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Double Sigma Models

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Abstract

This thesis investigates the properties of the interacting chiral boson model for the maximally doubled string. We first introduce and briefly review double field theory. We then review two world-sheet actions for the doubled string: Tseytlin's original duality symmetric string and a recent example of the double sigma model based on Hull's doubled formalism, which is constructed to be equivalent to Tseytlin's formulation. One feature of the latter concerns the underlying assumption of a base-fibre split, which means the fibre metric depends only on the base coordinates. In this thesis, we instead consider the case in which all coordinates are doubled. Taking the most general form of the Tseytlin action in which the fields possess arbitrary dependence on the full doubled geometry, we investigate whether a generic approach to the interacting chiral boson model - one which does not assume $O(D, D)$ invariance from the outset - satisfies the requirement of conformal invariance at the quantum level. This demands that the doubled beta-functionals of the sigma model couplings vanish in the maximally doubled space.

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Contents

Chapter 1

Introduction

1.1 History

String theory is the leading candidate for a self-consistent theory of quantum gravity. Its full realisation is expected to correlate with the unification of all theories of fundamental interaction. Crucial to its completion is to understand both the wealth of string symmetries and to what extent these symmetries (when made manifest) may be used to determine the fundamental structure of the underlying theory.

The nature of this wealth of string symmetries is owed primarily to the extended object of the string, which may be thought of as a generalisation of point particle theory. As late as 1995, it was believed that the five string theories - type I, type IIA, type IIB, and the two flavours of heterotic string theory (SO(32) and E8 $\times$ E8) were distinct. When it was eventually observed that the five string theories are in fact deeply related by non-trivial dualities, it was proposed by Edward Witten that rather than being distinct they actually represent different limits of an overarching theory. This overarching theory is known as M-theory, with the conception of a web of dualities based firstly on Witten's observation that the type IIA string and the E8$\times$E8 heterotic string are related to eleven-dimensional supergravity [1]. In that dualities find natural expression in the extended objects of string theory [2], it is rather the evocative unification principle in which this intricate web of string dualities are related that hints deeply at a unique theory of quantum gravity.

Target-space duality (T-duality) and strong-weak duality (S-duality) are two examples of string symmetries, with the former being fundamental in string theory. Together T-duality and S-duality unify all ten-dimensional superstring theories. When these two duality transformations are combined they then define the unified duality (U-duality), which is anticipated to be a fundamental symmetry of M-theory.

S-duality may be thought of in terms of a familiar description from classical physics, notably invariance of Maxwell's equations under the exchange of electric and magnetic fields: $\mathcal{E} \to \mathcal{B}$, $\mathcal{B} \to -\frac{1}{c^2}\mathcal{E}$. As suggested by its name, S-duality transformation displays physical equivalence between strong and weak couplings of a theory. The existence of S-duality in string theory was first proposed by Ashoke Sen [3], where he showed that the Type IIB string with g coupling was equivalent to the same theory with coupling constant $\frac{1}{g}$.

On the other hand, T-duality, first observed by Balachandran Sathiapalan [4], is a fundamental consequence of the existence of the string [5, 6]. This important duality famously constitutes an exact symmetry of the bosonic string, encoded by the transformations: $R \leftrightarrow \frac{\alpha'}{R}, k \leftrightarrow w$. This implies an equivalence between radius and inverse radius, with the exchange of momentum modes k and winding modes w in closed string theory, or in the case of the open string an exchange of Dirichlet and Neumann boundary conditions [7, 8, 9].

The reason that T-duality is considered a fundamental property of string theory has precisely to do with the existence of these intrinsically stringy winding modes. In that closed strings can wrap around non-contractible cycles in space-time, the winding states present in string theory have no analogue in point particle theory, and it is the existence of both momentum and winding states that allows T-duality. In that it is closely related to mirror symmetry in algebraic geometry, which in string theory is related to the important study of Calabi-Yau manifolds, T-duality in many cases enables us to observe how different geometries for the compact dimensions are physically equivalent.

Moreover, it is by this fundamental and intrinsic stringy duality that one of the most remarkable features of string theory may be stated: namely, the equivalence of string theories on dual backgrounds with very different geometries. More concisely, if under T-duality transformation the momentum mode p is exchanged with the winding mode w in which we also exchange R with α'/R, then it can be shown that the mass spectrum of the closed string $M^2 = (N + \bar{N} - 2) + p^2 \frac{\alpha'}{R^2} + w^2 \frac{R^2}{\alpha'}$ and the level-matching condition $N - \bar{N} = pw$ remain invariant. Thus, from the point of view of the string, it cannot distinguish whether it is propagating along a circle with radius R or $1/R$. In general, when d-dimensions are compactified on a n-torus, we may generalise T-duality transformations under the group $O(n, n, \mathbb{Z})$.

As it relates to this thesis, the importance of string dualities such as T-duality is amplified in the context of string field theory. In the early 1990s, field theory emerged as a complete gauge-invariant formulation of string dynamics [5, 10]. This led to the development of a precise spacetime action whose gauge symmetry arguably takes the most elegant possible form [11]. A key feature of string field theory is that the momentum and winding modes are treated symmetrically and on *equal footing*.

To better explain what this means, let us first denote the compact coordinates X^a and the non-compact coordinates X^μ, with $X^I = (X^a, X^\mu)$. Conventionally, we define the indices such that $I = 1, ..., D$, $\mu = 1, ..., d$, and $a = 1, ..., n$. If the string field gives component fields that depend on momentum p^a and winding w^a, then in position space we may assign the coordinates X^a conjugate to the momentum and new periodic dual coordinates $\tilde{X}_a$ conjugate to the winding modes. One consequence is therefore that, if one attempts to write the complete field theory of closed strings in coordinate space, the full theory depends naturally on dual coordinates X^a and $\tilde{X}_a$, which, again, is also to say that naturally the full phase space of the theory accompanies both the momentum and the winding modes. This is an incredibly interesting fact, because, to phrase it in a slightly different manner, for toroidal compactification there is a zero mode X^a and $\tilde{X}_a$, and the expansion of a string field provides component fields that depend on both momentum and winding. Thus, the arguments of all fields in such a theory are *doubled*. For the doubled fields $\phi(X^a, \tilde{X}_a, X^\mu)$ we may write the following seemingly simple action

$$S = \int dX^a \; d\tilde{X}_a \; dX^\mu \mathcal{L}(X^a, \tilde{X}_a, X^\mu). \tag{1.1}$$

For the integration measure $dX^a \; d\tilde{X}_a \; dX^\mu$, one sees that it runs over the flat d-dimensional spacetime as well as the full coordinate space. The advantages of obtaining a completely understood closed string field theory are vast. It is anticipated, for instance, that a fuller view of some of the fundamental properties of perturbative string theory will emerge in addition to fundamental aspects of the non-perturbative theory [11]. However, despite the appearance of the action (1.1), $\mathcal{L}$ is in fact incredibly complicated and our understanding of Lagrangians of this form is incomplete. In that one may argue that this doubled space - i.e., doubled fields depending on a doubled set of coordinates - is the true space of string theory[1], one issue has to do with how the physical content of the theory becomes buried underneath unphysical and computationally inaccessible data,

[1] In his 2010 series of lectures at the Ludwig Maximilian University of Munich, as part of the International School on Strings and Fundamental Physics, Zwiebach made explicit reference to the point that, however we interpret it, doubled space is the true home of the string.

with the full closed string field theory comprising an infinite number of fields. One simplification strategy is to therefore choose some finite subset of string fields. An obvious choice for such a subsector of the full theory is the massless sector. An important property of this sector of closed strings is of course that it consists of gravitational fields g_{IJ} with Riemann curvature $R(g)$, the Kalb-Ramond field b_{IJ} with the conventional definition for the field strength $H = db$, and a dilaton scalar field ϕ. The question is then to ask, if for the standard bosonic string the low-energy effective action is famously

$$S_{SUGRA} = \int dX \sqrt{-g}\, e^{-2\phi}[R + 4(\partial\phi)^2 - \frac{1}{12}H^2] \ + \ \text{higher derivative terms}, \tag{1.2}$$

where the integration measure dX is taken to be D-dimensional, what does this action become in the case of doubled coordinates on tori? What symmetries are present? Is T-duality manifest? Armed with these questions, the study of double field theory (DFT) may be motivated at its very foundations. Indeed, one way to think of DFT from first principles is by way of its proposal to develop a manifestly spacetime invariant T-duality theory, whose origins may be traced back to the important work of Tseytlin [12, 13] and Siegel [14, 15]. But it was in fact following Nigel Hitchin's introduction of generalised geometry [16, 17], itself inspired by the existence of T-duality, that serious efforts materialised to incorporate this mathematical insight into the study of the target-space geometry in which strings live [18, 19, 20, 21], beginning especially with the study of phase space and invariance of respective Hamiltonians. In 2009, C M Hull and Barton Zwiebach formulated such a T-duality invariant theory explicitly [18]. In general, the theory is constructed on the product manifold $\mathbb{R}^{d-1,1} \times T^n$ with coordinate space fields $\phi(X^\mu, X^a, \tilde{X}_a)$. The torus is naturally doubled, containing the spacetime torus and the torus parameterised by the winding modes, such that $(X^a, \tilde{X}_a)$ are periodic on T^{2n}. The spectrum for the massless fields is then described in terms of the supergravity limit of string theory.

With a fresh perspective on T-duality emerging in the last decade, the manifestly T-dual Lagrangians in DFT have been found to take on an intriguing structure, leading to a number of interesting applications. This has led to the development of deeper connections between frontier theoretical physics and mathematics through the appearance and use of Courant brackets and with the deepening role generalised geometry seems to play in string theory. Additionally, concepts and ideas to have emerged in the formalism have also been explored in the context of

M-theory [22, 23], where exceptional field theory seeks to promote the U-duality group to a manifest symmetry of the spacetime action. Parallel to the efforts of DFT is also the development of a T-duality invariant world-sheet description of string theory, which may be described as the doubled world-sheet and whose origins may be traced back to the notable work of Duff [24] and again Tseytlin [12, 13]. Much like the field theory, the doubled world-sheet theory was reinvigorated in the last decade following breakthrough work by Hull [25, 26], who used the formalism to define strings in a class of non-geometric backgrounds known as T-folds [27, 28, 29, 30, 31]. These are non-geometric manifolds where locally geometric regions are patched together such that the transition functions are T-duality transformations (see [32, 33, 34, 35] for review). Such non-geometric constructions are suspected to play an important role in moduli stabilisation, and hence have implications with regards to the string landscape. Finally, it may also be noted that all of these efforts with regards to the doubled string have raised interesting questions and inspired recent exciting research pathways in string cosmology.

1.2 Motivation: A return to Tseytlin and the duality symmetric string

From the spacetime perspective, T-duality is a solution generating symmetry of the low energy equations of motion. However, from a world-sheet point of view, T-duality is a non-perturbative symmetry. Given its importance in accessing a potentially fuller view of the perturbative theory and the still underdeveloped non-perturbative theory, the relatively recent realisation that T-duality (and other string dualities) enables an extension of string backgrounds with traditional geometry (and physical fluxes) to a much broader class of generalised and non-geometric objects - this has broken open a deeply exciting and fundamental area of string research (e.g., see [25, 33, 36, 37, 38]).

At the heart of such research, as far as the present thesis is concerned, is Tseytlin's 1990/91 formulation [12, 13] of the duality symmetric string and world-sheet theory for interacting chiral scalars. In [13], we observe a first-principle formulation of manifestly T-duality invariant closed string theory based on a double set of coordinates: $X = X_+ + X_-$, $\tilde{X} = X_+ - X_-$, where X_+ and X_- are left and right-movers respectively. An essential aspect of Tseytlin's formulation is its extension of string theory by which T-duality becomes an off-shell symmetry, implying manifest invariance of the scattering amplitudes and the effective action. Famously, Tseytlin's formulation of the closed string sigma model for interacting chiral scalars takes the form

$$S_{Tseytlin} = -\frac{T}{2} \int d^2\xi \; e[C_{IJ}\partial_0 X^I \partial_1 X^J + M_{IJ}\partial_1 X^I \partial_1 X^J]. \tag{1.3}$$

We will discuss this action and Tseytlin's approach to the duality symmetric string in Chapter 3. For now, however, we may note that T is the usual string tension. We may also consider $I = 1, ..., D$, and we may think of C_{IJ} and M_{IJ} as generic symmetric matrices. We take the conventional definition for the time and spatial derivatives to be $\partial_0 = \partial/\partial_\tau$ and $\partial_1 = \partial/\partial_\sigma$, and, as we will discuss later, e is the determinant of the zweibein that we pick up in the construction. Importantly, what should be emphasised here is how, in the last decade especially, this formulation and simple looking action has been refocused in various studies concerning the nature of the doubled string and its geometry.

One notable example, which serves as the structure for the main part of this thesis, pre-dates the first primary collection of DFT papers and, in many ways, can be interpreted to give a prediction to DFT. In 2008 David S. Berman, Neil B. Copland, and Daniel C. Thompson investigated the background field equations for the duality symmetric string using an action equivalent to that of Tseytlin's but constructed in the context of Hull's doubled formalism [39]. The details of this construction will also be reviewed in Chapter 3, where we will discuss the action from which the authors calculate the doubled beta-functionals for the interacting chiral boson model in the case where the background fields depend trivially on the doubled coordinates but non-trivially on the non-compact spacetime coordinates. In recent years, a series of publications on the doubled sigma model have appeared in [40, 41, 42, 43] in connection, where in [41] the double sigma model is directly related to DFT.

Another example may refer directly to DFT from a different perspective. In the years after 2009 when Hull and Zwiebach published their important paper, it was recognised that while a deep connection exists between DFT and generalised geometry, with DFT locally equivalent to the latter, it does not completely come into contact with its formal mathematical structures. In fact, an open research question remains motivated by the unmistakeable resemblance DFT has with generalised geometry and the formal gap that remains between them. Recent work in mathematics and physics has displayed some promise, suggesting that the use of para-Hermitian and para-Kähler manifolds may be the solution [44, 45, 46]. Related to these efforts is a recent reformulation of string theory under the heading metastring theory [45, 47, 48, 49, 50], which begins, similar to the studies on double sigma models, with a generalised version of the first-principle Tseytlin action for the duality symmetric string. The metastring is therefore a chiral T-duality invariant theory that, in many ways, wants to generalise from DFT and make direct connection with things like Born geometry [47], relying on the consistency of Tseytlin's formulation.

If a direct consequence of making T-duality manifest is that the winding modes are treated on equal footing with momentum, then for DFT all of these properties are incorporated into one field theory. The result, as mentioned, is a doubled coordinate space. In metastring theory, on the other hand, the target space of the world-sheet formulation is a phase space, much like in Tseytlin's original construction. The coordinates of this phase space are indeed doubled, but unlike in DFT they are also conjugate such that in this case the dual coordinates are related directly to energy-momentum coordinates. In other words, $\tilde{X}$ is now identified with p. This means that, instead of a physical spacetime formulation, the goal of metastring theory is to construct a sigma model as a phase space formulation of the string and its dynamics.

The implications of metastring theory, as they have so far been conjectured, are intriguing. For example, there have been claims toward obtaining a family of models with a 3+1-dimensional de Sitter spacetime, argued to be realised in the standard tree-level low-energy limit of string theory in the case of a non-trivial anisotropic axion-dilaton background [50]. A key statement here is that, while string theory has purely stringy degrees of freedom (from first principles consider simply the difference between the left and right-moving string modes), these are not captured by standard effective field theory approaches and their spacetime descriptions. Such approaches are usually employed when investigating de Sitter space. In the phase-space formulation of the metastring, these purely stringy degrees of freedom (generally *chiral* and *non-commutating*) are argued to be captured explictly. When it comes to the hope of obtaining an effective de Sitter background, one of the major claims in this non-commutative phase-space formalulation is how, in the doubled and generalised geometric description, the effective spacetime action translates directly into the see-saw formula for the cosmological constant. Furthermore, in this cosmic-string-like solution related to the concept of an emergent de Sitter space, it is argued that the metastring leads naturally to an expression of dark energy, represented by a positive cosmological constant to lowest order. Finally, it is argued that the intrinsic stringy non-commutativity provides a vital ingredient for an effective field theory that reproduces to lowest order the sequestering mechanism [50, 51] and thus a radiatively stable vacuum energy. Such claims are worthy of investigation.

As it relates to these two examples, in this dissertation our interest is to pursue a completely generic approach to the calculation of the doubled beta-functionals for the interacting chiral model following primarily the structure of [39]. To motivate such a calculation it is important to highlight that, because in [39] a form of the Tseytlin action is constructed from within the doubled formalism, what we will see is that one displeasing feature of such an approach is the need to separate, from the outset, the base of the fibration with the doubled fibration. This means the fibre metric depends only on the base coordinates. Rather than assuming a base-fibre split, there is an argument to be made that a more democratic approach is one in which *everything becomes doubled* [43].

What exactly does it mean to double everything and give the background fields arbitrary dependence on the full doubled geometry? What are the implications? Such are the questions we begin to observe in this thesis, and they are relevant not only to the referenced studies on double sigma models but also to DFT, which, as we will review in Chapter 2, is in its own way a highly constrained theory that depends on only half of the doubled coordinates. In the calculations featured in Chapter 4 we also seek to avoid a number of other assumptions made in [39] and elsewhere when calculating the beta-functionals and investigating the quantum consistency of the theory. For instance, $O(D, D)$ invariance is assumed from the outset and an important constraint, known as the chirality constraint, is applied at the level of the action. Instead, the action we shall use is the most general form the Tseytlin action can take and we shall calculate the doubled beta-functionals in a completely generic fashion for the completely doubled theory. Additionally, while in [39] the total divergence is described, comprising of both Weyl anomaly and Lorentz anomaly terms, the latter are not explicitly calculated. Instead, the authors leave the suggestion that due to 'an equal number of Bosons of each chirality' all occurrences of such terms cancel. In this dissertation all Lorentz anomaly terms are calculated explicitly for the totally doubled case.

In calculating the one-loop Weyl and Lorentz divergence for the background doubled metric, and in investigating whether the totally doubled theory for interacting chiral scalars satisfies the requirement of conformal invariance at the quantum level, the broader goal of this thesis is to lay important groundwork for planned future studies that investigate the full doubled geometry of the theory. Due to restrictions in length, such discussions could not be included here, although such investigations prove deeply interesting. Instead, by ensuring the fields have arbitrary dependence on the full doubled geometry from the outset, the first phase of the calculation presented in Chapter 4 is presented. And it will be noted that, in originally formulating this thesis, what we hoped to find was a quantum consistent theory that would give foundation for further investigation into possible effective spacetime theories that correspond to completely generic non-geometric geometries and also world-sheet theories underlying completely generic non-geometric string compactifications.

Finally, another advantage that this thesis offers is that, although we do not consider here the metastring formulation, a consequence of the calculations that proceed from Chapter 4 is the investigation into a number of intriguing claims. One example is how, from the quantum effective action and the calculation of the doubled beta-functionals we may investigate directly whether there are any hints at the presence of the sequestering equations of motion [52, 53] in the general formulation of the Tseytlin action. The presence of sequestering would be quite interesting and would raise a number of questions for future study of the duality symmetric string.

1.2.1 Structure of this thesis

This dissertation is organised as follows. In Chapter 2, we proceed with a lightning review of a few pertinent points in the construction of double field theory, followed by a review of the T-duality transformation group $O(D, D, \mathbb{Z})$, and finally we discuss briefly the geometrisation of the supergravity action. In Chapter 3, two approaches to the construction of the world-sheet action for non-geometric backgrounds is introduced. We review both Tseytlin's non-covariant formulation of the doubled string action as well as Hull's covariant formulation, concluding with a formalisation of the doubled sigma model as found in [39]. In chapter 4, we study the doubled sigma model action as it relates to the completely general version of Tseytlin's action for interacting chiral scalars. We perform a background field expansion to obtain the one-loop effective action, and then calculate the Weyl and Lorentz divergences. Finally, after lengthy calculation, we study the total divergence of the theory and obtain an expression for the generalised Ricci, followed with a study of the one-loop doubled beta-functionals. The thesis then concludes with a summary of results and several closing comments are offered in relation to a number of future sites of investigation.

Chapter 2

Target Space Duality, Double Field Theory, and $O(D, D, \mathbb{Z})$

In this chapter we review a few pertinent points in the construction of double field theory as formulated by Zwiebach, Hull, and Hohm [18, 19, 20, 21], with emphasis particularly on the generalisation of T-duality for toroidal backgrounds. The associated duality group $O(D, D, \mathbb{Z})$ is also studied.

