Interval Type–2 Defuzzification Using Uncertainty Weights

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Abstract

One of the most popular interval type–2 defuzzification methods is the Karnik–Mendel (KM) algorithm. Nie and Tan (NT) have proposed an approximation of the KM method that converts the interval type–2 membership functions to a single type–1 membership function by averaging the upper and lower memberships, and then applies a type–1 centroid defuzzification. In this paper we propose a modification of the NT algorithm which takes into account the uncertainty of the (interval type–2) memberships. We call this method the uncertainty weight (UW) method. Extensive numerical experiments motivated by typical fuzzy controller scenarios compare the KM, NT, and UW methods. The experiments show that (i) in many cases NT can be considered a good approximation of KM with much lower computational complexity, but not for highly unbalanced uncertainties, and (ii) UW yields more reasonable results than KM and NT if more certain decision alternatives should obtain a larger weight than more uncertain alternatives.

1 Introduction

A type–1 fuzzy set A [15] is characterized by a membership function $u_A : X \to [0, 1]$ which quantifies the degree of membership of each element of X in A. Here we will always consider fuzzy sets over one–dimensional continuous intervals $X = [x_{\min}, x_{\max}]$. Type–1 defuzzification is a function d that maps a type–1 fuzzy set to one representative crisp value in X.

$$d(u(x)) \in X \tag{1}$$

Numerous methods for type–1 defuzzification have been proposed in the literature. For an overview see [9, 10, 14]. A set of desirable properties of type–1 defuzzification operators has been proposed in [13]. A popular method for type–1 defuzzification is the centroid function, which will be described in more detail in section 2.

An interval type–2 fuzzy set [4, 6, 16] $\tilde{A}$ is characterized by two membership functions: a lower membership function $\underline{u}_{\tilde{A}} : X \to [0, 1]$ and an upper membership function $\overline{u}_{\tilde{A}} : X \to [0,1]$, where

$$\underline{u}_{\tilde{A}}(x) \leq \overline{u}_{\tilde{A}}(x) \tag{2}$$

for all $x \in X$. Interval type–2 fuzzy sets are known to be equivalent to interval–fuzzy sets [1, 2]. It was recently shown that type–2 fuzzy sets can be used to model risk in decision processes [12].

This paper deals with interval type–2 defuzzification, which is a function $\tilde{d}$ that maps an interval type–2 fuzzy set to one representative crisp value in X.

$$\tilde{d}(\underline{u}(x), \overline{u}(x)) \in X \tag{3}$$

A set of desirable properties of interval type–2 defuzzification operators has been proposed in [11]. A popular method for interval type–2 defuzzification is the Karnik–Mendel (KM) method [3], which will be described in more detail in section 2.

Nie and Tan (NT) [7] have proposed an approximation of the KM method that first converts the interval type–2 membership functions to a single type–1 membership function by averaging the upper and lower memberships, and then applies the standard type–1 centroid defuzzification. We will describe this method in more detail in section 3.

In this paper we propose a modification of the NT algorithm that takes into account the uncertainty of the (interval type–2) memberships. We call this method the *uncertainty weight* (UW) method. We compare the behavior of the KM, NT, and UW methods in extensive experiments motivated by fuzzy controller scenarios with different patterns of uncertainty.

This article is structured as follows: Sections 2 and 3 briefly review the KM and NT interval type–2 defuzzification methods. Section 4 introduces the UW interval type–2 defuzzification method. Section 5 presents our experiments to evaluate and compare the KM, NT, and UW methods. Section 6 summarizes the conclusions of this work and points out some future research questions.

2 Karnik–Mendel Interval Type–2 Defuzzification

One of the most popular methods for type–1 defuzzification [9, 10, 14] is the centroid.

$$d_C(u(x)) = \frac{\int\limits_{x_{\min}}^{x_{\max}} u(x) \cdot x \, dx}{\int\limits_{x_{\min}}^{x_{\max}} u(x) \, dx} \tag{4}$$

The KM defuzzification [3] is an extension of the centroid defuzzification to interval type–2 fuzzy sets. For any given interval type–2 fuzzy set with the lower and upper membership functions $\underline{u}(x)$ and $\overline{u}(x)$, each embedded type–1 fuzzy set with the membership function $u(x)$ with

