^n(V)) \to \infty$. We also know that since $\mathcal{Z}_\nu^n(V) \in B_{(0,0)}$, for all $n > N_0$, $\mathcal{Z}_\nu^n(V)$ must, by Lemma 4.2.9, lie below the surface S_0 . Since $m(\mathcal{Z}_\nu^n(V)) \to \infty$ then it must be true that for all $M_1 > 0$ there is a n_0 and a point z_0 in $\mathcal{Z}_\nu^{n_0}(V)$ with $p_3(z_0) < -M_1$. The pre-image of that point inside $B_{(0,0)}$ is a point $z^{(1)} = \left(z_1^{(1)}, z_2^{(1)}, z_3^{(1)}\right)$ for which $\nu e^{z_3^{(1)}} h_3\left(z_1^{(1)}, z_2^{(1)}\right) < -M_1$ and $h_3\left(z_1^{(1)}, z_2^{(1)}\right) < 0$. But since $h_3 > -\lambda$ we have that

$$e^{z_3^{(1)}} > \frac{M_1}{\nu\lambda} \Rightarrow z_3^{(1)} > \log\frac{M_1}{\nu\lambda}.$$

If we now take the pre-image in $B_{(0,0)}$ of that point, $z^{(2)} = \left(z_1^{(2)}, z_2^{(2)}, z_3^{(2)}\right)$ then $0 < h_3\left(z_1^{(2)}, z_2^{(2)}\right) < \lambda$ and

$$\nu e^{z_3^{(2)}} h_3\left(z_1^{(2)}, z_2^{(2)}\right) = z_3^{(1)} > \log\frac{M_1}{\nu\lambda},$$

which implies then

$$z_3^{(2)} > \log\frac{\log\frac{M_1}{\nu\lambda}}{\nu\lambda}.$$

Thus we have shown that for any $M > 0$ there is a point $z^{(2)}$ in $\mathcal{Z}_\nu^{n_0-2}(V)$ for which $z_3^{(2)} > M$ and also $\left|z_1^{(2)}\right|, \left|z_2^{(2)}\right| < \lambda$. This leads to a contradiction. To see why, note that by our assumptions $\mathcal{Z}_\nu^n(V)$ is ε_0 away from the x_3-axis and below the surface S_0 and thus all of its points, which are also inside the initial square beam $[-\lambda, \lambda]^2 \times \mathbb{R}$ (pink in figure 4.5), have a bounded x_3 coordinate. This is true because the surface S_0 together with any cylinder around the x_3 axis and the plane $x_3 = 0$ enclose a set inside $[-\lambda, \lambda]^2 \times \mathbb{R}$ and outside of the cylinder whose closure is compact.

For the second case now, where such an ε_0 does not exist, then there is a sequence $w_k \in \cup_{n>N_0}\mathcal{Z}_\nu^n(V)$ with $d(w_k, x_3\text{-axis}) \to 0$. If $p_3(w_k) \to \infty$ we are done. If on the other hand $p_3(w_k) \to -\infty$ then $y_k := \mathcal{Z}_\nu(w_k) \to (0,0,0)$ and thus $\mathcal{Z}_\nu^n(y_k) \to \mathcal{Z}_\nu^n(0) = (0, 0, E_{\nu\lambda}^n(0))$, where $E_{\nu\lambda}^n(0))$ converges to ∞ and again we are done. The remaining case to consider is when there is a subsequence w_{k_i} converging to some point $(0,0,a)$ in the x_3-axis. By relabelling we may assume that $w_k \to (0,0,a)$. Now choose an $N>0$ such that $E_{\nu\lambda}^N(a) > M$, where $E_{\nu\lambda}$ denotes the map $x \mapsto \nu\lambda e^x$. By continuity of $\mathcal{Z}_\nu^N$, for all $\varepsilon > 0$ we may find a δ such that if $|w_k - (0,0,a)| \leq \delta$, then

$$\left|\mathcal{Z}_\nu^N(w_k) - \mathcal{Z}_\nu^N(0,0,a)\right| = \left|\mathcal{Z}_\nu^N(w_k) - \left(0,0,E_{\nu\lambda}^N(a)\right)\right| \leq \varepsilon.$$

Hence, if we choose ε small enough we have that $x_3\left(\mathcal{Z}_\nu^N(w_k)\right) > M$ and $\mathcal{Z}_\nu^N(w_k)$ is within ε distance from the z-axis, when k is large enough. $\square$

Lemma 4.2.12. *Let $y_1, y_2 \in B(0,r)$, where $r>0$. Then for all $n \in \mathbb{N}$ it is true that*

$$|\mathcal{Z}_\nu^n(y_1) - \mathcal{Z}_\nu^n(y_2)| \leq \left(\frac{\max\{L,\lambda\}}{\lambda}\right)^n E_{\nu\lambda}(r) \cdots E_{\nu\lambda}^n(r)|y_1 - y_2|,$$

where $E_{\nu\lambda}$ denotes the exponential map $x \mapsto \nu\lambda e^x$ and L is the bi-Lipschitz constant we used in the construction of the Zorich maps.

Proof. The Zorich map is absolutely continuous on any line segment since it is locally Lipschitz. First we show that a version of the Finite Increment Theorem, see [142, 10.4.1, Theorem 1], is true for such functions.

Consider the map $g(t) = \mathcal{Z}_\nu(ty_1 + (1-t)y_2)$, $t \in [0,1]$, which is going to be absolutely continuous. By the chain rule we now have that

$$|g'(t)| = |D\mathcal{Z}_\nu(ty_1 + (1-t)y_2)(y_1 - y_2)| \leq \operatorname*{ess\,sup}_{x\in\gamma} |D\mathcal{Z}_\nu(x)||y_1 - y_2|.$$

Using the fundamental theorem of calculus for the Lebesgue integral and the above equality now we have that

$$|\mathcal{Z}_\nu(y_1) - \mathcal{Z}_\nu(y_2)| = |g(1) - g(0)| \leq \int_0^1 |g'(t)|dt \leq \operatorname*{ess\,sup}_{x\in\gamma} |D\mathcal{Z}_\nu(x)||y_1 - y_2|,$$

where γ is the line segment that connects y_1 to y_2. Remember that $|Df|$ denotes the operator norm of the total derivative. Hence, by the chain rule and elementary properties of linear maps we have that

$$|D\mathcal{Z}_\nu^n(x)| \leq |D\mathcal{Z}_\nu\left(\mathcal{Z}_\nu^{n-1}(x)\right)| \cdots |D\mathcal{Z}_\nu(x)|.$$

Hence by the above inequalities and because $y_1, y_2 \in B(0,r)$ we have that

$$\begin{aligned}|\mathcal{Z}_\nu^n(y_1) - \mathcal{Z}_\nu^n(y_2)| &\leq \operatorname*{ess\,sup}_{x\in\gamma} |D\mathcal{Z}_\nu\left(\mathcal{Z}_\nu^{n-1}(x)\right)| \cdots \operatorname*{ess\,sup}_{x\in\gamma} |D\mathcal{Z}_\nu(x)||y_1 - y_2| \\ &\leq \operatorname*{ess\,sup}_{x\in B(0,r)} |D\mathcal{Z}_\nu\left(\mathcal{Z}_\nu^{n-1}(x)\right)| \cdots \operatorname*{ess\,sup}_{x\in B(0,r)} |D\mathcal{Z}_\nu(x)||y_1 - y_2|. \quad (4.13)\end{aligned}$$

We also know that $D\mathcal{Z}_\nu(x) = e^{x_3} D\mathcal{Z}_\nu(x_1, x_2, 0)$. Moreover, we will prove that

$$|D\mathcal{Z}_\nu(x_1, x_2, 0)| \leq \nu \max\{L, \lambda\}.$$

Indeed, let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ then

$$|D\mathcal{Z}_\nu(x_1, x_2, 0)|^2 = \sup_{|v|=1} |D\mathcal{Z}_\nu(x_1, x_2, 0)(v)|^2 = \nu^2 \sup_{|v|=1} |v_1 A + v_2 B + v_3 C|^2,$$

where A, B, C are as in the proof of Lemma 4.2.6. Remember now that C is orthogonal to A and B and thus the above equation becomes

$$\begin{aligned}
|D\mathcal{Z}_\nu(x_1, x_2, 0)|^2 &= \nu^2 \sup_{|v|=1} \left(|v_1 A + v_2 B|^2 + |v_3 C|^2 \right) \\
&= \nu^2 \sup_{|v|=1} \left(|Dh(x)(v_1, v_2)|^2 + |v_3|^2 |C|^2 \right) \\
&\leq \nu^2 \sup_{|v|=1} \left(L^2 |(v_1, v_2)|^2 + \lambda^2 |v_3|^2 \right) \\
&\leq \nu^2 \max\{L^2, \lambda^2\},
\end{aligned}$$

where we have used the fact that h is locally bi-Lipschitz.

Hence, $|D\mathcal{Z}_\nu(x)| \leq \nu \max\{L, \lambda\} e^{x_3}$ and (4.13) becomes

$$\begin{aligned}
|\mathcal{Z}_\nu^n(y_1) - \mathcal{Z}_\nu^n(y_2)| &\leq \nu^n \max\{L, \lambda\}^n \sup_{x \in B(0,r)} e^{p_3(\mathcal{Z}_\nu^{n-1}(x))} \cdots \sup_{x \in B(0,r)} e^{p_3(x)} |y_1 - y_2| \\
&= \left(\frac{\max\{L, \lambda\}}{\lambda} \right)^n E_{\nu\lambda}(r) \cdots E_{\nu\lambda}^n(r) |y_1 - y_2|,
\end{aligned}$$

where we have used the fact that $\nu\lambda \sup_{x \in B(0,r)} e^{p_3(\mathcal{Z}_\nu^n(x))} = E_{\nu\lambda}^{n+1}(r)$ which can be easily proved by induction on n. $\square$

Proof of Theorem 4.1.3. First, let us note that by assumption $\nu > \sqrt{\frac{2L}{\lambda}} > \frac{1}{\lambda e}$. Let V be any open and connected set of $\mathbb{R}^3$. We want to show that $\mathcal{Z}_\nu^n(V)$ intersects one of the planes that belong to the Julia set for some n and thus V itself intersects the Julia set. Assume that this does not happen. By Lemma 4.2.10 now we can consider two cases

First Case

Suppose first that the sequence of iterates $\mathcal{Z}_\nu^n(V)$ does not eventually stay inside the square beam $B_{(0,0)} \cup B_{(0,-1)}$ but it also visits their complement infinitely often. Then we can find a subsequence n_j such that $\mathcal{Z}_\nu^{n_j}(V) \in B_{(0,0)} \cup B_{(0,-1)}$ and $\mathcal{Z}_\nu^{n_j+1}(V) \in B_{(k,l)}$ for some $(k, l) \neq (0,0), (0,-1)$. Without loss of generality we may assume that $\mathcal{Z}_\nu^{n_j}(V) \in B_{(0,0)}$.

Consider now the sets

$$V_{n_j}^+ = \{x \in V : |(p \circ \mathcal{Z}_\nu^{n_j})(x)| \geq \frac{\lambda}{2}\}$$

and

$$V_{n_j}^- = \{x \in V : |(p \circ \mathcal{Z}_\nu^{n_j})(x)| < \frac{\lambda}{2}\},$$

(λ is the scale factor by which we scaled up the initial square). Notice that $V = V_{n_j}^+ \cup V_{n_j}^-$. Since $\mathcal{Z}_\nu^{n_j+1}(V)$ is outside of $B_{(0,0)} \cup B_{(0,-1)}$ we will have that $\mathcal{Z}_\nu^{n_j}(V)$ lies between two "level surfaces" S_k and S_{k+1} but we know, from Lemma 4.2.9, that those surfaces enclose $\leq M_0$ volume between them, where M_0 is a constant. Thus $m(\mathcal{Z}_\nu^{n_j}(V)) \leq M_0$, where m denotes the Lebesgue measure. Also by Lemma 4.2.6 it is true that for almost all points in $V_{n_j}^+$

$$\det(D\mathcal{Z}_\nu^{n_j}) \geq \left(\frac{\lambda}{L^5}\right)^{n_j} \frac{1}{\lambda^3}|(p \circ \mathcal{Z}_\nu^{n_j})(x)|^3 \geq \left(\frac{\lambda}{L^5}\right)^{n_j} \frac{1}{8}.$$

Hence, because $\mathcal{Z}_\nu^{n_j}$ is a homeomorphism in V we have that

$$M_0 \geq m(\mathcal{Z}_\nu^{n_j}(V_{n_j}^+)) = \int_{V_{n_j}^+} |\det(D\mathcal{Z}_\nu^{n_j})| dm \geq \left(\frac{\lambda}{L^5}\right)^{n_j} \frac{1}{8} \cdot m(V_{n_j}^+).$$

This implies that $m(V_{n_j}^+) \to 0$ as $n_j \to \infty$. On the other hand, the set $\mathcal{Z}_\nu^{n_j}(V_{n_j}^-)$ is inside the initial square beam $[-\lambda, \lambda]^2 \times \mathbb{R}$. Thus $\mathcal{Z}_\nu^{n_j+1}(V_{n_j}^-)$ lies in the half space $x_3 > 0$ and outside the square beam $B_{(0,0)} \cup B_{(0,-1)}$. This same set also lies between some level surfaces or below all of them. Moreover, as we proved in Lemma 4.2.9 the volume enclosed by those successive surfaces and the volume enclosed by the first one and the plane $x_3 = 0$ is smaller than some constant M_0. Thus $m(\mathcal{Z}_\nu^{n_j+1}(V_{n_j}^-)) \leq M_0$. By arguing the same way as before now we have that

$$M_0 \geq m(\mathcal{Z}_\nu^{n_j+1}(V_{n_j}^-)) = \int_{V_{n_j}^-} |\det(D\mathcal{Z}_\nu^{n_j+1})| dm \geq \left(\frac{\lambda}{L^5}\right)^{n_j+1} \frac{1}{8} \cdot m(V_{n_j}^-).$$

This again implies that $m(V_{n_j}^-) \to 0$ as $n_j \to \infty$. But this is a contradiction since $m(V) = m(V_{n_j}^-) + m(V_{n_j}^+)$.

Second Case

Suppose now that $\mathcal{Z}_\nu^n(V) \in B_{(0,0)} \cup B_{(0,-1)}$, for all $n > N_0$. Observe that either $\mathcal{Z}_\nu(B_{(0,0)}) \subset \{(x_1, x_2, x_3) : x_2 \leq x_1\}$ or $\mathcal{Z}_\nu(B_{(0,0)}) \subset \{(x_1, x_2, x_3) : x_2 \geq x_1\}$. In the first case $\mathcal{Z}_\nu^n(V)$ stays in $B_{(0,0)}$ for all large n or it stays in $B_{(0,-1)}$ while in the second it alternates between $B_{(0,0)}$ and $B_{(0,-1)}$. For simplicity we will assume that the first case holds and thus $\mathcal{Z}_\nu^n(V) \in B_{(0,0)}$, for all $n > N_0$.

Let us now consider the inverse image under $\mathcal{Z}_\nu$ of the boundary of $B_{(0,0)}$, that lies inside $B_{(0,0)}$, namely the surface S_0 we had in the proof of Lemma 4.2.9. Remember that this surface is defined as $S_0 := \{(x_1, x_2, x_3) \in B_{(0,0)} : x_3 = f(x_1, x_2)\}$, where f is continuous on $B_{(0,0)} \cap \{x_3 = 0\}$ and extends continuously on the boundary of this set except at the points $(0, 0)$, $(2\lambda, 0)$, $(0, -2\lambda)$ where $f \to \infty$. Notice then that all the iterates $\mathcal{Z}_\nu^n(V)$ stay below the surface S_0.

Consider now a plane $x_3 = c$, with $c = E_{\nu\lambda}^N(0) - \lambda$ and N so large that this plane intersects this surface S_0 and also

$$(c + \lambda)^{\log(c+\lambda)+1} e^{c+\lambda} \nu^2 \lambda^2 e^{-\frac{\nu\lambda e^c}{2}} \leq \lambda. \tag{4.14}$$

We define now sets A_1, A_2 and A_3 as follows:

- $A_1 := \{(x_1, x_2, x_3) \in B_{(0,0)} : c < x_3 < f(x_1, x_2)$ and (x_1, x_2) in a neighbourhood of $(0, 0)\}$.
- $A_2 := \{(x_1, x_2, x_3) \in B_{(0,0)} : c < x_3 < f(x_1, x_2)$ and (x_1, x_2) in a neighbourhood of $(2\lambda, 0)\}$.
- $A_3 := \{(x_1, x_2, x_3) \in B_{(0,0)} : c < x_3 < f(x_1, x_2)$ and (x_1, x_2) in a neighbourhood of $(0, -2\lambda)\}$.

We now have that:

(i) All those sets lie below (in terms of x_3 coordinate) the surface S_0. By the definitions and Lemma 4.2.9 it is easy to see that the sets A_1, A_2 and A_3 are also of finite Lebesgue measure.

(ii) $\mathcal{Z}_\nu(A_2 \cup A_3) \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < -\frac{\nu\lambda e^c}{2}\}$ and thus $\mathcal{Z}_\nu^2(A_2 \cup A_3) \subset B(0, \delta)$, where $\delta = \nu\lambda e^{-\frac{\nu\lambda e^c}{2}}$. Note that $\delta < \nu\lambda = E_{\nu\lambda}(0)$.

(iii) It is easy to show by induction on N and since $\nu\lambda > 1/e$ that $E_{\nu\lambda}^N(0) \geq E_{\nu\lambda}(N - 1)$ and thus

$$E_{\nu\lambda}(N - 1) \leq c + \lambda \Rightarrow N \leq \log(c + \lambda) + 2. \tag{4.15}$$

By Lemma 4.2.12 and since $\lambda > L^5$ we will now have that for all $x \in B(0, \delta)$

$$\begin{aligned}
|\mathcal{Z}_\nu^N(x) - \mathcal{Z}_\nu^N(0)| \leq & E_{\nu\lambda}(\delta) \cdots E_{\nu\lambda}^N(\delta)|x| \\
\leq & E_{\nu\lambda}^2(0) \cdots E_{\nu\lambda}^{N+1}(0)\delta \\
\leq & (c + \lambda)^{N-1} E_{\nu\lambda}(\lambda + c)\delta \\
\leq & (c + \lambda)^{\log(c+\lambda)+1} \nu\lambda e^{\lambda+c}\delta.
\end{aligned}$$

Hence, by (4.14) we will have that

$$|\mathcal{Z}_\nu^N(x) - \mathcal{Z}_\nu^N(0)| \leq \lambda. \tag{4.16}$$

Equation (4.16) together with (ii) implies that $\mathcal{Z}_\nu^{N+2}(A_2 \cup A_3) \subset B(\mathcal{Z}_\nu^N(0), \lambda)$ and by the choice of c this last ball is contained in $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > c\}$. This implies that the part of $\mathcal{Z}_\nu^{N+2}(A_2 \cup A_3)$ that lies below S_0 is contained in A_1.

Recall that $\mathcal{Z}_\nu^n(V)$ stays below S_0 for $n > N_0$. By Lemma 4.2.11 now we know that there is a point $x_0 \in \mathcal{Z}_\nu^{n_0}(V)$ for some $n_0 > N_0$ such that $x_0 \in A_1$. Take such a point x_0. We now consider the iterates of this point. Let us examine the behaviour of those iterates more carefully. We can assume that A_1 is so close to the x_3-axis that if $y \in A_1$ then $p_3(\mathcal{Z}_\nu(y)) > p_3(y)$ by Lemma 4.2.7(a). Hence the points $\mathcal{Z}_\nu^n(x_0)$ go higher and higher up in the x_3 direction, while at the same time staying in A_1, until at some point the iterate $\mathcal{Z}_\nu^k(x_0)$, for some k, will lie in either A_2 or A_3 thanks to Lemma 4.2.7(b). Without loss of generality assume that $x_1 := \mathcal{Z}_\nu^k(x_0) \in A_2$ and take a small ball around x_1, $B(x_1, r)$ with $B(x_1, r) \subset A_2 \cap \mathcal{Z}_\nu^{n_0+k}(V)$. By what we have said in the previous paragraph now, we will have that $\mathcal{Z}_\nu^{N+2}(B(x_1, r)) \subset A_1$.

