ON THE CHARACTERISATION OF POLAR FIBROUS COMPOSITES WHEN FIBRES RESIST BENDING - PART II: CONNECTION WITH ANISOTROPIC POLAR LINEAR ELASTICITY

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Abstract

This continuation of Part I (Soldatos, 2018) aims to make a connection between the polar linear elasticity for fibre-reinforce materials due to (Spencer and Soldatos, 2007; Soldatos, 2014, 2015) with the anisotropic version and the principal postulates of its counterpart due to (Mindlin and Tiersten, 1963). The outlined analysis, comparison and discussions are purely theoretical, and aim to collect and classify valuable information regarding the nature of continuous as well as weak discontinuity solutions of relevant well-posed boundary value problems. Emphasis is given on the fact that the compared pair of theoretical models has a common theoretical background (Cosserat, 1909) but different kinds of origin. Some new concepts and features, introduced in Part I, in association with linear elastic behaviour of materials having embedded fibres resistant in bending, are thus shown relevant to more general linearly elastic, anisotropic, Cosserat-type material behaviour. The different routes followed for the origination of the compared pair of models is known to produce identical results in the case of conventional (non-polar) linear elasticity. The same is here found generally non true in the polar elasticity case, although considerable similarities are also observed. No definite answers are provided regarding the manner in which existing differences might be bridged or, if at all possible, eliminated. These are matters that require further study and thorough investigation.

Keywords: Clapeyron's theorem; Fibre-reinforced materials; Fibre bending resistance/stiffness; Polar linear elasticity; Potential energy; Orthotropic materials; Transverse isotropic materials.

1. Introduction

About twenty years after Adkins and Rivlin pioneered the non-linear theory of fibre-reinforced materials (Adkins 1951; Adkins and Rivlin, 1955), Spencer's (1972) monograph summarised the progress which had been made at the time in the subject. Figure 1 is extracted from that monograph (Spencer, 1972), and in its initial part (Figures 1(a, b)) illustrates a cantilevered block of fibre-reinforced material bent in a fully continuous manner. The fibres are noted as *a*-curves and are considered very stiff and strong. Each of the Figures 1(c) – 1(i) illustrates next one of many possible analogous deformation patterns that involve different kind of discontinuous fibre slope and/or fibre curvature, although the overall displacement field is still continuous. The example deformation patterns depicted in Figure 1 underpinned the applicability of the theory of ideal fibre-reinforced materials (Spencer, 1972). Today, these are felt as predictions that, within the elastic deformation regime, justify the term and the class of "weak discontinuity" deformations. The latter are deformation patterns which, due to micro-scale (fibre-thickness) material failure, are described by continuous displacements that possess discontinuous derivatives; see (Merodio and Ogden, 2002, 2003).

Existence of weak discontinuity deformations in non-polar and unconstrained non-linear elasticity did not become formally known before 1975 in the case of material isotropy (Knowles and Sternberg, 1975), and were not studied in connection with fibre-reinforced materials before 1983 (Triantafyllidis and Abeyarante, 1983). Such deformations occur in the form of material instability modes as soon as the influence that large deformation exerts on the elastic constitution of the material forces the equations of elasticity to lose ellipticity. These micro-mechanics failure modes are thus not observable in conventional (non-polar) linear elasticity, where the governing equations are always elliptic.

The same is not necessarily true in the case of polar linear elasticity (e.g., Mindlin and Tiersten, 1963; Spencer and Soldatos, 2007; Soldatos, 2014, 2015) where, still, the magnitude of the deformation does not affect material constitution but, due to the presence of couple-stress, the corresponding governing equations are generally non-elliptic. Weak discontinuity solutions of well-posed boundary value problems in polar linear elasticity may thus co-exist with their fully continuous counterpart(s). The latter are potential solutions described by continuous displacements possessing continuous derivatives of all orders, and, for simplicity, will be termed as “continuous solutions” in what follows.

Like the aforementioned monograph (Spencer, 1972), the polar linear elasticity presented by Mindlin and Tiersten (1963) was published before the pioneering work of Knowles and Sternberg (1975) on weak discontinuity elasticity solutions. Mindlin and Tiersten (1963) had thus every reason at the time to claim that a continuous solution to a well-posed mixed boundary value problem formulated in terms of their theory is unique. However, this claim is now disputable, at least because the non-elliptic nature of the relevant governing equations is already exposed and discussed (Gouriotis and Bigoni, 2016).

There exists no evidence suggesting that the anisotropic version of that theory (Mindlin and Tiersten, 1963) was motivated by potential applications on linearly elastic composites with embedded fibres resistant bending. Moreover, most of the polar linear elasticity analysis detailed in (Mindlin and Tiersten, 1963) deals with the isotropic version of that theory. Hence, a possible rational connection of that theory with applications referring to composites containing fibres resistant in bending would naturally be interesting as well as important (e.g., Asmanoglo and Menzel, 2017).

The present investigation aims to compare the anisotropic version of, and principal postulations stemming from the linear polar elasticity due to Mindlin and Tiersten (1963) with their counterparts presented in (Spencer and Soldatos, 2007; Soldatos, 2014, 2015). The comparison and relevant discussions are currently of purely theoretical nature and significance, and are associated with the search for continuous solutions of well-posed boundary value problems in polar linear elasticity. It is noted in this context that the compared polar elasticity models have a common theoretical background, namely that of the Cosserat (1909) couple-stress theory which is summarised in Section 2. However, they have different kind of origin and foundation.

Mindlin and Tiersten's (1963) polar linear elasticity is founded on constitutional considerations stemming from the observation that the internal energy function of the material is quadratic in the strains and the spin-gradients of the deformation. The same constitutive assumptions are thus employed and underpin the generally anisotropic polar linear elasticity formulated in Section 3.1. Nevertheless, several new concepts are introduced in the remaining of Section 3, where further relevant features are also developed and discussed. Some of those concepts and features were introduced in Part I (Soldatos, 2018), where were initially associated only with linearly elastic behaviour of fibre-reinforced materials (Soldatos, 2015). Their generalisation and connection with the (Mindlin and Tiersten, 1963) model, and, potentially, with other possible versions of Cosserat-type linearly elastic material behaviour is here considered interesting and important.

On the other hand, as is also described in Section 4, polar linear elasticity of unidirectional, transversely isotropic fibre-reinforced materials is founded on the proper linearisation of a corresponding non-linear theory of polar elasticity (Spencer and Soldatos, 2007). In the case of non-polar elasticity, this alternative formulation route produces identical results with those obtained through the route employed previously in Section 3 or, equivalently, used in (Mindlin and Tiersten, 1963). However, Sections 5 - 7 show that this is generally not true in the case of polar elasticity.

Section 5 makes thus initially understood that, by imposing some conditions on the Mindlin and Tiersten (1963) formulation, the latter may reduce to the restricted version of the theory presented in (Spencer and Soldatos, 2007; Soldatos, 2014) and used later in Part I (Soldatos, 2018) for transversely isotropic fibrous composites. This result clarifies thus the reason for which some of the new concepts introduced in Sections 3.2 and 3.3 are found already applicable in Part I.

However, Section 6 shows next that the appearance of some new kinematic variables, which are neither expressible in terms of the strains nor the spin-gradients of the deformation, prevents the Mindlin and Tiersten (1963) model from producing the unrestricted version of its counterpart presented in (Spencer and Soldatos, 2007). Moreover, Section 7 considers the case of a fibrous composite reinforced by two families of unidirectional fibres resistant in bending and shows that, for the same reason, the Mindlin and Tiersten (1963) framework is unable to produce in that case even the restricted version of its alternative, fibre-reinforced material counterpart (Soldatos, 2015).

It is re-emphasised that the attempted comparison is confined within bounds determined by the existence and applicability of continuous solutions of the governing equations of polar linear elasticity. No definite answers are provided regarding the manner in which the differences observed between the compared models might be exploited or, if at all possible, bridged/eliminated. These matters are further discussed in the closing Section 8, which also summarises the principal conclusions and provides directions for future relevant study.

2. Basic theoretical concepts of linearly elastic polar material behaviour

In a right-handed Cartesian co-ordinate system Ox_i, where subscripts take the values 1, 2 and 3, denote with $\boldsymbol{u}$ the displacement vector encountered during small elastic deformation of a solid material. In the usual manner, the linear elasticity strain and rotation tensors,

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad \omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right), \tag{2.1}$$

are defined as the symmetric and the antisymmetric part of the displacement-gradient tensor, respectively, where a comma between indices denotes partial differentiation with the corresponding co-ordinate parameter. Moreover, the spin vector, $\boldsymbol{\Omega}$, is related with the rotation tensor through the standard relationships

$$\Omega_{\mathrm{i}} = \frac{1}{2}\varepsilon_{ijk}\omega_{kj}, \quad \omega_{kj} = \varepsilon_{ijk}\Omega_i, \tag{2.2}$$

where $\boldsymbol{\varepsilon}$ is the three-dimensional alternating tensor and the summation notation applies over repeated indices.

Polar material behaviour is synonymous with the presence of a non-zero couple-stress tensor, $\boldsymbol{m}$. In turn, this makes the stress tensor, $\boldsymbol{\sigma}$, non-symmetric, in the following sense:

$$\sigma_{ij} = \sigma_{(ij)} + \sigma_{[ij]}, \quad \sigma_{[ij]} = \frac{1}{2}\varepsilon_{kji}m_{\ell k,\ell}, \tag{2.3}$$

where $\sigma_{(ij)}$ and $\sigma_{[ij]}$ represent the components of the symmetric and the antisymmetric part of $\boldsymbol{\sigma}$, respectively. The second of (2.3) is essentially the moment (or couple-stress) equilibrium equation, and is necessarily valid under the assumption that, like u_i, the components of $\boldsymbol{m}$ are differentiable functions of the spatial co-ordinates.

