Abstract—The synchronous reluctance motor works under heavy saturation. This paper presents a nonlinear analytical model of the reluctance machine, which is used to derive both average and torque harmonics as a function of the rotor geometry. Maps showing the torque harmonics as a function of the rotor barrier angles are derived. These maps are useful tools for the machine designer to get a proper rotor geometry. The torque maps are compared with those obtained from both linear analytical and finite element models. The maps computed analytically show good agreement with those derived by means of finite element analysis, and they are obtained in a much smaller computing time.

Index Terms—Electric machines, AC motors, Design tools, Synchronous reluctance machine, Analytical model, Saturation

I. INTRODUCTION

Both synchronous reluctance (REL) and permanent magnet assisted reluctance (PAREL) machines are more and more used, ranging from low- to medium-power applications as alternatives to the more expensive rare-earth magnet motors and to the less efficient induction motors [1]–[3]. Although the intrinsic advantages of these machines, a thoughtful design is key to reach the goals of good performance and high efficiency [4]–[6]. In particular, one of the most important design step is the choice of the number of flux-barriers and their end-angles [7], [8].

At first, the model with linear $B-H$ characteristic is described. Then the model is extended to include the saturation of some parts of the machine. Conversely to previous models, the saturation is taken into account following all the main flux lines paths inside the machine. Then a proper saturation coefficient is assigned to each of this path and applied at the air-gap. The analytical model proves to be fast and fairly accurate in any calculation. Therefore it is quick and easy to obtain the behavior of the average torque and torque ripple as a function of the rotor flux-barrier geometry. The result is presented using maps, which are essential for finding a proper combination of barrier angles which gives maximum average torque and minimum torque ripple.

II. ANALYTICAL MODEL

The analytical model considers REL machines with transversally laminated rotor. Furthermore, only integral-slot winding are considered. The electric loading of a symmetric three-phase distributed winding is [8]:

$$K_s(\vartheta_s, \vartheta_m) = \sum_{\nu=-6k+1}^{\infty} \tilde{K}_\nu \sin(\nu p \vartheta_s - p \vartheta_m - \alpha^e_s)$$ (1)

where $\nu$ is the space harmonic order whose values belong to the set $\{1, -5, 7, -11, 13, \ldots\}$, $\tilde{K}_\nu$ is the amplitude of the electric loading $\nu$-th harmonic, $p$ is the number of pole pairs, $\vartheta_s$ is the angular coordinate fixed to the stator, $\vartheta_m$ is the rotor angular position, and $\alpha^e_s$ is the current electric angle. It is worth noticing that (1) reproduces every space harmonic generated by the discretized winding.

The electric loading gives rise to the stator magnetic scalar potential, given by

$$U_s(\vartheta_s, \vartheta_m) = \int K_s(\vartheta_s, \vartheta_m) \frac{D}{2} \, d\vartheta_s$$ (2)
where $D$ is the stator diameter at the air-gap. For a three-phase machine it is

$$U_s(\vartheta_s, \vartheta_m) = -\frac{D}{2} \sum_{\nu = 6k + 1}^{\infty} \hat{K}_\nu \cos(\nu p \vartheta_s - \nu p \vartheta_m - \alpha_i^s)$$  \hspace{1cm} (3)$$

Similarly, the rotor magnetic scalar potential, which reacts to the stator potential, can also be expressed by means of its Fourier series expansion

$$U_r(\vartheta_r, t) = \sum_{\xi = 1}^{+\infty} \hat{U}_\xi \sin(\xi q \vartheta_r)$$  \hspace{1cm} (4)$$

It is different from zero when the flux flows in the $q$-axis direction, crossing the flux-barriers. The rotor magnetic scalar potentials and the rotor barrier angles are grouped in vectors:

$$\mathbf{u}_r = \{U_{r1}, U_{r2}, \ldots, U_{rn}\}^T ,$$ \hspace{1cm} (5)$$

$$\mathbf{\vartheta}_b = \{\vartheta_{b1}, \vartheta_{b2}, \ldots, \vartheta_{bm}\}^T$$  \hspace{1cm} (6)$$

