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Navigation and control based on integral-uncertainty observer for unmanned jet aircraft

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Abstract: A nonlinear integral-uncertainty observer is presented, which can estimate the integral of measurement output signal and the uncertainty in system, synchronously. In order to be satisfied with the existing hardware computational environments and to select the parameters more easily, a simplified linear version of the nonlinear integral-uncertainty observer is also developed. The effectiveness of the proposed observers are verified through the numerical simulations and experiments: i) through the integral-uncertainty observers, the attitude angle and the uncertainties in attitude dynamics are estimated synchronously from the measurements of angular velocity, and the estimate results by the two observers are compared; ii) a control law is designed based on the observers to drive the jet aircraft to track a reference trajectory.

Keywords: Integral-uncertainty observer, jet aircraft, attitude angle, uncertainty, synchronous estimate

I. INTRODUCTION

Unmanned jet aircraft control has been an active area of investigation for several years, and some inertial sensors were used and supplemented by GPS [1, 2, 3]. This interest was motivated by the enormous military and civil applications of such aircraft. It is one of the most interesting architectures because its dynamical system is characterized by the powerful thrust provision, high-speed flight, the payload augmentation and a high maneuverability.

Usually, controlling an unmanned jet aircraft needs the information of the attitude and position. For the system of an unmanned jet aircraft, we consider that no measurement of flying velocity and attitude angle is provided. Moreover, jet aircrafts are underactuated mechanical systems, which exhibit high nonlinear, time-varying and time-delay behaviors, meanwhile, the influences of aerodynamic disturbance, unmodelled dynamics and parametric uncertainties are not avoidable in modeling. These nonlinearities and uncertainties render great challenges in the design of flight control system.

On the one hand, an inertial measurement unit (IMU) can provide the attitude information. It contains three orthogonal rate-gyrosopes and three orthogonal accelerometers, measuring angular velocity and linear acceleration, respectively. The information at high sampling frequency is provided by this sensor. To calculate the attitude angles of the device, the angular velocity signals from the rate-gyrosopes are onefold integrated. The drift phenomenon of IMU is mainly brought by the usual integral methods [4, 5, 6, 7]: Romberg integration, Gaussian quadrature, extended Simpson’s rule, low-frequency integrator. They cannot restrain the effect of stochastic noise (especially non-white noise). Such noise leads to the accumulation of additional drift in the integrated signal. In [8], a fractional-order integrator is proposed to approximate the irrational fractional-order integrator \(1/s^m\). However, the condition of \(0 < m < 1\) limits the application of the fractional-order integrator. Obviously, the usual observers or differentiators [9, 10, 11, 12] only can estimate the derivatives of the signal. Recent years, Kalman filter is used to handle the separation of probabilistic noise and to estimate signal integral [13, 14]. However, for Kalman filter, the process noise covariance and measurement noise covariance are assumed to be zero-mean Gaussian distributed, and the process noise covariance is uncorrelated to the estimation error. These assumptions are different from the real noise in signal. The inaccurate noise information in sensed angular velocity may lead to the estimate drifts of attitude angle. In [15], a nonlinear double-integral observer with the abilities of noise rejection and drift correction was presented to estimate synchronously the onefold and double integrals of a signal. In [16], a generalized multiple integrator was designed to estimate the multiple integrals for a signal. In [17], a nonlinear integral-derivative observer was proposed to estimate synchronously the integral and derivative of a signal. However, these observer cannot be use to estimate the uncertainties in the flight dynamics directly.

On the other hand, some sensors provide usually the position-related information. Representative designs are: GPS positioning systems [18, 19]; GPS/INS systems [20, 21, 22]; ultrasonic rangers [23]; GPS module when outdoors and infrared rangers when indoors [24]; carrier phase differential GPS [25]; laser rangefinder [26]; vision system [27, 28, 29]; indoor motion capture system [30, 31]; laser rangefinder and vision system [32]. However, these strategies are dependent on the accurate model, and all the states are required to be known.

For the aircrafts with uncertainties, sliding-mode controls with intelligent estimate algorithms were proposed [33, 34]. The uncertainties in aircraft are approximated by some intelligent algorithms, such as radial-based-function (RBF) networks or fuzzy systems. It was concluded that RBF networks are capable of universal approximation [35], and fuzzy system can also provide universal approximation for a continuous function [36]. However, the uncertainty estimation of aircraft by neural network or fuzzy system requires that all the states are known. The main difficulties with estimating uncertainties by these algorithms are: 1) the parameters or the neural network weights are difficult to be regulated; 2) membership function and Gaussian function are selected by experiences; 3) all of the system states must be required for estimation; 4) high-frequency noise cannot be restrained. These disadvantages affect control performances of aircraft adversely.

Considering the problems above, the objective of this paper is to design an observer to estimate the unknown integral state of measurement output signal and the uncertainty in system, synchronously, in spite of the existence of measurement disturbance. Inspired by the theory of finite-time stability [37,38], singular perturbation technique [39,40] and our previous works [15, 16], a nonlinear integral-uncertainty observer is developed. Based on the theories in [15, 16],
an extended system is implemented after the uncertainty in system is taken as a new state.

Thus, a nonlinear integral-uncertainty observer is developed, which can estimate the unknown integral state of measurement output signal and the uncertainty in system, synchronously. The parameters selection is satisfied with Routh-Hurwitz Stability Criterion and the iterative equation relations. Furthermore, considering the adverse effects on the nonlinear system by the existing hardware computational environments in aircraft system, it is necessary to simplify the relatively complex nonlinear integral-uncertainty observer into a simple linear form, and the parameters selection needs to be more easier for some industrial applications. Fortunately, when some parameters in the nonlinear integral-uncertainty observer are given a particular value, the linear observer can be obtained, and it can still work. Although the nonlinear stability analysis no longer holds for the linear observer, the theory of linear system can be used to analyze it. The selection of parameters become relaxed, and it is only required to be satisfied with Routh-Hurwitz Stability Criterion. The parameters selection rules and robustness analysis for the two types of observers are presented based on frequency-domain analysis.

For industrial applications, the proposed observers are applied to an unmanned jet aircraft, and an experiment is presented to observe the performances of the proposed observers. In the jet aircraft system, the proposed observation presented to observe the performances of the proposed ob-

II. DESIGN OF INTEGRAL-UNCERTAINTY OBSERVERS

The following underactuated system has a minimum number of states and inputs but retains many of the features that must be considered when designing control laws for many mechanical systems:

\[
\begin{align*}
\dot{w}_1 &= w_2 \\
\dot{w}_2 &= \Pi(t) + \sigma(t) \\
y_{op} &= w_2 + d(t)
\end{align*}
\]  

(1)

where, \((w_1, w_2)\) is the state vector; \(y_{op} = w_2 + d(t)\) is the measurement output; \(d(t)\) is the bounded sensor error or stochastic noise, and \(\sup_{t \in [0, \infty)} |d(t)| \leq L_d < \infty\); \(w_1\) is the unknown state; \(\Pi(t) \in R\) is the known function; uncertainty \(\sigma(t) \in R\) includes the unknown parameters and nonlinearities, and it is relatively bounded.

In order to calculate the unknown state \(w_1\), the measurement signal \(y_{op}\) is integrated, i.e.,

\[
I(t) = \int_0^t y_{op}(\sigma) d\sigma = \int_0^t w_2(\sigma) d\sigma + \int_0^t d(\sigma) d\sigma
\]  

(2)

Owing to the integration of Eq. (2), a small noise \(d(t)\) (especially non-zero mean noise) in the measurement will grow rapidly in the computed final integration, i.e., the noise will be accumulated and accuracy in the computed final integration deteriorate with time. The error in the measurement signal is propagated to the integration. This results in integral drift. Moreover, the uncertainty \(\sigma(t)\) renders great challenges in the design of control systems.

**Assumption 1:** Suppose the frequency of the uncertainty \(\sigma(t)\) are far smaller than the system sampling frequency, and it has the following dynamics:

\[
\dot{\sigma}(t) = c_\sigma(t)
\]  

(3)

In fact, this assumption is satisfied with almost all engineering applications, for instance, the dynamics of crosswind or the uncertainties in the aircraft systems.

Let \(w_3 = \sigma(t)\), and \(w_3 = \dot{\sigma}(t) = c_\sigma(t)\), Eq. (1) can be augmented to

\[
\begin{align*}
\dot{w}_1 &= w_2 \\
\dot{w}_2 &= w_3 + \Pi(t) \\
\dot{w}_3 &= c_\sigma(t) \\
y_{op} &= w_2 + d(t)
\end{align*}
\]  

(4)

2.1 Design of nonlinear integral-uncertainty observer

In the following, considering sensor error and noise, finite-time stability and robustness [37,38] (The related concepts are introduced in Appendix A) and singular perturbation technique [39,40] will be used to present an integral-uncertainty observer with drift correction and strong robustness. The onefold integral of measurement output signal and the uncertainty in system can be estimated, and the effect of propagating the noise to the integral is rejected sufficiently.

**Theorem 1:** For system (4), if the following observer is designed,

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 + \Pi(t) \\
\varepsilon^4 \dot{x}_3 &= -k_1 |x_1|^{\alpha_1} \text{sign}(x_1) \\
&\quad -k_2 |x_2 - y_{op}|^{\alpha_2} \text{sign}(x_2 - y_{op}) \\
&\quad -k_3 |x_3|^{\alpha_3} \text{sign}(x_3)
\end{align*}
\]  

(5)

where \(\varepsilon \in (0, 1)\) is the perturbation parameter, \(\alpha_1, \alpha_2\) and \(\alpha_3\) satisfy:

\[
\alpha_3 \in (0, 1), \alpha_2 = \frac{\alpha_3}{2 - \alpha_3}, \alpha_1 = 1 - \frac{\alpha_3}{3 - 2\alpha_3}
\]  

(6)

and \(k_1, k_2, k_3 > 0\) are selected such that

\[
k_1 > 0, k_3 > 0, k_2 > \varepsilon^{2\alpha_2} k_1/k_3
\]  

(7)

then there exist \(\gamma > 1, \ L > 0, \ \delta_{di} \in (0, 1)\) and \(\Gamma > 0\), such that, for \(t \geq \varepsilon \Gamma(\Xi(\gamma)e(0))\),

\[
|x_i - w_i(t)| \leq L(\delta_{di})^\gamma, \ for \ i = 1, 2, 3
\]  

(8)
Ideally, if no sensor error and noise exist, i.e., $d(t) = 0$ and $L_d = 0$, then, for $t \geq \varepsilon \Gamma (\Xi(\varepsilon)e(0))$

$$|x_i - w_i(t)| \leq L e^{\alpha_i^2^\gamma - i}, i = 1, 2, 3$$ (9)

where $e_i = x_i - w_i(t)$, $i = 1, 2, 3$; $e = [e_1~ e_2~ e_3]^{T}$; $\Xi(\varepsilon) = diag\{\varepsilon, \varepsilon^2, \varepsilon^3\}$.