However, we know what happens in points inside A_1 when we iterate, they eventually leave A_1. Thus for some $k > N + 2$ we will have that $\mathcal{Z}_\nu^k(B(x_1, r)) \subset A_2 \cup A_3$ since $B(x_1, r)$ is a connected set and the sets A_1, A_2 and A_3 are disjoint. We can then repeat this whole argument, meaning take the set $\mathcal{Z}_\nu^k(B(x_1, r))$ which is now in A_2 or A_3 and thus will be mapped by $\mathcal{Z}_\nu$ to the lower half space $x_3 < -\frac{\nu\lambda e^c}{2}$ and by $\mathcal{Z}_\nu^{N+2}$ inside A_1. Now continue as above and then repeat. Eventually we obtain a sequence $n_j \to \infty$ with

$$\mathcal{Z}_\nu^{n_j}(B(x_1, r)) \subset A_2 \cup A_3.$$

Then by using Lemma 4.2.8 we will have that $m(\mathcal{Z}_\nu^{n_j}(B(x_1, r))) \to \infty$ but that is impossible since $m(A_2 \cup A_3)$ is finite. □

4.3 Escaping set of the Zorich maps

In this section we prove that the escaping set is connected for those Zorich maps, for which Theorem 4.1.3 holds. Note that we assume that $\lambda > L^5$ and $\nu > \sqrt{\frac{2L}{\lambda}}$.

The proof of this theorem closely follows Rempe's proof for the connectivity of the escaping set of the exponential family in [112]. Before we begin with the proof we need to define a few things. First, in this section, for simplicity and without loss of generality we will assume that our Zorich map sends $B_{(0,0)}$ in the half space $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 \le x_1\}$. Let

$$\mathbb{H}_0 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 < x_1 \text{ and } x_2 > -x_1\}$$

and similarly

$$\mathbb{H}_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > x_1 \text{ and } x_2 > -x_1\}$$

$$\mathbb{H}_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > x_1 \text{ and } x_2 < -x_1\}$$

$$\mathbb{H}_3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 < x_1 \text{ and } x_2 < -x_1\}.$$

Also, let $\mathbb{T}_0 = \mathbb{T}_{(0,0)} := B_{(0,0)} \cap \mathbb{H}_0$ and

$$\mathbb{T}_{(i,j)} = \mathbb{T}_{(0,0)} + i(\lambda, \lambda, 0) + j(\lambda, -\lambda, 0), \ \ i, j \in \mathbb{Z}.$$

Note that $\mathbb{T}_1 := \mathbb{T}_{(0,-1)} = B_{(0,-1)} \cap \mathbb{H}_1$, $\mathbb{T}_2 := \mathbb{T}_{(-1,-1)} = B_{(0,-1)} \cap \mathbb{H}_2$ and $\mathbb{T}_3 := \mathbb{T}_{(-1,0)} = B_{(0,0)} \cap \mathbb{H}_3$ and thus $\mathbb{T}_i \subset \mathbb{H}_i$, $i = 0, 1, 2, 3$. Define now $\Lambda_i : \mathbb{H}_i \to \mathbb{T}_i$, $i = 0, 1, 2, 3$ to be the inverse branches of $\mathcal{Z}_\nu$ in $\mathbb{T}_i$. We can extend those maps to $\overline{\mathbb{H}_i} \setminus \{0\}$ for $i = 0, 1, 2, 3$ and again those extended maps are injective. We will use the same symbols, Λ_i to denote these extended maps.

Now take $\gamma_0 := \{(0, 0, x_3) : x_3 < 0\}$ and inductively define

$$\gamma_k := \Lambda_0(\gamma_{k-1}),$$

for all $k \geq 1$. Each of the sets γ_k, $k \geq 1$, is an injective curve inside $\overline{\mathbb{T}_0}$.

We define now the set Γ_0 by

$$\Gamma_0 := \bigcup_{k \geq 0} \gamma_k.$$

Lemma 4.3.1. *If $U \subset \mathbb{R}^3$ is any open set with $U \cap \overline{\Gamma_0} \neq \emptyset$ then there is a $k_0 \in \mathbb{N}$ with $\gamma_k \cap U \neq \emptyset$ for all $k > k_0$. In particular, $\bigcup_{k \geq k_0} \gamma_k$ is dense in $\overline{\Gamma_0}$.*

Proof. Let $x_0 \in \overline{\Gamma_0}$, and U a neighbourhood of this point. We want to show that $\gamma_k \cap U \neq \emptyset$ for all sufficiently large k. We know, from the definition of $\overline{\Gamma_0}$, that there is a point $x_1 \in U \cap \Gamma_0$. This implies that $\mathcal{Z}_\nu^n(x_1)$ belongs to the x_3-axis for all $n \geq N_0$, for some $N_0 \in \mathbb{N}$ and in fact we can assume that $x_2 := \mathcal{Z}_\nu^{N_0}(x_1) \in H_{>M}$, where M is any positive number. Now taking $M > M_0$, where M_0 is the constant we used in Lemma 4.2.2 and applying that lemma for a ball $B(x_2, R) \subset \mathcal{Z}_\nu^{N_0}(U)$ n times we have

$$\mathcal{Z}_\nu^n \left(B(x_2, R) \cap H_{>M}\right) \supset B\left(\mathcal{Z}_\nu^n(x_2), \alpha^{-n} R\right) \cap H_{>E_{\nu\lambda}^n(M)}.$$

For all large enough n now the ball on the right hand side, $B\left(\mathcal{Z}_\nu^n(x_2), \alpha^{-n} R\right)$, intersects the line $\gamma_1 = \{(2\lambda, 0, t) : t \in \mathbb{R}\}$. Hence, for all large enough n, $\mathcal{Z}_\nu^n(U)$ intersects γ_1. Thus for each n large enough there is a point $x_3 \in \gamma_1$ whose backward orbit intersects U itself. This means that U contains a point in γ_k for all large enough k as we wanted. $\square$

Lemma 4.3.2. *The set Γ_0 is connected.*

Proof. Suppose $U \subset \mathbb{R}^3$ is an open set with $U \cap \Gamma_0 \neq \emptyset$ and $\Gamma_0 \cap \partial U = \emptyset$. We show that $\Gamma_0 \subset U$.

By Lemma 4.3.1 we have that $\gamma_k \cap U \neq \emptyset$, for all $k > k_0$. Since γ_k is a connected curve this implies that $\gamma_k \subset U$, for all $k > k_0$. Thus

$$\Gamma_0 \subset \overline{\Gamma_0} = \overline{\bigcup_{k \geq k_0} \gamma_k} \subset \overline{U}.$$

Hence, since $\Gamma_0 \cap \partial U = \emptyset$, we have that $\Gamma_0 \subset U$. $\square$

Similarly now we can define sets Γ_i, for $i = 1, 2, 3$ using this time Λ_i instead of Λ_0 and prove that Γ_i is also connected. This implies that the union $\Gamma := \bigcup_{i=0}^{3} \Gamma_i$ is a connected set. We define now the set

$$Y := \bigcup_{(k,l) \in \mathbb{Z}^2} (\Gamma + k(2\lambda, 2\lambda, 0) + l(2\lambda, -2\lambda, 0)),$$

which is connected since Γ contains the lines $\{(\pm 2\lambda, 0, t) : t \in \mathbb{R}\}$, $\{(0, \pm 2\lambda, t) : t \in \mathbb{R}\}$ and is a subset of $I(\mathcal{Z}_\nu)$ since the iterates of any point eventually land on the x_3-axis. Next we define, inductively, the sets $Y_j \subset I(\mathcal{Z}_\nu)$ by setting $Y_0 = Y$ and $Y_{j+1} = \mathcal{Z}_\nu^{-1}(Y_j) \cup Y_j$.

Lemma 4.3.3. *The sets Y_j are connected for all $j \geq 0$.*

Proof. We will prove this by induction on j. Let us define the inverse branches of $\mathcal{Z}_\nu$. By using the notation we introduced in the first paragraphs of this section define $\Lambda_{k,l} : \mathbb{H}_p \to \mathbb{T}_{(k,l)}$, with $p = 0, 1, 2, 3$ to be the inverse branches of $\mathcal{Z}_\nu$ that take values on the square beams $\mathbb{T}_{(k,l)}$. We also extend those maps to $\overline{\mathbb{H}_p} \setminus \{0\}$ and use the same symbol to denote those extensions. With that notation we have that

$$Y_{j+1} = \bigcup_{(k,l) \in \mathbb{Z}^2} \Lambda_{k,l}(Y_j) \cup Y_j.$$

By the inductive hypothesis now we know that Y_j is connected and because $\Lambda_{k,l}$ is continuous the set $\Lambda_{k,l}(Y_j)$ is also connected. Observe now that the point $x_{n,m} = (2\lambda, 0, 0) + n(2\lambda, 0, 0) + m(0, 2\lambda, 0)$ is inside $Y = Y_0$ and thus in Y_j, for all $n, m \in \mathbb{Z}$ and for all $j \in \mathbb{N}$. Also note that $\mathcal{Z}_\nu(x_{n,m}) = (0, 0, -\nu\lambda)$ or $(0, 0, \nu\lambda)$ which are both points in Y_j. This means that there are m, n depending on k, l such that $x_{n,m} \in \Lambda_{k,l}(Y_j)$. Hence $\Lambda_{k,l}(Y_j) \cap Y_j \neq \emptyset$. This implies that the set $\Lambda_{k,l}(Y_j) \cup Y_j$ is connected. Hence Y_{j+1} is connected as a union of connected sets with non-empty intersections with each other as we wanted. $\square$

Proof of Theorem 4.1.5. Consider the set

$$\bigcup_{j \geq 0} \mathcal{Z}_\nu^{-j}((0, 0, -1)) \subset \bigcup_{j \geq 0} Y_j.$$

The Zorich map is bounded on $\{(x_1, x_2, x_3) : x_3 < 0\}$ and thus it does not have the pits effect (see section 3.3 and [23]). Hence, by [23, Theorem 1.8] we will have that the set $\bigcup_{j\geq 0} \mathcal{Z}_\nu^{-j}((0,0,-1))$ is dense in $\mathcal{J}(\mathcal{Z}_\nu)$, which by Theorem 4.1.3 is $\mathbb{R}^3$, and thus also dense in $I(\mathcal{Z}_\nu)$. Thus the set $\bigcup_{j\geq 0} Y_j$ is a connected dense subset of $I(\mathcal{Z}_\nu)$ which implies that the escaping set itself is connected. $\square$

4.4 Density of periodic points

Proof of Theorem 4.1.4. First let $U_0 = B(x_0, r)$ be a ball centred at $x_0 \in \mathbb{R}^3$ of radius $r > 0$. We seek a periodic point of $\mathcal{Z}_\nu$ in U_0. Without loss of generality we may assume that $\overline{U_0}$ does not intersect any of the planes $x_1 = \pm x_2 + 2\lambda k$, $k \in \mathbb{Z}$.

We will follow the method of [57] where the authors prove that periodic points of a quasiregular version of the sine function are dense on $\mathbb{R}^3$. We will do this by finding an $N \in \mathbb{N}$ and a finite sequence of open sets U_j, $j = 0, \dots, N$ such that

(i) $U_{j+1} \subset \mathcal{Z}_\nu(U_j)$, $0 \leq j \leq N-1$.

(ii) $\mathcal{Z}_\nu$ is a homeomorphism on each U_j for $j \leq N-1$.

(iii) $\overline{U_0} \subset U_N$.

If these conditions are met then we can define a continuous inverse branch $\mathcal{Z}_\nu^{-N} : U_N \to U_0$. Thus by the Brouwer fixed point theorem the map $\mathcal{Z}_\nu^{-N}|_{U_0}$ has a fixed point in U_0.

We will now show how we can construct such a sequence. By Theorem 4.1.3 we know that $\mathcal{Z}_\nu^n(U_0)$ eventually covers $\mathbb{R}^3 \setminus \{0\}$. We set $U_j = \mathcal{Z}_\nu^j(U_0)$ for all j such that $\mathcal{Z}_\nu^j(U_0)$ does not intersect the set $P := \bigcup_{k\in\mathbb{Z}}\{(x_1, x_2, x_3) : x_1 = \pm x_2 + 2k\lambda\}$. Let n_0 be the biggest such j, so that we have defined $U_0, \dots, U_{n_0}$. Then take a point y_1 in $\mathcal{Z}^{n_0+1}(U_0) \cap P$ such that $y_1 \notin B_{\mathcal{Z}_\nu}$ and a ball $B(y_1, r) \subset \mathcal{Z}_\nu^{n_0+1}(U_0) \setminus B_{\mathcal{Z}_\nu}$, where we remind here that $B_{\mathcal{Z}_\nu}$ is the branch set. Set $U_{n_0+1} = B(y_1, r)$. We know that $\mathcal{Z}_\nu(U_{n_0+1})$ intersects one of the planes $x_1 = \pm x_2$ and it is easy to see that $\mathcal{Z}_\nu$ is a homeomorphism on U_{n_0+1}. Assume, without loss of generality that it intersects $x_1 = x_2$ and take $y_2 \in (\mathcal{Z}_\nu(U_{n_0+1}) \cap \{(x_1, x_2, x_3) : x_1 = x_2\}) \setminus B_{\mathcal{Z}_\nu}$. Set $U_{n_0+2} = B(y_2, r_2)$, where $r_2 > 0$ is such that $B(y_2, r_2) \subset \mathcal{Z}_\nu(U_{n_0+1}) \setminus B_{\mathcal{Z}_\nu}$.

Consider the set $V_0 = U_{n_0+2} \cap \{(x_1, x_2, x_3) : x_1 = x_2\}$ which is an open set of the plane $x_1 = x_2$ in the subspace topology. We define the sets V_n by induction as follows. Suppose that V_n has been defined and that $V_n \cap B_{\mathcal{Z}_\nu} = \emptyset$. We consider now two cases:

1. $\mathcal{Z}_\nu(V_n)$ intersects one of the lines $\{(x_1, x_2, x_3) : x_1 = x_2 = 2\lambda k\}$, $k \in \mathbb{Z}$ which are the pre-images of the x_3 axis on the plane $x_1 = x_2$.

2. $\mathcal{Z}_\nu(V_n)$ does not intersect any of those lines.

In the first case, let y_3 be a point in such an intersection. We define V_{n+1} to be an open ball around y_3 in the subspace topology of $x_1 = x_2$ of radius r_3 where r_3 is chosen in such a way that the ball does not contain branch points and such that $V_{n+1} \subset \mathcal{Z}_\nu(V_n)$.

In the second case, we define $V_{n+1} := \mathcal{Z}_\nu(V_n) \cap H_0$, where H_0 is the whole plane $x_1 = x_2$ in case $\mathcal{Z}_\nu(V_n) \cap B_{\mathcal{Z}_\nu} = \emptyset$ and it is an open half plane on $x_1 = x_2$, in any other case, which is defined as follows. Suppose that $\mathcal{Z}_\nu(V_n)$ intersects one of the lines of $B_{\mathcal{Z}_\nu}$ which we call ℓ_1. We set H_0 to be the half plane defined by this line and the property

$$m_2(\mathcal{Z}_\nu(V_n) \cap H_0) \geq \frac{1}{2} m_2(\mathcal{Z}_\nu(V_n)),$$

where m_2 is the 2 dimensional Lebesgue measure on $x_1 = x_2$. Note now that we have inductively defined the sets V_n.

We now claim that case (1) must occur for some n, otherwise notice that by construction $\mathcal{Z}_\nu$ is a homeomorphism on V_n, for all $n \in \mathbb{N}$. Hence,

$$m_2(V_{n+1}) = m_2(\mathcal{Z}_\nu(V_n) \cap H_0) \geq \frac{1}{2} m_2(\mathcal{Z}_\nu(V_n)). \tag{4.17}$$

Using the notation of section 4.2.1 we now have that

$$m_2(\mathcal{Z}_\nu(V_n)) = m_2((\phi^{-1} \circ g \circ \phi)(V_n)) = C m_2(g(\phi(V_n))),$$

where $C = |\det D\phi^{-1}|$ which is a constant since ϕ is linear. Combining with equation (4.17) this gives

$$m_2(V_{n+1}) \geq \frac{C}{2} m_2(g(\phi(V_n))).$$

Thus by Lemma 4.2.5 we have that $m_2(V_n) \to \infty$, as $n \to \infty$. This implies, just like in the proof of Theorem 4.2.3, that there is an m_0 such that V_{m_0} intersects the x_3 axis.

We now set U_{n_0+2+i} to be the open set

$$U_{n_0+2+i} := \bigcup_{x \in V_i} B(x, r_x),$$

where r_x is such that $B(x, r_x) \subset \mathcal{Z}_\nu(U_{n_0+1+i})$ and $B(x, r_x) \cap B_{\mathcal{Z}_\nu} = \emptyset$, for all $1 \leq i \leq m_0$.

Notice that the sets U_{n_0+2+i} satisfy the properties (i) and (ii). We have that $U_{n_0+2+m_0}$ intersects the x_3 axis, so let

$$U_{n_0+3+m_0} = \mathcal{Z}_\nu(U_{n_0+2+m_0}) \cap B_{(0,0)}.$$

Define also

$$Q_{(0,0)} = \{(x_1, x_2) : |x_1| + |x_2| < 2\lambda\},$$

and

$$Q_{(k,l)} = Q_{(0,0)} + k(2\lambda, 2\lambda) + l(2\lambda, -2\lambda), \;\; k, l \in \mathbb{Z}.$$

We now set

$$U_{n_0+2+m_0+j} = \mathcal{Z}_\nu(U_{n_0+1+m_0+j}) \cap B_{(0,0)},$$

for all $2 \leq j \leq m_1$, where m_1, depending on $M > 0$, is so large that $U_{n_0+2+m_0+m_1}$ contains a set of the form $Q_0 \times [R, R+M]$, where $Q_0 = Q_{(0,0)} \cap \{(x_1, x_2) : x_1 < x_2\}$ and some $R > 0$. We know that such an m_1 exists because the iterated image, under the Zorich map, of an open set that intersects the x_3-axis eventually contains a ball of radius as large as we want (see Lemma 4.2.2).

If M is large enough then $\mathcal{Z}_\nu(U_{n_0+2+m_0+m_1})$ will contain a set of the form

$$U_{n_0+3+m_0+m_1} := Q_{(k,l)} \times [-t_M, t_M],$$

for some $k, l \in \mathbb{Z}$ and $t_M \to \infty$, as $M \to \infty$. Note that $\mathcal{Z}_\nu$ is a homeomorphism on $U_{n_0+2+m_0+j}$ for all $2 \leq j \leq m_1 + 1$ and that $\mathcal{Z}_\nu(U_{n_0+3+m_0+m_1})$ will be the set

$$U_N := \{x \in \mathbb{R}^3 : \nu\lambda e^{-t_M} < |x| < \nu\lambda e^{t_M}\} \setminus W,$$

where $W = \{(x_1, x_2, x_3) : x_1 = \pm x_2, \;\; x_3 \leq 0\}$. If M is large enough U_N will contain the closure of our initial set U_0, since U_0 does not intersect any of the planes $x_1 = \pm x_2 + 2\lambda k$, $k \in \mathbb{Z}$ and we are done. $\square$

4.5 A theorem on measurable dynamics of Zorich maps

In this section we assume that $\nu = 1$ and λ as in Theorem 4.1.3. We will make some remarks on the Lebesgue measure of some sets. First we need to introduce symbolic dynamics. In order to do that we partition $\mathbb{R}^3$ in the rectangular beams

$$T_{(i,j)} = T_{(0,0)} + 2i(\lambda, \lambda, 0) + 2j(\lambda, -\lambda, 0),$$

where $i, j \in \mathbb{Z}$ and

$$\begin{aligned} T_{(0,0)} = B_{(0,0)} \cup B_{(0,-1)} &\cup \{(x_1, x_2, x_3) : x_1 = x_2 - 2\lambda, -2\lambda < x_1 \leq 0\} \\ &\cup \{(x_1, x_2, x_3) : x_1 = -x_2 + 2\lambda, 0 \leq x_1 < 2\lambda\} \\ &\cup \{(x_1, x_2, x_3) : x_1 = x_2, -\lambda < x_1 < \lambda\}. \end{aligned}$$

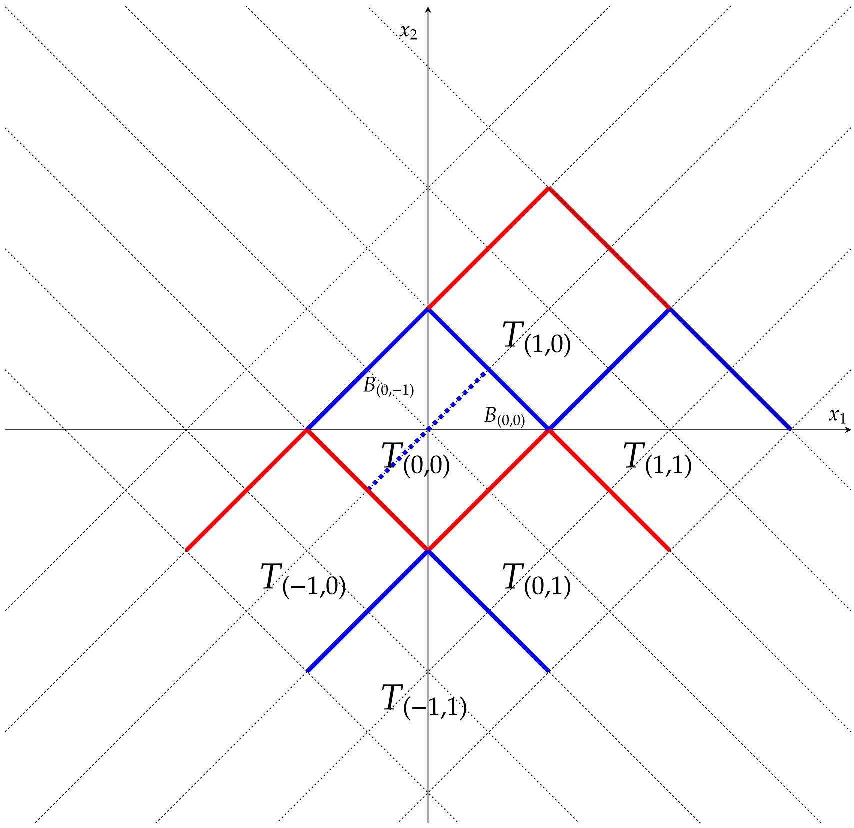

Figure 4.6: The squares that the rectangular beams that partition $\mathbb{R}^3$ create in $\mathbb{R}^2 \times \{0\}$.

