EPRL/FK Asymptotics and the Flatness Problem

José Ricardo Camões Oliveira

under supervision of Prof. John W. Barrett

Thesis submitted for the title of Doctor of Philosophy
at The University of Nottingham

November 2, 2016

Abstract

The main topic of this thesis is a key point in testing the viability of the EPRL/FK spin foam model as a quantum theory of gravity. While it is common knowledge that there are fundamental mathematical inconsistencies between Einstein's General Relativity and Quantum Mechanics, pointing, among other reasons, towards the necessity of such a theory, our current inability to observe the extremely high energies and/or small wavelengths at which quantum effects are expected to appear leaves us with mathematical consistency tests as the only, albeit incomplete, way of separating possibly viable models from incorrect ones. One of the most basic tests available is the study of the model's asymptotics in a semiclassical regime. Indeed, any quantum theory of gravity must be able to reproduce Einstein's model when quantum effects are negligible. With that in mind, we will discuss the asymptotics of spin foam models, in particular the EPRL/FK prescription, and note the non-trivial issues that arise in the course of that study.

In order to provide context to the discussion, first we will briefly introduce spin foam models as a state sum formulation of Loop Quantum Gravity, the canonical quantization program of Einstein's theory, giving a short review of the LQG formalism and the issues that led to the construction of spin foams. We will then briefly refer to some historical aspects of this line of study, starting with the original discussion based on BF theory that resulted in the Ponzano-Regge model for 3-dimensional gravity, and proceed to 4-dimensional models and the issues that led to the crafting of the EPRL/FK model. We will then review the calculation of the EPRL vertex amplitude in more detail, before moving on to the topic of asymptotics, the definition of an adequate semiclassical limit to work in, and existing results, with emphasis on the so-called "flatness problem" originally enunciated by Bonzom, as well as a critique of the reasonings that led to it, namely the concept of varying the EPRL action with respect to a discrete variable - the face areas in a given triangulation of spacetime geometry.

With the above in mind, and introducing our practical approach to the variation of the face areas, we move on to the main original work presented, a detailed calculation of the zero-order "classical" equations of motion and their solutions for a concrete triangulation of three 4-simplices, which has been named Δ_3. The goal of said calculation is to assess whether the flatness problem exists or not in a practical example, and ultimately check if the results obtained satisfy what is expected from Einstein gravity. A negative result would, in plain terms, "kill" the model, or at the very least show it needs modifications, while a positive result, though only a particular case, would be a small step towards the understanding of spin foam asymptotics and possibly hint towards more general properties of the model.

Acknowledgements

This thesis is the final product of a PhD program at the University of Nottingham, under a scholarship granted by the Fundação para a Ciência and Tecnologia (FCT) of the Government of Portugal. I would like to thank both entities for giving me this opportunity.

Within the University, an additional message of gratitude is in order, to Student Administration, to the Department of Mathematics and most of all to its Postgraduate Secretary, Helen Cunliffe, for the understanding and support provided throughout difficult situations that arose in the course of the program.

I am also thankful to the team of Prof. Fernando Magro Dias, from the Hospital de S. João, Porto, for invaluable medical care over the past year, without which I probably would not have been able to continue working.

Thank you to Krzysztof Polański, for being a great friend and one of the precious few that I counted on throughout the years in England.

My supervisor, Prof. John Barrett, was, with few exceptions, the only person that I could effectively discuss the relevant topics of this project with. I thank him for all the time and trouble spent with me.

And finally, thank you to my parents, for everything.

Contents

1 Introduction: Spin Foam Models for Quantum Gravity

1.1 Why Quantum Gravity?

The early 20th century saw the development of two extremely successful theories of physics. On one hand, General Relativity (GR), by and large the work of one man, Albert Einstein, which solved a number of known issues with the centuries-old Newtonian theory of gravity, introduced a radical shift in human understanding of such essential concepts as space, time, matter and energy, and made successful predictions that shaped the development of cosmology, astronomy and space travel ever since. At the time of writing of this thesis, LIGO's first scientifically reliable observation of gravitational waves is still recent news, and comes in as another resounding success of the theory.

On the other hand, Quantum Mechanics (QM), a work of a number of notable physicists throughout decades, which aimed to introduce a mathematical formalism capable of explaining the deeply puzzling phenomena involving elementary particles, such as the wave-particle duality, quantization of energy, and the inherent uncertainty to microscopic level measurements. From the Copenhagen interpretation and names such as Schrödinger, Heisenberg, Planck and Bohr at the helm, the community was led into the intricate world of subatomic particles and the strong and weak nuclear forces. They would eventually be unified with electromagnetism thanks to the development of quantum field theory - effectively a formulation of "classical" QM compatible with Special Relativity, where names such as Feynman, Weinberg, Salam and Glashow appear as some of the many contributors. The final result, dubbed the Standard Model, has been extensively tested and developed thanks to particle accelerators and colliders, and between the resulting mapping of subatomic particles and their possible interactions and the various other phenomena explained by QM in such diverse fronts as chemistry, condensed matter, optics, informatics, atomic and nuclear physics, it is a theory that, no matter how puzzling it may be at its mathematical and philosophical foundations, has accomplished immense feats for physics.

These theories are based on apparently contradictory mathematical views of spacetime and energy. QM has originally been enunciated, if we take the Heisenberg picture for the sake of clarity, with an equation of motion that describes the evolution of a unitary operator, the Hamiltonian, whose eigenvalues are the possible energies of the system, with time. This formulation assumed a Newtonian picture of spacetime, where time appears as an independent variable with no geometrical significance, while also introducing quantization of all dynamical fields involved, where a lot of the relevant physical quantities are discretized in terms of *quanta* and all of them are expressed in a probabilistic, rather than deterministic, fashion. This paradigm drastically contrasts with GR's concept of spacetime as a smooth, differentiable Lorentzian manifold, with time appearing simply as one of its dimensions rather than an independent variable, and its dynamics being purely deterministic, the spacetime metric $g_{ab}(x)$ being entirely specified by solutions of the Einstein equations. How can two theories with such seemingly incompatible basic principles both produce so many invaluable empiric results?

The problem is not that QM (or rather QFT) and GR contradict each other: they are simply valid in different regimes. GR's required assumption of continuity poses no issue in most situations of macroscopic systems. At the same time, QFT and its assumption of a Minkowski background (which can be extended to a fixed curved spacetime under certain conditions [1]) are not problematic as long as gravity's effects are negligible on the system being studied, which for the case of any subatomic particle interaction that could happen in a setup on Earth is easily verified. Indeed, since the gravitational "force" between two elementary particles with opposite electrical charge is around 40 orders of magnitude weaker than the corresponding electric "force"[1], the only situations in which gravity is relevant involve electrically neutral macroscopic bodies, which usually do not require any quantum considerations in their study.

It all boils down to a matter of scales, gravity being commonly relevant on the macroscopic level and quantum mechanics on the microscopic level. However, at sufficiently high energies or short wavelengths, it is expected that both quantum and relativistic effects are significant. The threshold is speculated to be (of the order of magnitude of) the Planck length[2] $l_P = \sqrt{\frac{\hbar G}{c^3}}$. There are at least two important instances from a cosmological point of view in which gravity at Planck scales or below comes into play. One is black holes, whose modelling according to GR involves a singularity at their center. This fact on its own is often cited [3] as a shortcoming of GR, since mathematical singularities imply that the assumption of smoothness is not verified in a black hole. The other is, of course, perhaps the most significant singularity of all, the Big Bang, for which there is plausible cosmological evidence but very little mathematical understanding. The generally accepted view is that GR should break down sufficiently close to these anomalous points, as quantum effects become significant within the Planck length radius. These scales are not, however, relevant in the particle interactions classified by the Standard Model, the size of elementary particles such as electrons and quarks being around 20 orders of magnitude larger than the Planck length.

Due to all of these observations, it has become plausible to the physics community that the incompatibilities between GR and QM are a symptom that the knowledge provided by the two, while not necessarily wrong, is incomplete, and there must exist a theory that reproduces their current results while describing the behaviour of gravity in systems with significant quantum effects. This is called quantum gravity (QG).

There is a vast number of more or less radical approaches to quantum gravity. Many of them, including the extensively studied and relatively successful string theory, attempt to write down a "Theory of Everything", which would be a field theory unifying gravity with the other fundamental forces under the same formalism. Others simply try to fit GR into a consistent QFT formulation, while others make more drastic departures from

[1]Term used loosely, as it implies the outdated Newtonian definition of force.

the existing theories. The research conducive to this thesis was made under a *spin foam* model, which is a line of study branched out from Loop Quantum Gravity (LQG). In the following subsection we will briefly discuss its main principles and the issues that inspired the inception of spin foams.

1.2 Loop Quantum Gravity and Spin Foams

1.2.1 Motivation for LQG

Loop Quantum Gravity (LQG) is an approach focused on finding a picture of quantum mechanics, and in particular QFT, that is compatible with the principles of general relativity. The intuitive prescription to carrying such a task would be to proceed as one does with the other interactions: define the field to be quantized, which for gravity would be the metric $g_{ab}(x)$, and perform quantization via perturbative theory, by defining the corresponding Fock space and ladder operators and expanding over Feynman diagrams. However, this method does not work for several reasons. While it is possible to partially circumvent QFT's original definition in Minkowski space and adapt it to curved spaces, Poincaré invariance, a well-defined notion of energy, and a time evolution generated by a non-vanishing unitary Hamiltonian operator need to be satisfied in order to even be able to define Fock space in the first place (for example, the vacuum state is defined as the lowest energy eigenvalue of the Hamiltonian). None of those conditions are satisfied in a general GR spacetime. Indeed, even without any quantum considerations, there isn't a local definition of gravitational energy in GR, for example [3]. Perhaps even more puzzling is the fact that even the notion of particle used by QFT seems to break down in general gravitational situations - it is known from QFT in curved spacetime that a change of coordinates can shift the vacuum state and therefore lead to creation or destruction of particles, with the Unruh effect [5], which predicts that the vacuum state for an inertial observer would be shifted to a thermodynamic equilibrium at a finite temperature for an observer subject to uniform acceleration, standing as a notorious example.

A tentative way of solving those difficulties would be to consider spacetimes that can be broken down in two parts,

$$g(x) = g_{background}(x) + h(x) \tag{1}$$

where $g_{background}$ defines a locally Poincaré-invariant spacetime consistent with the requirements of curved-space QFT, and $h(x)$ is a perturbation field around said background. There are two problems with this approach, once conceptual and one practical. Intuitively, defining a background spacetime for gravity to begin with seems to be at odds with the equivalence principle of GR, and it implies an (at least partial) separation of the gravitational field from spacetime, which does not agree with the picture painted by Einstein theory. More practically, attempts of carrying out perturbation theory under this method lead to manifest UV divergences that are non-renormalizable [4].

Motivated by these issues, the main principle of LQG is, in addition to all the basic principles of QM and GR, that it should be a *background-independent* theory, keeping with the spirit of the equivalence principle. In particular, this leads to doing away with the ladder operators a, $a^\dagger$ as the ones being quantized in the canonical quantization procedure, since they require the definition of positive and negative frequency modes, and in particular the vacuum state, which eventually lead to the necessity of a background. The idea of "loops" (which in LQG are holonomies of the spin connection) is inspired on the concept of Wilson loops in quantum electrodynamics, and, as will be succinctly described in the following section, gives rise to the usage of spin networks, which are a key tool in both LQG and spin foams.

1.2.2 Construction of LQG

Loop quantum gravity is constructed from the first order formalism of general relativity [6], since it lends itself more easily to a Hamiltonian formulation. This formalism is constructed from the action principle of GR, which states that the Einstein equations of motion can be derived from the Einstein-Hilbert action[2]

$$S = \frac{1}{2}\int R\sqrt{-g(x)}dx, \tag{2}$$

where $g = \det[g_{ab}]$ and R is the Ricci scalar. The Einstein equations are second-order differential equations on the metric. The goal of the reformulation is to write an equivalent set of first-order equations, much like what happens when passing from a Lagrangian to a Hamiltonian description of a classical mechanics system. To accomplish this, instead of using the metric g_{ab} as fundamental variables of geometry, we use the *tetrad* (colloquially referred to as a "square root" of the metric) defined in terms of g by

$$g_{ab}(x) = e^i_a(x)e^j_b(x)\eta_{ij}. \tag{3}$$

Geometrically, if space-time is a differential manifold $\mathcal{M}$, then the tetrad is a 1-form $e(x) : \mathcal{M} \to T_x\mathcal{M}$ mapping the coordinate system in $\mathcal{M}$ (explicitly defined by the metric) to the coordinate system in its tangent bundle $T\mathcal{M}$(whose metric at each point x is the Minkowski metric η_{ij}), so that $\{e^i_a(x)\}_{a\in\{0,\dots,3\}}$ can be interpreted as a basis of the tangent space $T_x\mathcal{M}$. In this notation the indices $i, j, \ldots$ are called internal indices, while $a, b, \ldots$ are external indices. The tetrad alone isn't enough to define the phase space of gravity, though: the *spin connection* ω^{ij}_a is also needed. This is the unique connection that defines a consistent covariant derivative, or in geometrical terms, the parallel transport of the frame fields defined by the tetrad across the manifold. With the new variables the action used in LQG, the Palatini-Holst action, is:

$$S[e,\omega] = \int_{\mathcal{M}} \left(\frac{1}{2}\epsilon_{ijkl}e^i \wedge e^j \wedge F^{kl}(\omega) + \frac{1}{\gamma}e^i \wedge e^j \wedge F_{ij}(\omega)\right) \tag{4}$$

where $F(\omega) = d\omega + \omega\wedge\omega$ is the curvature tensor. Rigorously, only the first term of this action is equivalent to the Einstein-Hilbert action, and the equations of motion it generates, $\epsilon_{ijkl}\wedge e^j \wedge F^{kl} = 0$, are first-order and equivalent to the Einstein equations provided

[2]From now on all notation assumes $c = 1$, $\hbar = 1$, $G = 1/(8\pi)$, and unless otherwise specified, the results stated are for vacuum: $T^{(\text{matter})}_{ab} = 0$.

there is no torsion. The second one is called the Holst term and is purely topological, making no change to the classical equations of motion if the tetrad (equivalently, the metric) is non-degenerate. It becomes relevant at the quantum level, though, as will become clearer below. γ is a dimensionless constant called the Immirzi parameter.

The next step is to perform a 3+1 ADM decomposition of spacetime [7], under the assumption that $\mathcal{M}$'s topology admits a decomposition $\mathcal{M} \approx \mathbb{R} \times \Sigma$ where Σ is a three-dimensional hypersurface representing space, and the remaining $\mathbb{R}$ represents time. However, instead of sticking to the original ADM fundamental variables, the induced metric in 3-space $h_{ab}(x)$ and the extrinsic curvature $K^{ab}(x)$, the decomposition for LQG uses the Ashtekar variables, the *triad* $E^a_i(x)$ and the *Ashtekar-Barbero connection* $A^j_b(x)$, which play similar roles to the first-order variables but reduced to 3-space: the triad satisfies $E^a_i(x)E^b_j(x)h_{ab}(x) = \delta_{ij}$, where δ_{ij} is the Cartesian metric of the tangent space $T_x\Sigma$, and the connection A is derived from the 3-space spin connection $\Gamma(E)$,

$$A^i_a = \Gamma^i_a(E) + \gamma\omega^{0i}_a. \tag{5}$$

In Hamiltonian terminology, A is a conjugate momentum of the triad, i.e. the following Poisson bracket is satisfied:

$$\left\{A^j_b(\vec{x}), E^a_i(\vec{y})\right\} = \gamma\delta^j_i\delta^a_b\delta^{(3)}(\vec{x} - \vec{y}) \tag{6}$$

Note the explicit dependency on the Immirzi parameter. It should be noted that a similar Poisson bracket that does not depend on said parameter could be obtained by taking the extrinsic curvature $K^i_a = \omega^{0i}_a$ as the conjugate momentum, but a particularly useful symmetry occurs with the Ashtekar-Barbero choice. When the ADM decomposition is performed using the time gauge, meaning that the time dimension normal to Σ is assigned to $n_t = (1, 0, 0, 0)$, the action's Lorentz symmetry under SO(3,1) translates to a simple SU(2) symmetry in Σ. It turns out that E and A are valued in SU(2)'s Lie algebra $\mathbf{su}(2)$[3], which means that under these specific choices of field and connection, gravity is expressed as a SU(2) gauge field theory. Rewritten in the new variables, the action admits a Hamiltonian form

$$S[E, A, \lambda, N] = \int dt \int_\Sigma d\vec{x}\frac{1}{\gamma}A^i_a\frac{dE^a_i}{dt} - H \tag{7}$$

where the Hamiltonian is entirely made of constraints,

$$H = \lambda^i G_i + N^a V_a + NS \tag{8}$$

and therefore vanishes when the classical equations of motion are satisfied. The SU(2) gauge transformations in particular are generated by the $G_i = D_a E^a_i$ (covariant derivative of the triad), which are called the Gauss law constraints, and take the form $E^a \to g(\vec{x})E^a g^{-1}(\vec{x})$,$A_a \to g(\vec{x})A_a g^{-1}(\vec{x}) + g(\vec{x})\partial_a g^{-1}(\vec{x})$, where $g : \Sigma \to \mathrm{SU}(2)$ parameterizes the gauge transformation. The remaining constraints are the vector constraints

[3] Since there is a double cover SU(2)→SO(3) and their Lie algebras are isomorphic, it is convenient to use SU(2) in this context.

V_a and the scalar constraint S, whose Lagrange multipliers are simply the ADM lapse and shift.

To summarise, the classical setup of LQG is first-order general relativity, using the Holst-Palatini action rewritten in the Ashtekar-Barbero variables E, A. Note the analogy to Maxwell field theory, a gauge theory where E as the electric field and A as the magnetic vector potential have similar roles as the triad and Ashtekar-Barbero connection in this context. The next step is the quantization programme. The wavefunctions to consider are $\psi(A)$, while E will be a differential operator acting on the wavefunctions. The "loops" are introduced with the definition of observables for this quantum theory: said observables are *holonomies* of the connection around paths $C \equiv \{c(s),\, s \in [0,1]\}$ in the 3-spatial slice Σ,

$$U_C[A] \equiv \mathcal{P}\exp\left(\int_C ds\, A_a^i J_i \frac{d}{ds} c^a\right), \tag{9}$$

where $\mathcal{P}\exp$ is the path-ordered exponential (i.e. the matrix exponential of the bracketed content with the series terms ordered over s), and the *flux observables*

$$E_S[f] \equiv \int_S E_i f^i \tag{10}$$

where the flux is that of a test function $f:\, S \to \mathbf{su}(2)$ through a 2-dimensional surface S. The definition of a working Hilbert space for LQG is somewhat more mathematically involved. Essentially, it consists in defining a set of wavefunctions, called *cylindrical wavefunctions*, based on the holonomy observables. If Γ is a graph, understood in this context as a set of edges e, then a generic cylindrical wavefunction is

$$\psi(A) \equiv f_\Gamma\left(\{U_e[A]\}_{e=1,\dots,L}\right). \tag{11}$$

where L is a finite number. Remembering that the wavefunctions of interest are ones that solve the constraints in the Hamiltonian (8), one can immediately do away with the Gauss law constraints if f_Γ are picked to be invariant under SU(2)-gauge transformations, which act on the holonomies in the usual way,

$$U_e \to h_{s(e)} U_e h_{t(e)}^{-1}, \tag{12}$$

where the orientation of the edge in question is from the source $s(e)$ to the target $t(e)$. This set of functions admits a scalar product, which is a generalization of the following scalar product between functions defined on the same graph:

$$\langle f_\Gamma |\, g_\Gamma \rangle = \int d^L U\, f^*\left(\{U_{e\in\Gamma}\}\right) g\left(\{U_{e\in\Gamma}\}\right). \tag{13}$$

where the measure dU is the Haar measure for SU(2). The scalar product is readily generalized for functions defined in different graphs. Note that if Γ' is a graph that shares no edges with Γ, a functional f_Γ can be defined as $\tilde{f}_{\Gamma\cup\Gamma'}$ by setting $\tilde{f}\left(\{U_{e\in\Gamma\cup\Gamma'}\}\right) =$

$f(\{U_{e\in\Gamma}\})$. It is straightforward to use this fact to prove that any two functionals $f_\Gamma, g_{\Gamma'}$ can be written as functionals of the same graph, by simply considering the union $\Gamma \cup \Gamma'$ (with the appropriate exceptions for when the two graphs share one or more edges). Then the scalar product can in general be represented by

$$\langle f_\Gamma | \, g_{\Gamma'} \rangle = \left\langle \tilde{f}_{\Gamma\cup\Gamma'} \mid \tilde{g}_{\Gamma\cup\Gamma'} \right\rangle \tag{14}$$

There is also a measure in the set of cylindrical wavefunctions. Refer to [11, 12, 13] for the mathematical details; for the purposes of this work it is enough to note that there exist a measure and scalar product that allow for the definition of a Hilbert space $\mathcal{H}_\Gamma$, which is the space of square-integrable cylindrical functions over the graph Γ. The kinematical Hilbert space of LQG is then a "direct sum" of the ones obtained from this definition, over all possible graphs:

$$\mathcal{H}_{kin} \equiv \text{"} \bigoplus_\Gamma \text{"} \, \mathcal{H}_\Gamma \tag{15}$$

where the quotes signal that we're not strictly performing a direct sum; indeed, as suggested by the scalar product above, a function $f_\Gamma \in \mathcal{H}_\Gamma$ is identified with the functions of the same mathematical form constructed over graphs that contain Γ as a subset: $\{f_{\Gamma'} \in \mathcal{H}_{\Gamma'}, \, \Gamma \subset \Gamma'\}$. This is called a projective limit, and the measure in the resultant space is called the Ashtekar-Lewandowski measure. A final important result, called the LOST theorem after Lewandowski, Okolow, Sahlmann and Thiemann, is a proof of uniqueness for the representations of the holonomy-flux algebra in $\mathcal{H}_{kin}$, assuming diffeomorphism invariance of the vacuum state. The theorem is more general than just LQG, applying to any gauge theory with a compact group over a manifold with a spin connection, and is considered one of the most fundamental ones in the theory.

With the Hilbert space constructed and the gauge symmetry explicitly included in that construction, "solving LQG" requires solving the vector constraints V_a and the scalar constraint S. While the Gauss law constraints generated SU(2) gauge transformations, the vector and scalar ones have clear geometrical significance: V_a generates diffeomorphisms in the spatial slice Σ, while S generates diffeomorphisms in the time slice $\mathcal{T} \sim \mathbb{R}$, amounting to time reparameterizations. Due to its relation to time evolution in the theory, S is called the *Hamiltonian constraint*.

The vector constraints are, at face-value, relatively easy to deal with, since a spatial diffeomorphism ϕ acts on a cylindrical wavefunction $f_\Gamma(\{U_e\})$ by simply shifting Γ to $\phi(\Gamma)$. Since the corresponding Hilbert spaces $\mathcal{H}_\Gamma$ and $\mathcal{H}_{\phi(\Gamma)}$ are isomorphic, one can consider a diffeomorphism-invariant Hilbert space by taking the quotient of $\mathcal{H}_{kin}$ with respect to the equivalence relation

$$\Gamma \sim \Gamma' \Leftrightarrow \Gamma' = \phi(\Gamma) \text{ for some diffeomorphism } \phi, \tag{16}$$

and taking the same projective limit over the equivalence classes $[\Gamma]$ (signalled by the direct sum between quotes as before):

$$\mathcal{H}_{diff} \equiv \text{"} \bigoplus_{[\Gamma]} \text{"} \, \mathcal{H}_\Gamma . \tag{17}$$

While there are some more technical issues with the topology of $\mathcal{H}_{diff}$ (namely separability), those issues can be solved by considering an appropriate choice of "basis" graphs Γ and class of diffeomorphisms to consider (maps differentiable everywhere but a finite number of points) for which $\mathcal{H}_{diff}$ is separable.

Solving the Hamiltonian constraint S is, however, one of the main difficulties in LQG, remaining at the moment as an open problem. The issue with it is two-fold. On one hand, there are issues with properly defining a meaningful dynamics, with time evolution, of a quantum system that is manifestly invariant under time reparameterizations (which is in some literature called the "problem of time" [53]). Indeed, at the *classical* level, within the scope of GR, one can define time according to what ADM decomposition $\mathcal{T} \times \Sigma$ is suitable, mark that time variable as independent and calculate evolution like that. Diffeomorphism invariance means that the theory is equally expressible in terms of a different time variable, and said variable is only locally meaningful rather than globally, but no matter what is the choice made, there is a clear notion of spacetime for the solutions and the dynamics is well-defined. In a quantum formulation of GR, however, "spacetime" as a solution of the quantum equations of motion does not exist, since said solutions determine probabilities of different spacetimes occurring. This problem is similar to what happens to the notion of particle trajectory in ordinary quantum mechanics: the wavefunction description of a particle implying a minimum uncertainty of position and momentum means that said trajectory is not well-defined, and solving the Schrödinger equation for the particle only provides us with a probabilistic map for its positions and momenta over time. It is conceptually difficult to define a time variable and time evolution when spacetime is quantized. There are ways of avoiding this problem, though, usually by introducing matter in the theory and using it to identify space points, as well as a working notion of "proper time" upon which evolution can be studied [14].

On the other hand, a much more serious problem is writing down the quantum Hamiltonian operator associated with the respective constraint. It turns out that there is a number of ambiguities in quantizing the Hamiltonian constraint, and while there are proposals for it, such as Thiemann's [15], it is a highly non-trivial problem to determine whether a proposal is correct, by verifying consistency and solving its asymptotics in the semi-classical regime to check that Einstein gravity can be recovered from it - a theme that would prove dominant in spin foam models as well. One of the motivations for the development of spin foam models is to circumvent these difficulties and introduce a LQG-inspired framework that lends itself to a simpler dynamical treatment.

Another, more conceptual, issue with the construction of the Hilbert space has to do with the graphs themselves. Indeed, having a structure of gravitational Hilbert space depending on graphs that are embedded on a spatial hypersurface of the manifold in question, even if diffeomorphism invariance is manifestly included in the model, can be somewhat uncomfortable for a theory that aims to be background-independent, and it raises some interest in working on an alternative formalism of the theory that completely does away with the 3+1 embedding, representing the topology of the spacelike

3-slice using only abstract graphs. This is another key idea that leads to the development of spin foam models, in addition to the basis of *spin network states* for the Hilbert space over a graph $\mathcal{H}_\Gamma$, which gives a nice geometric interpretation to those graphs and will be briefly described in the next subsection.

1.2.3 Spin Network States and the Area Spectrum

Consider an oriented graph Γ with edges $e = 1, ..., E$ and vertices $v = 1, ..., V$, and the SU(2) holonomies $U_e[A]$. Then the cylindrical wavefunctions $\psi_\Gamma(\{U_e\})$ can be defined by requiring SU(2) gauge invariance. With the definitions of scalar product and measure given in the previous subsection, the gravitational Hilbert space over the graph Γ is the space of square-integrable cylindrical wavefunctions with respect to the given measure,

$$\mathcal{H}_\Gamma = L^2\left(\mathrm{SU}(2)^E/\mathrm{SU}(2)^V\right) \tag{18}$$

which only depends on the combinatorics of the graph and is therefore isomorphic to any space $\mathcal{H}_{\phi(\Gamma)}$ for a diffeomorphism in the spatial slice ϕ. The goal is to determine a basis of $\mathcal{H}_\Gamma$. According to the Peter-Weyl theorem [16], any L^2 function with domain SU(2) admits a decomposition over the irreducible representations of SU(2), of the form

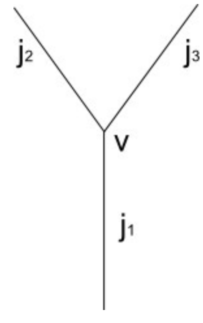
$$f(g) = \sum_{j\in\mathbb{N}/2} \mathrm{Tr}\left[f^j_{ab}\,\langle j,a|\,g\,|j,b\rangle_{ab}\right] \equiv \sum_{j\in\mathbb{N}/2} \mathrm{Tr}\left[f^j D^j(g)\right] \tag{19}$$

where j is the *spin* and $|j,b\rangle$ are basis vectors of the j-th irreducible representation of SU(2), living in the corresponding vector space V^j. $D^j(g)$ are the Wigner matrices that represent a group element $g \in$SU(2) in the irreducible representation of spin j and f^j_{ab} are analogous to Fourier components of f under this decomposition.

Using this theorem it can be shown that a basis of $\mathcal{H}_\Gamma$ can be constructed using spin network states $\psi_\Gamma^{j_e,i_v}$ where the labels are spins $j_e \in \mathbb{N}/2$ associated to each edge of Γ and intertwiners i_v defined on each vertex

$$i_v : \bigotimes_{s(e)=v} V^{j_e} \to \bigotimes_{t(e)=v} V^{j_e}, \tag{20}$$

which are SU(2)-invariant maps (i.e. commuting with SU(2) action $i_v g\,|\phi\rangle = g i_v\,|\phi\rangle$ for all states $|\phi\rangle$) that can be thought of as connector maps between the representations lying on all the edges that connect to the vertex v, mapping the edges that end at v to the ones that start from it. As a small example, if we consider a vertex with one edge coming in and two coming out, as in the figure below,



(where orientation is, by convention, assumed to be from the bottom to the top) the intertwiner map is given by the Clebsch-Gordan coefficients $C_{j_1}^{j_2,j_3}$, where $j_1 = j_2 + j_3$, and in fact more general intertwiners can always be decomposed into simple 3-valent ones such as the one above, that can then be expressed in terms of Clebsch-Gordan coefficients. A big advantage of spin network states is that they admit a very elegant and well-studied graphical calculus, that was studied originally by Penrose [17] in the context of ordinary quantum mechanics. We will discuss it further in the section about spin foams, since it is essential to their definition. The concrete expression for a spin network state $\psi_\Gamma^{j_e,i_v}$ is

$$\psi_\Gamma^{j_e,i_v}\left(\{g_e\}\right) \equiv \mathrm{Tr} \bigotimes_v i_v \bigotimes_e D^{j_e}\left(g_e\right), \tag{21}$$

and under a suitable basis of intertwiner states (essentially the ones defined from Clebsch-Gordan coefficients) they form a basis of the Hilbert space in question. The other key point towards the importance of spin network states is that they diagonalize geometric operators, most notably the areas and volumes. Classically, under the ADM formalism, the area of a parameterized surface $S \subset \Sigma$ with coordinates σ^1, σ^2 such that $x \equiv x^a(\sigma) \in S$ is

$$A_S = \int d^2\sigma \left(\epsilon_{abc}\partial_1 x^a \partial_2 x^b E_i^c(x) \cdot \epsilon_{de}^{\ f}\partial_1 x^d \partial_2 x^e E_f^i(x)\right)^{1/2}, \tag{22}$$

while the volume of a 3-dimensional region R is

$$V_R = \int d^3x \left(\frac{1}{3!}\epsilon_{abc}\epsilon^{ijk}E_i^a(x)E_j^b(x)E_k^c(x)\right)^{1/2}. \tag{23}$$

The process of quantization of these expressions as operators is somewhat technically involved [8], but the gist of it is first of all to quantize the triad as a derivative operator with respect to its conjugate momentum, the connection A, as expected from their Poisson bracket, $\hat{E}_i^a \equiv i\gamma\frac{\partial}{\partial A_a^i}$, and then to quantize the integral by taking it as the limit of a Riemann sum while regularizing the operator products within it. The eigenvalues of the resulting operator $\hat{A}_S$ can be studied by taking a particular spin network state ψ_Γ. The action of $\hat{A}$ in such a state gives non-zero contributions for the points at which the graph Γ intersects the surface S, and it turns out that, as the triad operators act very simply on spin network states by just introducing the J_i generators of the Lie algebra $\mathbf{su}(2)$, the area operator is diagonalized by the spin network states. If Γonly crosses S at a single point, the action of $\hat{A}_S$ on $\psi_\Gamma^{j_e,i_v}$ is

$$\hat{A}_S\psi_\Gamma^{j_e,i_v} = \gamma\sqrt{J_i^e J_i^e}\psi_\Gamma^{j_e,i_v} \tag{24}$$

and therefore the corresponding eigenvalues, using correct dimensional notation, are

$$A_S = \gamma l_P^2\sqrt{j_e\left(j_e+1\right)},\ j_e \in \frac{\mathbb{N}}{2} \tag{25}$$

where l_P is the Planck length. The full area spectrum amounts to sums of the eigenvalues above, for each point where the surface S is intersected by a general graph Γ.

This result demonstrates the *discretization of geometry* in LQG, with the appearance of quanta of area that fully determine the possible results of a measurement. The explicit dependence of this spectrum on the Immirzi parameter is suspect, manifesting itself in the famous LQG calculation of the black hole surface area fixing a concrete value for γ to fit the Bekenstein-Hawking entropy, which is a motive of controversy in the theory [18]. Although the derivation is a bit more complex, it is also possible to obtain a spectrum for the volume operator: The area spectrum and general concept of discretized geometry are also important motivation to the introduction of spin foams, where the spin network states take a more fundamental role of defining the dynamics of a triangulated geometry, instead of just surfacing as a convenient basis for the Hilbert space. In the following section we will discuss spin foam models and how they address some of LQG's issues.

1.3 Spin Foam Models as a dynamical realization of LQG

The concept of spin foam appeared as a response to the open problems and shortcomings found within loop quantum gravity, with its main motivation being to provide a clear picture of the quantum geometry of spacetime as a unit. Indeed, while the structure of quantum space in LQG is fairly well described thanks to the ADM decomposition and the structure that arises from it, the aforementioned problems with the Hamiltonian constraint and the associated lack of understanding of the time evolution of said space slice make it difficult to back to a proper general-relativistic vision of spacetime. The line of thought that leads to the development of spin foam models is the attempt to enunciate a path integral formalism for LQG. This is analogous to the path integral approach in quantum field theory, which leads to the expansion over Feynman diagrams: spin foams in this context are the equivalent of Feynman diagrams for quantum gravity, and can be thought of as representing the evolution over time of a spin network state.

A generic definition

Consider a spin network state $\psi_\Gamma(\{g_l\})$ over an arbitrary graph Γ embedded in a manifold $\mathcal{M}$ (more precisely, in its spatial slice if an space + time decomposition is made, although this is not necessary for the definition), with $g_l \in G$ for a Lie group G, which is the gauge group of the theory. In gravity G is the relativistic symmetry group of the manifold, but an array of models can be considered with different definitions. The edges l of Γ have spins j_l associated to them, corresponding to irreducible representations of G, while the graph's vertices v are labelled by intertwiners i_v. Now if we picture the extra time dimension and imagine the graph evolving into it, it will form a so-called *2-complex,* where the edges are foliated into faces f and the vertices into new edges e. The graph can change topologically with time, and there will be new vertices v, signalling points in spacetime where one edge breaks into several, or vice-versa with two or more edges joining into one. The "time-evolved" graph is a *spin foam.* Inspired by this construction, the ingredients to define a general spin foam are

- an arbitrary 2-complex;

- representation spins j_f for each face f of the 2-complex;
- intertwiners i_e for each edge e.

When a spin foam under this definition is spatially sliced, each slice defines a spin network state. Having spin foams defined in this fashion, they serve as states for the spin foam model, which is defined as a weighted sum over them: for a given set of 2-complexes to be summed over, the model is specified by a partition function of the form

$$Z = \sum_{\text{2-complexes}} \sum_{j_f, i_e} \prod_f W_f(j_f) \prod_e W_e(i_e, j_{f|e\in f}) \prod_v W_v(i_{e|v\in e}, j_{f|v\in f}). \tag{26}$$

where W_f, W_e, W_v are the face, edge and vertex amplitudes of each configuration, and it is considered that the amplitudes for each element depend only on the colourings (spins and intertwiners) that relate directly to them, i.e. W_f depends only on j_f, W_e depends on the associated intertwiner i_e but also on the spins of each face that edge belongs to, and W_v depends on the intertwiners and spins of edges and faces to which v belongs. There are some technicalities with the definition of the sum over 2-complexes, though - the sum often diverges and needs to be regularized, The difficulties are usually dealt with in a model by model basis, but in particular for the 4-dimensional models there are still some issues being discussed. Since a sum over 2-complexes in these models is computationally difficult to handle, most studies are made over a single one, associated to a triangulation of spacetime - but triangulation independence is manifestly not present, including in the more recent EPRL prescription.

Spin foams in BF theory and the Ponzano-Regge model

The first considerable success of spin foams in quantum gravity was the Ponzano-Regge model [10], which provides a consistent quantum theory of 3-dimensional Einstein gravity. It was a result of studying the application of the formalism to BF theory, a simpler version of Einstein theory that happens to coincide with it on 3 dimensions. In fact, as will be seen later, 4-dimensional Einstein gravity can also be described as a BF theory with constraints, which was essential to the development of 4-dimensional spin foam models for gravity.

In the classical setup, BF theory can be generally defined with a gauge (Lie) group G whose Lie algebra $\mathbf{g}$ admits an invariant nondegenerate bilinear form $\langle\cdot,\cdot\rangle$, which is used to define the trace. Taking a smooth orientable manifold $\mathcal{M}$ with dimension n, a (n-2)-form field B a connection ω both valued in $\mathbf{g}$, the action of the theory is given by

$$S[B,\omega] = \int_{\mathcal{M}} \mathrm{Tr}\,(B \wedge F[\omega])\,, \tag{27}$$

where the curvature tensor is, like in the similar definition for the Holst-Palatini action, $F = d\omega + \omega\wedge\omega$. Classically this is an extremely simple theory, since varying the action leads to the equations of motion

$$\begin{aligned} F &= 0 \\ d_\omega B &= 0, \end{aligned} \tag{28}$$

which mean that the connection is flat and the parallel transport of the B field is trivial. More than that, BF theory is a *topological field theory*, meaning that it has no local degrees of freedom and all solutions are related by gauge transformations. This is self-evident for the curvature equation since all flat connections are identical up to transformations under the gauge group G, but for the parallel transport equation the gauge symmetry in question is actually a property of the theory, where transforming B and ω via

$$\begin{aligned} \omega &\rightarrow \omega \\ B &\rightarrow B + d_\omega \eta, \end{aligned} \tag{29}$$

where η is a (n-3)-form field, leaves the action unchanged. This extra symmetry comes as a result of the structure of the phase space of the model, where it is straightforward to check that, if a ADM-like decomposition is done separating $\mathcal{M} = \mathbb{R} \times \Sigma$ where the dimension of the spatial slice Σ is $(n-1)$, and the time gauge is fixed (as was done in the LQG discussion), B is the conjugate momentum of ω,

$$\frac{\partial \mathcal{L}}{\partial \dot{\omega}} = B \tag{30}$$

and the equations of motion are constraints to the phase space which generate gauge transformations. In the symplectic geometry formalism for this theory it can be shown that indeed, while $d_\omega B = 0$ generates the G-gauge transformations, $F = 0$ generates transformations of the type (29) proving that they are also a gauge symmetry. Topological field theories, for their classical simplicity, are a good playground to study quantization tools. To see that 3-dimensional Einstein gravity is a BF theory, we can take $n = 3$, G be either SO(2,1) for Lorentzian gravity or SO(3) for Riemannian gravity (a lot of literature on spin foam models focuses on Riemannian spacetimes, both in 3 and 4 dimensions, since the corresponding representation theory is simpler) and the bilinear form $\langle \cdot, \cdot \rangle$ be minus the Killing form. If the field B is a one-to-one (n-2)-form, the spacetime metric can be defined as

$$g(v_1, v_2) = \langle E v_1, E v_2 \rangle \,. \tag{31}$$

The spin connection ω can also be pulled back to an affine connection Γ in the tangent bundle $T\mathcal{M}$, and with these definitions the equations of motion are equivalent to the statements that Γ is the Levi-Civita connection for the metric g on $\mathcal{M}$ (i.e. torsion vanishes) and the metric is flat. These are precisely the solutions of the Einstein equations in 3-dimensional gravity. In order to write BF theory as a spin foam theory, one must discretize the corresponding path integral[4]

$$Z = \int dB \, d\omega \, e^{iS_{\mathrm{BF}}} \tag{32}$$

as a weighted sum over spin foams. The equations of motion imply that this integral can be rewritten simply as

$$Z = \int d\omega \, \delta^{(3)} \left(F^i[\omega] \right) \tag{33}$$

[4]Working in Riemannian theory from now on. This means the gauge group in the following is SU(2).

which is a statement of connection flatness. To define the spin foams to sum over, consider 2-complexes that are dual to triangulations Δ of 3-space. Then triangles t_Δ are dual to edges e and edges l_Δ are dual to faces f. The natural discretization of B and ω over the triangulation is to consider discrete B-fields on each face, $B_f \in \mathbf{su}(2)$ and group elements associated with the spin connection on each edge, $U_e(\omega) \in$SU(2). Due to the enforcing of flatness in 3-dimensional gravity, the discretization of the path integral (33) is readily written as

$$Z = \int_{\mathrm{SU}(2)} \prod_e dU_e \prod_f \delta\left(\vec{\prod_{e\in f}} U_e\right), \tag{34}$$

where the oriented product $\vec{\prod}$ is used to represent a holonomy around a certain 2-complex face f, multiplying all the edge group elements in order. The path integral written in this form indicates that only configurations in which the holonomy equals identity are selected, which is geometrically related to flatness, since the holonomy can be associated with a deficit angle on the triangulated geometry. Note that there are some redundancies in the delta functions in the way that they are defined, leading to the divergence of the integral. However, those divergences have been regulated [19] by identifying the symmetry which causes the redundancies (it is actually related to a discrete version of diffeomorphism invariance), which allows one to eliminate the redundant deltas and make the integral converge. To write the partition function in spin foam formal it is then necessary to expand the delta functions over representations of SU(2). Once again the expansion uses the Peter-Weyl theorem,

$$\delta(g) = \sum_j d_j \mathrm{Tr}\left[D^j(g)\right] \tag{35}$$

so that the integral (34) can be done explicitly. $d_j = 2j+1$ is as before the dimension of the representation j, and D^j are the Wigner matrices. For a given triangulation there are three matrices in each holonomy, so the integrals will be of the form

$$\int_{\mathrm{SU}(2)} dg\, D^{j_1}_{a_1b_1}(g) D^{j_2}_{a_2b_2}(g) D^{j_3}_{a_3b_3}(g) = \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ a_1 & a_2 & a_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ b_1 & b_2 & b_3 \end{array} \right\} \tag{36}$$

where the quantities in brackets are called Wigner $3j$ symbols, sometimes called normalized Clebsch-Gordan coefficients. This expression related directly to the discussion of intertwiners in LQG since, when using a triangulation rather than a more complex cell decomposition for the spin foam discretization, all vertices are 3-valent and therefore the corresponding intertwiners are directly built from Clebsch-Gordan maps.

The above serves as a heuristic motivation for the discrete partition function of the Ponzano-Regge model, which is (for a specific triangulation)

$$Z = \sum_{\{j_f\}} \prod_f (-1)^{2j}(2j_f+1) \prod_e (-1)^{j_1+j_2+j_3} \prod_v \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \tag{37}$$

where the sum is performed over all possible values of j_f over the triangulation, and the bracketed quantity is called a Wigner $6j$ symbol, with the j's inside being the representations associated to the 6 edges of the tetrahedron dual to v. Explicitly, the $6j$ symbol is computed using $3j$ symbols as

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} = \sum_{\{a_i\}} \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ a_1 & a_2 & a_3 \end{array} \right\} \left\{ \begin{array}{ccc} j_3 & j_4 & j_5 \\ a_3 & a_4 & a_5 \end{array} \right\} \left\{ \begin{array}{ccc} j_5 & j_2 & j_6 \\ a_5 & a_2 & a_6 \end{array} \right\} \left\{ \begin{array}{ccc} j_6 & j_4 & j_1 \\ a_6 & a_4 & a_1 \end{array} \right\}, \tag{38}$$

and appears as a result of grouping the integral results (36) over each tetrahedron T (dual to a vertex v in the 2-complex). The sign factors are sometimes omitted but they are necessary to ensure that the state sum is triangulation-independent, a remarkable property of the model that unfortunately does not carry over to most 4-dimensional applications of this quantization programme. Study of triangulation independence uses Pachner's work [20], which demonstrated that any two triangulations of the same manifold $\mathcal{M}$ can be related by a finite number of elementary transformations called the Pachner moves. Proving the partition function's invariance under all Pachner moves is sufficient to prove triangulation independence.

It was also proven [21] that the Ponzano-Regge model has the adequate asymptotics to replicate Einstein 3d gravity in the zero-order semiclassical limit, making it into a full-fledged quantum theory of 3d gravity and one of the main successes of the spin foam formalism to date. Of course, asymptotics of this theory poses less of a problem than the analogous situation in 4 dimensions since it is a theory of flat space where flatness is immediately enforced in the path integral, whereas 4-dimensional gravity admits non-flat solutions. It is also possible to formulate a version of this model which includes the cosmological constant, the Turaev-Viro model [22].

