On a Family of Presentations Generalising Coxeter's $(l, m|n, k)$

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Abstract

We study the groups with presentation

$$(l, m|n, k|p, q) := \langle a, b|a^l, b^m, (ab)^n, (a^p b^q)^k \rangle,$$

in an attempt to characterise which parameter-sets give rise to a finite group, using a combination of geometric and computational methods, along with more elementary presentation manipulation techniques. We achieve a full characterisation for the case $l = 2p, m = 2q$, and a characterisation with a few families of exceptions under the simplifying assumption $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$.

Contents

1 Introduction

The presentations

$$(l,m|n,k) := \langle a,b|a^l,b^m,(ab)^n,(ab^{-1})^k\rangle,$$

first introduced by Coxeter [1], have been the subject of substantial study since their introduction, specifically pursuing the question of which values of (l,m,n,k) give rise to infinite groups. The answer to this question was completed by Edjvet and Thomas [6] in 1997. Some generalisations of these presentations have occurred, including Holt and Plesken [9] examining finiteness in a subfamily of the presentations $\langle a,b|a^l,b^m,(ab)^n,(ab^2)^k\rangle$. Dennis [2] extends this question to the presentations $\langle a,b|a^l,b^m,(ab)^n,(ab^q)^k\rangle$. We approach a family of presentations extending these in turn, defining the presentation

$$(l,m|n,k|p,q) := \langle a,b|a^l,b^m,(ab)^n,(a^pb^q)^k\rangle.$$

We shall generally restrict our attention to parameter-sets on which the simplifying assumption $\frac{1}{n}+\frac{1}{k}\leq\frac{1}{2}$ holds, giving a full account of which parameter-sets lead to infinite groups under this condition, with the exception of four three-parameter subfamilies. We are also able to abandon our simplifying assumption and give a full characterisation for the groups $(2p,2q|n,k|p,q)$, where p and q are greater than 1.

We note some symmetries of these presentations, and the simplifications that they allow. $(l,m|n,k|p,q)\cong(m,l|n,k|q,p)$, allowing us to choose, in cases where one of p or q is one, $p=1$ without loss of generality. We can also manipulate the value of p (and, by the previous symmetry, q), noting that $(l,m|n,k|p,q)\cong(l,m|n,k|-p,-q)$, and $(l,m|n,k|p,q)\cong(l,m|n,k|p\pm l,q)$, so that we can always choose p and q in the range $0\leq p\leq l-1$,$0\leq q\leq m-1$. We exclude the cases $p=0$ or $q=0$, as they are trivially triangle groups, and note that the case $(p,q)=(p,m-1)$ can be reduced to $(p,q)=(l-p,1)$, and $(p,q)=(1,m-1)$ can be reduced to $(p,q)=(1,-1)$, giving $(l,m|n,k|p,q)=(l,m|n,k)$, for which finiteness is characterised.

Expressing $(l,m|n,k|p,q)$ in compliance with the above simplifications, our main results, as follow, along with the already known characterisation of the groups $(l,m|n,k)$, cover all $(l,m|n,k|p,q)$ with $\frac{1}{n}+\frac{1}{k}\leq\frac{1}{2}$, apart from the four families explicitly excluded:

Main Theorem A: Let G be the group with presentation $(l,m|n,k|p,q)$, with $1<p<l-1$, $1<q<m-1$ and $\frac{1}{n}+\frac{1}{k}\leq\frac{1}{2}$. Then G is infinite.

Main Theorem B: Let G be the group with presentation $(l,m|n,k|1,q)$, with, $1<q<m-1$ and $\frac{1}{n}+\frac{1}{k}\leq\frac{1}{2}$. Then if $(l,m|n,k|1,q)$ is none of

- $(3,m|3,k|1,q):k\geq 6,\ q\neq 2,$
- $(3,m|n,5|1,q):n\geq 4,\ q\neq 2,$
- $(3,m|n,4|1,q):n\geq 4,\ q\neq 2,$
- $(3,m|n,3|1,q):n\geq 6\ q\neq 2,$

G is finite if and only if it is one of:

- $(1,m|n,k|p,q)$
- $(2,m|n,k|1,1):\frac{1}{m}+\frac{1}{gcd(n,k)}>\frac{1}{2},$
- $(2,m|3,k|1,2):m\geq 4,k\geq 6,gcd(m,k)\leq 5,$
- $(2,2\hat{m}+1|n,3|1,\hat{m}):\hat{m}\geq 2,k\geq 6,gcd(m,k)\leq 5,$
- $(2,4|4,k|1,2):k\geq 4,$
- $(2,4|5,k|1,2):4\leq k\leq 5,$

- $(2,4|n,3|1,2) : 6 \leq n \leq 9$,
- $(2,5|4,4|1,2) : k \geq 4$,
- $(2,5|4,5|1,2)$,
- $(2,5|5,4|1,2)$,
- $(2,6|3,k|1,3) : k \geq 6$,
- $(2,6|7,3|1,2)$,
- $(2,7|n,3|1,2) : 6 \leq n \leq 8$,
- $(2,7|3,k|1,3) : 6 \leq k \leq 8$,
- $(2,8|7,3|1,2)$.

Main Theorem A shall be proven in Section 3, along with the complete characterisation of finiteness in the $l = 2p, m = 2q$ case without the aid of our simplifying assumption. Section 4, based on a paper jointly written with Martin Edjvet, shall prove Main Theorem B, subject to the restriction $l = 2$, while Section 5 covers the $l > 2$ case. Section 2 gives an overview of the methodology used in Sections 3 and 4, and sets up notation and convention for the remainder of the thesis.

As an attribution, we note that the topological model used in Proposition 4.15, and the related models used in other sections, are a simplification of our original model, proposed by a referee to the paper that Section 4 is based upon.

2 Methods

Our approach shall largely be via the notion of *pictures*, graph-like objects similar to van-Kampen Diagrams that represent classes of maps from a disk or a sphere to a space related to the presentation complex of the presentation $\mathcal{P}$ in question. An analysis of these structures allows us to calculate the orders of certain elements of the group G that $\mathcal{P}$ represents, fairly directly, and to calculate the *rational Euler characteristic* of G, a quantity that must be equal to $\frac{1}{|G|}$ if G is finite, in terms of the rational Euler characteristics of the factors of an amalgamated product expression of G. We shall first introduce pictures, along with certain properties that they may possess and operations that may be performed upon them, and then give a brief account of some results that will be of use to us.

2.1 Pictures

In order to define pictures over a presentation $\mathcal{P} = \langle a, b | a^l, b^m, (ab)^n, (a^p b^q)^k \rangle$, we must express $\mathcal{P}$ as a quotient of a free product of cyclic groups (the same construction can be applied to quotients of more general free products, see for example [5], but for our purposes cyclic groups are the appropriate avenue), specifically, we view $\mathcal{P}$ as the quotient of the free product $A * B$, where $A := \langle a | a^l \rangle$ and $B = \langle b | b^m \rangle$, by the relators $(ab)^n$ and $(a^p b^q)^k$, which we shall denote α and β, respectively. Combinatorially, a picture consists of the following: A picture Π over the presentation $\mathscr{P}$ consists of the following:

- A disk D^2, including boundary δD^2
- A collection of disjoint closed disks in the interior of D^2 which we shall call *vertices*
- A finite collection of disjoint arcs in D^2 which we shall call *edges*,

each of which is either a simple closed curve interior to D^2 that does not meet any vertex, or an arc whose endpoints are each in either δD^2 or the boundary of a vertex, and which does not otherwise meet δD^2 or any vertex

- A *labelling*, associating to each vertex a label of $\alpha, \alpha^{-1}, \beta$ or β^{-1}, and to each segment of δD^2 and of the boundary of each vertex, as separated by edges, an element of $A \cup B$, such that reading the labels clockwise around any vertex yields the word assigned to that vertex, as a reduced word in $A * B$ (up to cyclic permutation), and the labels along the boundary of each *region* of Π (that is, each connected component of the complement of the edges and vertices of Π in D^2) are either all elements of A or all elements of B, and have product 1_A, 1_B, respectively.

We refer to vertices with label α as α-vertices, and likewise α^{-1}-,β- and β^{-1}-vertices, referring to vertices whose label is α or β as positively oriented, and vertices whose label is α^{-1} or β^{-1} as negatively oriented. In cases where we do not specify orientation, we refer to vertices with label $\alpha^{\pm 1}$ as vertices of type α, likewise vertices of type β.

We call a region whose boundary labels are all in A an A-region, likewise a B-region. A region that intersects δD^2 is called a *boundary* region, whilst one that does not is called an *interior* region. If no edge of Π meets δD^2, we call the picture *spherical*. In this case, there is only one boundary region, which we refer to as the *distinguished* region.

The boundary label of the picture Π is the cyclically reduced word given by reading the labels of the sections of δD^2 in an anticlockwise direction. In the case of a spherical picture, the boundary label is the inverse of the product of the other labels of the distinguished region. It shall be important to note that the boundary label of a picture is always a cyclically reduced word in $A * B$ representing the identity element of G, and that, conversely, given any such word w, a picture can be constructed whose boundary label is w.

We call a picture empty if it contains no vertex.

Given presentation $\mathcal{P}$, we define the space $Z_{\mathcal{P}}$ to be the space obtained by taking the wedge product of a pair of Eilenberg MacLane spaces of types $K(A, 1)$ and $K(B, 1)$ (which, in a moderate abuse of notation, we shall refer to themselves as A and B, where no ambiguity could arise, and adding a pair of 2-cells α and β with boundary $(\bar{a}\bar{b})^n$ and $(\bar{a}^p\bar{b}^q)^k$, where $\bar{a}$ and $\bar{b}$ are loops in A and B representing a and b, respectively. A map θ from a disk to $Z_{\mathcal{P}}$ can be represented by a picture Γ over $\mathcal{P}$ in the following manner: θ can be perturbed in such a manner that the preimages of α and β are a collection of disjoint disks, and the preimage of the basepoint is a collection of disjoint arcs. If an arc γ had neither endpoint on the boundary of a preimage of α or β, it is surrounded by the preimage of $A \cup B$. A and B only meet at the basepoint, so any loop surrounding γ must pass through a point mapped to the basepoint. Thus, every loop meets a preimage of α or β. Similarly, an arc with only one endpoint on such a preimage would provide us with a path from A to B in $A \cup B$ not passing through the basepoint, so every arc is either a closed curve or has both endpoints on the boundary of a preimage of α or β. Taking the preimages of α and β as our vertices, and labelling each segment of the boundary of each vertex by the element of $\pi_1(Z)$ represented by its image, taking the clockwise direction around the vertex as positive, the labelling requirements of a picture follow directly. This process can be reversed by taking a picture as an instruction set for a map from D to $Z_{\mathcal{P}}$, mapping the edges to the basepoint, and each vertex to α or β in a manner consistent with labelling. In the special case of a spherical picture whose boundary label is trivial in A or B, we can contract the boundary to a disk, giving a map from a sphere to Z. Conversely, given a map from a sphere to Z, we can perform an identical process to the disk case, choosing a region to serve as the distinguished region to construct a spherical picture. Due to this correspondence, elements of $\pi_2(Z)$ can be represented by spherical pictures with trivial boundary. Our approach shall be to use this fact to construct a generating set for $\pi_2(Z)$, in order to show that certain spaces containing, and sharing a 2-skeleton with, Z have trivial second homotopy groups.

We have a procedure for modifying a picture without altering the class of maps that it represents, or only altering this class by a known quantity.

Let γ be an arc whose endpoints are both on existing edges, and which is otherwise on the interior of a region Δ. If all connected components of $\Delta - \gamma$ have boundary labels whose product is still 1_A or 1_B, we may modify Π as per Fig. 1. We refer to this procedure as a *bridge-move*.

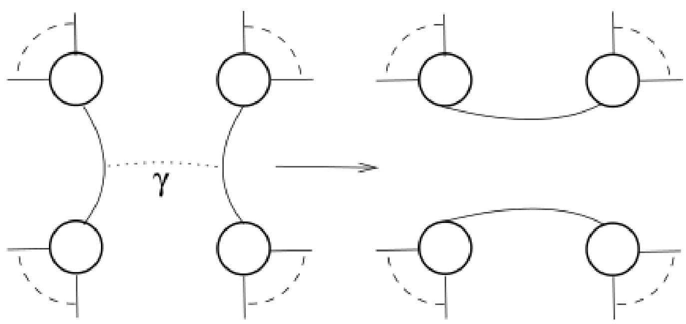

Figure 1: A bridge-move

Addition of or subtraction by an element represented by a spherical picture Γ in $\pi_2(Z)$ corresponds to adding or removing a copy of Γ from some other spherical picture - that is, replacing a simply connected portion of an A- or B-region, as appropriate, with a copy of Γ, as noted, for example, in [10]. Due to this, and the above, a set $\mathcal{S}$ of spherical pictures over $\mathcal{P}$ represent a generating set for $\pi_2(Z)$ if every spherical picture over $\mathcal{P}$ can be reduced to an empty picture (which corresponds to a map into the

Eilenberg MacLane space $A \vee B$) by bridge moves and the addition and removal of elements of $\mathcal{S}$.

Of note are a pair of spherical pictures that exist over every presentation within our scope: we call the picture consisting of an α-vertex and an α^{-1}-vertex, sharing all of their edges, a (proper) dipole over α, $\mathcal{D}_\alpha$. We define $\mathcal{D}_\beta$ likewise. We note that over presentations in which $l = 2p$ and $m = 2q$, a spherical picture consisting of two β-vertices connected in the same manner as $\mathcal{D}_\beta$ arises, which we shall refer to as an *improper* dipole, $\mathcal{D}'_\beta$. Given any edge connecting two vertices of the same type and opposite orientations, we can use bridge-moves to create a proper dipole of the appropriate type. We say that such a pair of vertices *cancels*. A fairly broad selection of the presentations we study give rise to $\pi_2(Z_\mathcal{P})$ generated by proper dipoles, and it shall always be convenient to include them in our set of potential generators, so we shall refer to a spherical picture that can be reduced to an empty picture by bridge moves and the addition and removal of copies of $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ as *dipole-reduced*, defining *dipole-reducible* pictures in a corresponding manner. Similarly, we call a spherical picture $\mathcal{S}$*-reduced* if it can be reduced to an empty picture by bridge moves and the addition and removal of copies of $\mathcal{D}_\alpha$, $\mathcal{D}_\beta$ and $\mathcal{S}$.

Our approach shall be to suppose that some $\mathcal{S}$/dipole-reduced spherical picture exists, and take an example with as few vertices as possible. This excludes from consideration all pictures containing cancelling pairs of vertices, as these could be transformed into a dipole by bridge-moves and removed, and all pictures containing a copy of part of $\mathcal{S}$ containing more than half of the vertices of $\mathcal{S}$, as this would allow us to complete the copy of $\mathcal{S}$ by introducing dipoles and connecting them to the copy via bridge-moves, and remove it, for a lower vertex-count.

2.2 The Associated Graph

We describe a pair of edges between two vertices as *parallel* if they are the only edges on the boundary of a simply connected region. We refer to a pair of parallel edges as a *double edge*, and an edge which is not part of such a pair a *single edge*. Given a connected spherical picture Π, we may form a connected planar graph Γ by identifying each pair of parallel edges into one edge, and contracting each vertex of Π from a disk to a point. In general, when visualising these graphs, we shall depict vertices as disks, labelled by their vertex-type in Π where relevant. Where convenient, we shall annotate labels in the corners of regions by the power of a or b in their original label.

We may define a notion of *curvature* upon the regions of this graph. Defining, for a region Δ whose boundary contains k vertices, of degrees $d_1, d_2, \ldots, d_k$ the curvature

$$c(\Delta) = (2-k)\pi + 2\pi \sum_{i=1}^{k} \frac{1}{d_i},$$

we note that Euler's formula ensures that the sum of the curvature of all regions Δ of Γ must be 4π (we may more directly relate the quantity $c(\Delta)$ to the notion of curvature by contracting δD^2 to a point, so that Γ is a graph on a sphere in which all regions are simply connected, assigning to each corner about a vertex of degree d the angle $\frac{2\pi}{d}$, and viewing $c(\Delta)$ as the discrepancy in total external angle around Δ).

Our method for proving a presentation $\mathscr{P}$ does not admit any nonempty $\mathcal{S}$-reduced spherical picture shall be to show that a vertex-minimal $\mathcal{S}$-reduced spherical picture over $\mathscr{P}$ cannot attain this required total curvature of 4π. We approach this locally, attempting to demonstrate that any positively curved interior region is necessarily accompanied by nearby negatively curved regions with enough negative curvature in total that all nearby positive curvature can be compensated for.

3 The initial case, $p, q \neq \pm 1$

In this section, we consider the presentations $(l, m|n, k|p, q)$, where $p, q > 1$. These presentations can be divided into three families, based on whether neither, one or two of the conditions $l = 2p$ and $m = 2q$ hold. The first family, in which $l = 2p$ and $m = 2q$, allow for a complete characterisation of the parameters for which the resulting groups are finite. The second, in which (by convention over a symmetry, as shall be discussed momentarily) $m = 2q$ but $l \neq 2p$, permits an accounting of finiteness in all but a few subfamilies of cases. The final family, those presentations in which $l \neq 2p$ and $m \neq 2q$, allow finiteness to be characterised under the assumption of large n and k, but for smaller values eludes a curvature-based argument.

As is the case in general, adding any multiple of l to p, or any multiple of m to q, gives an identical group, so we may assume without loss of generality that $p < l$ and $q < m$. Since $(l, m|n, k|l-1, q) \cong (l, m|n, k|1, -q)$, we exclude $p = l-1$, and likewise $q = m-1$, so that $l > p+1$ and $m > q+1$. If any of $l = 1, m = 1, n = 1$, then $(l, m|n, k|p, q)$ trivially provides a finite cyclic group. We also exclude the case $k = 1$ due to the following:

Lemma 3.1. *Let $G := \langle a, b | a^l, b^m, (ab)^n, a^p = b^{-q} \rangle$. Then G is infinite if and only if*

$$\frac{1}{hcf(l,p)} + \frac{1}{hcf(m,q)} + \frac{1}{n} \leq 1.$$

Proof. Let x be the highest common factor of l and p, and y the highest common factor of m and q, so that we can write

$$G = \langle a, b | (a^x)^{l'}, (b^y)^{m'}, (ab)^n, (a^x)^{p'} = (b^y)^{-q'} \rangle.$$

Then since l' and p' are coprime, (a^x) can be expressed as a power of $(a^x)^{p'}$, which is itself equal to a power of (b^y). Likewise, (b^y) can be expressed as a power of (a^x). Thus, $\langle (a^x) \rangle = \langle (b^y) \rangle =: N$. Since N is generated by a power of a, N commutes with a. Likewise N commutes with b. Thus N is normal in G.

The quotient G/N can be presented as $\langle a, b | a^x, b^y, (ab)^n \rangle$, so is the von Dyck group on parameters x, y, n, which is infinite precisely when $\frac{1}{x} + \frac{1}{y} + \frac{1}{n} \leq 1$, as required. $\square$

It may be of note that, since the subgroup N in this argument is cyclic, G is solvable precisely when the von Dyck group (x, y, n) is. In particular, the cases leading to a finite, insolvable G are precisely those in which x, y and n are some permutation of 2, 3 and 5.

Note also that this $k = 1$ case gives examples of what we shall later refer to as collapse: presentations that give rise to groups in which a and b need not have orders l and m. For instance, the presentation

$$\langle a, b | a^{20}, b^{20}, (ab)^6, a^4 = b^5 \rangle$$

gives rise to a group in which a^{16} is trivial.

3.1 The restricted case: $l = 2p$, $m = 2q$

Over these presentations, pairs of β- or β^{-1}-vertices can form improper dipoles. As noted in section 2, we therefore seek conditions on our parameters under which any spherical picture over these presentations can be reduced, by bridge moves and the addition and removal of proper and improper dipoles, to an empty picture. These conditions may be discovered by considering an example with fewest vertices amongst those that cannot be so reduced. It shall be useful to note that, in such a minimal picture, no edge may be found between any two vertices of type β, as this would allow the formation of a dipole (either proper or improper, depending on whether the two vertices had the same orientation) by bridge moves, which could then be removed to provide a picture with fewer vertices. As such, all edges are either between like-oriented vertices of type α or between a vertex of type α and a vertex of type β. Since none of these connections admit multiple edges, vertices of type α have valence $2n$, whilst vertices of type β have valence $2k$.

We shall begin with some preliminary results

3.1.1 Preliminary results

Lemma 3.2. *Let $q \geq 4$. Then the group G obtained from the presentation $(4, 2q|2, 2|2, q)$ is infinite.*

Proof. Take the quotient of G by $\langle\langle b^q \rangle\rangle$, to obtain the group $H := \langle a, b | a^4, b^q, (ab)^2 \rangle$. H is the triangle group $(4, q, 2)$. Thus, H (and so G) is infinite if $\frac{1}{4} + \frac{1}{q} + \frac{1}{2} \leq 1$, which is the case, since $q \geq 4$. $\square$

Lemma 3.3. *The group G with presentation $(4, 4|2, k|2, 2)$ is a finite solvable group of order $8k^2$, with derived subgroup $\mathbb{Z}_k^2$.*

Proof. This group is a quotient of the solvable triangle group $(4, 4, 2) = \langle a, b | a^4, b^4, (ab)^2 \rangle$ by the normal closure of the element $(a^2b^2)^k$, so is itself solvable. $(4, 4, 2)$ is a Euclidean triangle group, so has no infinite proper quotients, thus to establish finiteness of G it suffices to demonstrate that $(a^2b^2)^k$ is nontrivial in $(4, 4, 2)$. Consider the action of $(4, 4, 2)$ on $\mathbb{R}^2$ in which a acts by a rotation of $\pi/2$ anticlockwise about $(0, 0)$, and b by a rotation of $\pi/2$ anticlockwise about $(1, 1)$, so that (ab) is a rotation by π about $(1, 0)$. Then a^2b^2 acts as a translation by $(2, 2)$ Thus, $(a^2b^2)^k$ acts as a translation by $(2k, 2k)$, so $(a^2b^2)^k$ is nontrivial in $(4, 4, 2)$ for all positive k, as required.