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \tag{5}$$

will yield a centroid according to (4). The smallest and largest possible centroids of such embedded type–1 fuzzy sets are

$$\tilde{c}_l = \inf_{u(x)\in[\underline{u}(x),\overline{u}(x)]} \frac{\int\limits_{x_{\min}}^{x_{\max}} u(x)\cdot x\,dx}{\int\limits_{x_{\min}}^{x_{\max}} u(x)\,dx} \tag{6}$$

$$\tilde{c}_r = \sup_{u(x)\in[\underline{u}(x),\overline{u}(x)]} \frac{\int\limits_{x_{\min}}^{x_{\max}} u(x)\cdot x\,dx}{\int\limits_{x_{\min}}^{x_{\max}} u(x)\,dx} \tag{7}$$

These equations can be equivalently written as

$$\tilde{c}_l = \inf_{L\in[x_{\min},x_{\max}]} \frac{\int\limits_{x_{\min}}^{L} \overline{u}(x)\cdot x\,dx + \int\limits_{L}^{x_{\max}} \underline{u}(x)\cdot x\,dx}{\int\limits_{x_{\min}}^{L} \overline{u}(x)\,dx + \int\limits_{L}^{x_{\max}} \underline{u}(x)\,dx} \tag{8}$$

$$\tilde{c}_r = \sup_{R\in[x_{\min},x_{\max}]} \frac{\int\limits_{x_{\min}}^{R} \underline{u}(x)\cdot x\,dx + \int\limits_{R}^{x_{\max}} \overline{u}(x)\cdot x\,dx}{\int\limits_{x_{\min}}^{R} \underline{u}(x)\,dx + \int\limits_{R}^{x_{\max}} \overline{u}(x)\,dx} \tag{9}$$

The optimal switch points $L, R \in [x_{\min}, x_{\max}]$ can be found by the KM algorithm [3]. The result of the KM defuzzification is defined as the average of the smallest and largest possible centroids:

$$\tilde{d}(\underline{u}(x), \overline{u}(x)) = \frac{\tilde{c}_l + \tilde{c}_r}{2} \tag{10}$$

The next section provides the details of the Nie–Tan approach to interval type–2 defuzzification.

3 Nie–Tan Interval Type–2 Defuzzification

Nie and Tan (NT) [7] proposed an approximation of the KM method. The NT method first maps a given interval type–2 membership function to a type–1 membership function by averaging the upper and lower interval type–2 memberships.

$$u(x) = \frac{1}{2}(\underline{u}(x) + \overline{u}(x)) \tag{11}$$

Then NT computes the conventional type–1 centroid (4) of this type–1 membership function. Type–1 conversion (11) and computation of the type–1 centroid using (4) is computationally much cheaper than iteratively minimizing $\tilde{c}_l$ (8) and maximizing $\tilde{c}_r$ (9). Therefore, the NT method is a popular low effort approximation of the KM method. We now introduce our new UW method.

4 The Uncertainty Weight Method

Though NT is quite simple and straightforward, it is found that it may lose the information of uncertainty. Consider two data points x_1 and x_2 with the interval type–2 memberships $\underline{u}(x_1) = 0$, $\overline{u}(x_1) = 1$, $\underline{u}(x_2) = 0.5$, $\overline{u}(x_2) = 0.5$. For both data points the averaging function (11) will yield the same interval type–2 memberships $u(x_1) = u(x_2) = 0.5$, so both data points will have the same impact on the defuzzification result, although the membership of x_1 has a very high uncertainty reflected by the range of memberships from $\underline{u}(x_1) = 0$ to $\overline{u}(x_1) = 1$, and the membership of x_2 has a very low uncertainty reflected by the fact that the upper and lower memberships are equal, $\underline{u}(x_2) = \overline{u}(x_2) = 0.5$, so in this example the information about the uncertainty is lost by averaging the upper and lower memberships. We define the degree of certainty of the memberships $\underline{u}(x)$ and $\overline{u}(x)$ as