2.1 Double field theory

The standard formulation of DFT is known as the generalised metric formulation (for a complete review of the fundamentals see [54]). The effort begins with the NS-NS supergravity action (1.2). In the case of toroidal compactification defined by D-dimensional non-compact coordinates and d-dimensional compact directions, the target space manifold can be defined as a product between d-dimensional Minkowski spacetime and an n-torus, such that $\mathbb{R}^{d-1,1} \times T^n$ where $D = n + d$. We have for the full undoubled coordinates $X^I = (X^a, X^\mu)$ with $X^a = X^a + 2\pi$ being the internal coordinates on the torus. The background fields are $d \times d$ matrices taken conventionally to be constant with the properties:

$$G_{IJ} = \begin{pmatrix} \hat{G}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \; B_{IJ} = \begin{pmatrix} \hat{B}_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } G^{IJ} G_{JK} = \delta^I_K. \tag{2.1}$$

We define $\hat{G}_{ab}$ as a flat metric on the torus and $\eta_{\mu\nu}$ is simply the Minkowski metric on the d-dimensional spacetime. As usual, the inverse metric is defined with upper indices. In (2.1) we also have the antisymmetric Kalb-Ramond field.

Finally, for purposes of simplicity, we have dropped the dilaton. Of course one must include the dilaton at some point so as to obtain the correct form of the NS-NS supergravity action, and we will return to this point when discussing the duality invariant doubled dilaton scalar in Section 2.3. In the meantime, in conventional analysis, the dilaton may at first be dropped in what we will soon define as the $O(D, D)$ description of the theory, because the motivation for now is primarily to study the way in which G_{IJ} and B_{IJ} come together in a single generalised geometric entity, which we begin to construct with the internal metric denoted as

$$E_{IJ} = G_{IJ} + B_{IJ} = \begin{pmatrix} \hat{E}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix} \tag{2.2}$$

for the closed string background fields, with $\hat{E}_{ab} = \hat{G}_{ab} + \hat{B}_{ab}$ as first formulated by Narain et al [55]. It is important to note that the canonical momentum of the theory is $2\pi P_I = G_{IJ}\dot{X}^J + B_{IJ}X'^J$, where, in the standard way, $\dot{X}$ denotes a τ derivative and X' denotes a σ derivative. Famously, the Hamiltonian of the theory may then also be constructed from the expansion of the string modes for coordinate X^I, the canonical momentum, and from the Hamiltonian density to take the following form

$$H = \frac{1}{2} Z^T \mathcal{H}(E) Z + (N + \bar{N} - 2). \tag{2.3}$$

Or, to write it in terms of the mass operator,

$$M^2 = Z^T \mathcal{H}(E) Z + (N + \bar{N} - 2). \tag{2.4}$$

In summary, in an n-dimensional toroidal compactification, the momentum p^I and winding modes w_I become n-dimensional objects. So the momentum and the winding are combined in a single object known as the generalised momentum $Z = \begin{pmatrix} w_I \\ p^I \end{pmatrix}$. This generalised momentum Z is defined as a $2D$-dimensional column vector, and we will return to a discussion of its transformation symmetry in a moment. Meanwhile, in (2.3) and (2.4) N and $\bar{N}$ are the usual number operators counting the excitations familiar in the standard bosonic string theory. One typically derives these when obtaining the Virasoro operators. We also see the first appearance of the generalised metric $\mathcal{H}(E)$, which is a $2D \times 2D$ symmetric matrix constructed from G_{IJ} and B_{IJ} with $E = E_{IJ} = G_{IJ} + B_{IJ}$. We will discuss the generalised metric in just a few moments.

As is fundamental to closed string theory there is the Virasoro constraint $L_0 - \bar{L}_0 = 0$, where L_0 and $\bar{L}_0$ are the Virasoro operators. This fundamental constraint remains true in the case of DFT. Except in DFT this condition on the spectrum gives $N - \bar{N} = p_I w^I$ or, equivalently,

$$N - \bar{N} = \frac{1}{2} Z^T L Z, \tag{2.5}$$

where

$$L = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \tag{2.6}$$

Given some state and some oscillators, the fundamental constraint (2.5) must be satisfied, with the energy of such states computed using (2.3). For the time being, we treat L somewhat vaguely and simply consider it as a constant matrix; but in the next section it will prove important and its meaning will become explicit. We denote $\mathbb{I}$ as a $D \times D$ identity matrix.

Continuing with basic definitions, the generalised metric that appears in (2.3) and (2.4) is similar to what one finds using the Buscher rules [56] for T-duality transformations with the standard sigma model [9, 57]. That is to say, $\mathcal{H}$ takes a form in which there is clear mixing of the background fields. It is defined as follows,

$$\mathcal{H}(E) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{2.7}$$

One inuitive motivation for the appearance of the generalised metric is simply based on the fact that, if we decompose the supergravity fields into the metric G_{ij} and the Kalb-Ramond field B_{ij}, in DFT these then must assume the form of an $O(n,n)$ tensor. The generalised metric, constructed from the standard spacetime metric and the antisymmetric two-form serves this purpose. On the other hand, the appearance of the generalised metric can be approached from a more general perspective that offers a deeper view on toroidal compactifications. In (2.3) what we have is in fact an expression that serves to illustrate the underlying moduli space structure of toroidal compactifications [34, 55], which, for a general manifold $\mathcal{M}$ may be written as

$$\mathcal{M}^n = O(n) \times O(n)/O(n,n) \backslash \text{T-duality}. \tag{2.8}$$

The overall dimension of the moduli space is n^2 which follows from the parameters of the background matrix E_{ij}, with $n(n+1)/2$ for G_{ij} plus $n(n-1)/2$ for B_{ij}. The zero mode momenta of the theory define the Narain lattice $\Gamma_{n,n} \subset \mathbb{R}^{2n}$, and it can be proven that $\Gamma_{n,n}$ is even and also self-dual. These properties ensure that, in the study of 1-loop partition functions, the theory is modular invariant with the description enabling a complete classification of all possible toroidal compactifications (for free world-sheet theories). The feature of self-duality contributes $O(n,\mathbb{R}) \times O(n,\mathbb{R})$ in (2.8). The Hamiltonian (2.3) remains invariant from separate $O(n,\mathbb{R})$ rotations of the left and right-moving modes that then gives the quotient terms. As for the generalised metric, we may in fact define it as the $O(n,n)/O(n) \times O(n)$ coset form of the n^2 moduli fields.

2.2 $O(n,n,\mathbb{Z})$

In a lightning review of certain particulars of DFT, we may deepen our discussion of the T-duality group by returning first to the generalised momentum Z as it appears in (2.4). If we shuffle the quantum numbers w, p, which means we exchange w for p and vice versa, the transformation symmetry of Z is well known to be

$$Z \to Z = h^T Z'. \tag{2.9}$$

For now, h is considered generally as a $2D \times 2D$ invertible transformation matrix with integer entries, which mixes p^I and w_I after operating on the generalized momentum. It follows that h^{-1} should also have invertible entries, this will be shown to be true later on. Importantly, if we have a symmetry for the theory, this means a transformation in which we may take a set of states and, upon reshuffling the labels, we should obtain the same physics. Famously, it is indeed found that the level-matching condition and the Hamiltonian are preserved. If we take $Z \to Z'$ as a one-to-one correspondence, the level-matching condition (2.5) with the above symmetry transformation (2.9) gives

$$N - \bar{N} = \frac{1}{2} Z^T L Z = \frac{1}{2} Z^{T\prime} L Z'$$

$$= \frac{1}{2} Z^{T\prime} h L h^T Z'. \tag{2.10}$$

For this result to be true, it is necessary as a logical consequence that the transformation matrix h must preserve the constant matrix L. This means it is required that

$$hLh^T = L, \tag{2.11}$$

which also implies

$$h^T Lh = L. \tag{2.12}$$

These last two statements can be proven, producing several equations that give conditions on the elements of h. The full derivation will not be provided due to limited space (complete review of all items in this section is again found in [18, 19, 20, 21, 54]); however, to illustrate the logic, let a, b, c, d be $D \times D$ matrices, such that h may be represented in terms of these matrices

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.13}$$

The condition in which h preserves L demands that the elements a, b, c, d satisfy in the case of (2.11)

$$a^T c + c^T a = 0, \; b^T d + d^T b = 0, \; \text{and } a^T d + c^T b = 1. \tag{2.14}$$

Likewise, similar conditions are found for the case (2.12), for which altogether it is proven that h^{-1} has invertible entries. What this ultimately means, and why it is relevant in the context of later discussions in this thesis, is that although we previously considered h vaguely as some transformation matrix, it is in fact an element of $O(D, D, \mathbb{R})$ and L is an $O(D, D, \mathbb{R})$ invariant metric. Formally, an element $h \in O(D, D, \mathbb{R})$ is a $2D \times 2D$ matrix that preserves, by its nature, the $O(D, D, \mathbb{R})$ invariant metric L (2.6) such that

$$O(D, D, \mathbb{R}) = \Big\{ h \in GL(2D, \mathbb{R}) \;:\; h^T Lh = L \Big\}. \tag{2.15}$$

Finally, if the aim of DFT at this point is to completely fulfil the demand for the invariance of the massless string spectrum, it is required from (2.3) for the energy that, if the first term is invariant under $O(D, D, \mathbb{R})$ then we must have the following transformation property in the case $Z^T \mathcal{H}(E) Z \to Z'^T \mathcal{H}(E') Z'$:

$$Z'^T \mathcal{H}(E') Z' = Z^T \mathcal{H}(E) Z$$

$$= Z'^T h\mathcal{H}(E)h^T Z'. \tag{2.16}$$

Proposition 1. *By definition, given the principle requirement of (2.16) it is therefore also required that the generalised metric transforms as*

$$\mathcal{H}(E') = h\mathcal{H}(E)h^T. \tag{2.17}$$

The primary claim here is that for the transformation of E we find

$$(E') = h(E) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (E) \equiv (aE + b)(cE + d)^{-1}. \tag{2.18}$$

One should note that this is not matrix multiplication, and $h(E)$ is not a linear map. What we find in (2.18) is actually a well known transformation in string theory that appears often in different contexts, typically taking on the appearance of a modular transformation. Given the notational convention that $\mathcal{H}$ is acting on the background E, what we end up with is the following

$$(E'^T) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} (E^T) \equiv (aE^T - b)(d - cE^T)^{-1}, \tag{2.19}$$

where in the full derivation of this definition it is shown $(E'^T) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} E^T$.

Proof. To work out the full proposition with a proof of (2.17), we may also demonstrate the rather deep relation between (2.17) and (2.19). The basic idea is as follows: imagine creating E from the identity background $E' = \mathbb{I}$, where conventionally $E = G + B$ and $G = AA^T$. Recall, also, the definition for the generalised metric metric (2.7). Then for $E = h_E(\mathbb{I})$, what is $h_E \in O(D, D, \mathbb{R})$? To answer this, suppose we know some A such that

$$h_E = \begin{pmatrix} A & B(A^T)^{-1} \\ 0 & (A^T)^{-1} \end{pmatrix}. \tag{2.20}$$

It then follows

$$h_E(I) = (A \cdot \mathbb{I} + B(A^T)^{-1})(0 \cdot \mathbb{I} + (A^T)^{-1})^{-1}$$

$$= (A + B(A^T)^{-1})A^T = AA^T + B = E = G + B. \tag{2.21}$$

This means that the $O(D, D, \mathbb{R})$ transformation creates a $G + B$ background from the identity. Additionally, the transformation h_E is ambiguous because it is always possible to substitute h_E with $h_E \cdot g$, where we define $g(\mathbb{I}) = \mathbb{I}$ for $g \in O(D, D, \mathbb{R})$. In fact, it is known that g defines a $O(D) \times O(D)$ subgroup of $O(D, D)$ $g^T g = gg^T = I$.

In conclusion, one can show that $\mathcal{H}$ transforms appropriately, given that up to this point h_E was constructed in such a way that the metric G is split into the product A and A^T, with the outcome that only A is entered into h_E. To find G we simply now consider the product $h_E h_E^T$,

$$
\begin{aligned}
h_E h_E^T &= \begin{pmatrix} A & B(A^T)^{-1} \\ 0 & (A^T)^{-1} \end{pmatrix} \begin{pmatrix} A^T & 0 \\ -A^{-1}B & A^{-1} \end{pmatrix} \\
&= \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} = \mathcal{H}(E). \qquad (2.22)
\end{aligned}
$$

If we now suppose naturally E' is a transformation of E by h, such that $E' = h(E) = hh_E(\mathbb{I})$, we also have $E' = h_{E'}(\mathbb{I})$. Notice that this implies $h_{E'} = hh_{Eg}$ up to some ambiguous and so far undefined $O(D, D, \mathbb{R})$ subgroup defined by g. Putting everything together, we obtain the rather beautiful result

$$
\mathcal{H}(E') = h_{E'} h_{E'}^T = hh_{Eg}(hh_{Eg})^T = hh_E h_E^T h^T = h\mathcal{H}(E)h^T. \qquad (2.23)
$$

■

Thus ends the proof of (2.17). A number of other useful results can be obtained and proven in the formalism, including the fact that the number operators are invariant which gives complete proof of the invariance of the full spectrum under $O(D, D, \mathbb{R})$.

In conclusion, and to summarise, in DFT there is an explicit restriction on the winding modes w_I and the momenta p^I to take only discrete values and hence their reference up to this point as quantum numbers. The reason has to do with the boundary conditions of n-dimensional toroidal space, so that in the quantum theory the symmetry group is restricted to $O(n, n, \mathbb{Z})$ subgroup to $O(D, D, \mathbb{R})$. The group $O(n, n, \mathbb{Z})$ is as a matter of fact the T-duality symmetry group in string theory. It is conventional to represent the transformation matrix $h \in O(n, n, \mathbb{Z})$ in terms of $O(D, D, \mathbb{R})$ such that

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with,

$$a = \begin{pmatrix} \tilde{a} & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} \tilde{b} & 0 \\ 0 & 0 \end{pmatrix}, c = \begin{pmatrix} \tilde{c} & 0 \\ 0 & 0 \end{pmatrix}, d = \begin{pmatrix} \tilde{d} & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.24}$$

Each of $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are $n \times n$ matrices. They can be arranged in terms of $\tilde{h} \in O(n, n, \mathbb{Z})$ as

$$\tilde{h} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}. \tag{2.25}$$

In the remainder of this thesis, particularly as we begin to concentrate on the doubled formalism and the duality symmetric string, the representations $O(D, D)$, $O(d, d)$, and $O(n, n)$ are used. Invariance under the $O(D, D, \mathbb{Z})$ group of transformations is generated by the following transformations. To simplify matters, let us define generally the action of an $O(D, D)$ element as

$$\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathcal{O}^T L \mathcal{O}. \tag{2.26}$$

Residual diffeomorphisms: If $A \in GL(D, \mathbb{Z})$, then one can change the basis for the compactification lattice Γ by $A \Gamma A^T$. The action on the generalised metric is

$$\mathcal{O}_A = \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad A \in GL(D, \mathbb{Z}), \quad \det A = \pm 1. \tag{2.27}$$

B-field shifts: If we define Θ to be an antisymmetric matrix with integer entries, one can use Θ to shift the B-field producing no change in the path integral. For compact d-dimensions, this amounts to $B_{IJ} \to B_{IJ} + \Omega_{IJ}$. It follows that the $O(D, D)$ transformation acts on the generalised metric,

$$\mathcal{O}_\Omega = \begin{pmatrix} 1 & \Omega \\ 0 & 1 \end{pmatrix}, \quad \Omega_{IJ} = -\Omega_{JI} \in \mathbb{Z}. \tag{2.28}$$

Factorised dualities: We define a factorised duality as a $\mathbb{Z}_2$ duality corresponding to the $R \to \frac{1}{R}$ transformation for a single circular direction (i.e., radial inversion). It acts on the generalised metric as follows

$$\mathcal{O}_T = \begin{pmatrix} 1-e_i & e_i \\ e_i & 1-e_i \end{pmatrix}, \tag{2.29}$$

where e is a $D \times D$ matrix with 1 in the (i,i)-th entry, and zeroes elsewhere $(e_i)_{jk} = \delta_{ij}\delta_{ik}$. Altogether, these three essential transformations define the T-duality group $O(D,D,\mathbb{Z})$, as first established in [58, 59]. To calculate a T-dual geometry one simply performs the action (2.17) or (2.19) using an $O(D,D,\mathbb{R})$ transformation and, in general, one may view the formalism with the complete T-duality group as a canonical transformation on the phase space of a given system.

2.3 Geometrisation of the supergravity action and the generalised Ricci

We may once again recall the supergravity action (1.2). In anticipating what is to come in this thesis, it is important to emphasise that in DFT this action gets geometrised, which is to say that it ultimately becomes expressed in terms of a generalised Ricci scalar $\mathcal{R}$ and a generalised dilaton d (following conventional notation, d replaces ϕ). The action that corresponds with these developments on the doubled space looks similar to the Einstein-Hilbert action of general relativity,

$$S_{DFT} = \int d^D X e^{-2d} \mathcal{R}(\mathcal{H}, d). \tag{2.30}$$

In the background independent formulation, e^{-2d} is shown to be a generalised density owed to the way it transforms under gauge transformation establishing the identity $\sqrt{-g}e^{-2\phi} = e^{-2d}$.

The action (2.30) captures precisely the same dynamics as the supergravity action (1.2), given the definition for the generalised curvature scalar in 2D-dimensions

$$\begin{aligned}\mathcal{R} \equiv\ & 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \\ & \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK}.\end{aligned} \tag{2.31}$$

A derivation of $\mathcal{R}$ occurs naturally in DFT, and it is fairly straightforward to prove its gauge invariance in which it becomes clear that we may treat this generalised Ricci term as a scalar [54].

To achieve a T-duality invariant theory at the supergravity level, one relies on the fact that when n-dimensions are n-torus, the T-duality group is as previously stated $O(n, n, \mathbb{Z})$. In general, and in relation to what we have so far discussed, it proves more useful to think of it as the subgroup of the $O(D, D)$ transformation group, which is the global T-duality group. So in that the objective is to formulate an $O(D, D)$ invariant theory, it follows the entire action and the degrees of freedom at the supergravity level should be arranged in the form of an $O(D, D)$ tensor. Famously, we can write $\mathcal{R}$ as the combination of familiar scalars that is an $O(D, D)$ invariant generalised scalar. It takes the form

$$\mathcal{R} = R + 4\Big(\Box\phi - (\partial\phi)^2\Big) - \frac{1}{12}H^2. \tag{2.32}$$

In our study of the double sigma model constructed from Tseytlin's non-covariant approach, one objective will be to derive the generalised Ricci from the effective action.

2.4 Strong constraint

Finally, it is important to also introduce the definitions of the strong and weak forms of the level-matching constraint. These are vital ingredients of DFT, and references to them will surface many times throughout the remainder of this thesis. The level-matching constraint can be derived directly from the famous Virasoro constraint $L_0 - \bar{L}_0 = 0$. A more terse approach may express the weak form of the constraint in terms of the differential operator

$$L^{MN}\partial_M\partial_N = 2\partial_I \cdot \tilde{\partial}^I, \tag{2.33}$$

in which L is the $O(D, D)$ invariant constant metric and the tilde denotes derivatives on the dual tangent bundle. This is a fundamental constraint. In summary, the weak form of the constraint comes from closed string level-matching in a toroidal background. The role of this operator $\partial \cdot \tilde{\partial}$ is to annihilate all fields and gauge parameters of the theory [54]. But when we proceed to further generalise DFT toward the construction of $O(D, D)$ invariant actions, we come to see that not only all fields and gauge parameters must satisfy (2.33), what is required is an even stronger version that includes the product of two fields. In other words, that $\partial \cdot \tilde{\partial}$ annihilates all fields and all products of fields. Formally and generically put, if we let $A_I(x, \tilde{x})$ be in general fields or gauge parameters annihilated by the constraint $\partial^M\partial_M$, we now require all products $A_I A_J$ are killed such that

$$\partial_M A_I \partial^M A_J = 0, \forall\ I, J. \tag{2.34}$$

Here $\partial_M A_I \partial^M A_J$ is defined as an $O(D, D)$ scalar. The strong form of the level-matching constraint is required in writing a background independent form of the double field theory action. What makes this condition so strong is that it kills half of the fields of the theory, and we in fact lose a lot of physics. In the full doubled theory all of the coordinates are physical. Effectively, however, the above statement implies that our fields only depend on the real space-time coordinates, due to a theorem in $O(D, D)$ that states there is always some duality frame $(\tilde{x}'_I, \tilde{x}')$ in which the fields do not depend on $\tilde{x}'_I$. So we only have dependence on half of the coordinates.