To identify the order and derived subgroup of G, we consider its Abelianisation, G_{Ab}, a direct product of $\mathbb{Z}_4$ and $\mathbb{Z}_2$, which can be generated as such by the images of a and ab. As such, the derived subgroup G' of G has index 8. We consider the corresponding cover of the presentation complex of G.

The vertices of this cover correspond to the elements of G_{Ab}. We give the vertices corresponding to the images of e, a, a^2 and a^3 the labels v_0, v_1, v_2, v_3, respectively, and those corresponding to the images of ab, aba, aba^2 and aba^3 the labels V_0, V_1, V_2, V_3, respectively.

The 1-cells of the presentation complex of G corresponding to the generators a and b lift to edges:

$$\begin{array}{llll} a_0 := (v_0, v_1) & a_1 := (v_1, v_2) & a_2 := (v_2, v_3) & a_3 := (v_3, v_4) \\ A_0 := (V_0, V_1) & A_1 := (V_1, V_2) & A_2 := (V_2, V_3) & A_3 := (V_3, V_4) \\ b_0 := (v_0, V_3) & b_1 := (v_1, V_0) & b_2 := (v_2, V_1) & b_3 := (v_3, V_2) \\ B_0 := (V_0, v_3) & B_1 := (V_1, v_0) & B_2 := (V_2, v_1) & B_3 := (V_3, v_2) \end{array}$$

The relator a^4 lifts to $a_0a_1a_2a_3$ and $A_0A_1A_2A_3$. b^4 lifts to $b_0B_3b_2B_1$ and $B_0b_3B_2b_1$. $(ab)^2$ lifts to $a_0b_1A_0B_1$, $a_1b_2A_1B_2$, $a_2b_3A_2B_3$ and $a_3b_0A_3B_0$. $(a^2b^2)^k$ lifts to $(a_0a_1b_2B_1)^k$, $(a_1a_2b_3B_2)^k$, $(a_2a_3b_0B_3)^k$, $(a_3a_0b_1B_0)^k$, $(A_0A_1B_2b_1)^k$, $(A_1A_2B_3b_2)^k$, $(A_2A_3B_0b_3)^k$ and $(A_3A_0B_1b_0)^k$.

Taking $a_0, a_1, a_2, b_0, b_1, b_2, b_3$ as a spanning tree for the 1-skeleton of our cover, this gives us a presentation for G' with generators $a_3, A_0, A_1, A_2, A_3, B_0, B_1, B_2$ and B_3, and relators a_3, $A_0A_1A_2A_3$, B_3B_1, B_0B_2, A_0B_1, A_1B_2, A_2B_3, A_3B_0, B_1^k, B_2^k, B_3^k, B_0^k, $(A_0A_1B_2)^k$, $(A_1A_2B_3)^k$, $(A_2A_3B_0)^k$ and $(A_3A_0B_1)^k$. This reduces, via Tietze transformations, to the presentation $\langle A_0, A_1 | A_0^k, A_1^k, [A_0, A_1]\rangle$, establishing that $G' \cong \mathbb{Z}_k^2$, as required.

□

3.1.2 Curvature arguments

We begin by identifying the positively curved regions that may occur in pictures over these presentations.

Lemma 3.4. *Let $p, q, n, k \geq 2$, and let Γ be picture over the presentation $(2p, 2q|n, k|p, q)$. If Γ cannot be reduced to an empty picture by bridge moves and the introduction and removal of dipoles, both proper and improper, then* *the only possible positively curved interior regions in* Γ *are those consisting of a triangle between two like-oriented vertices of type* α *and one of type* β.

Proof. Let Δ be a positively curved interior region in a picture Γ satisfying the conditions above. Since vertices of type α and type β have valence $2n \geq 4$ and $2k \geq 4$, respectively, no region with more than four vertices could possibly be positively curved, so we may restrict our attention to triangular regions. A triangular region can have at most one vertex of type β, else two of them would share an edge. A triangular A-region or B-region with three α vertices could only exist if $2p|3$ or $2q|3$, respectively. Neither of these can occur, since $p, q \geq 2$. Thus the only possibility is a triangular region with precisely one vertex of type β, as required. $\square$

Observe that a such a region can only occur as an A-region when $2p|p\pm2$, so that $p = 2$. Likewise, such a region can only occur as a B-region when $q = 2$. Further, a region Δ consisting of two vertices of type α and one of type β has curvature $c(\Delta) = c(2n, 2n, 2k) = -\pi + 2\pi(\frac{1}{n} + \frac{1}{2k})$, which is positive, within our established scope, when either $n = 2$ or $(n, k) = (3, 2)$. As such, we have the following:

Lemma 3.5. *Let* $p, q, n, k \geq 2$. *If either* ($p \neq 2$ *and* $q \neq 2$) *or* ($n \neq 2$ *and* $(n, k) \neq (3, 2)$)*, then every spherical picture over the presentation* $(2p, 2q|n, k|p, q)$ *is* $\mathcal{D}'_\beta$*-reducible.*

Proof. Let Γ be a vertex-minimal example of a non-$\mathcal{D}'_\beta$-reducible spherical picture over $(2p, 2q|n, k|p, q)$. If the first condition, $p \neq 2$ and $q \neq 2$, holds, then no internal region of Γ can be a triangle consisting of two vertices of type α and one of type β, so by Lemma 3.4 no internal region of Γ is positively curved. If the second condition, $n \neq 2$ and $(n, k) \neq (3, 2)$, holds, then triangles consisting of two vertices of type α and one of type β are not positively curved, so by Lemma 3.4 Γ cannot have any positively curved interior regions. Thus, in either case, all 4π curvature of Γ must originate in the distinguished region. However, the curvature of a single region cannot exceed 2π. Thus, our hypothetical Γ cannot exist, and so, in the absence of a vertex-minimal example, there can be no non-$\mathcal{D}'_\beta$-reducible spherical picture over $(2p, 2q|n, k|p, q)$. □

We consider, therefore, the cases where these conditions fail. Since $n = 2$ and $(n, k) = (3, 2)$ are mutually exclusive, we deal with them separately. The cases in which $p = q = 2$ are dealt with by other approaches, so we assume that $p = 2, q \geq 3,$.

Lemma 3.6. *Let $k \geq 3$, and suppose that $p = 2, q \geq 3$ and $(k, q) \neq (3, 3)$. Then every spherical picture over $(2p, 2q|2, k|p, q)$ is $\mathcal{D}'_\beta$-reducible.*

Proof. Suppose we have a vertex-minimal counterexample Γ. Due to Lemma 3.4, every positively curved interior region of Γ is a triangular A-region with two vertices of type α and one of type β, and every interior B-region is at least a 4-region. Take a positively curved interior region Δ, and label adjacent regions as in Figure 2. Δ_L and Δ_R are both B-regions, so each must be either the distinguished region or a region of degree at least 4. We wish to empty Δ of its $2\pi(\frac{1}{n} + \frac{1}{2k} - \frac{1}{2}) = \frac{\pi}{k}$ curvature, so we transfer $\frac{\pi}{2k}$ to each of Δ_L and Δ_R.

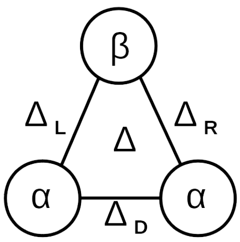

Figure 2: Neighbourhood of Δ

We first address the cases in which $k \geq 4$, so that q may be equal to 3.

A region with r vertices has curvature bounded above by $c(4,\ldots,4)$, which is nonpositive for $r \geq 4$, and swapping a vertex of type α, with valence 4, for one of type β, with valence $2k$, reduces a region's curvature by $\frac{\pi}{2} - \frac{\pi}{k}$. Thus, a region with $r \geq 4$ vertices, s of which are of type β, has curvature bounded above by $-s(\frac{\pi}{2} - \frac{\pi}{k})$. Since a B-region only receives curvature over edges between vertices of different types, any region that receives curvature over t edges must have at least $\frac{t}{2}$ vertices of type β. Thus, after the transfer of curvature, the curvature of a region that receives curvature over t edges is bounded above by $-\frac{t}{2}(\frac{\pi}{2} - \frac{\pi}{k}) + t\frac{\pi}{2k}$, which is nonpositive since $k \geq 4$.

For the remaining possibility, $k = 3$, so that $q > 3$, we begin our redistribution of curvature similarly, noting that the difference in curvature provided by a vertex of type β, with valence 6 and one of type α, of valence 4, is $\frac{\pi}{6}$, so since every B-region has degree at least four, we can transfer $\frac{\pi}{12}$ to each of Δ_L and Δ_R without any possibility of either of them becoming positively curved. In fact, the curvature of any B-region after this initial transfer of curvature is bounded above by the curvature of a region of the same degree whose vertices all have valence 4. We shall exploit this to establish that the $\frac{\pi}{6}$ remaining in Δ can be transferred across its remaining edge to Δ_D without Δ_D becoming positively curved.

Since $q > 3$, so that $2q \nmid 4$, Δ_D cannot be a square of four vertices of type α. Likewise, we have $2q \nmid q \pm 3$, so Δ_D cannot have one vertex of type β and three of type α. Since vertices of type β cannot share an edge, and Δ_D must have an edge between two vertices of type α, Δ_D cannot be a square with two or more vertices of type β, so we can rule out Δ_D having degree four.

The curvature of a region of degree 5 after the first round of curvature transfer is bounded above by $c(4,4,4,4,4) = \frac{-\pi}{2}$, so can receive curvature over up to three α-α-edges. The only case where such a region could be required to receive over four or more α-α-edges is if Δ_D has only vertices of type α. Since $2q \nmid 5$, such a region cannot exist.

After the first round of transfer, a region of degree $d \geq 6$ has curvature not exceeding $2\pi - \frac{d\pi}{2} \leq -d\frac{\pi}{6}$, so if Δ_D is such a region it can accept $\frac{\pi}{6}$ across each edge without becoming positively curved.

Thus, in both the $k \geq 4$ case and the $k = 3$ case, after all transfer of curvature, no interior region is positively curved, and so all 4π curvature of Γ must be in the distinguished region. The distinguished region has at most 2π intrinsic curvature, so this requires that another 2π curvature be received. This could only occur if the distinguished region had degree at least six, at which point the arguments establishing that interior regions receiving curvature remain nonpositively curved also apply to the distinguished region. Thus, a total curvature of 4π cannot be achieved, and so the hypothesised vertex-minimal counterexample Γ cannot exist. Thus, no counterexample can exist, so the lemma holds.

$\square$

Lemma 3.7. *Suppose $p = 2, q > 2$. Then every spherical picture over $(2p, 2q|3, 2|p, q)$ is $\mathcal{D}'_\beta$-reducible.*

Proof. As above, we choose, without loss of generality, $p = 2, q \neq 2$, and suppose a vertex-minimal counterexample Γ. By Lemma 3.4, all positively curved interior regions are A-regions forming a triangle with one vertex of type β and two of type α, and thus curvature $C(4, 4, 6) = \frac{\pi}{6}$, and all interior B-regions have degree at least 4. Let Δ be a positively curved interior region. By a similar argument to that used in the previous lemma, an interior B-region of degree r, with s α-vertices has curvature bounded above by $-s\frac{\pi}{6}$. Thus, if we transfer all $\frac{\pi}{6}$ curvature out of each positively curved region across the edge between its two vertices of type α, every receiving region will have as many vertices of type α as it has edges over which it receives curvature. As such, this transfer leaves no interior region with positive curvature, so all 4π curvature in Γ must be in the distinguished region. However, if the distinguished region has degree at least four, the same argument as applied to interior receiving regions demonstrates that its curvature after transfers is nonpositive, whilst if the distinguished region has degree less than four, it cannot receive more than $\frac{\pi}{2}$ curvature in transfers, so the total curvature of Γ is bounded above by $\frac{5\pi}{2} < 4\pi$. In either case, this contradicts our assumption that a vertex-minimal counterexample exists, so the lemma holds. □

With the equivalences between homotopy classes of maps and spherical pictures noted in section 2, this gives us the following:

Corollary 3.8. *Let the presentation $\mathcal{P} = (2p, 2q|n, k|p, q)$ satisfy the conditions of one of Lemma 3.5, Lemma 3.6 or Lemma 3.7. Then the second homotopy group of the space $Z_{\mathcal{P}}$, as defined in section 2, has a generating set whose elements correspond to the homotopy classes of maps represented by the proper dipoles $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ along with the improper dipole $\mathcal{D}'_\beta$.*

These results also allow us to obtain the orders of a, b, (ab) and $(a^p b^q)$ in the resulting groups:

Corollary 3.9. *Let p, q, n, k satisfy the conditions of one of Lemma 3.5 Lemma 3.6 or Lemma 3.7. Then the orders of a b, (ab) and $(a^p b^q)$ in the group with presentation $(2p, 2q|n, k|p, q)$ are precisely $2p$, $2q$, n and k, respectively.*

Proof. These lemmas establish that spherical pictures over the presentations in question are $\mathcal{D}'_\beta$-reduced, that is, any spherical picture over any such presentation can be reduced to an empty picture via bridge moves and the addition and removal of proper and improper dipoles. Note that none of these operations change the boundary label, as an element of $\langle a|a^{2p}\rangle * \langle b|b^{2q}\rangle$.

Suppose first, then, that a has order $r < 2p$ in the group with presentation $(2p, 2q|n, k|p, q)$. Then there exists some picture over $(2p, 2q|n, k|p, q)$ whose boundary label is equal to a^r, which is nontrivial in $\langle a|a^{2p}\rangle * \langle b|b^{2q}\rangle$. By bridge-moves at the boundary, we may convert this into a spherical picture whose boundary label is a^r. This picture is, by Lemma 3.5, Lemma 3.6 or Lemma 3.7, as appropriate, reducible by bridge moves and the addition and removal of proper and improper dipoles to an empty picture. The last step of this reduction must be the removal of a dipole (proper or improper), so must be performed upon a picture that consists entirely of that dipole. Since neither proper nor improper dipoles have boundary label nontrivial in $\langle a|a^{2p}\rangle * \langle b|b^{2q}\rangle$, however, so such a picture cannot be reached by our set of reductions from the spherical picture we began with. This contradiction shows that our assumption that $r < 2p$ cannot hold, so a must have order $2p$, as required. An identical argument applies to the order of b.

Suppose now, then, that (ab) had order $r < n$. We may therefore take a vertex-minimal picture Γ with boundary label $(ab)^r$, and construct a spherical picture Γ' by attaching n/r copies of Γ around an α^{-1}-vertex. Note that the number of α-vertices in Γ' minus the number of α^{-1}-vertices in Γ' is congruent to -1 modulo n/r, and in particular is nonzero. Neither bridge moves, nor the addition or removal of dipoles, change this quantity, so this spherical picture cannot be $\mathcal{D}'_\beta$-reducible, contradicting Lemma 3.5, Lemma 3.6 or Lemma 3.7, as appropriate. Thus, our initial assumption that $r < n$ cannot hold, so (ab) has order n.

We must approach a^pb^q differently. Suppose that the order of a^pb^q is not k. Then there exist pictures whose boundary label is some $(a^pb^q)^{k'}$, where $0 < k' < k$. We take a vertex-minimal picture with this property, Π. Π cannot have any connected component that does not share an edge with the boundary, as the boundary label of such a component would have to be either trivial in $A * B$, in which case it could be removed to obtain a picture with the same boundary and fewer vertices, or a nontrivial element of $\langle a|a^{2p}\rangle$ or $\langle b^{2q}\rangle$ (since an element of $\langle a|a^{2p}\rangle * \langle b^{2q}\rangle$ not in $\langle a|a^{2p}\rangle$ or $\langle b^{2q}\rangle$ would force the component to have an external edge, which would have to reach the boundary), which would require that the order of a or b not be $2p$ or $2q$, respectively, contradicting the results we have already obtained.

We contract the boundary of Π to a disk, to form a spherical picture Π', with one non-standard vertex whose label is $(a^p b^q)^{k'}$, which we shall call the distinguished vertex, v_0. Since each connected component of Π met the boundary, each connected component of $\Pi' - v_0$ shares an edge with v_0, so Π' is connected. If the distinguished vertex shares an edge with a vertex of type β, we may perform bridge moves as we would when forming a dipole, adding more edges between the two vertices until all of the distinguished vertices edges are to the same vertex of type β. Treating this vertex and the distinguished vertex as a single vertex, we obtain a new distinguished vertex, with boundary label $(a^p b^q)^{k-k'}$. Repeating this process for as long as possible (noting that the label of the distinguished vertex alternates between $(a^p b^q)^{k'}$ and $(a^p b^q)^{k-k'}$, and that each iteration of the process decreases the number of vertices of type β by one, so this procedure cannot continue indefinitely), Π' eventually reaches a state in which the distinguished vertex is not adjacent to any vertex of type β. Regions not incident to the distinguished vertex are subject to the same arguments made in Lemma 3.5, Lemma 3.6 or Lemma 3.7, as appropriate, so cannot be positively curved. We can apply the same arguments to regions incident to the distinguished vertex to see that all positive curvature in these regions arises from the difference in contribution between a vertex of degree $2k$ and one of degree $2k'$ or $2(k-k')$. Thus, if the distinguished vertex has label $(a^p b^q)^{k'}$, the $2k'$ regions that it is incident to have curvature bounded above by $2\pi(\frac{1}{2k'} - \frac{1}{2k}) < \frac{2\pi}{k'}$, for a total less than 2π, and likewise if the distinguished vertex has label $(a^p b^q)^{k-k'}$, the $2k - 2k' =: 2k''$ regions that it is incident to have curvature bounded above by $2\pi(\frac{1}{2k''} - \frac{1}{2k}) < \frac{2\pi}{2k''}$, giving, again, a total positive curvature less than 2π. As such, with interior curvature providing less than 2π curvature, and the distinguished region doing the same, the 4π curvature that Π' must have as a spherical picture is absent. Thus, our starting assumption that the order of $a^p b^q \neq k$ cannot hold. $\square$

3.1.3 The Homological Argument

After the previous segments of the argument, we have established that, if the presentation $\mathcal{P} = (2p, 2q|n, k|p, q)$ satisfies the conditions of Lemma 3.5, Lemma 3.6 or Lemma 3.7, all spherical pictures over $\mathcal{P}$ are $\mathcal{D}'_\beta$-reducible, and so the second homotopy group of the space $Z_\mathcal{P}$, as defined in section 2, is generated by elements represented by proper dipoles and $\mathcal{D}'_\beta$. We have also established that collapse does not occur in any of the parameters of these presentations (i.e. a and b have orders precisely $2p$ and $2q$, (ab) and $(a^p b^q)$ have orders precisely n and k).

We exploit these properties to identify the order of $G(\mathcal{P})$ in the following

Proposition 3.10. *Let G be the group with presentation $\mathcal{P} = (2p, 2q|n, k|p, q)$ satisfying the conditions of one of Lemma 3.5, Lemma 3.6 or Lemma 3.7. Then G is infinite.*

Proof. We construct G as the push-out of groups

$$\begin{array}{ccc} G_0 := \langle c, d | \rangle & \xrightarrow{\phi} & \langle c, d | c^2, d^2, (cd)^k \rangle =: G_1 \\ \downarrow{\scriptstyle\psi} & & \downarrow \\ G_2 := \langle a, b | (ab)^n \rangle & \longrightarrow & \langle a, b | a^{2p}, b^{2q}, (ab)^n, (a^p b^q)^k \rangle = G \end{array}$$

where ϕ maps c to c and d to d, whilst ψ maps c to a^p and d to b^q.

To obtain the desired result from this pushout, we seek to use the argument of [4]. This requires that we construct an aspherical space X, with trivial second homotopy group, that we can express as a union of two aspherical spaces X_1, X_2 with fundamental groups G_1, G_2, respectively, whose intersection X_0 has fundamental group G_0, with ϕ and ψ realised by the inclusions of X_0 into X_1 and X_2.

We begin by constructing a preliminary space X', which we shall extend via extensions of subspaces X_1' and X_2' satisfying the fundamental group requirements of X_1 and X_2 to form the required space.

Let X' be the presentation complex for the presentation

$$\mathcal{P}' := \langle a, b, c, d | c = a^p, d = b^q, c^2, d^2, (ab)^n, (cd)^k \rangle.$$

We shall denote the subspace of X' corresponding to a presentation $\mathcal{Q}$ whose generators and relators are found amongst those of $\mathcal{P}'$ by $X(\mathcal{Q})$. Let X_1' be $X(\langle c, d | c^2, d^2, (cd)^k \rangle)$ and X_2', $X(\langle a, b, c, d | a^p = c, b^q = d, (ab)^n \rangle)$. Then $X' = X_1' \cup X_2'$ and $X_0' := X_1' \cap X_2' = X(\langle c, d | \rangle)$. These spaces satisfy the conditions on first homotopy required of X, X_1, X_2 and X_0, so we proceed to extend them.

We extend X_1' in stages. First, we extend $X(\langle c, c^2 \rangle) \subset X_1'$, adding cells of dimension three and higher to eliminate all higher homotopy, and labeling the union of the added cells $Y_{1(a)}$. We then extend $X(\langle d, d^2 \rangle)$ likewise, labeling the union of the added cells $Y_{1(b)}$. In the same manner, we then extend the space $X_1 \cup Y_{1(a)} \cup Y_{1(b)}$, labeling the union of all cells added in this series of extensions Y_1.

Similarly, we extend X_2' in stages. First extending $X(\langle a, c | a^p = c \rangle)$, labeling the union of added cells $Y_{2(a)}$ then extending $X(\langle b, d | b^q = d \rangle)$, labeling the union of added cells $Y_{2(b)}$, and finally extending $X_2' \cup Y_{2(a)} \cup Y_{2(b)}$, labeling the union of all cells added in this series of extensions Y_2.

We define $X_1 := X_1' \cup Y_1$ and $X_2 := X_2' \cup Y_2$, taking $X := X_1 \cup X_2$ and $X_0 := X_1 \cap X_2 = X_0'$. Each X_i is an Eilenberg-MacLane space with 2-skeleton X_i', so we have an expression of X as a union of aspherical spaces satisfying the required conditions on first homotopy.

