OPTIMIZATION RELATED TO SOME NONLOCAL PROBLEMS OF KIRCHHOFF TYPE

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Abstract. In this paper we introduce two rearrangement optimization problems—one being a maximization and the other a minimization problem—related to a nonlocal boundary value problem of Kirchhoff type. Using the theory of rearrangements as developed by G. R. Burton we are able to show that both problems are solvable, and derive the corresponding optimality conditions. These conditions in turn provide information concerning the locations of the optimal solutions. The strict convexity of the energy functional plays a crucial role in both problems. The popular case in which the rearrangement class (i. e. the admissible set) is generated by a characteristic function is also considered. We show that in this case, the maximization problem gives rise to a free boundary problem of obstacle type, which turns out to be unstable. On the other hand, the minimization problem leads to another free boundary problem of obstacle type, which is stable. Some numerical results are included to confirm the theory.

1. Introduction

Consider the following Kirchhoff boundary value problem:

$$
\begin{cases} -M(\|u\|^p)\Delta_p u = f(x) & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1.1}
$$

where D is a smooth bounded domain in $\mathbb{R}^N$, $f \in L^q(D)$ is a non-negative (a. e.) and non-trivial (i. e. not identically zero) function in D, and $M : [0, \infty) \to (0, \infty)$ is continuous and strictly increasing. Henceforth, $p \in (1, \infty)$, $1/p + 1/q = 1$, and $\|u\| = (\int_D |\nabla u|^p dx)^{1/p}$. The differential operator Δ_p denotes the classical p-Laplace operator which is defined by

$$
\Delta_p u \coloneqq \nabla \cdot (|\nabla u|^{p-2}\nabla u).
$$

The problem (1.1) is called *nonlocal* because of the presence of $M(\|u\|^p)$, a quantity which is not measured pointwise.

The present paper is concerned with the role of the input function f in (1.1), in a specific way, which we elaborate in detail. To emphasize the dependence on f we denote the unique solution of (1.1) by u_f and define a quantity—which we will refer to as *energy*—associated with (1.1):

$$
\Phi(f) = \int_D f u_f \, dx - \frac{1}{p}\hat{M}(\|u_f\|^p),
$$

where $\hat{M}(t) = \int_0^t M(s)\, ds$. The key requirement is for the input function to be selected from the set $\mathcal{R}(f_0)$ of rearrangements generated by a prescribed function f_0. We ask whether it is

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possible to find a member of $\mathcal{R}(f_0)$ which generates the maximum energy. Mathematically speaking, we are investigating the solvability of the following maximization problem:

$$\sup_{f\in\mathcal{R}(f_0)} \Phi(f); \tag{1.2}$$

that is to say, whether or not we can find $\hat{f} \in \mathcal{R}(f_0)$ such that $\Phi(\hat{f}) = \sup_{f\in\mathcal{R}(f_0)} \Phi(f)$. As we will see the answer to this question is affirmative. In addition, we will also show that if $\hat{f}$ is a solution of (1.2), then the optimality condition leads to the *information*:

$$\hat{f} = \hat{\psi}(u_{\hat{f}}), \quad \text{a. e. in D},$$

for some increasing function $\hat{\psi}$. Incorporating this information into (1.1), we obtain an even more non-linear differential equation:

$$\begin{cases} -M(\|\hat{u}\|^p)\Delta_p\hat{u} \;=\; \hat{\psi}(\hat{u}) & \text{in } D \\ \qquad\qquad\quad \hat{u} \;=\; 0 & \text{on } \partial D, \end{cases} \tag{1.3}$$

where we have used $\hat{u}$ in place of $u_{\hat{f}}$.

We also address the minimization problem:

$$\inf_{f\in\mathcal{R}(f_0)} \Phi(f). \tag{1.4}$$

Indeed, we prove that (1.4) is also solvable. Moreover, thanks to the strict convexity of Φ, the minimizer $\check{f}$ is unique and satisfies the optimality condition:

$$\check{f} = \check{\psi}(\check{u}), \tag{1.5}$$

almost everywhere in D, for some decreasing function $\check{\psi}$. Here we have used $\check{u}$ in place of $u_{\check{f}}$. Whence we derive:

$$\begin{cases} -M(\|\check{u}\|^p)\Delta_p\check{u} \;=\; \check{\psi}(\check{u}) & \text{in } D \\ \qquad\qquad\quad \check{u} \;=\; 0 & \text{on } \partial D. \end{cases} \tag{1.6}$$

In case the generator f_0 is a characteristic function, we will see that the differential equation in (1.3) becomes a one-phase obstacle problem of *unstable* type whose free boundary has not been investigated before except for $p = 2$. On the other hand, the differential equation in (1.6) becomes a one-phase obstacle problem of *stable* type. For this type of free boundary problem there is an abundance of references in the literature. In the present paper we will not make any efforts in this direction. However, we have included some numerical examples to support the theory.

Let us present a physical motivation for studying the optimization problems (1.2) and (1.4) when $p = 2$, in which case (1.1) would be the steady state equation corresponding to the well known Kirchhoff equation:

$$u_{tt} - M\left(\int_D |\nabla u|^2 dx\right)\Delta u = f(t, x), \tag{1.7}$$

which was first introduced by Kirchhoff [20] as a generalization of the classical wave equation by adding the nonlinearity $M\left(\int_D |\nabla u|^2 dx\right)$ in front of the diffusion term. The function $f(t, x)$ in (1.7) stands for the external force. Usually, $M : \mathbb{R} \to \mathbb{R}$ is an affine function $M(s) = As + B$, in which both A and B are positive constants. The function u stands for the displacement of the elastic membrane that occupies the region D in $\mathbb{R}^2$.

Multiplying the differential equation in (1.1) by u and integrating the result over D, bearing in mind that u vanishes on the boundary (and p is assumed to be 2), leads to:

$$(A\|u\|^2 + B)\|u\|^2 = \int_D fu\,dx =: \gamma. \tag{1.8}$$

From (1.8), one gets:

$$\|u\|^2 = \frac{-B + \sqrt{B^2 + 4\gamma A}}{2A}. \tag{1.9}$$

From (1.9), we obtain the following formulation for the energy functional:

$$\Phi(f) = \xi\left(\int_D fu_f\,dx\right),$$

for an appropriate function $\xi : \mathbb{R}_+ \to \mathbb{R}$. Whence, the energy $\Phi(f)$ turns out to be a quantity that depends on the total displacement of the membrane, relative to the measure $f dx$.

Now let us consider the special case where f_0 is a characteristic function, thereby all members of the rearrangement class $\mathcal{R}(f_0)$ would also be characteristic functions supported on sets having the same measure as that of the support of f_0. In this case, the energy functional $\Phi(f)$ for $f \in \mathcal{R}(f_0)$ is simply a quantity that depends on the total displacement of the membrane only inside the region which is subject to the uniform external force $f(x) = 1$, i.e. $\Phi(E) = \xi\left(\int_E u_E dx\right)$, where we have used $\Phi(E)$ and u_E in place of $\Phi(\chi_E)$ and u_{χ_E}, respectively. In this setting, the maximization problem (1.2) amounts to that of a search for an optimal region $\hat{E}$ inside D such that $\Phi(\hat{E})$ is largest in comparison with all other competing regions E satisfying $|E| = |\hat{E}|$. The minimization problem (1.4) is interpreted similarly.