A quasi-diagonal matrix $\mathbf{G}$ can be built, given by

$$\mathbf{G} = \begin{bmatrix}
    +1 & -1 & -1 \\
    +1 & \ddots & \ddots \\
    +1 & -1 & +1 \\
  \end{bmatrix}$$  \hspace{1cm} (7)$$

with +1 in the main diagonal and -1 in the second right-hand diagonal, such that

$$\hat{U}_\xi = \frac{4}{\pi \xi} \sin(\frac{\xi \pi}{2}) [\sin(\xi q \vartheta_b^r) \mathbf{G} \mathbf{u}_r]$$  \hspace{1cm} (8)$$

where $\sin(\xi q \vartheta_b^r)$ is a row vector of sines, $\mathbf{G} \mathbf{u}_r$ is the product of the matrix (7) and the vector $\mathbf{u}_r$, defined in (5). The magnetic potentials of the islands can be calculated as the solution of the magnetic circuit shown in Fig. 2. The stator magnetic scalar potential, given in (3), can be expressed in the rotor reference frame. Since

$$\vartheta_s = \vartheta_r + \vartheta_m$$  \hspace{1cm} (9)$$

it results in

$$U_s(\vartheta_r, \vartheta_m) = -\sum_{\nu = 6k + 1}^{\infty} \frac{\hat{K}_\nu D}{2 \nu \varphi} \cos[(\nu - 1) \nu \vartheta_m - \alpha_i^s]$$  \hspace{1cm} (10)$$

It is split using sine and cosine as:

$$U_s(\vartheta_r, \vartheta_m) = -\sum_{\nu = 6k + 1}^{\infty} \frac{\hat{K}_\nu D}{2 \nu \varphi} \left[ \cos(\nu \vartheta_r) \cos((\nu - 1) \nu \vartheta_m - \alpha_i^s) + \sin(\nu \vartheta_r) \sin((\nu - 1) \nu \vartheta_m - \alpha_i^s) \right]$$  \hspace{1cm} (11)$$

$U_s(\vartheta_r, \vartheta_m)$ also contains harmonics multiple of three and they can be grouped apart. Then, it is easy to verify that $U_s$ is an even function with respect to the harmonic order $\xi$, so positive or negative indexes can be used indifferently.

$$U_s(\vartheta_r, \vartheta_m) = \sum_{\nu = 6k + 1}^{\infty} \hat{U}_s \nu \cos(\nu \vartheta_r) + \sum_{\mu = 6k + 3}^{\infty} \hat{U}_s \mu \sin(\mu \vartheta_r)$$  \hspace{1cm} (12)$$

Fig. 2: Magnetic circuit of one pole of the rotor of a REL machine. Circled labels represent nodes while underlined labels represent edges of the circuit.

The air-gap flux density can be expressed as the difference of the two magnetic scalar potentials [8], [9]:

$$B_g(\vartheta_r, \vartheta_m) = \frac{\mu_0}{g} \left[ -U_s(\vartheta_r, \vartheta_m) + U_r(\vartheta_r, \vartheta_m) \right]$$  \hspace{1cm} (13)$$

Introducing (11) and (12) in (13), the air-gap flux density results in

$$B_g = \sum_{\nu = 6k + 1}^{\infty} \frac{\mu_0 D \hat{K}_\nu}{2 \nu \varphi} \left[ \cos((\nu - 1) \nu \varphi t - \alpha_i^s) \cos(\nu \vartheta_r) \\
\sin((\nu - 1) \nu \varphi t - \alpha_i^s) \sin(\nu \vartheta_r) \right]$$

+ \sum_{\mu = 6h + 3}^{\infty} \frac{\mu_0}{g \mu} \hat{U}_s \mu \sin(\mu \vartheta_r)$$

and reordering

$$B_g(\vartheta_r, \vartheta_m) = \sum_{\nu} \left[ \frac{\alpha_\nu}{\nu} \cos(\nu \vartheta_r) + \frac{\beta_\nu}{\nu} \sin(\nu \vartheta_r) \right] + \sum_{\mu} \frac{\gamma_\mu}{\mu} \sin(\mu \vartheta_r)$$  \hspace{1cm} (14)$$

where $\alpha_\nu, \beta_\nu$ derives from the coefficients which multiply the cosine and sine functions under the summations in $\nu$, while $\gamma_\mu = \mu_0 \mu \hat{U}_s \mu / g$.

An example of the magnetic potentials and the flux density is reported in Fig. 3.
Remembering (13), it can be derived

\[
\tau_m(\vartheta_m) = -\frac{D}{2} \int_0^{2\pi} B_g K_s \frac{DL_{stk}}{2} d\vartheta,
\]

(15)

\[\tau_m = \frac{\mu_0 D^2 L_{stk}}{4g} \left[ \int_0^{2\pi} U_t K_s d\vartheta_A - \int_0^{2\pi} U_t K_s d\vartheta_B \right] \]

(16)

where \( K_s \) is the stator yoke height.