We now aim to use Theorem 4.2 of [10] to show that X is aspherical. For this approach to work, we require that the kernels of the maps in first homotopy induced by the inclusion of X_0, X_1 and X_2 into X (that is, of the maps from G_0, G_1 and G_2 to G) have homological dimension not exceeding 1, 2 and 2, respectively, and that $\pi_2(X) = 0$. The kernels can be dealt with fairly directly:

Since G_0 is free, the kernel K_0 of the map from G_0 to G must be free, and thus has homological dimension of at most 1.

Since, by the same corollary, the orders of a, b and $(a^p b^q)$ do not collapse in G, the kernel K_1 of the map from G_1 to G must not contain c, d or any nontrivial powers of (cd). However, every nontrivial element of $G_1 \cong D_{2k}$ is conjugate to one of these, so K_1 is trivial, and so certainly has homological dimension not exceeding 2.

Since, by the same corollary, the orders of a and (ab) do not collapse in G, the kernel K_2 of the map from G_2 to G cannot contain any nontrivial power of a or (ab). Since G_2 can be expressed as $\mathbb{Z} * \mathbb{Z}_n$, with generators a and (ab), every normal subgroup of G_2 must either contain some nontrivial power of one of a, (ab), or be free. Thus, K_2 is free, and so has homological dimension of at most 1.

To show that $\pi_2(X) = 0$, we note that Corollary 3.8, gives generators for the second homotopy group of Z, a space consisting of the wedge sum of A and B, where A is a $K(C_{2p}, 1)$ space whose fundamental group is generated by a loop $\bar{a}$ and B is a $K(C_{2q}, 1)$ space whose fundamental group is generated by a loop $\bar{b}$, augmented by 2-cells whose boundary labels are $(\bar{a}\bar{b})^n$ and $(\bar{a}^p \bar{b}^q)^k$.

We construct Z as a subspace of X. Let $A = X(\langle a, c | c = a^p, c^2 \rangle) \cup Y_{1(a)} \cup Y_{2,(a)}$, with the loop corresponding to a taking the role of $\bar{a}$, and similarly let $B = X(\langle b, d | d = b^q, d^2 \rangle) \cup Y_{1(b)} \cup Y_{2,(b)}$, with the loop corresponding to b taking the role of $\bar{b}$. Add the 2-cells associated to the relators $(ab)^n$

and $(a^p b^q)^k$ to complete Z. Note that Z so constructed contains the entire 2-skeleton of X, so that every element of $\pi_2(X)$ can be represented by an element of $\pi_2(Z)$. In particular, maps representing a generating set of $\pi_2(Z)$ also represent a generating set of $\pi_2(Z)$. Thus, if we can show that the generators of $\pi_2(Z)$ found in Corollary 3.8 all represent trivial elements of $\pi_2(X)$, then $\pi_2(X)$ is trivial as required. Noting that X_1 and X_2 are aspherical, it suffices to show that every element of $\pi_2(Z)$ can be represented by a map whose image is within either X_1 or X_2.

Recall from Section 2 the process for converting a spherical picture (with trivial boundary label) Γ over $\mathcal{P}$ into a map from S^2 to Z. Contracting the boundary of Γ to a point, to obtain a picture on a sphere rather than a disk, we map edges to the basepoint, corners labelled with powers of a and b to the loops $\bar{a}$ and $\bar{b}$, with appropriate orientation and multiplicity, depending on the power of a or b present, the interiors of vertices to the interior of the 2-cell corresponding to the same relator, and A and B regions to the subspaces A and B. Note that A- or B-regions whose corner labels are all of the form $a^{\pm p}$ or $b^{\pm q}$ can be mapped into the cells corresponding to c^2 d^2, respectively, in X_2.

The dipole over α contains A-regions, B-regions and vertices of type α, so can be represented by a map from S^2 whose image is contained in the union of A, B and the 2-cell corresponding to $(ab)^n$. Restricting our attention to a B-region Δ in this picture, we observe that the map in this region is a map from a rectangle to B, in which one pair of opposing sides are mapped to the basepoint, and the other pair mapped along $\bar{b}$ in parallel. As such, this restricted map corresponds to a homotopy between $\bar{b}$ and itself in B. Since B is an Eilenberg-MacLane space, any two homotopies between the same maps are themselves homotopic, so without changing the homotopy class of our map, we may use the trivial homotopy between $\bar{b}$ and itself as the interior of the rectangle, so that the entirety of Δ is mapped to the image of $\bar{b}$. Likewise, without any risk of changing the homotopy class of our map, we may assume that each A-region is mapped entirely to $\bar{a}$. Thus, the dipole over α can be represented by a map whose image lies within X_1, and as such must represent the trivial element of $\pi_2(X)$.

Dipoles over β, both proper and improper, are formed entirely of vertices of type β, A-regions whose corner labels are all $a^{\pm p}$, and B-regions whose corner-labels are all $b^{\pm q}$, so can be mapped in such a manner as to have image contained in the union of the cells associated to c, d, c^2, d^2 and $(cd)^k$, all of which are in X_1. Thus, these pictures represent trivial elements of $\pi_2(X)$.

Having found a generating set of $\pi_2(X)$ and shown that its elements are all trivial, we may conclude that $\pi_2(X)$ is trivial, as required, so following [10], X is aspherical. Thus, in the language of [4], our pushout is geometrically Mayer-Vietoris. If G is finite, this allows us to calculate

$$\frac{1}{|G|} = \chi_{\mathbb{Q}}(G) = \chi_{\mathbb{Q}}(G_1) + \chi_{\mathbb{Q}}(G_2) - \chi_{\mathbb{Q}}(G_0)$$

$$= \frac{1}{2k} + \left(\frac{1}{n} - 1\right) + 1.$$

This gives $\frac{1}{|G|} > \frac{1}{n}$, so that $|G| < n$, however by Corollary 3.9, G contains an element of order n. Thus, G cannot be finite. $\square$

3.1.4 Assembling Results

In the final part of this section, we combine our results to characterise the conditions under which $(2p, 2q|n, k|p, q)$ presents a finite group.

Theorem 3.11. *Let G be have presentation $\mathcal{P} = (2p, 2q|n, k|p, q)$ with $p, q \neq 1$. Then G is finite if and only if one of the following conditions hold:*

(i) $n = 1$

(ii) $k = 1$ *and* $\frac{1}{p} + \frac{1}{q} + \frac{1}{n} > 1$

(iii) $p = q = n = 2$

(iv) $p = n = 2$, $q = k = 3$

(v) $p = k = 3$, $q = n = 2$

Proof. If $n = 1$ then $\mathcal{P}$ satisfies condition (i), and is a finite cyclic group. If $k = 1$ then by Lemma 3.1 G is finite if and only if condition (ii) holds. Thus, we may restrict our attention to presentations in which $n, k > 1$.

If [$p > 2$ and $q > 2$] or [$n \neq 2$ and $(n, k) \neq (3, 2)$], then G is infinite, by Proposition 3.10 as applied to Lemma 3.5. The remaining presentations are those in which either one or both of p and q is 2, and either $n = 2$ or $(n, k) = (3, 2)$.

If $(n, k) = (3, 2)$ and both p and q are 2, then $(2p, 2q|n, k|p, q) = (4, 4|3, 2|2, 2)$, whose third derived subgroup, by calculation in [7], has infinite Abelianisation. If only $p = 2$, then Proposition 3.10 as applied to Lemma 3.7 shows that G is infinite. Likewise, noting the symmetry between p and q, if only $q = 2$ then G is infinite.

If $n = 2$ and both p and q are 2, then $\mathcal{P}$ satisfies (iii), and is finite by Lemma 3.3. If only p is 2, then if $k \neq 2$ and $(k, q) \neq (3, 3)$, Proposition 3.10 as applied to 3.6 shows that G is infinite. If $k = 2$, then G is infinite via Lemma 3.2, while if $(k, q) = (3, 3)$, $\mathcal{P} = (4, 6|2, 3|2, 3)$, satisfying (iv), whose second derived subgroup, of index 6, is isomorphic to $PSL(2, 11)$, by calculation in [7], giving an order of 3960. As above, noting the symmetry between p and q, if only q is 2, then G is infinite unless $(k, p) = (3, 3)$, satisfying (v), which gives the same group of order 3960.

$\square$

3.2 The Intermediate Case: $l \neq 2p$, $m = 2q$

We here consider the groups $(l, 2q|n, k|p, q)$. Every group with presentation $(2p, m|n, k|p, q)$ with $m \neq 2q$ is isomorphic to the group with presentation $(m, 2p|n, k|q, p)$, so this family of presentations covers, up to isomorphism, all presentations $(l, m|n, k|p, q)$ in which precisely one of $l = 2p$ or $m = 2q$ holds. Since $(l, m|n, k|l-1, q)$ is Coxeter's group $(l, m|n, k)$, we restrict our parameters in this section to $1 < p < l - 1$, $q \geq 2$, forcing $l > 4$, since otherwise p would have to be one of $1, \frac{l}{2}, l - 1$. We extend our conditions on n and k to $n \geq 3$, $k \geq 3$.

Over these presentations, no improper dipole can exist, but adjacent pairs of like-oriented vertices of type β can form double edges containing A-regions. Since a bridge-move in an A-region alters no other A-regions, so long as a picture Γ contains a like-oriented pair of vertices of type β connected by only a single edge, we can perform a bridge-move in Γ decreasing the number of such edges. Repeating this process until we are unable to continue, we find Γ in a configuration such that any two like-oriented adjacent vertices of type β share a double-edge. As in the previous subsection, we shall consider vertex-minimal spherical pictures that are reduced with regard to various sets of spheres, always including the proper dipoles. Thus, in the settings we consider, we may also assume that no edge exists between two vertices of the same type and opposite orientation. As such, the only types of edges present in pictures within the scopes we shall consider are single edges between like-oriented vertices of type α, double edges between like-oriented vertices of type β and single edges between vertices of differing type. This allows us to calculate that the valence of a vertex of type α is always $2n$, whilst the valence of a vertex of type β with r neighbours of type β is $2k - r$ In particular, a vertex of type β incident to at least s B-regions of degree greater than two (which we shall, for the remainder of

Section 3.2 refer to as 'proper' B-regions) and adjacent to at least t vertices of type β has valence between $k+s$ and $2k-t$.

3.2.1 Curvature Arguments

We begin by identifying the potentially positively curved regions that may occur in pictures over these presentations, and the conditions on our parameters required for them to exist, and for them to have positive curvature.

Lemma 3.12. *Let $1 < p < l-1$, $1 < q$, $n \geq 3$, $k \geq 3$. Let Δ be a positively curved interior region in a dipole-reduced spherical picture Γ over $\mathcal{P} = (l, 2q|n, k|p, q)$. Then Δ takes the form of one of the regions depicted in Figure 3. If Δ takes the form of $\Delta_{(j)}$ in Figure 3, then the conditions (j) and (j') below hold.*

(i) $q = 2$

(ii) $l = p \pm 2$ *or* $p = 2$

(iii) $l = 2p \pm 1$

(iv) $l \in \{3p, 3p/2\}$

(v) $l = 4p$

(vi) $l \in \{5p, \frac{5p}{2}, \frac{5p}{3}, \frac{5p}{4}\}$

(i') $n = 3$ *and* $k \in \{3, 4\}$

(ii') $(n, k) = (3, 3)$

(iii') *either* $k = 3$ *or*
$(n, k) \in \{(3, 4), (4, 4), (5, 4)\}$

(iv') $k \leq 5$

(v') $k = 3$

(vi') $k = 3$

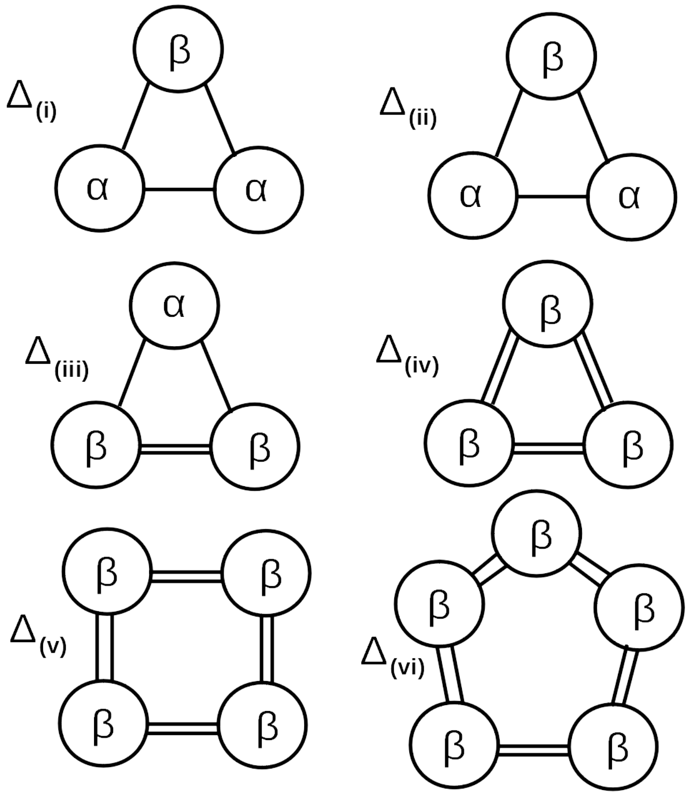

Figure 3: Possibly positively curved regions. $\Delta_{(i)}$ is a B-region, while the others are A-regions.

Proof. Regions of degree 2 are considered the interior of either a double-edge, in which case curvature is not assigned to them, or a dipole, which cannot occur since Γ is dipole-reduced. Thus, we first suppose that Δ has degree three.

If Δ is an A-region, then since $2p \nmid 3, 2p \pm 1, 3p$, Δ is a region of type $\Delta_{(i)}$. In order for such a region to be possible over $\mathcal{P}$, $\mathcal{P}$ must satisfy the conditions of (i). In order for such a region to be positively curved, $\mathcal{P}$ must satisfy the conditions of (i'). $l > p + 1 = 3$, so no triangular A-region of vertices of type α can exist over any picture $\mathcal{P}$ with parameters in the scope that the Lemma sets out. If Δ has one or two vertices of type β, then it is of type $\Delta_{(ii)}$, $\Delta_{(ii)}$, which can exist precisely when (ii), (iii) holds, and is positively curved precisely when (ii'), (iii') holds, respectively. If Δ is a triangular A-region with three vertices of type β, then it is of type $\Delta_{(iv)}$, which can exist precisely when (iv) holds and is positively curved precisely when (iv') holds.

We now suppose that Δ has degree four. If Δ is a B-region, then it cannot have any two adjacent vertices of type β, as they would share only a single edge. Thus, all vertices of type β have valence of at least $k+1 \geq 4$. Since vertices of type α have valence of $2n \geq 4$, any B-region over these presentations has curvature bounded above by $c(4,4,4,4) = 0$, so Δ cannot be a degree four B-region. Supposing, then, that A is an A-region of degree four, we note likewise that Δ cannot have 0, 1, 2 or 3 vertices of type β, else it must be nonpositively curved. If Δ is a B-region of degree four with all four vertices of type β, then it is of type $\Delta_{(v)}$, which can exist precisely when (v) holds and is positively curved precisely when (v') holds.

Supposing now that Δ has degree five, we note that if Δ has any vertex of type α, it cannot achieve positive curvature, so Δ must be a pentagon of five vertices of type β. Thus, Δ takes the form of $\Delta_{(vi)}$, which can exist precisely when (vi) holds and is positively curved precisely when (vi') holds.

Observe that Δ cannot have degree greater than five, as all vertices have valence of at least three, so any region with degree six or higher has curvature bounded above by $c(3,3,3,3,3,3) = 0$. Thus, all possibilities for a positively curved interior Δ require that it be one of $\Delta_{(i)}, \ldots, \Delta_{(vi)}$, and satisfy the corresponding conditions, as required. $\square$

Lemma 3.13. *Let $p, q, \geq 2$, $n, k \geq 3$ and $l > p+1$. If $\mathcal{P} = (l, 2q|n, k|p, q)$ satisfies none of the pairs of conditions $[(i), (i')], \ldots, [(vi), (vi')]$ of Lemma 3.12, then every spherical picture over $\mathcal{P}$ can be reduced to an empty picture by bridge moves and the introduction and removal of dipoles.*

Proof. Let Γ be a counterexample with as few vertices as possible, so that Γ is dipole-reduced. Then by Lemma 3.12, if any interior region Δ of Γ had positive curvature, then it would have to be of some form $\Delta_{(j)}$, and the conditions (j) and (j') would have to hold. Since this is, by hypothesis, not the case, Γ cannot have any positively curved interior region. The distinguished region cannot provide the 4π curvature that a spherical picture must hold, so we reach a contradiction, finding that no counterexample with fewest vertices - and thus no counterexample at all - can exist. $\square$

Lemma 3.14. *Let $p, q \geq 2$, $n, k \geq 3$ and $l > p+1$. If $\mathcal{P} = (l, 2q|n, k|p, q)$ satisfies precisely one of conditions $(i), (ii)$ or (iii) and none of $(iv), (v)$ or (vi), then every spherical picture over $\mathcal{P}$ can be reduced to an empty picture by bridge-moves and the introduction and removal of dipoles.*

Proof. If the corresponding condition $(i'), (ii')$ or (iii') does not hold, then the result follows by Lemma 3.13, so we assume that whichever (j) holds is accompanied by (j').

Let Γ be a counterexample with as few vertices as possible, so that Γ is dipole-reduced. Then by Lemma 3.12 all interior positively curved regions in Γ take the form associated to $\Delta_{(i)}$, $\Delta_{(ii)}$ or $\Delta_{(iii)}$, depending on which condition holds. We aim to show that all interior positively curved regions can be emptied of curvature, without any other regions becoming positively curved. We separate our argument by which of the conditions holds:

(i): Let Δ be a positively curved interior region of Γ. In this case, we have $q = 2$, so that Δ is a B-region with two vertices of type α and one of type β, which we shall label γ. Since (i') holds, $n = 3$ and $k \in \{3, 4\}$, so that Δ has curvature $\frac{2\pi}{r} - \frac{\pi}{3}$, which we aim to transfer to Δ' as denoted in Figure 4. As such, we must show that the curvature of Δ' remains nonpositive after receiving $c(\Delta) \leq \frac{\pi}{6}$ over each edge between two vertices of type α. Since conditions (ii) and (iii) do not hold and $l > 3$, Δ_L cannot have degree three, so must have degree $d \geq 4$.

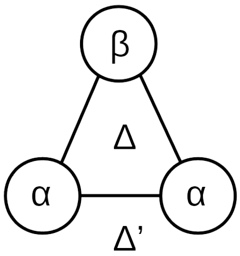

Figure 4: Neighbourhood of a region of type $\Delta_{(i)}$

Since Δ' receives curvature over edges between vertices of type α, it must have at least as many vertices of type α as it has edges over which it receives curvature. Suppose Δ' receives curvature over $r \geq 0$ edges. If all vertices of Δ' have valence at least four (which is necessarily the case as long as Δ' does not have three vertices of type β in a row), then we bound the curvature of Δ' by $c(\Delta') \leq c(4, \ldots, 4) + 2\pi r(\frac{1}{6} - \frac{1}{4}) + r\frac{\pi}{6} = c(4, \ldots, 4)$, which is nonpositive if Δ' has degree $d \geq 4$.

It is only possible for Δ' to have three vertices of type β in a row, so that the above argument fails, if Δ' consists of two vertices of type α and three of type β, or Δ' has degree $d \geq 6$. In the former case, $c(\Delta') \leq c(3, 4, 4, 6, 6) = \frac{-2\pi}{3}$ and Δ' has only one edge between two vertices of type α, so Δ' can certainly receive all required curvature without becoming positively curved. In the latter case, we use the same argument as in the previous paragraph, bounding $c(\Delta') \leq c(3,\ldots,3) + 2\pi r(\frac{1}{6} - \frac{1}{3}) + r\frac{\pi}{6} \leq c(3,\ldots,3)$, which is nonpositive since Δ' has degree $d \geq 6$.

Thus, with after the transfer of curvature, no interior regions of Γ holds a positive quantity of curvature.

(ii): Let Δ be a positively curved interior region of Γ, so that Δ is an A-region with one vertex of type β and two of type α. Since (ii') holds, $n = k = 3$, so that $c(\Delta) = \frac{2\pi}{k+2} \leq \frac{\pi}{15}$, which we aim to transfer to the region denoted Δ_D in Figure 5. As such, we seek to show that after receiving $\frac{\pi}{15}$ curvature over each edge that connects two vertices of type α, which we shall refer to as *receiving edges*, Δ_D remains nonpositively curved. Since (i) does not hold, $q > 2$, so that $2q > 4$. Thus, Δ_D has degree at least four.

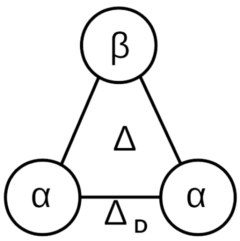

Figure 5: The neighbourhood of a region of type $\Delta_{(ii)}$

If Δ_D has degree four, it must have three vertices of type α and one vertex of type β, so has curvature $c(\Delta_D) \leq \frac{-\pi}{3}$, so can certainly accept $\frac{\pi}{15}$ over each of its receiving edges without becoming positively curved.

If Δ_D has degree five, it can have only a single vertex of type β, giving $c(\Delta_D) \leq \frac{-25\pi}{12}$, once more allowing it to receive $\frac{\pi}{15}$ curvature over each of its (three) receiving edges without becoming positively curved.

If Δ_D has degree $d \geq 6$, we follow our argument from case (i), noting that Δ_D has at least as many vertices of type α as it has receiving edges, so if Δ_D has r receiving edges, after transfer of curvature $c(\Delta_D) \leq c(3,\ldots,3) + 2\pi r(\frac{1}{6} - \frac{1}{3}) + r\frac{\pi}{15} \leq c(3,\ldots,3) \leq 0$.

Thus, after our transfer of curvature, $c(\Delta), c(\Delta_D) \leq 0$, so no interior region of Γ holds positive curvature

(iii): Let Δ be a positively curved interior region of Γ, so that Δ is an A-region with one vertex of type α and two of type β. Letting Δ_L, Δ_R be as depicted in Figure 6, we note that Δ_D may be another identical positively curved region, so we aim to split the curvature of Δ between Δ_L and Δ_R. By symmetry, it shall suffice to show that after receiving $c(\Delta)/2$ curvature over each edge between a vertex of type α and one of type β shared by a positively curved interior region (which now take on the label of *receiving* edges), Δ_L remains nonpositively curved.