Boundary value problems of Kirchhoff type have been investigated by many authors. The ones who inspired our research are [1, 8, 10] and the references therein. That said, we would like to stress that the aim of this article is to introduce a new type of rearrangement optimization problem that can be thoroughly analyzed using the well developed theory of rearrangements by G. R. Burton [5, 6]. The reader may refer to [13, 14, 15, 16, 9, 22, 12] which discuss other types of rearrangement optimization problems.

2. Preliminaries

In this section we collect the tools that we need to prove our claims. Let us start with the following:

Definition 2.1. We say $u \in W_0^{1,p}(D)$ is a solution of (1.1) provided that:

$$M(\|u\|^p)\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx = \int_D fv\,dx, \quad \forall v \in W_0^{1,p}(D). \tag{2.1}$$

The next result is a basic one:

Lemma 2.2. *Problem (1.1) has a unique solution u_f which is the unique maximizer of the functional:*

$$\gamma_f(u) := \int_D fu\,dx - \frac{1}{p}\hat{M}(\|u\|^p), \quad (\forall u \in W_0^{1,p}(D))$$

in which $\hat{M}(t) = \int_0^t M(s)\,ds$. Moreover, the following holds:

$$u_f = t^{\frac{1}{p}}\frac{w_f}{\|w_f\|}, \tag{2.2}$$

where $w_f \in W_0^{1,p}(D)$ *is the solution of:*

$$\begin{cases} -\Delta_p w = f(x), & \textit{in } D \\ \quad w = 0 & \textit{on } \partial D, \end{cases} \tag{2.3}$$

and t uniquely solves:

$$t^{\frac{1}{q}} M(t) = \|w_f\|^{\frac{p}{q}}. \tag{2.4}$$

Remark 2.3. Note that in equation (2.2) as f is assumed to be non-trivial, we have $w_f \neq 0$.

Remark 2.4. For equation (2.4) to be solvable in t we need the range of $t^{1/q}M(t)$ (for $t \geq 0$) to cover all of $[0, \infty)$. But this follows from our assumptions regarding the function M, i. e. M being positive, continuous and strictly increasing for $t \geq 0$. In particular, $\forall t > 0 : M(t) > M(0) > 0$, which combined with the fact that $\lim_{t\to\infty} t^{1/q} = \infty$, would imply that $\lim_{t\to\infty} t^{1/q}M(t) = \infty$. Hence:

$$\{t^{1/q}M(t) \mid t \geq 0\} = [0, \infty).$$

Uniqueness of the solution follows from the fact that $t^{1/q}M(t)$ is strictly increasing.

Proof. We set $\mathcal{K} = -\gamma_f$. Let us note that $\mathcal{K} \in C^1(X, \mathbb{R})$, where $X = W_0^{1,p}(D)$. Next, we show that $\mathcal{K}$ is coercive. For $\|u\| > 1$, we have:

$$\mathcal{K}(u) \geq \frac{1}{p}\int_1^{\|u\|^p} M(s)\,ds - \int_D fu\,dx \geq \frac{1}{p}M(1)(\|u\|^p - 1) - C\|f\|_q\|u\|,$$

where we have used the monotonicity of M in conjunction with the Hölder and Poincaré inequalities. Thus, $\mathcal{K}(u) \to \infty$ as $\|u\| \to \infty$, hence $\mathcal{K}$ is coercive. Finally, as $M(t)$ is strictly increasing then $\hat{M}$ is strictly convex, implying that $\mathcal{K}$ is also strictly convex. So, we can apply the direct method of calculus of variations to conclude that the minimization:

$$\inf_{u\in X} \mathcal{K}(u)$$

has a unique solution u_f. Whence:

$$\mathcal{K}'(u_f)(v) = M(\|u_f\|^p)\int_D |\nabla u_f|^{p-2}\nabla u_f \cdot \nabla v\,dx - \int_D fv\,dx = 0, \quad \forall v \in X.$$

Thus, u_f is a solution of (1.1). Conversely, any solution of (1.1) is a minimizer of $\mathcal{K}(u)$ relative to $u \in X$.

Note that since $t^{1/q}M(t)$ is continuous, strictly increasing and its range is $[0, \infty)$, the solution to (2.4) is unique. Verification of (2.2) is elementary. □

Let us fix a non-negative and non-trivial function $f_0 \in L^q(D)$ which is bounded on D. The rearrangement set generated by f_0 is defined as follows:

$$\mathcal{R}(f_0) := \{f \in L^q(D) \mid \forall\alpha \geq 0 : \lambda_f(\alpha) = \lambda_{f_0}(\alpha)\}, \tag{2.5}$$

where $\lambda_f(\alpha) = |\{x \in D \mid f(x) \geq \alpha\}|$, known as the distribution function of f. Henceforth, $|E|$ denotes the N-dimensional Lebesgue measure of $E \subset \mathbb{R}^N$. The tools that we need from rearrangement theory are stated in the next lemma; for the proof the reader is referred to [5, 6].

Lemma 2.5. *Consider a non-negative* $f_0 \in L^q(D)$ *and let* $\mathcal{R} \equiv \mathcal{R}(f_0)$ *be its generated rearrangement set as defined in (2.5) above. Then:*

(i) $\mathcal{R} \subseteq L^q(D)$ *and* $\forall f \in \mathcal{R} : \|f\|_q = \|f_0\|_q$.

(ii) The weak closure of $\mathcal{R}$ *in* $L^q(D)$*, denoted* $\overline{\mathcal{R}}$*, is weakly compact and convex.*

(iii) *For every $g \in L^p(D)$ the linear functional $L(f) = \int_D fg\,dx$ has a maximizer $\hat{f}$ relative to $f \in \mathcal{R}$.*

(iv) *If $\hat{f}$ is the unique maximizer of the linear functional L defined in (iii) relative to $\mathcal{R}$, then it is also the unique maximizer of L relative to $\overline{\mathcal{R}}$. Moreover, $\hat{f} = \hat{\psi}(g)$, almost everywhere in D, for some increasing function $\hat{\psi}$.*

The derivation of the optimality condition (1.5) crucially depends on the next result whose proof can be extracted from either of [5] or [6]:

Lemma 2.6. *Let f_0 and $\mathcal{R}$ be as in Lemma 2.5. Suppose $g \in L^p(D)$, and its graph has no significant flat zones; i. e. the sets $\{g = c\}$ have zero measure, for every $c \in \mathbb{R}$. Then, there exists a decreasing function $\check{\psi}$ such that $\check{\psi}(g) \in \mathcal{R}$, and $\check{f} = \check{\psi}(g)$ is the unique minimizer of the linear functional $L(f) = \int_D fg\,dx$ relative to $f \in \overline{\mathcal{R}}$.*

Let us set $K(t) = t^{1/q}M(t)$. Note that K is continuous and strictly increasing on $[0,\infty)$, and (by Remark 2.4) its range covers $[0,\infty)$. Whence, the function $\xi(t) = \frac{1}{t}\left(K^{-1}(t^{p/q})\right)^{1/p}$ is well defined on $(0,\infty)$. Thus, from (2.2) and (2.4), we obtain:

$$u_f = t^{1/p}\frac{w_f}{\|w_f\|} = \xi(\|w_f\|)w_f, \tag{2.6}$$

From (2.6) and [27], one gets $u_f \in C^{1,\alpha}(D) \cap C^1(\overline{D}) \cap W^{2,1}_{loc}(D)$. Moreover, as a consequence of Harnack's inequality u_f is strictly positive in D.

