# Measures on higher dimensional local fields and algebraic groups on them

**Wester van Urk**

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# Contents

## 1 Abstract

This thesis is about defining finitely additive measures on sets. The prototype for what we're doing is defining a $\mathbb{R}((X))$-valued measure on a 2-dimensional local field (such as $\mathbb{Q}_p\{\{t\}\}$). The thesis consists of three main parts.

The first part consists of defining finitely additive measures and integration in relatively high degree of generality so that we can not only integrate over 2-dimensional local fields but also higher dimensional local fields, $\mathbb{C}((t))$ and over algebraic groups.

The second part consists of applying this theory to obtain a sequence of more refined measures $\mu_n$ on a 2-dimensional local field which allow us to define the Fourier transform intrinsically.

The third and final part consists of applying the theory to coset measures on $GL(2)$ and $SL(2)$, including rigorously defining a local Hecke operator on $GL(2, \mathbb{C}((t)))$.

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# 3 Introduction

## 3.1 Overview of measure on 2-dimensional local fields

Higher dimensional local fields are never locally compact, so we know a priori that they can not have a Haar measure defined on them. At the same time they are not dissimilar to 1-dimensional local fields, so one might reasonably hope that we could still define some sort of measure on them, even if it behaves badly in some sense. This turns out to be true and is what Fesenko does in [1].

If we take concretely the example of $\mathbb{Q}_p((t))$, then the idea of this measure is that there is a natural projection from $O = \mathbb{Z}_p + t\mathbb{Q}_p[[t]]$ to $\mathbb{Z}_p$, so we have a natural notion that the measure $\mu(p^i O)$ should be $p^{-i}$. Then the first difficulty that presents itself is what kind of measure to assign to a set like $tO$ or $t^{-1}O$ which are infinitely smaller or larger respectively than the sets $p^i O$ which we could assign a value to. The solution to this problem is to introduce an auxiliary variable, $X$, which one might reasonable think of as an infinitesimal, and set $\mu(t^j O) = X^j$.

So to make this idea work, you have to allow your measure to take values in $\mathbb{R}((X))$ but Fesenko shows that you can obtain a meaningful, well-defined and translation-invariant finitely additive measure $\mu$ on some algebra $\mathcal{A}$ generated by sets of the form $a + p^i t^j O$ and equal to the following

$$\mu(a + p^i t^j O) = p^{-i} X^j$$

This measure satisfies the following properties:

1. $\mu(a + A) = \mu(A)$ for all $a \in K, A \in \mathcal{A}$
2. $\mu(\varepsilon t_1^i t_2^j A) = \mu(A) q^{-i} X^j$ for all $\varepsilon \in O^*, A \in \mathcal{A}$
3. $$\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i)$$

   if both sides are defined and the right hand side converges absolutely in $\mathbb{R}((X))$ (i.e. the series defining each coefficient converges absolutely and for sufficiently negative $N$, the series defining the $N$-th coefficient is identically zero).

We will often think the second property in the following equivalent form: there is a module $|\cdot| : K \to \mathbb{R}((X))$ such that $\mu(aA) = |a|\mu(A)$ for all $a \in K$ and $A \in \mathcal{A}$. The measure satisfies countable additivity only in a restricted sense, but this restricted sense is sufficient to allow you to define some sort of integration - the idea here is that the integral should satisfy

$$\int \sum_{i=0}^{\infty} \lambda_i \mathbb{1}_{A_i} d\mu = \sum_{i=0}^{\infty} \lambda_i \mu(A_i)$$

and one can use this as a definition if the $A_i$ are of a particular simple form, are pairwise disjoint and the series on the right hand side converges absolutely in $\mathbb{R}((X))$.

There's also a version of this theory developed by Matthew Morrow [4] which starts with the idea that we're lifting from the residue field (in the case of higher dimensional local fields, we consider the topmost/largest residue field) and develops a theory of integration from there. This covers slightly more functions but doesn't lend itself as well to the generalizations that we're interested in.

## 3.2 Summary of new work

We generalize Fesenko's approach to measure in 2 ways: we allow a slightly wider range of functions and at each step of the way to defining integration, we prove it in as much generality as we can so that for example we can also define a measure on $GL(2, \mathbb{C}((t)))$. We prove exactly when a premeasure defined on some collections of subsets extends to a finitely additive measure on an algebra of sets. One key problem is that finitely additive measures don't necessarily lead to natural theories of integration (since we have no a priori condition on how they behave with respect with countable operations); we resolve this problem by producing the concept of labelling and we prove a gluing lemma, allowing us to define a theory of integration on a larger set $K$ by means of defining it on subsets (or on quotients of subsets, i.e. we can lift a theory of integration from the residue field $E$ to a 2-dimensional field $F$).

Our next section uses the theory just developed to define a sequence of ever more refined measures $\mu_n$ where $\mu_1$ is our standard measure. If $\mu_1$ is related to the isomorphism $\mathcal{O}/t_2\mathcal{O} \simeq E$, then $\mu_n$ is related to the isomorphism $\mathcal{O}/t_2^n\mathcal{O} \simeq E^n$. The purpose of these measures is to define a Fourier transform intrinsically on a 2-dimensional local field and to do that, you need these more refined measures in order to define integrals of the form

$$\int f\chi d\mu$$

where $\chi$ is a character. The downside of these measures is that they are not inherently commensurable (e.g. $\mu_n(t_1^i A) = q^{in}\mu_n(A)$, so each $\mu_n$ behaves differently under scaling). We solve that problem with a limiting procedure, which allows us to define the Fourier transform intrinsically on the ring of functions we started with. We then show that this has all the expected properties such as Fourier inversion.

In the final section we discuss algebraic groups. We first prove a version of the Iwasawa decomposition that will be helpful for our purposes and then use that and our theory of measures to define a measure on the coset spaces $SL(2,F)/SL(2,O)$ for any 2-dimensional local field and $GL(2,\mathbb{C}((t)))/GL(2,\mathbb{C}[[t]])$. Using the latter measure we can give rigorous meaning to a local Hecke operator defined by Langlands [3] of the form

$$\theta_1 : f \mapsto \tilde{f}, \tilde{f}(g) = \int_{g\Delta_1/GL(2,\mathbb{C}[[X]])} f d\mu$$

and we show that we can define a ring of functions $M$ on $GL(2,\mathbb{C}((t)))$ such that $\theta_1$ acts on $M$.

# 4 Notation

Throughout this work we will take $F$ to be a 2-dimensional local field, e.g. $E((t))$ for a local field $E$ or $\mathbb{Q}_p\{\{t\}\}$. We write $E$ for the first residue field, $O$ for the ring of integers with respect to the rank 2 valuation and $\mathcal{O}$ for the rank 1 ring of integers, so e.g. if $K = E((t))$, $E = E$, $O = O_E + tE[[t]]$ and $\mathcal{O} = E[[t]]$; if $K = \mathbb{Q}_p\{\{t\}\}$, $E = \mathbb{F}_p((t))$, $O = \mathbb{Z}_p[[t]] + p\mathbb{Z}_p\{\{t\}\}$ and $\mathcal{O} = \mathbb{Z}_p\{\{t\}\}$. We will pick a 2-dimensional valuation $v$ and a pair of uniformizing elements $t_1, t_2$ with $v(t_1) = (1, 0)$ and $v(t_2) = (0, 1)$ (we order this lexicographically so that $v(t_2) > v(t_1)$). In our two examples $(t_1, t_2)$ may be chosen as $(\pi_E, t)$ and $(t, p)$ respectively.

We will also occasionally write $K$ to be any set, when we want to work in more generality. For the most part, this is still intended to be a 2-dimensional local field.

# 5 Creating a general additive measure theory

## 5.1 Philosophy

The goal of this section is to generalize Fesenko's approach to measure on 2-dimensional local fields. Consider first the simplest case, $\mathbb{Q}_p((t))$: if we want to define an additive measure on this field with $\mu(O) = 1$, then also we have $\mu(p^i O) = p^{-i}$ by additivity and since $tO \subset p^{-i}O$, there's no reasonable real-valued measure to assign $tO$. The solution is to choose $\mu(tO) = X$ for some new element $X$. If we want to have some sort of module (and thus a well-behaved theory of integration), we require $\mu(t^j O) = X^j$. Using this input (including specific features of this problem) one can then show that this does define a unique additive measure on $K$ on an algebra of sets and that it does lead to a reasonable theory of integration.

Now we want to view this more generally, so we take a set $K$, a collection of subsets $\mathcal{G}$ (where g stands for generating), an abelian group $B$ and a function $\mu : \mathcal{G} \to B$. One can naively extend $\mu$ to more subsets in the following two ways:

1. If $A, B \in \mathcal{G}$ are disjoint, then it should hold that $\mu(A \cup B) := \mu(A) + \mu(B)$
2. If $A, B \in \mathcal{G}$ satisfy $A \subset B$, then it should hold that $\mu(B \setminus A) = \mu(B) - \mu(A)$

If we want to extend $\mu$ to an additive measure on an algebra of subsets $\mathcal{A}$, then we will require firstly that we also have access to intersections and secondly, we need to identify under which conditions this naive procedure in fact produces a unique, well-defined measure. We give an answer to this question in full generality in the second subsection.