In conclusion, one of the important facets of DFT is the unification of B-field gauge transformations and diffeomorphisms acting on the spacetime manifold M. The result is a generalisation of diffeomorphisms acting on the doubled space P, which is a natural space of the full closed string field theory. But with the strong constraint, what we have is in fact a highly constrained theory despite the original notion of doubled coordinates, because, in a geometrical sense, the fields and gauge parameters may only depend on the undoubled slice of the doubled space.

Due to limitations in word length, we cannot discuss in detail these subtleties, nor how the strong constraint relates to a discussion of generalised geometry. But we may contextualise for the benefit of what is to come in Chapter 4 and its motivation. Crudely speaking, the basic mathematical statement of generalised geometry [16, 17] is that the tangent bundle TM of a manifold M is doubled in the sum of the tangent and co-tangent bundle $TM \oplus T\star M$. In this formalism we also replace the Lie bracket with a Courant bracket, which we may write as something of the form $[X + \xi, Y + \eta]_C = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)$ such that $X\xi, Y + \eta \in \Gamma(TM \oplus T\star M)$. Now, if in physics there is motivation to ask about the geometry of spacetime in which strings propagate, and if the existence of winding modes and the nature in which T-duality connects these winding modes to momentum hints that perhaps the fundamental geometry of spacetime should be doubled, one issue that we have is that the strong constraint does not offer a unique solution. This means there is no geometrical information that describes the remaining coordinates on which the fields depend. One interpretation of DFT therefore concerns how there is an arbitrariness in its construction, because there is a freedom to choose which submanifold P is the base M for the generalised geometry widely recognised to be deeply connected with what the doubled string hints.

In what follows from the calculations in Chapter 4 for the duality symmetric string on a completely generic doubled space, we hope to find deeper hints at these connections and their related concepts.

Chapter 3

The Doubled String: World-sheet Actions for Non-geometric Backgrounds

When it comes to the study of double sigma models, there are primarily two doubled string actions that we may consider: one which displays general covariance and one which displays general non-covariance. In that the two actions that we will briefly review satisfy the requirement of T-duality appearing as a manifest symmetry, they both have nuances that one must consider carefully.

The first world-sheet sigma model that we will describe is Tseytlin's first-principle formulation of the duality symmetric string [12, 13], which presents a direct stringy extension of the Floreanini-Jackiw Lagrangians [60] for chiral fields. In this approach, although explicit $O(D,D)$ invariance emerges rather organically as an intrinsic characteristic of the doubled string, the caveat is that we lose manifest Lorentz covariance on the string world-sheet. What one finds is that we must impose local Lorentz invariance on-shell.

The other primary action that is useful to study follows C M Hull's doubled formalism [25, 26]. In this formulation we have manifest 2-dimensional Lorentz invariance from the outset, and a notable advantage is that there is *a priori* doubling of the string coordinates in the target space. In other words, both the Tseytlin approach and the Hull approach are formulated such that both the string coordinates and their duals are treated on equal footing; hence one thinks of the coordinates being doubled. But in Hull's formulation, $O(D,D)$ invariance is effectively built in as a principle of construction. This is because for the covariant double sigma model action, the target space takes the form $R^{1,d-1} \otimes T^{2D}$, in which we have a non-compact spacetime and a doubled torus. From the torus identifications we have manifest $GL(2D;\mathbb{Z})$ symmetry. Then after imposing what is defined as the self-duality constraint of the theory, which contains the $O(D,D)$ metric, invariance of the theory reduces directly to $O(D,D;\mathbb{Z})$. In other words, while the doubled formalism starts with a covariant action that involves doubled coordinates, the invariance of this theory under $O(D,D)$ is generated by imposing this self-duality constraint, which, similar to the DFT case, effectively halves the degrees of freedom and ensures that the remaining fields are physical.

The equivalence of these two doubled string actions on a classical and quantum level has been shown in [61, 62, 63]. When it comes to the question of quantum consistency, in advance of the discussion in Chapter 4 in which we calculate the beta-functions of the doubled string with completely doubled coordinates, it is useful to first review both approaches to the double sigma model and detail a number of conventions.

3.1 Tseytlin's non-covariant duality symmetric string

As space is limited, we will not consider a comprehensive review of Tseytlin's non-covariant formulation of the doubled string. Instead, like in the previous chapter, we will cover a few pertinent points as they relate to the main investigations of this thesis beginning in Chapter 4. For a complete review see [12, 13, 64] as well as [61, 62, 63].

To begin, we note that directly from 2-dimensional scalar field theory constructed to be symmetric in ϕ and $\tilde{\phi}$, Tseytlin derives the Lagrangian density

$$\mathcal{L}_{sym} = \mathcal{L}_{+}(\phi_{+}) + \mathcal{L}_{-}(\phi_{-}) \tag{3.1}$$

with

$$\mathcal{L}_{\pm}(\phi_{\pm}) = \pm\frac{1}{2}\dot{\phi}_{\pm}\phi'_{\pm} - \frac{1}{2}\phi'^{2}_{\pm}. \tag{3.2}$$

Here $\mathcal{L}_{+}$ and $\mathcal{L}_{-}$ are the Floreanini-Jackiw [60] Lagrangian densities for chiral and anti-chiral fields, with $\dot{\phi} = \partial/\partial_{\tau}$ and $\phi' = \partial/\partial_{\sigma}$. The total Lagrangian $\mathcal{L}_{sym}$ is itself constructed so that it is manifestly invariant under the exchange of $\phi = \frac{1}{\sqrt{2}}(\phi_{+} + \phi_{-})$ with its Hodge dual $\tilde{\phi} = \frac{1}{\sqrt{2}}(\phi_{+} - \phi_{-})$. Directly from the equations of motion one can derive chirality conditions for this theory.

For our present purposes it is important to note that the goal for Tseytlin is to realise from 2-dimensional scalar field theory the corresponding formulation of string theory, which indeed proves general enough to incorporate the world-sheet dynamics of the winding sector. Writing the Lagrangian (3.1) for D scalar fields X^I and with a general background, in the Tseytlin approach we famously obtain the action

$$S[e^a_n, X^I] = -\frac{1}{2}\int_{\sum} d^2\xi \; e[\mathcal{C}^{ab}_{IJ}(\xi) \; \nabla_a X^I \nabla_b X^J]. \tag{3.3}$$

Here $I, J = 1, ..., D$. We define the coordinates on $\sum$ such that $\xi^0 \equiv \tau$ and $\xi^1 \equiv \sigma$. The two-dimensional scalar fields X^I depend on ξ and they are vectors in N-dimensional target space $\mathcal{M}$. The number N of embedding coordinates is kept general, because the purpose of this action is to be as generic as possible while minimising assumptions for its construction. We also note that C_{IJ} need not necessarily be symmetric and, from the outset, we can treat it completely generically. We also have the zweibein e^a_n, where $e = \det e^a_n$. This term appears in the definition of the covariant derivative of the scalar field $X^I : \nabla_a X^I \equiv e^a_n \partial_a X^I$, where a is a flat index and n is a curved index.

In its first principle construction, which occupies the earliest sections of [13], one can recover from this generic action (3.3) the standard manifestly Lorentz invariant sigma model action for strings propagating in a curved background. Furthermore, if we exclude the dilaton for simplicity we may define $\mathcal{C}^{ab}_{IJ} = T(\eta^{ab} G_{IJ} - \epsilon^{ab} B_{IJ})$, where we reintroduce explicit notation for the string tension T, G is the metric tensor on the target space, and B is the Kalb-Ramond field.

Keeping to a generic analysis with a general C, after a number of steps, one finds that they can finally rewrite (3.3) in the following way,

$$S = -\frac{1}{2}\int d^2\xi \; e[\mathbb{C}_{IJ}(\xi)\nabla_0 X^I \nabla_1 X^J + M_{IJ}\nabla_1 X^I \nabla_1 X_J]. \tag{3.4}$$

Here it is conventional to define $\mathbb{C}_{IJ} = C^{01}_{IJ} + C^{10}_{JI}$ and $M_{IJ} = M_{JI} = C^{11}_{IJ}$. The action is manifestly diffeomorphism $\xi^n \to \xi'^n(\xi)$ and Weyl $e^a_n \to \lambda(\xi) e^a_n$ invariant, but it is not manifestly invariant under local Lorentz transformations. Moreover, notice that (3.4) must be invariant for the finite transformation of the zweibein, because the physical theory should be independent of e^a_n. This means that if under such a transformation we have $e^a_n \to e'^a_n = \Lambda^a_b(\xi) e^b_n$, where one may recognise Λ^a_b is a Lorentz $SO(1,1)$ matrix dependent on ξ, we also have an induced infinitesimal transformation of the form $\delta e^a_n = \omega^a_b(\xi) e^b_n$ with $\omega_{ab} = -\omega_{ba}$. Now, substituting $\omega^a_b(\xi) = n(\xi)\epsilon^a_b$, we have an infinitesimal Lorentz transformation

$$\delta e^a_n = n(\xi)\epsilon^a_b(\xi)e^b_n. \tag{3.5}$$

Due to the fact that the action (3.4) is not invariant under this infinitesimal local Lorentz transformation, because, again, from first principles the chiral scalar action is of Floreanini-Jackiw variety, it follows as has been stated elsewhere in this thesis that the requirement of on-shell local Lorentz invariance is fundamental to the entire discussion. As Tseytlin comments in a footnote [13], alternatively we may prefer Siegel's [14, 15] manifestly Lorentz covariant formulation, but with that we obtain extra fields and gauge symmetries; whereas in extending the Floreanini-Jackiw formulation it is fairly simple to introduce interactions and, ultimately, we find that the condition in the Siegel approach that requires decoupling of the Lagrange multiplier corresponds to what we will review as the Lorentz invariance condition in the Floreanini-Jackiw approach.

For the action (3.4), a way to attack the requirement of local on-shell Lorentz invariance is by seeing in [13] that it demands we satisfy the condition

$$\epsilon^{ab}t_{ab} = 0, \text{where } t^b_a \equiv \frac{2}{\epsilon}\frac{\delta S}{\delta e^a_n}e^b_n. \tag{3.6}$$

The general idea is that the tree-level string vacua should be assumed to correspond to $S[X, \tilde{X}, e]$, which define the Weyl and Lorentz invariant quantum field theory. In performing the background field expansion, something we will cover in the general context in Chapter 4, we may take the expansion to be near the classical solution of the $(X, \tilde{X})$ equations of motion with the trace of the expectation value of the energy-momentum tensor as well as the ϵ^{ab} trace vanishing on-shell. In Tseytlin's formulation, $\hat{t}$ denotes precisely this epsilon trace such that $\hat{t} = \epsilon^a_b t^b_a$. The vanishing of $\hat{t}$ shows local Lorentz invariance. So let us now vary (3.4) under local Lorentz transformation, which is proportional to the equations of motion

$$\begin{aligned} t^b_a = &-\delta^b_a[\mathbb{C}_{IJ}(\xi)\nabla_0 X^I \nabla_1 X^J \\ &+ M_{IJ}\nabla_1 X^I \nabla_1 X^J] + \delta^b_0[C_{IJ}\nabla_a X^I \nabla_1 X^J] \\ &+ \delta^b_1[C_{IJ}\nabla_0 X^I \nabla_a X^J] + 2\delta^b_1 M_{IJ}\nabla_a X^I \nabla_1 X^J. \end{aligned} \tag{3.7}$$

This equation for t^b_a is equivalent to equation 4.3 in [13]. In order for the variation of the action to vanish under such a transformation, we derive the condition

$$\epsilon^{ab}t_{ab} = 0. \tag{3.8}$$

In other words, the condition that must be satisfied to recover local Lorentz invariance depends on the solution of the equations of motion for the zweibein. In fact, one will recognise that what is observed is completely analogous to the standard string theory formulation based on the Polyakov action, where one will recall that the equations of motion for the world-sheet metric determines the vanishing of the energy-momentum tensor [7].

This constraint must be imposed on a classical *and* quantum level. The key point is that now we can choose the flat gauge $e^a_n = \delta^a_n$, thanks to the invariances under diffeomorphisms, Weyl transformations, and finally local Lorentz invariance imposed on-shell. This is crucial for the formulation of the dual symmetric string in that, using the flat gauge for the zweibein, we are effectively performing the analogous procedure as when fixing the conformal gauge in standard string theory. Keeping C and M constant, we can compute the equations of motion for the field X^I to give

$$\nabla_1[e(C_{IJ}\nabla_0 X^J + M_{IJ}\nabla_1 X^J] = 0. \tag{3.9}$$

In the flat gauge this result becomes

$$\partial_1[C_{IJ}\partial_0\xi^J + M_{IJ}\partial_1\xi^J] = 0. \tag{3.10}$$

From (3.10) a now famous identity appears, where, in the flat gauge and along the equations of motion for ξ^I, the following constraint on C and M is obtained [13]:

$$C = MC^{-1}M. \tag{3.11}$$

One may recognise the tensor structure of (3.11) from an earlier discussion on the action of an $O(D, D, \mathbb{Z})$ element. The important thing to highlight is that throughout the lengthy calculation to get to this point, C and M are held constant. (When C and M are not treated as constant, a number of interesting questions arise which extend beyond the scope of the present discussion). What is also important is that, after rotating ξ^I, the matrix C can always be put into diagonal form such that

$$C = \ \mathbf{diag}\ (1, ..., 1, -1, ..., -1). \tag{3.12}$$

It remains to be said that $C = C^{-1}$, which means that the constraint (3.11) defines the indefinite orthogonal group $O(p,q)$ of $N \times N$ matrices M with $N = p+q$ in $\mathbb{R}^{p,q}$. The inner product may now be written as

$$C = MCM, \tag{3.13}$$

in which the matrix C eventually takes on the explicit definition of an $O(D,D,\mathbb{R})$ invariant metric in the 2D target space M. Although, admittedly, this cursory review has omitted many important and interesting details, the pertinent point in terms of this thesis is as follows. The action (3.4) turns out to describe rather precisely a mixture of D chiral ξ_-^μ and D anti-chiral ξ_+^μ scalars. In demanding local Lorentz invariance and the vanishing of the Lorentz anomaly, this requires that $p = q = D$ with $2D = N$. In working through the complete logic of the calculation, we observe quite explicitly that inasmuch the requirement of local Lorentz invariance is imposed through the condition (3.8), this leads one naturally to an interpretation of the matrix C as a 2D target space metric with coordinates

$$\xi^I = (\xi_-^\mu, \xi_+^\mu),\ ds^2 = dX^I C_{IJ} dX^J,\ I = 1, ..., 2D,\ \text{and}\ \mu = 1, ..., D. \tag{3.14}$$

To drive the point home in terms of our discussion in the last chapter on the T-duality symmetry group, if we make a change of coordinates in the target space, particularly by defining a set of new chiral coordinates, the matrix C takes on the off-diagonal form of the previous considered $O(D,D)$ constant metric L. This fact will be utilised a number of times throughout the remainder of this thesis. The chiral coordinates we define are

$$X^I = \frac{1}{\sqrt{2}}(X_+^\mu + X_-^\mu), \tilde{X}_I = \frac{1}{\sqrt{2}}(X_+^\nu - X_-^\nu). \tag{3.15}$$

In this frame, the matrix C is then shown to be

$$C_{IJ} = -\Omega_{IJ} = -\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \tag{3.16}$$

It follows that the condition (3.11) transforms into the constraint

$$M^{-1} = \Omega^{-1} M \Omega^{-1} \tag{3.17}$$

on the symmetric matrix M, which can be parametrised by a symmetric matrix

G and an antisymmetric matrix B. Therefore, remarkably, the symmetric matrix M takes the precise form of the generalised metric (2.7) in which M is found to be positive definite.

To conclude, in the chiral coordinates we arrive at a famous form of the Tseytlin action,

$$S = \frac{1}{2} \int d^2\xi \; e[\Omega_{IJ} \nabla_0 X^I \nabla_1 X^J - M_{IJ} \nabla_1 X^I \nabla_1 X^J]. \tag{3.18}$$

This action is manifestly $O(D,D)$ invariant. When $O(D,D)$ transformations are applied to (3.18), we obtain exactly what we would anticipate for the standard string in the sense of T-duality invariance under $X \to \tilde{X}$ and for the generalized metric $M \to M^{-1}$.

For completeness, from the action (3.4) in arriving at (3.18), it should be clear that what we are working with is a sigma model for the dual symmetric string. The generalised version of the celebrated action (3.18) is indeed often written as

$$S_{General} = \frac{1}{2} \int d^2\xi \; [-(C_{IJ} + \eta_{IJ}) \partial_0 X^I \partial_1 X^J + \mathcal{H}_{IJ} \partial_1 X^I \partial_1 X^J)]. \tag{3.19}$$

This final action can be argued to be a very natural generalisation for the standard string on a curved background. It not only contains the generalised metric $\mathcal{H}_{IJ}$, but also another symmetric metric η_{IJ} with (D,D) signature and an antisymmetric 2-tensor C_{IJ}. The coordinates are defined as before $X^I = \{X^I, \tilde{X}_I\}$ with the background fields in general depending on X^I.

It is this generalised version of the Tseytlin action to which we shall return in our investigations in Chapter 4.

3.2 The doubled formalism

The doubled formalism as developed recently by C M Hull [25, 26] may be viewed from its foundations as an alternative approach to DFT. It may be described as a duality symmetric approach akin to the Tseytlin approach in that the doubled formalism takes a world-sheet perspective. As expected it naturally employs a sigma model description of the string with the target space doubled.

Much like conventional DFT, the theory is constructed around the generalised metric. Typically, the target space for the sigma model is a torus fibration T^n over a base N. One may think of this as a description of string theory in which the target space is locally a T^n bundle, while N is some generic base manifold that may be thought of simply as a base space. The essential motivation is to double the torus by then adding $2n$ coordinates such that the fibre is T^{2n}; however, typically the fields depend only on the base coordinates. Finally, the strategy is generally to proceed with a patch-wise splitting $T^{2n} \to T^n \oplus \tilde{T}^n$ so that we have demarcated a strictly physical subspace T^n and its dual $\tilde{T}^n$. For a geometric background local patches are glued together with transition functions which include group $GL(n, \mathbb{Z})$ valued large diffeomorphisms of the fibre. For the non-geometric case, this is approached by gluing local patches with transition functions that take values in $GL(n, \mathbb{Z})$ as well as in the complete T-duality group, such that $O(D, D, \mathbb{Z})$ is a subgroup of $GL(2n, \mathbb{Z})$ large diffeomorphisms of the doubled torus.

Importantly, and as alluded in the previous sections, the doubled formalism has been found to be classically equivalent to the ordinary string sigma model, with the added feature that T-duality is manifest. Quantum equivalence of such sigma models has also been demonstrated using various methods in [26, 65, 66, 40, 67, 39]; however, it is a much more complicated endeavour. A key observation is that by doubling the fibre co-ordinates it is necessary to introduce a chirality (or duality) constraint to ensure no excess degrees of freedom. As mentioned at the start of the chapter, the constraint is enforced so that for the physical content in a certain frame only half the co-ordinates are left-moving chiral bosons and the other half are right-moving. We will discuss this in more detail later. For now the important thing to highlight is that in the primary paper [39] that we follow in Chapter 4, the constraint is incorporated into the action and the price is that although the dual symmetric string is formulated in the context of Hull's approach, Lorentz invariance is no longer manifest. This means that we have an action akin to Tseytlin's generalised action.

To understand what this means and to introduce the double sigma model from the doubled formalism, we primarily follow [39, 43].

