Coalition and Group Announcement Logic

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Dynamic epistemic logics which model abilities of agents to make various announcements and influence each other’s knowledge have been studied extensively in recent years. Two notable examples of such logics are Group Announcement Logic and Coalition Announcement Logic. They allow us to reason about what groups of agents can achieve through joint announcements in non-competitive and competitive environments. In this paper, we consider a combination of these logics – Coalition and Group Announcement Logic and provide its complete axiomatisation. Moreover, we partially answer the question of how group and coalition announcement operators interact, and settle some other open problems.

1 Introduction

To introduce the logics we will be working with in this paper, we start with an example loosely based on the one from [16]. Let us imagine that Ann, Bob, and Cath are travelling by train from Nottingham to Liverpool through Manchester. Cath was sound asleep all the way, and she has just woken up. She does not know whether the train passed Manchester, but Ann and Bob know that it has not. Now, if the train driver announces that the train is approaching Manchester, then Cath, as well as Ann and Bob, knows that they have not passed the city yet. To reason about changes in agents’ knowledge after public announcements, we can use Public Announcement Logic (PAL) [15]. Returning to the example, let us assume that the train driver does not announce anything, so that Cath is not aware of her whereabouts. Ann and Bob may tell her whether they passed Manchester. In other words, Ann and Bob have an announcement that can influence Cath’s knowledge. An extension of PAL, Group Announcement Logic (GAL) [2], deals with the existence of announcements by groups of agents that can achieve certain results. Now, let us assume that Ann does not want to disclose to Cath their whereabouts and Bob does, i.e. Ann and Bob have different goals. Then, it is clear that no matter what Ann says, the coalition of Bob and Cath can achieve the goal of Cath knowing that the train has not passed Manchester, that is, Bob can communicate this information to Cath. On the other hand, if Ann and Bob work together, then they have an announcement (for example, a tautology ‘It either rains in Liverpool or it doesn’t’), such that whatever Cath says, she remains unaware of her whereabouts. For this type of strategic behaviour, another extension of PAL – Coalition Announcement Logic (CAL) – has been introduced in [3].

CAL joins two logical traditions: Dynamic Epistemic Logic, of which PAL is a representative, and Coalition Logic (CL) [14]. The latter allows us to reason about whether a coalition of agents has a strategy to achieve some goal, no matter what the agents outside of the coalition do. CL essentially talks about concurrent games, and the actions that the agents execute are arbitrary actions (strategies in one-shot games). So, from this perspective, CAL is a coalition logic with available actions restricted to public announcements.

To the best of our knowledge, there is no complete axiomatisation of CAL [3, 11, 4, 5] or any other logic with coalition announcement operators. In this paper, we consider Coalition and Group
Announcement Logic (CoGAL), a combination of GAL and CAL, which includes operators for both group and coalition announcements. The main result of this paper is a sound and complete axiomatisation of CoGAL. As part of this result, we study the interplay between group and coalition announcement operators, and partially settle the question on their interaction that was stated as an open problem in [11][5].

2 Coalition and Group Announcement Logic

2.1 Syntax and Semantics

Throughout the paper, let a finite set of agents A, and a countable set of propositional variables P be given. The language of the logic is comprised of the language of classical propositional logic with added operators for agents’ knowledge $K_a \phi$ (reads ‘agent a knows $\phi$’), and public announcement $[\psi] \phi$ (reads ‘after public announcement that $\psi$, $\phi$ holds’), group $[G] \phi$ (‘after any public announcement by group of agents $G$, $\phi$ holds’), and coalition announcements $\langle G \rangle \phi$ (‘for every public announcement by coalition of agents $G$ there is an announcement by other agents $A \setminus G$, such that $\phi$ holds after joint simultaneous announcement’).

Definition 2.1. (Language) The language of coalition and group announcement logic $\mathcal{L}_{CoGAL}$ is as follows:

$$\phi, \psi ::= p \mid \neg \phi \mid (\phi \land \psi) \mid K_a \phi \mid [\psi] \phi \mid [G] \phi \mid \langle G \rangle \phi,$$

where $p \in P$, $a \in A$, $G \subseteq A$, and all the usual abbreviations of propositional logic (such as $\lor, \rightarrow, \leftrightarrow$) and conventions for deleting parentheses hold. The dual operators are defined as follows: $\hat{K}_a \phi \leftrightarrow \neg K_a \neg \phi$, $(\phi) \psi \leftrightarrow [\psi] \neg \phi$, $(G) \phi \leftrightarrow \neg [G] \neg \phi$, and $\langle G \rangle \phi \leftrightarrow \neg \langle [G] \rangle \neg \phi$. Observe that $(G) \phi$ means that $G$ has an announcement after which $\phi$ holds, and $\langle G \rangle \phi$ means that $G$ has an announcement such that after it is made simultaneously with any announcement by $A \setminus G$, $\phi$ holds. The latter corresponds to the Coalition Logic operator, but for announcements instead of arbitrary actions.

We define $\mathcal{L}_{GAL}$ as the language without the operator $\langle [G] \rangle$, $\mathcal{L}_{GAL}$ the language without $\langle G \rangle$ as well, and $\mathcal{L}_{EL}$ the purely epistemic language which in addition does not contain announcement operators $[\phi]$.

Next definition is needed for technical reasons in the formulation of infinite rules of inference in Definition 2.5. We want the rules to work for a class of different types of premises. Ultimately, we require premises to be expressions of depth $n$ of the type $\phi_1 \rightarrow \square_1 (\phi_2 \rightarrow \ldots (\phi_n \rightarrow \square_n \#)\ldots)$, where $\square_i$ is either $K_a$ or $[\psi]$ for some $a \in A$ and $\psi \in \mathcal{L}_{CoGAL}$, atom $\#$ denotes a placement of a formula to which a derivation is applied, and some $\phi$’s and $\square$’s can be omitted. This condition is captured succinctly by necessity forms originally introduced by Goldblatt in [13].

Definition 2.2. (Necessity forms) Let $\phi \in \mathcal{L}_{CoGAL}$, then necessity forms [13] are inductively defined as follows:

$$\eta ::= \sharp \mid \phi \rightarrow \eta(\sharp) \mid K_a \eta(\sharp) \mid [\phi] \eta(\sharp).$$

The atom $\sharp$ has a unique occurrence in each necessity form. The result of the replacement of $\sharp$ with $\phi$ in some $\eta(\sharp)$ is denoted as $\eta(\phi)$.

Whereas formulas of coalition logic [14] are interpreted in game structures, formulas of CoGAL are interpreted in epistemic models. Let us consider an example of such a model first. In Figure 1 there are three agents: $a$ (Ann), $b$ (Bob), and $c$ (Cath). Let $p$ denote the proposition that ‘The train has passed Manchester.’ There are two states in the model $M$: a state $w$ where $\neg p$ is true, and a state $v$ where $p$ is
true; and only one state in model $M^{-p}$ which denotes $M$ updated by the announcement $\neg p$ (the process of updating the model is described below). Let the $w$ be the actual state. Edges connect states that an agent cannot distinguish. In the actual state $w$ of $M$, Cath (agent $c$) does not know whether $p$ is true. Ann and Bob, on the contrary, know that $p$ is false. Now suppose that Bob announces that $\neg p$. This truthful public announcement ‘deletes’ all the states where $p$ is true, and the corresponding epistemic indistinguishability relations; in this example, $v$ is ‘deleted,’ and the resulting model is $M^{\neg p}$. After this announcement Cath knows $\neg p$, or, formally, $\neg p | M \neg p$. In this paper, within group and coalition announcements, we only quantify over announcements of formulas of the type $K_a \phi$. If a group consists only of Cath, who does not know $\neg p$ and hence cannot announce $K_c \neg p$, the following holds in state $w$ of $M$: $[c](\neg K_c \neg p \land \neg K_c p)$, i.e. whatever $c$ announces, she still does not know whether $p$ after the announcement.[1] Also, Ann and Bob can remain silent (or announce a tautology $\top$) and preclude Cath from knowing that $\neg p$. In other words, there is announcement by their group such that after it is made, agent $c$ does not know the value of $p$: $\langle\{a, b\}\rangle(\neg K_c \neg p \land \neg K_c p)$. Moreover, this holds whatever Cath announces at the same time: $(\{a, b\}\rangle(\neg K_c \neg p \land \neg K_c p)$. On the other hand, a coalition consisting of Ann and Cath does not have such a power, since Bob can always announce that $\neg p$: $\neg \langle\{a, c\}\rangle(\neg K_c \neg p \land \neg K_c p)$, or, equally, $\langle\{a, c\}\rangle(K_c \neg p \lor K_c p)$.