The next step then is to define an appropriate class of functions and an integration operator $\int_K d\mu$. The idea here is that the following formula should hold in as great a generality as possible:

$$\int \sum_i \lambda i \mathbb{1}_{A_i} d\mu = \sum_i \lambda_i \mu(A_i)$$

We now give an example to show that we can not hope this will hold in full generality:

**Example** Let $K = \mathbb{Q}_p((t))$ and pick three sequences $B_i$, $C_i$ and $D_i$ of subsets satisfying the following:

1. Each sequence consists of pairwise disjoint sets.

2. The $B_i$ and $C_i$ are also pairwise disjoint; $D_i$ is fully contained in $B_i$.

3. $\mu_K(B_i) = q^{-i}$

4. $\mu_K(D_i) = \mu_K(C_i) = X$

Then define $A_i = (B_i \setminus D_i) \sqcup C_i$ and $f_{\text{bad}} = \sum_{i=1}^{\infty} \mathbb{1}_{A_i}$; since $\mu(A_i) = q^{-i} - X + X = q^{-i}$, this representation would imply that

$$\int_n f_{\text{bad}} d\mu_n = \sum_{i=0}^{\infty} q^{-i} = \frac{1}{1 - \frac{1}{q}}$$

However, if we define $A'_i = (B_i \setminus D_i) \sqcup C_{i-1}$ for $i > 0$ and $A'_i = B_0 \setminus D_0$. Then we have that $f_{\text{bad}} = \sum_{i=0}^{\infty} \mathbb{1}_{A'_i}$ but also that:

$$\sum_{i=0}^{\infty} \mu_n(A'_i) = 1 - X + \sum_{i=1}^{\infty} q^{-i} = \frac{1}{1 - \frac{1}{q}} - X$$

In particular, this implies that our theory of integration will always be restricted in terms of what kind of functions it can handle and these restrictions will reflect features of the specific sets that we're working with. If we look back at how we define integration for functions on $\mathbb{Q}_p((t))$ (for example), we may observe that an important feature that we take advantage is that we know integration on $\mathbb{Q}_p$ is well-defined and under appropriate conditions we can lift this to the bigger field $\mathbb{Q}_p((t))$. Our goal therefore will be to generalize this procedure by means of a gluing lemma, which we do in the final subsection.

## 5.2 Existence of measures

Suppose we have a set $K$, a collection of subsets $\mathcal{G}$ (where g stands for generating), an abelian group $B$ and a function $\mu : \mathcal{G} \to B$. If we want to extend $\mu$ to a $B$-valued additive measure on an algebra of subsets $\mathcal{A}$ containing $\mathcal{G}$, then we can immediately note two necessary conditions:

1. We need to be able to extend $\mu$ to intersections of elements.

2. $\mu$ must satisfy the Inclusion-Exclusion principle on $\mathcal{G}$ (since any additive measure satisfies it).

We will prove in the following lemma that these conditions are also sufficient. Since there is no general method for defining the measure of an intersection, instead we'll assume that $\mathcal{G}_n$ is closed under intersections, which will be true in all our applications.

**Lemma 5.1** *Let $B$ be an abelian group and $K$ a set. Let $\mathcal{G}$ be a collection of subsets of $K$ equipped with a function $\mu : \mathcal{G} \mapsto B$ satisfying*

1. *Closure under intersection: $A \cap B \in \mathcal{G}$ for all $A, B \in \mathcal{G}$*
2. *Inclusion-Exclusion: If $A_1, ..., A_n$ and $\bigcup_{i=1}^n A_i$ are elements of $\mathcal{G}$, then*

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i) - \sum_{1 \leq i < j \leq n} \mu(A_i \cap A_j) + ... + (-1)^{n+1} \mu(\bigcap_{i=1}^{n} A_i)$$

*Then $\mu$ extends uniquely to a finitely additive $B$-valued measure on the algebra generated by $\mathcal{G}$, which we will call $\mathcal{A}$.*

**Proof** Since $\mathcal{G}$ is already closed under intersections, to obtain $\mathcal{A}$, we need to take the closure under set-theoretic difference and disjoint union. The idea is that we would like to define a measure $\tilde{\mu}$ on $\mathcal{A}$ by the following procedure:

1. If $\tilde{\mu}(A)$ and $\tilde{\mu}(B)$ are defined, then we define $\tilde{\mu}(A \sqcup B) := \tilde{\mu}(A) + \tilde{\mu}(B)$
2. If $\tilde{\mu}(A)$ and $\tilde{\mu}(B)$ are defined and $B \subset A$, then we define $\tilde{\mu}(A \setminus B) = \tilde{\mu}(A) - \tilde{\mu}(B)$

It is clear that this is the only way one can extend $\mu$ to $\mathcal{A}$; the problem is to show that $\tilde{\mu}(A)$ will be independent of the expression we have for $A$ in terms of sets $A_i \in \mathcal{G}$, $\setminus$ and $\sqcup$. So suppose that we have two expressions for the same set $A$ in terms of sets $A_i$ $(1 \leq i \leq k)$ in $\mathcal{G}$, $\setminus$ and $\sqcup$. Then we need to show that these two expressions lead to the same value for $\tilde{\mu}(A)$. The trick here is that since we're only dealing with finitely many sets $A_1, ..., A_k$, we can reduce the lemma to the case that $\mathcal{G}$ and $K$ are both finite sets as follows: define

$$\mathcal{G}' = \{\bigcap_{i \in I} A_i | I \subset \{1, ..., k\}\}$$

$$K' = K/_{\sim}$$

where $x \sim y$ if $x$ and $y$ belong to the exact same elements of $\mathcal{K}'$. Then $\mathcal{G}'$ is finite and closed under intersection by construction and the cardinality of $K'$ is at most $2^{\#\mathcal{G}'}$, so is also finite. Then we can naturally think of $\mathcal{G}'$

as consisting of subsets of $K'$ - this doesn't change the values of $\mu$ or the calculation of $\tilde{\mu}$, so we only have the prove the lemma in this specific case. So for the rest of the proof, we will assume $\mathcal{G}$ finite, $K = \{1, ..., n\}$ and that if $i$ and $j$ belong to the same elements of $\mathcal{G}$, then $i = j$. In particular, the latter implies that the only algebra of sets containing $\mathcal{G}$ is the full power set of $K$ (any algebra on a finite set is atomic and that assumption implies its atoms must be the singleton sets $\{i\}$).

Now define $n$ sets as follows:

$$A_i = \bigcap_{i \in A \in \mathcal{G}} A$$

Note that $A_i \in \mathcal{G}$ since it is closed under intersection, i.e. $A_i$ is the smallest set in $\mathcal{G}$ containing $i$. This definition implies that if $i \in A_j$, then $A_i \subset A_j$. So, if $i \in A_j$ and $j \in A_i$, then $A_i = A_j$ and $i = j$ since they belong to the same elements of $\mathcal{G}$. We also note that at least one of the $A_i$ has only one element - the $\subset$-relation gives us a partial ordering on the $A_i$ and if we take a minimal element $A_j$ with respect to this partial ordering, then $i \in A_j$ implies $A_i \subset A_j$, so $i = j$ is the unique element of $A_j$. Without loss of generality we may assume that $A_n$ is such a set with just one element. Then we consider the sets $A_i \setminus \{n\}$ for $1 \leq n - 1$ ordered by inclusion - we again find a minimal set which by possibly relabeling we may assume to be $A_{n-1} \setminus \{n\}$. Then $A_{n-1} \subset \{n-1, n\}$ and we can continue this procedure, relabelling as we go until our sets satisfy the following for all $1 \leq i \leq n$:

$$A_i \subset \{i, ..., n\}$$

Now define a $n$-by-$n$ matrix $M$ by the following:

$$M_{i,j} = \mathbb{1}(j \in A_i)$$

As a result of our relabelling procedure, this matrix is upper-triangular and it has 1's on the diagonal, so we conclude that $M$ is an invertible matrix. Now, if we have a measure $\tilde{\mu}$ on $\mathcal{P}(\{1, ..., n\})$, the matrix $M$ relates the values $\tilde{\mu}(\{i\})$ to the values $\tilde{\mu}(A_i)$ by means of the formula:

$$M(\tilde{\mu}(\{1\}), ..., \tilde{\mu}(\{n\})) = (\tilde{\mu}(A_1), ..., \tilde{\mu}(A_n))$$

So if we want to find a measure $\tilde{\mu}$ that agrees with $\mu$ on $\mathcal{G}$, it must at least satisfy the condition that $\tilde{\mu}(A_i) = \mu(A_i)$, which in terms of the matrix we can write as

$$M(\tilde{\mu}(\{1\}), ..., \tilde{\mu}(\{n\})) = (\mu(A_1), ..., \mu(A_n))$$

Since $M$ is an invertible matrix, there is a unique solution to this equation, which we'll call $\tilde{\mu}$ (note that a measure on $\mathcal{P}(\{1, ..., n\})$ is determined by the values $\tilde{\mu}(\{i\})$). We will now show that $\tilde{\mu}$ satisfies all the required properties.

Given $A \in \mathcal{G}$, we can write it as the following union (if $i \in A$, then $i \in A_i \subset A$ by definition of $A_i$):

$$A = \bigcup_{i \in A} A_i$$

Now we can apply the inclusion-exclusion principle to both $\mu$ and $\tilde{\mu}$ separately to express $\mu(A)$ in terms of $\mu(A_i)$ and $\tilde{\mu}(A)$ in terms of $\tilde{\mu}(A_i)$ (by construction $A_i \cap A_j$ is equal to $A_i$ or $A_j$, so the intersections appearing in the inclusion-exclusion principle are all equal to $A_i$'s) and we find that $\tilde{\mu}(A) = \mu(A)$ for all $A \in \mathcal{G}$. So $\tilde{\mu}$ is the unique measure that extends $\mu$ to a measure on the whole of $\mathcal{P}(\{1, ..., n\}$. ∎

## 5.3 Integration and gluing

Most functions on $K$ that we'll want to integrate will be real-valued but without any loss of generality we'll work with functions that are $B$-valued (where $\mu$ takes value in the abelian group $B$). Since we'll want to work with infinite sums and we'll want to work with absolute convergence, so we'll assume $B$ is a topological abelian group and define absolutely convergence in that generality:

**Definition** We say a sum $\sum_i b_i$ with $b_i \in B$ is absolutely convergent if it is convergent and any rearrangement of the sum converges to the same limit.