The first integral, labeled as \( A \), is zero since \( U_t \) and \( K_s \) are orthogonal functions. Therefore the torque is only due to the interaction of electric loading \( K_s \) and the magnetic scalar potential of the rotor \( U_t \).

The final expression of the torque is

\[
\tau_m(\vartheta_m) = -\frac{\mu_0 D^2 L_{stk}}{4g} \sum_{\nu,k} \hat{K}_\nu \sin \nu \frac{D}{2} \cdot \cos(\nu - 1) \alpha_\vartheta_m - \alpha_\nu^g \left[ \sin(\nu \theta^T) G_{\nu} \right].
\]

(17)

\[ B_y(\vartheta_m) = \frac{1}{2h_y k_{\text{pack}}} \int_{\gamma_a}^{\pi - \gamma_a} B_g(\vartheta, \vartheta_m) \frac{D}{2} d\vartheta
\]

(21)

and per phase, defined as \( q = Q/(m 2p) \). Thus the integration extremes can be expressed as

\[ \vartheta_d = (d - 1)\alpha_s - \gamma_a, \quad d = 1, \ldots, q \]

(20)

where \( \gamma_a \) identifies the angular position of the magnetic axis of phase \( a \) with respect to the first slot (see Fig. 1).

\[ B_{2\nu N_b + 2} \approx \frac{\varphi_{2\nu N_b + 2}}{w_{ch} k_{\text{pack}} L_{stk}} \]

\[ B_{cd} = \frac{\varphi_{cd}}{w_{ch} k_{\text{pack}} L_{stk}} \]

(22)

where \( w_{ch} \approx (1 - k_{air}) D \sin(\frac{\pi}{2\nu_s} - \beta_{b N_b}) \) is an approximation of the minimum width of the channel.

\[
B_{2\nu N_b + 2} \approx \frac{\varphi_{2\nu N_b + 2}}{w_{ch} k_{\text{pack}} L_{stk}}
\]

D. Rotor fluxes

In the linear case, the fluxes entering the rotor are directly obtained from the solution of the magnetic circuit in Fig 2. Both the island and the barrier fluxes are computed. The iron path closer to the shaft is referred to as channel. The remaining iron paths for the \( d \)-axis flux are referred to as islands. They assume a different potential, due to the \( q \)-axis flux, and they are numbered accordingly to the number of the barrier beneath them. The barrier fluxes are obtained solving the magnetic network depicted in Fig. 2. The \( d \)-axis channel flux comes from \( \varphi_{cd} = \varphi_{N_b + 1} - \varphi_{2N_b + 2} \) which simply sums the two fluxes to obtain the whole direct flux of the channel. Then

It can be demonstrated that the tooth flux density assumes \( q \) different behaviors, \( q \) being the number of slots per pole

\[
B_{2\nu N_b + 2} \approx \frac{\varphi_{2\nu N_b + 2}}{w_{ch} k_{\text{pack}} L_{stk}}
\]
For the \( q \)-axis flux of the main channel, just half of the last barrier flux can be considered, with \( \phi_{cq} = \phi_{h,Ns}/2 \). Thus
\[
B_{cq} = \frac{\phi_{cq}}{l_{cb}k_{pack}L_{stk}}
\]  
where \( l_{cb} = (D_{re} - D_{ri})/2 \). Then, it is
\[
|B_{c}| = \sqrt{B_{c1}^2 + B_{c2}^2}
\]  

The vector of rotor magnetic voltage drops, corresponding to the sequence of channel and islands starting from the origin of \( \vartheta_t \), is:
\[
\psi_{rp} = \{\psi_1, \psi_{1N_s}, \ldots, \psi_{1N_s+2}, \psi_{12N_s+2}, \psi_{c}\}^T
\]
where \( \psi_1 \) and \( \psi_y \) are the vectors of the first half pole (identified by the superscript \( N \) meaning North) and the second half pole (identified by the superscript \( S \) meaning South) voltage drops, respectively.

Numerically, this vector is distributed into a number of points accordingly to the angle spanned. In the end, the distribution of the rotor magnetic voltage drops is equivalent to the distribution of the scalar magnetic potentials along the air-gap.
\[
\psi_{\vartheta} = \{\psi_{e}, \psi_{1N_s}, \ldots, \psi_{1N_s+2}, \psi_{12N_s+2}, \psi_{11}, \ldots, \psi_{11}\}^T
\]

where \( Res \) is the number of points used for discretizing the space \([0, 2\pi]\) along the air-gap, and the subscript of \( N \) refers to the corresponding island.