The proof of Theorem 1 is presented in Appendix B.

In integral-uncertainty observer (5), $x_2$ tracks the state $w_2$; $x_1$ and $x_3$ estimate $w_1$ and the uncertainty $w_3$ (i.e., $\sigma(t)$ in system (1)), respectively. The usual integral algorithms inevitably suffer from unbounded errors in the calculations of integral when noise exists in measurement output signal $y_{op}$. However, from (8), the up-boundness of the estimate errors are bounded, and they are unrelated to integration time.

In fact, in the up-boundness of estimate error $L(\delta_{di})^\gamma$ ($i = 1, 2, 3$), $\gamma > 1$ holds, and from the proof of Theorem 1 in Appendix B, $L = \mu \delta_{d}$, where $\mu$ is a constant defined in Theorem 5.2 in [37]. For the low-level noise, from (88), $\delta_{di} \in (0, 1)$ holds. By selecting a suitable perturbation parameter $\varepsilon \in (0, 1)$, the up-boundness of estimate errors are sufficiently small. Therefore, the ultimate bound (8) on the estimation error is of higher order than the perturbation. Consequently, the presented double-integral observer (5) leads to perform rejection of low-level noise, i.e., almost no drift phenomenon happen in spite of the existence of the sensor error and non-white noise.

**2.2 Design of linear integral-uncertainty observer**

As we know, a linear system is easy to perform the analysis with respect to nonlinear one. In the following, based on the nonlinear integral-uncertainty observer (5), a simplified linear integral-uncertainty observer will be designed, and Theorem 2 is presented as follow.

**Corollary 1:** For system (4), if the following observer is designed,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3 + \Pi(t)$$
$$\varepsilon^4 \dot{x}_3 = -k_1 \varepsilon x_1 - k_2 (x_2 - w_2) - k_3 \varepsilon^3 x_3$$ (10)

where $\varepsilon \in (0, 1)$ is the perturbation parameter, and

$$k_1 > 0, k_3 > 0, k_2 > \varepsilon^2 k_1/k_3$$ (11)

then the following estimate relations hold:

$$\lim_{\varepsilon \to 0} x_i = w_i$$ (12)

where $i = 1, 2, 3$. The relevant analysis of Corollary 1 is presented in Appendix B.

In integral-uncertainty observer (10), $x_2$ tracks the state $w_2$; $x_1$ and $x_3$ estimate $w_1$ and the uncertainty $w_3$ (i.e., $\sigma(t)$ in system (1)), respectively. The proposed two observers (5) and (10) all can perform rejection of high-frequency noise. In the next section, the frequency-domain analysis will be presented for the nonlinear and linear observers, and the parameters selection rules will be given.

**III. ROBUSTNESS ANALYSIS AND PARAMETERS SELECTION**

In practice, high-frequency noises exist in measurement output $y_{op}$. In this paper, describing function method [40, 42] is used to approximately analyze and predict the nonlinear behaviors of the observer. Even though it is only an approximation method, the desirable properties it inherits from the frequency response method, and the shortage of other, systematic tools for nonlinear observer analysis, make it an indispensable component of the bag of tools of practicing control engineers. The describing function method succeeded in applications to analyze the frequency-domain characteristics for nonlinear differentiators and augmented observers [41]. By describing function method, it will be found that the presented integral-uncertainty observer leads to perform rejection of high-frequency noise. Alternatively, the frequency-sweep method [43, 44] can be used to approximately analyze and predict the nonlinear behaviors of these observers.

In addition, there exist seven parameters in nonlinear observer (5): $\alpha_1, \alpha_2, \alpha_3, k_1, k_3, k_1, \varepsilon$; and four parameters in linear observer (10): $k_1, k_3, k_1, \varepsilon$. How to select these parameters is critical for the estimate performances and robustness abilities.

For system (4), let $\xi_1 = w_1, \xi_2 = w_2, \xi_3 = w_3 + \Pi(t)$ and $\Pi(t) = \eta(t)$, then $\dot{w}_3 + \Pi(t) = \sigma(t) + \eta(t)$. Therefore, system (4) can be rewritten as

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = \xi_3$$
$$\dot{\xi}_3 = c_\sigma(t) + \eta(t)$$ (13)

$$y_{op} = \xi_2$$

Accordingly, for system (13), the observer (5) can be transferred to

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\varepsilon^4 \dot{x}_3 = -k_1 |\varepsilon x_1|^{\alpha_1} \text{sign}(x_1)$$
$$-k_2 |x_2 - y_{op}|^{\alpha_2} \text{sign}(x_2 - y_{op})$$
$$-k_3 |\varepsilon x_3|^{\alpha_3} \text{sign}(x_3)$$ (14)

where $x_2$ tracks the output signal $\xi_2$; $x_1$ and $x_3$ estimate $\xi_1$ and $\xi_3$, respectively.

The frequency characteristic of (14) is analyzed as follow.

Let $x_2 - y_{op} = A \sin(\omega t)$. For the nonlinear function $|A \sin(\omega t)|^{\alpha_1} \text{sign}(A \sin(\omega t))$, its describing functions can be obtained as follow:

$$N_i(A) = \frac{2}{A\pi} \int_0^\pi |A \sin(\omega t)|^{\alpha_1} \text{sign}(A \sin(\omega t)) \sin(\omega t) d\omega$$

$$= \frac{\Omega(\alpha_1)}{A^{1-\alpha_1}}$$

where $\Omega(\alpha_1) = \frac{2}{\pi} \int_0^\pi |\sin(\omega t)|^{\alpha_1+1} d\omega$. Therefore, the linearization of observer (14) is
\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_3 \\
\varepsilon^4 \dot{x}_3 = -k_1 \frac{\Omega(\alpha_1)}{A^{1-\alpha_1}} \varepsilon x_1 - k_2 \frac{\Omega(\alpha_2)}{A^{1-\alpha_2}} x_2 - y_{op} \\
- k_3 \frac{\Omega(\alpha_3)}{A^{1-\alpha_3}} \varepsilon^3 x_3 
\]

(15)

and the Laplace transformations of the linear system (15) can be written as

\[
sX_1(s) = X_2(s) \\
sX_2(s) = X_3(s) \\
\varepsilon^4 sX_3(s) = -k_1 \frac{\Omega(\alpha_1)}{A^{1-\alpha_1}} \varepsilon X_1(s) \\
- k_2 \frac{\Omega(\alpha_2)}{A^{1-\alpha_2}} (X_2(s) - Y_{op}(s)) \\
- k_3 \frac{\Omega(\alpha_3)}{A^{1-\alpha_3}} \varepsilon^3 X_3(s) 
\]

(16)

where \(X_i(s)\) and \(Y_{op}(s)\) denote the Laplace transformations of \(x_i\) and \(y_{op}\), respectively, and \(s\) denotes Laplace operator.

From Eq. (16), the following transfer functions are obtained:

\[
X_j(s) = \frac{k_2 \frac{\Omega(\alpha_2)}{A^{1-\alpha_2}} s^{j-1}}{\varepsilon^4 s^3 + \varepsilon^3 k_3 \frac{\Omega(\alpha_3)}{A^{1-\alpha_3}} s^2 + k_2 \frac{\Omega(\alpha_2)}{A^{1-\alpha_2}} s + \varepsilon k_1 \frac{\Omega(\alpha_1)}{A^{1-\alpha_1}}}, \quad j \in \{1, 2, 3\} 
\]

(17)

The transfer functions for linear observer (10) can be directly obtained as

\[
\frac{X_j(s)}{Y_{op}(s)} = \frac{k_2 s^{j-1}}{\varepsilon^4 s^3 + \varepsilon^3 k_3 s^2 + k_2 s + \varepsilon k_1}, \quad j \in \{1, 2, 3\} 
\]

(18)

The effects of the observer parameters on the robustness are analyzed as follows.

3.1 Frequency characteristic with different \(\varepsilon\) and \(\alpha_3\)

Selecting \(\alpha_3\) with different values, we obtain the following table of \(\Omega(\alpha_3)\), \(\Omega(\alpha_2)\) and \(\Omega(\alpha_1)\):

<table>
<thead>
<tr>
<th>(\alpha_3)</th>
<th>(\Omega(\alpha_3))</th>
<th>(\Omega(\alpha_2))</th>
<th>(\Omega(\alpha_1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>1.0410</td>
<td>1.0712</td>
<td>1.0944</td>
</tr>
<tr>
<td>0.5</td>
<td>1.1128</td>
<td>1.1596</td>
<td>1.1852</td>
</tr>
<tr>
<td>0.3</td>
<td>1.1697</td>
<td>1.2093</td>
<td>1.2270</td>
</tr>
</tbody>
</table>

Table 1 Values of \(\Omega(\alpha_3)\), \(\Omega(\alpha_2)\) and \(\Omega(\alpha_1)\) with respect to \(\alpha_3\)

For the transfer functions (17) and (18), the parameters are selected as follows: \(k_1 = 0.1, k_2 = 2, k_3 = 1; A = 1; \alpha_3 = 0.3, 0.5, 1\), respectively; \(R = 1/\varepsilon = 3, 4, 5\), respectively. The Bode plots of the frequency-domain characteristics with different \(\varepsilon\) and \(\alpha_3\) are described in Figs.1(a), 1(b) and 1(c), respectively: Fig.1(a) presents the frequency characteristics of the transfer functions of signal tracking; Figs.1(b) and 1(c) present the frequency characteristics of the transfer functions of integral and derivative estimations, respectively.

Particularly, in comparing with ideal integral operators \(1/s\) and \(s\), not only the observers can obtain their estimations precisely, but also the high-frequency noise is rejected sufficiently.
Parameter $\varepsilon$ affects the low-pass frequency bandwidth: Decreasing the perturbation parameter $\varepsilon$, the low-pass frequency bandwidth is larger, the estimation precision becomes better, and relatively higher frequency noise can be reduced; on the other hand, increasing perturbation parameter $\varepsilon$, the low-pass frequency bandwidth is smaller, much noise can be reduced sufficiently (See the cases of $R = 1/\varepsilon = 3, 4, 5$ in Fig.1, respectively). Parameter $\alpha_3 \in (0, 1)$ affects the decay speed of frequency characteristic curves near the cut-off frequency (See the cases of $\alpha_3 = 0.3, 0.5, 1$ in Fig.1, respectively): Smaller $\alpha_3 \in (0, 1]$ can obtain more precise estimations; Larger $\alpha_3 \in (0, 1]$ can reduce much noise, however, a bit estimation delay happens.

### 3.2 The proposed rules of parameters selection

For nonlinear integral-uncertainty observer (5), there are some rules suggested on the parameters selection:

1) The parameters $\alpha_1, \alpha_2, \alpha_3, k_1, k_2$ and $k_3$ are satisfied with the conditions (6) and (7).