Remark 2.7. In general for the boundary value problem:

$$\begin{cases} -\Delta_p Q = h(x), & \text{in } D \\ \quad Q = 0 & \text{on } \partial D, \end{cases}$$

in which h is bounded, we have $Q \in W^{2,1}_{loc}(D)$. To be more precise, if $p \geq 2$ then $Q \in W^{2,2}_{loc}(D)$, whereas for $p \in (1,2)$ one gets $Q \in W^{2,p}_{loc}(D)$.

We are heading towards tackling the maximization problem (1.2). First we ensure that the functional Φ is bounded above on $\mathcal{R}$. In fact Φ is bounded above on the larger set $\overline{\mathcal{R}}$. To see this, it suffices to observe that $Q(f) = \int_D f u_f\,dx$ is bounded above on $\overline{\mathcal{R}}$. For a fixed $f \in \overline{\mathcal{R}}$ equation (2.1) implies:

$$M(\|u_f\|^p)\|u_f\|^p = \int_D f u_f\,dx \leq \|f\|_q \|u_f\|_p \leq C\|f\|_q\|u_f\|, \tag{2.7}$$

where we have used the Hölder inequality in conjunction with the Poincaré inequality.[1] We know that there exists a sequence $\{f_n\}$ in $\mathcal{R}$ such that $f_n \rightharpoonup f$. So, by the weak lower semi-continuity of the L^q-norm, we obtain $\|f\|_q \leq \liminf_{n\to\infty}\|f_n\|_q = \|f_0\|_q$. Therefore, from (2.7) we get $M(\|u_f\|^p)\|u_f\|^{p-1} \leq C$. Setting $t = \|u_f\|^p$, yields $M(t)t^{1/q} \leq C$. Since $\lim_{t\to\infty} M(t)t^{1/q} \to \infty$, we infer $\|u_f\| \leq C$. Whence, we obtain $Q(f) \leq C\|f_0\|_q$ which is the desired result.

The boundedness of Φ from above implies that the maximization problem (1.2) is meaningful. We now start to show that the problem is in fact solvable:

Lemma 2.8. *The functional $\Phi : (\overline{\mathcal{R}}, \sigma(L^q, L^p)) \to \mathbb{R}$ is sequentially continuous. Here, $\sigma(L^q, L^p)$ denotes the weak topology on $L^q(D)$.*

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[1] We denote our constants by the symbol C, even though their values may vary from one place to another.

Proof. Let us consider $f_n \rightharpoonup f$ in $L^q(D)$; that is, the sequence $\{f_n\}$ converges weakly in $L^q(D)$ to f. We want to show that $\Phi(f_n) \to \Phi(f)$ as $n \to \infty$. To this end, first notice that from the weak convergence of $\{f_n\}$, we can infer that $\{f_n\}$ is bounded in $L^q(D)$; i. e. $\|f_n\|_q \leq C$. For simplicity, let us denote w_{f_n} as w_n. Multiplying the differential equation in (2.3) by w_n, and integrating the result over D would yield $\|w_n\|^p = \int_D f_n w_n \, dx \leq C\|w_n\|$, where we have applied the combination of the Hölder inequality and the Poincaré inequality. This implies that $\|w_n\| \leq C$, hence $\{w_n\}$ is bounded in $X := W_0^{1,p}(D)$. As a consequence, $\{w_n\}$ contains a subsequence—still denoted $\{w_n\}$—such that for some $w \in X$:

$$\begin{cases} w_n \rightharpoonup w \ \text{ in } \ X \\ w_n \to w \ \text{ in } \ L^p(D) \end{cases}$$

Therefore:

$$\begin{aligned} \int_D f w_f \, dx &= \frac{1}{p-1} \int_D \left(p f w_f - |\nabla w_f|^p \right) dx \\ &\geq \frac{1}{p-1} \int_D (p f w - |\nabla w|^p) \, dx \\ &\geq \frac{1}{p-1} \limsup_{n\to\infty} \int_D (p f_n w_n - |\nabla w_n|^p) \, dx \\ &\geq \frac{1}{p-1} \limsup_{n\to\infty} \int_D \left(p f_n w_f - |\nabla w_f|^p \right) dx \\ &= \frac{1}{p-1} \int_D \left(p f w_f - |\nabla w_f|^p \right) dx \\ &= \int_D f w_f \, dx, \end{aligned} \tag{2.8}$$

where the first inequality in (2.8) follows from the variational formulation of w_f, and in the second one we have used the weak lower semicontinuity of the X-norm. Clearly these inequalities must all be equalities, and as a result $w = w_f$.

Next we use the following equation:

$$\forall v \in X : \quad \int_D |\nabla w_n|^{p-2} \nabla w_n \cdot \nabla v \, dx = \int_D f_n v \, dx. \tag{2.9}$$

By setting $v = w_n$ in (2.9) we obtain $\int_D f_n w_n \, dx = \int_D |\nabla w_n|^p dx = \|w_n\|^p$. But, $\int_D f_n w_n \, dx \to \int_D f w_f \, dx$, since $f_n \rightharpoonup f$ in $L^q(D)$ and $w_n \to w_f$ in $L^p(D)$. Whence, $\|w_n\|^p \to \int_D f w_f \, dx = \|w_f\|^p$. Thus, we obtain $\|w_n\| \to \|w_f\|$ as $n \to \infty$.

Setting $u_n := u_{f_n}$, and recalling (2.6), we find $u_n = \xi(\|w_n\|) w_n$. This in turn implies that $\|u_n\| = \xi(\|w_n\|)\|w_n\| \to \xi(\|w_f\|)\|w_f\| = \|u_f\|$ as $n \to \infty$. So, we obtain $\hat{M}(\|u_n\|^p) \to \hat{M}(\|u_f\|^p)$ as $n \to \infty$. On the other hand:

$$Q(f_n) = \int_D f_n u_n \, dx = \xi(\|w_n\|) \int_D f_n w_n \to \xi(\|w_f\|) \int_D f w_f \, dx = Q(f)$$

as $n \to \infty$. Thus, $\Phi(f_n) \to \Phi(f)$ as $n \to \infty$. So, Φ is weakly continuous in $L^q(D)$. This completes the proof of the lemma. ◻

Remark 2.9. In the proof of Lemma 2.8, we have used the continuity of ξ at $\|w_f\|$, whose validity relies on establishing that $w_f \neq 0$. In fact, as $f \in \overline{\mathcal{R}}$ we have

$$|\{x \in D : f(x) > 0\}| \geq |\{x \in D : f_0(x) > 0\}|$$

a proof of which can be found in [5]. Hence, f is non-negative and non-trivial which guarantees $w_f \neq 0$.