Now, we provide formal definitions.

Definition 2.3. (Epistemic model) An epistemic model is a triple $M = (W, \sim, V)$, where

- $W$ is a non-empty set of states;
- $\sim: A \to \mathcal{P}(W \times W)$ assigns an equivalence relation to each agent; we will denote relation assigned to agent $a \in A$ by $\sim_a$;
- $V: P \to \mathcal{P}(W)$ assigns a set of states to each propositional variable.

A pair $(W, \sim)$ is called an epistemic frame, and a pair $(M, w)$ with $w \in W$ is called a pointed model. An announcement in a pointed model $(M, w)$ results in an updated pointed model $(M^\theta, w)$. Here $M^\theta = (W^\theta, \sim^\theta, V^\theta)$, and $W^\theta = [\phi]_M, \sim^\theta = \sim_a \cap ([\phi]_M \times [\phi]_M)$, and $V^\theta(p) = V(p) \cap [\phi]_M$. Generally speaking, an updated pointed model $(M^\theta, w)$ is a restriction of the original one to the states where $\phi$ holds.

Let $\mathcal{L}_{EL}^G$ denote the set of formulas of the type $\land_{i \in G} K_i \phi_i$, where for every $i \in G$ it holds that $\phi_i \in \mathcal{L}_{EL}$. These are the formulas we will be quantifying over in modalities of the form $[G]$ and $[[G]]$.

Definition 2.4. (Semantics) Let a pointed model $(M, w)$ with $M = (W, \sim, V)$, $a \in A$, and $\phi, \psi \in \mathcal{L}_{CoGal}$ be given.

$$(M, w) \models \phi \quad \text{iff} \quad w \in V(\phi)$$
$$(M, w) \models \neg \phi \quad \text{iff} \quad (M, w) \not\models \phi$$
$$(M, w) \models \phi \land \psi \quad \text{iff} \quad (M, w) \models \phi \text{ and } (M, w) \models \psi$$
$$(M, w) \models K_a \phi \quad \text{iff} \quad \forall v \in W \; w \sim_a v \text{ implies } (M, v) \models \phi$$
$$(M, w) \models [\phi]_\psi \quad \text{iff} \quad (M, w) \models \phi \text{ implies } (M^\phi, w) \models \psi$$
$$(M, w) \models [G] \phi \quad \text{iff} \quad \forall \psi \in \mathcal{L}_{EL}^G \; (M, w) \models [\psi] \phi$$
$$(M, w) \models [[G]] \phi \quad \text{iff} \quad \forall \psi \in \mathcal{L}_{EL}^G \; \exists \chi \in \mathcal{L}_{EL}^G \; (M, w) \models \psi \to (\psi \land \chi) \phi$$

[1] For readability, we use $[c]$ rather than $[[c]]$ for singleton coalitions.
Formula $\varphi$ is called valid if for any pointed model $(M, w)$ it holds that $(M, w) \models \varphi$.

The semantics for the ‘diamond’ versions of knowledge, public and group announcement operators $(\mathcal{K}_a \varphi, (\varphi)\psi$, and $\langle G\rangle \varphi$ respectively) are obtained by changing $\forall$ to $\exists$ and ‘implies’ to ‘and’ in the corresponding lines. The semantics for a dual of the coalition announcement operator is as follows:

$$(M, w) \models \langle G \rangle \varphi \iff \exists \psi \in \mathcal{L}_{EL}^G \forall \chi \in \mathcal{L}_{EL}^{A \setminus G} : (M, w) \models \psi \land [\psi \land \chi] \varphi;$$

which corresponds to ‘there is an announcement by agents from $G$, such that whatever other agents $A \setminus G$ announce at the same time, $\varphi$ holds.’

Note that following [8, 7, 2, 3, 5, 9, 11, 4] we restrict formulas which agents in a group or coalition can announce to formulas of $\mathcal{L}_{EL}$. This allows us to avoid circularity in the definition.

### 2.2 Axiomatisation and Some Logical Properties

In this section we present an axiomatisation of CoGAL and show its soundness.

**Definition 2.5.** Axiomatisation of CoGAL is a union of axiomatisation of GAL [2], interaction axiom for group and coalition announcements A11, rule of inference for coalition announcements R6, and necessitation R4.

<table>
<thead>
<tr>
<th>Axiom</th>
<th>Description</th>
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<tbody>
<tr>
<td>(A0)</td>
<td>instantiations of propositional tautologies, $\varphi \vdash [\varphi] \varphi$, where $\varphi \in \mathcal{L}_{EL}$,</td>
</tr>
<tr>
<td>(A1)</td>
<td>$\mathcal{K}_a \varphi \rightarrow (\mathcal{K}_a \varphi \rightarrow \mathcal{K}_a \psi)$,</td>
</tr>
<tr>
<td>(A2)</td>
<td>$\mathcal{K}_a \varphi \rightarrow \varphi$,</td>
</tr>
<tr>
<td>(A3)</td>
<td>$\mathcal{K}_a \varphi \rightarrow \mathcal{K}_a \mathcal{K}_a \varphi$,</td>
</tr>
<tr>
<td>(A4)</td>
<td>$\neg \mathcal{K}_a \varphi \rightarrow \mathcal{K}_a \neg \mathcal{K}_a \varphi$,</td>
</tr>
<tr>
<td>(A5)</td>
<td>$\varphi \models [\varphi \rightarrow p]$,</td>
</tr>
<tr>
<td>(A6)</td>
<td>$\varphi \models [\varphi \rightarrow [\varphi] \psi]$,</td>
</tr>
<tr>
<td>(A7)</td>
<td>$\varphi ([\varphi] \psi) \leftrightarrow ([\varphi] \psi \land [\varphi] \chi)$,</td>
</tr>
<tr>
<td>(A8)</td>
<td>$\varphi \models [\varphi \models [\psi \models [\varphi] \psi] \rightarrow [\varphi \models [\psi \models [\varphi] \psi] \land [\varphi] \psi] \chi)$,</td>
</tr>
<tr>
<td>(A9)</td>
<td>$\varphi \models [\varphi \models [\psi \models [\varphi] \psi] \land [\varphi] \psi] \chi)$,</td>
</tr>
<tr>
<td>(A10)</td>
<td>$[\mathcal{G}] \varphi \rightarrow [\varphi] \varphi$, where $\varphi \in \mathcal{L}_{EL}$,</td>
</tr>
<tr>
<td>(A11)</td>
<td>$[\mathcal{G}] \varphi \rightarrow [\mathcal{G}] [A \setminus G] \varphi$,</td>
</tr>
<tr>
<td>(R0)</td>
<td>$\vdash \varphi \models [\varphi \models [\psi \models [\varphi] \psi] \land [\varphi] \psi] \chi)$,</td>
</tr>
<tr>
<td>(R1)</td>
<td>$\vdash \varphi \models [\mathcal{G}] \varphi$,</td>
</tr>
<tr>
<td>(R2)</td>
<td>$\vdash \varphi \models [\mathcal{G}] \varphi$,</td>
</tr>
<tr>
<td>(R3)</td>
<td>$\vdash \varphi \models [\mathcal{G}] \varphi$,</td>
</tr>
<tr>
<td>(R4)</td>
<td>$\vdash \varphi \models [\mathcal{G}] \varphi$,</td>
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<tr>
<td>(R5)</td>
<td>$\vdash \varphi \models [\mathcal{G}] \varphi$,</td>
</tr>
<tr>
<td>(R6)</td>
<td>$\vdash \varphi \models [\mathcal{G}] \varphi$,</td>
</tr>
</tbody>
</table>

So, CoGAL is the smallest subset of $\mathcal{L}_{CoGAL}$ that contains all the axioms A0 – A11 and closed under rules of inference R0 – R6. Elements of CoGAL are called theorems. Note that R5 and R6 are infinitary rules: they require an infinite number of premises. Finding finite axiomatisations of any of APAL, GAL, or CAL is an open problem. Note also that CoGAL includes coalition logic [14], that is all the axioms of the latter are validities of CoGAL and a rule of inference preserves validity (see Appendix A).