This is the same as the usual definition in the case of $\mathbb{R}$. Our most common use-case will be the ring $\mathbb{R}((X))$, in which it corresponds to the following: suppose we have an infinite sum in $\mathbb{R}((X))$:

$$\sum_{j=1}^{\infty} \sum_{i=k_j}^{\infty} a_{i,j} X^i$$

Then it is absolutely convergent if each sum $\sum_{j=1}^{\infty} a_{i,j}$ is absolutely convergent (i.e. absolute convergence is coefficientwise) and the $k_j$ are all bounded from below (otherwise the infinite sum wouldn't convergence).

Our main idea will be to to integrate as many functions of the form $f = \sum_i \lambda_i \mathbb{1}_{A_i}$ as possible by means of the following formula:

$$\int \sum_i \lambda_i \mathbb{1}_{A_i} d\mu_K = \sum_i \lambda_i \mu_K(A_i)$$

For this to make sense and to be independent of our choice of representation for $f$, we will certainly require that the sum on the right hand side converges absolutely, so that will be one restriction on our function $f$. As our previous example shows, this condition will not be sufficient - essentially one can hide a conditionally convergent sum inside one that is nominally absolutely convergent by mixing the levels. Our solution is to explicitly label the levels. This may seem a little artificial but as we will see later, the theory of integration will depend on the labelling only in a nice way (i.e. a good labelling allows us to integrate more functions). The input datum that we'll end using consists of 5 pieces $(K, \mathcal{G}, \mu, B, L)$ satisfying the following:

1. $K$ is any set.
2. $\mathcal{G}$ is a collection of subsets of $K$ that is closed under intersections.
3. $\mu$, the measure, is a function from $\mathcal{G}$ to $B$ that satisfies the Inclusion-Exclusion principle.
4. $B$, the value group, is a topological abelian group.
5. $L$, the labelling, is a function with $\mathcal{G}$ as its domain

By the Key Lemma, this immediately gives us an algebra $\mathcal{A}$ containing $\mathcal{G}$ and an extension of $\mu$ to the algebra. The labelling $L$ doesn't immediately extend to $\mathcal{A}$, so we define the following concept:

**Definition** We call an element $A \in \mathcal{A}$ pure of label $l$ if it can be written in terms of $G_i \in \mathcal{G}$ with $L(G_i) = l$, $\sqcup$ and $\setminus$. That is to say:

1. Any element $G \in \mathcal{G}$ is pure of label $l$ if $L(G) = l$.
2. If $A, B \in \mathcal{A}$ are disjoint and pure of label $l$, then $A \sqcup B$ is pure of label $l$.
3. If $A, B \in \mathcal{A}$ satisfy $A \subset B$ and are pure of label $l$, then $B \setminus A$ is pure of label $l$

We will also simply say an element $A$ is pure when we don't care about the label.

With the notion of purity in hand, we are now in a position to say what the integrable functions will be:

**Definition** Given an appropriate input datum $(K, \mathcal{G}, \mu, B, L)$, define $M(K, \mathcal{G}, \mu, B, L)$, or $M(K, B)$ if unambiguous, to consist of those functions $f : K \to B$ such that:

1. $f$ can be written as $\sum_i \lambda_i \mathbb{1}_{A_i}$ with $\lambda_i \in B$ and each $A_i \in \mathcal{A}$ is pure.
2. The sum, i.e. the integral, $\sum_i \lambda_i \mu(A_i)$ is absolutely convergent.

**Remark** Since our decomposition does not require disjointness, the restriction that each $A_i$ is pure on its own is not a restriction on the function $f$. Its only in combination with the second condition that it's a restriction and as such one should see the whole labelling procedure as informing what kinds of absolutely convergent sums we allow (i.e. which functions have well-defined integral).

We then see that $\mu$ is integrable if we can define integration on $M(K, B)$ by means of the now familiar formula:

$$\int \sum_i \lambda_i \mathbb{1}_{A_i} d\mu = \sum_i \lambda_i \mu(A_i)$$

i.e. if this definition does not depend on the chosen representation of our function $f$.

**Example** Let $K$ be equal to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}_p$. Then we can define an integrable measure $\mu$ on $K$ based on the standard Haar measure as follows. We let $\mathcal{G}$ consist of all compact sets and $K$ itself (this is closed under intersections); for $A$ compact, we set $\mu(A) = \mu_{\text{Haar}}(A)$; $\mu(K)$ we set equal to 0. Then the Inclusion-Exclusion principle holds for $\mu$ because $K$ can not be written as the finite union as compact sets, so for any non-trivial equality of the form $A_0 = \cup_{i=1}^n A_i$, each $A_j$ must be compact and we inherit the Inclusion-Exclusion principle from the standard Haar measure. For the labelling, we simply pick the constant labelling (i.e. we are in a nice enough situation that we don't need to label sets). We'll write a lemma to show this is integrable.

**Lemma 5.2** *$\mu$, as just defined, is an integrable measure on $K = \mathbb{C}, \mathbb{R}$ or any finite extension of $\mathbb{Q}_p$.*

**Proof** Our definition of integrability is linear, so we only need to prove that if $\sum_i \lambda_i \mathbb{1}_{A_i} = 0$ with $A_i \in \mathcal{A}$ and $\sum_i \lambda_i \mu(A_i)$ absolutely convergent, then this sum is zero. Now observe that every element of $\mathcal{A}$ can be written as $A$ or $K \setminus A$ with $A$ a Haar-measurable set (one can check this collection is closed under $\cap, \cup, \setminus$) so we can split up this sum into two parts $\sum_i \lambda_i \mathbb{1}_{A_i} + \sum_i \nu_i \mathbb{1}_{K \setminus B_i}$ where $A_i$ and $B_i$ are Haar-measurable and both sums

$$\sum_i \lambda_i \mu(A_i), \sum_i \nu_i \mu(K \setminus B_i) = -\sum_i \nu_i \mu(B_i)$$

are absolutely convergent. Define $g = \sum_i \lambda_i \mathbb{1}_{A_i}$, $f = \sum_i \nu_i \mathbb{1}_{K \setminus B_i}$ and $\tilde{f} = \sum_i \nu_i \mathbb{1}_{B_i}$ - then $f + \tilde{f}$ is a constant function on $E$. $g$ and $\tilde{f}$ are integrable with respect to the original Haar measure on $E$ and thus $g - \tilde{f}$ is also integrable with respectable to the Haar measure on $E$ but also $g - \tilde{f} = g + f - (f + \tilde{f})$ is a constant function on $E$ since $g + f = 0$ and $f + \tilde{f}$ are both constant. The only integrable constant function on $E$ is 0, so $g = \tilde{f}$ and as we already know integration on $K$ with respect to the Haar measure is well-behaved, we obtain the equality:

$$\sum_i \lambda_i \mu(A_i) = \sum_i \nu_i \mu(B_i)$$

This is equivalent to the desired equality:

$$\sum_i \lambda_i \mu(A_i) + \sum_i \nu_i \mu(K \setminus B_i) = 0$$

∎

Now that was quite a bit of work even for a simple case, so we'd like some way to prove measures are integrable more easily. To do that, we have the following gluing lemma:

**Lemma 5.3** *Suppose we have integrable measures $(K, \mathcal{G}_i, \mu_i, B, L_i)$ with disjoint labellings $l_i$ satisfying the following three conditions:*

1. *For distinct $i, j \in I$, we have either that for all $A \in \mathcal{G}_i, B \in \mathcal{G}_j$, the intersection $A \cap B \in \{A, \emptyset\}$ or for all such $A, B$, $A \cap B \in \{B, \emptyset\}$.*
2. *Given an equality $A = \bigcup_{n \in \mathbb{N}} A_n$ with each $A_n \in \mathcal{A}_i$ for some $i$ and non-empty, then in fact each $i$ is equal.*
3. *Each measure $\mu_i$ satisfies the following: for each possible label $l$ there is a set $E_l \in \mathcal{G}_i$ which contains each set with label $l$ and has measure $\mu_i(E_l) = 0$.*

*Then we can glue together all these measures into a single integrable measure $(K, \mathcal{G}, \mu, B, L)$ with respect to the labelling $L$ which satisfies $L(A) = (L_i(A), i)$ for $A \in \mathcal{G}_i$.*

**Proof** Define $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}_i$ - this is closed under intersection by (1). Define $\mu$ on $\mathcal{G}$ by the formula $\mu(A) = \mu_i(A)$ for $A_i \in \mathcal{G}_i$ and the labelling $l$ as above - these are both well-defined by (2). Showing this defines a $\mathbb{R}((X))$-measure amounts to showing $\mu$ satisfies Inclusion-Exclusion. So suppose $A =$

$\bigcup_{k=1}^{n} A_k$ for some $A, A_k \in \mathcal{G}$. Suppose $A \in \mathcal{G}_i$, then define $K_i = \{k | A_k \in \mathcal{G}_i\}$ so that $B := A \setminus \bigcup_{k \in K_i} A_k \in \mathcal{A}_i$. It also holds that $B \subset \bigcup k \notin K_i A_k$ so tautologically we have $B = \bigcup_{k \notin K_i} A_k \cap B$; suppose $A_k \in \mathcal{G}_j$ and note that $A \cap A_k = A_k$, then by (1) intersecting $A_k$ with any element from $\mathcal{G}_i$ produces an element in $\mathcal{G}_j$. In particular, by distributivity of intersection, $A_k \cap B \in \mathcal{A}_j$. It follows that while $B = \bigcup_{k \notin K_i} A_k \cap B$ so by (2) the only possibility is that $B$ is empty and $A = \bigcup_{k \in K_i} A_k$ and Inclusion-Exclusion for $\mu$ follows from Inclusion-Exclusion for $\mu_i$.