Equation (2.3b) is accompanied by the moment of momentum (or stress) equilibrium equation

$$\sigma_{ij,i} = 0\,, \tag{2.4}$$

where body forces and body couples are neglected for simplicity, and the components of $\boldsymbol{\sigma}$ are assumed differentiable functions. The components of the traction and the couple-traction vectors acting on any internal or bounding surface of the material are respectively given as follows:

$$T_i^{(n)} = \sigma_{ji} n_j\,, \quad L_i^{(n)} = m_{ji} n_j\,, \tag{2.5}$$

where $\boldsymbol{n}$ denotes the outward unit normal of that surface.

Under the assumption that not only the components of $\boldsymbol{m}$, but also their derivatives appearing in (2.3b) are differentiable functions, the equilibrium equation (2.4) reduces to

$$\sigma_{(ij),i} + \frac{1}{2}\varepsilon_{kji}\overline{m}_{\ell k,\ell i} = 0\,, \tag{2.6}$$

where

$$\overline{m}_{\ell k} = m_{\ell k} - \frac{1}{3} m_{rr}\delta_{\ell k} \tag{2.7}$$

is the deviatoric part of the couple-stress tensor, and the appearing Kronecker's delta represents the components of the unit matrix, $\boldsymbol{I}$. It is recalled that the spherical part, m_{rr}, of the couple-stress tensor makes no contribution in the equilibrium or in the energy balance equation, while it remains unspecified/undetermined during the deformation of a polar material.

In the absence of body forces and body couples, the energy balance equation takes the form

$$\frac{dE}{dt} = \int_S \left(T_i^{(n)}\dot{u}_i + L_i^{(n)}\dot{\Omega}_i\right) dS\,, \tag{2.8}$$

where E represents the total work done by the tractions and couple-tractions acting on the surface S that surrounds the unstrained volume V of the continuum. Moreover, dS and dV represent the corresponding surface and volume element, respectively, and a dot denotes differentiation with respect to time, t.

Use of (2.5), followed by application of the divergence theorem and the product rule of differentiation, leads thus to

$$\frac{dE}{dt}=\int_V\left(\sigma_{ji}\dot{u}_{i,j}+m_{ji,j}\dot{\Omega}_i+m_{ji}\dot{\Omega}_{i,j}\right)dV\ ,$$

where (2.4) has also been accounted for. Use of (2.3a), (2.2a) and (2.7), along with the symmetry of the rate of strain tensor and the skew-symmetry and rate of rotation tensor leads next to

$$\frac{dE}{dt}=\int_V\left(\sigma_{(ji)}\dot{e}_{ji}+\sigma_{[ji]}\dot{\omega}_{ji}+m_{ji,j}\frac{1}{2}\varepsilon_{i\ell k}\dot{\omega}_{k\ell}+\overline{m}_{ji}\dot{\Omega}_{i,j}\right)dV,$$

and, by virtue of (2.3b), to

$$\frac{dE}{dt}=\int_V\left(\sigma_{(ji)}\dot{e}_{ji}+\overline{m}_{ji}\dot{\Omega}_{i,j}\right)dV\equiv\int_V\left(\dot{W}^e+\dot{W}^\Omega\right)dV=\int_V\dot{W}dV\ , \tag{2.9}$$

where $\dot{W}$, $\dot{W}^e$ and $\dot{W}^\Omega$ represent the rate of the internal, the strain and the spin-gradient energy per unit volume, respectively.

Connection of (2.8) with (2.9) provides finally the following mathematical expression for the principle of virtual work in polar continua:

$$\int_V\left(\sigma_{(ji)}\dot{e}_{ji}+\overline{m}_{ji}\dot{\Omega}_{i,j}\right)dV=\int_S\left(T_i^{(n)}\dot{u}_i+L_i^{(n)}\dot{\Omega}_i\right)dS\ . \tag{2.10}$$

The outlined derivations hold true regardless of the form of specific constitutive equations that determine precisely the linearly elastic behavior of the material of interest. In the absence of body forces and body couples, the set of equations (2.3), (2.4), (2.6) and (2.9) is in principle equivalent to its counterpart that, as is pointed out in (Mindlin and Tiersten, 1963), comprises the couple-stress theory as is essentially left by the Cosserats (1909). Mindlin and Tiersten (1963) have also identified precisely where each of these four equations can be found in the classical article of Truesdell and Toupin (1960), where non-mechanical terms are further introduced. Apart from the implied absence of body forces and body couples, the only principal difference encountered in this section comprises the fact that the deviatoric part (2.7) of the couple-stress tensor enters the implied theoretical formulation in advance of (2.6) and (2.9).

3. Generally anisotropic, polar linearly elasticity

3.1 *Conventional features of the constitutive equations in polar linear elasticity*

In linear elasticity, the internal energy function, W, is necessarily quadratic in the principal kinematic variables involved in the analysis. The outlined preliminary developments suggest that the set of these variables include the strains, e_{ij}, and the spin-gradients, $\Omega_{i,j}$. Mindlin and Tiersten (1963) have indeed employed the same kinematic variables during the linearisation process of a suitable three-term truncation of the energy polynomial expansion proposed by Toupin (1963), and concluded that the most general quadratic form possible for W is

$$W\left(e_{ij},\Omega_{l,k}\right)=\frac{1}{2}c_{ijkl}e_{ij}e_{kl}+b_{ijkl}e_{ij}\Omega_{l,k}+\frac{1}{2}a_{ijkl}\Omega_{j,i}\Omega_{l,k}\,, \tag{3.1a}$$

where the components of the fourth-order tensors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are regarded as appropriate material parameters. Nevertheless, the fact that e_{ij} and $\Omega_{i,j}$ are gradients of a proper vector and a pseudo-vector, respectively, implies further that W is invariant in the full orthogonal group only if

$$\boldsymbol{b}=\boldsymbol{0}\,. \tag{3.1b}$$

It is pointed out that the components of the spin-gradient tensor, $\Omega_{i,j}$, have dimensions of curvature, namely (length)$^{-1}$. They are accordingly associated in (Mindlin and Tiersten, 1963) with a tensor quantity termed the "curvature-twist dyadic". This terminology is not incorrect, but is here avoided because the term "curvature" will later be associated with actual curvature of fibres embedded in polar fibrous composites.

Expressions (3.1a, b) for W are in agreement with (2.9), which anticipates that

$$W=W^{e}+W^{\Omega}\,. \tag{3.2}$$

Due to the linearity of the constitutive equations sought, it is also anticipated that the appearing strain and the spin-gradient energy functions of the material are respectively defined as follows:

$$W^{e}=\frac{1}{2}\sigma_{(ij)}e_{ij}\,,\quad W^{\Omega}=\frac{1}{2}\bar{m}_{ji}\Omega_{i,j}\,. \tag{3.3}$$

It follows that

$$\begin{aligned} W^{e}\left(e_{ij}\right)&=\frac{1}{2}c_{ijkl}e_{kl}e_{ij}\,, & \sigma_{(ij)}&=\frac{\partial W^{e}}{\partial e_{ij}}=c_{ijkl}e_{kl}\,,\\ W^{\Omega}\left(\Omega_{i,j}\right)&=\frac{1}{2}a_{ijkl}\Omega_{j,i}\Omega_{l,k}\,, & \bar{m}_{ji}&=\frac{\partial W^{\Omega}}{\partial \Omega_{i,j}}=a_{jilk}\Omega_{k,l}\,. \end{aligned} \tag{3.4}$$

The linear constitutive equation (3.4b) is identical with the generalised Hooke's law met in non-polar linear anisotropic elasticity, while (3.4d) is a corresponding linear relationship between the couple-stresses and the spin-gradients.

It is now recalled that W represents stored internal energy and is, therefore, customarily considered positive definite in the appearing kinematic variables. In this context, consider initially any displacement field of the form

$$u_1 = f_1'(x_1) f_2(x_2) f_3(x_3), \quad u_2 = f_1(x_1) f_2'(x_2) f_3(x_3), \quad u_3 = f_1(x_1) f_2(x_2) f_3'(x_3), \tag{3.5}$$

where $f_i(x_i)$ are all continuous functions of their single argument, and a prime denotes ordinary differentiation with respect to that argument. It can be readily verified that this displacement field returns $\Omega_i = 0$, thus implying that $W^{\Omega} = 0$ is possible while $W^e \neq 0$. In a similar manner, $W^e = 0$ is also possible while $W^{\Omega} \neq 0$.

Indeed, by integrating the equations $e_{ij} = 0$, one obtains the displacement field

$$u_1 = c_1 x_2 x_3, \; u_2 = c_2 x_3 x_1, \; u_3 = c_3 x_1 x_2, \tag{3.6}$$

where the appearing constants are assumed to be such that $c_1 \neq c_2 \neq c_3$. This displacement field produces the non-zero spin-gradient field

$$\Omega_{1,1} = c_3 - c_2, \; \Omega_{2,2} = c_1 - c_3, \; \Omega_{3,3} = c_2 - c_1. \tag{3.7}$$

Hence, $W^e = 0$ is indeed possible while $W^{\Omega} \neq 0$.

