MATCH TESTS FOR NONPARAMETRIC ANALYSIS OF VARIANCE PROBLEMS

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P.W.

ABSTRACT

The thesis is presented in two parts,

(a) "Nonparametric Analysis of Variance"

and

(b) "An Asymptotic Expansion of the Null Distributions of Kruskal and Wallis's and Friedman's Statistics".

In the first part we present a number of new nonparametric tests designed for a variety of experimental situations. These tests are all based on a so-called "matching" principle. The range of situations covered by the tests are

(i) Two-way analysis of variance with a general alternative hypothesis (without interaction).

(ii) Two-way analysis of variance with an ordered alternative hypothesis (without interaction).

(iii) Interaction in two-way analysis of variance, both the univariate and multivariate cases.

(iv) Latin square designs.

(v) Second-order interaction in three-way analysis of variance.

(vi) Third-order interaction in four-way analysis of variance.

The validity of the tests is supported by a series of simulation studies which were performed with a number of different distributions.

In the second part of the thesis we develop an asymptotic expansion for the construction of improved approximations to the null distributions of Kruskal and Wallis's (1952) and Friedman's (1937) statistics. The approximation is founded on the method of steepest descents, a procedure that is better known in Numerical Analysis than in Statistics. In order to implement this approximation it was necessary to derive the third and fourth moments of the Kruskal-Wallis statistic and the fourth moment of Friedman's statistic.

Tables of approximate critical values based on this approximation are presented for both statistics.

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Part II - An Asymptotic Expansion of the Null Distributions of the Kruskal-Wallis and Friedman Statistic

CHAPTER 1

INTRODUCTION

1. Outline.

The thesis is divided into two main parts,

(a) "Nonparametric Analysis of Variance"

and

(b) "An Asymptotic Expansion of the Null Distributions of Kruskal and Wallis's and Friedman's Statistics".

Before proceeding it is appropriate to comment on the phrase "Analysis of Variance". This appears in the title more by common usage than by accuracy since "variance" is not considered in a nonparametric framework. Perhaps a more apt title would have been something like "Nonparametric Analysis of Multisample Experiments". However, the phrase "Analysis of Variance" is used because we are essentially producing procedures aimed at the same tasks and in similar situations as classical analysis of variance, but of course without the severe restriction of the normality assumption.

In the first part of the thesis we present a number of new nonparametric tests designed for a variety of experimental designs. These are all based on a so-called "matching" principle, which will be described in Chapter 2.

The second part is devoted to the development of an asymptotic expansion to be used in the construction of improved approximations to the null distributions of Kruskal and Wallis's (1952) and Friedman's (1937) statistics. The need for such approximations stems from the deficiency of exact critical values for even quite moderately-sized experiments. The most common approximation is the chi-square distribution although, as we shall see, several authors have attempted to produce improvements on this approximation. In view of these comments, we considered it quite suitable in a study on nonparametric analysis of variance to devise and include asymptotic expansions for these distributions.

2. Range of Experimental Situations.

The upsurge of interest in applying statistical methods to the biological and social sciences has resulted in users who are inexperienced in the complexities of classical analysis of variance. Often, perhaps because of lack of time or ability, they are prevented from acquiring the necessary expertise required to analyse experimental data. Such users as these will benefit greatly from our batch of "quick - and - simple" nonparametric tests designed for the wide range of experimental situations listed below.

(i) Two-way analysis of variance with a general alternative hypothesis (without interaction).

(ii) Two-way analysis of variance with an ordered alternative hypothesis (without interaction).

(iii) Interaction in two-way analysis of variance, both the univariate and multivariate cases.

(iv) Latin square designs.

(v) Second-order interaction in three-way analysis of variance.

(vi) Third-order interaction in four-way analysis of variance.

A notable absentee from this list is one-way analysis of variance which is one situation for which our technique is not applicable. However, it includes situations for which no useful nonparametric methods seem to have been previously developed.

3. The Simulation Studies.

A series of computer-simulated experiments was conducted in order to compare the virtues of our tests with some well-known competitors. A variety of symmetric and skewed distributions were used in the simulations to provide information regarding the performance of the tests under differing conditions. More precise details of the simulations are contained in Chapter 3.

Not all of the tests discussed in the various chapters were used in the simulations; for example, Hollander's (1967) test for ordered alternatives, Bhakpar and Gore's (1974) and Weber's (1972) tests for interactions in two-way layouts were considered unsuitable. The reason was that it is impossible to derive the exact null distributions for these tests and this obviously reduces their effectiveness in simulation studies. Bradley's (1979) test for second-order interactions was also not used; we felt that its reliance upon an arbitrary ordering to be too great a drawback.

4. Approximations to the Null Distributions of Kruskal and Wallis's and Friedman's Statistics.

As we have previously mentioned there is an embarrassing shortage of exact critical values for both the Kruskal-Wallis and Friedman's tests. In fact, for the Kruskal-Wallis test exact null distributions are available only for three treatments with a total sample size up to 24, four treatments with a total sample size up to 16 and five treatments with a total sample size up to 15.

Our task in the second part of the thesis was simply to "bridge the gap" between the exact null distributions and the chi-square and other approximations by developing a more accurate approximation.

The approximation is in fact a series expansion based upon a method that has been little-used in the statistical world, namely the method of steepest descents. In order to utilize this method we required an approximation to the characteristic functions of the statistics' null distributions. This in turn, required a knowledge of their third and fourth moments. The third moment of Friedman's statstic was derived in his paper of 1937. However, as the remaining moments (we believe) were hitherto unknown, these had to be derived.

Once we had obtained the approximations to the null distributions of these statistics we were able to compare the results with the few exact null distributions that have been computed and with the Beta and other approximations. The results from our expansions seem encouraging and certainly justify the large amount of computation that was required. We conclude the second part of the thesis by presenting our tables of critical values for the Kruskal-Wallis and Friedman statistics.

CHAPTER 2

NONPARAMETRIC ANALYSIS OF VARIANCE AND THE MATCHING PRINCIPLE

1. Introduction.

Before we introduce the matching principle and its application to analysis of variance problems we shall review some existing nonparametric tests appropriate for the experimental situations in which we are interested. The tests reviewed are perhaps the best-known of the nonparametric tests; below we present the main features of the tests and leave further detail to the relevant chapter.

2. Survey of Existing Nonparametric Tests for Analysis of Variance.

a. Two-way Analysis of Variance with a General Alternative Hypothesis (without interaction).

Friedman was the first to introduce a nonparametric test for the randomised block design with his χ^2_r - test of 1937. This test is now one of the best-known nonparametric tests thanks mainly to its computational ease. Since the introduction of Friedman's test many authors have presented alternative methods, notably Bell and Doksum (1965), Koch and Sen (1968), Gerig (1969) and Mack and Skillings (1981).

Bell and Doksum's novel idea was to replace the actual observations with a similarly-ranked random sample from a normal distribution and then proceed with the usual F-tests. Unfortuately, the resulting conclusion is, not surprisingly, very dependent upon the particular choice of random numbers. However, their test is certainly of value particularly since it can be applied to all designs.

The problems that may occur with tied observations were appreciated by Koch and Sen. They devised an extension of Friedman's procedure which provided a more adequate test for randomised blocks with ties than had hitherto existed. However, the computational complexities and the impossibility of deriving exact null distributions have resulted in their test being little used.

Gerig extended Friedman's test for the situation where there is more than one replication per cell. However, the weakness in this extension lies in its reliance on the replications possessing a natural order of occurrence. In practice such orderings would usually be obtained in quite an arbitrary manner which may lead to spurious conclusions being reached depending upon the particular choice of ordering.

Conover (1971) gave a procedure for analysing randomised block designs when there is equal number of replications per cell with no implied ordering. Mack and Skilling extended this idea to cater for unequal numbers per cell. Unfortuately, except in the case of proportional frequencies, their procedure seems to be rather involved.

b. Two-way Analysis of Variance with an Ordered Alternative Hypothesis (without interaction).

It was his involvement in psychological experiments that prompted Jonckheere (1954) to devise a test to accommodate ordered alternatives. His test is in fact based on Kendall's (1938) τ - statistic and is quite straightforward to apply.

Two more tests appeared in the 1960's; one in 1963 by Page and the other in 1967 by Hollander. Page's procedure is very similar, in terms of performance and computational work, to Jonckheere's test. However, Hollander's method is of limited practical use as it is neither even asymptotically distribution-free nor computationally straightforward.

c. Interaction in Two-way Analysis of Variance.

Interaction in two-way layouts may be classified in one of two ways. The replicates may be regarded either as possessing some natural ordering or as a random sample with no implied ordering. These two situations are sometimes referred to as the multivariate and univariate cases respectively.

Weber (1974), Bhapkar and Gore (1974) and Lin and Crump (1974) have all presented tests for the univariate situation. Weber's interesting procedure featured the use of normal scores. Bhapkar and Gore based their method on Hoeffding's (1948) generalised U-statistics while Lin and

Crump modified a procedure by Patel and Hoel (1973) which was based on the Mann-Whitney-Wilcoxon statistics. It is perhaps unfortunate that these tests suffer from one or more of the following drawbacks: (i) they are only asymptotically distribution-free, (ii) they are computationally complicated, (iii) their exact null distributions cannot be derived in general.

The situation with regard to the multivariate case is somewhat better. As early as 1949 Wilcoxon devised a simple and useful procedure based on Friedman's χ_r^2 - test. Although exact null distributions can be computed for his statistic, he recommends the use of chi-square approximations. Other procedures have been developed by Puri and Sen (1966), Mehra and Sen (1969) and Mehra and Smith (1970). However, their tests suffer from similar faults to those in the univariate case.

d. <u>Latin Squares Design</u>.

Surprisingly the Latin squares design has attracted no apparent attention from nonparametric statisticians. Clearly, the existence of a nonparametric procedure for such a popular design would be an asset to the experimenter.

e. <u>Second-order Interactions</u>.

In spite of being a fairly involved situation to analyse using classical methods, second-order interaction effects have not attracted much by way of simpler nonparametric procedures. Bradley (1979) did propose a test based on

Wilcoxon's (1949) procedure for first-order interaction. The use of this procedure is somewhat restricted by the conclusion being dependent on the particular assignment of ranks to observations.

f. Third-order Interactions.

Apparently the only nonparametric test for third-order interaction is Bradley's which can be extended to cover this situation.

3. The Matching Principle.

We shall now introduce the matching principle and illustrate its application in the analysis of experimental designs by an example relating to an experiment with an ordered alternative hypothesis.

The matching principle upon which our tests are founded is certainly not a recent innovation. As early as 1708 Montmort (see Feller 1968) presented a playing-card matching problem together with its solution. In this problem, two identical decks of N different cards are placed in random order alongside each other. The decks are then compared and where two identical cards occupy the same place in both decks there is a match. Clearly, matches may occur at any of the N places and at several places simultaneously. Out of this situation there arises the interesting problems of :

(i) What is the probability of having at least one match?

(ii) What are the probabilities of having exactly 0, 1, 2, ..., N-2, N matches?

The first problem has a particularly interesting answer, namely

Probability of at least one match

$$= 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots\ldots\ldots - \frac{(-1)^N}{N!}$$

$$\rightarrow 1 - e^{-1} = 0.6321, \quad \text{for } N \text{ sufficiently large.}$$

In other words, unless N is very small, the probability of having at least one match is just under 2/3, regardless of the number of cards. In fact, for $N \geqslant 7$ the result is correct to at least 4 decimal places.

The second problem, that of calculating the probability of exactly 0, 1, 2, ... ,N-2, N matches, will be encountered in Chapter 4. For the moment we shall content ourselves with showing how this ancient idea can be used to analyse modern experimental data.

These data are based on a subset of the data obtained by Fox and Randall (1970) in their study of forearm tremor. Each entry in the table is the mean of five experimental values of tremor frequency. The null hypothesis is that forearm tremor frequency is not affected by the weight applied at the wrist. The ordered alternative hypothesis is that tremor frequency decreases as the applied weight increases.

Forearm tremor frequency (Hz) as a function of weight (lb) applied to the wrist.

Subject	0	1.25	2.50	5.00	7.50
1	3.01	2.85	2.62	2.63	2.58
2	3.47	3.43	3.15	2.83	2.70
3	3.35	3.14	3.02	2.71	2.78
4	3.10	2.86	2.58	2.49	2.36
5	3.41	3.32	3.08	2.96	2.67
6	3.07	3.06	2.85	2.50	2.43

Once the table of intra-block rankings has been obtained, each row is compared with the ranks predicted under the alternative hypothesis. The number of matches with the predicted ranks is recorded for each row; the test statistic L1 is then the total number of matches. For the given data we have the following table of ranks.

Table of ranks

						Number of Matches
Predicted Order :	5	4	3	2	1	
	5	4	2	3	1	3
	5	4	3	2	1	5
Ranks	5	4	3	1	2	3
	5	4	3	2	1	5
	5	4	3	2	1	5
	5	4	3	2	1	5

Hence L1 = 3 + 5 +3 + 5 + 5 + 5 = 26. From the tables of exact probabilities in Chapter 4 we obtain

$P(L1 \geqslant 26) = 0$, to 6 decimal places, providing conclusive evidence to support the alternative hypothesis.

All our tests, ranging from this simplest case of ordered alternatives to the third-order interaction tests, are based on similar "matching" ideas, although a more powerful series of tests incorporates a concept of "near-matches".

CHAPTER 3

TWO-WAY ANALYSIS OF VARIANCE, GENERAL ALTERNATIVES

1. Introduction.

The quest for a nonparametric test for main effects in two-factor experiments is certainly not new. As early as 1937 Friedman proposed his now-famous χ^2_r - test. Since then many have been active in devising procedures which either rival or extend Friedman's work.

In 1965 Bell and Doksum introduced the idea of replacing the actual observations within a block by a similarly-ranked random sample from a normal distribution. The analysis is then completed by means of the usual F-test. Unfortunately this rather clever idea can result in different conclusions according to the particular choice of random sample. Nonetheless, their procedure is certainly worthy of note as it can be extended to other experimental designs.

Should ties occur in the data then it is common practice, provided the number of ties is small, to still proceed with the analysis using a conventional test, treating ties by average rank or similar compromise methods. However Koch and Sen's (1968) $\hat{\omega}_b$ - statistic is designed specifically to cater for the situation where ties do exist. Their statistic reduces to Friedman's χ^2_r - statistic when there are no ties.

Gerig (1969) extended Friedman's idea to cover the situation where, instead of having just one observation for each treatment-block combination, there is an ordered sequence of p (> 1) observations. This is perhaps a slightly artificial case since it is more likely that the observations will have no ordering; It is for this more practical situation that Mack and Skillings (1981) have developed a Friedman-type statistic which has the advantage of catering for unequal cell sizes. However, except for the case of proportional frequencies, their procedure does appear rather involved which might reduce its usefulness; particularly since Conover (1971) has presented a straightforward extension of Friedman's test for equal cell sizes.

The statistics to be introduced in this chapter are M1, based on the number of matches, and M2, based also on the number of "near-matches" between the successive intra-block rankings. Both tests may be considered to be of the quick and compact type in the sense of Tukey (1959), M1 being the easier of the two to apply while M2 has the greater power.

In the following sections we define the test statistics M1 and M2, and demonstrate their applicability to experimental data. In later sections we derive moment generating functions for the null distributions of these statistics which will enable us to discuss their asymptotic behaviour. In the final section we analyse the results of computer simulations.

2. Definition of M1 and M2.

The linear model on which we base our explorations is one in which the observations X_{ij} may be written as

$$X_{ij} = M + A_i + B_j + z_{ij} ,$$

$$i = 1, 2, \ldots\ldots , b$$

$$j = 1, 2, \ldots\ldots , c$$

where M represents the overall mean,

A_i represents the effect of the i^{th} block,

B_j represents the effect of the j^{th} treatment

and the z_{ij}'s are independent random variables having same continuous distribution, with $E(z_{ij}) = 0$.

We seek to test the null hypothesis

$$H_o : B_i = 0 \quad \text{for all } i$$

against the general alternative hypothesis

$$H_1 : B_i \neq 0 \quad \text{for some } i \; .$$

Our statistics M1 and M2 are obtained in the following manner.

First of all the observations within each block are ranked from 1 to c (as in Friedman's test). Then the ranks in the i_1^{th} block ($i_1 = 1, 2, \ldots\ldots, b-1$) are compared in turn with the ranks in the i_2^{th} block ($i_2 = i_1+1, i_1+2, \ldots \ldots, b$). From these comparisions we are able to define two scores m_{ij} and m^*_{ij} .

If $R(X_{ik})$ denotes the rank of the observation X_{ik} in the i^{th} block then we define

$$m_{ij} = \sum_{k=1}^{c} m_{ijk} \qquad \text{and } m^*_{ij} = \sum_{k=1}^{c} m^*_{ijk}$$

where

$$m_{ijk} = \begin{cases} 1 & \text{if } R(X_{ik}) = R(X_{jk}) \\ 0 & \text{otherwise} \end{cases}$$

and

$$m^*_{ijk} = \begin{cases} \frac{1}{2} & \text{if } \left| R(X_{ik}) - R(X_{jk}) \right| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus $m_{ijk} = 1$ corresponds to a "match" between $R(X_{ik})$ and $R(X_{jk})$ while $m^*_{ijk} = \frac{1}{2}$ corresponds to a "near-match" between the ranks so that m_{ij} and m^*_{ij} are simply the number of matches and near-matches between blocks i and j ($i = 1, 2, \ldots, b-1$; $j = i+1, i+2, \ldots, b$).

We now define the test statistics to be

$$M1 = \sum_{i=1}^{b-1} m_{i.}$$

and

$$M2 = \sum_{i=1}^{b-1} (m_{i.} + m^*_{i.}) \quad ,$$

where $$m_{i.} = \sum_{j=i+1}^{b} m_{ij} \quad \text{and} \quad m^*_{i.} = \sum_{j=i+1}^{b} m^*_{ij} \quad .$$

In other words, M1 is the sum of the matches between blocks i and j while M2 is the sum of M1 and the number of near-matches between blocks i and j ($i = 1, 2, \ldots, b-1$; $j = i+1, i+2, \ldots, b$).

3. The Problem of Ties.

With the majority of nonparametric tests the underlying theory depends on the assumption of having continuously-distributed populations, so that there is zero probability of ties occurring. In practice, populations may not be continuous or, even if they are, there is bound to be some physical limitation on the accuracy with which observations are recorded. In either case ties may occur which obviously poses problems when assigning ranks to the observations.

Since our tests are based on matches and near-matches, perhaps the most appropriate approach to the problem of ties is to calculate averages for M1 and M2 based on arrays of ranks generated from all possible permutations of "tied" ranks. Fortunately, it is fairly easy to calculate these averages without generating the permutations. This is acheived by writing down the range of ranks at all tied observations, and calculating the contributions to M1 and M2 as the proportion of matches or half the proportion of near-matches, respectively. The following example illustrates this procedure for two blocks (X and Y) and seven treatments.

<u>Raw Data</u>

X :	2	9	11	9	5	9	9
Y :	3	8	6	6	6	4	10

<u>Ranked Data</u>

X :	1	(3-6)	7	(3-6)	2	(3-6)	(3-6)
Y :	1	6	(3-5)	(3-5)	(3-5)	2	7

<u>Contribution to M1</u>

1	1/4	0	3/12	0	0	0

<u>Contribution to M2 (from near-matches)</u>

0	$\frac{1}{2}(1/4)$	0	$\frac{1}{2}(5/12)$	$\frac{1}{2}(1/3)$	$\frac{1}{2}(1/4)$	$\frac{1}{2}(1/4)$

Hence $\quad M1 = 1 + 1/4 + 3/12 = 1\frac{1}{2}$

and $\quad M2 = 1\frac{1}{2} + \frac{1}{2}(1/4 + 5/12 + 1/3 + 1/4 + 1/4) = 2\frac{1}{4}$.

To see how the contributions are obtained from the ranges of ranks consider the ranks in position 4, X : (3-6)

Y : (3-5)

There are 12 possibilities, 3 of which lead to matches and 5 of which lead to near-matches (these are $\{(x,y) = (3,4), (4,3), (4,5), (5,4) \text{ and } (6,5)\}$). So there is a contribution of 3/12 to M1 and $\frac{1}{2}(5/12)$ to M2.

This range method is clearly quicker than actually generating all the permutations. However for even quicker methods when dealing with ties we now examine ideas based on assigning to each tied observation the average of the ranks that would have been assigned had there been no ties.

Firstly we consider a possible approach for M1. Suppose that the two observations currently being compared have ranks R_1 and R_2, then the contribution to M1 is given by the following rule.

$$\text{If } |R_1 - R_2| = \begin{cases} 0 & \text{then contribute } 1 \\ \frac{1}{2} & \quad \text{..} \qquad \text{..} \qquad \frac{1}{2} \\ 1 & \quad \text{..} \qquad \text{..} \qquad 0 \end{cases}.$$

Applying this rule to the previous set of data where now average ranks are used where ties occur, we have

Ranked Data

X :	1	$4\frac{1}{2}$	7	$4\frac{1}{2}$	2	$4\frac{1}{2}$	$4\frac{1}{2}$
Y :	1	6	4	4	4	2	7

Contribution to M1

1	0	0	$\frac{1}{2}$	0	0	0

Hence M1 = $1\frac{1}{2}$ which by coincidence is the same result as given by the range method. This does not always happen; for example, had the data produced the following ranks :

X :	1	(3-6)	7	(3-6)	2	(3-6)	(3-6)
Y :	1	(3-5)	(3-5)	(3-5)	6	2	7 ,

then the range method would have given M1 = $1 + 3/12 + 3/12 = 1\frac{1}{2}$, as before, whereas the above method gives M1 = $1 + \frac{1}{2} + \frac{1}{2} = 2$.

The large number of different situations makes it difficult to produce precise information concerning those occasions when the two methods agree. However the simple case below will indicate that these methods are likely to produce results that are never very much in disagreement.

Consider two blocks, X and Y, of n observations where X contains no ties and Y contains k ($\leqslant n$) ties, the range of ranks covered by the ties being $r_1 - r_k$. Suppose the ranked data is (where $r_i = r_1 + i - 1$)

$$X : 1 \quad 2 \quad 3 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad n-1 \quad n$$

$$Y : a_1 \quad * \quad a_2 \quad * \quad \ldots\ldots * \quad \ldots\ldots \quad * \quad a_{n-k} \; ,$$

where a * represents one of the k-ties and the a_i's ($1 \leqslant i \leqslant n-k$) represent the other ranks.

The maximum contribution to M1 from the ties occurs when the k X-ranks, $r_1 \; \ldots\ldots \; r_k$, each coincide with a * . In this case the range method contributes $k \times 1/k = 1$ while the average rank method contributes $2 \times \frac{1}{2} = 1$ if k is even, or $1 \times 1 = 1$, if k is odd, to M1. When fewer than k of the X-ranks $r_1 \; \ldots\ldots\ldots \; r_k$ coincide with a * then the greatest discrepancy between the two methods is $1 - 2/k$ when k is even and 1 - 1/k when k is odd.

For M2 we propose two methods based on average ranks. Again suppose that the two observations currently being compared have ranks R_1 and R_2; then the contribution to M2 is given by :

Rule (a).

$$\text{If } |R_1 - R_2| = \begin{cases} 0 & \text{then contribute } 1 \\ \frac{1}{2} & \quad .. \quad .. \quad 3/4 \\ 1 & \quad .. \quad .. \quad \frac{1}{2} \\ 1\frac{1}{2} & \quad .. \quad .. \quad \frac{1}{4} \\ >1\frac{1}{2} & \quad .. \quad .. \quad 0\ . \end{cases}$$

This sliding scale of contributions caters for matches and near-matches where the amount of the contribution represents the closeness to a match or a near-match.

Rule (b).

$$\text{If } |R_1 - R_2| = \begin{cases} 0 & \text{then contribute } 1 \\ \left.\begin{matrix} \frac{1}{2} \\ 1 \\ 1\frac{1}{2} \end{matrix}\right\} & \quad .. \quad .. \quad \frac{1}{2} \\ >1\frac{1}{2} & \quad .. \quad .. \quad 0\ . \end{cases}$$

This is certainly an easy rule to remember. However it might be suggested that this system of weightings is somewhat unrepresentative of the relative importance of the near-matches. On the other hand, it can be argued that the contributions in rule (a) for near-matches of $\frac{1}{2}$ and $1\frac{1}{2}$ will often average to $\frac{1}{2}$ for each so that in practice there is likely to be little difference between the contributions from the two rules.