**Definition 2.6.** (Soundness and completeness) An axiomatisation is sound, if for any formula $\varphi$ of the language, it holds that $\varphi \in \text{CoGAL}$ implies $\varphi$ is valid. And vice versa for completeness.

Soundness of A0–A4, R0, and R1 is due to soundness of SS. Axioms A5–A9 and rule of inference R3 are sound, since PAL is sound [12]. Soundness of axiom A10 and rules of inference R3 and R5 was shown in [2]. We show soundness of R4, R6 in Proposition 2.7 and validity of A11 in Proposition 2.10.

**Proposition 2.7.** R4 and R6 are sound, that is, they preserve validity.

**Proof.** A proof is given in Appendix B (Proposition B.1).
formulae \( \chi \) which they did not know before the announcement. We need to make sure that these new formulae cannot allow them to make \( \phi \) false. However, since \( \psi \) is true in the initial model, and \( \chi \) in the updated one, agents in \( A \backslash G \) can always make an announcement that they know that after the announcement of \( \psi, \chi \) holds. This announcement, made simultaneously with the announcement by \( G \), ‘models’ the effect of announcing \( \chi \) later. Returning to our example (Figure 1), whichever formulae

\[ \psi \text{ can announce to enforce } \phi \text{ after the announcement by } G, \]

then they could have prevented it before.

Due to restriction of announcements to formulae of epistemic logic, we cannot directly employ public announcement operators in agents’ ‘utterances.’ In order to avoid this, we use the standard translation of \( \text{PAL} \) into epistemic logic.

**Definition 2.8.** Translation function \( t : \mathcal{L}_\text{PAL} \rightarrow \mathcal{L}_\text{EL} \) is defined as follows:

\[
\begin{align*}
 t(p) & = p, \\
 t(\neg \phi) & = \neg t(\phi), \\
 t(\phi \land \psi) & = t(\phi) \land t(\psi), \\
 t(K_a \phi) & = K_a t(\phi), \\
 t([\phi]p) & = t(\phi \rightarrow p),
\end{align*}
\]

Every \( \phi \in \mathcal{L}_\text{PAL} \) is equivalent to \( t(\phi) \in \mathcal{L}_\text{EL} \).

Now we show that for every announcement of agents’ knowledge in some updated pointed model \( (M', w) \) there is an equivalent announcement in the original one (i.e. in \( (M, w)) \).

**Lemma 2.9.** Let \( a, \ldots, b \in A \). The following formula is valid for all \( \psi, \chi_a, \ldots, \chi_b \in \mathcal{L}_\text{EL} \):

\[
[\psi \land K_a t([\psi]\chi_a) \land \ldots \land K_b t([\psi]\chi_b)] \phi \leftrightarrow [\psi] [K_a \chi_a \land \ldots \land K_b \chi_b] \phi
\]

**Proof.** Suppose that for some pointed model \( (M, w) \) it holds that \( (M, w) \models [\psi \land K_a t([\psi]\chi_a) \land \ldots \land K_b t([\psi]\chi_b)] \phi \). By propositional reasoning, it is equivalent to \( (M, w) \models [\psi \land (\psi \rightarrow K_a t([\psi]\chi_a)) \land \ldots \land (\psi \rightarrow K_b t([\psi]\chi_b))] \phi \), and, by equivalence of a formula and its translation, the latter is equivalent to \( (M, w) \models [\psi \land [\psi] K_a \chi_a \land \ldots \land [\psi] K_b \chi_b] \phi \). By A8, we have that \( (M, w) \models [\psi \land [\psi] K_a \chi_a \land \ldots \land [\psi] K_b \chi_b] \phi \). Finally, by A9, the latter is equivalent to \( (M, w) \models [\psi] [K_a \chi_a \land \ldots \land K_b \chi_b] \phi \).

We use Lemma 2.9 to show validity of axiom A11.

**Proposition 2.10.** \( \langle G \rangle \phi \rightarrow \langle G \rangle [A \backslash G] \phi \) is valid.

**Proof.** Suppose to the contrary that for some pointed model \( (M, w) \) it holds that \( (M, w) \models \langle G \rangle \phi \) and \( (M, w) \not\models \langle G \rangle [A \backslash G] \phi \). From \( (M, w) \models \langle G \rangle \phi \), by the semantics,

\[
\exists \psi \in \mathcal{L}_\text{EL} \forall \chi_a, \ldots, \chi_b \in \mathcal{L}_\text{EL} : (M, w) \models \psi \land [\psi \land K_a \chi_a \land \ldots \land K_b \chi_b] \phi.
\]

Let us call \( \psi_G \) the formula that \( G \) can announce to enforce \( \phi \). From \( (M, w) \not\models \langle G \rangle [A \backslash G] \phi \),

\[
\forall \psi' \in \mathcal{L}_\text{EL} \exists \chi_{a'}, \ldots, \chi_{b'} \in \mathcal{L}_\text{EL} : (M, w) \not\models \langle \psi' \rangle [K_a \chi_a \land \ldots \land K_b \chi_b] \phi.
\]
In particular, for \( \psi' = \psi_G \),
\[
\exists \chi'_a, \ldots, \chi'_b \in \mathcal{L}_{EL} : (M, w) \not\models \langle \psi_G \rangle \left[ K_a \chi'_a \land \ldots \land K_b \chi'_b \right] \phi.
\]
Since \( \psi_G \) is true in \((M, w)\), this is equivalent to
\[
\exists \chi'_a, \ldots, \chi'_b \in \mathcal{L}_{EL} : (M, w) \not\models [\psi_G \left[ K_a \chi'_a \land \ldots \land K_b \chi'_b \right] \phi].
\]
By Lemma 2.9, the latter is equivalent to
\[
\exists \chi'_a, \ldots, \chi'_b \in \mathcal{L}_{EL} : (M, w) \not\models [\psi \land K_a \psi] \left[ \psi_G \left[ \chi'_a \right] \right] \land \ldots \land K_b \psi \left[ \psi_G \left[ \chi'_b \right] \right] \phi.
\]
Since \( t(\psi_G \left[ \chi'_a \right], \ldots, t(\psi_G \left[ \chi'_b \right]) \) are in \( \mathcal{L}_{EL} \), we have the contradiction with
\[
\forall a, \ldots, b \in \mathcal{L}_{EL} : (M, w) \models [\psi \land K_a \chi_a \land \ldots \land K_b \chi_b] \phi.
\]

Proposition 2.10 allows us to prove Lindenbaum Lemma (Proposition 2.19) for CoGAL. But before that, let us show some properties of the logic. The following validity shows that if some formula \( \phi \) can be achieved by two coalition announcements, it can be achieved by a single joint coalition announcement as well. The validity was known only for the case of group announcements in GAL [2]. We show that this also holds for coalition announcements.

**Proposition 2.11.** \( \langle G \rangle \langle H \rangle \phi \rightarrow \langle G \cup H \rangle \phi \) is valid.

**Proof.** The proof is presented in Appendix B (Proposition 3.2).

**Corollary 2.12.** \( \langle G \rangle \langle G \rangle \phi \rightarrow \langle G \rangle \phi \) is valid.

The other direction of Proposition 2.11 does not hold. Whether \( \langle G \cup H \rangle \phi \rightarrow \langle G \rangle \langle H \rangle \phi \) is valid was posed as an open question in [5]. We settle this question by presenting a counterexample.

**Proposition 2.13.** \( \langle G \cup H \rangle \phi \rightarrow \langle G \rangle \langle H \rangle \phi \) is not valid.