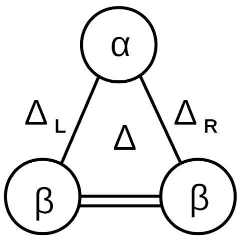

Figure 6: The neighbourhood of a region of type $\Delta_{(iii)}$

We first consider the case $n = k = 3$. In this case $c(\Delta) \leq \frac{\pi}{3}$, so we must transfer $\frac{\pi}{6}$ curvature to Δ_L. By the same argument as used in the previous case, Δ_L must have degree at least four.

If Δ_L has degree four, and has only one vertex of type β, it has curvature $c(\Delta_L) \leq \frac{-\pi}{2}$, so will certainly not be positively curved after receiving $\frac{\pi}{6}$ curvature over each of its two receiving edges.

If Δ_L has degree four, and has two vertices of type β, note that its vertices must alternate, else we have two vertices of type β sharing a single-edge. The vertices of type α around Δ_L must have opposite orientations, else we have $2q|2$. Choose one of the vertices of type β incident to Δ_L and label it γ. The edges of Δ_L incident to γ are to vertices of type α of differing orientations. If both of these edges were shared with positively curved regions, one of these regions would have label $2p+1$ and the other would have label $2p-1$. These cannot both be valid regions, else $l|2$. Thus, only one of the edges of Δ_L incident to γ can be shared with a positively curved region. Applying the same argument to the other vertex of type β in Δ_L, we see that at most two of the edges of Δ_L can be shared with a region of positive curvature. Thus, since $c(\Delta_L) \leq c(4,4,6,6) = \frac{-2\pi}{6}$, Δ_L can accept $\frac{\pi}{6}$ curvature over every edge it shares with a positively curved region, and remain nonpositively curved itself.

If Δ_L has degree four, it cannot have three vertices of type β, else we would find a single-edge between two vertices of type β.

If Δ_L has degree d, then the curvature of Δ_L after transfer is bounded above by $c(4,6,\ldots,4,6)+d\frac{\pi}{6} = 2\pi - d\pi + d\pi(\frac{1}{4}+\frac{1}{6}) + \frac{d\pi}{6} = 2\pi - \frac{5\pi d}{6} < 0$ whenever $d \geq 5$. Thus, if Δ_L has degree five or higher, it can accept $\frac{\pi}{6}$ curvature over each receiving edge.

With all possible Δ_L accounted for, by symmetry, transferring half of $c(\Delta)$ to Δ_L and half to Δ_R gives a distribution of curvature in which no interior region holds positive curvature.

As we have only used the degrees of vertices to bound curvature from above, and since the curvature of a region is decreasing in the valences of that region's vertices, an identical argument applies whenever $n, k \geq 3$, which covers all other n, k pairs satisfying (iii').

Having shown in all three cases that the curvature of Γ can be rearranged so that no interior region remains positive, without having moved more than $\frac{\pi}{6}$ across any edge, we note that the distinguished region, of degree d has curvature after transfer not exceeding $c(4, \ldots, 4) + d\frac{\pi}{6} = 2\pi - \frac{d\pi}{3} \leq 2\pi$, so the total curvature of Γ falls short of the 4π required of a spherical picture. Thus, in the absence of a vertex-minimal counterexample, every spherical picture must be reducible to an empty picture by means of bridge-moves and the introduction of dipoles. $\square$

Lemma 3.15. *Let $p, q \geq 2$, $n, k \geq 3$ and $l > p + 1$. If $\mathcal{P} = (l, 2q|n, k|p, q)$ satisfies precisely one of conditions $(iv), (v)$ or (vi) and none of $(i), (ii)$ or (iii), then every spherical picture over $\mathcal{P}$ can be reduced to an empty picture by bridge-moves, the introduction and removal of dipoles and the introduction and removal of one spherical picture denoted $S_{(iv)}$, $S_{(v)}$ or $S_{(vi)}$ in Figure 7, respectively.*

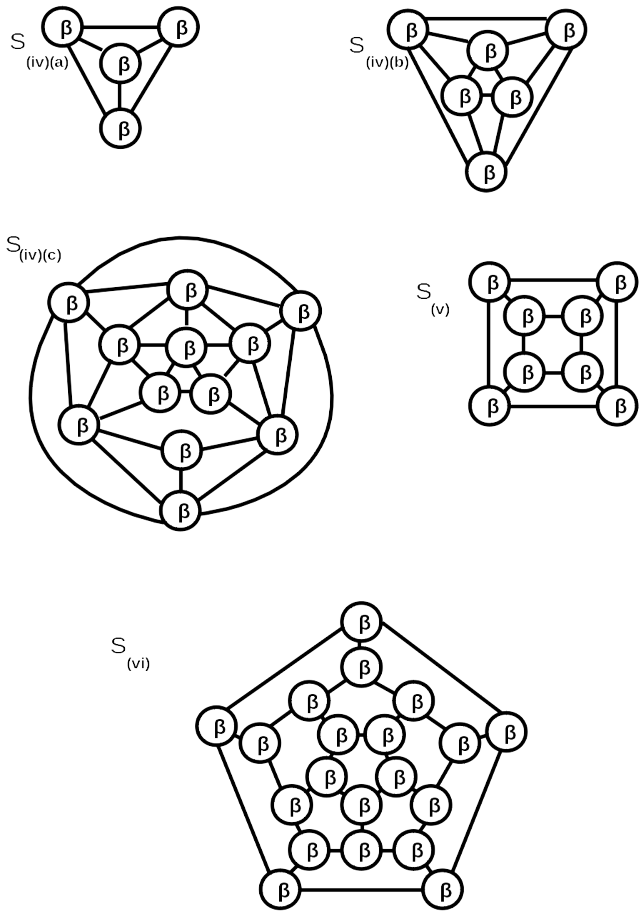

Figure 7: Possible spherical pictures. Since $S_{(iv)(a)}$, $S_{(iv)(b)}$ and $S_{(iv)(c)}$ require different values of k, and so no two of them can be permitted by the same presentation, we shall abuse notation by referring to all of them as $S_{(iv)}$, where convenient. All edges depicted are double-edges.

Proof. As in the previous lemma, we note that if the corresponding condition (iv'), (v') or (vi') does not hold then the result follows by Lemma 3.13, so we assume that whichever (j) holds is accompanied by (j').

Again, we let Γ be a counterexample with as few vertices as possible, so that Γ is $S_{(j)}$-reduced. Lemma 3.12, then, establishes that all interior positively curved regions of Γ take the form $\Delta_{(iv)}$, $\Delta_{(v)}$ or $\Delta_{(vi)}$, depending on which of $(iv), (v)$ or (vi) holds. Our aim, as previously, shall be to show that all interior positively curved regions can be emptied of curvature without any other regions becoming positively curved. We separate our argument by which of the conditions holds:

(iv): As noted above, we may assume that (iv') holds, so that $3 \leq k \leq 5$. Following our abuse of notation in Figure 7, the sphere that we suppose Γ is reduced relative to depends upon the value of k, so we split our argument further according to whether $k = 3, 4$ or 5.

Suppose $k = 3$. In this case Γ is $S_{(iv)(a)}$-reduced, so cannot contain a connected copy of more than half (by vertex-count) of $S_{(iv)(a)}$, a tetrahedron of vertices of type β. However, $\Delta_{(iv)}$, whose form all positively curved interior regions of Γ must take, by Lemma 3.12, is itself such a figure. Thus, no interior region of Γ can have positive curvature, so the total curvature of Γ is bounded above by that of its distinguished region, which cannot have more than 2π. Thus, no such $S_{(iv)}$-reduced spherical picture Γ can exist. In the absence of a counterexample with fewest vertices, all spherical pictures over $\mathcal{P}$ must be reducible to an empty picture by bridge-moves, dipoles and $S_{(iv)}$.

Suppose now that $k = 4$. Then Γ is $S_{(iv)(b)}$-reduced, so cannot contain a connected copy of more than half (by vertex-count) of $S_{(iv)(b)}$, an octahedron of vertices of type β. Let Δ be a positively curved interior region, which as above must take the form of a triangle of vertices of type β. Then no other vertex of type β can share an edge with any of the vertices of Δ, else we have a connected copy of five of the eight vertices of $S_{(iv)(b)}$. As such, each of the vertices of Δ has precisely two double-edges, and thus has valence six, so $c(\Delta) = c(6, 6, 6) = 0$. Thus,

Γ has no positively curved interior regions, so by the usual argument cannot be a spherical picture, and so every spherical picture over $\mathcal{P}$ are reducible to an empty picture via bridge-moves, dipoles and $S_{(iv)(b)}$.

Finally, suppose that $k = 5$. Then Γ is $S_{(iv)(c)}$-reduced, so cannot contain a connected copy of more than half (by vertex-count) of $S_{(iv)(c)}$, an icosahedron of vertices of type β. Let Δ be a positively curved interior region, which as above must take the form of a triangle of vertices of type β. If all three vertices of Δ have edges other than double-edges to vertices of type β, then they have valence of at least six, so $c(\Delta) \leq c(6, 6, 6) = 0$. Thus, at least one of the vertices of Δ must have five double-edges, giving us the arrangement depicted in Figure 8, potentially after some bridge moves. Observe, then, that if either of the vertices γ_1 or γ_2 have an edge to another vertex of type β, then we have a connected copy of part of $S_{(iv)(c)}$ containing seven of $S_{(iv)(c)}$'s twelve vertices, which cannot be present in Γ, so γ_1 and γ_2 must each have at least four edges to vertices of type α, giving $c(\Delta) = c(5, 7, 7) < 0$. Thus, Γ satisfying the conditions set out cannot have any positively curved interior regions, so by the usual argument every spherical picture over $\mathcal{P}$ is reducible to an empty picture via bridge-moves, dipoles and $S_{(iv)(c)}$.

(v): In this case Γ is $S_{(v)}$-reduced, so cannot contain a connected copy of more than half (by vertices) of $S_{(v)}$, a cube of vertices of type β. By assumption, (v') holds, so $k = 3$. Let Δ be an interior region of Γ with positive curvature. Then by Lemma 3.12 Δ takes the form $\Delta_{(v)}$, a square of vertices of type β. No vertex of Δ can have a third double edge, as such would lead to another vertex of type β, giving us a connected copy of five of the eight vertices of $S_{(v)}$. Thus, each

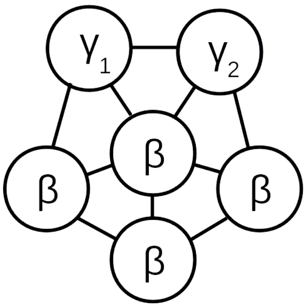

Figure 8: a neighbourhood of a region of type $\Delta_{(iv)}$

vertex of Δ is connected instead to a pair of vertices of type α, and therefore has valence 4, so that $c(\Delta) = c(4,4,4,4) = 0$.

Thus, Γ has no positively curved interior regions, and so has total curvature of at most 2π, falling short of the 4π required of a spherical picture. In the absence of a counterexample with fewest vertices, all spherical pictures over $\mathcal{P}$ must be reducible to an empty picture by bridge-moves, dipoles and $S_{(iv)}$.

(vi): In this case Γ is $S_{(vi)}$-reduced, so cannot contain a connected copy of more than half (by vertices) of $S_{(vi)}$, a dodecahedron of vertices of type β. By assumption, (vi') holds, so $k = 3$. Let Δ be an interior region of Γ with positive curvature. Then by Lemma 3.12 Δ takes the form $\Delta_{(v)}$, a pentagon of vertices of type β. Each vertex of Δ has valence three if and only if it has a double-edge to a vertex outside of Δ, and otherwise has valence four. We note that if more than one vertex of Δ had valence four, Δ would not be positively curved. Thus, we segment our argument according to whether Δ has any vertex of valence four.

Suppose first that all vertices of Δ have valence three, so that each vertex shares a double-edge with a β vertex outside of Δ. These external vertices cannot have any other neighbours of type β, else we have a copy of more than half of $\Delta_{(v)}$. In this case, $c(\Delta) = c(5,5,5,5,5) = \frac{\pi}{3}$, so we transfer $\frac{\pi}{15}$ curvature to each adjacent region. Letting Δ' be a region adjacent to Δ, note that since (iv) and (v) do not hold, Δ' cannot have degree three or four. Since the external vertices cannot be adjacent to vertices of type β other than their neighbour in Δ, the boundary of Δ' around the edge it shares with Δ must consist of a vertex of type α followed by four of type β and another of type α. Since the same must be found around any other edge that Δ' receives over, Δ' must have degree at least five times the number of edges over which it receives. Thus if Δ' receives curvature over r edges we can bound the pre-transfer curvature of Δ' below the curvature of a region whose boundary is r repetitions of a vertex of type α followed by four of type β, which is $2\pi - 5\pi r + 2\pi r(\frac{4}{3} + \frac{1}{6}) = 2\pi - 2\pi r$. Thus, after transfer, $c(\Delta') \leq 2\pi - \frac{11\pi r}{6}$, which is nonpositive for $r \geq 2$. For the $r = 1$ case, note that Δ' cannot have degree five, else we would have $l|p \pm 1$, so after transfer $c(\Delta') \leq c(3,3,3,3,6,6) + \frac{\pi}{6} = \frac{-\pi}{2}$. Thus, this transfer of curvature cannot result in Δ' becoming positively curved.

Suppose now that Δ has one vertex of valence four, so that $c(\Delta) = \frac{1}{6}$. The neighbourhood of Δ is, then, as depicted in Figure 9. If γ_1 and γ_2 were both of type β, we have a connected copy of part of $S_{(vi)}$ containing eleven of its twenty vertices, so at least one of the two must be of type α. Assume without loss of generality, then, that γ_1 is of type α. We transfer the curvature of Δ into Δ_L. A priori, it is possible that Δ_L has degree four - that is, γ_1 is one of the vertices of type α adjacent to Δ's vertex of valence four - however, this would require $l|3p \pm 1$, whose only solution for which (vi) holds is $l = 5, p = 3$, which is ruled out as it satisfies (iii). Thus, Δ_L has degree at least five. If Δ_L accepts curvature over only one edge, $c(\Delta_L) \leq c(3,4,4,6,6) = \frac{-2\pi}{3} < \frac{-\pi}{6}$, so Δ_L does not become positively curved after receiving curvature. If Δ_L receives curvature over $r \geq 2$ edges, its boundary must contain at least r segments consisting of a vertex of type α followed by three of type β and another of type α, so Δ_L has curvature $c(\Delta) \leq 2\pi - 2\pi r < \frac{-\pi r}{6}$. Thus, our transfer of curvature leaves no interior region positively curved.

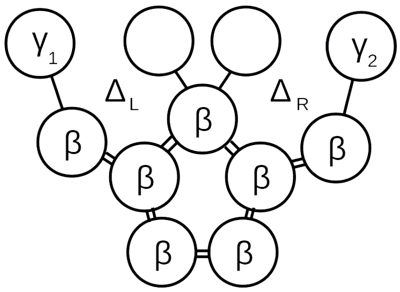

Figure 9: The neighbourhood of a region of type $\Delta_{(vi)}$

If some region Δ' receives curvature via both of these types of transfer, take the $\alpha - \beta - \beta - \beta - \beta$ segments over which Δ' receives curvature in the first type of transfer and concatenating them, to form a (hypothetical) region Δ'_1, and doing likewise for the second type of transfer to form another region Δ'_2. Then $c(\Delta') \leq c(\Delta'_1) + c(\Delta'_2) - 2\pi$. By the arguments above, if Δ'_1 received all the curvature that Δ' does by the first type of transfer, it would have curvature not exceeding $\frac{\pi}{15}$ (accounting for the one-receiving-edge case), and Δ'_2, upon similar receipt, would have no more than $\frac{\pi}{6}$ curvature. These excesses are more than balanced by the -2π in this expression, so Δ' is not positively curved after both types of transfer of curvature are performed.

Thus, after the transfer of curvature, every interior region of Γ holds nonpositive curvature. The intrinsic curvature of the distinguished region cannot exceed 2π, and no more than $\frac{\pi}{6}$ curvature is transferred over each edge, so in order to reach the 4π total curvature that Γ must hold, the distinguished region must receive curvature over at least twelve edges. However, a region with r edges has curvature not exceeding $2\pi - \frac{r\pi}{3} < -\frac{r\pi}{6}$ for $r \geq 12$. Thus, curvature transferred to the distinguished region cannot possibly make up the shortfall. In the absence of a counterexample with fewest vertices, all spherical pictures over $\mathcal{P}$ must be reducible to an empty picture by bridge-moves, dipoles and $S_{(v)}$.

$\square$

For the following, and the rest of this section, we must extend our assumption that $n, k \geq 3$ to $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$. This allows us to approach cases in which more than one of the conditions $(i), \ldots, (vi)$ hold.

Lemma 3.16. *Let $p, q \geq 2$, $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$ and $l > p + 1$. Then every spherical picture Γ over the presentation $\mathcal{P} = (l, 2q|n, k|p, q)$ is reducible to an empty picture by bridge-moves, the introduction and removal of dipoles and, if applicable, the introduction and removal of one of $S_(iv)$, $S_(v)$ or $S_(vi)$.*

Proof. By Lemma 3.13, we need only concern ourselves with cases in which at least one of the pairs of conditions $[(j), (j')]$ holds. If only one condition (j) holds, then Lemmas 3.14 and 3.15 suffice. Thus, we consider the case where multiple such conditions hold. Our aim is to show that, with $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$, the curvature redistribution from Lemmas 3.14 and 3.15 serves to eliminate all curvature. The conditions (i') and (ii') cannot hold, since

$\frac{1}{n}+\frac{1}{k}\leq\frac{1}{2}$, so positively curved regions of types $\Delta_{(i)}$ or $\Delta_{(ii)}$ cannot exist in pictures over these presentations. Thus, it suffices for us to show that any use of the assumption that other conditions do not hold in the arguments for $(iii),\ldots,(vi)$ are unnecessary given $\frac{1}{n}+\frac{1}{k}\leq\frac{1}{2}$, and that when n and k are such that more than one of $(iii),\ldots,(vi)$ can hold, no region receives curvature under both redistributions.

The assumptions that multiple conditions do not hold are used in the (iii), (iv), (v) or (vi) cases of Lemmas 3.14 and 3.15 to rule out the following possibilities, which we must now address:

(1) A region of type $\Delta_{(i)}$ being adjacent to one of type $\Delta_{(iii)}$ in case (iii) of Lemma 3.14.

(2) A region of type $\Delta_{(iv)}$ or type $\Delta_{(v)}$ being adjacent to one of type $\Delta_{(vi)}$ in the (vi) case of Lemma 3.15.

(3) A region with three vertices of type β and one of type α being adjacent to a region of type $\Delta_{(vi)}$ in the (vi) case of Lemma 3.15.

In dealing with case (1), (iii) holds and $\frac{1}{n}+\frac{1}{k}\leq\frac{1}{2}$, so that either $(n,k)\in\{(4,4),(4,5)\}$ or $k=3$. Case (iii) of Lemma 3.14 removes curvature from regions of type $\Delta_{(iii)}$ across two edges, and regions of type $\Delta_{(i)}$ have two edges over which they can receive curvature, so it suffices to show that if Δ is of type $\Delta_{(iii)}$ and Δ' is of type $\Delta_{(i)}$ then $c(\Delta)+c(\Delta')\leq 0$. The vertices of a region of type $\Delta_{(iii)}$ have valence $2n$, $k+1$ and $k+1$, whilst the vertices of a region of type $\Delta_{(i)}$ have valence $2n$, $2n$ and $k+1$. Direct calculation shows that $c(2n,k+1,k+1)+c(2n,2n,k+1)\leq 0$ for all permissible pairs (n,k).

Case (2) does not require any consideration of curvature, as if the regions ruled out were possible over the presentations in question, then we would have $l|5p$ and either $l|4p$ or $l|3p$, so that $l|p<l$.

In case (3), the region in question is required to receive $\frac{\pi}{6}$ curvature, and is in no danger of receiving over multiple edges. Since $k = 3$, we have $n \geq 6$, so the region has curvature $c(3, 4, 4, 2n) = \frac{\pi}{n} - \frac{\pi}{3} \leq \frac{-\pi}{6}$, as required.

Thus, the single-condition assumptions underlying the curvature redistribution from any given region in Lemmas 3.14 and 3.15 are not necessary when $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$.

We now consider how the redistribution of curvature under different schemes from Lemmas 3.14 and 3.15 may interact. Note first, that the only (l, p) pairs satisfying more than one of $(iii), (iv), (v)$ and (vi) are $(5, 2)$ and $(5, 3)$, satisfying (iii) and (vi). Curvature from a positively curved region Δ in case (vi) transfers curvature only to A-regions, and cannot transfer to a region of type $\Delta_{(iii)}$, as if Δ shared an edge with such a region, it would not be positively curved. Curvature in case (iii) is only transferred to B-regions. Thus, in a $S_{(vi)}$-reduced picture over presentations satisfying both (iii) and (vi), following the existing schemes for curvature transfer results in a picture with no positively curved interior regions, and a distinguished region carrying less than 4π curvature. Since this was the only problem pairing (and noting that the conditions (iv), (v) and (vi) are incompatible, so that there can be no ambiguity as to which sphere we reduce by), and dipole- or $S_{(j)}$-reduced spherical picture over a presentation satisfying the conditions of the lemma must fall short of the 4π curvature required of a spherical picture. Thus, every spherical picture over a presentation $\mathcal{P}$ satisfying the conditions of the lemma must be reducible to an empty picture by bridge-moves, dipoles and, as appropriate, one of $S_{(iv)}$, $S_{(v)}$ or $S_{(vi)}$. $\square$

Given the equivalence between homotopy classes of maps and spherical pictures noted in Section 2, this gives us the following:

Corollary 3.17. *Let $\mathcal{P} = (l, 2q|n, k|p, q)$, with $p, q \geq 2$, $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{k}$, and $l > p+1$. If $\mathcal{P}$ satisfies none of the pairs of conditions [$(iv), (iv')$], [$(v), (v')$]*

or $[(vi),(vi')]$, then the second homotopy group of the space Z associated to $\mathcal{P}$) is generated by the homotopy classes of maps $S^2 \to Z$ represented by the proper dipoles $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$. If $\mathcal{P}$ does satisfy some such pair, then the second homotopy group of Z is generated by the classes represented by $\mathcal{D}_\alpha$, $\mathcal{D}_\beta$ and the appropriate $S_{(j)}$.