Lemma 2.10. *The functional* $\Phi : L^q(D) \to \mathbb{R}$ *is strictly convex.*

Proof. Consider $u \in X \equiv W_0^{1,p}(D)$ and set $\gamma_f(u) = \int_D fu\,dx - \frac{1}{p}\hat{M}(\|u\|^p)$ $(\forall f \in L^q(D))$. Then, we have $\forall f \in L^q(D) : \Phi(f) = \sup_{u\in X} \gamma_f(u)$, i. e. the supremum of a family of affine functions, hence it is convex. Next, we show that Φ is in fact strictly convex. To this end, consider $f, g \in L^q(D)$, $f \neq g$, and suppose that for some $t \in (0, 1)$:

$$\Phi(tf + (1-t)g) = t\Phi(f) + (1-t)\Phi(g). \tag{2.10}$$

For simplicity, we set $u_t = u_{tf+(1-t)g}$. Using (2.10), with some straightforward calculations incorporating the fact that u_f and u_g are the maximizers of γ_f and γ_g, respectively, we obtain:

$$\int_D fu_t\,dx - \frac{1}{p}\hat{M}(\|u_t\|^p) = \int_D fu_f\,dx - \frac{1}{p}\hat{M}(\|u_f\|^p), \tag{2.11}$$

and

$$\int_D gu_t\,dx - \frac{1}{p}\hat{M}(\|u_t\|^p) = \int_D gu_g\,dx - \frac{1}{p}\hat{M}(\|u_g\|^p). \tag{2.12}$$

Once again, by the maximality of u_f and u_g, from (2.11) and (2.12) we deduce that $u_t = u_f = u_g$. On the other hand, we have:

$$-M(\|u_f\|^p)\Delta_p u_f = f(x) \quad \text{in } D,$$

and

$$-M(\|u_g\|^p)\Delta_p u_g = g(x) \quad \text{in } D.$$

This implies $f = g$, which is a contradiction. □

Our next aim is to prove that Φ is Gâteaux differentiable. First we need a generic estimate. So, consider $f, g \in L^q(D)$. Then

$$\begin{aligned}
\Phi(f) + \int_D (g-f)u_f\,dx &= \int_D gu_f\,dx - \frac{1}{p}\hat{M}(\|u_f\|^p) \\
&\leq \Phi(g) \\
&= \int_D fu_g\,dx - \frac{1}{p}\hat{M}(\|u_g\|^p) + \int_D (g-f)u_g\,dx \\
&\leq \Phi(f) + \int_D (g-f)u_g\,dx.
\end{aligned}$$

which implies:

$$\int_D (g-f)u_f\,dx \leq \Phi(g) - \Phi(f) \leq \int_D (g-f)u_g\,dx. \tag{2.13}$$

Lemma 2.11. *The functional* $\Phi : L^q(D) \to \mathbb{R}$ *is Gâteaux differentiable. Moreover, the Gâteaux derivative* $\Phi'(f)$ *of* Φ *at any* $f \in L^q(D)$ *can be identified with* $u_f \in L^p(D)$.

Proof. Consider $f, h \in L^q(D)$ and $t \in (0, 1)$. Set $g_t := f + t(h - f)$ and apply (2.13) to derive:

$$\int_D (h-f)u_f\,dx \leq \frac{\Phi(g_t) - \Phi(f)}{t} \leq \int_D (h-f)u_{g_t}\,dx. \tag{2.14}$$

By passing to the limit $t \to 0^+$ in (2.14) and keeping in mind that $u_{g_t} \to u_f$ in $L^p(D)$ (this is a standard result, see for example Lemma 4.2 in [11]), we obtain $\Phi'(f)(h - f) =$

$\int_D (h-f)u_f\,dx$. This shows that Φ is differentiable at f and its derivative can be identified with u_f. □

3. Main results

The main results regarding the maximization problem (1.2) and the minimization problem (1.4) are presented in this section.

3.1. **Maximization problem.**

We begin with the following:

Theorem 3.1. *The maximization problem (1.2) is solvable. Moreover, if $\hat{f} \in \mathcal{R}$ is a solution, then*

$$\hat{f} = \hat{\psi}(u_{\hat{f}}), \tag{3.1}$$

almost everywhere in D, for some a priori unknown increasing function $\hat{\psi}$.

Proof. We first relax the maximization problem (1.2) by extending the admissible set $\mathcal{R}$ to $\overline{\mathcal{R}}$ and consider:

$$\sup_{f\in\overline{\mathcal{R}}} \Phi(f). \tag{3.2}$$

From Lemmas 2.5 and 2.8 we infer that (3.2) is solvable. Let us suppose $\overline{f} \in \overline{\mathcal{R}}$ is a solution of (3.2). Set $g = \Phi'(\overline{f})$ (i. e. $u_{\overline{f}}$ as in Lemma 2.11) and consider the linear functional $L(f) = \int_D fg\,dx$. From item (iii) of Lemma 2.5 we infer existence of $\hat{f} \in \mathcal{R}$ that maximizes $L(f)$ relative to $f \in \mathcal{R}$. Since L is weakly continuous, we then deduce $L(\overline{f}) \leq L(\hat{f})$. Next, by convexity of Φ we get:

$$\Phi(\hat{f}) \geq \Phi(\overline{f}) + L(\hat{f} - \overline{f}) \geq \Phi(\overline{f}) \geq \Phi(\hat{f}), \tag{3.3}$$

where the second inequality in (3.3) is a consequence of $L(\overline{f}) \leq L(\hat{f})$, already observed above. Hence, all inequalities in (3.3) are in fact equalities and we obtain $\Phi(\hat{f}) = \Phi(\overline{f})$. This, in turn, implies that $\hat{f}$ solves (1.2), as desired.

Now we proceed to derive (3.1). To this end, we assume $\hat{f}$ is any solution of (1.2). By Lemma 2.10, Φ is strictly convex. Hence, for any $f \in \mathcal{R} \setminus \{\hat{f}\}$:

$$\Phi(f) > \Phi(\hat{f}) + \int_D (f - \hat{f})\hat{u}\,dx, \tag{3.4}$$

where $\hat{u} = u_{\hat{f}} = \Phi'(\hat{f})$. Since $\Phi(\hat{f}) \geq \Phi(f)$ for every $f \in \mathcal{R}$, we find from (3.4) that $\int_D f\hat{u}\,dx < \int_D \hat{f}\hat{u}\,dx$, for all $f \in \mathcal{R} \setminus \{\hat{f}\}$. So we can apply item (iv) of Lemma 2.5 which guarantees the existence of an increasing function $\hat{\psi}$ for which (3.1) holds. This completes the proof of the theorem. □

In the introduction we referred to the optimality condition as "information". Our next result partly explains the choice of this terminology. Henceforth, by the support $S(f)$ of a non-negative function f we mean the set of points where f is positive,[2] i. e.

$$S(f) := \{x \in D : f(x) > 0\}.$$

[2] Note that this is different from the usual topological definition of the support of a function where the closure of this set is taken.