To illustrate the application of these rules we again consider the data whose ranks (averaged where appropriate) are given by

X :	1	$4\frac{1}{2}$	7	$4\frac{1}{2}$	2	$4\frac{1}{2}$	$4\frac{1}{2}$
Y :	1	6	4	4	4	2	7

<u>Contributions to M2</u>

Rule (a) :	1	$\frac{1}{4}$	0	3/4	0	0	0
Rule (b) :	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0

giving M2 = 2 in each case. We recall that the range method gave M2 = $2\frac{1}{4}$ for these data. Had the ranks being given by

X :	1	$4\frac{1}{2}$	7	$4\frac{1}{2}$	2	$4\frac{1}{2}$	$4\frac{1}{2}$
Y :	1	4	4	4	6	2	7

then the values of M2 by rules (a) and (b) are $2\frac{1}{2}$ and 2 respectively while the range method gives a value of $2\frac{1}{6}$.

We now consider the same simple general case as for M1. The ranked data are

$$\begin{array}{llllllll} X : & 1 & 2 & 3 & \cdots\cdots\cdots & n-1 & n \\ Y : & a_1 & * & a_2 & * \cdot\cdot * \cdot\cdot & * & a_{n-k} \end{array}$$

where as before a $*$ represents one of the k ties and the a_i's ($1 \leqslant i \leqslant n-k$) represent the other ranks. The maximum contribution to M2 is

$$\text{from the range method : } \frac{3}{k} + \frac{2(k-2)}{k} = 2 - \frac{1}{k} ,$$

$$\text{rule (a) : } 2(3/4) + 2(1/4) = 2\,, \quad (\,k \text{ even}\,)$$
$$1 \times 1 + 2(1/2) = 2\,, \quad (\,k \text{ odd}\,)$$
$$\text{rule (b) : } 2(1/2) + 2(1/2) = 2\,, \quad (\,k \text{ even}\,)$$
$$1 \times 1 + 2(1/2) = 2\,, \quad (\,k \text{ odd}\,)\,.$$

So, as with M1, we have some indication that methods based on average ranks and the range of ranks are not likely to differ muoh.

Whenever ties occur in examples in this and future chapters we shall give the values of the test statistic obtained by using all methods. This will supply further insight into differences in the test statistic brought about by using average ranks and the range of ranks methods. Of course, no matter which method is used when dealing with ties, the distributions of the statistics so obtained will be different from the correct null distributions.

4. Examples.

To illustrate the use of M1 and M2 we shall apply them to the two case studies that appear in Koch and Sen's paper of 1968. It is interesting to note that to apply the $\tilde{\omega}_n$ - statistic of that paper it is necessary to rely on asymptotic theory and the authors admit to having no idea concerning the level of accuracy of this approximation. They write " In cases II and IV this approximation should be satisfactory." ; their case II corresponds to the randomised block experiment.

Example 1 - a situation in which the null hypothesis is not rejected.

Sixteen animals were randomly placed into one of two equal groups - an experimental group receiving ethionine in their diets and a pair-fed control group (i.e. a control animal was given the same amount of food as the experimental animal with which it was paired). The data for each animal consisted of a measurement of the amount of radioactive iron among various subcellular fractions from liver cells. The cell fractions used were nuclei (N), mitochondria (Mit), microsomes (Mic) and supernatant (S). One question of interest to the experimenters was whether the ratio of the measurements for the experimental group to those for the control group was the same for all cell fractions. If matched pairs of animals are regarded as blocks and cell fractions are regarded as treatments then we have a randomised block experiment. The ratios were as follows.

Pair	N	Mit	Mic	S
1	1.73	1.08	2.60	1.67
2	2.50	2.55	2.51	1.80
3	1.17	1.47	1.49	1.47
4	1.54	1.75	1.55	1.72
5	1.53	2.71	2.51	2.25
6	2.61	1.37	1.15	1.67
7	1.86	2.13	2.47	2.50
8	2.21	1.06	0.95	0.98

The hypotheses under investigation are

H_0 : there is no difference between the cell fractions

H_1 : there is some difference between the cell fractions .

The table of within -block rankings for the above data is given below, range of ranks being quoted where ties occur.

Table of Ranks

Pair	N	Mit	Mic	S
1	3	1	4	2
2	2	4	3	1
3	1	(2-3)	4	(2-3)
4	1	4	2	3
5	1	4	3	2
6	4	2	1	3
7	1	2	3	4
8	4	3	1	2
Rank sums	17	22.5	25	19.5

Tests (i) - the match tests

The critical values for M1 and M2 are from the approximations given in sections 9 and 10 respectively.

For the M1 test, the null hypothesis will be rejected at the 5 % and 1 % levels of significance if $M1 \geqslant 40$ and $M1 \geqslant 45$, respectively; while for the M2 test rejection at the same levels of significance will occur if $M2 \geqslant 57$ and $M2 \geqslant 60.5$.

The astute reader will observe that if the frequency, f, of each rank in each column is counted then M1 can be obtained by summing the binomial coefficients $\binom{f}{2}$ ($f > 1$). However this procedure does not facilitate the calculation of M2 and furthermore does not help to develop the pattern for subsequent developments in sections 5 and 6. So we shall calculate the values of M1 and M2 in the manner described in section 2.

By comparing the ranks in the various blocks we obtain the following tables of matches and near-matches.

Table of Matches for M1

Method for Ties	Matches						
	$m_{1\cdot}$	$m_{2\cdot}$	$m_{3\cdot}$	$m_{4\cdot}$	$m_{5\cdot}$	$m_{6\cdot}$	$m_{7\cdot}$
Average Ranks	$3\frac{1}{2}$	4	$6\frac{1}{2}$	4	3	3	0
Range	$3\frac{1}{2}$	4	$6\frac{1}{2}$	4	3	3	0

Both methods for ties give M1 = 24, a value which clearly does not provide any evidence to support the alternative hypothesis.

The table of near-matches for M2 is given overleaf.

Table of Contributions for M2 from Near-matches

Method for Ties		$m^*_{1.}$	$m^*_{2.}$	$m^*_{3.}$	$m^*_{4.}$	$m^*_{5.}$	$m^*_{6.}$	$m^*_{7.}$
Average Ranks	(a)	$6\frac{1}{2}$	5	7	4	1	$1\frac{1}{2}$	$\frac{1}{2}$
	(b)	$6\frac{1}{2}$	$5\frac{1}{2}$	6	4	1	$1\frac{1}{2}$	$\frac{1}{2}$
Range		6	5	$3\frac{1}{2}$	4	1	$1\frac{1}{2}$	$\frac{1}{2}$

The values of M2 from each of the three methods of dealing with ties are found by calculating $M1 + \sum_{i=1}^{7} m^*_{i.}$ in each case to give $49\frac{1}{2}$, 49 and $45\frac{1}{2}$ respectively. Clearly, M2 does not provide evidence in support of the alternative hypothesis.

Test (ii) - Friedman's χ^2_r - test

The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $\chi^2_r \geqslant 7.65$ and $\chi^2_r \geqslant 10.50$ respectively, these being the best conservative critical values from the exact null distribution of χ^2_r.

Using $\chi^2_r = \frac{12}{bc(c+1)} \sum_{i=1}^{c} R_i^2 - 3b(c+1)$ we

obtain $$\chi^2_r = \frac{12}{160}(17^2 + 22.5^2 + 21^2 + 19.5^2) - 120$$
$$= 1.24 .$$

Again we have a result which does not support the alternative hypothesis.

Test (iii) - Koch and Sen's test

In view of the fact that Koch and Sen's test reduces to Friedman's test when there are no ties, we shall clearly obtain the same conclusion as above as we have only one tie in the data. However since we are demonstrating test procedures rather than simply comparing results, we shall proceed to illustrate Koch and Sen's procedure.

Their test statistic is defined by

$$\tilde{\omega}_b = \frac{b(c-1)}{c\,\sigma_R^2} \sum_{j=1}^{c} \left(T_{b,j} - \frac{c+1}{2} \right)^2$$

where $$\sigma_R^2 = \frac{1}{cb} \sum_{i=1}^{b} \sum_{j=1}^{c} \left(R_{ij} - \frac{c+1}{2}\right)^2 ,$$

$$T_{b,j} = \frac{1}{c} \sum_{i=1}^{c} R_{ij}$$

and R_{ij} denotes the within-block rank of the ij^{th} observation, average ranks being used for ties.

Koch and Sen showed that $\tilde{\omega}_b$ is asymptotically distributed as chi-square with $c - 1$ degrees of freedom. Accordingly the null hypothesis will be rejected at approximately the 5 % and 1 % levels of significance if $\tilde{\omega}_b > 7.815$ and $\tilde{\omega}_b > 11.34$ respectively.

The procedure adopted by Koch and Sen involves computing

(1) $$s_t^2 = \frac{1}{b} \sum_{j=1}^{c} \sum_{i=1}^{b} R_{ij}^{\;2} - \frac{bc}{4}(c+1)^2$$

(2) $$s_e^2 = \frac{cb\,\sigma_R^2}{b(c-1)}$$

(3) $$\tilde{\omega}_b = s_t^2 / s_e^2$$

The results obtained are $s_t^2 = 2.06$ and $s_e^2 = 1.65$ giving $\tilde{\omega}_b = 1.25$. Again there is no evidence at all to support the alternative hypothesis.

Test (iv) - the classical F-test

The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $F > 3.07$ and $F > 4.87$ respectively, the critical values being obtained from the F-distribution with (3,21) degrees of freedom.

Performing the usual analysis of variance calculations produces $F = 0.21$, a result which is quite consistent with the previous tests in not supporting the alternative hypothesis.

Example 2 - a situation in which the null hypothesis is rejected.

In the second experiment the liver of each animal was split into two parts, one of which was treated with radioactive iron and oxygen, and the other with radioactive iron and nitrogen. The data consist of the amounts of iron absorbed by the variously treated liver-halves. If matched pairs of animals are regarded as blocks and the combinations ethionine-oxygen (EO), ethionine-nitrogen (EN), control-oxygen (CO) and control-nitrogen (CN) are regarded as treatments then the hypothesis that neither diet nor gas has any effect may be tested. The data are as follows.

Pair	EO	EN	CO	CN
1	38.43	31.47	36.09	32.53
2	36.09	29.89	34.01	27.73
3	34.49	34.50	36.54	29.51
4	37.44	38.86	39.87	33.03
5	35.53	32.69	33.38	29.88
6	32.35	32.69	36.07	29.29
7	31.54	31.89	35.88	31.53
8	33.37	33.26	34.17	30.16

The hypotheses under investigation are

H_0 : the different diets have no effect

H_1 : the different diets do have some effect .

The table of within-block rankings for the data is given below.

Table of Ranks

Pair	EO	EN	CO	CN
1	4	1	3	2
2	4	2	3	1
3	2	3	4	1
4	2	3	4	1
5	4	2	3	1
6	2	3	4	1
7	2	3	4	1
8	3	2	4	1
Rank sums	23	19	29	9

Tests (i) - the match tests

For the M1 test, the null hypothesis will be rejected at the 5 % and 1 % levels of significance if $M1 \geqslant 40$ and $M1 \geqslant 45$, respectively; while for the M2 test rejection at the same levels of significance will occur if $M2 \geqslant 57$ and $M2 \geqslant 60.5$.

As before, comparing the ranks in the various blocks produces tables of matches and near-matches.

Table of Matches for M1

$m_{1.}$	$m_{2.}$	$m_{3.}$	$m_{4.}$	$m_{5.}$	$m_{6.}$	$m_{7.}$
4	10	15	11	4	6	2

Hence M1 = 52, a result which strongly supports the alternative hypothesis.

Table of Contributions for M2 from Near-matches

$m^*_{1.}$	$m^*_{2.}$	$m^*_{3.}$	$m^*_{4.}$	$m^*_{5.}$	$m^*_{6.}$	$m^*_{7.}$
8	5	1	2	3	1	1

Hence M2 = 52 + 21 = 73 which also provides strong evidence to support the alternative hypothesis.

Test (ii) - Friedman's χ^2_r - test

The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $\chi^2_r \geqslant 7.65$ and $\chi^2_r \geqslant 10.50$ respectively.

With the above data we obtain

$$\chi_r^2 = \frac{12}{160} \left(23^2 + 19^2 + 29^2 + 9^2\right) - 120$$

$$= 15.9 \ .$$

Clearly, this result provides strong evidence to support the alternative hypothesis.

Test (iii) - Koch and Sen's test

As there are no ties in the data, the test becomes identical to Friedman's test.

Test (iv) - the classical F-test

The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $F > 3.07$ and $F > 4.87$ respectively, the critical values being obtained from the F-distribution with (3,21) degrees of freedom.

Performing the usual analysis of variance calculations produces $F = 15.47$ which clearly strongly supports the alternative hypothesis.

It is quite obvious that the above examples are so extreme that any worthwhile test would return the correct verdict. The simulation studies will highlight the behaviour of the tests (excluding Koch and Sen's) in the region where the support for H_0 or H_1 is not so clear.

5. A Note on Situations with More Than One Observation per Cell

As mentioned in the introduction, some work has already been produced on the case of two-way layouts without interaction but with more than one observation per cell.

To analyse such situations using the matching principle we recommend replacing each cell of observations by some appropriate measure of location such as the mean or median. Thereafter the usual procedure mey be followed.

6. Moment Generating Function of M1

We shall see that the first three moments of M1 lead to interesting conjectures concerning its asymptotic behaviour. These are obtained by means of a type of moment generating function, the derivation of which is based on a modification of Battin's (1942) work on multiple matchings.

In order to explain the idea behind the generating function we shall consider the simple case where there are three treatments and two blocks.

Consider the function

$$\phi \equiv u^3 = \left\{ \sum_{i=1}^{3} \sum_{j=1}^{3} x_i y_j \, e^{\delta_{ij} \theta_{12}} \right\}^3$$

$$= \left\{ x_1 y_1 e^{\theta_{12}} + x_1 y_2 + x_1 y_3 + x_2 y_1 + x_2 y_2 e^{\theta_{12}} + x_2 y_3 + x_3 y_1 + x_3 y_2 + x_3 y_3 e^{\theta_{12}} \right\}^3$$

where

$$\delta_{ij} = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}$$

x_i and y_j relate to blocks 1 and 2 respectively and θ_{12} is a parameter associated with blocks 1 and 2. Since in this case we have only two blocks, θ_{12} is the only such parameter; in general there are similar parameters for all pairs of blocks.

A term such as $x_1 y_1 e^{\theta_{12}}$ corresponds to a match between the two blocks with both ranks equal to 1 whereas a term such as $x_1 y_3$ corresponds to no match between the blocks as the ranks are then 1 and 3. So in the expansion of $\phi \equiv u^3$ the coefficient of $x_1 x_2 x_3 y_1 y_2 y_3$ will contain information concerning the number of possible matches and their frequency. In the above function ϕ, the coefficient of $x_1 x_2 x_3 y_1 y_2 y_3$ is

$$1.e^{3\theta_{12}} + 3.e^{\theta_{12}} + 2.e^{0\theta_{12}} \quad .$$

The coefficients of θ_{12} give the values of the possible number of matches between blocks 1 and 2; these are 3, 1 and 0 respectively. The number of ways in which these values can occur, out of the total of $3! = 6$ possible arrangements, is given by 1, 3 and 2 from the appropriate coefficient of the exponentials. Of course, setting $\theta_{12} = 0$ produces the sum 1 + 3 + 2 which is the total number of possible arrangements.

If we now define the operator K by

$$K \text{ expression} = \text{coefficient of } x_1 x_2 x_3 y_1 y_2 y_3 \text{ in the expression,}$$

we may express a number of important quantities in a concise manner. For instance, the total number of possible arrangements is given by $K\,\phi\big|_{\theta_{12}=0}$. Also, the probability of obtaining exactly 3 matches (for example) is given by

$$\frac{\text{coefficient of } e^{3\theta_{12}} \text{ in } K\phi}{K\phi\big|_{\theta_{12}=0}}$$, under the assumption

resulting from the null hypothesis that all permutations are equally likely. The probabilities of obtaining exactly 1 or 0 matches may be similarly written.

If we now recall from section 2 that m_{12} represents the number of matches between blocks 1 and 2 then

$$P(\,m_{12} = s\,) = \frac{\text{coefficient of } e^{s\theta_{12}} \text{ in } K\phi}{K\phi\big|_{\theta_{12}=0}}\,, \quad s \geqslant 0$$

and so

$$E(\,m_{12}\,) = \frac{K\dfrac{\partial\phi}{\partial\theta_{12}}\Big|_{\theta_{12}=0}}{K\phi\big|_{\theta_{12}=0}}$$

and, more generally,

$$E(\,m_{12}{}^{p}\,) = \frac{K\dfrac{\partial^{p}\phi}{\partial\theta_{12}^{p}}\Big|_{\theta_{12}=0}}{K\phi\big|_{\theta_{12}=0}}$$

We now proceed to obtain the mean, variance and the third moment of M1. In the first instance we consider the case of c treatments and just 3 blocks.

The function ϕ is now defined as

$$\phi \equiv u^c = \left\{ \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k \, e^{\delta_{ij}\theta_{12} + \delta_{ik}\theta_{13} + \delta_{jk}\theta_{23}} \right\}^c$$

The operator K is defined by

K expression = coefficient of $x_1 x_2 \ldots\ldots x_c y_1 y_2 \ldots\ldots y_c z_1 z_2 \ldots z_c$ in the expression.

Now $$K\phi\big|_{\underline{\theta} = \underline{0}} = K\left\{ \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k \right\}^c$$

$$= (c!)^3 ,$$

where $\underline{\theta} = \underline{0}$ denotes $\theta_{rs} = 0$ for all r, s.

Hence by a direct extension of the ideas presented above we have

$$E(m_{ij}^{\,p}) = \frac{K \dfrac{\partial^p \phi}{\partial \theta_{ij}^p}\Big|_{\underline{\theta} = \underline{0}}}{K\phi\big|_{\underline{\theta} = \underline{0}}}$$

$$= K \frac{\partial^p \phi}{\partial \theta_{ij}^p}\Big|_{\underline{\theta} = \underline{0}} \Big/ (c!)^3 \qquad \ldots\ldots (1),$$

where m_{ij} is the number of matches between blocks i and j.

The expected value of M1 is given by

$$E(\ M1\ ) = \sum_{1 \leq i < j \leq 3} \sum E(\ m_{ij}\ )$$

$$= 3E(\ m_{12}\ )$$ by virtue of the independence of the blocks.

From (1) the mean value of m_{12} is given by

$$E(\ m_{12}\ ) = K\frac{\partial \phi}{\partial \theta_{12}}\bigg|_{\underline{\theta} = \underline{0}} \bigg/ (\ c!\ )^3 \qquad \ldots\ldots (2)$$

Now

$$\frac{\partial \phi}{\partial \theta_{12}} = cu^{c-1}\left\{ \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k \delta_{ij}\, e^{\delta_{ij}\theta_{12} + \delta_{ik}\theta_{13} + \delta_{jk}\theta_{23}} \right\}$$

Hence $$\frac{\partial \phi}{\partial \theta_{12}}\bigg|_{\underline{\theta} = \underline{0}} = cu_o^{c-1}\left\{ \sum_{i=1}^{c} \sum_{k=1}^{c} x_i y_i z_k \right\},$$

where $$u_o = \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k\ .$$

So, $$K\frac{\partial \phi}{\partial \theta_{12}}\bigg|_{\underline{\theta} = \underline{0}} = c\ (c-1\ )!^3\ c^2 = (\ c!\ )^3\ .$$

Hence (2) gives $E(\ m_{12}\ ) = 1$ from which we have $E(\ M1\ ) = 3$.

To calculate the variance of M1 we require $E(\ M1^2\ )$, which is given by

$$E(\,M1^2\,) = \sum_{1\le i<j\le 3}\sum E(\,m_{ij}^{\;2}\,) + \sum_{\substack{1\le i,k<j,l\le 3\\(i,j)\ne(k,l)}}\sum\sum\sum E(\,m_{ij}m_{kl}\,)$$

$$= 3E(\,m_{12}^{\;2}\,) + 6E(\,m_{12}m_{13}\,)\,,$$

where

$$E(\,m_{12}^{\;2}\,) = K\,\frac{\partial^2\phi}{\partial\theta^2_{12}}\bigg|_{\underline{\theta}=\underline{0}}\bigg/(\,c!\,)^3$$

and

$$E(\,m_{12}m_{13}\,) = K\,\frac{\partial^2\phi}{\partial\theta_{12}\partial\theta_{13}}\bigg|_{\underline{\theta}=\underline{0}}\bigg/(\,c!\,)^3 .$$

Now

$$\frac{\partial^2\phi}{\partial\theta^2_{12}} = c(c-1)u^{c-2}\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c}x_iy_jz_k\delta_{ij}\,e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}^2$$

$$+\; cu^{c-1}\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c}x_iy_jz_k\delta^2_{ij}\,e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}.$$

Hence

$$\frac{\partial^2\phi}{\partial\theta^2_{12}}\bigg|_{\underline{\theta}=\underline{0}} = c(c-1)u_o^{c-2}\left\{\sum_{i=1}^{c}\sum_{k=1}^{c}x_iy_iz_k\right\}^2 + cu_o^{c-1}\sum_{i=1}^{c}\sum_{k=1}^{c}x_iy_iz_k$$

so that

$$K\,\frac{\partial^2\phi}{\partial\theta^2_{12}}\bigg|_{\underline{\theta}=\underline{0}} = c(c-1)\;(c-2)!^{\,3}\,c^2(c-1)^2 + c\;(c-1)!^{\,3}\,c^2$$

$$= 2(\,c!\,)^3 .$$

Thus $E(m_{12}^2) = 2 = E(m_{13}^2) = E(m_{23}^2)$.

Next,

$$\frac{\partial^2\phi}{\partial\theta_{12}\partial\theta_{13}} = c(c-1)u^{c-2}\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ik}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}} \cdot \sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ij}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}$$

$$+ cu^{c-1}\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ij}\delta_{ik}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}$$

Hence

$$\left.\frac{\partial^2\phi}{\partial\theta_{12}\partial\theta_{13}}\right|_{\underline{\theta}=\underline{0}} = c(c-1)u_o^{c-2}\sum_{i=1}^{c}\sum_{j=1}^{c} x_i y_j z_i \sum_{i=1}^{c}\sum_{k=1}^{c} x_i y_i z_k$$

$$+ cu_o^{c-1}\sum_{i=1}^{c} x_i y_i z_i$$

so that

$$K\left.\frac{\partial^2\phi}{\partial\theta_{12}\partial\theta_{13}}\right|_{\underline{\theta}=\underline{0}} = c(c-1)\,(c-2)!^{\,3}\, c(c-1) + c^2\,(c-1)!^{\,3}$$

$$= (c!)^3$$

Thus $E(m_{12}m_{13}) = 1$

By symmetry, then also $E(m_{12}m_{23}) = E(m_{13}m_{23}) = 1$.