**Proof.** Let \( G = \{a\}, H = \{b\} \), and \( \phi := K_b (p \land q \land r) \land \neg K_a (p \land q \land r) \land \neg K_c (p \land q \land r) \). Informally, \( \phi \) says that agent \( b \) knows that the given propositional variables are true, and agents \( a \) and \( c \) do not. Consider the model \( M \) in Figure 2 (reflexive arrows are omitted for convenience). By the semantics, \( (M, \bullet) \models \langle \{a, b\} \rangle \phi \) iff \( \exists (K_a \psi \land K_b \chi, \psi, \chi) \in \mathcal{L}_{EL}, \forall K, \tau \in \mathcal{L}_{EL} : (M, \bullet) \models K_a \psi \land K_b \chi \land (K_a \psi \land K_b \chi \land K_c \tau) \phi \).

Let \( \psi \) be \( q \), and \( \chi \) be \( T \). Observe that \( (M, \bullet) \models K_a q \land K_b T \). Moreover, \( c \) does not know any formula that she can announce to avoid \( \phi \). An informal argument is as follows. Whatever \( c \) announces in this situation, she cannot avoid \( b \) learning \( p \land q \land r \). In order to make \( a \) learn that \( p \land q \land r \), \( c \) has to announce something of the form \( \psi \rightarrow p \), since she does not know the value of \( p \) herself. Formula \( \psi \) can be neither \( r \) nor \( q \), because \( c \) does not know their truth values. Also, it cannot be a statement of \( b \)’s knowledge, since in every \( q \)-world accessible by \( c \), \( b \)’s knowledge is only a reflexive arrow. It cannot be \( a \)’s or \( c \)’s knowledge either, since in this case \( a \) would have known \( p \) herself, and \( c \)’s relation between \( q \)-states is universal.

In the consequent, we have \( (M, \bullet) \models \langle \{a\} \langle b \rangle \phi \). By the semantics, \( \exists \psi \in \mathcal{L}_{EL}, \forall \chi \in \mathcal{L}_{EL} : (M, \bullet) \models \psi \land [\psi \land \chi] \langle \{b\} \phi \). Let us fix such a \( \psi \), and let \( \chi := K_b \psi \land K_c T \). Then \( (M, \bullet) \models \psi \land [\psi \land K_b \psi \land K_c T] \langle \{b\} \phi \).

Observe that no matter what \( a \) announces, \( K_b \psi \) ‘forces’ \( a \) to learn that \( p \land q \land r \), and whatever is announced in the updated model \( (M \psi \land K_b \psi \land K_c T, \bullet) \), \( a \)’s knowledge of \( p \land q \land r \) and, hence, falsity of \( \phi \) remains. So, \( (M, \bullet) \not\models \langle \{a\} \langle b \rangle \phi \).
2.3 Completeness

In order to prove completeness of CoGAL, we expand and modify the completeness proof for APAL [7, 9, 6]. Although the proof is partially based upon the classic canonical model approach, we have to ensure that construction of maximal consistent theories (Proposition 2.19) allows us to include infinite amount of formulas for cases of coalition announcements. This corresponds to ‘whatever other agents may say.’ But before we start, let us state two auxiliary lemmas.

**Lemma 2.14.** Let \( \varphi, \psi \in \mathcal{L}_{CoGAL} \). If \( \varphi \rightarrow \psi \) is a theorem, then \( \eta(\varphi) \rightarrow \eta(\psi) \) is a theorem as well.

**Proof.** A simple induction on the complexity of \( \eta \). \( \square \)

**Lemma 2.15.** \( \langle \chi \rangle \langle \psi \rangle \rightarrow \langle \chi \rangle \langle G \rangle \varphi \), where \( \chi \in \mathcal{L}_{CoGAL} \), and \( \psi \in \mathcal{L}_{EL}^G \), is a theorem.

**Proof.** Let \( \psi \in \mathcal{L}_{EL}^G \). By A10, \( [G] \neg \varphi \rightarrow [\psi] \neg \varphi \) is a theorem. By contraposition, we have that \( \langle \psi \rangle \varphi \rightarrow \langle G \rangle \varphi \) is also a theorem. By R2 and distribution over implication, we infer \( \langle \chi \rangle \langle \psi \rangle \varphi \rightarrow \langle \chi \rangle \langle G \rangle \varphi \). \( \square \)

Now, the first part of the proof up to Proposition 2.19 is based on [7].

**Definition 2.16.** A set of formulae \( x \) is called a theory if and only if it contains CoGAL, and is closed under R0, R5, and R6. A theory \( x \) is consistent if and only if \( \bot \not\in x \), and is maximal if and only if for all \( \varphi \in \mathcal{L}_{CoGAL} \) it holds that either \( \varphi \in x \) or \( \neg \varphi \in x \).

**Proposition 2.17.** Let \( x \) be a theory, \( \varphi, \psi \in \mathcal{L}_{CoGAL} \), and \( a \in A \). The following are theories: \( x + \varphi = \{ \psi : \varphi \rightarrow \psi \in x \}, K_a x = \{ \varphi : K_a \varphi \in x \}, \) and \( \{ \varphi \} x = \{ \psi : [\varphi] \psi \in x \} \).

**Proof.** The proof is an extension of the one from [7] (see Appendix B, Proposition B.3). \( \square \)

**Proposition 2.18.** Let \( \varphi \in \mathcal{L}_{CoGAL} \). Then, CoGAL + \( \varphi \) is consistent iff \( \neg \varphi \not\in \text{CoGAL} \).

**Proof.** The proof is given in Appendix B (Proposition B.4). \( \square \)

The following proposition is a variation of Lindenbaum Lemma. Validity of axiom A11 allows us to expand the corresponding proof for APAL, and to deal with having two different quantifiers at the same time.

**Proposition 2.19.** Every consistent theory \( x \) can be extended to a maximal consistent theory \( y \).

**Proof.** Let \( \psi_0, \psi_1, \ldots \) be an enumeration of formulae of the language, and let \( y_0 = x \). Suppose that for some \( n \geq 0 \), \( y_n \) is a consistent theory, and \( x \subseteq y_n \). If \( y_n + \psi_n \) is consistent, then \( y_{n+1} = y_n + \psi_n \). Otherwise, if \( \psi_n \) is not a conclusion of either R5 or R6, \( y_{n+1} = y \). If \( \psi_n \) is a conclusion of R5, we enumerate all the subformulae of \( \psi_n \) which contain group announcement modalities [\( G \)]. Let \( \eta_1([G] \varphi_1), \ldots, \eta_k([G] \varphi_k) \) be all these subformulae. Then \( y_n^0, \ldots, y_n^k \) is a sequence of consistent theories, where \( y_n^0 = y_n \), and for some
\textit{i < k, }y_n^i\text{ is a consistent theory containing } y_n \text{ and } \neg \eta_i((G)\varphi_i). \text{ Since } y_n^i \text{ is closed under } R5, \text{ there exists } 
abla \in L_{EL}^{G} \text{ such that } \eta_i((\nabla)\varphi_i) \notin y_n^i. \text{ Hence, } y_n^{i+1} = y_n^i + \neg \eta_i((\nabla)\varphi_i), \text{ and } y_{n+1} = y_n^{i+1}.

Now, we consider the case when } y_n \text{ is a conclusion of } R6. \text{ We enumerate all the subformulæ of } y_n, \text{ which contain coalition announcement modalities } ((G). \text{ Let } \eta_1((G)\varphi_1), \ldots, \eta_k((G)\varphi_k) \text{ be all these subformulæ. Then } y_n^0, \ldots, y_n^k \text{ is a sequence of consistent theories, where } y_n^0 = y_n, \text{ and for some } i < k, y_n^i \text{ is a consistent theory containing } y_n \text{ and } \neg \eta_i((G)\varphi_i). \text{ By } A11, \text{ this means that } \neg \eta_i((G)(A \setminus G)\varphi_i) \in y_n^i. \text{ Since } y_n^i \text{ is closed under } R5, \text{ there exists } 
abla \in L_{EL}^{G} \text{ such that } \eta_i((\nabla)(A \setminus G)\varphi_i) \notin y_n^i. \text{ Hence, } y_n^{i+1} = y_n^i + \neg \eta_i((\nabla)(A \setminus G)\varphi_i), \text{ and } y_{n+1} = y_n^{i+1}. \text{ Note that since for all } \tau \in L_{EL}^{A \setminus G} \eta((\nabla)(\tau)\varphi) \rightarrow \eta((\nabla)(A \setminus G)\varphi) \text{ are theorems (by Lemmas 2.14 and 2.15), they and their contrapositions are already in } y_n^i \text{ (since } y_n^i \text{ is a theory). Thus, adding } \neg \eta_i((\nabla)(A \setminus G)\varphi_i) \text{ to } y_n^i \text{ adds all the } \neg \eta_i((\nabla)(\tau)\varphi) \text{ for } \tau \in L_{EL}^{A \setminus G} \text{ as well.}