It follows that displacement fields that produce either strains or spin-gradients alone, as the only non-zero kinematic variables, do exist. Hence, positive definiteness of W requires from both W^e and W^{Ω} to be positive definite, namely

$$W^e > 0, \quad W^{\Omega} > 0. \tag{3.8}$$

The latter arguments are not detailed in (Mindlin and Tiersten, 1963) where most of the outlined linear elasticity analysis is based on a combination of the isotropic elasticity counterpart of (3.4b) with the isotropic equivalent of (3.4d). However, Mindlin and Tiersten (1963) have clearly and correctly required from the isotropic versions of both W^e and W^{Ω} to be positive definite and, hence, to satisfy (3.8). They thus concluded that the polar material equivalent of the Lamé elastic moduli, λ and μ, should still satisfy the well-known conditions that guarantee positive definiteness of W^e in non-polar isotropic linear elasticity.

In a similar manner, positive definiteness of the generally anisotropic form (3.4a) of W^e leads here to the conclusion that the elastic moduli c_{ijkl} should still satisfy those conditions that guarantee positive definiteness of W^e in non-polar anisotropic linear elasticity (e.g., Ting, 1996;

Jones, 1998). The remaining of Section 3 is thus enabled to further present several new concepts, theorems and results which are relevant and, hence, complement their counterparts met in to the polar linear elasticity model introduced in (Mindlin and Tiersten, 1963). Some of these new developments are already introduced in Part I (Soldatos, 2018) where, however, are found valid and applicable only in connection with the restricted version of the linearly elastic model presented and discussed in (Spencer and Soldatos, 2007; Soldatos, 2015) for transversely isotropic fibrous composites with embedded fibres resistant in bending.

3.2 *The displacement-gradient and the rotation energy functions*

In a close connection with the definition of the strain and the spin-gradient energy functions (3.3), Soldatos (2018) introduced the concept of the displacement-gradient energy function

$$U(e_{ij},\omega_{ij}) = \frac{1}{2}\sigma_{ji}u_{i,j} = \frac{1}{2}(\sigma_{(ji)} + \sigma_{[ji]})(e_{ij} + \omega_{ij}) = \frac{1}{2}(\sigma_{(ij)}e_{ij} + \sigma_{[ji]}\omega_{ij}) = W^{e}(e_{ij}) + W^{\omega}(\omega_{ij}). \qquad (3.9)$$

It is observed that the displacement-gradient and the strain energy functions coincide in non-polar linear elasticity, where there is no rotation energy stored in the material ($W^{\omega} = 0$). Use of the displacement field (3.6) shows that, like W^{Ω}, $W^{\omega} \neq 0$ is also possible while $W^{e} = 0$. Hence, in general, W^{ω} should necessarily also be regarded as a positive definite function ($W^{\omega} > 0$).

Under the aforementioned assumption that requires from all components of $\boldsymbol{m}$ to be differentiable functions, use of (2.2b) and (2.3b) yields the rotation energy into the following equivalent forms:

$$W^{\omega}(\omega_{ij}) = \frac{1}{2}\sigma_{[ji]}\omega_{ij} = \frac{1}{4}\varepsilon_{kij}m_{\ell k,\ell}\omega_{ij} = \frac{1}{4}\varepsilon_{kij}(m_{\ell k}\omega_{ij})_{,\ell} - \frac{1}{4}\varepsilon_{kij}\left(\overline{m}_{\ell k} + \frac{1}{3}m_{rr}\delta_{\ell k}\right)\omega_{ij,\ell}, \qquad (3.10)$$

or

$$W^{\omega}(\omega_{ij}) = W^{\Omega}(\Omega_{i,j}) - \frac{1}{2}(m_{ij}\Omega_{j})_{,i}, \qquad (3.11)$$

where use is also made of (2.2a) and (3.3b).

Integrating (3.11) over the body of the polar solid of interest and making use of the divergence theorem, one obtains

$$\int_V W^{\omega}(\omega_{ij})\,dV = \int_V W^{\Omega}(\Omega_{i,j})\,dV - \frac{1}{2}\int_S L_i^{(n)}\Omega_i\,dS, \qquad (3.12)$$

which shows that the total rotation energy stored in the material equals the total spin-gradient energy minus one half of the work done by the external couple-tractions acting through their ultimate spin vector field, $\boldsymbol{\Omega}$. Alternatively, the work done by the field of external moments equals twice the difference of the total spin and rotation energies.

It is now recalled that, with use of (3.4d), (2.2a) and (2.1b), the equilibrium equations (2.6) produce an equivalent set of displacement partial differential equation (PDEs) which is generally non-elliptic (e.g., Soldatos, 2014, 2015). As is already mentioned in the Introduction, lack or loss of ellipticity of that Navier-type set of PDEs is associated with existence of potential weak discontinuity solutions of well-posed boundary value problems. Namely, solutions for which the components of $\boldsymbol{m}$ and/or their spatial derivatives are discontinuous and, therefore, non-differentiable, while the components of $\boldsymbol{u}$, $\boldsymbol{e}$ and $\boldsymbol{\omega}$ may still be differentiable functions. The outlined connection of weak discontinuity solutions with potential non-differentiability of the couple-stress tensor, $\boldsymbol{m}$, makes impossible some of the differentiations implied in (3.10) and, hence, invalidates (3.11) in that case. It is thus re-emphasised that validity of (3.11) in well-posed boundary value problems of polar linear elasticity is associated only with potential continuous, though not necessarily unique relevant solutions.

3.3 *Fundamental Theorems in generally anisotropic polar linear elasticity*

The polar elasticity extension of Clapeyron's theorem, noted as Theorem 1 in (Soldatos 2017), made use of an early definition of W^{Ω} that accounts twice the amount of the spin-gradient energy noted in the right hand side of (3.3b). By replacing that definition of W^{Ω} with (3.3b), the theorem is refined as follows:

Theorem 1 (polar material extension of Clapeyron's theorem)

If a polar linearly elastic body of volume V is in equilibrium under the action of tractions $\boldsymbol{T}$ and couple-tractions $\boldsymbol{L}$ applied externally on its bounding surface S, then the sum of the total strain and spin-gradient energies of deformation equals one half of the work done by the external forces and moments acting through their ultimate displacement and spin vector fields, $\boldsymbol{u}$ and $\boldsymbol{\Omega}$, respectively. Namely,

$$\int_V \left[W^e\left(e_{ij}\right) + W^{\Omega}\left(\Omega_{i,j}\right)\right] dV = \frac{1}{2}\int_S \left(T_i^{(n)} u_i + L_i^{(n)} \Omega_i \right) dS \,. \tag{3.13}$$

The proof of this revised version of the theorem is essentially identical to its counterpart presented in (Soldatos, 2018). For self-sufficiency of this study, Appendix A outlines an alternative, briefer form of that proof.

It is emphasised that the outlined polar material extension of Clapeyron's theorem applies not only on polar fibre-reinforced materials of the type considered in (Soldatos, 2018), but, more generally, on any kind of isotropic or anisotropic linearly elastic solid consistent with the constitutive equations (3.4). Moreover, use of (3.12) converts (3.13) into

$$\int_V \left[W^e\left(e_{ij}\right) + W^{\omega}\left(\omega_{ij}\right)\right] dV = \frac{1}{2}\int_S T_i^{(n)} u_i \, dS \,, \tag{3.14}$$

thus leading to the following

Alternative form of Theorem 1:

If a polar linearly elastic body of volume V is in equilibrium under the action of tractions $\boldsymbol{T}$ and couple-tractions $\boldsymbol{L}$ applied externally on its bounding surface S, then the sum of the total strain and rotation energies of deformation equals one half of the work done by the external forces acting through their ultimate displacement field, $\boldsymbol{u}$.

In view of these results, it is worth noting that (3.12) underpins the following

Theorem 2:

If a polar linearly elastic body is in equilibrium under the action of homogeneous couple-traction boundary conditions ($\boldsymbol{L} = \mathbf{0}$), then the total rotation energy stored in its material equals its spin-gradient counterpart.

Appropriate combination of the polar material extension of Clapeyron's theorem (3.13) or, equivalently (3.14) with the linear constitutive equations (3.4) leads next to the following

Theorem 3:

A well-posed boundary value problem in generally anisotropic polar linear elasticity can have only a single continuous solution.

The proof of this theorem is briefly detailed in Appendix B, mostly for self-sufficiency of this study. In a slightly different form, this proof is also outlined in (Gourgiotis and Bigoni, 2016) where, however, no mention is made of the observation that potential continuous and weak discontinuity solutions of a well-posed boundary value problem may co-exist in polar linear elasticity.

The fact that, when underpinned by (3.4), a well-posed boundary value problem in polar linear elasticity admits a single continuous solution suggests that the latter may be sought by minimising some relevant potential energy functional. Indeed, the following energy minimisation theorem holds:

Theorem 4 (Theorem of minimum potential energy in generally anisotropic polar linear elasticity):

Of all continuous and differentiable displacement fields ***u**** which (i) satisfy the displacement boundary conditions on S^u, and (ii) possess up to third-order continuous and differentiable derivatives, the field ***u*** that represents the single continuous solution of a well-posed boundary value problem in the polar linear elasticity underpinned by the constitutive equations (3.14) yields a minimum value of the potential energy functional

$$P(u_i)=\int_V\left[\,W^e\left(e_{ij}\right)+2W^{\Omega}\left(\Omega_{i,j}\right)\right]dV-\int_{S^T}\left(T_i^B u_i+L_i^B\Omega_i\right)dS\,, \tag{3.15a}$$

which, by virtue of (3.12), is equivalent to

$$P(u_i)=\int_V\left[\,W^e\left(e_{ij}\right)+2W^{\omega}\left(\omega_{ij}\right)\right]dV-\int_{S^T}T_i^B u_i\,dS\,. \tag{3.15b}$$

Here S^T represents the part of the bounding surface of the solid that boundary tractions, T_i^B, and couple-tractions, L_i^B, are prescribed on. The remaining of the bounding surface, which boundary displacements, u_i^B, and boundary spins, Ω_i^B, are prescribed on, is denoted by S^u (see also relevant notation in Appendix B).