As in the $l = 2p, m = 2q$ case, we also obtain a result on the orders of elements:

Corollary 3.18. *Let $\mathcal{P} = (l, 2q|n, k|p, q)$, with $p, q \geq 2$, $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$, and $l > p + 1$. Then the orders of a b, (ab) and $(a^p b^q)$ in the group with presentation $\mathcal{P}$ are precisely l, $2q$, n and k, respectively.*

Proof. The arguments used in Corollary 3.9 suffice to establish the orders of a, b and (ab). Applying the argument that Corollary 3.9 uses for (ab) to $(a^p b^q)$ establishes this element as having order k, as required. □

3.2.2 Completing the Argument

Proposition 3.19. *Let G be the group with presentation $\mathcal{P} = (l, 2q|n, k|p, q)$ with $p, q \geq 2$, $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$ and $l > p + 1$, satisfying one of the (mutually exclusive) pairs of conditions $[(iv),(iv')]$, $[(v),(v')]$ or $[(vi),(vi')]$. Then G is infinite.*

Proof. Suppose that G is finite. Corollary 3.17 establishes that $\pi_2(Z_{\mathcal{P}}) = \langle \mathcal{D}_\alpha, \mathcal{D}_\beta, S_{(j)} \rangle$, where $\mathcal{P}$ satisfies $[(j),(j')]$. Corollary 3.18 establishes that a, b (ab) and $(a^p b^q)$ have orders l, $2q$, n and k, respectively. Observe that these presentations can be expressed as

$$\langle a, b | a^{\frac{x}{y}p}, b^{2q}, (ab)^n, (a^p b^q)^k \rangle$$

with x, y coprime and $\frac{x}{y}p$ an integer, so that $y|p$, and in each case $\frac{1}{x} + \frac{1}{k} > \frac{1}{2}$. Expressing $\mathcal{P}$ in this way, we use the pushout

$$\begin{array}{ccc} G_0 := \langle c,d|\rangle & \xrightarrow{\phi} & \langle c,d|c^x,d^2,(c^yd)^k\rangle =: G_1 \\ \downarrow \psi & & \downarrow \\ G_2 := \langle a,b|(ab)^n\rangle & \longrightarrow & \langle a,b|a^{\frac{x}{y}p},b^{2q},(ab)^n,(a^pb^q)^k\rangle = G \end{array}$$

where ϕ maps c to c and d to d, whilst ψ maps c to $a^{\frac{p}{y}}$ and d to b^q.

We use the same topological construction as used in Lemma 3.10, replacing factors of 2 with factors of $\frac{p}{y}$, where appropriate. Since $S_{(iv)}$, $S_{(v)}$ and $S_{(vi)}$ can all be mapped entirely into X_1, the same argument as used in Lemma 3.10 shows that this pushout is geometrically Mayer-Vietoris, so that

$$\frac{1}{|G|} = \chi_{\mathbb{Q}}(G) = \chi_{\mathbb{Q}}(G_1) + \chi_{\mathbb{Q}}(G_2) - \chi_{\mathbb{Q}}(G_0).$$

As in Lemma 3.10, $\chi_{\mathbb{Q}}(G_2) - \chi_{\mathbb{Q}}(G_0) = \frac{1}{n}$. Since x and y are coprime, G_1 is isomorphic to the finite triangle group $(2,x,k)$, and thus has positive rational Euler characteristic. Therefore, $\frac{1}{|G|} > \frac{1}{n}$, which cannot be the case, as G contains an element of order n. Thus, G must be infinite. □

Proposition 3.20. *Let G be the group with presentation $\mathcal{P} = (l, 2q|n,k|p,q)$ with $p,q \geq 2$, $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$ and $l > p+1$, satisfying none of the pairs of conditions [$(iv),(iv')$], [$(v),(v')$] or [$(vi),(vi')$]. Then G is infinite.*

Proof. For this result, we require only non-collapse. A result of R.M. Thomas [12] establishes that if G has presentation $\langle x_1,\dots,x_i|r_1^{n_1},\dots,r_j^{n_j}\rangle$, and $r_1,\dots,r_j$ have order $n_1,\dots,n_j$ in G, then G can only be finite if

$$\frac{1}{n_1} + \dots + \frac{1}{n_j} - i + 1 > 0.$$

In the case of the presentation $\mathcal{P}$, we note that by Corollary 3.18, a, b, (ab) and a^pb^q have orders l, $2q$, n and k, so that $\mathcal{P}$ satisfies the conditions of [12], with $i=2, j=4$ and n_1,n_2,n_3 and n_4 equal to l, $2q$, n and k. Thus, for G to be finite would require

$$\frac{1}{l} + \frac{1}{2q} + \frac{1}{n} + \frac{1}{k} > 1,$$

but, by hypothesis, $l > p+1 \geq 3$, $2q \geq 4$, and $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$, so that this cannot be the case. Therefore, G is infinite. $\square$

Combining these two cases, we reach our result for this subsection:

Theorem 3.21. *Let G be the group with presentation $\mathcal{P} = (l, 2q|n, k|p, q)$ with $p, q \geq 2$, $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$ and $l > p+1$. Then G is infinite.*

3.3 The Broad Case: $l \neq 2p, m \neq 2q$

These presentations may be dispatched rather more quickly. We maintain our assumption that $\frac{1}{n} + \frac{1}{k} \geq \frac{1}{2}$, and continue to exclude those cases reducible to $p = 1$ or $q = 1$, so that $1 < p < l-1$ and $1 < q < m-1$. In a dipole-reduced picture over such a presentation, note that no region of degree two can arise. Thus, double-edges do not occur over these presentations, so vertices of type α and β have valence $2n$ and $2k$, respectively. Since n and k are both, by hypothesis, at least three, all vertices in pictures over these presentations have valence at least six. Therefore, any interior region Δ of a picture over these presentations has curvature $c(\Delta) \leq c(6,6,6) = 0$. Thus, pictures over these presentation cannot have any interior curvature, and so admit no spherical picture. Applying the arguments from Corollary 3.18 and Proposition 3.20, we reach

Theorem 3.22. *Let G be the group with presentation $\mathcal{P} = (l, m|n, k|p, q)$ with $p, q \geq 2$, $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$, $l > p+1$ and $m > q+1$. Then G is infinite.*

4 The $p = 1$ case: $l = 2$

We shall first approach these presentations via the method of pictures, as described above, and then fill in remaining cases piecemeal via other methods.

Our primary result from this section shall be the following:

Theorem 4.1. *If $m, n, k > 1$, $1 < q \leq \frac{m}{2}$, and $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$, then the presentation $(2, m|n, k|1, q)$ represents a finite group if and only if it is one of the following.*

$(2, m|n, k|1, 1) : \frac{1}{m} + \frac{1}{gcd(n,k)} > \frac{1}{2}$

$(2, m|3, k|1, 2) : m \geq 4, k \geq 6, gcd(m, k) \leq 5$

$(2, 2\hat{m} + 1|n, 3|1, \hat{m}) : \hat{m} \geq 2, k \geq 6, gcd(m, k) \leq 5$

$(2, 4|4, k|1, 2) : k \geq 4$

$(2, 4|5, k|1, 2) : 4 \leq k \leq 5$

$(2, 4|n, 3|1, 2) : 6 \leq n \leq 9$

$(2, 5|4, 4|1, 2) : k \geq 4$

$(2, 5|4, 5|1, 2)$

$(2, 5|5, 4|1, 2)$

$(2, 6|3, k|1, 3) : k \geq 6$

$(2, 6|7, 3|1, 2)$

$(2, 7|n, 3|1, 2) : 6 \leq n \leq 8$

$(2, 7|3, k|1, 3) : 6 \leq k \leq 8$

$(2, 8|7, 3|1, 2)$

This result relies upon a variety of preliminary results, and two core lemmas using the method of pictures discussed in previous chapters. We will begin with the largely computational preliminary results, obtained with [7], and then proceed to the core lemmas, which establish non-collapse and certain homological conditions upon the presentation complexes of the groups in question, allowing us to rule out finiteness via a combination of a push-out construction, and arithmetic regarding the orders of the groups in question. Before these, however, we note the following isomorphisms:

Lemma 4.2. *(i)* $(2, m|n, k|1, q) \cong (2, m|n, k|1, m - q)$

(ii) $(2, \alpha q \pm 1|n, k|1, q) \cong (2, \alpha q \pm 1|k, n|1, \alpha)$

(iii) $(2, m|n, k|1, 2) \cong (n, m|k, 2|1, m - 1)$

Proof. (i) Since, in $G_t := \langle a, b | a^2, b^m, (ab)^n \rangle$, $(a^{-1}b^{-q})$ is conjugate to $(b^{-q}a^{-1}) = (ab^q)^{-1}$, adding the relator $(a^{-1}b^{-q})^k$ gives the same group as adding the relator $(ab^q)^k$. Since the relations $a = a^{-1}$ and $b^{-q} = b^{m-q}$ also hold in G_t, adding the relator $(ab^{m-q})^k$ also gives the same group.

(ii) Setting $c := b^q$, so that $b = c^{\pm\alpha}$, and replacing all occurrences of b in each relator with this expression gives the required presentation on a and c.

(iii) Set $c := ab$, so that $a = cb^{-1}$ and replacing all occurrences of a in each relator with this expression gives the required presentation on b and c.

□

These identities allow us to make the following assumptions about m, n, k and q in the groups in question:

(A1) $0 \leq q \leq \frac{m}{2}$

Since adding any multiple of m to q gives an identical group, we may minimise the absolute value of q, and use Lemma 4.2(i) to replace q with $m-q$ if negative, which after another subtraction of m gives a non-negative value that is at most $\frac{m}{2}$.

(A2) $q > 2$

Under **(A1)** we may already assume that q is non-negative, so we need only rule out the values of 0, 1 and 2. If $q = 0$, then our group is either the triangle group $(2, m, n)$ (and thus is finite if and only if $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$), if k is even, or the cyclic group on $hcf(m, n)$ elements, if k is odd. If $q = 1$, then our group is the triangle group $(2, m, hcf(n, k))$, and so is finite if and only if $\frac{1}{m} + \frac{1}{hcf(n,k)} > \frac{1}{2}$. If $q = 2$, then Lemma 4.2(iii) allows us to replace it with $(n, m|k, 2|1, m-1)$, which is the group $(n, m|k, 2)$ in the notation of [6], in which finiteness amongst that family of groups was characterised.

Note that **(A1)** and **(A2)** imply $m \geq 6$

(A3) $m \neq 2q + 1$

If $m = 2q + 1$, Lemma 4.2(ii) allows us to transform $(2, m|n, k|1, q)$ to $(2, m|k, n|1, 2)$, which via the argument for **(A2)** we can already characterise as finite or infinite.

4.1 Preliminary Results

Lemma 4.3. *If* $\frac{1}{m} + \frac{1}{n} + \frac{1}{k} > \frac{1}{2}$, $n \geq 4$, $k \geq 4$ *and* $(n,k) \neq (4,4)$ *then* $G(\mathcal{P})$ *is infinite*

Proof. 70 triples (n,k,m) satisfy these conditions, specifically $(4,5,m)$: $6 \leq m \leq 19$; $(4,6,m) : 6 \leq m \leq 11$; $(4,7,m) : 6 \leq m \leq 9$; $(4,k,m)$: $8 \leq k \leq 9, 6 \leq m \leq 7$; $(4,k,6) : 10 \leq k \leq 11$; $(5,4,m) : 6 \leq m \leq 19$; $(5,5,m) : 6 \leq m \leq 9$; $(5,6,m) : 6 \leq m \leq 7$; $(5,7,6)$;$(6,4,m) : 6 \leq m \leq 11$; $(6,5,m) : 6 \leq m \leq 7$; $(7,4,m) : 6 \leq m \leq 9$; $(7,5,6)$; $(n,4,m) : 8 \leq n \leq 9, 6 \leq m \leq 7$; and $(n,4,6) : 10 \leq n \leq 11$. Accounting for possible values of q, given that $3 \leq q \leq \frac{m}{2}$, this translates to 140 presentations, each of which is infinite-automatic via [7][8].

$\square$

Lemma 4.4. *If* $n = 3$ *and* $(k, m; q)$ *is one of* $(7, 21, q) : q = 6$ *or* 9; $(7, 28, q) : q = 5, 7, 11$ *or* $13)$; $(7, 36, q) : q = 5, 6, 7, 9, 11, 13$ *or* 17; $(7, 39, q) : q = 6, 9$ *or* 12; $(7, 40, q) : q = 5, 7, 9, 11, 15$ *or* 17; $(8, 12, 5)$; $(8, 16, q) : q = 6$ *or* 7; $(8, 18, q) : q = 5, 6, 7$ *or* 8; $(8, 20, q) : q = 5, 6, 8$ *or* 9; $(8, 21, q) : q = 6, 7, 8$ *or* 9; $(8, 22, q) : q = 5, 6, 8, 9$ *or* 10; $(9, 12, 5)$; $(9, 15, q) : q = 5$ *or* 6;

$(9,16,7); (10,12,5); (10,14,6); (11,12,5); (7,36,12); (7,36,15); (7,40,19)$, *then* $G(\mathcal{P})$ *is infinite.*

Proof. The groups $(2,18|3,7|1,12)$ and $(2,18|3,7|1,15)$ are infinite automatic, and are quotients of $(2,18|3,7|1,12)$ and $(2,18|3,7|1,15)$, respectively. $(2,40|3,7|1,19)$ has a subgroup of index 8 whose core, of index 336, has infinite Abelianisation. The remaining 49 groups are infinite automatic. $\square$

Lemma 4.5. *(i)* $(2,m|3,k|1,3)$ *is infinite for* $m \geq 8$ *and* $k \geq 6$.

(ii) $(2,8|n,3|1,3)$ *is infinite for* $n \geq 6$

(iii) $(2,10|n,3|1,3)$ *is infinite for* $n \geq 6$

Proof. (i) Since we have the relators $a^2 = 1$ and $(ab)^3$, $bab = ab^{-1}a$ is an identity in each of the groups in this family. We may therefore rewrite the relator $(ab^3)^k$, already expressible as $(bab^2)^k$, as $(ab^{-1}ab)^k$, or $(a^{-1}b^{-1}ab)^k$, so that $(2,m|3,k|1,3)$ can be identified with what [3] denotes as $(2,m,3;k)$, which in the same paper is shown to be infinite.

(ii) Since we have the relators $a^2 = 1$ and $(ab^3)^3)$, $ab^3 = b^{-3}ab^{-3}a^{-1}$ is an identity in each of the groups in this family. Since we also have the relator b^8, we may write the relator $(ab)^n$ as $(ab^3b^6)^n$. Writing ab^3 in terms of our identity above, this becomes $(b^{-3}ab^{-3}a^{-1}b^6)^n$, which is conjugate to $(b^3ab^{-3}a^{-1})^n$. Writing the resulting presentation in terms of a and $c := b^3$, we obtain the presentation denoted in [3] as $(2,8,3;n)$, which the same paper shows to be infinite for $n \geq 6$.

(iii) Expressing the presentation in terms of a and $c := b^3$ shows that $(2,10|n,3|1,3)$ is isomorphic to $(2,10|3,n|1,3)$, which is infinite, by part (i) of this lemma.

$\square$

Lemma 4.6. *If $k = 3$ and (m, n, q) is one of $(12, n, 5) : 8 \leq n \leq 11$; $(14, 10, 6)$; $(16, 9, 7)$; $(17, 9, 5)$; $(16, 8, q) : q = 6$ or 7; $(18, 8, q) : q = 5, 7$ or 8; $(20, 8, q) : q = 6$ or 9; $(21, 8, q) : q = 6, 8$ or 9; $(22, 8, q) : q = 5, 6, 8$ or 9; $(28, 7, q) : q = 5$ or 11; $(36, 7, q) : q = 5, 7, 11, 13$ or 15; $(40, 7, q) : q = 3, 7, 9, 11, 15, 17$ or 19. Then $G(\mathcal{P})$ is infinite.*

Proof. The groups $(2, 36|7, 3|1, 11)$, $(2, 36|7, 3|1, 13)$, $(2, 36|7, 3|1, 15)$, $(2, 40|7, 3|1, 9)$, $(2, 40|7, 3|1, 11)$, $(2, 40|7, 3|1, 15)$ and $(2, 40|7, 3|1, 17)$ have as quotients $(2, 18|7, 3|1, 11)$, $(2, 18|7, 3|1, 13)$, $(2, 18|7, 3|1, 15)$, $(2, 20|7, 3|1, 9)$, $(2, 20|7, 3|1, 11)$, $(2, 20|7, 3|1, 15)$ and $(2, 20|7, 3|1, 17)$, which are infinite automatic. The group $(2, 40|7, 3|1, 19)$ has a subgroup of index 8 whose core, of index 336, has infinite Abelianisation. The remaining 27 cases yield infinite automatic groups. □

Lemma 4.7. *The group $(2, 6|3, k|1, 3)$ is finite of order $6k^2$ for $k \geq 6$ with derived subgroup $\mathbb{Z}_k^2$, and is solvable.*

Proof. For any k, the Abelianisation $\mathcal{G}_{Ab}$ of this group $\mathcal{G}$ is cyclic on 6 elements, generated by the image of b, whose cube is the image of a. Thus, the derived subgroup $\mathcal{G}'$ of $\mathcal{G}$ has index 6 in $\mathcal{G}$. To find the order of this subgroup, we consider the corresponding cover of the presentation complex of $\mathcal{G}$. The vertices of this cover correspond to the elements of $\mathcal{G}_{Ab}$, and shall be denoted $v_0, \ldots, v_5$. The 1-cell of the presentation complex corresponding to b lifts to edges $y_0 := (v_0, v_1), y_1 := (v_1, v_2), y_2 := (v_2, v_3), y_3 := (v_3, v_4), y_4 := (v_4, v_5)$ and $y_5 := (v_5, v_0)$, whilst the 1-cell corresponding to a lifts to edges $x_0 := (v_0, v_3), x_1 := (v_1, v_4), x_2 := (v_2, v_5), x_3 := (v_3, v_0), x_4 := (v_4, v_1)$ and $x_5 := (v_5, v_2)$. The 2-cells corresponding to the relators of $\mathcal{G}$ lift to 2-cells $x_i x_{i+3}$, $y_i y_{i+1} y_{i+2} y_{i+3} y_{i+4} y_{i+5}$, $x_i y_{i+3} x_{i+4} y_{i+1} x_{i+2} y_{i+5}$, $(x_i y_{i+3} y_{i+4} y_{i+5})^k$ (for all $0 \leq i \leq 5$, taking subscripts modulo 6). The edges y_0, y_1, y_2, y_3 and y_4 form a spanning tree, which we collapse to obtain a space with one vertex, giving us a presentation (omitting y_5, since each of the cells containing only edges of type y now renders it trivial) is

$$\mathcal{G}' = \langle x_0, x_1, x_2, x_3, x_4, x_5 | x_0x_3, x_1x_4, x_2x_5, x_0x_4x_2, x_1x_5x_3, x_i^k (0 \leq i \leq 5)\rangle$$

$$= \langle x_0, x_1, x_2 | x_0^k, x_1^k, x_2^k, x_0x_1^{-1}x_2, x_1x_2^{-1}x_0^{-1}\rangle$$

$$= \langle x_0, x_1 | x_0^k, x_1^k, [x_0, x_1]\rangle$$

To observe that $\mathcal{G}$ is solvable, note that it is a quotient of the solvable von Dyck group $(2, 6, 3)$. □

4.2 Curvature argument

As set out in Section 2, our approach to finiteness via the rational Euler characteristic requires that we characterise the pictures that exist over a given presentation. As related in the same section, we aim to assign to each presentation $\mathcal{P}$ under consideration a set $\mathcal{S}$ of spherical pictures over $\mathcal{P}$ such that any spherical picture over $\mathcal{P}$ can be reduced to an empty picture by bridge moves, the addition and removal of dipoles, and the addition and removal of elements of $\mathcal{S}$. As noted, this suffices to provide us with the generating set for the fundamental group of $Z_{\mathcal{P}}$.

Since we are dealing with the case $l = 2$, it is possible to encounter double-edges with interior label a^2 in pictures over these presentations. To simplify later argument, we shall accumulate as many edges as possible into such double-edges: So long as a picture Γ contains an A-region with more than two corners, we can pick a pair of adjacent corners, and use a bridge move to form them into a double-edge, as in Figure 10. This procedure does not disassemble any double-edge containing an A-region, so strictly increases the number of 'a' labels inside double-edges, and so since the total number of 'a' labels in the picture, must decrease the number of 'a' labels not inside double-edges. As such, we can repeat this process for as long as Γ has any A-regions with more than two corners, so must reach an arrangement in which the only A-regions not inside double-edges have only a single corner. Since $m \neq 1$, this can only be the distinguished region. Since this region must have precisely one edge and one vertex, its edge must meet its vertex twice. Thus, after undergoing this procedure, all α-vertices have precisely n edges, up to parallelism, and all β-vertices (excluding the case of an improper dipole, which shall be discussed below) have precisely k, up to the same. We shall assume that all pictures we deal with have undergone this procedure, and so dispense with the notation of single, double, etc edges, simply referring to each double-edge as an edge.

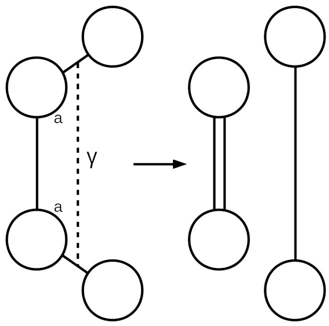

Figure 10: Creating a double-edge via the arc γ

As it allows some measure of simplification, we shall cover the cases in which it can be shown that $\mathcal{S}$ consists only of proper dipoles before those in which some other spherical picture is required.

