Mixed Discretization of the Time Domain MFIE at Low Frequencies

H. Arda Ülkü, Member, IEEE, Ignace Bogaert, Kristof Cools, Francesco P. Andriulli, Senior Member, IEEE, and Hakan Bağcı, Senior Member, IEEE

Abstract—Solution of the magnetic field integral equation (MFIE), which is obtained by the classical marching on-in-time (MOT) scheme, becomes inaccurate when the time step is large, i.e., under low-frequency excitation. It is shown here that the inaccuracy stems from the classical MOT scheme’s failure to predict the correct scaling of the current’s Helmholtz components for large time steps. A recently proposed mixed discretization strategy is used to alleviate the inaccuracy problem by restoring the correct scaling of the current’s Helmholtz components under low-frequency excitation.

Index Terms—Marching on-in-time (MOT) method, magnetic field integral equation (MFIE), transient analysis, low-frequency analysis, Buffa-Christiansen functions, mixed discretization.

I. INTRODUCTION

The classical marching on-in-time (MOT) scheme developed for solving time-domain electric and magnetic field integral equations (TD-EFIE and TD-MFIE) use Rao-Wilton-Glisson (RWG) and polynomial basis functions to expand the current in space and time, respectively. This expansion is inserted into the TD-EFIE and TD-MFIE and the resulting equations are tested with RWG functions at discrete times. The conditioning and accuracy of this classical MOT-TD-EFIE matrix system becomes ill-conditioned and cannot be solved using iterative schemes. This problem, which is known as the “low-frequency breakdown” of the EFIE, stems from the fact that the Helmholtz decomposed MOT-TD-EFIE matrix is not balanced in scaling with \( \Delta t \to \infty \) and can be remedied with loop/star [1] and hierarchical preconditioning techniques [2]–[4].

On the other hand, behavior of the MOT-TD-MFIE matrix system as \( \Delta t \to \infty \) has never been investigated. This work, for the first time, studies this behavior. Its contribution is twofold: (i) It rigorously shows that the solution of the classical MOT-TD-MFIE matrix system does not scale correctly in \( \Delta t \) as \( \Delta t \to \infty \). The non-solenoidal component of the current scales as \( O(1) \), which does not yield a finite value for the charge when integrated in time. Consequently, the accuracy of the solution deteriorates regardless of the integration rule used for computing the MOT matrix entries. (ii) It shows that the mixed discretization scheme, which has been originally developed in [5] for solving the frequency domain MFIE, restores the correct scaling, i.e., the non-solenoidal component of the current scales as \( O(\Delta t^{-1}) \). Consequently, the solution of the mixed-discretized TD-MFIE maintains its accuracy for large \( \Delta t \). This is also shown by numerical results.

Mixed discretization [5] uses divergence conforming RWG and curl conforming Buffa-Christiansen (BC) functions [6], i.e., \( \hat{n} \times BC \), as basis and testing functions. Unlike the classical discretization, this scheme conforms with respect to the function spaces of the MFIE operator’s both domain and range [5]–[8] and preserves the correct frequency scaling of the solution’s Helmholtz components [9], [10]. As a result, solution of the mixed-discretized MFIE is more accurate than the classically discretized MFIE especially at low frequencies [5]–[10].

II. FORMULATION

A. TD-MFIE and MOT Scheme

Let \( S \) represent the surface of a perfect electric conductor residing in an unbounded homogeneous background medium. A magnetic field \( \mathbf{H}(r, t) \) bandlimited to \( f_{\text{max}} \) is incident on the conductor. Enforcing the boundary condition on the total magnetic field on \( S \) yields the TD-MFIE [11], [12]:

\[
\mathbf{n}(r) \times \mathbf{H}(r, t) = \frac{1}{2} \mathbf{J}(r, t) - \mathbf{n}(r) \times \nabla \times \int_S \frac{\mathbf{J}(r', t-R/c)}{4\pi R} \, dr'.
\]

(1)