Proof:

Multiplying both sides of (2.4) by the vector $\boldsymbol{u} - \boldsymbol{u}^*$ and, then, integrating over the volume V, one obtains

$$\int_V \sigma_{ji,j}\left(u_i - u_i^*\right) dV = 0 \,. \tag{3.16}$$

Through a process similar to that described in Appendix A, (3.16) leads to

$$\int_S n_j \sigma_{ji}\left(u_i - u_i^*\right) dS = \int_V \left[\sigma_{(ji)}\left(e_{ij} - e_{ij}^*\right) + \sigma_{[ji]}\left(\omega_{ij} - \omega_{ij}^*\right)\right] dV \,,$$

where all quantities marked with a star relate to $\boldsymbol{u}^*$ in the same manner that their unmarked counterparts relate to $\boldsymbol{u}$. By virtue of (2.3b), (2.5a) and (3.4b), one obtains next

$$\int_S T_i^{(n)}\left(u_i - u_i^*\right) dS = \int_V c_{ijk\ell} e_{k\ell}\left(e_{ij} - e_{ij}^*\right) dV - \frac{1}{2}\varepsilon_{kji} \int_V m_{\ell k,\ell}\left(\omega_{ij} - \omega_{ij}^*\right) dV \,,$$

which, after appropriate use of the product rule of differentiation, the divergence theorem and (2.7), leads to

$$\int_S T_i^{(n)}\left(u_i - u_i^*\right) dS = \int_V c_{ijk\ell} e_{k\ell}\left(e_{ij} - e_{ij}^*\right) dV - \frac{1}{2}\varepsilon_{kji} \left\{\int_S n_\ell m_{\ell k}\left(\omega_{ij} - \omega_{ij}^*\right) dS - \int_V m_{\ell k}\left(\omega_{ij} - \omega_{ij}^*\right)_{,\ell} dV\right\} . \tag{3.17}$$

Through direct use of (2.2a), (3.17) is seen equivalent to

$$\begin{aligned}\int_{S^T} T_i^{(n)}\left(u_i - u_i^*\right) dS &= \int_V c_{ijk\ell} e_{k\ell}\left(e_{ij} - e_{ij}^*\right) dV - \left\{\int_{S^T} L_k^{(n)}\left(\varOmega_k - \varOmega_k^*\right) dS - 2\int_V \left[W^\varOmega\left(\varOmega_{i,j}\right) - W^\varOmega\left(\varOmega_{i,j}^*\right)\right] dV\right\} \\ &= \int_V c_{ijk\ell} e_{k\ell}\left(e_{ij} - e_{ij}^*\right) dV + 2\int_V \left[W^\omega\left(\omega_{ij}\right) - W^\omega\left(\omega_{ij}^*\right)\right] dV,\end{aligned} \tag{3.18}$$

where use is also made of (3.12), and of the fact $\boldsymbol{u} - \boldsymbol{u}^* = \boldsymbol{\varOmega} - \boldsymbol{\varOmega}^* = 0$ on S^u. The following identity is now noted:

$$c_{ijk\ell} e_{k\ell}\left(e_{ij} - e_{ij}^*\right) = \frac{1}{2} c_{ijk\ell}\left[e_{k\ell} e_{ij}^* + \left(e_{k\ell} - e_{kl}^*\right)\left(e_{ij} - e_{ij}^*\right) - e_{kl}^* e_{ij}\right] = W^e\left(e_{ij}\right) + W^e\left(e_{ij} - e_{ij}^*\right) - W^e\left(e_{ij}^*\right), \tag{3.19}$$

and applied in the first integral on the right hand side of (3.18). Appropriate rearrangement of the resulting equation with simultaneous use of (3.15) leads then to

$$P\left(u_i^*\right) - P\left(u_i\right) = W^e\left(e_{ij} - e_{ij}^*\right) \geq 0 \,, \tag{3.20}$$

where, due to the positive definiteness of W^e, equality holds only when $\boldsymbol{u}$ and $\boldsymbol{u}^*$ produce identical continuous deformation fields ($\boldsymbol{e} = \boldsymbol{e}^*$). Hence, the theorem. □

Consider now the particular class of linear elasticity boundary value problems for which

$$W^{\omega}\left(\omega_{ij}\right)=0\,, \tag{3.21}$$

and, hence, the displacement and the strain energy functions coincide by virtue of (3.9). For this class of problems, (3.12) yields

$$\int_V W^{\Omega}\left(\Omega_{i,j}\right)dV=\frac{1}{2}\int_S L_i^{(n)}\Omega_i\,dS\,, \tag{3.22}$$

and, hence, the total spin-gradient energy stored in the material equals one half of the work done by the external moments acting through their ultimate spin-vector field. In that case, (3.13) or, equivalently, (3.14) reduces to its conventional form met in non-polar linear elasticity. Hence, all known non-polar linear elasticity theorems and relevant results still apply when $W^{\omega}=0$. In particular:

Theorem 5:

In linear elasticity, a well-posed mixed boundary value problem that stores no rotation energy is free from weak discontinuity solutions and, therefore, possesses a unique continuous solution.

It is worth noting that, in view of this Theorem, the plane strain solutions and applications presented in Part I are essentially underpinned by the following

Corollary:

By virtue of (3.10), a well-posed linear elasticity boundary value problem that creates a constant couple-stress field throughout a continuum stores no rotation energy and, therefore, possesses a unique continuous solution.

Part I (Soldatos, 2018) has already shown that the fundamental theorem that underpins solution uniqueness of well-posed boundary value problems in non-polar linear elasticity is a particular case of this Corollary which, in turn, becomes now a particular case of Theorem 5 above.

In the light of these observations, it is concluded that boundary value problems in linear elasticity can be divided into two principal classes, namely (i) the class of problems that do store, and (ii) the class of problems that do not store rotation energy, W^{ω}, in the continuum of interest. Problems involved in class (i) may possess one or more weak discontinuity solutions and a single continuous solution. By virtue of (3.9), class (ii) involves boundary value problems that make no distinction between the displacement and the strain energy function, and possess a unique continuous solution (Theorem 5). Class (ii) can further be divided into two subclasses. Namely, a subclass (ii_1) of problems involving couple-stresses that vary with the spatial co-ordinates, and a subclass (ii_2) of problems that generate some constant couple-stress field throughout the continuum of interest.

Non-polar linear elasticity emerges as a particular case of subclass (ii_2), when the implied constant value of the couple-stress field is zero. The widely used identification of non-polar linear elasticity with the evidently much wider term "linear elasticity" is thus now seen as far too general, if not as misleading.

The polar linear elasticity presented in (Mindlin and Tiersten, 1963) made no distinction between the displacement-gradient and the strain energy functions. At the time, this referred to a continuous solution as the unique solution of a relevant well-posed boundary value problem, leaving today the impression that it is essentially referring to the problem class (ii) only. However, the presented new developments (a) clarify the existing difference between the displacement-gradient and the strain energy functions, and (b) lend the theory ability to capture that single continuous solution by minimising of the new potential energy functional (3.15). The present augmented development of Mindlin and Tiersten's (1963) model enables thus the theory to embrace all polar elasticity boundary value problems underpinned by the constitutive equations (3.14).

4. Polar linear elasticity of fibre-reinforced materials when fibres resist bending

Both the definition and the form (3.9) of the displacement-gradient energy function refer to deformable solids that respond in a general, polar linearly elastic manner. It makes thus no distinction between materials exhibiting the generally anisotropic polar material behavior described in the preceding section and the linearly elastic fibrous composites considered in

(Spencer and Soldatos, 2007; Soldatos, 2014, 2015, 2018), where the embedded fibres possess bending resistance. However, the same is not true with the definition and the form of the corresponding internal energy function.

Rather than (3.2), the linearisation process of the equations of polar non-linear elasticity detailed in (Spencer and Soldatos, 2007) requires use of an internal energy function of the form

$$W = W^e + W^K, \tag{4.1}$$

where W^e still coincides with the strain energy function (3.4a) and, hence, depends on the degree of anisotropy involved in the material. As a result, the symmetric strain constitutive equation (3.4b) still holds in this case, along with the requirement (3.8a) for the positive definiteness of W^e. These observations are evidently in line with the fact that, in the non-polar material case, a proper linearisation of the equations of a non-linear elasticity produces identical results with the direct linear elasticity formulation route employed in Section 3.

However, the polar part, W^K, of the internal energy of a composite having embedded N unidirectional fibre families makes no direct use of the spin-gradient tensor $\Omega_{i,j}$. Instead, W^K is required to be quadratic in a set of agents formed by (i) the direction vectors, $\boldsymbol{a}^{(n)}$, of those families, and (ii) kinematic variables stemming from the tensor quantity

$$\kappa_{ij}^{(n)} = \left(u_{i,k} a_k^{(n)}\right)_{,j}, \quad (n = 1, 2 \ldots, N). \tag{4.2}$$

This represents the gradient of the directional derivative of the displacement vector along $\boldsymbol{a}^{(n)}$ and, like its spin-gradient counterpart, has dimensions of $(\text{length})^{-1}$. For convenience, its components are loosely referred to as "curvature-strains" of the n-th fibre family.