$$w(x) = (1 + \underline{u}(x) - \overline{u}(x))^{\alpha} \tag{12}$$

with a suitable parameter $\alpha > 0$. Smaller values of α will lead to a higher weight for medium uncertainties, and larger values of α will lead to a lower weight for medium uncertainties. In this paper we will always use $\alpha = 1$, which corresponds to a linear weight of the uncertainties. For our example above, equation (12) yields the certainty values $w(x_1) = 1 + 0 - 1 = 0$ and $w(x_2) = 1 - 0.5 + 0.5 = 1$, so data point x_1 is considered very uncertain, and data point x_2 is considered very certain. We want to reflect the (un)certainty of the different data points in defuzzification by using the certainty as a weight for each data point. This means that a relatively certain alternative has a large weight and that a relatively uncertain alternative has a small weight. Including the weights (12) in the averaging function (11) yields the weighted averaging function

$$u(x) = \frac{1}{2}(\underline{u}(x) + \overline{u}(x)) \cdot (1 + \underline{u}(x) - \overline{u}(x))^{\alpha} \tag{13}$$

We combine weighted averaging (13) with type–1 centroid defuzzification (4) and call this the *uncertainty weight* (UW) method. Just as NT, UW is computationally much cheaper than KM. The main motivation for UW, however, is not only the computational cost, but also the explicit consideration of uncertainties.

5 Experiments

In this section we illustrate and compare the behavior of the KM, NT, and UW interval type–2 defuzzification methods. The results of the KM method are shown as solid lines, the results of the NT method are shown as dotted lines, and the results of the UW method are shown as dashed lines.

The considered examples are motivated by a fuzzy controller [5] with (for simplicity) two rules, Gaussian membership functions, and sum–product inference, so the fuzzy controller output is a weighted sum of two Gaussian membership functions. An application example for this setup is a controller of an autonomous vehicle avoiding an obstacle, where one rule triggers the action turn left and the other rule triggers turn right [8]. To mimic this scenario we consider the unit range $X = [0, 1]$ and construct interval type–2 membership functions by adding pairs of weighted Gaussian functions.

$$\underline{u}(x) = \underline{y}_1 \cdot e^{\left(-\frac{x-\mu_1}{2\sigma_1^2}\right)} + \underline{y}_2 \cdot e^{\left(-\frac{x-\mu_2}{2\sigma_2^2}\right)} \tag{14}$$

$$\overline{u}(x) = \overline{y}_1 \cdot e^{\left(-\frac{x-\mu_1}{2\sigma_1^2}\right)} + \overline{y}_2 \cdot e^{\left(-\frac{x-\mu_2}{2\sigma_2^2}\right)} \tag{15}$$

In our first set of experiments we investigate the effect of varying uncertainty on the results of the considered defuzzification methods. We keep one Gaussian constant and perform different variations of the uncertainty of the second Gaussian: difference between upper and lower memberships, the lower memberships only, and the upper memberships only.

In our first experiment we consider variations of the difference between the upper and the lower memberships. To do so, we set $\mu_1 = 1/4$, $\sigma_1 = 1/8$, $\mu_2 = 3/4$, $\sigma_2 = 1/8$, $\underline{y}_1 = 0.5$, $\overline{y}_1 = 1$, $\underline{y}_2 = 0.75 - \Delta/2$, $\overline{y}_2 = 0.75 + \Delta/2$, where the parameter Δ is varied in $[0, 0.5]$. The top left graph in Fig. 1 shows an example of this membership function(s) for $\Delta = 0.3$. Here, the uncertainty of the right Gaussian is a little smaller that the uncertainty of the left Gaussian. The different defuzzification results are marked by vertical lines. In this case, KM (solid) and NT (dotted) yield almost the same results, and UW (dashed) yields a slightly higher defuzzification result which takes into account the fact that the certainty on the right is higher than the certainty on the left. The top right graph in Fig. 1 shows the defuzzification results d for KM, NT, and UW as the uncertainty of the right Gaussian Δ is changed from 0 to 0.5. For $\Delta = 0.5$ both Gaussians are equal, and so for reasons of symmetry all three methods yield $\tilde{d} = 0.5$. As pointed out above, NT (dotted) ignores the different levels of uncertainty and therefore *always* yields the output $\tilde{d} = 0.5$. KM (solid) is only very slightly different from NT (dotted), so here NT is a good approximation of KM with a much lower computational effort. Only UW (dashed) takes into account the (un)certainty and yields a much higher output (closer to the right Gaussian) when the uncertainty of the right Gaussian is lower (for smaller values of Δ).