2) When the up-boundness of the integral, the uncertainty $\sigma(t)$ or the derivative $c_\sigma(t)$ of signal $\sigma(t)$ increase, i.e., $h_1, h_3$ or $L_a$ increase, $\delta_6$ in (80) will increase, and $L = \mu\sigma_3$ also increases, where $\mu$ is a constant defined in Theorem 5.2 in [37]. Perturbation parameter $\varepsilon \in (0, 1)$ should decrease to improve the estimation precisions.

3) When the magnitude of the noise increases, i.e., $L_d$ increases, in order to decrease $L_d^{a_2}$ in (80), $\alpha_2 \in (0, 1)$ should increase to improve the estimate precisions. In fact, for $L_4 \in (0, 1)$ and $\alpha_21, \alpha_22 \in (0, 1)$, if $\alpha_22 > \alpha_21$, then $L_d^{a_2} < L_d^{a_2}$ holds.

The parameters selection of linear observer (10) is easier than that of nonlinear observer (5): Parameters $k_1, k_2$ and $k_3$ decide the observer stability, and they should be satisfied with the conditions (11). The selection of $\varepsilon$ decides the estimate precision and robustness: when the up-boundness of the integral, the uncertainty $\sigma(t)$ or the derivative $c_\sigma(t)$ of signal $\sigma(t)$ increase, i.e., $h_1, h_3$ or $L_a$ increase, $\varepsilon \in (0, 1)$ should decrease to improve the estimation precisions; if much noise exists, $\varepsilon$ should increase, the low-pass frequency bandwidth is smaller, much noise can be reduced sufficiently.

To evaluate the theory of the proposed integral-uncertainty observers, they will be applied to an unmanned jet aircraft.

### IV. APPLICATION TO AN UNMANNED JET AIRCRAFT

#### 4.1 Modeling of jet aircraft

The proposed observers are applied to control an unmanned jet aircraft, which is shown in Fig.2, and the forces and torques of the aircraft are denoted in Fig.3.

1) Coordinates and frames

Let $\Xi_a = (E_x, E_y, E_z)$ denote the right handed inertial frame and $\Xi_b = (E_x^b, E_y^b, E_z^b)$ denote the frame attached to the aircraft’s fuselage whose origin is located at its center of gravity. $\Theta = (\psi, \theta, \phi)$ describes the aircraft orientation expressed in the classical yaw, pitch and roll angles. We use $c_\theta$ for $\cos \theta$ and $s_\theta$ for $\sin \theta$. $R$ is the transformation matrix representing the orientation of the aircraft from frame $\Xi_b$ to $\Xi_g$, i.e.,

$$R = 
\begin{bmatrix}
c_\psi c_\theta & c_\psi s_\theta s_\phi - s_\psi c_\phi & c_\psi s_\theta c_\phi + s_\psi s_\phi \\
c_\theta s_\psi & s_\theta s_\phi + c_\phi c_\theta & s_\theta c_\phi - c_\phi s_\phi \\
-s_\theta & s_\phi & c_\theta
\end{bmatrix}
$$

Let $\alpha$ and $\beta$ be the angle of attack of the fixed wing and the sideslip angle, respectively, thus

$$\alpha = \theta - \arctan^{-1}(\dot{z}_s/\dot{x}_s), \beta = \arcsin^{-1}(\dot{y}_s/v_s)$$

where, and the relative wind speed $(\dot{x}_s, \dot{y}_s, \dot{z}_s)$ is measured by the airspeed tube, and $v_s = \sqrt{\dot{x}_s^2 + \dot{y}_s^2 + \dot{z}_s^2}$.

Define $(x, y, z)$ and $(\dot{x}, \dot{y}, \dot{z})$ as the position of center of gravity and the velocity to frame $\Xi_g$, body force $F \in \mathbb{R}^3$ and torque $\tau \in \mathbb{R}^3$.

The total external force $F$ consists of the thrust $F_c$ generated by the jet engine, aerodynamic forces on the fixed wing $F_w$, aerodynamic forces on the fuselage $F_f$, the forces created by the verticals $F_v$, the forces created by the elevators $F_e$, and uncertainties and external disturbances $F_d$. These forces are expressed in body frame $\Xi_b$, and they are transformed by $R$ to be expressed in the inertial frame $\Xi_g$ as follows:

$$F = R(F_c + F_w + F_f + F_v + F_e + F_d)$$

The total moment $\tau$ consists of the moments created by the fixed wings $\tau_w$, the moments created by the verticals $\tau_v$, the moments created by the elevators $\tau_e$, and moments due to the uncertainties and external disturbances $\tau_d$:

$$\tau = \tau_w + \tau_v + \tau_e + \tau_d$$

![Fig. 2. An unmanned jet aircraft.](image)

![Fig. 3. Forces and torques illustration of the unmanned jet aircraft.](image)
The parameters of fuselage lift and drag are presented as follows:

\[
L_f = 0.5 \rho C_{1f} S_f (\dot{x}_a^2 + \dot{z}_a^2), \quad D_f = 0.5 \rho C_{df} S_f (\dot{x}_a^2 + \dot{z}_a^2),
\]

\[
C_{1f} = C_{1f0} \alpha, \quad C_{df} = C_{df0} + C_{df} \alpha
\]  

(26)

where \( L_f \) and \( D_f \) are the lift and drag forces generated by the fuselage, respectively; \( C_{1f} \) is the lift coefficient; \( C_{df} \) is the drag coefficient; \( C_{df0} \) is the constant in the coefficient of drag force. Then the forces on the fuselage \( F_f \) in body frame are written as

\[
F_f = \begin{bmatrix}
L_f \sin \alpha - D_f \cos \alpha & 0 \\
0 & -L_f \cos \alpha - D_f \sin \alpha
\end{bmatrix}
\]

(27)

4) The aerodynamic parameters of level stabilizer

The parameters of elevator lift and drag are presented as follows:

\[
L_e = 0.5 \rho C_{le} S_e (\dot{x}_a^2 + \dot{z}_a^2), \quad D_e = 0.5 \rho C_{de} S_e (\dot{x}_a^2 + \dot{z}_a^2)
\]

\[
C_{le} = C_{le0} \alpha + C_{la} \delta_e, \quad C_{de} = C_{de0} + C_{le}^2 \alpha e_e
\]

\[
e_e = 1.78(1 - 0.045 A_{w0}^{0.68}) - 0.46
\]

(28)

where \( S_e \) is the area of the level stabilizer, \( \delta_e \) is the bias angle of elevator, and \( C_{le0} \), \( C_{la} \) are the lift coefficient due to the bias angle \( \delta_e \), \( C_{le0} \) is the lift coefficient due to the angle of attack \( \alpha \) and the normal bias angle \( \delta_e \). \( A_e \) is the aspect ratio of the level stabilizer. \( e_e \) is the value of the Oswald’s efficiency factor. Then the force \( F_e \) on the level stabilizer in body frame is written as

\[
F_e = \begin{bmatrix}
L_e \sin (\alpha + \delta_e) - D_e \cos (\alpha + \delta_e) & 0 \\
0 & -L_e \cos (\alpha + \delta_e) - D_e \sin (\alpha + \delta_e)
\end{bmatrix}
\]

(29)

and the moment \( \tau_e \) created by the aerodynamic forces produced by the level stabilizer is

\[
\tau_e = \begin{bmatrix}
-l_e [L_e \cos (\alpha + \delta_e) + D_e \sin (\alpha + \delta_e)] & 0 \\
0 & 0
\end{bmatrix}
\]

(30)

5) The aerodynamic parameters of vertical stabilizer

The lift and drag forces generated by the vertical stabilizer, respectively

\[
L_v = 0.5 \rho C_{lv} S_v (\dot{x}_a^2 + \dot{z}_a^2), \quad D_v = 0.5 \rho C_{dv} S_v (\dot{x}_a^2 + \dot{z}_a^2)
\]

\[
C_{lv} = C_{lv0} \beta + C_{lv} \delta_v, \quad C_{dv} = C_{dv0} + C_{lv}^2 \alpha e_v
\]

\[
e_v = 1.78(1 - 0.045 A_{w0}^{0.68}) - 0.46
\]

(31)

where \( S_v \) is the area of the half wing, \( C_{lv0} \) is the lift coefficient due to the angle of attack \( \alpha \), \( \delta_v \) is the bias angle of rudder, and \( C_{lv0} \) is the lift coefficient due to the bias angle \( \delta_v \). \( A_e \) is the aspect ratio of the fixed wing. \( e_v \) is the value of the Oswald’s efficiency factor.

Then the aerodynamic force \( F_v \) on the vertical stabilizer in body frame can be written as
and the moment $\tau_v$ created by the aerodynamic forces produced by the vertical stabilizer is

$$
\tau_v = \begin{bmatrix}
    l_{v1} L_v \cos \alpha + D_v \sin \beta \\
    0
\end{bmatrix} \Delta \alpha
$$

(33)

6) Motion equation of the jet aircraft

The equations of motion written in terms of the centre of mass $C$ in the fixed axes of coordinate $(x, y, z)$ are then

$$
m\ddot{x} = c_\theta c_\psi F_c + F_{wx} + F_{fx} + F_{vx} + F_{ex} + \Delta x
$$

$$
m\ddot{y} = c_\theta s_\psi F_c + F_{wy} + F_{fy} + F_{vy} + F_{ey} + \Delta y
$$

$$
m\ddot{z} = -s_\theta F_c - mg + F_{wz} + F_{fz} + F_{vz} + F_{ez} + \Delta z
$$

(34)

$$
\begin{bmatrix}
    J_x \dddot{\psi} \\
    J_y \dddot{\theta} \\
    J_z \dddot{\phi}
\end{bmatrix} = \begin{bmatrix}
    \dot{\psi}(J_x - J_y) - k_\theta \dot{\psi} + \tau_\psi + \Delta \psi \\
    \dot{\theta}(J_y - J_z) - k_\psi \dot{\theta} + \tau_\theta + \Delta \theta \\
    \dot{\phi}(J_z - J_x) - k_\phi \dot{\phi} + \tau_\phi + \Delta \phi
\end{bmatrix}
$$

(35)

where $J_x$, $J_y$ and $J_z$ are the three-axis moment of inertias; $k_\psi$, $k_\theta$ and $k_\phi$ are the drag coefficients; $m$ is the mass of the aircraft; $g$ is the gravity acceleration; $\Delta x$, $\Delta y$ and $\Delta z$ are the bounded disturbances or uncertainties in the position dynamics; $\Delta \psi$, $\Delta \theta$ and $\Delta \phi$ are the bounded disturbances or uncertainties in the attitude dynamics; $\tau_\psi$, $\tau_\theta$ and $\tau_\phi$ are the control torques for yaw, pitch and roll dynamics, respectively, and they are selected as

$$
\begin{align*}
\tau_\psi &= 0.5l_{v1}(z^2 + s^2)\rho s \sigma_c \cos \theta \delta_v \\
\tau_\theta &= -0.5l_{v2}(z^2 + s^2)\rho s \sigma_c \cos \theta \delta_v \\
\tau_\phi &= l_v \sigma_c \cos \theta \delta_v
\end{align*}
$$

(36)

where $\delta_1 = -\delta_2 = \delta_{1,2}$, and

$$
\begin{bmatrix}
    F_{wx} + F_{fx} + F_{vx} + F_{ex} \\
    F_{wy} + F_{fy} + F_{vy} + F_{ey} \\
    F_{wz} + F_{fz} + F_{vz} + F_{ez}
\end{bmatrix} = R(F_w + F_f + F_v + F_e)
$$

(37)

The tracking control problem for the jet aircraft can be stated mathematically as: For the reference trajectory $(\dot{x}_d, \dot{y}_d, \dot{z}_d)$ in the inertial frame, find the control laws such that the following statements hold:

i) $x \rightarrow x_d, \dot{x} \rightarrow \dot{x}_d, y \rightarrow y_d, \dot{y} \rightarrow \dot{y}_d, z \rightarrow z_d, \dot{z} \rightarrow \dot{z}_d$ as $t \rightarrow \infty$; ii) The whole closed-loop system is stable.