The Barrett-Crane model

The Barrett-Crane model [46, 47] was the first widely studied attempt at extending the spin foam framework to 4-dimensional gravity, building on the work of Crane and Yetter [30] and Ooguri [29] among others. It was originally enunciated for Riemannian theory with the gauge group SO(4)≈SU(2)×SU(2), where the double-covering group is used for its simpler and vastly studied representation theory. As was previously mentioned, this and other 4-dimensional spin foam models are based on the idea that Einstein gravity is a modified BF theory with constraints, so the basic principles of quantizing the spin foam version of BF theory still apply. Indeed, one can see that if B is defined in terms of the tetrad e by

$$B = \star(e \wedge e) + \frac{1}{\gamma} e \wedge e, \tag{39}$$

where $\star$ is the Hodge dual, and the gauge group is $G = \mathrm{SU}(2)\times\mathrm{SU}(2)$ for Riemannian theory or $G = \mathrm{SL}(2,\mathbb{C})$ for Lorentzian theory (using the double cover of the restricted Lorentz group SO*(3,1) by $\mathrm{SL}(2,\mathbb{C})$), the Holst-Palatini action can be written in a BF form, although modified by constraints. This BF formulation of gravity was introduced originally by Plebanski [23] but later generalized in [24] to be

$$S = \int_{\mathcal{M}} \left(B^{ij} \wedge F_{ij}(\omega) - \frac{1}{2}\phi_{ijkl} B^{ij} \wedge B^{kl} \right) + \mu \left(a_1 \phi_{ij}^{\;ij} + a_2 \phi_{ijkl} \epsilon^{ijkl} \right), \tag{40}$$

where all indices are internal, ϕ_{ijkl} is a scalar field of Lagrange multipliers obeying the symmetries $\phi_{ijkl} = -\phi_{jikl} = -\phi_{ijlk} = \phi_{klij}$, μ is a 4-form Lagrange multiplier which enforces an additional constraint of the vanishing of the term it pre-factors, and a_1, a_2 are constants. The second term in (40) is directly related to the introduction of the Holst term and therefore the Immirzi parameter, but the Barrett-Crane approach considered the original Plebanski action, without said term, and therefore with a field B given simply by $\star(e \wedge e)$.

The geometrical picture of 4-dimensional spin foam gravity can be intuitively understood by the duality between 2-complexes and triangulations of a 4-dimensional manifold. Indeed, if a simplicial complex Δ composed of 4-simplices σ_v, tetrahedra τ_e and triangles δ_f is a triangulation, its dual spin foam 2-complex is established by the following correspondence:

simplicial complex	dual 2-complex
σ_v	vertex v
τ_e	edge e
δ_f	face f

This is particularly interesting because if the 2-complex is defined so that its faces are triangles, then those faces' duals in the 4-dimensional picture are also triangles with spins j_f corresponding to irreducible representations of SU(2)×SU(2) (or SL(2,ℂ)), which allows us to establish a clear analogy between the model and Regge calculus [43], a formalism for (non-quantum) GR originally designed with numeric applications in mind, which showed that a solution to the Einstein equations can be approximated by a sufficiently fine triangulation of 4d spacetime with the fundamental quantities in triangulated geometry being the face areas (which can also be expressed in terms of edge lengths) A_f and deficit angles Θ_f which encode curvature, with the LQG area spectrum establishing a direct correspondence between j_f and A_f. As will be seen below, Regge calculus is used extensively in asymptotics as a term of comparison for the semiclassical approximations of 4d spin foam models.

The essential idea behind the Barrett-Crane model is that when discretizing the integral in the Plebanski action over a triangulated geometry in 4d, the result is a sum of integrals over triangles f, and assuming that the tetrad e used in defining the B field gives a linear embedding of the triangle in $\mathbb{R}^4$, the integral of B over the given triangle is a *simple* bivector, i.e.

$$\int_f e \wedge e = f \wedge g \tag{41}$$

for some $f,\, g \in \mathbb{R}^4 \wedge \mathbb{R}^4$. The geometrical picture associated with it is then usually built over a single vertex (i.e. 4-simplex) v. This 4-simplex has 10 triangles dual to faces f of the spin foam, and each of them is assigned a simple bivector b_f. It can be shown that, in the classical setup, the set of bivectors $\{b_f\}$ uniquely specifies a geometric 4-simplex up to parallel translations and inversions through the origin (and conversely, every geometric 4-simplex determines a set of bivectors) provided that b_f have the following properties:

- If the triangle f's orientation is changed, b_f changes sign.
- If f_1 and f_2 share a common edge, the sum $b_{f_1} + b_{f_2}$ is also a simple bivector.
- For a given tetrahedron e, the sum of its face bivectors vanishes, $\sum_{f\in e} b_f = 0$. This is called the closure condition.
- Non-degeneracy: a vertex in the tetrahedron is shared by 6 triangles. Their corresponding set of face bivectors must be linearly independent.
- Regarding b_f as linear operators using the Euclidean metric in $\mathbb{R}^4$, if 3 triangles share a certain vertex of the tetrahedron, their bivectors must satisfy $\mathrm{Tr}\left(b_1\left[b_2, b_3\right]\right) > 0$. This sign condition is orientation-related and associates the correct order of the bivectors in the formula with the orientation of the boundary of the tetrahedron, from which the orientation of the interior elements is derived.[5]

In the quantum setup it is necessary to slightly relax these conditions to allow for degenerate geometries and vanishing volumes so that the fourth condition is dropped and in the fifth the $>$ is replaced by $\geq$. Apart from that the quantization procedure is at its core analogous to Ponzano-Regge, but the additional constraints generated by the Lagrange multipliers in (40) (which are the *simplicity constraints*, necessary to enforce the geometric conditions on the bivectors above) change the structure of the phase space and therefore the path integral. The bivectors b_f relate directly to the discretized B field in this model, and the group elements $U_e[\omega] \in$SU(2) are now defined on 2-complex edges, which are dual to tetrahedra.

The spin foam colourings in the BC model in Riemannian signature are sets of spins $\left(j_f^+, j_f^-\right)$ corresponding to the SU(2)$\times$SU(2) representations and intertwiners i_e on the tetrahedra, whose main "ingredients" are $15j$ symbols, which arise from the 4-dimensional equivalent of the integrals (36). Indeed, in the 4-dimensional setup, since a tetrahedron has 4 triangles, the intertwiners are 4-valent and application of the Peter-Weyl decomposition leads to integrals of the form

$$\int_{\mathrm{SU}(2)} dg \prod_{i=1}^{4} D^{j_i}_{a_i b_i}(g) = \sum_J \left\{ \begin{array}{cccc} j_1 & j_2 & j_3 & j_4 \\ a_1 & a_2 & a_3 & a_4 \end{array} \right\}^J \left\{ \begin{array}{cccc} j_1 & j_2 & j_3 & j_4 \\ b_1 & b_2 & b_3 & b_4 \end{array} \right\}^J \tag{42}$$

where J is an internal index labelling the basis of 4-valent intertwiners between the representations of spins $j_1, ..., j_4$ (they can be rewritten in terms of 3-valent intertwiners

[5]Indeed, with the correct orientations in place, we have $\mathrm{Tr}\left(b_1\left[b_2, b_3\right]\right) = \frac{9}{8}V^2$ where V is the tetrahedron volume, so it is required to have a stable sign.

and therefore Clebsch-Gordan coefficients using the properties of SU(2) representation theory). Grouping up these integrals over a 4-simplex leads to the $15j$ symbols. The true difficulty in quantizing the BC model in comparison to BF theory is discretizing the simplicity constraints, and indeed that's where the problems in this particular approach surge. Such problems would lead to the inception of the EPRL/FK model, which is the topic of the next section.

2 The EPRL/FK model

2.1 Motivation

The EPRL model was created as a joint work by Engle, Perera, Rovelli and Livine [32, 34] as an attempt to improve on the results obtained by the BC model and correct some of its known issues at the time, the most significant one being the overspecification of its bivector constraints at the quantum level, and the issues with the graviton propagator from linearized gravity [25], which the new model claimed to solve. An independent work by Freidel and Krasnov [33] was developed in parallel to it, and it was eventually shown that under certain conditions on the Immirzi parameter the two groups' approaches were equivalent, which led to the model being dubbed EPRL/FK in subsequent literature. For the gist of our work it is only important to know the vertex amplitude of the model whose asymptotics we will be studying, but it is worth noting that, in conceptual terms, that the main step taken from the BC model was, instead of implementing the simplicity constraints strongly at the classical level, doing so only on the quantum level under a suitable definition of expectation value.

2.2 The EPRL vertex amplitude

We will now state the prescriptions for the different amplitudes in the EPRL/FK model[32],[33] in Riemannian signature.

Vertex amplitude W_v

We follow the construction of W_v given in [38]. The colourings for the Euclidean EPRL/FK model are SU(2) quantum numbers j_f for each face and SU(2) intertwiners $\hat{\iota}_e$ for each edge, given by

$$\hat{\iota}_e\left(k_{ef},\, n_{ef}\right) = \int_{SU(2)} dh_e \bigotimes_{f\in e} h_e \left|k_{ef},\, n_{ef}\right\rangle \tag{43}$$

where $|k,\, n\rangle \equiv |k,\, \vec{n}, \theta_n\rangle$ are the Livine-Speziale coherent states[34] in the spin-k representation of SU(2)[6]. They minimize the uncertainty $\Delta(J^2) = \left|\left\langle \vec{J}^2 \right\rangle - \left\langle \vec{J} \right\rangle^2\right|$ in the direction of angular momentum $\vec{n}$, and their definition is

$$|k,\, n\rangle \equiv G(\vec{n})\, |k,\, k\rangle_{\vec{z}} \tag{44}$$

[6]Note that a priori $k_f \neq j_f$.

where $|k,\, k\rangle_{\vec{z}}$ is the maximum[7] angular momentum eigenstate of $\hat{J}_z$ and $G(\vec{n}) \in SU(2)$ rotates $\vec{z}$ into $\vec{n}$. There is a phase ambiguity in this definition that cannot be resolved in a canonical way, since the information about it is lost in the projection of the state vector $|n\rangle \in S^3 \subset \mathbb{C}^2$ to S^2 to obtain the rotation axis $\vec{n}$. It will become apparent in a later section that this ambiguity is not reflected in any calculations, as all related phase factors cancel out.

For the intertwiner definition to make sense there must be an ordering of the faces in a tetrahedron[35]. Setting an ordering for the points in a 4-simplex, $\sigma_v = (p_1, p_2, p_3, p_4, p_5) \equiv (1, 2, 3, 4, 5)$, is equivalent to doing the same for the tetrahedra in it, since the tetrahedron t_{e_i} can be defined as the one that does not contain the point i. The operation

$$\begin{aligned} \partial_i(v_1, ..., v_n) &\equiv (-1)^i(v_1, ..., \hat{v}_i, ..., v_n) \\ \partial_{n+1}(v_1, ..., v_n) &\equiv \partial_n(v_1, ..., v_n) \end{aligned} \tag{45}$$

induces an ordering in a $(n-1)$-simplex from that of a n-simplex. Using it, we can establish a coherent ordering of tetrahedra and triangles starting from what was defined for the 4-simplex. We can also define the orientation of a simplex - $(v_1, ..., v_n)$ is positively oriented if it is an even permutation of $(1, ..., n)$, and negatively oriented otherwise. Since ∂ satisfies $\partial_i\partial_j = -\partial_j\partial_i$, a consequence of the definition is that if $f = t_{e_1} \cap t_{e_2}$, then the orientations of f induced by t_{e_1} and t_{e_2} are opposite. This has an intuitive explanation if one considers the normal vectors to each tetrahedron.

The construction of the 4-vertex amplitude is based on the spin network basis states of Loop Quantum Gravity[37], and it relies on defining a Spin(4) (that is, the Euclidean isometry group SO(4)) intertwiner ι_e from $\hat{\iota}_e$, using the decomposition $\mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Spin}(4)$. First note that

$$\hat{\iota}_e \in \mathrm{Hom}_{\mathrm{SU}(2)}\left(\mathbb{C},\, \bigotimes_{f\in e} V_{k_{ef}}\right), \tag{46}$$

since it is a SU(2)-invariant vector of $\bigotimes_{f\in e} V_{k_{ef}}$, where $V_{k_{ef}}$ is the vector space associated with the k_{ef}-spin (irreducible unitary) representation of SU(2). One can construct an injection

$$\phi:\, \mathrm{Hom}_{\mathrm{SU}(2)}\left(\mathbb{C},\, \bigotimes_{f\in e} V_{k_{ef}}\right) \to \mathrm{Hom}_{\mathrm{Spin}(4)}\left(\mathbb{C},\, \bigotimes_{f\in e} V_{j_f^-, j_f^+}\right) \tag{47}$$

such that $\phi(\hat{\iota}_e) = \iota_e$ is the Spin(4) intertwiner. This is done by using the Clebsch-Gordan maps $C_{k_{ef}}^{j_f^-, j_f^+} : V_{k_{ef}} \to V_{j_f^-} \otimes V_{j_f^+} \approx V_{j_f^-, j_f^+}$ and constraining the values of $j_f^\pm$ via the Immirzi parameter: $j_f^\pm = \frac{1}{2}|1 \pm \gamma|\, j_f$ relates them to the original SU(2) quantum number (which is itself constrained by this relation, since $j_f^\pm \in \frac{\mathbb{N}}{2}$).

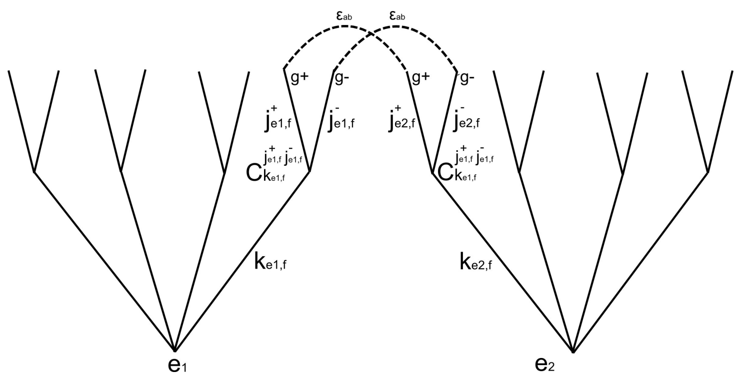
$$\iota_e(j_f,\, n_{ef}) \equiv \sum_{k_{ef}} \int_{\mathrm{Spin}(4)} dg(\pi_{j_f^-} \otimes \pi_{j_f^+})(g) \circ \bigotimes_{f\in e} C_{k_{ef}}^{j_f^-, j_f^+} \circ \hat{\iota}_e(k_{ef},\, n_{ef}), \tag{48}$$

[7] Assuming we've fixed the orientation of the z axis for convenience. The maximum angular momentum state for z would be the minimum one for $-z$.

where $g = (g^+, g^-)$, $g^\pm \in \mathrm{SU}(2)$ and $\pi_{j_f^\pm} : \mathrm{Spin}(4) \to V_{j_f^\pm}$, such that $(\pi_{j_f^-} \otimes \pi_{j_f^+})(g) : V_{j_f^-} \otimes V_{j_f^+} \to V_{j_f^-, j_f^+}$. The integration over Spin(4) is there, once again, to ensure group invariance of the intertwiner.[8]

The vertex amplitude W_v is then a closed spin network (more details on graphical calculus in [40] for the Lorentzian case) constructed by taking $\bigotimes_{e=1}^{5} \iota_e$ and "joining the extremities", for each face, of the two edges that share it, as illustrated in the figure below (each face corresponds to 2×2 of the extremities, for a total of 40, since a 4-simplex has 10 tetrahedra) by using the so-called ϵ-inner product

$$\epsilon_k : V_k \otimes V_k \to \mathbb{C}. \tag{49}$$



The inner product is constructed by linearity from the $\epsilon_{1/2}$, given in our convention by the matrix $\epsilon_{ab} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. The spin network diagram can now be evaluated using the Kaufmann bracket[39] with parameter $A = -1$. In practice this means that each pair of crossing lines with spins k_1, k_2 adds a sign $(-1)^{4k_1k_2}$. These signs result in an overall sign $(-1)^\chi$ in the amplitude.

Finally, W_v takes the form (now introducing the dependence in v)

$$W_v = (-1)^\chi \sum_{\{k_{ef}\}} \int_{\mathrm{Spin}(4)^5} \prod_{e \in v} dg^+_{ve} dg^-_{ve} \int_{(S^3)^{20}} \prod_{ef} dn_{ef} \left(\bigotimes_f \mathcal{K}_{vf} \right) \circ \left(\bigotimes_e \hat{\iota}_e \right) \tag{50}$$

where

$$\mathcal{K}_{vf} = \left(\epsilon_{j_f^-} \otimes \epsilon_{j_f^+}\right) \circ \left[\left(\left(\pi_{j_f^-}(g^-_{ve_f}) \otimes \pi_{j_f^+}(g^+_{ve_f}) \right) \circ C^{j_f^-, j_f^+}_{k_{ef}} \right) \otimes \left(\left(\pi_{j_f^-}(g^-_{ve'_f}) \otimes \pi_{j_f^+}(g^+_{ve'_f}) \right) \circ C^{j_f^-, j_f^+}_{k_{e'f}} \right) \right]. \tag{51}$$

In this expression, e, e' are the edges that share the face f.

Edge amplitude W_e

The edge amplitude is taken in modern models to be a selection rule for the values of k_{ef}, and is the only difference between the EPRL and FK models. Its choice depends on the value of the Immirzi parameter.

[8] The sum over k_{ef} is there because the edge amplitude has the practical effect of selecting these numbers. For a general W_e, they are summed over (as happens in the FK model for $\gamma > 1$)

- for $\gamma < 1$, both EPRL and FK select the choice $k_{ef} = j_f = j_f^+ + j_f^-$:

$$W_e^{\gamma<1} = d_{\hat{\iota}_e} \prod_{f\in e} \delta_{k_{ef}, j_f^+ + j_f^-} \tag{52}$$

- for $\gamma > 1$, EPRL select $k_{ef} = j_f = j_f^+ - j_f^-$,

$$W_e^{\mathrm{EPRL},\,\gamma<1} = d_{\hat{\iota}_e} \prod_{f\in e} \delta_{k_{ef}, j_f^+ - j_f^-} \tag{53}$$

  while FK's amplitude is a weighed sum over all possible values of k_{ef}, peaking at $k_{ef} = j_f = j_f^+ - j_f^-$(the expression in brackets is a squared 3j-symbol):

$$W_e^{\mathrm{EPRL},\,\gamma<1} = d_{\hat{\iota}_e} \prod_{f\in e} \sum_{k_{ef}} d_{k_{ef}} \left[\begin{pmatrix} j_f^+ & j_f^- & k_{ef} \\ j_f^+ & -j_f^- & j_f^- - j_f^+ \end{pmatrix} \right]^2 . \tag{54}$$

Face amplitude W_f

Fixing the face amplitude has been an open problem since the inception of spin foam models, since the structure of Loop Quantum Gravity does not seem to impose any particular choice for it. It is often associated with the quantized area of a triangle (see for example [13]). While several choices have been proposed in the literature, the most common being simply the dimension of the SU(2) representation associated to the face, $W_f = 2j_f{+}1$ (indeed, in [41] it is argued it is the correct choice), in the following we shall keep it as general as possible depending only on the face quantum numbers, $W_f \equiv \mu(j_f)$.

For the rest of this study we will use the EPRL prescription, so that the partition function is (considering a manifold with boundary and fixed boundary data satisfying Regge-like conditions[38])

$$\begin{aligned} Z(j_{f_B}, g_{ve_B}, n_{ef_B}) \;=\; & (-1)^\chi \sum_{j_f} \prod_f \mu(j_f) \int \prod_{ve} dg_{ve}^+ dg_{ve}^- \int \prod_{ef} dn_{ef} \int \prod_e dh_e \,\cdot \\ & \cdot \left(\bigotimes_f \mathcal{K}_f \right) \circ \left(\bigotimes_e \hat{\iota}_e (j_f^+ \pm j_f^-,\, n_{ef}) \right) \end{aligned} \tag{55}$$

It can now be established that the *de facto* variables of the model are the face SU(2) quantum numbers, Spin(4) elements for each half-edge (ve) and the coherent state vectors $\left| j_f^+ \pm j_f^-,\, n_{ef} \right\rangle$ for each edge connected to the vertex containing f, for each f.

2.3 Path integral formalism

In order to study the asymptotics of the model, we use the partition function written in a path integral form,

$$Z = \sum_c e^{S[c]}. \tag{56}$$

We will review the derivation of this form for the EPRL/FK model[44], but it is worth noting that Bonzom[42] has extended the process for any SFM under some general assumptions.

Introducing in (55) the expressions for $\hat{\iota}_e$ and $\mathcal{K}_f$, ϵ-inner products of coherent states appear. They can be written in terms of the standard Hilbert inner product by introducing the antilinear structure map $\mathcal{J} : V_k \to V_k$ defined by

$$\epsilon_k(v_k, v'_k) = \langle \mathcal{J} v_k |\, v'_k \rangle \,. \tag{57}$$

$\mathcal{J}$ has several properties: it commutes with SU(2) group elements, satisfies $\mathcal{J}^2 = (-1)^{2k}$ and, since $\mathcal{J}(\vec{n}\cdot\vec{J}) = -(\vec{n}\cdot\vec{J})\mathcal{J}$, it takes a coherent state for the vector $\vec{n}$ to one for $-\vec{n}$. We should also notice that the orientation requirements described above (45) are the basis for a supplementary requirement on the n_{ef}, which we will call here the weak gluing condition,

$$|n_{ef}\rangle_v = \mathcal{J}\,|n_{ef}\rangle_{v'} \tag{58}$$

for a tetrahedron that is shared by two vertices. Using this notation the partition function becomes

$$Z = (-1)^{\chi'} \sum_{j_f} \prod_f \mu(j_f) \int \prod_{ve} dg^+_{ve} dg^-_{ve} \int \prod_{ef} dn_{ef} \prod_e dh_e \prod_{vf} P_{vf} \tag{59}$$

where

$$P_{vf} = \langle k_{ef}, \mathcal{J} n_{ef}|\, \pi_{k_{ef}}(h_e^{-1}) C^{k_{ef}}_{j_f^- j_f^+} \pi_{j_f^-}(g^-_{ev} g^-_{ve'}) \pi_{j_f^+}(g^+_{ev} g^+_{ve'}) C^{j_f^- j_f^+}_{k_{e'f}} \pi_{k_{e'f}}(h_{e'})\, |k_{e'f}, n_{e'f}\rangle \tag{60}$$

can be interpreted as a propagator between two coherent states in the two edges sharing the face f. Now the Clebsch-Gordan maps are SU(2)-invariant, which means that the h_e can be commuted with the C's into the Spin(4) terms, which take the form $\pi_{j_f^\pm}(h_e^{-1} g^\pm_{ev} g^\pm_{ve'} h_{e'})$. The h_e can then be eliminated by a change of variables $\tilde{g}^\pm_{ve} = g^\pm_{ve} h_e$, and the corresponding integrations over them add up to a prefactor $\mathrm{Vol}(\mathrm{SU}(2))^{\#}$.

The action of the Clebsch-Gordan maps is simple in the EPRL prescription. In particular for $\gamma < 1$ (the case $\gamma > 1$ is slightly more complicated in analysis but similar in result), we have $k_{ef} = k_{e'f} = j_f^- + j_f^+$: the C-G maps project to the highest spin subspace of $V_{j_f^-} \otimes V_{j_f^+}$. Remembering the property of coherent states that

$$|k,\, n\rangle \sim \otimes^{2k} \left| \frac{1}{2}, n \right\rangle \equiv \otimes^{2k} |n\rangle \,, \tag{61}$$

which is a fully symmetric state and that the highest spin subspace is precisely the one obtained by full symmetrization, we conclude that

$$C^{j_f^- j_f^+}_{k_{ef}} |k_{ef}, n_{ef}\rangle = |k_{ef}, n_{ef}\rangle = \otimes^{2k} |n_{ef}\rangle \,. \tag{62}$$

Therefore the propagator simplifies to

$$P_{vf} = \langle \mathcal{J} n_{ef} | \, g^{-}_{ev} g^{-}_{ve'} \, | n_{e'f} \rangle^{2j^{-}_{f}} \, \langle \mathcal{J} n_{ef} | \, g^{+}_{ev} g^{+}_{ve'} \, | n_{e'f} \rangle^{2j^{+}_{f}} \,, \tag{63}$$

and with some simple algebra we can now write

$$Z = (-1)^{\chi'} \sum_{j_f} \mu(j_f) \int \prod_{ve} dg^{+}_{ve} dg^{-}_{ve} \int \prod_{ef} dn_{ef} e^{S}, \tag{64}$$

where the "action" is

$$\begin{aligned} S &= \sum_{f} \sum_{v \in f} 2 j^{\pm}_{f} \log \langle \mathcal{J} n_{ef} | \, g^{\pm}_{ev} g^{\pm}_{ve'} \, | n_{e'f} \rangle \\ &\equiv \sum_{f} S_f \end{aligned} \tag{65}$$

Since, by the discussion above, the boundary data are considered to be fixed for the "path-integral" approach, while only the interior data are dynamical, it is important to separate the action into its boundary and interior parts, $S = S_I + S_B = \sum_{f_I} S_f + \sum_{f_B} S_f$. In section 3 we will see how the action here written can be related to that of Regge calculus in the large-j regime, the base point of the asymptotics discussion.

3 Asymptotics: general considerations and past work

The semiclassical limit in quantum gravity is commonly taken in the literature as the limit of large areas, since the discrete area spectrum of LQG is asymptotically indistinguishable from the continuous classical spectrum when the corresponding quantum number j_f is large (i.e. $\frac{\Delta j}{j} \underset{j \to \infty}{\to} 0$) . Mathematically this is imposed by making the transformation $j_f \to \lambda j_f$, $\forall f$ in the regime $\lambda \to \infty$. For the EPRL model this means that its action is proportional to λ, so that the partition function is (roughly) of the form

$$I_\lambda = \int d^n z \, g(z) e^{\lambda F(z)}, \; \lambda \to \infty. \tag{66}$$

This suggests the use of the stationary phase method to derive an approximation of I_λ in the large λ limit.

3.1 The stationary phase method

The main principle of the stationary phase method is that due to the large argument of the exponential in the integrand, the contributions to the integral near certain *critical points* are much larger than everywhere else, and the integral can be estimated by considering the function only near those points. Critical points are given by the following conditions:

- $\Re(F(z))$ is at its absolute maximum, so that $\left| e^{\lambda F(z)} \right|$ is maximized;

- the oscillation is minimized, i.e. the variation of $\arg\left(e^{\lambda F(z)}\right)$ in a neighbourhood of the point in question is the slowest. At a first order level this is obtained by extremizing the action, i.e. $\partial_i f(z) = 0$, $\forall i.$, so that the variation of $\Im(F(z))$ near a critical point z_0 is at least second order in $z - z_0$, rather than first.

While not a rigorous proof (see [48, 49] for more detailed mathematical treatment), the essentials of the method can be understood with the following argument. That we need to maximize the real part of $F(z)$ should be obvious in the large λ regime, so assume in the following that $F(z) = if(z)$, $f \in \mathbb{R}$, and for simplicity $g(z) \equiv 1$ (the only condition on g is that it allows for convergence of the integral, which won't be a problem in the cases we are interested in considering). Take a Taylor expansion of f around an arbitrary point z_0:

$$\begin{aligned} f(z) &\approx f(z_0) + \left.\frac{\partial f}{\partial z^i}\right|_{z_0} (z-z_0)^i + \frac{1}{2}\left.\frac{\partial^2 f}{\partial z^i \partial z^j}\right|_{z_0} (z-z_0)^i (z-z_0)^j \\ &+ \frac{1}{3!}\left.\frac{\partial^3 f}{\partial z^i \partial z^j \partial z^k}\right|_{z_0} (z-z_0)^i (z-z_0)^j (z-z_0)^k + \mathcal{O}(z^4) \\ &\equiv f(z_0) + D_i(z_0)(z-z_0)^i + H_{ij}(z_0)(z-z_0)^i (z-z_0)^j \\ &+ T_{ijk}(z_0)(z-z_0)^i (z-z_0)^j (z-z_0)^k + \mathcal{O}(z^4) \end{aligned} \tag{67}$$

The stationary phase method assumes that when z_0 are critical points, the integral (66) is estimated by the formula

$$I_\lambda \approx \int dz_0 \int_{U(z_0)} d^n z \, e^{i\lambda f(z)} \tag{68}$$

where $U(z_0)$ is a neighbourhood of z_0. Now suppose we only took the first order term in the Taylor expansion of f. Then

$$\begin{aligned} I_\lambda^1 &\approx \int dz_0 \int_{U(z_0)} d^n z \, \exp[i\lambda(f(z_0) + D_i(z_0)(z-z_0)^i)] \\ &= \int dz_0 \exp[i\lambda(f(z_0) + D_i(z_0) z_0^i)] \int_{U(z_0)} d^n z \, e^{i\lambda D_i(z_0) z^i} \end{aligned} \tag{69}$$

If we further assume that the contribution away from a critical point is (after taking the Taylor approximation) so small that the integral above can be extended to the whole z-space, the integral over z is directly related to the delta "function":

$$\int d^n z \, e^{i\lambda D_i(z_0) z^i} = \frac{1}{2\pi\lambda} \delta(D_i(z_0)) \tag{70}$$

in this extremely crude approximation, divergences show up when $D_i(z_0) = 0$. While this points to the necessity of refining the method, which happens by taking the Taylor expansion to second order (enough in most applications), it also serves as a very simple justification that the contributions of points z_0 satisfying $D_i(z_0) = 0$ are dominant, justifying the definition of critical point above. Taking the second order expansion of f, then, we get the more accurate formula

$$\begin{aligned} I_\lambda^2 &= \int d^n z_0 \exp[i\lambda(f(z_0) + D_i(z_0) z_0^i)] \cdot \\ &\quad \cdot \int d^n z \exp[i\lambda(D_i(z_0) z^i) + H_{ij}(z_0)(z-z_0)^i (z-z_0)^j] \prod_i \delta(D_i(z_0)) \\ &= \int_{\Sigma_C} d^n z_0 e^{i\lambda f(z_0)} \int d^n z\, e^{i\lambda H_{ij}(z_0)(z-z_0)^i(z-z_0)^j} \end{aligned} \tag{71}$$

where Σ_C, the *critical surface*, is the hypersurface[9] of z-space formed by all critical points. Using analytic continuation of the standard formula $\int d^n x\, e^{-\frac{1}{2}A_{\alpha\beta}x^\alpha x^\beta} = \sqrt{\frac{(2\pi)^n}{det\, A}}$ to complex A, we can solve the integral over z:

$$\int d^n z\, e^{i\lambda H_{ij}(z_0)(z-z_0)^i(z-z_0)^j} = \left(\frac{2\pi}{i\lambda}\right)^{n/2} \frac{1}{\sqrt{\det H_r(z_0)}} \tag{72}$$

where H_r is the restriction of H to the orthogonal complement of its null space, as the conditions imposed on the z_0 constrain some degrees of freedom of H.

3.2 EPRL asymptotics: the reconstruction theorem

In the context of state sum models the critical point equations can be interpreted as classical equations of motion for the interior variables of the simplicial complex (boundary data is fixed). Considering the action (65) for the Euclidean EPRL model with $0 < \gamma < 1$, the equations of motion are

$$\Re(S_I) = R_{max} \tag{73}$$
$$\delta_{g_{ve}} S_I = 0 \tag{74}$$
$$\delta_{n_{ef}} S_I = 0 \tag{75}$$
$$\delta_{j_{f_I}} S_I = 0 \tag{76}$$

Or are they? (76) in particular has rarely been considered in existing literature. The main reason is simple - unlike the other spin foam variables in play, the $j_f \in \frac{\mathbb{N}}{2}$ are discrete, and it is unclear whether there is an extension of the stationary phase method applying to sums over general discrete variables. The only work in this direction that we are aware of is Lachaud's[50] results for sums over finite fields, which is in general not the case of the j_f sums.

The other equations of motion can be written explicitly, and are as follows:

- (73) gives the *gluing condition:* $R(g_{ve}^{\pm})\vec{n}_{ef} = -R(g_{ve'}^{\pm})\vec{n}_{e'f}$, where $R(g)$ is the rotation matrix associated to g by the 2-1 surjective homomorphism SU(2) $\to$ SO(3);

[9] The critical surface is in fact a submanifold of z-space iff $\det H_r(z_0) \neq 0\, \forall z_0 \in \Sigma_C$.

- (74) gives the *closure condition:* $\sum_{f\in e}\sum_{\pm} 2j_f^{\pm}\epsilon_{ef}(v)R(g_{ve}^{\pm})\vec{n}_{ef}=0$, where $\epsilon_{ef}(v)$ is defined to be 1 if the orientation of f agrees with the one induced from e according to (45), and -1 otherwise. $\epsilon_{ef}(v)$ are also subject to the orientation conditions, $\epsilon_{ef}(v')=-\epsilon_{ef}(v)=-\epsilon_{e'f}(v')$.
- if the previous two conditions are met, (75) is automatically satisfied.

The main existing result for EPRL asymptotics is the *reconstruction theorem*, proven originally by JWB et al[38] for the case of one single 4-simplex, and more recently extended by Han and Zhang[10][35, 36] for a general simplicial complex with boundary. Essentially, the reconstruction theorem states that given a set of boundary data satisfying a number of conditions guaranteeing their geometricity, called "Regge-like", and non-degenerate interior spin foam variables j_f, g_{ve}, n_{ef} satisfying the equations of motion, then it is possible to construct a classical, non-degenerate geometry which matches them and is unique up to global symmetries. The proof is constructive and involves defining bivectors $X_{ef}(j_f,n_{ef})$ which are interpreted as area bivectors of the discrete geometry, while the g_{ve} are identified with the spin connection (in both cases up to sign factors). Additionally, the Regge deficit angles Θ_f can be identified within the bivector formalism, such that the semiclassical action is found to be

$$S=\sum_f i\epsilon\left[j_f\mathcal{N}_f\pi-\gamma j_f\mathrm{sign}(V_4)\Theta_f\right] \tag{77}$$

where $\mathcal{N}_f\in\mathbb{N}$ and V_4 is the 4-volume of the connected component of the discrete manifold that contains f , its sign depending on the orientation induced from spin foam variables. Since the first term is a half-integer times $i\pi$ and only gives a $\pm$ sign when exponentiated, it is mostly ignored, so this "classical" form for S bears an uncanny resemblance to the discrete Einstein-Hilbert action in Regge calculus[43]:

$$S_{Regge}=\sum_f A_f\Theta_f \tag{78}$$

where A_f is the area of the triangle f, which coincides with γj_f in the reconstructed geometry.

3.3 The j-equation and the Flatness Problem

Given (77), it is readily seen how the j-equation (76) was the original motivation to the "flatness problem" mentioned by Freidel and Conrady[45] and later Bonzom[42]. The result shows that the EPRL action (65) can be written as

$$S \;=\; \sum_{f_I} j_f\tilde{\Theta}_f(g_{ve},n_{ef})$$

where $\tilde{\Theta}_f$ is a quantity that is proportional, in the semiclassical limit, to the Regge-like deficit angle, $\tilde{\Theta}_f\underset{\lambda\to\infty}{\to}\pm\gamma\Theta_f$. If we were to ignore the discreteness of the j_f and carry

[10]Han and Zhang developed their results for both the Euclidean and Lorentzian signature versions of the EPRL model. We will focus on Euclidean signature for this paper.

out the derivation as if it were continuous, the j-equation would be simply $\tilde{\Theta}_f = 0, \forall f$, therefore showing that the classical geometries reproduced by the model are restricted to be flat - a result that puts the model in question, since GR in four dimensions admits curved spacetime solutions. However, the applicability of this equation is questionable, not only because of the issues with the discreteness of j_f, but due to an ambiguity in the way the semiclassical limit is taken - taking the limit of large j_f, while at the same time summing over them. In the following we consider a slight reformulation.

Assume that in the semiclassical limit the boundary face quantum numbers are given by $j_{f_B} = \lambda j'_{f_B}, \forall f_B$ where $j'_{f_B} \in \frac{\mathbb{N}}{2}$ and $\lambda \to \infty$. Then, define new interior variables $x_{f_I} = \frac{j_{f_I}}{\lambda} \in \frac{\mathbb{N}}{2\lambda}$ (and $x^{\pm}_{f_I}$ accordingly). The partition function then takes the form

$$Z(\lambda j'_{f_B}, g_{ve_B}, n_{ef_B}) = \sum_{x_{f_I}} \int \prod_{ve} dg_{ve} \int \prod_{ef} dn_{ef} e^{i\lambda(S_I + S_B)} \tag{79}$$

with

$$\begin{aligned} S_I &= -i \sum_{f_I} \sum_{v \in f} \sum_{\pm} 2x^{\pm}_f \log \langle \mathcal{J} n_{ef} | \, g^{\pm}_{ev} g^{\pm}_{ve'} \, | n_{e'f} \rangle \equiv \sum_{f_I} x_{f_I} \tilde{\Theta}_{f_I}(g_{ve}, n_{ef}) \\ S_B &= -i \sum_{f_B} \sum_{v \in f} \sum_{\pm} 2j'^{\pm}_f \log \langle \mathcal{J} n_{ef} | \, g^{\pm}_{ev} g^{\pm}_{ve'} \, | n_{e'f} \rangle \end{aligned} \tag{80}$$

(we factor out i to explicit the fact that the argument of the exponential becomes pure imaginary when the gluing condition is satisfied). With this prescription, we don't have to assume anything about the x_{f_I}'s, eliminating ambiguities, and the dependence of the partition function on λ is completely explicit. Additionally, we can propose a workaround to the discreteness issue, consisting of a continuum approximation for the x_f. Since the $\Delta x_{f_I} = \frac{1}{2\lambda}$ tend to zero for large λ, it makes sense to consider replacing the sum over x_f by an integral:

$$\frac{1}{\Delta x_{f_I}} \sum_{x_{f_I}} f(x_{f_I}) \Delta x_{f_I} \approx \frac{1}{\Delta x_{f_I}} \int_0^{\infty} f(x_{f_I}) dx_{f_I} \tag{81}$$

and therefore the "semiclassical" partition function would be

$$Z_{SC}(\lambda j'_{f_B}) = (2\lambda)^{\# f_I} \int \prod_{f_I} dx_f \int \prod_{(ve)_I} dg_{ve} \int \prod_{(ef)_I} dn_{ef} e^{i\lambda(S_I + S_B)} \tag{82}$$

Of course, one must be careful with the errors incurring from this approximation, which is essentially the rectangle method of numerical integration "done backwards". It can be shown[11] that the difference between the sum and the integral is of order $\frac{1}{\lambda}$, making

[11] Consider the difference $\int_{x_0}^{x_0+\Delta x} f(x)dx - f(x_0)\Delta x$. For $\Delta x = 1/2\lambda$ the difference is of order $1/\lambda^2$. In practical semiclassical calculations the integral will not extend to infinity because triangle inequalities limit the maximum value of j. The cutoff will be of order λ, so the error in approximating the sum by an integral is of order $1/\lambda$.

the continuum approximation unreliable to compute any quantum corrections to the zero-order, $\lambda = \infty$ results. It could still be argued that that it can be used safely in the zero-order situation, but we will try to progress as much as possible without using it. The problem is to estimate the integral

$$\sum_{j_f} \mu(j_f) \int dY \, e^{\sum_f i\lambda x_f \tilde{\Theta}_f(Y)} \tag{83}$$

where we used Y as short for the set of g_{ve}, n_{ef} integration variables. Using the stationary phase method for the integral over Y, we obtain

$$\int dY \, e^{\sum_f i\lambda x_f \tilde{\Theta}_f(Y)} \approx \int_{\Sigma_C(x_f)} dY_C \prod_f e^{i\lambda x_f \tilde{\Theta}_f(Y_C)} \left(\frac{-2\pi i}{\lambda}\right)^{\#Y_C/2} \frac{1}{\sqrt{\det\left[\sum_f x_f H_r^f(Y_C)\right]}} \tag{84}$$

where Y_C are the critical points that solve the equations of motion, and Σ_C the submanifold of Y-space they form. Ideally, if we use the continuum approximation, we could think of reversing order of integration and doing the x integral first, but this is not possible for the general case because not only there is an x dependence on the determinant factor, which is a priori arbitrary, but due to the closure condition the critical surface Σ_C also depends on x. This makes the integral seemingly intractable without further assumptions. There are some heuristic considerations that can be made on this form of Z that lead to something suggestive of the flatness problem, but the apparent "dead end" we reach here leads us to consider a concrete example in which a full calculation is possible, the Δ_3 manifold studied in section 4.

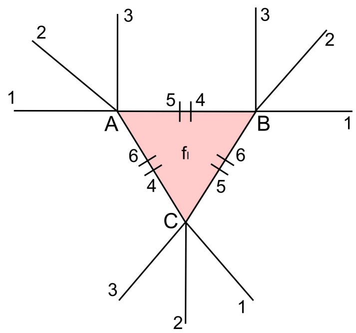
More recently, a different approach to asymptotics devised by Hellmann and Kaminski [27] derived a result similar to the flatness problem. Their main idea is to introduce the concept of wavefront sets for a distribution, which are designed with asymptotics in mind and represent the subspace of phase space where the distribution is peaked in the limit of large λ. The wavefront sets of partition functions of various models like BC and EPRL can be written using the holonomy (or operator) representation of spin foams [26] and their main result regarding asymptotics is an accidental curvature constraint acting on the deficit angles Θ_f,

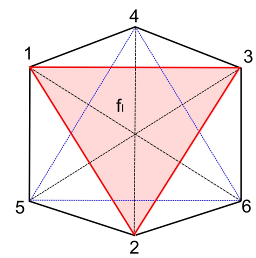
$$\gamma \Theta_f = 0 \mod 2\pi, \tag{85}$$

which is not strictly flatness (the dependence on the Immirzi parameter is somewhat puzzling) but still a worrying result in terms of the accuracy of the theory's asymptotics in respect to Einstein theory. It is noteworthy that for the BC model, which can essentially be obtained form EPRL by taking the limit $\gamma \to \infty$, the wavefront approach leads to an exact flatness constraint.

4 An example: Δ_3

In the following we will attempt to compute the asymptotic EPRL partition function for the case of the three 4-simplex manifold Δ_3, which is represented in the figure below together with its 2-complex dual. This particular manifold is chosen as a simple example of a semiclassical calculation, since it has only one interior face f_I. Therefore, assuming the boundary data are fixed, Regge-like, and non-degenerate, the classical Euclidean geometry of Δ_3 is completely determined by the area $j = \lambda x$ and the deficit angle Θ of f_I, two quantities that are easily seen to be completely determined by the boundary geometry. We will now define the EPRL model in this triangulation.





Boundary faces are notated f_{ij}^v, $i, j \in \{1, ..., 5\}$ where f_{ij}^v is the triangle that does not contain the points i, j of the 4-simplex v it belongs to, and has the area variable x_{ij}^v. Edges are labelled e_k^v, $k \in \{1, ..., 5\}$ and e_k^v is the tetrahedron that does not contain the point k of v. We will call the n_{ef} as $|n_{e,f}\rangle_v$, $v \in \{A, B, C\}$ for clarity, while the interior g_{ve} are labelled g_{A5}, g_{A6}, g_{B5}, g_{B6}, g_{C5}, g_{C6} according to the figure. The partition function is (proportional to, with extra pre-factors not being of importance in the analysis)

$$Z = \sum_{x=j/\lambda} \frac{\mu(\lambda x)}{x^{\#Y_C}} \int_{\Sigma_C(x)} dY_c \frac{e^{i\lambda x\tilde{\Theta}(Y_C)}}{\sqrt{\det H_r(Y_C)}} \tag{86}$$

noting that the dimension $\#Y$ of Y-space is that of 12 copies of S^3 associated to the interior g_{ve} and other 6 copies associated to the interior n_{ef}. The dimension $\#Y_C$ of the critical surface is the number of degrees of freedom unconstrained by the equations of motion.

