A note on “Anisotropic Total Variation Regularized $L^1$-Approximation and Denoising/Deblurring of 2D Bar Codes”

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Abstract

This note addresses an error in [1].

In this short note, we address an error in [1, Lemma 4.2 and Theorem 6.4] which was pointed out to YvG by ND in April 2016. In this note we assume familiarity with the notation from [1]. That paper erroneously argues that the only binary signals $F_1$ and $F_3$ are faithful to are clean 2D bar codes.

Lemma 4.2 stated that: if $f \in BV(\mathbb{R}^2; \{0,1\})$ is both the measured signal in $F_1$ and a minimizer of $F_1$ over $BV(\mathbb{R}^2)$, then $f \in B$. This statement is false, as can be seen from the following counterexample, which was provided in a private communication by ND.

Let $h \in (1/\sqrt{2}, 1)$ and let $\Omega := B(0, 1) \cap [-h,h]^2$ be a truncated circle. Let $f = \chi_{\Omega} \in L^1(\mathbb{R})$ be the characteristic function of $\Omega$. Define $s := \sqrt{1-h^2}$ and let $^1\lambda > s$. Define, for $r \in \mathbb{R}$, $w(r) := \min\{1, \max\{-1, r/s\}\}$ and let, for $(x,y) \in \mathbb{R}^2$,

$$v(x,y) := \begin{pmatrix} w(x) \\ w(y) \end{pmatrix}.$$ 

We will now show that $v \in V(f)$ and hence, by [1, Theorem 3.2], $F_1$ is faithful to $f$.

It can be verified by direct computation that, for $x \in \mathbb{R}^2$, $|v(x)|_{\infty} \leq 1$ and $\|\text{div} \ v\|_{L^\infty(\mathbb{R}^2)} \leq \frac{2}{s}$. Furthermore, we have for all $z \in \partial\Omega$ that $v(z) \cdot n_{\partial\Omega}(z) = |n_{\partial\Omega}(z)|_1$, where $n_{\partial\Omega}$ is the outward normal vector to the boundary $\partial\Omega$. By the definition of the anisotropic total

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^1Numerical simulations by Nils Dabrock suggest this condition can be weakened to $\lambda > \frac{1}{s}$.
variation in [1, Formula (1)] and [1, Appendix A, Corollary 3] we then find

\[
\int_{\mathbb{R}^2} |f_x| + |f_y| = \sup \left\{ \int_{\mathbb{R}^2} f \ \text{div} \ \varphi : \varphi \in C^1_c(\mathbb{R}^2; \mathbb{R}^2), \forall z \ |\varphi(z)|_\infty \leq 1 \right\}
\]

\[
= \sup \left\{ \int_{\Omega} \text{div} \ \varphi : \varphi \in C^1_c(\mathbb{R}^2; \mathbb{R}^2), \forall z \ |\varphi(z)|_\infty \leq 1 \right\}
\]

\[
= \sup \left\{ \int_{\partial \Omega} \varphi \cdot n_{\partial \Omega} : \varphi \in C^1_c(\mathbb{R}^2; \mathbb{R}^2), \forall z \ |\varphi(z)|_\infty \leq 1 \right\}
\]

\[
\leq \int_{\partial \Omega} |n_{\partial \Omega}(z)|_1 = \int_{\partial \Omega} v \cdot n_{\partial \Omega} = \int_{\mathbb{R}^2} f \ \text{div} \ v
\]

\[
\leq \sup \left\{ \int_{\mathbb{R}^2} f \ \text{div} \ \varphi : \varphi \in L^\infty(\mathbb{R}^2; \mathbb{R}^2), \forall z \ |\varphi(z)|_\infty \leq 1 \ a.e. \right\}
\]

\[
= \int_{\mathbb{R}^2} |f_x| + |f_y|.
\]

Since the second part of Theorem 6.4 was based directly on Lemma 4.2, that result is also incorrect (part 1 of Theorem 6.4 is unaffected).

For a more general treatment of this topic by ND, including the abovementioned counterexample, we refer to [2].

References