In figure 4.5 we have drawn the squares that this partition creates in $\mathbb{R}^2 \times \{0\}$.

For each point $x \in \mathbb{R}^3$ we associate a sequence on $\mathbb{Z} \times \mathbb{Z}$, $S(x) := (s_1, s_2, \dots)$ which we call its *itinerary* and the $s_k = (s_{k,1}, s_{k,2})$ are chosen such that $\mathcal{Z}^k(x) \in T_{s_k}$. We denote the space of all sequences by Σ so S is a map from $\mathbb{R}^3$ to Σ. This procedure of course can be done to the exponential map in a similar manner. Consider now the set of all points with a given itinerary $(s_1, s_2, \dots)$. Namely the set

$$\{x \in \mathbb{R}^3 : S(x) = (s_1, s_2, \dots)\}.$$

Ghys, Sullivan and Goldberg in [59] proved that the analogous set for the exponential map has Lebesgue measure zero. With a bit more work we can see that in our proof of Theorem 4.1.3 we have actually proven the same result for the Zorich maps. Phrasing it in the same way as in [59] we have proven

Theorem 4.5.1. *The fibers of the map S have Lebesgue measure zero.*

Proof. Let V be a set with $m(V) > 0$ and all points in V have the same itinerary s. Remember that the planes $x_1 = \pm x_2$ together with their parallel translates form a forward invariant set so any point in V which lands on one of those planes stays on those planes. Those points will have zero Lebesgue measure since the planes have zero Lebesgue measure and quasiregular maps have Luzin's N property (see [118, I.Proposition 4.14]). Hence, we can assume that V does not contain such points and it always stays on the interior of the square beams under iteration. Thus we find ourselves in the same two cases as in the proof of Theorem 4.1.3. Note here that Lemmas 4.2.8, 4.2.10 require the set V to be connected. However, it is easy to see in their proofs that this hypothesis can be weakened to all points in V have the same itinerary which is exactly what we have here.

The first case now of Theorem 4.1.3 is exactly the same. Assuming that points in V have an itinerary in which we can find a subsequence s_{n_k} with $s_{n_k} \neq (0,0)$ we arrive at a contradiction due to the fact that $m(V) > 0$.

On the second case we assume that the itinerary of points in V is eventually $(0,0)$ and without losing generality in fact equal to $((0,0),(0,0),\dots)$. We may assume that all points in V are density points since by Lebesgue's density theorem this is true for almost all points. Thus if $x \in V$ then we know that

$$\frac{m(B(x,\varepsilon) \cap V)}{m(B(x,\varepsilon))} > 0,$$

for all $\varepsilon > 0$.

We now claim that $m\left(B(x,\varepsilon) \cap V\right) > 0$, for all $\varepsilon > 0$ small enough if and only if $m(B(\mathcal{Z}(x),\varepsilon) \cap \mathcal{Z}(V)) > 0$ for all $\varepsilon > 0$ small enough. Indeed, this follows by Lusin's N property and the fact that the Zorich map is locally invertible in $\mathcal{Z}(V)$.

This implies that all points in $\mathcal{Z}^n(V)$ have the property $m(B(y,\varepsilon) \cap \mathcal{Z}^n(V)) > 0$ for all $\varepsilon > 0$ small enough. Hence, by Lemma 4.2.11, we may assume that x lies in A_1 (otherwise just consider an iterate of V and rename that as V) and fix a small ε so that

$$U := B(x,\varepsilon) \cap V \subset A_1.$$

We can now repeat the argument in the proof of the second case of Theorem 4.1.3 and conclude that there is a subsequence n_j with $\mathcal{Z}^{n_j}(U) \subset A_2 \cup A_3$ but $m(\mathcal{Z}^{n_j}(U)) \to \infty$ which is a contradiction due to the fact that $m(A_2 \cup A_3) < \infty$. $\square$

4.6 Generalized Zorich Maps

In this section we discuss a more general construction of Zorich maps. The goal of this section is to sketch how to prove Theorem 4.1.6 by following the same methods we used in the proof of Theorem 4.1.3 and highlight the most significant differences

between the two cases.
We start again with the square

$$Q = \Big\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \le 1, |x_2| \le 1\Big\}.$$

We suppose that the L bi-Lipschitz function $\mathfrak{h}_{gen} : Q \to \mathbb{R}^3$ maps this square to a surface $\mathcal{S}$ which satisfies the following:

1. The surface lies in the half space $\{(x_1, x_2, x_3) : x_3 \ge 0\}$.

2. The boundary of $\mathcal{S}$ lies on the plane $x_3 = 0$.

3. The ray that connects $(0, 0, 0)$ with $\mathfrak{h}_{gen}(x)$, $x \in Q$, intersects the surface $\mathcal{S}$ only at $\mathfrak{h}_{gen}(x)$.

4. There is a $\theta_{\mathcal{S}} \in (0, \pi/2)$ and $\varepsilon > 0$ such that for all points $w, z \in \mathcal{S}$ such that $|w - z| \le \varepsilon$ the acute angle between the lines connecting 0 with z and w with z is greater than $\theta_{\mathcal{S}}$. We will call this property the *non-tangential position vector* property.

5. $\min_{x\in Q} |\mathfrak{h}_{gen}(x)| > 0$.

Remark. We make two observations on the non-tangential position vector property that we are going to need later.

First we note that it implies that for all points $x \in \mathcal{S}$ for which a tangent plane to $\mathcal{S}$ is defined at x (we know that this includes Lebesgue almost all points of $\mathcal{S}$) the angle between the vector $\mathfrak{h}_{gen}(x)$ and the plane is at least $\theta_{\mathcal{S}}$.

Second, consider any straight line segment inside Q, which we can parametrize by $\phi(t)$, $t \in [0, 1]$ and ϕ linear, and consider $h(\phi([0, 1]))$ which is a curve in $\mathcal{S}$ that admits a tangent line almost everywhere. The non-tangential position vector property now implies that the angle between the vector $h(\phi(t))$ and the tangent line at that point on the surface is again at least $\theta_{\mathcal{S}}$.

We also note here that a similar condition to the non-tangential position vector property was used by Nicks and Sixsmith in [98] on the boundary of a domain in $\mathbb{R}^d$ in order to prove an extension theorem on bi-Lipschitz maps between domains.

Again if $\mathfrak{h}_{gen} = (\mathfrak{h}_{gen,1}, \mathfrak{h}_{gen,2}, \mathfrak{h}_{gen,3})$ we require that $\mathfrak{h}_{gen,1}(x_1, x_1) = \mathfrak{h}_{gen,2}(x_1, x_1)$ and $\mathfrak{h}_{gen,1}(x_1, -x_1) = -\mathfrak{h}_{gen,2}(x_1, -x_1)$. For simplicity we will also assume that

$$\mathfrak{h}_{gen}(0, 0) = (0, 0, 1) \text{ and that } \sup_{x\in Q} |\mathfrak{h}_{gen}(x)| = 1. \tag{4.18}$$

Although the last two conditions are not needed for our methods to work, they make the arguments less arduous and more similar with the arguments we used in the more classical setting.

We also rescale our map $\mathfrak{h}_{gen}$ by defining

$$h_{gen}(x_1, x_2) = \lambda \mathfrak{h}_{gen}\left(\frac{1}{\lambda}(x_1, x_2)\right), \quad (x_1, x_2) \in \lambda Q.$$

We then define

$$\mathcal{Z}_{gen}(x_1, x_2, x_3) = e^{x_3} h_{gen}(x_1, x_2),$$

on $\lambda Q \times \mathbb{R}$ and extend this map to the whole $\mathbb{R}^3$ through reflections.

4.6.1 Proof of Theorem 4.1.6

We are ready now to discuss the proof of Theorem 4.1.6.

First we have to show that the x_3-axis belongs to the Julia set which is proven in exactly the same way as for the spherical Zorich maps (see Proposition 4.2.1) so we omit the proof. Then we have to study our maps in the planes $x_1 = \pm x_2$. Again in those planes our map is conjugate through $\phi(x_1, x_2, x_3) = \frac{1}{\lambda}(x_3 + ix_1)$ to the map

$$\hat{g}(z) := \begin{cases} \hat{\psi}\left(\bar{z} + 2i\right), & \operatorname{Im}(z) \in [(4k+1), (4k+3)] \\ \\ \hat{\psi}(z) & \operatorname{Im}(z) \in [(4k-1), (4k+1)] \end{cases},$$

where $z = x + iy \in \mathbb{C}$, $k \in \mathbb{Z}$ and $\hat{\psi}(x + iy) = e^{\lambda x}\left(\mathfrak{h}_{gen,3}(y, y) + i\mathfrak{h}_{gen,1}(y, y)\right)$. We again set $\mathfrak{a}(y) = \mathfrak{h}_{gen,3}(y, y)$ and $\mathfrak{b}(y) = \mathfrak{h}_{gen,1}(y, y)$.

We can then prove that Theorem 4.2.3 holds in this setting as well.

Lemma 4.6.1. *For $\lambda > \frac{2L^2}{\sin\theta_{\mathcal{S}} \min_{x\in Q}|\mathfrak{h}_{gen}(x)|}$ if V is a connected set of the complex plane with $m(V) > 0$ then $\hat{g}^n(V)$ intersects the real axis for some $n \in \mathbb{N}$.*

This of course implies that the planes $x_1 = \pm x_2$ and all their parallel translate planes $x_1 = \pm x_2 + 2\lambda k$, $k \in \mathbb{Z}$ are in $\mathcal{J}(\mathcal{Z}_{gen})$. Again all those planes partition $\mathbb{R}^3$ in square beams whose boundaries are in the Julia set and in which our Zorich map is a homeomorphism.

We will not give the proof of the above lemma here since it is very similar with the proof of Theorem 4.2.3. The only significant difference in the proof of the above lemma in this more general setting is in the corresponding Lemma 4.2.4 which we prove below.

Lemma 4.6.2.

$$|\det(D\hat{g}(z))| \geq \frac{\sin\theta_{\mathcal{S}} \min_{x\in Q} |\mathfrak{h}_{gen}(x)|\lambda e^{2\lambda \operatorname{Re}(z)}}{2L} \quad a.e.$$

Proof. The only difference with the proof of Lemma 4.2.4 is in finding a lower bound for

$$\left|\det \begin{pmatrix} \mathfrak{a}(y) & \frac{d\mathfrak{a}}{dy}(y) \\ \\ \mathfrak{b}(y) & \frac{d\mathfrak{b}}{dy}(y) \end{pmatrix}\right|.$$

This time we know that the absolute value of the determinant equals

$$|(\mathfrak{a}(y), \mathfrak{b}(y))| \left|\left(\frac{d\mathfrak{a}}{dy}(y), \frac{d\mathfrak{b}}{dy}(y)\right)\right| |\sin\theta(y)|,$$

where $\theta(y)$ is the angle between the vectors $(\mathfrak{a}(y), \mathfrak{b}(y))$ and $(\frac{d\mathfrak{a}}{dy}(y), \frac{d\mathfrak{b}}{dy}(y))$. Hence using the non-tangential position vector property and the fact that

$$\left|\left(\frac{d\mathfrak{a}}{dy}(y), \frac{d\mathfrak{b}}{dy}(y)\right)\right| \geq \frac{1}{\sqrt{2}L} \quad \text{and} \quad |(\mathfrak{a}(y), \mathfrak{b}(y))| \geq \frac{\min_{x\in Q} |\mathfrak{h}_{gen}(x)|}{\sqrt{2}}$$

we have that

$$\left|\det \begin{pmatrix} \mathfrak{a}(y) & \frac{d\mathfrak{a}}{dy}(y) \\ \\ \mathfrak{b}(y) & \frac{d\mathfrak{b}}{dy}(y) \end{pmatrix}\right| \geq \frac{\sin\theta_{\mathcal{S}} \min_{x\in Q} |\mathfrak{h}_{gen}(x)|}{2L}$$

and thus

$$|\det D\hat{g}(z)| \geq \frac{\sin\theta_{\mathcal{S}} \min_{x\in Q} |\mathfrak{h}_{gen}(x)|\lambda e^{2\lambda \operatorname{Re} z}}{2L} \quad \text{a.e.}$$

□

Next we need the Misiurewicz type Lemma 4.2.6 which in this case becomes

Lemma 4.6.3.

$$\det\left(D\mathcal{Z}^n_{gen}(x)\right) \geq \left(\frac{\lambda \min_{x\in Q} |\mathfrak{h}_{gen}(x)| \sin\theta_{\mathcal{S}}}{L^5}\right)^n \frac{1}{\lambda^3} \left|\left(p \circ \mathcal{Z}^n_{gen}\right)(x)\right|^3 \quad a.e.$$

Proof. Again the only difference is in obtaining a lower bound for the determinant $\det D\mathcal{Z}_{gen}(x)$. Define

$$\mathcal{H} = \begin{pmatrix} \frac{\partial h_{gen,1}}{\partial x_1}(p(x)) & \frac{\partial h_{gen,1}}{\partial x_2}(p(x)) & h_{gen,1}(p(x)) \\ \\ \frac{\partial h_{gen,2}}{\partial x_1}(p(x)) & \frac{\partial h_{gen,2}}{\partial x_2}(p(x)) & h_{gen,2}(p(x)) \\ \\ \frac{\partial h_{gen,3}}{\partial x_1}(p(x)) & \frac{\partial h_{gen,3}}{\partial x_2}(p(x)) & h_{gen,3}(p(x)) \end{pmatrix}$$

and set

$$\mathcal{A} = \begin{pmatrix} \frac{\partial h_{gen,1}}{\partial x_1}(p(x)) \\ \\ \frac{\partial h_{gen,2}}{\partial x_1}(p(x)) \\ \\ \frac{\partial h_{gen,3}}{\partial x_1}(p(x)) \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \frac{\partial h_{gen,1}}{\partial x_2}(p(x)) \\ \\ \frac{\partial h_{gen,2}}{\partial x_2}(p(x)) \\ \\ \frac{\partial h_{gen,3}}{\partial x_2}(p(x)) \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} h_{gen,1}(p(x)) \\ \\ h_{gen,2}(p(x)) \\ \\ h_{gen,3}(p(x)) \end{pmatrix}.$$

Then $\det \mathcal{H} = \langle \mathcal{A} \times \mathcal{B}, \mathcal{C} \rangle = |\mathcal{A} \times \mathcal{B}||\mathcal{C}| \cos\phi$, where ϕ is the angle between $\mathcal{A} \times \mathcal{B}$ and $\mathcal{C}$. Using now the fact that $|\mathcal{C}| \geq \lambda \min_{x \in Q} |\mathfrak{h}_{gen}(x)|$ and $|\mathcal{A} \times \mathcal{B}| \geq \frac{1}{L^2}$ together with the non-tangential position vector property we can show that

$$\det \mathcal{H} \geq \frac{\lambda \min_{x \in Q} |\mathfrak{h}_{gen}(x)| \sin\theta_{\mathcal{S}}}{L^2}.$$

Hence

$$\det D\mathcal{Z}_{gen}(x) \geq e^{3x_3} \frac{\lambda \min_{x \in Q} |\mathfrak{h}_{gen}(x)| \sin\theta_{\mathcal{S}}}{L^2}$$

and the rest follows in exactly the same way as in the proof of Lemma 4.2.6. □

Versions of Lemmas 4.2.7, 4.2.8, 4.2.9, 4.2.10, 4.2.11, 4.2.12 now follow with only slight modifications on their proofs. Hence, the proof of Theorem 4.1.6 now follows with the same arguments as the proof of Theorem 4.1.3. Let us briefly sketch how all this should work.

Lemma 4.6.4. *(a) There are $\delta > 0$ and $c > 0$ such that for all $x \in C_\delta$, where C_δ is the cylinder around x_3-axis with δ radius, we have that $p_3(\mathcal{Z}_{gen}(x)) > p_3(x) + c$.*

(b) For δ as in (a) and for every $x \in C_\delta$, with $p(x) \neq (0,0)$, there is an $n \in \mathbb{N}$ such that $\mathcal{Z}^n_{gen}(x) \notin C_\delta$.

Proof. The proof of (a) goes word for word as Lemma 4.2.7. For (b) again the proof is almost the same. The difference here is the lower bound for the angle θ used in the proof of Lemma 4.2.7 where instead of $\frac{\pi}{4}$ is now some constant larger than 0. □

Lemma 4.6.5. *Assume $\lambda > \frac{L^5}{\min_{x \in Q} |\mathfrak{h}_{gen}(x)| \sin\theta_{\mathcal{S}}}$. Let $V \subset \mathbb{R}^3$ be a connected set with $m(V) > 0$ and whose iterates do not intersect any of the planes $x_1 = \pm x_2 + 2k\lambda$, $k \in \mathbb{Z}$. Suppose also that there is a sequence of integers $n_j > 0$ with $\mathcal{Z}^{n_j}_{gen}(V) \cap C_a = \emptyset$, where C_a is a cylinder around x_3-axis of any radius $a > 0$. Then $m(\mathcal{Z}^{n_j}_{gen}(V)) \to \infty$ as $n_j \to \infty$, where m is the 3-dimensional Lebesgue measure.*

Proof. The proof is the same as in Lemma 4.2.8 only now we use Lemma 4.6.3 in place of Lemma 4.2.6. □

In the same way, as for the Zorich map defined using spheres, we can define the surfaces S_n and K_n lying inside the rectangular beams $B_{(0,0)}$ and $B_{(0,-1)}$ respectively. Again those surfaces, together with the plane $x_3 = 0$ and the boundaries of the beams, define sets of finite volume.

The next three lemmas are the corresponding ones to Lemmas 4.2.9, 4.2.10, 4.2.11 respectively. Their proofs almost go word for word with the proofs of the lemmas we just mentioned and are therefore omitted.

Lemma 4.6.6. *Let I_n be the volume that the surface S_n encloses together with the plane $x_3 = 0$ and inside the beam $B_{(0,0)}$. Then I_n is finite for all $n \in \mathbb{N}$. Furthermore, if $T_n := I_{n+1} - I_n$ is the volume between S_n and S_{n+1} then T_n is a decreasing sequence.*

Lemma 4.6.7. *Assume $\lambda > \frac{L^5}{\min_{x\in Q} |\mathfrak{h}_{gen}(x)| \sin \theta_{\mathcal{S}}}$. Let V be a connected subset of $\mathbb{R}^3$ with $m(V) > 0$ and such that $\mathcal{Z}^n_{gen}(V)$ does not intersect any of the planes $x_1 = \pm x_2 + 2k\lambda$, $k \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Then $\mathcal{Z}^n_{gen}(V)$ visits infinitely often one of the two rectangular beams $B_{(0,0)}$, $B_{(0,-1)}$, that have the x_3-axis in their boundary.*

Lemma 4.6.8. *Assume $\lambda > \frac{L^5}{\min_{x\in Q} |\mathfrak{h}_{gen}(x)| \sin \theta_{\mathcal{S}}}$. Let V be a connected set of $\mathbb{R}^3$ with $m(V) > 0$ and such that $\mathcal{Z}^n_{gen}(V)$ does not intersect any of the planes $x_1 = \pm x_2 + 2k\lambda$, $k \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Suppose that there is an $N_0 \in \mathbb{N}$ such that $\mathcal{Z}^n_{gen}(V) \subset B_{(0,0)}$, for all $n > N_0$. Then for all $M > 0$ and $\varepsilon > 0$ there is some $n_0 > N_0$ and a point $x \in \mathcal{Z}^{n_0}_{gen}(V)$ such that $p_3(x) > M$ and $d(x, x_3\text{-axis}) < \varepsilon$, where d is the Euclidean distance.*

The next Lemma is the analogue of Lemma 4.2.12 in this new setting.