4.1 Solving the equations of motion

We will now study the equations of motion for Δ_3. For starters, n_{ef}^v and $n_{ef}^{v'}$ are related by the weak gluing equations (58):

$$\begin{aligned} |n_{6,56}\rangle_A &= \mathcal{J}\,|n_{4,45}\rangle_C \\ |n_{5,45}\rangle_C &= \mathcal{J}\,|n_{6,46}\rangle_B \\ |n_{4,46}\rangle_B &= \mathcal{J}\,|n_{5,56}\rangle_A \end{aligned} \tag{87}$$

We can choose a simpler notation for the interior n_{ef} so that (87) reads

$$\begin{aligned} |n_{AC}\rangle &= \mathcal{J}\,|n_{CA}\rangle \\ |n_{CB}\rangle &= \mathcal{J}\,|n_{BC}\rangle \\ |n_{BA}\rangle &= \mathcal{J}\,|n_{AB}\rangle \end{aligned} \tag{88}$$

Stationary phase computation on the g, n integrals results in 6 interior gluing conditions,

$$\begin{aligned}
R(g^{\pm}_{C4}) \triangleright \vec{n}_{CA} &= -R(g^{\pm}_{C5}) \triangleright \vec{n}_{CB} \\
R(g^{\pm}_{B6}) \triangleright \vec{n}_{BC} &= -R(g^{\pm}_{B4}) \triangleright \vec{n}_{BA} \\
R(g^{\pm}_{A5}) \triangleright \vec{n}_{AB} &= -R(g^{\pm}_{A6}) \triangleright \vec{n}_{AC}
\end{aligned} \tag{89}$$

36 interior-boundary gluing conditions,

$$\begin{aligned}
R(g^{\pm}_{A5}) \triangleright \vec{n}^{A}_{5,i5} &= -R(g^{\pm}_{Ai}) \triangleright \vec{n}^{A}_{i,i5} \\
R(g^{\pm}_{A6}) \triangleright \vec{n}^{A}_{6,i6} &= -R(g^{\pm}_{Ai}) \triangleright \vec{n}^{A}_{i,i6} \\
R(g^{\pm}_{B6}) \triangleright \vec{n}^{B}_{6,i6} &= -R(g^{\pm}_{Bi}) \triangleright \vec{n}^{B}_{i,i6} \\
R(g^{\pm}_{B4}) \triangleright \vec{n}^{B}_{4,i4} &= -R(g^{\pm}_{Bi}) \triangleright \vec{n}^{B}_{i,i4} \\
R(g^{\pm}_{C4}) \triangleright \vec{n}^{C}_{4,i4} &= -R(g^{\pm}_{Ci}) \triangleright \vec{n}^{C}_{i,i4} \\
R(g^{\pm}_{C5}) \triangleright \vec{n}^{C}_{5,i5} &= -R(g^{\pm}_{Ci}) \triangleright \vec{n}^{C}_{i,i5},\ i \in \{1,2,3\}
\end{aligned} \tag{90}$$

and 6 closure conditions,

$$\begin{aligned}
x\left[(1+\gamma)R(g^{+}_{C4}) + (1-\gamma)R(g^{-}_{C4})\right] \triangleright \vec{n}_{CA} + b.t._{(C+)} &= 0 \\
x\left[(1+\gamma)R(g^{+}_{A6}) + (1-\gamma)R(g^{-}_{A6})\right] \triangleright \vec{n}_{AC} + b.t._{(A+)} &= 0 \\
x\left[(1+\gamma)R(g^{+}_{B6}) + (1-\gamma)R(g^{-}_{B6})\right] \triangleright \vec{n}_{BC} + b.t._{(B+)} &= 0 \qquad (91) \\
-x\left[(1+\gamma)R(g^{+}_{C5}) + (1-\gamma)R(g^{-}_{C5})\right] \triangleright \vec{n}_{CB} + b.t._{(C-)} &= 0 \\
-x\left[(1+\gamma)R(g^{+}_{A5}) + (1-\gamma)R(g^{-}_{A5})\right] \triangleright \vec{n}_{AB} + b.t._{(A-)} &= 0 \\
-x\left[(1+\gamma)R(g^{+}_{B4}) + (1-\gamma)R(g^{-}_{B4})\right] \triangleright \vec{n}_{BA} + b.t._{(B-)} &= 0 \qquad (92)
\end{aligned}$$

where the $b.t.$ represents terms depending exclusively on boundary variables. Indeed, the closure conditions contain sums over edges in each vertex, so each of them contains exactly one term corresponding to the interior edge, and the rest of the sum depends on the boundary edge variables. The boundary terms are labelled by the edges they pertain to.

First off, we will note that Eqs. (90) determine all the interior g_{ve} uniquely in terms of boundary data. Indeed, consider the first equation referring to $g^{\pm}_{A5}$. The only term in this equation that is not a boundary variable is $R(g^{\pm}_{A5})$, and the indices 1,2,3 can be grouped in a matrix form equation:

$$R(g^{\pm}_{A5}) \triangleright \underbrace{\left[\ \vec{n}^{A}_{5,15} \quad \vec{n}^{A}_{5,25} \quad \vec{n}^{A}_{5,35}\ \right]}_{\equiv N_{A5}} = -\underbrace{\left[\ R(g^{\pm}_{A1}) \triangleright \vec{n}^{A}_{1,15} \quad R(g^{\pm}_{A2}) \triangleright \vec{n}^{A}_{2,25} \quad R(g^{\pm}_{A2}) \triangleright \vec{n}^{A}_{2,25}\ \right]}_{\equiv V^{\pm}_{A5}} \tag{93}$$

Note that the non-degeneracy assumption on the boundary data implies that, since all tetrahedra are non-degenerate, any set of three out of the four $\vec{n}_{ef}$ that define a tetrahedron must be linearly independent. This means that N_{A5} is invertible in the equation above, which can then immediately be solved:

$$R(g^{\pm}_{A5}) = -N^{-1}_{A5} V^{\pm}_{A5} \tag{94}$$

and similar solutions are derived for the remaining g_{ve}. This result means that the purely interior gluing conditions (89), if consistent (consistency should be guaranteed by the boundary data being Regge-like), are redundant, however we will analyse them together with the closure conditions in the following, as they have valuable physical content for the problem.

It is possible to eliminate three of the closure equations by using the gluing ones: indeed, substituting (89) on (92), we obtain (91) while being forced to impose that $b.t._{(A+)} = -b.t._{(A-)}$ (and similar for the $B\pm$ and $C\pm$ boundary terms). Conditions on boundary variables are not problematic if they can be related to the equations for Regge-like data. To elaborate on this and to properly solve the closure conditions we need to specify the boundary data. The equations (91) in their full form are

$$\begin{aligned}
\left[(1+\gamma)R(g^{+}_{C4})+(1-\gamma)R(g^{-}_{C4})\right]\triangleright\left(x\vec{n}_{CA}+x^{C}_{41}\vec{n}^{C}_{4,41}+x^{C}_{42}\vec{n}^{C}_{4,42}+x^{C}_{43}\vec{n}^{C}_{4,43}\right) &= 0\\
\left[(1+\gamma)R(g^{+}_{B6})+(1-\gamma)R(g^{-}_{B6})\right]\triangleright\left(x\vec{n}_{BC}+x^{B}_{61}\vec{n}^{B}_{6,61}+x^{B}_{62}\vec{n}^{B}_{6,62}+x^{B}_{63}\vec{n}^{B}_{6,63}\right) &= 0\\
\left[(1+\gamma)R(g^{+}_{A5})+(1-\gamma)R(g^{-}_{A5})\right]\triangleright\left(x\vec{n}_{AB}+x^{A}_{51}\vec{n}^{A}_{5,51}+x^{A}_{52}\vec{n}^{A}_{5,52}+x^{A}_{53}\vec{n}^{A}_{5,53}\right) &= 0
\end{aligned} \tag{95}$$

The solution of these equations is simple to obtain, noting that they are of the form $M\triangleright\vec{v}=0$, a condition satisfied if and only if $\vec{v}=0$ or M has a vanishing determinant. The second possibility can be ruled out, though, by proving that $M=(1+\gamma)G+(1-\gamma)H$ has nonzero determinant for all $G,H\in SO(3)$ and $0<\gamma<1$. Proof starts with noting that $(\det M)^2=\det(M^tM)$. It is possible to get a general expression for $\det(M^tM)$:

$$\begin{aligned}
M^tM &= \left[(1+\gamma)G^t+(1-\gamma)H^t\right]\left[(1+\gamma)G+(1-\gamma)H\right]\\
&= 2(1+\gamma^2)\mathbf{1}+(1-\gamma^2)(G^tH+H^tG)\\
&= 2(1+\gamma^2)\mathbf{1}+(1-\gamma^2)(A+A^t)
\end{aligned} \tag{96}$$

defining $A\equiv G^tH\in SO(3)$. We can compute the determinant in a basis where $A+A^t$ is diagonal - note that the identity matrix is basis-invariant and $A+A^t$ is a symmetric real matrix, hence diagonalizable. To do so we need its eigenvalues, which can be found using one of the several possible parameterizations of $SO(3)$. Here we use a parameterization by Janaki and Rangarajan[51]:

$$A=\begin{bmatrix}\cos\theta_1\cos\theta_2 & \sin\theta_1\cos\theta_3-\cos\theta_1\sin\theta_2\sin\theta_3 & \sin\theta_1\sin\theta_3+\cos\theta_1\sin\theta_2\cos\theta_3\\ -\sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3 & \cos\theta_1\sin\theta_3-\sin\theta_1\sin\theta_2\cos\theta_3\\ -\sin\theta_2 & -\cos\theta_2\sin\theta_3 & \cos\theta_2\cos\theta_3\end{bmatrix} \tag{97}$$

where $\theta_i\in[0,2\pi]$ are angles for simple rotations. $A+A^t$ can then be diagonalized, being a symmetric real matrix. There is a basis in which $A+A^t=\begin{bmatrix}a & & \\ & b & \\ & & c\end{bmatrix}$, where

$$\begin{aligned}
a &= 2\\
b=c &= \sin\theta_1\sin\theta_2\sin\theta_3+\cos\theta_1(\cos\theta_2+\cos\theta_3)+\cos\theta_2\cos\theta_3-1
\end{aligned} \tag{98}$$

are its eigenvalues. In this basis,

$$
\begin{aligned}
M^t M &= 2(1+\gamma^2)\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + (1-\gamma^2)\begin{bmatrix} 2 & & \\ & b & \\ & & b \end{bmatrix} \\
&= \begin{bmatrix} 4 & & \\ & 2(1+\gamma^2)+b(1-\gamma^2) & \\ & & 2(1+\gamma^2)+b(1-\gamma^2) \end{bmatrix} \qquad (99)
\end{aligned}
$$

so that $(\det M)^2 = 4\left[2(1+\gamma^2)+b(1-\gamma^2)\right]^2$. Therefore,

$$\det M = 0 \Leftrightarrow b = -2\frac{1+\gamma^2}{1-\gamma^2} \qquad (100)$$

It is straightforward to verify that $-2 \leq b \leq 2$ for all values of θ_i, which makes the above condition impossible in the $0 < \gamma < 1$ range we are working on. Hence, M is always invertible in the conditions of our study, and the closure conditions are simplified:

$$
\begin{aligned}
x\vec{n}_{CA} + x^C_{41}\vec{n}^C_{4,41} + x^C_{42}\vec{n}^C_{4,42} + x^C_{43}\vec{n}^C_{4,43} &= 0 \\
x\vec{n}_{BC} + x^B_{61}\vec{n}^B_{6,61} + x^B_{62}\vec{n}^B_{6,62} + x^B_{63}\vec{n}^B_{6,63} &= 0 \\
x\vec{n}_{AB} + x^A_{51}\vec{n}^A_{5,51} + x^A_{52}\vec{n}^A_{5,52} + x^A_{53}\vec{n}^A_{5,53} &= 0 \qquad (101)
\end{aligned}
$$

Notice that these are precisely the necessary and sufficient conditions for the 3 tetrahedra of Δ_3 that contain the interior face f to be geometrical in the Euclidean sense, which shows that the large areas limit for this manifold imposes a discrete classical geometry on it. Also, the partition function is considerably simplified, since it is immediately verified that there exists only one value of x that makes solving the closure equations possible,

$$x = \left| x^C_{41}\vec{n}^C_{4,41} + x^C_{42}\vec{n}^C_{4,42} + x^C_{43}\vec{n}^C_{4,43} \right|, \qquad (102)$$

and all the interior $\vec{n}_{ef}$ are fixed:

$$\vec{n}_{BA} = -\frac{x^C_{41}\vec{n}^C_{4,41} + x^C_{42}\vec{n}^C_{4,42} + x^C_{43}\vec{n}^C_{4,43}}{\left| x^C_{41}\vec{n}^C_{4,41} + x^C_{42}\vec{n}^C_{4,42} + x^C_{43}\vec{n}^C_{4,43} \right|} \qquad (103)$$

with similar expressions for $\vec{n}_{AC}$ and $\vec{n}_{CB}$. In particular, note that the j-equation (76) seems not to apply in this example: x is fixed in terms of boundary data by the gluing/closure conditions, without need of an extra equation for it. This is to say that if there is an issue with the distribution of the action over x, it does not appear in the determination of critical points that we are doing here, but at most it can appear in the partition function's behaviour in a neighbourhood of the critical points. Note that the other two closure conditions also give expressions for x at critical points, leading to additional constraints on boundary data:

$$\left| x^A_{13}\vec{n}^A_{1,13} + x^A_{14}\vec{n}^A_{1,14} + x^A_{15}\vec{n}^A_{1,15} \right| = \left| x^B_{13}\vec{n}^B_{1,13} + x^B_{14}\vec{n}^B_{1,14} + x^B_{15}\vec{n}^B_{1,15} \right| = \left| x^C_{13}\vec{n}^C_{1,13} + x^C_{14}\vec{n}^C_{1,14} + x^C_{15}\vec{n}^C_{1,15} \right|. \qquad (104)$$

Additionally, the relations between (91) and (92) make it so that

$$
\begin{aligned}
\vec{n}_{CA} &= -\vec{n}_{CB} \\
\vec{n}_{BC} &= -\vec{n}_{BA} \\
\vec{n}_{AC} &= -\vec{n}_{AB} \qquad (105)
\end{aligned}
$$

and together with weak gluing, we obtain that $\vec{n}_{AB} = \vec{n}_{BC} = \vec{n}_{CA} \equiv \vec{n}$. Considering only for the moment the term of the sum over x that includes the critical surface, its corresponding partition function is now reduced to

$$Z = \frac{\mu(\lambda x)}{x^5} \int_{\Sigma_C} dY_c \frac{e^{i\lambda x \tilde{\Theta}(Y_C)}}{\sqrt{\det H_r(Y_C)}} \tag{106}$$

where, with x and $\vec{n}_{ef}$ fixed, the only integrations remaining are over group elements and the phases α_{ef}, and the face amplitude μ becomes no more than a pre-factor. The critical surface Σ_C in this new expression is $S^2 \times U(1)^3$, corresponding to the one free vector $\vec{n} \in S^2$ and the three free phases α_{AB}, α_{BC}, α_{CA} necessary to define the respective coherent states.

4.2 Geometric interpretation

We will attempt to find a compact expression for the deficit angle $\tilde{\Theta}$ using the new data. The "quantum deficit angle" for Δ_3 is

$$\begin{aligned}
\tilde{\Theta} &= \pm 2i \sum_{\pm} (1 \pm \gamma) \left[\log \langle \mathcal{J} n_{CA} | \left(g^{\pm}_{C4}\right)^{\dagger} g^{\pm}_{C5} | n_{CB} \rangle + \log \langle \mathcal{J} n_{BC} | \left(g^{\pm}_{B6}\right)^{\dagger} g^{\pm}_{B4} | n_{BA} \rangle + \log \langle \mathcal{J} n_{AB} | \left(g^{\pm}_{A5}\right)^{\dagger} g^{\pm}_{A6} | n_{AC} \rangle \right] \\
&= \pm 2i \sum_{\pm} (1 \pm \gamma) \left[\log \langle n_{AC} | \left(g^{\pm}_{C4}\right)^{\dagger} g^{\pm}_{C5} | n_{CB} \rangle + \log \langle n_{CB} | \left(g^{\pm}_{B6}\right)^{\dagger} g^{\pm}_{B4} | n_{BA} \rangle + \log \langle n_{BA} | \left(g^{\pm}_{A5}\right)^{\dagger} g^{\pm}_{A6} | n_{AC} \rangle \right]
\end{aligned} \tag{107}$$

We will focus on the first of the three matrix elements in the above expression. The results for the other two can be easily extrapolated by symmetry. In order to perform the necessary computations, we will use the following parameterizations of $SU(2)$ and the Hilbert space $\mathcal{H}^{1/2}$ of spin $\frac{1}{2}$ states:

- For the SU(2) variables, we use the decomposition

  $$\forall g \in \mathrm{SU}(2),\ g = z^{\alpha} \Sigma_{\alpha},\ \left(z^0\right)^2 + \left(z^1\right)^2 + \left(z^2\right)^2 + \left(z^3\right)^2 = 1 \tag{108}$$

  where $\Sigma_0 = \mathbf{1}$ and $\Sigma_i = i\sigma_i$ for $i = 1, 2, 3$ (σ_i are the Pauli matrices). SU(2) is therefore diffeomorphic to S^3, and considering the change of variables

  $$\begin{aligned}
  z^0 &= \cos\gamma \cos\beta^1 \\
  z^3 &= \cos\gamma \sin\beta^1 \\
  z^1 &= \sin\gamma \cos\beta^2 \\
  z^2 &= \sin\gamma \sin\beta^2,
  \end{aligned} \tag{109}$$

  with Jacobian $\frac{\sin(2\gamma)}{2}$, where $0 < \beta^i < 2\pi$ and $0 < \gamma < \frac{\pi}{2}$, it follows that a general SU(2) matrix can be written as

  $$g = \begin{bmatrix} \cos\gamma e^{i\beta^1} & i\sin\gamma e^{-i\beta^2} \\ i\sin\gamma e^{i\beta^2} & \cos\gamma e^{-i\beta^1} \end{bmatrix}. \tag{110}$$

- For the $\mathcal{H}^{1/2}$ variables, naively, one could parametrize them as follows:

$$\forall\, |n\rangle \in \mathcal{H}^{1/2},\ |n\rangle = \begin{bmatrix} w^0 + iw^1 \\ w^2 + iw^3 \end{bmatrix},\ \left(w^0\right)^2 + \left(w^1\right)^2 + \left(w^2\right)^2 + \left(w^3\right)^2 = 1 \tag{111}$$

obtaining $\int_{\mathcal{H}^{1/2}} dn = \int_{S^3} dw$. However, it is advantageous to consider a change of variables that reflects the construction of a coherent state. Recall that

$$|n\rangle = e^{i\alpha} G(\vec{n})\,|+\rangle \tag{112}$$

where $\vec{n} \in S^2$, α is an undetermined phase and $|+\rangle = (1,0)$ is the eigenstate of J_z with eigenvalue $+\frac{1}{2}$. The SU(2) element $G(\vec{n})$ is the rotation that takes $\vec{z}$ to $\vec{n}$ and is readily calculated. Consider the parameterization of S^2 in spherical coordinates

$$\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{113}$$

To go from $\vec{z}$ to $\vec{n}$ we perform a rotation of angle θ around the axis $\vec{n}_\perp = (-\sin\phi, \cos\phi, 0)$. From this we get

$$\begin{aligned} G(\vec{n}) &= \exp\left(\frac{i\theta}{2}\vec{\sigma}\cdot\vec{n}_\perp\right) \\ &= \exp\left(\frac{i\theta}{2}\left(\cos\phi\sigma_y - \sin\phi\sigma_x\right)\right) \\ &= \begin{bmatrix} \cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}. \end{aligned} \tag{114}$$

and therefore

$$|n\rangle = e^{i\alpha}\begin{bmatrix} \cos\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} \end{bmatrix}. \tag{115}$$

The Jacobian of the change of coordinates from $\vec{w}$ to (θ, ϕ, α) is $\frac{\sin(\theta)}{2}$.

Since the matrix element $\langle n_{AC}|\left(g^{\pm}_{C4}\right)^{\dagger} g^{\pm}_{C5}\,|n_{CB}\rangle$ is a scalar, it does not depend on the choice of basis in $\mathcal{H}^{1/2}$. Since the vector part for each of the coherent states present is the same, we will choose a basis in which $\vec{n}_{AB} = \vec{n} = (0,0,1)$ to carry out computations[12]. This translates to

$$|n_i\rangle = e^{i\alpha_i}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{116}$$

for $i \in \{BA, CB, AC\}$. Notice that due to each of the coherent states appearing exactly once as a bra and a ket in (122), the contribution of the phases α_i will cancel out and we can just consider $|n\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ from now on. With the coherent states taken care of, we can move on to $g^{\pm}_{C4}$ and $g^{\pm}_{C5}$. We need to use the gluing conditions (89) to relate the two in order to exhaust the constraints incurring from them, so we will also

[12] There appears to be an ambiguity with this choice, coming from the parameterization of S^2 in spherical coordinates - $\vec{n} = (0,0,1)$ is obtained when $\theta = 0$, which makes ϕ undefined. But it is evident from (114) that $G(0,0,1) = \mathbf{1}$.

need an expression for $R(g)$ for $g \in SU(2)$. Westra[13] gives us a parameterization for $g = \begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix}$, $|x|^2 + |y|^2 = 1$:

$$R(g) = \begin{bmatrix} \Re(x^2 - y^2) & \Im(x^2 + y^2) & -2\Re(xy) \\ -\Im(x^2 - y^2) & \Re(x^2 + y^2) & 2\Im(xy) \\ 2\Re(x\bar{y}) & 2\Im(x\bar{y}) & |x|^2 - |y|^2 \end{bmatrix} \tag{117}$$

In our set of coordinates for SU(2), $x = \cos\gamma e^{i\beta^1}$ and $y = i\sin\gamma e^{-i\beta^2}$, hence we can write

$$R(g) = \begin{bmatrix} \cos^2\gamma\cos(2\beta^1) + \sin^2\gamma\cos(2\beta^2) & \cos^2\gamma\sin(2\beta^1) + \sin^2\gamma\sin(2\beta^2) & \sin(2\gamma)\sin(\beta^1 - \beta^2) \\ -\cos^2\gamma\sin(2\beta^1) + \sin^2\gamma\sin(2\beta^2) & \cos^2\gamma\cos(2\beta^1) - \sin^2\gamma\cos(2\beta^2) & \sin(2\gamma)\cos(\beta^1 - \beta^2) \\ \sin(2\gamma)\sin(\beta^1 + \beta^2) & -\sin(2\gamma)\cos(\beta^1 + \beta^2) & \cos(2\gamma) \end{bmatrix} \tag{118}$$

While daunting at first, this expression becomes more tractable within the context of the gluing condition and the basis choice we made for $\vec{n}_{AB}$. The gluing condition is reduced to

$$\begin{bmatrix} \sin(2\gamma_A)\sin(\beta^1_A - \beta^2_A) \\ \sin(2\gamma_A)\cos(\beta^1_A - \beta^2_A) \\ \cos(2\gamma_A) \end{bmatrix} = \begin{bmatrix} \sin(2\gamma_B)\sin(\beta^1_B - \beta^2_B) \\ \sin(2\gamma_B)\cos(\beta^1_B - \beta^2_B) \\ \cos(2\gamma_B) \end{bmatrix} \tag{119}$$

where the variables labelled A pertain to g_{A2} and the ones labelled B pertain to g_{B1}, and we omit the $\pm$ index for simplicity. It is clear that the gluing condition does not fix g_{A2} completely given g_{B1}, since they only depend on the differences $\beta^1_{A,B} - \beta^2_{A,B} \equiv \delta_{A,B}$. Analysing the equations,

- the third equation implies $\gamma_A = \gamma_B = \gamma$, since $2\gamma_{A,B} \in [0, \pi]$ and the cosine function is injective in this domain;

- given that $\gamma_A = \gamma_B$, the first and second equations read $\sin\delta_A = \sin\delta_B$ and $\cos\delta_A = \cos\delta_B$, which for $\delta_{A,B} \in [0, 2\pi]$ is enough to infer $\delta_A = \delta_B$.

Hence, we have that, in our chosen basis for $\mathcal{H}^{1/2}$, if $g^{\pm}_{C4}$ is given by the coordinates $(\gamma_\pm, \beta^1_\pm, \beta^2_\pm)$, then $g^{\pm}_{C5}$ must have the form $(\gamma_\pm, \beta^1_\pm + \epsilon_\pm, \beta^2_\pm + \epsilon_\pm)$ where $\epsilon_\pm \in [0, 2\pi[$. We can now compute $\langle n| \left(g^{\pm}_{C4}\right)^\dagger g^{\pm}_{C5} |n\rangle$:

$$\begin{aligned} \langle n| \left(g^{\pm}_{C4}\right)^\dagger g^{\pm}_{C5} |n\rangle &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\gamma e^{-i\beta^1} & -i\sin\gamma e^{-i\beta^2} \\ -i\sin\gamma e^{i\beta^2} & \cos\gamma e^{i\beta^1} \end{bmatrix} \cdot \\ &\quad \cdot \begin{bmatrix} \cos\gamma e^{i(\beta^1+\epsilon)} & i\sin\gamma e^{-i(\beta^2+\epsilon)} \\ i\sin\gamma e^{i(\beta^2+\epsilon)} & \cos\gamma e^{-i(\beta^1+\epsilon)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \qquad (120) \\ &= \begin{bmatrix} \cos\gamma e^{-i\beta^1} & \sin\gamma e^{-i\beta^2} \end{bmatrix} \begin{bmatrix} \cos\gamma e^{i(\beta^1+\epsilon)} \\ \sin\gamma e^{i(\beta^2+\epsilon)} \end{bmatrix} \\ &= e^{i\epsilon} \qquad (121) \end{aligned}$$

[13] http://www.mat.univie.ac.at/~westra/so3su2.pdf

Taking logarithms, we get simply $i\epsilon$, and substituting (with proper labels) on the expression for $\tilde{\Theta}$ and repeating the process for the other two inner products in $\tilde{\Theta}$ (we shall identify the variables pertaining to each of these terms with an index $i \in \{1, 2, 3\}$), we obtain

$$\tilde{\Theta} \;=\; \pm 2 \sum_{\pm} (1 \pm \gamma) \sum_{i=1}^{3} \epsilon_i^{\pm}. \tag{122}$$

Remember that all g_{ve} have been determined earlier using the interior-boundary conditions. Therefore, the $\epsilon_i^{\pm}$ can be expressed in terms of the boundary data through some simple algebra. We give an example. $R(g_{A5}^{\pm})$ and $R(g_{A6}^{\pm})$ are known. Let's call them A, B for simplicity. Using the parameterization (118), we want to find either β^1 or β^2 for each matrix, and take their difference to obtain ϵ. Step by step:

- γ is obtained through $\cos(2\gamma) = A_{33}$. Since $2\gamma \in [0, \pi]$, the cosine function is injective in this domain and we can write $\gamma = \frac{1}{2}\cos^{-1}(A_{33})$. There will be three cases to consider due to the possibility of $\sin(2\gamma)$ being zero.
- If $0 < \gamma < \pi/2$, it's easy to extract the sine and cosine of $\beta^1 \pm \beta^2$ through A_{31}, A_{32} and A_{12}, A_{13} respectively. The angles can then be obtained using the angle function $\mathcal{A}_1(x, y) \equiv 2\tan^{-1}\left(\frac{x}{1+y}\right)$. The result for β^1 is

$$\beta^1 = \frac{1}{2}\left[\mathcal{A}_1\left(\frac{A_{13}}{\sqrt{1 - A_{33}^2}}, \frac{A_{23}}{\sqrt{1 - A_{33}^2}}\right) + \mathcal{A}_1\left(\frac{A_{31}}{\sqrt{1 - A_{33}^2}}, \frac{A_{32}}{\sqrt{1 - A_{33}^2}}\right)\right] \tag{123}$$

- If $\gamma = 0$, it is readily seen that $R(g)$ does not depend on β^2 but β^1 has a simple expression

$$\beta^1 = \frac{1}{2}\mathcal{A}_1\left(A_{12},\, A_{11}\right) \tag{124}$$

- If $\gamma = \pi/2$, $R(g)$ does not depend on β^1 instead. β^2 is found to be

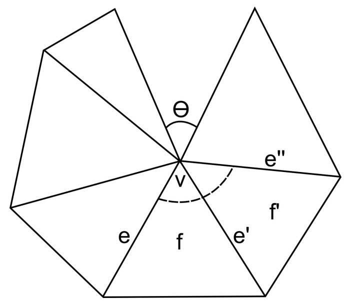
$$\beta^2 = \frac{1}{2}\mathcal{A}_1\left(A_{12},\, A_{11}\right) \tag{125}$$

  so we can combine the two extremal cases into one, as they give the same formal expression for ϵ.

Why the emphasis on determining the $\epsilon_i^{\pm}$? As seen in (122), the deficit angle $\tilde{\Theta}$ has a very simple expression in terms of them, and they can be interpreted geometrically. Indeed, note that the expression for $\tilde{\Theta}_f$ in a general face can be written as a sum over vertices, $\tilde{\Theta}_f = \sum_{v \in f} \tilde{\Theta}_{vf}$. We know from Han/Zhang's work (among others) that the action is interpreted as a holonomy around a certain face, going through all the vertices it belongs to. And in the expression for $\tilde{\Theta}_{vf}$,

$$\begin{aligned} \tilde{\Theta}_{vf} \;&=\; \sum_{\pm} 2(1 \pm \gamma) \log \langle \mathcal{J} n_{ef} |\, g_{ev}^{\pm} g_{ve'}^{\pm} \,| n_{e'f} \rangle && (126) \\ &\sim\; \sum_{\pm} 2(1 \pm \gamma) \epsilon_i^{\pm} && (127) \end{aligned}$$

the inner product clearly illustrates the parallel transport between the two tetrahedra in v which contain f. Therefore $\tilde{\Theta}_{vf}$ can be associated to the internal angle $\angle(e, e')_{vf}$, as illustrated by the figure below, a two dimensional sketch of the geometric structure around a vertex.



The sum of all internal angles is equal to 2π minus the deficit angle Θ_{Regge}, while the sum of all the $\tilde{\Theta}_{vf}$ should tend asymptotically to a sign factor times $i\gamma\Theta_{Regge}$. Hence, the correct identification which relates the ϵ to the internal angles is

$$\pm\frac{i}{\gamma}\tilde{\Theta}_{vf} = \pm\frac{2i}{\gamma}\sum_{\pm}\sum_{i}(1 \pm \gamma)\epsilon_i^{\pm} \sim \angle(e, e')_{vf} \tag{128}$$

The results obtained in this section seem positive towards the consistency of EPRL/FK asymptotics with Regge calculus, in contradiction with the flatness problem, since we are able to obtain geometrically consistent values for the key quantities in this problem, the area γj and the deficit angle Θ of the only interior triangle in the manifold. In fact, a similar result has been claimed by Perini and Magliaro[52], although the paper in question does not treat the problem in detail and fails to address one important difficulty which we will now mention: the behaviour of the state contributions when j is varied. This is a problem because j is discrete, and while we get equations of motion that guarantee the nonexistence of a critical point when j is different from the unique value j_0 found above, it has not been properly justified that the contribution from this point is dominant over certain non-critical configurations with different values of j, since it is unclear how to vary the action over it. Additionally, the value of j that solves exactly the closure conditions will in general be a non-integer, therefore there is some uncertainty in this calculation which is important to address. The closure conditions will, in general, not be exactly satisfied, because of the discreteness feature.

4.3 Variation over j

To address the issue, we will use results from Chapter 7 of [49] related to the stationary phase method. In particular we are interested in the following theorem about the study of the stationary phase integral when the functions that define it depend on free parameters.

Theorem: *Let $f(x, y)$ be a complex valued C^∞ function in a neighbourhood K of* $(0, 0) \in$

$\mathbb{R}^{n+m}$, *such that* $\Im(f) \geq 0$, $\Im(f(0,0)) = 0$, $D_x f(0,0) = 0$ *and* $\det D_x^2 f(0,0) \neq 0$. *Let* u *be a* C^∞ *function with compact support in* K. *Then*

$$\int u(x,y) e^{i\lambda f(x,y)} dx \underset{\lambda\to\infty}{\sim} e^{i\lambda f^0} \left(\frac{2\pi i}{\lambda}\right)^{n/2} \sqrt{\frac{1}{\det D_x^2 f(0,y)^0}} \tag{129}$$

where the superscript 0 in front of the determinant signals that the corresponding function is specified modulo the ideal I *of functions generated by the derivatives* $D_x f(x,y)$.

Essentially, what the theorem states is that if $x = 0$ is a critical point of f when the free parameter y is zero, then when y is non-zero the point is "moved", and is in general not a critical point any more, but its contribution to the full integral is approximated by the formula above. The key point is that if f^0 has an imaginary part, this contribution is suppressed by a factor $e^{-\lambda\Im(f^0)}$. We are interested in this suppression factor for the integral we are studying, where the free parameter y is taken to be $x - x_0$, x_0 being the critical value of x. But what is f^0? The proof of the theorem above uses the Malgrange preparation theorem, also explained in Chapter 7 of [49]. Basically, one can choose a set of functions $X^i(y)$ satisfying $X^i(0) = 0$ such that the ideal I of functions generated by $\frac{\partial f}{\partial x^i}$ is also generated by $\{x^i - X^i(y)\}_i$, and using the Malgrange preparation theorem it is possible to write the following expansion for $f(x,y)$ near the critical point:

$$f(x,y) \approx \sum_{|\alpha|<N} \frac{f^\alpha(y)}{\alpha!}(x - X(y))^\alpha \,\mathrm{mod}\, I^N,\ \forall N \tag{130}$$

f^0 is the term independent of x in this expansion. It is also noted that the $f_i^1(y)$ belong to I^N for any N, so that they can be chosen to vanish - which is an intuitive result when compared to a Taylor expansion around a critical point. Since we are only looking for the leading term of f^0 to be able to obtain the suppression factor, we will consider an expansion to second order ($N = 2$), and to compute the different functions in play we will use the well-known Taylor series for f:

$$f(x,y) \approx f(0,0) + \underbrace{\left.\frac{\partial f}{\partial x^i}\right|_{(0,0)}}_{=0} x^i + \underbrace{\left.\frac{\partial f}{\partial y}\right|_{(0,0)}}_{\equiv\delta_1} y \tag{131}$$

$$+ \frac{1}{2}\underbrace{\left.\frac{\partial^2 f}{\partial y \partial x^i}\right|_{(0,0)}}_{\equiv K_i} y x^i + \frac{1}{2}\underbrace{\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)}}_{\equiv\delta_2} y^2 + \frac{1}{2}\underbrace{\left.\frac{\partial^2 f}{\partial x^i \partial x^j}\right|_{(0,0)}}_{\equiv H_{ij}} x^i x^j \tag{132}$$

The second order Malgrange expansion for $f(x,y)$ is (setting $f^1 = 0$)

$$f(x,y) \approx f^0(y) + \frac{1}{2} f_{ij}^2(y)(x^i - X^i(y))(x^j - X^j(y)) \tag{133}$$

Equating both expansions and gathering terms independent, linear and quadratic in x, we get

$$\begin{aligned}
f(0,0)+\delta_1 y+\frac{1}{2}\delta_2 y^2 &= f^0+\frac{1}{2}f^2_{ij}X^iX^j \\
\frac{1}{2}K_i x^i y &= -\frac{1}{2}\left(f^2_{ij}+f^2_{ji}\right)x^iX^j \\
\frac{1}{2}H_{ij}x^ix^j &= \frac{1}{2}f^2_{ij}x^ix^j
\end{aligned} \tag{134}$$

which we solve to obtain (H^{ij} is the inverse matrix of H_{ij}. Remember we assumed $\det H \neq 0$)

$$\begin{aligned}
f^0 &= f(0,0)+\delta_1 y+\frac{1}{2}\delta_2 y^2-\frac{1}{2}K_iH^{ij}K_j y^2 \\
-H^{ij}K_i y &= X^j \\
f^2_{ij} &= H_{ij}
\end{aligned} \tag{135}$$

Applying to the Δ_3 case, remembering that we chose $y = x - x_0$, we see that $f(0,0)$ is the action at the critical point S_C, $\delta_1 = -i\tilde{\Theta}_C \sim \pm\gamma\Theta_{Regge}$ and $\delta_2 = 0$. Note that δ_1 is real. We are only interested in the imaginary part of f^0, which is quadratic in $(x - x_0)$, and gives us the suppressing factor as

$$\exp\left(\frac{\lambda}{2}\Im\left(K_iH^{ij}K_j\right)(x-x_0)^2\right) \tag{136}$$

Note that the variation of x has to be discrete. We would set $j = j_0 + \frac{n}{2}$, $n \in \mathbb{Z}$, so that $x - x_0 = \frac{n}{2\lambda}$. This allows us to write the partition function as a sum over n in terms of the term corresponding to $n = 0$, the critical term:

$$Z = Z_C\sum_n \exp\left(-\frac{A}{4\lambda}n^2\right) \tag{137}$$

where $A = -\Im\left(K_iH^{ij}K_j\right)$. If x is thought of as an approximately continuous variable, the distribution of x values follows a Gaussian curve with standard deviation $\sigma = \sqrt{\frac{1}{\lambda A}}$. This is a sufficiently small deviation, assuming A finite, to conclude that the distribution of the (j_f, g_{ve}, n_{ef}) variables is sufficiently peaked around the critical surface. Since A does not have any λ dependence, the positive result should be guaranteed simply by $A \neq 0$. However, the most rigorous approach to this problem is to compute the sum of the series in (137) and obtain the statistics of the discrete variable n (note, in particular, that j_0 as given by the closure equations might not be a semi-integer, so the dominant contribution would come from the semi-integer closest to it). The EPRL/FK action

$$S = -2i\sum_f\sum_{v\in f}\sum_{\pm} j_f(1\pm\gamma)\log\langle\mathcal{J}n_{ef}|\left(g^{\pm}_{ve}\right)^{\dagger}g^{\pm}_{ve'}|n_{e'f}\rangle \tag{138}$$

can be interpreted in terms of this stationary phase method by setting $j_f \equiv y$ as the free parameter, and $x_i \equiv \left(\{g_{ve}\}_a, \{n_{ef}\}_b\right)$ as the dependent variables, where a, b signal an appropriate coordinate system in which to express the interior g_{ve}, n_{ef} (which can be, for example, the parameterizations of SU(2) and $\mathcal{H}^{1/2}$ specified in section 4.2). The quantities necessary to compute the approximate partition function (137) are

$$K_i = \left.\frac{\partial^2 S}{\partial j_f \partial x_i}\right|_{\text{critical}} = \left.\frac{\partial \tilde{\Theta}_f}{\partial x_i}\right|_{\text{critical}} \tag{139}$$

$$H_{ij} = \left.\frac{\partial^2 S}{\partial x_i \partial x_j}\right|_{\text{critical}} \tag{140}$$

where "critical" means the derivatives are computed at the unique critical point for Δ_3 determined in section 4.1, and K_i is simplified due to the action being linear in j, being reduced to first derivatives of the quantum deficit angle of the interior face $\tilde{\Theta}_f$. The conditions of theorem (129) require that det $H \neq 0$ for the stationary phase method to be applicable. However, explicit computation of this determinant, even using algebraic computation software, proves to be a bit too cumbersome (refer to appendix 1 for some notes on this) because of the dependence of the derivatives in question on a high number of *a priori* arbitrary boundary variables, $\{g_{ve}, n_{ef}\}_B$ - even though it is possible to compute $det\, H$ explicitly in terms of them, and obtain a numeric answer if numeric data are introduced for the EPRL variables, it is not clear at the moment whether, for example, it is nonzero for all their possible values. For that reason, we will analyse the determination of EPRL boundary data from geometric constructions, in order to obtain values for H in concrete cases.

While showing consistency of the EPRL behaviour with Einstein theory in such examples is in no way a proof for the general case even within Δ_3, it would nevertheless be an interesting result, and on the flipside, an inconsistency would be a significant result on its own, albeit a negative one. To summarize the possible outcomes:

- $det\, H = 0$: then the stationary phase method is not valid (in particular the quantity A is not defined), and we must find a different method to evaluate the asymptotics;

- $det\, H \neq 0$ and $A = 0$: in that case the Gaussian distribution (137) has infinite standard deviation and as such will not specify the semiclassical value of x, failing to reproduce the expected classical result;

- $det\, H \neq 0$ and $A \neq 0$: the Gaussian distribution around the semiclassical value of x should guarantee reproduction of the expected geometric values. In particular, if one can verify this to happen for a certain boundary configuration, continuity conditions assure that the EPRL asymptotics match the expected classic solutions in a certain open neighbourhood of that configuration, which would give us some confidence that the semiclassical limit is correct for a significant range of boundary data. It does not, however, discard the possibility of there existing isolated points in the critical surface for which one of the two situations above happen, and it is unclear how this would affect the overall statistics.

4.4 Constructing EPRL spin foam variables from geometrical data

Obtaining the spins j_f and the Livine-Speziale coherent states $|n_{ef}\rangle$ for a triangulated geometry Δ explicitly defined by its coordinates is straightforward. Indeed, it has already been established that j_f are directly related to the triangle areas via $A_f = \gamma j_f$, while the $|n_{ef}\rangle$ are expressed in terms of $\vec{n}_{ef} \in S^2$, the normal vectors to the tetrahedron faces' Euclidean images in the tangent spaces $T_e\Delta \approx \mathbb{R}^3$ and phases $\alpha_{ef} \in \mathrm{U}(1)$ which can be consistently defined by imposing Regge boundary conditions but are of no consequence to the dynamics of the model, and can therefore safely be ignored. Obtaining the g_{ve} is somewhat less trivial, though. The first step is to identify what they represent geometrically. Indeed, g_{ve} are SO(4) group elements related to the triangulated equivalent of the spin connection, which in the geometrical setup translates to mapping the geometrical tetrahedron $e \in v$ to its image in the tangent space $T_e\Delta$. We have to define what this means, though.

Consider a 4-simplex $v \in \Delta$ and a tetrahedron $e \in v$ defined by points $p_1, ..., p_4$. Note that for a general triangulation each 4-simplex lives on its own copy of $\mathbb{R}^4$: if the entire triangulation can be embedded isometrically in $\mathbb{R}^4$ that implies all the deficit angles are zero and the triangulation is flat. We will define the tetrahedron's geometric matrix M_{ve} and projected matrix $M_{ve}^{(3)}$:

- to construct M_{ve}, consider an oriented trivector $\tau_{ve} = \{\tau_{ve}^1, \tau_{ve}^2, \tau_{ve}^3\}$ consisting of the three edge vectors coming out of a previously defined pivot point. For example, if p_1 is chosen as the pivot, a possible trivector is $\{p_2 - p_1, p_3 - p_1, p_4 - p_1\}$. If e is non-degenerate, the trivector defines a (non-orthonormal) basis of the 3-dimensional hyperplane e lives on, which can be equated to $T_e\Delta$. Compute the normal to this hyperplane, N_{ve}, which is the normal to the tetrahedron. Note that there are two possible orientations for this normal, so we will establish as a convention that the orientation to choose is the one that makes $\det M_{ve} > 0$. The full matrix is then
$$M_{ve} = \left\{N_{ve}, \tau_{ve}^1, \tau_{ve}^2, \tau_{ve}^3\right\}. \tag{141}$$
Note that this matrix is, by construction, invertible, since its 4 columns are linearly independent.

- for $M_{ve}^{(3)}$, write down an orthonormal basis of $T_e\Delta$ as defined above, for example using the Gram-Schmidt orthonormalization algorithm, and determine the coordinates of the vectors in τ_{ve} on that basis. Call them $\tau_{ve}^{(3)}$. We will regard $T_e\Delta$ as a subspace of $\mathbb{R}^4$ normal to $(1, 0, 0, 0)$, since it will help with decomposing g_{ve} into its SU(2) components $g_{ve}^{\pm}$. The projected tetrahedron matrix is then
$$M_{ve}^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & (\tau_{ve}^1)^{(3)} & (\tau_{ve}^2)^{(3)} & (\tau_{ve}^3)^{(3)} \\ 0 & & & \end{bmatrix} \tag{142}$$
which is also invertible by the same reasons as above.