Now we show that $\mu$ is integrable. Our definition of integrability is linear, so we may restrict to the case of the zero function, i.e. if $\sum_{n \in \mathbb{N}} \lambda_n \mathbb{1}_{A_n} = 0$, each $A_n$ is pure and the sum $\sum_{n \in \mathbb{N}} \lambda_n \mu(A_n)$ is absolutely convergent, then this sum is zero. Let $L$ denote the set of all labels and for $l \in L$ define $N_l$ to be those indices such that $A_n$ is labelled by $l$. Let $f_l = \sum_{n \in N_l} \lambda_n \mathbb{1}_{A_n}$ which is integrable. Then we get the following equality:

$$0 = \sum_{n \in \mathbb{N}} \lambda_n \mathbb{1}_{A_n} = \sum_{l \in L} \sum_{n \in N_l} \lambda_n \mathbb{1}_{A_n} = \sum_{l \in L} f_l$$

Now suppose for some label $m$ we have $\int f_m d\mu_i \neq 0$ (where $i$ is the appropriate index). Note that $f_m \mathbb{1}_{E_m} = f_m$ by definition of $E_m$ (see (3)). So multiplying the zero function by $\mathbb{1}_{E_m}$ and subtracting $f_m$ we obtain:

$$-f_m = \sum_{l \neq m} \sum_{n \in N_l} \lambda_n \mathbb{1}_{A_n \cap E_m}$$

Now we note that by condition (1) and distributivity of intersections, we have that if $A_n \in \mathcal{A}_j$, then $A_n \cap E_m$ is either $E_m$ or again an element of $\mathcal{A}_j$ - so we can divide $\mathbb{N} \setminus N_m$ into two parts: $N_+$ where $A_n \cap E_m = E_m$, $N_0$ and $N_-$ where $A_n \cap E_m \in \mathcal{A}_j$ for some $j \neq i$. Moving the sum with $N_+$ to the left-hand side we find the following:

$$-f_m - \sum_{n \in N_+} \lambda_n \mathbb{1}_{E_m} = \sum_{n \in N_-} \lambda_n \mathbb{1}_{A_n \cap E_m}$$

So the left-hand side is $g := -f_m - C\mathbb{1}_{E_m}$ for some constant $C$ - then $g$ still has non-zero integral as $\mu_i(E_m) = 0$, so we can find some set $A \in \mathcal{A}_i$ such that $g\mathbb{1}_A$ is non-vanishing. Then we can multiply our previous equality by $\mathbb{1}_A$:

$$g\mathbb{1}_A = \sum_{n \in N_-} \lambda_n \mathbb{1}_{A_n \cap A}$$

For $n \in N_-$, it holds that $A_n \cap A \in \mathcal{A}_j$ with $j \neq i$ as before, so looking at the support of the left-hand side and the right-hand side we get that

$A \subset \cup_{n\in N_-}(A_n \cap A)$ but also tautologically we have that $\cup_{n\in N_-}(A_n \cap A) \subset A$ so we obtain a contradiction with (2). ∎

**Corollary 5.4** *If we have integrable measures* $(K, \mathcal{G}_i, \mu_i, B, L_i)$ *with disjoint labellings* $l_i$ *that also have disjoint support in the sense that for* $A \in \mathcal{G}_i, B \in \mathcal{G}_j$ *(with* $i \neq j$*,* $A \cap B = \emptyset$*, then we can glue the measures to a single integrable measure* $(K, \mathcal{G}, \mu, B, L)$ *with respect to the labelling* $L$ *which satisfies* $L(A) = (L_i(A), i)$*.*

# 6 Fourier Transform on 2-dimensional local fields

## 6.1 A more refined measure

To define a Fourier Transform on a 2-dimensional local field $F$, we need to be able to integrate sufficiently many functions so that we can make sense of expressions such as $\int f\chi d\mu$ where $\chi$ is a character on $F$. To do that, as it turns out, we need to refine the measure and gain access to more measurable sets. To do so, we fix a natural number $n$; the case $n = 1$ corresponds to the standard measure on $F$. For each $n$, we'll define a measure $\mu_n$ on a collection $\mathcal{G}_n$ and in the end we'll relate each measure to the $n = 1$ case by means of a limiting procedure. We note that we can think of our existing measure as gluing together the pullbacks of the measure on $E$ through the projections $\rho_i : t_2^i\mathcal{O} \to t_2^i\mathcal{O}/t_2^{i+1} \simeq E$ and for any measurable subset $Z$ of $E$, we have the following:

$$\mu(a + \rho_i^{-1}(Z)) = \mu_E(Z)X^i$$

Similarly we can construct projections $p_i : t_2^i\mathcal{O} \to t_2^i\mathcal{O}/t_2^{i+n} \to E^n$ where the second arrow is a continuous bijection and we can try to generalize this definition to define a measure $\mu_n$ by the following formula for a measurable subset $Z$ of $E^n$:

$$\mu_n(a + p_i^{-1}(Z)) = \mu_{E^n}(Z)X^i$$

**Remark** These projections are related to our goal of harmonic analysis as follows: if we have a character $\chi$ on $F$ with conductor equal to $t_2^nO$, then the function $\chi\mathbb{1}_O$ factors through the projection $p_0$.

There's no canonical choice of projections $p_i$ in the mixed-characteristic case - this is essentially because there are many sections to the quotient map $\mathcal{O} \to \mathcal{O}/t_2\mathcal{O}$ and although they agree mod $t_2$ by definition, they don't agree mod $t_2^n$. Let $\pi_E$ be a uniformizer of $E$, then we choose a lift $s : E \to \mathcal{O}$ satisfying $s(\pi_E^n u) = t_1^n s(u)$ (in equal characteristic $s$ will be an inclusion map). Then, having fixed $s$, we know we can write any $x \in t_2^i$ in the following form:

$$x = \sum_{j=i}^{\infty} x_j t_2^j$$

where each $x_j$ lies in the image of $s$. Then we define $p_i$ by the following equality:

$$p_i(x) = (x_i, ..., x_{i+n-1})$$

In particular, our projections glue together to give a map $P : K \to E^{\mathbb{Z}}$ in the sense that $p_i(x) = (P(x)(i), ..., P(x)(i+n-1))$.

To show that this leads to a well-defined finitely additive $\mathbb{R}((X))$-valued measure on a ring $\mathcal{A}_n$ as in the case $n = 1$, we will first restrict ourselves to a smaller collection than all sets of the form $a + p_i^{-1}(Z)$, to simplify the proofs. To every $\vec{z} = (z_1, ..., z_n) \in E^n$, $\vec{r} = (r_1, ...r_n) \in \mathbb{R}^n_{>0}$, we can associate the multiball:

$$B(\vec{z}, \vec{r}) = \{(x_1, ..., x_n) \in E^n | \; |x_i - z_i| < r_i \text{ for all } i\}$$

We also call $i$ the level of the set $a + p_i^{-1}(Z)$. Then our initial collection of objects $\mathcal{G}_n$ consists of translates of pullbacks of multiballs $a + p_i^{-1}(B(\vec{z}, \vec{r}))$ where we require that the constant $a \in K$ satisfies $P(a)(j) = 0$ for $i \leq j \leq i+n-1$ (this restriction simplifies the exposition a bit and lets us postpone dealing with a key difficulty in the mixed-characteristic case until a more appropriate time) and in addition we also have the empty set.

We'd like to apply our Key Lemma to this scenario to produce a measure, so we first check that $\mathcal{G}_n$ is closed under intersection:

**Lemma 6.1** *$\mathcal{G}_n$ is closed under intersection.*

**Proof** Suppose we have two sets in $\mathcal{G}_n$, $a + p_i^{-1}(B(\vec{z}, \vec{r}))$ and $a' + p_j^{-1}(B(\vec{z'}, \vec{r'}))$. We can rewrite the condition $x \in a + p_i^{-1}(B(\vec{z}, \vec{r}))$ in terms of the map $P$ - it is equivalent to the requirement that $P(x)(k) = P(a)(k)$ for $k < i$, that $P(x)(k) \in B(z_{k+1-i}, r_{k+1-i})$ for $i \leq k \leq i+n-1$ and $P(x)(k)$ can be anything for $k \geq i+n$. We can write this more uniformly as:

1. $P(x)(k) \in \{P(a)(k)\}$ for $k < i$
2. $P(x)(k) \in B(z_{k+1-i}, r_{k+1-i})$ for $i \leq k \leq i+n-1$
3. $P(x)(k) \in E$ for $k \geq i+n$

We note that the intersection of two open balls in $E$ is again an open ball in $E$ or empty. Assume first that $i = j$, then $x$ is in the intersection $(a + p_i^{-1}(B(\vec{z}, \vec{r}))) \cap (a' + p_i^{-1}(B(\vec{z'}, \vec{r'})))$ if and only if the following hold