In our second experiment we consider variations of the lower memberships only, and set $\mu_1 = 1/4$, $\sigma_1 = 1/8$, $\mu_2 = 3/4$, $\sigma_2 = 1/8$, $\underline{y}_1 = 0.5$, $\overline{y}_1 = 1$, $\underline{y}_2 = h$, $\overline{y}_2 = 1$, where the parameter h is varied in $[0, 1]$. The second row of Fig. 1 shows the results of this experiment. On the left we see two Gaussians again, both with a maximum upper membership of one. The left Gaussian has a maximum lower membership of 0.5, and the right Gaussian has a maximum lower membership of h, in this case $h = 0.3$. Here, KM (solid vertical line) and NT (dotted vertical line) yield very similar results, and UW (dashed vertical line) yields a slightly smaller result. The right diagram in row 2 shows the results of the three methods for $h \in [0, 1]$. For $h \approx 0.5$ all three methods produce (almost) the same result around $\tilde{d} \approx 0.5$, as expected for symmetry reasons. As the upper limit of

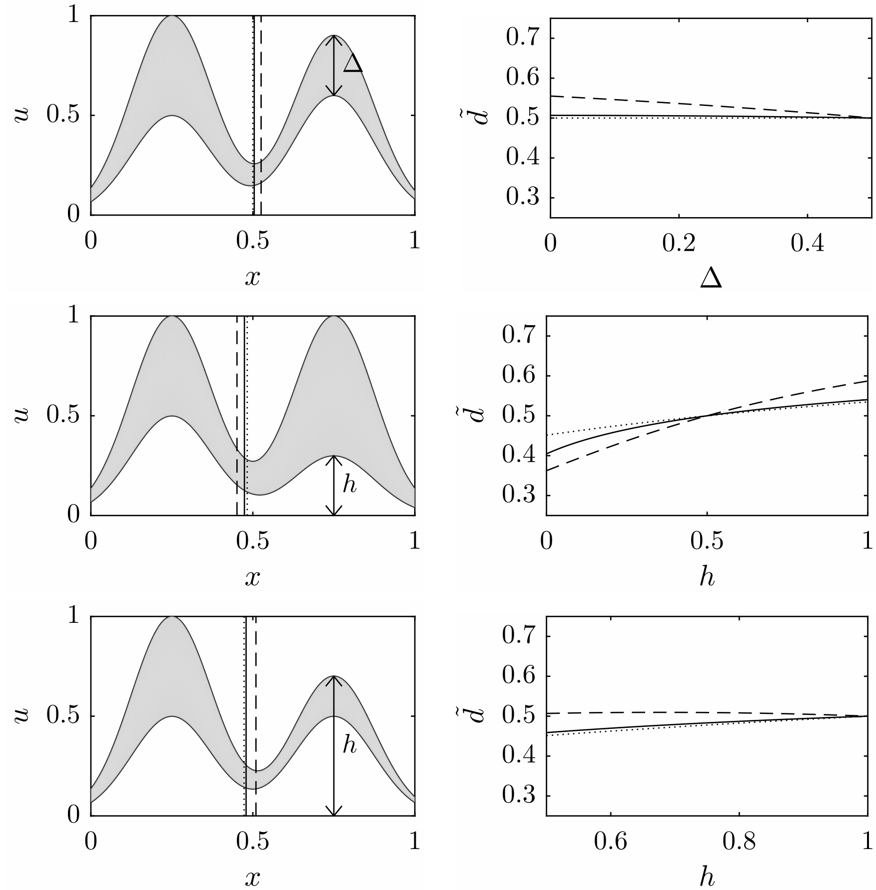

Figure 1: Gaussians with different uncertainty patterns (solid: KM, dotted: NT, dashed: UW).

the lower membership function decreases (lower h), UW (dashed) decreases the result most, then KM (solid), and NT (dotted) decreases the result least. For lower values of h, i.e. for quite unbalanced uncertainty patterns, KM (solid) and NT (dotted) yield significantly different results. This is an example, where NT does not approximate KM well. As the upper limit of the lower membership function increases (higher h), KM (solid) and NT (dotted) stay almost the same, but UW (dashed) increases the result much more, which reflects the higher certainty of the right Gaussian(s). For $h \in [0.5, 1]$ NT can again be considered a good low effort approximation of KM, but not for $h \in [0, 0.5]$, and UW better takes into account the varying uncertainty of the type–2 memberships.