Here, \( R = |r - r'| \) is the distance between observer and source points, \( r \) and \( r' \), and \( \mathbf{n}(r) \) is the outward pointing unit normal vector on \( S \). To numerically solve (1), \( \mathbf{J}(r, t) \) is approximated using spatial and temporal basis functions, \( f_n(r) \) and \( T_i(t) \):
Here, \( \{ \mathbf{I}_m \}_n = I_{m,n} \), \( \{ \mathbf{V}_j \}_m = V_j \{ t_m \} \), and \( \{ \mathbf{Z}_k \}_m,n = Z_k \{ t_m,f_n \} \), where the operators \( V_j \{ t_m \} \), \( G_k \{ t_m,f_n \} \), \( k_k \{ t_m,f_n \} \), \( Z_k \{ t_m,f_n \} \) are defined as

\[
V_j \{ t_m \} = \int_{S_m} t_m(r) \cdot \hat{n} \times \mathbf{H}^i(r, \rho) \, d\rho \tag{4}
\]

\[
Z_k \{ t_m,f_n \} = 1 \frac{1}{2} G_k \{ t_m,f_n \} - 1 \frac{1}{2} \mu_k \{ t_m,f_n \} \tag{5}
\]

\[
G_k \{ t_m,f_n \} = \int_{S_m} (t_m(r) \cdot f_n(r)) T(k \Delta t) \, d\rho \tag{6}
\]

\[
k_k \{ t_m,f_n \} = \int_{S_m} (t_m(r) \cdot \hat{n} \times \nabla \left[ \frac{T(k \Delta t - R/c)}{R} \right]) \times f_n(r') \, d\rho' \, d\rho \tag{7}
\]

The choice of \( t_m(r) \) determines the spatial discretization schemes termed “classical” and “mixed” as detailed in Sections II-B and II-C, respectively. \( \Delta t \) depends only on \( f_{\text{max}} \): 
\( \Delta t = 1/(\alpha f_{\text{max}}) \), where \( \alpha \) is the over-sampling factor and \( 5 \leq \alpha \leq 20 \). For a given spatial discretization, choosing a high value for \( \alpha \) increases \( N_u \) unnecessarily without any additional gain in accuracy. When \( f_{\text{max}} \) is small, \( \Delta t \) should ideally be chosen large.

### B. Classical Discretization

It has been shown in [5] and [8] that to obtain accurate results, the discretization of an integral equation should be conforming with respect to the function spaces, where the range and domain of the integral operator reside, and the resulting matrix system should be well-conditioned. For the MFIE, conforming discretization means that the testing functions \( t_m(r) \) should reside in the dual space of the divergence conforming RWG basis functions \( f_n(r) \). Curl conforming RWG testing functions \( t_m(r) = \hat{n}(r) \times f_m(r) \) satisfy this condition. However, the resulting Gram matrix with entries \( G_0 \{ \hat{n} \times f_m,f_n \} \) is singular, which makes the solution of (3) impossible. Therefore, in the literature, the choice \( t_m(r) = f_m(r) \) is adopted. In this work the scheme resulting from this choice of testing function is termed the classical discretization scheme. In what follows here, it is rigorously shown that the current obtained by solving the classically-discretized TD-MFIE has incorrect scaling in \( \Delta t \) under plane-wave excitation. It should be noted here that the results of the analysis carried out in Sections II-B and II-C are valid for any excitation that can be represented by a plane-wave expansion.

The behavior of the Helmholtz components of the current as \( \Delta t \to \infty \) can be analyzed by decomposing the RWG space into two components spanned by loop and star functions. Assume that \( \mathbf{J}(r,t) \) is approximated as

\[
\mathbf{J}(r,t) = \sum_{n=1}^{N} \sum_{i=1}^{N} \mathbf{I}^i_n(t) f^i_n(r) \tag{8}
\]

Here, \( N_1 + N_s = N \) and \( f^i_n(r) \) and \( f^s_n(r) \) are loop and star basis functions, which are constructed from linear combinations of RWG functions [13]. Inserting (8) into (2) and testing the resulting equation with \( \mathbf{f}^i_n(r) \), \( m = 1 : N_1 \) and \( \mathbf{f}^s_n(r) \), \( m = 1 : N_s \), at times \( t = j \Delta t \) yield

\[
\mathbf{Z}^i_{0j} \mathbf{I}^i_j = \mathbf{V}^i_j - \sum_{i=0}^{j-1} \sum_{i=1}^{N} \mathbf{Z}^i_{j-i} \mathbf{I}^s_i, \quad j = 1 : N_i. \tag{9}
\]