1. $P(x)(k) \in \{P(a)(k)\} \cap \{P(a')(k)\}$ for $k < i$
2. $P(x)(k) \in B(z_{k+1-i}, r_{k+1-i}) \cap B(z'_{k+1-i}, r_{k+1-i})'$ for $i \leq k \leq i+n-1$
3. $P(x)(k) \in E$ for $k \geq i+n$

So we see that the intersection is equal to an element of $\mathcal{G}_n$ of level $i$ or the empty set. Now we assume wlog that $i < j$. Then we see that $x$ is in the intersection $(a + p_i^{-1}(B(\vec{z}, \vec{r}))) \cap (a' + p_j^{-1}(B(\vec{z'}, \vec{r'})))$ if and only if the following hold:

1. $P(x)(k) \in \{P(a)(k)\} \cap \{P(a')(k)\}$ for $k < i$
2. $P(x)(k) \in B(z_{k+1-i}, r_{k+1-i}) \cap \{P(a')(k)\}$ for $i \leq k \leq j - 1$
3. $P(x)(k) \in B(z_{k+1-i}, r_{k+1-i}) \cap B(z'_{k+1-i}, r'_{k+1-i})$ for $j \leq k \leq i + n - 1$
4. $P(x)(k) \in E \cap B(z'_{k+1-i}, r_{k+1-i})'$ for $i + n \leq k \leq j + n - 1$
5. $P(x)(k) \in E$ for $k \geq i + n$

We see that we either get a condition of the form $P(x)(k) \in \emptyset$ and the intersection is empty or the conditions are of the form $P(x)(k) \in A_k$ where $|A_k| = 1$ if $k < j$, $A_k$ is an open ball in $E$ for $j \leq k \leq j + n - 1$ and $A_k = E$ for $k \geq j + n$, in which case the intersection is an element of $\mathcal{G}_n$ of level $j$. $\blacksquare$

Now we just need to check the Inclusion-Exclusion principle. We do that as follows: suppose we have $A_0, ..., A_l \in \mathcal{G}_n$ with $A_0 = \bigcup_{j=1}^{l} A_j$. Write each $A_j$ as $a_j + p_{i_j}^{-1}(B(\vec{z}_j, \vec{r}_j))$. It is true by assumption that $A_j \cap A_0 = A_j$, so following the proof that $\mathcal{G}_n$ is closed under intersection, we can conclude that $i_j \geq i$ and $P(a_j)(k) = P(a_0)(k)$ for $k < i_0$ (otherwise the intersection would be empty and we're assuming non-triviality). So we can subtract $a_0$ from each $a_j$ and reduce to the case that $A_j \subset t_2^{i_0}\mathcal{O}$ for all $0 \leq j \leq l$. In particular, we can apply the projection $p_{i_0}$ and work in $E_n$. If $i_j = i_0$, then $A_j$ corresponds to a multiball in $E^n$; if $i_j > i_0$, then $A_j$ corresponds to a subset of $E^n$ that is at most $(n-1)$-dimensional. Since the set-theoretical difference of two open sets in $E^n$ will always be empty or $n$-dimensional, by non-triviality we conclude that $i_j = i_0$ for all $j$. Then the inclusion-exclusion principle in this case reduce to the inclusion-exclusion principle for $\mu_{E^n}$ which we know holds. $\blacksquare$

Thus, the lemma gives us a ring, which we will call $\mathcal{A}_n$ and a measure, which by abuse of notation we still refer to as $\mu_n$. $\mu_n$ satisfies similar properties to those of $\mu_1$ we previously mentioned, which merits a proposition:

**Proposition 6.2**

1. *For $a \in K, A \in \mathcal{A}_n$, $a + A \in \mathcal{A}_n$ and $\mu_n(a + A) = \mu_n(A)$*
2. *$\mu_n(\varepsilon t_1^j t_2^k A) = \mu(A) q^{-nj} X^k$ for all $\varepsilon \in O^\times, A \in \mathcal{A}_n$*

**Proof** For the first and second statement, we can immediately reduce to the case that $A \in \mathcal{G}_n$.

We will prove these first in the relatively easy equal-characteristic case. In that case, $s$ is just the inclusion map and we can write $A$ in the following more explicit form:

$$A = b + B(z_1, r_1)t_2^i + ... + B(z_n, r_n)t_2^{i+n-1} + t_2^{i+n}\mathcal{O}$$

where $P(b)(k) = 0$ for $i \leq k \leq i + n - 1$ and conversely any set that can be written in such a form is in $\mathcal{G}_n$. Then we can write $a = \sum_{j=N}^{\infty} a_j t_2^j$ for some $N \in \mathbb{Z}$ and with $a_j \in E$. For $e \in E$, it holds that $B(z, r) + e = B(z + e, r)$, so we can calculate that

$$a+A = (b+\sum_{j=N}^{i-1} a_j t_2^j)+B(z_1+a_i, r_1)t_2^i+...+B(z_n+a_{i+n-1}, r_n)t_2^{i+n-1}+t_2^{i+n}\mathcal{O}$$

which is an element of $\mathcal{G}_n$ with the same measure as $A$. $\blacksquare$

To prove the second statement in the equal-characteristic case, using translation invariance we can eliminate the constant to reduce to the case that $A = \sum_{j=0}^{n-1} B(z_j, r_j)t_2^{i+j} + t_2^{i+n}\mathcal{O}$ (we switch to this notation so that we can manipulate things a little easier). Then we can immediately note the following:

$$t_1^j t_2^k A = \sum_{j=0}^{n-1} t_1^j B(z_j, r_j)t_2^{i+k+j} + t_2^{n+k}\mathcal{O}$$

Since $\mu_E(t_1^j B(z_i, r_i)) = q^{-j}$, this implies that $\mu(t_1^j t_2^k A) = q^{-nj} X^k \mu(A)$. To show that $\mu(\varepsilon A) = \mu(A)$ for any $\varepsilon \in O^*$, we make a further reduction step. We can split up a ball $B(z, r)$ into a finite disjoint union of balls with strictly smaller radius, so any multiball can be written as a finite disjoint union of multiballs satisfying the condition that $r_i \ll r_j$ whenever $i < j$ - we use $\ll$ here because the exact nature of the condition depends on $\varepsilon$. We can write $\varepsilon \equiv \varepsilon_0 + \sum_{j=1}^{n-1} \varepsilon_j t_2^j \mod t_2^n$ where $\varepsilon_0 \in \mathcal{O}_E^\times$ by assumption. Then we have

$$\varepsilon A = \sum_{j=0}^{n-1}[\varepsilon_0 B(z_j, r_j) + \sum_{k=1}^{n-1} \varepsilon_k B(z_{n-1-k}, r_{n-1-k})]t_2^{i+j} + t_2^n\mathcal{O}$$

Now, on the level of $E$, we note the following two identities: $eB(z, r) = B(ez, r|e|_E)$ for any $e \in E$ and $B(z, r) + B(z', r') = B(z + z', r)$ if $r' < r$. Using our assumption that $r_i \ll r_j$ whenever $i < j$, this lets us calculate the previous expression explicitly as being equal to the following:

$$\varepsilon A = \sum_{j=0}^{n-1} B(\sum_{k+l=j} \varepsilon_k z_l, r_j) t_2^{i+j} + t_2^n \mathcal{O}$$

Thus, it follows that $\varepsilon A \in \mathcal{A}_n$ with $\mu(\varepsilon A) = \mu(A)$. ∎

To prove the first and second statement in the mixed-characteristic case, similarly as above, we reduce to the case that $A \in \mathcal{G}_n$ and that if we write $A = b + p_i^{-1}(B(\vec{z}, \vec{r}))$, then $r_i \ll r_j$ whenever $i < j$ although now the exact nature of $\ll$ will also depend on $a$. So let $a \in K$, then we can write $a + b$ as $c + d$ where $P(c)(k) = 0$ for $k < i$ and $d \in t_2^i \mathcal{O}$. Now we wish to understand the set $d + p_i^{-1}(B(\vec{z}, \vec{r}))$, so we note that the condition $x \in d + p_i^{-1}(B(\vec{z}, \vec{r}))$ is equivalent to $x \in t_2^i \mathcal{O}$ and $P(x - d)(i - 1 + k) \in B(z_k, r_k)$ for $1 \leq k \leq n$. We will now prove by induction on $k$ that $P(x - d)(i - 1 + k) \in B(z_k, r_k)$ is equivalent to a condition of the form $P(x) \in B(c_k, r_k)$ for some constants $c_k$. For $x, d \in t_2^i \mathcal{O}$, it holds that $P(x - d)(i) = P(x)(i) - P(d)(i)$ since the map $\rho_i = P(\cdot)(i)|_{t_2^i \mathcal{O}}$ is additive, so the condition $P(x - d)(i) \in B(z_1, r_1)$ is equivalent to $P(x)(i) \in B(c_1, r_1)$ where $c_1 = z_1 + P(d)(1)$, which establishes the base case. Now suppose the claim holds for all $j < k$ and we'll prove it for $k$. The induction step is a little involved, so we'll prove it do it in the specific case $n = 2$, $k = 2$, $i = 0$ and then say how to generalize it. We can write $x \equiv x_0 + x_1 t_2 \mod t_2^2$ where $x_j = s(P(x)(j))$ for $j = 0, 1$. Similarly, if we define $e_j = s(P(x_0 - d)(j))$, then $x_0 - d \equiv e_0 + e_1 \mod t_2^2$ for all $x \in \mathcal{O}$. It follows that $x - d \equiv e_0 + (e_1 + x_1)t_2 \mod t_2^2$ and $P(x - d)(1) = \rho_1(e_1 + x_1) = \rho_1(e_1) + P(x)(1)$ and $P(x - d)(1) \in B(z_2, r_2)$ becomes equivalent to $P(x)(1) \in B(c_2, r_2)$ for $c_2 = z_2 - \rho_1(e_1)$. The only issue here is that $\rho_1(e_1)$ is not constant; however we can think of it a continuous function $E \to E$ in the variable $P(x)(0)$ by writing it as $\rho_1(s(P(s(P(x)(0)) - d)))$, so since $r_1 \ll r_2$ by assumption and $P(x)(0) \in B(c_1, r_1)$ by the induction hypothesis, continuity implies that we get a unique well-defined $B(c_2, r_2)$.