First, however, we establish the following:

Lemma 4.8. *If Δ is an interior, positively curved region of a dipole-reduced picture Γ over the presentation $(2, m|n, k|1, q)$ satisfying **(A1)**, **(A2)**, **(A3)**, $m \neq 2q$, and $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$ then Δ is one of $\{T, S_j(1 \leq j \leq 5), P\}$, as depicted in Figure 11.*

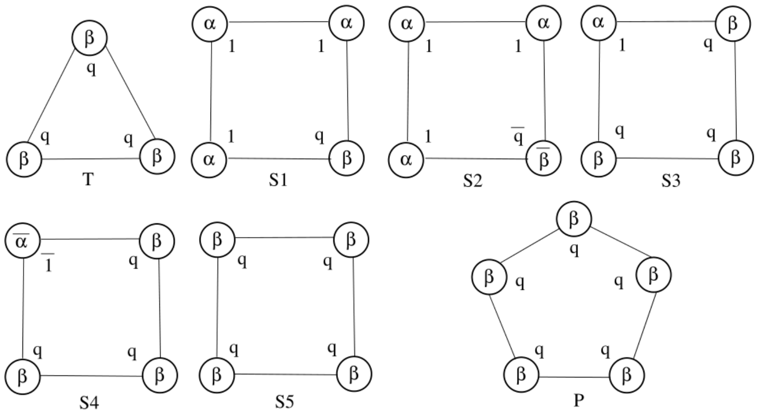

Figure 11: Possible positively curved regions

Proof. Since both $\frac{1}{n} + \frac{1}{k} \geq \frac{1}{2}$, both n and k must be at least 3, so α-vertices and β-vertices must both have at least 3 double-edges. Thus, we may exclude all regions with at least 6 vertices from consideration. As such, we proceed by the degree of the region. We may also rule out $d(\Delta) = 1$, as this would require $m = 1$ or $m = q$, neither of which are permitted by **(A1)** and **(A2)**.

If $d(\Delta) = 2$, then Δ must have label $0, 2, \pm q \pm 1$, or $2q$. Whichever of these values is taken must be a multiple of m, but the only values that permit this are 0 and $2q$. If the label of Δ is 0, then the two vertices bounding it must be of the same type, and in opposite orientation, so can be transformed into a dipole by bridge moves. On the other hand, if the label is $2q$, this forces $m = 2q$, contradicting the hypotheses of the lemma.

If $d(\Delta) = 3$, then since $m \geq 6$, the only possible label is $3q$, which can only be achieved via the region T.

If $d(\Delta) = 4$ then since $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$ rules out the possibility that Δ has two $\pm\alpha$-vertices and two $\pm\beta$-vertices. S_1 through S_4 cover all four possibilities in which Δ has one vertex of one type, and three of the other. Since $m \geq 6$, Δ cannot have four $\alpha^{\pm 1}$-vertices, which leaves S_5 as the only remaining possibility.

If $d(\Delta) = 5$, then since $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$ Δ can only have positive curvature if all vertices are of the same type. $m \geq 6$ again rules out the all-$\alpha^{\pm 1}$-vertex case, leaving us only P.

□

Lemma 4.9. *If any of the following hold, then no dipole-reduced picture exists over* $(2, m|n, k|1, q)$*.*

(i) $m > 2q \geq 10$, $m \notin \{2q + 1, 3q \pm 1\}$ *and either*

- *(a)* $n \geq 4$, $k \geq 4$ *and either* $m \neq 3q$, *or* $k \geq 6$,
- *(b)* $n = 3$, $k \geq 6$,

  or

- *(c)* $n \geq 6$, $k = 3$ *and* $m \notin \{3q, 4q, \frac{5q}{2}, 5q\}$

(ii) $m > 2q = 8$, $m \notin \{9, 11, 13\}$ *and either*

- *(a)* $n \geq 4$, $k \geq 4$, $m \neq 8$, *and either* $m \neq 12$ *or* $k \geq 6$,
- *(b)* $n = 3$, $k \geq 6$ *and* $m \neq 8$,

  or

- *(c)* $n \geq 6$, $k = 3$ *and* $m \notin \{8, 10, 12, 16, 20\}$

(iii) $m > 2q = 6$, $m \neq 7$ *and either*

- *(a)* $n \geq 4$, $k \geq 4$ *and either* $m \neq 9$ *or* $k \geq 6$ *and* $m \neq 6$,

  or

- *(b)* $n \geq 6$, $k = 3$ *and* $m \notin \{8, 9, 10, 12, 15\}$

Proof. Dividing cases as in the statement of the lemma, (i)-(ii) Since $q \geq 4$ and $m \neq 3q \pm 1$, Lemma 4.8 permits only regions of the forms T, S_5 and P as positively curved interior regions of a spherical picture Γ over $(2, m|n, k|1, q)$. If T occurs then $m = 3q$ which, within conditions (i) and (ii), forces $n \geq 6$, whereas T is positively curved only when $n < 6$. The region S_5 occurs only when $m|4q$, so since $m > 2q$ in both cases it can only occur when $m = 4q$. Within the conditions of (i) and (ii), $m = 4q$ implies $k \geq 4$, however S_5 is only positively curved when $k < 4$. P only occurs when m divides $5q$. Given that $m > 2q$, this is only the case when $m = 5q$ or $m = \frac{5q}{2}$. This condition can only coincide with one of (i) or (ii) if $k \geq 4$, however P is only positively curved when $k < 4$.

Since (i) and (ii) permit T, S_5 and P as interior regions only in conditions under which they are not positively curved, and allow no other potentially positively curved interior region, we must conclude that the 4π total curvature that Γ must have, being spherical, lies entirely within its distinguished region. However, the curvature of a single region is bounded above by 2π, so this region must fall short, contradicting our notion of a spherical picture.

(iii) For the cases in (iii), any of $T, S_1, S_2, S_3, S_4, S_5$ and P are possible. However, following the same argumentative pattern as for (i)-(ii), we note that (iii) only permits $m = 3q$, and thus the existence of regions of type T, when $m = 9$ so that $k \geq 6$, precisely the condition in which regions of type T are non-positively curved. S_1 and S_2 are only positively curved when $n = 3$ or $(n, k) = (4, 3)$, neither of which are permitted by (iii). For S_3 or S_4 to appear requires $m = 3q \pm 1 = 9 \pm 1$, which under (iii) requires $n \geq 4$, $k \geq 4$, which suffices to rule out S_3 and S_4 being positively curved. Internal regions can only take the form of S_5 when $m = 12$, which can only occur under (iii) if $k \geq 4$, and regions of this form are only positively curved when $k < 4$, so internal regions in the form of S_5 cannot contribute positive curvature. Similarly, P can only appear if $m = 15$, which is only compatible with (iii) if $k \geq 4$, whilst P is only positively curved if $k < 4$.

Thus, by an argument identical to that of the (i)-(ii) case, no dipole-reduced spherical picture can exist over $(2, m|n, k|1, q)$ where m, n, k, q satisfy the conditions of (iii).

$\square$

With the equivalences between homotopy classes of maps and spherical pictures noted in section 2, this gives us the following:

Corollary 4.10. *If m, n, k, q satisfy the conditions of Lemma 4.9, then the second homotopy group of the space Z as defined in section 2 has a generating set whose elements correspond to the homotopy classes of maps represented by the dipoles $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$.*

We can also obtain some information about the structure of the groups themselves:

Corollary 4.11. *Let $\mathcal{P}$ be the presentation $(2, m|n, k|1, q)$, where m, n, k, q satisfy one of the conditions of Lemma 4.9. Then in $\mathcal{G}(\mathcal{P})$, the orders of a, b, (ab) and (ab^p) are precisely 2, m, n and k, respectively.*

Proof. For a and b, this follows fairly immediately from Lemma Suppose a^i were trivial in $\mathcal{G}(\mathcal{P})$. Then we could construct a picture Π over $\mathcal{P}$ whose boundary label is this element. Since this label is an element of A of $A * B$, any boundary B-regions must have a trivial label on the boundary. Thus, using a bridge move, we can reduce any boundary B-region to a region bounded only by the boundary of the picture, and a single edge which only meets the boundary. Having done so, we can remove all such edges, obtaining a picture whose boundary regions are all A-regions. Since an edge cannot divide an A-region from another A-region, it must be the case that our picture has only one boundary region, and so is spherical. Lemma 4.9, therefore, establishes that Π can be reduced to an empty picture by bridge moves and the addition and removal of dipoles. Observe that neither of these operations can change the element of $A * B$ represented by the boundary word. Thus, considering the final step before reaching the empty picture - that in which only a dipole remains - the boundary of a dipole, which is trivial in $A * B$ must represent the same element of $A * B$ as the boundary word of Π. Thus a^i must be trivial in $A * B$, so $2 \mid i$. Thus, the order of a in $\mathcal{G}(\mathcal{P})$ is 2. An identical argument applies to b to establish that the order of b in $\mathcal{G}(\mathcal{P})$ must be m.

Suppose now that the order of ab were n', with $n = n'r$ and $r > 1$. Then we can construct a picture Π over $\mathcal{P}$ whose boundary label is $(ab)^{n'}$. We use this picture to construct a spherical picture by taking one α^{-1} vertex and surrounding it with r copies of Π, to obtain a new picture Π'. Define $C_\alpha(P)$ to be the number of α-vertices in P minus the number of α^{-1}-vertices in P. Note that bridge-moves do not change vertex-count, and the addition or removal of dipoles only adds or removes a pair of inverses, so neither procedure changes C_α. Observe that, since Π' is formed of one α^{-1}-vertex and r copies of another picture, $C_\alpha(\Pi') \cong -1 \bmod r$, and $r > 1$, so $C_\alpha(\Pi') \neq 0$. Thus, Π' cannot be reducible to an empty picture by bridge-moves and addition and removal of dipoles, contradicting Lemma 4.9. Thus, it must be the case that the order of ab is n. An identical argument establishes that the order of ab^q is k.

□

We now approach the cases in which an additional sphere, beyond dipoles, is required.

Lemma 4.12. *If any of the following conditions on m, n, k, q hold, then all spherical pictures over $(2, m|n, k|1, q)$ are reducible to an empty picture by bridge-moves, the addition and removal of dipoles, and the addition and removal of one other spherical picture. If the condition (i),... (vi) holds, then this additional spherical picture is that denoted $\mathcal{S}_j$ in Figure 12, $j =$ $1, \ldots 6$, respectively.*

(i) $m = 2q$ *and one of*

(a) $n \geq 4, k \geq 4$ *and* $q \geq 3$

(b) $n = 3, k \geq 6$ *and* $q \geq 4$

or

(c) $n \geq 6, k = 3$ *and* $q \geq 3$

(ii) $m = 3q, n \geq 4, k = 4$ *and* $q \geq 3$

(iii) $m = 3q, n \geq 4, k = 5$ *and* $q \geq 3$

(iv) $m = 3q, n \geq 6, k = 3$ *and* $q \geq 3$

(v) $m = 4q, n \geq 6, k = 3$ *and* $q \geq 3$

(vi) *(a)* $m = 5q, n \geq 6, k = 3$ *and* $q \geq 3$

(b) $m = \frac{5q}{2}, n \geq 6, k = 3, q \geq 3$ *and if* $m = 10$ *and* $q = 4$ *then* $n \geq 8$.

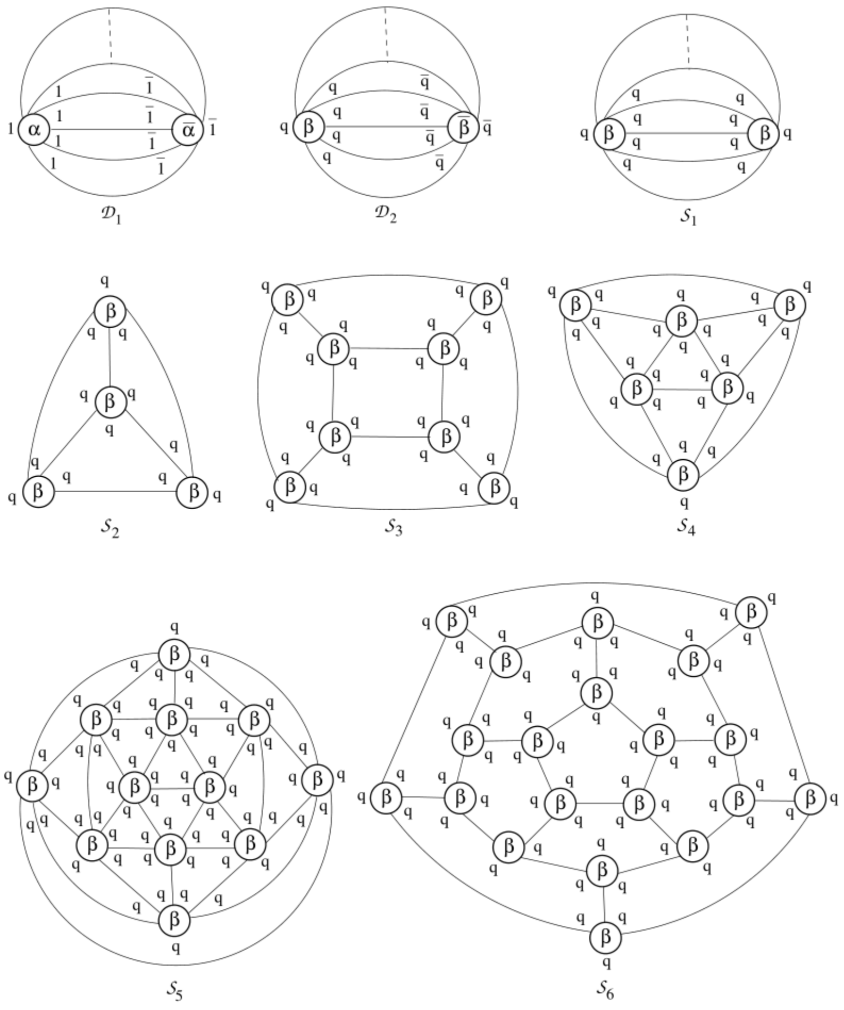

Figure 12: Possible spherical pictures.

Proof. We shall approach these proofs by considering a minimal (by number of vertices) counterexample. This allows us to assume that the pictures we consider do not contain any dipoles, or pairs of vertices that can be transformed into dipoles by bridge moves. We can also conclude that no connected figure corresponding to an induced subgraph on any majority of the vertices of the $\mathcal{S}_j$ in question can be present in the pictures we are considering, else we would be able to add the remaining vertices of a copy of $\mathcal{S}_j$ by introduction of dipoles followed by bridge moves, adding fewer vertices to the picture than are present in $\mathcal{S}_j$, and remove the copy, attaining a new picture with fewer vertices than the original. If we show that the curvature of any picture satisfying these conditions can be distributed in such a way that the total curvature must not reach 4π, then we establish that no vertex-minimal counterexample to the claim of the lemma exists, and so the lemma must hold.

(i) Since $m = 2q$, β-vertices and β^{-1}-vertices have the same boundary, and so for $k \geq 3$ we can construct spherical pictures of the form $\mathcal{S}_1$, which following previous sections we shall denote 'improper dipoles'. Now consider a vertex-minimal spherical picture Γ over $(2, m|n, k|1, q)$ satisfying condition (i) of the lemma. If Γ contained any two vertices of type β with an edge between them, we could use bridge moves to form either a dipole or an improper dipole, which we could then remove to reach a spherical picture with fewer vertices. Thus, any interior region of Γ must be of form S_1 or S_2, by Lemma 4.8. But for S_1 or S_2 to be present and positively curved requires $q = 3$ and either $n = 3$ or $n = 4$ and $k = 3$, conditions ruled out by (i). Thus, since no interior region contributes to curvature, and as noted in the proof of Lemma 4.9 the distinguished region cannot provide 4π curvature itself, Γ cannot be the minimal counterexample we supposed. As such, every spherical picture over these presentations is reducible to an empty picture by dipoles and $\mathcal{S}_1$.

(ii) Presentations satisfying this condition admit the sphere $\mathcal{S}_2$ of Figure 12. Letting Γ be, as above, a minimal counterexample to the claim of the lemma, a positively curved interior region Δ of Γ takes the form of T, from Figure 11, and has curvature $c(\Delta) = c(4, 4, 4) = \pi/2$

If such a region Δ is adjacent to a vertex of type β, then either a dipole can be formed or a majority of $\mathcal{S}_2$ can be formed, and in either case, Γ

fails to be vertex-minimal. Thus, the neighbourhood of Δ is given by Figure 13(i). Distributing $c(\Delta)/3 = \pi/6$ to each of the $\hat{\Delta}_i$ as shown removes all curvature from Δ. Let $\hat{\Delta}$ be one of the $\hat{\Delta}_i$, and assume $\hat{\Delta}$ is interior. Since $q \geq 3$ we have $m = 3q \nmid 2q \pm 2$, so $d := d(\hat{\Delta}) \geq 5$ If $d = 5$ then $c(\hat{\Delta}) \leq c(4,4,4,4,4) = -\pi/2$, whereas $\hat{\Delta}$ receives curvature only once (as two disjoint edges of a hexagon account for four of its vertices, so there must be another edge connecting them). If $d \geq 6$ then $c(\hat{\Delta}) + d\frac{\pi}{2} \leq 2 - d/3 \leq 0$. Thus, the only possible region with positive curvature after the redistribution of curvature is the distinguished region. However, to reach the 4π total required, since the distinguished region has at most 2π curvature of its own, we must have $d \geq 12$. If $d = 12$ then $c(\hat{\Delta}) \leq c(4,4,4,4,4,4,4,4,4,4,4,4) < -2\pi$, and each additional vertex decreases the curvature of the region by a further $\frac{\pi}{2}$, which overwhelms the effect of any transferred curvature. Thus, no such minimal counterexample Γ can exist, and so the claim of the lemma holds.

(iii) These presentations admit the sphere $\mathcal{S}_3$ of Figure 12. Taking, as above, Δ to be a positively curved interior region of a vertex-minimal spherical picture Γ not reducible to the empty picture by bridge-moves, dipoles and $\mathcal{S}_3$, we note that Δ is of type T as denoted in Figure 11, and $c(\Delta) = c(5,5,5) = \frac{\pi}{5}$. If Δ has a neighbouring region $\hat{\Delta}$ such that $d(\hat{\Delta}) > 3$ or $\hat{\Delta}$ is the distinguished region, then transfer all $c(\Delta) = \pi/5$ to $\hat{\Delta}$. In the former case, note that since $m = 3q$, no 4-region with two adjacent vertices of type β exists. If $d(\hat{\Delta}) = 5$, then $c(\hat{\Delta}) \leq c(4,4,4,5,5) = \frac{-7\pi}{10}$, so the transfer does not make $\hat{\Delta}$ positively curved. If $d(\hat{\Delta}) = 6$, $c(\hat{\Delta}) \leq c(4,4,4,4,4,4) = -\pi$, allowing $\hat{\Delta}$ to receive curvature across up to five edges. If $\hat{\Delta}$ receives curvature across all six edges, its vertices must all be of type β, so

$c(\hat{\Delta}) \leq c(5,5,5,5,5,5) = -\frac{8\pi}{5}$, so no new positively curved region is created by this transfer of curvature. For $d := d(\hat{\Delta}) > 6, c(\hat{\Delta}) \leq c(4,\ldots,4) = 2\pi - d\pi + 2d\pi/4 < -\frac{d\pi}{5}$, so these regions can also accept all curvature they might have to without becoming positive.

If Δ has no neighbour of degree greater than three, and is not adjacent to the distinguished region, then it is Δ_1 of the configuration shown in Figure 13(ii). Note that none of the adjoining regions $\hat{\Delta}_1, \ldots \hat{\Delta}_6$ can be of form T, else we would have a majority of $\mathcal{S}_3$, and be able to reduce to a smaller counterexample. We have a total of $4\frac{\pi}{5}$ curvature across $\Delta_1, \ldots \Delta_4$ to distribute, and do so as Figure 13 depicts, splitting the excess curvature evenly between $\hat{\Delta}_i; i = 1, \ldots, 6$ (and reverting any previous transfer of curvature from Δ_2, Δ_3 and Δ_4). Since $m = 3q$ does not divide any of $2q, 2q \pm 1, 2q \pm 2, 3q \pm 1, 3q \pm 2$ or $4q$, it follows that if $\hat{\Delta} = \hat{\Delta}_i$ for some $i = 1, \ldots, 6$, and $\hat{\Delta}$ is an interior region, then $d(\hat{\Delta}) \geq 5$. Given that this transfer of curvature only occurs across $\beta - \beta$-edges, and the recipient regions must have degree exceeding three, the argument from the case in which Δ does have a neighbour of degree greater than three applies, and suffices to demonstrate that this transfer of curvature cannot result in a positively curved region.

Our argument for the distinguished region is as in the previous case: Since we have eliminated all positive curvature from the interior of Γ, the only way that the 4π curvature of a spherical picture could be present is if it were all in the distinguished region. This region's own curvature is bounded above by 2π, so if Γ is the spherical picture we have supposed it to be, the distinguished region must have received at least another 2π curvature. Our distribution rule only ever sends $\pi/5$ across any edge, so in order to have received so much curvature, the distinguished region must have at least ten edges. However, a region of degree d, which has received $\frac{\pi}{5}$ additional curvature across each edge has curvature bounded above by $2\pi - d\pi + 2\pi(d/4) + \pi d/5 = (2 - 3d/10)\pi$, which for $d \geq 10$ is negative. Thus, the distinguished region cannot possibly be left holding enough curvature for Γ to be a spherical picture, and thus, lacking a vertex-minimal counterexample, the statement of the lemma must hold.

(iv) In this case, our identified spherical picture is $\mathcal{S}_4$. These presentations admit only T as a positively curved region, however since $\mathcal{S}_4$ has only four vertices, the region T itself constitutes a vertex-majority of $\mathcal{S}_4$. Therefore, a spherical picture not reducible to an empty picture via bridge-moves, dipoles and $\mathcal{S}_4$ would have to have no positively curved interior regions. Thus, all positive curvature in such a picture would have to come from the distinguished region, which can provide at most 2π of the 4π required of a spherical picture. Thus, as with the above cases, this section of the lemma has no vertex-minimal counterexample, and so must hold.