Here, we are interested in adopting the presented observers to estimate the attitude angle $(\psi, \theta, \phi)$ and the uncertainties in the attitude dynamics from the measurement of the angular rate $(\dot{\psi}, \dot{\theta}, \dot{\phi})$ of IMU. Moreover, our previous augmented observer [41] will be used to estimate the velocity $(\dot{x}, \dot{y}, \dot{z})$ and the uncertainties in the position dynamics from the measurement of the position $(x, y, z)$. Finally, the controller will be designed to implement the tracking control for the jet aircraft.

4.2 Design of integral-uncertainty observers for the jet aircraft

For Eqs. (34) and (35), it is considered that $(\dot{x}, \dot{y}, \dot{z})$ and $(\psi, \theta, \phi)$ are not measured directly, $(k_x, k_y, k_z, \xi, \eta, \phi)$ and $(\Delta x, \Delta y, \Delta z, \Delta \psi, \Delta \theta, \Delta \phi)$ are bounded and unknown. Let $w_{1,1} = x, w_{1,2} = \dot{x}, w_{2,1} = y, w_{2,2} = \dot{y}, w_{3,1} = z, w_{3,2} = \dot{z}, w_{4,1} = \psi, w_{4,2} = \dot{\psi}, w_{5,1} = \theta, w_{5,2} = \dot{\theta}, w_{6,1} = \phi, w_{6,2} = \dot{\phi},$ and

$$
\Pi_1(t) = c_\theta c_\psi F_c + F_{wx} + F_{fx} + F_{vx} + F_{ex} \\
\Pi_2(t) = c_\theta s_\psi F_c + F_{wy} + F_{fy} + F_{vy} + F_{ey} \\
\Pi_3(t) = -s_\theta F_c - mg + F_{wz} + F_{fz} + F_{vz} + F_{ez} \\
\sigma_1(t) = \frac{\Delta x}{m}, \sigma_2(t) = \frac{\Delta y}{m}, \sigma_3(t) = \frac{\Delta z}{m} \\
\Pi_4(t) = \frac{\dot{\psi}(J_x - J_y) + \tau_\psi}{J_x}, \Pi_5(t) = \frac{\dot{\theta}(J_y - J_z) + \tau_\theta}{J_y}, \Pi_6(t) = \frac{-k_\theta \dot{\psi} + \Delta \psi}{J_x} \\
\sigma_4(t) = \frac{-k_\psi \dot{\theta} + \Delta \theta}{J_y}, \sigma_5(t) = \frac{-k_\phi \dot{\phi} + \Delta \phi}{J_z}
$$

(38)

Then, Eqs. (34) and (35) can be rewritten as

$$
\dot{w}_{i,1} = w_{i,2} \\
\dot{w}_{i,2} = \Pi_i(t) + \sigma_i(t)
$$

(40)

where $i = 1, \cdots, 6$. Based on Assumption 1, uncertainties $\sigma_i(t), i = 1, \cdots, 6,$ have the following dynamics:

$$
\dot{\sigma}_i(t) = c_\sigma \sigma_i(t)
$$

(41)

In fact, this assumption is satisfied with aircraft dynamic systems.

Let $w_{i,3} = \sigma_i(t), \dot{w}_{i,3} = \sigma_i(t) = c_\sigma(t), \dot{w}_{i,1} = \Pi_i(t)$. Eq. (40) can be augmented to

$$
\dot{w}_{i,1} = w_{i,2} \\
\dot{w}_{i,2} = w_{i,3} + \Pi_i(t) \\
\dot{w}_{i,3} = c_\sigma(t)
$$

(42)

1) Integral-uncertainty observers for attitude estimation

Based on Theorem 1 or Corollary 1, the following corollary gives the design of integral-uncertainty observers for the jet aircraft.

Corollary 2: The integral-uncertainty observers are designed for aircraft attitude Eq. (35) as follows:
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\[ \dot{x}_{1,1} = x_{1,1}, \]
\[ \dot{x}_{1,2} = x_{1,2} + \Pi(t), \]
\[ \dot{\varepsilon}_{1,1,3} = -k_{1,1} \varepsilon_{1,1,1}^{3} \alpha_{1} \text{sign}(x_{1,1}) - k_{1,2} \varepsilon_{1,1,2}^{2} \alpha_{2} \text{sign}(x_{1,2} - w_{1,2}) - k_{1,3} \varepsilon_{1,1,3}^{3} \alpha_{3} \text{sign}(x_{1,3}) \]  
(43)

or

\[ \dot{x}_{2,1,1} = x_{2,1,1}, \]
\[ \dot{x}_{2,1,2} = x_{2,1,2} + \Pi(t), \]
\[ \dot{\varepsilon}_{2,1,2,3} = -k_{1,1} \varepsilon_{2,1,1} - k_{1,2} (x_{2,1,2} - w_{1,2}) - k_{1,3} \varepsilon_{2,1,2,3}^{3} \alpha_{3} \text{sign}(x_{2,1,3}) \]  
(44)

where \( i = 4, 5, 6. \) We can estimate \( \Theta = (\psi, \theta, \phi) \) and the uncertainties, i.e., for \( t \geq t_s, \)

\[ |x_{1,1} - \psi| \leq L_{x_{1,1}}^{\alpha_{1} - 1}, |x_{1,2} - \psi| \leq L_{x_{1,2}}^{\alpha_{2} - 2}, \]
\[ |x_{1,3} - \Theta| \leq L_{x_{1,3}}^{\alpha_{3} - 3}, |x_{1,5} - \Theta| \leq L_{x_{1,5}}^{\alpha_{5} - 1}, \]
\[ |x_{1,6} - \Theta| \leq L_{x_{1,6}}^{\alpha_{6} - 2}, \]
\[ |x_{1,6,3} - \Theta| \leq L_{x_{1,6,3}}^{\alpha_{6,3} - 3} \]  
(45)

2) Augmented observers in [41] for velocity estimate in position dynamics

Based on [41], the following corollary gives the observers to estimate \((\dot{x}, \dot{y}, \dot{z})\) and uncertainties in the position dynamics in the jet aircraft.

Corollary 3: The augmented observers in [41] are designed for aircraft position Eq. (34) as follows:

\[ \dot{x}_{i,1} = x_{i,1} - k_{i,1} \varepsilon_{i}^{3} \text{sign}(x_{i,1} - w_{i,1}), \]
\[ \dot{x}_{i,2} = x_{i,2} - k_{i,2} \varepsilon_{i}^{2} \text{sign}(x_{i,1} - w_{i,1}) + \Pi(t), \]
\[ \dot{x}_{i,3} = k_{i,3} \varepsilon_{i} x_{i,1} - w_{i,1} \alpha_{3} \text{sign}(x_{i,1} - w_{i,1}) \]  
(46)

where \( i = 1, 2, 3. \) From \( X = (x, y, z), \) we can estimate \( \dot{X} = (\dot{x}, \dot{y}, \dot{z}) \) and \( \sigma(t) (i = 1, 2, 3) \) by the above augmented observers, i.e., for \( t \geq t_s, \)

\[ |x_{1,1} - x| \leq L_{x_{1,1}}^{x_{1,1}}, |x_{1,2} - \dot{x}| \leq L_{x_{1,2}}^{x_{1,2}}, \]
\[ |x_{1,3} - \Theta| \leq L_{x_{1,3}}^{\alpha_{3}}, |x_{2,2} - \dot{y}| \leq L_{x_{2,2}}^{\alpha_{2}}, \]
\[ |x_{3,1} - z| \leq L_{x_{3,1}}^{\alpha_{1}}, |x_{3,2} - \dot{z}| \leq L_{x_{3,2}}^{\alpha_{2}}, \]
\[ |x_{3,3} - \Theta| \leq L_{x_{3,3}}^{\alpha_{3}} \]  
(47)

If the attitude angle measurement is made by three encoders, the augmented observers in [41] can be selected to estimate the angular rates and the uncertainties in the attitude dynamics.

4.3 Controller design

In this section, a control law is derived for the purpose of attitude stabilization and trajectory tracking of the jet aircraft. The unknown states and uncertainties are reconstructed by the presented observers.

Suppose the reference trajectory and its finite order derivatives are bounded, and can be directly generated.