To prove the induction step in the general case, you give the same definition for $e_i$ but $e_k$ will now be a continuous function of the $k - 1$ variables $P(x)(i + j)$ for $0 \leq j \leq k - 2$, but since $P(x)(i + j) \in B(c_j, r_j)$ by the induction hypothesis and $r_j \ll r_k$, we can still conclude that we get a unique well-defined ball $B(c_k, r_k)$. It follows that $d + p_i^{-1}(B(\vec{z}, \vec{r}))$ is equal to $p_i^{-1}(B(\vec{c}, \vec{r})$, so we have

$$a + A = c + p_i^{-1}(B(\vec{c}, \vec{r}))$$

and $a + A \in \mathcal{A}_n$ and furthermore $\mu_n(a + A) = \mu_n(A)$, so we have translation invariance.

Now to prove the second statement in the mixed-characteristic case. Firstly, as in the equal-characteristic case, we can eliminate the constant now that we have translation invariance and write $A = p_i^{-1}(B(\vec{z}, \vec{r}))$ with $r_i \ll r_j$ whenever $i < j$. Then, as $s(\pi_E) = t_1$, we have

$$t_1^j t_2^k A = p_{i+k}^{-1}(\pi_E^j B(\vec{z}, \vec{r}))$$

so $t_1^j t_2^k A \in \mathcal{A}_n$ and $\mu_n(t_1^j t_2^k A) = \mu_n(A) q^{-jn} X^k$ since $\mu_E(\pi_E^j A) = q^{-j} A$ for any measurable set $A$ in $E$. Now let $\varepsilon^{-1} \in O^*$ (we take the inverse purely to simply the notation) and we want to show that $\varepsilon^{-1} p_i^{-1}(B(\vec{z}, \vec{r})) = p_i^{-1}(B(\vec{c}, \vec{r}))$ for some $\vec{c} \in E^n$. The condition $x \in \varepsilon^{-1} p_i^{-1}(B(\vec{z}, \vec{r}))$ is again equivalent to $x \in t_2^i \mathcal{O}$ and $P(x\varepsilon^{-1})(i - 1 + k) \in B(z_k, r_k)$ for $1 \le k \le n$ and we will prove the claim by induction on $k$. For $k = 1$, $P(\cdot)(i)$ coincides with $\rho_i$ on $t_2^i \mathcal{O}$, so $P(x\varepsilon)(i) = P(x)(i)\rho_0(\varepsilon)$. Now assume we've proved it for all $j < k$, then define $x_l = s(P(x)(l))$ for $i \le l \le i - 1 + k$ and $e_k = s(P(\sum_{l=i}^{i+k-2} x_l \varepsilon)(k))$. Then, by construction of $e_k$, we have $P(x\varepsilon)(k) = P(x)(k)\rho_0(\varepsilon) + \rho_{i+k-1}(e_k)$ where $\rho_{i+k-1}(e_k)$ is a continuous function of the $k-1$ variables $P(x)(l)$ for $i \le l \le i + k - 2$, so by the induction hypothesis and our assumption on the radii, the ball $B(c_k, r_k)$ is unique and well-defined where $c_k = \frac{z_k - \rho_{i+k-1}(e_k)}{\rho_0(\varepsilon)}$. ∎

## 6.2 Integration with respect to this measure

We have a measure $\mu_n$. To talk about integration, we need a labelling. Our labelling will be $L(a + p_i^{-1}(B(\vec{z}, \vec{r})) = (a, i)$.

**Proposition 6.3** *$\mu_n$ is integrable with respect to this labelling.*

**Proof** Apply the gluing lemma.

**Remark** A very important class of functions $f : F \to \mathbb{C}$ that belongs to $M(F, \mu_n, \mathbb{C}((X)))$ is the following: if $f$ factors through a map of the form $x \mapsto p_i(x - a)$ with $P(a)(k) = 0$ for $i \le k \le i + n - 1$ and the induced function $\tilde{f}$ from $E^n$ to $\mathbb{C}((X))$ is integrable and has countable image, then $f \in M(F, \mu_n, \mathbb{C}((X)))$. This follows simply from writing

$$f = \sum_{y \in f(F)} y \mathbb{1}_{a + p_i^{-1}((\tilde{f}^{-1}(\{y\}))}$$

and observing that $\tilde{f}$ being integrable means that we have the following equality and the sum on the left hand side converges absolutely:

$$\sum_{y \in f(F)} y \mu_n(a + p_i^{-1}((\tilde{f}^{-1}(\{y\}))) = \int_{E^n} \tilde{f} d\mu_{E^n} X^i$$

We also note that our integral has the same change-of-variable property as the measure:

**Lemma 6.4** *For any $f \in M(F, \mu_n, \mathbb{C}((X)))$, $\varepsilon \in O^*$ and $b \in F$, we have:*

$$\int_n f(\varepsilon t_1^i t_2^j x + b) d\mu_n(x) = q^{-ni} X^j \int_n f d\mu_n$$

**Proof** This follows straightforwardly from the same property of the measure.

**Example** Since our goal is to define a Fourier transform, we want to calculate integrals of the following type

$$\int_n \mathbb{1}_{a+A}(x)\chi(\alpha x) d\mu_n$$

where $a \in K$ and $A = p_i^{-1}(B(\vec{0}, \vec{r})$ (it being centered at $\vec{0}$ means exactly that for every $y \in A$, $y + A = A$), $\mathrm{ord}_{t_2}(\alpha) = -i - n$ and $\chi$ the standard character with conductor equal to $O$ - then the integral exists by the remark preceding the lemma. We will need to define the following set

$$\widehat{A} = \{\alpha \in K | \ \chi(\alpha x) = 1 \ \forall x \in A\}$$

Then we can calculate the integral in the classical way: take any $y \in A$ and apply our assumed equality $y + A = A$:

$$\int_n \mathbb{1}_{a+A}(x)\chi(\alpha x) d\mu_n = \chi(\alpha y) \int_n \mathbb{1}_{a+A}(x)\chi(\alpha x)$$

and we conclude that the integral is non-zero if and only if $\alpha \in \mathbb{1}_{\widehat{A}}$, which means we obtain the following

$$\int_n \mathbb{1}_{a+A}(x)\chi(\alpha x) d\mu_n = \mathbb{1}_{\widehat{A}} \chi(\alpha a)$$

One can also show that $\widehat{A} \in \mathcal{A}_n$ and that $\widehat{\widehat{A}} = A$ but we will not need these facts.

## 6.3 Connecting the theories of integration

Although we have now defined provisional integration theories for each $\mu_n$ that allow us to do some calculations, for $n \geq 2$, they are not as relevant to what we want to be doing as the case $n = 1$ - they do not satisfy the right scaling and for example, the most basic function of the theory $\mathbb{1}_O$ lies only in $M(F, \mu_n, \mathbb{C}((X)))$ and not in $M(F, \mu_n, \mathbb{C}((X)))$ when $n \geq 2$. So our aim now is to repair that defect. We introduce the following notation:

1. If $A \in \mathcal{G}_n$, i.e. $A$ is of the form $a + p_{i,n}^{-1}(B(\vec{z}, \vec{r}))$ and $Z$ a subset of $E^m$ of finite measure, then

$$A * Z = a + p_{i,n+m}^{-1}(B(\vec{z}, \vec{r}) \times Z)$$

2. If $A_1 * Z$ and $A_2 * Z$ are defined, then $(A_1 \sqcup A_2) * Z = (A_1 * Z) \sqcup (A_2 * Z)$ and $(A_1 \setminus A_2) * Z = (A_1 * Z) \setminus (A_2 * Z)$.