Note that M_{ve} is not unique to a tetrahedron, but the g_{ve} rotation will be well defined provided that the orientations of both are consistent with respect to the considerations of section 2, that is, deriving the orientation of each tetrahedron from the 4-simplex v by (45) and permuting the edge vectors in τ_{ve} to guarantee the same sign for all M_{ve} associated with v. With these definitions in place, g_{ve} is the SO(4) matrix that rotates the projected matrix into the geometric matrix, i.e.

$$\begin{aligned} g_{ve} \cdot M_{ve}^{(3)} &= M_{ve} \\ \Leftrightarrow g_{ve} &= M_{ve} \left(M_{ve}^{(3)}\right)^{-1} \end{aligned} \tag{143}$$

Next step is to find g_{ve}'s SU(2) components. To do this we will use a result of van Elfrinkhof [54] which gives an algorithm for decomposition of a SO(4) rotation into left- and right-isoclinic rotations, which can each be associated to SU(2) elements. Given a matrix $g \in \mathrm{SO}(4)$, define the associate matrix

$$\mathrm{Asc}(g) = \frac{1}{4}\begin{bmatrix} g_{00}+g_{11}+g_{22}+g_{33} & g_{10}-g_{01}-g_{32}+g_{23} & g_{20}+g_{31}-g_{02}-g_{13} & g_{30}-g_{21}+g_{12}-g_{03} \\ g_{10}-g_{01}+g_{32}-g_{23} & -g_{00}-g_{11}+g_{22}+g_{33} & g_{30}-g_{21}-g_{12}+g_{03} & -g_{20}-g_{31}-g_{02}-g_{13} \\ g_{20}-g_{31}-g_{02}+g_{13} & -g_{30}-g_{21}-g_{12}-g_{03} & -g_{00}+g_{11}-g_{22}+g_{33} & g_{10}+g_{01}-g_{32}-g_{23} \\ g_{30}+g_{21}-g_{12}-g_{03} & g_{20}-g_{31}+g_{02}-g_{13} & -g_{10}-g_{01}-g_{32}-g_{23} & -g_{00}+g_{11}+g_{22}-g_{33} \end{bmatrix}. \tag{144}$$

van Elfrinkhof's theorem states that $\mathrm{Asc}(g)$ has rank one and is normalized under the Euclidean norm, $\sum_{ij}\left(\mathrm{Asc}(g)_{ij}\right)^2 = 1$, and that there exists a duo of vectors (a,b,c,d) and (p,q,r,s) in $S^3 \times S^3$ such that

$$\mathrm{Asc}(g) = \begin{bmatrix} ap & aq & ar & as \\ bp & bq & br & bs \\ cp & cq & cr & cs \\ dp & dq & dr & ds \end{bmatrix}. \tag{145}$$

More precisely, there are exactly two vector pairs in $S^3 \times S^3$ that satisfy this, since for a given $\{(a,b,c,d),\ (p,q,r,s)\}$, their opposites $\{(-a,-b,-c,-d),\ (-p,-q,-r,-s)\}$ also constitute a solution. Since there is a group isomorphism between S^3 and SU(2) given by

$$\begin{aligned} \phi :\ S^3 &\rightarrow \mathrm{SU}(2) \\ (a,b,c,d) &\rightarrow a\mathbf{1} + i\left(b\sigma_1 + c\sigma_2 + d\sigma_3\right), \end{aligned} \tag{146}$$

where σ_i are the Pauli matrices and $\mathbf{1}$ is the identity matrix in SU(2), the aforementioned vector duos are directly mapped to SU(2) group elements. The decomposition is made explicit within SO(4) by the formula

$$g = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \cdot \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}. \tag{147}$$

where the left and right matrices are left-isoclinic and right-isoclinic, respectively. (147) can also be specified neatly in quaternion notation. Consider the set of quaternions

$\mathbb{H} \approx \mathbb{R}^4$ with the basis vectors $\mathbf{1}$, I, J, K. $\mathbb{H}$ can also be defined in $\mathbb{C}^{2\times 2}$ by extending the domain of the map ϕ in (146) to all of $\mathbb{R}^4$. Using the latter formulation, the SU(2)×SU(2) action on a vector $v \in \mathbb{H}$ is neatly written as

$$(g^+,\, g^-) \cdot v = g^+ v \left(g^-\right)^{-1} \tag{148}$$

and translates to the action of the SO(4) matrix with $(g^+,\, g^-)$ as its left and right isoclinic components according to the van Elfrinkhof formula. We will use these results to establish the correspondence

$$\begin{aligned} g^+_{ve} &= \phi(a, b, c, d) \\ g^-_{ve} &= [\phi(p, q, r, s)]^{-1}\,. \end{aligned} \tag{149}$$

Now there is an issue with this definition, which is the ambiguity between which of the two vector pairs that solve the van Elfrinkhof theorem to choose for each g_{ve} in order to maintain consistency, since SU(2)×SU(2) double covers SO(4). We will address this problem by establishing an algorithm. For notation simplicity write $M \equiv \mathrm{Asc}(g)$. First analyze cases where $M_{11} \neq 0$ (resulting that $a,\, p \neq 0$). Define

$$K = \sqrt{M_{11}^2 + M_{12}^2 + M_{13}^2 + M_{14}^2} \tag{150}$$

Since, using (145),

$$p = \frac{M_{11}}{a};\ q = \frac{M_{12}}{a};\ r = \frac{M_{13}}{a};\ s = \frac{M_{14}}{a} \tag{151}$$

and $p^2 + q^2 + r^2 + s^2 = 1$, it follows that $a = \pm\sqrt{M_{11}^2 + M_{12}^2 + M_{13}^2 + M_{14}^2} = \pm K$. For the sake of consistency we will always take the positive root $a = K$. It is then straightforward to obtain

$$\begin{aligned} p &= \frac{M_{11}}{K};\ q = \frac{M_{12}}{K};\ r = \frac{M_{13}}{K};\ s = \frac{M_{14}}{K} \\ a &= K;\ b = K\frac{M_{21}}{M_{11}};\ c = \frac{M_{31}}{M_{11}};\ d = \frac{M_{41}}{M_{11}} \end{aligned} \tag{152}$$

Whenever $M_{11} \neq 0$ this algorithm provides a consistent definition of the g^+ and g^-, but when $M_{11} = 0$ a similar process can be carried out by choosing a non-zero entry M_{ij} (it exists since both parameter vectors are non-zero) and defining

$$K = \sqrt{\sum_{l=1}^{4} M_{il}^2}. \tag{153}$$

If we use the notation $(a, b, c, d) \equiv (x_1, x_2, x_3, x_4)$ and $(p, q, r, s) = (y_1, y_2, y_3, y_4)$ then we can define a solution for them as follows:

$$\begin{aligned} x_i &= K \\ y_l &= \frac{M_{il}}{K},\, l \in \{1, 2, 3, 4\} \\ x_l &= K\frac{M_{lj}}{M_{ij}}, l \neq i. \end{aligned} \tag{154}$$

To finalize this section we will mention the two geometrical examples that are being worked on. Given the circumstances of the flatness problem, it was deemed appropriate to consider a flat and a non-flat version of Δ_3 in calculations. As mentioned above, a flat triangulation is easily defined by considering an embedding of it in $\mathbb{R}^4$, but it's somewhat less trivial to define a non-flat one. For the latter we will consider a figure analogous to a triangulation of S^4 by taking an embedding of Δ_3 into $\mathbb{R}^5$ given by an equilateral 5-simplex centered at the origin. This embedding is defined by assigning the 6 points of Δ_3 into the 6 points of the 5-simplex. Refer to appendix 2 for the current status and notes on this.

Let us define the equilateral 5-simplex by building it "from the ground up" from an equilateral triangle centered at the origin. A triangle in $\mathbb{R}^2$ with the desired characteristics is given by

$$\{A_2, B_2, C_2\} = \left\{\left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right), \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right), \left(0, \frac{1}{\sqrt{3}}\right)\right\}. \tag{155}$$

Adding the third axis x^2 we see that if a fourth point is $D_3 = (0, 0, a_3)$, then the tetrahedron formed by

$$\{A_3, B_3, C_3, D_3\} = \left\{\left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}, -\frac{a_3}{3}\right), \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}, -\frac{a_3}{3}\right), \left(0, \frac{1}{\sqrt{3}}, -\frac{a_3}{3}\right), (0, 0, a_3)\right\} \tag{156}$$

is centered in the origin and a_3 can be fixed to make it equilateral by forcing $\overline{C_3D_3} = 1$. (Note that if O_3 is the centre of the triangle $A_3B_3C_3$ then O_3D_3 is normal to said triangle and therefore $\overline{A_3D_3} = \overline{B_3D_3} = \overline{C_3D_3}$.) Solving that constraint gives $a_3 = \sqrt{\frac{3}{8}}$. Similarly, we construct a 4-simplex under the same conditions by adding the axis x^3, defining the point $E_4 = (0, 0, 0, a_4)$ and considering the 4-simplex

$$\begin{aligned}\{A_4, B_4, C_4, D_4, E_4\} \quad = \quad & \left\{\left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}, -\frac{1}{3}\sqrt{\frac{3}{8}}, -\frac{a_4}{4}\right), \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}, -\frac{1}{3}\sqrt{\frac{3}{8}}, -\frac{a_4}{4}\right),\right. \\ & \left(0, \frac{1}{\sqrt{3}}, -\frac{1}{3}\sqrt{\frac{3}{8}}, -\frac{a_4}{4}\right), \left(0, 0, \sqrt{\frac{3}{8}}, -\frac{a_4}{4}\right), \\ & \left.(0, 0, 0, a_4)\right\}. \end{aligned} \tag{157}$$

By analogous argument to what we used for the tetrahedron, this 4-simplex is centered in the origin and will be equilateral if $\overline{D_4E_4} = 1$, which is solved to give $a_4 = \sqrt{\frac{2}{5}}$. Finally, we add the axis x^4, define $F_5 = (0, 0, 0, 0, a_5)$ and consider the 5-simplex

$$\begin{aligned}\{A_5, B_5, C_5, D_5, E_5, F_5\} \quad = \quad & \left\{\left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}, -\frac{1}{3}\sqrt{\frac{3}{8}}, -\frac{1}{4}\sqrt{\frac{2}{5}}, -\frac{a_5}{5}\right), \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}, -\frac{1}{3}\sqrt{\frac{3}{8}}, -\frac{1}{4}\sqrt{\frac{2}{5}}, -\frac{a_5}{5}\right),\right. \\ & \left(0, \frac{1}{\sqrt{3}}, -\frac{1}{3}\sqrt{\frac{3}{8}}, -\frac{1}{4}\sqrt{\frac{2}{5}}, -\frac{a_5}{5}\right), \left(0, 0, \sqrt{\frac{3}{8}}, -\frac{1}{4}\sqrt{\frac{2}{5}}, -\frac{a_5}{5}\right), \\ & \left.\left(0, 0, 0, \sqrt{\frac{2}{5}}, -\frac{a_5}{5}\right), (0, 0, 0, 0, a_5)\right\}. \end{aligned} \tag{158}$$

The 5-simplex has the characteristics we need if $\overline{E_5F_5} = 1$, which is satisfied when $a_5 = \sqrt{\frac{5}{12}}$. The coordinates of the equilateral 5-simplex to be used are therefore

$$\begin{aligned}\{A_5, B_5, C_5, D_5, E_5, F_5\} \quad = \quad & \left\{\left(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}, -\frac{1}{3}\sqrt{\frac{3}{8}}, -\frac{1}{4}\sqrt{\frac{2}{5}}, -\frac{1}{5}\sqrt{\frac{5}{12}}\right), \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}, -\frac{1}{3}\sqrt{\frac{3}{8}}, -\frac{1}{4}\sqrt{\frac{2}{5}}, -\frac{1}{5}\sqrt{\frac{5}{12}}\right),\right. \\ & \left(0, \frac{1}{\sqrt{3}}, -\frac{1}{3}\sqrt{\frac{3}{8}}, -\frac{1}{4}\sqrt{\frac{2}{5}}, -\frac{1}{5}\sqrt{\frac{5}{12}}\right), \left(0, 0, \sqrt{\frac{3}{8}}, -\frac{1}{4}\sqrt{\frac{2}{5}}, -\frac{1}{5}\sqrt{\frac{5}{12}}\right), \\ & \left.\left(0, 0, 0, \sqrt{\frac{2}{5}}, -\frac{1}{5}\sqrt{\frac{5}{12}}\right), \left(0, 0, 0, 0, \sqrt{\frac{5}{12}}\right)\right\}. \end{aligned} \tag{159}$$

This example is particularly simple in numeric terms since the construction implies that all triangles have the same area $A_f = \sqrt{3}/4$, and the normal vectors $\vec{n}_{ef}$ can all be derived from the same equilateral tetrahedron in $\mathbb{R}^3$, taking only care to match their orientations correctly.

Conclusions and future work

Presently the main unfinished work of this research is the example calculations of EPRL variables and stationary phase-approximated partition function for twoΔ_3-like configurations, a curved one based off an equilateral 5-simplex and a flat one with its 6 points embedded in $\mathbb{R}^4$, methods used to carry them out having been detailed in Section 4.4. However, there are already a few remarks that we would like to convey with this work.

The first one is that varying the asymptotic EPRL action with respect to j_f, with these being discrete, is a delicate issue, and one that we do not believe can be tackled by simply ignoring discreteness and taking some *ad hoc* continuum approximation to be able to differentiate with respect to those spins. Although that line of thought was what originally lead to the enunciation of the flatness problem, Hellmann/Kaminski seem to have recovered it under a more rigorous approach with their holonomy spin foam formalism. In this work we attempted to explicitly acknowledge the discreteness of j and study its effects on the statistics of the partition function, by using the Malgrange preparation theorem and its corollaries to apply the stationary phase method, and explicit the distribution with respect to j in a neighbourhood of the critical point. However, the validity of this method is dependent on the A quantity defined in section 4.3 being finite and mathematically meaningful, which essentially comes down to whether the Hessian determinant of the action is non-zero at the (singular) critical point for any possible boundary configuration. It is a highly non-trivial task from a computational point of view to verify this, so for the time being we would settle with finishing the calculation for the example cases proposed.

At the time of final submission of this thesis, we were indeed able to numerically compute the Hessian of the action and the quantity A for the example of a configuration embedded in an equilateral 5-simplex, and we assert they are both non-zero for this configuration. This is a positive, albeit incomplete, sign of consistency of the spin foam model in this example, since it allows us to assert by continuity arguments that the same is valid in a neighbourhood of the critical point considered. It would be helpful to conduct a more detailed statistical analysis of the behaviour of this example's partition function for values of j near the geometric one, and that is a question to be considered in subsequent work.

The second remark is the positive result that, for this Δ_3 configuration, containing only one interior face whose data are entirely specified at the classical level by boundary data, it is possible to recover the expected critical point of the action, corresponding to the values of area and deficit angle for the interior triangle that ensure proper geometric gluing. Incidentally, this result also allows us to perform the converse of the reconstruction theorem and recover EPRL variables from geometric variables in concrete realizations of the triangulation. The assertion that the critical point for a given boundary configuration is unique and corresponds to the expected classical geometry had already been verified by Perini and Magliaro in [52], but the subtleties regarding the statistics of the partition function's distribution over j are not addressed in their work (it is just assumed that non-critical configurations are exponentially suppressed), in particular the fact that the classical j_0 may not be an integer, and in general the range of j near j_0 that contributes significantly to the partition function (even in the circumstances where stationary phase applies correctly with $A \neq 0$) is dictated by a Gaussian distribution whose width increases with λ, although the relative uncertainty $\Delta j/j \approx \Delta j/\lambda$ is suppressed for large λ. We hope that our analysis (with the work that is yet to be done) will bring some more clarity to those issues.

References

[1] L.H. Ford, "Quantum Field Theory in Curved Spacetime" [gr-qc/9707062]

[2] L. Garay, "Quantum Gravity and Minimum Length" [gr-qc/9403008v2]

[3] Wald, Robert M. (1984). General Relativity. University of Chicago Press. ISBN 978-0-226-87033-5

[4] M. Goroff, A. Sagnotti, "The ultraviolet behavior of Einstein gravity", Nuclear Physics, Section B, Volume 266, Issue 3, p. 709-736.

[5] W.G. Unruh, "Notes on black-hole evaporation", Physical Review D 14 (4): 870

[6] S. Holst, "Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action" [gr-qc/9511026]

[7] R. Arnowitt, S. Deser, C. Misner, "Dynamical Structure and Definition of Energy in General Relativity". Physical Review 116 (5): 1322–1330

[8] C. Rovelli, "Loop Quantum Gravity", Living Reviews in Relativity Vol. 1 (1998) [http://www.livingreviews.org/lrr-1998-1]

[9] E. Livine, “A Short and Subjective Introduction to the Spinfoam Framework for Quantum Gravity” [arXiv:1101.5061]

[10] G. Ponzano, T. Regge, “Semiclassical limit of Racah coefficients”, Spectroscopic and Group Theoretical Methods in Physics, pp.1-58

[11] T. Thiemann, “Introduction to Modern Canonical Quantum General Relativity” [gr-qc/0110034]

[12] A. Ashtekar, J. Lewandowski, “Projective Techniques and Functional Integration” [gr-qc/9411046]

[13] J. Lewandowski, A. Okolow, H. Sahlmann, T. Thiemann, “Uniqueness of diffeomorphism invariant states on holonomy-flux algebras” [gr-qc/0504147]

[14] K. Giesel, S. Hofmann, T. Thiemann and O. Winkler, “Manifestly Gauge-Invariant General Relativistic Perturbation Theory”, [arXiv:0711.0115], [arXiv:0711.0117]

[15] T. Thiemann, “Quantum Spin Dynamics” [arXiv:gr-qc/9606089] [arXiv:gr-qc/9606090]

[16] F. Peter, H. Weyl (1927), "Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe", Math. Ann. 97: 737–755

[17] R. Penrose, “Angular momentum: an approach to combinatorial spacetime”, Cambridge University Press 1971, pp. 151–180

[18] A. Ashtekar, J. Baez, A. Corichi, K. Krasnov, "Quantum Geometry and Black Hole Entropy". [arXiv:gr-qc/9710007]

[19] L. Freidel, D. Louapre, E. Livine, “Ponzano-Regge model revisited” [arXiv:hep-th/0401076] [arXiv:gr-qc/0410141] [arXiv:hep-th/0502106]

[20] U. Pachner, ”P.L. Homeomorphic Manifolds are Equivalent by Elementary Shellings”, doi:10.1016/S0195-6698(13)80080-7

[21] R. Gurau, “The Ponzano-Regge asymptotic of the 6j symbol: an elementary proof” [arXiv:0808.3533]

[22] V. Turaev, O. Viro, “State sum invariants of 3-manifolds and quantum $6j$ symbols”, Topology Vol. 31, No. 4, pp. 865–902, 1992

[23] J.F. Plebanski, J. Math. Phys. 16, 2395 (1975)

[24] R. Capovilla, M. Montesinos, V. A. Prieto, E. Rojas, ”BF gravity and the Immirzi parameter” [gr-qc/0102073]

[25] E. Alesci, C. Rovelli, ”The complete LQG propagator I. Difficulties with the Barrett-Crane vertex” [arXiv:0708.0883]

[26] B. Bahr, F. Hellmann, W. Kaminski, M. Kisielowski, J. Lewandowski, ”Operator Spin Foam Models” [arXiv:1010.4787]

[27] F. Hellmann, W. Kaminski, ”Holonomy spin foam models: Asymptotic geometry of the partition function” [arXiv:1307.1679]

[28] S. Carlip, “Lectures in (2+1)-dimensional gravity”, J.Korean Phys.Soc.28:S447-S467,1995 [gr-qc/9503024]

[29] H. Ooguri, “Topological lattice models in four-dimensions”, Mod. Phys. Lett. A7, 2799–2810 (1992), [hep-th/9205090]

[30] L. Crane and D. Yetter, “A categorical construction of 4d TQFTs”, Quantum Topology eds. L. Kauff- man and R. Baadhio, World Scientific, Singapore, 1993, pp. 120–130. [hep-th/9301062]

[31] J.C. Baez, “An introduction to spin foam models of quantum gravity and BF theory”, Lect. Notes Phys., vol 543, p. 25, 2000 [gr-qc/9905087]

[32] J. Engle, E. Livine, R. Pereira, C. Rovelli, “LQG vertex with finite Immirzi parameter”, Nucl. Phys. vol. B799, pp. 136-149, 2008 [gr-qc/0711.0146]

[33] L. Freidel, K. Krasnov, “A New Spin Foam Model for 4D Gravity”, Class. Quant. Grav. vol. 25, p. 125018, 2008 [gr-qc/0708.1595]

[34] E. Livine, S. Speziale, “A new spinfoam vertex for quantum gravity”, Phys. Rev. D vol. 76, p. 084028, 2007 [gr-qc/0705.0674]

[35] M. Han, M. Zhang, “Asymptotics of Spin Foam Amplitude on Simplicial Manifold: Euclidean Theory” [gr-qc/1109.0500]

[36] M. Han, M. Zhang, “Asymptotics of Spin Foam Amplitude on Simplicial Manifold: Lorentzian Theory” [gr-qc/1109.0499]

[37] C. Rovelli, L. Smolin, “Loop space representation of quantum general relativity”, Nucl. Phys. B vol. 331, issue 1, pp. 80-152.

[38] J.W. Barrett, R. Dowdall, W. Fairbairn, H. Gomes, F. Hellmann, “Asymptotic analysis of the EPRL four-simplex amplitude”, J. Math. Phys. 50:112504, 2009 [gr-qc/0902.1170]

[39] L. Kauffman, “State models and the Jones polynomial”, Topology 26 (1987), no. 3, pp. 395-407.

[40] J.W. Barrett, R. Dowdall, W. Fairbairn, F. Hellmann, R. Pereira, “Lorentzian spin foam amplitudes: graphical calculus and asymptotics” [gr-qc/0907.2440]

[41] E. Bianchi, D. Regoli, C. Rovelli, “Face amplitude of spinfoam quantum gravity”, Class. Quant. Grav. 27:185009, 2010 [gr-qc/1005.0764]

[42] V. Bonzom, “Spin foam models for quantum gravity from lattice path integrals”, Phys. Rev. D 80:064028, 2009 [gr-qc/0905.1501]

[43] T. Regge, “General Relativity Without Coordinates”, Il Nuovo Cimento, Vol. 19, N. 3, p. 558

[44] F. Conrady, L. Freidel, “Path integral representation of spin foam models of 4D gravity”, Class. Quant. Grav. 25, 245010, 2008 [gr-qc/0806.4640]

[45] F. Conrady, L. Freidel, “On the semiclassical limit of 4D spin foam models”, Phys. Rev. D 78, 104023, 2008 [gr-qc/0809.2280]

[46] J.W. Barrett, L. Crane, “Relativistic Spin Networks and Quantum Gravity”, J.Math.Phys. 39, 3296-3302, 1998 [gr-qc/9709028]

[47] J.W. Barrett, L. Crane, A Lorentzian Signature Model for Quantum General Relativity [arXiv:gr-qc/9904025]

[48] V. Guillemin, S. Sternberg, “Symplectic Techniques in Physics”, Cambridge University Press 1990

[49] L. Hörmander, “The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis”, Springer-Verlag, 2nd Edition.

[50] G. Lachaud, “Exponential Sums as Discrete Fourier Transform with Invariant Phase Functions”, Proceedings of the 10th International Symposium on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, pp. 231-243

[51] International Journal of Mathematics and Mathematical Sciences, Volume 2003 (2003), Issue 49, Pages 3091-3099 [http://math.iisc.ernet.in/~rangaraj/docs/ijmms_so_n.pdf]

[52] E. Magliaro, C. Perini, ”Curvature in spinfoams”, Class.Quant.Grav. 28 (2011) 145028 [gr-qc/1103.4602]

[53] E. Anderson ,”The Problem of Time in Quantum Gravity” [arXiv:1009.2157]

[54] L. van Elfrinkhof, “Eene eigenschap van de orthogonale substitutie van de vierde orde”, Handelingen van het 6e Nederlandsch Natuurkundig en Geneeskundig Congres, Delft, 1897.

Appendix 1 - Sketching the Hessian determinant calculation

The following is a Mathematica script for computing the relevant derivatives of the Δ_3 EPRL action to determine the asymptotic distribution,

$$\begin{aligned} K_i &= \left.\frac{\partial^2 S}{\partial j_f \partial x_i}\right|_{\text{critical}} = \left.\frac{\partial \tilde{\Theta}_f}{\partial x_i}\right|_{\text{critical}} \\ H_{ij} &= \left.\frac{\partial^2 S}{\partial x_i \partial x_j}\right|_{\text{critical}} \end{aligned}$$

for a given set of boundary data. The action can be separated into the following components

$$S = j_f \tilde{\Theta}_f + S_{\text{coupling}} + S_{\text{pure boundary}} \tag{160}$$

where $\tilde{\Theta}_f$ is the "quantum deficit angle" on the interior face f and the first term defines the interior action S_I. The other two terms make up the boundary action, but S_{coupling} groups up the terms depending in interior g_{ve} by matrix elements of the form

$$\langle \mathcal{J} n_{e_B f} | \left(g^{\pm}_{ve_B}\right)^{-1} \left(g^{\pm}_{ve_I}\right) | n_{e_I f} \rangle , \tag{161}$$

corresponding to faces f shared by one interior edge e_I and one boundary edge e_B. These terms also contribute towards the Hessian matrix via the second derivatives

$$\left.\frac{\partial^2 S}{\partial \left(g_{ve}\right)_i \partial \left(g_{ve}\right)_j}\right|_{\text{critical}} . \tag{162}$$

In the Mathematica file the contributions from S_I and S_{coupling} are computed separately, using these facts. Some results are not shown because they are cumbersome and add little to the discussion.

```
(* Computing the Hessian matrix and its determinant for geometrical
 constructions of Δ3. First we introduce some definitions. *)
(* Our parameterization of SU(2) by angles 0 < γ < Pi/2, 0 < β1, β2 < 2 Pi *)
```

```
In[3157]:= g[γ_, β1_, β2_] = {{Cos[γ] Exp[I β1], I Sin[γ] Exp[-I β2]},
    {I Sin[γ] Exp[I β2], Cos[γ] Exp[-I β1]}}; g[γ, β1, β2] // MatrixForm
```

Out[3157]//MatrixForm=

$$\begin{pmatrix} e^{i\beta 1}\,\mathrm{Cos}[\gamma] & i\,e^{-i\beta 2}\,\mathrm{Sin}[\gamma] \\ i\,e^{i\beta 2}\,\mathrm{Sin}[\gamma] & e^{-i\beta 1}\,\mathrm{Cos}[\gamma] \end{pmatrix}$$

```
In[3158]:= (* Definition of a coherent state |
  n> depending on the spherical coordinates of the vector n (θ,ϕ) *)
 (* and an arbitrary phase α *)

 n[α_, θ_, ϕ_] = Exp[I α] {{Cos[θ / 2]}, {-Exp[I ϕ] Sin[θ / 2]}}; n[α, θ, ϕ] // MatrixForm
```

Out[3158]//MatrixForm=

$$\begin{pmatrix} e^{i\alpha}\,\mathrm{Cos}\left[\frac{\theta}{2}\right] \\ -e^{i\alpha+i\phi}\,\mathrm{Sin}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

```
In[3159]:= (* Using the definition for the antilinear operator J in the 2 qubit space,
 J(a,b) = (-I b, I a) *)
 (* we obtain an expression for Jn based on the parameterization above *)
 Jn[α_, θ_, ϕ_] = Exp[I α] {{I Exp[I ϕ] Sin[θ / 2]}, {I Cos[θ / 2]}}; Jn[α, θ, ϕ] // MatrixForm
```

Out[3159]//MatrixForm=

$$\begin{pmatrix} i\,e^{i\alpha+i\phi}\,\mathrm{Sin}\left[\frac{\theta}{2}\right] \\ i\,e^{i\alpha}\,\mathrm{Cos}\left[\frac{\theta}{2}\right] \end{pmatrix}$$

In[3160]:=
```
(* the cartesian formulation is also useful *)
Jn2[n1_, n2_] = {-I n2, I n1}
```

Out[3160]=
```
{-ⅈ n2, ⅈ n1}
```

In[3161]:=
```
(* The Hessian comes in the form H = Hpure + Hcoupling, *)
(* where Hpure comes from derivatives of the interior action and *)
(* Hcoupling comes from the dependency of
 the boundary action on the interior group elements *)

(* We start with computing Hpure. The interior action will be called Spure here. *)
spure =
  Simplify[-2 I (1 + γ) (Log[ConjugateTranspose[n[αBA, θBA, ϕBA]].ConjugateTranspose[g[
           γA5P, β1A5P, β2A5P]].g[γA6P, β1A6P, β2A6P].n[αAC, θAC, ϕAC]] +
        Log[ConjugateTranspose[n[αCB, θCB, ϕCB]].ConjugateTranspose[
           g[γB6P, β1B6P, β2B6P]].g[γB4P, β1B4P, β2B4P].n[αBA, θBA, ϕBA]] +
        Log[ConjugateTranspose[n[αAC, θAC, ϕAC]].ConjugateTranspose[
           g[γC4P, β1C4P, β2C4P]].g[γC5P, β1C5P, β2C5P].n[αCB, θCB, ϕCB]]) +
     -2 I (1 - γ) (Log[ConjugateTranspose[n[αBA, θBA, ϕBA]].ConjugateTranspose[
           g[γA5M, β1A5M, β2A5M]].g[γA6M, β1A6M, β2A6M].n[αAC, θAC, ϕAC]] +
        Log[ConjugateTranspose[n[αCB, θCB, ϕCB]].ConjugateTranspose[
           g[γB6M, β1B6M, β2B6M]].g[γB4M, β1B4M, β2B4M].n[αBA, θBA, ϕBA]] +
        Log[ConjugateTranspose[n[αAC, θAC, ϕAC]].ConjugateTranspose[
           g[γC4M, β1C4M, β2C4M]].g[γC5M, β1C5M, β2C5M].n[αCB, θCB, ϕCB]]),
   Assumptions → Element[{γ, αBA, θBA, ϕBA, αCB, θCB, ϕCB, αAC, θAC, ϕAC, γA5P,
      β1A5P, β2A5P, γA6P, β1A6P, β2A6P, γB6P, β1B6P, β2B6P, γB4P,
      β1B4P, β2B4P, γC4P, β1C4P, β2C4P, γC5P, β1C5P, β2C5P, γA5M,
      β1A5M, β2A5M, γA6M, β1A6M, β2A6M, γB6M, β1B6M, β2B6M, γB4M,
      β1B4M, β2B4M, γC4M, β1C4M, β2C4M, γC5M, β1C5M, β2C5M}, Reals]];
```

In[3162]:=
```
Spure = spure[[1, 1]];
```

In[3163]:=
```
(* Define the list of variables we will differentiate with respect to *)
var = {θBA, ϕBA, θCB, ϕCB, θAC, ϕAC, γA5P, β1A5P, β2A5P, γA6P,
   β1A6P, β2A6P, γB6P, β1B6P, β2B6P, γB4P, β1B4P, β2B4P, γC4P, β1C4P, β2C4P,
   γC5P, β1C5P, β2C5P, γA5M, β1A5M, β2A5M, γA6M, β1A6M, β2A6M, γB6M, β1B6M,
   β2B6M, γB4M, β1B4M, β2B4M, γC4M, β1C4M, β2C4M, γC5M, β1C5M, β2C5M};

(* The phases α don't contribute to the action and need not be considered. *)
```

```
Length[var]
```

```
42
```

```
In[3207]:= (* Hpure is a 42 x 42 matrix which will
             be defined by explicit differentiation now. *)
           Hpure = Table[0, {i, 42}, {j, 42}];
           i = 1; j = 1;
           While[j < 43, While[i < 43, Hpure[[i, j]] = D[Spure, var[[i]], var[[j]]]; i++]; j++; i = 1];
```

```
(* The derivatives are computed at the critical point,
which we will define by inputting spin foam
 data (g, n) derived from geometric configurations. *)

(* Resulting matrix is very large and not shown in this printout,
but inputting numeric values for the given variables
 at a critical point will produce a numeric matrix. *)

(* we will also need K, which is a list of first derivatives of the pure action,
as specified on the paper. *)
```

```
In[3168]:= K = Table[0, {i, 42}];

In[3169]:= i = 1; While[i < 43, K[[i]] = D[Spure, var[[i]]]; i++];
```

```
(* Now we need to compute the derivatives for the interior-
 boundary coupling of the action. *)

(* We will compute them explicitly for a portion of the coupling
 depending on just one of the elements, namely GA5+. This is a 3x3 matrix,
and all the other portions will have similar forms by symmetry of the problem. *)
```

```
In[3173]:= SA5 =
   Simplify[-2 (xa15 ( I (1 + γ) Log[ {a11P, a12P}.g[γA5P, β1A5P, β2A5P].n[0, θA15, ϕA15]] +
            I (1 - γ) Log[ {a11M, a12M}.g[γA5M, β1A5M, β2A5M].n[0, θA15, ϕA15]] ) +
         xa25 (I (1 + γ) Log[{a21P, a22P} .g[γA5P, β1A5P, β2A5P].n[0, θA25, ϕA25]] +
            I (1 - γ) Log[{a21M, a22M} .g[γA5M, β1A5M, β2A5M].n[0, θA25, ϕA25]] ) +
         xa35 (I (1 + γ) Log[ {a31P, a32P}.g[γA5P, β1A5P, β2A5P].n[0, θA35, ϕA35]] +
            I (1 - γ) Log[ {a31M, a32M}.g[γA5M, β1A5M, β2A5M].n[0, θA35, ϕA35]] )),
      Assumptions → Element[{γ, γA5P, β1A5P, β2A5P, γA6P, β1A6P, β2A6P,
         θA15, ϕA15, θA25, ϕA25, θA35, ϕA35, a11P, a11M, a12P, a12M,
         a21P, a21M, a22P, a22M, a31P, a31M, a32P, a32M}, Reals]][[1]];
```

```
(* where a_iP = Jn_ii5 (gAiP)^-1 and a_iM =
  Jn_ii5 (gAiM)^-1 depend only on boundary data that we fill in later *)
```

```
In[3195]:= varSA5P = {γA5P, β1A5P, β2A5P};

In[3196]:= HSA5P = Table[0, {i, 3}, {j, 3}];

In[3197]:= i = 1; j = 1; While[j < 4,
             While[i < 4, HSA5P[[i, j]] = D[SA5, varSA5P[[i]], varSA5P[[j]]]; i++]; j++; i = 1];
```

```
(* HSA5P is a matrix of second derivatives of
  the coupling action with respect to the element gA5+ *)
(* Note that the boundary terms on the left have been put together into the "a"
 quantities, which can be computed for a given choice of boundary data later. *)
(* Again the resulting matrix is very large and the output is omitted,
but inputting numeric values at a critical point will produce a numeric matrix. *)
```

```
(* Preparing to plug in numeric data. Since we calculated the normals in 3-
 space we will find the corresponding coherent states in terms of them *)

Ncoherent[n1_, n2_, n3_] =
 {Sqrt[(1 + n3) / 2], -Sqrt[(1 - n3) / 2] (n1 + I n2) / Sqrt[n1^2 + n2^2]}
(* this formula doesn't work for the extreme cases of n =
 (0,0,1) corresponding to (1,0) and n = (0,0,-1) corresponding to (0,1) *)
(* but we can deal with those individually *)
```

Out[3181]= $\left\{\frac{\sqrt{1+n3}}{\sqrt{2}}, -\frac{(n1 + i\, n2)\sqrt{1-n3}}{\sqrt{2}\sqrt{n1^2+n2^2}}\right\}$

```
(* In particular for the GA5 normals, nA551,
nA552, nA553 don't include the exception case. *)

(* filling out the boundary data of HSA5P *)
```

In[3237]:=
```
aA51P = Jn2[Ncoherent[nA115[[1]], nA115[[2]], nA115[[3]]][[1]],
    Ncoherent[nA115[[1]], nA115[[2]], nA115[[3]]][[2]]].
   ConjugateTranspose[GA1P] // FullSimplify
```

Out[3237]= $\left\{\frac{1}{6}\,\mathrm{Root}\left[150\,544 + 88\,464\,\#1 + 28\,060\,\#1^2 + 4260\,\#1^3 + 21\,\#1^4 - 138\,\#1^5 + 7\,\#1^6 + 6\,\#1^7 + \#1^8\ \&,\ 2\right],\right.$
$\left.\frac{1}{6}\,\mathrm{Root}\left[10\,000 - 1400\,\#1^2 + 609\,\#1^4 - 14\,\#1^6 + \#1^8\ \&,\ 6\right]\right\}$

In[3238]:=
```
aA51M = Jn2[Ncoherent[nA115[[1]], nA115[[2]], nA115[[3]]][[1]],
    Ncoherent[nA115[[1]], nA115[[2]], nA115[[3]]][[2]]].
   ConjugateTranspose[GA1M] // FullSimplify
```

Out[3238]= $\left\{\frac{1}{6}\,\mathrm{Root}\left[38\,416 + 2156\,\#1^2 + 141\,\#1^4 + 11\,\#1^6 + \#1^8\ \&,\ 5\right],\right.$
$\left.\frac{1}{6}\,\mathrm{Root}\left[234\,256 + 6776\,\#1^2 - 63\,\#1^4 + 14\,\#1^6 + \#1^8\ \&,\ 1\right]\right\}$

In[3239]:=
```
aA52P = Jn2[Ncoherent[nA225[[1]], nA225[[2]], nA225[[3]]][[1]],
    Ncoherent[nA225[[1]], nA225[[2]], nA225[[3]]][[2]]].
   ConjugateTranspose[GA2P] // FullSimplify
```

Out[3239]= $\left\{\frac{1}{6}\,\mathrm{Root}\left[38\,416 + 2156\,\#1^2 + 141\,\#1^4 + 11\,\#1^6 + \#1^8\ \&,\ 4\right],\right.$
$\left.\frac{1}{6}\,\mathrm{Root}\left[234\,256 + 6776\,\#1^2 - 63\,\#1^4 + 14\,\#1^6 + \#1^8\ \&,\ 8\right]\right\}$

In[3240]:=
```
aA52M = Jn2[Ncoherent[nA225[[1]], nA225[[2]], nA225[[3]]][[1]],
    Ncoherent[nA225[[1]], nA225[[2]], nA225[[3]]][[2]]].
   ConjugateTranspose[GA2M] // FullSimplify
```

Out[3240]= $\left\{\frac{1}{6}\,\mathrm{Root}\left[150\,544 - 88\,464\,\#1 + 28\,060\,\#1^2 - 4260\,\#1^3 + 21\,\#1^4 + 138\,\#1^5 + 7\,\#1^6 - 6\,\#1^7 + \#1^8\ \&,\ 7\right],\right.$
$\left.\frac{1}{6}\,\mathrm{Root}\left[10\,000 - 1400\,\#1^2 + 609\,\#1^4 - 14\,\#1^6 + \#1^8\ \&,\ 3\right]\right\}$

In[3241]:=
```
aA53P = Jn2[Ncoherent[nA335[[1]], nA335[[2]], nA335[[3]]][[1]],
    Ncoherent[nA335[[1]], nA335[[2]], nA335[[3]]][[2]]].
   ConjugateTranspose[GA3P] // FullSimplify
```

Out[3241]= $\left\{\frac{1}{6}\,\mathrm{Root}\left[4 + \#1^2 + \#1^4\ \&,\ 2\right],\ \frac{1}{6}\,\mathrm{Root}\left[1\,336\,336 + 2312\,\#1^2 - 2007\,\#1^4 + 2\,\#1^6 + \#1^8\ \&,\ 4\right]\right\}$

In[3242]:=
```
aA53M = Jn2[Ncoherent[nA335[[1]], nA335[[2]], nA335[[3]]][[1]],
    Ncoherent[nA335[[1]], nA335[[2]], nA335[[3]]][[2]]].
   ConjugateTranspose[GA3M] // FullSimplify
```

Out[3242]= $\left\{\frac{1}{6}\sqrt{\frac{1}{2}\left(-\mathrm{i}+12\sqrt{2}\right)\left(\mathrm{i}+\sqrt{15}\right)}\right.$, Root$\left[1-9\,\#1^2+324\,\#1^4\,\&,\,4\right]\Big\}$

```
(* next step is to grab the S^2 coordinates for the n's attached to gA5P *)
```

In[3188]:=
```
Theta[n_] := ArcCos[n[[3]]] // FullSimplify
Phi[n_] :=
 2 ArcTan[n[[2]] / Sqrt[n[[1]]^2 + n[[2]]^2] / (1 + n[[1]] / Sqrt[n[[1]]^2 + n[[2]]^2])] //
  FullSimplify
```

```
(* again these formulas don't work for n =
 (0,0,+/-1) but those cases are dealt with separately *)

(* the final step is to grab SU(2) angle parameters from the GA5P,
GA5M matrices. The formula below works for γ different from {0,Pi/2}. The
 case of it being zero can be treated separately when it comes up. *)
```

In[3190]:=
```
Gama[g_] := ArcCos[Sqrt[g[[1, 1]] g[[2, 2]]]] // FullSimplify;
Beta1[g_] := -I / 2 Log[g[[1, 1]] / g[[2, 2]]] // FullSimplify;
Beta2[g_] := -I / 2 Log[g[[2, 1]] / g[[1, 2]]] // FullSimplify;
```

```
(* So now we can plug it all into the derivative matrix *)
```

In[3243]:=
```
HSA5Pn = HSA5P /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
    a11P → aA51P[[1]], a12P → aA51P[[2]], a21P → aA52P[[1]], a22P → aA52P[[2]],
    a31P → aA53P[[1]], a32P → aA53P[[2]], a11M → aA51M[[1]], a12M → aA51M[[2]],
    a21M → aA52M[[1]], a22M → aA52M[[2]], a31M → aA53M[[1]], a32M → aA53M[[2]],
    θA15 → Theta[nA551], ϕA15 → Phi[nA551], θA25 → Theta[nA552], ϕA25 → Phi[nA552],
    θA35 → Theta[nA553], ϕA35 → Phi[nA553], γA5P → Gama[GA5P], β1A5P → Beta1[GA5P],
    β2A5P → Beta2[GA5P], γA5M → Gama[GA5M], β1A5M → Beta1[GA5M], β2A5M → Beta2[GA5M]};
```

In[3244]:=

```
HSA5Pn // N
```

Out[3244]=

$$\left\{\left\{\frac{(10.4595+50.0885\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(16.3368-20.6955\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(25.4995-1.26999\,i)\,(1.+\gamma)}{\gamma}\right\},\ \left\{-\frac{(16.3368-20.6955\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(13.3757+2.06307\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(10.1514+15.3504\,i)\,(1.+\gamma)}{\gamma}\right\},\ \left\{-\frac{(25.4995-1.26999\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(10.1514+15.3504\,i)\,(1.+\gamma)}{\gamma},\ \frac{(8.14589-20.3831\,i)\,(1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* expressing result numerically just as a
 check. the Immirzi parameter is the only unspecified one. *)
(* we can now add this contribution in the proper place in the total Hessian matrix
  H = Hpure + Hcoupling. For example looking at the variables list of Spure,
the HA5Pn matrix should be added to the (7,7) to (9,9)
 matrix block in the pure Hessian. *)
```

In[3385]:=

```
Hfinal = Hpure;
i = 1; j = 1;
While[j < 4, While[i < 4, Hfinal[[6 + i, 6 + j]] += HSA5Pn[[i, j]]; i++]; j++; i = 1];
```

```
(* In the following we compute the remaining
 HS matrices for the remaining contributions. *)
(* They are easy to compute because the form of the corresponding
 action terms is the same as HSA5P and HSA5M (to be computed below),
and it is only necessary to replace boundary terms in the appropriate places. *)
```

```
(* HSA5M - terms depending on gA5- *)

In[3387]:= varSA5M = {γA5M, β1A5M, β2A5M};

In[3388]:= HSA5M = Table[0, {i, 3}, {j, 3}];
 (* computing derivatives with respect to the gA5- parameters *)
 i = 1; j = 1; While[j < 4,
  While[i < 4, HSA5M[[i, j]] = D[SA5, varSA5M[[i]], varSA5M[[j]]]; i++]; j++; i = 1];

In[3389]:= HSA5Mn = HSA5M /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
    a11P → aA51P[[1]], a12P → aA51P[[2]], a21P → aA52P[[1]], a22P → aA52P[[2]],
    a31P → aA53P[[1]], a32P → aA53P[[2]], a11M → aA51M[[1]], a12M → aA51M[[2]],
    a21M → aA52M[[1]], a22M → aA52M[[2]], a31M → aA53M[[1]], a32M → aA53M[[2]],
    θA15 → Theta[nA551], ϕA15 → Phi[nA551], θA25 → Theta[nA552], ϕA25 → Phi[nA552],
    θA35 → Theta[nA553], ϕA35 → Phi[nA553], γA5P → Gama[GA5P], β1A5P → Beta1[GA5P],
    β2A5P → Beta2[GA5P], γA5M → Gama[GA5M], β1A5M → Beta1[GA5M], β2A5M → Beta2[GA5M]};

In[3390]:= HSA5Mn // N
```

Out[3390]=

$$\left\{\left\{-\frac{(0.0360505+1.51782\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.925819-0.798833\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.234327+1.18223\,i)\,(-1.+\gamma)}{\gamma}\right\},\right.$$
$$\left\{\frac{(0.925819-0.798833\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(1.07089+1.74089\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.736023-0.861773\,i)\,(-1.+\gamma)}{\gamma}\right\},$$
$$\left.\left\{\frac{(0.234327+1.18223\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.736023-0.861773\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(1.08891-0.0982808\,i)\,(-1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* the position for this matrix to be summed is (25,25) to (27,27) *)

In[3391]:= i = 1; j = 1;
 While[j < 4, While[i < 4, Hfinal[[24 + i, 24 + j]] += HSA5Mn[[i, j]]; i++]; j++; i = 1];
```

```
(* HSA6P - terms depending on gA6+ *)

(* as said earlier we can use the same derivatives we computed earlier,
just recalculating boundary-dependent terms *)
```

In[3252]:=
```
nA116
nA226
nA336
```

Out[3252]= {0, 0, -1}

Out[3253]= {0, 0, -1}

Out[3254]= {0, 0, -1}

```
(* The necessary n's for this section are all equal to
 one of the Ncoherent formula's exceptions. (0,0,-1) corresponds
 to the coherent state (0,1) which we replace manually *)
```

In[3256]:=
```
aA61P = Jn2[0, 1].ConjugateTranspose[GA1P] // FullSimplify
aA62P = Jn2[0, 1].ConjugateTranspose[GA2P] // FullSimplify
aA63P = Jn2[0, 1].ConjugateTranspose[GA3P] // FullSimplify
```

Out[3256]= $$\left\{\frac{1}{24}\left(-3+3\sqrt{5}-2\sqrt{30}-i\sqrt{6\left(19+3\sqrt{5}-2\sqrt{6}\left(3+\sqrt{5}\right)\right)}\right),\ \frac{1}{24}\left(-3+3\sqrt{5}+2\sqrt{30}+i\sqrt{6\left(19+3\sqrt{5}+2\sqrt{6}\left(3+\sqrt{5}\right)\right)}\right)\right\}$$