3.2.1 Lagrangian and double sigma model

If the objective is to describe a sigma model in the doubled formalism, we first must establish local coordinates on the torus T^{2n} which we denote $\mathbb{X}^I$. We also have coordinates on the base N conventionally denoted Y^a. For a sigma model description the string world-sheet is then mapped into T^{2n} by $\mathbb{X}^I(\sigma)$. The starting point of the double sigma model is the Lagrangian (an overall factor of

2π is dropped here, following conventions in [39])

$$\mathcal{L} = \frac{1}{2}\mathcal{H}_{IJ}(Y)\hat{\mathcal{P}}^I \wedge \star\hat{\mathcal{P}}^J - \frac{1}{2}L_{IJ}\mathcal{P}^I \wedge \mathcal{A}^J + \mathcal{L}(Y) + \mathcal{L}_{top}(\mathbb{X}). \tag{3.20}$$

We note the following conventions. $\mathcal{H}_{IJ}$ is the generalised metric. The 1-form on the world-sheet is defined $\mathcal{P}^I = d\mathbb{X}^I$. From the definition of the 1-form a connection on the bundle may be defined in the form $\mathcal{A}^I = \mathcal{A}^I_a dY^a$, with the covariant momenta $\hat{\mathcal{P}}^I = d\mathbb{X}^I + \mathcal{A}^I$. In the Lagrangian (3.20) we have a strictly topological term $\mathcal{L}_{top}(\mathbb{X})$, which will feature later in our examination. We also have $\mathcal{L}(Y)$ term, which is a standard sigma model on the base N described by the following:

$$\mathcal{L}(Y) = \frac{1}{2}g_{ab}dY^a \wedge \star dY^b + \frac{1}{2}b_{ab}dY^a \wedge dY^b. \tag{3.21}$$

3.2.2 Simplified Lagrangian and constraint

The Lagrangian (3.20) that is conventionally employed at the outset of the construction is usually supplemented by the following constraint

$$\hat{\mathcal{P}}^I = L^{IK}\mathcal{H}_{KJ} \star \hat{\mathcal{P}}^J, \tag{3.22}$$

where L_{IJ} is the constant and invariant $O(D,D)$ metric introduced in (2.6). This constraint is imposed on-shell to eliminate half of the degrees of freedom and to ensure reduction to a physical subspace. We will discuss the role of this constraint later in this section. Meanwhile, we can use L_{IJ} to raise and lower indices on $\mathcal{H}$, and from the constraint (3.22) it is demanded for consistency purposes that

$$L^{IK}\mathcal{H}_{KJ}L^{JL}\mathcal{H}_{LM} = \delta^I_M. \tag{3.23}$$

This is true for the $O(D,D)$ coset form of the doubled fibre metric $\mathcal{H}_{IJ}$. Just like before in the context of DFT, we may package the n^2 moduli fields on the fibre into coset form such that

$$\mathcal{H}_{IJ}(Y) = \begin{pmatrix} (G - BG^{-1}B)_{ij} & (BG^{-1})_i^{\ j} \\ -(G^{-1}B)^i_{\ j} & G^{ij} \end{pmatrix}, \tag{3.24}$$

with the primary difference being that the moduli now rely on the base coordinates, although this isn't strictly necessary. As it is currently written, and as it appears in (3.20), the generalised metric for the doubled fibre depends on the base. This is something that again we will have to revisit. Meanwhile, we can still arrive at a more simple Lagrangian then in (3.20) following [43]. Making the simplifying assumption that the fibre is trivial we can set the connection $\mathcal{A}^I = 0$. This then means that for the physical fibration in $T^{2n} \to T^n \oplus \tilde{T}^n$ over base N, we should demand for the background that the off-diagonal components have an index in the torus and that the others set in the base are zero such that $E_{ai} = E_{ia} = 0$. Also assuming that the antisymmetric 2-form b_{ab} on the base vanishes, the Lagrangian describing the double sigma model finally becomes

$$\mathcal{L} = \frac{1}{4}\mathcal{H}_{IJ}(Y) d\mathbb{X}^I \wedge \star d\mathbb{X}^J + \mathcal{L}(Y) + \mathcal{L}_{top}(\mathbb{X}). \tag{3.25}$$

A consequence of constructing this simplified Lagrangian is that the constraint placed on the fields also simplifies to the form,

$$d\mathbb{X}^I = L^{IJ} H_{JK} \star d\mathbb{X}^K. \tag{3.26}$$

These assumptions which lead to the above constraint, although perhaps seemingly innocuous at first, prove fairly important and serve as a notable motivation for the original calculations in the second half of this thesis. To understand why, we follow the conventional approach [39, 41, 68, 43] by first introducing a vielbein to enable a change to the chiral frame (denoted by barred indices for the remainder of this dissertation) where

$$\mathcal{H}_{\bar{A}\bar{B}}(y) = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix}, \quad \mathcal{L}_{\bar{A}\bar{B}} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}. \tag{3.27}$$

In the chiral frame, the constraint (3.26) becomes a chirality constraint. As already mentioned, it is employed to ensure that half of the fields $\mathbb{X}^{\bar{A}}$ are chiral bosons and the other half anti-chiral bosons. In the literature on chiral boson models, there are several ways in which one may introduce this chirality constraint. Limiting to the formalism of the duality symmetric string, one approach is to calculate the partition function using holomorphic factorisation [69, 70]. Another approach is canonical quantisation, using Dirac brackets and promoting (3.27) as a second-class constraint. In Hull [26], the approach taken is to impose the constraint by way of gauging the associated current.

To achieve the action used in [39] for the purpose of our investigations in Chapter 4, at the classical level the chirality constraint is imposed at the level of the action using the method of Pasti, Sorokin and Tonin (PST) [71, 72, 73], otherwise known as the PST procedure. In its standard formulation, consider the simple example of a one-dimensional target space modelled as a circle [43]. Given this target space has a constant radius R, on the fibre the action may be written

$$S_{1-dim} = \frac{1}{4}R^2 \int dX \wedge \star dX + \frac{1}{4}R^{-2} \int d\tilde{X} \wedge \star d\tilde{X}. \tag{3.28}$$

Now, by making a change of basis so that the fields become chiral, we have the following definitions:

$$\mathbb{X}_+ = RX + R^{-1}\tilde{X},\ \partial_- \mathbb{X}_+ = 0 \tag{3.29}$$

and

$$\mathbb{X}_- = RX - R^{-1}\tilde{X},\ \partial_+ \mathbb{X}_- = 0. \tag{3.30}$$

As a result of asserting that the radius of the target space is constant, the chirality constraint takes a simple form and the action becomes

$$S_{1-dim} = \frac{1}{8} \int d\mathbb{X}_+ \wedge \star d\mathbb{X}_+ + \frac{1}{8} \int d\mathbb{X}_- \wedge \star d\mathbb{X}_-. \tag{3.31}$$

In the PST approach, one may then implement the constraint by defining the 1-forms:

$$\mathcal{P} = d\mathbb{X}_+ - \star d\mathbb{X}_+,\ \mathcal{Q} = d\mathbb{X}_- + \star d\mathbb{X}_-. \tag{3.32}$$

These vanish on the constraint, which therefore allows the incorporation of (3.26) into the action by way of two auxiliary closed 1-forms u and v. The final PST action takes the form

$$S_{PST} = \frac{1}{8} \int d\mathcal{P} \wedge \star d\mathcal{P} + \frac{1}{8} \int d\mathcal{Q} \wedge \star d\mathcal{Q} - \frac{1}{8} \int d^2\sigma \Big(\frac{(\mathcal{P}_m u^m)^2}{u^2} + \frac{(\mathcal{Q}_m v^m)^2}{v^2} \Big), \tag{3.33}$$

which works by introducing a new gauge symmetry, namely the PST symmetry. Here $m = 0, 1, 2, 3$. This introduction of a new symmetry enables the gauging away of any and all fields that do not obey the chirality constraint. It is not so dissimilar to the rationale followed in the construction of DFT, where the only remaining fields in the theory are physical.

From this point, one may either gauge fix the PST action at the cost of Lorentz invariance or invoke covariant quantisation and employ ghosts to manage to the PST symmetry. Following the first prescription, this noncovariant option results in a Floreanini-Jackiw style action. What is useful about this approach is that, by then defining the auxiliary fields to be time-like, one obtains a chiral and anti-chiral copy of the FJ action. As established in the literature, such an action takes the form

$$S_{FJ} = \frac{1}{4} \int d^2\sigma (\partial_1 \mathcal{P} \partial_- \mathcal{P} - \partial_1 \mathcal{Q} \partial_+ \mathcal{Q}). \tag{3.34}$$

Crucially, especially for the purposes of the present thesis, if one re-expands this action in the non-chiral basis, Tseytlin's first-principle duality symmetric string action is recovered [39],

$$S_{Tseytlin} = \frac{1}{2} \int d^2\sigma [-(R\partial_1 X)^2 - (R^{-1}\partial_1 \tilde{X})^2 + 2\partial_0 X \partial_1 \tilde{X}]. \tag{3.35}$$

The constraints can now be written as

$$\partial_0 \tilde{X} = R^2 \partial_1 X \quad \text{and} \quad \partial_0 X = R^{-2} \partial_0 \tilde{X}, \tag{3.36}$$

which arise after integrating the equations of motion. Interestingly, the string wave equation is given by combining the constraint equations.

As for the general case, the PST procedure yields the action

$$S_{PST} = \frac{1}{2} \int d^2\sigma [-\mathcal{G}_{\alpha\beta} \partial_1 X^\alpha \partial_1 X^\beta + \mathcal{L}_{\alpha\beta} \partial_1 X^\alpha \partial_0 X^\beta + \mathcal{K}_{\alpha\beta} \partial_0 X^\alpha \partial_0 X^\beta]. \tag{3.37}$$

In this case, we define the fields similar to the standard formulation of DFT, where $X^\alpha = (\mathbb{X}^A, Y^a) = (X^i, \tilde{X}_j, Y^a)$. For the background fields, including the topological term $\mathcal{K}_{\alpha\beta}$, we have

$$\mathcal{G} = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & g \end{pmatrix}, \ \mathcal{L} = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{K} = \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}. \tag{3.38}$$

One will notice that $\mathcal{G}$ contains the generalised metric $\mathcal{H}$ on the fibre and the standard sigma model metric g on the base. Likewise, the topological term $\mathcal{K}$ also contains the standard sigma model metric on the base.

On the fibre, the second-order equation of motion takes the form

$$\partial_1(\mathcal{H}\partial_1\mathbb{X}) = L\partial_1\partial_0\mathbb{X}. \tag{3.39}$$

Using the gauge invariance of the action under $\mathbb{X}^A \to \mathbb{X}^A + f(\tau)$ an integration function of τ can be removed, and the equations of motion can be integrated to recover the chirality constraint imposed on the action.

Chapter 4

Double Everything: Generalising the Interacting Chiral Boson Model

Following an introductory review of several important concepts in the study of the doubled string, the goal of this chapter is to work toward calculating the beta-functionals for the duality symmetric string in the context of the interacting chiral boson model. Given that the model of interacting chiral scalars is classically consistent, solving the classical Lorentz invariance constraints [62], we face the question of whether a completely generic treatment of the most general form of the Tseytlin action is consistent at the quantum level. In comparison to past studies, notably [39] which uses the action (3.37) and relies on a split between base and fibre, the question of quantum consistency is all the more pertinent when working from an action in which *all coordinates are doubled*. This means that we will consider fields with arbitrary dependence on the full doubled geometry, which is necessary in order to obtain effective spacetime theories corresponding to generic non-geometric geometries and interesting from the perspective of studying world-sheet theories on generic non-geometric backgrounds. In calculating the total divergence of the theory, what we will achieve is in fact a completely general tensor structure for the Weyl and Lorentz anomalies. And from this, we will then investigate whether this generic approach to the interacting chiral boson model satisfies the requirement of conformal invariance at the quantum level. Such a study should stand out when compared to our review of other approaches in the last two chapters.

To work toward calculating the beta-functionals for the completely doubled theory, this amounts to computing all of the relevant contributions to the quantum effective action at one-loop, which, as we shall see in this chapter, include two component terms that comprise the total divergence: Weyl anomaly pieces and global Lorentz anomaly pieces. In contrast to the standard sigma model, the global Lorentz anomaly terms arise because we are working with chiral bosons. They signify phase dependence. On the other hand, the Weyl anomaly terms are just like with the standard string, signifying the scale dependence of the theory. In order for the Weyl invariance of the sigma model to be preserved at one-loop, it is necessary that the beta-functional vanishes.

To this end, we follow the structure and basic strategy of [39]. However, as mentioned in the introduction of this thesis, we would like to avoid a number of assumptions the authors made in calculating the background field equations, including avoiding the enforcement of the self-duality constraint that comes with the doubled formalism discussed in Section 3.2.

We will set-up the lengthy calculation in several steps. In Section 4.1, we introduce the form of the action that we will use and offer several opening comments with regards to some of the subtleties that come with the fully doubled theory. In Section 4.2, we employ the background field method to obtain the one-loop effective action for the background fields of the model. In Section 4.3, we then derive the general structure of the quantum effective action Γ at one-loop. Following this result, we detail the general strategy used to ensure a completely generic calculation of the final total divergence. We conclude with an updated version of the second-order Lagrangian for the background fields. In Section 4.4, we introduce the kinetic terms of the theory and derive the master expression for the effective action. Then, in Section 4.5, we introduce the propagators of the theory and calculate the fluctuation contractions described in Section 4.4. To conclude the chapter, in Section 4.6 we combine all of our results to show the final tensor structure for the total divergence of theory. We then proceed to calculate the tensor algebra to give a final result for the Weyl and Lorentz anomalies. In Section 4.7, we close the chapter with a calculation of the doubled beta-functionals.

4.1 General action with arbitrary dependence on full doubled geometry

In [39], the action (3.37) is used to calculate the background field equations. As we saw in Section 3.2.2, one displeasing feature of this approach to calculating the chiral boson model (as it is formulated in the doubled formalism) is the requirement that the base and the fibre are separated, with the fibre doubled and the base remaining undoubled. A more democratic approach, one could argue, is one in which *everything becomes doubled*. But in return to the questions posed in the introduction of this thesis: what exactly does it mean to double everything and give the background fields arbitrary dependence on the full doubled geometry? What are the implications?

To start, it means that we choose not to assume a base-fibre split. So we drop any dependence on the base coordinates Y, such that for the doubled coordinates we now have $X^A = \mathbb{X}^A = (X^i, \tilde{X}_j)$ with $\tilde{X}_j$ the dual coordinates. We also drop the topological term $\mathcal{K}_{\alpha\beta}$, since, although it is useful to show double gauge invariance, in [41] one sees that the topological term is Lorentz invariant and does not affect the equations of motion, so it plays no role in calculating the background field equations [39]. Thus, one immediate implication is that we shall begin with an action of the tentative form

$$S = \frac{1}{2}\int d^2\sigma[-\mathcal{G}_{AB}(X^A)\partial_1 X^A \partial_1 X^B + \mathcal{L}_{AB}(X^A)\partial_1 X^A \partial_0 X^B],$$

where $\mathcal{G}$ now contains only the generalised metric on the doubled space and $\mathcal{L}$ resumes its familiar form

$$\mathcal{G} = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & 0 \end{pmatrix}, \; \mathcal{L} = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.1}$$

Given that we have dropped base dependence, all other quadrants are now zero except for the first in both $\mathcal{G}$ and $\mathcal{L}$. Therefore, we may avoid unnecessary redundancy and write for both $\mathcal{H}$ and L explicitly. When plugging (4.1) into the action we obtain

$$S = \frac{1}{2}\int d^2\sigma[-\mathcal{H}_{AB}(X^A)\partial_1 X^A \partial_1 X^B + L_{AB}(X^A)\partial_1 X^A \partial_0 X^B]. \tag{4.2}$$

Although for the time being we will keep the definition of L generic, we note that this is the sort of action considered by Tseytlin [12, 13] as well as more recently by others in [41, 42]. It is an action for the dual symmetric chiral string that takes the form proposed to lead directly to DFT [41]. One will notice that its basic structure is that of (3.19), with the exclusion of the topological term. It should be re-emphasised that this completely doubled action remains equivalent to that which comes from Hull's doubled formalism, and as it is equivalent to the Tseytlin action (3.19) it is also therefore equivalent to the standard Polyakov action.

In principle, (4.2) is the most general doubled action we can write without manifest Lorentz invariance, because it allows us to calculate the background fields in a way in which the fields maintain arbitrary dependence on the full doubled geometry. In that sense, in reading and following the structure of [39], in this chapter the goal is in fact to employ a similar strategy of [42] in which we're ultimately seeking to set the foundation for an effective spacetime theory that corresponds to completely generic non-geometric geometries.

The issue that we have, however, is that (4.2) is arguably too general. With all of the coordinates doubled - and given that we want to ensure that the matrices $\mathcal{H}$ and L maintain arbitrary coordinate dependence throughout the entire procedure - the conventional argument is that this sort of action and approach ultimately requires the implementation of some form of constraint [43] so that the theory possesses: 1) first order equations of motion allowing half the degrees of freedom to be eliminated; 2) emergent on-shell Lorentz invariance; and 3) an off-shell invariance under a set of modified Lorentz transformations.

In general, there are two established pathways to solving these conditions. The first is to use the DFT constraint, which means we set $L(X^A)$ to take the definition of an $O(D,D)$ invariant constant metric like we saw in Section 2.1. Indeed, as we reviewed in Chapter 2, and as Tseytlin [13] originally observed, taking this approach would provide a consistent model. But this approach ultimately means that we set the matrix $\mathcal{H}(X^A)$ to depend on only half of the coordinates, and this defeats our primary motivation. Moreover, if $\mathcal{H} = \mathcal{H}(X^A)$ so that it can depend on any of the doubled co-ordinates, including the dual coordinates, in the DFT approach the freedom to have metric dependence on any of the doubled coordinates follows with the restriction of the strong form of the level-matching constraint. This is because one can always perform an $O(D,D)$ rotation to find the duality frame so that the fields only depend on X^i and not $\tilde{X}_j$. In [39, 41, 42], this is the path that is generally followed. The alternative to the DFT approach, conventionally, is to use the Scherk-Schwarz ansatz assuming some underlying group structure. We will not discuss here this solution to the above conditions.

In this thesis, neither of these options are satisfactory, as we wish to maintain course in calculating the background field equations in a completely generic way (i.e., without assuming $O(D,D)$ invariance from the outset and without imposition of the chirality constraint). To achieve this, the strategy that we will employ requires some subtle clarification and careful description, which we will establish and define over the next few sections. First, it is helpful to show the Lagrangian for the expanded background fields from which we can go on to introduce the most basic structure of the effective action. This will lead to a derivation of the master expression for the effective action, which will enable us to finally describe the strategy we undertake to ensure a completely generic calculation.

In what remains of this thesis we adopt the conventions of [39]. We will reserve the Greek characters μ and ν to denote worldsheet indices. The worldsheet signature is $(+,-)$, with time and spatial derivatives $\partial_\pm = \partial_0 \pm \partial_1$. We shall also use barred notation to denote chiral frame indices, which will be introduced more formally later on.

4.2 Background field expansion

In proceeding to analyse the quantum behaviour of the doubled action (4.2), we employ the background field method [74, 75, 76, 77]. This means we perform a background field expansion in quantum fluctuations around a classical background $X^A = X_0^A + \pi^A$, where $X_0 = X_{classical}$ is the stationary part of the action and thus a solution of the classical equations of motion. The procedure is the same as that which may be used for the standard sigma model, allowing the perturbative study of UV divergences.

In expanding the background fields, the quantum field π^A is generally defined as a coordinate difference $X^A - X_0^A$. This means that, under general coordinate transformations, π^A does not transform as a vector and therefore yields a non-covariant expansion of the action. To rectify this, we follow the convention of defining geodesic coordinates with ξ^A defined as the tangent to the geodesic from X_0^A to $X_0^A + \pi^A$. The arc length of the geodesic is equivalent to the distance between these two points. As is easily verified from first-principles, the benefit of this construction is that the ξ^A field transforms covariantly as a vector and so we can expand the background fields covariantly as well. This means that we shall perform a perturbative expansion in powers of ξ^A.

4.2.1 Algorithmic method

In calculating the background field expansion, we use the algorithmic method developed in [77]. This allows us to obtain up to nth order the appropriately expanded action by acting on the Lagrangian (4.2) with the operator

$$\int d^2\sigma \xi^A(\sigma)\mathcal{D}^\sigma_A. \tag{4.3}$$

As is understood from the derivation of this method, $\mathcal{D}^\sigma_A$ is a covariant functional derivative with respect to the fields $X^A(\sigma)$. It is obtained when we account for infinitesimal changes in X_0^A along the geodesic map ξ^A to another vector under parallel transport, enabling the operator (4.3) to be covariantly constant along the geodesic path. The calculation of the background field equations then proceeds by simply acting with the operator n times and then dividing by $n!$.