3. It follows that $\mu_{n+m}(A * Z) = \mu_n(A)\bar{\mu}^m(Z)$

Let's now consider the example of an indicator function of a set $A \in \mathcal{A}_n$. If we take an increasing sequence $B_i$ such that $\cup_i B_i = E$, then $\mathbb{1}_{A*B_i} \in M(F, \mu_{n+1}, \mathbb{C}((X)))$ and the sequence converges pointwise to $\mathbb{1}_A$. If we consider the sequence of integrals, we get

$$\int_{n+1} \mathbb{1}_{A*B_i} d\mu_{n+1} = \mu_n(A)\bar{\mu}(B_i)$$

This implies that the correct quantity to consider if one wants to obtain something commensurate with $\mu_n$ is the quotient $\frac{\int_{n+1} \mathbb{1}_{A*B_i} d\mu_{n+1}}{\bar{\mu}(B_i)}$. This then leads us to note the following somewhat tautological identity:

$$\int_n \mathbb{1}_A d\mu_n = \lim_{i\to\infty} \frac{\int_{n+1} \mathbb{1}_{A*B_i} d\mu_{n+1}}{\bar{\mu}(B_i)}$$

This motivates the following definition. First, for the increasing sequence, we take the most straightforward option of $B_i = t_1^{-i} O_E^{n-1}$. If we have some function $f : F \mapsto \mathbb{C}$ of the form $\sum_{j\in\mathbb{N}} f_j \mathbb{1}_{A_j}$ with $A_j \in \mathcal{A}_1$ and $f_j$ functions from $F$ to $\mathbb{C}$, then $\sum_{j\in\mathbb{N}} f_j \mathbb{1}_{A_j * B_i}$ converges pointwise to $f$. Now we define a set of integrable functions $L_n(\mathbb{C})$ as follows: for $f$ as above, $f \in L_n(\mathbb{C})$ if $\sum_{j\in\mathbb{N}} f_j \mathbb{1}_{A_j * B_i} \in M(F, \mu_2, \mathbb{C}((X)))$ for all $B_i$ and $\lim_{i\to\infty} \frac{\int_n \sum_{j\in\mathbb{N}} f_j \mathbb{1}_{A_j * B_i} d\mu_2}{\bar{\mu}(B_i)}$ converges. In that case, we say that

$$\int f d\mu = \lim_{i\to\infty} \frac{\int_n \sum_{j\in\mathbb{N}} f_j \mathbb{1}_{A_j * B_i} d\mu_2}{\bar{\mu}(B_i)}$$

If we also let $L_1(\mathbb{C}) = M(F, \mu_1, \mathbb{C}((X)))$, then we obtain the following lemma:

**Lemma 6.5** *1.* $L_1(\mathbb{C}) \subset L_2(\mathbb{C}) \subset L_3(\mathbb{C})$...

*2. For any* $a, b \in K$, $f \in L_n(\mathbb{C})$,

$$\int f(ax+b) d\mu(x) = \frac{1}{|a|} \int f d\mu$$

**Proof** Let $n \geq 1$ and $f \in L_n(\mathbb{C})$, we will show that $f \in L_{n+1}(\mathbb{C})$. Suppose first that $f$ can be written as $f\mathbb{1}_A$ with $A$ a generating set of $\mathcal{A}_1$ such that $f\mathbb{1}_{A*B_i^{n-1}} \in M(F, \mu_n, \mathbb{C}((X)))$ and $\lim_{i\to\infty} \frac{\int_n f\mathbb{1}_{A*B_i^{n-1}} d\mu_n}{q^{(n-1)i}}$ converges. If $A = a + t_1^N t_2^j O_E + t_2^{j+1} O$, define $\mathrm{env}(A) = a + O_E t_2^{j-1} + t_2^j O$. Then by construction we have that $\mathrm{env}(A) * B_i^n \cap A * B_i^{n-1} = A * B_i^{n-1}$ if $i \geq N$ and $A \subset \mathrm{env}(A)$. It follows that $f = f\mathbb{1}_A \mathbb{1}_{\mathrm{env}(A)}$ and we have the following equality for $i \geq N$

$$\frac{\int_n f\mathbb{1}_{A*B_i^{n-1}} d\mu_n}{q^{(n-1)i}} = \frac{\int_{n+1} f\mathbb{1}_A \mathbb{1}_{\mathrm{env}(A)*B_i^n} d\mu_n}{q^{(n-1)i} q^i}$$

This then immediately implies that $f \in L_{n+1}(\mathbb{C})$. The case that $f = \sum_{i=0}^k f_i \mathbb{1}_{A_i}$ with $A_i \in \mathcal{A}_1$ follows from this one using linearity and the general case then follows by taking limits.

**Definition** We define the set $L(\mathbb{C})$ of all integrable functions on $F$ as follows:

$$L(\mathbb{C}) = \bigcup_{i=1}^{\infty} L_i(\mathbb{C})$$

**Example** Let $\alpha \in K$ and $\chi$ the standard character on $K$, then $x \mapsto \mathbb{1}_O(x)\chi(\alpha x) \in L(\mathbb{C})$ and we have the following equality:

$$\int \mathbb{1}_O(x)\chi(\alpha x) d\mu = \mathbb{1}_O(\alpha)$$

The equality is immediate when $\alpha \in O$ as then the integrand is then simply $\mathbb{1}_O(x)$. Now assume $\alpha \in t_2^{-n} O$, then we've already calculated the following integral:

$$\int_{n+1} \mathbb{1}_{O*(t_1^{-j} O_E^n)}(x)\chi(\alpha x) d\mu_{n+1} = \mathbb{1}_{\widehat{O*(t_1^{-j} O_E^n)}}(\alpha)$$

We observe that $\widehat{O} = O$ and that $O = \cup_j O * (t_1^{-j} O_E^n)$, so if $\alpha \notin O$, then there is some $N$ such that for all $j > N$, $\alpha \notin \widehat{O * (t_1^{-j} O_E^n)}$. That is to say, for $\alpha \notin O$, we have

$$\lim_{j\to\infty} \frac{\int_{n+1} \mathbb{1}_{O*(t_1^{-j} O_E^n)}(x)\chi(\alpha x) d\mu_{n+1}}{q^{nj}} = 0$$

which finishes the calculation.

## 6.4 Defining the Fourier transform

**Definition** For any $f \in L(\mathbb{C})$, we define, if it converges, the Fourier transform as follows:

$$\mathcal{F}(f)(\alpha) = \int f(x)\chi(\alpha x)d\mu(x)$$

**Proposition 6.6** *For any $f \in L_1(\mathbb{C})$, $\mathcal{F}(f)$ is well-defined, integrable and furthermore Fourier inversion holds:*

$$\mathcal{F}^2(f)(x) = f(-x)$$

Before proving the proposition in full generality, we first show it holds in the case of indicator functions $\mathbb{1}_A$ of sets $A \in \mathcal{A}_1$.

**Example** We know from the previous example that $\mathcal{F}(\mathbb{1}_O) = \mathbb{1}_O$. Since we know how our integral behaves under linear transformations, this can immediately be generalized to the following:

$$\int \mathbb{1}_{a+t_1^i t_2^j O}(x)\chi(\alpha x)d\mu = |t_1^i t_2^j| \mathbb{1}_{t_1^{-i} t_2^{-j} O}(\alpha)\chi(\alpha a)$$

That is to say:

$$\mathcal{F}(\mathbb{1}_{a+t_1^i t_2^j O})(\alpha) = |t_1^i t_2^j| \mathbb{1}_{t_1^{-i} t_2^{-j} O}(\alpha)\chi(\alpha a)$$

It is now to easy to check that Fourier inversion holds. Since any indicator function $\mathbb{1}_A$ can be written as a linear combination of indicator functions of generating functions and the Fourier transform is linear, it follows that

$$\mathcal{F}^2(\mathbb{1}_A)(x) = \mathbb{1}_A(-x)$$

Now we prove the proposition:

**Proof** First we will show that $\mathcal{F}(f)$ exists; we will only need to show this for $\mathcal{F}(f)(1)$ as $\mathcal{F}(f)(0)$ exists by assumption and the rest follows from linear change of variables. Write $f = \sum_{i=0}^{\infty} \lambda_i \mathbb{1}_{A_i}$ where the $A_i$ are disjoint differences of two sets of pure level in $\mathcal{A}_1$ and $\sum_{j=0}^{\infty} \lambda_i \mu(A_i)$ converges absolutely. We observe as before that we have a generating set $a + A \in \mathcal{A}_1$ where $A$ is centered at 0, then by standard properties of characters one of the two following equalities holds:

$$\int \chi(x)\mathbb{1}_{a+A}d\mu_n = \chi(a)\mu(A_i)$$

$$\int \chi(x)\mathbb{1}_{a+A}d\mu_n = 0$$

Now if we have a set $D \in \mathcal{A}_n$ of pure level, on the one hand $D$ corresponds to a subset of $E^n$, so the equality $|\chi(x)| = 1$ implies the following inequality:

$$|\int \chi(x)\mathbb{1}_D d\mu_n| \leq \mu(D)$$

Conversely, we can write the indicator function $\mathbb{1}_D$ in terms of indicator functions of generating sets $\mathbb{1}_{a_i+A_i}$ to which we can apply the previous dichotomy. So $\int \chi(x)\mathbb{1}_D d\mu_n = c\mu(D)$ for some complex number $c$ with $|c| \leq 1$. It follows that if we write $A_i = C_i \setminus D_i$ with $C_i$ and $D_i$ of pure level, there exists sequences $c_i$ and $d_i$ of complex numbers of modulus at most 1 such that

$$\int \sum_{i=0}^{\infty} \lambda_i \chi(\alpha x)\mathbb{1}_{A_i} d\mu = \sum_{i=0}^{\infty} \lambda_i (c_i \mu(C_i) - d_i \mu(D_i))$$

The latter sum converges absolutely in $\mathbb{C}((X))$ so $\mathcal{F}(f)(1)$ converges and it makes sense to talk of a Fourier transform. Fourier inversion for arbitrary functions in $L_1(\mathbb{C})$ follows immediately from Fourier inversion for indicator functions as shown in the example.