Out[3257]= {√Root[1 + 2 #1 - 201 #1^2 - 82 #1^3 + 13 849 #1^4 + 14 088 #1^5 + 26 352 #1^6 + 17 280 #1^7 + 20 736 #1^8 &, 5], √Root[1 + 2 #1 - 201 #1^2 - 82 #1^3 + 13 849 #1^4 + 14 088 #1^5 + 26 352 #1^6 + 17 280 #1^7 + 20 736 #1^8 &, 2]}

Out[3258]= {Root[9 + 6 #1^2 - 47 #1^4 + 24 #1^6 + 144 #1^8 &, 5], Root[9 + 6 #1^2 - 47 #1^4 + 24 #1^6 + 144 #1^8 &, 8]}

In[3259]:=
```
aA61M = Jn2[0, 1].ConjugateTranspose[GA1M] // FullSimplify
aA62M = Jn2[0, 1].ConjugateTranspose[GA2M] // FullSimplify
aA63M = Jn2[0, 1].ConjugateTranspose[GA3M] // FullSimplify
```

Out[3259]= {Root[1 + 2 #1^2 - 201 #1^4 - 82 #1^6 + 13 849 #1^8 + 14 088 #1^10 + 26 352 #1^12 + 17 280 #1^14 + 20 736 #1^16 &, 6], $\frac{-6 i\sqrt{2}-2 i\sqrt{3}-6\sqrt{5}+5\sqrt{6}+3 i\sqrt{10}}{12\sqrt{19+3\sqrt{5}-2\sqrt{6}\left(3+\sqrt{5}\right)}}$}

Out[3260]= {Root[1 - 6 #1 + 17 #1^2 - 66 #1^3 + 151 #1^4 - 60 #1^5 + 12 #1^6 - 144 #1^7 + 144 #1^8 &, 6], Root[1 - 6 #1 + 17 #1^2 - 66 #1^3 + 151 #1^4 - 60 #1^5 + 12 #1^6 - 144 #1^7 + 144 #1^8 &, 1]}

Out[3261]= {Root[3 + 6 #1 + 5 #1^2 + 12 #1^3 + 12 #1^4 &, 3], Root[3 + 6 #1 + 5 #1^2 + 12 #1^3 + 12 #1^4 &, 1]}

In[3262]:=
```
nA661
nA662
nA663
```

Out[3262]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3263]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3264]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3265]:=
```
GA6P
GA6M
```

Out[3265]= $\left\{\left\{\frac{i}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right\}, \left\{\frac{i}{\sqrt{2}}, -\frac{i}{\sqrt{2}}\right\}\right\}$

Out[3266]= $\left\{\left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \left\{-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}\right\}$

```
(* these are not (0,0,+/- 1) so the formulas for theta and phi work *)
```

In[3394]:=
```
HSA6Pn = HSA5P /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
    a11P → aA61P[[1]], a12P → aA61P[[2]], a21P → aA62P[[1]], a22P → aA62P[[2]],
    a31P → aA63P[[1]], a32P → aA63P[[2]], a11M → aA61M[[1]], a12M → aA61M[[2]],
    a21M → aA62M[[1]], a22M → aA62M[[2]], a31M → aA63M[[1]], a32M → aA63M[[2]],
    θA15 → Theta[nA661], ϕA15 → Phi[nA661], θA25 → Theta[nA662], ϕA25 → Phi[nA662],
    θA35 → Theta[nA663], ϕA35 → Phi[nA663], γA5P → Gama[GA6P], β1A5P → Beta1[GA6P],
    β2A5P → Beta2[GA6P], γA5M → Gama[GA6M], β1A5M → Beta1[GA6M], β2A5M → Beta2[GA6M]};
HSA6Pn //
 N
```

Out[3395]= $\left\{\left\{-\frac{(3.58707-10.2585\,i)\,(1.+\gamma)}{\gamma}, -\frac{(7.40026-1.44089\,i)\,(1.+\gamma)}{\gamma}, -\frac{(2.61285+3.54453\,i)\,(1.+\gamma)}{\gamma}\right\}, \left\{-\frac{(7.40026-1.44089\,i)\,(1.+\gamma)}{\gamma}, -\frac{(2.20041+3.35849\,i)\,(1.+\gamma)}{\gamma}, \frac{(1.05247-4.15464\,i)\,(1.+\gamma)}{\gamma}\right\}, \left\{-\frac{(2.61285+3.54453\,i)\,(1.+\gamma)}{\gamma}, \frac{(1.05247-4.15464\,i)\,(1.+\gamma)}{\gamma}, \frac{(3.99395+0.827336\,i)\,(1.+\gamma)}{\gamma}\right\}\right\}$

```
(* this matrix goes in the positions (10,10) to (12,12) *)
```

In[3396]:=
```
i = 1; j = 1;
While[j < 4, While[i < 4, Hfinal[[9 + i, 9 + j]] += HSA6Pn[[i, j]]; i++]; j++; i = 1];
```

In[3272]:=
```
(* HSA6M - terms depending on gA6- *)

HSA6Mn = HSA5M /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
    a11P → aA61P[[1]], a12P → aA61P[[2]], a21P → aA62P[[1]], a22P → aA62P[[2]],
    a31P → aA63P[[1]], a32P → aA63P[[2]], a11M → aA61M[[1]], a12M → aA61M[[2]],
    a21M → aA62M[[1]], a22M → aA62M[[2]], a31M → aA63M[[1]], a32M → aA63M[[2]],
    θA15 → Theta[nA661], ϕA15 → Phi[nA661], θA25 → Theta[nA662], ϕA25 → Phi[nA662],
    θA35 → Theta[nA663], ϕA35 → Phi[nA663], γA5P → Gama[GA6P], β1A5P → Beta1[GA6P],
    β2A5P → Beta2[GA6P], γA5M → Gama[GA6M], β1A5M → Beta1[GA6M], β2A5M → Beta2[GA6M]};
HSA6Mn //
 N
```

Out[3273]=

$$\left\{\left\{-\frac{(1.02762+5.73053\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(3.38073-1.77939\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.996288+0.184347\,i)\,(-1.+\gamma)}{\gamma}\right\},\right.$$
$$\left\{\frac{(3.38073-1.77939\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(1.58325+1.0477\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.153278+1.73469\,i)\,(-1.+\gamma)}{\gamma}\right\},\ \left\{\frac{(0.996288+0.184347\,i)\,(-1.+\gamma)}{\gamma},\right.$$
$$\left.\left.\frac{(0.153278+1.73469\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(1.06944+0.780515\,i)\,(-1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* this matrix goes in the positions (28,28) to (30,30) *)
```

In[3397]:=
```
i = 1; j = 1;
While[j < 4, While[i < 4, Hfinal[[27 + i, 27 + j]] += HSA6Mn[[i, j]]; i++]; j++; i = 1];
```

```
(* HSB4P - terms depending on gB4+ *)

In[3275]:= nB114
           nB224
           nB334
           nB441
           nB442
           nB443 (* tracking exceptions *)
```

Out[3275]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3276]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3277]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3278]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3279]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3280]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

```
In[3295]:= aB41P = Jn2[Ncoherent[nB114[[1]], nB114[[2]], nB114[[3]]][[1]],
              Ncoherent[nB114[[1]], nB114[[2]], nB114[[3]]][[2]]].
            ConjugateTranspose[GB1P] // Simplify;

In[3296]:= aB42P = Jn2[Ncoherent[nB224[[1]], nB224[[2]], nB224[[3]]][[1]],
              Ncoherent[nB224[[1]], nB224[[2]], nB224[[3]]][[2]]].
            ConjugateTranspose[GB2P] // Simplify;

In[3297]:= aB43P = Jn2[Ncoherent[nB334[[1]], nB334[[2]], nB334[[3]]][[1]],
              Ncoherent[nB334[[1]], nB334[[2]], nB334[[3]]][[2]]].
            ConjugateTranspose[GB3P] // FullSimplify;

In[3298]:= aB41M = Jn2[Ncoherent[nB114[[1]], nB114[[2]], nB114[[3]]][[1]],
              Ncoherent[nB114[[1]], nB114[[2]], nB114[[3]]][[2]]].
            ConjugateTranspose[GB1M] // FullSimplify;

In[3299]:= aB42M = Jn2[Ncoherent[nB224[[1]], nB224[[2]], nB224[[3]]][[1]],
              Ncoherent[nB224[[1]], nB224[[2]], nB224[[3]]][[2]]].
            ConjugateTranspose[GB2M] // FullSimplify;

In[3300]:= aB43M = Jn2[Ncoherent[nB334[[1]], nB334[[2]], nB334[[3]]][[1]],
              Ncoherent[nB334[[1]], nB334[[2]], nB334[[3]]][[2]]].
            ConjugateTranspose[GB3M] // FullSimplify;
```

```
GB4P
GB4M (* tracking exceptions *)
```

Out[3301]= $\{\{\mathrm{Root}[1-3\#1+5\#1^2-6\#1^3+4\#1^4\ \&,4],\mathrm{Root}[1+3\#1+5\#1^2+6\#1^3+4\#1^4\ \&,1]\},\{\mathrm{Root}[1-3\#1+5\#1^2-6\#1^3+4\#1^4\ \&,3],\mathrm{Root}[1-3\#1+5\#1^2-6\#1^3+4\#1^4\ \&,3]\}\}$

Out[3302]= $$\left\{\left\{\frac{1}{8}\left(-i\left(-3+\sqrt{5}\right)+\sqrt{6\left(3+\sqrt{5}\right)}\right),\frac{i+2\sqrt{3}+\sqrt{15}}{6+2\sqrt{5}}\right\},\left\{-\frac{-i+2\sqrt{3}+\sqrt{15}}{2\left(3+\sqrt{5}\right)},\frac{1}{8}\left(i\left(-3+\sqrt{5}\right)+\sqrt{6\left(3+\sqrt{5}\right)}\right)\right\}\right\}$$

In[3308]:=
```
HSB4Pn = HSA5P /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
    a11P → aB41P[[1]], a12P → aB41P[[2]], a21P → aB42P[[1]], a22P → aB42P[[2]],
    a31P → aB43P[[1]], a32P → aB43P[[2]], a11M → aB41M[[1]], a12M → aB41M[[2]],
    a21M → aB42M[[1]], a22M → aB42M[[2]], a31M → aB43M[[1]], a32M → aB43M[[2]],
    θA15 → Theta[nB441], ϕA15 → Phi[nB441], θA25 → Theta[nB442], ϕA25 → Phi[nB442],
    θA35 → Theta[nB443], ϕA35 → Phi[nB443], γA5P → Gama[GB4P], β1A5P → Beta1[GB4P],
    β2A5P → Beta2[GB4P], γA5M → Gama[GB4M], β1A5M → Beta1[GB4M], β2A5M → Beta2[GB4M]};
```

In[3309]:=
```
HSB4Pn // N
```

Out[3309]= $$\left\{\left\{\frac{(11.6472+45.8324\,i)(1.+\gamma)}{\gamma},-\frac{(17.5169-19.9046\,i)(1.+\gamma)}{\gamma},-\frac{(28.5474+0.0456836\,i)(1.+\gamma)}{\gamma}\right\},\left\{-\frac{(17.5169-19.9046\,i)(1.+\gamma)}{\gamma},-\frac{(13.2085+1.92642\,i)(1.+\gamma)}{\gamma},-\frac{(10.0986+13.7563\,i)(1.+\gamma)}{\gamma}\right\},\left\{-\frac{(28.5474+0.0456836\,i)(1.+\gamma)}{\gamma},-\frac{(10.0986+13.7563\,i)(1.+\gamma)}{\gamma},\frac{(7.38487-18.3917\,i)(1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* this matrix goes in the positions (16,16) to (18,18) *)
```

In[3398]:=
```
i = 1; j = 1;
While[j < 4, While[i < 4, Hfinal[[15 + i, 15 + j]] += HSB4Pn[[i, j]]; i++]; j++; i = 1];
```

```
(* HSB4M - terms depending on gB4- *)

In[3310]:= HSB4Mn = HSA5M /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
      a11P → aB41P[[1]], a12P → aB41P[[2]], a21P → aB42P[[1]], a22P → aB42P[[2]],
      a31P → aB43P[[1]], a32P → aB43P[[2]], a11M → aB41M[[1]], a12M → aB41M[[2]],
      a21M → aB42M[[1]], a22M → aB42M[[2]], a31M → aB43M[[1]], a32M → aB43M[[2]],
      θA15 → Theta[nB441], ϕA15 → Phi[nB441], θA25 → Theta[nB442], ϕA25 → Phi[nB442],
      θA35 → Theta[nB443], ϕA35 → Phi[nB443], γA5P → Gama[GB4P], β1A5P → Beta1[GB4P],
      β2A5P → Beta2[GB4P], γA5M → Gama[GB4M], β1A5M → Beta1[GB4M], β2A5M → Beta2[GB4M]};

In[3311]:= HSB4Mn // N
```

Out[3311]=

$$\left\{\left\{-\frac{(0.184765+1.57391\, i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.496794+0.434657\, i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.574416+0.0935196\, i)\,(-1.+\gamma)}{\gamma}\right\},\right.$$
$$\left\{\frac{(0.496794+0.434657\, i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.107499-1.08858\, i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.0852812-0.437964\, i)\,(-1.+\gamma)}{\gamma}\right\},$$
$$\left.\left\{\frac{(0.574416+0.0935196\, i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.0852812-0.437964\, i)\,(-1.+\gamma)}{\gamma},\ -\frac{(0.0151166+0.722535\, i)\,(-1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* this matrix goes in the positions (34,34) to (36,36) *)

In[3399]:= i = 1; j = 1;
While[j < 4, While[i < 4, Hfinal[[33 + i, 33 + j]] += HSB4Mn[[i, j]]; i++]; j++; i = 1];
```

```
(* HSB6P - terms depending on gB6+ *)
```

In[3313]:=
```
nB116
nB226
nB336
nB661
nB662
nB663 (* tracking exceptions *)
```

Out[3313]= $\{0, 0, -1\}$

Out[3314]= $\{0, 0, -1\}$

Out[3315]= $\{0, 0, -1\}$

Out[3316]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3317]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3318]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3319]:=
```
aB61P = Jn2[0, 1].ConjugateTranspose[GB1P] // Simplify;
```

In[3323]:=
```
aB62P = Jn2[0, 1].ConjugateTranspose[GB2P] // Simplify;
```

In[3321]:=
```
aB63P = Jn2[0, 1].ConjugateTranspose[GB3P] // FullSimplify;
```

In[3324]:=
```
aB61M = Jn2[0, 1].ConjugateTranspose[GB1M] // Simplify;
```

In[3325]:=
```
aB62M = Jn2[0, 1].ConjugateTranspose[GB2M] // Simplify;
```

In[3326]:=
```
aB63M = Jn2[0, 1].ConjugateTranspose[GB3M] // FullSimplify;
```

In[3327]:=
```
GB6P
GB6M (* tracking exceptions *)
```

Out[3327]= $\left\{\left\{\frac{1}{4}\left(\sqrt{2}+i\sqrt{6}\right), -\frac{i\left(-i+\sqrt{3}\right)}{2\sqrt{2}}\right\}, \left\{\frac{1}{4}\left(\sqrt{2}-i\sqrt{6}\right), \frac{1}{4}\left(\sqrt{2}-i\sqrt{6}\right)\right\}\right\}$

Out[3328]= $\left\{\left\{\frac{-i+\sqrt{3}}{2\sqrt{2}}, \frac{-i+\sqrt{3}}{2\sqrt{2}}\right\}, \left\{-\frac{i+\sqrt{3}}{2\sqrt{2}}, \frac{i+\sqrt{3}}{2\sqrt{2}}\right\}\right\}$

In[3329]:=
```
HSB6Pn = HSA5P /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
    a11P → aB61P[[1]], a12P → aB61P[[2]], a21P → aB62P[[1]], a22P → aB62P[[2]],
    a31P → aB63P[[1]], a32P → aB63P[[2]], a11M → aB61M[[1]], a12M → aB61M[[2]],
    a21M → aB62M[[1]], a22M → aB62M[[2]], a31M → aB63M[[1]], a32M → aB63M[[2]],
    θA15 → Theta[nB661], ϕA15 → Phi[nB661], θA25 → Theta[nB662], ϕA25 → Phi[nB662],
    θA35 → Theta[nB663], ϕA35 → Phi[nB663], γA5P → Gama[GB6P], β1A5P → Beta1[GB6P],
    β2A5P → Beta2[GB6P], γA5M → Gama[GB6M], β1A5M → Beta1[GB6M], β2A5M → Beta2[GB6M]};
```

In[3330]:= **HSB6Pn // N**

Out[3330]=
$$\left\{\left\{-\frac{(3.41651-2.44059\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(0.265756-1.07703\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(1.77634-0.984371\,i)\,(1.+\gamma)}{\gamma}\right\},\ \left\{-\frac{(0.265756-1.07703\,i)\,(1.+\gamma)}{\gamma},\ \frac{(0.309891+2.28359\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(0.224269-0.252841\,i)\,(1.+\gamma)}{\gamma}\right\},\ \left\{-\frac{(1.77634-0.984371\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(0.224269-0.252841\,i)\,(1.+\gamma)}{\gamma},\ \frac{(1.39836-0.90581\,i)\,(1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* this matrix goes in the positions (13,13) to (15,15) *)
```

In[3400]:=
```
i = 1; j = 1;
While[j < 4, While[i < 4, Hfinal[[12 + i, 12 + j]] += HSB6Pn[[i, j]]; i++]; j++; i = 1];
```

```
(* HSB6M - terms depending on gB6- *)

In[3332]:= HSB6Mn = HSA5M /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
     a11P → aB61P[[1]], a12P → aB61P[[2]], a21P → aB62P[[1]], a22P → aB62P[[2]],
     a31P → aB63P[[1]], a32P → aB63P[[2]], a11M → aB61M[[1]], a12M → aB61M[[2]],
     a21M → aB62M[[1]], a22M → aB62M[[2]], a31M → aB63M[[1]], a32M → aB63M[[2]],
     θA15 → Theta[nB661], ϕA15 → Phi[nB661], θA25 → Theta[nB662], ϕA25 → Phi[nB662],
     θA35 → Theta[nB663], ϕA35 → Phi[nB663], γA5P → Gama[GB6P], β1A5P → Beta1[GB6P],
     β2A5P → Beta2[GB6P], γA5M → Gama[GB6M], β1A5M → Beta1[GB6M], β2A5M → Beta2[GB6M]};

In[3333]:= HSB6Mn // N
```

Out[3333]=

$$\left\{\left\{-\frac{(0.204865+1.92141\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.233356+0.212422\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(1.60235+0.658294\,i)\,(-1.+\gamma)}{\gamma}\right\},\right.$$
$$\left\{\frac{(0.233356+0.212422\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.146157-1.76052\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(0.552981+0.134742\,i)\,(-1.+\gamma)}{\gamma}\right\},\ \left\{-\frac{(1.60235+0.658294\,i)\,(-1.+\gamma)}{\gamma},\right.$$
$$\left.\left.-\frac{(0.552981+0.134742\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(0.0437244-0.12315\,i)\,(-1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* this matrix goes in the positions (31,31) to (33,33) *)

In[3401]:= i = 1; j = 1;
  While[j < 4, While[i < 4, Hfinal[[30 + i, 30 + j]] += HSB6Mn[[i, j]]; i++]; j++; i = 1];
```

```
(* HSC5P - terms depending on gC5+ *)

In[3335]:= nC115
           nC225
           nC335
           nC551
           nC552
           nC553 (* tracking exceptions *)
```

Out[3335]= $\{0, 0, -1\}$

Out[3336]= $\{0, 0, -1\}$

Out[3337]= $\{0, 0, -1\}$

Out[3338]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3339]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3340]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

```
In[3343]:= aC51P = Jn2[0, 1].ConjugateTranspose[GC1P] // Simplify;

In[3344]:= aC52P = Jn2[0, 1].ConjugateTranspose[GC2P] // Simplify;

In[3345]:= aC53P = Jn2[0, 1].ConjugateTranspose[GC3P] // FullSimplify;

In[3346]:= aC51M = Jn2[0, 1].ConjugateTranspose[GC1M] // Simplify;

In[3347]:= aC52M = Jn2[0, 1].ConjugateTranspose[GC2M] // Simplify;

In[3349]:= aC53M = Jn2[0, 1].ConjugateTranspose[GC3M] // Simplify;

In[3350]:= GC5P
           GC5M (* tracking exceptions *)
```

Out[3350]= $\left\{\left\{\frac{1}{4}\left(\sqrt{2}+i\sqrt{6}\right), -\frac{i\left(-i+\sqrt{3}\right)}{2\sqrt{2}}\right\}, \left\{\frac{1}{4}\left(\sqrt{2}-i\sqrt{6}\right), \frac{1}{4}\left(\sqrt{2}-i\sqrt{6}\right)\right\}\right\}$

Out[3351]= $\left\{\left\{\frac{-i+\sqrt{3}}{2\sqrt{2}}, \frac{-i+\sqrt{3}}{2\sqrt{2}}\right\}, \left\{-\frac{i+\sqrt{3}}{2\sqrt{2}}, \frac{i+\sqrt{3}}{2\sqrt{2}}\right\}\right\}$

```
In[3352]:= HSC5Pn = HSA5P /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
              a11P → aC51P[[1]], a12P → aC51P[[2]], a21P → aC52P[[1]], a22P → aC52P[[2]],
              a31P → aC53P[[1]], a32P → aC53P[[2]], a11M → aC51M[[1]], a12M → aC51M[[2]],
              a21M → aC52M[[1]], a22M → aC52M[[2]], a31M → aC53M[[1]], a32M → aC53M[[2]],
              θA15 → Theta[nC551], ϕA15 → Phi[nC551], θA25 → Theta[nC552], ϕA25 → Phi[nC552],
              θA35 → Theta[nC553], ϕA35 → Phi[nC553], γA5P → Gama[GC5P], β1A5P → Beta1[GC5P],
              β2A5P → Beta2[GC5P], γA5M → Gama[GC5M], β1A5M → Beta1[GC5M], β2A5M → Beta2[GC5M]};
```

In[3353]:= **HSC5Pn // N**

Out[3353]=
$$\left\{\left\{-\frac{(0.0549911-1.04997\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(1.40984+1.29333\,i)\,(1.+\gamma)}{\gamma},\ \frac{(0.646454+0.970998\,i)\,(1.+\gamma)}{\gamma}\right\},\left\{-\frac{(1.40984+1.29333\,i)\,(1.+\gamma)}{\gamma},\ \frac{(0.857259+0.145612\,i)\,(1.+\gamma)}{\gamma},\ \frac{(0.955005+0.662343\,i)\,(1.+\gamma)}{\gamma}\right\},\left\{\frac{(0.646454+0.970998\,i)\,(1.+\gamma)}{\gamma},\ \frac{(0.955005+0.662343\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(0.829763-1.92748\,i)\,(1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* this matrix goes in the positions (22,22) to (24,24) *)
```

In[3402]:=
```
i = 1; j = 1;
While[j < 4, While[i < 4, Hfinal[[21 + i, 21 + j]] += HSC5Pn[[i, j]]; i++]; j++; i = 1];
```

```
(* HSC5M - terms depending on gC5- *)

In[3355]:= HSC5Mn = HSA5M /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
      a11P → aC51P[[1]], a12P → aC51P[[2]], a21P → aC52P[[1]], a22P → aC52P[[2]],
      a31P → aC53P[[1]], a32P → aC53P[[2]], a11M → aC51M[[1]], a12M → aC51M[[2]],
      a21M → aC52M[[1]], a22M → aC52M[[2]], a31M → aC53M[[1]], a32M → aC53M[[2]],
      θA15 → Theta[nC551], ϕA15 → Phi[nC551], θA25 → Theta[nC552], ϕA25 → Phi[nC552],
      θA35 → Theta[nC553], ϕA35 → Phi[nC553], γA5P → Gama[GC5P], β1A5P → Beta1[GC5P],
      β2A5P → Beta2[GC5P], γA5M → Gama[GC5M], β1A5M → Beta1[GC5M], β2A5M → Beta2[GC5M]};

In[3356]:= HSC5Mn // N
```

Out[3356]=
$$\left\{\left\{\frac{(0.145624+2.87475\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.0188819+0.623814\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(3.31999+2.91133\,i)\,(-1.+\gamma)}{\gamma}\right\},\right.$$
$$\left\{\frac{(0.0188819+0.623814\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(0.00337522+2.39695\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(0.903533+1.05053\,i)\,(-1.+\gamma)}{\gamma}\right\},\ \left\{-\frac{(3.31999+2.91133\,i)\,(-1.+\gamma)}{\gamma},\right.$$
$$\left.\left.-\frac{(0.903533+1.05053\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(0.0694368+1.63851\,i)\,(-1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* this matrix goes in the positions (40,40) to (42,42) *)

In[3403]:= i = 1; j = 1;
      While[j < 4, While[i < 4, Hfinal[[39 + i, 39 + j]] += HSC5Mn[[i, j]]; i++]; j++; i = 1];
```

```
(* HSC4P - terms depending on gC4+ *)
```

In[3358]:=
```
nC114
nC224
nC334
nC441
nC442
nC443 (* tracking exceptions *)
```

Out[3358]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3359]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3360]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3361]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3362]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3363]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3364]:=
```
aC41P = Jn2[Ncoherent[nC114[[1]], nC114[[2]], nC114[[3]]][[1]],
     Ncoherent[nC114[[1]], nC114[[2]], nC114[[3]]][[2]]].
    ConjugateTranspose[GC1P] // Simplify;
```

In[3365]:=
```
aC42P = Jn2[Ncoherent[nC224[[1]], nC224[[2]], nC224[[3]]][[1]],
     Ncoherent[nC224[[1]], nC224[[2]], nC224[[3]]][[2]]].
    ConjugateTranspose[GC2P] // Simplify;
```

In[3366]:=
```
aC43P = Jn2[Ncoherent[nC334[[1]], nC334[[2]], nC334[[3]]][[1]],
     Ncoherent[nC334[[1]], nC334[[2]], nC334[[3]]][[2]]].
    ConjugateTranspose[GC3P] // FullSimplify;
```

In[3367]:=
```
aC41M = Jn2[Ncoherent[nC114[[1]], nC114[[2]], nC114[[3]]][[1]],
     Ncoherent[nC114[[1]], nC114[[2]], nC114[[3]]][[2]]].
    ConjugateTranspose[GC1M] // FullSimplify;
```

In[3368]:=
```
aC42M = Jn2[Ncoherent[nC224[[1]], nC224[[2]], nC224[[3]]][[1]],
     Ncoherent[nC224[[1]], nC224[[2]], nC224[[3]]][[2]]].
    ConjugateTranspose[GC2M] // FullSimplify;
```

In[3369]:=
```
aC43M = Jn2[Ncoherent[nC334[[1]], nC334[[2]], nC334[[3]]][[1]],
     Ncoherent[nC334[[1]], nC334[[2]], nC334[[3]]][[2]]].
    ConjugateTranspose[GC3M] // FullSimplify;
```

In[3370]:=
```
GC4P
GC4M (* tracking exceptions *)
```

Out[3370]= {{Root[1 + #1^2 + 4 #1^4 &, 3], Root[1 + #1^2 + 4 #1^4 &, 3]},
{Root[1 + #1^2 + 4 #1^4 &, 1], Root[1 + #1^2 + 4 #1^4 &, 4]}}

Out[3371]= {{Root[1 - #1^2 + 4 #1^4 &, 3], Root[1 - #1^2 + 4 #1^4 &, 2]},
{Root[1 - #1^2 + 4 #1^4 &, 4], Root[1 - #1^2 + 4 #1^4 &, 4]}}

In[3372]:=
```
HSC4Pn = HSA5P /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
    a11P → aC41P[[1]], a12P → aC41P[[2]], a21P → aC42P[[1]], a22P → aC42P[[2]],
    a31P → aC43P[[1]], a32P → aC43P[[2]], a11M → aC41M[[1]], a12M → aC41M[[2]],
    a21M → aC42M[[1]], a22M → aC42M[[2]], a31M → aC43M[[1]], a32M → aC43M[[2]],
    θA15 → Theta[nC441], ϕA15 → Phi[nC441], θA25 → Theta[nC442], ϕA25 → Phi[nC442],
    θA35 → Theta[nC443], ϕA35 → Phi[nC443], γA5P → Gama[GC4P], β1A5P → Beta1[GC4P],
    β2A5P → Beta2[GC4P], γA5M → Gama[GC4M], β1A5M → Beta1[GC4M], β2A5M → Beta2[GC4M]};
```

In[3373]:=
```
HSC4Pn // N
```

Out[3373]=
$$\left\{\left\{-\frac{(0.201729-2.46202\,i)\,(1.+\gamma)}{\gamma},\ \frac{(0.221534+0.486315\,i)\,(1.+\gamma)}{\gamma},\ \frac{(0.371758+0.612978\,i)\,(1.+\gamma)}{\gamma}\right\},\right.$$
$$\left\{\frac{(0.221534+0.486315\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(0.668128-0.0920215\,i)\,(1.+\gamma)}{\gamma},\ -\frac{(0.351444-0.0259157\,i)\,(1.+\gamma)}{\gamma}\right\},\ \left\{\frac{(0.371758+0.612978\,i)\,(1.+\gamma)}{\gamma},\right.$$
$$\left.\left.-\frac{(0.351444-0.0259157\,i)\,(1.+\gamma)}{\gamma},\ \frac{(0.768992+1.27505\,i)\,(1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* this matrix goes in the positions (19,19) to (21,21) *)
```

In[3404]:=
```
i = 1; j = 1;
While[j < 4, While[i < 4, Hfinal[[18 + i, 18 + j]] += HSC4Pn[[i, j]]; i++]; j++; i = 1];
```

```
(* HSC4M - terms depending on gC4- *)
```

In[3375]:=
```
HSC4Mn = HSA5M /. {xa15 → Sqrt[3] / (4 γ), xa25 → Sqrt[3] / (4 γ), xa35 → Sqrt[3] / (4 γ),
    a11P → aC41P[[1]], a12P → aC41P[[2]], a21P → aC42P[[1]], a22P → aC42P[[2]],
    a31P → aC43P[[1]], a32P → aC43P[[2]], a11M → aC41M[[1]], a12M → aC41M[[2]],
    a21M → aC42M[[1]], a22M → aC42M[[2]], a31M → aC43M[[1]], a32M → aC43M[[2]],
    θA15 → Theta[nC441], ϕA15 → Phi[nC441], θA25 → Theta[nC442], ϕA25 → Phi[nC442],
    θA35 → Theta[nC443], ϕA35 → Phi[nC443], γA5P → Gama[GC4P], β1A5P → Beta1[GC4P],
    β2A5P → Beta2[GC4P], γA5M → Gama[GC4M], β1A5M → Beta1[GC4M], β2A5M → Beta2[GC4M]};
```

In[3376]:=
```
HSC4Mn // N
```

Out[3376]=
$$\left\{\left\{\frac{(2.63152-4.25579\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(0.506432+2.20533\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(0.721224+0.931728\,i)\,(-1.+\gamma)}{\gamma}\right\},\right.$$
$$\left\{-\frac{(0.506432+2.20533\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(0.393506-2.19287\,i)\,(-1.+\gamma)}{\gamma},\ \frac{(0.0300262+1.83665\,i)\,(-1.+\gamma)}{\gamma}\right\},\ \left\{-\frac{(0.721224+0.931728\,i)\,(-1.+\gamma)}{\gamma},\right.$$
$$\left.\left.\frac{(0.0300262+1.83665\,i)\,(-1.+\gamma)}{\gamma},\ -\frac{(0.922256+2.66305\,i)\,(-1.+\gamma)}{\gamma}\right\}\right\}$$

```
(* this matrix goes in the positions (37,37) to (39,39) *)
```

In[3405]:=
```
i = 1; j = 1;
While[j < 4, While[i < 4, Hfinal[[36 + i, 36 + j]] += HSC4Mn[[i, j]]; i++]; j++; i = 1];
```

```
(* While the considerations about possible simplifications to the Hessian
 at the critical point done upon defining it could prove useful later on,
for the strict purpose of computing a numeric determinant we
 have used plainly Hpure rather than the simplified version. *)

(* We now need to substitute the G, n values in the "pure" part of
 this Hessian matrix to compute the determinant with the geometric G,
n data. For simplicity the phases α can be taken as zero. *)
```

```
In[3409]:= nB446
nC554
nA665 (* consistency check for the interior n's *)

Out[3409]= {0, 0, -1}

Out[3410]= {0, 0, -1}

Out[3411]= {0, 0, -1}
```

```
(* as expected from the main text,
the three relevant interior n's are all equal. Fixing the basis to make them equal
 to (0,0,1) rather than (0,0,-1) is not necessary at this point because we are
 not using the simplifications that were derived from that basis choice. *)

(* the spherical coordinates for (0,0,-1) are Theta = Pi , Phi = 0 *)
```

```
In[3442]:= Hfinal1 =
  Hfinal /. {αBA → 0, θBA → Pi, ϕBA → 0, αCB → 0, θCB → Pi, ϕCB → 0, αAC → 0, θAC → Pi, ϕAC → 0,
    γA6P → Gama[GA6P], γA5P → Gama[GA5P], β1A6P → Beta1[GA6P], β1A5P → Beta1[GA5P],
    β2A6P → Beta2[GA6P], β2A5P → Beta2[GA5P], γB4P → Gama[GB4P], γB6P → Gama[GB6P],
    β1B4P → Beta1[GB4P], β1B6P → Beta1[GB6P], β2B4P → Beta2[GB4P] , β2B6P → Beta2[GB6P] ,
    γC5P → Gama[GC5P], γC4P → Gama[GC4P], β1C5P → Beta1[GC5P], β1C4P → Beta1[GC4P] ,
    β2C5P → Beta2[GC5P], β2C4P → Beta2[GC4P], γA6M → Gama[GA6M], γA5M → Gama[GA5M],
    β1A6M → Beta1[GA6M], β1A5M → Beta1[GA5M], β2A6M → Beta2[GA6M], β2A5M → Beta2[GA5M],
    γB4M → Gama[GB4M], γB6M → Gama[GB6M], β1B4M → Beta1[GB4M], β1B6M → Beta1[GB6M],
    β2B4M → Beta2[GB4M] , β2B6M → Beta2[GB6M] , γC5M → Gama[GC5M], γC4M → Gama[GC4M],
    β1C5M → Beta1[GC5M], β1C4M → Beta1[GC4M] , β2C5M → Beta2[GC5M], β2C4M → Beta2[GC4M]};
```

In[3443]:=
```
Hfinal1 // N
```

A very large output was generated. Here is a sample of it:

Out[3443]=
```
{{(0. - 2. i)
   ((0.0786893 + 1.32884 × 10^-16 i) (-1. + γ) + (1.14536 + 3.95558 × 10^-15 i) (1. + γ)),
  (0. - 2. i) ((0.57735 - 1.12429 × 10^-16 i) (-1. + γ) - (1.61015 + 1.58797 × 10^-15 i) (1. + γ)),
  <<38>>, 0., 0.}, {<<1>>}, <<38>>, {<<1>>},
 {0., <<40>>, (2.77556 × 10^-17 - 0.509288 i) (-1. + γ) - ((<<20>> + <<19>> i) <<1>> (<<1>> <<1>>))/γ}}
```

Show Less | Show More | Show Full Output | Set Size Limit...

```
(* Printing out a sample of the output to show that the resulting Hessian matrix
 for this configuration depends only on the unspecified Immirzi parameter. *)
```

In[3469]:=
```
Hfinaltest = Hfinal1 /. {γ → Sqrt[3] / 2};
```

```
(* Using a numeric value for the Immirzi parameter for testing purposes,
since the 42x42 matrix is too large for the
 determinant to be computed algebraically. As per the main text,
the acceptable values are γ=Sqrt[3]/(4j) where j is a semi-integer. *)
```

In[3491]:=
```
HfinaltestN = N[Hfinaltest];
```

```
(* Large numeric result omitted. *)
```

In[3492]:=
```
Det[HfinaltestN]
```

Out[3492]= $9.86934 \times 10^{13} - 1.69985 \times 10^{14}\, i$

```
(* Testing with different levels of numeric precision on Mathematica
 seems to indicate a robust result. Hessian determinant is nonzero. *)

(* We will now compute the K derivatives using the same prescription used in
 Hfinaltest, including the same test value for the Immirzi parameter. *)
(* We need this to compute the constant A referred to in the main text. *)
```

In[3477]:=
```
Kfinaltest =
  K /. {αBA → 0, θBA → Pi, ϕBA → 0, αCB → 0, θCB → Pi, ϕCB → 0, αAC → 0, θAC → Pi, ϕAC → 0,
    γA6P → Gama[GA6P], γA5P → Gama[GA5P], β1A6P → Beta1[GA6P], β1A5P → Beta1[GA5P],
    β2A6P → Beta2[GA6P], β2A5P → Beta2[GA5P], γB4P → Gama[GB4P], γB6P → Gama[GB6P],
    β1B4P → Beta1[GB4P], β1B6P → Beta1[GB6P], β2B4P → Beta2[GB4P] ,
    β2B6P → Beta2[GB6P] , γC5P → Gama[GC5P], γC4P → Gama[GC4P], β1C5P → Beta1[GC5P],
    β1C4P → Beta1[GC4P] , β2C5P → Beta2[GC5P], β2C4P → Beta2[GC4P],
    γA6M → Gama[GA6M], γA5M → Gama[GA5M], β1A6M → Beta1[GA6M], β1A5M → Beta1[GA5M],
    β2A6M → Beta2[GA6M], β2A5M → Beta2[GA5M], γB4M → Gama[GB4M], γB6M → Gama[GB6M],
    β1B4M → Beta1[GB4M], β1B6M → Beta1[GB6M], β2B4M → Beta2[GB4M] ,
    β2B6M → Beta2[GB6M] , γC5M → Gama[GC5M], γC4M → Gama[GC4M], β1C5M → Beta1[GC5M],
    β1C4M → Beta1[GC4M] , β2C5M → Beta2[GC5M], β2C4M → Beta2[GC4M], γ → Sqrt[3] / 2};
```

In[3493]:=
```
KfinaltestN = N[Kfinaltest];
```

```
(* Large numeric result omitted. *)

(* Now we can numerically compute the constant A via
 the expression given in the main text (inline, after eq. 137) *)
```

In[3494]:=
```
A = -Im[KfinaltestN.Inverse[HfinaltestN].KfinaltestN]
```

Out[3494]=
```
6.62021
```

```
(* Testing with different levels of numeric precision on Mathematica seems to
  indicate a robust result as far as order of magnitude goes - not shown here,
but can be reproduced by running the code. *)
(* A is nonzero for this configuration. *)
```

Appendix 2 - Sketching EPRL variable calculation for a geometric Δ_3 based on equilateral 5-simplex

The following is the Mathematica script used for carrying out the calculation specified in section 4.4 for the Δ_3 built from the equilateral 5-simplex centered at the origin, with the goal of obtaining numeric data to test the Hessian of the previous appendix.. The "gluing matrices" referred to in the script are a consistency check for the geometricity of the variables found. Indeed, considering a gluing equation

$$R(g^{\pm}_{ve})\vec{n}_{ef} = -R(g^{\pm}_{ve'})\vec{n}_{e'f}\,, \tag{163}$$

notice that the $+$ and $-$ equations contained in it can both be manipulated to give the value of $\vec{n}_{e'f}$, and therefore

$$\left(R^{-1}(g^{+}_{ve'})R(g^{+}_{ve}) - R^{-1}(g^{-}_{ve'})R(g^{-}_{ve})\right)\vec{n}_{ef} = 0. \tag{164}$$

Defining the matrix in brackets as the gluing matrix between two tetrahedra, $R_{ee'}$, $\vec{n}_{ef}$ must lie in its null space.

```
In[2392]:= (* Determining 4d coordinates for each of the 4-
             simplices in the equilateral 5-simplex Delta3 *)
            a = {-1 / 2, -1 / (2 Sqrt[3]), -1 / 3 Sqrt[3 / 8], -1 / 4 Sqrt[2 / 5], -1 / 5 Sqrt[5 / 12]};
            b = {1 / 2, -1 / (2 Sqrt[3]), -1 / 3 Sqrt[3 / 8], -1 / 4 Sqrt[2 / 5], -1 / 5 Sqrt[5 / 12]};
            c = {0, 1 / Sqrt[3], -1 / 3 Sqrt[3 / 8], -1 / 4 Sqrt[2 / 5], -1 / 5 Sqrt[5 / 12]};
            d = {0, 0, Sqrt[3 / 8], -1 / 4 Sqrt[2 / 5], -1 / 5 Sqrt[5 / 12]};
            e = {0, 0, 0, Sqrt[2 / 5], -1 / 5 Sqrt[5 / 12]};
            f = {0, 0, 0, 0, Sqrt[5 / 12]};

In[2398]:= (* Out of these 6 points the 4-simplices we are interested in are: A = abcef,
            B = abcdf, C = abcde *)

In[2399]:= (* We want to find the 4d hyperplane in which each of them lives, and obtain a
             sensible orthonormal basis for that plane in order to get our coordinates *)

In[2400]:= (* Since the vectors connecting one point of the 4-
             simplex to the other 4 are necessarily linearly independent,
            for a non-degenerate figure, we can use them to form a (non-orthonormal) basis. *)
            vb = b - a;
            vc = c - a;
            vd = d - a;
            ve = e - a;
            vf = f - a;

In[2405]:= (* To obtain an orthonormal basis we use the Orthogonalize command in
             Mathematica. Transpose is necessary so the basis vectors will show as columns.*)
            BasisD = Transpose[Orthogonalize[{vb, vc, ve, vf}]] // FullSimplify;
            BasisE = Transpose[Orthogonalize[{vb, vc, vd, vf}]] // FullSimplify;
            BasisF = Transpose[Orthogonalize[{vb, vc, vd, ve}]] // FullSimplify;

In[2408]:= (* Solving the equation BasisMatrix . X = Coordinates in R^5 gives
              you the coordinates in the chosen basis, so we use LinearSolve *)

In[2409]:= (* we also need the R3 tetrahedron matrix continued to
             R4. For the equilateral case it's a "standard tetrahedron" *)

In[2410]:=
            (* For the equilateral 5-simplex the tetrahedra's images in R^3 are all equal,
            which means for all of these matrices we can just consider the one given below. *)

In[2411]:= A3 = {-1 / 2, -1 / (2 Sqrt[3]), -1 / 3 Sqrt[3 / 8]};
            B3 = {1 / 2, -1 / (2 Sqrt[3]), -1 / 3 Sqrt[3 / 8]};
            C3 = {0, 1 / Sqrt[3], -1 / 3 Sqrt[3 / 8]};
            D3 = {0, 0,  Sqrt[3 / 8]}; (* points for standard tetrahedron in R3 *)
```

In[2415]:=
```
vB3 = Join[{0}, B3 - A3];
vC3  = Join[{0}, C3 - A3];
vD3 = Join[{0}, D3 - A3];
M3 = {{1, 0, 0, 0}, vB3, vC3, vD3} // Transpose;
M3minus = {{-1, 0, 0, 0}, vB3, vC3, vD3} // Transpose;
M3 // MatrixForm
(* use either M3 or M3minus to have consistent determinants *)
```

Out[2420]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & \frac{\sqrt{\frac{3}{2}}}{2} + \frac{1}{2\sqrt{6}} \end{pmatrix}$$

In[2421]:=
```
(* compute normals of standard tetrahedron *)
```

In[2422]:=
```
ND = -Cross[B3 - A3, C3 - A3] // Normalize // Simplify
```

Out[2422]= $\{0, 0, -1\}$

In[2423]:=
```
NC = Cross[B3 - A3, D3 - A3] // Normalize // Simplify
```

Out[2423]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2424]:=
```
NB = -Cross[C3 - A3, D3 - A3] // Normalize // Simplify
```

Out[2424]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2425]:=
```
NA = Cross[C3 - B3, D3 - B3] // Normalize // Simplify
```

Out[2425]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2426]:=
```
M3 // FullSimplify // MatrixForm
Det[M3]
```

Out[2426]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[2427]= $\frac{1}{\sqrt{2}}$

In[2428]:=
```
(* The test is to determine g_ve for vertex A,
the corresponding n_ef and verify consistency. *)
(* g_A6 *)
MA6 = {LinearSolve[BasisD, vb], LinearSolve[BasisD, vc], LinearSolve[BasisD, ve]} ;
MA6 // MatrixForm
MA6N = Normalize[NullSpace[MA6][[1]]]
MA6final =
 Transpose[{-MA6N, Normalize[MA6[[1]]], Normalize[MA6[[2]]], Normalize[MA6[[3]]]}];
MA6final // MatrixForm
Det[MA6final]
```

Out[2428]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \end{pmatrix}$$

Out[2429]= {0, 0, 0, 1}

Out[2430]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Out[2431]= $\frac{1}{\sqrt{2}}$

In[2432]:=
```
GA6 = MA6final.Inverse[M3] // FullSimplify; GA6 // MatrixForm
GA6.Transpose[GA6] // FullSimplify // MatrixForm
Det[GA6] // FullSimplify
```

Out[2432]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Out[2433]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2434]= 1

In[2435]:=
```
(* gA5 *)
```

In[2436]:=
```
(* Determining GA5 for tetrahedron ABCF : pivot triangle is ABC *)
```