Thus the variance of M1 is given by

$$\begin{aligned}\operatorname{var}(M1) &= E(M1^2) - E(M1)^2\\ &= 3E(m_{12}^2) + 6E(m_{12}m_{13}) - E(M1)^2\\ &= 3\end{aligned}$$

While we are discussing the case of three blocks it will be of interest to consider also the third moment of M1. Now

$$E(M1^3) = E\Big(\sum_{1\le i<j\le 3} \sum m_{ij}^3 + \sum_{1\le i,k<j,l\le 3} \sum \sum \sum m_{ij} m_{kl}^2$$

$$+ 6m_{12}m_{13}m_{23} \Big)$$

$$= 3E(m_{12}^3) + 18E(m_{12}m_{13}^2) + 6E(m_{12}m_{13}m_{23}),$$

where

$$E(m_{12}^3) = K \frac{\partial^3 \phi}{\partial \theta_{12}^3} \Big|_{\underline{\theta} = \underline{0}} \Big/ (c!)^3$$

$$E(m_{12}m_{13}^2) = K \frac{\partial^3 \phi}{\partial \theta_{12} \partial \theta_{13}^2} \Big|_{\underline{\theta} = \underline{0}} \Big/ (c!)^3$$

and $$E(m_{12}m_{13}m_{23}) = K \frac{\partial^3 \phi}{\partial \theta_{12} \partial \theta_{13} \partial \theta_{23}} \Big|_{\underline{\theta} = \underline{0}} \Big/ (c!)^3$$

Now

$$\frac{\partial^3 \phi}{\partial \theta_{12}^3} = c(c-1)(c-2)u^{c-3} \left\{ \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k \delta_{ij} e^{\delta_{ij}\theta_{12} + \delta_{ik}\theta_{13} + \delta_{jk}\theta_{23}} \right\}^3$$

$$+ 3c(c-1)u^{c-2} \left\{ \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k \delta_{ij}^2 e^{\delta_{ij}\theta_{12} + \delta_{ik}\theta_{13} + \delta_{jk}\theta_{23}} \right. .$$

$$\left. \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k \delta_{ij} e^{\delta_{ij}\theta_{12} + \delta_{ik}\theta_{13} + \delta_{jk}\theta_{23}} \right\}$$

$$+ cu^{c-1} \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k \delta_{ij}^3 e^{\delta_{ij}\theta_{12} + \delta_{ik}\theta_{13} + \delta_{jk}\theta_{23}}$$

Hence

$$\frac{\partial^3 \phi}{\partial \theta_{12}^3} \Big|_{\underline{\theta} = \underline{0}} = c(c-1)(c-2)u_o^{c-3} \left\{ \sum_{i=1}^{c} \sum_{k=1}^{c} x_i y_i z_k \right\}^3$$

$$+ 3c(c-1)u_o^{c-2} \left\{ \sum_{i=1}^{c} \sum_{k=1}^{c} x_i y_i z_k \right\}^2 + cu_o^{c-1} \sum_{i=1}^{c} \sum_{k=1}^{c} x_i y_i z_k$$

whence

$$K \frac{\partial^3 \phi}{\partial \theta_{12}^3} \Big|_{\underline{\theta} = \underline{0}} = 5(c!)^3$$

Thus $E(m_{12}^3) = 5.$

Next,

$$\frac{\partial^3 \phi}{\partial \theta_{12} \partial \theta_{13}^2} = c(c-1)(c-2)u^{c-3} \left[\sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k \delta_{ik} e^{\delta_{ij}\theta_{12} + \delta_{ik}\theta_{13} + \delta_{jk}\theta_{23}} \right]^2 \times$$

$$\left\{ \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k \delta_{ij} e^{\delta_{ij}\theta_{12} + \delta_{ik}\theta_{13} + \delta_{jk}\theta_{23}} \right\}$$

$$+ c(c-1)u^{c-2} \left\{ \sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} x_i y_j z_k \delta_{ik}^2 e^{\delta_{ij}\theta_{12} + \delta_{ik}\theta_{13} + \delta_{jk}\theta_{23}} \right\} .$$

$$\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ij}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}$$

$$+\ 2c(c-1)u^{c-2}\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ik}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}.$$

$$\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ij\ ik}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}$$

$$+\ cu^{c-1}\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ij}\delta_{ik}^{2}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}$$

Hence

$$\frac{\partial^3 \phi}{\partial\theta_{12}\partial\theta_{13}^2}\bigg|_{\underline{\theta}=\underline{0}} =$$

$$c(c-1)(c-2)u_o^{c-3}\left\{\sum_{i=1}^{c}\sum_{j=1}^{c} x_i y_j z_i\right\}^2 \sum_{i=1}^{c}\sum_{k=1}^{c} x_i y_i z_k$$

$$+\ c(c-1)u_o^{c-2}\sum_{i=1}^{c}\sum_{j=1}^{c} x_i y_j z_i \sum_{i=1}^{c}\sum_{k=1}^{c} x_i y_i z_k$$

$$+\ 2c(c-1)u_o^{c-2}\sum_{i=1}^{c}\sum_{j=1}^{c} x_i y_j z_i \sum_{i=1}^{c} x_i y_i z_i \ +\ cu_o^{c-1}\sum_{i=1}^{c} x_i y_i z_i$$

so that

$$K\frac{\partial^3 \phi}{\partial\theta_{12}\partial\theta_{13}^2}\bigg|_{\underline{\theta}=\underline{0}} = 2(c!)^3 .$$

Hence $E(m_{12}m_{13}^2) = 2$.

Finally,

$$\frac{\partial^3 \phi}{\partial \theta_{12}\partial\theta_{13}\partial\theta_{23}} =$$

$$c(c-1)(c-2)u^{c-3}\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{jk}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}.$$

$$\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ik}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}.$$

$$\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ij}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}$$

$$+ 3c(c-1)u^{c-2}\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ik}\delta_{jk}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}.$$

$$\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ij}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}$$

$$+ cu^{c-1}\left\{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c} x_i y_j z_k \delta_{ij}\delta_{ik}\delta_{jk}\, e^{\delta_{ij}\theta_{12}+\delta_{ik}\theta_{13}+\delta_{jk}\theta_{23}}\right\}$$

Hence

$$\left.\frac{\partial^3 \phi}{\partial \theta_{12}\partial\theta_{13}\partial\theta_{23}}\right|_{\underline{\theta} = \underline{0}} =$$

$$c(c-1)(c-2)u_o^{c-3} \sum_{i=1}^{c}\sum_{j=1}^{c} x_i y_j z_j \sum_{i=1}^{c}\sum_{j=1}^{c} x_i y_j z_i \sum_{i=1}^{c}\sum_{k=1}^{c} x_i y_i z_k$$

$$+ 3c(c-1)u_o^{c-2} \sum_{i=1}^{c} x_i y_i z_i \sum_{i=1}^{c} \sum_{k=1}^{c} x_i y_i z_k + cu_o^{c-1} \sum_{i=1}^{c} x_i y_i z_i$$

whence

$$K \frac{\partial^3 \phi}{\partial \theta_{12} \partial \theta_{13} \partial \theta_{23}} \Bigg|_{\underline{\theta} = \underline{0}} = \frac{3}{2} (c!)^3$$

Hence $E(m_{12} m_{13} m_{23}) = 3/2$.

So collecting together these results,

$$E(M1^3) = 3E(m_{12}^3) + 18E(m_{12} m_{13}^2) + 6E(m_{12} m_{13} m_{23})$$

$$= 60$$

and so $E\left[M1 - E(M1)\right]^3 = 6$.

It is interesting to observe that these 1st, 2nd and 3rd moments are exactly those of a Poisson distribution with mean 3 ($= b(b-1)/2$). To reinforce this observation we now consider the general case with b blocks.

Let the variable $x_{\alpha i_\alpha}$ ($\alpha = 1, 2, \ldots, b$) relate to the α^{th} block. It will be to our advantage to abbreviate the exponent of e; so we shall set

$$f(\delta ; \theta) = \sum_{p,q} \delta_{i_p i_q} \theta_{pq} .$$

Then as before we define the function ϕ by

$$\phi \equiv u^c = \left\{ \sum_{i_1=1}^{c} \sum_{i_2=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_2} \cdots x_{bi_b} e^{f(\delta ; \theta)} \right\}^c .$$

In the same way as before we find that

$$K\,\phi\Big|_{\underline{\theta}=\underline{0}} = (\,c!\,)^b\ .$$

Now $$E(\,M1\,) = \sum_{i=1}^{b-1}\ \sum_{j=i+1}^{b} E(\,m_{ij}\,) = \beta E(\,m_{12}\,)\ ,$$

where $\beta = b(b-1)/2$,

and, as before,

$$E(\,m_{12}\,) = K\frac{\partial\phi}{\partial\theta_{12}}\Big|_{\underline{\theta}=\underline{0}} \Big/ (\,c!\,)^b\ .$$

Now

$$\frac{\partial\phi}{\partial\theta_{12}} = cu^{c-1}\left\{\sum_{i_1=1}^{c}\ \sum_{i_2=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_2}\cdots x_{bi_b}\,\delta_{12}\,e^{f(\,\delta\,;\,\theta\,)}\right\}$$

Hence

$$\frac{\partial\phi}{\partial\theta_{12}}\Big|_{\underline{\theta}=\underline{0}} = cu_o^{c-1}\left\{\sum_{i_1=1}^{c}\ \sum_{i_3=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}\cdots x_{bi_b}\right\}$$

from which $$K\frac{\partial\phi}{\partial\theta_{12}}\Big|_{\underline{\theta}=\underline{0}} = c\,(c-1)!^{\,b}\,c^{b-1}$$

$$= (\,c!\,)^b\ .$$

Hence $E(\,m_{12}\,) = 1$ giving $E(\,M1\,) = \beta E(\,m_{12}\,) = \beta.$

For the variance of M1 we require

$$E(\,M1^2\,) = \sum\sum_{1\leq i<j\leq b} E(\,m_{ij}^{\;2}\,) + \sum\sum\sum\sum_{\substack{1\leq i,j<k,l\leq b\\(i,j)\neq(k,l)}} E(\,m_{ij}m_{kl}\,)$$

which becomes

$$E(\, M1^2\,) = \beta E(\, m_{12}^{\;2}\,) + \beta(\beta-1)E(\, m_{12}m_{13}\,)\,,$$

where $$E(\, m_{12}^{\;2}\,) = K \frac{\partial^2\phi}{\partial\theta_{12}^2}\bigg|_{\underline{\theta}=\underline{0}} \bigg/ (\, c!\,)^b$$

and $$E(\, m_{12}m_{13}\,) = K \frac{\partial^2\phi}{\partial\theta_{12}\partial\theta_{13}}\bigg|_{\underline{\theta}=\underline{0}} \bigg/ (\, c!\,)^b \,.$$

Now

$$\frac{\partial^2\phi}{\partial\theta_{12}^2} = c(c-1)u^{c-2}\left\{\sum_{i_1=1}^{c}\sum_{i_2=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_2}\cdots x_{bi_b}\delta_{i_1i_2}e^{f(\delta\,;\theta)}\right\}^2$$

$$+ cu^{c-1}\left\{\sum_{i_1=1}^{c}\sum_{i_2=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_2}\cdots x_{bi_b}\delta^{\,2}_{i_1i_2}e^{f(\delta\,;\theta)}\right\}$$

Hence

$$\frac{\partial^2\phi}{\partial\theta_{12}^2}\bigg|_{\underline{\theta}=\underline{0}} = c(c-1)u_o^{c-2}\left\{\sum_{i_1=1}^{c}\sum_{i_3=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}\cdots x_{bi_b}\right\}^2$$

$$+ cu_o^{c-1}\left\{\sum_{i_1=1}^{c}\sum_{i_3=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_3}\cdots x_{bi_b}\right\}$$

giving

$$K\frac{\partial^2\phi}{\partial\theta_{12}^2}\bigg|_{\underline{\theta}=\underline{0}} = c(c-1)\;(c-2)!^{\;b}\;c^{b-1}(c-1)^{b-1} + c\;(c-1)!^{\;b}c^{b-1}$$

$$= 2(\, c!\,)^b \,.$$

So $E(m_{12}^{\ 2}) = 2$.

Also

$$\frac{\partial^2\phi}{\partial\theta_{12}\partial\theta_{13}} = c(c-1)u^{c-2}\left\{\sum_{i_1=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b}\delta_{i_1i_3}\, e^{f(\delta;\theta)}\right\} \times$$

$$\left\{\sum_{i_1=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b}\delta_{i_1i_2}\, e^{f(\delta;\theta)}\right\}$$

$$+ \; cu^{c-1}\left\{\sum_{i_1=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b}\delta_{i_1i_2}\delta_{i_1i_3}\, e^{f(\delta;\theta)}\right\}$$

Hence

$$\left.\frac{\partial^2\phi}{\partial\theta_{12}\partial\theta_{13}}\right|_{\underline{\theta}=\underline{0}} = c(c-1)u_o^{c-2} \sum_{i_1=1}^{c} \sum_{i_2=1}^{c} \sum_{i_4=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1}x_{2i_2}x_{3i_1}\cdots x_{bi_b} \times$$

$$\sum_{i_1=1}^{c} \sum_{i_3=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_3} \cdots\cdots x_{bi_b}$$

$$+ \; cu_o^{c-1} \sum_{i_1=1}^{c} \sum_{i_4=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_1}x_{4i_4} \cdots x_{bi_b}$$

so that

$$K\left.\frac{\partial^2\phi}{\partial\theta_{12}\partial\theta_{13}}\right|_{\underline{\theta}=\underline{0}} = c(c-1) \;\; (c-2)!\; {}^bc^{b-2}(c-1)^b \; + \; c\,(c-1)!\; {}^bc^{b-2}$$

$$= (c!)^b \; .$$

Thus $E(m_{12}m_{13}) = 1$.

So $E(M1^2) = \beta E(m_{12}^2) + \beta(\beta-1)E(m_{12}m_{13})$

$= \beta(\beta+1)$.

Finally, $\mathrm{var}(M1) = \beta(\beta+1) - \beta^2 = \beta$.

In order to speculate on the asymptotic behaviour of the null distribution of M1 we further calculate its third moment. For this we require

$$E(M1^3) = \beta E(m_{12}^3) + 3\beta(\beta-1)E(m_{12}m_{13}^2)$$

$$+ 2\beta(b-2)E(m_{12}m_{13}m_{23}) + (\beta(\beta-1)(\beta-2) - 2\beta(b-2))E(m_{12}m_{13}m_{24})$$

To calculate $E(m_{12}^2)$ we require $K \left.\dfrac{\partial^3\phi}{\partial\theta_{12}^3}\right|_{\underline{\theta}=\underline{0}}$.

$$\frac{\partial^3\phi}{\partial\theta_{12}^3} = c(c-1)(c-2)u^{c-3}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b}\delta_{i_1 i_2}\, e^{f(\delta;\theta)}\right\}^3$$

$$+ 2c(c-1)u^{c-2}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b}\delta^2_{i_1 i_2}\, e^{f(\delta;\theta)}\right\} .$$

$$\left\{\sum_{i_1=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b}\delta_{i_1 i_2}\, e^{f(\delta;\theta)}\right\}$$

$$+ c(c-1)u^{c-2}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b}\delta^2_{i_1 i_2}\, e^{f(\delta;\theta)}\right\}^2$$

$$+ cu^{c-1}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b}\delta^3_{i_1 i_2}\, e^{f(\delta;\theta)}\right\}$$

from which

$$\left.\frac{\partial^3\phi}{\partial\theta_{12}^3}\right|_{\underline{\theta}=\underline{0}} = c(c-1)(c-2)u_o^{c-3}\left\{\sum_{i_1=1}^{c}\sum_{i_3=1}^{c}..\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_3}..x_{bi_b}\right\}^3$$

$$+\ 3c(c-1)u_o^{c-2}\left\{\sum_{i_1=1}^{c}\sum_{i_3=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_3}\cdots x_{bi_b}\right\}^2$$

$$+\ cu_o^{c-1}\sum_{i_1=1}^{c}\sum_{i_3=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_3}\cdots x_{bi_b}$$

thereby producing

$$K\left.\frac{\partial^3\phi}{\partial\theta_{12}^3}\right|_{\underline{\theta}=\underline{0}} = 5(\ c!\ )^b \quad .$$

Hence $E(\ m_{12}{}^3\ ) = 5$.

For $E(\ m_{12}m_{13}{}^2\ )$ we require $K\left.\dfrac{\partial^3\phi}{\partial\theta_{12}\partial\theta_{13}^2}\right|_{\underline{\theta}=\underline{0}}$.

Now $\dfrac{\partial^3\phi}{\partial\theta_{12}\partial\theta_{13}^2} =$

$$c(c-1)(c-2)u^{c-3}\left\{\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}\ \cdots\ x_{bi_b}\delta_{i_1i_3}\ e^{f(\delta\,;\theta)}\right\}^2 \times$$

$$\left\{\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}\cdots x_{bi_b}\delta_{i_1i_2}\ e^{f(\delta\,;\theta)}\right\}$$

$$+ \; c(c-1)u^{c-2}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b} \delta^2_{i_1 i_3} e^{f(\delta;\theta)}\right\}^2 \times$$

$$\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b} \delta_{i_1 i_2} e^{f(\delta;\theta)}\right\}$$

$$+2c(c-1)u^{c-2}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b} \delta_{i_1 i_3} e^{f(\delta;\theta)}\right\} \times$$

$$\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b} \delta_{i_1 i_2} \delta_{i\, i_3} e^{f(\delta;\theta)}\right\}$$

$$+ \; cu^{c-1}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots x_{bi_b} \delta_{i_1 i_2} \delta^2_{i\, i_3} e^{f(\delta;\theta)}\right\},$$

so that

$$\frac{\partial^3 \phi}{\partial \theta_{12} \partial \theta_{13}^2} \Bigg|_{\underline{\theta} = \underline{0}} =$$

$$c(c-1)(c-2)u_o^{c-3}\left\{\sum_{i_1=1}^{c} \sum_{i_2=1}^{c} \sum_{i_4=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_2} x_{3i_1} x_{4i_4} \cdots x_{bi_b}\right\}^2 \times$$

$$\left\{\sum_{i_1=1}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1} x_{3i_3} \cdots x_{bi_b}\right\}$$

$$+ \; c(c-1)u_o^{c-2}\left\{\sum_{i_1=1}^{c} \sum_{i_2=1}^{c} \sum_{i_4=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_2} x_{3i_3} \cdots x_{bi_b}\right\} \times$$

$$\left\{\sum_{i_1=1}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1} \cdots x_{bi_b}\right\}$$

$$+ 2c(c-1)u_o^{c-2}\left\{\sum_{i_1=1}^{c}\sum_{i_2=1}^{c}\sum_{i_4=1}^{c}\cdots\sum_{i_b=1}^{}x_{1i_1}x_{2i_2}x_{3i_1}\cdots x_{bi_b}\right\}\times$$

$$\left\{\sum_{i_1=1}^{c}\sum_{i_4=1}^{c}\cdots\cdots\sum_{i_b=1}^{c}x_{1i_1}x_{2i_1}x_{3i_1}\cdots\cdots x_{bi_b}\right\}$$

$$+ cu_o^{c-1}\left\{\sum_{i_1=1}^{c}\sum_{i_4=1}^{c}\cdots\sum_{i_b=1}^{c}x_{1i_1}x_{2i_1}x_{3i_1}\cdots x_{bi_b}\right\} .$$

from which $K\left.\dfrac{\partial^3\phi}{\partial\theta_{12}\partial\theta_{13}^2}\right|_{\underline{\theta}=\underline{0}} = 2(\,c!\,)^b$.

Hence $E(\,m_{12}m_{13}^{\;2}\,) = 2$.

For $E(\,m_{12}m_{13}m_{23}\,)$ we require $K\left.\dfrac{\partial^3\phi}{\partial\theta_{12}\partial\theta_{13}\partial\theta_{23}}\right|_{\underline{\theta}=\underline{0}}$.

Now $\dfrac{\partial^3\phi}{\partial\theta_{12}\partial\theta_{13}\partial\theta_{23}} =$

$$c(c-1)(c-2)u^{c-3}\left\{\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c}x_{1i_1}\cdots x_{bi_b}\delta_{i_1i_2}\,e^{f(\delta\,;\theta)}\right\}\times$$

$$\left\{\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c}x_{1i_1}\cdots x_{bi_b}\delta_{i_1i_3}\,e^{f(\delta\,;\theta)}\right\}\times$$

$$\left\{\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c}x_{1i_1}\cdots x_{bi_b}\delta_{i_2i_3}\,e^{f(\delta\,;\theta)}\right\}$$

$$+ c(c-1)u^{c-2}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b}\delta_{i_1i_3}\delta_{i_2i_3}\, e^{f(\delta\,;\theta)}\right\} \times$$

$$\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b}\delta_{i_1i_2}\, e^{f(\delta\,;\theta)}\right\}$$

$$+ c(c-1)u^{c-2}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b}\delta_{i_1i_3}\, e^{f(\delta\,;\theta)}\right\} \times$$

$$\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b}\delta_{i_1i_2 i_2i_3}\, e^{f(\delta\,;\theta)}\right\}$$

$$+ c(c-1)u^{c-2}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b\; i_2i_3}\, e^{f(\delta\,;\theta)}\right\}$$

$$\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b}\delta_{i_1i_2}\delta_{i_1i_3}\, e^{f(\delta\,;\theta)}\right\}$$

$$+ cu^{c-1}\left\{\sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b}\delta_{i_1i_2}\delta_{i_1i_3}\delta_{i_2i_3}\, e^{f(\delta\,;\theta)}\right\}$$

so that $\dfrac{\partial^3\phi}{\partial\theta_{12}\partial\theta_{13}\partial\theta_{23}}\Big|_{\underline{\theta}=\underline{0}}$ =

$$c(c-1)(c-2)u_o^{c-3}\left\{\sum_{i_1=1}^{c}\sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_3} \cdots x_{bi_b}\right\} \times$$

$$\left\{\sum_{i_1=1}^{c}\sum_{i_2=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} x_{1i_1}x_{2i_2}x_{3i_1} \cdots\cdots x_{bi_b}\right\} \times$$

$$\left\{\sum_{i_1=1}^{c}\sum_{i_2=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1}x_{2i_2}x_{3i_2} \cdots x_{bi_b}\right\}$$

$$+ c(c-1)u_o^{c-2}\left\{\sum_{i_1=1}^{c}\sum_{i_4=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_1}x_{4i_4}\cdots x_{bi_b}\right\} \times$$

$$\left\{\sum_{i_1=1}^{c}\sum_{i_3=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_3}\cdots x_{bi_b}\right\}$$

$$+ c(c-1)u_o^{c-2}\left\{\sum_{i_1=1}^{c}\sum_{i_4=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_1}x_{4i_4}\cdots x_{bi_b}\right\} \times$$

$$\left\{\sum_{i_1=1}^{c}\sum_{i_2=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_2}x_{3i_1}\cdots x_{bi_b}\right\}$$

$$+ c(c-1)u_o^{c-2}\left\{\sum_{i_1=1}^{c}\sum_{i_4=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_1}x_{4i_4}\cdots x_{bi_b}\right\} \times$$

$$\left\{\sum_{i_1=1}^{c}\sum_{i_2=1}^{c}\sum_{i_4=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_2}x_{3i_2}\cdots x_{bi_b}\right\} \times$$

$$+ cu_o^{c-1}\left\{\sum_{i_1=1}^{c}\sum_{i_4=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1}x_{3i_1}x_{4i_4}\cdots x_{bi_b}\right\}.$$

from which

$$K\frac{\partial^3\phi}{\partial\theta_{12}\partial\theta_{13}\partial\theta_{23}}\bigg|_{\underline{\theta}=\underline{0}} = \frac{c}{c-1}(c!)^b .$$

Hence $E(m_{12}m_{13}m_{23}) = \dfrac{c}{c-1}$.

For $E(m_{12}m_{13}m_{24})$ we need $K\dfrac{\partial^3\phi}{\partial\theta_{12}\partial\theta_{13}\partial\theta_{24}}\bigg|_{\underline{\theta}=\underline{0}}$.

Since the derivation of this is similar to that for $E(m_{12}m_{13}m_{23})$

we simply quote the result; $E(m_{12}m_{13}m_{24}) = 1$.

Combining the above results we obtain

$$E(M1^3) = 5\beta + 6\beta(\beta-1) + 2\beta(b-2)\frac{c}{c-1}$$

$$+ \beta(\beta-1)(\beta-2) - 2\beta(b-2) \quad .$$

Hence

$$E(M1 - \mu_{M1})^3 = \beta + \frac{2\beta(b-2)}{c-1} \quad .$$

Using the standard measure of skewness $\mu_3 / \mu_2^{3/2}$ we obtain

$$\text{skewness of M1} = \frac{1}{\beta^{\frac{1}{2}}}\left\{1 + \frac{2(b-2)}{c-1}\right\} \quad .$$

We can now comment on the asymptotic behaviour of the null distribution of M1. The first two moments are consistent with those of the Poisson distribution with mean β, as is the third moment as $c \rightarrow \infty$. Furthermore, as c, $b \rightarrow \infty$ the skewness of $M1 \rightarrow 0$. This affinity with the Poisson distribution will enable us to quote approximate critical values for various values of b, independent of the number of treatments c. The limiting value of the skewness, coupled with the Poissionian behaviour, is an indication of M1 having asymptotic normal properties.

7 . Moment Generating Function of M2

We now seek the moment generating function of M2 with a view to obtaining its first three moments, knowledge of which will again enable us to make speculations regarding the asymptotic behaviour of its distribution.

We will proceed directly to the general case of b blocks and c treatments. To take into account the "near-matches" we need the following definitions.

Define $$\delta^*_{ij} = \begin{cases} \frac{1}{2} & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $$m^*_{ij} = \tfrac{1}{2}\,(\text{number of near-matches between blocks } i \text{ and } j)$$

Thus $$M2 = M1 + M^* ,$$

where $$M^* = \sum_{i=1}^{b} \sum_{j=i+1}^{b-1} m^*_{ij} .$$

Hence $E(\ M2\ ) = E(\ M1\ ) + E(\ M^*\ )$

and $E(\ M2^2\ ) = E(\ M1^2\ ) + E(\ M^{*2}\ ) + 2E(\ M1.M^*\ )$,

where $E(\ M1\ )$ and $E(\ M1^2\ )$ are already known. In order to calculate the remaining terms we define a generating function ϕ^* by

$$\equiv u^{*c} = \left\{ \sum_{i_1=1}^{c} \sum_{i_2=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_2} \cdots x_{bi_b}\, e^{f(\delta,\delta^*;\theta,\theta^*)} \right\}^c$$

where $$f \equiv f(\delta,\delta^*;\theta,\theta^*) = \sum_{i_k<i_l} \delta_{i_k i_l}\,\theta_{kl} + \sum_{i_k<i_l} \delta^*_{i_k i_l}\,\theta^*_{kl} .$$

The operator K is defined and used in the same manner as before.

So we have immediately that $K\,\phi^{*}\big|_{\underline{\theta},\underline{\theta}^{*}=\underline{0}} = (\,c!\,)^{b}$.