In our third experiment we consider variations of the upper memberships only, and set $\mu_1 = 1/4$, $\sigma_1 = 1/8$, $\mu_2 = 3/4$, $\sigma_2 = 1/8$, $\underline{y}_1 = 0.5$, $\overline{y}_1 = 1$, $\underline{y}_2 = 0.5$, $\overline{y}_2 = h$, where the parameter h is varied in $[0.5, 1]$, see the third row of Fig. 1. On the left we see that the maximum upper membership of the right Gaussian is h, here $h = 0.7$. Also in this case KM (solid vertical line) and NT (dotted vertical line) yield almost the same results, but UW (dashed vertical line) yields a slightly higher result. The right diagram shows the results for $h \in [0.5, 1]$. For $h = 1$ we obtain the symmetric case again and all three methods yield $\tilde{d} = 0.5$. For $h < 1$ UW (dashed) stays almost constant at $\tilde{d} = 0.5$, because the reduction of the memberships is approximately compensated by the increased certainty. In contrast to that, KM (solid) and NT (dotted) are almost the same again and decrease with decreasing h. Again, NT is a good approximator for KM, but UW handles the varying uncertainties in an intuitively more reasonable way than KM and NT.

In our second set of experiments we investigate the effect of variations of horizontal widths σ_2, horizontal positions μ_2, and vertical scales $\overline{y}_2 = 2 \cdot \overline{y}_2$. In [11, 13] the corresponding transformations are called *x–scaling*, *x–translation*, and *u–scaling*, respectively.

The first row of Fig. 2 shows the effects of variations in the horizontal width σ_2 of the right Gaussian. We set $\mu_1 = 1/4$, $\sigma_1 = 1/8$, $\mu_2 = 3/4$, $\underline{y}_1 = 0.5$, $\overline{y}_1 = 1$, $\underline{y}_2 = 0.5$, $\overline{y}_2 = 1$ and vary the parameter σ_2 in $[0, 0.5]$. The left diagram shows the case $\sigma_2 = 1/16$, where all three methods (solid, dotted, and dashed vertical lines) yield almost the same results. The right diagram shows the results of the three defuzzification methods for $\sigma_2 = [0, 0.5]$. For $\sigma_2 = 1/8$ we have the symmetric case and all three methods yield $\tilde{d} = 0.5$. For smaller σ_2 all three methods yield almost the same results: As the width of the right Gaussian is decreased, the defuzzification result decreases as well. For smaller σ_2 KM (solid) and NT (dotted) yield almost the same results: As the width of the right Gaussian is increased, both Gaussians overlap and the left Gaussian gets larger memberships, so also here the defuzzification result becomes lower. For UW (dashed) the results stays close to $\tilde{d} \approx 0.5$ because the increasing memberships of the left Gaussian are approximately compensated by an increasing uncertainty (difference between upper and lower memberships of the left Gaussians). Also here, NT is a good approximator for KM, but UW better take into account the uncertainties.