Finally, } y \text{ is a maximal consistent theory, and } x \subseteq y. \qed

The rest of the proof is an expansion of the one from [9]. \text{ It employs induction on complexity of formulae to prove Truth Lemma (Proposition 2.25) and, ultimately, completeness (Proposition 2.26) of CoGAL.}

**Definition 2.20.** The size of some formula } \varphi \in L_{CoGAL} \text{ is defined as follows:}

1. \text{ Size}(p) = 1,
2. \text{ Size}(-\varphi) = \text{ Size}(K_\varphi) = \text{ Size}((G)\varphi) = \text{ Size}((\nabla)\varphi) = \text{ Size}(\varphi) + 1,
3. \text{ Size}(\varphi \land \psi) = \text{ Size}(\varphi) + \text{ Size}(\psi) + 1,
4. \text{ Size}((\nabla)\varphi) = \text{ Size}(\varphi) + 3 \cdot \text{ Size}(\varphi).

The \textit{[]}\text{-depth} is defined as follows:

1. \text{ d}_{[]}([p]) = 0,
2. \text{ d}_{[]}([-\varphi]) = \text{ d}_{[]}([K_\varphi]) = \text{ d}_{[]}([G]) = \text{ d}_{[]}([\nabla]) = \text{ d}_{[]}([\varphi]),
3. \text{ d}_{[]}([\varphi \land \psi]) = \max\{\text{ d}_{[]}([\varphi]), \text{ d}_{[]}([\psi])\},
4. \text{ d}_{[]}([\nabla]) = \text{ d}_{[]}([\varphi]) = \text{ d}_{[]}([\psi]) + 1.

The \textit{[]}\text{-depth} is the same as [\textit{]}, with the following exceptions.

1. \text{ d}_{[]}([G]) = \text{ d}_{[]}([\varphi]),
2. \text{ d}_{[]}([G]) = \text{ d}_{[]}([\varphi]) + 1.

**Definition 2.21.** The binary relation \textit{< Size} between } \varphi, \psi \in L_{CoGAL} \text{ is defined as follows: } \varphi <_{\text{ Size}} \psi \text{ iff } \text{ d}_{[]}([\varphi]) < \text{ d}_{[]}([\psi]), \text{ or, otherwise, } \text{ d}_{[]}([\varphi]) = \text{ d}_{[]}([\psi]), \text{ and either } \text{ d}_{[]}([\varphi]) < \text{ d}_{[]}([\psi]), \text{ or } \text{ d}_{[]}([\varphi]) = \text{ d}_{[]}([\psi]) \text{ and } \text{ Size}(\varphi) < \text{ Size}(\psi). \text{ The relation is a well-founded strict partial order between formulae.}

Now, we ensure that the order of complexity is preserved. Case \text{ [C \land G K_i \varphi_i] <_{\text{ Size}} [G] \varphi \text{ is obvious, since the public announcement on the left-hand side of the inequality is epistemic, and for any epistemic formula } \psi, \text{ d}_{[]}([\psi]) = 0. \text{ Case } \text{ [C] [C \land G K_i \varphi_i] <_{\text{ Size}} [C \land G] [\varphi_i] \text{ holds for the same reason. The cases for coalitions are identical: } \text{ [C \land G K_i \varphi_i] <_{\text{ Size}} [C \land G K_i \varphi_i] \land [C \land G K_i \varphi_i] \varphi \text{ <_{\text{ Size}} [C \land G] [\varphi_i].}

**Definition 2.22.** The canonical model is the model } M_C = (W_C, C, V_C), \text{ where}

- \text{ W_C is the set of all maximal consistent theories,}
• $\sim^C$ is defined as $x \sim^C y$ iff $K_x \subseteq y$.

• $x \in V^C$ iff $p \in x$.

Relation $\sim^C$ is equivalence due to axioms A2, A3, and A4.

**Definition 2.23.** Let $\varphi \in \mathcal{L}_{\text{CoGAL}}$. Condition $P(\varphi)$: for all maximal consistent theories $x$, $\varphi \in x$ iff $x \in \llbracket \varphi \rrbracket_{M_C}$. Condition $H(\varphi)$: for all $\psi \in \mathcal{L}_{\text{CoGAL}}$, if $\psi \prec_{\text{Size}_{[0,1]}} \varphi$, then $P(\psi)$.

**Proposition 2.24.** For all $\psi \in \mathcal{L}_{\text{CoGAL}}$, if $H(\varphi)$, then $P(\varphi)$.

**Proof.** Suppose $H(\varphi)$ holds, and let $x$ be a maximal consistent theory. The proof is by induction on $\prec_{\text{Size}_{[0,1]}}$-complexity of formulae. Most of the cases were proved in [9]. We prove here only remaining instances involving group and coalition announcements.

**Case $\varphi = \llbracket G \rrbracket \psi.** Suppose that $\llbracket G \rrbracket \psi \in x$. Since $x$ is closed under $R_5$, this is equivalent to $\forall \chi \in \mathcal{L}_{\text{EL}}^G$: $\llbracket \chi \rrbracket \psi \in x$. By the fact that $\llbracket \chi \rrbracket \psi \prec_{\text{Size}_{[0,1]}} \llbracket G \rrbracket \psi$, the latter holds if and only if $x \in \llbracket \llbracket \chi \rrbracket \psi \rrbracket_{M_C}$ for all $\chi \in \mathcal{L}_{\text{EL}}^G$, which is equivalent to $x \in \llbracket \llbracket G \rrbracket \psi \rrbracket_{M_C}$ by the semantics.

**Case $\varphi = \llbracket \chi \rrbracket G \psi.** Suppose that $\llbracket \chi \rrbracket G \psi \in x$. Since $\llbracket \chi \rrbracket G \psi$ is a necessity form and $x$ is closed under $R_5$, this is equivalent to $\forall \tau \in \mathcal{L}_{\text{EL}}^G$: $\llbracket \chi \rrbracket \llbracket \tau \rrbracket \psi \in x$. By the fact that $\llbracket \chi \rrbracket \llbracket \tau \rrbracket \psi \prec_{\text{Size}_{[0,1]}} \llbracket \chi \rrbracket G \psi$, the latter holds if and only if $x \in \llbracket \llbracket \chi \rrbracket \llbracket \tau \rrbracket \psi \rrbracket_{M_C}$ for all $\tau \in \mathcal{L}_{\text{EL}}^G$, which is equivalent to $x \in \llbracket \llbracket \chi \rrbracket G \psi \rrbracket_{M_C}$ by the semantics.

**Case $\varphi = \llbracket (\theta) \rrbracket (G) \psi.** Suppose that $\llbracket (\theta) \rrbracket (G) \psi \in x$. Since $x$ is closed under $R_6$, this is equivalent to $\forall \chi \in \mathcal{L}_{\text{EL}}^G \exists \tau \in \mathcal{L}_{\text{EL}}^{A(G)}: (\chi \rightarrow (\chi \wedge \tau)) \psi \in x$. By the fact that $\llbracket (\theta) \rrbracket (\llbracket (\chi \rightarrow (\chi \wedge \tau)) \psi \rrbracket \prec_{\text{Size}_{[0,1]}} \llbracket (G) \rrbracket \psi$, the latter holds if and only if $\forall \chi \in \mathcal{L}_{\text{EL}}^G \exists \tau \in \mathcal{L}_{\text{EL}}^{A(G)}: (\theta \llbracket (\chi \rightarrow (\chi \wedge \tau)) \psi \rrbracket) \in x$. By the fact that $\llbracket (\theta) \rrbracket (\chi \rightarrow (\chi \wedge \tau)) \psi \prec_{\text{Size}_{[0,1]}} (\theta) \llbracket (G) \rrbracket \psi$, the latter holds if and only if $\forall \chi \in \mathcal{L}_{\text{EL}}^G \exists \tau \in \mathcal{L}_{\text{EL}}^{A(G)}: x \in \llbracket (\theta) \llbracket (\chi \rightarrow (\chi \wedge \tau)) \psi \rrbracket \rrbracket_{M_C}$, which is equivalent to $x \in \llbracket (\theta) \llbracket (G) \rrbracket \psi \rrbracket_{M_C}$ by the semantics.