For the reference trajectory \( X_d = (x_d, y_d, z_d), \) let \( e_1 = x - x_d, e_2 = \dot{x} - \dot{x}, e_3 = y - y_d, e_4 = \dot{y} - \dot{y}_d, e_5 = z - z_d, \) and \( e_6 = \dot{z} - \dot{z}_d, \) then the system error for position dynamics (34) is

\[ \dot{e}_p = m^{-1}(u_p + \Xi_p + \Omega_p + \Delta_p) \]  
(48)

where

\[ u_p = \begin{bmatrix} u_{px} \\ u_{py} \\ u_{pz} \end{bmatrix}, \]
\[ \Xi_p = \begin{bmatrix} \Delta_x \\ \Delta_y \\ \Delta_z \end{bmatrix}, \]
\[ \Omega_p = \begin{bmatrix} F_{ux} + F_{fx} + F_{ux} + F_{ex} \\ F_{uy} + F_{fy} + F_{uy} + F_{ey} \\ F_{uz} + F_{fz} + F_{uz} + F_{ez} \end{bmatrix} \]  
(49)

For the desired attitude angle \( \Theta_d = (\psi_d, \theta_d, \phi_d), \) let \( e_7 = \psi - \psi_d, e_8 = \psi - \psi_d, e_9 = \theta - \theta_d, e_{10} = \theta - \theta_d, e_{11} = \phi - \phi_d, e_{12} = \phi - \phi_d, \) then the system error for attitude dynamics (35) is

\[ \dot{e}_a = J^{-1}(u_a + \Xi_a + \Omega_a + \Delta_a) \]  
(50)

where

\[ e_a = \begin{bmatrix} e_7 \\ e_9 \\ e_{11} \end{bmatrix}, \]
\[ \Xi_a = \begin{bmatrix} -J_x \dot{\psi}_d \\ -J_y \dot{\theta}_d \\ -J_z \dot{\phi}_d \end{bmatrix}, \]
\[ \Omega_a = \begin{bmatrix} \Delta_\psi - k_{\phi} \dot{\psi} \\ \Delta_\theta - k_{\phi} \dot{\phi} \\ \Delta_\phi - k_{\phi} \dot{\phi} \end{bmatrix}, \]
\[ u_a = \begin{bmatrix} \tau_\psi \\ \tau_\theta \\ \tau_\phi \end{bmatrix}, \]
\[ J = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix} \]  
(51)

1) Controller design for position dynamics

Theorem 2: For position dynamics (34), to track the reference trajectory \( X_d = (x_d, y_d, z_d), \) if the controller is designed as

\[ u_p = -\Xi_p - \Omega_p - \Delta_p - m(k_{p1} \dot{e}_p + k_{p2} \dot{e}_p) \]  
(52)

where \( \hat{e}_1 = \hat{x} - x_d, \hat{e}_2 = \hat{x} - \dot{x}_d, \hat{e}_3 = \hat{y} - y_d, \hat{e}_4 = \hat{y} - \dot{y}_d, \hat{e}_5 = \hat{z} - z_d, \hat{e}_6 = \hat{z} - \dot{z}_d, \) \( k_{p1}, k_{p2} > 0, \) and

\[ \hat{e}_p = \begin{bmatrix} \hat{e}_1 \\ \hat{e}_3 \\ \hat{e}_5 \end{bmatrix}, \]
\[ \hat{e}_p = \begin{bmatrix} \hat{e}_2 \\ \hat{e}_4 \\ \hat{e}_6 \end{bmatrix} \]  
(53)
the control system is described in Fig.5, then the position error dynamic system (48) rendering by controller (52) will converge asymptotically to the origin, i.e., the tracking errors $e_p \to 0$ and $\dot{e}_p \to 0$ as $t \to \infty$.

**Proof:** In the light of Corollary 3, for $t \geq ts = \varepsilon \Gamma$, the observation signals $\left\| \hat{X} - X \right\| \leq L e^{3\gamma t}$, $\left\| \hat{X} - X \right\| \leq L e^{3\gamma t-1}$, $\left\| \hat{\Delta}_p - \Delta_p \right\| \leq L e^{3\gamma t-2}$. Considering controller (52), the closed-loop error system for the attitude dynamics is

$$\dot{e}_p = -k_{p1} e_p - k_{p2} \dot{e}_p - k_{p1} (\hat{X} - X) - k_{p2} (\hat{\Delta}_p - \Delta_p) \tag{54}$$

For $t \geq \varepsilon \Gamma$ and sufficiently small $\varepsilon$, selecting the Lyapunov function be $V_1 = k_{p1}^2 \dot{e}_p^2 + k_{p2}^2 \dot{e}_p^2$, we can obtain that $e_p \to 0$ and $\dot{e}_p \to 0$ as $t \to \infty$. This concludes the proof. $\blacksquare$

From (49), we know that

$$F_c = \left\| u_p \right\|_2 = \sqrt{u_{px}^2 + u_{py}^2 + u_{pz}^2} \tag{55}$$

The desired pitch and yaw angles are, respectively,

$$\theta_d = -\arcsin\left(\frac{u_{pz}}{F_c}\right), \psi_d = \left\{ \begin{array}{ll} \arctan\left(\frac{u_{wy}}{u_{wx}}\right), & u_{wx} \neq 0 \\ \pi/2, & u_{px} = 0 \quad \text{and} \quad \dot{x}_d > 0 \\ 3\pi/2, & u_{px} = 0 \quad \text{and} \quad \dot{x}_d < 0 \\ \end{array} \right. \tag{56}$$

and the desired roll angle $\phi_d$ can be set to a reference value.  

2) **Controller design for attitude dynamics**

**Theorem 3:** For attitude dynamics (35), to track the desired attitude $\Theta_d = (\psi_d, \theta_d, \phi_d)$, if the controller is designed as

$$u_a = -\sigma_a - \sigma - J[k_{a1} \hat{e}_a + k_{a2} e_a + L_c \text{sign}(e_a)] \tag{57}$$

where, $k_{a1}, k_{a2}, L_c > 0; \hat{e}_7 = \hat{\psi} - \psi_d, \hat{e}_9 = \hat{\theta} - \theta_d, \hat{e}_{11} = \hat{\phi} - \phi_d; e_a = [\hat{e}_7 \hat{e}_9 \hat{e}_{11}]^T; \hat{e}_a = [\hat{e}_8 \hat{e}_{10} \hat{e}_{12}]^T$; and the control system is described in Fig.5, then the attitude error dynamic system (50) rendering by controller (57) will converge asymptotically to the origin, i.e., the tracking errors $e_a \to 0$ and $\dot{e}_a \to 0$ as $t \to \infty$.

**Proof:** In the light of Corollary 2, for $t \geq ts = \varepsilon \Gamma$, the observation signals $\left\| \hat{\Theta} - \Theta \right\| \leq L e^{\gamma t-1}$ and $\left\| \hat{\Delta}_a - \Delta_a \right\| \leq L e^{\gamma t-2}$. Considering controller (57), the closed-loop error system for the attitude dynamics is

$$\dot{e}_a = -k_{a1} e_a - k_{a2} \dot{e}_a - L_c \text{sign}(\dot{e}_a) - k_{a1} (\hat{\Theta} - \Theta) - (\hat{\Delta}_a - \Delta_a) \tag{58}$$

For $t \geq \varepsilon \Gamma$ and sufficiently small $\varepsilon$, selecting the Lyapunov function be $V_2 = k_{a1}^2 e_a^2 + L_c^2 \hat{e}_a^2 + k_{a1}^2 (\hat{\Theta} - \Theta)^2$, we can obtain that $e_a \to 0$ and $\dot{e}_a \to 0$ as $t \to \infty$. This concludes the proof. $\blacksquare$

V. **COMPUTATIONAL ANALYSIS AND SIMULATIONS**

In this section, we use a simulation on the jet aircraft to illustrate the effectiveness of the proposed estimate and control methods. The goal is to force the aircraft to track a reference trajectory. Here, the aircraft tracks a given trajectory $(x_d, y_d, z_d)$ without the information of $(\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}, \dot{d}_1, \dot{d}_2, \dot{d}_3, \dot{d}_4, \dot{d}_5, \dot{d}_6)$. The estimation performances of the presented integral-uncertainty observers are compared with Kalman filter [14].

In this simulation, without the information of velocity, attitude angle and uncertainties, the aircraft is controlled to track the reference trajectory. The position is obtained from the GPS receiver, and the altitude information is from the altimeter. The angular velocity $(\dot{\psi}, \dot{\theta}, \dot{\phi})$ is measured by the IMU. The integral-uncertainty observer (43) (or (44)) is used to estimate the attitude angle $(\psi, \theta, \phi)$ and uncertainties $(d_1, d_2, d_3)$ in the attitude dynamics from the angular velocity $(\dot{\psi}, \dot{\theta}, \dot{\phi})$. The augmented observer (46) is adopted to estimate the velocity $(\dot{x}, \dot{y}, \dot{z})$ and uncertainties $(d_1, d_2, d_3)$ in the position dynamics from the position $(x, y, z)$. Controllers (52) and (57) are presented to stabilize the flight dynamics. Reference trajectory: The given waypoints generate the reference trajectory $(x_d, y_d, z_d)$ shown in Fig.6.

The initial states of aircraft are: $(x(0), \dot{x}(0), y(0), \dot{y}(0), z(0), \dot{z}(0), \psi(0), \dot{\psi}(0), \theta(0), \dot{\theta}(0), \phi(0), \dot{\phi}(0)) = (x_0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$; the initial states of the observers are: $(x_{11}(0), x_{12}(0), x_{13}(0), x_{21}(0), x_{22}(0), x_{23}(0), x_{31}(0), x_{32}(0), x_{33}(0), x_{41}(0), x_{42}(0), x_{43}(0), x_{51}(0), x_{52}(0), x_{53}(0), x_{61}(0), x_{62}(0), x_{63}(0)) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. Let the uncertainties be: $\Delta_y = 0.5 \sin(t)$, $\Delta_y = 0.5 \sin(t)$, $\Delta_y = 0.5 \sin(t)$, $\Delta_y = 0.5 \sin(t)$, $\Delta_y = 0.5 \sin(t)$. The position measurement outputs are $y_{op} = u_{i1} + n_i$, where $i = 1, 2, 3$, and $u_{i1} = x$, $u_{21} = y$, $u_{31} = z$; the attitude measurement outputs are $y_{op} = u_{i2} + n_i$, where $i = 4, 5, 6$, and $u_{42} = \psi$, $u_{52} = \theta$, $u_{62} = \phi$; $n_i$ (where $i = 1, \ldots, 6$) are the disturbances.

The disturbances $n_i$ (where $i = 1, \ldots, 6$) include two types of noises: Random number with Mean=0, Variance=0.001, Initial speed=0, and Sample time=0; Pulses with Amplitude=0.001, Period=1s, Pulse width=1, and Phase delay=0.

The aerodynamic parameters of the unmanned jet aircraft were obtained through the wind tunnel tests. The parameters are given as follows:

Jet aircraft: $m = 35.6 kg$, $g = 9.8 ms^{2}$, wingspan=1.92m, wing area=1.58m², Fuselage length=2.35m, and $C_{L10} = C_{L20} = 0.3145$, $C_{L11} = C_{L21} = 0.5122$, $C_{L14} = C_{L24} = 0.1183$, $C_{L41} = C_{L52} = 0.1634$, $C_{D10} = C_{D20} = 0.0083$.  

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Fig. 6. Aircraft control based on nonlinear integral-uncertainty observer. (a) Estimation in x coordinate. (b) Estimation in y coordinate. (c) Estimation in z coordinate. (d) Estimation in yaw dynamics.

Fig. 6. (continued): Aircraft control based on nonlinear integral-uncertainty observer. (e) Estimation in pitch dynamics. (f) Estimation in roll dynamics. (g) Thrust force. (h) Control angles of rudder.
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Fig. 7. Aircraft control based on linear integral-uncertainty observer. (a) Estimation in x coordinate. (b) Estimation in y coordinate. (c) Estimation in z coordinate. (d) Estimation in yaw dynamics.