Clearly

$$E(\,M^{*}\,) = \sum_{i=1}^{b-1}\ \sum_{j=i+1}^{b} m^{*}_{ij}$$

$$= \beta E(\,m^{*}_{12}\,) \quad , \text{ by symmetry,}$$

where

$$E(\,m^{*}_{12}\,) = K\frac{\partial\phi^{*}}{\partial\theta^{*}_{12}}\Bigg|_{\underline{\theta},\underline{\theta}^{*}=\underline{0}} \Bigg/ (\,c!\,)^{b} \quad .$$

Now $$\frac{\partial\phi^{*}}{\partial\theta^{*}_{12}} = cu^{*\,c-1}\left\{\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_2}\ \cdots\ x_{bi_b}\delta^{*}_{12}\,e^{f}\right\}$$

so that $$\frac{\partial\phi^{*}}{\partial\theta^{*}_{12}}\Bigg|_{\underline{\theta},\underline{\theta}=\underline{0}} =$$

$$\tfrac{1}{2}cu_{o}^{*\,c-1}\left\{\sum_{i_1=1}^{c-1}\ \sum_{i_3=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1+1}x_{3i_3}\cdots\ x_{bi_b}\right.$$

$$\left. + \sum_{i_1=2}^{c}\ \sum_{i_3=1}^{c}\cdots\sum_{i_b=1}^{c} x_{1i_1}x_{2i_1-1}x_{3i_3}\cdots\ x_{bi_b}\right\}$$

Thus $$K\frac{\partial\phi^{*}}{\partial\theta^{*}_{12}}\Bigg|_{\underline{\theta},\underline{\theta}=\underline{0}} = c\,(c-1)!^{\,b}(c-1)c^{b-2}$$

$$= \frac{(c-1)}{c}\ (\,c!\,)^{b}$$

Hence $E(\,m^{*}_{12}\,) = 1 - \frac{1}{c}$.

Since $E(M2) = E(M1) + E(M^*)$ we have

$$E(M2) = \beta + \beta(1 - \frac{1}{c})$$

$$= \beta(2 - \frac{1}{c}) \;.$$

For the variance of M2 we require $E(M^{*2})$ and $E(M1 . M^*)$. $E(M^{*2})$ is given by

$$E(M^{*2}) = \sum_{i=1}^{b-1} \sum_{j=i+1}^{b} E(m_{ij}^{*2}) + \sum_{i=1}^{b-1} \sum_{j=i+1}^{b} \sum_{\substack{k=i+1 \\ j \neq k}}^{b} E(m_{ij}^* m_{ik}^*)$$

$$= \beta E(m_{12}^{*2}) + \beta(\beta-1) E(m_{12}^* m_{13}^*) \quad \text{,by symmetry,}$$

where

$$E(m_{12}^{*2}) = K \frac{\partial^2 \phi^*}{\partial \theta_{12}^{*2}} \Big|_{\underline{\theta},\underline{\theta}^* = \underline{0}} \Big/ (c!)^b$$

and

$$E(m_{12}^* m_{13}^*) = K \frac{\partial^2 \phi^*}{\partial \theta_{12}^* \partial \theta_{13}^*} \Big|_{\underline{\theta},\underline{\theta}^* = \underline{0}} \Big/ (c!)^b \;.$$

Now $\dfrac{\partial^2 \phi^*}{\partial \theta_{12}^{*2}} =$

$$c(c-1)u^{*(c-2)} \left\{ \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b} \delta^*_{i_1 i_2} e^f \right\}^2$$

$$+ cu^{*(c-1)} \left\{ \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b} \delta^{*2}_{i_1 i_2} e^f \right\}$$

so that $\left.\dfrac{\partial^2 \phi^*}{\partial \theta_{12}^{*\,2}}\right|_{\underline{\theta},\underline{\theta}^* = \underline{0}} =$

$$\tfrac{1}{4} c(c-1) u_o^{*(c-2)} \left\{ \sum_{i_1=1}^{c-1} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1+1} x_{3i_3} \cdots\cdots x_{bi_b} \right.$$

$$\left. + \sum_{i_1=2}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1-1} x_{3i_3} \cdots\cdots x_{bi_b} \right\}^2$$

$$+ \tfrac{1}{4} c u_o^{*(c-1)} \left\{ \sum_{i_1=1}^{c-1} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1+1} x_{3i_3} \cdots\cdots x_{bi_b} \right.$$

$$\left. + \sum_{i_1=2}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1-1} x_{3i_3} \cdots\cdots x_{bi_b} \right\} .$$

Hence, after some simplification,

$$K \left.\frac{\partial^2 \phi^*}{\partial \theta_{12}^{*\,2}}\right|_{\underline{\theta},\underline{\theta}^* = \underline{0}} = \frac{(3c^2 - 9c + 8)}{2(c-1)} . \,(c!)^b$$

which gives

$$E(m_{12}^{*\,2}) = \frac{(3c^2 - 9c + 8)}{2c(c-1)} .$$

Now $\dfrac{\partial^2 \phi^*}{\partial \theta_{12}^* \partial \theta_{13}^*} =$

$$c(c-1) u^{*(c-2)} \left\{ \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b} \delta^*_{i_1 i_3} e^f \right\} .$$

$$\left\{ \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b} \delta^{*}_{i_1 i_2} e^{f} \right\}$$

$$+ \; cu^{*(c-1)} \left\{ \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} \cdots\cdots x_{bi_b} \delta^{*}_{i_1 i_2} \delta^{*}_{i_1 i_3} e^{f} \right\}$$

so that $$\left.\frac{\partial^2 \phi^*}{\partial \theta^*_{12} \partial \theta^*_{13}}\right|_{\underline{\theta}, \underline{\theta}^* = \underline{0}} =$$

$$\tfrac{1}{4} c(c-1) u_o^{*(c-2)} \left\{ \sum_{i_1=1}^{c-1} \sum_{i_3=1}^{c} \sum_{i_4=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{3i_1+1} x_{2i_2} \cdots\cdots x_{bi_b} \right.$$

$$\left. + \sum_{i_1=2}^{c} \sum_{i_3=1}^{c} \sum_{i_4=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{3i_1-1} x_{2i_2} \cdots\cdots x_{bi_b} \right\} \times$$

$$\left\{ \sum_{i_1=1}^{c-1} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1+1} x_{3i_3} \cdots\cdots x_{bi_b} \right.$$

$$\left. + \sum_{i_1=2}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1-1} x_{3i_3} \cdots\cdots x_{bi_b} \right\}$$

$$+ \tfrac{1}{4} cu_o^{*(c-1)} \left\{ \sum_{i_1=2}^{c-1} x_{1i_1} (x_{2i_1+1} x_{3i_1-1} + x_{2i_1-1} x_{3i_1+1}) \cdot \right.$$

$$\left(\sum_{i_4=1}^{c} \cdots \sum_{i_b=1}^{c} x_{4i_4} \cdots\cdots x_{bi_b} \right)$$

$$+ \sum_{i_1=1}^{c-1} x_{1i_1} x_{2i_1+1} x_{3i_1+1} \left(\sum_{i_4=1}^{c} \cdots \sum_{i_b=1}^{c} x \quad \cdots\cdots x_{bi_b} \right)$$

$$\left. + \sum_{i_1=2}^{c} x_{1i_1} x_{2i_1-1} x_{3i_1-1} \left(\sum_{i_4=1}^{c} \cdots \sum_{i_b=1}^{c} x \quad \cdots\cdots x_{bi_b} \right) \right\}$$

Hence, after some simplification,

$$K \frac{\partial^2 \phi^*}{\partial \theta^*_{12} \partial \theta^*_{13}} \Big|_{\underline{\theta}, \underline{\theta}^* = \underline{0}} = \frac{(c-1)^2}{c^2} \ (c!)^b$$

from which we have $E(m^*_{12} m_{13}) = \dfrac{(c-1)^2}{c^2}$

Thus $E(M^{*2}) = \dfrac{\beta(3c^2 - 9c + 8)}{2c(c-1)} + \dfrac{\beta(\beta-1)(c-1)^2}{c^2}$.

$$\text{Next, } E(M1 . M^*) = \sum_{i=1}^{b-1} \sum_{j=i+1}^{b} E(m_{ij} m^*_{ij})$$

$$+ \sum \sum_{1 \leq i,k < j,l \leq b} \sum \sum E(m_{ij} m^*_{kl})$$

$$= \beta E(m_{12} m^*_{12}) + \beta(\beta-1) E(m_{23} m^*_{12}) \text{ ,by symmetry,}$$

where

$$E(m_{12} m^*_{12}) = K \frac{\partial^2 \phi^*}{\partial \theta_{12} \partial \theta^*_{12}} \Big|_{\underline{\theta}, \underline{\theta}^* = \underline{0}} \Big/ (c!)^b$$

and

$$E(m_{23} m^*_{12}) = K \frac{\partial^2 \phi^*}{\partial \theta_{23} \partial \theta^*_{12}} \Big|_{\underline{\theta}, \underline{\theta}^* = \underline{0}} \Big/ (c!)^b .$$

Now $\dfrac{\partial^2 \phi^*}{\partial \theta^*_{12} \partial \theta_{12}} =$

$$c(c-1) u^{*(c-2)} \left\{ \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_2} \cdots\cdots x_{bi_b} \delta^*_{i_1 i_2} e^f \right\} \times$$

$$\left\{ \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_2} \cdots\cdots x_{bi_b} \delta_{i_1 i_2} e^{f} \right\}$$

$$+ c u^{*(c-1)} \left\{ \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_2} \cdots\cdots x_{bi_b} \delta^{*}_{i_1 i_2} \delta_{i_1 i_2} e^{f} \right\}$$

Thus $\dfrac{\partial^2 \phi^*}{\partial \theta^*_{12} \partial \theta_{12}} \Big|_{\underline{\theta}, \underline{\theta}^* = \underline{0}} =$

$$\tfrac{1}{2} c(c-1) u_o^{*(c-2)} \left\{ \sum_{i_1=1}^{c-1} \sum_{i_3=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1+1} x_{3i_3} \cdots\cdots x_{bi_b} \right.$$

$$\left. + \sum_{i_1=2}^{c} \sum_{i_3=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1-1} x_{3i_3} \cdots\cdots x_{bi_b} \right\} \times$$

$$\left\{ \sum_{i_1=1}^{c} \sum_{i_3=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} x_{1i_1} x_{2i_1} x_{3i_3} \cdots\cdots x_{bi_b} \right\},$$

since $\delta^{*}_{i_1 i_2} \delta_{i_1 i_2} = 0$.

Hence

$$K \frac{\partial^2 \phi^*}{\partial \theta^*_{12} \partial \theta_{12}} \Big|_{\underline{\theta}, \underline{\theta}^* = \underline{0}} = \left(1 - \frac{2}{c}\right) (c!)^b \;.,$$

after some simplification. From this we obtain

$$E(m_{12} m^*_{12}) = 1 - \frac{2}{c} \;.$$

For $E(m^*_{12} m_{23})$ we require $K \dfrac{\partial^2 \phi^*}{\partial \theta^*_{12} \partial \theta_{23}} \Big|_{\underline{\theta}, \underline{\theta}^* = \underline{0}} \quad (c!)^b$.

Sincs the derivation of this is similar to that for $E(m_{12} m^*_{12})$

we simply quote the result; $E(\ m^*_{12}m_{12}\ )\ =\ 1 - \frac{1}{c}$.

Combining the above results gives

$$E(\ M1.M^*\ )\ =\ \beta(1 - \frac{2}{c})\ +\ \beta(\beta - 1)(1 - \frac{1}{c})\ .$$

Hence we may now calculate $E(\ M2^2\ )$:

$$E(\ M2^2\ )\ =\ E(\ M1^2\ )\ +\ E(\ M^{*\,2}\ )\ +\ 2E(\ M1.M^*\ )$$

$$=\ \beta(\beta + 1)\ +\ \frac{\beta(3c^2 - 9c + 8)}{2c(c - 1)}\ +\ \frac{\beta(\beta - 1)(c - 1)^2}{c^2}$$

$$+\ 2\beta(1 - \frac{2}{c})\ +\ 2\beta(\beta - 1)(1 - \frac{1}{c})\ .$$

Finally we obtain the variance of M2 as

$$\mathrm{var}(\ M2\ )\ =\ \frac{\beta(3c^3 - 9c^2 + 6c + 2)}{2c^2(c - 1)}\ .$$

To aid our investigation of the asymptotic behaviour of M2 we shall also calculate the third moment of M2.

Clearly $E(\ M2^3\ )\ =\ E(\ M1^3\ ) + 3(\ E(\ M1^2.M^*\ ) + E(\ M1.M^*\ )\ )$

$$+\ E(\ M^{*\,3}\ )\ ,$$

where as before $M2\ =\ M1 + M^*$.

First we calculate $E(\ M^{*\,3}\ )$ where

$$E(\ M^{*\,3}\ )\ =\ \beta E(\ m^{*\,3}_{12}\ ) + 3\beta(\beta-1)E(\ m^*_{12}m^{*\,2}_{13}\ ) + 2\beta(b-2)E(\ m^*_{12}m^*_{13}m^*_{23})$$

$$+\ (\ \beta(\beta-1)(\beta-2)\ -\ 2\beta(b-2)\ )E(\ m^*_{12}m^*_{13}m_{24}\ )\ .$$

By performing similar computations as before we obtain

$$E(M^{*\,3}) = \frac{1}{8} \quad \frac{2\beta(11c^3 - 67c^2 + 148c - 122)}{c(c-1)(c-2)}$$

$$+ \quad 12\beta(\beta - 1)\frac{(3c^3 - 12c^2 + 17c - 8)}{c^2(c-1)}$$

$$+ \quad 4\beta(b - 2)\frac{(4c^4 - 20c^3 + 36c^2 - 34c + 26)}{c^2(c-1)^2}$$

$$+ \quad 8(\beta(\beta - 1)(\beta - 2) - 2\beta(b - 2))\frac{(c^3 - 3c^2 + 3c - 1)}{c^3} \quad .$$

Next $E(M1.M^{*\,2}) = \beta E(m_{12}m^*_{12}) + \beta(\beta-1)E(m_{12}m^{*\,2}_{13})$

$+ 2\beta(\beta-1)E(m_{12}m^*_{12}m^*_{13}) + 2\beta(b-2)E(m_{12}m^*_{12}m^*_{23})$

$+ (\beta(\beta-1)(\beta-2) - 2\beta(b-2))E(m_{12}m^*_{13}m^*_{24})$

for which we obtain

$$E(M1.M^{*\,2}) = \frac{1}{4} \quad \frac{2\beta(3c^2 - 14c + 19)}{c(c-1)}$$

$$+ \quad 2\beta(\beta - 1)\frac{(3c^2 - 9c + 8)}{c(c-1)} + \quad 8\beta(\beta - 1)\frac{(c^3 - 4c^2 + 5c - 2)}{c^2(c-1)}$$

$$+ \quad 4\beta(b - 2)\frac{(2c^4 - 7c^3 + 9c^2 - 6c + 4)}{c^2(c-1)^2}$$

$$+ \ 4(\ \beta(\beta - 1)(\beta - 2) - 2\beta(b - 2)\ )\frac{(c - 1)^2}{c^2} \quad .$$

Similarly, $E(\ M1^2.M^*\ ) \ = \ \beta E(\ m_{12}^2 m_{12}^* \ ) + \beta(\beta-1)E(\ m_{12}^2 m_{13}^* \ )$

$$+ \ 2\beta(\beta-1)E(\ m_{12}m_{13}m_{12}^* \ ) + 2\beta(b-2)E(\ m_{12}m_{23}m_{12}^* \ )$$

$$+ \ (\ \beta(\beta-1)(\beta-2) - 2\beta(b-2)\ )E(\ m_{13}m_{24}m_{12}^* \ )$$

for which we obtain

$$E(\ M1^2.M^*\ ) \ = \ \frac{1}{2} \quad 2\beta\frac{(2c - 5)}{c} + 4\beta(\beta - 1)\frac{(c - 2)}{c}$$

$$+ \ 4\beta(b - 2)\frac{(c - 2)}{c - 1} + 2(\beta(\beta - 1)(\beta - 2) - 2\beta(b - 2)\ )\frac{(c - 1)}{c} \quad .$$

Combining these results with those for $E(\ M1^3\ )$, we finally obtain

$$E(\ M2^3\ ) \ = \ \beta\frac{(5c^5 - 37c^4 + 88c^3 - 34c^2 - 72c - 16)}{4c^3(c - 1)(c - 2)}$$

$$+ \ \beta^2\frac{(18c^4 - 63c^3 + 63c^2 - 6c - 6)}{2c^3(c - 1)} + \beta^3\frac{(8c^3 - 12c^2 + 6c - 1)}{c^3}$$

$$+ \ \beta(b - 2)\frac{(10c^4 - 38c^3 + 30c^2 + 18c + 4)}{2c^3(c - 1)^2} \quad .$$

Using this result we find

$$E(\ M2 - \mu_{M2})^3 \ = \ \beta b\frac{(10c^4 - 38c^3 + 30c^2 + 18c + 4)}{2c^3(c-1)^2}$$

$$+ \ \beta\frac{(5c^6 - 82c^5 + 357c^4 - 546c^3 + 130c^2 + 184c + 48)}{4c^3(c-1)^2(c-2)}$$

We can now comment on the asymptotic behaviour of the distribution of M2. As $c \to \infty$ we see that

$$E(\ M2\ ) \longrightarrow 2\beta\ ,$$

$$\mathrm{var}(\ M2\ ) \longrightarrow 3\beta/2\ ,$$

$$E(\ M2 - E(\ M2\ )\ )^3 \longrightarrow 5\,\beta/4$$

and the skewness of M2 $\longrightarrow 5(\ 6\beta\ )^{-\frac{1}{2}} / 3$ which tends to zero as $b \to \infty$.

Since M2 is the sum of the b-1 dependent variables $m_{i.}$ (i = 1, 2, ... ,b-1) we may invoke a version of the central limit theorem given by Erdős and Renyi (1959) to show that as $b \to \infty$ the distribution of M2 is normal with mean 2β and variance $3\beta/2$.

Actually, examination of exact null distributions of M2 indicates that, for moderate values of b, a truncated normal distribution may be more appropriate. This is indeed the case as we shall see in section 9.

7. Upper Tail Probabilities for the Null Distribution of M1

Below we give the probabilities P(M1 ⩾ x) for c = 3, b = 3 to 9; c = 4, b = 3 to 5; c = 5, b = 3. These were derived by the enummeration of all possible arrays.

c = 3 b = 3

x	P(M1 ⩾ x)
0	1
2	.944444
3	.444444
5	.277778
9	.027778

c = 3 b = 4

x	P(M1 ⩾ x)
3	1
4	.833333
6	.666667
7	.305556
9	.138889
10	.101852
12	.060185
18	.004630

c = 3 b = 5

x	P(M1 ⩾ x)
6	1
7	.884259
9	.745370
10	.405864
12	.336420
13	.182099
15	.089506
18	.043210
22	.012346
30	.000772

c = 3 b = 6

x	P(M1 ⩾ x)
9	1
11	.980710
12	.772377
14	.664352
15	.479167
17	.340278
18	.166667
20	.144805
21	.098508
23	.048354
26	.025206
27	.013632
29	.009774
30	.003987
35	.002443
45	.000129

c = 3 b = 7

x	P(M1 ⩾ x)
15	1
16	.927984
18	.846965
19	.600909
21	.506387
22	.384859
24	.303841
25	.146305
27	.119299

x	P(M1 ⩾ x)
28	.090792
30	.063786
31	.040381
33	.026878
36	.012474
39	.007073
40	.004823
43	.002122
45	.000772
51	.000472
63	.000021

c = 3   b = 8

x	P(M1 ⩾ x)
21	1
22	.943987
24	.871971
25	.657422
27	.591407
28	.457376
30	.353852
31	.233825
33	.197817
34	.139603
36	.109596
37	.052333
39	.039731

x	P(M1 ⩾ x)
40	.039330
42	.031229
43	.018026
45	.012024
48	.007023
52	.003222
54	.001647
57	.001047
60	.000447
63	.000146
70	.000089
84	.000004

c = 3   b = 9

x	P(M1 ⩾ x)
27	1
29	.990665
30	.873638
32	.809123
33	.676343
35	.568318
36	.388277
38	.367947
39	.291430
41	.218813
42	.141395
44	.114389

x	P(M1 ⩾ x)
45	.083107
47	.075605
48	.043198
50	.034646
51	.027444
53	.016642
54	.013041
56	.012941
57	.008440
59	.006640
62	.003039
63	.002139
66	.001796
68	.000853
71	.000628
72	.000370
77	.000220
80	.000092
84	.000027
92	.000017
108	.000001

c = 4   b = 3

x	P(M1 ⩾ x)
0	1
1	.958333
2	.833333

x	P(M1 ⩾ x)
3	.583333
4	.277778
5	.199653
6	.074653
8	.032986
12	.001736

c = 4   b = 4

x	P(M1 ⩾ x)
0	1
2	.998264
3	.956597
4	.887153
5	.684028
6	.548611
7	.350694
8	.246528
9	.128689
10	.080078
12	.043620
13	.015842
15	.005425
16	.003111
18	.001808
24	.000072

c = 4   b = 5

x	P(M1 ⩾ x)
4	1
5	.989511
6	.954789
7	.911386
8	.769604
9	.654586
10	.496166
11	.402850
12	.250940
13	.194878
14	.126881
15	.089265
16	.053096
17	.037905
18	.023437
19	.016927
20	.008729
21	.006739
22	.005293
23	.002881
24	.001435
25	.001118
28	.000395
32	.000093
40	.000003

c = 5   b = 3

x	P(M1 ⩾ x)
0	1
1	.961667
2	.811667
3	.582500
4	.324167
5	.161667
6	.094167
7	.034444
8	.018819
9	.006319
11	.002153
15	.000069

8. Upper Tail Probabilities for the Null Distribution of M2

Below we give the probabilities P(M2 ⩾ x) for c = 3, b = 3 to 9; c = 4, b = 3 to 5; c = 5, b = 3. These were obtained by the enummeration of all the possible arrays. Note that for c = 3, M2 is always integral since for this case near-matches can only occur in pairs.

c = 3 b = 3

x	P(M2 ⩾ x)
3	1
4	.944444
5	.611111
7	.194444
9	.027778

c = 3 b = 4

x	P(M2 ⩾ x)
8	1
9	.777778
10	.555556
11	.291667
12	.180556
14	.069444
15	.041667
18	.004630

c = 3 b = 5

x	P(M2 ⩾ x)
14	1
15	.837963
16	.652778
17	.405864
18	.282407
19	.189815
20	.128086
22	.050926
24	.023920
26	.008488
30	.000772

c = 3 b = 6

x	P(M2 ⩾ x)
21	1
22	.980710
23	.841821
24	.579475
25	.533179
26	.324846
27	.216821
28	.170525
29	.116512
31	.068287
32	.037423
35	.014275
36	.008102
37	.005530
40	.001672
45	.000129

c = 3 b = 7

x	P(M2 ⩾ x)
31	1
32	.900977
33	.774949
34	.573903

x	P(M2 ≥ x)
35	.465878
36	.326346
37	.278335
38	.188314
39	.134302
40	.093043
41	.075039
42	.056134
43	.039330
45	.024477
46	.013975
47	.010374
51	.004072
53	.001222
57	.000322
63	.000021

c = 3 b = 8

x	P(M2 ≥ x)
42	1
43	.919982
44	.811957
45	.632916
46	.530893
47	.407865
48	.337849
49	.227824

x	P(M2 ≥ x)
50	.197817
51	.142404
52	.120399
53	.079615
54	.063611
55	.050208
56	.035405
57	.026402
58	.021201
59	.015800
60	.012999
61	.008798
62	.006798
64	.003597
65	.002797
68	.001197
69	.000947
70	.000547
72	.000261
77	.000061
84	.000004

c = 3 b = 9

x	P(M2 ≥ x)
54	1
55	.990665
56	.912647

x	P(M2 ≥ x)
57	.746859
58	.710851
59	.524808
60	.413783
61	.369448
62	.292630
63	.202609
64	.200609
65	.137294
66	.097685
67	.086883
68	.074880
69	.051625
70	.046824
71	.028220
72	.023418
73	.022268
74	.016267
75	.010866
76	.009065
77	.006515
78	.005314
79	.004971
80	.004286
82	.001821
85	.000920
86	.000749

x	P(M2⩾ x)
88	.000363
90	.000213
92	.000113
94	.000054
100	.000011
108	.000001

c = 4 b = 3

x	P(M2⩾ x)
3.0	1
3.5	.916667
4.0	.833333
4.5	.734375
5.0	.560764
5.5	.467014
6.0	.342014
6.5	.225694
7.0	.163194
8.0	.090278
10.0	.017361
12.0	.001736

c = 4 b = 4

x	P(M2⩾ x)
6.0	1
7.0	.998264
7.5	.973958
8.0	.963542
8.5	.856120
9.0	.774523
9.5	.668620
10.0	.585286
10.5	.470269
11.0	.416450
11.5	.318359
12.0	.250651
12.5	.190249
13.0	.155527
13.5	.102575
14.0	.079427
14.5	.054688
15.0	.046007
15.5	.027778
16.0	.022569
16.5	.013455
17.0	.011719
18.0	.006510
20.0	.001591
21.0	.000940
24.0	.000072

c = 4 b = 5

x	P(M2⩾ x)
13.0	1
13.5	.980107
14.0	.956597
14.5	.919343
15.0	.850622
15.5	.782986
16.0	.707031
16.5	.627068
17.0	.538695
17.5	.453698
18.0	.394381
18.5	.341514
19.0	.284849
19.5	.240240
20.0	.191653
20.5	.151777
21.0	.129232
21.5	.099633
22.0	.077570
22.5	.062470
23.0	.049449
23.5	.037815
24.0	.030581
24.5	.021192
25.0	.016731
25.5	.013295
26.0	.011125

x	P(M2 ⩾ x)	x	P(M2 ⩾ x)
26.5	.006634	8.0	.086528
27.0	.006152	8.5	.054861
27.5	.004162	9.0	.038194
28.0	.003801	9.5	.019028
28.5	.002279	10.0	.012361
29.0	.001917	11.0	.006528
30.0	.001435	13.0	.000903
32.0	.000440	15.0	.000069
34.0	.000139		
36.0	.000048		
40.0	.000003		

$c = 5 \quad b = 3$

x	P(M2 ⩾ x)
2.0	1
2.5	.994583
3.0	.975417
3.5	.921667
4.0	.848333
4.5	.739167
5.0	.611667
5.5	.492917
6.0	.380417
6.5	.278611
7.0	.191944
7.5	.137361

9. Approximate Critical Values for M1

By exploiting its near-Poissonian behaviour we can easily obtain approximate critical values of M1 that are independent of the number of treatments c.