In general, the action of the operator yields several identities. The first identity is

$$\int d^2\sigma\ \xi^A(\sigma)\mathcal{D}^\sigma_A\xi^B(\sigma') = 0. \tag{4.4}$$

This means that when the functional derivative acts on the vectors $\xi^A(\sigma)$ there is no contribution. The functional derivative can of course also act on other objects. When, for example, we act on tensors $T_{A_1\ldots A_n}(X)$ we obtain

$$\int d^2\sigma\xi^A(\sigma)\mathcal{D}^\sigma_A\ T_{A_1\ldots A_n}(X(\sigma')) = \mathcal{D}_B T_{A_1\ldots A_n}(X(\sigma'))\xi^B(\sigma'). \tag{4.5}$$

For vectors $\partial_\mu X^A$ the action of the operator gives the identity,

$$\int d^2\sigma\xi^A(\sigma)\mathcal{D}^\sigma_A(\partial_\mu X^B(\sigma')) = \mathcal{D}_\mu\xi^B(\sigma'). \tag{4.6}$$

And finally for the vector $\mathcal{D}_\mu\xi^B$ we have,

$$\int d^2\sigma\xi^A(\sigma)\mathcal{D}^\sigma_A(\mathcal{D}_\mu\xi^B(\sigma')) = R^B_{ACD}\partial_\mu X^D\xi^A\xi^C(\sigma'). \tag{4.7}$$

Given the added subtleties of the doubled formalism, we note here that R^B_{ACD} in (4.7) is the target space Riemann curvature tensor. It is evaluated at X_0^A. So, too, is the arbitrary tensor $T_{A_1\ldots A_n}$ of rank n evaluated at the classical solution.

By expanding to second order in ξ^A fluctuations, this will be sufficient to calculate the one-loop background field equations. In the algorithmic computation, this amounts to setting $n = 2$. The terms linear in ξ provide no contributions to the one-loop effective action since, in general, they are proportional to the classical equations of motion, which may be also observed when deriving the structure of the quantum effective action (we will discuss the effective action in

Section 4.3). The terms quadratic in ξ provide the kinetic terms for the fluctuations as well as the interaction pieces. The equations we obtain at one-loop give important information about the conditions on the background necessary for world-sheet Weyl invariance to be preserved. The calculation to this order will also provide sufficient information in order to analyse world-sheet Lorentz invariance. It is through the background field equations that the connection to double field theory can also be made [41].

4.2.2 Expansion of the background fields

The only assumption we make in expanding the background fields of (4.2) is that we treat $\mathcal{H}$ as playing the role of the metric, although other choices may be made [42]. Therefore, expanding the first term in the action (4.2), we obtain at first order in ξ fluctuations

$$-\mathcal{H}_{AB}\partial_1 X^A D_1 \xi^B, \tag{4.8}$$

and at second order

$$-\frac{1}{2}(\mathcal{H}_{AB}D_1\xi^A D_1\xi^B + R_{CABD}\partial_1 X^C \partial_1 X^D \xi^A \xi^B). \tag{4.9}$$

We observe in (4.9) the presence of the Riemann curvature tensor, which again should be emphasised that this is constructed from the metric $\mathcal{H}$.

For L_{AB}, we recall that the original plan is to treat this 2-tensor generically; hence its expansion carries extra terms without much simplification

$$\begin{aligned}
&\frac{1}{2}(L_{AB}\partial_0 X^A D_1\xi^B + L_{AB}D_0\xi^A\partial_1 X^B + D_K L_{AB}\xi^K \partial_0 X^A \partial_1 X^B)\\
&+\frac{1}{2}[L_{AB}D_0\xi^A D_1\xi^B + \frac{1}{2}(D_A D_B L_{KD} + L_{KC}R^C_{ABD} + L_{DC}R^C_{ABK})\partial_0 X^K \partial_1 X^D \xi^A \xi^B\\
&+ D_K L_{AB}\xi^K(\partial_0 X^A D_1\xi^B + D_0\xi^A \partial_1 X^B)].
\end{aligned}$$

As it is convention, we may at this point invoke the classical equations of motion of X_0 and drop the terms linear in ξ. When collecting terms quadratic in ξ we also move an overall factor of $\frac{1}{2}$ to the left-hand side of the equality for the following second order Lagrangian

$$
\begin{aligned}
&2L^{(2)} = -\mathcal{H}_{AB}D_1\xi^A D_1\xi^B + L_{AB}D_0\xi^A D_1\xi^B \\
&- R_{KABD}\partial_1 X^K\partial_1 X^D\xi^A\xi^B + D_K L_{AB}\xi^K(\partial_0 X^A D_1\xi^B + D_0\xi^A\partial_1 X^B) \\
&+ \frac{1}{2}D_A D_B L_{KD}\partial_0 X^K\partial_1 X^D\xi^A\xi^B + \frac{1}{2}(L_{KC}R^C_{ABD} + L_{DC}R^C_{ABK})\partial_0 X^D\partial_1 X^K\xi^A\xi^B.
\end{aligned}
\tag{4.10}
$$

The effective action for the background fields at second order is given by (4.10). We can now either simplify it by using the simplification method in [39] or proceed as usual. In the present case, the Wick contractions are indeed found to made more manageable if we follow the simplification procedure. It should be noted that it does not fundamentally make a difference which path is chosen; although in other approaches to the construction of double sigma models, the simplification strategy that we will describe may in the end yield a more complicated form of the effective action.

4.2.3 Simplification of the expanded background field action

Before introducing vielbeins and moving the indices on the ξ fields, we first use the equation of motion

$$
D_1(\mathcal{H}_{AB}\partial_1 X^B) = L_{AB}\partial_1\partial_0 X^B \tag{4.11}
$$

to eliminate all L terms in (4.10) with exception of the L fluctuation kinetic piece. In taking this step the calculation is lengthy (see especially eqn. 29 in [39]), including the use of integration by parts and the use of multiple copies of the equations of motion. Also, by expanding covariant derivatives and simplifying the result using the equations of motion, what we end up with is of course a second order Lagrangian no longer expressed in terms of covariant derivatives. It is worth noting that this approach differs from [42], in which the authors choose to maintain throughout explicit expression in terms of covariant derivatives. Regardless of approach at this point, one will obtain an equivalent result [42, 68]. The reduced Lagrangian that we obtain when performing the above is as follows

$$
\begin{aligned}
2L^{(2)} &= -\mathcal{H}_{AB}\partial_1\xi^A\partial_1\xi^B + L_{AB}\partial_0\xi^A\partial_1\xi^B \\
&- 2\partial_A\mathcal{H}_{DB}\partial_1 X^D\xi^A\partial_1\xi^B - \frac{1}{2}\partial_A\partial_B\mathcal{H}_{KD}\partial_1 X^K\partial_1 X^D\xi^A\xi^B. \qquad (4.12)
\end{aligned}
$$

Interestingly, in using the simplification procedure the result we obtain is just as the one we would get if, in a more direct way, we had performed a non-covariant expansion from the outset. In any case, one can see in (4.12) how things simplify as we are now only carrying four terms, two of which are the kinetic terms of the theory. At this point, we may proceed to introduce the vielbein formalism. The price we pay, however, is that we must introduce derivatives acting on vielbeins such that

$$
\partial_\mu\xi^A = \mathcal{V}^A_{\bar{A}}\partial_\mu\xi^{\bar{A}} + \partial_\mu\mathcal{V}^A_{\bar{A}}\xi^{\bar{A}}. \qquad (4.13)
$$

Generally, it also means we replace the connection in covariant derivatives of fluctuations by the spin connection

$$
\begin{aligned}
D_\mu\xi^A &= \partial_\mu\xi^A + \Gamma^A_{\mu B}\xi^B \to \partial_\mu\xi^A + \Gamma^A_{\mu B}\xi^B + \partial_\mu\mathcal{V}^A_{\bar{A}}\mathcal{V}^{\bar{A}}_B\xi^B \\
&= \partial_\mu\xi^A + A^A_{\mu B}\xi^B. \qquad (4.14)
\end{aligned}
$$

There is a subtly with the connection that must be described carefully. Depending on the approach one takes, the pieces with derivatives acting on the vielbeins usually are addressed by exchanging the standard connection for the spin connection [76]. That is, in the standard string formulation it is well-known that the pull back of this world-sheet spin connection transforms as a gauge field, and it is well established for the undoubled sigma model that this gauge field does not contribute at one-loop to the Weyl anomaly because it is minimally coupled. But what about in the doubled formalism for the dual symmetric chiral string? Things are now slightly different. The gauge connection is no longer minimally coupled [39, 41], because in the doubled case the connection carries $O(D, D)$ rather than $O(D)$ indices. So this argument does not stand and we should find contributions from the gauge terms. Such contributions were for example explicitly found in [68]. However, taking into consideration the fact that we do not wish to assume $O(D, D)$ invariance from the outset, we must be careful in our approach. For now, we proceed cautiously with the generic calculation.

Finally, what we now want to do is replace all ξ^A fluctuations with their chiral frame counterparts $\xi^A = \mathcal{V}^A_{\bar{A}}\xi^{\bar{A}}$ in (4.12) and then pull the vielbeins through the derivatives. The cost being the addition of a few extra terms. The outcome is that, in addition to (4.12) in the chiral frame, we obtain the following extra terms to the second order Lagrangian

$$\begin{aligned}
&2L_{\mathcal{V}} = \\
&-2\mathcal{H}_{\bar{A}B}\partial_1\mathcal{V}^B_{\bar{B}}\xi^{\bar{B}}\partial_1\xi^{\bar{A}} - \mathcal{H}_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^A_{\bar{A}}\xi^{\bar{B}}\xi^{\bar{A}} - 2\partial_{\bar{A}}\mathcal{H}_{KB}\partial_1 X^K\partial_1\mathcal{V}^B_{\bar{B}}\xi^{\bar{A}}\xi^{\bar{B}} \\
&+ L_{\bar{A}B}\partial_1\mathcal{V}^B_{\bar{B}}\xi^{\bar{B}}\partial_0\xi^{\bar{A}} + L_{\bar{A}B}\partial_0\mathcal{V}^B_{\bar{B}}\xi^{\bar{B}}\partial_1\xi^{\bar{A}} + L_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_0\mathcal{V}^A_{\bar{A}}\xi^{\bar{B}}\xi^{\bar{A}}.
\end{aligned} \tag{4.15}$$

The outcome is therefore that we have two Lagrangians comprising the total second-order action for the background fields, one with vielbeins and one without. The main purpose and advantage of this strategy is to aid in finding the fluctuation propagators. In the chiral frame it is generally understood how to define and treat the propagators using the same techniques as in [12, 13]. One of the other advantages is that, as discussed in Section 4.3.1, the vielbein formalism assists in putting $\mathcal{H}(X)$ and $L(X)$ in canonical form. Finally, the chiral frame proves useful because we may approach the fluctuation contractions that will need to be calculated later as if the indices are in the metric frame [39].

Compared with the result for the background fields in [39], when including chiral frame indices in (4.12) in addition to the vielbein Lagrangian (4.15) we find a term for term match, with the exception that we don't have any base dependent terms.

4.3 Structure of the effective action

There are a number of ways to compute the quantum effective action. As we are using the background field method to perturbatively study ultra-violet divergences at one-loop, it follows that we may begin by thinking of the effective action with the standard formula [1]

$$e^{i\Gamma[X_0]} = \int \mathcal{D}\xi\ e^{i(S_2[X;\xi])}. \tag{4.16}$$

The effective action Γ is itself taken to be $\Gamma = S_{cl} + \Gamma_1$, where we define Γ_1 as the one-loop corrections. Here S_2 denotes the second-order action, and we may suppress the classical term in the effective action because ultimately it gets

[1] For simplicity we suppress $\hbar$.

pulled out of the integrand. We also recall as an aid that ξ is the tangent vector to the geodesic between X_0 and π, with ξ our fluctuation field. Additionally, from (4.16) we may obtain the loop propagator in momentum space. But for our present purposes, we focus here on how we can decompose S_2 into kinetic $S_{2,k}$ and interacting $S_{2,i}$ parts such that, schematically, $S_2 = S_{2,k} + S_{2,i}$. Doing so gives

$$e^{i\Gamma[X_0]} = \int \mathcal{D}\xi e^{i(S_{2,k}+S_{2,i})}. \tag{4.17}$$

We may now expand the interaction terms $e^{iS_{2,i}}$

$$e^{i\Gamma[X_0]} = \int \mathcal{D}\xi (1 + iS_{2,i} + \frac{i^2}{2} S_{2,i}^2 + ...) e^{iS_{2,k}}. \tag{4.18}$$

From (4.18) we can compute the equation explicitly for Γ_1. In doing so we drop the classical piece as well as any unneeded log terms that arise from the standard procedure of derivation in Quantum Field Theory. We find as a result

$$\Gamma_1 = \langle S_{2,i} \rangle + \frac{i}{2} \langle S_{2,i}^2 \rangle_{connected}. \tag{4.19}$$

It should be highlighted that we are only interested in *connected diagrams*, and the terms in brackets $\langle ... \rangle$ denote the expectation value with respect to the kinetic term $S_{2,k}$.

For the calculation of the Weyl and Lorentz anomalies we only need the divergent contributions to (4.19). We obtain these by calculating $\langle S_{2,i} \rangle$ which gives single contractions $\langle \xi\xi \rangle$, and $\langle S_{2,i}^2 \rangle_{connected}$ which corresponds with double contractions of the general schematic form $\langle \xi \partial_\mu \xi \xi \partial_\nu \xi \rangle$. For the double contractions, we will have different combinations of world-sheet derivatives, as will become clear later. Any terms of the form $\xi \partial \xi$ will contribute only to the logarithmic divergence at second order in this expansion of the effective action, and we are only interested in these logarithmic divergences.

4.3.1 General strategy

Now that we have expanded the background fields and derived the basic structure for the quantum effective action at one-loop, it is an opportune time to lay out our general strategy for calculating the beta-functionals. To achieve this, given the generic approach of this thesis, we shall need to return to the discussion about moving to the chiral frame in Section 4.2.3.

To begin let us note that in calculating the background field expansion to obtain (4.12) and (4.15), it was reasonable that we viewed $\mathcal{H}_{AB}$ as the metric on the totally doubled space. On the other hand, if in the approaches that use the DFT constraint the object L_{AB} is treated as an $O(D,D)$ invariant constant metric, we instead chose to treat L_{AB} as a generic 2-tensor. This means we have to be careful when it comes to arranging the kinetic terms for the fields ξ.

Moreover, if the approach in [39, 41, 42] is from the outset to take L (from the first quadrant in $\mathcal{L}$ as per 4.1) to have off-diagonal form such that $L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the motivation for doing so pertains to the fact that this definition of L aids in enabling $O(2D)$ rotations. Conventionally, the aim then is to use the Lorentz invariance condition to diagonalise L with ± 1 entries. The final advantage in taking these steps is that, after obtaining the background field expanded action, with $\mathcal{H}$ and L diagonal in the chiral frame, as alluded before it is well-known in the literature how to calculate the propagators for the fluctuations from the kinetic terms in the Lagrangian (4.12).

We do not wish to make these same assumptions. So we need to therefore obtain the same structure in principle - that is, we have to ensure $\mathcal{H}$ is diagonal and then also put this generic 2-tensor L into diagonal form - albeit without assuming $O(D,D)$ and without applying the chirality constraint. To this end, we pursue the following argument. It is observed in the Lagrangian (4.12) that the quadratic form $-\mathcal{H}_{AB}\partial_1\xi^A\partial_1\xi^B$ is not in canonical form, as we recall that $\mathcal{H}$ depends on X. The same is true for $L_{AB}\partial_0\xi^A\partial_1\xi^B$. Given that H and L can be simultaneously put into canonical form if we assume $O(D,D)$, but not otherwise, we simply put H in that form and for L we write

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{plus corrections.} \tag{4.20}$$

This is precisely of the same form L as found in the literature, with exception that it now carries corrections that vanish when reinstating the assumption of $O(D,D)$ invariance.

This means, firstly, that the vielbein Lagrangian for the background fields (4.15) should be updated. If we want to be as generic as possible, then the strategy on the level of the action is to ensure that for every instance in which $L_{\bar{A}\bar{B}}$ deviates from $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we treat it as a perturbation. We therefore define

$$L_{\bar{A}\bar{B}} = L^{(0)}_{\bar{A}\bar{B}} + \delta L_{\bar{A}\bar{B}}. \tag{4.21}$$

That we arrive at equation (4.21) is important. As we will see, an outcome of our approach will mean that we pick up δL correction terms in the total divergence of the theory. One may also speculate about the form and properties of this perturbation $\delta L_{\bar{A}\bar{B}}$; in this thesis we maintain a generic treatment and a study of its properties will be saved for future work. The immediate result of these steps is that the vielbein portion of the second-order background field expanded Lagrangian now takes on an additional correction term

$$\begin{aligned}
&2L_{\mathcal{V}} = \\
&-2\mathcal{H}_{\bar{A}B}\partial_1\mathcal{V}^B_{\bar{B}}\xi^{\bar{B}}\partial_1\xi^{\bar{A}} - \mathcal{H}_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^A_{\bar{A}}\xi^{\bar{B}}\xi^{\bar{A}} - 2\partial_{\bar{A}}\mathcal{H}_{KB}\partial_1 X^K\partial_1\mathcal{V}^B_{\bar{B}}\xi^{\bar{A}}\xi^{\bar{B}} \\
&L_{\bar{A}B}\partial_1\mathcal{V}^B_{\bar{B}}\xi^{\bar{B}}\partial_0\xi^{\bar{A}} + L_{\bar{A}B}\partial_0\mathcal{V}^B_{\bar{B}}\xi^{\bar{B}}\partial_1\xi^{\bar{A}} + L_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_0\mathcal{V}^A_{\bar{A}}\xi^{\bar{B}}\xi^{\bar{A}} \\
&+\delta L_{\bar{A}\bar{B}}\partial_1\xi^{\bar{A}}\partial_0\xi^{\bar{B}}.
\end{aligned} \tag{4.22}$$

With the inclusion of δL this final expression for the vielbein Lagrangian is clearly different than in [39].

4.4 Master expression

4.4.1 Kinetic terms and interactions terms

Key to the stratgey described in the previous section concerns how the definition of the L in (4.20) means that we obtain the correct kinetic terms of the theory, given that in using the vielbein basis we may decompose the action into kinetic and interaction terms for the tangent-space fields $\xi^{\bar{A}}$. The kinetic term for the fluctuations reads [39] as follows

$$S_{2,k} = \frac{1}{2}\int d^2\sigma(-\mathcal{H}_{\bar{A}\bar{B}}\partial_1\xi^{\bar{A}}\partial_1\xi^{\bar{B}} + L_{\bar{A}\bar{B}}\partial_1\xi^{\bar{A}}\partial_0\xi^{\bar{B}}), \tag{4.23}$$

with $\mathcal{H}$ and L defined as described above. This kinetic term is the sum of Floreanini-Jackiw style Lagrangians (3.34) for n chiral and n anti-chiral bosons.

As for the interaction terms $S_{2,i}$ observed in the quantum effective action (4.19), we approach their calculation generically following [42]. In doing so, we are concerned for the time being only with the updated vielbein Lagrangian (4.22), because as we see the Lagrangian (4.12) in the chiral frame gives only two interaction terms without vielbeins. When calculating the square of $S_{2,i}$ these non-vielbein terms do not cross multiply with the purely vielbein pieces found in (4.22). So for now, we put the non-vielbein pieces to the side and note that it will prove straightforward to read them off and substitute them when later calculating the general tensor structure for the total divergence. Henceforth, we write $S_{2,i}$ in schematic form by reading off the structure from the updated second order vielbein Lagrangian (4.22) and include $\delta\mathcal{L}$ corrections

$$S_{2,i} = \frac{1}{2}\int d^2\sigma \; (S_{\bar{A}\bar{B}}\xi^{\bar{A}}\xi^{\bar{B}} + Q_{\bar{A}\bar{B}}\xi^{\bar{A}}\partial_1\xi^{\bar{B}} + P_{\bar{A}\bar{B}}\xi^{\bar{A}}\partial_0\xi^{\bar{B}} \\ + \delta L_{\bar{A}\bar{B}}\partial_1\xi^{\bar{A}}\partial_0\xi^{\bar{B}}), \tag{4.24}$$

where we may make explicit the following definitions:

$$S_{\bar{A}\bar{B}} \coloneqq S^{11}_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^A_{\bar{A}} + \partial_{\bar{A}}S^{11}_{\gamma B}\partial_1 X^\gamma\partial_1\mathcal{V}^B_{\bar{B}} + S^{01}_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_0\mathcal{V}^A_{\bar{A}},$$

$$Q_{\bar{A}\bar{B}} \coloneqq Q^1_{\bar{A}B}\partial_1\mathcal{V}^B_{\bar{B}} + Q^0_{\bar{A}B}\partial_0\mathcal{V}^B_{\bar{B}},$$

$$P_{\bar{A}\bar{B}} \coloneqq P^1_{\bar{A}B}\partial_1\mathcal{V}^B_{\bar{B}}. \tag{4.25}$$

The use of upper indices in the definitions of $S, Q,$ and P will aid in keeping track of the derivative terms when we make the appropriate substitutions for $S_{2,i}$.