**Proposition 2.24** implies the following fact.

**Proposition 2.25.** Let $\varphi \in \mathcal{L}_{\text{CoGAL}}$, and $x$ be a maximal consistent theory. Then $\varphi \in x$ iff $x \in \llbracket \varphi \rrbracket_{M_C}$.

Finally, we prove the completeness of CoGAL.

**Proposition 2.26.** For all $\varphi \in \mathcal{L}_{\text{CoGAL}}$, if $\varphi$ is valid, then $\varphi \in \text{CoGAL}$.

**Proof.** Towards a contradiction, suppose that $\varphi$ is valid and $\varphi \notin \text{CoGAL}$. Since CoGAL is a consistent theory, and by Propositions 2.17 and 2.18 we have that $\text{CoGAL} + \neg \varphi$ is a consistent theory. Then, by Proposition 2.19 there exists a maximal consistent theory $x \supseteq \text{CoGAL} + \neg \varphi$, such that $\neg \varphi \in x$. By Proposition 2.25 this means that $x \notin \llbracket \varphi \rrbracket_{M_C}$, which contradicts $\varphi$ being a validity.

**3 Conclusion**

We presented CoGAL and provided a complete axiomatisation for it. The proof of completeness hinges on the validity of the axiom $(\llbracket G \rrbracket \varphi \rightarrow \llbracket G \rrbracket (A \setminus G) \varphi$. Validity of the other direction of the axiom, however, is still an open question. Answering it either way, positively, or negatively, will allow us to understand better mutual expressivity of CAL and GAL. The axiomatisation of CoGAL we presented is infinitary and employs necessity forms. Finding a finitary axiomatisation is yet another open problem. An interesting avenue of further research is adding common and distributed knowledge operators to CoGAL in the
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vein of [1]. Additionally, since it is known that \( \text{GAL} \), \( \text{CAL} \) [5], and hence \( \text{CoGAL} \), are undecidable, a search for decidable fragments of these logics is another research question. We would also like to investigate applicability of logics with group and coalition announcements to epistemic planning [10]. Finally, a complete axiomatisation of \( \text{CAL} \) without group announcement operators has not been provided yet, and it is an intriguing direction of further research.

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References


A Coalition and Group Announcement Logic Subsumes Coalition Logic

Definition A.1. Axiomatisation of CL is as follows:

(C0) all instantiation of propositional tautologies,
(C1) \( \neg \langle G \rangle \top \),
(C2) \( \langle G \rangle \top \),
(C3) \( \neg \langle \emptyset \rangle \neg \varphi \rightarrow \langle A \rangle \varphi \),
(C4) \( \langle G \rangle \varphi \land \psi \rightarrow \langle G \rangle \varphi \),
(C5) \( \langle G \rangle \varphi \land \langle H \rangle \psi \rightarrow \langle G \cup H \rangle (\varphi \land \psi) \),
(RO) \( \vdash \varphi, \varphi \rightarrow \psi \equiv \psi \),
(R1) \( \vdash \varphi \iff \langle G \rangle \varphi \iff \langle G \rangle \psi \).

Proposition A.2. CoGAL contains CL.

Proof. C0 and R0 are already in CoGAL.

R1: \( \vdash \varphi \iff \psi \Rightarrow \langle G \rangle \varphi \iff \langle G \rangle \psi \). Assume that \( \vDash \varphi \iff \psi \). This means that for any pointed model \((M, w)\) the following holds: \( (M, w) \models \varphi \iff (M, w) \models \psi \) (1). Now, suppose that for some pointed model \((M, v)\) it holds that \( (M, v) \models \langle G \rangle \varphi \). By the semantics, \( \exists \chi \in \mathcal{L}_{EL}^G \forall \tau \in \mathcal{L}_{EL}^A : (M, v) \models \chi \land [\chi \land \tau] \varphi \), which is equivalent to the following: \( (M, v) \models \chi \land (\langle M, v \rangle \models (M^{X \land \tau}, v) \models \varphi) \). By (1), we have that \( \exists \chi \in \mathcal{L}_{EL}^G \forall \tau \in \mathcal{L}_{EL}^A : (M, v) \models \chi \land (\langle M, v \rangle \models (M^{X \land \tau}, v) \models \varphi) \), which is \( (M, v) \models \langle G \rangle \varphi \) by the semantics. The same argument holds in the other direction.

C1: \( \neg \langle G \rangle \bot \). From \( \top \) we derive \( \langle G \rangle \bot \) by R4. Using the dual of box, we have \( \neg \langle G \rangle \neg \top \), or \( \neg \langle G \rangle \bot \).

C2: \( \langle G \rangle \top \). In any state, there exists a true announcement by \( G \) (each agent in \( G \) announces their knowledge of a tautology) and after any joint announcement, \( \top \) is true, hence, the axiom is valid.

C3: \( \neg \langle \emptyset \rangle \neg \varphi \rightarrow \langle A \rangle \varphi \). By the semantics, \( \neg \langle \emptyset \rangle \neg \varphi \), which is \( \langle \emptyset \rangle \varphi \), means that there exists some \( \psi \in \mathcal{L}_{EL}^A \), such that \( \langle \psi \rangle \varphi \). This is precisely the meaning of \( \langle A \rangle \varphi \).

C4: \( \langle G \rangle \varphi \land \psi \rightarrow \langle G \rangle \varphi \). Suppose that \( \langle G \rangle \varphi \land \psi \) holds. By the semantics, \( \exists \chi \in \mathcal{L}_{EL}^G \forall \tau \in \mathcal{L}_{EL}^A : (M, v) \models \chi \land [\chi \land \tau] \varphi \land [\chi \land \tau] \psi \). Then, by A7, we have \( \exists \chi \in \mathcal{L}_{EL}^G \forall \tau \in \mathcal{L}_{EL}^A : (M, v) \models \chi \land [\chi \land \tau] \varphi \land [\chi \land \tau] \psi \). The latter implies \( \exists \chi \in \mathcal{L}_{EL}^G \forall \tau \in \mathcal{L}_{EL}^A : (M, v) \models \chi \land [\chi \land \tau] \varphi \), which is \( \langle G \rangle \varphi \) by the semantics.

C5: \( \langle G \rangle \varphi_1 \land \langle H \rangle \varphi_2 \rightarrow \langle G \cup H \rangle (\varphi_1 \land \varphi_2) \), if \( \langle G \cap H = \emptyset \) holds. Assume that \( \langle G \rangle \varphi_1 \land \langle H \rangle \varphi_2 \). Let us consider the first conjunct. By the semantics, we have \( \exists \psi \in \mathcal{L}_{EL}^G \forall \chi \in \mathcal{L}_{EL}^G : (\psi \land [\psi \land \chi]) \varphi_1 \). The latter implies \( \exists \psi \in \mathcal{L}_{EL}^G \forall \chi \in \mathcal{L}_{EL}^G : (\psi \land [\psi \land \chi]) \varphi_1 \). In order to show this let us assume to the contrary that \( \exists \psi \in \mathcal{L}_{EL}^G : (\psi \land [\psi \land \chi]) \varphi_1 \). Then, we have \( \forall \psi \in \mathcal{L}_{EL}^G : (\psi \land [\psi \land \chi]) \varphi_1 \). Next, by (1), we fix some \( \psi \). Then, we have \( \forall \psi \in \mathcal{L}_{EL}^G : (\psi \land [\psi \land \chi]) \varphi_1 \). Similarly, we can prove that \( \exists \psi \in \mathcal{L}_{EL}^G : (\psi \land [\psi \land \chi]) \varphi_1 \).
∀A \backslash G, H \in \mathcal{L}_{EL}^{A \backslash G, H}: \psi_{G, H} \land [\psi_{G, H} \land \chi_{A \backslash G, H}] \varphi_2. By distributivity of [] over \land, we get \exists \psi_{G, H} \in \mathcal{L}_{EL}^{G, H} \forall A \backslash G, H \in \mathcal{L}_{EL}^{A \backslash G, H}: \psi_{G, H} \land [\psi_{G, H} \land \chi_{A \backslash G, H}] (\varphi_1 \land \varphi_2). Hence \langle (G \cup H) \rangle (\varphi_1 \land \varphi_2). \hfill \Box

\section{Proofs of Propositions}

\textbf{Proposition B.1.} \( R4 \) and \( R6 \) are sound, that is, they preserve validity.

\begin{proof}
\((R4)\) Assume that \( \models \varphi \). By \( R2 \), for an arbitrary \( \psi \), \( \models [\psi] \varphi \). Since \( \psi \) is arbitrary, \( \models (G) \varphi \); in other words, whatever agents announce, they cannot make a valid formula false.