Lemma 4.6.9. *Let $y_1, y_2 \in B(0, r)$, where $r > 0$. Then for all $n \in \mathbb{N}$ it is true that*

$$|\mathcal{Z}^n_{gen}(y_1) - \mathcal{Z}^n_{gen}(y_2)| \leq \left(\frac{\sqrt{L^2 + \lambda^2}}{\lambda} \right)^n E_\lambda(r) \cdots E^n_\lambda(r) |y_1 - y_2|,$$

where E_λ denotes the exponential map λe^x.

Proof. The proof almost goes word for word with the proof of Lemma 4.2.12. Note however that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are not orthogonal. Still when estimating $|D\mathcal{Z}_{gen}(x_1, x_2, 0)|$ (see proof of Lemma 4.2.12) we can argue as follows

$$\begin{aligned}
|D\mathcal{Z}_{gen}(x_1, x_2, 0)|^2 &= \sup_{|v|=1} \left(|v_1\mathcal{A} + v_2\mathcal{B} + v_3\mathcal{C}|^2 \right) \\
&\leq \sup_{|v|=1} \left(|v_1\mathcal{A} + v_2\mathcal{B}| + |v_3\mathcal{C}| \right)^2 \\
&\leq \sup_{|v|=1} \left(L|(v_1, v_2)| + \lambda|v_3| \right)^2 \\
&\leq L^2 + \lambda^2
\end{aligned}$$

We also note that to argue here as in the last few lines of the proof of Lemma 4.2.12 we use the two conditions in equation (4.18). □

Proof of Theorem 4.1.6. Let V be any open and connected set of $\mathbb{R}^3$. Assuming that

$$\lambda > C_{h_{gen}} := \frac{\max\{L^5, 2L\}}{\min_{x\in Q} |\mathfrak{h}_{gen}(x)| \sin\theta_{\mathcal{S}}}$$

we want to show that $\mathcal{Z}^n_{gen}(V)$ intersects one of the planes that belong to the Julia set for some n and thus V itself intersects the Julia set.

The proof now proceeds in the same way as the proof of Theorem 4.1.3. We consider the same two cases:

(i) The iterates $\mathcal{Z}^n_{gen}(V)$ do not eventually stay inside the beam $B_{(0,0)} \cup B_{(0,-1)}$. In this case the proof is the same almost word for word.

(ii) The iterates $\mathcal{Z}^n_{gen}(V)$ eventually stay inside $B_{(0,0)} \cup B_{(0,-1)}$. The idea in this case will be the same. We leave the details, which will be slightly different, to the interested reader.

□

4.6.2 Pyramidic Zorich maps

A case of particular interest in the above discussion is when the surface $\mathcal{S}$ is a square based pyramid. In this case we can be much more explicit and define the function $h_{pyr} : \lambda Q \to \mathbb{R}^3$,

$$h_{pyr}(x_1, x_2) := (x_1, x_2, \lambda - \max\{|x_1|, |x_2|\})$$

which sends the square λQ to a pyramid with base λQ and height λ. We then define on $\lambda Q \times \mathbb{R}$

$$\mathcal{Z}_{pyr}(x_1, x_2, x_3) = e^{x_3} h_{pyr}(x_1, x_2)$$

and extend this map to all $\mathbb{R}^3$ in the same way we did with the classical Zorich map.

Also let us mention that in [98] the authors used those kind of Zorich maps to construct a quasiregular function in $\mathbb{R}^3$ which resembles $e^z + z$.

For those maps we can prove, using the same methods, the corresponding result to Theorem 4.1.3 where we have a more explicit value for the scale factor. Again we assume $\nu = 1$ for simplicity.

Theorem 4.6.10. *For $\lambda > 2$ the Julia set $\mathcal{J}(\mathcal{Z}_{pyr})$ is the entire $\mathbb{R}^3$.*

We note that if we use the results of the previous subsection the lower bound we obtain for λ is worse than 2. For this reason we briefly sketch how the value $\lambda > 2$ comes up. To obtain the value 2 as a lower bound for λ we use in a significant way the special form that the bi-Lipschitz map h_{pyr} has.

Following the same reasoning as in the proofs of Theorems 4.1.6 and 4.1.3 we first study the map $\mathcal{Z}_{pyr}$ on the planes $x_1 = \pm x_2$ where the map is conjugate to

$$\hat{g}_{pyr}(z) := \begin{cases} \hat{\psi}\left(\bar{z} + 2i\right), & \mathrm{Im}(z) \in [(4k+1), (4k+3)] \\ \\ \hat{\psi}(z), & \mathrm{Im}(z) \in [(4k-1), (4k+1)] \end{cases},$$

where $z = x + iy \in \mathbb{C}$, $k \in \mathbb{Z}$ and $\hat{\psi}(x+iy) = e^{\lambda x}\left(1 - |y - 4k| + i(y - 4k)\right)$.

First we can show directly, without referring to Lemma 4.6.1, the following.

Lemma 4.6.11. *For $\lambda > 2$ if V is a connected set of the complex plane with $m(V) > 0$ then $\hat{g}^n_{pyr}(V)$ intersects the real axis for some $n \in \mathbb{N}$.*

We note here that the lower bound 2 for λ essentially comes from the analogue of Lemma 4.2.5 in this case.

Next we need the Misiurewicz type Lemma 4.2.6 which in this case, again without referring to Lemma 4.6.3, becomes

Lemma 4.6.12.

$$\det\left(D\mathcal{Z}^n_{pyr}(x)\right) \geq \frac{\lambda^{n-3}}{2\sqrt{2}} \left|\left(p \circ \mathcal{Z}^n_{pyr}\right)(x)\right|^3 \quad a.e.$$

Proof. First note that

$$|p(\mathcal{Z}^n_{pyr}(x))| = e^{x_3(\mathcal{Z}^{n-1}_{pyr})} \cdot |(p \circ h_{pyr} \circ p \circ \mathcal{Z}^{n-1}_{pyr})(x)|,$$

where p is again the projection map. Also, note that if $x = (x_1, x_2) \in Q_{(k,l)}$ then

$$|p \circ h_{pyr}(x)| = \left|((-1)^k(x_1 - 2k\lambda), (-1)^l(x_2 - 2l\lambda)\right| \leq |(x_1, x_2)|.$$

Hence

$$\begin{aligned} |p \circ \mathcal{Z}^n_{pyr}(x)| =& e^{x_3(\mathcal{Z}^{n-1}_{pyr})} \left|\left(p \circ h_{pyr} \circ p \circ \mathcal{Z}^{n-1}_{pyr}\right)(x)\right| \\ \leq& e^{x_3(\mathcal{Z}^{n-1}_{pyr})} \left|\left(p \circ \mathcal{Z}^{n-1}_{pyr}\right)(x)\right| \leq \cdots \\ \leq& e^{x_3(\mathcal{Z}^{n-1}_{pyr})} e^{x_3(\mathcal{Z}^{n-2}_{pyr})} \cdots e^{x_3} |p \circ h_{pyr}| \\ \leq& e^{x_3(\mathcal{Z}^{n-1}_{pyr})} e^{x_3(\mathcal{Z}^{n-2}_{pyr})} \cdots e^{x_3} \sqrt{2}\lambda. \end{aligned} \tag{4.19}$$

Notice also that

$$\det D\mathcal{Z}_{pyr}(x) = \lambda e^{3x_3}.$$

Hence, by (4.19) we have that

$$|p(\mathcal{Z}_{pyr}^{n}(x))|^3 \leq 2\sqrt{2}\frac{\lambda^3}{\lambda^n} \det\left(D\mathcal{Z}_{pyr}(\mathcal{Z}_{pyr}^{n-1}(x))\right) \cdots \det\left(D\mathcal{Z}_{pyr}(x)\right).$$

Which by the chain rule gives us what we wanted. □

Again versions of Lemmas 4.2.7, 4.2.8, 4.2.9, 4.2.10, 4.2.11, 4.2.12 now follow with only slight modifications on their proofs.

Lemma 4.6.13. *(a) For $0 < \delta < \lambda - \frac{1}{e}$ there exists $c > 0$ such that for all $x \in C_\delta$, where C_δ is the cylinder around x_3-axis with δ radius, we have that $x_3(\mathcal{Z}_{pyr}(x)) > x_3(x) + c$.*

(b) For δ as in (a) and for every $x \in C_\delta$, with $p(x) \neq 0$, there is an $n \in \mathbb{N}$ such that $\mathcal{Z}_{pyr}^{n}(x) \notin C_\delta$.

Proof. (a)Note that when $x \in C_\delta$ then

$$x_3(\mathcal{Z}_{pyr}(x)) = e^{x_3}(\lambda - \max\{|x_1|, |x_2|\}) \geq e^{x_3}(\lambda - \delta).$$

The desired inequality follows by minimizing $e^t(\lambda - \delta) - t$.

(b)Suppose that $\mathcal{Z}_{pyr}^{n}(x) \in C_\delta$ for all large n. Then by (a) we would have that $x_3(\mathcal{Z}_{pyr}^{n}(x)) \to \infty$ as $n \to \infty$. Note that

$$|(p \circ \mathcal{Z}_{pyr}^{n+1})(x)| = e^{x_3(\mathcal{Z}_{pyr}^{n})}|\left(p \circ \mathcal{Z}_{pyr}^{n}\right)(x)|.$$

Hence, for large enough n,

$$|(p \circ \mathcal{Z}_{pyr}^{n+1})(x)| \geq 2|\left(p \circ \mathcal{Z}_{pyr}^{n}\right)(x)|,$$

which leads to a contradiction. □

Lemma 4.6.14. *Assume $\lambda > 1$. Let $V \subset \mathbb{R}^3$ be a connected set with $m(V) > 0$ and whose iterates do not intersect any of the planes $x_1 = \pm x_2 + 2k\lambda$, $k \in \mathbb{Z}$. Suppose also that there is a sequence of integers $n_j > 0$ with $\mathcal{Z}_{pyr}^{n_j}(V) \cap C_a = \emptyset$, where C_a is a cylinder around x_3-axis of any radius $a > 0$. Then $m(\mathcal{Z}_{pyr}^{n_j}(V)) \to \infty$ as $n_j \to \infty$, where m is the 3-dimensional Lebesgue measure.*

The surfaces S_n and K_n from Lemma 4.2.9 can again be defined in the same way.

Lemma 4.6.15. *Let I_n be the volume that the surface S_n encloses together with the plane $x_3 = 0$ and inside the beam $B_{(0,0)}$. Then I_n is finite for all $n \in \mathbb{N}$. Furthermore, if $T_n := I_{n+1} - I_n$ is the volume between S_n and S_{n+1} then T_n is a decreasing sequence.*

Lemma 4.6.16. *Assume $\lambda > 1$. Let V be a connected subset of $\mathbb{R}^3$ with $m(V) > 0$ and such that $\mathcal{Z}_{pyr}^n(V)$ does not intersect any of the planes $x_1 = \pm x_2 + 2k\lambda$, $k \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Then $\mathcal{Z}_{pyr}^n(V)$ visits infinitely often the two rectangular beams $B_{(0,0)}$, $B_{(0,-1)}$, that have the x_3-axis in their boundary.*

Lemma 4.6.17. *Assume $\lambda > 1$. Let V be a connected set of $\mathbb{R}^3$ with $m(V) > 0$ and such that $\mathcal{Z}_{pyr}^n(V)$ does not intersect any of the planes $x_1 = \pm x_2 + 2k\lambda$, $k \in \mathbb{Z}$ for all $n \in \mathbb{N}$. Suppose that there is an $N_0 \in \mathbb{N}$ such that $\mathcal{Z}_{pyr}^n(V) \subset B_{(0,0)}$, for all $n > N_0$. Then for all $M > 0$ and $\varepsilon > 0$ there is some $n_0 > N_0$ and a point $x \in \mathcal{Z}_{pyr}^{n_0}(V)$ such that $x_3(x) > M$ and $d(x, x_3\text{-axis}) < \varepsilon$, where d is the Euclidean distance.*

Lemma 4.6.18. *Let $y_1, y_2 \in B(0,r)$, where $r > 0$. Then for all $n \in \mathbb{N}$ it is true that*

$$|\mathcal{Z}_{pyr}^n(y_1) - \mathcal{Z}_{pyr}^n(y_2)| \leq \left(\sqrt{18}\lambda\right)^n E_\lambda(r) \cdots E_\lambda^n(r)|y_1 - y_2|,$$

where E_λ denotes the exponential map λe^x.

Proof. Just like in the proof of Lemma 4.2.12 we can show that

$$\left|\mathcal{Z}_{pyr}^n(y_1) - \mathcal{Z}_{pyr}^n(y_2)\right| \leq \operatorname*{ess\,sup}_{x \in B(0,r)} \left|D\mathcal{Z}_{pyr}\left(\mathcal{Z}_{pyr}^{n-1}(x)\right)\right| \cdots \operatorname*{ess\,sup}_{x \in B(0,r)} |D\mathcal{Z}_{pyr}(x)|\, |y_1 - y_2|.$$

Now note that we now have an explicit formula for $D\mathcal{Z}_{pyr}$ and we can calculate that

$$|D\mathcal{Z}_{pyr}(x)| \leq \sqrt{18}\lambda e^{x_3}.$$

Hence,

$$\begin{aligned}|\mathcal{Z}_{pyr}^n(y_1) - \mathcal{Z}_{pyr}^n(y_2)| &\leq \left(\sqrt{18}\lambda\right)^n \sup_{x \in B(0,r)} e^{x_3(\mathcal{Z}_{pyr}^{n-1}(x))} \cdots \sup_{x \in B(0,r)} e^{x_3(x)}|y_1 - y_2| \\ &= \left(\sqrt{18}\lambda\right)^n E_\lambda(r) \cdots E_\lambda^n(r)|y_1 - y_2|\end{aligned}$$

as we wanted. ☐

Finally the proof of Theorem 4.6.10 follows by combining the above lemmas and arguing as in Theorems 4.1.3 and 4.1.6.

4.7 Preliminaries and main idea of the proof of Theorem 4.1.8

The main idea of the proof of Theorem 4.1.8 is to first show that the Julia set $\mathcal{J}(Z_\nu)$ is a so called *Lelek fan*. Let us first define Lelek fans before we explain why we need this.

Definition 4.7.1 (Fans)**.** Let X be a non-degenerate continuum (compact and connected metric space which is not a single point). Then X is a fan with *top* x_0 when the following conditions are satisfied.

(i) X is hereditarily unicoherent, meaning that $K \cap L$ is connected for every pair of subcontinua K, L.

(ii) X is arcwise connected. This together with (i) implies that X is uniquely arcwise connected.

(iii) x_0 is the only point that is the common endpoint of at least three different arcs that are otherwise disjoint.

Let X be a fan and $x, y \in X$. With $[x, y]$ we will denote the unique arc connecting x and y. Also if a point $x \in X$ is an endpoint of every arc in X containing it then we call this point an *endpoint*. We will use the same symbol, as in the introduction of this section, to denote the set of endpoints $\mathcal{E}(X)$ of a fan.

Definition 4.7.2 (Lelek fans)**.** A fan with top x_0 is called a Lelek fan if it has the following two properties.

(i) **Smoothness:** For any sequence $y_n \in X$ converging to $y \in X$ the arcs $[x_0, y_n]$ converge to $[x_0, y]$ in the Hausdorff metric.

(ii) **Density of endpoints:** The endpoints of X are dense in X.

With this terminology we have the following.

Theorem 4.7.1. *Let $0 < \nu < e^{-(\log L + L)}$. Then $\mathcal{J}(Z_\nu) \cup \{\infty\}$ is a Lelek fan with top at ∞.*

The reason why this theorem is important for what we want to prove is that Lelek in [77] gave an example of a Lelek fan with top $\{x_0\}$ and showed that x_0 is an explosion point for the set of endpoints $\mathcal{E}(X)$. Much later Charatonik and independently Bula and Oversteegen proved the following.

Theorem 4.7.2 (Charatonik, Bula-Oversteegen [28, 31])**.**
Any two Lelek fans are homeomorphic.

The above theorem combined with Theorem 4.7.1 allows us to prove Theorem 4.1.8.

Proof of Theorem 4.1.8. Since, by Theorem 4.7.1, $\mathcal{J}(Z_\nu) \cup \{\infty\}$ is a Lelek fan it follows by Theorem 4.7.2 that it is homeomorphic to the one that Lelek constructed. Hence, because endpoints get mapped to endpoints and the top gets mapped to the top we have that $\mathcal{E}(\mathcal{J}(Z_\nu)) \cup \{\infty\}$ is connected while $\mathcal{E}(\mathcal{J}(Z_\nu))$ is totally separated. □

So the only thing left to prove now is Theorem 4.7.1. For that we will need to construct a topological model for the Julia set for which it will be easier to show that it is a Lelek fan. For the exponential map a topological model for the Julia set is given by the so called *straight brush* which was first introduced by Aarts and Oversteegen in [1]. Following them, we define a three dimensional version of a straight brush.

Definition 4.7.3 (3-d Straight Brush)**.**
A 3-d Straight Brush B is a subset of

$$\{(y, a_1, a_2) \in \mathbb{R}^3 : y \geq 0, (a_1, a_2) \in (\mathbb{R} \setminus \mathbb{Q})^2\}$$

with the following properties:

(i) **Hairiness:** For every $(a_1, a_2) \in \mathbb{R}^2$, there is a $t_{(a_1,a_2)} \in [0, \infty]$ such that $(t, a_1, a_2) \in B$ if and only if $t \geq t_{(a_1,a_2)}$

(ii) **Density:** The set of (a_1, a_2) with $t_{(a_1,a_2)} < \infty$ is dense in $(\mathbb{R} \setminus \mathbb{Q})^2$. Also, for any such (a_1, a_2) there exist sequences $(a_1, a_{n,2})$, $(a_1, b_{n,2})$, $(c_{n,1}, a_2)$, $(d_{n,1}, a_2)$, such that $a_{n,2} \uparrow a_2$, $b_{n,2} \downarrow a_2$, $c_{n,1} \uparrow a_1$, $d_{n,1} \downarrow a_1$. Moreover it is true that $t_{(a_1,a_{n,2})} \to t_{(a_1,a_2)}$ and similarly for the other sequences.

(iii) **Compact Sections:** B is a closed subset of $\mathbb{R}^3$.

In order to prove Theorem 4.7.1 we will need to prove Theorem 4.1.9 first.

Here let us also introduce some notation that we will need later. For $(r_1, r_2) \in \mathbb{Z}^2$ we set

$$P(r_1, r_2) := \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - 2r_1| < 1, |x_2 - 2r_2| < 1\}.$$

For any $c \in \mathbb{R}$ we also define the half space

$$H_{>c} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > c\}.$$

Note now that Z_ν maps $P(r_1, r_2) \times \mathbb{R}$ bijectively onto $H_{>0}$ or $H_{<0}$ depending on whether $r_1 + r_2$ is even or odd. Also we mention here that from Bergweiler's and Nicks' work in [16, 23] it follows that for the values of ν they consider

$$\overline{P(r_1, r_2)} \text{ belongs to the quasi-Fatou set when } r_1 + r_2 = \text{odd} . \tag{4.20}$$

We will also need the notion of the *itinerary* of a point $x \in \mathcal{J}(Z_\nu)$. To each such point we can associate a sequence

$$\Delta(x) = n_0 n_1 n_2 \ldots ,$$

where $n_k = (n_{k,1}, n_{k,2}) \in \mathbb{Z} \times \mathbb{Z}$ and $n_{k,1} + n_{k,2} = \text{even}$, in such a way that

$$Z_\nu^k(x) \in P(n_k) \times \mathbb{R},$$

for all $k \in \mathbb{N}$. That sequence we will call the itinerary of x.

We also continue to define $p : \mathbb{R}^3 \to \mathbb{R}^2$, $p_3 : \mathbb{R}^3 \to \mathbb{R}$ to be the projection maps defined by

$$p(x_1, x_2, x_3) = (x_1, x_2) \;\text{ and }\; p_3(x_1, x_2, x_3) = x_3.$$

Let us note here again that in [16] it is shown that Theorem 4.1.7 holds for all sufficiently small values of the parameter ν without an explicit estimate for those values. In what follows we make this more precise.