### B. Stator teeth and yoke magnetic voltage drops

The instantaneous tooth flux density in the saturation case is obtained through the numerical integration of (19). Once the flux density for every tooth of interest (typically \( Q/p \) teeth for an integer-slot winding) is computed, the magnetic voltage drops are derived, as follows:
\[
|B_{i,j}| \to H_{i,j} \quad \text{then} \quad \psi_{i,j} = H_{i,j} l_{stk,j} \frac{1}{2}
\]

where \( h_s \) is the tooth height.

The back-iron corresponding to a pole pair is split into \( Q/p \) parts, each one covering a slot angle, as illustrated in Fig. 5 (dotted lines). The flux density can be derived for each part, as
\[
\tilde{\phi}_y(k) = w_tL_{stk}\sum_{j=1}^{k} B_{i,j} \quad k = 1, \ldots, Q/p
\]

\[
\phi_y(k) = \tilde{\phi}_y(k) - \frac{1}{Q/p} \sum_{j=1}^{Q/p} \tilde{\phi}_y(j)
\]

\[
B_y(k) = \frac{\phi_y(k)}{h_sL_{stk}} \to H_y(k), \quad \psi_y(k) = H_y(k) \Delta l
\]
C. Total magnetic voltage drop and saturation factor

In order to combine the rotor and stator magnetic voltage drop distributions, the rotor position and the first tooth displacement have to be taken into account through the shift of one of the two distributions, according to the adopted reference frame. The air-gap magnetic voltage drop can be easily computed from the flux density obtained in the previous iteration.

\[ \Psi_g = H_g g = \frac{B_g}{\mu_0} g \]  

(39)

All the magnetic voltage drops are summed to obtain the total voltage drop, which is again referred to the air-gap.

\[ \Psi_{tot} = \Psi_g + \Psi_s + \Psi_r \]

(40)

where \( \Psi_r \) is the vector of stator voltage drops in the rotor reference frame. Then

\[ k_{sat} = \frac{\Psi_{tot}}{\Psi_g} \]

(41)

This saturation factor is different from the usually adopted factor. In fact (41) is a distribution of saturation factors along the air-gap that better represents the saturation of the machine.

The adopted iteration scheme is fixed-point like, with a random relaxation to improve the stability of the convergence:

\[ k_{sat}^{(m+1)} \leftarrow k_{sat}^{(m)} + 0.5 \text{rand} \left( k_{sat}^{(m)} - k_{sat}^{(m)} \right) \]

(42)

For the next iteration the updated air-gap flux density is simply

\[ B_g^{(m+1)} = \mu_0 \frac{U_s + U_r}{k_{sat}^{(m+1)} g} \]

(43)

and the iteration cycle restarts. The error of the method was evaluated through

\[ \epsilon_{sat} = \left| k_{sat}^{(m+1)} - k_{sat}^{(m)} \right| \]

(44)

IV. TORQUE MAPS

The nonlinear model is used to compute the impact of the rotor geometry on both the average torque and torque ripple. In particular, the impact of the flux-barrier-end angles is analyzed, since they heavily affect the torque ripple [8]. The average torque and some torque ripple harmonics are evaluated from the flux density obtained in the previous iteration.

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<table>
<thead>
<tr>
<th>TABLE I: Parameters of the reference motor.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q = 36 )</td>
</tr>
<tr>
<td>( 2p = 4 )</td>
</tr>
<tr>
<td>( y_q = 9 )</td>
</tr>
<tr>
<td>( D_e = 200 \text{ mm} )</td>
</tr>
<tr>
<td>( D = 125 \text{ mm} )</td>
</tr>
<tr>
<td>( l_{stk} = 40 \text{ mm} )</td>
</tr>
<tr>
<td>( g = 0.35 \text{ mm} )</td>
</tr>
<tr>
<td>( S_{slot} = 100 \text{ mm}^2 )</td>
</tr>
<tr>
<td>( J = 3 \text{ A/mm}^2 )</td>
</tr>
<tr>
<td>( k_{fill} = 0.45 )</td>
</tr>
<tr>
<td>( k_{pack} = 0.95 )</td>
</tr>
<tr>
<td>( k_{sat} = 0.35 )</td>
</tr>
</tbody>
</table>
For the sake of an easy comparison, Fig. 6 reports three columns, which refer to the results obtained by means of:

i. the analytical linear model [8],
ii. the nonlinear model, described above,
iii. the FE method applied on the same motor geometries [13]–[15].

The results on synchronous REL motor achieved through FE analysis have been compared with experimental test several times by the authors, obtaining satisfactory agreements [8], [16]–[18] but also in other works [19]–[21]. For this reason, in the following comparison, the results computed analytically are considered to be valid if they agree with the FE results.