The table below lists the 5 %, 1 % and 0.1 % approximate critical values. These values have been obtained from the Poisson distribution with mean $\beta = b(b - 1)/2$, together with the assumption that c is large.

b	Approximate Critical Values 5 %	1 %	0.1 %
3	7	9	11
4	11	13	16
5	16	19	22
6	23	26	29
7	30	33	37
8	38	42	47
9	47	52	57
10	57	62	68

The adequacy of these approximations may be judged by considering the case of c = 5 and b = 3. The true critical values (best conservative) for the 5 %, 1 % and 0.1 % significance levels are 7, 9 and 15 (though it should be noted that the last value has a probability of 0.000069 of occurring) while the appropriate approximate values are 7, 9 and 11.

An alternative method of deriving approximate critical values for M1 is by use of the normal distribution. As is well-known, for large values of the mean, the Poisson distribution can be approximated by a normal distribution which in this instance is $N(\beta, \beta)$. Thus for large values of β, approximate critical values of M1 may be obtained using the following table.

	Significance Level		
	5 %	1 %	0.1 %
Critical Value	$1.65\sqrt{\beta} + \beta + \frac{1}{2}$	$2.33\sqrt{\beta} + \beta + \frac{1}{2}$	$3.09\sqrt{\beta} + \beta + \frac{1}{2}$

To indicate the adequacy of these values consider the case of $c = 4$ and $b = 5$ (giving $\beta = 10$). The approximate critical values are 16, 18 and 20 at the 5 %, 1 % and 0.1 % levels compared with the true (best conservative) values of 16, 20 and 25 .

10. Approximate Critical Values for M2

In section 5 we concluded that as $b \longrightarrow \infty$ the distribution of M2 tends to normality. However for moderate values of b a truncated normal distribution is a more apt description of the distribution of M2 in view of the truncation brought about by the minimum value of M2.

Accordingly, approximate critical values for M2 have been derived from truncated normal distributions using a method credited to Fisher (1931). To implement the method it is necessary to know the truncation point T_b of the distribution which is, of course, the minimum value of M2. A recurrence relation for T_b was determined by examining the effect on the truncation point of increasing the number of blocks for various number of treatments. The relation is given by

$$T_b = T_{b-1} + \alpha(c - 1) + (b - 1) ,$$

where T_b is the truncation point of the distribution with b blocks and c treatments ($T_1 = 0$),

and α is the integer part of $(b - 1)/c$.

In order to judge the effectiveness of Fisher's method we calculated the approximate critical values for the known distribution of $c = 4$ and $b = 5$. The true (best conservative) critical values at the 5 % and 1 % significance levels are 23.0 and 26.5 respectively while the appropriate approximate values are 22.5 and 26.0 .

A table of approximate critical values for M2, based on the above method, is given overleaf.

Table of Approximate Critical Values for M2

c	b	Significance Level 5 %	1 %	0.1 %
3	10	85.5	91.5	98.5
4	6	33.0	36.0	40.0
	7	44.0	47.0	51.0
	8	57.0	60.5	64.5
	9	72.5	76.5	81.5
	10	89.5	94.5	99.5
5	4	15.0	16.5	18.5
	5	23.0	25.0	27.5
	6	33.5	36.0	39.0
	7	46.0	48.5	52.0
	8	59.0	63.0	67.0
	9	74.5	78.5	83.0
	10	91.5	96.0	101.0
6	4	15.5	17.5	19.5
	5	24.0	26.5	29.0
	6	34.0	36.5	39.5
	7	46.5	50.0	53.5
	8	60.5	64.5	68.5
	9	76.5	81.0	85.5
	10	94.0	99.0	104.0

12. General Description of the Simulation Studies.

We now provide some background information on the simulation studies in this and subsequent chapters.

In the general and ordered alternatives cases both linear and non-linear models were investigated. The two-way linear model (without interaction) has the form $X_{ij} = M + A_i + B_j + z_{ij}$ while the non-linear model used was basically of the form $X_{ij} = M + A_i + B_j z_{ij}$, where M represents the overall mean, A_i ($i = 1, 2, \ldots, b$) and B_j ($j = 1, 2, \ldots, c$) represent the main effects with $\Sigma A_i = \Sigma B_j = 0$ and z_{ij} is a random variable having some specified continuous distribution.

Five distributions of various shapes were selected. Thus it was hoped to produce valuable information regarding the behaviour of all of the tests under a variety of conditions, some of which in the case of the F-test are far removed from theoretical assumptions. All the distributions, apart from of course the Cauchy distribution, were constructed to have approximately the same variance so that the effect of difference in shape could be more fully observed. The actual distributions were as follows.

1. The normal distribution N(0,1).
2. The uniform distribution over (0, 3.5).
3. The Cauchy distribution

$$f(x) = \frac{2}{\pi(1 + 4x^2)} , \quad -\infty < x < \infty .$$

4. The exponential distribution

$$f(x) = e^{-x} , \quad x \geq 0 .$$

5. The double exponential distribution

$$f(x) = \tfrac{1}{4}e^{-2|x|}, \quad -\infty < x < \infty$$

Departures from the null hypothesis $H_0 : B_j = 0$ $(j = 1, 2, \dots, c)$ were obtained by varying the parameter θ over the range 0 to 1 in the model $X_{ij} = M + A_i + B_j\theta + z_{ij}$; thus when $\theta = 0$ the null hypothesis is valid, whilst $\theta = 1$ indicates that an alternative hypothesis is more appropriate. The powers of the tests in each situation were estimated from 4000 replications.

Not all the tests discussed in the various chapters have been used in the simulations; for example, we avoided the use of Hollander's (1967) test for ordered alternatives, Bhakpar and Gore's (1974) and Weber's (1974) tests for interactions. For these and other tests not included their use in simulations, as in practice, is limited by the non-availability of their exact null distributions.

A practical difficulty encountered when comparing the powers of tests with discrete-valued statistics is the general impossibility of achieving a specified significance level. For example, with $c = 4$ and $b = 4$ the tables in section 6 give

$$P(M1 \geq 10) = 0.080078$$
$$P(M1 \geq 12) = 0.043620,$$

so that to use 10 as the 5 % critical value would give far too large a probability of rejection while 12 would give a probability that is too small. To overcome this difficulty we set up a randomized test (see for example Lindgren (1968)) at the desired level. Thus suppose the desired level were 100α per cent and that

$$P(M1 \geqslant r) = p_1 > \alpha$$
$$P(M1 \geqslant r + 1) = p_2 < \alpha .$$

Then H_0 is rejected whenever $M1 \geqslant r + 1$ and is rejected with a probability $= (\alpha - p_2)/(p_1 - p_2)$ when $M1 = r$. The overall probability of rejection of H_0 is then exactly α. In our simulations, the number of rejections of H_0 is the number of occasions that $M1 > r + \lambda$ [the number of occasions that $M1 = r$]. The same procedure was adopted for all the other discrete-valued statistics.

With regard to the graphs there are two general points to observe. Firstly, in an attempt to represent the information as clearly as possible, two scales for the power were used; one for when the power did not exceed 0.6 and the other for when the power exceeded this value. Secondly, the smoothing of the graphs was performed by a standard procedure inherent in the Nottingham University software.

13. Comments and Results of the Simulations

The simulations were performed with four treatments and four blocks.

(i) Results from the linear model $X_{ij} = M + A_i + B_j + z_{ij}$.

Normal Distribution. As might be expected the F-test reigned supreme when subjected to the normal distribution. However, it is encouraging to see M2 performing almost as well as Friedman's test and even M1 gives quite a respectable account of itself.

Uniform Distribution. The best overall performer is the F-test. Note the behaviour of the tests in the region of $\theta = 0.25$; here, in both the 5 % and 1 % cases, the three nonparametric tests have superior performances to the F-test.

Cauchy Distribution. The poor performance of the F-test under the Cauchy distribution is no surprise. Not only does it achieve a low maximum power but it also exhibits extremely poor robustness properties. The best overall performance is produced by M2, closely followed by Friedman's test.

Double Exponential Distribution. Perhaps the notable feature here is the superior performance of M2, closely followed by Friedman's test and M1, over the range $0 \leqslant \theta \leqslant 0.5$. Looking at the 1 % case, we see that there is little to choose between the F, M2 and Friedman's tests.

Exponential Distribution. Not surprisingly, the F-test proved to be the worst performer while M2 and Friedman's tests are the best.

(ii) Results from the non-linear model $X_{ij} = (M + A_i + B_j)z_{ij}$.

Normal Distribution. Compared to the linear model all tests have a much reduced maximum power.

Uniform Distribution. Somewhat surprisingly, all the tests exhibited good robustness features and even the maximum power is reasonable.

Exponential Distribution. The F-test gave a poor robustness performance. Overall, Friedman's and the M2 tests are the best performers.

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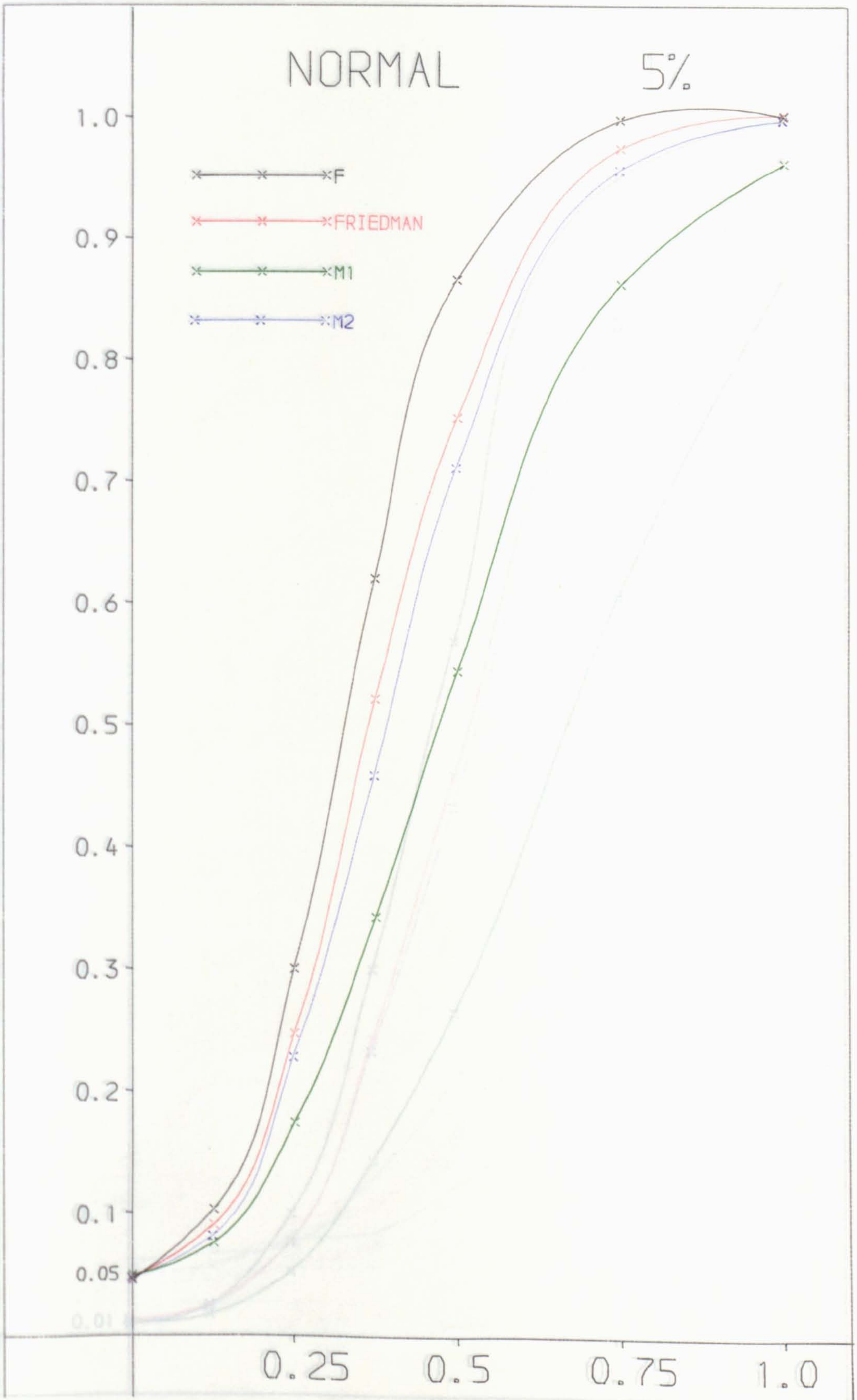

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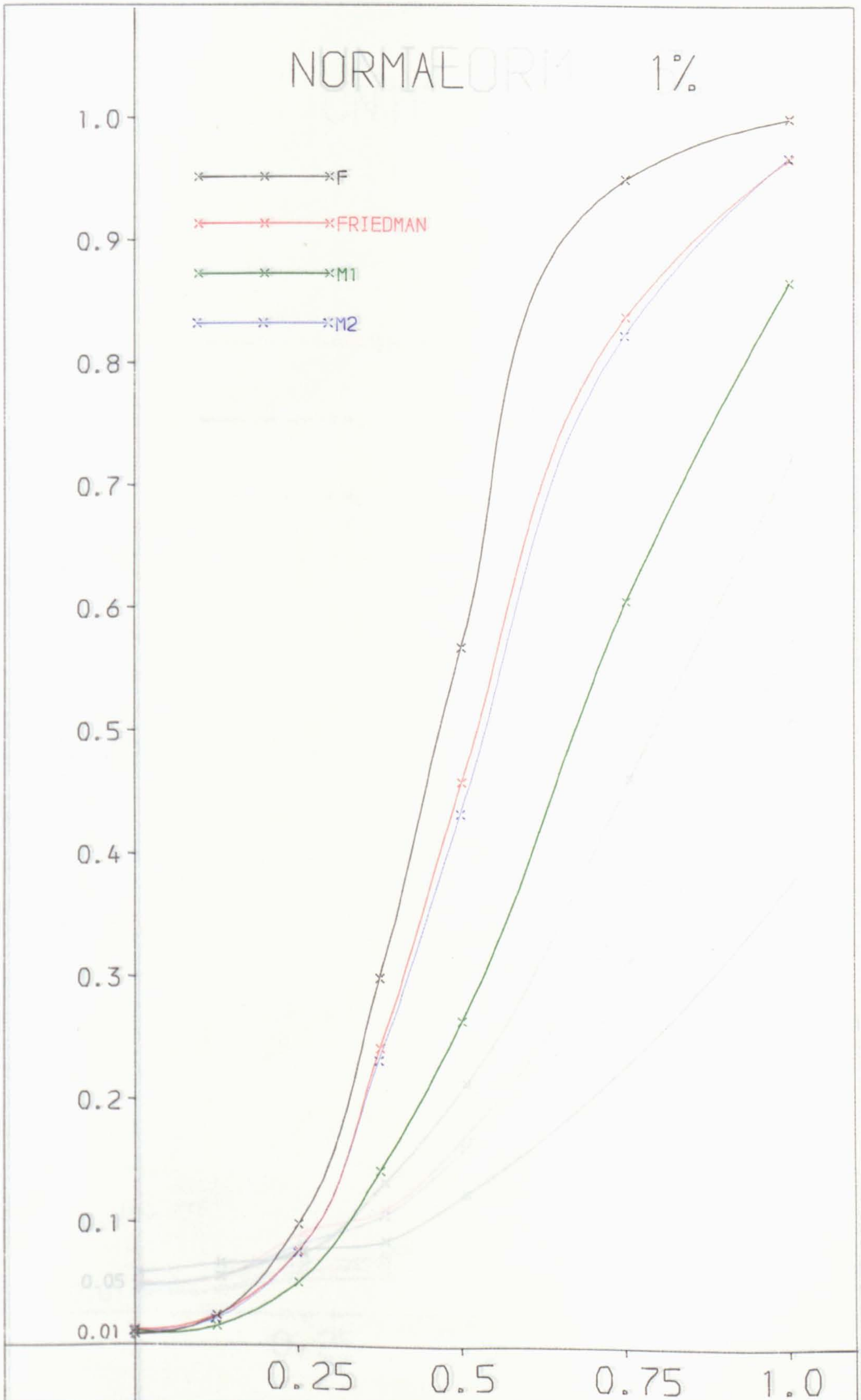

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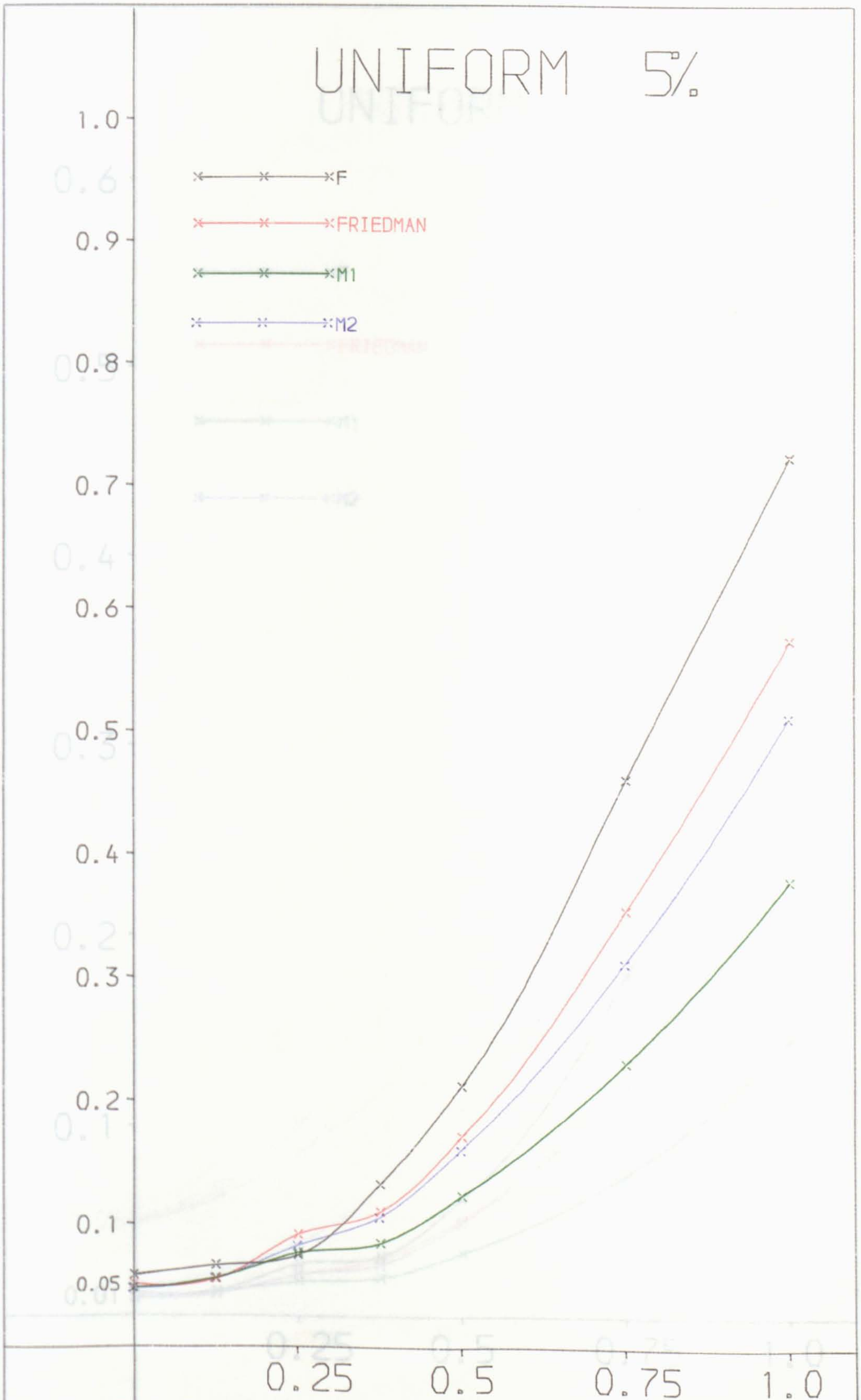

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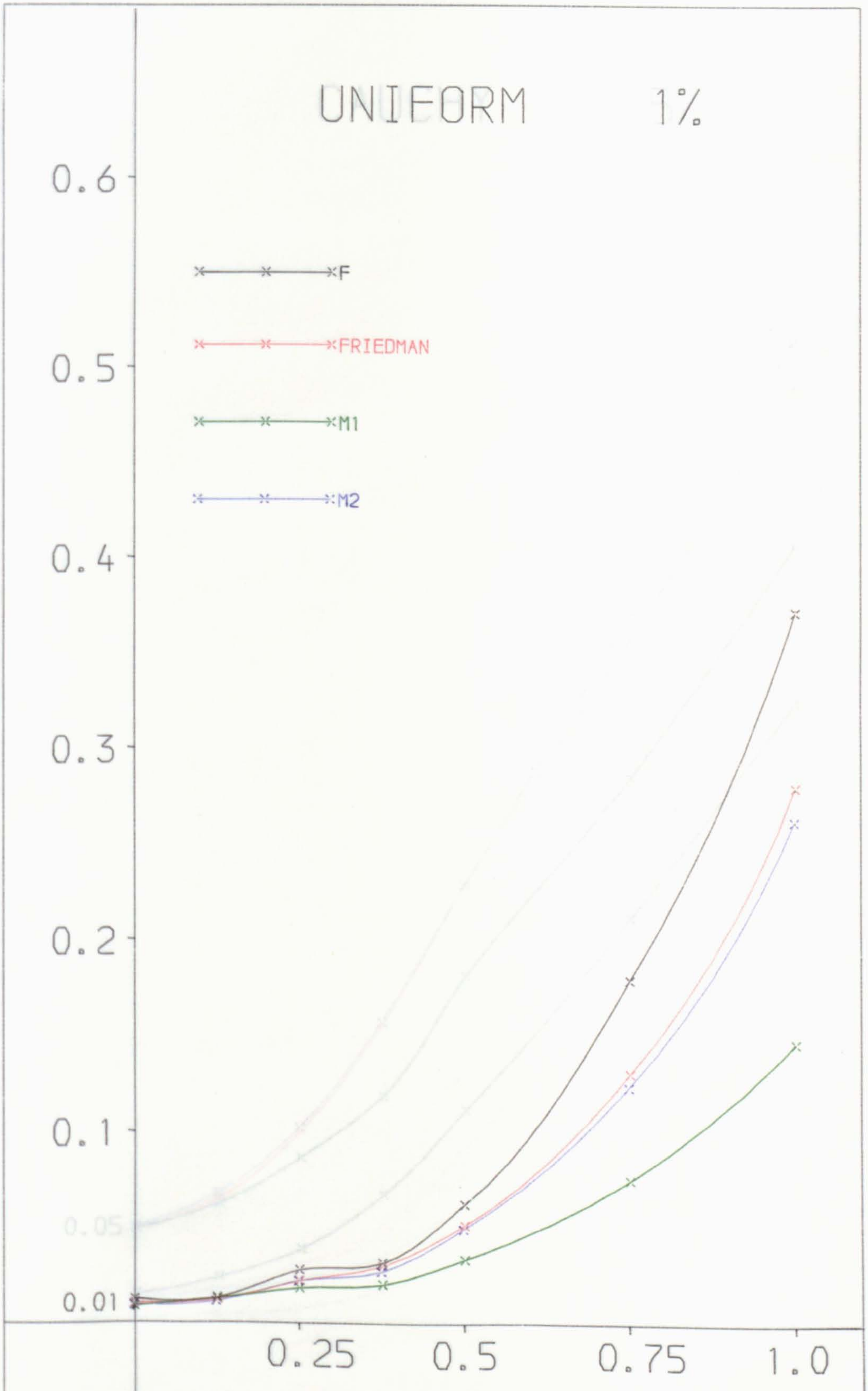