The second row of Fig. 2 shows the effects of variations in the horizontal

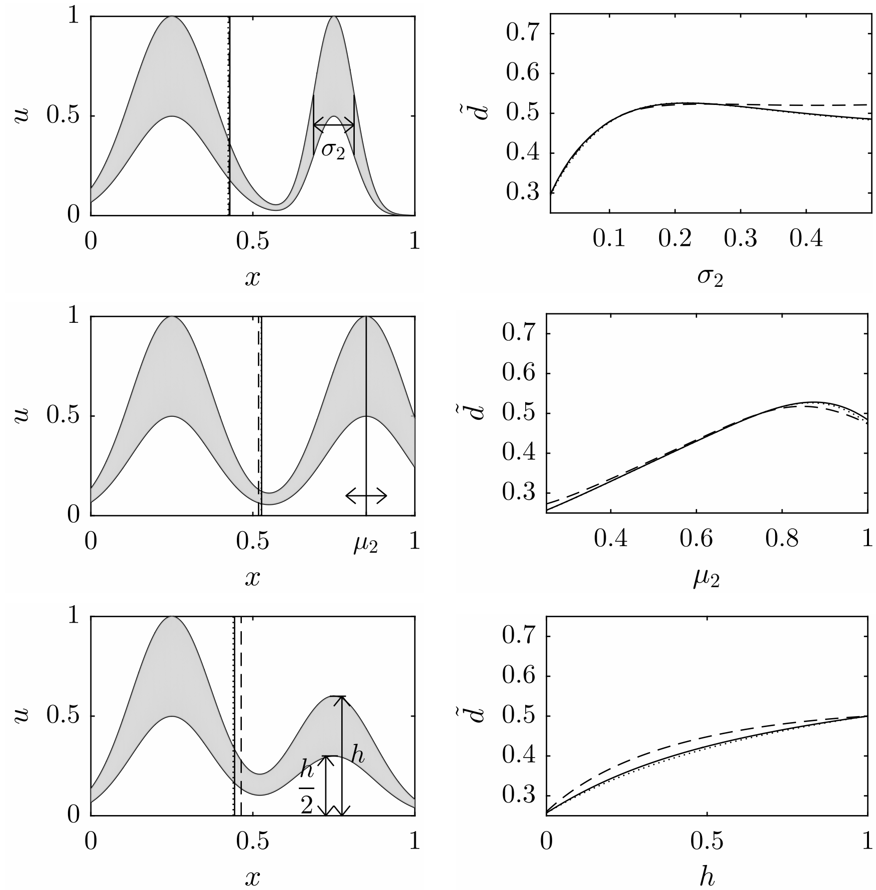

Figure 2: Gaussians with different horizontal widths, horizontal positions, and vertical scales (solid: KM, dotted: NT, dashed: UW).

position μ_2 of the right Gaussian. We set $\mu_1 = 1/4$, $\sigma_1 = 1/8$, $\sigma_2 = 1/8$, $\underline{y}_1 = 0.5$, $\overline{y}_1 = 1$, $\underline{y}_2 = 0.5$, $\overline{y}_2 = 1$ and vary the parameter μ_2 in $[0.5, 1]$. For $\mu_2 = 0.85$ (left diagram) all three methods (solid, dotted, and dashed vertical lines) yield almost the same results. The right diagram shows the results for $\mu_2 = [0.5, 1]$. For $\mu_2 = 3/4$ we have the symmetric case and all three methods yield $\tilde{d} = 0.5$. Also for all other values of $\mu_2 = [0.5, 1]$, all three methods yield almost the same results, so in this case both NT and UW are good approximators for KM.

The third row of Fig. 2 shows the effects of variations in the vertical scale of the right Gaussian. In contrast to the experiments in the third row of Fig. 1 we not only scale the upper membership function of the right Gaussian, $\overline{y}_2$, but also the lower membership function of the right Gaussian, $\underline{y}_2$, but keep the ratio between upper and lower memberships equal to 2, so that $\overline{y}_2 = 2 \cdot \overline{y}_2$. This simulates the situation that the first rule fires with strength 1 (yielding the left Gaussian) and the second rule fires with strength $h \in [0, 1]$ (yielding the right Gaussian), so in our experiments we can observe the behavior of the output when a rule fades out (or fades in). We set $\mu_1 = 1/4$, $\sigma_1 = 1/8$, $\mu_2 = 3/4$, $\sigma_2 = 1/8$, $\underline{y}_1 = 0.5$, $\overline{y}_1 = 1$, $\underline{y}_2 = h/2$, $\overline{y}_2 = h$ and vary the parameter h in $[0, 1]$. For $h = 0.6$ (left diagram) we have $\underline{y}_2 = 0.3$ and $\overline{y}_2 = 0.6$, and KM (solid vertical line) and NT (dotted vertical line) yield very similar results, whereas UW (dashed vertical line) yields a slightly higher result. The right diagram shows the results for $h \in [0, 1]$. For $h = 1$ we obtain the symmetric case and all three methods yield $\tilde{d} = 0.5$. For $h = 0$ the second Gaussian disappears and all three methods yield the center of the first Gaussian $\tilde{d} = 0.25$. The transition between the two extremes $h = 0$ (first rule completely active and second rule completely inactive) and $h = 1$ (both rules completely active) simulates a gradual increase of the firing strength of the second rule from zero to one. During this transition all three methods smoothly move from $\tilde{d} = 0.25$ at $h = 0$ to $\tilde{d} = 0.5$ at $h = 1$. KM (solid) and NT (dotted) yield almost the same results, but UW (dashed) yields slightly higher values, because the certainty of the right Gaussian is higher than the certainty of the left Gaussian. Here again, NT is a good approximator for KM but UW handles uncertainties in more plausible way.