Here, the entries of the blocks of the matrix \( \mathbf{Z}^i_{j-i} \) and the vectors \( \mathbf{V}^i_j \) and \( \mathbf{I}^i_j \) are given by

\[
\{ \mathbf{Z}^i_{j-i} \} = \begin{cases}
\mathbf{Z}_k \{ f^i_n,f^i_n \}, m = 1 : N_1, n = 1 : N_1 \\
\mathbf{Z}_k \{ f^s_{n-N^i_1}, f^i_n \}, m = N_1 + 1 : N, n = 1 : N_1 \\
\mathbf{Z}_k \{ f^s_{n-N^i_1}, f^s_{n-N^s_1} \}, m = 1 : N, n = N_1 + 1 : N \\
\mathbf{Z}_k \{ f^s_{n-N^i_1} \}, N_1 + 1 : N, n = N_1 + 1 : N \\
\end{cases}
\]

\[
\{ \mathbf{V}^i_j \} = \begin{cases}
V_j \{ f^i_n \}, m = 1 : N_1 \\
V_j \{ f^s_{n-N^i_1} \}, m = N_1 + 1 : N \\
\end{cases}
\]

\[
\{ \mathbf{I}^i_j \} = \begin{cases}
\mathbf{I}^i_n, m = 1 : N_1 \\
\mathbf{I}^s_{n-N^i_1}, m = N_1 + 1 : N \\
\end{cases}
\]

The scaling of \( \mathbf{I}^i_j \) as \( \Delta t \to \infty \) can be obtained from the scaling of \( \mathbf{Z}^i_{0j} \) and \( \mathbf{V}^i_j \). It is obvious that \( G_0 \{ f^i_n,f^i_n \} \sim O(1) \), \( p,q \in \{ 1,s \} \). Consider the Taylor series expansion of the gradient term in (7) with \( k = 0 \):

\[
\nabla \left[ \frac{T(-R/c)}{R} \right] = \nabla \left[ \sum_{p=0}^{\infty} \frac{(-1)^p T^{(p)}(0) R^{p-1}}{p!} \right] \tag{11}
\]

\[
= T(0) \nabla R^{-1} + \hat{\mathbf{R}} \sum_{p=2}^{\infty} \frac{(-1)^p T^{(p)}(0) R^{p-2}}{p!}.
\]

Here, \( T^{(p)}(0) = \partial^p T(t) \bigg|_{t=0} \) and \( \hat{\mathbf{R}} = (r - r')/R \). The first term in the right hand side (RHS) of (11) scales as \( O(1) \). The terms in the summation scale as \( O(\Delta t^{-p}) \), \( p \geq 2 \), due to the derivatives \( T^{(p)}(0) \). By inserting (11) into the expression of \( K_0 \{ f^i_n,f^i_n \} \), \( p,q \in \{ 1,s \} \), one can show that, its dominant terms all scale as \( O(1) \). From expressions of \( V_j \{ f^i_n \} \) and \( V_j \{ f^s_{n-N^i_1} \} \), it can be concluded that \( V_j \{ f^i_n \} \) and \( V_j \{ f^s_{n-N^i_1} \} \) scale as \( O(1) \). Consequently, one obtains

\[
\mathbf{Z}^i_{0j} \sim \begin{cases}
O(1) & \{ O(1) \}
\{ O(1) \}
\end{cases}
, \quad \mathbf{V}^i_j \sim \begin{cases}
O(1) & \{ O(1) \}
\{ O(1) \}
\end{cases}
, \quad \mathbf{I}^i_j \sim \begin{cases}
O(1) & \{ O(1) \}
\{ O(1) \}
\end{cases}
\]

Inserting (8) into the equation of continuity \( \rho(r,t) = -\int_0^t \nabla \cdot \mathbf{J}(r,t) \, dt \) and using the fact that \( \nabla \cdot f^i_n(r) = 0 \), one can obtain

\[
\rho(r,t) = -\sum_{n=1}^{N_1} \sum_{i=1}^{N_1} \int_0^t \mathbf{J}(t)(r) \, dt. \tag{13}
\]

Here, \( \rho(r,t) \) is the charge density. The integral in (13) scales as \( O(\Delta t) \). Therefore, \( \mathbf{I}^i_j \) has to scale as \( O(\Delta t^{-1}) \) to obtain a finite value for \( \rho(r,t) \). This immediately shows that \( \mathbf{I}^i_j \) obtained by solving (9) does not scale correctly as \( \Delta t \to \infty \).