Fig. 7. (continued): Aircraft control based on linear integral-uncertainty observer. (e) Estimation in pitch dynamics. (f) Estimation in roll dynamics. (g) Thrust force. (h) Control angles of rudder.
$C_{v1} = C_{v2} = 0.0057$, $C_{L30} = C_{L40} = 0.3029$, $C_{L50} = 0.4753$, $C_{4\psi} = C_{4\phi} = 0.1067$, $C_{L43} = C_{L44} = 0.1446$, $C_{D30} = C_{D40} = 0.0062$, $C_{3\psi} = C_{4\psi} = 0.0034$; $C_{L54} = 0.0812$, $C_{D50} = 0.0003$, $C_{D55} = 0.0094$, $J_z = 18Nm$, $J_y = 18Nm$, $J_z = 34Nm$, $k_\psi = k_\theta = k_\phi = 0.52Ns/rad$.

Augmented observer: $k_{i,1} = 6$, $k_{i,2} = 11$, $k_{i,3} = 6$, $\varepsilon_1 = 0.2$, $\alpha_{i,1} = 0.8$, $i = 1, 2, 3$.

Nonlinear integral-uncertainty observer: $k_{i,1} = 0.1$, $k_{i,2} = 2$, $k_{i,3} = 1$, $\varepsilon_{1,2} = 1/3$, $\alpha_{i,3} = 0.3$, $i = 4, 5, 6$.

Linear integral-uncertainty observer: $k_{2,i,1} = 0.1$, $k_{2,i,2} = 2$, $k_{2,i,3} = 1$, $\varepsilon_{2,2} = 1/3$, $i = 4, 5, 6$.

Controllers: $k_{p1} = 3.2$, $k_{p2} = 1.6$, $k_{n1} = 1.5$, $k_{n2} = 1.0$, $L_c = 0.15$.

The sampling time is $1\text{ms}$. The data of flying test were presented in Figs.6 and 7. Fig.6 shows the aircraft control based on nonlinear integral-uncertainty observer. Fig.6(a) describes the estimate of $x$, $dx/dt$ and $d_k(t)$; Fig.6(b) describes the estimate of $y$, $dy/dt$ and $d_1(t)$; Fig.6(c) presents the estimate of $z$, $dz/dt$ and $d_2(t)$; Fig.6(d) presents the estimate of the yaw angle $\psi$, yaw rate $d\psi/dt$ and $d_3(t)$; Fig.6(e) presents the estimate of the pitch angle $\theta$, pitch rate $d\theta/dt$ and $d_4(t)$; Fig.6(f) presents the estimate of the roll angle $\phi$, roll rate $d\phi/dt$ and $d_5(t)$; Figs.6(g) and 6(h) present the controller curves of $F_c, \varepsilon_1, \varepsilon_2$ and $\delta_{1,2}$, respectively. Fig.7 presents the aircraft control based on linear integral-uncertainty observer. In the simulation figures, the green lines denote measured values, the blue lines are the estimated values, and the red dot lines are the desired lines. In the Figs. 6 and 7, high-frequency noise and the bounded uncertainties exist. Position tracking, the estimations of velocity, attitude angle and the uncertainties are described, respectively. Black lines are the real values, blue lines are the estimated values, and the red dot lines are the desired values.

In the simulation above, although high-frequency stochastic noises exist in the measurement signals, the uncertainties exist in the aircraft dynamics, and only the angular velocities are considered in the IMU outputs, the attitude estimations by the presented integral-uncertainty observer, the velocity-uncertainty estimations by the augmented observer and the control results by the designed controller have satisfying qualities. The stochastic noises are rejected sufficiently by the observers.

Figs. 6 and 7 also describe the estimate comparisons of attitude estimations by the integral-uncertainty observers and Kalman filter. From Figs. 6 (d), (e), (f) and Figs. 7 (d), (e), (f), the obvious estimation drifts of attitude angles exist by Kalman filter. Furthermore, from Figs.6 and 7, no drift phenomena happen for the integral-uncertainty observers. In the tracking outputs, not only the dynamical performances are fast, but also the tracking precisions are accurate.

We found that, for the simulation computational environment, the nonlinear and linear integral-uncertainty observers can obtain almost the same estimate performances.

VI. EXPERIMENTAL VERIFICATION OF THE PROPOSED OBSERVERS ON THE UNMANNED JET AIRCRAFT

In this section, in order to verify the effectiveness of the proposed observation algorithms and controller proposed in the previous sections, we present a real-time experiment on the flight of an unmanned jet aircraft. The jet aircraft prototype has been designed and shown in Fig.2. In the aircraft, a jet engine (JetCat P200-SX) is adopted to provide the thrust, and its starter includes: Jet-tronic ECU (fuel control electronics); electronic valve; electronic starting gas valve; electronic fuel valve; fuel tubing, tubing connector set, filters, and cable set; 2 cell, 3300mA LiPoly battery pack; starting gas tank. The thrust is $220 \text{N}$ ($52 \text{lbs}$) @ $112000 \text{RPM}$, and RPM range: $3300 \sim 112000 \text{RPM}$. An IMU (XsensMTI AHRS) is used to measure the angular velocity, whose sampling frequency is $10 \text{kHz}$. Arduino Mega 2560 (sampling frequency: $16\text{MHz}$) → (CPU clock rate (or speed)): $16\text{MHz}$). Gumstix microcomputer and an Arduino Mega 2560 are taken as the driven boards, which have multiple PWM output channels (See Fig.8). The input voltage is $7 \sim 12\text{V}$. The control update time is $5\text{ms}$. The rudders (FUTABA S3001) are adopted to control the bias angles of elevators, verticals and ailerons, respectively. GPS MT3329 (10Hz update rate) is selected as the GPS receiver. A VTI Technologies SCP1000 altimeter with $10 \text{cm}$ resolution is utilized for above the sea level altitude measurements at higher altitudes, and its sampling frequency is $9 \text{Hz}$. A SF02-F Laser altimeter is used for altitude measurements at lower altitudes with $40 \text{m}$ range (sampling frequency: $12 \text{Hz}$), and it is fine for landings. A kpiot 32 digital air speed sensor is utilized to obtain the relative wind speed (sampling frequency: $192 \text{kHz}$). A 4239-01 AOA sensor is used to measure the angle of attack (sampling frequency: $100 \text{kHz}$). The flight control system implementation on the hardware is shown in Fig.8.

The aerodynamic parameters of the unmanned jet aircraft were obtained through the wind tunnel tests. The parameters have been given in the section of simulation. Here, the discrete forms of the observers use the 4th-order Runge–Kutta Method. The presented method is suitable to the several sampling times (i.e., $0.001$ in the simulation and $0.005$ in the experiment, respectively). The discrete-form analysis of a similar continuous differentiator has been described in [12].

In this experiment, without the information of velocity, attitude angle and uncertainties, the unmanned jet aircraft
controlled to track the reference trajectory. The position is obtained from the GPS receiver, and the altitude information is from the altimeter. The angular velocity \((\psi, \theta, \phi)\) is measured by the IMU.

Integral-uncertainty observer (43) (or (44)) is used to estimate the attitude angle \((\psi, \theta, \phi)\) and uncertainties in the attitude dynamics from the angular velocity \((\dot{\psi}, \dot{\theta}, \dot{\phi})\). The augmented observer (46) is adopted to estimate the velocity \((\dot{x}, \dot{y}, \dot{z})\) and uncertainties in the position dynamics from the position \((x, y, z)\). Controllers (52) and (57) are designed to control the jet aircraft to track the reference trajectory.

The flying test scenario for the jet aircraft is shown in Fig.9, and the data of flying test are presented in Figs.10, 11, 12 and 13. In Fig.10, Fig.10(a) shows the trajectory tracking for the unmanned jet aircraft. Fig.10(b) describes the position estimate in x-direction; Fig.10(c) describes the position estimate in z-direction. In Fig.11, Fig.11(a) describes the velocity estimate in x-direction; Fig.11(b) describes the velocity estimate in y-direction; Fig.11(c) describes the velocity estimate in z-direction. The attitude angles estimate by the integral-uncertainty observers are shown in Fig.12, where Fig.12(a) shows the yaw angle estimate; Fig.12(b) shows the pitch angle estimate; Fig.12(c) shows the roll angle estimate. Fig. 13 presents the controller curves of the thrust force \(F_c\) and the duty ratios of rudders control \(\delta_1, \delta_2, \text{ and } \delta_{1,2}\), respectively. The pulse control of the rudder is shown in Table 2 (The time period is 20ms).

<table>
<thead>
<tr>
<th>Duty ratio</th>
<th>Pulse width</th>
<th>output angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.025</td>
<td>0.5(ms)</td>
<td>-90°</td>
</tr>
<tr>
<td>0.05</td>
<td>1(ms)</td>
<td>-45°</td>
</tr>
<tr>
<td>0.075</td>
<td>1.5(ms)</td>
<td>0°</td>
</tr>
<tr>
<td>0.1</td>
<td>2(ms)</td>
<td>45°</td>
</tr>
<tr>
<td>0.125</td>
<td>2.5(ms)</td>
<td>90°</td>
</tr>
</tbody>
</table>

Table 2 Pulse control of rudder

In the experiment above, although the stochastic noise exists in the GPS signal, the uncertainties exist in the aircraft dynamics, and only the angular velocity is considered in the IMU output, the attitude estimations by the presented integral-uncertainty observer, the velocity-uncertainty estimations by our previous augmented observer [41] and the control results by the designed controller have satisfying qualities. The estimation results of attitude angles are satisfying and are better than those of IMU (XsensMTI AHRS) (See Fig.12).

Fig. 9. Flight test for jet aircraft.

Fig. 10. Position navigation. (a) Position trajectory. (b) Position in x-direction. (c) Position in y-direction. (d) Position in z-direction.
Furthermore, no drift phenomena happen, and trajectory taking are accurate. From Fig.12, we also found that, the linear integral-uncertainty observer ($\alpha_3 = 1$) can obtain the better estimate results than those of nonlinear one ($\alpha_3 = 0.5, 0.8$). It is more easier to regulate the parameters of the linear integral-uncertainty observer. The linear observer is more suitable to the existing hardware computational environment in the aircraft system, and it exhibited more satisfying estimate performances.

Fig. 11. Velocity estimate by augmented observer (46). (a) Velocity in x-direction. (b) Velocity in y-direction. (c) Velocity in z-direction.

Fig. 12. Attitude angles estimate by integral-uncertainty observers (43) and (44). (a) Yaw angle estimate. (b) Pitch angle estimate. (c) Roll angle estimate.