We repeated the same experiments with triangular instead of Gaussian membership functions, where for comparability we chose the triangle widths as $4\sigma_1$ and $4\sigma_2$.

$$\underline{u}(x) = \underline{y}_1 \cdot \max\left(0, 1 - \left|\frac{x - \mu_1}{2\sigma_1}\right|\right) + \underline{y}_2 \cdot \max\left(0, 1 - \left|\frac{x - \mu_2}{2\sigma_2}\right|\right) \quad (16)$$

$$\overline{u}(x) = \overline{y}_1 \cdot \max\left(0, 1 - \left|\frac{x - \mu_1}{2\sigma_1}\right|\right) + \overline{y}_2 \cdot \max\left(0, 1 - \left|\frac{x - \mu_2}{2\sigma_2}\right|\right) \quad (17)$$

Fig. 3 shows the results of the triangle experiments corresponding to the Gaussian experiments shown in Fig. 1. The results of the triangular case are very similar to the results of the Gaussian case. Fig. 4 shows the results of the triangle experiments corresponding to the Gaussian experiments shown in Fig.

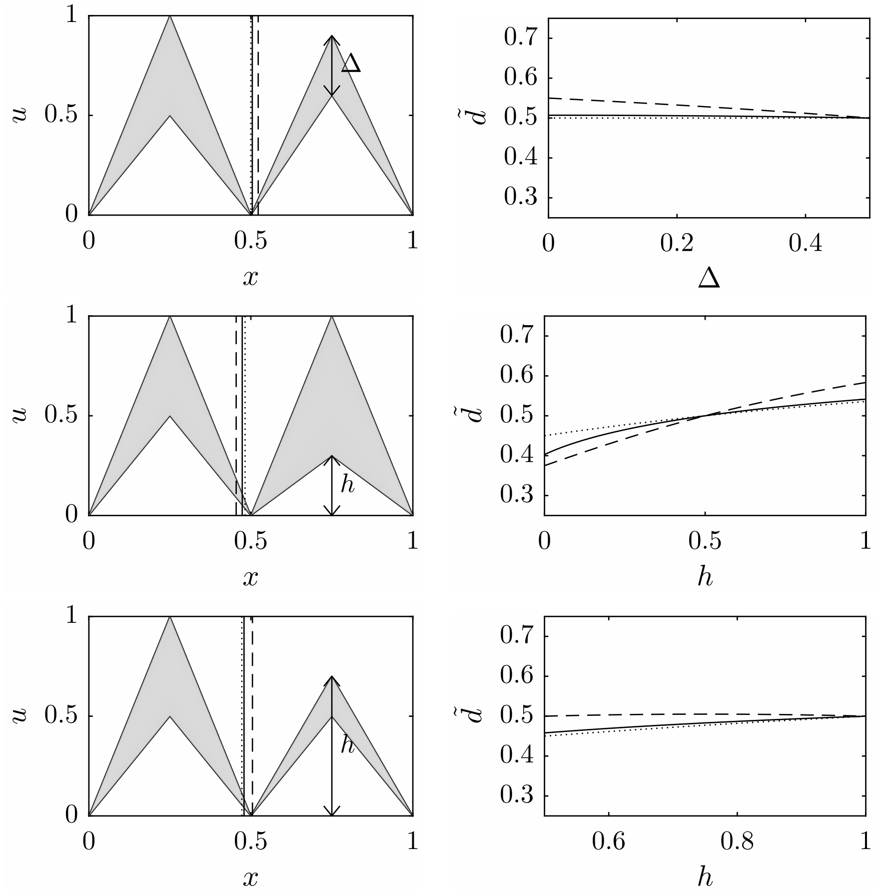

Figure 3: Triangles with different uncertainty patterns (solid: KM, dotted: NT, dashed: UW).