Corollary 3.2. *In addition to the hypotheses of Theorem 3.1, assume that the support of f_0 is an essentially proper subset of D, i. e. $|S(f_0)| < |D|$. Also, suppose that $\hat{f}$ is a solution of the maximization problem (1.2), and $\hat{u} = u_{\hat{f}}$. Then, $\hat{u}$ attains its largest values on the support of $\hat{f}$, in the following sense:*

$$\alpha \equiv \operatorname{ess\,inf}_{S(\hat{f})}\hat{u} \geq \operatorname{ess\,sup}_{D\setminus S(\hat{f})}\hat{u} \equiv \beta. \tag{3.5}$$

Proof. In order to derive a contradiction, let us assume that the assertion (3.5) is false. So, we can find constants α_1 and β_1 such that $\alpha < \alpha_1 < \beta_1 < \beta$. From the definitions of α and β we infer existence of two measurable sets $A \subseteq S(\hat{f})$ and $B \subseteq D \setminus S(\hat{f})$ such that $|A|\,|B| > 0$, and:

$$\begin{cases} \hat{u} \leq \alpha_1 & \text{on } A \\ \hat{u} \geq \beta_1 & \text{on } B. \end{cases}$$

Without loss of generality we may assume $|A| = |B|$, otherwise we consider subsets of them. From classical measure theory (e. g. [25]) we know that there exists a measure preserving bijection $\eta : A \to B$. Using η we define a new element of $\mathcal{R}$ as follows:

$$\tilde{f}(x) = \begin{cases} \hat{f}(\eta(x)) & x \in A \quad \text{(Note that here } \hat{f}(\eta(x)) = 0) \\ \hat{f}(\eta^{-1}(x)) & x \in B \\ \hat{f}(x) & x \notin A \cup B. \end{cases}$$

A close inspection of the proof of Theorem 3.1 confirms that $\hat{f}$ maximizes the linear functional $L(f) = \int_D f\hat{u}\,dx$, relative to $f \in \mathcal{R}$. Since $\tilde{f} \in \mathcal{R}$, we have $L(\tilde{f}) \leq L(\hat{f})$. On the other hand,

$$\begin{aligned} \int_D \tilde{f}\hat{u}\,dx - \int_D \hat{f}\hat{u}\,dx &= \int_A \tilde{f}\hat{u}\,dx + \int_B \tilde{f}\hat{u}\,dx - \int_A \hat{f}\hat{u}\,dx - \int_B \hat{f}\hat{u}\,dx \\ &= \int_B \tilde{f}\hat{u}\,dx - \int_A \hat{f}\hat{u}\,dx \\ &= \int_B \hat{f}(\eta^{-1}(x))\hat{u}\,dx - \int_A \hat{f}\hat{u}\,dx \\ &= \int_A \hat{f}(\hat{u}(\eta(x)) - \hat{u}(x))\,dx \\ &\geq (\beta_1 - \alpha_1)\int_A \hat{f}\,dx \\ &> 0. \end{aligned}$$

Thus, $L(\hat{f}) < L(\tilde{f})$, which is a contradiction. □

From Theorem 3.1 we see that if $\hat{f}$ is a solution of (1.2), then $\hat{u} = u_{\hat{f}}$ satisfies (1.3). Let us briefly look into (1.3) for the case when the rearrangement class $\mathcal{R}$ is generated by a characteristic function $f_0 = \chi_{E_0}$. Here E_0 is a measurable subset of D. If $|E_0| = \beta < |D|$, then it is easy to see that

$$\mathcal{R} = \{\chi_E \mid E \subseteq D \text{ is measurable and } |E| = \beta\}.$$

It is well-known that $\overline{\mathcal{R}} = \{f : 0 \leq f \leq 1, \int_D f\,dx = \beta\}$. Let us use the notation $\phi(E) = \Phi(\chi_E)$. Identifying χ_E with E, the maximization problem in this new setting becomes:

$$\sup_{|E|=\beta} \phi(E). \tag{3.6}$$

The maximization problem (3.6) can be thought of as a *shape optimization* problem, where each E indicates a shape.

As a consequence of Theorem 3.1, we know that there exists $\hat{E} \subseteq D$ which solves the maximization problem (3.6). Moreover, from (3.1), since $\hat{\psi}$ is increasing, we infer the existence of a constant δ (which is necessarily positive) such that $\hat{E} = \{\hat{u} \geq \delta\}$ or $\hat{E} = \{\hat{u} > \delta\}$, modulo a set of measure zero. Note that as $\hat{u} \in W^{2,1}_{loc}(D)$, the set $\{\hat{u} = \delta\} \cap \hat{E}$ has measure zero. Thus, from (1.1) we obtain:

$$\begin{cases} -M(\|\hat{u}\|^p)\Delta_p \hat{u} = \chi_{\{\hat{u}>\delta\}} & \text{in } D \\ \hat{u} = 0 & \text{on } \partial D. \end{cases} \tag{3.7}$$

By setting $v = \delta - \hat{u}$, the differential equation in (3.7) becomes:

$$\Delta_p v = \frac{1}{M(\|v\|^p)}\chi_{\{v<0\}}, \quad \text{in } D. \tag{3.8}$$

The differential equation (3.8) is an example of a one-phase obstacle problem of unstable type. The boundary of the set $\{v < 0\}$ is known as the free boundary since it is not known a priori.

As far as we know, however, for the p-Laplace operator the free boundary of the unstable obstacle problem has not been investigated when $p \neq 2$. Therefore, we have decided to include some numerical experiments in Section 4 to provide some ground for future work on this subject, both for us and the interested reader. For the case $p = 2$ and $N = 2$ the reader may refer to [26] where the singularities of the free boundary are discussed, whilst for the higher dimensions one may refer to [2, 3, 4].

3.2. **Minimization problem.** The main result regarding the minimization problem (1.4) is stated below. In order to avoid obscuring the main ideas with technicalities we assume that the generator f_0 is positive throughout the domain D. However, we stress that the result of the theorem below stands valid even if f_0 is non-negative, but the proof would require more technical steps.

Theorem 3.3. *The minimization problem (1.4) has a unique solution $\check{f}$. Moreover, setting $\check{u} = u_{\check{f}}$, we will have:*

$$\check{f} = \check{\psi}(\check{u}) \tag{3.9}$$

almost everywhere in D, for some decreasing function $\check{\psi}$, unknown a priori.