Lemma 4.7.3.

$$\frac{\nu e^{x_3}}{L} \leq \ell\left(DZ_\nu(x_1, x_2, x_3)\right) \leq |DZ_\nu(x_1, x_2, x_3)| \leq \nu L e^{x_3} \;\; a.e. \tag{4.21}$$

Proof. We assume that $(x_1, x_2) \in Q$ since the other case can be handled similarly. We note that

$$DZ_\nu(x) = e^{x_3} DZ_\nu(x_1, x_2, 0). \tag{4.22}$$

Then $DZ_\nu(x_1, x_2, 0)$ is the linear map induced by the matrix

$$\nu \begin{pmatrix} \frac{\partial \mathfrak{h}_1}{\partial x_1}(p(x)) & \frac{\partial \mathfrak{h}_1}{\partial x_2}(p(x)) & \mathfrak{h}_1(p(x)) \\ \\ \frac{\partial \mathfrak{h}_2}{\partial x_1}(p(x)) & \frac{\partial \mathfrak{h}_2}{\partial x_2}(p(x)) & \mathfrak{h}_2(p(x)) \\ \\ \frac{\partial \mathfrak{h}_3}{\partial x_1}(p(x)) & \frac{\partial \mathfrak{h}_3}{\partial x_2}(p(x)) & \mathfrak{h}_3(p(x)) \end{pmatrix},$$

where $\mathfrak{h} = (\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$ is the bi-Lipschitz map we used in the construction of the Zorich map.

We now set

$$A = \begin{pmatrix} \frac{\partial \mathfrak{h}_1}{\partial x_1}(p(x)) \\ \\ \frac{\partial \mathfrak{h}_2}{\partial x_1}(p(x)) \\ \\ \frac{\partial \mathfrak{h}_3}{\partial x_1}(p(x)) \end{pmatrix}, \quad B = \begin{pmatrix} \frac{\partial \mathfrak{h}_1}{\partial x_2}(p(x)) \\ \\ \frac{\partial \mathfrak{h}_2}{\partial x_2}(p(x)) \\ \\ \frac{\partial \mathfrak{h}_3}{\partial x_2}(p(x)) \end{pmatrix}, \quad C = \begin{pmatrix} \mathfrak{h}_1(p(x)) \\ \\ \mathfrak{h}_2(p(x)) \\ \\ \mathfrak{h}_3(p(x)) \end{pmatrix}$$

and $v=(v_1,v_2,v_3)\in\mathbb{R}^3$. Then by using (4.22)

$$\ell\left(DZ_\nu(x)\right)^2=\inf_{|v|=1}|DZ_\nu(x)(v)|^2=e^{2x_3}\nu^2\inf_{|v|=1}|v_1A+v_2B+v_3C|^2.$$

Notice now that C is orthogonal to A and B and thus the above equation becomes

$$\begin{aligned}\ell\left(DZ_\nu(x)\right)^2=&e^{2x_3}\nu^2\inf_{|v|=1}\left(|v_1A+v_2B|^2+|v_3C|^2\right)\\=&e^{2x_3}\nu^2\inf_{|v|=1}\left(|Dh(p(x))(v_1,v_2)|^2+|v_3|^2|C|^2\right)\\\geq&e^{2x_3}\nu^2\inf_{|v|=1}\left(\frac{1}{L^2}|(v_1,v_2)|^2+|v_3|^2\right)\\\geq&e^{2x_3}\frac{\nu^2}{L^2}\ \text{a.e.}\end{aligned}$$

Similarly we can show that $|DZ_\nu(x_1,x_2,x_3)|^2\leq\nu^2L^2e^{2x_3}$ so that (4.21) readily follows. $\square$

Lemma 4.7.4. *For all $0<\nu<e^{-(\log L+L)}$ Theorem 4.1.7 is true.*

Proof. First we note that in [16, 23] a different parametrization for the Zorich family is used, namely $\mathscr{Z}_\kappa(x)=Z(x)+(0,0,\kappa)$, $\kappa\in\mathbb{R}$. Moreover, Theorem 4.1.7 is shown to hold for all

$$\kappa\leq M_2-e^{M_1},$$

where $M_1:=M_1(\alpha)>0$ is such that $x_3\geq M_1$ implies $\ell\left(DZ(x_1,x_2,x_3)\right)\geq\frac{1}{\alpha}$, $M_2:=M_2(\alpha)$ is such that $x_3\leq M_2$ implies that $|DZ(x_1,x_2,x_3)|\leq\alpha$ and α is any number in $(0,1)$ (see [16, (1.4), (1.5)]). It is easy to see now using Lemma 4.7.3, for $\nu=1$, that we can take

$$M_1=\log\frac{L}{\alpha}\ \text{ and }\ M_2=\log\frac{\alpha}{L}.$$

Of course the two parametrizations are conjugate with $\kappa=\log\nu$. Converting the facts of the above paragraph in our setting we obtain that Theorem 4.1.7 holds for

$$0<\nu\leq e^{M_2-e^{M_1}}=e^{\log\left(\frac{\alpha}{L}\right)-\frac{L}{\alpha}}.$$

Taking $\alpha\to1$ we obtain that Theorem 4.1.7 is true for all $0<\nu<e^{-(\log L+L)}$. $\square$

Later we will also need the fact that the Zorich maps are expansive in a suitable upper half space. This is analogous to the fact that the exponential map is expansive in a right half plane. The next lemma makes this precise.

Lemma 4.7.5. *For any $0<\nu<e^{-(\log L+L)}$ there is an $0<\alpha<1$ such that if $M=\log\frac{L}{\nu\alpha}$ and $\Lambda:H_{\geq M}\to P(r_1,r_2)$, for some $(r_1,r_2)\in\mathbb{Z}^2$, is an inverse branch of the Zorich map Z_ν defined in $H_{\geq M}$, then $Z_\nu(H_{\leq M})\subset H_{<M}$ and for all $x,y\in H_{\geq M}$*

$$|\Lambda(x)-\Lambda(y)|\leq\alpha|x-y|.\tag{4.23}$$

Proof. First note that

$$D\Lambda(x) = DZ_\nu(\Lambda(x))^{-1}.$$

Using the finite increment theorem (see [142, 10.4.1, Theorem 1] and the proof of Lemma 4.2.12) for Λ we can now show that if γ is the segment that connects x and y then

$$|\Lambda(x) - \Lambda(y)| \le \operatorname*{ess\,sup}_{z\in\gamma} |D\Lambda(z)||x-y|. \tag{4.24}$$

It is true that

$$\operatorname*{ess\,sup}_{z\in\gamma} |D\Lambda(z)| \le \frac{1}{\operatorname{ess\,inf}_{z\in\gamma} \ell\left(DZ_\nu(\Lambda(z))\right)}. \tag{4.25}$$

By Lemma 4.7.3 we have that

$$\ell\left(DZ_\nu(\Lambda(x))\right) \ge \frac{e^{p_3(\Lambda(x))}\nu}{L} \quad \text{a.e.} \tag{4.26}$$

Hence for any $0 < \alpha < 1$ and for all x such that

$$p_3(\Lambda(x)) \ge \log\frac{L}{\nu\alpha} \tag{4.27}$$

we have $\ell(DZ_\nu(\Lambda(x))) \ge 1/\alpha$.

We now claim that for $M = \log\frac{L}{\nu\alpha}$ we have that for all $x \in H_{\ge M}$ equation (4.27) holds. Indeed if $x = (x_1, x_2, x_3) \in H_{\le M}$ we have that

$$p_3\left(Z_\nu(x)\right) = p_3\left(\nu e^{x_3}\mathfrak{h}(x_1, x_2)\right) \le \nu\frac{L}{\nu\alpha} = \frac{L}{\alpha}. \tag{4.28}$$

Notice now that by continuity we can choose $0 < \alpha < 1$ so that

$$\nu < e^{-(\log L + L)} = \frac{e^{-L}}{L} \le \frac{e^{-\frac{L}{\alpha}}L}{\alpha},$$

which after rearranging and using (4.28) implies that

$$p_3\left(Z_\nu(x)\right) < \log\frac{L}{\nu\alpha}.$$

Hence $Z_\nu(H_{\le M}) \subset H_{<M}$ as we wanted. Moreover, this implies that whenever $x \in H_{\ge M}$ we have that $\Lambda(x) \in H_{\ge M}$ so that (4.27) is true.

Putting everything together we have that for $x \in H_{\ge M}$

$$|D\Lambda(x)| \le \frac{1}{\ell(DZ_\nu(\Lambda(x)))} \le \alpha \ \text{ a.e.}$$

Combined with (4.24) and (4.25) this gives us what we wanted. □

The next lemma introduces a number p_ν which is going to play an important role in the subsequent sections.

Lemma 4.7.6. *Let $0 < \nu < e^{-(\log L+L)}$ and α, M be as in Lemma 4.7.5. Then there is a number $p_\nu > 1$ such that (4.23) holds for all $x, y \in H_{\geq p_\nu}$ and such that all planes $x_3 = c$, where $c > p_\nu + 1$, intersect the Julia set $\mathcal{J}(\mathcal{Z}_\nu)$.*

Proof. From Lemma 4.7.5 we know that $Z_\nu(H_{\leq M}) \subset H_{<M}$ so that $H_{\leq M}$ is in the quasi-Fatou set. Moreover, Theorem 4.1.7 implies that there is a minimum number J such that for all $c \geq J$ the plane $\{(x_1, x_2, x_3) : x_3 = c\}$ intersects the Julia set $\mathcal{J}(Z_\nu)$ and thus $J \geq M$. Also all the points in $H_{<J}$ are in the quasi-Fatou set and converge to a fixed point.

We set

$$p_\nu = \max\{J - 1, M\}.$$

Also notice that

$$p_\nu \geq M = \log \frac{L}{\nu\alpha} \geq \log \frac{L}{e^{-(\log L+L)}\alpha} \geq 2\log L + L - \log\alpha > 1,$$

where in the last inequality we used the fact that $L > 1$ and $0 < \alpha < 1$. □

4.8 Proving Theorem 4.1.9

Our proof will closely follow that of the corresponding result for the exponential map which is due to Aarts and Oversteegen (see [1, Theorem 1.4]).

4.8.1 Construction of the 3-d straight brush

First we will need to define a correspondence between the set $(\mathbb{R} \setminus \mathbb{Q})^2$ and $\prod_{i=0}^{\infty} \mathbb{Z} \times \mathbb{Z}$. This can be done in many ways but let us mention here a method used by Devaney based on Farey trees. In [38, Section 5.3] Devaney finds for every irrational number ζ a sequence of integers $n_0 n_1 n_2 \ldots$ by doing the following. We break the real line into intervals $I_k = (k, k+1)$ for each $k \in \mathbb{Z}$. Then we further subdivide each I_k in intervals I_{kl}, $l \in \mathbb{Z}$ in a certain way and so on. Specifically, assuming that $I_{n_0 n_1 \ldots n_k}$ has been defined we define $I_{n_0 \ldots n_k j}$ as follows. Let

$$I_{n_0 n_1 \ldots n_k} = \left(\frac{a}{b}, \frac{c}{d}\right)$$

and $\frac{p_0}{q_0} = \frac{a+c}{b+d}$. We then define $\frac{p_n}{q_n} = \frac{p_{n-1}+c}{q_{n-1}+d}$ and $\frac{p_{-n}}{q_{-n}} = \frac{p_{-n+1}+a}{q_{-n+1}+b}$, for all $n \in \mathbb{N}$. Finally, define

$$I_{n_0 \ldots n_k j} = \left(\frac{p_j}{q_j}, \frac{p_{j+1}}{q_{j+1}}\right).$$

We refer to [38] for more details but the whole construction implies that the intervals we obtain satisfy the following.

1. $I_{n_0 n_1 \dots n_{k+1}} \subset I_{n_0 n_1 \dots n_k}$.

2. The endpoints of each interval $I_{n_0 n_1 \dots n_k}$ are rational.

3. $\{\zeta\} = \bigcap_{k=1}^{\infty} I_{n_0 n_1 \dots n_k}$ and every irrational can be obtained this way.

In our case for any pair of irrational numbers (a_1, a_2) we can apply this method twice and obtain a pair of sequences $\{(n_{k,1}, n_{k,2})\}_{k\in\mathbb{N}} \in \prod_{i=0}^{\infty} \mathbb{Z}\times\mathbb{Z}$. We also equip $\prod_{i=0}^{\infty} \mathbb{Z}\times\mathbb{Z}$ with the product topology of the spaces $\mathbb{Z}\times\mathbb{Z}$ equipped with the discrete topology. The correspondence between $(\mathbb{R}\setminus\mathbb{Q})^2$ and $\prod_{i=0}^{\infty} \mathbb{Z}\times\mathbb{Z}$ can be shown to be a homeomorphism.

Let now $x \in [0,\infty)$ and $(s_1, s_2) \in \mathbb{Z}^2$ with $s_1 + s_2 = \text{even}$. We define the cubes

$$S(x, s_1, s_2) := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x \le x_3 \le x+1, (x_1, x_2) \in \overline{P(s_1, s_2)}\}.$$

Note that because the Zorich map is doubly periodic with periods $(4,0,0)$, $(0,4,0)$ and because $Z \circ R = Z$, where R is a half-turn around the lines $x_1 = 2n+1, x_2 = 2m+1$, $n, m \in \mathbb{Z}$ the Julia set will be periodic and invariant under R as well. Also note that by (4.20) the Julia set lies in the square beams $P(s_1,s_2)\times\mathbb{R}$ with $s_1+s_2 =$ even and thus when $s_1 + s_2$ is even

$$S(x, s_1, s_2) \cap \mathcal{J}(Z_\nu) \neq \emptyset, \text{ for all } x \ge p_\nu, \tag{4.29}$$

where p_ν was defined in Lemma 4.7.6.

Let us now construct the 3-d Straight Brush B of Theorem 4.1.9 which will be homeomorphic to our Julia set. We will suitably modify the construction done in [1]. Let $x \ge p_\nu$ and $(a_1, a_2) \in (\mathbb{R}\setminus\mathbb{Q})^2$ with the corresponding sequence $\{(n_{k,1}, n_{k,2})\}_{k\in\mathbb{N}\cup\{0\}}$. Set $x_0 = x$ and $R_0(x, a_1, a_2) = S(x, n_{0,1}, n_{0,2})$. By induction on k now we define $R_k = R_k(x, a_1, a_2)$ and x_k. We consider now two cases:

(i) $R_k \neq \emptyset$ and there is a ξ with

$$S(\xi, n_{k+1,1}, n_{k+1,2}) \subset Z_\nu(R_k). \tag{4.30}$$

Let $\xi' = \min\{\xi : \xi \text{ satisfies } (4.30)\}$. If $\xi' \ge p_\nu$, we set $x_{k+1} = \xi'$ and

$$R_{k+1} = S(\xi', n_{k+1,1}, n_{k+1,2}).$$

If $\xi' < p_\nu$, we set $x_{k+1} = x_k$ and $R_{k+1} = \emptyset$.

(ii) If the case (i) does not apply, we set $x_{k+1} = x_k$ and $R_{k+1} = \emptyset$.

For each $k \in \mathbb{N}$ now we set

$$B_k = B_k(x) := \{y \in R_0 : Z_\nu^j(y) \in R_j, \text{ for } 1 \leq j \leq k\}.$$

The 3-d straight brush is then defined as

$$B := \{(x, a_1, a_2) : R_k \neq \emptyset, \text{ for all } k \in \mathbb{N}\}.$$

The proof that this construction defines a 3-d straight brush and that this is homeomorphic to the Julia set will be now split in several lemmas. First we need to show that the set B we defined is indeed a 3-d straight brush.

Lemma 4.8.1. *B is a 3-d straight brush.*

Proof. First we prove hairiness.

Hairiness.

Let $(x, a_1, a_2) \in B$ with $\{(n_{k,1}, n_{k,2})\}_{k\in\mathbb{N}}$ the corresponding sequence of (a_1, a_2). Suppose that $x < y$ then $(y, a_1, a_2) \in B$. Indeed, by induction $x_k < y_k$ and $R_k(y, a_1, a_2) \neq \emptyset$ for all $k \in \mathbb{N}$. Hence $(y, a_1, a_2) \in B$. Suppose now that $(z, a_1, a_2) \notin B$. Consider the smallest k with $R_k(z, a_1, a_2) \neq \emptyset$ but $R_{k+1}(z, a_1, a_2) = \emptyset$. Then either there is no ξ with

$$S(\xi, n_{k+1,1}, n_{k+1,2}) \subset Z_\nu(R_k(z, a_1, a_2))$$

or there is such a ξ but for the minimal ξ_m we have that $\xi_m < p_\nu$. In both cases the same holds for all y slightly larger than z.

Combining now what we have proven we see that the set $\{t : (t, a_1, a_2) \in B\}$ is of the form $[t_{(a_1,a_2)}, \infty)$.

Secondly, we prove that B is closed.

Compact Sections.

Let $(x, a_1, a_2) \in B^{\complement}$. We will show that $B^{\complement}$ is an open set by constructing an open box S such that $(x, a_1, a_2) \in S \subset \mathbb{R}^3 \setminus B$. The case where $(a_1, a_2) \notin (\mathbb{R} \setminus \mathbb{Q})^2$ is the hardest one and the one we will prove here.

Without loss of generality suppose that $a_1 \in \mathbb{Q}$. Consider the minimal $k \in \mathbb{N}$ such that

$$(a_1, a_2) \notin \bigcup_{i,j\in\mathbb{N}} I_{n_0\ldots n_{k-1}i} \times J_{m_0\ldots m_{k-1}j},$$

where $I_{n_0\ldots n_k}$ and $J_{m_0\ldots m_k}$ are the intervals we obtain using the method we described at the start of this section. There is a n_k such that for all $m \in \mathbb{Z}$, if we denote by ζ and ζ' the irrationals that correspond to the sequences $(n_0, \ldots, n_k, m, \ldots)$ and $(n_0, \ldots, n_k + 1, m, \ldots)$ respectively then

$$\zeta < a_1 < \zeta'. \tag{4.31}$$

We can now choose N large enough so that no point (x, γ_1, γ_2) belongs to B, where γ_1 satisfies

$$(n_0, \ldots, n_k, N, \ldots) < \gamma_1 < (n_0, \ldots, n_k + 1, -N, \ldots), \tag{4.32}$$

$\gamma_2 \in \mathbb{R}$ and we have abused the notation in the obvious way. The reason we can do this is because for such (γ_1, γ_2) we have that $R_{k+1}(x, \gamma_1, \gamma_2) = \emptyset$. Notice that by (4.31) we have that a_1 satisfies (4.32). It follows that there is a $\delta > 0$ such that for all y with $|y - x| < \delta$ we have that $R_{k+1}(y, \gamma_1, \gamma_2) = \emptyset$, for all γ_1 satisfying equations (4.32) and $\gamma_2 \in \mathbb{R}$.

Finally we prove density.

Density.

We want to show that

$$A := \{(a_1, a_2) : t_{a_1, a_2} < \infty\}$$

is dense in $(\mathbb{R} \setminus \mathbb{Q})^2$. It is enough to show that the corresponding sequences in $\prod_{i=0}^{\infty} \mathbb{Z}^2$ of such points are dense in the space of all sequences. We know that the periodic sequences, meaning sequences with $(n_{k,1}, n_{k,2}) = (n_{k+N,1}, n_{k+N,2})$, for some $N \in \mathbb{N}$ and all $k \in \mathbb{N}$, are dense in the space of all sequences. Hence, it is enough to prove that periodic sequences correspond to points in A. Indeed, for any periodic sequence it not hard to see that when x is large enough the sequence x_k is increasing and $R_k(x, a_1, a_2) \neq \emptyset$ for $k \leq N + 1$ and as a result, since the sequence $(n_{k,1}, n_{k,2})$ is periodic, for all $k \in \mathbb{N}$. This implies that $t_{(a_1, a_2)} < \infty$ for that (a_1, a_2).

Suppose now that $(x, a_1, a_2) \in B$ and (a_1, a_2) has a corresponding sequence $\{(n_{k,1}, n_{k,2})\}_{k \in \mathbb{N}}$. Choose y_i with $x < y_i < x + \frac{1}{i}$, for all $i \in \mathbb{N}$. Then for all i the inner radius of the shell $Z_\nu(R_{k-1}(y_i, a_1, a_2))$ will be much larger than the inner radius of $Z_\nu(R_{k-1}(x, a_1, a_2))$, for k large enough. It follows that $(y_i, a_1, a_{i,2}) \in B$ where $(a_1, a_{i,2})$ is chosen so that it has a corresponding sequence

$$(n_{0,1}, n_{0,2}), \ldots (n_{i,1}, n_{i,2} - 1), (n_{i+1,1}, n_{i+1,2}), (n_{i+2,1}, n_{i+2,2}), \ldots$$

and thus $a_{i,2}$ is an increasing sequence. Hence, $y_i \to x$ while $a_{i,2} \uparrow a_2$ as $i \to \infty$. Note that closedness of B implies now that $t_{(a_1, a_{n,2})} \to t_{(a_1, a_2)}$.