Non-polar linearly elastic response of many structural fibrous composites, such as transverse isotropic, orthotropic and monoclinic plate-like structures, is adequately described with the involvement and use of one or, at most, two straight directions of material preference ($N = 2$). For simplicity, these are also the only cases of principal interest employed in what follows. Both vectors $\boldsymbol{a}^{(n)}$ ($n = 1, 2$) involved in the analysis are thus assumed constant and, as a result, (4.2) simplifies into the following:

$$\kappa_{ij}^{(n)} = u_{i,jk} a_k^{(n)}. \tag{4.3}$$

Each fibre family gives rise to its own couple-stress field. Hence, while the symmetric stress constitutive equation of the fibrous composite is still represented by (3.4b), its couple-stress counterpart is as follows (Soldatos, 2014):

$$\bar{m}_\ell = \sum_{n=1}^{N} \bar{m}_\ell^{(n)} = \frac{2}{3}\sum_{n=1}^{N}\left(\frac{\partial W^K}{\partial K_{i\ell}} a^{(n)} + \frac{\partial W^K}{\partial K_{is}} a_\ell^{(n)}\right). \tag{4.4}$$

The particular case of a single family of fibres (N = 1), where this couple-stress constitutive equation becomes easier to manage mathematically, has studied into considerable depth in (Soldatos, 2014). In this context, (Soldatos, 2014) gives also details of the manner that potential weak discontinuity surfaces that exist in the fibrous composite are sought and found. However, the mathematical complexity involved in (4.4) may reach such an overwhelming level when N > 1, that the introduction of some physically meaningful simplification would be helpful; and welcome in those cases.

Instead of employing the full set (4.3) of curvature strains, Soldatos' (2015) analysis for N = 2 employed the restricted version of the theory, which requires from the curvature-strain energy function, W^K, to be quadratic only in the components of the vectors

$$K_i^{(n)} = u_{i,kj} a_k^{(n)} a_j^{(n)} . \tag{4.5}$$

Being the second directional derivative of the displacement along $\boldsymbol{a}^{(n)}$, $K_i^{(n)}$ represents the curvature vector of the n-th fibre family and, and as such, posseses naturally components with dimensions of (length)$^{-1}$.

The couple-stress constitutive equation then simplifies and, rather than (4.4), obtains the following form (Soldatos, 2015):

$$\bar{m}_\ell = \frac{4}{3}\sum_{n=1}^{2} \frac{\partial W^K}{\partial K_i} a^{(n)} a_\ell^{(n)} . \tag{4.6}$$

It is recalled that, along with the (N = 2)-case, the particular case of a single family of fibres (N = 1) was also considered and studied separately in Soldatos (2015) with use of this restricted version of the theory.

In each of these cases, W^K obtains some different form that depends not only on the degree of the observed material anisotropy, but also on the manner that anisotropy is affected by fibre bending resistance. These observations and other relevant issues will be clarified better in the next three sections, where the polar linear elasticity concepts detailed in the preceding sections are connected with transversely isotropic and orthotropic polar material behaviour due to fibre resistance in bending.

5. Transverse isotropy - Restricted theory

The restricted version of the theory was also employed in Part I (Soldatos, 2018) and handles transverse isotropy by dropping in (4.5) and (4.6) the influence of the second family of fibres ($n = N = 1$). By further choosing the x_1-direction parallel to the remaining single family of embedded fibres, one has $\boldsymbol{a} \equiv \boldsymbol{a}^{(1)} = (1,0,0)^T$ and, hence, (4.5) simplifies as follows:

$$K_i \equiv K_i^{(1)} = u_{i,11} \,. \qquad (5.1)$$

The most general form of W^K, which is also quadratic in the kinematic variables (5.1), is as follows (Soldatos, 2015):

$$W^K = \gamma K_j K_j + \tilde{\gamma} \qquad \tilde{} \qquad , \qquad (5.2)$$

where the appearing coefficients represent appropriate material moduli having dimensions of force. By retaining only the first term in the summation noted in (4.6), the corresponding couple-stress constitutive equation provides only two non-zero couple-stress components, namely

$$\bar{m}_{12} = -d^f u_{3,11}, \quad \bar{m}_{13} = d^f u_{2,11}, \quad \left(d^f = 8\gamma/3\right), \qquad (5.3)$$

and involves d^f as the only active fibre bending stiffness parameter. The second material parameter appearing in (5.2), $\tilde{\gamma}$, exerts no influence on these constitutive equations.

Connection of the particular displacement field (3.6) with the fibre curvature vector (5.1) yields $\boldsymbol{K} = \boldsymbol{0}$. Hence, along which $W^e = 0$ and $W^{\Omega} \neq 0$, (3.6) returns $W^K = 0$. It follows that there exist no displacement field that makes the value of W^K non-zero in the complete absence of strains. Positive definiteness of the internal energy (4.1) requires thus from W^K to be positive semi-definite, namely

$$W^K\left(K_i\right) \geq 0 \,. \qquad (5.4)$$

An attempt to connect these results with the generally anisotropic polar material analysis detailed earlier in Section 3 begins, necessarily, with a comparison of (4.1) and (3.2). This comparison reveals that W^{Ω} and the present form (5.2) of W^K are dissimilar, at least because the former is positive definite while the latter is positive semi-definite; see also (3.8b).

However, by excluding in this comparison the displacement field (3.6), one can temporarily lend W^{Ω} properties of a positive semi-definite function. A special form of W^{Ω} can then be sought that relates in such a manner to the form (5.2) of W^K that the corresponding couple-stress fields, obtained with use of (3.4d) and (5.3), respectively, resemble each other as closely as possible.

To this end, use of (4.6) with $N = 1$ and $\boldsymbol{a} \equiv \boldsymbol{a}^{(1)} = (1,0,0)^T$ leads to

$$\bar{m}_\ell = \frac{4}{3}a_i\frac{\partial W^K}{\partial K_i}a_\ell = \frac{4}{3}a_i\frac{\partial W^K}{\partial \Omega_{m,n}}\frac{\partial \Omega_{m,n}}{\partial K_i}a_\ell, \tag{5.5}$$

which, by virtue of (2.2a), (2.1b) and (4.5), and after the use of the intermediate result

$$\frac{\partial \Omega_{i,j}}{\partial K_\ell} = \frac{1}{4}\varepsilon_{imn}\frac{\partial\left(u_{n,mj} - u_{m,nj}\right)}{\partial K_\ell} = \frac{1}{4}\varepsilon_{imn}\left(\delta_{n\ell}\delta_{m1} - \delta_{m\ell}\delta_{n1}\right)\delta_{j1} = \frac{1}{2}\varepsilon_{i\ell 1}\delta_{j1}, \tag{5.6}$$

leads to

$$\bar{m}_\ell = \frac{4}{3}a_i\frac{\partial W^K}{\partial K_i}a_\ell = \frac{2}{3}\varepsilon_{mi1}a_i\frac{\partial W^K}{\partial \Omega_{m,1}}a_\ell = \frac{4}{3}\frac{\partial W^K}{\partial \Omega_{r,1}}\varepsilon_{r1\ell}. \tag{5.7}$$

Because $\boldsymbol{a} = (1,0,0)^T$, (5.7) returns the following non-zero couple-stress components

$$\bar{m}_{12} = \frac{4}{3}\frac{\partial W^K}{\partial \Omega_{2,1}}, \quad \bar{m}_{13} = \frac{4}{3}\frac{\partial W^K}{\partial \Omega_{3,1}}, \tag{5.8}$$

which are the same with their (5.3) counterparts. In view of (3.4d), (5.8) then suggests that the special form sought for W^{Ω} is

$$W^{\Omega} = \frac{4}{3}W^K, \tag{5.9}$$

provided that the inactive material coefficient appearing in (5.2) is set equal to zero ($\tilde{\zeta} = 0$). Hence, a comparison of (5.8) and (3.4d), with simultaneous use of (5.3) and (5.9), suggests that the special form sought for (3.4c) involves only two nonzero parameters a_{ijkl}, namely

$$a_{1212} = a_{1313} = d^f. \tag{5.10}$$

In that case, use of (3.4c, d) produces, as closest possible resemblance of (5.3), the following constitutive equations:

$$\bar{m}_{12} = \frac{\partial W^{\Omega}}{\partial \Omega_{2,1}} = a_{1212}\Omega_{2,1} = d^{f}\left(u_{1,31} - u_{3,11}\right), \quad \bar{m}_{13} = \frac{\partial W^{\Omega}}{\partial \Omega_{3,1}} = a_{1313}\Omega_{3,1} = d^{f}\left(u_{2,11} - u_{1,21}\right). \tag{5.11}$$

These still differ to their (5.3) counterparts, to which become however identical if

$$u_{1,31} \simeq \qquad \simeq \quad . \tag{5.12}$$

The latter additional requirements happen to be consistent with a fundamental postulate of the present restricted theoretical framework. That postulate implies that the derivatives of the longitudinal normal strain, e_{11}, have negligible influence on the couple-stress constitutive equations and becomes evident through a careful comparison of the equations (9.21) and (9.23) presented in (Spencer and Soldatos, 2007).

By proposing (5.10) as the only non-zero material parameters retained in (3.4c, d), the present analysis shows thus that appropriate "filtering" of the generally anisotropic constitutive equations of the polar linear elasticity proposed by Mindlin and Tiersten (1963) enables that theory to account for the bending resistance of a single family of unidirectional straight fibres. This filtering process requires from W^{Ω} to (i) be regarded as positive semi-definite, rather than as positive definite function, and (ii) possess only the pair (5.10) of non-zero material moduli. Moreover, it requires from the model of (Mindlin and Tiersten, 1963) to (iii) adopt a postulate of the restricted version of the present model (Spencer and Soldatos, 2007; Soldatos, 2014) which supports the approximation (5.12).