Proof. Let us first address the uniqueness. Suppose f_1 and f_2 solve the minimization problem (1.4). Since Φ is weakly continuous, it follows that f_1 and f_2 also solve the problem:

$$\inf_{f\in\overline{\mathcal{R}}} \Phi(f) \equiv m.$$

As $\overline{\mathcal{R}}$ is convex, we get $(f_1 + f_2)/2 \in \overline{\mathcal{R}}$. Thus, the strict convexity of Φ implies:

$$\Phi\left(\frac{f_1 + f_2}{2}\right) < \frac{1}{2}\Phi(f_1) + \frac{1}{2}\Phi(f_2) = m,$$

which is a contradiction.

Let us now address the existence. Again, we start by relaxing the problem:

$$\inf_{f\in\overline{\mathcal{R}}} \Phi(f). \tag{3.10}$$

Since Φ is weakly continuous and $\overline{\mathcal{R}}$ is weakly compact in $L^q(D)$, problem (3.10) is solvable. Let us denote the solution by $\check{f} \in \overline{\mathcal{R}}$. The following relation, satisfied by $\check{f}$, is called the optimality condition:

$$0 \in \Phi'(\check{f}) + \partial\xi_{\overline{\mathcal{R}}}(\check{f}), \tag{3.11}$$

in which $\xi_{\overline{\mathcal{R}}}$ denotes the indicator function supported on $\overline{\mathcal{R}}$:

$$\xi_{\overline{\mathcal{R}}}(f) = \begin{cases} 0 & f \in \overline{\mathcal{R}} \\ \infty & f \notin \overline{\mathcal{R}}. \end{cases}$$

Here $\partial\xi_{\overline{\mathcal{R}}}(\check{f})$ denotes the subdifferential of $\xi_{\overline{\mathcal{R}}}$ at $\check{f}$:

$$\partial\xi_{\overline{\mathcal{R}}}(\check{f}) := \left\{ g \in L^p(D) \mid \forall f \in L^q(D) : \xi_{\overline{\mathcal{R}}}(f) \geq \xi_{\overline{\mathcal{R}}}(\check{f}) + \int_D (f - \check{f}) g \, dx \right\} \tag{3.12}$$

Since $\Phi'(\check{f}) = \check{u} = u_{\check{f}}$, (3.11) implies the existence of $g \in \partial\xi_{\overline{\mathcal{R}}}(\check{f})$ such that $\check{u} + g = 0$. Whence, we obtain:

$$\forall f \in \overline{\mathcal{R}} : \int_D (\check{u} + g)(f - \check{f}) \, dx = 0 \tag{3.13}$$

On the other hand, from (3.12) we infer $\forall f \in \overline{\mathcal{R}} : \int_D g(f - \check{f}) \, dx \leq 0$, which together with (3.13) yields:

$$\forall f \in \overline{\mathcal{R}} : \int_D \check{u}(f - \check{f}) \, dx \geq 0$$

Thus, $\check{f}$ minimizes the linear functional $L(f) = \int_D f\check{u} \, dx$ over $\overline{\mathcal{R}}$.

Note that $\check{u} \in W^{2,1}_{loc}(D)$. So the differential equation

$$-M(\|\check{u}\|^p)\Delta_p\check{u} = \check{f} \quad \text{in } D, \tag{3.14}$$

holds almost everywhere in D. On the other hand, recalling the assumption that f_0 is positive, we infer that $\check{f}$ is also positive (see [6, Lemma 2.14]). Therefore, (3.14) in conjunction with [18, Lemma 7.7] implies that the graph of $\check{u}$ has no significant flat zones in D. This makes it possible to apply Lemma 2.6 to deduce that there exists a decreasing function $\check{\psi}$ such that $\check{\psi}(\check{u}) \in \mathcal{R}$ is the unique minimizer of the linear functional $L(f)$ relative to $f \in \overline{\mathcal{R}}$. Recalling that $\check{f}$ is a minimizer of the same linear functional, we obtain:

$$\check{f} = \check{\psi}(\check{u}),$$

almost everywhere in D. So, the proof of the theorem is complete. □

As mentioned earlier, the assertion of Theorem 3.3, in particular the optimality condition (3.9), still holds even when the generator f_0 is merely *non-negative*. As a result, in case $f_0 = \chi_{E_0}$ the Kirchhoff boundary value problem (1.1) in conjunction with Theorem 3.3 would give rise to the following:

$$\begin{cases} -M(\|\check{u}\|^p)\Delta_p\check{u} = \chi_{\{\check{u}<\check{\delta}\}} & \text{in } D \\ \check{u} = 0 & \text{on } \partial D. \end{cases}$$

Introducing $\check{v} = \check{\delta} - \check{u}$, this differential equation becomes:

$$\Delta_p\check{v} = \frac{1}{M(\|\check{v}\|^p)}\chi_{\{\check{v}>0\}}.$$

This equation is a one phase obstacle problem of stable type. There is a vast literature addressing qualitative properties of the free boundary $\partial\{\check{v} > 0\}$, but we specifically mention the book [24].

4. Numerical experiments

We have developed a numerical algorithm to get a better view of the qualitative nature of the solutions to some of the free boundary problems mentioned so far. As these experiments are primarily meant to provide us with further insight into the problems at hand, the minute details of the implementation will not be presented here. Rather, an overview of the main ingredients of the experiments will be provided so that the interested reader can replicate the results and experiment with other sample problems of their own choice.

4.1. **Setting the parameters.** First of all, we only consider problems over two-dimensional domains, and focus on the case where the rearrangement class $\mathcal{R}(f_0)$ is generated by a characteristic function, i. e.

$$\exists E_0 \subseteq D : f_0 = \chi_{E_0}.$$

It should be clear that every function $f \in \mathcal{R}(f_0)$ is also the characteristic function of some $E_f \subseteq D$ with the same Lebesgue measure as E_0, i. e. $|E_f| = |E_0|$. By considering the rearrangement class of a characteristic function, we are able to restrict our search to subsets of the domain D with a given Lebesgue measure, rather than a function space.

Each of our optimization problems is specified by a few parameters:

(i) For the function M we consider $M(t) = 1 + \sqrt{t}$.
(ii) We set the p-Laplace parameter to 3, i. e. $p = 3$.
(iii) The Lebesgue measure of E_0 is determined by its ratio $\rho \in (0, 1)$ to the area of the reference domain D. We consider a few different values. Specifically, we are interested in cases where ρ is relatively small (for maximization) or relatively large (for minimization).
(iv) Regarding the domain D, we will consider a few different cases, including non-convex domains.

In each experiment, we are essentially interested in the *location* of the set E_f whose characteristic function f is the optimizer of the problem. This way we can provide evidence of radial symmetry in case the domain D is a disc, and symmetry breaking for a couple of non-convex domains.