Similarly we will have that $(y_i, a_1, b_{i,2})$, $(y_i, c_{i,1}, a_2)$, $(y_i, d_{i,1}, a_2) \in B$, where $(a_1, b_{i,2})$, $(c_{i,1}, a_2)$, $(d_{i,1}, a_2)$ have corresponding sequences

$$(n_{0,1}, n_{0,2}), \ldots (n_{i,1}, n_{i,2} + 1), (n_{i+1,1}, n_{i+1,2}), (n_{i+2,1}, n_{i+2,2}), \ldots,$$

$$(n_{0,1}, n_{0,2}), \dots (n_{i,1} - 1, n_{i,2}), (n_{i+1,1}, n_{i+1,2}), (n_{i+2,1}, n_{i+2,2}), \dots,$$

$$(n_{0,1}, n_{0,2}), \dots (n_{i,1} + 1, n_{i,2}), (n_{i+1,1}, n_{i+1,2}), (n_{i+2,1}, n_{i+2,2}), \dots$$

respectively and $b_{i,2}$ is decreasing, $c_{i,1}$ is increasing and $d_{i,1}$ is decreasing.

□

Next comes the construction of a suitable homemorphism φ from B to the Julia set $\mathcal{J}(Z_\nu)$.

4.8.2 Proving that B is homeomorphic to $\mathcal{J}(Z_\nu)$

For each $(x, a_1, a_2) \in B$ we define

$$\varphi(x, a_1, a_2) := \bigcap_{k=0}^{\infty} B_k.$$

Note that $B_k \subset \cdots \subset B_0$, for all $k \in \mathbb{N}$ and also that $\operatorname{diam}(B_k) \to 0$ since all boxes $R_k(x, a_1, a_2)$ are inside the half space $\{(x_1, x_2, x_3) : x_3 \geq p_\nu\}$ on which, by Lemma 4.7.5 and by Lemma 4.7.6, the Zorich map is expanding. Thus

$$\operatorname{diam}(B_k) = \sup_{z,y \in R_k(x,a_1,a_2)} \left|\Lambda^k(z) - \Lambda^k(y)\right| \leq \alpha^k \operatorname{diam}(R_k(x, a_1, a_2)).$$

Lemma 4.8.2. *φ is an injective continuous map from B into the Julia set $\mathcal{J}(Z_\nu)$.*

Proof. Let us first show that φ is injective. Let $(x, a_1, a_2), (y, b_1, b_2)$ be two different points of B and let $\{(n_{k,1}, n_{k,2})\}_{k\in\mathbb{N}}$ and $\{(m_{k,1}, m_{k,2})\}_{k\in\mathbb{N}}$ their corresponding sequences. Now either $x \neq y$ or $x = y$ and $(a_1, a_2) \neq (b_1, b_2)$. In the first case we may assume $x < y$ and thus $x_k < y_k$ for all $k \in \mathbb{N}$. We can easily see by the mapping properties of the Zorich map that in fact $y_k - x_k \to \infty$ for $k \to \infty$. Thus for large enough k we can find cubes $R_k(x, a_1, a_2) \cap R_k(y, b_1, b_2) = \emptyset$, which implies that $\varphi(x, a_1, a_2) \neq \varphi(y, b_1, b_2)$. In the second case, we have that $(n_{k,1}, n_{k,2}) \neq (m_{k,1}, m_{k,2})$ for some $k \in \mathbb{N}$ and thus $R_k(x, a_1, a_2) \cap R_k(y, b_1, b_2) = \emptyset$.

Now for the continuity of φ let (x, a_1, a_2) and (y, b_1, b_2) be two points in B that are "close" meaning that their corresponding sequences satisfy $n_{k,i} = m_{k,i}$, $i = 1, 2$, for $k = 0, \dots, N$ where $N \in \mathbb{N}$ and $|x - y| < \delta$ for some small $\delta > 0$. If δ is small enough this implies that

$$R_k(x, a_1, a_2) \cap R_k(y, b_1, b_2) \neq \emptyset, \text{ for all } k \leq N.$$

Hence, $B_k(x) \cap B_k(y) \neq \emptyset$ for all $k \leq N$. This implies that

$$|\varphi(x, a_1, a_2) - \varphi(y, b_1, b_2)| \leq \operatorname{diam} B_N(x) + \operatorname{diam} B_N(y)$$

and because those diameters tend to zero we have that when N is large enough $\varphi(x, a_1, a_2)$ and $\varphi(y, b_1, b_2)$ are close.

Finally, we want to show that $\varphi(x, a_1, a_2) \in \mathcal{J}(Z_\nu)$. If $\{(n_{k,1}, n_{k,2})\}_{k\in\mathbb{N}}$ is the corresponding sequence of (a_1, a_2) then, by construction and (4.29), we will have that

$$R_k(x, a_1, a_2) \cap \mathcal{J}(Z_\nu) \neq \emptyset, \text{ for all } k \in \mathbb{N}.$$

This implies that $d\left(\varphi(x, a_1, a_2), \mathcal{J}(Z_\nu)\right) = 0$ and because $\mathcal{J}(Z_\nu)$ is closed we have that $\varphi(x, a_1, a_2) \in \mathcal{J}(Z_\nu)$. □

Next we need to find the inverse of φ and show that it is continuous. Let us first define this function and then show that indeed it is the inverse of φ.

Let $w = (w_1, w_2, w_3) \in \mathcal{J}(Z_\nu)$. Remember now that with each w in the Julia set we can associate its itinerary $\Delta(w) = n_0 n_1 n_2 \ldots$ and $Z_\nu^k(w) \in P(n_k) \times \mathbb{R}$. For any $k \in \mathbb{N}$ define the boxes $T_j(k)$, $j \in \mathbb{N}$ as follows. First we set $T_k(k) = S(u, n_{k,1}, n_{k,2})$, where u is minimal with respect to the properties $u \geq p_\nu$ and $Z_\nu^k(w) \in T_k(k)$. We now define $T_j(k)$ for $j = 0, \ldots, k-1$ inductively as follows. Suppose that $T_j(k)$ has been defined, let $T_{j-1}(k) = S(v, n_{j-1,1}, n_{j-1,2})$, where v is maximal with respect to the property

$$T_j(k) \subset Z_\nu(T_{j-1}(k)). \tag{4.33}$$

Note that a v for which (4.33) holds exists since $u \geq p_\nu > 1$. Moreover, (4.33) thanks to the continuity of Z_ν, implies that the box $T_j(k)$ hits the inner radius of the half-shell $Z_\nu(T_{j-1}(k))$ in exactly one point. Also it implies that the lowest (in terms of x_3 coordinate) side of the box $T_{j-1}(k)$ has a third coordinate at least p_ν for all j. This can be shown by first noting that it is true for $T_{k-1}(k)$ and then we can use induction on j to prove it for all j. To see why it is true for $T_{k-1}(k)$ note that if that was not the case and $p_\nu = J - 1$ (see Lemma 4.7.6) then $T_{k-1}(k)$ would not contain any points in the Julia set and since $T_k(k)$ does (by Lemma 4.7.6) we obtain a contradiction by (4.33) and the invariance of the Julia set. If on the other hand $p_\nu = M$ then the lowest side of the box $T_{k-1}(k)$ would be below M and since $Z_\nu(H_{\leq M}) \subset H_{<M}$ we obtain a contradiction by the maximality condition on (4.33). Thus in any case the lowest (in terms of x_3 coordinate) side of the box $T_{k-1}(k)$ has a third coordinate at least p_ν.

We now define z_k by the condition $T_0(k) = S(z_k, n_{0,1}, n_{0,2})$ and note that $w \in T_0(k)$. This implies that

$$w_3 - 1 \leq z_0 \leq z_k \leq w_3, \tag{4.34}$$

for all $k \in \mathbb{N}$. We can also prove that

$$z_k \leq z_{k+1}. \tag{4.35}$$

This follows by the fact that $T_k(k+1)$ is higher than $T_k(k)$ (in terms of x_3 coordinate) and thus by the inductive construction (4.35) holds.

Finally, we set $z_\infty := \lim_{k\to\infty} z_k$ and define ψ by

$$\psi(w) = (z_\infty, a_1, a_2)\,,$$

where (a_1, a_2) is the pair of irrationals associated with the pair of sequences $\Delta(w)$. The next lemma concludes the proof of Theorem 4.1.9.

Lemma 4.8.3. *The map ψ is the inverse of φ and φ is a homeomorphism. Moreover, φ extends to a homeomorphism between $B \cup \{\infty\}$ and $\mathcal{J}(Z_\nu) \cup \{\infty\}$.*

Proof. We will first show that

$$\varphi \circ \psi = \mathrm{id}_{\mathcal{J}(Z_\nu)}. \tag{4.36}$$

Let $w \in \mathcal{J}(Z_\nu)$ and we continue using notation as above. We define $T_0(\infty) = S(z_\infty, n_{0,1}, n_{0,2})$ and note that $T_0(\infty) = \lim_{k\to\infty} T_0(k)$ in the Hausdorff metric. Define also $T_j(\infty) := \lim_{k\to\infty} T_j(k)$, where the limit is again in the Hausdorff metric. Notice that by construction the box $T_j(\infty)$ hits the inner radius of the half-shell $Z_\nu(T_{j-1}(\infty))$ in exactly one point and that point has a third coordinate at least p_ν. Applying now the construction of φ to the point $\psi(w) = (z_\infty, a_1, a_2)$, where (a_1, a_2) is the pair of irrationals associated with the pair of sequences $\Delta(w)$ and using the notation of the construction of B we have that $x_0 = z_\infty$ and

$$R_k(x, a_1, a_2) = T_k(\infty),$$

for all $k \in \mathbb{N}$. Hence, $\psi(w) \in B$ and

$$\varphi(\psi(w)) = \varphi(z_\infty, a_1, a_2) = \bigcap_{k\in\mathbb{N}} Z_\nu^{-k}(T_k(\infty)),$$

where Z_ν^{-k} is the composition of inverse branches of Z_ν to appropriate square beams (see construction of φ). Now note that by equation (4.34) taking limits we have that $w \in T_0(\infty)$. We can prove similar equations with (4.34), (4.35) for all boxes $T_j(k)$. Hence, we will also have that $Z_\nu^k(w) \in T_k(\infty)$. This implies that

$$w \in \bigcap_{k\in\mathbb{N}} Z_\nu^{-k}(T_k(\infty))$$

and since that intersection contains only one point we have that $\varphi(\psi(w)) = w$.

Equation (4.36) implies now that φ is onto the Julia set and thus a bijection. Since we have already proven that φ is continuous we can now extend φ to a continuous map $\hat{\varphi}$, from $B \cup \{\infty\}$ to the one point compactification $\mathcal{J}(Z_\nu) \cup \{\infty\}$ of

$\mathcal{J}(Z_\nu)$, with the spherical metric by setting $\hat{\varphi}(\infty) = \infty$, since $\varphi(b) \to \infty$ as $b \to \infty$, $b \in B$. We can now use a well known lemma (see for example [125, Theorem 4.17]) and immediately conclude that in fact $\hat{\varphi}$ is a homeomorphism (with the spherical metric) and $\hat{\psi}$, defined as the extension of ψ on $\mathcal{J}(Z_\nu) \cup \{\infty\}$ by setting $\hat{\psi}(\infty) = \infty$, is its inverse. This implies that ψ is continuous with the Euclidean metric. □

4.9 Proving Theorem 4.7.1

First we will show that the model of the Julia set, namely the 3-d straight brush, is a Lelek fan.

Lemma 4.9.1. *The one point compactification $B \cup \{\infty\}$ of a 3-d straight brush B is a Lelek fan with top at ∞.*

Proof. Any subcontinuum of $B \cup \{\infty\}$ is a collection of segments of straight lines together with ∞ or a segment of one such line. Obviously the intersection of any two such subcontinua is again of that form and thus connected. Properties (ii) and (iii) of the definition of a fan are obviously satisfied for $B \cup \{\infty\}$. Smoothness is also quite easy to prove since arcs $[y_n, \infty]$ in $B \cup \{\infty\}$ are straight lines and thus converge to a straight line $[y, \infty]$ in the Hausdorff metric, when $y_n \to y$.

Finally, we prove that the set of endpoints $\mathcal{E}(B)$ is dense in B. Choose a point $(x, b_1, b_2) \in B$. We will show that there are hairs with

$$(t_{b_{1,n}, b_{2,n}}, b_{1,n}, b_{2,n}) \to (x, b_1, b_2), \tag{4.37}$$

as $n \to \infty$. In other words their endpoints converge to (x, b_1, b_2) and this obviously gives us the density of endpoints.

Consider the length function $\mathcal{L} : (\mathbb{R} \setminus \mathbb{Q})^2 \to [0, \infty)$ which measures the spherical length of the hair at (a_1, a_2). It is easy to see that thanks to the closedness of B, $\mathcal{L}$ is upper semi-continuous. Let

$$M := \sup\{\mathcal{L}(a_1, a_2) : (a_1, a_2) \in (\mathbb{R} \setminus \mathbb{Q})^2\}.$$

Take now any $c \in (0, M)$ and consider the set

$$V := \{(a_1, a_2) \in (\mathbb{R} \setminus \mathbb{Q})^2 : \mathcal{L}(a_1, a_2) > c\}.$$

Fix now a point (b_1, b_2) in V and consider the set

$$W := \{(b_1, a_2) \in \overline{V} : |a_2| \leq t, \mathcal{L}(b_1, a_2) > c\},$$

where $t > 0$ a constant.

From the upper semi-continuity of $\mathcal{L}$ we conclude that the set

$$\{(a_1, a_2) \in (\mathbb{R} \setminus \mathbb{Q})^2 : \mathcal{L}(a_1, a_2) \geq c\}$$

is closed and thus contains $\overline{W}$.

The set $\overline{W}$ is compact and lies on a line. By property (ii) of the definition of a 3-d straight brush now we will have that it is also perfect. Moreover, it is a totally disconnected set, since it is a subset of a totally disconnected set. Hence, by the characterization of the Cantor set (see for example [29, Theorem 6.17] or the more general [72, Theorem 7.4]) we have that $\overline{W}$ is a Cantor set (i.e. homeomorphic to the standard ternary Cantor set $\mathcal{C}$). In fact, the sets $\overline{W}$ and $\mathcal{C}$ are ambiently homeomorphic (meaning here that the homeomorphism extends to the line on which $\overline{W}$ lies). This implies that the obvious order in the sets is either preserved or reversed under the homeomorphism. This in turn implies, that a point $(b_1, a_2) \in \overline{W}$ for which $\mathcal{L}(b_1, a_2) > c$ cannot get mapped to an endpoint of the Cantor set since, thanks to property (ii) of a 3-d straight brush, it can be approximated from both sides by other points of $\overline{W}$ (unless of course $a_2 = \pm t$ but those are just two points). Hence, only points with $\mathcal{L}(b_1, a_2) = c$ get mapped to endpoints and since endpoints are dense in the Cantor set so are points with $\mathcal{L}(b_1, a_2) = c$ dense in $\overline{W}$.

Since this argument works for any $(b_1, b_2) \in V$ we have that points with $\mathcal{L}(a_1, a_2) = c$ are dense in $\overline{V}$.

Take now any point $(x, b_1, b_2) \in B$, which is not an endpoint, and let c be the spherical length of the hair segment $\{(t, b_1, b_2) : t \geq x\}$. This implies that $\mathcal{L}(b_1, b_2) > c$. As we have shown then there is a sequence of hairs $(t_{b_{1,n}, b_{2,n}}, b_{1,n}, b_{2,n})$ of length c for which $(b_{1,n}, b_{2,n}) \to (b_1, b_2)$ and thus the endpoints of the hairs $(t_{b_{1,n}, b_{2,n}}, b_{1,n}, b_{2,n})$ converge to (x, b_1, b_2) as we wanted. □

Proof of Theorem 4.7.1. From Theorem 4.1.9 we know that there a homeomorphism

$$\hat{\varphi} : B \cup \{\infty\} \to \mathcal{J}(Z_\nu) \cup \{\infty\}.$$

By Lemma 4.9.1 we know that $B \cup \{\infty\}$ is a Lelek fan. It is quite easy to see that all the properties of a Lelek fan are preserved under a homeomorphism and thus $\mathcal{J}(Z_\nu)$ will also be a Lelek fan with top at ∞. □

4.10 Hairy squares and hairy surfaces

In this section we generalize the notion of a hairy arc that Aarts and Oversteegen first introduced in [1]. Our exposition closely follows theirs but as we shall see some things are different in three dimensions.

First we will introduce the notion of a *straight one-sided hairy square* (abbreviated soshs) which will be a generalization of *straight one-sided hairy arcs* (abbreviated sosha) on the plane (see [1] for that definition). Let $I = [0, 1]$.

Definition 4.10.1 (Straight one-sided hairy square)**.** A straight one-sided hairy square X is a compact subset of I^3 satisfying the following properties. There is a function $\ell : I^2 \to I$, called the length function, such that

(i) For all $(x, y, z) \in I^3$ we have $(x, y, z) \in X$ if and only if $0 \leq z \leq \ell(x, y)$.

(ii) The sets

$$\{(x, y) \in I^2 : \ell(x, y) > 0\},\ \{x \in I : \ell(x, y) = 0, \forall y \in I\}$$

$$\text{and } \{y \in I : \ell(x, y) = 0, \forall x \in I\}$$

are dense in I^2, I and I respectively and $\ell(0, t) = \ell(1, t) = \ell(t, 0) = \ell(t, 1) = 0$ for all $t \in [0, 1]$.

(iii) For each $(x, y) \in I^2$ with $\ell(x, y) > 0$ there exist sequences a_n, b_n, c_n, d_n such that $a_n \uparrow y$, $b_n \downarrow y$, $c_n \uparrow x$, $d_n \downarrow x$. Moreover it is true that $\ell\left(x, a_n\right) \to \ell(x, y)$ and similarly for the other sequences.

For each $(x, y) \in I^2$, the set $\{(x, y, z) : 0 \leq z \leq \ell(x, y)\}$ will be called the *hair* at (x, y) while the set $I^2 \times \{0\}$ will be called the *base*.

The usefulness of the above object lies in the fact that when we suitably embed a 3-d straight brush to I^3 with the usual topology and then compactify that embedding we obtain a soshs.

Indeed, let $\mathcal{H} : \mathbb{R}^3 \to I^3$ be defined as

$$\mathcal{H}(x, y, z) = \left(\frac{\arctan y}{\pi} + \frac{1}{2}, \frac{\arctan z}{\pi} + \frac{1}{2}, \frac{\mathcal{L}\left([x, \infty) \times \{(y, z)\}\right)}{\pi}\right) \tag{4.38}$$

where $\mathcal{L}$ is once again the spherical length of the half line $[x, \infty) \times \{(y, z)\}$. We note here that we are viewing $\overline{\mathbb{R}^3}$ as the sphere of centre $(0, 0, 1/2)$ and radius $1/2$ so the spherical length of any straight line is less or equal than π.

It is easy to see now that $\mathcal{H}$ is an embedding of $\mathbb{R}^3$ to I^3 and that if B is a 3-d straight brush then the compactification of $\mathcal{H}(B)$ in I^3, with usual topology is a soshs.

After they defined straight one-sided hairy arcs, Aarts and Oversteegen went on to define the notions of a *hairy arc* and a *one-sided hairy arc* which are a homeomorphic image of a sosha and a homeomorphic image of a sosha on the plane with all of the hairs attached on the same side respectively. The importance of those objects becomes apparent when we notice that compactified versions of Julia sets of many entire transcendental functions are one-sided hairy arcs. Initially, Aarts and Oversteegen showed this for some functions in the exponential family and some functions in the sine and cosine families. That was extended to much larger classes of transcendental entire maps by Baranski, Jarque and Rempe in [8].

Generalizing in $\mathbb{R}^3$ we define the notions of a *hairy surface* and a *one-sided hairy surface*.

Definition 4.10.2 (Hairy Surface)**.** A hairy surface is any homeomorphic image of a soshs. The base of the hairy surface is the image of $I^2 \times \{0\}$ under that homeomorphism and the hairs are the images of the hairs of the soshs. A one-sided hairy surface is an embedding φ of a hairy surface X, with base D, in $\mathbb{R}^3$ such that all hairs are attached to the same side of the base $\varphi(D)$.

The notion of same sidedness is intuitively clear. Rigorously we can define it as follows. There is a surface $\mathcal{S}$ which contains $\varphi(D)$ and all of the hairs are contained in the same bounded complementary component of $\mathcal{S}$.