(v) These presentations admit the sphere $\mathcal{S}_5$ of Figure 12. As in the previous cases, we suppose that Γ is a vertex-minimal spherical picture not reducible to the empty picture by way of dipoles, bridge-moves and $\mathcal{S}_5$. Letting Δ be a positively curved interior region of Γ, Lemma 4.8 suffices to establish that Δ can only be of the form $S5$. Notably, this figure constitutes precisely half (by vertices) of a copy of $\mathcal{S}_5$, so wherever it occurs in Γ, it must be surrounded by vertices of type α, else we could reduce Γ to a smaller picture by either removing a dipole or introducing three of them to complete the copy of $\mathcal{S}_5$, and then removing it. As shown in Figure 13, we split the $\frac{2\pi}{3}$ curvature of Δ evenly between adjacent regions, sending $\frac{\pi}{6}$ to each. Let $\hat{\Delta}$ be an interior region that receives curvature in this way. Since $m = 4q$

divides none of $2q \pm 1$ or $2q \pm 2$, $d(\hat{\Delta})$ must have degree at least five, with at least two non-adjacent vertices of type α. Thus, there must be at least four different edges of $\hat{\Delta}$ across which no curvature is received. Thus, a receiving region with degree d has curvature $c(\hat{\Delta} \leq c(3,\ldots,3,6,6) = (4-d)\pi/3$ and receives $(d-4)\pi/6$, so since $d > 4$, the total curvature of Δ after receiving must be negative. Having eliminated all positive curvature from internal regions, we consider the distinguished region. As in previous cases, the distinguished region has at most 2π curvature of its own, so must receive at least another 2π curvature for there to be any possibility of it carrying the 4π curvature required of a spherical picture. A maximum of $\pi/6$ curvature can be received over each edge, so this requirement demands that the distinguished region have at least 12 edges. However, even if it were to receive curvature across every edge, a region $\hat{\Delta}$ with degree d has curvature $c(\hat{\Delta}) \leq c(3,\ldots,3) = 2\pi - d\pi/3 \leq -d\pi/6$ for $d \geq 12$. Thus, the distinguished region cannot possibly provide the positive curvature necessary for Γ to be a spherical picture. Therefore, as with the previous cases, this case of the lemma lacks a vertex-minimal counterexample, so must hold.

(vi) This final collection of cases admit the spherical picture $\mathcal{S}_6$. As with the other cases, we suppose that Γ is a vertex-minimal spherical picture not reducible to an empty picture by bridge moves, the addition and removal of dipoles, and the addition and removal of copies of $\mathcal{S}_6$. Let Δ be a positively curved interior region of Γ. Following Lemma 4.8, Δ must be of the form P, as depicted in Figure 11, and so has curvature $c(\Delta) = c(3,3,3,3,3) = \pi/3$. If Δ is adjacent to a vertex of type α, as in Figure 13(iv), split the curvature of Δ evenly between the regions marked as $\hat{\Delta}_1$ and $\hat{\Delta}_2$, adding $\pi/6$ to each. Otherwise,

Δ must be surrounded by β-vertices, as a β^{-1}-vertex would allow the formation and removal of a dipole. Such a figure contains precisely half of a copy of $\mathcal{S}_5$, so each of the adjacent vertices must in turn be adjacent to vertices of type α, as depicted in Figure 13(v). In this case, we divide the curvature between all adjacent regions, adding $\pi/15$ to each as Figure 13 suggests.

Now let $\hat{\Delta}$ be an internal region of Γ that receives curvature in the above scheme. Since neither $5q$ nor $5q/2$ divides $rq \pm 1$, $\hat{\Delta}$ must contain more than one vertex of type α. Thus, since curvature is only transferred across $\beta-\beta$-edges, if $\hat{\Delta}$ has degree d, it receives curvature over at most $d-3$ edges. It follows that $c(\hat{\Delta})$ plus all curvature added to it is bounded above by $c(6,6,3,\ldots,3)+(d-3)\pi/6 = (5-d)\frac{\pi}{6}$, so for $d \geq 5$ there is no possibility that $\hat{\Delta}$ has positive curvature after redistribution. Observe that if $m = 5q$ there is no possibility of a 4-region with two adjacent vertices of type β, while if $m = \frac{5q}{2}$ such a region can occur, with label $2q+2$, requiring $q = 4$, so that $m = 10$, and thus by condition (b), $n \geq 8$. This gives $c(\hat{\Delta}) \leq c(3,3,8,8) < \frac{-\pi}{6}$, so the region can accept $\frac{\pi}{6}$ curvature over its single $\beta-\beta$-edge without becoming positively curved.

Having arranged curvature in such a way that no interior region has positive curvature, we approach the distinguished region. As in previous cases, a spherical picture under these conditions requires that the distinguished region have received 2π curvature. With a maximum transfer over a single edge being $\pi/6$, this requires that the distinguished region have 12 edges. However, if the degree of the distinguished region is $d \geq 12$, then it has curvature after transfers not exceeding $c(3,\ldots,3)+n\pi/6 = 2\pi - d\frac{\pi}{6} < 0$. Thus, the distinguished region cannot provide the curvature required for Γ to be a spherical picture, contradicting our assumption that a vertex-minimal spherical picture satisfying the conditions of (vi) that is not reducible to an empty picture via bridge moves, dipoles and $\mathcal{S}_6$. In the absence of a vertex-minimal counterexample, the lemma holds on these presentations.

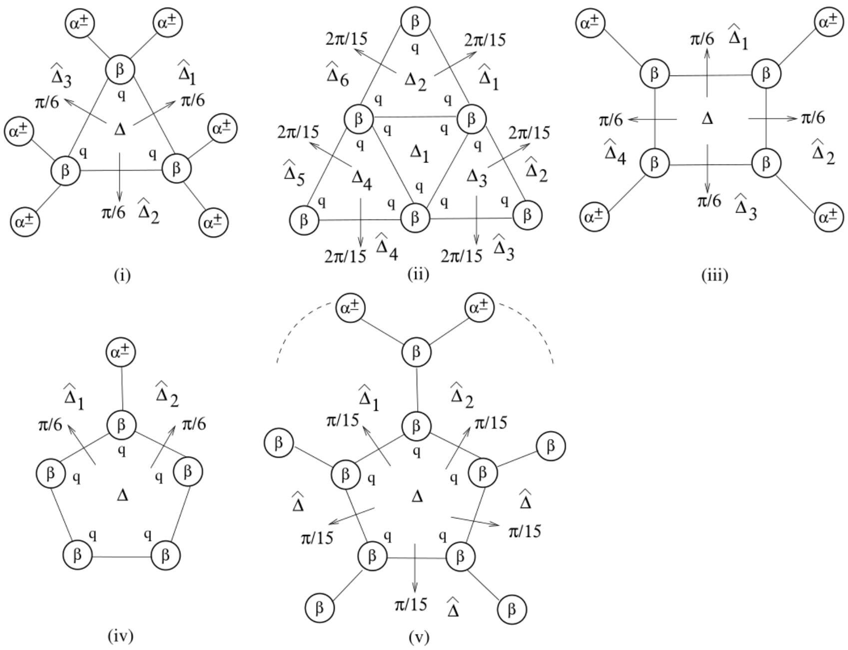

Figure 13: Redistribution of curvature

□

As with Lemma 4.9, this can be interpreted in homological terms to give the following:

Corollary 4.13. *If m, n, k, q satisfy condition (i),...,(vi), respectively, of Lemma 4.12, then the second homotopy group of the space $Z_{\mathcal{P}}$ as defined in section 2 has a generating set whose elements correspond to the homotopy classes of maps represented by the dipoles $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$, along with $\mathcal{S}_j$; $j = 1, \ldots, 6$, respectively.*

We can also obtain the same non-collapse condition on the presentations satisfying a condition of Lemma 4.12 as we could on those satisfying a condition of Lemma 4.9:

Corollary 4.14. *Let $\mathcal{P}$ be the presentation $(2, m|n, k|1, q)$, with m, n, k, q satisfying one of the conditions of Lemma 4.12. Then the orders of a, b, ab and ab^q are precisely $2, m, n$ and k, respectively.*

Proof. Since $\mathcal{S}_1, \ldots, \mathcal{S}_6$ all have boundary-labels trivial in $A*B$, and contain no vertices of type α, the arguments regarding the orders of a, b and ab proceed identically to those in Corollary 4.11. Establishing that the order of ab^q is k, however, requires some additional reasoning.

In cases (iii), (iv), (v) and (vi), non-collapse in ab^q follows from non-collapse in a and b, as k is prime, so any collapse would force a and b^q to have the same order, which is incompatible with the established orders of a and b.

In case (ii), even if collapse occurs, ab^q cannot be trivial, for the same reason as in the above cases. The only remaining possibility that we must rule out, then, is that ab^q has order 2. However, this is amenable to the same form of argument as was used to rule out collapse in ab: If ab^q has order 2, then we may construct a picture with boundary $(ab^q)^2$. Attaching two such pictures around a β^{-1}-vertex gives a spherical picture. This picture necessarily has an odd number of vertices of type β. Since $\mathcal{S}_2$ has an even number of β-vertices, none of performing bridge moves, adding and removing dipoles, or adding and removing copies of $\mathcal{S}_2$ change the parity of the number of vertices of type β in a picture. This contradicts Lemma 4.12(ii), which requires that we be able to reach the empty picture, which has 0 vertices of type β, with these moves.

We approach the order of ab^q in case (i) as we did that of a^pb^q in Corollary 3.9, supposing that the order of ab^q is $k' < k$ and using a picture

Π witness to this fact to build a connected spherical picture Π' with a distinguished vertex whose label is $(ab^q)^{k'}$. As in said corollary, the distinguished vertex can 'absorb' adjacent vertices of type β, with its label alternating between $(ab^q)^{k'}$ and $(ab^q)^{k-k'}$ as it does so. Having reduced Π' to the smallest picture that can be reached by these absorptions along with the usual reductions, we note that the distinguished vertex is not adjacent to any vertex of type β. The arguments made in Lemma 4.12(i) establish that regions of Π' not incident to the distinguished vertex cannot have positive curvature, and that regions that include the distinguished vertex have curvature bounded above by the difference in curvature contribution between a vertex of valence k and one of valence k' or $k-k'$, depending on which the distinguished vertex arrived at. Thus, if the distinguished vertex has valence k', then the total positive curvature of Π' is bounded above by $k' * 2\pi(\frac{1}{k'} - \frac{1}{k}) < 2\pi < 4\pi$, and if the distinguished vertex has valence $k-k'$, then the total positive curvature of Π' is bounded above by $(k-k') * 2\pi(\frac{1}{k-k'} - \frac{1}{k}) < 2\pi < 4\pi$. However, Π' is a spherical picture, so must have total curvature of 4π. This contradiction demonstrates that our assumption $k' < k$ was false, so the order of ab^q must be k.

□

4.3 Pushout diagrams and the homological argument

The results we have obtained in this section so far allow us to obtain fairly strong results on the order of the group $\mathcal{G}(\mathcal{P})$, where $\mathcal{P}$ satisfies one of the conditions of lemmas 4.9 and 4.12, using the methods laid out in Section 2.

We shall begin with presentations satisfying one of the conditions of Lemma 4.12.

Proposition 4.15. *Let $\mathcal{P}$ be the presentation $(2, m|n, k|1, q)$. If m, n, k, q*

satisfy one of the conditions of Lemma 4.12 other than (vi)(b), then $\mathcal{G}(\mathcal{P})$ is infinite

Proof. These presentations can all be expressed as $(2, pq|n, k|1, q)$, where (p, k) is one of $(2, k), (3, 3), (3, 4), (4, 3), (3, 5), (5, 3)$. In any of these cases, the triangle group $(2, p, k)$ is finite.

To try to exploit this structure, we approach these presentations via the pushout

$$\begin{array}{ccc} C_2 * \mathbb{Z} \cong \langle a, d | a^2 \rangle & \xrightarrow{\phi} & \langle a, d | a^2, d^p, (ad)^k \rangle \cong (2, p, k) \\ \downarrow{\scriptstyle\psi} & & \downarrow \\ C_2 * C_n \cong \langle a, b | a^2, (ab)^n \rangle & \longrightarrow & \langle a, b | a^2, b^{pq}, (ab)^n, (ab^q)^k \rangle = G(\mathcal{P}) \end{array}$$

with ϕ mapping a to a and d to d, and ψ mapping a to a and d to b^q.

To obtain the required results from this pushout via Theorem 4.2 of [10], we must build a corresponding pushout of topological spaces. Specifically, we require a space X, with trivial second homotopy group, which we can express as a union of two aspherical subspaces X_1 and X_2, with fundamental groups $\langle a, b | a^2, (ab)^n \rangle$ and $\langle a, d | a^2, d^p, (ad)^k \rangle$, respectively, which intersect in such a manner that $X_0 := X_1 \cap X_2$ has fundamental group $\langle a, d | a^2 \rangle$, and the inclusions of X_0 into X_1 and X_2 induce the maps ϕ and ψ, respectively, in homotopy.

In pursuit of such a space X, we construct a preliminary space X', with subspaces exhibiting the correct behaviour in homotopy, extending those subspaces to Eilenberg-MacLane spaces to obtain X.

Take as X' the presentation complex of $\langle a, b, d | a^2, (ab)^n, d = b^q, d^p, (ad)^k \rangle$. For X'_1, take the subspace of X' consisting of the basepoint, along with the cells corresponding to $a, b, d, a^2, (ab)^n$, and $d = b^q$. Let X'_2 be the subspace consisting of the basepoint along with the cells corresponding to a, d, a^2, d^p and $(ad)^k$. Thus, X'_1 is the presentation complex for the presentation $\langle a, b, d | a^2, (ab)^n, d = b^p \rangle$, X'_2 is the presentation complex for the presentation $\langle a, d | a^2, d^p, (ad)^k \rangle$. Since all cells in X' are accounted for, $X' = X_1' \cup X_2'$, whilst $X_0' := X_1' \cap X_2'$ is a presentation complex for the presentation $\langle a, d | a^2 \rangle$.

X_1' and X_2' have the required fundamental groups, and the inclusions of X_0' into each induce the correct maps in homotopy, so now we extend our subspaces to eliminate higher homotopy. We begin by adding 3-cells and higher to the subspace of X_0' corresponding to $\langle a | a^2 \rangle$, denoting union of added cells as Y_0. We then perform the same process on the spaces $X_1' \cup Y_0$ and $X_2' \cup Y_0$, denoting the unions of added cells Y_1 and T_2, respectively. Finally, we define $X_1 := X_1' \cup Y_0 \cup Y_1$, $X_2 := X_2' \cup Y_0 \cup Y_2$, so that $X_0 := X_0' \cup Y_0$. These are Eilenberg-MacLane spaces with 2-skeletons X_1', X_2', X_0', respectively, so satisfy our requirements on homotopy.

Taking $X := X_1 \cup X_2$, we have the required space and subspaces. To show that X is aspherical, we use Theorem 4.2 of [10]. This gives the required result, so long as the kernels of the maps in homotopy induced by the inclusions of X_0, X_1 and X_2 into X have homological dimension not exceeding 1, 2 and 2, respectively, and $\pi_2(X) = 0$. The kernel of the map induced by the inclusion $X_0 \subset X$ cannot contain any nontrivial power of a or d, as this would mean the order of a or b in $\mathcal{G}(\mathcal{P})$ was less than 2 or pq, respectively, contradicting Corollary 4.14, and thus, as a normal subgroup of a free product of two groups, not containing any nontrivial element of either, must be free. Thus, this kernel has homological dimension of at most 1. Likewise, since $\pi_1(X_1)$ can be expressed as a free product of $\langle a \rangle$ and $\langle ab \rangle$, and both a and ab must have in $\mathcal{G}(\mathcal{P})$ the same order as they do in $\pi_1(X_1)$, the kernel of the map induced by the inclusion of X_1 into X must be free, and so has homological dimension of at most 1. Since $\pi_1(X_2)$ is a triangle group, its only torsion elements are those conjugate to a power of one of a, d or ad. Thus, the kernel of the map in first homotopy induced by the inclusion $X_2 \subset X$ is torsion-free. Since $\pi_1(X_2)$ is finite, this establishes that the kernel is trivial, so certainly does not have homological dimension greater than 2. With these conditions fulfilled, if $\pi_2(X) = 0$, then X is aspherical.

To show that $\pi_2(X)$ is trivial, note that Lemma 4.13 gives generators for the second homotopy group of Z, the wedge sum of a $\mathrm{K}(C_2, 1)$ space A (in which we have chosen a loop $\bar{a}$ representing the generator a of C_2) and a $\mathrm{K}(C_{pq}, 1)$ space B (similarly, having selected a loop $\bar{b}$ representing a generator b of C_{pq}), to which 2-cells have been attached along paths corresponding to $(ab)^n$ and $(ab^q)^k$, to give a space whose fundamental group has presentation $(2, pq|n, k|1, q)$. We construct Z as a subspace of X by taking A as the union of the basepoint, the cells associated to a and a^2, and Y_0, letting $\bar{a}$ be the path along the 1-cell associated to a, taking as B the basepoint, the cells associated to b, d, d^p, and $d = b^q$, along with subspaces of Y_1 and Y_2 sufficient to eliminate all higher homotopy, and letting $\bar{b}$ be the path along the 1-cell associated to b. For the 2-cells of Z we take the remaining 2-cells in X, those corresponding to the relators $(ab)^n$ and $(ad)^k$. Note that this subspace contains the 2-skeleton of X. As a result, every element of $\pi_2(X)$ can be represented by a map from S^2 to Z. In particular, the maps representing a generating set of $\pi_2(Z)$ also represent a generating set of $\pi_2(X)$. Since X_1 and X_2 are aspherical, it shall suffice to show that the elements of $\pi_2(Z)$ can all be represented by maps to $Z \cap X_1$ or $Z \cap X_2$.

Recall from Section 2 the process for converting a spherical picture (with trivial boundary label) Γ over $\mathcal{P}$ into a map from S^2 to Z. Contracting the boundary of Γ to a point, to obtain a picture on a sphere rather than a disk, we map edges to the basepoint, corners labelled with powers of a and b to the loops $\bar{a}$ and $\bar{b}$, with appropriate orientation and multiplicity, depending on the power of a or b present, the interiors of vertices to the interior of the 2-cell corresponding to the same relator, and A and B regions to the subspaces A and B. Note that B-regions whose corner labels are all of the form $b^{\pm q}$ can be mapped into the cell corresponding to d^p, in X_2.

The dipole over α contains A-regions, B-regions and vertices of type α, so can be represented by a map from S^2 whose image is contained in the union of A, B and the 2-cell corresponding to $(ab)^n$. Restricting our attention to a B-region Δ in this picture, we observe that the map in this region is a map from a rectangle to B, in which one pair of opposing sides are mapped to the basepoint, and the other pair mapped along $\bar{b}$ in parallel. As such, this restricted map corresponds to a homotopy between $\bar{b}$ and itself. Since B is an Eilenberg-MacLane space, any two homotopies between the same maps are themselves homotopic, so without changing the homotopy class of our map, we may use the trivial homotopy between $\bar{b}$ and itself as the interior of the rectangle, so that the entirety of Δ is mapped to the image of $\bar{b}$. Likewise, without any risk of changing the homotopy class of our map, we may assume that each A-region is mapped entirely to $\bar{a}$. Thus, the dipole over α can be represented by a map whose image lies within X_1, and as such must represent the trivial element of $\pi_2(X)$.

The dipoles over β, both proper and improper, along with the remaining spheres $\mathcal{S}_i$; $i = 2, \ldots, 6$, are formed entirely of β-vertices, A-regions, and B-regions whose corner labels are all of the form $b^{\pm q}$, so represent maps whose image is contained in the cells corresponding to $a, a^2, (ad)^k$, and d^p, all of which are in X_2. As such, these spherical pictures all represent the trivial element of $\pi_2(X)$.

Having found a generating set of $\pi_2(X)$ and shown that all of its elements are trivial, we can conclude that $\pi_2(X)$ is trivial, and so following [10], X is aspherical. Thus, our pushout of groups is geometrically Mayer-Vietoris. If $\mathcal{G}(\mathcal{P})$ is finite, this allows us to calculate

$$\frac{1}{\mathcal{G}(\mathcal{P})} = \chi_{\mathbb{Q}}(G(\mathcal{P})) = \chi_{\mathbb{Q}}(C_2 * C_n) + \chi_{\mathbb{Q}}((2, p, k)) - \chi_{\mathbb{Q}}(C_2 * \mathbb{Z}).$$

The groups $C_2 * C_n$ and $C_2 * \mathbb{Z}$ have rational Euler characteristics $\frac{1}{n} +$

$\frac{1}{2} - 1 = \frac{1}{n} - \frac{1}{2}$ and $-\frac{1}{2}$, respectively, and $(2, p, k)$ is a finite group, so has positive rational Euler characteristic. Thus,

$$\frac{1}{|G(\mathcal{P})|} > \frac{1}{n} - \frac{1}{2} + \frac{1}{2} = \frac{1}{n}$$

so that $|G(\mathcal{P})| < n$, but Corollary 4.14 establishes that $G(\mathcal{P})$ has an element, ab, of order n. Thus, $G(\mathcal{P})$ cannot be finite.

$\square$

Proposition 4.16. *If $\mathcal{P}$ satisfies condition (vi)(b) of Lemma 4.12, then $G(\mathcal{P})$ is infinite.*

Proof. This follows identically to the above argument, using the pushout

$$\begin{tikzcd}
C_2 * \mathbb{Z} \cong \langle a, d | a^2 \rangle \arrow[r, "\phi"] \arrow[d, "\psi"] & \langle a, d | a^2, d^5, (ad^2)^3 \rangle \cong A_5 \arrow[d] \\
C_2 * C_n \cong \langle a, b | a^2, (ab)^n \rangle \arrow[r] & \langle a, b | a^2, b^{\frac{5q}{2}}, (ab)^n, (ab^q)^3 \rangle = G(\mathcal{P})
\end{tikzcd}$$

with ϕ mapping a to a and d to d, and ψ mapping a to a and d to $b^{\frac{q}{2}}$ $\square$

Proposition 4.17. *If $\mathcal{P}$ satisfies the conditions of Lemma 4.9, then either $G(\mathcal{P})$ is infinite, or it has order N, where*

$$\frac{1}{N} = \frac{1}{m} + \frac{1}{n} + \frac{1}{k} - \frac{1}{2}$$

Proof. In this case we use the pushout

$$\begin{tikzcd}
F(s,t) \cong \langle s, t | \rangle \arrow[r, "\phi"] \arrow[d, "\psi"] & \langle s, t | s^n, t^k \rangle \cong C_n * C_k \arrow[d] \\
C_2 * C_m \cong \langle a, b | a^2, b^m \rangle \arrow[r] & \langle a, b | a^2, b^m, (ab)^n, (ab^q)^k \rangle = G(\mathcal{P})
\end{tikzcd}$$

with ϕ mapping s to s and t to t, and ψ mapping s to ab and t to ab^q.