### C. Mixed Discretization

Mixed discretization scheme uses the rotated BC functions as testing functions, i.e., \( t_m(r) = \hat{n}(r) \times g_m(r) \), \( m = 1 : N \), where \( g_m(r) \) denote the divergence conforming BC functions [6]. Since \( \hat{n}(r) \times g_m(r) \) are curl conforming, the mixed discretization is conforming with respect to the function space of the MFIE operator’s range. Additionally, the resulting Gram matrix with entries \( G_0 \{ \hat{n} \times g_m, f_n \} \) is well-conditioned [14]. In what follows here, it is shown that the current obtained by
Here, \( \phi \) is the surface gradient operator. It should be noted that the expressions obtained by replacing \( \hat{n} \) \( \times \) \( g_m(r) \) \( \times \) \( f_n(r) \) \( / (4\pi) \) in (10) into the expression of \( K_0(\hat{n} \times g_m, f_n) \) yields:

\[
K_0(\hat{n} \times g_m, f_n) = \int_{S_m} g_m(r) \int_{S_n} \nabla R^{-1} \times f_n(r') dr' dr
\]

\[
+ \int_{S_m} g_m(r) \sum_{p=2}^{\infty} (-1)^p T_p(0) R^{p-2} \hat{R} \times f_n(r') dr' dr.
\]

The first term of the RHS in (18) represents the static magnetic field due to a loop source tested by a loop function and is zero [15]. Terms of the summation in the RHS scale as \( O(\Delta t^{-p}) \), \( p \geq 2 \), due to derivatives \( T_p(0) \). Therefore, the dominant term in \( K_0(\hat{n} \times g_m, f_n) \) and hence \( Z_0(\hat{n} \times g_m, f_n) \) scales as \( O(\Delta t^{-2}) \). As a result,

\[
Z_0^{1s} \sim \frac{O(\Delta t^{-2})}{O(1)} \cdot O(1).
\]

The scaling of \( V_j^{1s} \) can be determined using \( V_j(\hat{n} \times g_m) \) and \( V_j(\hat{n} \times g_m) \). It can be easily seen that \( V_j(\hat{n} \times g_m) \sim O(1) \). Inserting (16) into the expression of \( V_j(\hat{n} \times g_m) \) and applying the chain rule and several vector manipulations to the resulting equation yield:

\[
\int_{S_m} \nabla \times [\partial_m(\hat{n} \times H(r,t))] dr
\]

Surface divergence theorem is applied to the first term of the RHS. The second term is simplified assuming \( H(r,t) \) is the magnetic field of a plane wave. These operations yield:

\[
\int_{S_m} \nabla \times [\partial_m(\hat{n} \times H(r,t))] dr
\]

\[
+ \int_{S_m} c^{-1} (\hat{n} \times \hat{k}) \cdot \partial_m(\hat{n} \times H(r,t)) dr
\]

\[
\sim [O(\Delta t^{-1})] \cdot I_j^{1s} \sim \frac{O(1)}{O(\Delta t^{-1})} \quad \text{(22)}
\]

Indeed, the scaling of \( I_j^{1s} \) obtained by solving (14) matches that predicted by the continuity equation as \( \Delta t \to \infty \).

III. NUMERICAL RESULTS

In this section, the MOT-TD-MFIE solver, which uses classical and mixed discretization schemes, is applied to the characterization of transient scattering from a unit sphere that resides in free space and is centered at the origin. Retarded-time source integrals and the test integrals in the MOT matrix entries in (7) are computed using the semi-analytical integration scheme described in [11], [12] and the Gauss-Legendre quadrature rule, respectively. Two levels of numerical integration are used: (i) “lev=1" uses seven quadrature points. (ii) "lev=2" first divides the triangles into four and uses seven quadrature points in each sub-triangle. The

\[
\nabla_S \times \nabla_S \varphi_n(r) = 0, \quad G_0(\hat{n} \times g_m, l_1) = 0 \quad \text{and} \quad Z_0(\hat{n} \times g_m, l_1) = -K_0(\hat{n} \times g_m, l_1) / (4\pi).\n\]

Inserting (10) into the expression of \( K_0(\hat{n} \times g_m, l_1) \) yields:

\[
K_0(\hat{n} \times g_m, l_1) = \int_{S_m} g_m(r) \int_{S_n} \nabla R^{-1} \times l_1(r') dr' dr
\]

\[
+ \int_{S_m} g_m(r) \sum_{p=2}^{\infty} (-1)^p T_p(0) R^{p-2} \hat{R} \times l_1(r') dr' dr.
\]