# 7 Coset measure on $SL(2,F)$ and $GL(2,F)$

## 7.1 Iwasawa decomposition

**Lemma 7.1** *Let $A \in GL(n,K)$. Then there exists a matrix $H \in GL(n,O)$, a diagonal matrix $D$ and a unipotent upper-triangular matrix $U$ such that*

$$A = HUD$$

**Proof** This is equivalent to showing that there exist $D$ and $U$ with $ADU \in GL(n,O)$ (since both diagonal matrices and u.u.t. matrices are closed under inversion). If we think of $A$ as consisting of $n$ row-vectors $\vec{x}_i$, then for each $\vec{x}_i$, there exist a scalar $d_i \in K$ such that $d_i\vec{x}_i \in O^n$ but also $\vec{x}_i \not\equiv 0 \bmod M^n$; we define $D$ as the diagonal matrix with $d_i$ as the $i$-th diagonal element. Then $AD$ has entries in $O$ and it makes sense to talk of the reduction of $AD$ mod $M$ which we'll denote $\bar{A}\bar{D}$; by the determinant criterion for invertibility, $AD \in GL(n,O)$ if and only if $\bar{A}\bar{D} \in GL(n,E)$. The latter is equivalent to the $\vec{x}_i$ being linearly independent mod $M$, whereas our assumption that $A$ is invertible merely implies that there exist a $N$ such that $\vec{x}_i$ are linearly independent over $M^N$. Using induction, we will show we can make the $\vec{x}_i$ linearly independent by a u.u.t. transformation as follows: suppose $\vec{x}_j$ are linearly independent for $i+1 \le j \le n$. If $\vec{x}_i$ is linearly independent from those, we are done. Otherwise we have a relationship of the form:

$$\vec{x}_i \equiv \sum_{j=i}^{n} \lambda_j \vec{x}_j \bmod M$$

where $\lambda_j$ correspond to elements in $E$. So, if we multiply the matrix $AD$ by the unipotent upper-triangular matrix $U_i$ whose only non-zero entries above the diagonal consist of $\lambda_j$'s in the $(i,j)$-th spot, then the new $\vec{x}_i$ will be 0 mod $M$ but then we can adjust $d_i$ so that again $\vec{x}_i$ will be in $O$ but non-zero mod $M$. This process must eventually halt within $N$ steps since the original $\vec{x}_i$ were independent mod $M^N$, so after at most $(n-1)N$ unipotent upper-triangular transformations, the matrix $AD$ will be in $GL(n,O)$. Since unipotent upper-triangular transformations form a group, this completes the proof.

## 7.2 Coset measure on $SL(2,F)$

Our goal is to define rigorously a $\mathbb{R}((X))$-valued measure on $SL(2,F)$ that will allow us to define integration. Matt Waller has defined a measure on $SL(2,O)$ rather than on the full group $SL(2,F)$ (e.g. his theory has no sets bigger than $X^{-1}$), so it seems natural focus on the coset space $SL(2,F)/SL(2,O)$.

**Proposition 7.2** *For every coset of $SL(2,O)$, we can find a representative of the form*

$$\begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

*where $u$ is unique up to $t_1^{-2i} t_2^{-2j} O$.*

**Proof** By the Iwasawa decomposition, we can write any element of $SL(2,F)$ as a product $HDU$ where $H \in SL(2,O)$, $D$ is a diagonal matrix and $U$ is an unipotent upper-triangular matrix. Write $d_{i,j} = \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix}$ (any diagonal matrix in $SL(2,F)$ can be factored as a diagonal matrix in $SL(2,O)$ multiplied by $d_{i,j}$ for some $(i,j)$) and we can then simply calculate that any element of $SL(2,O)d_{i,j} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ can be written in the form

$$\begin{pmatrix} t_1^i t_2^j a & t_1^i t_2^j au + t_1^{-i} t_2^{-j} b \\ t_1^i t_2^j c & t_1^i t_2^j cu + t_1^{-i} t_2^{-j} d \end{pmatrix}$$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,O)$. Then we can immediately observe that the minimal valuation of the top-left entry is exactly $(i,j)$ so $(i,j)$ is uniquely determined by the coset. Now suppose that for some $u, u' \in F$ we have an equality of cosets

$$SL(2,O)d_{i,j} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = SL(2,O)d_{i,j} \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix}$$

then by picking the identity element on the left, we deduce that there exists $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,O)$ such that the following equality holds

$$\begin{pmatrix} t_1^i t_2^j & t_1^i t_2^j u \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix} = \begin{pmatrix} t_1^i t_2^j a & t_1^i t_2^j au' + t_1^{-i} t_2^{-j} b \\ t_1^i t_2^j c & t_1^i t_2^j cu' + t_1^{-i} t_2^{-j} d \end{pmatrix}$$

We first read off that $a = 1$ and $c = 0$ and then it follows that $d = 1$ and $b = t_1^{2i} t_2^{2j}(u - u')$. This defines an element of $SL(2,O)$ if and only if $u - u' \in t_1^{-2i} t_2^{-2j} O$. ∎

Thus we have a bijection from $SL(2,F)/SL(2,O)$ to $\bigsqcup_{(i,j)\in\mathbb{Z}^2} F/t_1^{-2i}t_2^{-2j}O$. Since we defined a measure on $F$ in section 2, we are now almost done.

We use the corollary of the gluing lemma to lift this measure to the disjoint union $\bigsqcup_{(i,j)} F/t_1^{-2i}t_2^{-2j}O$ as follows. The index set is $\mathbb{Z}^2$ in the obvious way, the measurable sets of $F/t_1^{-2i}t_2^{-2j}O$ are simply those elements of $\mathcal{G}_F$ that are invariant under the natural $t_1^{-2i}t_2^{-2j}O$-action and for such an set $A$ we have

$$\mu_{F/t_1^{-2i}t_2^{-2j}O}(A) = \frac{\mu(A)}{\mu_F(t_1^{-2i}t_2^{-2j}O)}$$

so that in particular $t_1^{-2i}t_2^{-2j}O/t_1^{-2i}t_2^{-2j}O$ will end up having measure 1 which is exactly right as it corresponds to a single coset of $SL(2,O)$. These measures again have disjoint support, so we can glue them together to an integrable $\mathbb{R}((X))$-measure $\mu_{SL(2,F)/SL(2,O)}$.

## 8 An application to Langlands' work

In Langlands' work[3] he wants to define a local Hecke operator on $GL(2,\mathbb{C}((X)))$ by the following formula:

$$\theta_1 : f \mapsto \tilde{f}, \tilde{f}(g) = \int_{g\Delta_1/GL(2,\mathbb{C}[[X]])} f d\mu$$

where $\Delta_1 = \begin{pmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{pmatrix} GL(2,\mathbb{C}[[X]])$ and the goal is to show that this operator forms a commutative algebra with another local Hecke operator. Langlands gives this as an intended definition but doesn't have any class of functions for which he can prove this operator exists. We provide such a class of functions.

To do so, we first define a measure on $\mathbb{C}((X))$. We defined an integrable measure on $\mathbb{C}$ in section 5.3. For each $\lambda \in \mathbb{C}$ and $i \in \mathbb{Z}$, we have the projection map $p_{a,i} : a + X^i\mathbb{C}[[X]] \to \mathbb{C}$, each of which gives us a measure defined on $\mathbb{C}((X))$ which we can glue together by the gluing lemma.

By the Iwasawa decomposition we have an isomorphism from the coset space of $GL(2,\mathbb{C}[[X]])$ in $GL(2,\mathbb{C}((X))$ which states that it is equal to

$$\bigsqcup_{i,j} \mathbb{C}((X))/X^{i-j}\mathbb{C}[[X]]$$

Using the quotient measure on each $\mathbb{C}((X))/X^j\mathbb{C}[[X]]$ (satisfying$\mu(X^j\mathbb{C}[[X]]/X^j\mathbb{C}[[X]]) = 1$), we can glue those together to get an integrable measure on $GL(2,\mathbb{C}((X))/GL(2,\mathbb{C}[[X]]$, so we have a class of functions $M(GL(2,\mathbb{C}((X))/GL(2,\mathbb{C}[[X]],\mathbb{C}((X)))$ we can integrate. Of course, we want our Hecke operator to act on functions on $GL(2,\mathbb{C}((X)))$, so let $M$ be the corresponding ring of functions defined on $GL(2,\mathbb{C}((X)))$ and invariant under the $GL(2,\mathbb{C}[[X]])$-action.

So we know that $\theta_1$ is well-defined on $M$ and the next step is to show that it acts on $M$, i.e. that $\theta_1(f) \in M$ for all $f$ in $M$. We first consider the case that $f = \mathbb{1}_A$ for some measurable $A$ invariant under $GL(2,\mathbb{C}[[X]])$, so we're interested in the expression

$$\tilde{f}(g) = \mu(g\Delta_1 \cap A/GL(2,\mathbb{C}[[X]])$$

Since $g\Delta_1$ is a single coset of $GL(2,\mathbb{C}[[X]])$, this is either 0 or 1. At this point it's helpful to reduce to the case that $A$ is of the simple form:

$$A = \bigsqcup_{u\in U} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^i & 0 \\ 0 & X^j \end{pmatrix} GL(2,\mathbb{C}[[X]])$$

where $U$ contains $X^{i-j}\mathbb{C}[[X]]$ since our measurable sets are generated by subsets of this type. If we similarly write out the Iwasawa decomposition for $g$:

$$g = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^k & 0 \\ 0 & X^l \end{pmatrix} h)$$

Then $\tilde{f}(g) = 0$ unless $k = i+1$, $l = j+1$ and $v \in U$ (note that $k-l = i-j$). So if we define $A_{+1}$ by the following formula:

$$A = \bigsqcup_{u \in U} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^{i+1} & 0 \\ 0 & X^{j+1} \end{pmatrix} GL(2, \mathbb{C}[[X]])$$

then we obtain the following equality

$$\theta_1(\mathbb{1}_A) = \mathbb{1}_{A_{+1}}$$

so we conclude that $\theta_1(\mathbb{1}_A)$ is again in $M$. Now we know that $\theta_1$ sends indicator functions to elements of $M$ but with this explicit expression, we can make the more precise claim that $\theta_1$ sends indicators of pure sets to indicators of pure sets of the same measure, from which it follows immediately that $\theta_1$ maps $M$ to $M$.

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