\((R6)\) Let \((M, w)\) be an arbitrary pointed model. We proceed by induction on \( \eta \).

\textbf{Base case.} \( \forall \psi \in \mathcal{L}_{EL}^{G} \exists \chi \in \mathcal{L}_{EL}^{A \backslash G}: \psi \rightarrow (\psi \land \chi) \varphi \) is valid. Therefore, by the semantics, we infer validity of \( (G) \varphi \).

\textbf{Induction hypothesis.} Assume the rule preserves validity for \( n(\eta (\psi \rightarrow (\psi \land \chi) \varphi)) = k \). We show that it holds for \( k + 1 \).

\textbf{Case 1.} \( \forall \psi \in \mathcal{L}_{EL}^{G} \exists \chi \in \mathcal{L}_{EL}^{A \backslash G}: \tau \rightarrow \eta (\psi \rightarrow (\psi \land \chi) \varphi) \) is valid. This means that \( (M, w) \models \tau \rightarrow \eta (\psi \rightarrow (\psi \land \chi) \varphi) \) iff \( (M, w) \models \neg \tau \) or \( (M, w) \models \eta (\psi \rightarrow (\psi \land \chi) \varphi) \), which is \( (M, w) \models \neg \tau \) or \( (M, w) \models (G) \varphi \) by Induction hypothesis. Hence, \( (M, w) \models \tau \rightarrow (G) \varphi \).

\textbf{Case 2.} \( \forall \psi \in \mathcal{L}_{EL}^{G} \exists \chi \in \mathcal{L}_{EL}^{A \backslash G}: K_\alpha \eta (\psi \rightarrow (\psi \land \chi) \varphi) \) is valid. This means that \( (M, w) \models K_\alpha \eta (\psi \rightarrow (\psi \land \chi) \varphi) \). By the semantics, for every \( v \in W : (w, v) \in \sim_a \) implies \( (M, v) \models \eta (\psi \rightarrow (\psi \land \chi) \varphi) \). By Induction hypothesis, for every \( v \in W : (w, v) \in \sim_a \) implies \( (M, v) \models \eta ((G) \varphi) \). And, by the semantics, \( (M, w) \models K_\alpha (G) \varphi \). Finally, by the semantics, \( (M, w) \models (\tau (G)) \varphi \). \hfill \Box

\textbf{Proposition B.2.} \( (G) \langle H \rangle \varphi \rightarrow (G \cup H) \varphi \) is valid.

\begin{proof}
Let \( \psi := \bigwedge_{i \in G} K_i \psi_i \), \( \psi' := \bigwedge_{j \in A \backslash G} K_j \psi_j \), \( \chi := \bigwedge_{k \in H} K_i \chi_k \), and \( \chi' := \bigwedge_{l \in A \backslash H} K_l \chi'_l \). Suppose \( (M, w) \models ((G) \langle H \rangle \varphi) \) for some \( M \) and \( w \in W \). By the semantics,

\( \exists \psi \in \mathcal{L}_{EL}^{G} \forall \psi' \in \mathcal{L}_{EL}^{A \backslash G} \exists \chi \in \mathcal{L}_{EL}^{H} \forall \chi' \in \mathcal{L}_{EL}^{A \backslash H}: (M, w) \models \psi \land [\psi \land \psi'] (\chi \land [\chi \land \chi']) \varphi. \)

By A7, we have: \( (M, w) \models \psi \land [\psi \land \psi'] (\chi \land [\psi \land \psi'] (\chi \land \chi')) \varphi. \) We are interested now in the third conjunct: \( [\psi \land \psi'] (\chi \land \chi') \varphi. \) By A9, we have that \( (M, w) \models [\psi \land \psi'] (\chi \land \chi') \varphi. \) Now, let us examine the following conjunction: \( \exists \psi \in \mathcal{L}_{EL}^{G} \forall \psi' \in \mathcal{L}_{EL}^{A \backslash G}: \psi \land \psi' \), which is

\[ \exists \bigwedge_{i \in G} K_i \psi_i \land \bigwedge_{j \in A \backslash G} K_j \psi_j : \bigwedge_{i \in G} K_i \psi_i \land \bigwedge_{j \in A \backslash G} K_j \psi_j \]

in the full form. We can present the set of agents \( A \backslash G \) as a union of \( G \cup H \) and \( H \backslash G \) by expanding the right conjunct. So, we have \( \exists \bigwedge_{i \in G} K_i \psi_i \land \bigwedge_{m \in H \backslash G} K_m \psi'_m \land \bigwedge_{j \in A \backslash G \backslash H} K_j \psi'_j \):

\[ \bigwedge_{i \in G} K_i \psi_i \land \bigwedge_{m \in H \backslash G} K_m \psi'_m \land \bigwedge_{j \in A \backslash G \backslash H} K_j \psi'_j, \] (1)
Since none of the universal quantifiers here is vacuous, there are particular $ψ''$ for which the conjunction holds. Formally, $∃i ∈ G \cup H \quad K_i ψ_i \quad ∃m ∈ H \cup G \quad K_m ψ_m'' \quad ∀j ∈ A \setminus G \cup H \quad K_j ψ_j'$. Therefore, combining $G$ and $H \setminus G$, we have

$$∃ \bigwedge_{i ∈ G \cup H} K_i ψ_i \quad ∀ \bigwedge_{j ∈ A \setminus G \cup H} K_j ψ_j' : ∃ \bigwedge_{i ∈ G \cup H} K_i ψ_i \quad ∀ \bigwedge_{j ∈ A \setminus G \cup H} K_j ψ_j'.$$

The same argument holds for the conjunction $∃χ ∈ \mathcal{L}_{EL}^G \quad ∀χ' ∈ \mathcal{L}_{EL}^G : χ \land χ'$. Let us redefine our auxiliary formulae: $ψ := \bigwedge_{i ∈ G \cup H} K_i ψ_i$, $ψ' := \bigwedge_{j ∈ A \setminus G \cup H} K_j ψ_j'$, $χ := \bigwedge_{i ∈ G \cup H} K_i \chi_i$, and $χ' := \bigwedge_{j ∈ A \setminus G \cup H} K_j \chi_j'$. Thus, we have that $∃ψ ∈ \mathcal{L}_{EL}^G \quad ∀ψ' ∈ \mathcal{L}_{EL}^G : χ ∈ \mathcal{L}_{EL}^G \quad ∀χ' ∈ \mathcal{L}_{EL}^G : (M, w) \models [ψ \land ψ' \land [ψ \land ψ'](χ \land χ')]φ$. In the full form, the latter is

$$∃ \bigwedge_{i ∈ G \cup H} K_i ψ_i \quad ∀ \bigwedge_{j ∈ A \setminus G \cup H} K_j ψ_j' : \quad ∃ \bigwedge_{i ∈ G \cup H} K_i χ_i \quad ∀ \bigwedge_{j ∈ A \setminus G \cup H} K_j χ_j' :$$

$$(M, w) \models [ \bigwedge_{i ∈ G \cup H} K_i ψ_i \land \bigwedge_{j ∈ A \setminus G \cup H} K_j ψ_j' ] \land [ \bigwedge_{i ∈ G \cup H} K_i ψ_i \land \bigwedge_{j ∈ A \setminus G \cup H} K_j ψ_j' ]( \bigwedge_{i ∈ G \cup H} K_i χ_i \land \bigwedge_{j ∈ A \setminus G \cup H} K_j χ_j' ) ) φ.$$