Next we prove that when we suitably compactify the Julia sets of some Zorich maps we do get hairy surfaces.

Proof of Theorem 4.1.10. We know from section 4.2.1 that there is a homeomorphism $\psi : \mathcal{J}(Z_\nu) \to B$, where B is a 3-d straight brush. Consider now the embedding $f : \mathcal{J}(Z_\nu) \to I^3$, with $f = \mathcal{H} \circ \psi$, where $\mathcal{H} : B \to I^3$ was defined in (4.38). The compactification we are looking for is the one induced by the embedding f (see for example [96, Chapter 5.3]). Also, by construction, f extends to a homeomorphism $\widetilde{f}$ between the compactification of $\mathcal{J}(Z_\nu)$, $\widetilde{\mathcal{J}(Z_\nu)}$ and $\overline{\mathcal{H}(B)}$ the closure of $\mathcal{H}(B)$ in the usual topology of I^3. Hence, $\widetilde{\mathcal{J}(Z_\nu)}$ is a hairy surface. □

4.11 Wild one-sided hairy surfaces

When studying embeddings of a subset X of $\mathbb{R}^d$ in $\mathbb{R}^d$ an important notion is that of *tameness* of X. In other words, whether or not there is a homeomorphism $h : \mathbb{R}^d \to \mathbb{R}^d$ sending X to $H(X)$, where $H : X \to \mathbb{R}^d$ is a homeomorphism onto its image. We also say that X and $H(X)$ are then ambiently homeomorphic.

Aarts and Oversteegen in [1] showed that all one-sided hairy arcs are tame. In other words any embedding $\varphi : X \to \mathbb{R}^2$ of a sosha X for which the hairs of $\varphi(X)$ are all on the same side extends to a homeomorphism of the whole plane. This is reminiscent of the tameness of the Cantor set and the arc in the plane. Once we move however to higher dimensions the situation is different. It is well known that in $\mathbb{R}^3$, for example, there are wild arcs, wild Cantor sets and wild spheres meaning sets that are homeomorphic images of $[0,1]$, the standard ternary Cantor set and the unit sphere respectively and yet they are not ambiently homeomorphic to those sets. Classical examples of such sets are the wild arc, Antoine's necklace and Alexander's horned sphere (see for example [95] and references therein).

The situation is similar for soshs' and one-sided hairy surfaces as Theorem 4.1.11 shows. Before we proceed with the proof of that theorem we will give a brief description of how a wild arc is constructed since we are going to use them in the proof. We start with an arc A in $\mathbb{R}^3$, i.e. a homeomorphic image of the unit interval, which is knotted as in figure 4.7. Consider now the sequence of arcs $2^{-n}A$ for $n \in \mathbb{N}$ and stack each term of the sequence on top of the previous while also "gluing" the bottom of each term of this sequence with the top of the previous arc. The object W we obtain will be another arc with endpoints a_0 and a_1 with a_0 being also an endpoint of A. Now take a line segment L and glue it on top of W, see figure 4.7. The final object we obtain is a wild arc, see [95, Chapter 19, Theorem 4] for the proof.

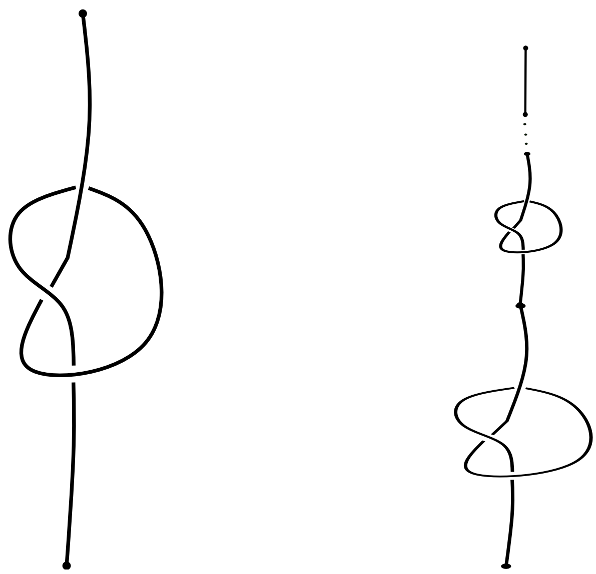

Figure 4.7: A knotted arc and a wild arc

In order to prove our theorem we will first construct a soshs S and then a homeomorphism which takes that soshs to a wild one-sided hairy surface.

4.11.1 Constructing a soshs

For convenience we will construct our soshs as a subset of $I^2 \times [0, 3/2]$ instead of I^3.

Our soshs will be constructed as an intersection of subsets T_n of $I^2 \times [0, 3/2]$ each of which comprises of a square base I^2 and a set of cuboids $R(n, i, j)$. The cuboids will have a square base on I^2 and will have their sides parallel with the axis x, y and z. The construction proceeds inductively. Set $T_0 = T_1 = I^2 \times [0, 3/2]$ and partition I^2 in 9 equal squares $Q(2, i, j)$, $i, j = 1, 2, 3$ which will be the base of our cuboids $R(2, i, j)$. On each of those squares we erect a cuboid of height as in Figure 4.8. The bottom left corner square is $Q(2, 1, 1)$ and the top right one is $Q(2, 3, 3)$.

1/2	1/2	1/2
1/2	3/2	1/2
1/2	1/2	1/2

Figure 4.8: The squares $Q(2, i, j)$ and the heights of the corresponding cuboids

We now name $T_2 = \bigcup_{i,j\in\{1,2,3\}} R(2, i, j)$. Suppose now that we have constructed T_n and it is a finite union of cuboids $R(n, i, j)$ with bases $Q(n, i, j)$. Take each square base $Q(n, i, j)$ and partition it in $(2n + 1)^2$ equal squares, which we will now call $Q(n, i, j, k, l)$. After we define T_{n+1} we relabel those squares as $Q(n + 1, i, j)$. Notice that we have used the same indices i, j although they take different values for different values of n. Also, let us denote by $h(n, i, j)$ the height of the cuboid $R(n, i, j)$.

We now erect cuboids $R(n, i, j, k, l)$ with bases $Q(n, i, j, k, l)$ with heights

$$h(n,i,j,k,l) = \begin{cases} \frac{1}{n+1} h(n,i,j), & \text{if } l = 1 \text{ or } l = 2n+1 \text{ or } k = 1 \text{ or } k = 2n+1 \\ \\ h(n,i,j), & \text{if } Q(n,i,j,k,l) \text{ is the central square of } Q(n,i,j) \\ \\ \frac{n}{n+1} h(n,i,j), & \text{otherwise.} \end{cases}$$

Then we define $T_{n+1} = \bigcup_{i,j,k,l} R(n, i, j, k, l)$.

1/2	1/2	1/2	1/2	1/2
1/2	1	1	1	1/2
1/2	1	3/2	1	1/2
1/2	1	1	1	1/2
1/2	1/2	1/2	1/2	1/2

Figure 4.9: The squares $Q(2,2,2,k,l)$ and the heights $h(2,2,2,k,l)$ of the corresponding cuboids

Our soshs will then be defined as $S = \bigcap_{n=0}^{\infty} T_n$. Since every T_n is a continuum and $T_{n+1} \subset T_n$, S will also be a continuum and it is easy to see that it will satisfy all of the properties of a soshs. We will only prove here that property (iii) of the definition of a soshs is satisfied.

Indeed, for any hair $\delta : [0,1] \to S$ of S at the base point (x, y) it will be true that there is a sequence of cuboids $R(n, i_n, j_n)$ with $R(k+1, i_{k+1}, j_{k+1}) \subset R(k, i_k, j_k)$, for all $k \in \mathbb{N}$ and

$$\bigcap_n R(n, i_n, j_n) = \delta\left([0,1]\right).$$

It is true now that there is a subsequence n_k such that $R(n_k, i_{n_k}, j_{n_k})$ is the central square of $R(n_k - 1, i_{n_k-1}, j_{n_k-1})$ since otherwise there would not be a hair at (x, y).

We now find a sequence of hairs δ_k, $k = 1, 2, \ldots$ with base points (x_k, y) such that $x_k \uparrow x$ and $\ell(x_k, y) \to \ell(x, y)$. We choose the hair δ_k as the one defined by the sequence of cuboids

$$\begin{aligned} R(1, i_1, j_1), \ldots, R(n_k - 1, i_{n_k-1}, j_{n_k-1}), R(n_k, i_{n_k} - 1, j_{n_k}), \\ R(n_k + 1, m_{1,k}, j_{k+1}), R(n_k + 2, m_{2,k}, j_{n_k+2}), \ldots, \end{aligned}$$

where $m_{i,k}$, $i \in \mathbb{N}$ is a sequence of integers such that the sequence of squares

$$Q(n_k + 1, m_{1,k}, j_{k+1}), Q(n_k + 2, m_{2,k}, j_{n_k+2}), \ldots$$

is the sequence of squares

$$Q(n_k + 1, i_{n_k+1}, j_{k+1}), Q(n_k + 2, i_{n_k+2}, j_{n_k+2}), \ldots$$

translated to the left by the length of a side of a square at the n_k level.

It is now easy to see that δ_k has its base point at (x_k, y) with $x_k \uparrow x$. Also the hairs δ_k have length

$$\ell(x_k, y) = \frac{n_k}{n_k + 1}\ell(x, y)$$

so that $\ell(x_k, y) \to \ell(x, y)$ as $k \to \infty$.

Similarly we can construct the other sequences of points from property (iii) of the definition of a soshs.

4.11.2 Proof of Theorem 4.1.11

Before we proceed with the proof of Theorem 4.1.11 let us first introduce some standard terminology taken from [95].

Let $V = \{u_0, u_1, \ldots, u_n\}$ be a set of $n+1$ points in $\mathbb{R}^d$ which are affinely independent, meaning $u_1 - u_0$, $u_2 - u_0$, $\ldots$, $u_n - u_0$ are linearly independent. Then the *n-simplex* is defined as the convex hull of V. We will denote the n-simplex defined by those points by σ^n. The convex hull τ of a non empty subset W of V will be called a face of σ^n. A *(Euclidean) complex* is a collection $\mathcal{K}$ of simplexes in $\mathbb{R}^d$ such that

1. $\mathcal{K}$ contains all faces of all elements of $\mathcal{K}$.
2. If σ, $\tau \in \mathcal{K}$ are simplexes and $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau$ is a face of both σ and τ.
3. Every σ in $\mathcal{K}$ lies in an open set U which intersects only a finite number of members of $\mathcal{K}$.

If $\mathcal{K}$ is a complex then with $|\mathcal{K}|$ we denote the union of the elements of $\mathcal{K}$. Such a set is called a *polyhedron*. Note that a polyhedron can be seen as a manifold with boundary with the subspace topology induced from the standard topology of $\mathbb{R}^d$. An *n-cell* is a space homeomorphic to an n-simplex. A *polyhedral n-cell* is a polyhedron homeomorphic to an n-simplex.

Let M be a manifold with boundary. Then by $\operatorname{Bd} M$ and $\operatorname{Int} M$ we denote its boundary and its interior respectively.

Lemma 4.11.1. *[95, Theorem 1, Chapter 19] Let A be a polyhedral 1-cell in $\mathbb{R}^3$ with endpoints P and Q. Then there is a polyhedral 3-cell C such that (1) $\operatorname{Int} A \subset \operatorname{Int} C$, (2) $P, Q \in \operatorname{Bd} C$, and (3) there is a homeomorphism $\phi : C \to \sigma^2 \times [0,1]$, such that $A \mapsto R \times [0,1]$, for some $R \in \operatorname{Int} \sigma^2$.*

If A and C satisfy the conditions of the above lemma, then we say that A is *unknotted* in C.

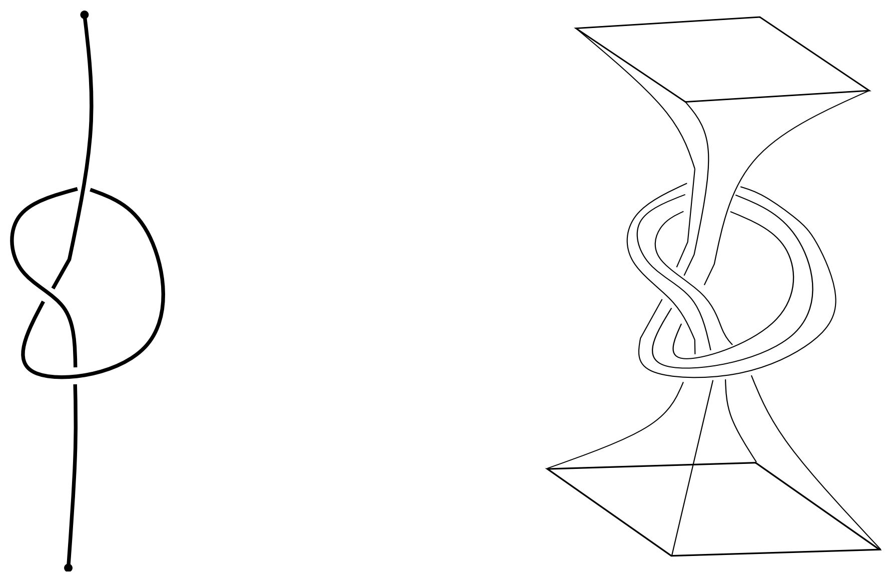

Figure 4.10: The polyhedral 1-cells A_n (left) and the polyhedral 3-cells C_n (right) used in the Proof of Theorem 4.1.11.

Proof of Theorem 4.1.11. Consider the soshs S we constructed in the previous subsection. We will construct a homeomorphism by defining it on the sets $R(n, i, j)$ for all n. Consider now cuboids B_k, $k \in \mathbb{N}$ with sides parallel to the x, y and z axis which are constructed as follows:

- B_1 has a square base $I^2 \times \{0\}$ and height $h_1 = 1/2$.
- B_2 has as base $Q(2, 2, 2) \times \{1/2\}$ and height $h_2 = \frac{1}{4}$.

  $\vdots$

- B_n has as base $Q(n, i_n, j_n) \times \{\sum_{k=1}^{n-1} h_k\}$, where $Q(n, i_n, j_n)$ is the central square of $Q(n-1, i_{n-1}, j_{n-1})$ and height $h_n = \frac{1}{2^n}$.

  $\vdots$

Consider now the polyhedral 1-cells A_n inside B_n that are overhand knotted (see Figure 4.10) with their endpoints at $(\frac{1}{2}, \frac{1}{2}, h_{n-1})$ and $(\frac{1}{2}, \frac{1}{2}, h_n)$. From Lemma 4.11.1 now, for each of those 1-cells, we can find a polyhedral 3-cell C_n, inside B_n, in which A_n is unknotted and $\operatorname{Bd} C_n$ contains the bottom and the top square bases of B_n.

Lemma 4.11.1 also gives us that there are homeomorphisms $\phi_n : C_n \to \sigma^2 \times [0, 1]$. Since $\sigma^2 \times [0, 1]$ is homeomorphic to B_n this implies that there are homeomorphisms $\varphi_n : B_n \to C_n$ which can also be arranged so that they fix the square bases of B_n.

Define now the map $H : S \to \mathbb{R}^3$ as

$$H(x) := \begin{cases} \varphi_n(x), & \text{for } x \in B_n, \\ x, & \text{otherwise.} \end{cases}$$

It is easy to see, by our construction, that this map is a homeomorphism of S onto $H(S)$.

Suppose now that there exists a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(S) = H(S)$. Any homeomorphism that sends S to $H(S)$ will map the base of S, $I^2 \times \{0\}$ to the base of $H(S)$ and the hairs of S to the hairs of $H(S)$. Let $\gamma(t)$, $t \in [0, 1]$ be a parametrization of the longest hair of S attached at $(1/2, 1/2, 0)$. By [95, Theorem 4, Chapter 19] and our construction we have that $H(\gamma(t))$ will be a wild arc and since it is a hair of $H(S)$ it will be the image of a hair of S under h. However this is a contradiction since a wild arc cannot be the image of an arc under a homeomorphism of $\mathbb{R}^3$ such as h. $\square$

4.12 Open questions

As we have already seen, the Zorich maps resemble in a lot of ways the exponential family. The literature on exponential dynamics is vast and there are many striking phenomena. It is expected that Zorich maps, given the higher dimensional setting and the greater flexibility, should have an even more intricate nature. In this section we will mention some problems that require further study.

4.12.1 Dynamics for different values of λ

We saw in Theorem 4.1.3 that when λ is large enough then the Julia set of the Zorich map is the entire $\mathbb{R}^3$ assuming that ν is large enough. It is interesting to ask what happens in the case when the scale factor λ is not large. In that case we do not have enough expansion in the sense of Lemma 4.2.6 in order for our argument to work. Nonetheless, it seems that the dynamics in this case are also chaotic. So we ask

Question 4.12.1. *Let $\lambda > 0$. Does there always exist a constant c_λ depending on λ such that for all $\nu > c_\lambda$ the Julia set $\mathcal{J}(\mathcal{Z}_\nu)$ is the whole $\mathbb{R}^3$?*

We can even ask this question in the complex plane. If we rescale the complex exponential family we obtain the maps

$$f_\nu(x + iy) = \nu\lambda e^x \left(\cos\left(\frac{y}{\lambda}\right) + i\sin\left(\frac{y}{\lambda}\right)\right),$$

for $\lambda > 0$ and $\nu \in \mathbb{R}$. Note that for $\lambda = 1$ we obtain the exponential family. Of course those maps are no longer holomorphic but they are quasiregular and we can define their Julia set. A similar approach to that used for the Zorich maps should give us that the Julia set of those maps for λ large enough and $\nu > c'_\lambda$ is the entire complex plane, where c'_λ constant depending on λ. But we can ask

Question 4.12.2. *Let $\lambda > 0$. Assuming that $\nu > c'_\lambda$ is $\mathcal{J}(f_\nu)$ the whole complex plane?*

Closely related to the above question and worth mentioning here is the paper [34] where the authors study families of functions such as f_ν in the complex plane. The functions they study are not necessarily quasiregular. However their results show that if we choose a $\lambda > 0$ then for small values of ν the Julia set of f_ν is a "Cantor bouquet".

4.12.2 Measurable dynamics of Zorich maps

Another quite interesting question is that of the typical behaviour of an orbit of the exponential map. Lyubich in [84] proved that for Lebesgue almost all points of the complex plane the limit set of their orbit $E^n(z)$ is the orbit of 0, $\{E^n(0)\}_{n\in\mathbb{N}}$ plus ∞. Thus a typical point will follow closely the orbit of 0 for some time and then "break off" for some iterates until it goes back to following the orbit of 0 for more iterates this time. Hence, almost all points in the complex plane belong to the bungee set of the exponential map (see [105]), namely the set of points that neither escape to infinity nor remain bounded under iteration. The bungee set can be also defined for quasiregular maps (see [101]). So we ask

Question 4.12.3. *What is the typical behaviour of an orbit of a point $x \in \mathbb{R}^3$ under the Zorich maps of Theorem 4.1.3?*

Question 4.12.4. *For the same Zorich maps, do almost all points of $\mathbb{R}^3$ belong to the bungee set?*

It is also known that the escaping set of the exponential map E_κ, for any value of the parameter $\kappa \in \mathbb{C} \setminus \{0\}$, has zero Lebesgue measure (see [42, Theorem 7]).

Question 4.12.5. *Is the Lebesgue measure of the escaping set of any Zorich map equal to zero?*

Another interesting question that was answered by Lyubich is that of ergodicity of the exponential. Ergodicity here means that there is no partition of the plane in two invariant sets of positive Lebesgue measure. We have that

Theorem 4.12.6 (Lyubich [84])**.** *$E(z)$ is not ergodic.*

In the same sense we can ask

Question 4.12.7. *Is the Zorich map $\mathcal{Z}$ ergodic?*

4.12.3 Indecomposable continua in Zorich maps

Another fascinating and well-studied phenomenon in exponential dynamics is the presence of *indecomposable continua* in the dynamic plane. A non empty metric space is called a *continuum* if it is compact and connected. A continuum X is called *decomposable* if there exist two subcontinua $A \neq X$ and $B \neq X$ such that $X = A \cup B$. A continuum which is not decomposable is called indecomposable. Indecomposable continua have a long history and appear quite often in the study of dynamical systems, see for example [74].

It was Devaney in [36] who first studied such sets in the context of exponential dynamics. The way to construct them in the complex plane is as follows. Consider the strip

$$S = \{z \in \mathbb{C} : 0 \leq \operatorname{Im} z \leq \pi\}.$$

Now take any $\kappa > 1/e$ and consider the set

$$\Lambda := \{z \in \mathbb{C} : E_\kappa^n(z) \in S \text{ for all } n \in \mathbb{N}\}.$$

By suitably compactifing this set then Devaney shows that we obtain a curve that accumulates everywhere on itself but does not separate the plane. Then by applying a theorem of Curry, [35, Theorem 8] he concludes that this curve must be an indecomposable continuum.