4.2. **Parametrization of the domain E_f.** The domain E_f is the support of a characteristic function and may be parametrized in various ways. One option—sometimes referred to as the Lagrangian approach—is to consider a parametrization of the boundary Γ_{E_f} via (say) some $\gamma : \mathbb{S}^{n-1} \to \Gamma_{E_f}$ in which $\mathbb{S}^{n-1}$ is the n-dimensional unit sphere (see e. g. [17]). This would provide smoother looking figures, but unfortunately, it does not suit our approach.

The main challenge we face is when the optimal domain is neither connected, nor symmetric. Thus, to make our algorithm work for different domain shapes, we work directly on the mesh grids that we make over the domain D. This is sometimes referred to as the Eulerian approach. In our algorithm we triangulate the domain, but the arguments could work just as well for any other type of mesh grid.

4.3. **Computational complexity and heuristics.** Suppose that we have a mesh $\mathcal{D} = \{D_i \mid 1 \le i \le P\}$ (for some $P \in \mathbb{N}$) over the domain $D \subseteq \mathbb{R}^2$ such that $D = \cup\mathcal{D}$. In this discretization, let $\mathcal{D}_E = \{D_{i_j} \mid 1 \le i_j \le P\}$ be the best outer approximation of the set $E \subseteq D$, i. e. $\mathcal{D}_E$ is a subgrid of $\mathcal{D}$ with the smallest area for which $E \subseteq \cup\mathcal{D}_E$. We define

$\#E$ as the number of indices k such that $D_k \in \mathcal{D}_E$. Assuming that the subregions D_i $(1 \leq i \leq P)$ have roughly the same area, then $\#E \approx \rho P$. In other words, the ratio of $\#E$ to the total number of subregions P in the grid is close to the ratio ρ of the area of E to that of D, and the finer the mesh is, the closer these values will be.

In these types of problems, one usually needs to refine the mesh by bisecting along all dimensions to obtain one more digit of accuracy. In other words, for a given parameter $n \in \mathbb{N}$, one needs a mesh of size $P \approx \Omega(2^n)$ to provide a satisfactory result to within n digits of accuracy. Remember that the search space $\mathbb{S}$ is essentially the set of subsets of D of a given (fixed) Lebesgue measure, which in the discretized version would have the cardinality

$$|\,\mathbb{S}\,| \approx \binom{P}{\#E_0}. \tag{4.1}$$

One can obtain an estimate of the growth of this value with respect to n using Stirling's approximation of the factorial function [23]:

$$\forall k \in \mathbb{N}: \sqrt{2\pi}\, k^{k+\frac{1}{2}} e^{-k} \leq k! \leq e\, k^{k+\frac{1}{2}} e^{-k}. \tag{4.2}$$

Assuming that $\#E_0 \approx \rho P$ we can use inequalities (4.2) and formula (4.1), together with some routine calculations, to obtain:

$$\frac{\sqrt{2\pi}}{e^2\sqrt{\rho(1-\rho)}}\frac{\zeta^P}{\sqrt{P}} \leq |\,\mathbb{S}\,| \leq \frac{e}{2\pi\sqrt{\rho(1-\rho)}}\frac{\zeta^P}{\sqrt{P}}$$

in which

$$\zeta := \frac{1}{\rho^\rho} \times \frac{1}{(1-\rho)^{(1-\rho)}}.$$

Note that as $\rho \in (0,1)$ we have $1 < \zeta \leq 2$.

Thus, for a constant C_1 which depends only on ρ, one gets:

$$C_1 \frac{\zeta^P}{\sqrt{P}} \leq |\,\mathbb{S}\,|,$$

which together with the fact that $P \approx \Omega(2^n)$ implies that for some constant C which depends on ρ and the domain D, and for some $\zeta > 1$ (also depending on ρ):

$$C \frac{\zeta^{2^n}}{2^{\frac{n}{2}}} \leq |\,\mathbb{S}\,|.$$

This is an instance of an intractable computational problem which should not be attacked by unguided search. Hence, we devise a numerical method which employs a combination of tools and methods, as follows:

Gradient: Lemma 2.11 provides a gradient to guide us with generating an optimizing sequence.

Stopping conditions: These are provided by the formulae (3.1) and (3.9).

Escaping local optima and saddle points: Even though the gradient formula helps us with generating an optimizing sequence, we do not know whether the algorithm is converging toward the real global optimum or a local non-global one. It is also possible for the algorithm to follow gradient and converge on a saddle point. Unfortunately, we have not proven the non-existence of non-global local optima or saddle points in the problems we have considered so we have to assume that there could exist such points.

Hence, we employ a combination of certain heuristic methods including Simulated Annealing [21] and Tabu Search [19] to tackle the problem. The algorithm uses Simulated Annealing to escape local optima and saddle points by adding some elements of randomness to the search, while Tabu Search is used to increase efficiency by avoiding unnecessary repetition of operations that are unlikely to improve the result.

4.4. **Examples.** Our experiments are primarily classified according to the shape of the domain D:[3]

4.4.1. *Disc.* For a disc shaped domain D of radius R, we get a maximizing domain E which seems to be a disc of radius r concentric with D where $r = R\sqrt{\rho}$ (Fig. 1 below).

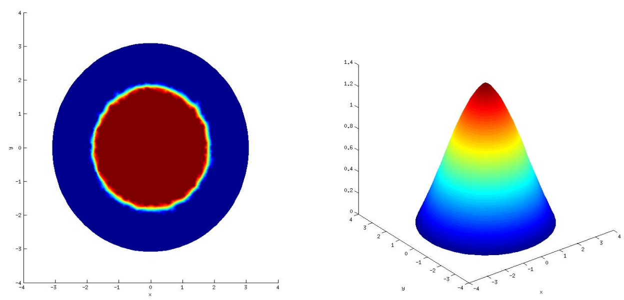

FIGURE 1. Disc: maximizing domain E (left) and the corresponding solution u (right). It seems that we get radial symmetry in the optimizing domain, which is the brown colored concentric disc on the left.

For the minimization problem, the minimizing domain is in fact a strip around the boundary (Fig. 2 below).

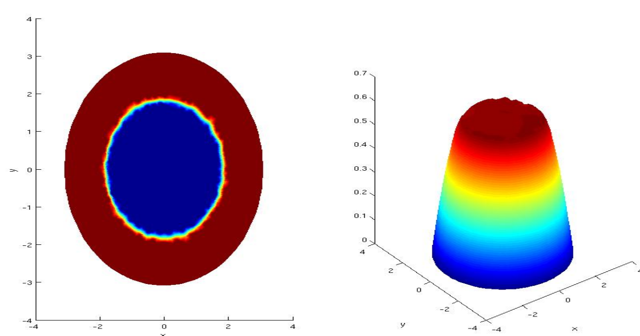

FIGURE 2. Disc: minimizing domain E (brown colored strip along the border on the left) and the corresponding solution u (right).

[3]All of the figures have been produced in MATLAB®.

4.4.2. *Ellipse.* Let us squeeze the circle 'a bit' to obtain an ellipse. Again we see that the maximizing domain E is located in the middle, with a shape that indicates preservation of some symmetries of the outer domain D (Fig. 3 below).