We see already a general tensor structure emerging in (4.25). When we take the square of the terms in (4.24) to obtain the second term in (4.19), it is clear that we will obtain combinations of $S, Q, P,$ and δL as well as their associated fluctuations. One can therefore foresee how in the product of these terms we will eventually obtain fluctuation contractions in the form of 4-point functions with mixed spacetime derivatives.

4.4.2 Master expression for the effective action

With all of the results from the last few sections at hand, we can now start putting things together. When we substitute (4.24) into (4.19) we obtain a lengthy list of terms, not all of which are relevant. Only $\xi^{\bar{A}}\xi^{\bar{B}}$ terms appear at linear order in (4.19), so we only pick up $S_{\bar{A}\bar{B}}$ terms here. This is because at second order in (4.19) the square of $S_{\bar{A}\bar{B}}$ appears and comes with $\xi^{\bar{A}}\xi^{\bar{B}}\xi^{\bar{C}}\xi^{\bar{D}}$, but, upon explicitly introducing the propagator in the next section, we see this amounts to its square. Additionally, we are not interested in the expansion in higher order derivatives, so we drop these and other similar terms. The result is that at quadratic order in (4.19) only combinations of $\xi^{\bar{A}}\partial_\mu\xi^{\bar{B}}\xi^{\bar{C}}\partial_\mu\xi^{\bar{D}}$ appear.

Full definition of the 4-point correlation functions that we obtain will be provided in the next section. For now, as an aid, we introduce the following short hand

$$\langle\xi^{\bar{A}}\xi^{\bar{B}}\rangle \equiv \langle\xi^{\bar{A}}(\sigma)\xi^{\bar{B}}(\sigma)\rangle$$

and

$$\langle\xi^{\bar{A}}\partial_\mu\xi^{\bar{B}}\xi^{\bar{C}}\partial_\nu\xi^{\bar{D}}\rangle \equiv i\int d^2\sigma\ \langle\xi^{\bar{A}}(\sigma)\partial_\mu\xi^{\bar{B}}(\sigma)\xi^{\bar{C}}(\sigma')\partial_\nu\xi^{\bar{D}}(\sigma')\rangle.$$

Now, finally making the substitution of (4.24) at first order $\langle S_{2,i}\rangle$ and second order $\langle S_{2,i}\rangle$ in the effective action (4.19), we obtain the generic structure

$$\Gamma := \frac{1}{2}\int d^2\sigma\ \Gamma^{00} + \Gamma^{01} + \Gamma^{11} + \delta L \text{ corrections} + \Gamma_{Non-vielbein}. \tag{4.26}$$

After organising each collection of relevant terms, the master expression for the effective action can be written as follows

$$\Gamma^{00} := \frac{1}{4} Q^0_{\bar{A}B} Q^0_{\bar{C}D} \partial_0 \mathcal{V}^B_{\bar{B}} \partial_0 \mathcal{V}^D_{\bar{D}} \langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \xi^{\bar{C}} \partial_1 \xi^{\bar{D}} \rangle,$$

$$\Gamma^{01} :=$$
$$S^{01}_{AB} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_0 \mathcal{V}^A_{\bar{A}} \langle \xi^{\bar{A}} \xi^{\bar{B}} \rangle + \frac{1}{2} Q^1_{\bar{A}B} Q^0_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_0 \mathcal{V}^D_{\bar{D}} \langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \xi^{\bar{C}} \partial_1 \xi^{\bar{D}} \rangle$$
$$+ \frac{1}{2} Q^0_{\bar{A}B} P^1_{\bar{C}D} \partial_0 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} \langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle,$$

$$\Gamma^{11} :=$$
$$S^{11}_{AB} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^A_{\bar{A}} \langle \xi^{\bar{A}} \xi^{\bar{B}} \rangle + \partial_{\bar{K}} S^{11}_{AB} \partial_1 X^A \partial_1 \mathcal{V}^B_{\bar{B}} \langle \xi^{\bar{K}} \xi^{\bar{B}} \rangle$$
$$+ \frac{1}{4} Q^1_{\bar{A}B} Q^1_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} \langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \xi^{\bar{C}} \partial_1 \xi^{\bar{D}} \rangle$$
$$+ \frac{1}{2} Q^1_{\bar{A}B} P^1_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} \langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle$$
$$+ \frac{1}{4} P^1_{\bar{A}B} P^1_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} \langle \xi^{\bar{A}} \partial_0 \xi^{\bar{B}} \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle,$$

$$\delta L_{corrections} :=$$
$$\frac{1}{2} Q^1_{\bar{A}B} \delta L_{\bar{C}\bar{D}} \partial_1 \mathcal{V}^B_{\bar{B}} \langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle + \frac{1}{2} Q^0_{\bar{A}B} \delta L_{\bar{C}\bar{D}} \partial_0 \mathcal{V}^B_{\bar{B}} \langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle$$
$$+ \frac{1}{2} P^1_{\bar{A}B} \delta L_{\bar{C}\bar{D}} \partial_1 \mathcal{V}^B_{\bar{B}} \langle \xi^{\bar{A}} \partial_0 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle + \frac{1}{4} \delta L_{\bar{A}\bar{B}} \delta L_{\bar{C}\bar{D}} \langle \partial_1 \xi^{\bar{A}} \partial_0 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle,$$

and lastly

$$\Gamma_{Non-vielbein} :=$$
$$\partial_{\bar{N}} \mathcal{H}_{G\bar{B}} \partial_1 X^G \partial_{\bar{K}} \mathcal{H}_{D\bar{E}} \partial_1 X^D \langle \xi^{\bar{N}} \partial_1 \xi^{\bar{B}} \xi^{\bar{K}} \partial_1 \xi^{\bar{E}} \rangle$$
$$- \frac{1}{2} \partial_N \partial_K \mathcal{H}_{GD} \partial_1 X^G \partial_1 X^D \langle \xi^N \xi^K \rangle. \tag{4.27}$$

The total list of terms comprising (4.27) is the most completely general expression for the quantum effective action that we can write. To make complete sense of what this master expression is saying, we will need to calculate all of the fluctuation contractions denoted by the 4-point functions for each combination of world-sheet derivatives. We also have simple propagator contractions to calculate for the S_{AB} terms. What aids us in calculating the more complication 4-point functions is that they can be broken down into a pair of propagator contractions with derivatives acting on the fluctuation fields. A number of these contractions were originally calculated in [39], and we calculate them again to ensure accuracy. The inclusion of δL corrections also give completely new correlation functions with three and four derivatives that we will have to calculate.

After calculating all of the correlation functions, we will also need to substitute for the explicit values of $S, Q,$ and P by reading them off from the final form of the purely vielbein Lagrangian (4.22). Then, as a final step, we need to substitute all of the results into the master expression (4.27), which will leave some lengthy tensor algebra that we will need to calculate. We will start by first calculating the fluctuation contractions, which means we need to finally introduce the propagators of the theory.

4.5 Propagators

The propagators of the theory were first derived by Tseytlin [13] and were more recently reviewed in [34, 39]. A full derivation is not offered here due to lack of space; but we note that the quickest pathway is to generalise from the Floreanini-Jackiw style kinetic terms, beginning with a standard 2-dimensional boson with a kinetic Lagrangian of the form $\mathcal{L}_0 = \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi$. From this one can basically read off the propagator.

As verified in the literature, the sum of a chiral and anti-chiral propagator is directly proportional to the standard boson propagator

$$\Delta_0(\sigma - \sigma') = i\int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2}e^{ip(\sigma-\sigma')}.$$

The difference, on the other hand, gives a phase θ, the definition of which is also taken to Δ_0 [39].

To calculate the contractions appearing in (4.27) we require the boson propagator corresponding to the kinetic Lagrangian (4.23). Here one makes use of the diagonal form of the $\mathcal{H}$ and L matrices as discussed in Section 4.3.1. The full propagator takes the form

$$\begin{aligned}\Delta^{\bar{A}\bar{B}}(\sigma - \sigma') &= \langle \xi^{\bar{A}}(\sigma)\xi^{\bar{B}}(\sigma')\rangle = \mathcal{H}^{\bar{A}\bar{B}}\Delta(\sigma - \sigma') + L^{\bar{A}\bar{B}}\theta(\sigma - \sigma') \\ &= \frac{1}{2}(\mathcal{H} + L)^{\bar{A}\bar{B}}\Delta_+ + \frac{1}{2}(\mathcal{H} - L)^{\bar{A}\bar{B}}\Delta_-,\end{aligned} \tag{4.28}$$

where

$$\Delta(\sigma-\sigma') = \frac{1}{2}(\Delta_+(\sigma-\sigma')+\Delta_-(\sigma-\sigma')) = -\frac{1}{4\pi}\ln(\sigma-\sigma')^2 = 2\Delta_0(\sigma,\sigma'),$$

$$\theta(\sigma-\sigma') = \frac{1}{2}(\Delta_+(\sigma-\sigma')-\Delta_-(\sigma-\sigma')) = -\frac{1}{2\pi}\operatorname{arctanh}\frac{\sigma-\sigma'}{\bar\sigma-\bar\sigma'} = 2\theta(\sigma,\sigma'). \tag{4.29}$$

In this thesis we also follow the conventional notation in which

$$\Delta_+(\sigma,\sigma') = -\frac{1}{2\pi}\ln(\sigma-\sigma'),\ \Delta_-(\sigma,\sigma') = -\frac{1}{2\pi}\ln(\bar\sigma-\bar\sigma'). \tag{4.30}$$

Explicitly stated, in (4.28) what we have are odd and even combinations of the chiral propagators [13]. In (4.29) we observe a fairly intuitive structure. The coefficients of Δ_0 contain the UV divergence that requires regularisation and normalisation, which means that in the path integral of the effective action the term proportional to Δ_0 will contribute to the Weyl anomaly (these terms are scale sensitive and therefore related to the breakdown of Weyl invariance). On the other hand, θ does not contain any divergence. Terms proportional to it will correspond to a breakdown in world-sheet Lorentz invariance (these terms are sensitive to rotation by phase shifts and hence contribute to parametrising the Lorentz anomaly). In [39], although not explicitly computed the claim is that all occurrences of θ in the effective action ultimately cancel, which means there is no trace of a Lorentz anomaly. The explanation is that this has to do with having 'an equal number of Bosons of each chirality', which is owed to the assumption of $O(D,D)$ invariance. In this thesis we are of course not assuming $O(D,D)$ and so we will preserve θ throughout and calculate these terms explicitly.

4.6 Fluctuation contractions

Given the definition of the propagator in (4.28), the single contraction terms $\langle\xi\xi\rangle$ that appear in (4.27) are simple propagator contractions. So we may make a straightforward substitution when the time comes. The double contraction terms, on the other hand, are found using Wick's theorem. Following the definitions in [39], when calculating the contractions we use the fact that in momentum space $\Delta_+ = \frac{1}{p_1 p_-}$ and $\Delta_- = -\frac{1}{p_1 p_+}$. Then when calculating the momentum integrals we use the identity

$$\frac{p_0}{p_1}\frac{1}{p_\pm^2} = \frac{1}{p_1 p_\pm} \mp \frac{1}{p_\pm^2},$$

and we discard integrals of the form $\frac{1}{p_\pm^2}$.

4.6.1 Original fluctuation contractions

Due to limited space, we will not rederive here all of the fluctuation contractions originally calculated in [39]. However, we should note that in recalculating these 4-point functions we did find a number of corrections in the form of missing factors. The impact of these corrections is that we have picked up additional factors as featured in the following corrected results:

$$\langle \xi^{\bar{A}}\partial_1\xi^{\bar{B}}\xi^{\bar{C}}\partial_1\xi^{\bar{D}}\rangle \sim (\mathcal{H}^{\bar{A}[\bar{C}}\mathcal{H}^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\Delta_0,$$

$$\langle \xi^{\bar{A}}\partial_0\xi^{\bar{B}}\xi^{\bar{C}}\partial_0\xi^{\bar{D}}\rangle \sim -(\mathcal{H}^{\bar{A}[\bar{C}}\mathcal{H}^{\bar{D}]\bar{B}} + 3L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\Delta_0 - 2(\mathcal{H}^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}} + L^{\bar{A}[\bar{C}}\mathcal{H}^{\bar{D}]\bar{B}})\theta,$$

$$\langle \xi^{\bar{A}}\partial_1\xi^{\bar{B}}\xi^{\bar{C}}\partial_0\xi^{\bar{D}}\rangle \sim -(\mathcal{H}^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}} + L^{\bar{A}[\bar{C}}\mathcal{H}^{\bar{D}]\bar{B}})\Delta_0 - 2L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}}\theta. \tag{4.31}$$

where $\sim$ represents up to finite terms. We use the convention $Q^{[C}Q^{D]} := \frac{Q^CQ^D - Q^DQ^C}{2}$. As the above contractions involve propagators of fields evaluated at a coincident point, this means one requires a regularisation prescription to handle any pathologies. We note that the propagators can be written in z-space upon Wick rotation. We follow the conventions of [13, 39], where $z = \sigma + i\tau = \sigma + \tau$ and $\partial_\sigma = \partial + \bar{\partial}$, $\partial_\tau = \partial - \bar{\partial}$. Then using $z \to 0$ regularisation, for the propagator we have the standard result $\bar{\partial}z^{-1} = \pi\delta^{(2)}(z)$.

4.6.2 New fluctuation contractions

In comparison with [39], we must also calculate the 3 new propagator contractions that come with the inclusion of δL correction terms. We offer one example in detail, using the procedure to calculate the correlation functions in (4.35), and then quote the other two results.

Example: Calculate $\langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle$

For example, consider the following 4-point function as found in the δL corrections in (4.27)

$$i \int d^2\sigma' \langle \xi^{\bar{A}}(\sigma_A) \partial_1 \xi^{\bar{B}}(\sigma_B) \partial_1 \xi^{\bar{C}}(\sigma'_C) \partial_0 \xi^{\bar{D}}(\sigma'_D) \rangle. \tag{4.32}$$

Let us for the time being drop the bar notation on the chiral frame indices. We will reinstate this notation after. To calculate this 4-point function, we use the standard Wick procedure and we are only interested in *connected diagrams*. Additionally, when using Wick contraction notation we will simply state the indices to denote which terms are being contracted. We also use $\sigma_{A,B,C,D}$ notation in the argument for the fluctuation fields as a bookkeeping device. Therefore, in using the Wick procedure we find

$$i \int d^2\sigma' \langle \xi^{A}(\sigma_A) \partial_1 \xi^{B}(\sigma_B) \partial_1 \xi^{C}(\sigma'_C) \partial_0 \xi^{D}(\sigma'_D) \rangle = ACBD + ADBC, \tag{4.33}$$

where $ACBD + ADBC$ is the sum of contractions we must compute.

Let us first consider the $ACBD$ contractions. As we may pull out the derivatives, what we end up with are simple propagator contractions. We may also at this point substitute for the definition of the propagator. To simplify notation, we use $\int_p$ to denote the more complete expression for the integration measure $i \int \frac{d^2p}{(2\pi)^2}$, understanding also that each integral comes with a factor of i. Taking the above steps the result is

$$ACBD = \partial\sigma_1^C \partial\sigma_1^B \partial\sigma_0^D \; i^3 \int_{\sigma'} \int_p \int_q \Delta^{AC}(p) \Delta^{BD}(q) e^{ip\cdot(\sigma_A - \sigma'_C)} e^{iq\cdot(\sigma_B - \sigma'_D)}. \tag{4.34}$$

When computing the derivatives, we should be careful keeping track of the i's that come down from the exponential. We also obtain i terms from the two propagator integrals, hence the factor of i^3 out front. After calculating the derivatives in (4.34) we find

$$\overbracket{AC}\,\overbracket{BD} = i^3 \int_{\sigma'} \int_p \int_q \Delta^{AC}(p)\Delta^{BD}(q)\,(-ip_1)(iq_1)(-iq_0)\, e^{ip\cdot(\sigma_A - \sigma'_C)} e^{iq\cdot(\sigma_B - \sigma'_D)}. \tag{4.35}$$

The σ' integral sitting out front gives a delta function $(2\pi)^2\delta(p+q)$, just as it does for the two derivative correlation functions calculated in (4.31). In fact, what we obtain is the same general structure as that when calculating the left-hand side of (4.31), with the only exception being that we have an extra derivative which brings down an extra momentum term from the exponential.

Since we are only calculating propagators around a single loop, we can change the variable $\Delta^{BD}(q) = \Delta^{BD}(-p)$. This results in a cancellation of the negatives signs on the p's. Let us also collect all i's and move them outside the integrand. We end up with,

$$\overbracket{AC}\,\overbracket{BD} = i^6 \int_p \Delta^{AC}(p)\Delta^{BD}(-p)\,(p_1 p_1 p_0). \tag{4.36}$$

We may perform the same procedure for $\overbracket{AD}\,\overbracket{BC}$ in (4.33). We end up with,

$$\overbracket{AD}\,\overbracket{BC} = i^6 \int_p \Delta^{AD}(p)\Delta^{BC}(-p)\,(p_0 p_1 p_1). \tag{4.37}$$

The next step is to collect all terms and then substitute for the propagators given the definition (4.28). Dropping Wick contraction notation and making the appropriate substitutions we find

$$ACBD + ADBC =$$
$$i^6 \int_p [\Delta^{AC}(p)\Delta^{BD}(-p)\,(p_1^2 p_0)] + i^6 \int_p [\Delta^{AD}(p)\Delta^{BC}(-p)\,(p_0 p_1^2)]$$

$$= i^6 \int_p \left[\frac{1}{2}(\mathcal{H}+L)^{AC}\Delta_+ + \frac{1}{2}(\mathcal{H}-L)^{AC}\Delta_-\right] \times \left[\frac{1}{2}(\mathcal{H}+L)^{BD}\Delta_+ + \frac{1}{2}(\mathcal{H}-L)^{BD}\Delta_-\right] p_1^2 p_0$$
$$+ i^6 \int_p \left[\frac{1}{2}(\mathcal{H}+L)^{AD}\Delta_+ + \frac{1}{2}(\mathcal{H}-L)^{AD}\Delta_-\right] \times \left[\frac{1}{2}(\mathcal{H}+L)^{BC}\Delta_+ + \frac{1}{2}(\mathcal{H}-L)^{BC}\Delta_-\right] p_0 p_1^2$$

$$
\begin{aligned}
&= i^6 \int_p \frac{1}{4} \Big[(\mathcal{H}+L)^{AC}(\mathcal{H}+L)^{BD}\Delta_+\Delta_+ + (\mathcal{H}+L)^{AC}(\mathcal{H}-L)^{BD}\Delta_+\Delta_- \\
&\quad + (\mathcal{H}+L)^{BD}(\mathcal{H}-L)^{AC}\Delta_+\Delta_- + (\mathcal{H}-L)^{AC}(\mathcal{H}-L)^{BD}\Delta_-\Delta_-\Big] p_1^2 p_0 \\
&\quad + i^6 \int_p \frac{1}{4} \Big[(\mathcal{H}+L)^{AD}(\mathcal{H}+L)^{BC}\Delta_+\Delta_+ + (\mathcal{H}+L)^{AD}(\mathcal{H}-L)^{BC}\Delta_+\Delta_- \\
&\quad + (\mathcal{H}+L)^{BC}(\mathcal{H}-L)^{AD}\Delta_+\Delta_- + (\mathcal{H}-L)^{AD}(\mathcal{H}-L)^{BC}\Delta_-\Delta_-\Big] p_0 p_1^2. \qquad (4.38)
\end{aligned}
$$