In[2437]:=
```
MA5 = {LinearSolve[BasisD, vc], LinearSolve[BasisD, vb], LinearSolve[BasisD, vf]} ;
MA5 // MatrixForm
```

Out[2437]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

In[2438]:=
```
MA5N = Normalize[NullSpace[MA5][[1]]];
MA5final =
 Transpose[{MA5N, Normalize[MA5[[1]]], Normalize[MA5[[2]]], Normalize[MA5[[3]]]}];
MA5final // MatrixForm
Det[MA5final]
```

Out[2439]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2\sqrt{3}} \\ -\frac{\sqrt{15}}{4} & 0 & 0 & \frac{1}{2\sqrt{6}} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2440]= $\frac{1}{\sqrt{2}}$

In[2441]:=
```
GA5 = MA5final.Inverse[M3] // FullSimplify; GA5 // MatrixForm
GA5.Transpose[GA5] // FullSimplify // MatrixForm
Det[GA5] // FullSimplify
```

Out[2441]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{\sqrt{15}}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2442]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2443]= 1

In[2444]:=
```
(* gA3 *)
```

In[2445]:=
```
MA3 = {LinearSolve[BasisD, vb], LinearSolve[BasisD, ve], LinearSolve[BasisD, vf]} ;
MA3 // MatrixForm
```

Out[2445]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

In[2446]:=
```
MA3N = Normalize[NullSpace[MA3][[1]]];
MA3final =
  Transpose[{MA3N, Normalize[MA3[[1]]], Normalize[MA3[[2]]], Normalize[MA3[[3]]]}];
MA3final // MatrixForm
Det[MA3final]
```

Out[2447]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ -\sqrt{\frac{5}{6}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{\frac{5}{3}}}{4} & 0 & \sqrt{\frac{2}{3}} & \frac{1}{2\sqrt{6}} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2448]= $\frac{1}{\sqrt{2}}$

In[2449]:=
```
GA3 = MA3final.Inverse[M3] // FullSimplify; GA3 // MatrixForm
GA3.Transpose[GA3] // FullSimplify // MatrixForm
Det[GA3] // FullSimplify
```

Out[2449]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\sqrt{\frac{5}{6}} & 0 & \frac{1}{3} & \frac{1}{3\sqrt{2}} \\ \frac{\sqrt{\frac{5}{3}}}{4} & 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{12} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2450]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2451]= 1

In[2452]:=
```
(* gA2 *)
```

In[2453]:=
```
MA2 = {LinearSolve[BasisD, ve], LinearSolve[BasisD, vc], LinearSolve[BasisD, vf]} ;
MA2 // MatrixForm
```

Out[2453]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

In[2454]:=
```
MA2N = Normalize[NullSpace[MA2][[1]]];
MA2final =
 Transpose[{MA2N, Normalize[MA2[[1]]], Normalize[MA2[[2]]], Normalize[MA2[[3]]]}];
MA2final // MatrixForm
Det[MA2final]
```

Out[2455]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{\frac{5}{2}}}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{\frac{5}{6}}}{2} & \frac{1}{2\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{\frac{5}{3}}}{4} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{2\sqrt{6}} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2456]= $\frac{1}{\sqrt{2}}$

In[2457]:=
```
GA2 = MA2final.Inverse[M3] // FullSimplify; GA2 // MatrixForm
GA2.Transpose[GA2] // FullSimplify // MatrixForm
Det[GA2] // FullSimplify
```

Out[2457]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{\frac{5}{2}}}{2} & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} \\ \frac{\sqrt{\frac{5}{6}}}{2} & \frac{1}{2\sqrt{3}} & \frac{5}{6} & -\frac{1}{6\sqrt{2}} \\ \frac{\sqrt{\frac{5}{3}}}{4} & \sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{3} & -\frac{1}{12} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2458]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2459]= 1

In[2460]:=
```
(* gA1 *)
```

In[2461]:=
```
MA1 = {LinearSolve[BasisD, e - b], LinearSolve[BasisD, c - b], LinearSolve[BasisD, f - b]} ;
MA1 // MatrixForm
```

Out[2461]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

In[2462]:=
```
MA1N = Normalize[NullSpace[MA1][[1]]];
MA1final =
  Transpose[{-MA1N, Normalize[MA1[[1]]], Normalize[MA1[[2]]], Normalize[MA1[[3]]]}];
MA1final // MatrixForm
Det[MA1final] // Simplify
```

Out[2463]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{\frac{5}{2}}}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{\sqrt{\frac{5}{6}}}{2} & \frac{1}{2\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{2\sqrt{6}} \\ -\frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2464]= $\frac{1}{\sqrt{2}}$

In[2465]:=
```
GA1 = MA1final.Inverse[M3] // FullSimplify; GA1 // MatrixForm
GA1.Transpose[GA1] // FullSimplify // MatrixForm
Det[GA1] // FullSimplify
```

Out[2465]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{\frac{5}{2}}}{2} & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{6}} \\ -\frac{\sqrt{\frac{5}{6}}}{2} & \frac{1}{2\sqrt{3}} & \frac{5}{6} & -\frac{1}{6\sqrt{2}} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & \sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{3} & -\frac{1}{12} \\ -\frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2466]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2467]= 1

In[2468]:=
```
(* now we have to get the SU(2) components out of this. For that
  purpose we need to find the left and right-isoclinic rotation components
  of the SO(4) matrix as described in Van Elfrinkhof's formula. *)

(* First define the associate matrix. *)
```

In[2469]:=
```
Asc[M_] :=
 1 / 4 {{M[[1, 1]] + M[[2, 2]] + M[[3, 3]] + M[[4, 4]], M[[2, 1]] - M[[1, 2]] - M[[4, 3]] +
     M[[3, 4]], M[[3, 1]] + M[[4, 2]] - M[[1, 3]] - M[[2, 4]], M[[4, 1]] - M[[3, 2]] +
     M[[2, 3]] - M[[1, 4]]}, {M[[2, 1]] - M[[1, 2]] + M[[4, 3]] - M[[3, 4]],
    -M[[1, 1]] - M[[2, 2]] + M[[3, 3]] + M[[4, 4]], M[[4, 1]] + M[[1, 4]] -
     M[[2, 3]] - M[[3, 2]], -M[[3, 1]] - M[[4, 2]] - M[[1, 3]] - M[[2, 4]]},
   {M[[3, 1]] - M[[1, 3]] + M[[2, 4]] - M[[4, 2]], -M[[4, 1]] - M[[3, 2]] -
     M[[2, 3]] - M[[1, 4]], -M[[1, 1]] + M[[2, 2]] - M[[3, 3]] + M[[4, 4]],
    M[[2, 1]] + M[[1, 2]] - M[[4, 3]] - M[[3, 4]]},
   {M[[4, 1]] - M[[1, 4]] + M[[3, 2]] - M[[2, 3]],
    M[[3, 1]] + M[[1, 3]] - M[[4, 2]] - M[[2, 4]], -M[[2, 1]] - M[[1, 2]] -
     M[[4, 3]] - M[[3, 4]], -M[[1, 1]] + M[[2, 2]] + M[[3, 3]] - M[[4, 4]]}}
```

In[2470]:=
```
(* gA6 *)
```

In[2471]:=
```
AGA6 = Asc[GA6] // FullSimplify; AGA6 // MatrixForm
```

Out[2471]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{pmatrix}$$

In[2472]:=
```
(* use the exception algorithm from eqs. 151-152 of the thesis, with i=2, j=1 *)
KGA6 = Sqrt[AGA6[[2, 1]]^2 + AGA6[[2, 2]]^2 + AGA6[[2, 3]]^2 + AGA6[[2, 4]]^2] // Simplify
```

Out[2472]= $\frac{1}{\sqrt{2}}$

In[2473]:=
```
pGA6 = AGA6[[2, 1]] / KGA6 // Simplify;
qGA6 = AGA6[[2, 2]] / KGA6 // Simplify;
rGA6 = AGA6[[2, 3]] / KGA6 // Simplify;
sGA6 = AGA6[[2, 4]] / KGA6 // FullSimplify;
aGA6 = KGA6 AGA6[[1, 1]] / AGA6[[2, 1]] // FullSimplify;
bGA6 = KGA6;
cGA6 = KGA6 AGA6[[3, 1]] / AGA6[[2, 1]] // FullSimplify;
dGA6 = KGA6 AGA6[[4, 1]] / AGA6[[2, 1]] // FullSimplify;
```

In[2481]:=
```
(* operations to figure out if we're doing everything right *)

aGA6^2 + bGA6^2 + cGA6^2 + dGA6^2 // FullSimplify
```

Out[2481]= 1

In[2482]:=
```
pGA6^2 + qGA6^2 + rGA6^2 + sGA6^2 // FullSimplify
```

Out[2482]= 1

In[2483]:=
```
{{aGA5 pGA5, aGA5 qGA5, aGA5 rGA5, aGA5 sGA5}, {bGA5 pGA5, bGA5 qGA5, bGA5 rGA5, bGA5 sGA5},
    {cGA5 pGA5 , cGA5 qGA5, cGA5 rGA5, cGA5 sGA5},
    {dGA5 pGA5, dGA5 qGA5, dGA5 rGA5, dGA5 sGA5}} - AGA5 // FullSimplify // MatrixForm
```

Out[2483]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2484]:=
```
(* (a,b,c,d) and (p,q,r,s) are S^3 vectors representing
 the left and right SU(2) components of the rotation, *)
(* which we can compose now. Attribute + to (a,b,c,d) and - to (p,q,r,s) *)
```

In[2485]:=
```
GA6P = (aGA6 IdentityMatrix[2] + I bGA6 PauliMatrix[1] +
    I cGA6 PauliMatrix[2] + I dGA6 PauliMatrix[3]) // Simplify; GA6P // MatrixForm
GA6M = (pGA6 IdentityMatrix[2] + I qGA6 PauliMatrix[1] + I rGA6 PauliMatrix[2] +
     I sGA6 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GA6M // MatrixForm
```

Out[2485]//MatrixForm=

$$\begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

Out[2486]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

In[2487]:=
```
(* gA5 *)
```

In[2488]:=
```
AGA5 = Asc[GA5] // FullSimplify; AGA5 // MatrixForm
```

Out[2488]//MatrixForm=

$$\begin{pmatrix} \frac{1}{16}\sqrt{3}\left(2+\sqrt{5}\right) & -\frac{1}{16} & -\frac{1}{16}\sqrt{3}\left(2+\sqrt{5}\right) & -\frac{1}{16} \\ -\frac{3}{16} & \frac{1}{16}\sqrt{3}\left(-2+\sqrt{5}\right) & \frac{3}{16} & \frac{1}{16}\sqrt{3}\left(-2+\sqrt{5}\right) \\ -\frac{1}{16}\sqrt{3}\left(2+\sqrt{5}\right) & \frac{1}{16} & \frac{1}{16}\sqrt{3}\left(2+\sqrt{5}\right) & \frac{1}{16} \\ \frac{3}{16} & -\frac{1}{16}\sqrt{3}\left(-2+\sqrt{5}\right) & -\frac{3}{16} & -\frac{1}{16}\sqrt{3}\left(-2+\sqrt{5}\right) \end{pmatrix}$$

In[2489]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[2490]:=
```
KGA5 = Sqrt[AGA5[[1, 1]]^2 + AGA5[[1, 2]]^2 + AGA5[[1, 3]]^2 + AGA5[[1, 4]]^2] // Simplify;
```

In[2491]:=
```
pGA5 = AGA5[[1, 1]] / KGA5 // Simplify;
qGA5 = AGA5[[1, 2]] / KGA5 // Simplify;
rGA5 = AGA5[[1, 3]] / KGA5 // Simplify;
sGA5 = AGA5[[1, 4]] / KGA5 // FullSimplify;
aGA5 = KGA5;
bGA5 = KGA5 AGA5[[2, 1]] / AGA5[[1, 1]] // FullSimplify;
cGA5 = KGA5 AGA5[[3, 1]] / AGA5[[1, 1]] // FullSimplify;
dGA5 = KGA5 AGA5[[4, 1]] / AGA5[[1, 1]] // FullSimplify;
```

In[2499]:=
```
(* operations to figure out if we're doing everything right *)
```

In[2500]:=
```
aGA5^2 + bGA5^2 + cGA5^2 + dGA5^2 // FullSimplify
```

Out[2500]= 1

In[2501]:=
```
pGA5^2 + qGA5^2 + rGA5^2 + sGA5^2 // FullSimplify
```

Out[2501]= 1

In[2502]:=
```
{{aGA5 pGA5, aGA5 qGA5, aGA5 rGA5, aGA5 sGA5}, {bGA5 pGA5, bGA5 qGA5, bGA5 rGA5, bGA5 sGA5},
    {cGA5 pGA5 , cGA5 qGA5, cGA5 rGA5, cGA5 sGA5},
    {dGA5 pGA5, dGA5 qGA5, dGA5 rGA5, dGA5 sGA5}} - AGA5 // FullSimplify // MatrixForm
```

Out[2502]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2503]:=
```
GA5P = (aGA5 IdentityMatrix[2] + I bGA5 PauliMatrix[1] +
    I cGA5 PauliMatrix[2] + I dGA5 PauliMatrix[3]) // Simplify; GA5P // MatrixForm
GA5M = (pGA5 IdentityMatrix[2] + I qGA5 PauliMatrix[1] + I rGA5 PauliMatrix[2] +
     I sGA5 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GA5M // MatrixForm
```

Out[2503]//MatrixForm=

$$\begin{pmatrix} \frac{1}{8}\left(\sqrt{14+6\sqrt{5}}+2\,i\,\text{Root}\left[9-9\,\#1^2+\#1^4\,\&,\,3\right]\right) & \frac{1}{8}\left(-3-\sqrt{5}+2\,i\,\text{Root}\left[9-9\,\#1^2+\#1^4\,\&,\,2\right]\right) \\ \frac{1}{8}\left(3+\sqrt{5}+2\,i\,\text{Root}\left[9-9\,\#1^2+\#1^4\,\&,\,2\right]\right) & \frac{1}{8}\left(\sqrt{14+6\sqrt{5}}-2\,i\,\text{Root}\left[9-9\,\#1^2+\#1^4\,\&,\,3\right]\right) \end{pmatrix}$$

Out[2504]//MatrixForm=

$$\begin{pmatrix} \frac{1}{8}\left(-i\left(-3+\sqrt{5}\right)+\sqrt{6\left(7-3\sqrt{5}\right)}\left(2+\sqrt{5}\right)\right) & \frac{i+2\sqrt{3}+\sqrt{15}}{2\sqrt{14+6\sqrt{5}}} \\ -\frac{-i+2\sqrt{3}+\sqrt{15}}{2\sqrt{14+6\sqrt{5}}} & \frac{1}{8}\left(i\left(-3+\sqrt{5}\right)+\sqrt{6\left(7-3\sqrt{5}\right)}\left(2+\sqrt{5}\right)\right) \end{pmatrix}$$

In[2505]:=
```
(* gA3 *)
```

In[2506]:=
```
AGA3 = Asc[GA3] // FullSimplify; AGA3 // MatrixForm
```

Out[2506]//MatrixForm=

$$\begin{pmatrix} \frac{1}{48}\left(8\sqrt{2}+3\sqrt{15}\right) & \frac{1}{48}\left(-13-2\sqrt{30}\right) & \frac{1}{48}\left(-2\sqrt{2}+\sqrt{15}\right) & \frac{7}{48} \\ \frac{1}{48}\left(-11-2\sqrt{30}\right) & \frac{1}{48}\left(8\sqrt{2}+3\sqrt{15}\right) & -\frac{1}{48} & -\frac{1}{48}\sqrt{23+4\sqrt{30}} \\ \frac{1}{48}\left(2\sqrt{2}+\sqrt{15}\right) & -\frac{7}{48} & \frac{1}{48}\left(-8\sqrt{2}+3\sqrt{15}\right) & \frac{1}{48}\left(13-2\sqrt{30}\right) \\ -\frac{1}{48} & \frac{1}{48}\left(-2\sqrt{2}+\sqrt{15}\right) & \frac{1}{48}\left(-11+2\sqrt{30}\right) & \frac{1}{48}\left(8\sqrt{2}-3\sqrt{15}\right) \end{pmatrix}$$

In[2507]:=
```
KGA3 = Sqrt[AGA3[[1, 1]]^2 + AGA3[[1, 2]]^2 + AGA3[[1, 3]]^2 + AGA3[[1, 4]]^2] // Simplify
```

Out[2507]= $\frac{1}{4}\sqrt{\frac{1}{3}\left(13+2\sqrt{30}\right)}$

In[2508]:=
```
pGA3 = AGA3[[1, 1]] / KGA3 // Simplify;
qGA3 = AGA3[[1, 2]] / KGA3 // Simplify;
rGA3 = AGA3[[1, 3]] / KGA3 // Simplify;
sGA3 = AGA3[[1, 4]] / KGA3 // Simplify;
aGA3 = KGA3;
bGA3 = KGA3 AGA3[[2, 1]] / AGA3[[1, 1]] // Simplify;
cGA3 = KGA3 AGA3[[3, 1]] / AGA3[[1, 1]] // Simplify;
dGA3 = KGA3 AGA3[[4, 1]] / AGA3[[1, 1]] // Simplify;
```

In[2516]:=
```
aGA3^2 + bGA3^2 + cGA3^2 + dGA3^2 // FullSimplify
```

Out[2516]= 1

In[2517]:=
```
pGA3^2 + qGA3^2 + rGA3^2 + sGA3^2 // FullSimplify
```

Out[2517]= 1

In[2518]:=
```
{{aGA3 pGA3, aGA3 qGA3, aGA3 rGA3, aGA3 sGA3}, {bGA3 pGA3, bGA3 qGA3, bGA3 rGA3, bGA3 sGA3},
    {cGA3 pGA3 , cGA3 qGA3, cGA3 rGA3, cGA3 sGA3},
    {dGA3 pGA3, dGA3 qGA3, dGA3 rGA3, dGA3 sGA3}} - AGA3 // FullSimplify // MatrixForm
```

Out[2518]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2519]:=
```
GA3P = (aGA3 IdentityMatrix[2] + I bGA3 PauliMatrix[1] +
    I cGA3 PauliMatrix[2] + I dGA3 PauliMatrix[3]) // Simplify; GA3P // MatrixForm
GA3M = (pGA3 IdentityMatrix[2] + I qGA3 PauliMatrix[1] + I rGA3 PauliMatrix[2] +
     I sGA3 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GA3M // MatrixForm
```

Out[2519]//MatrixForm=

$$\begin{pmatrix} \frac{\left(-i\sqrt{3}+9\sqrt{5}+8\sqrt{6}\right)\sqrt{13+2\sqrt{30}}}{96\sqrt{2}+36\sqrt{15}} & \frac{\left(-11i\sqrt{3}+3\sqrt{5}+2\sqrt{6}-6i\sqrt{10}\right)\sqrt{13+2\sqrt{30}}}{96\sqrt{2}+36\sqrt{15}} \\ -\frac{\left(11i\sqrt{3}+3\sqrt{5}+2\sqrt{6}+6i\sqrt{10}\right)\sqrt{13+2\sqrt{30}}}{96\sqrt{2}+36\sqrt{15}} & \frac{\left(i\sqrt{3}+9\sqrt{5}+8\sqrt{6}\right)\sqrt{13+2\sqrt{30}}}{96\sqrt{2}+36\sqrt{15}} \end{pmatrix}$$

Out[2520]//MatrixForm=

$$\begin{pmatrix} \frac{-7i\sqrt{3}+9\sqrt{5}+8\sqrt{6}}{12\sqrt{13+2\sqrt{30}}} & \frac{13i\sqrt{3}-3\sqrt{5}+2\sqrt{6}+6i\sqrt{10}}{12\sqrt{13+2\sqrt{30}}} \\ \frac{13i\sqrt{3}+3\sqrt{5}-2\sqrt{6}+6i\sqrt{10}}{12\sqrt{13+2\sqrt{30}}} & \frac{7i\sqrt{3}+9\sqrt{5}+8\sqrt{6}}{12\sqrt{13+2\sqrt{30}}} \end{pmatrix}$$

In[2521]:=
```
GA3M.Conjugate[Transpose[GA3M]] // FullSimplify // MatrixForm
GA3P.Conjugate[Transpose[GA3P]] // FullSimplify // MatrixForm
```

Out[2521]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Out[2522]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In[2523]:=
```
(* gA2 *)
```

In[2524]:=
```
AGA2 = Asc[GA2] // FullSimplify; AGA2 // MatrixForm
```

Out[2524]//MatrixForm=

$$\begin{pmatrix} \frac{1}{48}\left(-4\sqrt{2}+2\sqrt{3}-3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-7+\sqrt{30}\right) & \frac{1}{48}\left(\sqrt{2}-2\sqrt{3}+\sqrt{15}\right) & \frac{1}{48}\left(13-5\sqrt{6}\right) \\ \frac{1}{48}\left(-5+\sqrt{30}\right) & \frac{1}{48}\left(-4\sqrt{2}-2\sqrt{3}+3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-7-3\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}-2\sqrt{3}-\sqrt{15}\right) \\ \frac{1}{48}\left(-\sqrt{2}-2\sqrt{3}+\sqrt{15}\right) & \frac{1}{48}\left(-13-5\sqrt{6}\right) & \frac{1}{48}\left(4\sqrt{2}+2\sqrt{3}+3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(7+\sqrt{30}\right) \\ \frac{1}{48}\left(-7+3\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}+2\sqrt{3}+\sqrt{15}\right) & \frac{1}{48}\left(-5-\sqrt{30}\right) & \frac{1}{48}\left(-4\sqrt{2}+2\sqrt{3}+3\sqrt{10}-3\sqrt{15}\right) \end{pmatrix}$$

In[2525]:=
```
KGA2 = Sqrt[AGA2[[1, 1]]^2 + AGA2[[1, 2]]^2 + AGA2[[1, 3]]^2 + AGA2[[1, 4]]^2] // Simplify
```

Out[2525]= $\frac{1}{4}\sqrt{\frac{1}{6}\left(29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}\right)}$

In[2526]:=
```
pGA2 = AGA2[[1, 1]] / KGA2 // Simplify;
qGA2 = AGA2[[1, 2]] / KGA2 // Simplify;
rGA2 = AGA2[[1, 3]] / KGA2 // Simplify;
sGA2 = AGA2[[1, 4]] / KGA2 // Simplify;
aGA2 = KGA2;
bGA2 = KGA2 AGA2[[2, 1]] / AGA2[[1, 1]] // Simplify;
cGA2 = KGA2 AGA2[[3, 1]] / AGA2[[1, 1]] // Simplify;
dGA2 = KGA2 AGA2[[4, 1]] / AGA2[[1, 1]] // Simplify;
```

In[2534]:=
```
aGA2^2 + bGA2^2 + cGA2^2 + dGA2^2 // FullSimplify
```

Out[2534]= 1

In[2535]:=
```
pGA2^2 + qGA2^2 + rGA2^2 + sGA2^2 // FullSimplify
```

Out[2535]= 1

In[2536]:=
```
{{aGA2 pGA2, aGA2 qGA2, aGA2 rGA2, aGA2 sGA2}, {bGA2 pGA2, bGA2 qGA2, bGA2 rGA2, bGA2 sGA2},
    {cGA2 pGA2 , cGA2 qGA2, cGA2 rGA2, cGA2 sGA2},
    {dGA2 pGA2, dGA2 qGA2, dGA2 rGA2, dGA2 sGA2}} - AGA2 // FullSimplify // MatrixForm
```

Out[2536]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2537]:=
```
GA2P = (aGA2 IdentityMatrix[2] + I bGA2 PauliMatrix[1] +
     I cGA2 PauliMatrix[2] + I dGA2 PauliMatrix[3]) // Simplify; GA2P // MatrixForm
GA2M = (pGA2 IdentityMatrix[2] + I qGA2 PauliMatrix[1] + I rGA2 PauliMatrix[2] +
      I sGA2 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GA2M // MatrixForm
```

Out[2537]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4}\sqrt{\frac{1}{6}\left(29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}\right)}\left(1+\frac{i\left(-7+3\sqrt{6}\right)}{-4\sqrt{2}+2\sqrt{3}-3\sqrt{10}+3\sqrt{15}}\right) & \frac{\left(6\sqrt{2}+2\sqrt{3}-6i\sqrt{5}+5i\sqrt{6}-3\sqrt{10}\right)\sqrt{29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}}}{24\left(4\sqrt{2}-2\sqrt{3}+3\sqrt{10}-3\sqrt{15}\right)} \\ \frac{\left(-6\sqrt{2}-2\sqrt{3}-6i\sqrt{5}+5i\sqrt{6}+3\sqrt{10}\right)\sqrt{29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}}}{24\left(4\sqrt{2}-2\sqrt{3}+3\sqrt{10}-3\sqrt{15}\right)} & \frac{1}{4}\sqrt{\frac{1}{6}\left(29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}\right)}\left(1+\frac{i\left(-7+3\sqrt{6}\right)}{4\sqrt{2}-2\sqrt{3}+3\sqrt{10}-3\sqrt{15}}\right) \end{pmatrix}$$

Out[2538]//MatrixForm=

$$\begin{pmatrix} \frac{30i+6\sqrt{2}-8\sqrt{3}-13i\sqrt{6}+9\sqrt{10}-6\sqrt{15}}{12\sqrt{29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}}} & \frac{6\sqrt{2}-2\sqrt{3}-6i\sqrt{5}+7i\sqrt{6}-3\sqrt{10}}{12\sqrt{29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}}} \\ \frac{-6\sqrt{2}+2\sqrt{3}-6i\sqrt{5}+7i\sqrt{6}+3\sqrt{10}}{12\sqrt{29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}}} & \frac{-30i+6\sqrt{2}-8\sqrt{3}+13i\sqrt{6}+9\sqrt{10}-6\sqrt{15}}{12\sqrt{29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}}} \end{pmatrix}$$

In[2539]:=
```
GA2M.Conjugate[Transpose[GA2M]] // FullSimplify // MatrixForm
```

Out[2539]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In[2540]:=
```
(* gA1 *)
```

In[2541]:=
```
AGA1 = Asc[GA1] // FullSimplify; AGA1 // MatrixForm
```

Out[2541]//MatrixForm=

$$\begin{pmatrix} \frac{1}{48}\left(-4\sqrt{2}+2\sqrt{3}-3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(5-\sqrt{30}\right) & \frac{1}{48}\left(\sqrt{2}+2\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(7-3\sqrt{6}\right) \\ \frac{1}{48}\left(7-\sqrt{30}\right) & \frac{1}{48}\left(-4\sqrt{2}-2\sqrt{3}+3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-13-5\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}+2\sqrt{3}+\sqrt{15}\right) \\ \frac{1}{48}\left(-\sqrt{2}+2\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(-7-3\sqrt{6}\right) & \frac{1}{48}\left(4\sqrt{2}+2\sqrt{3}+3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-5-\sqrt{30}\right) \\ \frac{1}{48}\left(-13+5\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}-2\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(7+\sqrt{30}\right) & \frac{1}{48}\left(-4\sqrt{2}+2\sqrt{3}+3\sqrt{10}-3\sqrt{15}\right) \end{pmatrix}$$

In[2542]:=
```
KGA1 =
  Sqrt[AGA1[[1, 1]]^2 + AGA1[[1, 2]]^2 + AGA1[[1, 3]]^2 + AGA1[[1, 4]]^2] // FullSimplify
```

Out[2542]=

$$\frac{1}{4}\sqrt{\frac{1}{6}\left(19+3\sqrt{5}-2\sqrt{6}\left(3+\sqrt{5}\right)\right)}$$

In[2543]:=
```
pGA1 = AGA1[[1, 1]] / KGA1 // Simplify;
qGA1 = AGA1[[1, 2]] / KGA1 // Simplify;
rGA1 = AGA1[[1, 3]] / KGA1 // Simplify;
sGA1 = AGA1[[1, 4]] / KGA1 // Simplify;
aGA1 = KGA1;
bGA1 = KGA1 AGA1[[2, 1]] / AGA1[[1, 1]] // FullSimplify;
cGA1 = KGA1 AGA1[[3, 1]] / AGA1[[1, 1]] // FullSimplify;
dGA1 = KGA1 AGA1[[4, 1]] / AGA1[[1, 1]] // FullSimplify;
```

In[2551]:=
```
aGA1^2 + bGA1^2 + cGA1^2 + dGA1^2  // FullSimplify
```

Out[2551]= 1

In[2552]:=
```
pGA1^2 + qGA1^2 + rGA1^2 + sGA1^2  // FullSimplify
```

Out[2552]= 1

In[2553]:=
```
{{aGA1 pGA1, aGA1 qGA1, aGA1 rGA1, aGA1 sGA1}, {bGA1 pGA1, bGA1 qGA1, bGA1 rGA1, bGA1 sGA1},
    {cGA1 pGA1 , cGA1 qGA1, cGA1 rGA1, cGA1 sGA1},
    {dGA1 pGA1, dGA1 qGA1, dGA1 rGA1, dGA1 sGA1}} - AGA1 // FullSimplify // MatrixForm
```

Out[2553]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2554]:=
```
GA1P = (aGA1 IdentityMatrix[2] + I bGA1 PauliMatrix[1] +
     I cGA1 PauliMatrix[2] + I dGA1 PauliMatrix[3]) // Simplify; GA1P // MatrixForm
GA1M = (pGA1 IdentityMatrix[2] + I qGA1 PauliMatrix[1] + I rGA1 PauliMatrix[2] +
      I sGA1 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GA1M // MatrixForm
```

Out[2554]//MatrixForm=

$$\begin{pmatrix} \frac{1}{24}\left(\sqrt{6\left(19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}\right)}+i\left(3-3\sqrt{5}+2\sqrt{30}\right)\right) & \frac{1}{24}\left(-i\left(-3+3\sqrt{5}+2\sqrt{30}\right)+\sqrt{6\left(19+3\sqrt{5}+6\sqrt{6}+2\sqrt{30}\right)}\right) \\ -\frac{1}{24}i\left(-3+3\sqrt{5}+2\sqrt{30}-i\sqrt{6\left(19+3\sqrt{5}+6\sqrt{6}+2\sqrt{30}\right)}\right) & \frac{1}{24}\left(\sqrt{6\left(19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}\right)}-i\left(3-3\sqrt{5}+2\sqrt{30}\right)\right) \end{pmatrix}$$

Out[2555]//MatrixForm=

$$\begin{pmatrix} \frac{18i+6\sqrt{2}-8\sqrt{3}-7i\sqrt{6}+9\sqrt{10}-6\sqrt{15}}{12\sqrt{19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}}} & \frac{-6\sqrt{2}-2\sqrt{3}+6i\sqrt{5}-5i\sqrt{6}+3\sqrt{10}}{12\sqrt{19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}}} \\ \frac{6\sqrt{2}+2\sqrt{3}+6i\sqrt{5}-5i\sqrt{6}-3\sqrt{10}}{12\sqrt{19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}}} & \frac{-18i+6\sqrt{2}-8\sqrt{3}+7i\sqrt{6}+9\sqrt{10}-6\sqrt{15}}{12\sqrt{19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}}} \end{pmatrix}$$

In[2556]:=
```
(* Now we need the SO(3) matrices R(g) to do
 consistency checks and find n's from gluing equations. *)
```

In[2557]:=
```
su2[x_, y_] = {{x, y}, {-Conjugate[y], Conjugate[x]}} ; su2[x, y] // MatrixForm
(* general SU(2) matrix *)
```

Out[2557]//MatrixForm=

$$\begin{pmatrix} x & y \\ -\text{Conjugate}[y] & \text{Conjugate}[x] \end{pmatrix}$$

In[2558]:=
```
R[x_, y_] =
  {{Re[x^2 - y^2], Im[x^2 + y^2], -2 Re[x y]}, {-Im[x^2 - y^2], Re[x^2 + y^2], 2 Im[x y]},
   {2 Re[x Conjugate[y]], 2 Im[x Conjugate[y]], x Conjugate[x] - y Conjugate[y]}};
R[x, y] // MatrixForm (* SO(3) image of the general SU(2) matrix defined as above *)
```

Out[2558]//MatrixForm=

$$\begin{pmatrix} \text{Re}\left[x^2 - y^2\right] & \text{Im}\left[x^2 + y^2\right] & -2\,\text{Re}[x\,y] \\ -\text{Im}\left[x^2 - y^2\right] & \text{Re}\left[x^2 + y^2\right] & 2\,\text{Im}[x\,y] \\ 2\,\text{Re}[x\,\text{Conjugate}[y]] & 2\,\text{Im}[x\,\text{Conjugate}[y]] & x\,\text{Conjugate}[x] - y\,\text{Conjugate}[y] \end{pmatrix}$$

```
(* step needed to fix an inconsistency between
 two different conventions used for these matrices *)
```

In[2560]:=
```
R[x_, y_] = R[x, y] /. x → -Conjugate[x]
```

Out[2560]=

$$\left\{\left\{\text{Re}\left[-y^2 + \text{Conjugate}[x]^2\right], \text{Im}\left[y^2 + \text{Conjugate}[x]^2\right], 2\,\text{Re}[y\,\text{Conjugate}[x]]\right\},\right.$$
$$\left\{-\text{Im}\left[-y^2 + \text{Conjugate}[x]^2\right], \text{Re}\left[y^2 + \text{Conjugate}[x]^2\right], -2\,\text{Im}[y\,\text{Conjugate}[x]]\right\},$$
$$\{-2\,\text{Re}[\text{Conjugate}[x]\,\text{Conjugate}[y]],$$
$$\left.-2\,\text{Im}[\text{Conjugate}[x]\,\text{Conjugate}[y]], x\,\text{Conjugate}[x] - y\,\text{Conjugate}[y]\}\right\}$$

In[2561]:=
```
(* gA6 *)
```

In[2562]:=
```
RA6P = R[GA6P[[1, 1]], GA6P[[1, 2]]] // FullSimplify; RA6P // MatrixForm
RA6P.Transpose[RA6P] // FullSimplify // MatrixForm
Det[RA6P] // FullSimplify
```

Out[2562]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Out[2563]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2564]= 1

In[2565]:=
```
RA6M = R[GA6M[[1, 1]], GA6M[[1, 2]]] // Simplify; RA6M // MatrixForm
RA6M.Transpose[RA6M] // FullSimplify // MatrixForm
Det[RA6M] // FullSimplify
```

Out[2565]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Out[2566]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2567]= 1

In[2568]:=
```
(* gA5 *)
```

In[2569]:=
```
RA5P = R[GA5P[[1, 1]], GA5P[[1, 2]]] // FullSimplify; RA5P // MatrixForm
RA5P.Transpose[RA5P] // FullSimplify // MatrixForm
Det[RA5P] // FullSimplify
```

Out[2569]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 \\ \frac{1}{4}\sqrt{\frac{3}{2}\left(3+\sqrt{5}\right)} & \frac{1}{8}\left(-1+3\sqrt{5}\right) & 0 \\ \frac{1}{8}\left(-1+3\sqrt{5}\right) & -\frac{1}{4}\sqrt{\frac{3}{2}\left(3+\sqrt{5}\right)} & 0 \end{pmatrix}$$

Out[2570]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2571]= 1

In[2572]:=
```
RA5M = R[GA5M[[1, 1]], GA5M[[1, 2]]] // Simplify; RA5M // MatrixForm
RA5M.Transpose[RA5M] // FullSimplify // MatrixForm
Det[RA5M] // FullSimplify
```

Out[2572]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{\sqrt{3}\left(8+4\sqrt{5}-5\sqrt{70-30\sqrt{5}}-11\sqrt{14-6\sqrt{5}}\right)}{16\left(7+3\sqrt{5}\right)} & \frac{54\sqrt{7-3\sqrt{5}}+24\sqrt{5\left(7-3\sqrt{5}\right)}-\sqrt{2}\left(-3+\sqrt{5}\right)}{16\sqrt{7+3\sqrt{5}}} \\ \frac{\sqrt{3}\left(8+4\sqrt{5}+5\sqrt{70-30\sqrt{5}}+11\sqrt{14-6\sqrt{5}}\right)}{16\left(7+3\sqrt{5}\right)} & \frac{1}{8}\left(1+3\sqrt{5}\right) & \frac{\sqrt{6}+\sqrt{30}-4\sqrt{21-9\sqrt{5}}-2\sqrt{15\left(7-3\sqrt{5}\right)}}{16\sqrt{7+3\sqrt{5}}} \\ \frac{-54\sqrt{7-3\sqrt{5}}-24\sqrt{5\left(7-3\sqrt{5}\right)}-\sqrt{2}\left(-3+\sqrt{5}\right)}{16\sqrt{7+3\sqrt{5}}} & \frac{\sqrt{6}+\sqrt{30}+4\sqrt{21-9\sqrt{5}}+2\sqrt{15\left(7-3\sqrt{5}\right)}}{16\sqrt{7+3\sqrt{5}}} & 0 \end{pmatrix}$$

Out[2573]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2574]= 1

In[2575]:=
```
(* gA3 *)
```

In[2576]:=
```
RA3P = R[GA3P[[1, 1]], GA3P[[1, 2]]] // FullSimplify; RA3P // MatrixForm
RA3P.Transpose[RA3P] // FullSimplify // MatrixForm
Det[RA3P] // FullSimplify
```

Out[2576]//MatrixForm=

$$\begin{pmatrix} \sqrt{\frac{5}{6}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & \frac{1}{12} & \frac{2\sqrt{2}}{3} \\ -\frac{1}{4} & -\frac{\sqrt{15}}{4} & 0 \end{pmatrix}$$

Out[2577]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2578]= 1

In[2579]:= $\frac{1}{4}$ Root[9 - 9 #1^2 + #1^4 &, 2] // N

Out[2579]= -0.267617

In[2580]:=

In[2581]:=
```
RA3M = R[GA3M[[1, 1]], GA3M[[1, 2]]] // FullSimplify; RA3M // MatrixForm
RA3M.Transpose[RA3M] // FullSimplify // MatrixForm
Det[RA3M] // FullSimplify
```

Out[2581]//MatrixForm=

$$\begin{pmatrix} \sqrt{\frac{5}{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & -\frac{1}{12} & -\frac{2\sqrt{2}}{3} \\ -\frac{1}{4} & \frac{\sqrt{15}}{4} & 0 \end{pmatrix}$$

Out[2582]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2583]= 1

In[2584]:=
```
(* gA2 *)
```

In[2585]:=
```
RA2P = R[GA2P[[1, 1]], GA2P[[1, 2]]] // FullSimplify; RA2P // MatrixForm
RA2P.Transpose[RA2P] // FullSimplify // MatrixForm
Det[RA2P] // FullSimplify
```

Out[2585]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2}\sqrt{\frac{1}{3}\left(3+\sqrt{5}\right)} & -\frac{1}{6}\sqrt{23-3\sqrt{5}} & \frac{1}{3} \\ -\frac{1}{4}\sqrt{\frac{1}{6}\left(63-5\sqrt{5}\right)} & \frac{1}{24}\left(5+3\sqrt{5}\right) & -\frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(-1+\sqrt{5}\right) & -\frac{1}{4}\sqrt{\frac{1}{6}\left(23+3\sqrt{5}\right)} & -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[2586]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2587]= 1

In[2588]:=
```
RA2M = R[GA2M[[1, 1]], GA2M[[1, 2]]] // FullSimplify; RA2M // MatrixForm
RA2M.Transpose[RA2M] // FullSimplify // MatrixForm
Det[RA2M] // FullSimplify
```

Out[2588]//MatrixForm=

$$\begin{pmatrix} \text{Root}\left[1-18\,\#1^2+36\,\#1^4\,\&,\,2\right] & -\frac{1}{6}\sqrt{23+3\sqrt{5}} & -\frac{1}{3} \\ -\frac{1}{4}\sqrt{\frac{1}{6}\left(63+5\sqrt{5}\right)} & \frac{1}{24}\left(-5+3\sqrt{5}\right) & \frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(-1-\sqrt{5}\right) & \frac{1}{4}\sqrt{\frac{1}{6}\left(23-3\sqrt{5}\right)} & -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[2589]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2590]= 1

In[2591]:=
```
(* gA1 *)
```

In[2592]:=
```
RA1P = R[GA1P[[1, 1]], GA1P[[1, 2]]] // FullSimplify; RA1P // MatrixForm
RA1P.Transpose[RA1P] // FullSimplify // MatrixForm
Det[RA1P] // FullSimplify
```

Out[2592]//MatrixForm=

$$\begin{pmatrix} \frac{1}{12}\left(\sqrt{6}-\sqrt{30}\right) & -\frac{1}{6}\sqrt{23+3\sqrt{5}} & -\frac{1}{3} \\ -\frac{1+5\sqrt{5}}{8\sqrt{3}} & \frac{1}{24}\left(-5+3\sqrt{5}\right) & \frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(-1-\sqrt{5}\right) & \frac{-1+3\sqrt{5}}{8\sqrt{3}} & -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[2593]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2594]= 1

In[2595]:=
```
RA1M = R[GA1M[[1, 1]], GA1M[[1, 2]]] // FullSimplify; RA1M // MatrixForm
RA1M.Transpose[RA1M] // FullSimplify // MatrixForm
Det[RA1M] // FullSimplify
```

Out[2595]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2}\sqrt{\frac{1}{3}\left(3+\sqrt{5}\right)} & -\frac{1}{6}\sqrt{23-3\sqrt{5}} & \frac{1}{3} \\ -\frac{1}{4}\sqrt{\frac{1}{6}\left(63-5\sqrt{5}\right)} & \frac{1}{24}\left(5+3\sqrt{5}\right) & -\frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(-1+\sqrt{5}\right) & -\frac{1}{4}\sqrt{\frac{1}{6}\left(23+3\sqrt{5}\right)} & -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[2596]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2597]= 1

In[2598]:=
```
RAP = {RA1P, RA2P, RA3P, RA4P, RA5P, RA6P};
RAM = {RA1M, RA2M, RA3M, RA4M, RA5M, RA6M};
```

In[2600]:=
```
RA[i_, j_] := Transpose[RAP[[j]]].RAP[[i]] - Transpose[RAM[[j]]].RAM[[i]]
(* gluing matrices. The null space of these define the normal vectors n_ef.  *)
```

In[2601]:=
```
(* define the normal vectors tetrahedron by tetrahedron *)
```

In[2602]:=
```
(* due to our basis choices they coincide with the
 normals of the standard tetrahedron up to permutation *)
```

In[2603]:=
```
NA
NB
NC
ND
```

Out[2603]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2604]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2605]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2606]= $\{0, 0, -1\}$

In[2607]:=
```
(* Tetrahedron (1235) or (abce) *)
```

In[2608]:=
```
NullSpace[RA[6, 5]] // FullSimplify
NullSpace[RA[6, 5]][[1]] // FullSimplify // Normalize
```

Out[2608]= $\{\{0, 0, 1\}\}$

Out[2609]= $\{0, 0, 1\}$

In[2610]:=
```
nA665 = ND
```

Out[2610]= $\{0, 0, -1\}$

In[2611]:=
```
NullSpace[RA[6, 3]] // FullSimplify
NullSpace[RA[6, 3]][[1]] // FullSimplify // Normalize
```

Out[2611]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[2612]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2613]:=
```
nA663 = NC
```

Out[2613]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2614]:=
```
NullSpace[RA[6, 2]] // FullSimplify
NullSpace[RA[6, 2]][[1]] // FullSimplify // Normalize
```

Out[2614]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2615]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2616]:=
```
nA662 = NB
```

Out[2616]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2617]:=
```
NullSpace[RA[6, 1]] // FullSimplify
NullSpace[RA[6, 1]][[1]] // FullSimplify // Normalize
```

Out[2617]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2618]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2619]:=
```
nA661 = NA
```

Out[2619]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2620]:=
```
(* Tetrahedron (1236) or (abcf) *)
```

In[2621]:=
```
NullSpace[RA[5, 6]] // FullSimplify
NullSpace[RA[5, 6]][[1]] // FullSimplify // Normalize
```

Out[2621]= {{0, 0, 1}}

Out[2622]= {0, 0, 1}

In[2623]:=
```
nA556 = ND
```

Out[2623]= {0, 0, -1}

In[2624]:=
```
NullSpace[RA[5, 3]] // FullSimplify
NullSpace[RA[5, 3]][[1]] // FullSimplify // Normalize
```

Out[2624]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2625]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2626]:=
```
nA553 = NB
```

Out[2626]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2627]:= `RA52 = RA[5, 2] // FullSimplify`

Out[2627]= $\left\{\left\{-\frac{\sqrt{5}}{4}, \frac{\sqrt{\frac{5}{3}}}{4}, \sqrt{\frac{5}{6}}\right\}, \left\{\frac{\sqrt{\frac{5}{3}}}{4}, \frac{\sqrt{5}}{4}, \sqrt{\frac{5}{2}}\right\}, \left\{-\sqrt{\frac{10}{3}}, 0, 0\right\}\right\}$

In[2628]:=
```
NullSpace[RA52] // FullSimplify
NullSpace[RA52][[1]] // FullSimplify // Normalize
```

Out[2628]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[2629]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2630]:= `nA552 = NC`

Out[2630]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2631]:=
```
NullSpace[RA[5, 1]] // FullSimplify
NullSpace[RA[5, 1]][[1]] // FullSimplify // Normalize
```