The first term of the RHS in (18) represents the static magnetic field due to a loop source tested by a loop function and is zero [15]. Terms of the summation in the RHS scale as \( O(\Delta t^{-p}) \), \( p \geq 2 \), due to derivatives \( T_p(0) \). Therefore, the dominant term in \( K_0(\hat{n} \times g_m, l_1) \) and hence \( Z_0(\hat{n} \times g_m, l_1) \) scales as \( O(\Delta t^{-2}) \). As a result,

\[
Z_0^{1s} \sim \frac{O(\Delta t^{-2})}{O(1)} \cdot O(1).
\]

The scaling of \( V_j^{1s} \) can be determined using \( V_j(\hat{n} \times g_m) \) and \( V_j(\hat{n} \times g_m) \). It can be easily seen that \( V_j(\hat{n} \times g_m) \sim O(1) \). Inserting (16) into the expression of \( V_j(\hat{n} \times g_m) \) and applying the chain rule and several vector manipulations to the resulting equation yield:

\[
V_j(\hat{n} \times g_m) = -\int_{S_m} \nabla \cdot [\partial_n(\hat{n} \times H(r,t))] dr
\]

\[
- \int_{S_m} \partial_m(\hat{n} \times H(r,t)) dr.
\]
excitation is a plane wave: \( \mathbf{H}^\text{s}(\mathbf{r}, t) = -\eta^{-1} \mathbf{E}^\text{r}(t) \times \mathbf{r} / c \) where \( G(t) = \cos(2\pi f_0 t - \varphi_0) \) is a Gaussian pulse with modulation frequency \( f_0 = 0.66 f_{\text{max}} \), duration \( \sigma = 3.34 / f_{\text{max}} \), and delay \( t_0 = 21.7 / f_{\text{max}} \). To investigate the scaling of \( I^s_\text{r} \) as \( \Delta t \to \infty \), \( f_{\text{max}} \) is swept in the interval \([24 \text{ kHz-192 MHz}]\), and \( \Delta t = 1 / (10 f_{\text{max}}) \). \( \mathbf{J}(\mathbf{r}, t) \) induced on the sphere is discretized using \( N = 750 \) RWG basis functions.

Fig. 1 plots the \( L_2 \) norm of the star current coefficients at \( t = \Delta t \), i.e., \( \| I^s_\text{r} \| = \sqrt{\sum_{n=1}^{N_x} |I^s_{n, n}|^2} \), vs. \( \Delta t \) as \( f_{\text{max}} \) and \( \Delta t \) are swept. Note that \( I^s_\text{r} \) are computed via loop/star decomposition after the MOT systems in (9) or (14). Fig. 1 clearly demonstrates that \( \| I^s_\text{r} \| \) obtained by solving the classically-discretized TD-MFIE saturates and scales as \( O(1) \) as \( \Delta t \) gets larger. On the other hand, \( \| I^s_\text{r} \| \) obtained by solving the mixed-discretized TD-MFIE scales as \( O(\Delta t^{-1}) \). These results verify the analysis carried out in Sections II-B and II-C. Fig. 1 also shows that the classical discretization produces the wrong scaling regardless of the integration accuracy (due to non-conforming testing). On the other hand, higher integration accuracy helps the mixed discretization scheme achieve the correct scaling especially as \( \Delta t \) gets larger.

Figs. 2(a)-(c) plot the \( x \)-component of the (range-corrected) scattered electric field obtained from the three sets of MOT solutions with \( f_0 = 0 \) Hz and \( f_{\text{max}} \in \{75, 50, 25\} \) Hz (\( \Delta t \in \{1.33, 2, 4\} \) ms). The figures clearly show that, as \( \Delta t \) gets larger, the accuracy of the classically-discretized TD-MFIE’s solution deteriorates while the solution of the mixed-discretized TD-MFIE maintains its accuracy.

**IV. Conclusion**

The TD-MFIE discretized using RWG basis and testing functions produces inaccurate results when the \( \Delta t \) is large because this discretization scheme can not predict the correct scaling of the current’s Helmholtz components as \( \Delta t \to \infty \). This can be avoided by using the mixed discretization scheme with RWG basis and BC testing functions that are conforming with respect to the function spaces of the MFIE operator.

**References**