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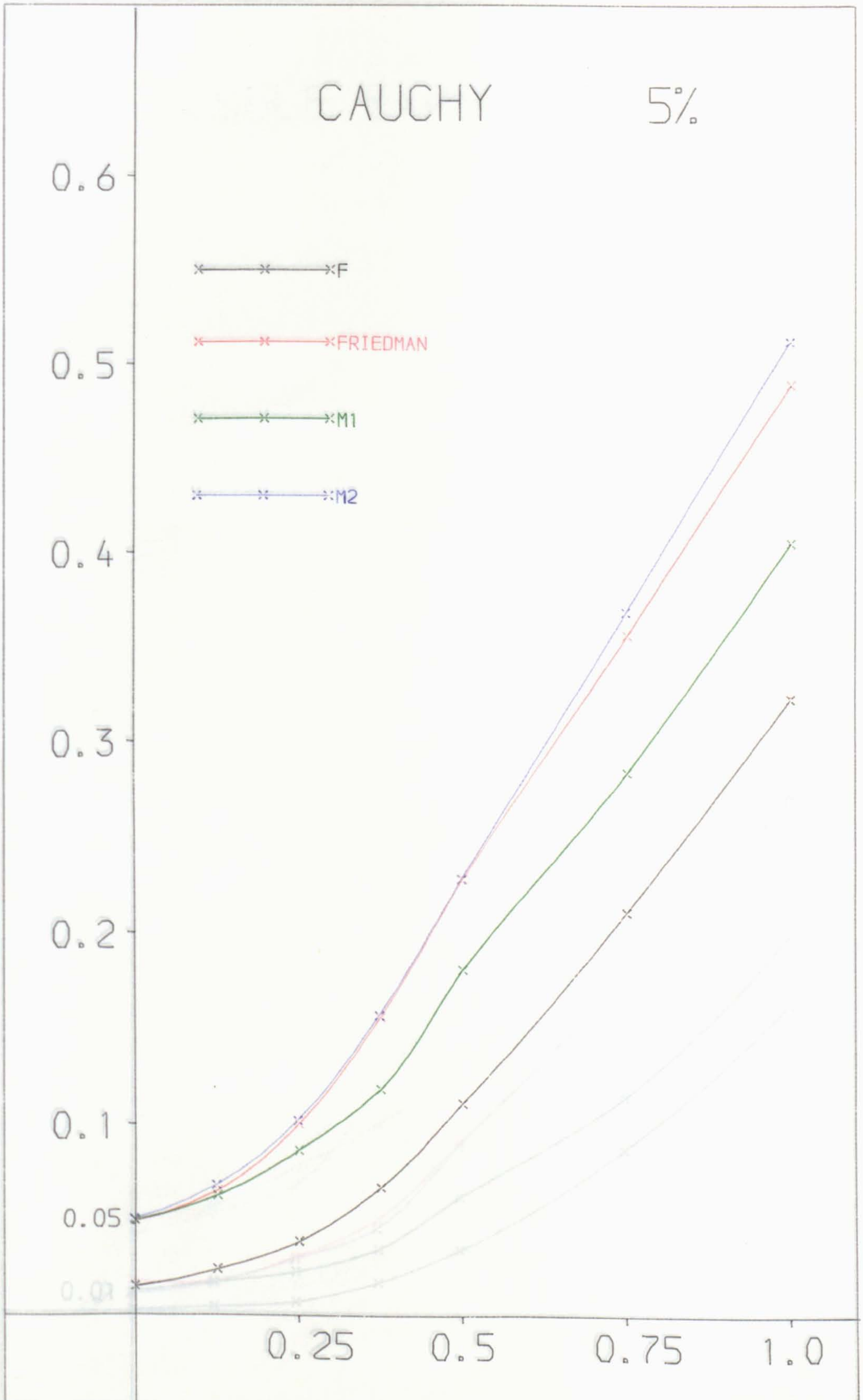

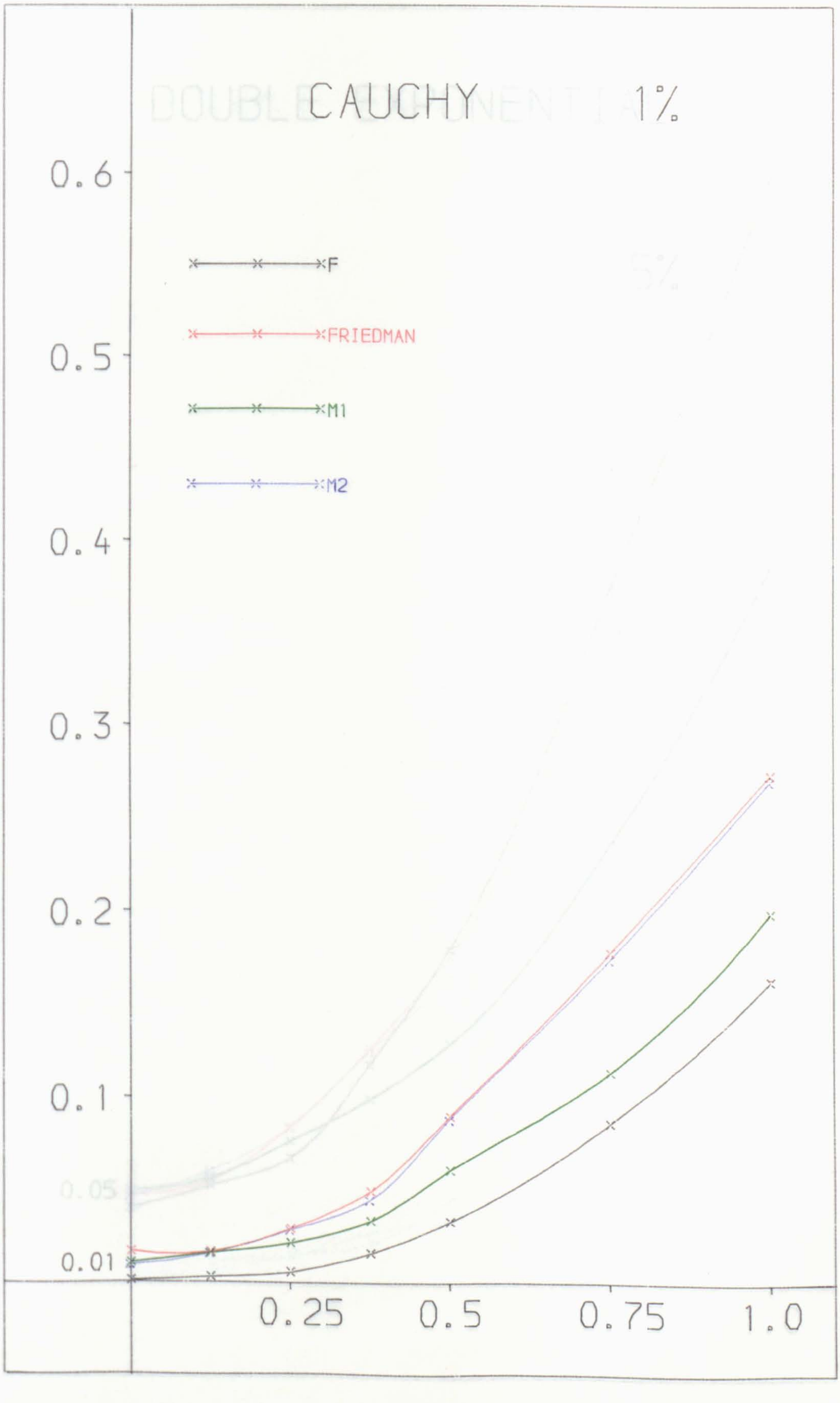
CAUCHY 1%
F
FRIEDMAN
M1
M2
0.6
0.5
0.4
0.3
0.2
0.1
0.01
0.25
0.5
0.75
1.0

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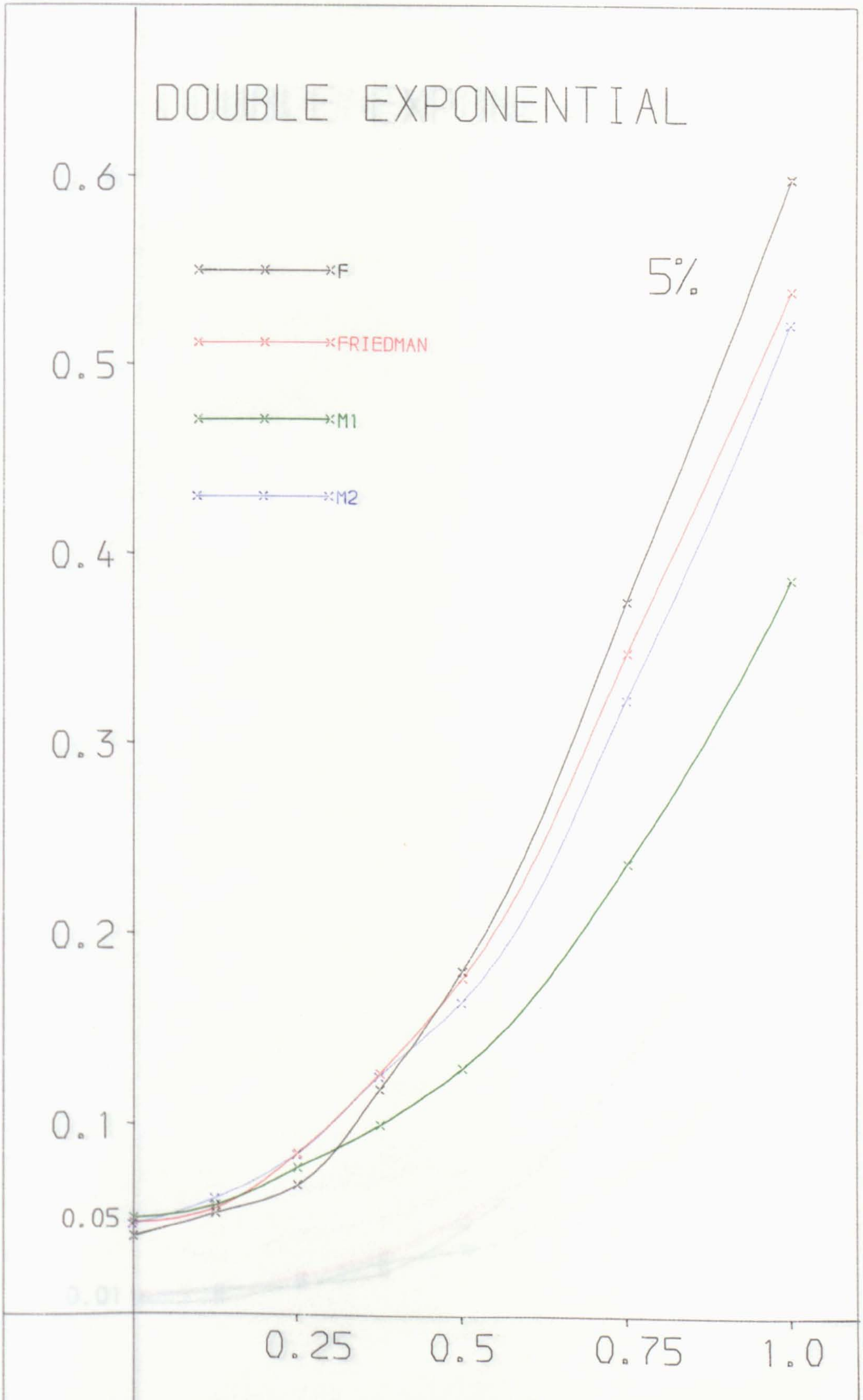
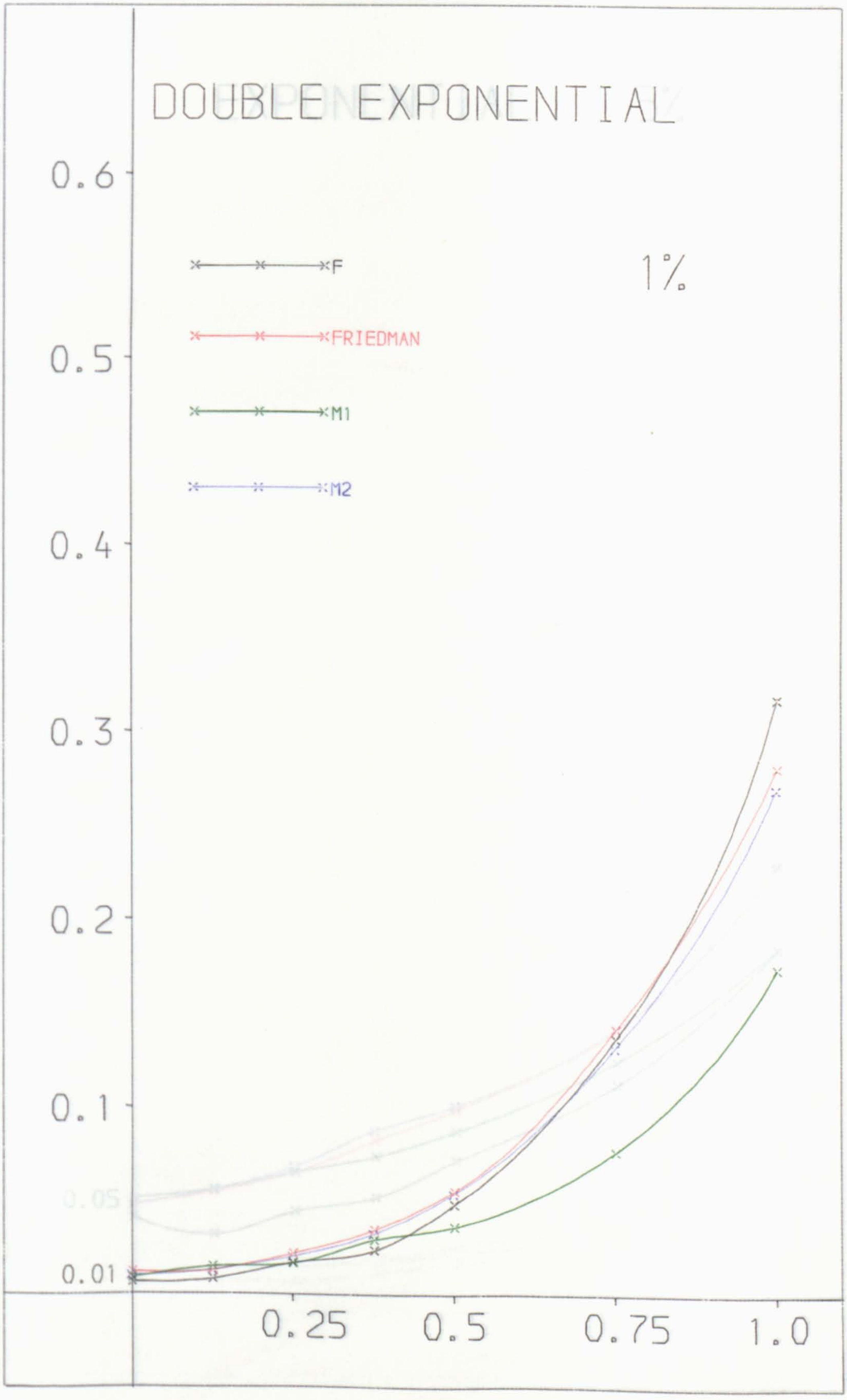
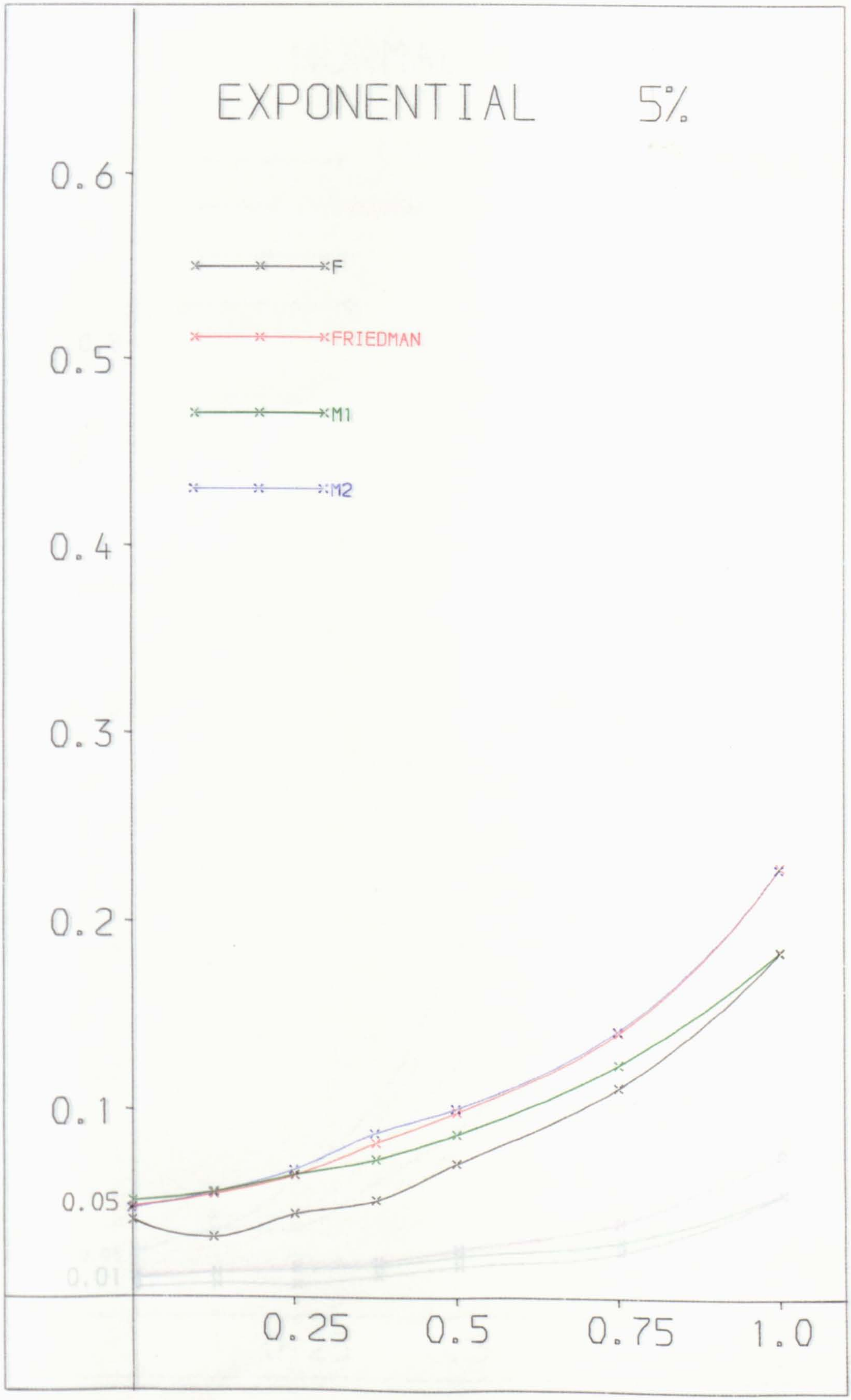
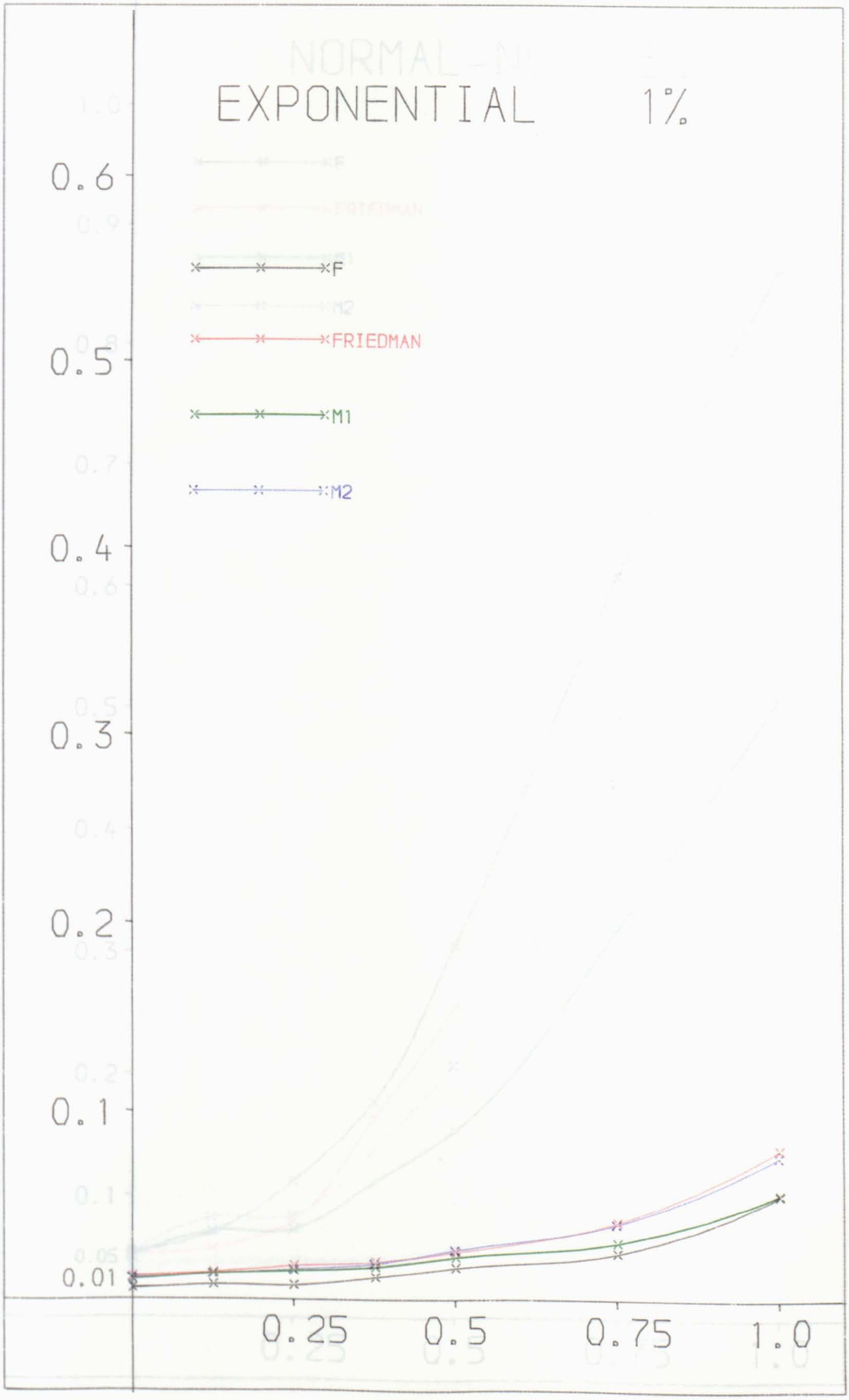
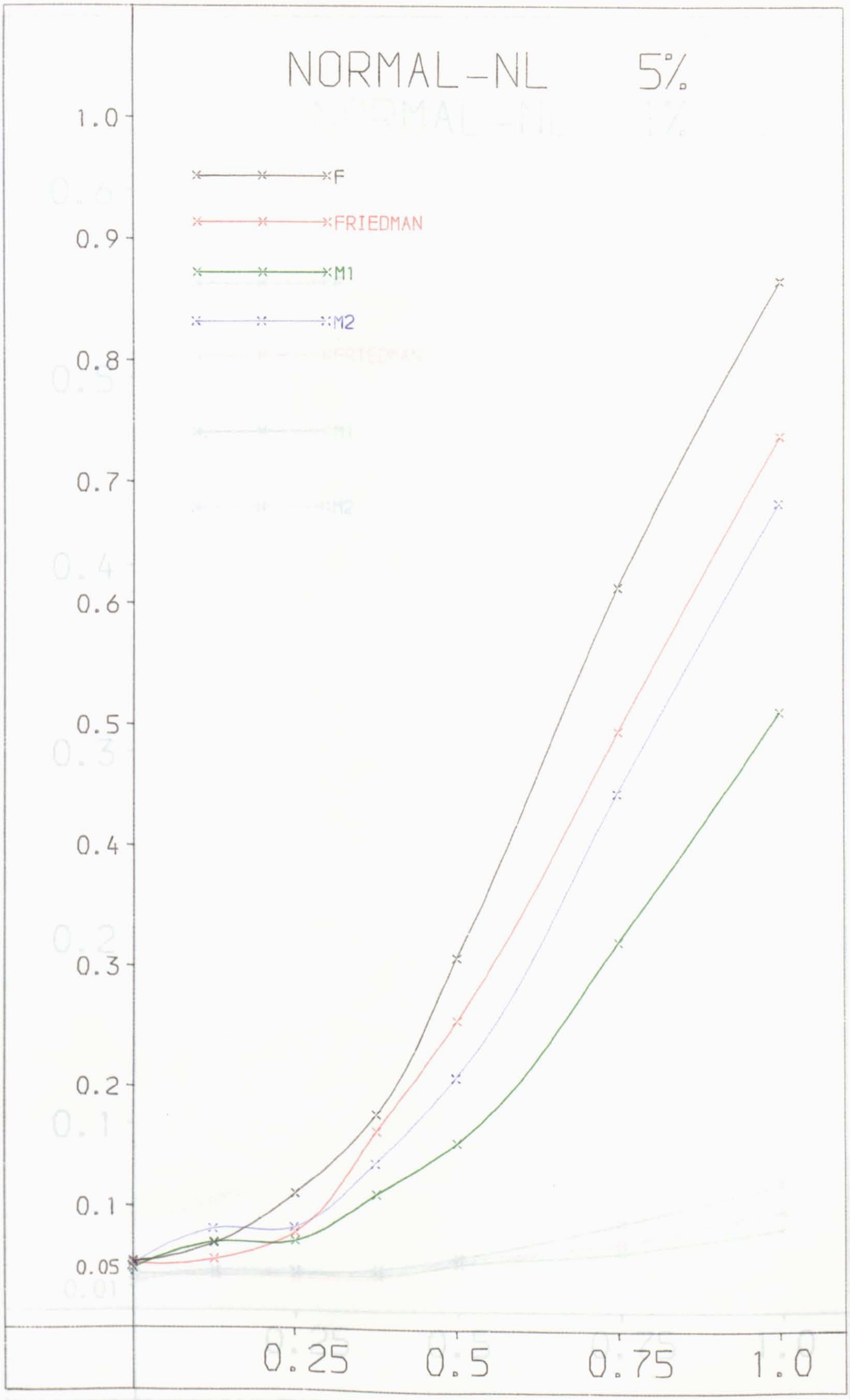
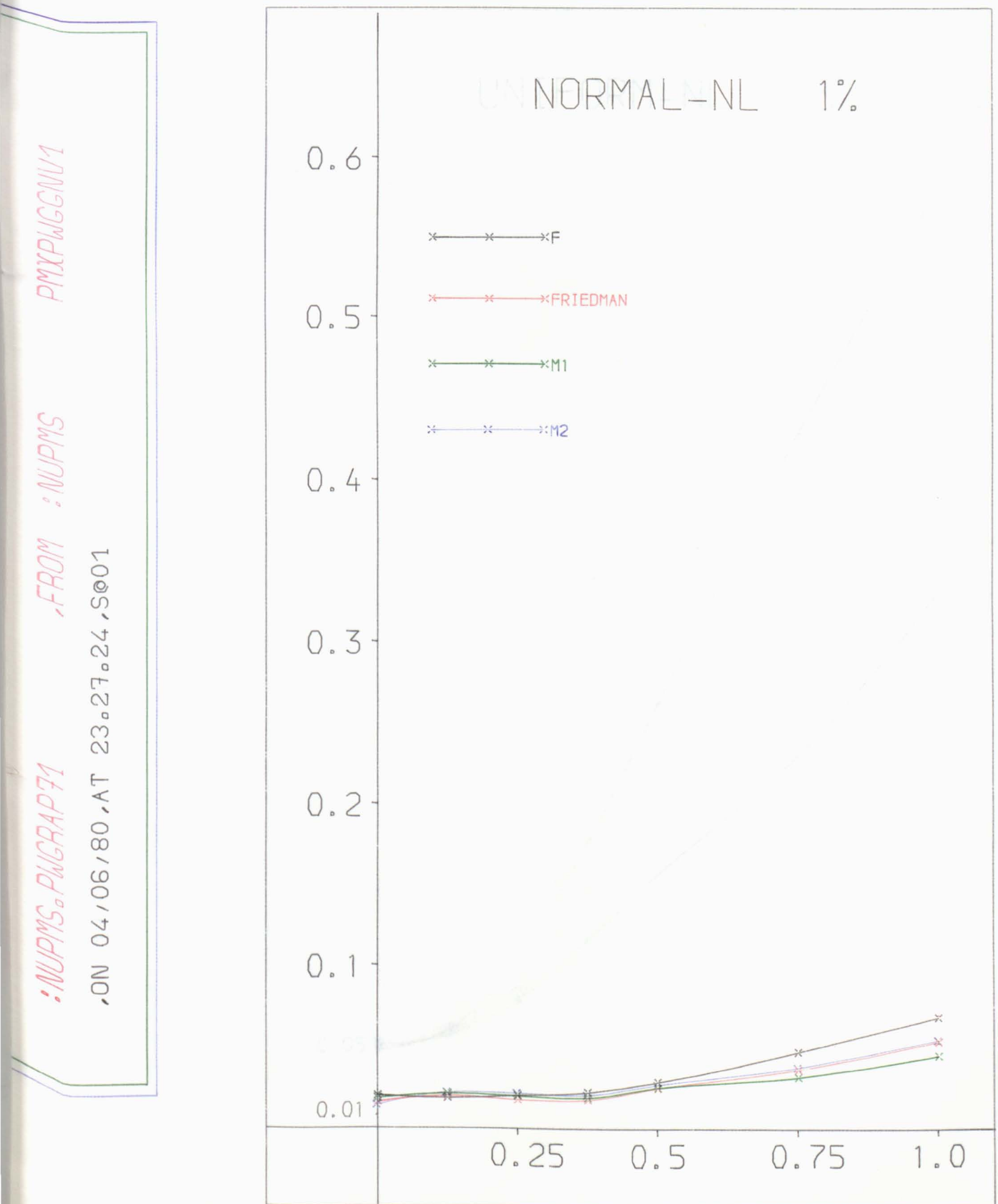