VII. CONCLUSIONS

In this paper, two types of integral-uncertainty observers have been developed, which all can estimate the integral state of measurement output signal and the uncertainty in system, synchronously. The effectiveness of the proposed integral-uncertainty observers was shown by the simulations and experiments on an unmanned jet aircraft: i) they succeeded in estimating the attitude angle and uncertainties in attitude dynamics from the angular velocity measurement; ii) The flying test also verified the validity of our previous augmented observer, which estimated the velocity and uncertainties in
parameters of the presented methods. Those of the nonlinear one. Our future work is to optimize the ease of parameters regulation, the proposed linear integral-uncertainty observers include the synchronous estimation of attitude angles and position from the GPS receiver signals. iii) The satisfying estimate precision and the strong robustness of the integral-uncertainty observers make the selected control law very simple. The merits of the presented integral-uncertainty observers include the synchronous estimation of attitude angles and uncertainties, ease of parameters selection, sufficient stochastic noise rejection and almost no drift phenomenon. Although high-frequency stochastic noises and measurement errors exist, the attitude angle and uncertainty estimations by the presented observers and the tracking results by the designed controller have satisfying qualities. Moreover, due to the limitation of hardware computational environment and the requirement of ease of parameters regulation, the proposed linear integral-uncertainty observer can obtain the better estimate results than those of the nonlinear one. Our future work is to optimize the parameters of the presented methods.

ACKNOWLEDGEMENT
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APPENDIX A

The related concepts of finite-time stability of autonomous systems are presented here.

Definition 1 in [37]: Let us consider a time-invariant system in the form of

\[ \dot{x} = f(x), f(0) = 0, x \in \mathbb{R}^n, \]  

(59)

where \( f : D \rightarrow \mathbb{R}^n \) is continuous on open neighborhood \( D \subseteq \mathbb{R}^n \) of the origin. The origin is said to be a finite-time-stable equilibrium of the above system if there exists an open neighborhood \( N \subseteq D \) of the origin and a function \( T_f : N \setminus \{0\} \rightarrow (0, \infty) \), called the settling-time function, such that the following statements hold: (i) Finite-time-convergence: For every \( x \in N \setminus \{0\} \), \( \psi^x \) is the flow starting from \( x \) and defined on \([0, T_f(x)) \), \( \psi^x(t) \in N \setminus \{0\} \) for all \( t \in [0, T_f(x)) \), and \( \lim_{t \to T_f(x)} \psi^x(t) = 0 \).

(ii) Lyapunov stability: For every open neighborhood \( U_\varepsilon \) of 0 there exists an open subset \( U_\delta \) of \( N \) containing 0 such that, for every \( x \in U_\delta \setminus \{0\} \), \( \psi^x(t) \in U_\varepsilon \) for all \( t \in [0, T_f(x)) \).

The origin is said to be a globally finite-time-stable equilibrium if it is a finite-time-stable equilibrium with \( D = N = \mathbb{R}^n \). Then the system is said to be finite-time-convergent with respect to the origin.

Assumption in [11]: For a system depicted by Eq. (59), there exists \( \rho_i \in (0, 1], i = 1, \cdots, n \), and a nonnegative constant \( \rho \) such that

\[ |f_j(\bar{z}_1, \cdots, \bar{z}_n) - f_j(\bar{x}_1, \cdots, \bar{x}_n)| \leq \bar{a} \sum_{i=1}^n |\bar{z}_i - \bar{x}_i|^{\rho_i} \]  

(60)

where \( \bar{z}_i, \bar{x}_i \in \mathbb{R}, i = 1, \cdots, n, j = 1, \cdots, n \).

There are many nonlinear functions capable of satisfying this assumption. For example, one such function is \( x^{\rho_i} \) since

\[ |x^{\rho_i} - \bar{x}^{\rho_i}| \leq 2|1 - \rho_i| |x - \bar{x}|^{\rho_i}, \rho_i \in (0, 1]. \]

Theorem 4.2 in [37]: Suppose there exists a continuous function \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) such that the following conditions holds:

(i) \( V \) is positive definite;
(ii) There exist real numbers \( c > 0 \) and \( \beta_c \in (0, 1) \) such that

\[ V(x) + c(V(x))^{\beta_c} \leq 0 \]  

(61)

Then (59) is globally finite-time stable. Moreover, if \( N \) is as in Definition 1 and \( T_f \) is the setting time function, then

\[ T_f(x) \leq \frac{1}{c(1 - \beta_c)} V(x)^{1 - \beta_c} \]  

(62)

Proposition 8.1 in [38]: Let \( k_1, \cdots, k_n > 0 \) be such that \( s^n + k_n s^{n-1} + \cdots + k_2 s + k_1 \) is Hurwitz, and consider the system

\[ \begin{align*}
\dot{x}_i &= x_{i+1}; i = 1, \cdots, n-1, \\
\dot{x}_n &= -\sum_{i=1}^n k_i |x_i|^{\alpha_i} sign(x_i)
\end{align*} \]

(63)

there exists \( \xi_c \in (0, 1) \) such that, for every \( \alpha \in (1 - \xi_c, 1) \), the origin is globally finite-time-stable equilibrium for Eq. (63) where \( \alpha_1, \cdots, \alpha_n \) satisfy

\[ \alpha_{i-1} = \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}, i = 2, \cdots, n \]  

(64)

with \( \alpha_{n+1} = 1 \) and \( \alpha_n = \alpha \).
Theorem 5.2 in [37]: Consider the perturbed system of (59) following:

\[ \dot{x} = f(x) + g(t, x(t)), x(0) = x_0 \]  

(65)
suppose there exists a function \( V : D \to R \) such that \( V \) is positive definite and Lipschitz continuous on \( D \), and satisfies (61), where \( \nu \subseteq D \) is an open neighborhood of the origin, \( c > 0 \) and \( \beta_0 \in (0, \frac{1}{2}) \). Then there exist \( \delta_0 > 0, \mu > 0, \Gamma > 0 \), and an open neighborhood \( U \) of origin such that, for every continuous function \( g : R_+ \times D \to R^n \) with

\[ \delta = \sup_{R_+ \times D} \|g(t, x(t))\| \leq \delta_0 \]  

(66)
every right maximally defined solution \( x \) of (65) with \( x(0) \in U \) is defined on \( R_+ \) and satisfies \( x(t) \in U \) for all \( t \in R_+ \) and

\[ \|x(t)\| \leq \mu \delta \gamma, t \geq \Gamma \]  

(67)
where \( \gamma = (1 - \beta_\varepsilon)/\beta_\varepsilon > 1 \).

APPENDIX B

Proof of Theorem 1: The system error between Eqs. (5) and (4) is given by:

\[ \dot{e}_1 = e_2; \dot{e}_2 = e_3; \]
\[ \dot{\varepsilon}^3 e_3 = -k_1 |\varepsilon e_1 + \varepsilon w_1(t)|^{\alpha_1} \text{sign} (e_1 + w_1(t)) \]
\[ -\frac{k_2}{\varepsilon z_{\alpha_2}} |\varepsilon^2 e_2 - \varepsilon^2 d(t)|^{\alpha_2} \text{sign} (e_2 - d(t)) \]
\[ -k_3 |\varepsilon^3 e_3 + \varepsilon^3 w_3(t)|^{\alpha_3} \text{sign} (e_3 + w_3(t)) \]
\[ -\varepsilon^4 e_\sigma(t) \]  

(68)

Eq. (68) can be rewritten as:

\[ \frac{d\varepsilon e_1}{dt/\varepsilon} = \varepsilon^2 e_2; \frac{d\varepsilon e_2}{dt/\varepsilon} = \varepsilon^3 e_3; \]
\[ \frac{d\varepsilon^3 e_3}{dt/\varepsilon} = -k_1 |\varepsilon e_1 + \varepsilon w_1(t)|^{\alpha_1} \text{sign} (e_1 + w_1(t)) \]
\[ -\frac{k_2}{\varepsilon z_{\alpha_2}} |\varepsilon^2 e_2 - \varepsilon^2 d(t)|^{\alpha_2} \text{sign} (e_2 - d(t)) \]
\[ -k_3 |\varepsilon^3 e_3 + \varepsilon^3 w_3(t)|^{\alpha_3} \text{sign} (e_3 + w_3(t)) \]
\[ -\varepsilon^4 e_\sigma(t) \]  

(69)

Let

\[ \tau = \frac{t}{\varepsilon}; z_i (\tau) = e^i e_i(t), \bar{w}_i (\tau) = e^i w_i(t), \]
\[ i = 1, 2, 3; z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T; \]
\[ \bar{w}_4 (\tau) = e^4 e_\sigma(t), \bar{d}(\tau) = e^2 d(t) \]  

(70)
we obtain \( z = \Xi(\varepsilon)e \). Eq. (69) can be written as

\[ \frac{dz_1}{d\tau} = z_2; \frac{dz_2}{d\tau} = z_3; \]
\[ \frac{dz_3}{d\tau} = -k_1 |z_1 + \bar{w}_1(\tau)|^{\alpha_1} \text{sign} (z_1 + \bar{w}_1(\tau)) \]
\[ -\frac{k_2}{\varepsilon z_{\alpha_2}} |z_2 - \bar{d}(\tau)|^{\alpha_2} \text{sign} (z_2 - \bar{d}(\tau)) \]
\[ -k_3 |z_3 + \bar{w}_3(\tau)|^{\alpha_3} \text{sign} (z_3 + \bar{w}_3(\tau)) \]
\[ -\bar{w}_4(\tau) \]  

(71)
Furthermore, Eq. (71) can be rewritten as

\[ \frac{dz_1}{d\tau} = z_2; \frac{dz_2}{d\tau} = z_3; \]
\[ \frac{dz_3}{d\tau} = -k_1 |z_1|^{\alpha_1} \text{sign} (z_1) - \frac{k_2}{\varepsilon z_{\alpha_2}} |z_2|^{\alpha_2} \text{sign} (z_2) \]
\[ -k_3 |z_3|^{\alpha_3} \text{sign} (z_3) \]
\[ -k_1 |z_1 + \bar{w}_1(\tau)|^{\alpha_1} \text{sign} (z_1 + \bar{w}_1(\tau)) \]
\[ -|z_1|^{\alpha_1} \text{sign} (z_1) \]
\[ -\bar{w}_4(\tau) \]  

(72)

Let

\[ g_2(\tau, z(\tau)) = -k_1 |z_1 + \bar{w}_1(\tau)|^{\alpha_1} \text{sign} (z_1 + \bar{w}_1(\tau)) \]
\[ -|z_1|^{\alpha_1} \text{sign} (z_1) \]
\[ -\frac{k_2}{\varepsilon z_{\alpha_2}} |z_2 - \bar{d}(\tau)|^{\alpha_2} \text{sign} (z_2 - \bar{d}(\tau)) \]
\[ -|z_2|^{\alpha_2} \text{sign} (z_2) \]
\[ -k_3 |z_3 + \bar{w}_3(\tau)|^{\alpha_3} \text{sign} (z_3 + \bar{w}_3(\tau)) \]
\[ -|z_3|^{\alpha_3} \text{sign} (z_3) \]
\[ -\bar{w}_4(\tau) \]  