Using A7 and A8, we can ‘push’ announcements into the scope of knowledge operators:

$$(M, w) \models [ψ \land ψ' \land ( \bigwedge_{i ∈ G \cup H} \bigwedge_{j ∈ A \setminus G \cup H} (K_i ψ_i \land K_j ψ_j' \rightarrow K_i [K_i ψ_i \land K_j ψ_j'] χ_i ) \land (K_i ψ_i \land K_j ψ_j' \rightarrow K_j [K_i ψ_i \land K_j ψ_j'] χ_j' ) ) φ.$$ 

By propositional reasoning, the latter is equivalent to $∀i ∈ G \cup H \quad K_i ψ_i \land K_j ψ_j' \land K_i [K_i ψ_i \land K_j ψ_j'] \chi_i \land K_j [K_i ψ_i \land K_j ψ_j'] \chi_j' φ$. Finally, we have

$$∃ \bigwedge_{i ∈ G \cup H} K_i ψ_i \quad ∀ \bigwedge_{j ∈ A \setminus G \cup H} K_j ψ_j' : \quad ∃ \bigwedge_{i ∈ G \cup H} K_i χ_i \quad ∀ \bigwedge_{j ∈ A \setminus G \cup H} K_j χ_j' :$$

$$(M, w) \models [ \bigwedge_{i ∈ G \cup H} \bigwedge_{j ∈ A \setminus G \cup H} (K_i (ψ_i \land [K_i ψ_i \land K_j ψ_j'] χ_i ) \land K_j (ψ_j' \land [K_i ψ_i \land K_j ψ_j'] χ_j' ) ) φ.$$ 

Conjuncts of the form $K_i (ψ_i \land [K_i ψ_i \land K_j ψ_j'] χ_i )$ mean that agent $i$ can announce $ψ_i$, i.e. what she knows now, or $[K_i ψ_i \land K_j ψ_j'] χ_i$ (which is equivalent to $t([K_i ψ_i \land K_j ψ_j'] χ_i )$, i.e. what she will know after announcements of other agents but not necessarily knows now, or both. Since all the variants comprise $\mathcal{L}_{EL}^{G \cup H}$, we rewrite the notation. Hence, $∃i ∈ G \cup H \quad K_i τ_i \land ∀j ∈ A \setminus G \cup H \quad K_j τ_j' : (M, w) \models [ \bigwedge_{i ∈ G \cup H} K_i τ_i \land \bigwedge_{j ∈ A \setminus G \cup H} K_j τ_j' ] φ$, and at the same time $(M, w) \models [ \bigwedge_{i ∈ G \cup H} K_i τ_i \land \bigwedge_{j ∈ A \setminus G \cup H} K_j τ_j' φ$, where agents from $H$ announce $τ_i$. And, by the semantics, this is $(M, w) \models (G \cup H) φ$.

**Proposition B.3.** Let $x$ be a theory, $ψ, φ ∈ \mathcal{L}_{CoGal}$, and $a ∈ A$. The following are theories: $x + φ = \{ψ : φ → ψ ∈ x\}$, $K_a x = \{φ : K_a φ ∈ x\}$, and $[φ] x = \{ψ : [φ] ψ ∈ x\}$. 

**Proof.** We just expand the proof from [7] by showing that corresponding theories are closed under R5 and R6. 

Suppose that $η( [ψ] χ ) ∈ x + φ$ for all $ψ ∈ \mathcal{L}_{EL}^G$. It means that $φ → η( [ψ] χ ) ∈ x$ for all $ψ ∈ \mathcal{L}_{EL}^G$. Since $φ → η( [ψ] χ )$ is a necessity form, and $x$ is closed under R5, we infer that $φ → η( [G] χ ) ∈ x$, and, consequently, $η( [G] χ ) ∈ x + φ$. So, $x + φ$ is closed under R5. Now, let $∀ψ ∈ \mathcal{L}_{EL}^G \quad ∀τ ∈ \mathcal{L}_{EL}^{A \setminus G} : η(ψ → (ψ → τ) χ ) ∈ x + φ$. It means that $∀ψ ∈ \mathcal{L}_{EL}^G \quad ∀τ ∈ \mathcal{L}_{EL}^{A \setminus G} : φ → η(ψ → (ψ → τ) χ ) ∈ x$. Since
\( \varphi \to \eta(\psi \to \langle \psi \land \tau \rangle \chi) \) is a necessity form, and \( x \) is closed under R6, we infer that \( \varphi \to \eta([G]\chi) \in x \), and, consequently, \( \eta([G]\chi) \in x + \varphi \). So, \( x + \varphi \) is closed under R6.

Suppose that \( \eta([G]\chi) \in K_\varphi x \) for all \( \psi \in L^G_{EL} \). It means that \( K_\varphi \eta([G]\chi) \in x \) for all \( \psi \in L^G_{EL} \). Since \( K_\varphi \eta([G]\chi) \) is a necessity form, and \( x \) is closed under R5, we infer that \( K_\varphi \eta([G]\chi) \in x \), and, consequently, \( \eta([G]\chi) \in K_\varphi x \). So, \( K_\varphi x \) is closed under R5. Now, let \( \forall \psi \in L^G_{EL} \exists \tau \in L^A_{EL} : \eta(\psi \to \langle \psi \land \tau \rangle \chi) \in K_\varphi x \). It means that \( \forall \psi \in L^G_{EL} \exists \tau \in L^A_{EL} : K_\varphi \eta(\psi \to \langle \psi \land \tau \rangle \chi) \in x \). Since \( K_\varphi \eta(\psi \to \langle \psi \land \tau \rangle \chi) \) is a necessity form, and \( x \) is closed under R6, we infer that \( K_\varphi \eta([G]\chi) \in x \), and, consequently, \( \eta([G]\chi) \in K_\varphi x \). So, \( K_\varphi x \) is closed under R6.

Finally, suppose that \( \eta([G]\chi) \in [\varphi]x \) for all \( \psi \in L^G_{EL} \). It means that \( [\varphi] \eta([G]\chi) \in x \) for all \( \psi \in L^G_{EL} \). Since \( [\varphi] \eta([G]\chi) \) is a necessity form, and \( x \) is closed under R5, we infer that \( [\varphi] \eta([G]\chi) \in x \), and, consequently, \( \eta([G]\chi) \in [\varphi]x \). So, \( [\varphi]x \) is closed under R5. Now, let \( \forall \psi \in L^G_{EL} \exists \tau \in L^A_{EL} : \eta(\psi \to \langle \psi \land \tau \rangle \chi) \in [\varphi]x \). It means that \( \forall \psi \in L^G_{EL} \exists \tau \in L^A_{EL} : [\varphi] \eta(\psi \to \langle \psi \land \tau \rangle \chi) \in x \). Since \( [\varphi] \eta(\psi \to \langle \psi \land \tau \rangle \chi) \) is a necessity form, and \( x \) is closed under R6, we infer that \( [\varphi] \eta([G]\chi) \in x \), and, consequently, \( \eta([G]\chi) \in [\varphi]x \). So, \( [\varphi]x \) is closed under R6.

Proposition B.4. Let \( \varphi \in L^A_{CoGAL} \). Then \( \text{CoGAL} + \varphi \) is consistent iff \( \neg \varphi \notin \text{CoGAL} \).

Proof. From left to right. Suppose to the contrary that \( \text{CoGAL} + \varphi \) is consistent and \( \neg \varphi \in \text{CoGAL} \). Then, having both \( \varphi \) and \( \neg \varphi \) means that \( \bot \in \text{CoGAL} + \varphi \), which contradicts to \( \text{CoGAL} + \varphi \) being consistent.

From right to left. Let us consider the contrapositive: if \( \text{CoGAL} + \varphi \) is inconsistent, then \( \neg \varphi \in \text{CoGAL} \). Since \( \text{CoGAL} + \varphi \) is inconsistent, \( \bot \in \text{CoGAL} + \varphi \), or, by Proposition B.3, \( \varphi \to \bot \in \text{CoGAL} \). By consistency of \( \text{CoGAL} \) and propositional reasoning, we have that \( \neg \varphi \notin \text{CoGAL} \). \( \square \)