When we substitute for the propagators to obtain (4.38), the structure is precisely the same as that which we will always find for any combination of world-sheet derivatives. To simplify matters, we may observe that the second integrand is the same as the first just with $A \leftrightarrow B$. Finally, we can also substitute for the combinations of $\mathcal{H}$ and L. The result as follows

$$
\begin{aligned}
ACBD + ADBC = \frac{i^6}{4}\int_p [&\mathcal{H}^{AC}\mathcal{H}^{BD}(\frac{1}{p_1 p_-}\frac{1}{p_1 p_-}p_1^2 p_0) \qquad (4.39)\\
&+ \mathcal{H}^{AC}L^{BD}(-\frac{1}{p_1 p_-}\frac{1}{p_1 p_+}p_1^2 p_0) \\
&+ \mathcal{H}^{BD}L^{AC}(-\frac{1}{p_1 p_-}\frac{1}{p_1 p_+}p_1^2 p_0) \\
&+ L^{AC}L^{BD}(\frac{1}{p_1 p_+}\frac{1}{p_1 p_+}p_1^2 p_0)] \\
&+ \ (A \leftrightarrow B).
\end{aligned}
$$

The mixed pieces in (4.39) become negative. We can also account for the $i^6 = -1$ out front, so the signs in the integrand get flipped.

$$
\begin{aligned}
= \frac{1}{4}\int_p [&-\mathcal{H}^{AC}\mathcal{H}^{BD}(\frac{1}{p_1 p_-}\frac{1}{p_1 p_-}p_1^2 p_0) \qquad (4.40)\\
&+\mathcal{H}^{AC}L^{BD}(\frac{1}{p_1 p_-}\frac{1}{p_1 p_+}p_1^2 p_0) \\
&+\mathcal{H}^{BD}L^{AC}(\frac{1}{p_1 p_-}\frac{1}{p_1 p_+}p_1^2 p_0) \\
&-L^{AC}L^{BD}(\frac{1}{p_1 p_+}\frac{1}{p_1 p_+}p_1^2 p_0)] \\
&+ (A \leftrightarrow B).
\end{aligned}
$$

In (4.40) we find that the momentum integrals for the non-mixed terms vanish. For instance, if $p_-^2 = (p_0 - p_1)^2$ then $\int_{-\infty}^{\infty} dp_0 dp_1 \frac{p_0}{(p_0 - p_1)^2} = 0$ with the integral odd in p. The same is true for the case of p_+^2. So we are left only with momentum integrals for the mixed pieces,

$$
\begin{aligned}
= \frac{1}{4}\int_p [\mathcal{H}^{AC} L^{BD} (\frac{1}{p_1 p_-}\frac{1}{p_1 p_+} p_1^2 p_0) \\
+\mathcal{H}^{BD} L^{AC} (\frac{1}{p_1 p_-}\frac{1}{p_1 p_+} p_1^2 p_0)] \\
+ (A \leftrightarrow B)
\end{aligned}
\tag{4.41}
$$

$$
\begin{aligned}
\Longrightarrow \int_p \frac{1}{2}\mathcal{H}^{AC} L^{BD} \ \frac{1}{p_1 p_-}\frac{1}{p_1 p_+} p_1^2 p_0 \\
+ (A \leftrightarrow B).
\end{aligned}
\tag{4.42}
$$

However, in the analysis of this remaining momentum integral it is found that

$$
\begin{aligned}
&\int_p \frac{1}{p_1 p_-}\frac{1}{p_1 p_+} p_1^2 p_0 \\
&= \int_p \frac{p_0}{p_- p_+} \\
&= \int_p \frac{p_0}{p_0^2 - p_1^2} = 0.
\end{aligned}
\tag{4.43}
$$

In conclusion $\langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle = 0$, so there are no corrections from this 4-point function that contribute to the effective action. For similar reasons, we find that $\langle \xi^{\bar{A}} \partial_0 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle = 0$, while the 4-point function $\langle \partial_1 \xi^{\bar{A}} \partial_0 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle$ with 4 derivatives does not trivially cancel. In summary, we find only one of the correlation functions that arise as a result of the inclusion of δL to contribute to the effective action. The full list of results is given below:

$\langle \xi^{\bar{A}} \partial_1 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle = 0,$

$$\langle \xi^{\bar{A}} \partial_0 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle = 0,$$

$$\langle \partial_1 \xi^{\bar{A}} \partial_0 \xi^{\bar{B}} \partial_1 \xi^{\bar{C}} \partial_0 \xi^{\bar{D}} \rangle = \frac{1}{2} (H^{A[C} H^{D]B} - L^{A[C} L^{D]B}) \int_p \frac{1}{p_1^2 p_- p_+} p_1^2 p_0^2. \tag{4.44}$$

For the final 4-point function in (4.44) we have a remaining non-trivial divergent momentum integral that we will need to calculate. We will address this in Section 4.7.1.

4.7 The Weyl anomaly and the Lorentz anomaly

4.7.1 Final tensor structure

With several steps in the larger calculation complete, we now have everything that we need to begin making our final substitutions. This will result in a final generic tensor structure from which we will need to sort through all of the algebra in order to then obtain a conclusive expression for the Weyl and Lorentz anomaly terms. Importantly, in addition to the factors that we picked up in arriving at (4.27), when we substitute for the propagator contractions we must be careful to pay attention to any changes in sign and any additional factors. One sees, moreover, that with some of the fluctuation contractions (4.31) and (4.44) we pick up an overall negative sign for some of the Weyl anomaly pieces and some of the Lorentz anomaly pieces.

Substituting (4.31) and (4.44) into (4.27) we find

$$\Gamma^{00} := \frac{1}{4}Q^{0}_{\bar{A}B}Q^{0}_{\bar{C}D}\partial_0\mathcal{V}^{B}_{\bar{B}}\partial_0\mathcal{V}^{D}_{\bar{D}}(H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\Delta_0,$$

$$\begin{aligned}\Gamma^{01} &:= S^{01}_{AB}\partial_1\mathcal{V}^{B}_{\bar{B}}\partial_0\mathcal{V}^{A}_{\bar{A}}(H^{\bar{A}\bar{B}}\Delta_0 + L^{\bar{A}\bar{B}}\theta)\\ &+ \frac{1}{2}Q^{1}_{\bar{A}B}Q^{0}_{\bar{C}D}\partial_1\mathcal{V}^{B}_{\bar{B}}\partial_0\mathcal{V}^{D}_{\bar{D}}(H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\Delta_0\\ &+ \frac{1}{2}Q^{0}_{\bar{A}B}P^{1}_{\bar{C}D}\partial_0\mathcal{V}^{B}_{\bar{B}}\partial_1\mathcal{V}^{D}_{\bar{D}}[-(H^{\bar{A}[\bar{C}}L^{\bar{D}]B} + L^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}})\Delta_0 - 2L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}}\theta],\end{aligned}$$

$$\begin{aligned}\Gamma^{11} &:= S^{11}_{AB}\partial_1\mathcal{V}^{B}_{\bar{B}}\partial_1\mathcal{V}^{A}_{\bar{A}}(H^{\bar{A}\bar{B}}\Delta_0 + L^{\bar{A}\bar{B}}\theta) + \partial_{\bar{K}}S^{11}_{AB}\partial_1 X^{A}\partial_1\mathcal{V}^{B}_{\bar{B}}(H^{\bar{K}\bar{B}}\Delta_0 + L^{\bar{K}\bar{B}}\theta)\\ &+ \frac{1}{4}Q^{1}_{\bar{A}B}Q^{1}_{\bar{C}D}\partial_1\mathcal{V}^{B}_{\bar{B}}\partial_1\mathcal{V}^{D}_{\bar{D}}(H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\Delta_0\\ &+ \frac{1}{2}Q^{1}_{\bar{A}B}P^{1}_{\bar{C}D}\partial_1\mathcal{V}^{B}_{\bar{B}}\partial_1\mathcal{V}^{D}_{\bar{D}}[-(H^{\bar{A}[\bar{C}}L^{\bar{D}]B} + L^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}})\Delta_0 - 2L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}}\theta]\\ &+ \frac{1}{4}P^{1}_{\bar{A}B}P^{1}_{\bar{C}D}\partial_1\mathcal{V}^{B}_{\bar{B}}\partial_1\mathcal{V}^{D}_{\bar{D}}[-(H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} + 3L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\Delta_0 - 2(H^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}} + L^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}})\theta],\end{aligned}$$

$$\delta L_{corrections} := \frac{1}{2}\delta L_{\bar{A}\bar{B}}\delta L_{\bar{C}\bar{D}}\left[(H^{A[C}H^{D]B} - L^{A[C}L^{D]B})\int_p \frac{1}{p_{-}p_{+}}p_0^2\right],$$

$$\begin{aligned}\Gamma_{non-vielbein} &:= -\frac{1}{2}\partial_{\bar{N}}\partial_{\bar{K}}\mathcal{H}_{GD}\partial_1 X^{G}\partial_1 X^{D}(H^{\bar{N}\bar{K}}\Delta_0 + L^{\bar{N}\bar{K}}\theta)\\ &+ \partial_{\bar{N}}\mathcal{H}_{G\bar{B}}\partial_1 X^{G}\partial_{\bar{K}}\mathcal{H}_{D\bar{E}}\partial_1 X^{D}(H^{\bar{N}[\bar{K}}H^{\bar{E}]\bar{B}} - L^{\bar{N}[\bar{K}}L^{\bar{E}]\bar{B}})\Delta_0. \qquad (4.45)\end{aligned}$$

From the definition of the propagator of the theory we may now decompose the above list of terms such that its coefficients are written in terms of parts that relate to the Weyl anomaly $\Delta(0)$ (breakdown of scale invariance) and the Lorentz anomaly $\theta(0)$ (breakdown of Lorentz invariance). Schematically, the effective action coefficients may be organised in the form

$$\Gamma^{\mu\nu}_{IJ} := \frac{1}{2}\int d^2\sigma\ \mathbb{W}^{\mu\nu}_{IJ}\Delta(0) + \frac{1}{2}\int d^2\sigma\ \mathbb{L}^{\mu\nu}_{IJ}\theta(0), \qquad (4.46)$$

where $\mathbb{W}$ and $\mathbb{L}$ are the total Weyl and Lorentz divergences, respectively. The implication is that if the present model (or any double string model) is to describe a consistent string background, it is required that the corresponding parts of the effective action vanish on-shell. We find that the Weyl anomaly is therefore given by

$$\mathbb{W}^{00}_{IJ} \coloneqq \frac{1}{4} Q^0_{\bar{A}\bar{B}} Q^0_{\bar{C}D} \partial_0 \mathcal{V}^B_{\bar{B}} \partial_0 \mathcal{V}^D_{\bar{D}} (H^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}}) \Delta_0,$$

$$\begin{aligned}
\mathbb{W}^{01}_{IJ} \coloneqq & S^{01}_{AB} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_0 \mathcal{V}^A_{\bar{A}} \; H^{\bar{A}\bar{B}} \Delta_0 \\
& + \frac{1}{2} Q^1_{\bar{A}B} Q^0_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_0 \mathcal{V}^D_{\bar{D}} (H^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}}) \Delta_0 \\
& - \frac{1}{2} Q^0_{\bar{A}B} P^1_{\bar{C}D} \partial_0 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} (H^{\bar{A}[\bar{C}} L^{\bar{D}]B} + L^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}}) \Delta_0,
\end{aligned}$$

$$\begin{aligned}
\mathbb{W}^{11}_{IJ} \coloneqq & S^{11}_{AB} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^A_{\bar{A}} \; H^{\bar{A}\bar{B}} \Delta_0 + \partial_{\bar{K}} S^{11}_{AB} \partial_1 X^A \partial_1 \mathcal{V}^B_{\bar{B}} H^{\bar{K}\bar{B}} \Delta_0 \\
& + \frac{1}{4} Q^1_{\bar{A}B} Q^1_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} (H^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}}) \Delta_0 \\
& - \frac{1}{2} Q^1_{\bar{A}B} P^1_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} [(H^{\bar{A}[\bar{C}} L^{\bar{D}]B} + L^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}}) \Delta_0 \\
& - \frac{1}{4} P^1_{\bar{A}B} P^1_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} (H^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}} + 3 L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}}) \Delta_0 \\
& - \frac{1}{2} \partial_{\bar{N}} \partial_{\bar{K}} \mathcal{H}_{GD} \partial_1 X^G \partial_1 X^D \; H^{\bar{N}\bar{K}} \Delta_0 \\
& + \partial_{\bar{N}} \mathcal{H}_{G\bar{B}} \partial_1 X^G \partial_{\bar{K}} \mathcal{H}_{D\bar{E}} \partial_1 X^D (H^{\bar{N}[\bar{K}} H^{\bar{E}]\bar{B}} - L^{\bar{N}[\bar{K}} L^{\bar{E}]\bar{B}}) \Delta_0.
\end{aligned} \tag{4.47}$$

For the Lorentz anomaly we find $\mathbb{L}^{00}_{IJ} = 0$ and

$$\begin{aligned}
\mathbb{L}^{01}_{IJ} \coloneqq & S^{01}_{AB} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_0 \mathcal{V}^A_{\bar{A}} L^{\bar{A}\bar{B}} \theta \\
& - Q^0_{\bar{A}\bar{B}} P^1_{\bar{C}\bar{D}} \partial_0 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^B_{\bar{B}} \; L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}} \theta,
\end{aligned}$$

$$\begin{aligned}
\mathbb{L}^{11}_{IJ} \coloneqq & S^{11}_{AB} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^A_{\bar{A}} L^{\bar{A}\bar{B}} \theta \\
& - Q^1_{\bar{A}B} P^1_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} \; L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}} \theta \\
& - \frac{1}{2} P^1_{\bar{A}B} P^1_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} (H^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}} + L^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}}) \theta \\
& + \partial_{\bar{K}} S^{11}_{AB} \partial_1 X^A \partial_1 \mathcal{V}^B_{\bar{B}} \; L^{\bar{K}\bar{B}} \theta \\
& - \frac{1}{2} \partial_{\bar{N}} \partial_{\bar{K}} \mathcal{H}_{GD} \partial_1 X^G \partial_1 X^D \; L^{\bar{N}\bar{K}} \theta.
\end{aligned} \tag{4.48}$$

Finally, we have the remaining δL corrections which we will look at separately,

$$\delta L_{corrections} \coloneqq \frac{1}{2} \delta L_{\bar{A}\bar{B}} \delta L_{\bar{C}\bar{D}} [(H^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}}) \int_p \frac{1}{p_- p_+} p_0^2]. \tag{4.49}$$

The next task is to substitute for the explicit values of $S, Q,$ and P by using (4.24) and (4.25), reading off the terms from the purely vielbein Lagrangian (4.22). To avoid cluttered notation, let us drop the derivative indices on the 2-tensors. We start with the Weyl anomaly terms:

$$\mathbb{W}^{00}_{IJ} \coloneqq \frac{1}{4} L_{\bar{A}B} L_{\bar{C}D} \partial_0 \mathcal{V}^B_{\bar{B}} \partial_0 \mathcal{V}^D_{\bar{D}} (H^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}}) \Delta_0,$$

$$\begin{aligned}
\mathbb{W}^{01}_{IJ} \coloneqq\, & L_{AB} \partial_0 \mathcal{V}^A_{\bar{A}} \partial_1 \mathcal{V}^B_{\bar{B}}\ H^{\bar{A}\bar{B}} \Delta_0 \\
& - \mathcal{H}_{\bar{A}B} L_{\bar{C}D} \partial_0 \mathcal{V}^D_{\bar{D}} \partial_1 \mathcal{V}^B_{\bar{B}} (H^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}}) \Delta_0 \\
& - \frac{1}{2} L_{\bar{A}B} L_{\bar{C}D} \partial_0 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} (H^{\bar{A}[\bar{C}} L^{\bar{D}]B} + L^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}}) \Delta_0,
\end{aligned}$$

$$\begin{aligned}
\mathbb{W}^{11}_{IJ} \coloneqq\, & -\mathcal{H}_{AB} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^A_{\bar{A}}\ H^{\bar{A}\bar{B}} \Delta_0 - 2 \partial_{\bar{K}} \mathcal{H}_{BG} \partial_1 X^G \partial_1 \mathcal{V}^B_{\bar{B}} H^{\bar{K}\bar{B}} \Delta_0 \\
& + \mathcal{H}_{\bar{A}B} \mathcal{H}_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}} (H^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}}) \Delta_0 \\
& + \mathcal{H}_{\bar{A}B} L_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}}\ (H^{\bar{A}[\bar{C}} L^{\bar{D}]B} + L^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}}) \Delta_0 \\
& - \frac{1}{4} L_{\bar{A}B} L_{\bar{C}D} \partial_1 \mathcal{V}^B_{\bar{B}} \partial_1 \mathcal{V}^D_{\bar{D}}\ (H^{\bar{A}[\bar{C}} H^{\bar{D}]\bar{B}} + 3 L^{\bar{A}[\bar{C}} L^{\bar{D}]\bar{B}}) \Delta_0 \\
& - \frac{1}{2} \partial_{\bar{N}} \partial_{\bar{K}} \mathcal{H}_{GD} \partial_1 X^G \partial_1 X^D\ H^{\bar{N}\bar{K}} \Delta_0 \\
& + \partial_{\bar{N}} \mathcal{H}_{G\bar{B}} \partial_1 X^G \partial_{\bar{K}} \mathcal{H}_{D\bar{E}} \partial_1 X^D (H^{\bar{N}[\bar{K}} H^{\bar{E}]\bar{B}} - L^{\bar{N}[\bar{K}} L^{\bar{E}]\bar{B}}) \Delta_0. \qquad (4.50)
\end{aligned}$$

We see that after we make all of the appropriate substitutions for the Weyl anomaly terms, what we end up with has precisely the same structure as found in [39]. All factors and signs agree, and when we compute the tensor algebra we should end up with a similar expression for the total Weyl divergence.