Out[2631]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2632]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2633]:= `nA551 = NA`

Out[2633]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2634]:= `(* Tetrahedron (1256) or (abef) *)`

In[2635]:=
```
NullSpace[RA[3, 6]] // FullSimplify
NullSpace[RA[3, 6]][[1]] // FullSimplify // Normalize
```

Out[2635]= $\{\{0, 0, 1\}\}$

Out[2636]= $\{0, 0, 1\}$

In[2637]:= `nA336 = ND`

Out[2637]= $\{0, 0, -1\}$

In[2638]:=
```
NullSpace[RA[3, 5]] // FullSimplify
NullSpace[RA[3, 5]][[1]] // FullSimplify // Normalize
```

Out[2638]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[2639]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2640]:= `nA335 = NC`

Out[2640]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2641]:=
```
NullSpace[RA[3, 2]] // FullSimplify
NullSpace[RA[3, 2]][[1]] // FullSimplify // Normalize
```

Out[2641]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2642]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2643]:=
```
nA332 = NB
```

Out[2643]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2644]:=
```
NullSpace[RA[3, 1]] // FullSimplify
NullSpace[RA[3, 1]][[1]] // FullSimplify // Normalize
```

Out[2644]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2645]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2646]:=
```
nA331 = NA
```

Out[2646]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2647]:=
```
(* Tetrahedron (1356) or (acef) *)
```

In[2648]:=
```
NullSpace[RA[2, 6]] // FullSimplify
NullSpace[RA[2, 6]][[1]] // FullSimplify // Normalize
```

Out[2648]= {{0, 0, 1}}

Out[2649]= {0, 0, 1}

In[2650]:=
```
nA226 = ND
```

Out[2650]= {0, 0, -1}

In[2651]:=
```
RA25 = RA[2, 5] // FullSimplify
NullSpace[RA25][[1]] // FullSimplify // Normalize
```

Out[2651]= $\left\{\left\{-\frac{\sqrt{5}}{4}, \frac{\sqrt{\frac{5}{3}}}{4}, -\sqrt{\frac{10}{3}}\right\}, \left\{\frac{\sqrt{\frac{5}{3}}}{4}, \frac{\sqrt{5}}{4}, 0\right\}, \left\{\sqrt{\frac{5}{6}}, \sqrt{\frac{5}{2}}, 0\right\}\right\}$

Out[2652]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2653]:=
```
nA225 = NB
```

Out[2653]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2654]:=
```
NullSpace[RA[2, 3]] // FullSimplify
NullSpace[RA[2, 3]][[1]] // FullSimplify // Normalize
```

Out[2654]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[2655]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2656]:=
```
nA223 = NC
```

Out[2656]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2657]:=
```
NullSpace[RA[2, 1]] // FullSimplify
NullSpace[RA[2, 1]][[1]] // FullSimplify // Normalize
```

Out[2657]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2658]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2659]:=
```
nA221 = NA
```

Out[2659]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2660]:=
```
(* Tetrahedron (2356) or (bcef) *)
```

In[2661]:=
```
NullSpace[RA[1, 6]] // FullSimplify
NullSpace[RA[1, 6]][[1]] // FullSimplify // Normalize
```

Out[2661]= {{0, 0, 1}}

Out[2662]= {0, 0, 1}

In[2663]:=
```
nA116 = ND
```

Out[2663]= {0, 0, -1}

In[2664]:=
```
NullSpace[RA[1, 5]] // FullSimplify
NullSpace[RA[1, 5]][[1]] // FullSimplify // Normalize
```

Out[2664]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2665]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2666]:=
```
nA115 = NB
```

Out[2666]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2667]:=
```
NullSpace[RA[1, 3]] // FullSimplify
NullSpace[RA[1, 3]][[1]] // FullSimplify // Normalize
```

Out[2667]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[2668]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2669]:=
```
nA113 = NC
```

Out[2669]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2670]:=
```
NullSpace[RA[1, 2]] // FullSimplify
NullSpace[RA[1, 2]][[1]] // FullSimplify // Normalize
```

Out[2670]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2671]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2672]:=
```
nA112 = NA
```

Out[2672]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2673]:=
```
(* Time to move on to vertex B (abcdf). *)
```

In[2674]:=
```
(* Computing g's first. *)
(* g_B6 *)
MB6 = {LinearSolve[BasisE, vc], LinearSolve[BasisE, vb], LinearSolve[BasisE, vd]} ;
MB6 // MatrixForm
MB6N = Normalize[NullSpace[MB6][[1]]]
MB6final =
 Transpose[{MB6N, Normalize[MB6[[1]]], Normalize[MB6[[2]]], Normalize[MB6[[3]]]}];
MB6final // MatrixForm
Det[MB6final]
```

Out[2674]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \end{pmatrix}$$

Out[2675]= {0, 0, 0, 1}

Out[2676]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Out[2677]= $\frac{1}{\sqrt{2}}$

In[2678]:=
```
GB6 = MB6final.Inverse[M3] // FullSimplify; GB6 // MatrixForm
GB6.Transpose[GB6] // FullSimplify // MatrixForm
Det[GB6] // FullSimplify
```

Out[2678]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Out[2679]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2680]= 1

In[2681]:=
```
(* g_B4 *)
MB4 = {LinearSolve[BasisE, vc], LinearSolve[BasisE, vb], LinearSolve[BasisE, vf]} ;
MB4 // MatrixForm
MB4N = Normalize[NullSpace[MB4][[1]]]
MB4final =
 Transpose[{MB4N, Normalize[MB4[[1]]], Normalize[MB4[[2]]], Normalize[MB4[[3]]]}];
MB4final // MatrixForm
Det[MB4final]
GB4 = MB4final.Inverse[M3] // FullSimplify; GB4 // MatrixForm
GB4.Transpose[GB4] // FullSimplify // MatrixForm
Det[GB4] // FullSimplify
```

Out[2681]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2682]= $\left\{0, 0, -\frac{\sqrt{15}}{4}, \frac{1}{4}\right\}$

Out[2683]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2\sqrt{3}} \\ -\frac{\sqrt{15}}{4} & 0 & 0 & \frac{1}{2\sqrt{6}} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2684]= $\frac{1}{\sqrt{2}}$

Out[2685]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{\sqrt{15}}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2686]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2687]= 1

In[2688]:=
```
(* g_B3 *)
MB3 = {LinearSolve[BasisE, vb], LinearSolve[BasisE, vd], LinearSolve[BasisE, vf]} ;
MB3 // MatrixForm
MB3N = Normalize[NullSpace[MB3][[1]]]
MB3final =
 Transpose[{MB3N, Normalize[MB3[[1]]], Normalize[MB3[[2]]], Normalize[MB3[[3]]]}];
MB3final // MatrixForm
Det[MB3final]
GB3 = MB3final.Inverse[M3] // FullSimplify; GB3 // MatrixForm
GB3.Transpose[GB3] // FullSimplify // MatrixForm
Det[GB3] // FullSimplify
```

Out[2688]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2689]= $\left\{0, -\sqrt{\frac{5}{6}}, \frac{\sqrt{\frac{5}{3}}}{4}, \frac{1}{4}\right\}$

Out[2690]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ -\sqrt{\frac{5}{6}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{\frac{5}{3}}}{4} & 0 & \sqrt{\frac{2}{3}} & \frac{1}{2\sqrt{6}} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2691]= $\frac{1}{\sqrt{2}}$

Out[2692]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\sqrt{\frac{5}{6}} & 0 & \frac{1}{3} & \frac{1}{3\sqrt{2}} \\ \frac{\sqrt{\frac{5}{3}}}{4} & 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{12} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2693]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2694]= 1

In[2695]:=
```
(* g_B2 *)
MB2 = {LinearSolve[BasisE, vd], LinearSolve[BasisE, vc], LinearSolve[BasisE, vf]} ;
MB2 // MatrixForm
MB2N = Normalize[NullSpace[MB2][[1]]]
MB2final =
 Transpose[{MB2N, Normalize[MB2[[1]]], Normalize[MB2[[2]]], Normalize[MB2[[3]]]}];
MB2final // MatrixForm
Det[MB2final]
GB2 = MB2final.Inverse[M3] // FullSimplify; GB2 // MatrixForm
GB2.Transpose[GB2] // FullSimplify // MatrixForm
Det[GB2] // FullSimplify
```

Out[2695]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2696]= $\left\{-\frac{\sqrt{\frac{5}{2}}}{2}, \frac{\sqrt{\frac{5}{6}}}{2}, \frac{\sqrt{\frac{5}{3}}}{4}, \frac{1}{4}\right\}$

Out[2697]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{\frac{5}{2}}}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{\sqrt{\frac{5}{6}}}{2} & \frac{1}{2\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{\frac{5}{3}}}{4} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{2\sqrt{6}} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2698]= $\frac{1}{\sqrt{2}}$

Out[2699]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{\frac{5}{2}}}{2} & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} \\ \frac{\sqrt{\frac{5}{6}}}{2} & \frac{1}{2\sqrt{3}} & \frac{5}{6} & -\frac{1}{6\sqrt{2}} \\ \frac{\sqrt{\frac{5}{3}}}{4} & \sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{3} & -\frac{1}{12} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2700]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2701]= 1

In[2702]:=
```
(* g_B1 *)
MB1 = {LinearSolve[BasisE, c - b], LinearSolve[BasisE, d - b], LinearSolve[BasisE, f - b]} ;
MB1 // MatrixForm
MB1N = Normalize[NullSpace[MB1][[1]]]
MB1final =
 Transpose[{MB1N, Normalize[MB1[[1]]], Normalize[MB1[[2]]], Normalize[MB1[[3]]]}];
MB1final // MatrixForm
Det[MB1final]
GB1 = MB1final.Inverse[M3] // FullSimplify; GB1 // MatrixForm
GB1.Transpose[GB1] // FullSimplify // MatrixForm
Det[GB1] // FullSimplify
```

Out[2702]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2703]= $\left\{\frac{\sqrt{\frac{5}{2}}}{2}, \frac{\sqrt{\frac{5}{6}}}{2}, \frac{\sqrt{\frac{5}{3}}}{4}, \frac{1}{4}\right\}$

Out[2704]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{\frac{5}{6}}}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{\frac{5}{3}}}{4} & 0 & \sqrt{\frac{2}{3}} & \frac{1}{2\sqrt{6}} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2705]= $\frac{1}{\sqrt{2}}$

Out[2706]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{\frac{5}{2}}}{2} & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{6}} \\ \frac{\sqrt{\frac{5}{6}}}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{6} & -\frac{1}{6\sqrt{2}} \\ \frac{\sqrt{\frac{5}{3}}}{4} & 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{12} \\ \frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2707]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2708]= 1

In[2709]:=
```
(* determining the SU(2) components via Van Elfrinkhof's formula *)

(* gB6 *)
AGB6 = Asc[GB6] // FullSimplify; AGB6 // MatrixForm (* associate matrix *)
```

Out[2709]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{3}}{8} & \frac{1}{8} & -\frac{\sqrt{3}}{8} & \frac{1}{8} \\ -\frac{3}{8} & -\frac{\sqrt{3}}{8} & \frac{3}{8} & -\frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{8} & -\frac{1}{8} & \frac{\sqrt{3}}{8} & -\frac{1}{8} \\ \frac{3}{8} & \frac{\sqrt{3}}{8} & -\frac{3}{8} & \frac{\sqrt{3}}{8} \end{pmatrix}$$

In[2710]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[2711]:=
```
KGB6 = Sqrt[AGB6[[1, 1]]^2 + AGB6[[1, 2]]^2 + AGB6[[1, 3]]^2 + AGB6[[1, 4]]^2] // Simplify;
```

In[2712]:=
```
pGB6 = AGB6[[1, 1]] / KGB6 // Simplify;
qGB6 = AGB6[[1, 2]] / KGB6 // Simplify;
rGB6 = AGB6[[1, 3]] / KGB6 // Simplify;
sGB6 = AGB6[[1, 4]] / KGB6 // FullSimplify;
aGB6 = KGB6;
bGB6 = KGB6 AGB6[[2, 1]] / AGB6[[1, 1]] // FullSimplify;
cGB6 = KGB6 AGB6[[3, 1]] / AGB6[[1, 1]] // FullSimplify;
dGB6 = KGB6 AGB6[[4, 1]] / AGB6[[1, 1]] // FullSimplify;
```

In[2720]:=
```
(* operations to figure out if we're doing everything right *)
```

In[2721]:=
```
aGB6^2 + bGB6^2 + cGB6^2 + dGB6^2 // FullSimplify
```

Out[2721]= 1

In[2722]:=
```
pGB6^2 + qGB6^2 + rGB6^2 + sGB6^2 // FullSimplify
```

Out[2722]= 1

In[2723]:=
```
{{aGB6 pGB6, aGB6 qGB6, aGB6 rGB6, aGB6 sGB6}, {bGB6 pGB6, bGB6 qGB6, bGB6 rGB6, bGB6 sGB6},
    {cGB6 pGB6 , cGB6 qGB6, cGB6 rGB6, cGB6 sGB6},
    {dGB6 pGB6, dGB6 qGB6, dGB6 rGB6, dGB6 sGB6}} - AGB6 // FullSimplify // MatrixForm
```

Out[2723]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2724]:=
```
GB6P = (aGB6 IdentityMatrix[2] + I bGB6 PauliMatrix[1] +
    I cGB6 PauliMatrix[2] + I dGB6 PauliMatrix[3]) // Simplify; GB6P // MatrixForm
GB6M = (pGB6 IdentityMatrix[2] + I qGB6 PauliMatrix[1] + I rGB6 PauliMatrix[2] +
      I sGB6 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GB6M // MatrixForm
```

Out[2724]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4}\left(\sqrt{2}+i\sqrt{6}\right) & -\frac{i\left(-i+\sqrt{3}\right)}{2\sqrt{2}} \\ \frac{1}{4}\left(\sqrt{2}-i\sqrt{6}\right) & \frac{1}{4}\left(\sqrt{2}-i\sqrt{6}\right) \end{pmatrix}$$

Out[2725]//MatrixForm=

$$\begin{pmatrix} \frac{-i+\sqrt{3}}{2\sqrt{2}} & \frac{-i+\sqrt{3}}{2\sqrt{2}} \\ -\frac{i+\sqrt{3}}{2\sqrt{2}} & \frac{i+\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$$

In[2726]:=
```
(* gB4 *)
AGB4 = Asc[GB4] // FullSimplify; AGB4 // MatrixForm
```

Out[2726]//MatrixForm=

$$\begin{pmatrix} \frac{1}{16}\sqrt{3}\left(2+\sqrt{5}\right) & -\frac{1}{16} & -\frac{1}{16}\sqrt{3}\left(2+\sqrt{5}\right) & -\frac{1}{16} \\ -\frac{3}{16} & \frac{1}{16}\sqrt{3}\left(-2+\sqrt{5}\right) & \frac{3}{16} & \frac{1}{16}\sqrt{3}\left(-2+\sqrt{5}\right) \\ -\frac{1}{16}\sqrt{3}\left(2+\sqrt{5}\right) & \frac{1}{16} & \frac{1}{16}\sqrt{3}\left(2+\sqrt{5}\right) & \frac{1}{16} \\ \frac{3}{16} & -\frac{1}{16}\sqrt{3}\left(-2+\sqrt{5}\right) & -\frac{3}{16} & -\frac{1}{16}\sqrt{3}\left(-2+\sqrt{5}\right) \end{pmatrix}$$

In[2727]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[2728]:=
```
KGB4 = Sqrt[AGB4[[1, 1]]^2 + AGB4[[1, 2]]^2 + AGB4[[1, 3]]^2 + AGB4[[1, 4]]^2] // Simplify;
```

In[2729]:=
```
pGB4 = AGB4[[1, 1]] / KGB4 // Simplify;
qGB4 = AGB4[[1, 2]] / KGB4 // Simplify;
rGB4 = AGB4[[1, 3]] / KGB4 // Simplify;
sGB4 = AGB4[[1, 4]] / KGB4 // FullSimplify;
aGB4 = KGB4;
bGB4 = KGB4 AGB4[[2, 1]] / AGB4[[1, 1]] // FullSimplify;
cGB4 = KGB4 AGB4[[3, 1]] / AGB4[[1, 1]] // FullSimplify;
dGB4 = KGB4 AGB4[[4, 1]] / AGB4[[1, 1]] // FullSimplify;
```

In[2737]:=
```
(* operations to figure out if we're doing everything right *)
```

In[2738]:=
```
aGB4^2 + bGB4^2 + cGB4^2 + dGB4^2 // FullSimplify
pGB4^2 + qGB4^2 + rGB4^2 + sGB4^2 // FullSimplify
```

Out[2738]= 1

Out[2739]= 1

In[2740]:=
```
{{aGB4 pGB4, aGB4 qGB4, aGB4 rGB4, aGB4 sGB4}, {bGB4 pGB4, bGB4 qGB4, bGB4 rGB4, bGB4 sGB4},
    {cGB4 pGB4 , cGB4 qGB4, cGB4 rGB4, cGB4 sGB4},
    {dGB4 pGB4, dGB4 qGB4, dGB4 rGB4, dGB4 sGB4}} - AGB4 // FullSimplify // MatrixForm
```

Out[2740]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2741]:=
```
GB4P = (aGB4 IdentityMatrix[2] + I bGB4 PauliMatrix[1] +
    I cGB4 PauliMatrix[2] + I dGB4 PauliMatrix[3]) // FullSimplify; GB4P // MatrixForm
GB4M = (pGB4 IdentityMatrix[2] + I qGB4 PauliMatrix[1] + I rGB4 PauliMatrix[2] +
     I sGB4 PauliMatrix[3]) // Conjugate // Transpose // FullSimplify; GB4M // MatrixForm
```

Out[2741]//MatrixForm=

$$\begin{pmatrix} \text{Root}\left[1-3\,\#1+5\,\#1^2-6\,\#1^3+4\,\#1^4\,\&,\,4\right] & \text{Root}\left[1+3\,\#1+5\,\#1^2+6\,\#1^3+4\,\#1^4\,\&,\,1\right] \\ \text{Root}\left[1-3\,\#1+5\,\#1^2-6\,\#1^3+4\,\#1^4\,\&,\,3\right] & \text{Root}\left[1-3\,\#1+5\,\#1^2-6\,\#1^3+4\,\#1^4\,\&,\,3\right] \end{pmatrix}$$

Out[2742]//MatrixForm=

$$\begin{pmatrix} \frac{1}{8}\left(-i\left(-3+\sqrt{5}\right)+\sqrt{6\left(3+\sqrt{5}\right)}\right) & \frac{i+2\sqrt{3}+\sqrt{15}}{6+2\sqrt{5}} \\ -\frac{-i+2\sqrt{3}+\sqrt{15}}{2\left(3+\sqrt{5}\right)} & \frac{1}{8}\left(i\left(-3+\sqrt{5}\right)+\sqrt{6\left(3+\sqrt{5}\right)}\right) \end{pmatrix}$$

In[2743]:=
```
(* gB3 *)
AGB3 = Asc[GB3] // FullSimplify; AGB3 // MatrixForm
```

Out[2743]//MatrixForm=

$$\begin{pmatrix} \frac{1}{48}\left(8\sqrt{2}+3\sqrt{15}\right) & \frac{1}{48}\left(-13-2\sqrt{30}\right) & \frac{1}{48}\left(-2\sqrt{2}+\sqrt{15}\right) & \frac{7}{48} \\ \frac{1}{48}\left(-11-2\sqrt{30}\right) & \frac{1}{48}\left(8\sqrt{2}+3\sqrt{15}\right) & -\frac{1}{48} & -\frac{1}{48}\sqrt{23+4\sqrt{30}} \\ \frac{1}{48}\left(2\sqrt{2}+\sqrt{15}\right) & -\frac{7}{48} & \frac{1}{48}\left(-8\sqrt{2}+3\sqrt{15}\right) & \frac{1}{48}\left(13-2\sqrt{30}\right) \\ -\frac{1}{48} & \frac{1}{48}\left(-2\sqrt{2}+\sqrt{15}\right) & \frac{1}{48}\left(-11+2\sqrt{30}\right) & \frac{1}{48}\left(8\sqrt{2}-3\sqrt{15}\right) \end{pmatrix}$$

In[2744]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[2745]:=
```
KGB3 = Sqrt[AGB3[[1, 1]]^2 + AGB3[[1, 2]]^2 + AGB3[[1, 3]]^2 + AGB3[[1, 4]]^2] // Simplify;
```

In[2746]:=
```
pGB3 = AGB3[[1, 1]] / KGB3 // Simplify;
qGB3 = AGB3[[1, 2]] / KGB3 // Simplify;
rGB3 = AGB3[[1, 3]] / KGB3 // Simplify;
sGB3 = AGB3[[1, 4]] / KGB3 // FullSimplify;
aGB3 = KGB3;
bGB3 = KGB3 AGB3[[2, 1]] / AGB3[[1, 1]] // FullSimplify;
cGB3 = KGB3 AGB3[[3, 1]] / AGB3[[1, 1]] // FullSimplify;
dGB3 = KGB3 AGB3[[4, 1]] / AGB3[[1, 1]] // FullSimplify;
```

In[2754]:=
```
(* operations to figure out if we're doing everything right *)
```

In[2755]:=
```
aGB3^2 + bGB3^2 + cGB3^2 + dGB3^2 // FullSimplify
pGB3^2 + qGB3^2 + rGB3^2 + sGB3^2 // FullSimplify
```

Out[2755]= 1

Out[2756]= 1

In[2757]:=
```
{{aGB3 pGB3, aGB3 qGB3, aGB3 rGB3, aGB3 sGB3}, {bGB3 pGB3, bGB3 qGB3, bGB3 rGB3, bGB3 sGB3},
    {cGB3 pGB3 , cGB3 qGB3, cGB3 rGB3, cGB3 sGB3},
    {dGB3 pGB3, dGB3 qGB3, dGB3 rGB3, dGB3 sGB3}} - AGB3 // FullSimplify // MatrixForm
```

Out[2757]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2758]:=
```
GB3P = (aGB3 IdentityMatrix[2] + I bGB3 PauliMatrix[1] +
    I cGB3 PauliMatrix[2] + I dGB3 PauliMatrix[3]) // Simplify; GB3P // MatrixForm
GB3M = (pGB3 IdentityMatrix[2] + I qGB3 PauliMatrix[1] + I rGB3 PauliMatrix[2] +
     I sGB3 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GB3M // MatrixForm
```

Out[2758]//MatrixForm=

$$\begin{pmatrix} \frac{-i+\sqrt{263+48\sqrt{30}}}{4\sqrt{33+6\sqrt{30}}} & \frac{1}{12}\left(-3+\sqrt{30}+i\,\text{Root}\left[9-66\,\#1^2+\#1^4\,\&,1\right]\right) \\ \frac{1}{12}\left(3-\sqrt{30}+i\,\text{Root}\left[9-66\,\#1^2+\#1^4\,\&,1\right]\right) & \frac{i+\sqrt{263+48\sqrt{30}}}{4\sqrt{33+6\sqrt{30}}} \end{pmatrix}$$

Out[2759]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{12}i\left(-3+\sqrt{30}\right)+\frac{8\sqrt{2}+3\sqrt{15}}{4\sqrt{39+6\sqrt{30}}} & \frac{13i\sqrt{3}-3\sqrt{5}+2\sqrt{6}+6i\sqrt{10}}{12\sqrt{13+2\sqrt{30}}} \\ \frac{13i\sqrt{3}+3\sqrt{5}-2\sqrt{6}+6i\sqrt{10}}{12\sqrt{13+2\sqrt{30}}} & \frac{1}{12}i\left(-3+\sqrt{30}\right)+\frac{8\sqrt{2}+3\sqrt{15}}{4\sqrt{39+6\sqrt{30}}} \end{pmatrix}$$

In[2760]:=
```
(* gB2 *)
AGB2 = Asc[GB2] // FullSimplify; AGB2 // MatrixForm
```

Out[2760]//MatrixForm=

$$\begin{pmatrix} \frac{1}{48}\left(-4\sqrt{2}+2\sqrt{3}-3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-7+\sqrt{30}\right) & \frac{1}{48}\left(\sqrt{2}-2\sqrt{3}+\sqrt{15}\right) & \frac{1}{48}\left(13-5\sqrt{6}\right) \\ \frac{1}{48}\left(-5+\sqrt{30}\right) & \frac{1}{48}\left(-4\sqrt{2}-2\sqrt{3}+3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-7-3\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}-2\sqrt{3}-\sqrt{15}\right) \\ \frac{1}{48}\left(-\sqrt{2}-2\sqrt{3}+\sqrt{15}\right) & \frac{1}{48}\left(-13-5\sqrt{6}\right) & \frac{1}{48}\left(4\sqrt{2}+2\sqrt{3}+3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(7+\sqrt{30}\right) \\ \frac{1}{48}\left(-7+3\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}+2\sqrt{3}+\sqrt{15}\right) & \frac{1}{48}\left(-5-\sqrt{30}\right) & \frac{1}{48}\left(-4\sqrt{2}+2\sqrt{3}+3\sqrt{10}-3\sqrt{15}\right) \end{pmatrix}$$

In[2761]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[2762]:=
```
KGB2 = Sqrt[AGB2[[1, 1]]^2 + AGB2[[1, 2]]^2 + AGB2[[1, 3]]^2 + AGB2[[1, 4]]^2] // Simplify;
```

In[2763]:=
```
pGB2 = AGB2[[1, 1]] / KGB2 // Simplify;
qGB2 = AGB2[[1, 2]] / KGB2 // Simplify;
rGB2 = AGB2[[1, 3]] / KGB2 // Simplify;
sGB2 = AGB2[[1, 4]] / KGB2 // FullSimplify;
aGB2 = KGB2;
bGB2 = KGB2 AGB2[[2, 1]] / AGB2[[1, 1]] // FullSimplify;
cGB2 = KGB2 AGB2[[3, 1]] / AGB2[[1, 1]] // FullSimplify;
dGB2 = KGB2 AGB2[[4, 1]] / AGB2[[1, 1]] // FullSimplify;
```

In[2771]:=
```
(* operations to figure out if we're doing everything right *)
```

In[2772]:=
```
aGB2^2 + bGB2^2 + cGB2^2 + dGB2^2 // FullSimplify
pGB2^2 + qGB2^2 + rGB2^2 + sGB2^2 // FullSimplify
```

Out[2772]= 1

Out[2773]= 1

In[2774]:=
```
{{aGB2 pGB2, aGB2 qGB2, aGB2 rGB2, aGB2 sGB2}, {bGB2 pGB2, bGB2 qGB2, bGB2 rGB2, bGB2 sGB2},
    {cGB2 pGB2 , cGB2 qGB2, cGB2 rGB2, cGB2 sGB2},
    {dGB2 pGB2, dGB2 qGB2, dGB2 rGB2, dGB2 sGB2}} - AGB2 // FullSimplify // MatrixForm
```

Out[2774]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2775]:=
```
GB2P = (aGB2 IdentityMatrix[2] + I bGB2 PauliMatrix[1] +
    I cGB2 PauliMatrix[2] + I dGB2 PauliMatrix[3]) // Simplify; GB2P // MatrixForm
GB2M = (pGB2 IdentityMatrix[2] + I qGB2 PauliMatrix[1] + I rGB2 PauliMatrix[2] +
      I sGB2 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GB2M // MatrixForm
```

Out[2775]//MatrixForm=

$$\begin{pmatrix} \frac{1}{24}\left(\sqrt{6\left(29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}\right)}-i\sqrt{6\left(19-3\sqrt{5}-6\sqrt{6}+2\sqrt{30}\right)}\right) & \frac{1}{24}\left(3+3\sqrt{5}+2\sqrt{30}-i\sqrt{6\left(19-3\sqrt{5}+6\sqrt{6}-2\sqrt{30}\right)}\right) \\ \frac{1}{24}\left(-3-3\sqrt{5}-2\sqrt{30}-i\sqrt{6\left(19-3\sqrt{5}+6\sqrt{6}-2\sqrt{30}\right)}\right) & \frac{1}{24}\left(\sqrt{6\left(29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}\right)}+i\sqrt{6\left(19-3\sqrt{5}-6\sqrt{6}+2\sqrt{30}\right)}\right) \end{pmatrix}$$

Out[2776]//MatrixForm=

$$\begin{pmatrix} \frac{-4\sqrt{2}+2\sqrt{3}-3\sqrt{10}+3\sqrt{15}}{2\sqrt{6\left(29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}\right)}}-\frac{1}{24}i\left(3-3\sqrt{5}+2\sqrt{30}\right) & \frac{6\sqrt{2}-2\sqrt{3}-6i\sqrt{5}+7i\sqrt{6}-3\sqrt{10}}{12\sqrt{29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}}} \\ \frac{-6\sqrt{2}+2\sqrt{3}-6i\sqrt{5}+7i\sqrt{6}+3\sqrt{10}}{12\sqrt{29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}}} & \frac{-4\sqrt{2}+2\sqrt{3}-3\sqrt{10}+3\sqrt{15}}{2\sqrt{6\left(29+3\sqrt{5}-10\sqrt{6}-2\sqrt{30}\right)}}+\frac{1}{24}i\left(3-3\sqrt{5}+2\sqrt{30}\right) \end{pmatrix}$$

In[2777]:=
```
(* gB1 *)
AGB1 = Asc[GB1] // FullSimplify; AGB1 // MatrixForm
```

Out[2777]//MatrixForm=

$$\begin{pmatrix} \frac{1}{48}\left(8\sqrt{2}+6\sqrt{3}+3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(5+\sqrt{30}\right) & \frac{1}{48}\left(\sqrt{2}+2\sqrt{3}+\sqrt{15}\right) & \frac{1}{48}\left(1+\sqrt{6}\right) \\ \frac{1}{48}\left(7+\sqrt{30}\right) & \frac{1}{48}\left(8\sqrt{2}-6\sqrt{3}-3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(5-\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}+2\sqrt{3}-\sqrt{15}\right) \\ \frac{1}{48}\left(-\sqrt{2}+2\sqrt{3}+\sqrt{15}\right) & \frac{1}{48}\left(-1+\sqrt{6}\right) & \frac{1}{48}\left(-8\sqrt{2}+6\sqrt{3}-3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-5+\sqrt{30}\right) \\ \frac{1}{48}\left(5+\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}-2\sqrt{3}+\sqrt{15}\right) & \frac{1}{48}\left(7-\sqrt{30}\right) & \frac{1}{48}\left(8\sqrt{2}+6\sqrt{3}-3\sqrt{10}-3\sqrt{15}\right) \end{pmatrix}$$

In[2778]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[2779]:=
```
KGB1 = Sqrt[AGB1[[1, 1]]^2 + AGB1[[1, 2]]^2 + AGB1[[1, 3]]^2 + AGB1[[1, 4]]^2] // Simplify;
```

In[2780]:=
```
pGB1 = AGB1[[1, 1]] / KGB1 // Simplify;
qGB1 = AGB1[[1, 2]] / KGB1 // Simplify;
rGB1 = AGB1[[1, 3]] / KGB1 // Simplify;
sGB1 = AGB1[[1, 4]] / KGB1 // FullSimplify;
aGB1 = KGB1;
bGB1 = KGB1 AGB1[[2, 1]] / AGB1[[1, 1]] // FullSimplify;
cGB1 = KGB1 AGB1[[3, 1]] / AGB1[[1, 1]] // FullSimplify;
dGB1 = KGB1 AGB1[[4, 1]] / AGB1[[1, 1]] // FullSimplify;
```

In[2788]:=
```
(* operations to figure out if we're doing everything right *)
```

In[2789]:=
```
aGB1^2 + bGB1^2 + cGB1^2 + dGB1^2 // FullSimplify
pGB1^2 + qGB1^2 + rGB1^2 + sGB1^2 // FullSimplify
```

Out[2789]= 1

Out[2790]= 1

In[2791]:=
```
{{aGB1 pGB1, aGB1 qGB1, aGB1 rGB1, aGB1 sGB1}, {bGB1 pGB1, bGB1 qGB1, bGB1 rGB1, bGB1 sGB1},
    {cGB1 pGB1 , cGB1 qGB1, cGB1 rGB1, cGB1 sGB1},
    {dGB1 pGB1, dGB1 qGB1, dGB1 rGB1, dGB1 sGB1}} - AGB1 // FullSimplify // MatrixForm
```

Out[2791]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2792]:=
```
GB1P = (aGB1 IdentityMatrix[2] + I bGB1 PauliMatrix[1] +
    I cGB1 PauliMatrix[2] + I dGB1 PauliMatrix[3]) // Simplify; GB1P // MatrixForm
GB1M = (pGB1 IdentityMatrix[2] + I qGB1 PauliMatrix[1] + I rGB1 PauliMatrix[2] +
     I sGB1 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GB1M // MatrixForm
```

Out[2792]//MatrixForm=

$$\begin{pmatrix} \frac{1}{24}\left(i\sqrt{6\left(25-9\sqrt{5}+8\sqrt{6}-4\sqrt{30}\right)}+\sqrt{6\left(23+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}\right)}\right) & \frac{1}{24}\left(-3-3\sqrt{5}+3\sqrt{6}+\sqrt{30}+i\sqrt{6\left(25-9\sqrt{5}-8\sqrt{6}+4\sqrt{30}\right)}\right) \\ \frac{1}{24}\left(3+3\sqrt{5}-3\sqrt{6}-\sqrt{30}+i\sqrt{6\left(25-9\sqrt{5}-8\sqrt{6}+4\sqrt{30}\right)}\right) & \frac{1}{24}\left(-i\sqrt{6\left(25-9\sqrt{5}+8\sqrt{6}-4\sqrt{30}\right)}+\sqrt{6\left(23+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}\right)}\right) \end{pmatrix}$$

Out[2793]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{24}i\left(-3+3\sqrt{5}-3\sqrt{6}+\sqrt{30}\right)+\frac{8\sqrt{2}+6\sqrt{3}+3\sqrt{10}+3\sqrt{15}}{2\sqrt{6\left(23+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}\right)}} & \frac{-6\sqrt{2}-2\sqrt{3}-6i\sqrt{5}-5i\sqrt{6}-3\sqrt{10}}{12\sqrt{23+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}}} \\ \frac{6\sqrt{2}+2\sqrt{3}-6i\sqrt{5}-5i\sqrt{6}+3\sqrt{10}}{12\sqrt{23+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}}} & \frac{1}{24}i\left(-3+3\sqrt{5}-3\sqrt{6}+\sqrt{30}\right)+\frac{8\sqrt{2}+6\sqrt{3}+3\sqrt{10}+3\sqrt{15}}{2\sqrt{6\left(23+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}\right)}} \end{pmatrix}$$

In[2794]:=
```
(* Computing the SO(3) matrices R(g) for the g_Bi *)
```

In[2795]:=
```
(* gB6 *)
```

In[2796]:=
```
RB6P = R[GB6P[[1, 1]], GB6P[[1, 2]]] // FullSimplify; RB6P // MatrixForm
RB6P.Transpose[RB6P] // FullSimplify // MatrixForm
Det[RB6P] // FullSimplify
RB6M = R[GB6M[[1, 1]], GB6M[[1, 2]]] // FullSimplify; RB6M // MatrixForm
RB6M.Transpose[RB6M] // FullSimplify // MatrixForm
Det[RB6M] // FullSimplify
```

Out[2796]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

Out[2797]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2798]= 1

Out[2799]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

Out[2800]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2801]= 1

In[2802]:=
```
(* gB4 *)
```

In[2803]:=
```
RB4P = R[GB4P[[1, 1]], GB4P[[1, 2]]] // FullSimplify; RB4P // MatrixForm
RB4P.Transpose[RB4P] // FullSimplify // MatrixForm
Det[RB4P] // FullSimplify
RB4M = R[GB4M[[1, 1]], GB4M[[1, 2]]] // FullSimplify; RB4M // MatrixForm
RB4M.Transpose[RB4M] // FullSimplify // MatrixForm
Det[RB4M] // FullSimplify
```

Out[2803]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 \\ \frac{1}{4}\sqrt{\frac{3}{2}\left(3+\sqrt{5}\right)} & \frac{1}{8}\left(-1+3\sqrt{5}\right) & 0 \\ \frac{1}{8}\left(-1+3\sqrt{5}\right) & -\frac{1}{4}\sqrt{\frac{3}{2}\left(3+\sqrt{5}\right)} & 0 \end{pmatrix}$$

Out[2804]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2805]= 1

Out[2806]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{4}\,\mathrm{Root}\left[9-9\,\sharp1^2+\sharp1^4\,\&,\,3\right] & \frac{1}{8}\left(1+3\sqrt{5}\right) & 0 \\ \frac{1}{8}\left(-1-3\sqrt{5}\right) & \frac{1}{4}\,\mathrm{Root}\left[9-9\,\sharp1^2+\sharp1^4\,\&,\,3\right] & 0 \end{pmatrix}$$

Out[2807]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2808]= 1

In[2809]:=
```
(* gB3 *)
```

In[2810]:=
```
RB3P = R[GB3P[[1, 1]], GB3P[[1, 2]]] // FullSimplify; RB3P // MatrixForm
RB3P.Transpose[RB3P] // FullSimplify // MatrixForm
Det[RB3P] // FullSimplify
RB3M = R[GB3M[[1, 1]], GB3M[[1, 2]]] // FullSimplify; RB3M // MatrixForm
RB3M.Transpose[RB3M] // FullSimplify // MatrixForm
Det[RB3M] // FullSimplify
```

Out[2810]//MatrixForm=

$$\begin{pmatrix} \sqrt{\frac{5}{6}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & \frac{1}{12} & \frac{2\sqrt{2}}{3} \\ -\frac{1}{4} & -\frac{\sqrt{15}}{4} & 0 \end{pmatrix}$$

Out[2811]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2812]= 1

Out[2813]//MatrixForm=

$$\begin{pmatrix} \sqrt{\frac{5}{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & -\frac{1}{12} & -\frac{2\sqrt{2}}{3} \\ -\frac{1}{4} & \frac{\sqrt{15}}{4} & 0 \end{pmatrix}$$

Out[2814]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2815]= 1

In[2816]:=
```
(* gB2 *)
```

In[2817]:=
```
RB2P = R[GB2P[[1, 1]], GB2P[[1, 2]]] // FullSimplify; RB2P // MatrixForm
RB2P.Transpose[RB2P] // FullSimplify // MatrixForm
Det[RB2P] // FullSimplify
RB2M = R[GB2M[[1, 1]], GB2M[[1, 2]]] // FullSimplify; RB2M // MatrixForm
RB2M.Transpose[RB2M] // FullSimplify // MatrixForm
Det[RB2M] // FullSimplify
```

Out[2817]//MatrixForm=

$$\begin{pmatrix} -\frac{1+\sqrt{5}}{2\sqrt{6}} & \frac{1-3\sqrt{5}}{6\sqrt{2}} & \frac{1}{3} \\ -\frac{1}{4}\sqrt{\frac{1}{6}\left(63-5\sqrt{5}\right)} & \frac{1}{24}\left(5+3\sqrt{5}\right) & -\frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(-1+\sqrt{5}\right) & -\frac{1}{4}\sqrt{\frac{1}{6}\left(23+3\sqrt{5}\right)} & -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[2818]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2819]= 1

Out[2820]//MatrixForm=

$$\begin{pmatrix} \text{Root}\left[1-18\,\#1^2+36\,\#1^4\,\&,\,2\right] & -\frac{1}{6}\sqrt{23+3\sqrt{5}} & -\frac{1}{3} \\ -\frac{1}{4}\sqrt{\frac{1}{6}\left(63+5\sqrt{5}\right)} & \frac{1}{24}\left(-5+3\sqrt{5}\right) & \frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(-1-\sqrt{5}\right) & \frac{1}{4}\sqrt{\frac{1}{6}\left(23-3\sqrt{5}\right)} & -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[2821]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2822]= 1

In[2823]:=
```
(* gB1 *)
```

In[2824]:=
```
RB1P = R[GB1P[[1, 1]], GB1P[[1, 2]]] // FullSimplify; RB1P // MatrixForm
RB1P.Transpose[RB1P] // FullSimplify // MatrixForm
Det[RB1P] // FullSimplify
RB1M = R[GB1M[[1, 1]], GB1M[[1, 2]]] // FullSimplify; RB1M // MatrixForm
RB1M.Transpose[RB1M] // FullSimplify // MatrixForm
Det[RB1M] // FullSimplify
```

Out[2824]//MatrixForm=

$$\begin{pmatrix} \sqrt{\frac{5}{6}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3} \\ \frac{3+\sqrt{5}}{8\sqrt{3}} & \frac{1}{24}\left(-1+9\sqrt{5}\right) & -\frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(1-\sqrt{5}\right) & \sqrt{\frac{23}{96}+\frac{\sqrt{5}}{32}} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[2825]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2826]= 1

Out[2827]//MatrixForm=

$$\begin{pmatrix} \sqrt{\frac{5}{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3} \\ \frac{1}{12}\,\mathrm{Root}\left[9-21\,\#1^2+\#1^4\,\&,\,2\right] & \frac{1}{24}\left(1+9\sqrt{5}\right) & \frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(1+\sqrt{5}\right) & -\frac{1}{4}\sqrt{\frac{1}{6}\left(23-3\sqrt{5}\right)} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[2828]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[2829]= 1

In[2830]:=
```
(* gluing matrices for vertex B. The null
 space of these define the normal vectors n_ef.  *)
```

In[2831]:=
```
RBP = {RB1P, RB2P, RB3P, RB4P, RB5P, RB6P};
RBM = {RB1M, RB2M, RB3M, RB4M, RB5M, RB6M};
RB[i_, j_] := Transpose[RBP[[j]]].RBP[[i]] - Transpose[RBM[[j]]].RBM[[i]] ;
```

In[2834]:=
```
(* standard tetrahedron normals for reference *)
```

In[2835]:=
```
NA
NB
NC
ND
```

Out[2835]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2836]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2837]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2838]= $\{0, 0, -1\}$

In[2839]:=
```
(* Tetrahedron (1234) or (abcd) *)

NullSpace[RB[6, 4]] // FullSimplify
NullSpace[RB[6, 4]][[1]] // FullSimplify // Normalize
NullSpace[RB[6, 3]] // FullSimplify
NullSpace[RB[6, 3]][[1]] // FullSimplify // Normalize
NullSpace[RB[6, 2]] // FullSimplify
NullSpace[RB[6, 2]][[1]] // FullSimplify // Normalize
NullSpace[RB[6, 1]] // FullSimplify
NullSpace[RB[6, 1]][[1]] // FullSimplify // Normalize
```

Out[2839]= $\{\{0, 0, 1\}\}$

Out[2840]= $\{0, 0, 1\}$

Out[2841]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2842]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2843]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[2844]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2845]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2846]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2847]:=
```
nB664 = ND
nB663 = NB
nB662 = NC
nB661 = NA
```

Out[2847]= $\{0, 0, -1\}$

Out[2848]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2849]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2850]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2851]:=
```
(* Tetrahedron (1236) or (abcf) *)

NullSpace[RB[4, 6]] // FullSimplify
NullSpace[RB[4, 6]][[1]] // FullSimplify // Normalize
NullSpace[RB[4, 3]] // FullSimplify
NullSpace[RB[4, 3]][[1]] // FullSimplify // Normalize
NullSpace[RB[4, 2]] // FullSimplify
NullSpace[RB[4, 2]][[1]] // FullSimplify // Normalize
NullSpace[RB[4, 1]] // FullSimplify
NullSpace[RB[4, 1]][[1]] // FullSimplify // Normalize
```

Out[2851]= $\{\{0, 0, 1\}\}$

Out[2852]= $\{0, 0, 1\}$

Out[2853]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2854]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2855]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[2856]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2857]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2858]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2883]:=
```
nB446 = ND
nB443 = NB
nB442 = NC
nB441 = NA
```

Out[2883]= $\{0, 0, -1\}$

Out[2884]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2885]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2886]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2859]:=
```
(* Tetrahedron (1246) or (abdf) *)

NullSpace[RB[3, 6]] // FullSimplify
NullSpace[RB[3, 6]][[1]] // FullSimplify // Normalize
NullSpace[RB[3, 4]] // FullSimplify
NullSpace[RB[3, 4]][[1]] // FullSimplify // Normalize
NullSpace[RB[3, 2]] // FullSimplify
NullSpace[RB[3, 2]][[1]] // FullSimplify // Normalize
NullSpace[RB[3, 1]] // FullSimplify
NullSpace[RB[3, 1]][[1]] // FullSimplify // Normalize
```

Out[2859]= $\{\{0, 0, 1\}\}$

Out[2860]= $\{0, 0, 1\}$

Out[2861]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[2862]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2863]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2864]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2865]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2866]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2891]:=
```
nB336 = ND
nB334 = NC
nB332 = NB
nB331 = NA
```

Out[2891]= $\{0, 0, -1\}$

Out[2892]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2893]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2894]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2867]:=
```
(* Tetrahedron (1346) or (acdf) *)

NullSpace[RB[2, 6]] // FullSimplify
NullSpace[RB[2, 6]][[1]] // FullSimplify // Normalize
NullSpace[RB[2, 4]] // FullSimplify
NullSpace[RB[2, 4]][[1]] // FullSimplify // Normalize
NullSpace[RB[2, 3]] // FullSimplify
NullSpace[RB[2, 3]][[1]] // FullSimplify // Normalize
NullSpace[RB[2, 1]] // FullSimplify
NullSpace[RB[2, 1]][[1]] // FullSimplify // Normalize
```