DOUBLE EXPONENTIAL
1%
F
FRIEDMAN
M1
M2
0.6
0.5
0.4
0.3
0.2
0.1
0.01
0.25
0.5
0.75
1.0


EXPONENTIAL 5%
0.6
0.5
0.4
0.3
0.2
0.1
0.05
F
FRIEDMAN
M1
M2
0.25
0.5
0.75
1.0


EXPONENTIAL 1%
F
FRIEDMAN
M1
M2
0.6
0.5
0.4
0.3
0.2
0.1
0.01
0.25
0.5
0.75
1.0


NORMAL-NL 5%
F
FRIEDMAN
M1
M2
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.05
0.25
0.5
0.75
1.0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NORMAL-NL 1%
F
FRIEDMAN
M1
M2
0.6
0.5
0.4
0.3
0.2
0.1
0.01
0.25
0.5
0.75
1.0

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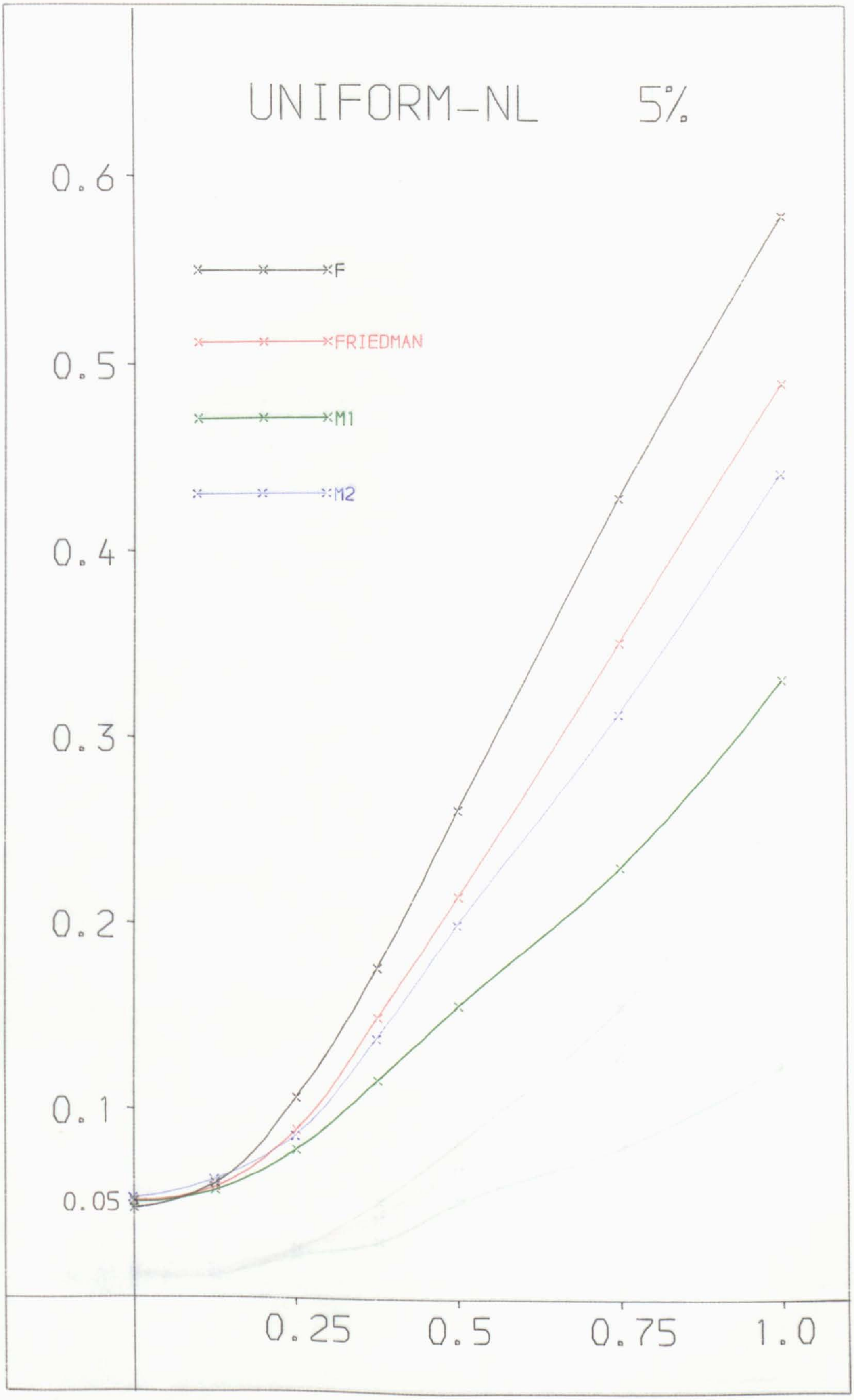
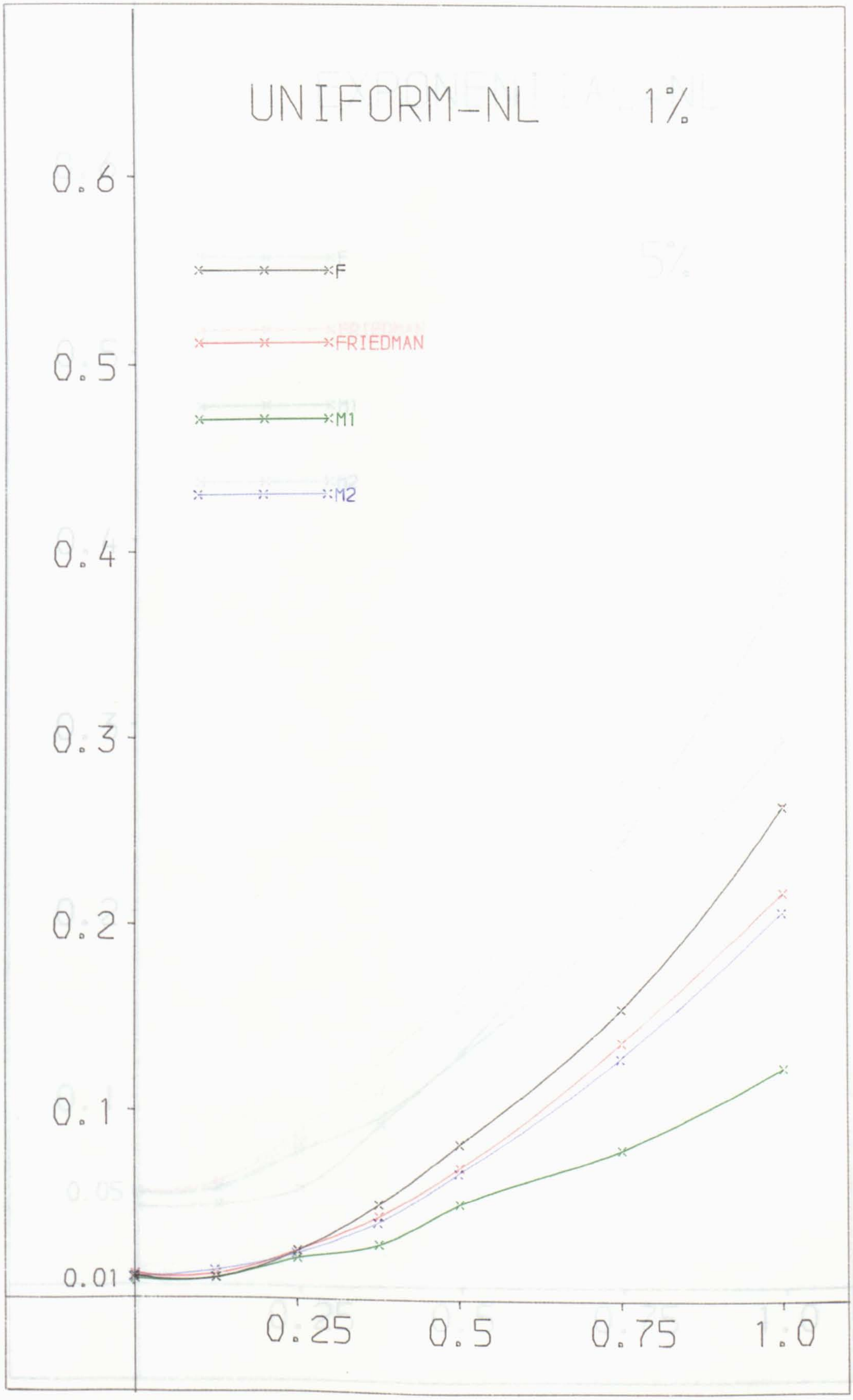
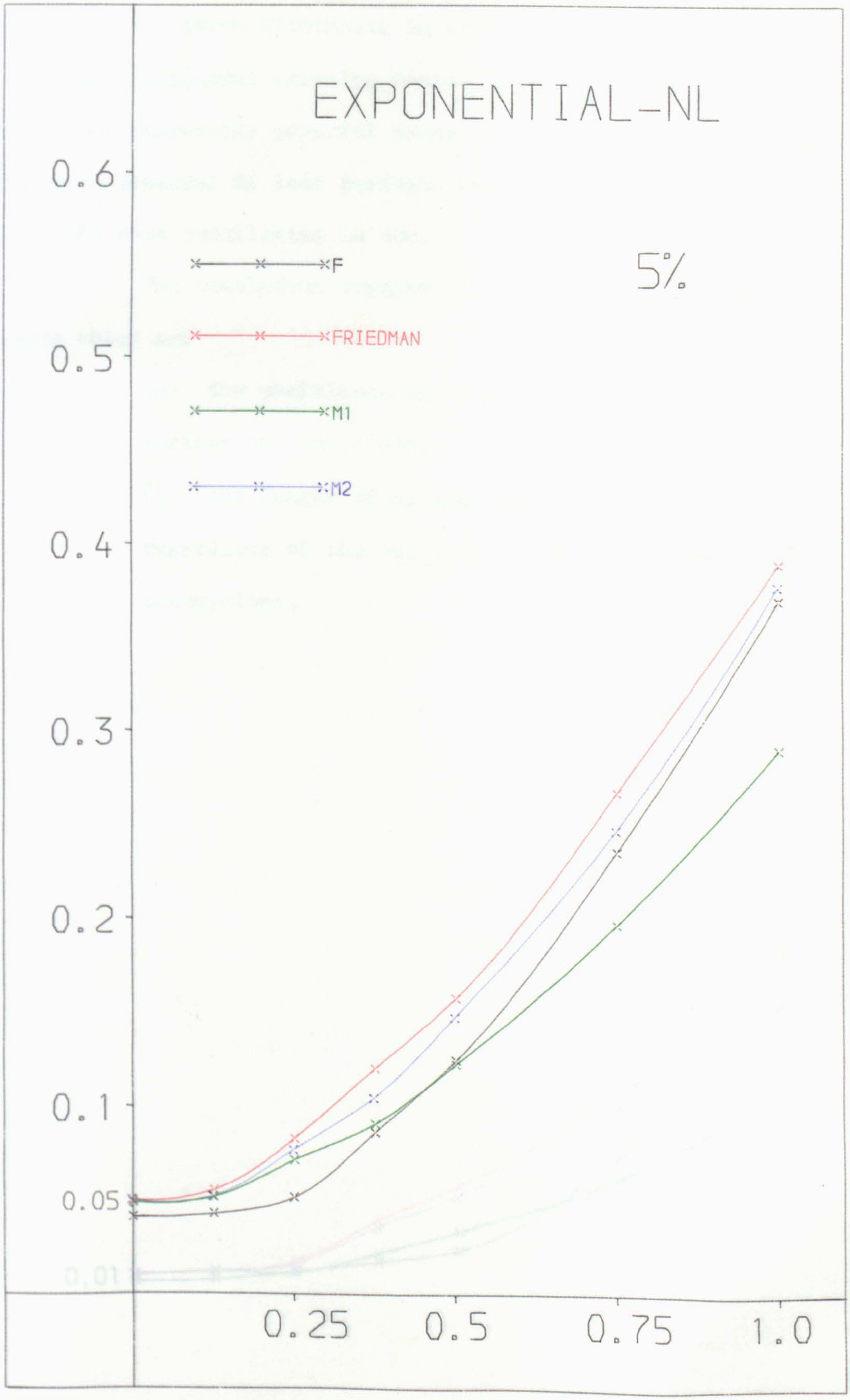



UNIFORM-NL 1%
F
FRIEDMAN
M1
M2
0.6
0.5
0.4
0.3
0.2
0.1
0.01
0.25
0.5
0.75
1.0


EXPONENTIAL-NL
5%
F
FRIEDMAN
M1
M2
0.6
0.5
0.4
0.3
0.2
0.1
0.05
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1.0

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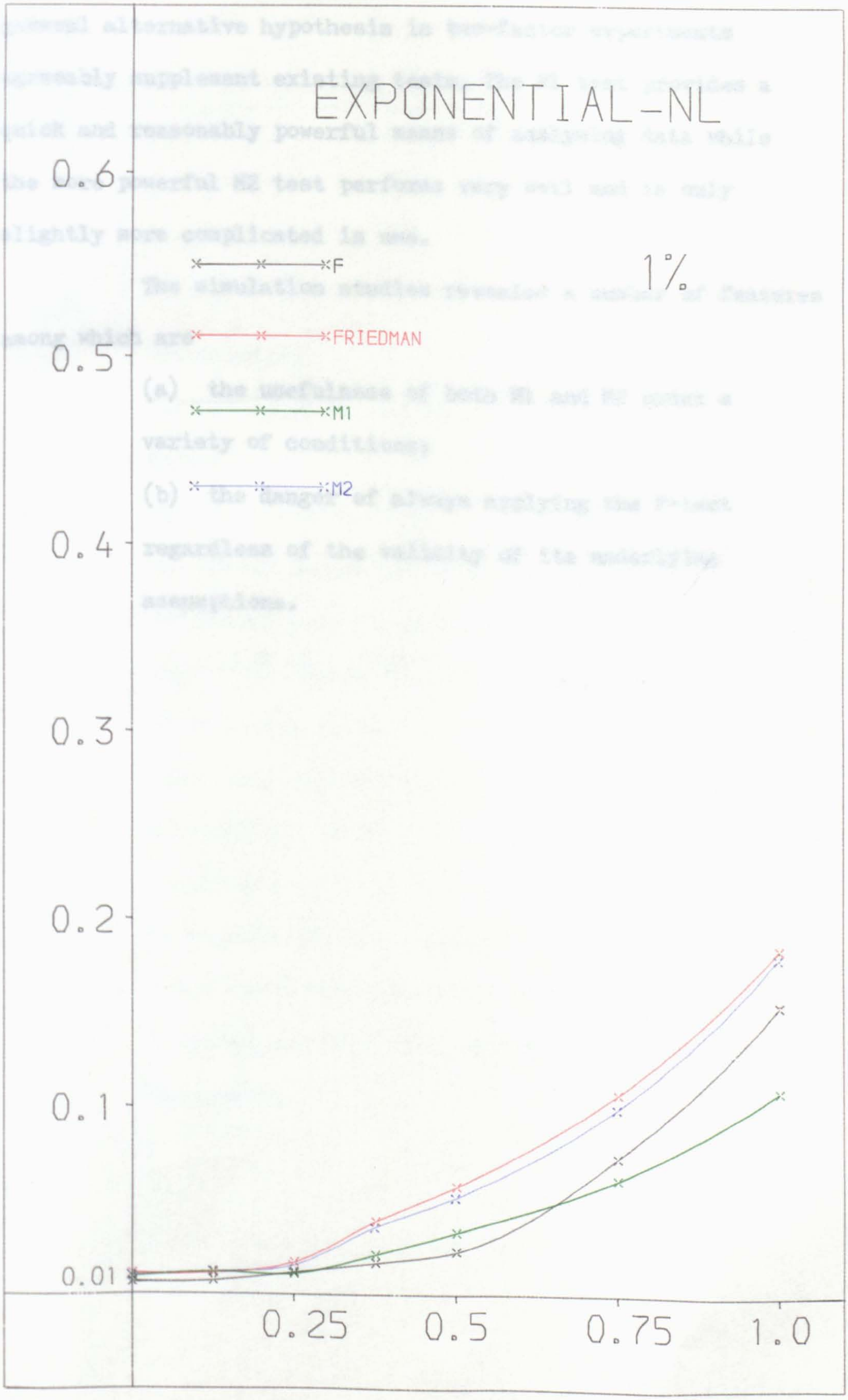


14. Conclusion.

The two procedures we have presented for the general alternative hypothesis in two-factor experiments agreeably supplement existing tests. The M1 test provides a quick and reasonably powerful means of analysing data while the more powerful M2 test performs very well and is only slightly more complicated in use.

The simulation studies revealed a number of features among which are

(a) the usefulness of both M1 and M2 under a variety of conditions;

(b) the danger of always applying the F-test regardless of the validity of its underlying assumptions.

CHAPTER 4

TWO-WAY ANALYSIS OF VARIANCE, ORDERED ALTERNATIVES

1. Introduction.

Many statisticians feel that in two-sample situations a two-sided test should always be used, regardless of the circumstances. However, there are numerous occasions when the experimenter argues, usually on the basis of experience or the demands of the experiment, that a one-sided test is more appropriate.

A similar choice in the type of alternative hypothesis exists even with randomised block experiments. The particular choice of alternative hypothesis is again left partially to the subjective reasoning of the experimenter. Thus with two-way experiments we speak of the general alternative and the ordered alternatives hypotheses which correspond to the two-sided and one-sided hypotheses of two-sample experiments.

Before presenting our statistics, L1 and L2, for the case of ordered alternative hypotheses, we shall briefly review the history of the development of nonparametric tests for such situations.

Jonckheere (1954) was the first to present such a test for ordered alternatives in randomised block designs. His motive was to analyse a frequently-occurring situation in education and social psychology investigations where c objects are ranked for some characteristic by b judges. The investigator wishes to determine whether the b sets of rankings from the judges agree with rank-order specified by the alternative hypothesis. Jonckheere's statistic is based on Kendall's τ and is given by

$$J = \tfrac{1}{4}c(c - 1)\sum_{i=1}^{b} \tau_i + \tfrac{1}{4}bc(c - 1) \quad ,$$

where τ_i is Kendall's rank correlation coefficient between the predicted order and the observed order in the i^{th} block. No tables of critical values were given; instead, he relied on J being asymptotically ($b \to \infty$) normal with a mean of $bc(c - 1)$ and a variance of $bc(c - 1)(2c + 5)/72$. In the simulation study we have used an equivalent statistic, namely

$$I = \sum_{i=1}^{b} v_i \quad ,$$

where v_i is the number of inversions in the i^{th} block when it is compared to the predicted ranking.