For the Lorentz anomaly terms we now also have

$$
\begin{aligned}
&\mathbb{L}^{01}_{IJ} \coloneqq L_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_0\mathcal{V}^A_{\bar{A}}L^{\bar{A}\bar{B}}\theta \\
&-\ L_{\bar{A}B}L_{\bar{C}D}\partial_0\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^D_{\bar{D}}L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}}\theta, \\
&\mathbb{L}^{11}_{IJ} \coloneqq -\mathcal{H}_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^A_{\bar{A}}L^{\bar{A}\bar{B}}\theta \\
&+ 2\ \mathcal{H}_{\bar{A}B}\partial_1\mathcal{V}^B_{\bar{B}}L^1_{\bar{C}D}\partial_1\mathcal{V}^D_{\bar{D}}\ L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}}\theta \\
&- \frac{1}{2}\ L_{\bar{A}B}L_{\bar{C}D}\partial_1\mathcal{V}^D_{\bar{D}}\partial_1\mathcal{V}^B_{\bar{B}}(H^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}} + L^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}})\theta \\
&- 2\partial_{\bar{K}}\mathcal{H}_{BG}\partial_1X^G\partial_1\mathcal{V}^B_{\bar{B}}\ L^{\bar{K}\bar{B}}\theta \\
&- \frac{1}{2}\partial_{\bar{N}}\partial_{\bar{K}}\mathcal{H}_{GD}\partial_1X^G\partial_1X^D\ L^{\bar{N}\bar{K}}\theta.
\end{aligned}
\tag{4.51}
$$

And, again, the δL corrections, which we will analyse in a few moments:

$$
\delta L_{corrections} \coloneqq \frac{1}{2}\delta L_{\bar{A}\bar{B}}\delta L_{\bar{C}\bar{D}}[(H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\int_p \frac{1}{p_1p_-}\frac{1}{p_1p_+}p_1^2p_0^2]. \tag{4.52}
$$

We are now in position to calculate the tensor algebra for each collection of terms in $\mathbb{W}^{\mu\nu}_{IJ}$ and $\mathbb{L}^{\mu\nu}_{IJ}$. We begin with the Weyl anomaly pieces. We will give the results term for term moving down the list appearing in (4.50):

$$
\begin{aligned}
&\mathbb{W}^{00}_{IJ} := \\
&\frac{1}{4}L_{\bar{A}B}L_{\bar{C}D}\partial_0\mathcal{V}^B_{\bar{B}}\partial_0\mathcal{V}^D_{\bar{D}}(H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\Delta_0 = (-\frac{1}{8}\partial_0\mathcal{H}_{AB}\partial_0\mathcal{H}^{AB})\Delta_0,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{W}^{01}_{IJ} := \\
&L_{AB}\partial_0\mathcal{V}^A_{\bar{A}}\partial_1\mathcal{V}^B_{\bar{B}}\ H^{\bar{A}\bar{B}}\Delta_0 = (\partial_1\mathcal{V}^{\bar{A}}_A\partial_0\mathcal{V}^A_{\bar{B}}\delta^{\bar{A}}_{\bar{D}})\Delta_0, \\
&-\mathcal{H}_{\bar{A}B}L_{\bar{C}D}\partial_0\mathcal{V}^D_{\bar{D}}\partial_1\mathcal{V}^B_{\bar{B}}(H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\Delta_0 = 0, \\
&-\frac{1}{2}L_{\bar{A}B}L_{\bar{C}D}\partial_0\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^D_{\bar{D}}(H^{\bar{A}[\bar{C}}L^{\bar{D}]B} + L^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}})\Delta_0 = (-\partial_1\mathcal{V}^{\bar{A}}_A\partial_0\mathcal{V}^A_{\bar{B}}\delta^{\bar{B}}_{\bar{A}})\Delta_0,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{W}^{11}_{IJ} := \\
&-\mathcal{H}_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^A_{\bar{A}}\ H^{\bar{A}\bar{B}}\Delta_0 = (-\partial_1\mathcal{V}^A_{\bar{A}}\partial_1\mathcal{V}^{\bar{A}}_A + \frac{1}{2}\partial_1\mathcal{H}_{AB}\partial_1\mathcal{H}^{AB})\Delta_0, \\
&-2\partial_{\bar{K}}\mathcal{H}_{BG}\partial_1X^G\partial_1\mathcal{V}^B_{\bar{B}}H^{\bar{K}\bar{B}}\Delta_0 = 0 \\
&+\mathcal{H}_{\bar{A}B}\mathcal{H}_{\bar{C}D}\partial_1\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^D_{\bar{D}}(H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\Delta_0 = 0, \\
&+2\mathcal{H}_{\bar{A}B}L_{\bar{C}D}\partial_1\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^D_{\bar{D}}\ (H^{\bar{A}[\bar{C}}L^{\bar{D}]B} + L^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}})\Delta_0 \\
&\qquad\qquad = (2\partial_1\mathcal{V}^A_{\bar{A}}\partial_1\mathcal{V}^{\bar{A}}_A - \frac{1}{2}\partial_1\mathcal{H}_{AB}\partial_1\mathcal{H}^{AB})\Delta_0, \\
&-\frac{1}{4}L_{\bar{A}B}L_{\bar{C}D}\partial_1\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^D_{\bar{D}}\ (H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} +\ 3L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\Delta_0 \\
&\qquad\qquad = (-\partial_1\mathcal{V}^A_{\bar{A}}\partial_1\mathcal{V}^{\bar{A}}_A + \frac{1}{8}\partial_1\mathcal{H}_{AB}\partial_1\mathcal{H}^{AB})\Delta_0 \\
&-\frac{1}{2}\partial_{\bar{N}}\partial_{\bar{K}}\mathcal{H}_{GD}\partial_1X^G\partial_1X^D\ H^{\bar{N}\bar{K}}\Delta_0 = -\frac{1}{2}\partial_K\partial^K\mathcal{H}_{GD}\partial_1X^G\partial_1X^D\Delta_0 \\
&+\partial_{\bar{N}}\mathcal{H}_{G\bar{B}}\partial_1X^G\partial_{\bar{K}}\mathcal{H}_{D\bar{E}}\partial_1X^D(H^{\bar{N}[\bar{K}}H^{\bar{E}]\bar{B}} - L^{\bar{N}[\bar{K}}L^{\bar{E}]\bar{B}})\Delta_0 \\
&\qquad\qquad = \frac{1}{2}\partial_K\mathcal{H}_{GB}\mathcal{H}^{BC}\partial^K\mathcal{H}_{CD}\partial_1X^G\partial_1X^D\Delta_0. \qquad (4.53)
\end{aligned}
$$

We will return to these results shortly. In the meantime, the results for the Lorentz anomaly terms may be similarly summarised:

$$\mathbb{L}^{01}_{IJ} :=$$
$$L_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_0\mathcal{V}^A_{\bar{A}}L^{\bar{A}\bar{B}}\theta \;=\; (-\partial_1\mathcal{V}^{\bar{A}}_B\partial_0\mathcal{V}^A_{\bar{A}}\delta^B_A)\theta,$$
$$-\,L_{\bar{A}B}L_{\bar{C}D}\partial_0\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^D_{\bar{D}}L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}}\theta \;=\; (\partial_0\mathcal{V}^{\bar{D}}_B\partial_1\mathcal{V}^D_{\bar{D}}\delta^B_D)\theta,$$

$$\mathbb{L}^{11}_{IJ} :=$$
$$-\,\mathcal{H}_{AB}\partial_1\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^A_{\bar{A}}L^{\bar{A}\bar{B}}\theta \;=\; (\partial_1\mathcal{V}^{\bar{A}}_D\partial_1\mathcal{V}^A_{\bar{A}}\delta^D_A)\theta,$$
$$2\,\mathcal{H}_{\bar{A}B}L_{\bar{C}D}\partial_1\mathcal{V}^B_{\bar{B}}\partial_1\mathcal{V}^D_{\bar{D}}\;L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}}\theta = (-2\partial_1\mathcal{V}^{\bar{D}}_B\partial_1\mathcal{V}^D_{\bar{D}}\delta^B_D)\theta,$$
$$-\,\frac{1}{2}\;L_{\bar{A}B}L_{\bar{C}D}\partial_1\mathcal{V}^D_{\bar{D}}\partial_1\mathcal{V}^B_{\bar{B}}(H^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}} + L^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}})\theta = (\partial_1\mathcal{V}^{\bar{D}}_A\partial_1\mathcal{V}^D_{\bar{D}}\delta^A_D)\theta.$$
$$-\,2\partial_{\bar{K}}\mathcal{H}_{BG}\partial_1X^G\partial_1\mathcal{V}^B_{\bar{B}}\;L^{\bar{K}\bar{B}}\theta = 2\partial_{\bar{K}}\mathcal{H}_{\bar{A}G}\partial_1\mathcal{V}^{\bar{A}}_C\partial_1X^GL^{\bar{K}C}\theta$$
$$-\,\frac{1}{2}\partial_{\bar{N}}\partial_{\bar{K}}\mathcal{H}_{GD}\partial_1X^G\partial_1X^D\;L^{\bar{N}\bar{K}}\theta = -\frac{1}{2}\partial_{\bar{N}}\partial_{\bar{K}}\mathcal{H}_{GD}\partial_1X^G\partial_1X^D\;L^{\bar{N}\bar{K}}\theta. \tag{4.54}$$

Lastly, we have the δL corrections

$$\delta L_{corrections} := \frac{1}{2}\delta L_{\bar{A}\bar{B}}\delta L_{\bar{C}\bar{D}}[(H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\int_p \frac{1}{p_1p_-}\frac{1}{p_1p_+}p_1^2p_0^2]. \tag{4.55}$$

It is now time to calculate the momentum integral on the far right-hand side of (4.55). So let us consider it separately and start by making appropriate cancellations to the momentum terms,

$$\int_p \frac{1}{p_1p_-}\frac{1}{p_1p_+}p_1^2p_0^2 \tag{4.56}$$
$$= \int_p \frac{p_0^2}{p_+p_-}.$$

Using similar methods as those employed to calculate the momentum integrals that appear in the fluctuation contractions discussed in Section 4.6.1 and 4.6.2, we can go to Euclidean space and make careful use of a regulator as well as invoke the identity $p_\pm = p_0 \pm p_1 \implies p_0 = \frac{p_+ \pm p_-}{2}$. The calculation also requires that we compute the appropriate residue, typically for $z_+ > 0$, which results in a sgn function [7]. When we substitute for p_0 in (4.56) we get

$$= \frac{1}{4}\int_p (\frac{p_+}{p_-} + 2 + \frac{p_-}{p_+})e^{ip\cdot z}. \tag{4.57}$$

We may now use the standard formula $S(z) = -\frac{i}{2}\int \frac{dp_+ dp_-}{(2\pi)^2}\frac{1}{p_\mp^2}e^{i(\frac{p_+ z_-}{2}+\frac{p_- z_+}{2})}$ that one can derive from the definition of the propagator for chiral bosons. Plugging (4.57) into this formula and now also writing explicitly for $\int_p$ as defined in Section 4.6.2 we obtain,

$$= -\frac{i}{8}\int \frac{dp_+ dp_-}{4\pi}\ (\frac{p_+}{p_-} + 2 + \frac{p_-}{p_+})\ e^{i(\frac{p_+ z_-}{2}+\frac{p_- z_+}{2})}. \tag{4.58}$$

Let us drop the factors and focus specifically on the case for $\frac{p_+}{p_-}$. What we end up with is the following integral that can be calculated in a fairly straightforward manner,

$$\begin{aligned} &= \int dp_+ dp_-\ \frac{p_+}{p_-}\ e^{i(\frac{p_+ z_-}{2}+\frac{p_- z_+}{2})} \\ &= \int dp_+\ p_+\ e^{\frac{ip_+ z_-}{2}} \int dp_-\ \frac{1}{p_-}\ e^{\frac{ip_- z_+}{2}} \\ &= \delta(z_-)\,\mathrm{sgn}(z_+). \end{aligned} \tag{4.59}$$

The same procedure can then be made for the case of $\frac{p_-}{p_+}$ in (4.57). What we observe, as found in (4.59), is that the divergence is in fact not logarithmic. Considering that we are *only interested here in logarithmic divergences*, this means that there are no $\delta\mathcal{L}$ corrections coming from (4.55) and so we denote this by its vanishing.

$$\delta L_{corrections} \coloneqq \frac{1}{2}\delta L_{\bar{A}\bar{B}}\delta L_{\bar{C}\bar{D}}[(H^{\bar{A}[\bar{C}}H^{\bar{D}]\bar{B}} - L^{\bar{A}[\bar{C}}L^{\bar{D}]\bar{B}})\int_p \frac{1}{p_1 p_-}\frac{1}{p_1 p_+}p_1^2 p_0^2] = 0. \tag{4.60}$$

4.7.2 Final total Weyl divergence and Lorentz anomaly

As we near the conclusion of this chapter, the final total Weyl divergence is given by summing all of the contributions from (4.53), in which the coefficient of $\Delta(0)$ in (4.46) is obtained. Likewise, for the final total Lorentz anomaly, the coefficient of $\theta(0)$ is given by summing all of the contributions from (4.54).

For the Weyl anomaly we find,

$$\mathbb{W} = (\frac{1}{8}\partial_1\mathcal{H}_{AB}\partial_1\mathcal{H}^{AB} - \frac{1}{8}\partial_0\mathcal{H}_{AB}\partial_0\mathcal{H}^{AB} - \frac{1}{2}\partial_K\partial^K\mathcal{H}_{GD}\partial_1X^G\partial_1X^D$$
$$+ \frac{1}{2}\partial_K\mathcal{H}_{GB}\mathcal{H}^{BC}\partial^K\mathcal{H}_{CD}\partial_1X^G\partial_1X^D). \tag{4.61}$$

And for the Lorentz anomaly, interestingly, we find two remaining terms

$$\mathbb{L} = 2\partial_{\bar{K}}\mathcal{H}_{\bar{A}G}\partial_1\mathcal{V}_C^{\bar{A}}\partial_1X^G L^{\bar{K}C} - \frac{1}{2}\partial_{\bar{N}}\partial_{\bar{K}}\mathcal{H}_{GD}\partial_1X^G\partial_1X^D\ L^{\bar{N}\bar{K}}. \tag{4.62}$$

It is possible to simplify $\mathbb{L}$ a bit more, finding in the process that it carries δL corrections. As in [41], it is suspected that the equations of motion may also be used to show that the Lorentz anomaly vanishes. As space is restricted, such an analysis must be saved for another time. Instead, we focus the remainder of this chapter on the Weyl anomaly. After some brief calculation, where we use integration by parts, the first two terms in $\mathbb{W}$ are shown to vanish. So the final total Weyl divergence becomes

$$\mathbb{W} = -(\frac{1}{2}\partial_K\partial^K\mathcal{H}_{GD} - \frac{1}{2}\partial_K\mathcal{H}_{GB}\mathcal{H}^{BC}\partial^K\mathcal{H}_{CD})\partial_1X^G\partial_1X^D. \tag{4.63}$$

The structure for these two terms has some semblance of the structure we observed in (2.31) for the generalised curvature scalar, albeit we have not included the dilaton. Additionally, the main difference with our result when compared to the findings in [39] is that we're missing a connection term constructed from the base metric. It is possible that we may find a complete description for the generalised Ricci scalar on the maximally doubled space, however this pursuit must again be saved for future study.

4.8 Doubled beta-functionals

The fluctuation contractions calculated earlier in this chapter involved propagators of fields, and these are evaluated in the limit $\sigma \to \sigma'$. In this limit there will be divergences at one-loop, because we are evaluating the propagators of fields at the same point. For physical intuition, in terms of scalar particles one may think of this limit at one-loop according to a field running the loop, where the beginning and end-point ultimately coincide. This means, recalling the definitions of the propagators in (4.29), $\Delta(\sigma - \sigma') \to \Delta(0)$ clearly diverges while

$\theta(\sigma - \sigma') \to \theta(0)$ takes on an ambiguity such that arctanh $\frac{\sigma-\sigma'}{\bar{\sigma}-\bar{\sigma}'}$ can be taken to equal some arbitrary constant.

For $\theta(0)$, following [42] if we set arctanh $\frac{\sigma-\sigma'}{\bar{\sigma}-\bar{\sigma}'} = \tanh\delta$ then upon regularisation the limiting expression takes the form

$$\theta(0) \to -\frac{1}{2\pi}\delta, \tag{4.64}$$

where we simply interpret δ as the boost parameter.

Now, for the divergence coming from $\Delta(0)$ at one-loop, which again reflects the UV divergence we would observe in the momentum integral around the loop, to regularise and renormalise it the strategy is exactly the same as with the ordinary non-doubled string [76], wherein we simply reduce the problem to an interacting quantum field theory. That is, we regulate the divergence by introducing a UV cut-off so that, after we renormalise the theory, the background fields of the sigma model depend on a mass scale λ. In isolating the UV divergence of the theory, we also introduce appropriate counter-terms. In fact, we've been utilising this strategy from the outset, when we first chose to invoke the background field method and to calculate the quantum fluctuations to quartic interaction. The result of the overall procedure requires us to look at how the couplings of the theory depend on λ, which is described by the beta-functionals.

Regularising Δ by sending $\ln(\sigma - \sigma')^2 \to \ln((\sigma - \sigma')^2 + \lambda^2)$ results in the limiting expression

$$\Delta(0) \to -\frac{1}{2\pi}\ln\lambda, \tag{4.65}$$

where $\lambda = \frac{\mu}{\alpha'}$. We define $\alpha' = l_s^2$, where l_s^2 is the string length scale.

These two limiting expressions (4.64) and (4.65) describe precisely how we've written the effective action (4.46), where for $\Delta(0)$ it is clear we have scale dependence and for $\theta(0)$ we instead have phase dependence. It is worth noting that by introducing a mass scale λ and then counter-terms, scale dependence is ultimately absorbed in the definition of the renormalised couplings

$$\{\mathcal{H}, L\} \to \{\mathcal{H}(\lambda), L(\lambda)\}. \tag{4.66}$$

Then from (4.66) we can define a renormalised action, which, recalling our original action (4.2), takes the form

$$S^{(R)} = \frac{1}{2}\int d^2\sigma[-\mathcal{H}^{(R)}_{AB}(\lambda)\partial_1 X^A \partial_1 X^B + L^{(R)}_{AB}(\lambda)\partial_1 X^A \partial_0 X^B]. \tag{4.67}$$

Schematically, in string theory the beta-functionals will typically have the following structure

$$\beta[\mathcal{H}_{AB}] \sim \frac{\partial \mathcal{H}_{AB}(X;\lambda)}{\partial\lambda}, \quad \beta[L_{AB}] \sim \frac{\partial L_{AB}(X;\lambda)}{\partial\lambda}, \tag{4.68}$$

where we take the derivative of the background fields in the renormalised action (4.67) with respect to the log of the mass scale. Here, the logarithmic derivative of the coupling constant with respect to the scale is just the beta-function. In this case, we have the following beta-functionals for the doubled geometry metric couplings:

$$\beta[\mathcal{H}_{AB}] = -\mathbb{W}_{IJ} \quad \text{and} \quad \beta[L_{AB}] = \mathbb{L}_{IJ}. \tag{4.69}$$

By the requirements of conformal invariance we demand these beta-functionals to vanish on-shell

$$\beta[\mathcal{H}_{AB}] = -W_{IJ} = 0 \quad \text{and} \quad \beta[L_{AB}] = \mathbb{L}_{IJ} = 0. \tag{4.70}$$

It then follows that, given the vanishing of the beta-functionals, this is equivalent to showing conformal invariance of the theory at the quantum level. The result is that one obtains the background field equations.

Chapter 5

Concluding Remarks

Let us now summarise the primary results of this thesis and point toward future directions.

We began by utilising the background field method for the completely doubled action. Having laid out our general strategy to ensure a generic calculation with minimal assumptions, we derived the master expression for the effective action at one-loop before calculating the final total divergence of the theory. The tensor structure that we derived for the Weyl and Lorentz anomalies is completely general, valid for any expansion of the doubled action (e.g., if instead one chooses to expand (4.2) with respect to L instead of $\mathcal{H}$). For the Weyl anomaly, we found a number of elegant cancellations in the algebra, leaving only two terms comprising the coefficient of $\mathbb{W}$. We are heartened to note that this result matches that found in [39] with the exclusion of any base dependent objects. For the Lorentz anomaly, interestingly, we have two remaining terms which are expected to cancel upon further study. It is also interesting to report that, for the δL corrections, we didn't find any logarithmic divergences. This result should perhaps be expected with the use of dimensional analysis, but it is nonetheless curious.

There is much more to be said about the doubled beta-functionals and how we might go about working with them directly in the future. In a sense, one could view the doubled space of the duality symmetric string as an emergent phenomenon insofar that it presents itself in clear view when T-duality is made to be a manifest symmetry of the action. Taking a generic approach with the completely doubled action for the interacting chiral boson model, do the beta-functionals provide a fuller view of the linear and quadratic equations of motion in DFT [18] as similarly conjectured in [42]?

The primary results of this thesis also pave the way for future work in a number of other directions. If past studies (e.g., [39, 41, 42] to list a few) employ formulations that depend on a doubled torus background or other types of doubled geometry such as twisted doubled tori, it is important to investigate generalisations of chiral boson models for generic doubled geometries. The primary motivation to date for using such torus-based doubled geometries has to do with how the theory on a torus is generally tractable, with the extension to a twisted torus a natural step to a more complicated case. But exploration of other doubled geometries is in many ways still an uncharted savannah of stringy physics. These geometries are certainly non-trivial and it is quite possible that mathematically we remain at a point where the calculations are too unyielding. In any case, such is no doubt a fundamental direction of study, with the challenge to construct more exotic Lorentz invariant theories facing us head-on.

Belonging to such important efforts is the requirement to construct models that realise completely the full web of string dualities. This point refers back to our opening discussion in the introduction. When it comes to such frontier pursuits, a natural first step would be to incorporate manifest S-duality in the manifestly T-duality invariant theory, if possible. The purpose, among a large list of objectives, would be to probe a deeper view of non-geometric string backgrounds. Finally, we have to ask: in these models for interacting chiral scalars, how do we come into complete contact with generalised geometry, if we take the view of understanding such geometry in terms of a study of conventional geometry with a metric and B-field on some D-dimensional manifold M on which $O(D, D)$ finds natural action. What are the implications, if any, for string cosmology given such intriguing pathways of investigation?

The results of this paper for the maximally doubled interacting chiral boson model have also left a number of other open questions. From the total Weyl divergence, one such question relates to the derivation of some generalised scalar curvature. It is also interesting to ask if, and how, we may build from the results in this paper in comparison with Tseytlin's original results for the effective action for the duality symmetric string [12, 13], including investigation into any evidence for sequestering. In phase space formulations, such as the metastring, what are the implications of the connection to Born geometries? Additionally, in [64] Tseytlin comments on how the doubled action takes the Heterotic string to its maximal logical completion, and this idea offers a few other interesting directions one may probe, given the importance of the Heterotic idea in string theory. Finally, in using the most general form of the Tseytlin action, what happens when we switch on the B-field and the dilaton for the case where the background fields have arbitrary dependence on the generic doubled geometry? Indeed, the entire approach to the primary calculation of the effective action in this thesis is positioned in such a way to explore effective spacetime theories for completely generic non-geometries.

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