Since these presentations admit no spheres beyond (proper) dipoles, they are quasi-aspherical. Following, for example, [5], we note that it suffices to take the space obtained by taking a $K(G, 1)$ spaces A, B and C

corresponding to $F(s,t)$, $C_n{*}C_k$ and $C_2{*}C_m$, with $\hat{\phi}$ and $\hat{\psi}$ being maps from A to B and A to C that induce ϕ and ψ, taking the union of the mapping cylinders $M(\hat{\phi})$ and $M(\hat{\psi})$, and identifying the 'tops' of both, homeomorphic to A. The resulting space X is aspherical, so long as our presentations are quasi-aspherical, and suffer no collapse. Since both of these conditions hold, we use the same result on rational Euler characteristics to conclude that, if $G(\mathcal{P})$ is finite, then it has order N given by

$$\frac{1}{N} = \left(\frac{1}{2} + \frac{1}{m} - 1\right) + \left(\frac{1}{n} + \frac{1}{k} - 1\right) - (-1) = \frac{1}{m} + \frac{1}{n} + \frac{1}{k} - \frac{1}{2}$$

□

4.4 Assembling Results

We now combine the results of the previous subsection with our preliminary results to obtain this section's theorem.

Proposition 4.18. *If $n \geq 4$, $k \geq 4$ and any of the following hold, then $G(\mathcal{P})$ is infinite.*

(i) $m \geq 2q \geq 10$ *and* $m \notin \{2q \pm 1, 3q \pm 1, 4q \pm 1\}$*;*

(ii) $m \geq 2q = 8$ *and* $m \notin \{9, 11, 13\}$*;*

(iii) $m \geq 2q = 6$ *and* $m \neq 7$*.*

Proof. If $m = 2q$, or if $m = 3q$ and $k \in \{4, 5\}$, then applying Proposition 4.15 to conditions (i)(a), (ii) or (iii) of Lemma 4.12 suffices to show that $G(\mathcal{P})$ is infinite. If $m \notin \{2q, 3q\}$, or if $m = 3q$ and $k \geq 6$, then applying Proposition 4.17 to conditions (i)(a), (ii)(a) or (iii)(a) of Lemma 4.9 establishes that $\frac{1}{|G(\mathcal{P})|} = \frac{1}{m} + \frac{1}{n} + \frac{1}{k} - \frac{1}{2}$. If $(n, k) \neq (4, 4)$, then $\frac{1}{|G(\mathcal{P})|} > 0$ forces $\frac{1}{m} + \frac{1}{n} + \frac{1}{k} > \frac{1}{2}$, so the conditions of Lemma 4.3 hold, and so $G(\mathcal{P})$ must be infinite. On the other hand, if $(n, k) = (4, 4)$, then we have

$\frac{1}{|G(\mathcal{P})|} = \frac{1}{m} + \frac{2}{4} - \frac{1}{2}$ forcing $G(\mathcal{P})$ to be a cyclic group of order m, generated by b. This, however, cannot be the case, as if $G(\mathcal{P})$ is Abelian, then the relator $(ab)^4$ reduces to b^4, contradicting Lemma 4.17, which establishes that the order of b is $m \geq 6$.

□

The remaining results require one additional trick. Observe that given a presentation $\mathcal{P}$ one may fairly trivially construct the Abelianisation of $G(\mathcal{P})$. If $G(\mathcal{P})$ is known to have order, if finite, of N, and if N is small, we may narrow the set of finite candidates for $G(\mathcal{P})$ from all those of order N to only those of order N with $N^{Ab} \cong G(\mathcal{P})^{Ab}$. If no such groups exist, then we can conclude that $G(\mathcal{P})$ is infinite. Our requirements only involve the groups for which $G(\mathcal{P})^{Ab}$ is trivial, that is, the cases in which $G(\mathcal{P})$ is perfect. As such, we define the following:

Definition 4.19. *The presentation $\mathcal{P}$ fails the perfect test if all three of the following hold:*

1. *$G(\mathcal{P})$ is perfect,*
2. *The order of $G(\mathcal{P})$, if finite, must be N,*
3. *No group of order N is perfect.*

Clearly, a group that fails the perfect test must be infinite. We use this, amongst other approaches, for the following results

Lemma 4.20. *(i) Let $n = 3$. If $G(\mathcal{P})$ has order, if finite, of $1/\left(\frac{1}{m} + \frac{1}{k} - \frac{1}{6}\right)$ and (m, k, q) is one of $(13, 11, 5)$; $(16, 9, 6)$; $(17, 9, 5)$; $(17, 9, 7)$; $(23, 8, q)$: $q = 5, 7, 9$ or 10; $(21, 7, 7)$; $(21, 7, 8)$; $(28, 7, q)$: $q = 6, 8, 10$ or 12; $(35, 7, q)$: $5 \leq q \leq 8$ or $10 \leq q \leq 11$ or $13 \leq q \leq 16$; $(36, 7, q)$: $q = 8, 10, 14$ or 16; $(39, 7, q)$: $q = 5, 7, 8, 11, 13, 14, 16$ or 17; $(40, 7, q)$: $q = 6, 8, 10, 12, 14, 16$ or 18; $(41, 7, q)$: $5 \leq q \leq 9$ or $11 \leq q \leq 13$ or $15 \leq q \leq 19$; then $\mathcal{P}$ fails the perfect test.*

(ii) Let $k = 3$*. If* $G(\mathcal{P})$ *has order, if finite, of* $1/\left(\frac{1}{m} + \frac{1}{n} - \frac{1}{6}\right)$ *and* (m, n, q) *is one of* $(13, 11, 5)$*;* $(16, 9, 6)$*;* $(17, 9, 7)$*;* $(23, 8, q) : q = 5, 7, 9$ *or* 10*;* $(21, 7, q) : q = 6, 8$ *or* 9*;* $(28, 7, q) : q = 6, 8, 10$ *or* 12*;* $(35, 7, q) : q = 5, 6, 8, 10, 11, 13, 15$ *or* 16*;* $(36, 7, q) : q = 6, 8, 10, 14$ *or* 16*;* $(39, 5, q) : 5 \le q \le 9$ *or* $11 \le q \le 12$ *or* $14 \le q \le 18$*;* $(40, 7, q) : q = 6, 12, 14$ *or* 18*;* $(41, 7, q) : 5 \le q \le 9$ *or* $11 \le q \le 13$ *or* $15 \le q \le 19$*; then* $\mathcal{P}$ *fails the perfect test.*

Proof. In each of these 112 cases it can be verified, by checking the Abelianisations of the presentations in question and verifying with [7] that no perfect group of the given order exists, that the conditions for $\mathcal{P}$ to fail the perfect test are satisfied.

$\square$

Proposition 4.21. *If* $n = 3$*,* $k \ge 6$ *and any one of the following holds, then* $G(\mathcal{P})$ *is infinite:*

(i) $m \ge 2q \ge 10$ *and* $m \notin \{2q + 1, 3q \pm 1, 4q \pm 1\}$*;*

(ii) $m \ge 2q = 8$ *and* $m \notin \{9, 11, 13\}$*;*

(iii) $m \ge 2q = 6$ *and* $m \ge 8$*.*

Proof. (i) If $m = 2q$, then applying Proposition 4.15 to Lemma 4.12(i)(b), $G(\mathcal{P})$ is infinite, so we may assume $m \ne 2q$. In particular, given $m \ne 2q + 1$, $m \ge 12$. In this case, applying Proposition 4.17 to Lemma 4.9(i)(b), we see that if $G(\mathcal{P})$ were finite, it would have order N such that $\frac{1}{N} = \frac{1}{m} + \frac{1}{k} - \frac{1}{6}$. If $k = 6$, then this forces $G(\mathcal{P})$ to be cyclic of order m, generated by b. In this case, m must be even, since the element a must have order 2, which can only be $b^{\frac{m}{2}}$. With this established, the relator $(ab)^3$ forces $b^{3+\frac{m}{2}} = 1$, which can only occur if $m = 6$, which lies below our lower bound for m. Thus, if $k = 6$ $G(\mathcal{P})$

is infinite. For the remaining (k, m), the only values which give an integer value for N divisible by 6, m and k are $k = 11, 12 \leq m \leq 13$; $k = 10, m \in \{12, 14\}$; $k = 9, m \in \{12, 16, 18, 20, 21, 22, 23\}$; and $k = 7, m \in \{21, 28, 35, 36, 39, 40, 41\}$. Taking all values of q for these cases consistent with the conditions of (i) gives 110 groups. 56 of these fail the perfect test, by Lemma 4.20(i), and 52 more give infinite groups by Lemma 4.4. Of the remaining two groups, the first is $(2, 39|3, 7|1, 15)$, which has an order if finite of 546, but can be shown by [7] to have a subgroup of index 39 whose derived subgroup has index 117, and the second is $(2, 39|3, 7|1, 18)$, which has the same order if finite, but which maps onto PSL(2,701). Thus, both remaining groups are infinite.

(ii) If $m = 2q$ then by the same argument as in (i), $G(\mathcal{P})$ is infinite, so again we assume $m > 2q$. Applying Proposition 4.17 to Lemma 4.9(ii)(b) again lets us conclude that if $G(\mathcal{P})$ has finite order N, then $\frac{1}{N} = \frac{1}{m} + \frac{1}{k} - \frac{1}{6}$. If $k = 6$ we can rule out a finite $G(\mathcal{P})$ by the same argument as in (i). Amongst the remaining (k, m), the only candidates that give integer N with the required divisors are $k = 7, m \in \{21, 28, 35, 36, 39, 40, 41\}$; $k = 8, m \in \{12, 16, 18, 20, 21, 22, 23\}$; $k = 9, m \in \{12, 15, 16, 17\}$; $k = 10, m \in \{10, 12, 14\}$; $k = 11, m = 12$; $k = 12, m = 10$; and $k = 14, m = 10$. Each of the resulting groups can be shown with [7],[8] to be infinite-automatic.

(iii) Follows directly from Lemma 4.5(i).

□

Proposition 4.22. *If* $n \geq 6, k = 3$ *and any one of the following conditions hold, then* $G(\mathcal{P})$ *is infinite.*

(i) $m \geq 2q \geq 10$ *and* $m \notin \{2q + 1, 3q \pm 1, 4q \pm 1\}$*;*

(ii) $m \geq 2q = 8$ *and* $m \notin \{9, 11, 13\}$*;*

(iii) $m \geq 2q = 6$ *and* $m \neq 7$*.*

Proof. (i) If $m \in \{2q, 3q, 4q, 5q/2, 5q\}$ then $G(\mathcal{P})$ is infinite by Propositions 4.15 and 4.16 as applied to Lemma 4.12, and so we assume otherwise. If $G(\mathcal{P})$ is finite, of order N, then by applying Proposition 4.17 to Lemma 4.9(i)(c) we find $\frac{1}{N} = \frac{1}{m} + \frac{1}{n} - \frac{1}{6}$. In the case $n = 6$, this forces $G(\mathcal{P})$ to be cyclic of order m, generated by b. However in this case, the relator $(ab)^6$ requires that $b^6 = 1$, contradicting our condition from Corollary 4.11 that $m \geq 10$ is the order of b. The only remaining values of n and m that give an integer order for $G(\mathcal{P})$ with the appropriate divisors are: $n = 11, 12 \leq m \leq 13$; $n = 10, m \in \{12, 14\}$; $n = 9, m \in \{12, 15, 16, 17\}$; $n = 8, m \in \{12, 16, 18, 20, 21, 22, 23\}$; and $n = 7, m \in \{21, 28, 35, 36, 39, 40, 41\}$. Taking all values of q consistent with the conditions of (i) gives 94 groups, 56 of which fail the perfect test by Lemma 4.20(ii) and 35 of which are infinite by Lemma 4.6. This leaves the groups $(2, 22|8, 3|1, 10)$, $(2, 28|7, 3|1, 13)$ and $(2, 36|7, 3|1, 17)$. The first of these, if finite, has order 264, but has a derived subgroup, of index 2, which is perfect, and no perfect group of order 132 exists. The second, if finite, has order 84, but maps onto PSL(2,187). The third, if finite, has order 252, but has a subgroup of index 14 whose core has index 2184. Thus none of these groups can be finite.

(ii) If $m \in \{8, 10, 12, 16, 20\}$ then applying Propositions 4.15 and 4.16 to Lemma 4.12 suffices to show that $G(\mathcal{P})$ is infinite, except in the case where $m = 10$ and $6 \leq n \leq 7$. This exception covers two groups, $(2, 10|6, 3|1, 4)$ and $(2, 10|7, 3|1, 4)$, the first of which has a subgroup of index 18 with infinite Abelianisation, and the second of which is infinite automatic. Thus, if $m \in \{8, 10, 12, 16, 20\}$ then $G(\mathcal{P})$

is infinite, so we consider the case where $m \notin \{8, 10, 12, 16, 20\}$. In this case, if $G(\mathcal{P})$ is finite, then applying Lemma 4.9(ii)(c) gives us $\frac{1}{N} = \frac{1}{m} + \frac{1}{n} - \frac{1}{6}$. $n = 6$ can be ruled out by the same argument as in (i). The remaining values of n and m which give an integer N with all required divisors are $n = 7, m \in \{21, 28, 35, 36, 39, 40, 41\}$; $n = 8, m \in \{18, 21, 22, 23\}$; $n = 9, m \in \{15, 17\}$; and $n = 10, m = 14$. Each of these possibilities give an infinite automatic group, by [7],[8], with the possible exception of $(2, 18|8, 3|1, 4)$. This final group has a subgroup of index 8 whose core, of index 336, has infinite Abelianisation.

(iii) If m is 8 or 10, then by parts (ii) or (iii), respectively, of Lemma 4.5, $G(\mathcal{P})$ is infinite. If $m \in \{6, 9, 12, 15\}$, then applying Proposition 4.15 to Lemma 4.12 establishes that $G(\mathcal{P})$ is infinite. Thus, we assume otherwise, that $m \notin \{6, 8, 9, 10, 12, 15\}$, so that $\mathcal{P}$ is within the scope of Lemma 4.9(iii)(b), and so by Proposition 4.17 $\frac{1}{N} = \frac{1}{m} + \frac{1}{n} - \frac{1}{6}$. As in the previous cases, $n = 6$ would force $|G(\mathcal{P})| = m$ and force collapse. Taking $n \geq 7$, then, the values of n and m that give integer N with all required divisors are $n = 7, m \in \{21, 28, 35, 36, 39, 40, 41\}$; $n = 8, m \in \{16, 18, 20, 21, 22, 23\}$; $n = 9, m \in \{16, 17\}$; $n = 10, m = 14$; $n = 11, m \in \{11, 13\}$; and $n = 12, m = 11$, each of which yields an infinite automatic group by [7],[8].

$\square$

With these results in hand, we assemble the main result of the section. Since every $(2, m|n, k|1, q)$ can be expressed in a form with $m \geq 2q$, we maintain our assumption **(A1)**. **(A2)** and **(A3)**, however, are based on reductions to other groups, so for the sake of completeness, we suspend them for this final step.

Proof of Theorem 4.1: Let $\mathcal{P} = (2, m|n, k|1, q)$ with, as given above, $m \geq 2q$. First, assume $m \neq 2q + 1$ and that $2 < q \leq \frac{m}{2}$. If one of $(q \geq 5, m \notin$

$\{3q \pm 1, 4q \pm 1\}$), ($q = 4, m \notin \{11, 13\}$), or $q = 3$, then $G(\mathcal{P})$ is infinite, by Propositions 4.18, 4.21 and 4.22, with the exception of $(2, 6|3, k|1, 3)$, which has order $6k^2$, by Lemma 4.7. Otherwise, we have $m = 3q \pm 1$ or $4q \pm 1$, and by Lemma 4.2(ii) we can reduce $(2, 3q \pm 1|n, k|1, q)$ to $(2, 3q \pm 1|k, n|1, 3)$ and $(2, 4q \pm 1|n, k|1, q)$ to $(2, 3q \pm 1|k, n|1, 4)$, both of which satisfy the conditions above, and so are infinite.

Suppose now that $q = 1$. Then $G(\mathcal{P})$ is the triangle group $(2, m, \gcd(n, k))$, and so is finite precisely when $\frac{1}{m} + \frac{1}{\gcd(n,k)} > \frac{1}{2}$, as required.

Finally, let $m = 2q+1$ or $q = 2$, so that by Lemma 4.2(ii) and (iii) $G(\mathcal{P})$ is isomorphic to one of the groups studied in [6]. Following the reductions required, we see that $(2, 2q+1|n, k|1, q)$ reduces first to $(2, 2q+1|k, n|1, 2)$, which in turn reduces to $(k, 2q + 1|n, 2|1, 2q) \cong (k, 2q + 1|n, 2)$, whilst $(2, m|n, k|1, 2)$ reduces to $(n, m|k, 2|1, m - 1) \cong (n, m|k, 2)$. Theorem 1.4 of [6] enumerates the groups $(l, m|n, k)$ that are finite, and the remaining presentations listed in Theorem 4.1 are precisely those that reduce to those presentations. Of these, the presentations $(2, 7|8, 3|1, 2)$, $(2, 7|3, 8|1, 3)$ and $(2, 8|7, 3|1, 2)$ yield groups (in fact, a single group) originally found to be finite (specifically, identified as a perfect group of order 10752) in [11], whilst the rest yield groups that were originally identified as finite in [1], and whose structures are identified in the same, and listed in Proposition 2.1 of [4]. $\square$

5 The $p = 1$ case: $l > 2$

These presentations are rather more resistant to our methods, but have been studied in some depth in the Thesis of Mark Dennis [2]. We recount results of his that establish certain presentations within our $\frac{1}{n}+\frac{1}{k} \leq \frac{1}{2}$ scope are infinite, including a reduction that allows us to entirely characterise the parameters within this scope, with the added assumption that $l \neq 3$, that lead to a group of finite order. In the case $l = 3$, we note reductions that allows us to settle all cases where $n \neq 3$ and $k \geq 6$.

Theorem 5.1. *Let $l > 3$, $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$, $1 < q < m - 1$. If G is the group with presentation $(l, m|n, k|1, q)$, then G is infinite.*

Proof. [2] proves that the group with presentation $(l, m|n, k|1, q)$ is infinite whenever any of the following conditions occurs:

- $k \geq 11$
- $n \geq 10$ and $k \geq 4$,
- $n \geq 6$ and $k \geq 5$,
- $n \geq 5$ and $k \geq 6$,
- $n = 3$ and $k \geq 6$,
- $n \geq 4$, $k \geq 4$ and $m \neq 2q - 1$,
- $k = 3$ and $m \neq 2q - 1$

With our condition $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$, this leaves only the cases in which $m = 2q - 1$ and (n, k) is one of: $(4, 4)$, $(5, 5)$, $(4, k) : k \geq 5$, $(n, 4) : n \geq 5$, $(n, 3) : n \geq 6$. Note that q and $2q - 1$ are coprime, so that in the resulting presentation $\langle a, b|a^l, b^{2q-1}, (ab)^n, (ab^q)^k\rangle$, $b \in \langle b^q\rangle$, and in particular $b = (b^q)^2$. Thus, expressing our presentation in terms of a and $c := b^q$, we have $G \cong \langle a, c | a^l, c^{2q-1}, (ac)^k, (ac^2)^n \rangle = (l, 2q-1 | k, n | 1, 2)$. [2] notes that $(l, m | n, k | 1, 2)$ is reducible to $(n, m | k, l)$. Thus, our remaining groups can be expressed as $(k, 2q-1 | n, l)$, with n and k satisfying conditions as above, all of which are infinite, via [6]. $\square$

This leaves only the presentations with $l = 3$. As noted above, some of these are beyond the reach of our analysis, but we can perform a pair of reductions that reduces the space of groups in this family whose finiteness is unknown from a four-parameter family to four three-parameter subfamilies.

We first note that, by the same reduction as used above, the presentation $(3, m | n, k | 1, 2)$ can be transformed into the presentation $(3, n | k, m)$ of [6], which is infinite if and only if

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{2\pi}{k}\right) + \cos\left(\frac{2\pi}{m}\right) \geq 0.$$

Theorem 5.2. *Let G be the group with presentation $\mathcal{P}(3, m | n, k | 1, q)$, with $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$ and $2 < q < m - 1$. If $n \neq 3$ and $k \geq 6$, G is infinite.*

Proof. Expressing $\langle a, b | a^3, b^m, (ab)^n, (ab^q)^k \rangle$ in terms of $c := ab$ and $d := b^{-1}$, we obtain the presentation $\langle c, d | c^n, d^m, (cd)^3, (cd^{m+1-q})^k \rangle$, which is the presentation $(n, m | 3, k | 1, m+1-q)$. Since $k \geq 6$, $\frac{1}{3} + \frac{1}{k} \leq \frac{1}{2}$. Since $n \neq 3$ and $\frac{1}{n} + \frac{1}{k} \leq \frac{1}{2}$, $n > 3$. Finally, since $2 < q < m-1$, $1 < m+1-q < m-1$. Thus, $(n, m | 3, k | 1, m+1-q)$ satisfies the conditions of Theorem 5.1, and so must represent an infinite group. $\square$

This leaves as unsolved only the families $(3, m | 3, k | 1, q) : k \geq 6$, $(3, m | n, 5 | 1, q) : n \geq 4$, $(3, m | n, 4 | 1, q) : n \geq 4$ and $(3, m | n, 3 | 1, q) : n \geq 6$, with $2 < q < m - 1$.

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