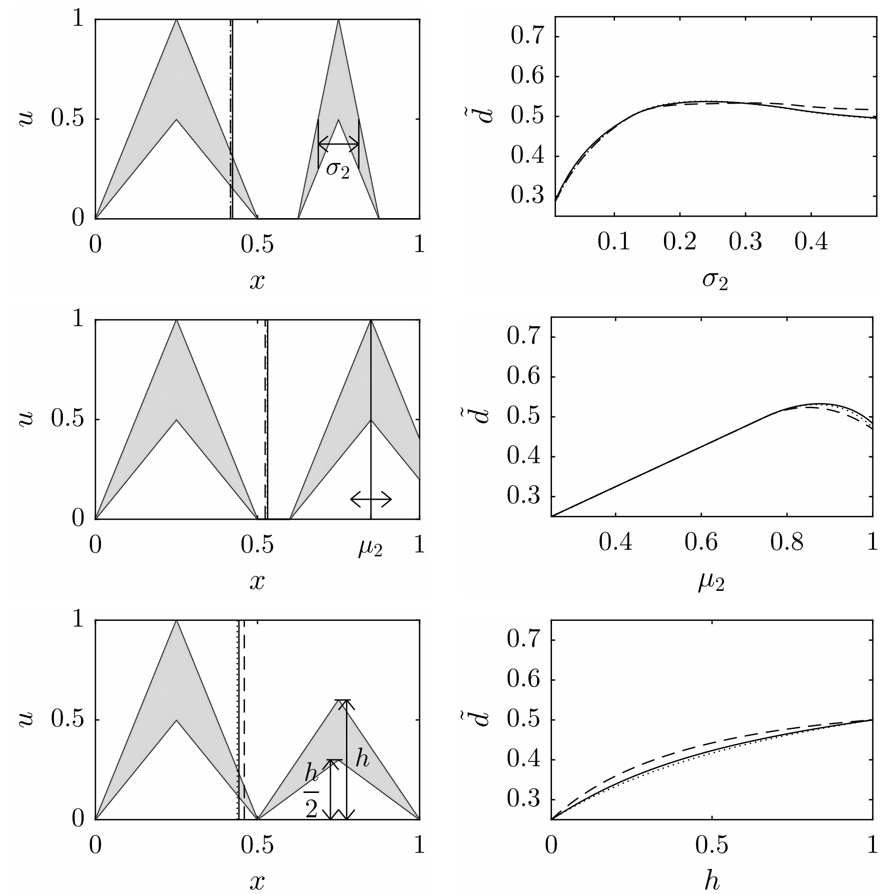

Figure 4: Triangles with different horizontal widths, horizontal positions, and vertical scales (solid: KM, dotted: NT, dashed: UW).

2. Again, the results of the triangular case are very similar to the results of the Gaussian case.

6 Conclusions

We have proposed UW, a modification of the NT interval type–2 defuzzification method that takes into account the uncertainties of the (upper and lower) type–2 membership values. We performed extensive experiments comparing the standard KM method with NT and UW. All experiments were motivated by fuzzy controller scenarios with (for simplicity) two rules, where we investigated the effect of different uncertainty patterns and of different horizontal widths, horizontal positions, and vertical scales of the membership functions on the defuzzification results.

To summarize, our experiments show the following: For the considered scenarios KM and NT mostly yield very similar results, except when parts of the interval type–2 membership function have very different levels of uncertainty. The computational complexity of KM is much higher than NT. Therefore, NT can often be considered a good approximation of KM with low complexity but only for well balanced uncertainties. UW also has a much lower computational complexity than KM, but in addition explicitly takes into account the uncertainty of the interval type–2 memberships, so it yields more reasonable results if more certain decision alternatives should obtain a larger weight than more uncertain alternatives.

This work is a first step in the explicit consideration of uncertainty in type–2 defuzzification. We have to leave many points open for future research, for example:

- We used the weighting scheme in equation (13) to implement the uncertainty weighting, with $\alpha = 1$. What are other equations or values of α will lead to a intuitively plausible treatment of the uncertainties in the type–2 memberships?
- We have applied the uncertainty weighting to the NT method. How could different levels of uncertainty be considered in the KM method?
- How does the behavior of all three methods change if we replace the centroid by other (type–1) defuzzification methods?

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