It can thus readily be verified that, as soon as $W^{K} \geq 0$ (with $\tilde{\;}$ in (5.2)) is connected with W^{Ω} through (5.9) and, further, (2.2) is modified in the manner proposed by (5.12), the main theoretical developments outlined in Section 3 become directly applicable to the present restricted theoretical framework which is also employed in Part I (Soldatos 2018). These new theoretical developments include (i) validity of the polar material extension of Clapeyron's theorem, (ii) applicability of the subsequent *Theorem 3* regarding "uniqueness" of the continuous solution of a well-posed boundary value problem, and (iii) the fact that minimisation of the potential energy functional (3.15) captures that single (rather than unique) continuous solution. Additional weak discontinuity solutions of the type detailed in (Soldatos, 2014) may still be present in this case although, (iv) by virtue of *Theorem 5*, these are certainly not observed/activated in boundary value problems that do not store rotation energy in the material ($W^{\omega} = 0$); e.g., (Soldatos, 2018).

The outlined analysis and observations may lead to a feeling that analogous situations occur when material anisotropy due to fibre resistant in bending exceeds the implied bounds of transverse isotropy. However, it is seen next that this is not true even in a relatively simple case of special orthotropy (Section 7). As is shown next, in Section 6, this is not true even in the case that transverse isotropy is handled with use of the unrestricted version of the present theory.

6. Transverse isotropy - Unrestricted theory

When transverse isotropy of the type discussed in the preceding Section is modelled by means of the unrestricted version of the theory (Spencer and Soldatos, 2007; Soldatos, 2014), the relevant curvature-strain tensor, namely (4.2), obtains the following simplified form:

$$\kappa_{ij} = u_{i,j1} . \tag{6.1}$$

W^e in (4.1) is still in the form of its non-polar transverse isotropic material counterpart.

However, the procedure detailed in (Soldatos 2014) revealed that the curvature-strain part of (4.1) can be described as follows:

$$W^K = W^\Omega + W^E , \tag{6.2}$$

where the parts

$$W^\Omega = D_{55}\Omega_{1,1}^2 + D_{77}\left(\Omega_{2,1}^2 + \Omega_{3,1}^2\right), \tag{6.3}$$

and

$$W^E = \left(e_{11,1},\ e_{22,1},\ e_{33,1}\right)\begin{bmatrix} D_{11} & D_{12} & D_{12} \\ D_{12} & D_{22} & D_{23} \\ D_{12} & D_{23} & D_{22} \end{bmatrix}\begin{pmatrix} e_{11,1} \\ e_{22,1} \\ e_{33,1} \end{pmatrix} + D_{44}e_{23,1}^2 + D_{66}\left(e_{31,1}^2 + e_{12,1}^2\right), \tag{6.4}$$

of the internal energy depend on the directional derivatives of the spin and the strain components, respectively, along the fibre direction. The appearing *D*-coefficients have dimensions of force and are regarded as material parameters.

Soldatos (2014) considering W^Ω and W^E as parts of the internal energy stored in the material and presented a set of non-strict inequalities which, when satisfied by the *D*-coefficients, guarantee positive semi-definiteness of (6.2). However, the aforementioned role of the displacement field (3.6) suggests now that, due to the involvement of W^{Ω}, (6.2) should rather be positive definite. Hence, the implied non-strict inequalities (Soldatos, 2014) should slightly be modified and replaced by their strict inequality counterparts.

The couple-stress constitutive equation stemming from (6.2) is obtained by retaining only the first term in the summation noted in (4.4), and provides the following non-zero couple-stress components:

$$\begin{pmatrix} \bar{m}_{11} \\ \bar{m}_{22} \\ \bar{m}_{33} \end{pmatrix} = \frac{2}{3}\begin{bmatrix} 2D_{55} & 0 \\ -D_{55} & -D_{44} \\ -D_{55} & D_{44} \end{bmatrix}\begin{pmatrix} \Omega_{1,1} \\ e_{23,1} \end{pmatrix}, \quad \begin{pmatrix} \bar{m}_{23} \\ -\bar{m}_{32} \end{pmatrix} = \frac{4}{3}\begin{bmatrix} D_{12} & D_{22} & D_{23} \\ D_{12} & D_{23} & D_{22} \end{bmatrix}\begin{pmatrix} e_{11,1} \\ e_{22,1} \\ e_{33,1} \end{pmatrix},$$
$$\bar{m}_{12} = 2D_{77}\Omega_{2,1} - \frac{2}{3}D_{66}e_{31,1}, \quad \bar{m}_{13} = -2D_{77}\Omega_{3,1} + \frac{2}{3}D_{66}e_{12,1}. \tag{6.5}$$

It is worth noting that, in accordance with (2.7), the trace of this tensor is zero. Moreover, through appropriate rearrangement of the appearing terms and coefficients, (6.5) can be brought into their alternative form detailed in Section 9 of (Spencer and Soldatos, 2007).

It is observed that either of the parts W^{Ω} and W^{E} of W^{K} exerts its own different influence on the constitutive equations (6.5). In accordance with its Section 3 counterpart, W^{Ω} depends on spin-gradients only. Its first term contributes to deformations that resemble the so-called twist mode in the mechanics of liquid crystals (Stewart, 2004), while its second term to deformation resembling the corresponding bending mode. However, W^{E} is expressed in terms of additional kinematic variables, $e_{ij,1}$, which are not met in Section 3. Accordingly, W^{E} consists of three terms that contribute to deformation modes that resemble splay, twist and bending modes, respectively, met in the mechanics of liquid crystals; see also (Spencer and Soldatos, 2007).

The appearing additional kinematic variables, $e_{ij,1}$, are neither involved in (Mindlin and Tiersten, 1963) nor in the relevant conventional theoretical analysis detailed in Section 3. Hence, unlike their spin-gradient counterparts, their involvement in the constitutive equations (6.5) is not, and cannot be captured through the direct influence that externally applied tractions, $\boldsymbol{T}$, and couple-tractions, $\boldsymbol{L}$, exert on the internal energy stored in the material. Instead, the appearance of $e_{ij,1}$ is evidently inflicted by second-gradient deformation effects that represent changes of the strain field along the fibre direction. The scale of those changes is apparently comparable to the scale of the fibre thickness and, in this regard, W^E seems connected with micro-scale deformation modes, including fibre-damage modes of the type implied in Figure 1.

It is thus concluded that positive semi-definiteness of W^E suffices to guarantee the required positive definiteness of W^K. However, in that case, the theoretical framework of the Mindlin and Tiersten (1963) model or its generalised counterpart detailed in Section 3 becomes incompatible with the unrestricted version of the present theory.

Alternatively, one could find that some, if not all of the analysis detailed in Section 3 is still applicable in the present case, provided that the W^{Ω}-part of W^K retains positive definiteness while, at the same time, no limitations are imposed on the sign of W^E. In such a case, a comparison of (3.2) with (4.1) and (6.2) would suggest that (3.4d) can produce the part of (6.5) that depends on the spin-gradients if the only non-zero parameters appearing in the spin-gradient function (3.4c) were

$$a_{1111} = -2a_{2211} = -2a_{3311} = 4D_{55}/3, \quad a_{1212} = -a_{1313} = 2D_{77}\,. \tag{6.6}$$

These particularly interesting observations require considerable and careful further consideration, which, however, fall beyond the purposes of the present study.

7. Advanced anisotropy due to a pair of fibre families resistant in bending – Restricted theory

Orthotropy is the immediate higher step of advanced anisotropy, and is characterised by two mutually orthogonal families of fibres (N = 2). The so-called case of "special orthotropy" refers to the relatively simplest possible situation, where both families are made of straight fibres, and their directions define the directions of two co-ordinate axes. If the x_1- and x_2-axes are chosen parallel to those fibre directions, so that $\boldsymbol{a}^{(1)} = (1,0,0)^T$ and $\boldsymbol{a}^{(2)} = (0,1,0)^T$, then (4.5) requires from the restricted theory to employ the fibre curvature vectors

$$K_i^{(1)} = u_{i,11}, \quad K_i^{(2)} = u_{i,22}\,. \tag{7.1}$$

In accordance with the analysis presented in (Soldatos, 2015), (5.2) is next replaced by the following expression:

$$
\begin{aligned}
W^K\left(K_i^{(1)}, K_i^{(2)}\right) = \gamma_1 K_j^{(1)} K_j^{(1)} + 2\gamma_{12} K_j^{(1)} K_j^{(2)} + \gamma_2 K_j^{(2)} K_j^{(2)} + \cdots \\
+ 2\gamma_3 K_1^{(1)} K_1^{(2)} + 2\bar{\gamma}_3 K_2^{(1)} K_2^{(2)} + \gamma_4 \left(K_2^{(1)}\right)^2 + \bar{\gamma}_4 \left(K_1^{(2)}\right)^2,
\end{aligned}
\tag{7.2}
$$

while (4.6) yields the following non-zero couple-stress components

$$
\begin{aligned}
&\bar{m}_{12} = -d_{11} u_{3,11} - d_{12} u_{3,22}, \quad \bar{m}_{21} = d_{12} u_{3,11} + d_{22} u_{3,22}, \\
&\bar{m}_{13} = d_{31} u_{2,11} + d_{32} e_{22,2}, \quad \bar{m}_{23} = -d_{13} e_{11,1} - d_{23} u_{1,22},
\end{aligned}
\tag{7.3}
$$

where the appearing material moduli are given in terms of the coefficients of (7.2) as follows:

$$
\begin{aligned}
&\left(d_{11}, d_{12}, d_{22}\right) = \frac{8}{3}\left(\gamma_1, \gamma_{12}, \gamma_2\right), \quad d_{31} = \frac{8}{3}\left(\gamma_1 + \gamma_4\right), \\
&d_{32} = \frac{8}{3}\left(\gamma_{12} + \bar{\gamma}_3\right), \quad d_{13} = \frac{8}{3}\left(\gamma_{12} + \gamma_3\right), \quad d_{23} = \frac{8}{3}\left(\gamma_2 + \bar{\gamma}_4\right).
\end{aligned}
\tag{7.4}
$$

In attempting to connect (7.3) with the constitutive equations (3.4d), one can follow similar steps to those detailed in Section 5. Accordingly, W^{Ω} needs again to be temporary associated with the class of positive semi-definite functions and, hence, the displacement field (3.6) is again temporarily excluded from the analysis. A form of W^{Ω} is next sought that enables the couple-stress fields (3.4d) and (7.3) to resemble each other as closely as possible.