Out[2867]= $\{\{0, 0, 1\}\}$

Out[2868]= $\{0, 0, 1\}$

Out[2869]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2870]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2871]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[2872]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2873]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2874]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2895]:=
```
nB226 = ND
nB224 = NB
nB223 = NC
nB221 = NA
```

Out[2895]= $\{0, 0, -1\}$

Out[2896]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2897]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2898]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2875]:=
```
(* Tetrahedron (2346) or (bcdf) *)

NullSpace[RB[1, 6]] // FullSimplify
NullSpace[RB[1, 6]][[1]] // FullSimplify // Normalize
NullSpace[RB[1, 4]] // FullSimplify
NullSpace[RB[1, 4]][[1]] // FullSimplify // Normalize
NullSpace[RB[1, 3]] // FullSimplify
NullSpace[RB[1, 3]][[1]] // FullSimplify // Normalize
NullSpace[RB[1, 2]] // FullSimplify
NullSpace[RB[1, 2]][[1]] // FullSimplify // Normalize
```

Out[2875]= $\{\{0, 0, 1\}\}$

Out[2876]= $\{0, 0, 1\}$

Out[2877]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[2878]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2879]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2880]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2881]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[2882]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[2907]:=
```
nB116 = ND
nB114 = NC
nB113 = NB
nB112 = NA
```

Out[2907]= $\{0, 0, -1\}$

Out[2908]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2909]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[2910]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

```
(* Time to move on to vertex C (abcde). *)
```

In[2911]:=
```
(* Computing g's first. *)
(* g_C5 *)
MC5 = {LinearSolve[BasisF, vc], LinearSolve[BasisF, vb], LinearSolve[BasisF, vd]} ;
MC5 // MatrixForm
MC5N = Normalize[NullSpace[MC5][[1]]]
MC5final =
 Transpose[{MC5N, Normalize[MC5[[1]]], Normalize[MC5[[2]]], Normalize[MC5[[3]]]}];
MC5final // MatrixForm
Det[MC5final]
```

Out[2911]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \end{pmatrix}$$

Out[2912]= {0, 0, 0, 1}

Out[2913]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2\sqrt{3}} \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Out[2914]= $\frac{1}{\sqrt{2}}$

In[2915]:=
```
GC5 = MC5final.Inverse[M3] // FullSimplify; GC5 // MatrixForm
GC5.Transpose[GC5] // FullSimplify // MatrixForm
Det[GC5] // FullSimplify
```

Out[2915]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Out[2916]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2917]= 1

In[2925]:=
```
(* g_C4 *)
MC4 = {LinearSolve[BasisF, vb], LinearSolve[BasisF, vc], LinearSolve[BasisF, ve]} ;
MC4 // MatrixForm
MC4N = Normalize[NullSpace[MC4][[1]]]
MC4final =
  Transpose[{-MC4N, Normalize[MC4[[1]]], Normalize[MC4[[2]]], Normalize[MC4[[3]]]}];
MC4final // MatrixForm
Det[MC4final]
GC4 = MC4final.Inverse[M3] // FullSimplify; GC4 // MatrixForm
GC4.Transpose[GC4] // FullSimplify // MatrixForm
Det[GC4] // FullSimplify
```

Out[2925]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2926]= $\left\{0, 0, -\frac{\sqrt{15}}{4}, \frac{1}{4}\right\}$

Out[2927]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{15}}{4} & 0 & 0 & \frac{1}{2\sqrt{6}} \\ -\frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2928]= $\frac{1}{\sqrt{2}}$

Out[2929]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{15}}{4} & 0 & 0 & \frac{1}{4} \\ -\frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2930]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2931]= 1

In[2932]:=
```
(* g_C3 *)
MC3 = {LinearSolve[BasisF, vd], LinearSolve[BasisF, vb], LinearSolve[BasisF, ve]} ;
MC3 // MatrixForm
MC3N = Normalize[NullSpace[MC3][[1]]]
MC3final =
  Transpose[{-MC3N, Normalize[MC3[[1]]], Normalize[MC3[[2]]], Normalize[MC3[[3]]]}];
MC3final // MatrixForm
Det[MC3final]
GC3 = MC3final.Inverse[M3] // FullSimplify; GC3 // MatrixForm
GC3.Transpose[GC3] // FullSimplify // MatrixForm
Det[GC3] // FullSimplify
```

Out[2932]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2933]= $\left\{0, -\sqrt{\frac{5}{6}}, \frac{\sqrt{\frac{5}{3}}}{4}, \frac{1}{4}\right\}$

Out[2934]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ \sqrt{\frac{5}{6}} & \frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{2\sqrt{6}} \\ -\frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2935]= $\frac{1}{\sqrt{2}}$

Out[2936]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \sqrt{\frac{5}{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{6} & \frac{1}{3\sqrt{2}} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & \sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{3} & -\frac{1}{12} \\ -\frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2937]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2938]= 1

In[2939]:=
```
(* g_C2 *)
MC2 = {LinearSolve[BasisF, vc], LinearSolve[BasisF, vd], LinearSolve[BasisF, ve]} ;
MC2 // MatrixForm
MC2N = Normalize[NullSpace[MC2][[1]]]
MC2final =
 Transpose[{-MC2N, Normalize[MC2[[1]]], Normalize[MC2[[2]]], Normalize[MC2[[3]]]}];
MC2final // MatrixForm
Det[MC2final]
GC2 = MC2final.Inverse[M3] // FullSimplify; GC2 // MatrixForm
GC2.Transpose[GC2] // FullSimplify // MatrixForm
Det[GC2] // FullSimplify
```

Out[2939]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2940]= $\left\{-\frac{\sqrt{\frac{5}{2}}}{2}, \frac{\sqrt{\frac{5}{6}}}{2}, \frac{\sqrt{\frac{5}{3}}}{4}, \frac{1}{4}\right\}$

Out[2941]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{\frac{5}{2}}}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{\frac{5}{6}}}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & 0 & \sqrt{\frac{2}{3}} & \frac{1}{2\sqrt{6}} \\ -\frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2942]= $\frac{1}{\sqrt{2}}$

Out[2943]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{\frac{5}{2}}}{2} & \frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} \\ -\frac{\sqrt{\frac{5}{6}}}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{6} & -\frac{1}{6\sqrt{2}} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{12} \\ -\frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2944]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2945]= 1

In[2946]:=
```
(* g_C1 *)
MC1 = {LinearSolve[BasisF, d - b], LinearSolve[BasisF, c - b], LinearSolve[BasisF, e - b]} ;
MC1 // MatrixForm
MC1N = Normalize[NullSpace[MC1][[1]]]
MC1final =
 Transpose[{-MC1N, Normalize[MC1[[1]]], Normalize[MC1[[2]]], Normalize[MC1[[3]]]}];
MC1final // MatrixForm
Det[MC1final]
GC1 = MC1final.Inverse[M3] // FullSimplify; GC1 // MatrixForm
GC1.Transpose[GC1] // FullSimplify // MatrixForm
Det[GC1] // FullSimplify
```

Out[2946]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{6}} & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2947]= $\left\{\frac{\sqrt{\frac{5}{2}}}{2}, \frac{\sqrt{\frac{5}{6}}}{2}, \frac{\sqrt{\frac{5}{3}}}{4}, \frac{1}{4}\right\}$

Out[2948]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{\frac{5}{2}}}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{\sqrt{\frac{5}{6}}}{2} & \frac{1}{2\sqrt{3}} & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & \sqrt{\frac{2}{3}} & 0 & \frac{1}{2\sqrt{6}} \\ -\frac{1}{4} & 0 & 0 & \frac{\sqrt{\frac{5}{2}}}{2} \end{pmatrix}$$

Out[2949]= $\frac{1}{\sqrt{2}}$

Out[2950]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{\frac{5}{2}}}{2} & -\frac{1}{2} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{6}} \\ -\frac{\sqrt{\frac{5}{6}}}{2} & \frac{1}{2\sqrt{3}} & \frac{5}{6} & -\frac{1}{6\sqrt{2}} \\ -\frac{\sqrt{\frac{5}{3}}}{4} & \sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{3} & -\frac{1}{12} \\ -\frac{1}{4} & 0 & 0 & \frac{\sqrt{15}}{4} \end{pmatrix}$$

Out[2951]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[2952]= 1

In[2953]:=
```
(* determining the SU(2) components via Van Elfrinkhof's formula *)

(* gC5 *)
AGC5 = Asc[GC5] // FullSimplify; AGC5 // MatrixForm (* associate matrix *)
```

Out[2953]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{3}}{8} & \frac{1}{8} & -\frac{\sqrt{3}}{8} & \frac{1}{8} \\ -\frac{3}{8} & -\frac{\sqrt{3}}{8} & \frac{3}{8} & -\frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{8} & -\frac{1}{8} & \frac{\sqrt{3}}{8} & -\frac{1}{8} \\ \frac{3}{8} & \frac{\sqrt{3}}{8} & -\frac{3}{8} & \frac{\sqrt{3}}{8} \end{pmatrix}$$

In[2710]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[2962]:=
```
KGC5 = Sqrt[AGC5[[1, 1]]^2 + AGC5[[1, 2]]^2 + AGC5[[1, 3]]^2 + AGC5[[1, 4]]^2] // Simplify;
```

In[2963]:=
```
pGC5 = AGC5[[1, 1]] / KGC5 // Simplify;
qGC5 = AGC5[[1, 2]] / KGC5 // Simplify;
rGC5 = AGC5[[1, 3]] / KGC5 // Simplify;
sGC5 = AGC5[[1, 4]] / KGC5 // FullSimplify;
aGC5 = KGC5;
bGC5 = KGC5 AGC5[[2, 1]] / AGC5[[1, 1]] // FullSimplify;
cGC5 = KGC5 AGC5[[3, 1]] / AGC5[[1, 1]] // FullSimplify;
dGC5 = KGC5 AGC5[[4, 1]] / AGC5[[1, 1]] // FullSimplify;
```

In[2720]:=
```
(* operations to figure out if we're doing everything right *)
```

In[2971]:=
```
aGC5^2 + bGC5^2 + cGC5^2 + dGC5^2 // FullSimplify
```

Out[2971]= 1

In[2972]:=
```
pGC5^2 + qGC5^2 + rGC5^2 + sGC5^2 // FullSimplify
```

Out[2972]= 1

In[2973]:=
```
{{aGC5 pGC5, aGC5 qGC5, aGC5 rGC5, aGC5 sGC5}, {bGC5 pGC5, bGC5 qGC5, bGC5 rGC5, bGC5 sGC5},
    {cGC5 pGC5 , cGC5 qGC5, cGC5 rGC5, cGC5 sGC5},
    {dGC5 pGC5, dGC5 qGC5, dGC5 rGC5, dGC5 sGC5}} - AGC5 // FullSimplify // MatrixForm
```

Out[2973]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2974]:=
```
GC5P = (aGC5 IdentityMatrix[2] + I bGC5 PauliMatrix[1] +
    I cGC5 PauliMatrix[2] + I dGC5 PauliMatrix[3]) // Simplify; GC5P // MatrixForm
GC5M = (pGC5 IdentityMatrix[2] + I qGC5 PauliMatrix[1] + I rGC5 PauliMatrix[2] +
      I sGC5 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GC5M // MatrixForm
```

Out[2974]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4}\left(\sqrt{2}+i\sqrt{6}\right) & -\frac{i\left(-i+\sqrt{3}\right)}{2\sqrt{2}} \\ \frac{1}{4}\left(\sqrt{2}-i\sqrt{6}\right) & \frac{1}{4}\left(\sqrt{2}-i\sqrt{6}\right) \end{pmatrix}$$

Out[2975]//MatrixForm=

$$\begin{pmatrix} \frac{-i+\sqrt{3}}{2\sqrt{2}} & \frac{-i+\sqrt{3}}{2\sqrt{2}} \\ -\frac{i+\sqrt{3}}{2\sqrt{2}} & \frac{i+\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$$

In[2976]:=
```
(* gC4 *)
AGC4 = Asc[GC4] // FullSimplify; AGC4 // MatrixForm
```

Out[2976]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{15}}{16} & -\frac{3}{16} & \frac{\sqrt{15}}{16} & \frac{3}{16} \\ -\frac{5}{16} & \frac{\sqrt{15}}{16} & -\frac{5}{16} & -\frac{\sqrt{15}}{16} \\ \frac{\sqrt{15}}{16} & -\frac{3}{16} & \frac{\sqrt{15}}{16} & \frac{3}{16} \\ -\frac{5}{16} & \frac{\sqrt{15}}{16} & -\frac{5}{16} & -\frac{\sqrt{15}}{16} \end{pmatrix}$$

In[2727]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[2977]:=
```
KGC4 = Sqrt[AGC4[[1, 1]]^2 + AGC4[[1, 2]]^2 + AGC4[[1, 3]]^2 + AGC4[[1, 4]]^2] // Simplify;
```

In[2978]:=
```
pGC4 = AGC4[[1, 1]] / KGC4 // Simplify;
qGC4 = AGC4[[1, 2]] / KGC4 // Simplify;
rGC4 = AGC4[[1, 3]] / KGC4 // Simplify;
sGC4 = AGC4[[1, 4]] / KGC4 // FullSimplify;
aGC4 = KGC4;
bGC4 = KGC4 AGC4[[2, 1]] / AGC4[[1, 1]] // FullSimplify;
cGC4 = KGC4 AGC4[[3, 1]] / AGC4[[1, 1]] // FullSimplify;
dGC4 = KGC4 AGC4[[4, 1]] / AGC4[[1, 1]] // FullSimplify;
```

In[2737]:=
```
(* operations to figure out if we're doing everything right *)
```

In[2986]:=
```
aGC4^2 + bGC4^2 + cGC4^2 + dGC4^2 // FullSimplify
pGC4^2 + qGC4^2 + rGC4^2 + sGC4^2 // FullSimplify
```

Out[2986]= 1

Out[2987]= 1

In[2988]:=
```
{{aGC4 pGC4, aGC4 qGC4, aGC4 rGC4, aGC4 sGC4}, {bGC4 pGC4, bGC4 qGC4, bGC4 rGC4, bGC4 sGC4},
    {cGC4 pGC4 , cGC4 qGC4, cGC4 rGC4, cGC4 sGC4},
    {dGC4 pGC4, dGC4 qGC4, dGC4 rGC4, dGC4 sGC4}} - AGC4 // FullSimplify // MatrixForm
```

Out[2988]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[2989]:=
```
GC4P = (aGC4 IdentityMatrix[2] + I bGC4 PauliMatrix[1] +
    I cGC4 PauliMatrix[2] + I dGC4 PauliMatrix[3]) // FullSimplify; GC4P // MatrixForm
GC4M = (pGC4 IdentityMatrix[2] + I qGC4 PauliMatrix[1] + I rGC4 PauliMatrix[2] +
     I sGC4 PauliMatrix[3]) // Conjugate // Transpose // FullSimplify; GC4M // MatrixForm
```

Out[2989]//MatrixForm=

$$\begin{pmatrix} \text{Root}\left[1 + \#1^2 + 4\,\#1^4\ \&,\ 3\right] & \text{Root}\left[1 + \#1^2 + 4\,\#1^4\ \&,\ 3\right] \\ \text{Root}\left[1 + \#1^2 + 4\,\#1^4\ \&,\ 1\right] & \text{Root}\left[1 + \#1^2 + 4\,\#1^4\ \&,\ 4\right] \end{pmatrix}$$

Out[2990]//MatrixForm=

$$\begin{pmatrix} \text{Root}\left[1 - \#1^2 + 4\,\#1^4\ \&,\ 3\right] & \text{Root}\left[1 - \#1^2 + 4\,\#1^4\ \&,\ 2\right] \\ \text{Root}\left[1 - \#1^2 + 4\,\#1^4\ \&,\ 4\right] & \text{Root}\left[1 - \#1^2 + 4\,\#1^4\ \&,\ 4\right] \end{pmatrix}$$

In[2991]:=
```
(* gC3 *)
AGC3 = Asc[GC3] // FullSimplify; AGC3 // MatrixForm
```

Out[2991]//MatrixForm=

$$\begin{pmatrix} \frac{1}{48}\left(-4\sqrt{2}+2\sqrt{3}+3\sqrt{15}\right) & \frac{1}{48}\left(-7+2\sqrt{30}\right) & \frac{1}{48}\left(-2\sqrt{2}-6\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(-5-4\sqrt{6}\right) \\ \frac{1}{48}\left(-5+2\sqrt{30}\right) & \frac{1}{48}\left(-4\sqrt{2}-2\sqrt{3}+3\sqrt{15}\right) & \frac{1}{48}\left(-1-4\sqrt{6}\right) & \frac{1}{48}\left(-2\sqrt{2}-6\sqrt{3}+\sqrt{15}\right) \\ \frac{1}{48}\left(2\sqrt{2}-6\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(5-4\sqrt{6}\right) & \frac{1}{48}\left(4\sqrt{2}+2\sqrt{3}+3\sqrt{15}\right) & \frac{1}{48}\left(7+2\sqrt{30}\right) \\ \frac{1}{48}\left(-1+4\sqrt{6}\right) & \frac{1}{48}\left(-2\sqrt{2}+6\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(-5-2\sqrt{30}\right) & \frac{1}{48}\left(-4\sqrt{2}+2\sqrt{3}-3\sqrt{15}\right) \end{pmatrix}$$

In[2744]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[2992]:=
```
KGC3 = Sqrt[AGC3[[1, 1]]^2 + AGC3[[1, 2]]^2 + AGC3[[1, 3]]^2 + AGC3[[1, 4]]^2] // Simplify;
```

In[2993]:=
```
pGC3 = AGC3[[1, 1]] / KGC3 // Simplify;
qGC3 = AGC3[[1, 2]] / KGC3 // Simplify;
rGC3 = AGC3[[1, 3]] / KGC3 // Simplify;
sGC3 = AGC3[[1, 4]] / KGC3 // FullSimplify;
aGC3 = KGC3;
bGC3 = KGC3 AGC3[[2, 1]] / AGC3[[1, 1]] // FullSimplify;
cGC3 = KGC3 AGC3[[3, 1]] / AGC3[[1, 1]] // FullSimplify;
dGC3 = KGC3 AGC3[[4, 1]] / AGC3[[1, 1]] // FullSimplify;
```

In[2754]:=
```
(* operations to figure out if we're doing everything right *)
```

In[3001]:=
```
aGC3^2 + bGC3^2 + cGC3^2 + dGC3^2 // FullSimplify
pGC3^2 + qGC3^2 + rGC3^2 + sGC3^2 // FullSimplify
```

Out[3001]= 1

Out[3002]= 1

In[3003]:=
```
{{aGC3 pGC3, aGC3 qGC3, aGC3 rGC3, aGC3 sGC3}, {bGC3 pGC3, bGC3 qGC3, bGC3 rGC3, bGC3 sGC3},
    {cGC3 pGC3 , cGC3 qGC3, cGC3 rGC3, cGC3 sGC3},
    {dGC3 pGC3, dGC3 qGC3, dGC3 rGC3, dGC3 sGC3}} - AGC3 // FullSimplify // MatrixForm
```

Out[3003]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[3004]:=
```
GC3P = (aGC3 IdentityMatrix[2] + I bGC3 PauliMatrix[1] +
    I cGC3 PauliMatrix[2] + I dGC3 PauliMatrix[3]) // Simplify; GC3P // MatrixForm
GC3M = (pGC3 IdentityMatrix[2] + I qGC3 PauliMatrix[1] + I rGC3 PauliMatrix[2] +
     I sGC3 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GC3M // MatrixForm
```

Out[3004]//MatrixForm=

$$\begin{pmatrix} \frac{1}{24}\left(\sqrt{6\left(25+3\sqrt{5}+2\sqrt{6}-2\sqrt{30}\right)}-i\left(-3-3\sqrt{5}-3\sqrt{6}+\sqrt{30}\right)\right) & \frac{1}{24}i\left(3+3\sqrt{5}-3\sqrt{6}+\sqrt{30}+i\sqrt{6\left(25+3\sqrt{5}-2\sqrt{6}+2\sqrt{30}\right)}\right) \\ \frac{1}{24}\left(i\left(3+3\sqrt{5}-3\sqrt{6}+\sqrt{30}\right)+\sqrt{6\left(25+3\sqrt{5}-2\sqrt{6}+2\sqrt{30}\right)}\right) & \frac{1}{24}\left(\sqrt{6\left(25+3\sqrt{5}+2\sqrt{6}-2\sqrt{30}\right)}+i\left(-3-3\sqrt{5}-3\sqrt{6}+\sqrt{30}\right)\right) \end{pmatrix}$$

Out[3005]//MatrixForm=

$$\begin{pmatrix} \frac{6\sqrt{2}-8\sqrt{3}+9\sqrt{10}+i\sqrt{726+240\sqrt{6}}}{12\sqrt{25+3\sqrt{5}+2\sqrt{6}-2\sqrt{30}}} & \frac{18\sqrt{2}+4\sqrt{3}-12i\sqrt{5}+7i\sqrt{6}+3\sqrt{10}}{12\sqrt{25+3\sqrt{5}+2\sqrt{6}-2\sqrt{30}}} \\ \frac{-18\sqrt{2}-4\sqrt{3}-12i\sqrt{5}+7i\sqrt{6}-3\sqrt{10}}{12\sqrt{25+3\sqrt{5}+2\sqrt{6}-2\sqrt{30}}} & \frac{6\sqrt{2}-8\sqrt{3}+9\sqrt{10}-i\sqrt{726+240\sqrt{6}}}{12\sqrt{25+3\sqrt{5}+2\sqrt{6}-2\sqrt{30}}} \end{pmatrix}$$

In[3006]:=
```
(* gC2 *)
AGC2 = Asc[GC2] // FullSimplify; AGC2 // MatrixForm
```

Out[3006]//MatrixForm=

$$\begin{pmatrix} \frac{1}{48}\left(8\sqrt{2}+6\sqrt{3}+3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-7-\sqrt{30}\right) & \frac{1}{48}\left(\sqrt{2}-2\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(-5-\sqrt{6}\right) \\ \frac{1}{48}\left(-5-\sqrt{30}\right) & \frac{1}{48}\left(8\sqrt{2}-6\sqrt{3}-3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-1+\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}-2\sqrt{3}+\sqrt{15}\right) \\ \frac{1}{48}\left(-\sqrt{2}-2\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(5-\sqrt{6}\right) & \frac{1}{48}\left(-8\sqrt{2}+6\sqrt{3}-3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(7-\sqrt{30}\right) \\ \frac{1}{48}\left(-1-\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}+2\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(-5+\sqrt{30}\right) & \frac{1}{48}\left(8\sqrt{2}+6\sqrt{3}-3\sqrt{10}-3\sqrt{15}\right) \end{pmatrix}$$

In[2761]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[3007]:=
```
KGC2 = Sqrt[AGC2[[1, 1]]^2 + AGC2[[1, 2]]^2 + AGC2[[1, 3]]^2 + AGC2[[1, 4]]^2] // Simplify;
```

In[3008]:=
```
pGC2 = AGC2[[1, 1]] / KGC2 // Simplify;
qGC2 = AGC2[[1, 2]] / KGC2 // Simplify;
rGC2 = AGC2[[1, 3]] / KGC2 // Simplify;
sGC2 = AGC2[[1, 4]] / KGC2 // FullSimplify;
aGC2 = KGC2;
bGC2 = KGC2 AGC2[[2, 1]] / AGC2[[1, 1]] // FullSimplify;
cGC2 = KGC2 AGC2[[3, 1]] / AGC2[[1, 1]] // FullSimplify;
dGC2 = KGC2 AGC2[[4, 1]] / AGC2[[1, 1]] // FullSimplify;
```

In[2771]:=
```
(* operations to figure out if we're doing everything right *)
```

In[3016]:=
```
aGC2^2 + bGC2^2 + cGC2^2 + dGC2^2 // FullSimplify
pGC2^2 + qGC2^2 + rGC2^2 + sGC2^2 // FullSimplify
```

Out[3016]= 1

Out[3017]= 1

In[3018]:=
```
{{aGC2 pGC2, aGC2 qGC2, aGC2 rGC2, aGC2 sGC2}, {bGC2 pGC2, bGC2 qGC2, bGC2 rGC2, bGC2 sGC2},
    {cGC2 pGC2 , cGC2 qGC2, cGC2 rGC2, cGC2 sGC2},
    {dGC2 pGC2, dGC2 qGC2, dGC2 rGC2, dGC2 sGC2}} - AGC2 // FullSimplify // MatrixForm
```

Out[3018]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[3497]:=
```
GC2P = (aGC2 IdentityMatrix[2] + I bGC2 PauliMatrix[1] +
    I cGC2 PauliMatrix[2] + I dGC2 PauliMatrix[3]) // Simplify; GC2P // MatrixForm
GC2M = (pGC2 IdentityMatrix[2] + I qGC2 PauliMatrix[1] + I rGC2 PauliMatrix[2] +
     I sGC2 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GC2M // MatrixForm
```

Out[3497]//MatrixForm=

$$\begin{pmatrix} \frac{1}{24}\left(-i\left(-3+3\sqrt{5}-3\sqrt{6}+\sqrt{30}\right)+\sqrt{6\left(25+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}\right)}\right) & \frac{1}{24}i\left(3-3\sqrt{5}-3\sqrt{6}+\sqrt{30}+i\sqrt{6\left(25+9\sqrt{5}-8\sqrt{6}-4\sqrt{30}\right)}\right) \\ \frac{1}{24}\left(\sqrt{6\left(25+9\sqrt{5}-8\sqrt{6}-4\sqrt{30}\right)}+i\left(3-3\sqrt{5}-3\sqrt{6}+\sqrt{30}\right)\right) & \frac{1}{24}\left(i\left(-3+3\sqrt{5}-3\sqrt{6}+\sqrt{30}\right)+\sqrt{6\left(25+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}\right)}\right) \end{pmatrix}$$

Out[3498]//MatrixForm=

$$\begin{pmatrix} \frac{6i+18\sqrt{2}+16\sqrt{3}+5i\sqrt{6}+9\sqrt{10}+6\sqrt{15}}{12\sqrt{25+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}}} & \frac{6\sqrt{2}-2\sqrt{3}+6i\sqrt{5}+7i\sqrt{6}+3\sqrt{10}}{12\sqrt{25+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}}} \\ \frac{-6\sqrt{2}+2\sqrt{3}+6i\sqrt{5}+7i\sqrt{6}-3\sqrt{10}}{12\sqrt{25+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}}} & \frac{-6i+18\sqrt{2}+16\sqrt{3}-5i\sqrt{6}+9\sqrt{10}+6\sqrt{15}}{12\sqrt{25+9\sqrt{5}+8\sqrt{6}+4\sqrt{30}}} \end{pmatrix}$$

In[3021]:=
```
(* gC1 *)
AGC1 = Asc[GC1] // FullSimplify; AGC1 // MatrixForm
```

Out[3021]//MatrixForm=

$$\begin{pmatrix} \frac{1}{48}\left(-4\sqrt{2}+2\sqrt{3}-3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(5-\sqrt{30}\right) & \frac{1}{48}\left(\sqrt{2}+2\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(7-3\sqrt{6}\right) \\ \frac{1}{48}\left(7-\sqrt{30}\right) & \frac{1}{48}\left(-4\sqrt{2}-2\sqrt{3}+3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-13-5\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}+2\sqrt{3}+\sqrt{15}\right) \\ \frac{1}{48}\left(-\sqrt{2}+2\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(-7-3\sqrt{6}\right) & \frac{1}{48}\left(4\sqrt{2}+2\sqrt{3}+3\sqrt{10}+3\sqrt{15}\right) & \frac{1}{48}\left(-5-\sqrt{30}\right) \\ \frac{1}{48}\left(-13+5\sqrt{6}\right) & \frac{1}{48}\left(\sqrt{2}-2\sqrt{3}-\sqrt{15}\right) & \frac{1}{48}\left(7+\sqrt{30}\right) & \frac{1}{48}\left(-4\sqrt{2}+2\sqrt{3}+3\sqrt{10}-3\sqrt{15}\right) \end{pmatrix}$$

In[2778]:=
```
(* As long as the first line of this matrix has a non-
 zero entry we can copy-paste the algorithm from the main text. *)
```

In[3022]:=
```
KGC1 = Sqrt[AGC1[[1, 1]]^2 + AGC1[[1, 2]]^2 + AGC1[[1, 3]]^2 + AGC1[[1, 4]]^2] // Simplify;
```

In[3023]:=
```
pGC1 = AGC1[[1, 1]] / KGC1 // Simplify;
qGC1 = AGC1[[1, 2]] / KGC1 // Simplify;
rGC1 = AGC1[[1, 3]] / KGC1 // Simplify;
sGC1 = AGC1[[1, 4]] / KGC1 // FullSimplify;
aGC1 = KGC1;
bGC1 = KGC1 AGC1[[2, 1]] / AGC1[[1, 1]] // FullSimplify;
cGC1 = KGC1 AGC1[[3, 1]] / AGC1[[1, 1]] // FullSimplify;
dGC1 = KGC1 AGC1[[4, 1]] / AGC1[[1, 1]] // FullSimplify;
```

In[2788]:=
```
(* operations to figure out if we're doing everything right *)
```

In[3031]:=
```
aGC1^2 + bGC1^2 + cGC1^2 + dGC1^2 // FullSimplify
pGC1^2 + qGC1^2 + rGC1^2 + sGC1^2 // FullSimplify
```

Out[3031]= 1

Out[3032]= 1

In[3033]:=
```
{{aGC1 pGC1, aGC1 qGC1, aGC1 rGC1, aGC1 sGC1}, {bGC1 pGC1, bGC1 qGC1, bGC1 rGC1, bGC1 sGC1},
    {cGC1 pGC1 , cGC1 qGC1, cGC1 rGC1, cGC1 sGC1},
    {dGC1 pGC1, dGC1 qGC1, dGC1 rGC1, dGC1 sGC1}} - AGC1 // FullSimplify // MatrixForm
```

Out[3033]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In[3034]:=
```
GC1P = (aGC1 IdentityMatrix[2] + I bGC1 PauliMatrix[1] +
    I cGC1 PauliMatrix[2] + I dGC1 PauliMatrix[3]) // Simplify; GC1P // MatrixForm
GC1M = (pGC1 IdentityMatrix[2] + I qGC1 PauliMatrix[1] + I rGC1 PauliMatrix[2] +
     I sGC1 PauliMatrix[3]) // Conjugate // Transpose // Simplify; GC1M // MatrixForm
```

Out[3034]//MatrixForm=

$$\begin{pmatrix} \frac{1}{24}\left(\sqrt{6\left(19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}\right)}+i\left(3-3\sqrt{5}+2\sqrt{30}\right)\right) & \frac{1}{24}\left(-i\left(-3+3\sqrt{5}+2\sqrt{30}\right)+\sqrt{6\left(19+3\sqrt{5}+6\sqrt{6}+2\sqrt{30}\right)}\right) \\ -\frac{1}{24}i\left(-3+3\sqrt{5}+2\sqrt{30}-i\sqrt{6\left(19+3\sqrt{5}+6\sqrt{6}+2\sqrt{30}\right)}\right) & \frac{1}{24}\left(\sqrt{6\left(19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}\right)}-i\left(3-3\sqrt{5}+2\sqrt{30}\right)\right) \end{pmatrix}$$

Out[3035]//MatrixForm=

$$\begin{pmatrix} \frac{6\sqrt{2}-8\sqrt{3}+9\sqrt{10}-6\sqrt{15}+i\sqrt{618-252\sqrt{6}}}{12\sqrt{19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}}} & \frac{-6\sqrt{2}-2\sqrt{3}+6i\sqrt{5}-5i\sqrt{6}+3\sqrt{10}}{12\sqrt{19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}}} \\ \frac{6\sqrt{2}+2\sqrt{3}+6i\sqrt{5}-5i\sqrt{6}-3\sqrt{10}}{12\sqrt{19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}}} & \frac{6\sqrt{2}-8\sqrt{3}+9\sqrt{10}-6\sqrt{15}-i\sqrt{618-252\sqrt{6}}}{12\sqrt{19+3\sqrt{5}-6\sqrt{6}-2\sqrt{30}}} \end{pmatrix}$$

```
(* Computing the SO(3) matrices R(g) for the g_Ci *)

(* gC5 *)
```

In[3036]:=
```
RC5P = R[GC5P[[1, 1]], GC5P[[1, 2]]] // FullSimplify; RC5P // MatrixForm
RC5P.Transpose[RC5P] // FullSimplify // MatrixForm
Det[RC5P] // FullSimplify
RC5M = R[GC5M[[1, 1]], GC5M[[1, 2]]] // FullSimplify; RC5M // MatrixForm
RC5M.Transpose[RC5M] // FullSimplify // MatrixForm
Det[RC5M] // FullSimplify
```

Out[3036]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

Out[3037]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[3038]= 1

Out[3039]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

Out[3040]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[3041]= 1

```
(* gC4 *)
```

In[3042]:=
```
RC4P = R[GC4P[[1, 1]], GC4P[[1, 2]]] // FullSimplify; RC4P // MatrixForm
RC4P.Transpose[RC4P] // FullSimplify // MatrixForm
Det[RC4P] // FullSimplify
RC4M = R[GC4M[[1, 1]], GC4M[[1, 2]]] // FullSimplify; RC4M // MatrixForm
RC4M.Transpose[RC4M] // FullSimplify // MatrixForm
Det[RC4M] // FullSimplify
```

Out[3042]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 \\ -\frac{\sqrt{15}}{4} & -\frac{1}{4} & 0 \\ \frac{1}{4} & -\frac{\sqrt{15}}{4} & 0 \end{pmatrix}$$

Out[3043]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[3044]= 1

Out[3045]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & -1 \\ -\frac{\sqrt{15}}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{\sqrt{15}}{4} & 0 \end{pmatrix}$$

Out[3046]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[3047]= 1

```
(* gC3 *)

In[3048]:= RC3P = R[GC3P[[1, 1]], GC3P[[1, 2]]] // FullSimplify; RC3P // MatrixForm
RC3P.Transpose[RC3P] // FullSimplify // MatrixForm
Det[RC3P] // FullSimplify
RC3M = R[GC3M[[1, 1]], GC3M[[1, 2]]] // FullSimplify; RC3M // MatrixForm
RC3M.Transpose[RC3M] // FullSimplify // MatrixForm
Det[RC3M] // FullSimplify
```

Out[3048]//MatrixForm=

$$\begin{pmatrix} \frac{1}{12}\left(\sqrt{6}-\sqrt{30}\right) & -\frac{1}{6}\sqrt{23+3\sqrt{5}} & -\frac{1}{3} \\ \frac{-1+\sqrt{5}}{8\sqrt{3}} & \frac{1}{24}\left(1+3\sqrt{5}\right) & -\frac{2\sqrt{2}}{3} \\ \frac{1}{8}\left(1+3\sqrt{5}\right) & -\frac{1}{8}\sqrt{3}\left(-1+\sqrt{5}\right) & 0 \end{pmatrix}$$

Out[3049]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[3050]= 1

Out[3051]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2}\sqrt{\frac{1}{3}\left(3+\sqrt{5}\right)} & -\frac{1}{6}\sqrt{23-3\sqrt{5}} & \frac{1}{3} \\ \frac{1}{4}\sqrt{\frac{1}{6}\left(3+\sqrt{5}\right)} & \frac{1}{24}\left(-1+3\sqrt{5}\right) & \frac{2\sqrt{2}}{3} \\ \frac{1}{8}\left(1-3\sqrt{5}\right) & \frac{1}{4}\sqrt{\frac{3}{2}\left(3+\sqrt{5}\right)} & 0 \end{pmatrix}$$

Out[3052]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[3053]= 1

```
(* gC2 *)
```

In[3054]:=
```
RC2P = R[GC2P[[1, 1]], GC2P[[1, 2]]] // FullSimplify; RC2P // MatrixForm
RC2P.Transpose[RC2P] // FullSimplify // MatrixForm
Det[RC2P] // FullSimplify
RC2M = R[GC2M[[1, 1]], GC2M[[1, 2]]] // FullSimplify; RC2M // MatrixForm
RC2M.Transpose[RC2M] // FullSimplify // MatrixForm
Det[RC2M] // FullSimplify
```

Out[3054]//MatrixForm=

$$\begin{pmatrix} \sqrt{\frac{5}{6}} & \frac{1}{3\sqrt{2}} & -\frac{1}{3} \\ \frac{-3+\sqrt{5}}{8\sqrt{3}} & \frac{1}{24}\left(1+9\sqrt{5}\right) & \frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(1+\sqrt{5}\right) & \frac{1-3\sqrt{5}}{8\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[3055]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[3056]= 1

Out[3057]//MatrixForm=

$$\begin{pmatrix} \sqrt{\frac{5}{6}} & -\frac{1}{3\sqrt{2}} & \frac{1}{3} \\ \sqrt{\frac{7}{96}+\frac{\sqrt{5}}{32}} & \frac{1}{24}\left(-1+9\sqrt{5}\right) & -\frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(1-\sqrt{5}\right) & \sqrt{\frac{23}{96}+\frac{\sqrt{5}}{32}} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[3058]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[3059]= 1

```
(* gC1 *)
```

In[3060]:=
```
RC1P = R[GC1P[[1, 1]], GC1P[[1, 2]]] // FullSimplify; RC1P // MatrixForm
RC1P.Transpose[RC1P] // FullSimplify // MatrixForm
Det[RC1P] // FullSimplify
RC1M = R[GC1M[[1, 1]], GC1M[[1, 2]]] // FullSimplify; RC1M // MatrixForm
RC1M.Transpose[RC1M] // FullSimplify // MatrixForm
Det[RC1M] // FullSimplify
```

Out[3060]//MatrixForm=

$$\begin{pmatrix} \frac{1}{12}\left(\sqrt{6}-\sqrt{30}\right) & -\frac{1}{6}\sqrt{23+3\sqrt{5}} & -\frac{1}{3} \\ -\frac{1+5\sqrt{5}}{8\sqrt{3}} & \frac{1}{24}\left(-5+3\sqrt{5}\right) & \frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(-1-\sqrt{5}\right) & \frac{-1+3\sqrt{5}}{8\sqrt{3}} & -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[3061]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[3062]= 1

Out[3063]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{2}\sqrt{\frac{1}{3}\left(3+\sqrt{5}\right)} & -\frac{1}{6}\sqrt{23-3\sqrt{5}} & \frac{1}{3} \\ -\frac{1}{4}\sqrt{\frac{1}{6}\left(63-5\sqrt{5}\right)} & \frac{1}{24}\left(5+3\sqrt{5}\right) & -\frac{\sqrt{2}}{3} \\ \frac{1}{8}\left(-1+\sqrt{5}\right) & -\frac{1}{4}\sqrt{\frac{1}{6}\left(23+3\sqrt{5}\right)} & -\sqrt{\frac{2}{3}} \end{pmatrix}$$

Out[3064]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[3065]= 1

```
(* gluing matrices for vertex C. The null
 space of these define the normal vectors n_ef.  *)
```

In[3066]:=
```
RCP = {RC1P, RC2P, RC3P, RC4P, RC5P, RC6P};
RCM = {RC1M, RC2M, RC3M, RC4M, RC5M, RC6M};
RC[i_, j_] := Transpose[RCP[[j]]].RCP[[i]] - Transpose[RCM[[j]]].RCM[[i]] ;
```

In[2834]:=
```
(* standard tetrahedron normals for reference *)
```

In[3069]:=
```
NA
NB
NC
ND
```

Out[3069]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3070]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3071]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3072]= $\{0, 0, -1\}$

In[3073]:=
```
(* Tetrahedron (1234) or (abcd) *)

NullSpace[RC[5, 4]] // FullSimplify
NullSpace[RC[5, 4]][[1]] // FullSimplify // Normalize
NullSpace[RC[5, 3]] // FullSimplify
NullSpace[RC[5, 3]][[1]] // FullSimplify // Normalize
NullSpace[RC[5, 2]] // FullSimplify
NullSpace[RC[5, 2]][[1]] // FullSimplify // Normalize
NullSpace[RC[5, 1]] // FullSimplify
NullSpace[RC[5, 1]][[1]] // FullSimplify // Normalize
```

Out[3073]= $\{\{0, 0, 1\}\}$

Out[3074]= $\{0, 0, 1\}$

Out[3075]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[3076]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3077]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[3078]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3079]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[3080]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3081]:=
```
nC554 = ND
nC553 = NB
nC552 = NC
nC551 = NA
```

Out[3081]= $\{0, 0, -1\}$

Out[3082]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3083]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3084]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3093]:=
```
(* Tetrahedron (1235) or (abce) *)

NullSpace[RC[4, 5]] // FullSimplify
NullSpace[RC[4, 5]][[1]] // FullSimplify // Normalize
NullSpace[RC[4, 3]] // FullSimplify
NullSpace[RC[4, 3]][[1]] // FullSimplify // Normalize
NullSpace[RC[4, 2]] // FullSimplify
NullSpace[RC[4, 2]][[1]] // FullSimplify // Normalize
NullSpace[RC[4, 1]] // FullSimplify
NullSpace[RC[4, 1]][[1]] // FullSimplify // Normalize
```

Out[3093]= $\{\{0, 0, 1\}\}$

Out[3094]= $\{0, 0, 1\}$

Out[3095]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[3096]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3097]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[3098]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3099]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[3100]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3105]:=
```
nC445 = ND
nC443 = NC
nC442 = NB
nC441 = NA
```

Out[3105]= $\{0, 0, -1\}$

Out[3106]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3107]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3108]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3109]:=
```
(* Tetrahedron (1245) or (abde) *)

NullSpace[RC[3, 5]] // FullSimplify
NullSpace[RC[3, 5]][[1]] // FullSimplify // Normalize
NullSpace[RC[3, 4]] // FullSimplify
NullSpace[RC[3, 4]][[1]] // FullSimplify // Normalize
NullSpace[RC[3, 2]] // FullSimplify
NullSpace[RC[3, 2]][[1]] // FullSimplify // Normalize
NullSpace[RC[3, 1]] // FullSimplify
NullSpace[RC[3, 1]][[1]] // FullSimplify // Normalize
```

Out[3109]= $\{\{0, 0, 1\}\}$

Out[3110]= $\{0, 0, 1\}$

Out[3111]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[3112]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3113]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[3114]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3115]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[3116]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3121]:=
```
nC335 = ND
nC334 = NB
nC332 = NC
nC331 = NA
```

Out[3121]= $\{0, 0, -1\}$

Out[3122]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3123]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3124]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3125]:=
```
(* Tetrahedron (1345) or (acde) *)

NullSpace[RC[2, 5]] // FullSimplify
NullSpace[RC[2, 5]][[1]] // FullSimplify // Normalize
NullSpace[RC[2, 4]] // FullSimplify
NullSpace[RC[2, 4]][[1]] // FullSimplify // Normalize
NullSpace[RC[2, 3]] // FullSimplify
NullSpace[RC[2, 3]][[1]] // FullSimplify // Normalize
NullSpace[RC[2, 1]] // FullSimplify
NullSpace[RC[2, 1]][[1]] // FullSimplify // Normalize
```

Out[3125]= $\{\{0, 0, 1\}\}$

Out[3126]= $\{0, 0, 1\}$

Out[3127]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[3128]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3129]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[3130]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3131]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[3132]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3137]:=
```
nC225 = ND
nC224 = NC
nC223 = NB
nC221 = NA
```

Out[3137]= $\{0, 0, -1\}$

Out[3138]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3139]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3140]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3141]:=
```
(* Tetrahedron (2345) or (bcde) *)

NullSpace[RC[1, 5]] // FullSimplify
NullSpace[RC[1, 5]][[1]] // FullSimplify // Normalize
NullSpace[RC[1, 4]] // FullSimplify
NullSpace[RC[1, 4]][[1]] // FullSimplify // Normalize
NullSpace[RC[1, 3]] // FullSimplify
NullSpace[RC[1, 3]][[1]] // FullSimplify // Normalize
NullSpace[RC[1, 2]] // FullSimplify
NullSpace[RC[1, 2]][[1]] // FullSimplify // Normalize
```

Out[3141]= $\{\{0, 0, 1\}\}$

Out[3142]= $\{0, 0, 1\}$

Out[3143]= $\left\{\left\{-\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[3144]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3145]= $\left\{\left\{0, -2\sqrt{2}, 1\right\}\right\}$

Out[3146]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3147]= $\left\{\left\{\sqrt{6}, \sqrt{2}, 1\right\}\right\}$

Out[3148]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

In[3153]:=
```
nC115 = ND
nC114 = NB
nC113 = NC
nC112 = NA
```

Out[3153]= $\{0, 0, -1\}$

Out[3154]= $\left\{-\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3155]= $\left\{0, -\frac{2\sqrt{2}}{3}, \frac{1}{3}\right\}$

Out[3156]= $\left\{\sqrt{\frac{2}{3}}, \frac{\sqrt{2}}{3}, \frac{1}{3}\right\}$