The first row reports the comparison among average torque maps. The map computed with the linear model shows higher average torques with respect to the other two maps. This is obvious since the model does not take into account the iron saturation. However, it is worth noticing that the behavior of the torque curves is almost identical.

Comparing the maps of the second and third column, it is possible to note that the nonlinear analytical model predicts correctly not only the behavior of the torque maps as a function of \( b_1 \) and \( b_2 \), but also its amplitude. Therefore, the nonlinear model can be used as an alternative to FE simulation to derive such a map.

For this machine configuration, the average torque is almost independent of the first barrier angle. It can be noted that the average torque reaches its maximum in a wide region. For instance, for \( b_2^e > 60^\circ \) the maps are quite flat. Thus the designer is free to move within this space looking for torque ripple minima. This behavior is reflected also by the FE map.

The second row shows the maps of the torque harmonic of order 6, which is the lowest order one. Independently from the model used, it appears that there is an evident minimum—highlighted by the black dot—corresponding to the angle combination \( (b_1, b_2^e) = (36^\circ, 72^\circ) \). Such a point is coincident in the maps obtained from the linear and nonlinear analytical models, while it is a bit shifted when FE is used. This is caused by the local saturation of tooth tips and iron parts, which is not considered in the analytical models. In addition, it can be noted that the torque ripple contours obtained by the three models are in a satisfactory agreement in the whole region.

The same considerations can be made comparing the maps for the third row, which report the torque harmonic of order 18. Such a torque harmonic corresponds to the first magnetic scalar potential (also referred to as MMF) slot harmonic. They are the MMF harmonics produced by the winding discretized inside the slots \( (18 = 36 \text{ slots}/2 \text{ pole pairs}) \), which are characterized by a winding factor equal to the fundamental one. They typically cause the highest ripple. It can be noted that the number of peaks and valleys is increased with respect to the sixth torque harmonic. This trend is general: the higher the harmonic order, the higher the number of maxima and minima. Once more, peaks and valleys predicted by the linear analytical model and nonlinear analytical models are in a good agreement with those found by FE.

Finally, in the last row the maps compare the average torque harmonic distortion (THD), defined as

\[
\text{THD} = \frac{\sqrt{\sum_{h \neq 0} \tau_h^2}}{\tau_0}
\]

where \( \tau_0 \) is the average torque. The filled map shows THD contours, while the white superimposed contours refer to the average torque map. The brighter the color, the higher the THD. Thus, as far as the torque ripple is concerned, the better combinations of \( (b_1, b_2^e) \) are those corresponding to the darker areas.

Furthermore, by comparing the results of the third and fourth rows, it can be observed that bright colors correspond to the peaks of torque harmonic of 18th order, which is due to the MMF slot harmonics. This highlights the heavy impact of the MMF slot harmonics on the overall torque ripple.

Finally, as an overall conclusion, the linear and nonlinear analytical models produce a behavior of average torque map very similar to that obtained through FE, with the difference that the linear map has higher values than the other two models. Therefore, the nonlinear analytical model has to be adopted to predict the average torque. On the other hand, both the analytical models are able to find the position of maxima and minima of the torque THD. Thus, it can be stated that even the linear model can be used in spite of the nonlinear one when searching for the flux-barrier angle combinations exhibiting a minimum torque ripple, with the advantage of a higher speed and similar accuracy. The slight shifts of maxima and minima found by the FE maps are mainly due to the local saturation of iron, which is not taken into account in the linear analytical model.

V. CONCLUSIONS

This paper has shown an accurate nonlinear analytical model for the synchronous reluctance machine. A good agreement between analytical and FE simulations has been achieved, even in highly saturated machines.

Thanks to the speed of the analytical model, it is possible to quickly obtain some maps of torque harmonics as a function of the barrier-end angles. These maps can be a useful design tool for the design of a reluctance machine. Overall, the analytical maps are able to properly estimate the FE maps.

The average torque behavior is correctly predicted by the nonlinear analytical model, while the linear model overestimates it. However, even if there are some differences in the amplitude prediction, the angle combinations corresponding to the minima and maxima are correctly estimated, by means of both the linear and nonlinear models. This fact is quite significant because it suggests that the analytical linear model can be employed to get good design points in the barrier-angle plane for a specific motor in a small amount of time (some minutes). On the other hand, the nonlinear analytical model is used to properly predict the average torque.

REFERENCES

Fig. 6: Main torque maps of the 2-barrier rotor. The darker and cooler colors represent the valleys of the corresponding quantity.