After use is mad of of (4.6), (5.5) and (5.6) are thus replaced by the following:

$$
\begin{aligned}
&\bar{m}_\ell \quad \frac{4}{3}\frac{\partial W^K}{\partial \Omega_{m,n}}\left(\quad \frac{\partial \Omega_{m,n}}{\partial K_i} K_\ell^{(1)} \quad \frac{\partial \Omega_{m,n}}{\partial K_i} K_\ell^{(2)} \right) \\
&\frac{\partial \Omega_{m,n}}{\partial K_i^{(1)}} = \frac{1}{2}\varepsilon_{m1i}\delta_{n1}, \quad \frac{\partial \Omega_{m,n}}{\partial K_i^{(2)}} = \frac{1}{2}\varepsilon_{m2i}\delta_{n2},
\end{aligned}
\tag{7.5}
$$

which lead to the constitutive equation

$$
\bar{m}_\ell \quad \frac{4}{3}\left(\frac{\partial W^K}{\partial \Omega_{r,1}} K_\ell^{(1)} \quad \frac{\partial W^K}{\partial \Omega_{r,2}} K_\ell^{(2)} \right). \tag{7.6}
$$

With the use of (5.9), this constitutive equation returns the following non-zero couple-stress components:

$$
\bar{m}_{12} = \frac{\partial W^{\Omega}}{\partial \Omega_{2,1}}, \quad \bar{m}_{21} = \frac{\partial W^{\Omega}}{\partial \Omega_{1,2}}, \quad \bar{m}_{13} = \frac{\partial W^{\Omega}}{\partial \Omega_{3,1}}, \quad \bar{m}_{23} = \frac{\partial W^{\Omega}}{\partial \Omega_{3,2}}, \tag{7.7}
$$

which are the same with their counterparts shown in (7.3).

The closest resemblance of (7.3) sought, through the use of (3.4c, d), is thus observed by (i) retaining in (3.4c) only the following non-zero a_{ijkl}-parameters:

$$a_{1212} = d_{11}, \quad a_{1221} = a_{2121} = -d_{12}, \quad a_{2112} = d_{22}, \quad a_{1313} = d_{31}, \quad a_{2323} = d_{23}, \tag{7.8}$$

and (ii) by requiring from (5.12) still to hold, along (iii) with their x_2-direction counterparts

$$u_{2,32} \simeq \qquad \simeq \quad . \tag{7.9}$$

In this manner, (7.7) yields

$$\begin{aligned}
\bar{m}_{12} &= \frac{\partial W^{\Omega}}{\partial \Omega_{2,1}} = a_{1212}\Omega_{2,1} + a_{1221}\Omega_{1,2} = -d_{11}u_{3,11} - d_{12}u_{3,22}, \\
\bar{m}_{21} &= \frac{\partial W^{\Omega}}{\partial \Omega_{1,2}} = a_{2112}\Omega_{2,1} + a_{2121}\Omega_{1,2} = d_{12}u_{3,11} + d_{22}u_{3,22}, \\
\bar{m}_{13} &= \frac{\partial W^{\Omega}}{\partial \Omega_{3,1}} = a_{1313}\Omega_{3,1} = d_{31}u_{2,11}, \\
\bar{m}_{23} &= \frac{\partial W^{\Omega}}{\partial \Omega_{3,2}} = a_{2323}\Omega_{3,2} = -d_{23}u_{1,22}.
\end{aligned} \tag{7.10}$$

However, this set of constitutive equations is still dissimilar to (7.3), which makes also use of the additional kinematic variables $e_{11,1}$ and $e_{22,2}$. The latter represent changes of normal strain (extension or contraction) along the direction of the first and second fibre family, respectively. They make thus the present, restricted version of the theory to look more similar to its unrestricted theory counterpart discussed in the preceding Section rather than to the Mindlin and Tiersten version (1963) detailed in Section 3. Indeed, unlike their spin-gradient counterparts, $\Omega_{i,j}$, the additional variables $e_{11,1}$ and $e_{22,2}$ are neither involved in the Mindlin and Tiersten, (1963) model nor in the relevant analysis detailed in Section 3. The appearance of $e_{11,1}$ and $e_{22,2}$ is again inflicted by second-gradient deformation effects, but these strain changes have now a seemingly simpler origin.

Accordingly, deformation effects due to bending of one fibre family influences the normal strain measured in its perpendicular (curvature) direction, which, in turn, is the initial direction of the other fibre family. Such interactions between bending and extension modes of the involved pair of fibre families are obviously not present in the corresponding case of transverse isotropy (Section 5), and in the present situation do not solicit unconditional neglect of changes that fibre extension/contraction experience throughout the fibrous composite of interest. Approximations of the type (5.12) and (7.9) are, however, still acceptable in the presence of two families of embedded fibres.

Under these considerations, the polar part (7.2) of the internal energy function is again found susceptible to a decomposition of the form (6.2), where

$$
\begin{aligned}
W^{\Omega} &= \gamma_1 K_j^{(1)} K_j^{(1)} + 2\gamma_{12} K_3^{(1)} K_3^{(2)} + \gamma_2 K_j^{(2)} K_j^{(2)} + \ldots \left(K_2^{(1)}\right)^2 + \overline{\gamma}_4 \left(K_1^{(2)}\right)^2, \\
W^{E} &= 2\left(\gamma_{12} + \gamma_3\right) K_1^{(1)} K_1^{(2)} + 2\left(\gamma_{12} + \overline{\gamma}_3\right) K_2^{(1)} K_2^{(2)}.
\end{aligned}
\tag{7.11}
$$

If the outlined analysis is considered consistent and comparable with the conventional analysis detailed in Section 3, then W^{Ω} is required to be positive definite. As is claimed towards the end of the preceding Section, limitations on the sign of the W^{E} might be found unnecessary in that case, though this matter requires considerable and careful further investigation. If, on the other hand, W^{E} is conveniently declared positive semi-definite, then the theoretical framework detailed in Section 3 becomes in this case incompatible even with the restricted version of the present theory. The situation remains essentially unchanged in cases that anisotropy advances beyond the bounds of special orthotropy.

8. Conclusions

All new concepts, theorems and features presented in Section 3 in association with generally anisotropic, polar, linearly elastic materials are found consistent not only with the formalism due to Mindlin and Tiersten (1963), but also with the analysis presented in Part I (Soldatos, 2018). This is because, by imposing certain conditions on the spin-gradient energy function employed in (Mindlin and Tiersten, 1963), that function becomes reducible to the curvature-strain energy function of the restricted version of the model detailed in (Spencer and Soldatos, 2007, Soldatos, 2015) and used in Part I for transversely isotropic composites with embedded fibres resistant in bending.

The implied new concepts include those of the displacement-gradient and the rotation energy functions. A refined version is also provided in Section 3 of the polar material extension of Clapeyron's theorem, introduced initially in Part I, along with a proof of a theorem (Theorem 3), according which a well-posed boundary value problem in polar, generally anisotropic linear elasticity can have only a single continuous solution. That solution can be captured either by solving the relevant non-elliptic governing differential equations, or by minimising an appropriately refined version of the potential energy met in non-polar linear elasticity (Theorem 4).

It is further concluded that well-posed boundary value problems in generally anisotropic linear elasticity can be divided into two principal classes: (i) these problems that do store, and (ii) those that do not store rotation energy, W^{ω}, in the solid of interest. Problems in class (i) may possess one or more weak discontinuity solutions, in addition to the aforementioned continuous solution. Class (ii) involves boundary value problems that make no distinction between the displacement and the strain energy functions and are unable to possess/activate weak discontinuity solutions (Theorem 5).

Class (ii) can further be divided into two subclasses, namely a subclass (ii_1) of problems involving couple-stresses that vary with the spatial co-ordinates, and a subclass (ii_2) of problems that generate some constant couple-stress field throughout the solid of interest. Non-polar linear elasticity emerges as a particular case of subclass (ii_2), in which the implied constant value of the couple-stress field is zero. The widely used identification of non-polar linear elasticity with the evidently much wider term “linear elasticity” is thus seen too general, if not misleading.