(73)
Therefore, we obtain

\[ \delta = \sup_{(\tau, z) \in R^4} |g_2(\tau, z(\tau))| \leq 2^{1-\alpha_1} k_1 h_{\alpha_1} \varepsilon^{\alpha_1} \]
\[ + 2^{1-\alpha_3} k_3 h_{\alpha_3} \varepsilon^{3\alpha_3} + \varepsilon^4 L_a + 2^{1-\alpha_2} k_2 L_{d\alpha_2} \]
\[ \leq \varepsilon^\rho \delta_0 + 2^{1-\alpha_2} k_2 L_{d\alpha_2} \]  

(74)
where \( \delta_0 = 2^{1-\alpha_1} k_1 h_{\alpha_1} \varepsilon^{\alpha_1} + 2^{1-\alpha_3} k_3 h_{\alpha_3} \varepsilon^{3\alpha_3} + L_a \), and \( \rho = \min_{i \in \{1, 3\}} \{ \min(4, i \alpha_i) \} = \alpha_1 \). In fact, when \( n = 3 \), it is checked that the recursive form of (64) may be rewritten in the non-recursive form

\[ \alpha_i = \frac{\alpha_3}{(4 - i) - (3 - i) \alpha_3}, i = 1, 2, 3 \]  

(75)
Defining the following function

\[ g_3(\zeta) = \frac{\zeta \alpha_3}{(4 - \zeta) - (3 - \zeta) \alpha_3}, \zeta \in (0, 4) \]  

(76)
and taking derivative of \( g_3(\zeta) \) with respect to \( \zeta \), we obtain
where $\zeta \in (0, 1)$, function $g_3(\zeta)$ is monotone increasing for $\zeta \in (0, 4)$. Moreover, the sequence $\{1, 2, 3\}$ is monotone increasing in open interval $(0, 4)$. Therefore,

$$\min_{\zeta \in (1, 3)} \{i_{\alpha_z}\} = \alpha_1$$

Furthermore, because $\varepsilon \in (0, 1)$ and $\alpha_i \in (0, 1), i = 1, 2, 3.$, we obtain

$$\max_{\zeta \in (1, 3)} \{\varepsilon^{in}\} = \varepsilon^\alpha = \varepsilon^{\alpha_1}$$

From Proposition 8.1 in [38], Theorem 5.2 in [37] and Eq. (74), for Eq. (72), there exist positive constants $\mu$ and $\Gamma (z(0))$, such that, for $\forall \tau \in [\Gamma (z(0)), \infty),$

$$\|z(\tau)\| \leq \mu \delta\tau \leq \mu(\varepsilon_{\alpha_1}\delta_0 + 2^{-1/2} k_2 L_{d_a}\gamma)^\gamma$$

where $\mu$ is a constant defined in Theorem 5.2 in [37]. Therefore, from coordinate transformation (70), we obtain

$$\|\varepsilon_{\varepsilon_1}\varepsilon_{\varepsilon_2}\varepsilon_{\varepsilon_3}\| \leq \mu(\varepsilon_{\alpha_1}\delta_0 + 2^{-1/2} k_2 L_{d_a}\gamma)^\gamma$$

for $\forall \tau \in [\varepsilon\Gamma (\Xi (\varepsilon) \varepsilon (0)), \infty)$. Thus, the following inequality holds:

$$\|\varepsilon_{\varepsilon_1}\varepsilon_{\varepsilon_2}\| \leq L(\delta_{\varepsilon_1})^\gamma, i = 1, 2, 3, \forall \tau \in [\varepsilon\Gamma (\Xi (\varepsilon) \varepsilon (0)), \infty)$$

where $L = \mu \delta_0; \delta_{\varepsilon_1} = \varepsilon_{\alpha_1}^{\gamma_1} + \varepsilon_{\alpha_2}^{\gamma_2} k_2 L_{d_a}\gamma$. If $\varepsilon \in (0, 1)$ and $L_{d_a} < \left(\frac{1}{2(\varepsilon_{\alpha_1}^{\gamma_1} + \varepsilon_{\alpha_2}^{\gamma_2})}\right)^\frac{1}{\gamma_2},$

$$0 < \varepsilon_{\alpha_1}^{\gamma_1} + \varepsilon_{\alpha_2}^{\gamma_2} k_2 L_{d_a}\gamma < 1$$

Furthermore, from Theorem 4.3 in [37], $\beta_c$ can be chosen to be arbitrarily small. Hence, the requirement that $\beta_c$ lies on

$$\beta_c \in \left(0, \min \left\{ \frac{1}{\log (\varepsilon_{\alpha_1}^{\gamma_1} + \varepsilon_{\alpha_2}^{\gamma_2} k_2 L_{d_a}\gamma)}, \frac{1}{2} \right\} \right)$$

is not restrictive. Accordingly, we can obtain

$$\gamma = \frac{1 - \beta_c}{\beta_c} > \max \left\{ \frac{4 \log \varepsilon}{\log (\varepsilon_{\alpha_1}^{\gamma_1} + \varepsilon_{\alpha_2}^{\gamma_2} k_2 L_{d_a}\gamma), 1} \right\}$$

Therefore,

$$\gamma \log (\varepsilon_{\alpha_1}^{\gamma_1} + \varepsilon_{\alpha_2}^{\gamma_2} k_2 L_{d_a}\gamma) < 4 \log \varepsilon$$

i.e.,

$$\varepsilon_{\alpha_1}^{\gamma_1} + \varepsilon_{\alpha_2}^{\gamma_2} k_2 L_{d_a}\gamma < \varepsilon_{\gamma}^4$$

From Eq. (85), $\gamma > 4$ holds. Therefore, from $\varepsilon \in (0, 1), we can obtain $\varepsilon_{\gamma}^4 < \varepsilon_{\gamma}, i = 1, 2, 3.$ Then
\[ \varepsilon^4 s \{ sX_2(s) - sW_2(s) + \frac{1}{s} c_\sigma(s) \} = -k_1 \frac{1}{s} X_2(s) - k_2 (X_2(s) - W_2(s)) -k_3 \varepsilon^3 \{ sX_2(s) - sW_2(s) + \frac{1}{s} c_\sigma(s) \} \]  

(94)

Then, it follows that

\[ X_2(s) = \frac{\varepsilon^4 s^3 + k_3 \varepsilon^3 s^2 + k_2 s + k_1 \varepsilon}{\varepsilon^4 s^3 + k_3 \varepsilon^3 s^2 + k_2 s + k_1 \varepsilon} W_2(s) \]

\[ = \frac{\varepsilon^4 s^3 + k_3 \varepsilon^3 s^2 + k_2 s + k_1 \varepsilon}{\varepsilon^4 s^3 + k_3 \varepsilon^3 s^2 + k_2 s + k_1 \varepsilon} \frac{1}{s} c_\sigma(s) \]  

(95)

Therefore,

\[ \lim_{\varepsilon \to 0} X_2(s) = 1 \]  

(96)

2) \( x_1 \) estimates \( w_1 \): From Eqs. (91) and (92), we obtain

\[ W_2(s) = sW_1(s) \]

\[ W_3(s) = \frac{1}{s} c_\sigma(s) \]

\[ \Pi(s) = s^2 W_1(s) - \frac{1}{s} c_\sigma(s) \]  

(97)

and

\[ X_2(s) = sX_1(s) \]

\[ X_3(s) = s^2 X_1(s) - s^2 W_1(s) + \frac{1}{s} c_\sigma(s) \]

\[ \varepsilon^4 sX_3(s) = -k_1 \varepsilon X_1(s) - k_2 (X_2(s) - W_2(s)) -k_3 \varepsilon^3 X_3(s) \]  

(98)

Therefore, Eqs. (97) and (98) can be written as

\[ \varepsilon^4 s \{ s^2 X_1(s) - s^2 W_1(s) + \frac{1}{s} c_\sigma(s) \} = -k_1 \varepsilon X_1(s) - k_2 (sX_1(s) - sW_1(s)) -k_3 \varepsilon^3 \{ s^2 X_1(s) - s^2 W_1(s) + \frac{1}{s} c_\sigma(s) \} \]  

(99)

Then, it follows that

\[ X_1(s) = \frac{\varepsilon^4 s^3 + k_3 \varepsilon^3 s^2 + k_2 s + k_1 \varepsilon}{\varepsilon^4 s^3 + k_3 \varepsilon^3 s^2 + k_2 s + k_1 \varepsilon} W_1(s) \]

\[ = \frac{\varepsilon^4 s^3 + k_3 \varepsilon^3 s^2 + k_2 s + k_1 \varepsilon}{\varepsilon^4 s^3 + k_3 \varepsilon^3 s^2 + k_2 s + k_1 \varepsilon} \frac{1}{s} c_\sigma(s) \]  

(100)

Therefore,

\[ \lim_{\varepsilon \to 0} X_1(s) = 1 \]  

(101)

3) \( x_3 \) estimates \( w_3 \): From Eqs. (91) and (92), we obtain

\[ W_2(s) = \frac{1}{s} W_3(s) + \frac{1}{s} \Pi(s) \]

\[ X_1(s) = \frac{1}{s} X_2(s) \]

\[ X_2(s) = \frac{1}{s} X_3(s) + \frac{1}{s} \Pi(s) \]

\[ \varepsilon^4 sX_3(s) = -k_1 \varepsilon X_1(s) - k_2 (X_2(s) - W_2(s)) -k_3 \varepsilon^3 X_3(s) \]  

(102)

Therefore, Eq. (102) can be written as

\[ \varepsilon^4 s^3 X_3(s) = -k_1 \varepsilon X_3(s) - k_1 \Pi(s) - k_2 sX_3(s) -k_2 s\Pi(s) + k_2 s W_3(s) + k_2 s \Pi(s) -k_3 \varepsilon^3 s^2 X_3(s) \]  

(103)

Then, it follows that

\[ X_3(s) = \frac{k_2 s}{\varepsilon^4 s^3 + k_3 \varepsilon^3 s^2 + k_2 s + k_1 \varepsilon} W_3(s) \]

\[ = \frac{k_1 \varepsilon}{\varepsilon^4 s^3 + k_3 \varepsilon^3 s^2 + k_2 s + k_1 \varepsilon} \Pi(s) \]  

(104)

Therefore,

\[ \lim_{\varepsilon \to 0} X_3(s) = 1 \]  

(105)

It means that \( x_i \) approximates \( w_i \) for \( i = 1, 2, 3 \). Furthermore, the denominator of Eq. (95) (also in Eqs. (100) and (104)) is required to be Hurwitz, i.e., \( s^3 + \frac{k_3}{k_2} s^2 + \frac{k_2}{k_1} s + k_1 \) is Hurwitz. It is equivalent that \( s^3 + k_3 s^2 + \frac{k_2}{k_1} s + k_1 \) should be Hurwitz. For arbitrary \( \varepsilon \in (0, 1) \), from the Routh-Hurwitz Stability Criterion, this polynomial is Hurwitz if \( k_1 > 0 \), \( k_3 > 0 \), \( k_2 > \varepsilon^2 k_1/k_3 \). This concludes the proof. ■

REFERENCES


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