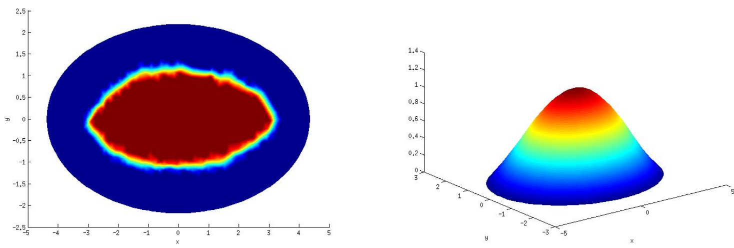

FIGURE 3. Ellipse: maximizing domain E (left) and the corresponding solution u (right).

4.4.3. *Dumbbell.* We squeeze the circle even more to get a dumbbell shape. We consider the case in which D is a dumbbell shaped domain comprising two large discs connected with a narrow channel. When the ratio ρ is taken to be relatively small, we can observe the breaking of symmetry in the maximizing zone (Fig. 4 below). It is interesting to note that for the dual problem, i. e. minimization with relatively large ρ, symmetry is preserved (Fig. 5 on the next page).

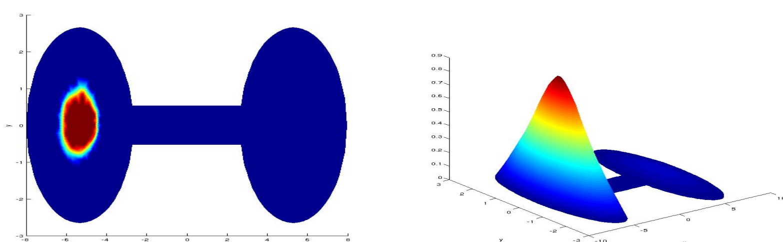

FIGURE 4. Dumbbell domain: maximizing domain E (left) and the corresponding solution u (right). Notice the breaking of symmetry in the optimizing domain.

4.4.4. *Annulus.* Finally, we consider an annulus $\{(x, y) \in \mathbb{R}^2 \mid R^2 \leq x^2 + y^2 \leq (R + h)^2\}$ in which R is large and h is quite small. For maximization we take the area ratio ρ to be also quite small. Again we can observe symmetry breaking in Fig. 6 on the following page.

However, as in the case of the dumbbell shaped domain, for the dual minimization problem with relatively large ρ, the symmetry is preserved (Fig. 7 on page 17).

5. CONCLUDING COMMENTS

We end the paper with some comments and questions.

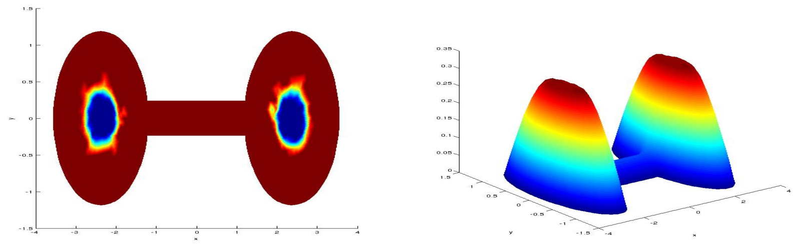

Figure 5. Dumbbell domain: minimizing domain E (left) and the corresponding solution u (right). Notice that symmetry is preserved in the optimizing domain.

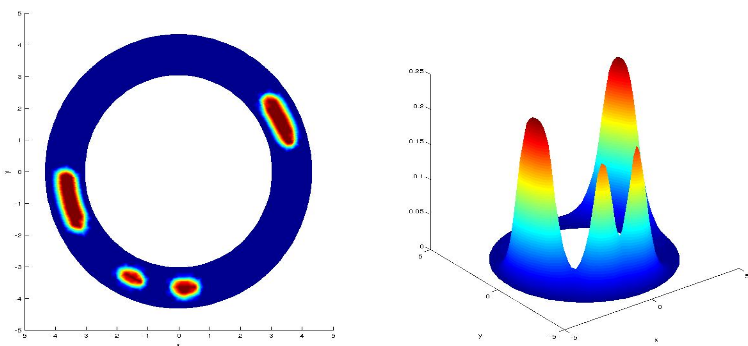

Figure 6. Annulus: maximizing domain E (the red colored patches on the left) and the corresponding solution u (right). Notice the breaking of symmetry in the optimizing domain.

1. A popular case of study in rearrangement optimization problems is when D is a disk (ball). In this case one wonders if the optimal solutions are radial or not. In the context of the present paper, if M is a constant function, then the functional Φ reduces to:
$$\Phi(f) = \left(1 - \frac{1}{p}\right) \int_D f u_f \, dx,$$
which is essentially the operator considered in [7] in which the authors prove that f^*, the decreasing Schwarz symmetrization of f_0, is the unique solution to the maximization problem (1.2), and f_*, the increasing Schwarz symmetrization of f_0, is the unique solution to the minimization problem (1.4).

   It is easy to see that for $M(t)$ of type:
$$M(t) = \begin{cases} c & (0 < t \leq t_1) \\ m(t) & (t_1 \leq t), \end{cases}$$
and $\|f_0\|_q$ sufficiently small, again f^* and f_* will be the unique optimal solutions of (1.2) and (1.4), respectively.
2. It is intriguing to see what happens if we relax the sign restriction on f_0 and allow it to change sign.

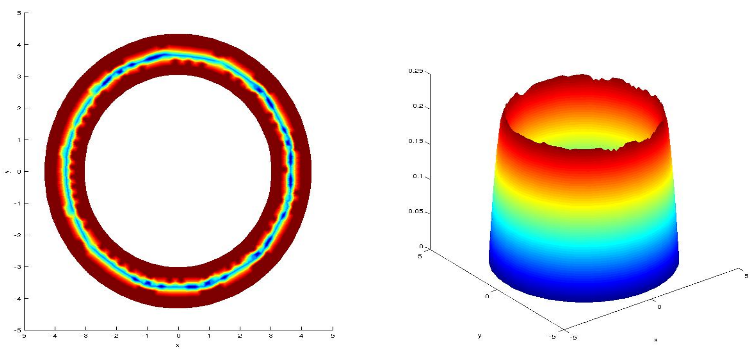

FIGURE 7. Annulus: minimizing domain E (left) and the corresponding solution u (right). Notice that symmetry is preserved in the optimizing domain.

3. Consider the eigenvalue problem:

$$\begin{cases} -M(\|u\|^p)\Delta_p u = \lambda w(x)|u|^{p-2}u & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{5.1}$$

where $w(x)$ is a non-negative bounded weight function. Let $\lambda_1(w)$ be the principal eigenvalue of (5.1). It is interesting to investigate the following problems:

$$\sup_{w\in\mathcal{R}(w_0)} \lambda_1(w) \quad \text{and} \quad \inf_{w\in\mathcal{R}(w_0)} \lambda_1(w),$$

where $\mathcal{R}(w_0)$ is the rearrangement class generated by a prescribed weight function w_0.

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