The compared pair of theoretical formalisms, namely those stemming from (Mindlin and Tiersten, 1963) and (Spencer and Soldatos, 2007), fail to agree and, hence, lose mutual consistency as soon as either (i) transverse isotropy is modelled by the unrestricted version of the theory due to (Spencer and Soldatos, 2007; Soldatos, 2014) or (ii) the restricted version of the latter theory is associated with modelling fibrous composites with embedded two or more unidirectional families of fibres resistant in bending. This disagreement is due to the appearance of additional kinematic variables (Spencer and Soldatos, 2007; Soldatos, 2014) which are seemingly inflicted by second-gradient deformation effects that represent changes of the strain field along the fibre direction(s). Those variables depend neither on the strains nor on the spin-gradient variables employed in the (Mindlin and Tiersten, 1963) model. As their scale is apparently comparable with the scale of the fibre thickness. these can thus be connected with micro-scale deformation modes, including fibre-damage modes of the type implied in Figure 1.

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Appendix A: Proof of the polar material extension of Clapeyron's Theorem (Theorem 1)

Multiplying both sides of (2.4) by the ultimate displacement vector of the deformation, $\boldsymbol{u}$, and then integrating the result over the volume V, one obtains

$$\int_V \sigma_{ji,j} u_i \, dV = 0 , \qquad \text{(A.1)}$$

or, equivalently,

$$\int_V \left[\left(\sigma_{ji} u_i \right)_{,j} - \sigma_{ji} u_{i,j} \right] dV = 0 . \qquad \text{(A.2)}$$

Applying the divergence theorem on the first term of the integrant, one obtains

$$\int_S n_j \sigma_{ji} u_i \, dS = \int_V \left(\sigma_{(ji)} e_{ij} + \sigma_{[ji]} \omega_{ij} \right) dV = \int_V \left(2W^e + 2W^\omega \right) dV ,$$

or, equivalently,

$$\int_S T_i^{(n)} u_i dS = 2\int_V \left(W^e + W^\omega\right) dV \, . \tag{A.3}$$

This is the alternative form (3.14) of the theorem which, by virtue of (3.12), is equivalent to (3.13). □

Appendix B: Proof of Theorem 3

Consider a mixed boundary value problem in linear elasticity, governed by the general polar material constitutive law (3.4), and denote: (*i*) with S^u the part of the bounding surface, *S*, on which the boundary displacement, u_i^B, and boundary spin, Ω_i^B, are prescribed; and (*ii*) with S^T that part of *S* on which boundary tractions, T_i^B, and couple-tractions, L_i^B, are prescribed ($S^u \cup S^T = S$).

Suppose that there exist two differentiable displacement fields, ***u*** and ***u****, which (i) satisfy the boundary conditions

$$\boldsymbol{u} = \boldsymbol{u}^* = \boldsymbol{u}^{\boldsymbol{B}}, \; \boldsymbol{\Omega} = \boldsymbol{\Omega}^* = \boldsymbol{\Omega}^{\boldsymbol{B}} \text{ on } S^u ; \tag{B.1}$$

and (ii) have corresponding differentiable stress fields and doubly differentiable couple-stress fields that respectively satisfy the boundary conditions

$$\boldsymbol{T} = \boldsymbol{T}^* = \boldsymbol{T}^{\boldsymbol{B}}, \quad \boldsymbol{L} = \boldsymbol{L}^* = \boldsymbol{L}^{\boldsymbol{B}} \text{ on } S^T , \tag{B.2}$$

as well as the equilibrium equations

$$\sigma_{ij,i} = \sigma^*_{ij,i} = 0 \text{ in } V. \tag{B.3}$$

Consider next that, due to linearity of all equations involved, the difference fields

$$\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}^*, \quad \hat{\boldsymbol{e}} = \boldsymbol{e} - \boldsymbol{e}^*, \quad \hat{\boldsymbol{\Omega}} = \boldsymbol{\Omega} - \boldsymbol{\Omega}^*, \quad \hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^*, \text{ etc.} \tag{B.4}$$

satisfy the equilibrium equations

$$\hat{\sigma}_{ij,i} = 0 \text{ in } V, \tag{B.5}$$

and the homogeneous set of boundary conditions

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{\Omega}} = \boldsymbol{0} \text{ on } S^u , \tag{B.6a}$$

$$\hat{\boldsymbol{T}} = \hat{\boldsymbol{L}} = \boldsymbol{0} \text{ on } S^T . \tag{B.6b}$$

Then, application of the polar material extension (3.13) of Clapeyron's theorem, in connection with (3.4), yields

$$\int_V \left[W^e\left(\hat{e}_{ij}\right) + W^{\Omega}\left(\hat{\Omega}_{i,j}\right)\right] dV = \frac{1}{2}\left\{ \int_{S^u} \left(\hat{T}_i \hat{u}_i + \hat{L}_i \hat{\Omega}_i\right) dS + \int_{S^T} \left(\hat{T}_i \hat{u}_i + \hat{L}_i \hat{\Omega}_i\right) dS \right\} = 0 \cdot \qquad \text{(B.7)}$$

When combined with the positive definiteness of both W^e and W^{Ω}, (B.7) necessarily requires

$$\hat{e}_{ij} = \hat{\Omega}_{i,j} = 0 . \qquad \text{(B.8)}$$

Hence, in line with the corresponding non-polar linear elasticity result, the displacement field $\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}^*$ represents, at most, a rigid body motion. The corresponding strain, spin, stress and couple-stress fields produced by $\boldsymbol{u}$ and $\boldsymbol{u}^*$ are identical and, hence, there exists only a single continuous solution to the well-posed mixed boundary value problem of interest.

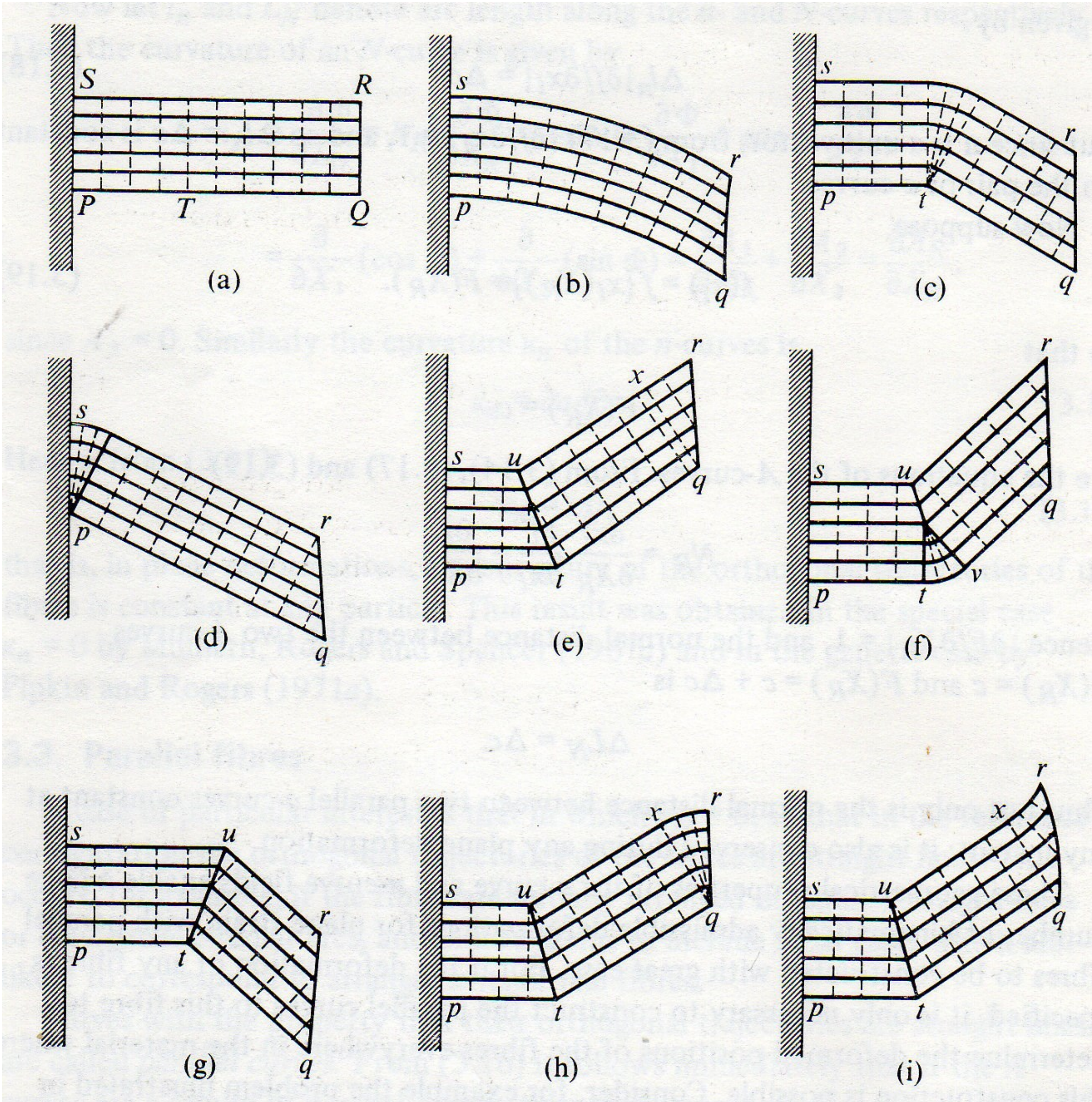

Fig. 3.2. Kinematically admissible deformations of a rectangular block. Solid lines denote a-curves, broken lines denote n-curves.

Figure 1: Scanned image of Fig. 3.2 of (Spencer, 1972) showing: (a) an un-deformed cantilevered rectangular block reinforced by a unidirectional family of straight fibres (the so-called a-curves); and (b -i) a number of different deformation patterns, which are kinematically admissible under the theory of ideal fibre-reinforced materials.