Assuming that $\nu = 1$, we can try and construct a similar set in the case of Zorich maps. The role of the strip S is played now by the rectangular beam $\overline{B_{(0,0)}}$. Thus we can consider the set

$$\Lambda_{\mathcal{Z}} := \{x \in \mathbb{R}^3 : \mathcal{Z}^n(x) \in \overline{B_{(0,0)}}, \text{ for all } n \in \mathbb{N}\}.$$

We can also, just like Devaney, suitably compactify this set and obtain a surface, let us call it Γ, that accumulates everywhere on itself. However the criterion of Curry is no longer available in this higher dimensional setting so Devaney's argument does not work here.

Question 4.12.8. *Is Γ an indecomposable continuum?*

If the answer to the above question is yes we can then consider the same continua for different values of ν. Let us call those continua Γ_{ν_1} and Γ_{ν_2}, with $\nu_1, \nu_2 \geq 1$.

Question 4.12.9. *If $\nu_1 \neq \nu_2$ are Γ_{ν_1} and Γ_{ν_2} homeomorphic?*

Let us also remark here that the points in the set $\Lambda_{\mathcal{Z}}$ all have the same itinerary so by the results of section 4.5 we have that the three dimensional Lebesgue measure of this set is zero.

Finally, let us mention [39] where the authors prove the existence of many more indecomposable continua in the dynamical plane of the exponential map and ask many more questions. Such considerations also make sense for Zorich maps.

4.12.4 Explosion points for subsets of the endpoints

As we have already mentioned explosion points have also been studied for subsets of the set of endpoints $\mathcal{E}(\mathcal{J}(E_\kappa))$. Specifically in [2, Theorem 1.3] it is shown that the escaping endpoints, namely the set $\mathcal{E}(\mathcal{J}(E_\kappa)) \cap I(E_\kappa)$, has ∞ as an explosion point assuming that the parameter κ is chosen appropriately. That result includes the case when $\kappa \in (0, 1/e)$. We can ask the same question about Zorich maps.

Question 4.12.10. *Does the set of escaping endpoints, $\mathcal{E}(\mathcal{J}(Z_\nu)) \cap I(Z_\nu)$ have infinity as an explosion point for $0 < \nu < e^{-(\log L + L)}$?*

Moreover, in [46] the authors show that for the same values of the parameter κ the set of endpoints that do not escape, namely $\mathcal{E}(\mathcal{J}(E_\kappa)) \cap I(E_\kappa)^c$, does not have ∞ as an explosion point and in fact $\mathcal{E}(\mathcal{J}(E_\kappa)) \cap I(E_\kappa)^c \cup \{\infty\}$ is totally separated. They also show that even the set of endpoints that do not escape fast, $\mathcal{E}(\mathcal{J}(E_\kappa)) \cap \mathcal{A}(E_\kappa)^c$ have the property that $\mathcal{E}(\mathcal{J}(E_\kappa)) \cap \mathcal{A}(E_\kappa)^c \cup \{\infty\}$ is totally separated. All those results make sense for Zorich maps as well.

Question 4.12.11. *For $0 < \nu < e^{-(\log L + L)}$, is $\mathcal{E}(\mathcal{J}(Z_\nu)) \cap I(Z_\nu)^c \cup \{\infty\}$ totally separated?*

Question 4.12.12. *Is $\mathcal{E}(\mathcal{J}(Z_\nu)) \cap \mathcal{A}(Z_\nu)^c \cup \{\infty\}$ totally separated?*

Finally, let us mention the papers [79–81] where the author studies various topological questions about the Julia sets of exponential maps. For example in [81, Corollary 9] it is shown that the sets $\mathcal{E}(\mathcal{J}(E_\kappa)) \cap \mathcal{A}(E_\kappa)^c$, $\mathcal{E}(\mathcal{J}(E_\kappa)) \cap \mathcal{A}(E_\kappa)^c \cup \{\infty\}$ are both homeomorphic to the set of irrationals. Many of those results again make sense in the higher dimensional setting of Zorich maps.

Question 4.12.13. *Are $\mathcal{E}(\mathcal{J}(Z_\nu)) \cap \mathcal{A}(Z_\nu)^c$, $\mathcal{E}(\mathcal{J}(Z_\nu)) \cap \mathcal{A}(Z_\nu)^c \cup \{\infty\}$ homeomorphic to $(\mathbb{R} \setminus \mathbb{Q})^2$?*

Chapter 5

Mapping properties of domains in $\mathbb{R}^d$ under quasiregular maps

5.1 Introduction

Suppose that we are given a domain V and a meromorphic function f in the complex plane. Let U now be a connected component of $f^{-1}(V)$. A simple and important question to ask then would be how does the image of U cover the set V? In other words how large can the set $V \setminus f(U)$ be and how many preimages of each point in V are there in U? This question has been answered by Heins in [61]. In particular Heins showed that

Theorem 5.1.1 (Heins). *Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, V a domain in $\overline{\mathbb{C}}$ and U a connected component of $f^{-1}(V)$ then either*

1. *$V \setminus f(U) = \emptyset$ and all points in V have the same finite number of preimages in U (counting multiplicities) or*

2. *the number of points in V with a finite number of preimages in U is at most two in which case $V \setminus f(U)$ contains at most two points.*

The above theorem has been generalized to a wider class of maps by Bolsch in [25]. It would also be quite useful to have such a theorem for quasiregular maps. The main goal of this chapter is to prove such a theorem.

Theorem 5.1.2. *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}^d}$ be a K-quasimeromorphic map. If V is a domain in $\overline{\mathbb{R}^d}$ and U is a connected component of $f^{-1}(V)$ then either*

1. *$V \setminus f(U) = \emptyset$ and every point in V has the same finite number of preimages in U or*

2. *the set of points in V with a finite number of preimages in U has conformal capacity zero. In this case $V \setminus f(U)$ is of conformal capacity zero.*

In fact in the second case of the above theorem we can prove something more. Recall here that a point $x \in \mathbb{R}^d$ is called an asymptotic value for f if there is a path $\gamma : (0,1) \to \mathbb{R}^d$ with $\gamma(t) \to \infty$ as $t \to 1$ such that $f(\gamma(t)) \to x$ as $t \to 1$.

Theorem 5.1.3. *Let f, U and V be as in Theorem 5.1.2 and suppose that some point in V has infinitely many preimages in U. Then all the points in V with finitely many preimages in U are asymptotic values of f.*

Theorem 5.1.1 is of significant importance in the field of complex dynamics since it answers the following important question. Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function and let U_1 be a Fatou component. Then since the Fatou set is invariant, $f(U_1)$ must be inside another or possibly the same Fatou component U_2.

Question: How big can $U_2 \setminus f(U_1)$ be?

Using Theorem 5.1.1 we can answer that question. Either $U_2 \setminus f(U_1)$ will be empty or it will contain at most two points. For rational functions it is quite easy to see that the former case holds. Indeed, suppose that $f(U_1) \subsetneq U_2$ and let $x \in \partial f(U_1) \cap U_2$. Notice that $\partial f(U_1) \subset f(\partial U_1)$ due to the openness of f. Since every point in $\mathbb{C}$ has a non empty inverse image under a rational map, there will be a point $y \in \partial U_1$ such that $f(y) = x$. Take a sufficiently small neighbourhood N around y so that $f(N) \subset U_2$. Hence $N \subset f^{-1}(U_2)$ which is clearly impossible since U_1 is a connected component of $f^{-1}(U_2)$ and N intersects that component and its complement.

However for the general case of meromorphic functions the latter case will hold. Although Heins' result completely answers our question about Fatou components, it is also worth mentioning here that the same question has been studied by Herring, [62] who used different methods than those of Heins to obtain the same conclusion. Moreover, those methods were used by Bolsch in [25] to obtain a version of Theorem 5.1.1 for a more general class of maps which quite often in the literature is called Bolsch class (see [25] for more details).

In the higher dimensional setting of quasiregular maps now, as we have already mentioned, there is a sensible way to define the Julia and quasi-Fatou set of a function (see Chapter 2 for more details). Again the quasi-Fatou set is an open and completely invariant set (see Theorem 2.3.9). Hence the same question makes sense in this setting as well: If U_1, U_2 are quasi-Fatou components with $f(U_1) \subset U_2$ then how big can $U_2 \setminus f(U_1)$ be?

Applied in the setting of quasi-Fatou components the two theorems that we will prove immediately yield

Corollary 5.1.4. *Let $f : \mathbb{R}^d \to \overline{\mathbb{R}^d}$ be a quasimeromorphic map and U_1, U_2 be quasi-Fatou components of f such that $f(U_1) \subset U_2$. Then $U_2 \setminus f(U_1)$ can be at most of conformal capacity zero and all of its points are asymptotic values of f.*

5.2 Preliminaries

First we need to introduce the notions of the *cluster set* and the *boundary cluster set* which play an important role in the study of boundary behaviour of analytic maps. We refer to the books [32, 102] for an introduction to these concepts and their relation with geometric function theory.

Let G be an arbitrary domain in $\mathbb{R}^d$ and ∂G its boundary on $\overline{\mathbb{R}^d}$. We will always consider boundaries with regard to $\overline{\mathbb{R}^d}$ unless otherwise stated. Let also f be a quasimeromorphic map on G and x_0 be a non-isolated point of ∂G. Then we define the cluster set at x_0 as

$$C(f, G, x_0) := \{a \in \mathbb{R}^d : \exists x_n \in G,\ x_n \to x_0,\ f(x_n) \to a\}.$$

The boundary cluster set at x_0 with respect to $F \subset \partial G$ is defined as

$$\begin{aligned} C_F(f, x_0) : &= \{a \in \mathbb{R}^d : \text{there is } \zeta_n \in F \text{ and } w_n \in C(f, G, \zeta_n) \text{ such that } \zeta_n \to x_0 \text{ and } w_n \to a\} \\ &= \bigcap_{r>0} \overline{\bigcup_{x \in F \cap D_r} C(f, G, x)}, \end{aligned}$$

where D_r is a ball of centre x_0 and radius $r > 0$ in the spherical distance.

Next we have some preliminary theorems related to cluster sets and boundary cluster sets. The first one is a well known result of Iversen and Tsuji generalized for quasiregular maps by Martio and Rickman [86, Theorem 3.10]. There is an even more general version of that theorem, which we will not need here, due to Vuorinen [135].

Theorem 5.2.1 (Martio-Rickman, [86])**.** *Let $f : G \to \overline{\mathbb{R}^d}$ be a quasimeromorphic mapping. Let $E \subset \partial G$ be a compact set of capacity zero and let $y \in E \cap \overline{\partial G \setminus E}$ then*

$$\partial C(f, G, y) \subset \partial C_{\partial G \setminus E}(f, y).$$

The next theorem generalizes a result of Tsuji for analytic maps on the plane, see [102, II.§4 Theorem 5].

Theorem 5.2.2 (Vuorinen, [136]). *Let $f : G \to \overline{\mathbb{R}^d}$ be a quasimeromorphic function and let $E \subset \partial G$ be a compact set of conformal capacity zero. Suppose that $x_0 \in E \cap \overline{\partial G \setminus E}$. If*

$$\mathcal{K} := C(f, G, x_0) \setminus C_{\partial G \setminus E}(f, x_0)$$

is not empty then every point in $\mathcal{K}$ is assumed by f an infinite number of times in a neighbourhood of x_0, except possibly a set of conformal capacity zero.

To prove Theorem 5.1.3 we are going to need the notion of maximal path lifts. Let G be a domain and $f : G \to \overline{\mathbb{R}^d}$ be a quasimeromorphic map. Let $\beta : [a, b) \to \overline{\mathbb{R}^d}$ be a path and let $x_0 \in G$ be such that $f(x_0) = \beta(a)$. A path $\gamma : [a, c) \to G$ is said to be a *maximal f-lifting of β starting at x_0* if

1. $\gamma(a) = x_0$.
2. $f \circ \gamma = \beta_{|[a,c)}$.
3. If $c < c' \leq b$, then there does not exist a path $\gamma' : [a, c') \to G$ such that $\gamma = \gamma'_{|[a,c)}$ and $f \circ \gamma' = \beta_{|[a,c')}$.

Let $x_1, \ldots, x_k$ be k different points of $f^{-1}(\beta(a))$ and let $m = \sum_{i=1}^{k} i(x_i, f)$, where $i(x, f)$ denotes the local topological index, i.e.

$$i(x, f) = \inf_{N} \sup_{y \in \overline{\mathbb{R}^d}} \operatorname{card}(f^{-1}(y) \cap N)$$

and the inf is taken over all neighbourhoods N of x. We say that the sequence of paths $\gamma_1, \ldots \gamma_m$ is a *maximal sequence of f-liftings of β starting at the points* $x_1, \ldots, x_k$ if

(i) each γ_j is a maximal f-lifting of β,

(ii) $\operatorname{card}\{j : \gamma_j(a) = x_i\} = i(x_i, f)$, $1 \leq i \leq k$,

(iii) $\operatorname{card}\{j : \gamma_j(t) = x\} \leq i(x, f)$ for all $x \in G$ and all t.

We also say that the paths $\gamma_1, \ldots \gamma_m$ are *essentially separate* if (iii) is satisfied.

If $\beta : [a, b) \to \overline{\mathbb{R}^d}$ is a path and $C \subset \overline{\mathbb{R}^d}$, we say that $\beta(t) \to C$ as $t \to b$ if the spherical distance between $\beta(t)$ and C goes to zero as $t \to b$.

For a thorough discussion on path lifts we refer to Rickman's monograph [118]. Here we will need the following lemma which is a combination of [118, II.3 Theorem 3.2] and [87, Lemma 3.12]:

Lemma 5.2.3. *Suppose that $f : G \to \overline{\mathbb{R}^d}$ is a discrete, open and sense-preserving map and that $x_i \in G$, $i = 1, \dots k$. Let $\beta : [a, b) \to \overline{\mathbb{R}^d}$ be a path such that $f(x_i) = \beta(a)$ for all $i = 1, \dots, k$ and such that either $\lim_{t\to b} \beta(t)$ exists or $\beta(t) \to \partial f(G)$ as $t \to b$. Then β has a maximal sequence of f-liftings $\gamma_j : [a, c_j) \to G$, $j = 1, \dots m$ starting at $x_1, \dots x_k$. If $\gamma_j(t) \to y_j \in G$ as $t \to c_j$ then $c_j = b$ and $f(y_j) = \lim_{t\to b} \beta(t)$. Otherwise $\gamma_j(t) \to \partial G$ as $t \to c_j$.*

5.3 Proof of Theorems 5.1.2, 5.1.3

The proof of Theorem 5.1.2 is a combination of the following two lemmas and Theorem 5.2.2. In this section, using the notation of Theorem 5.2.2, we take $E = \{\infty\}$, $G = U$ and $x_0 = \infty$ so that $\mathcal{K} = C(f, U, \infty) \setminus C_{\partial U\setminus\{\infty\}}(f, \infty)$.

Lemma 5.3.1. *Let f, V and U be as in Theorem 5.1.2. If $\infty \in \partial U$ and $\mathcal{K} \neq \emptyset$ then $V \subset \mathcal{K}$.*

Proof. Since U is a connected component of $f^{-1}(V)$ we have that

$$C(f, U, \infty) \subset \overline{f(U)} \subset \overline{V}. \tag{5.1}$$

Moreover, for the same reason, we have that

$$f(\partial U \setminus \{\infty\}) \cap V = \emptyset.$$

Also, since f is defined on $\partial U \setminus \{\infty\}$ we have that for $y \in \partial U \setminus \{\infty\}$ we have $C(f, U, y) = \{f(y)\}$. Hence

$$C_{\partial U\setminus\{\infty\}}(f, \infty) \subset f(\partial U) \subset \partial V. \tag{5.2}$$

Note now that since $\mathcal{K}$ is open and by Theorem 5.2.1

$$\partial\mathcal{K} = \overline{\mathcal{K}} \setminus \mathcal{K} \subset C(f, U, \infty) \setminus \mathcal{K} \subset C_{\partial U\setminus\{\infty\}}(f, \infty). \tag{5.3}$$

Equations (5.1) and (5.2) now imply $\mathcal{K} \subset \overline{V}$ and by using again the fact that $\mathcal{K}$ is open we obtain that $\mathcal{K} \cap V \neq \emptyset$.

This implies that $V \subset \mathcal{K}$. Indeed if not then there would be a point $x \in V$ such that $x \in \partial\mathcal{K}$, but (5.2) and (5.3) would then imply that $x \in \partial V$ which is a contradiction. □

Before we state our next lemma let us introduce some terminology. We are going to need the notion of a *normal domain*. Let $G \subset \overline{\mathbb{R}^d}$ be a domain and $f : G \to \overline{\mathbb{R}^d}$ be a quasimeromorphic map. A domain U with $\overline{U} \subset G$ is said to be a normal domain of f if $f(\partial U) = \partial f(U)$. If a domain is normal then every point in $f(U)$ has the same number of preimages in U counting multiplicities. For a comprehensive discussion on this we refer to [118, section I.4].

Lemma 5.3.2. *Let f, V and U be as in Theorem 5.1.2. If $\overline{U}$ is compact subset of $\mathbb{R}^d$ then $f(U) = V$, U is a normal domain and every point in V has the same number of preimages in U.*

Proof. This is basically [118, I.4 Lemma 4.7] and [118, I.4 Proposition 4.10]. $\square$

Proof of Theorem 5.1.2. Either the set $\overline{U}$ is compact in $\mathbb{R}^d$ or it is not. In the first case Lemma 5.3.2 gives us that assertion 1 of the theorem holds. In the second case Lemma 5.3.1 together with Theorem 5.2.2 give us that assertion 2 holds in the case that $\mathcal{K}$ is not empty.

If $\overline{U}$ is not compact and $\mathcal{K}$ is empty then this implies that for any sequence $z_n \in U$ with $z_n \to \infty$ we have that $f(z_n) \to \partial V$. This implies that $f : U \to V$ is a proper map and thus for any point $y \in V$ and any compact set $B \subset V$ containing y have that $f^{-1}(B) \subset U$ is a compact set. Hence, using [118, I.4 Lemma 4.7] and [118, I.4 Proposition 4.10], for any open neighbourhood $N \subset \overline{N} \subset V$ of y we have that any connected component of $f^{-1}(N)$ is a normal domain and all points in that neighbourhood have the same number of preimages in U. Since this is true for any neighbourhood we have that all points have the same finite number of preimages. $\square$

Proof of Theorem 5.1.3. Let z_0 be a point in V with a finite number of preimages $\{x_1, \dots, x_k\}$ in U. Since we know by Theorem 5.1.2 that the set of points with finite preimage is of capacity zero, we can find a non-constant path $\beta : [0,1] \to V$ inside V that avoids all those points other than z_0 and also $\beta(1) = z_0$.

Let $m = \sum_{i=1}^{k} i(x_i, f)$. Choose now $m+1$ preimages of $\beta(0)$ (we know that there are infinitely many), $\{y_1, \dots, y_{m+1}\}$ so that $f(y_j) = \beta(0)$, $j = 1, \dots, m+1$. By Lemma 5.2.3 we will now have that there are $m+1$ maximal liftings, $\gamma_j : [a_j, c_j) \to V$ of β. By the same Lemma we also know that the endpoints of the paths will either be in the set $\{x_1, \dots, x_k\}$ or $\gamma_j(t) \to \partial U$ as $t \to c_j$. From (iii) of the definition of maximal sequences of paths we now have that there are at most m f-liftings ending on points of the set $\{x_1, \dots, x_k\}$. Since we have more liftings than m we will have that there is a path γ_j, for some $j \in \{1, \dots, m+1\}$ with $\gamma_j(t) \to \partial U$. But from Lemma 5.3.1 we know that for all points $x \neq \infty$ in ∂U we have that $f(x) \in \partial V$ and thus $f(\gamma_j(c_j)) \in \partial V$ which is impossible since $f(\gamma_j(t)) = \beta(t)$ and $\beta(t) \in V$ for all t. $\square$

Finally let us finish by noting that the second case in Theorem 5.1.2 could be upgraded to $V \setminus f(U)$ being at most a finite set if we could generalize Theorem 5.2.2. We write this as a question.

Question 5.3.3. *If Ω is any connected component of the set $\mathcal{K}$ from Theorem 5.2.2 is it true that every point of Ω, except finitely many, is assumed by f an infinite number of times in any neighbourhood of x_0?*

Such a theorem, if true, would be the higher dimensional counterpart to a theorem due to Noshiro, see [102, II.§4 Theorem 6].

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