The subject of ordered alternatives was taken up again by Page (1963). In his paper, Page remarks on the inappropriateness of the well-trusted Friedman statistic for situations that are in essence the equivalent of "one-sided" tests in the two-sample situation. His statistic for an alternative hypothesis of the form

$$H_1 : t_1 < t_2 < \ldots\ldots < t_c \quad ,$$

where t_i denotes the effect of the i^{th} treatment, is

$$G = \sum_{j=1}^{c}\left[j \sum_{i=1}^{b} R_{ij} \right] ,$$

R_{ij} being the within-block rank of X_{ij}. Actually, this statistic was shown by Hollander (1967) to be equivalent to

$$\rho = \sum_{i=1}^{b} \rho_i \quad ,$$

where ρ_i is Spearman's rank correlation coefficient between the predicted order and the observed order in the i^{th} block. Page's paper contains exact critical values for $c = 3, 4, .. , 8$ and $b = 2, 3, ... , 12$ and relies on G being asymptotically normal for other critical values.

In his paper of 1967, Hollander also presented his Y-statistic which is based on a sum of Wilcoxon signed-rank statistics. Unfortunately, Y is shown to be neither distribution-free for finite c nor asymptotically distribution-free. The Y-statistic is defined in the following manner. Let

$$Y_{uv}^{(i)} = |X_{iu} - X_{iv}|$$

and

$$R_{uv}^{(i)} = \text{the within-block rank of } Y_{uv}^{(i)}, \quad (i = 1, .., b)$$

Also, let

$$T_{uv} = \sum_{i=1}^{b} R_{uv}^{(i)} \psi_{uv}^{(i)} ,$$

where

$$\psi_{uv}^{(i)} = \begin{cases} 1 & \text{if } X_{iu} < X_{iv} \\ 0 & \text{otherwise} \end{cases} .$$

Then

$$Y = \sum_{1<u<v<c} \sum T_{uv} .$$

In the following sections we introduce our match statistics for the ordered alternatives situation and demonstrate their ease of applicability to experimental data. In later sections we derive the exact null distributions and the moment generating functions for both statistics which will yield information concerning their asymptotic behaviour. In the final section we analyse the results of computer simulations.

2. Definition of L1 and L2

The linear model under consideration is expressed by

$$X_{ij} = M + A_i + B_j + z_{ij} \quad , (i = 1, 2, \ldots , b \quad j = 1, 2, \ldots , c)$$

where M represents the overall mean,

A_i represents the effect of the i^{th} block and $\Sigma A_i = 0$,

B_j represents the effect of the j^{th} treatment and $\Sigma B_j = 0$,

and z_{ij}'s are independent random variables having some continuous distribution.

We seek to test the null hypothesis

$$H_o : B_1 = B_2 = \ldots\ldots\ldots = B_c$$

against the ordered alternative hypothesis

$$H_1 : B_1 < B_2 < \ldots\ldots\ldots < B_c \quad .$$

Our statistics L1 and L2 are obtained in the following manner.

First of all the observations within each block are ranked from 1 to c. Then the ranks in each block are compared to the ranks predicted according to H_1 . From these comparisons we define two sets of scores l_{ij} and l^*_{ij}. If $R(X_{ij})$ denotes the rank of X_{ij} then we define

$$l_{ij} = \begin{cases} 1 & \text{if } R(X_{ij}) = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$l^*_{ij} = \begin{cases} \frac{1}{2} & \text{if } |R(X_{ij}) - j| = 1 \\ 0 & \text{otherwise} \end{cases} .$$

So l_{ij} corresponds to a match between $R(X_{ij})$ and the predicted rank j, while l^*_{ij} corresponds to a near-match between $R(X_{ij})$ and j.

The test statistics are now defined as

$$L1 = \sum_{i=1}^{b} l_i ,$$

where

$$l_i = \sum_{j=1}^{c} l_{ij}$$

and

$$L2 = \sum_{i=1}^{b} (l_i + l^*_i) ,$$

where

$$l^*_i = \sum_{j=1}^{c} l^*_{ij}$$

In other words, L1 is the total number of matches obtained when each block is compared to the ranks predicted under H_1, and L2 is the sum of L1 and the number of near-matches obtained from the comparison.

3. Example.

To illustrate the procedure of testing an ordered alternative hypothesis using L1 and L2 we analyse the results of an investigation by Syme and Pollard (1972) into the feeding behaviour of rats.

Their experiment consisted of eight naive male hooded rats subjected to various food deprivation schedules. The rats were observed once for each of three deprivation conditions in the following order : (a) after 24 hours ad lib food; (b) after 24 hours food deprivation; (c) after 72 hours food deprivation. The aim was to investigate how the feeding behaviour altered with these manipulations. Data were collected on the amount of food eaten by each rat and is shown in the table below.

Amount of Food (grams) Eaten by Eight Rats under Three Levels of Food Deprivation

	Hours of Food Deprivation		
Rat	0	24	72
1	3.5	5.9	13.9
2	3.7	8.1	12.6
3	1.6	8.1	8.1
4	2.5	8.6	6.8
5	2.8	8.1	14.3
6	2.0	5.9	4.2
7	5.9	9.5	14.5
8	2.5	7.9	7.9

If we denote the average amount of food eaten under the three levels of deprivation by f_0, f_{24} and f_{72} respectively, then the hypotheses may be written as

$$H_o : f_0 = f_{24} = f_{72}$$

$$H_1 : f_o < f_{24} < f_{72} .$$

The table of ranks for the above data is given below with range of ranks being quoted when ties occur.

Rat	0	24	72
1	1	2	3
2	1	2	3
3	1	(2-3)	(2-3)
4	1	3	2
5	1	2	3
6	1	3	2
7	1	2	3
8	1	(2-3)	(2-3)
Rank sum	8	19	21

Tests (i) - the match tests

The critical values (best conservative) for L1 and L2 are obtained from the exact distributions given in sections 7 and 8 respectively.

For the L1 test, the null hypothesis will be rejected at the 5 % and 1 % levels of significance if L1 $\geqslant$ 14 and L1 $\geqslant$ 16 respectively; while for the L2 test, rejection at the same levels of significance will occur if L2 $\geqslant$ 18 and L2 $\geqslant$ 19.

Comparing the ranks in the various blocks with the ranks predicted under H_1 produces tables of matches and near-matches.

Table of Matches for L1

Method for Ties	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8
Average Ranks	3	3	2	1	3	1	3	2
Range	3	3	2	1	3	1	3	2

The value of L1 is found by summing the l_i ; this produces the value of 18 in each case. Clearly this value of L1 strongly supports the alternative hypothesis; in fact P(L1 $\geqslant$ 18) = 0.0013.

Table of Contributions for L2 from Near-matches

Method for Ties		l_1^*	l_2^*	l_3^*	l_4^*	l_5^*	l_6^*	l_7^*	l_8^*
Average Ranks	(a)	0	0	$1\frac{1}{2}$	1	0	1	0	$1\frac{1}{2}$
	(b)	0	0	1	1	0	1	0	1
Range		0	0	$\frac{1}{2}$	1	0	1	0	$\frac{1}{2}$

The values of L2 from each of the methods of dealing with ties are found by calculating $L1 + \sum_{i=1}^{8} l_i^*$ in each case to give $20\frac{1}{2}$, 20 and $19\frac{1}{2}$ respectively. Clearly, all three values are consistent in their support for the alternative hypothesis.

Test (ii) - Page's test

The critical values, being obtained from the exact distribution, are best conservative values.

Rejection of the null hypothesis occurs at the 5 % and 1 % levels of significance if $G \geqslant 104$ and $G \geqslant 106$ respectively.

Using $G = \sum_{j=1}^{3} \left\{ j \sum_{i=1}^{8} R_{ij} \right\}$ we obtain

$G = 109$, a result which also strongly supports the alternative hypothesis.

4. The Distribution of L1

The null distribution of L1 is readily obtained by using a well-known result concening the probability of having exactly m matches out of c. Feller (1968) derives the following result

$$P_{[m]} = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} - \ldots\ldots\ldots \pm \binom{c}{m} S_c ,$$

where $P_{[m]}$ is the probability of having exactly m matches out of c,

and $S_m = \Sigma\, P\left\{A_{i_1} A_{i_2} \ldots . A_{i_m}\right\}$ with the A_i's being just m of c possible events $(S_0 = 1)$.

Now Feller shows that for the matching problem $S_m = 1/m!$. Hence on subsituting this into the above expression for $P_{[m]}$, we obtain the following distribution of probabilities for the number of matches in the i^{th} $(i = 1, 2, \ldots . , b)$ block.

$$P_{[0]} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + \frac{(-1)^c}{c!}$$

$$\cdots$$

$$\cdots$$

$$P_{[m]} = \frac{1}{m!}\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots + \frac{(-1)^{c-m}}{(c-m)!}\right)$$

$$\cdots$$

$$\cdots$$

$$P_{[c-2]} = \frac{1}{(c-2)!}\left(1 - 1 + \frac{1}{2!}\right)$$

$$P_{[c-1]} = 0 \quad \text{and} \quad P_{[c]} = \frac{1}{c!} \; .$$

Clearly, as $c \to \infty$ $\quad P_{[m]} \to \frac{e^{-m}}{m!} \quad (m = 0, 1, 2, \ldots.)$

so that asymptotically m has the Poisson distribution with mean 1. In fact, in the next section, we show that the exact mean = 1 . So, in view of the independence of the blocks, L1 is asymptotically distributed as a Poisson variable with méan b.

5. The Moment Generating Function of L1

The generating function for L1 is defined in a slightly different manner from that for M1, although the method is still based on Battin's (1942) idea. We shall first explain its structure by considering the simple case where there are three treatments and only one block.

Consider the function

$$\phi \equiv u^3 = \left\{ \sum_{r=1}^{3} \sum_{i=1}^{3} R_r x_i \, e^{\delta_{ri} \theta_1} \right\}^3$$

where $\delta_{ri} = \begin{cases} 1 & \text{if } r = i \\ 0 & \text{otherwise ,} \end{cases}$

R_r represents the predicted order under H_1 of the effect of the r^{th} treatment (w.l.o.g. we assume a natural order of the ranks),

x_i is a parameter relating to block 1 and the i^{th} treatment, (a second block would use y rather than x, etc.)

and θ_1 is a parameter associated with the predicted order of ranks and block 1 (with b blocks there would be b such parameters θ_1, θ_2, ... θ_b).

A term such as $R_1 x_1 e^{\theta_1}$ corresponds to a match between block 1 and the predicted ranks, the rank being equal to 1. Likewise, a term such as $R_2 x_1$ indicates a non-match between block 1 and the predicted rank.

In the expansion of $\phi \equiv u^3$ the coefficient of $R_1 R_2 R_3 x_1 x_2 x_3$ contains information concerning the numbers of possible matches and their frequency. In the above function ϕ, the coefficient is

$$1.e^{3\theta_1} + 3.e^{1\theta_1} + 2.e^{0\theta_1} = \sum_{m=0}^{3} f(m) \, e^{m\theta_1} .$$

The coefficients m of θ_1 give the values of the possible number of matches between the block and the predicted ranks. The number of ways in which these values can occur, out of the total of $3! = 6$ possible arrangements, are given by $f(m) = 1, 3$ and 2 from the coefficients in the appropriate exponential terms (note that $m = 2$ is not possible). Of course, setting $\theta_1 = 0$ produces $\Sigma f(m) = 1 + 3 + 2$ which is the total number of arrangements.

We define the operator K by

$$K \text{ expression } = \text{ coefficient of } R_1R_2R_3x_1x_2x_3 \text{ in the expression.}$$

This operator enables us to concisely express a number of important quantities. For instance, the total number of arrangements ($3!$) is given by $K\phi|_{\theta_1 = 0}$. Also the probability of obtaining exactly 3 matches (for example) is

$$\frac{\text{coefficient of } e^{3\theta_1} \text{ in } K\phi}{K\phi|_{\theta_1 = 0}},$$

in the situation resulting from the null hypothesis that all permutations are equally likely.

If we recall from section 2 that l_1 represents that number of matches in block 1 then

$$P(\,l_1 = s\,) = \frac{\text{coefficient of } e^{s\theta_1} \text{ in } K\phi}{K\phi|_{\theta_1 = 0}}, \quad 0 \leqslant s \leqslant 3$$

and so

$$E(\,l_1\,) = \frac{K\partial\phi/\partial\theta_1|_{\theta_1 = 0}}{K\phi|_{\theta_1 = 0}}$$

and more generally,

$$E(l_1^p) = \frac{K \partial^p \phi / \partial \theta_1^p |_{\theta_1 = 0}}{K \phi |_{\theta_1 = 0}} .$$

In the case of three treatments and two blocks

$$\phi = \left\{ \sum_{r=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{3} R_r x_i y_j \, e^{\delta_{ri}\theta_1 + \delta_{rj}\theta_2} \right\}^3$$

The coefficient of $R_1R_2R_3x_1x_2x_3y_1y_2y_3$ is $\sum_{m_1=0}^{3} \sum_{m_2=0}^{3} f(m_1,m_2)e^{m_1\theta_1+m_2\theta_2}$

A typical term in this coefficient is $3e^{3\theta_1 + \theta_2}$ where the coefficient of e indicates that there are 3 arrangements, namely

$$\begin{matrix} 1\ 2\ 3 & 1\ 2\ 3 & 1\ 2\ 3 \\ 1\ 3\ 2 & 3\ 2\ 1 & 2\ 1\ 3 \end{matrix}$$

, giving rise to 3 matches between block 1 and the predicted ranks and 1 match between block 2 and the predicted ranks. Likewise, in the general term $f(m_1,m_2)e^{m_1\theta_1 + m_2\theta_2}$, $f(m_1,m_2)$ is the number of arrangements out of $(3!)^2 = 36$ possibilities in which there are m_1 and m_2 matches between blocks 1 and 2 and the predicted ranks respectively. Setting $\theta_1 = \theta_2 = 0$ (i.e. $\underline{\theta} = \underline{0}$) produces $\sum_{m_1=0}^{3} \sum_{m_2=0}^{3} f(m_1,m_2) = 36 = (3!)^2$, the total number of arrangements. This is also obtained from $K \phi |_{\underline{\theta} = \underline{0}} = (3!)^2$ with the K operator defined as above. Thus, for example, the probability of obtaining exactly 3 matches in block 2 is

$$\frac{\text{coefficient of } e^{3\theta_2} \text{ in } K\phi}{K\phi\,\big|_{\underline{\theta}=\underline{0}}},$$

in the situation resulting from the null hypothesis that all permutations are equally likely.

Furthermore with l_i representing the number of matches in block i (i = 1, 2) then

$$P(\, l_i = s \,) = \frac{\text{coefficient of } e^{s\theta_i} \text{ in } K\phi}{K\phi\,\big|_{\underline{\theta}=\underline{0}}}, \quad 0 \leqslant s \leqslant 3$$

and

$$E(\, l_i^p \,) = \frac{K\,\partial^p\phi/\partial\theta_i\,\big|_{\underline{\theta}=\underline{0}}}{K\phi\,\big|_{\underline{\theta}=\underline{0}}}.$$

We now proceed to obtain the mean and variance of Li for the case of c treatments and b blocks using a generating function similar to that considered above.

The function ϕ is now defined as

$$\phi \equiv u^c = \left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} R_r x_{1i_1} x_{2i_2} \cdots\cdots x_{bi_b} f(\delta, \underline{\theta}) \right\}^c$$

where $\quad f(\delta, \underline{\theta}) = \exp(\, \sum_{j=1}^{b} \delta_{rj}\, \theta_j \,)$.

The operator K is defined by

$$K \text{ expression} \equiv \text{ coefficient of } \prod_{i=1}^{c} R_i \prod_{j=1}^{b} x_{ji} \text{ in the expression.}$$

Now $K\phi\Big|_{\underline{\theta}=\underline{0}} = K\left\{\sum_{r=1}^{c}\sum_{i_1=1}^{c}\dots\sum_{i_b=1}^{c} R_r x_{1i_1} x_{2i_2} \dots x_{bi_b}\right\}^c$

$$= (c!)^b ,$$

where, as before, $\underline{\theta} = \underline{0}$ denotes $\theta_s = 0$ for all s.

Hence by a direct extension of the ideas presented above we have

$$E(l_i^p) = \frac{K\,\partial^p\phi/\partial\theta_i^p\Big|_{\underline{\theta}=\underline{0}}}{K\phi\Big|_{\underline{\theta}=\underline{0}}}$$

$$= K\,\partial^p\phi/\partial\theta_i^p\Big|_{\underline{\theta}=\underline{0}} \Big/ (c!)^b \quad \dots . (1) ,$$

where l_i is the number of matches between the i^{th} block and the predicted ranks.

The expected value of Ll is given by

$E(Ll) = \sum_{i=1}^{b} E(l_i) = bE(l_1)$ by virtue of the independence of the blocks.

From (1) the mean value of l_1 is given by

$$E(l_1) = K\frac{\partial\phi}{\partial\theta_1}\Big|_{\underline{\theta}=\underline{0}} \Big/ (c!)^b \quad \dots .. (2) .$$

Now $\dfrac{\partial\phi}{\partial\theta_1} = cu^{c-1}\left\{\sum_{r=1}^{c}\sum_{i_1=1}^{c}\dots\sum_{i_b=1}^{c} R_r x_{1i_1} x_{2i_2}\dots x_{bi_b}\,\delta_{r1} f(\delta;\underline{\theta})\right\}$

Hence $\dfrac{\partial\phi}{\partial\theta_1}\Big|_{\underline{\theta}=\underline{0}} = cu_o^{c-1}\left\{\sum_{r=1}^{c}\sum_{i_2=1}^{c}\dots\sum_{i_b=1}^{c} R_r x_{1r} x_{2i_2}\dots x_{bi_b}\right\}$

where $u_o = u\Big|_{\underline{\theta}=\underline{0}}$.

So $$K \frac{\partial \phi}{\partial \theta_1}\bigg|_{\underline{\theta}=\underline{0}} = c\,(c-1)!^{\,b}\, c^{b-1} = (\,c!\,)^b \,.$$

Hence (2) gives $E(\,l_1\,) = 1$ from which we have $E(\,Ll\,) = b$.

To calculate the variance of Ll we require $E(\,Ll^2\,)$.

Now

$$\begin{aligned} E(\,Ll^2\,) &= \sum_{i=1}^{b} E(\,l_i^2\,) + \sum_{\substack{i=1\\ i\neq j}}^{b}\sum_{j=1}^{b} E(\,l_i l_j\,) \\ &= bE(l_1^2\,) + b(b-1\,)E(\,l_1 l_2\,) \quad , \end{aligned}$$

by symmetry and the independence of the blocks where

$$E(\,l_1^2\,) = K\frac{\partial \phi^2}{\partial \theta_1^2}\bigg|_{\underline{\theta}=\underline{0}} \bigg/ (\,c\,!\,)^b$$

and $$E(\,l_1 l_2\,) = K\frac{\partial \phi^2}{\partial \theta_1 \partial \theta_2}\bigg|_{\underline{\theta}=\underline{0}} \bigg/ (\,c!\,)^b \,.$$

Now,

$$\frac{\partial \phi^2}{\partial \theta_1^2} = c(c-1)u^{c-2}\left\{\sum_{r=1}^{c}\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c} R_r x_{1i_1}\cdots\; x_{bi_b}\delta_{1r}\, f(\delta;\underline{\theta})\right\}^2$$

$$+ \quad cu^{c-1}\left\{\sum_{r=1}^{c}\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c} R_r x_{1i_1}\cdots\; x_{bi_b}\delta_{1r}^2\, f(\delta;\underline{\theta})\right\}$$

so that $$\frac{\partial \phi^2}{\partial \theta_1^2}\bigg|_{\underline{\theta}=\underline{0}} =$$

$$c(c-1)u_o^{c-2}\left\{\sum_{r=1}^{c}\sum_{i_2=1}^{c}\cdots\sum_{i_b=1}^{c} R_r x_{1r} x_{2i_2}\cdots\; x_{bi_b}\right\}$$

$$+ \quad cu_o^{c-1}\left\{ \sum_{r=1}^{c} \sum_{i_2=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1r} x_{2i_2} \cdots\cdots x_{bi_b} \right\} .$$

Hence, after some simplification,

$$K \frac{\partial \phi^2}{\partial \theta_1^2} \Big|_{\underline{\theta} = \underline{0}} = 2(\ c!\ )^b$$

and so,

$$E(\ l_1^2\ ) = K \frac{\partial \phi^2}{\partial \theta_1^2} \Big|_{\underline{\theta} = \underline{0}} \Big/ (\ c!\ )^b$$

$$= 2 .$$

Next, $$\frac{\partial \phi^2}{\partial \theta_1 \partial \theta_2} =$$

$$c(c-1)u^{c-2}\left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} R_r x_{1i_1} \cdots\cdots x_{bi_b} \delta_{1r} \ f(\delta;\underline{\theta}\ ) \right\} \times$$

$$\left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} R_r x_{1i_1} \cdots\cdots x_{bi_b} \delta_{r2} \ f(\delta;\underline{\theta}\ ) \right\}$$

$$+ \ cu^{c-1}\left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} R_r x_{1i_1} \cdots\cdots x_{bi_b} \delta_{r1} \delta_{r2} \ f(\delta;\underline{\theta}\ ) \right\}$$

so that $$\frac{\partial \phi^2}{\partial \theta_1 \partial \theta_2} \Big|_{\underline{\theta} = \underline{0}} =$$

$$c(c-1)u_o^{c-2}\left\{ \sum_{r=1}^{c} \sum_{i_2=1}^{c} \cdots\cdots \sum_{i_b=1}^{c} R_r x_{1r} x_{2i_2} \cdots\cdots x_{bi_b} \right\} \times$$

$$\left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1i}\ x_{2r} x_{3i} \cdots\cdots x_{bi_b} \right\}$$

$$+ \; c u_o^{c-1} \left\{ \sum_{r=1}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1r} x_{2r} x_{3i_3} \cdots\cdots x_{bi_b} \right\}$$

Hence, after some simplification,

$$K \frac{\partial^2 \phi}{\partial \theta_1 \partial \theta_2} \bigg|_{\underline{\theta} = \underline{0}} = (c!)^b$$

which gives $E(l_1 l_2) = 1$.

Thus

$$E(L1^2) = 2b + b(b - 1) = b^2 + b \, .$$

Hence

$$\begin{aligned} \mathrm{var}(L1) &= E(L1^2) - (E(L1))^2 \\ &= b^2 + b - b^2 \\ &= b \, . \end{aligned}$$

Both the moments we have obtained, $E(L1)$ and $\mathrm{var}(L1)$ are consistent with our previous results concerning the asymptotic behaviour of L1.

6. The Moment Generating Function of L2

To obtain the moments of L2 we define the function ϕ^* by

$$\phi^* \equiv u^{*c} = \left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{i_1} \cdots\cdots x_{i_b} f(\delta, \delta^*; \underline{\theta}, \underline{\theta}^*) \right\}^c$$

where $\quad f(\delta, \delta^*; \underline{\theta}, \underline{\theta}^*) \equiv f = \exp(\sum_{\alpha=1}^{b} \delta_{r\alpha} \theta_\alpha + \sum_{\alpha=1}^{b} \delta^*_{r\alpha} \theta^*_\alpha) \quad ,$

with $$\delta^{*}_{r\alpha} = \begin{cases} \frac{1}{2} & \text{if } |r-\alpha| = 1 \\ 0 & \text{otherwise} \end{cases} .$$

The operator K is defined and used in the same manner as before. So it follows immediately that

$$K\,\phi^{*}\,\Big|_{\underline{\theta},\underline{\theta}^{*}=\underline{0}} = (\,c!\,)^{b} .$$

Since $L2 = L1 + \sum_{i=1}^{b} l_i^{*}$, where l_i^{*} is equal to half the number of near-matches between the i^{th} block and the predicted ranks, we immediately have

$$E(\,L2\,) = E(\,L1\,) + bE(\,l_1^{*}\,) .$$

The expected value of l_1^{*} is given by

$$E(\,l_1^{*}\,) = K\,\frac{\partial\phi^{*}}{\partial\theta_1^{*}}\Big|_{\underline{\theta},\underline{\theta}^{*}=\underline{0}} \Big/ (\,c!\,)^{b} .$$

Now

$$\frac{\partial\phi^{*}}{\partial\theta_1^{*}} = cu_o^{*(c-1)}\left\{\sum_{r=1}^{c}\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c} R_r x_{1i_1}\cdots\cdots x_{bi_b}\delta^{*}_{r1}\,f\right\}$$

so that $$\frac{\partial\phi^{*}}{\partial\theta_1^{*}}\Big|_{\underline{\theta},\underline{\theta}^{*}=\underline{0}} =$$

$$\tfrac{1}{2}\,cu_o^{*\,(c-1)}\left\{\sum_{r=1}^{c-1}\sum_{i_2=1}^{c}\cdots\cdots\sum_{i_b=1}^{c} R_r x_{1r+1}x_{2i_2}\cdots\cdots x_{bi_b} + \sum_{r=2}^{c}\sum_{i_2=1}^{c}\cdots\cdots\sum_{i_b=1}^{c} R_r x_{1r-1}x_{2i_2}\cdots\cdots x_{bi_b}\right\}$$

where $$u_o^{*} = u\Big|_{\underline{\theta},\underline{\theta}^{*}=\underline{0}} .$$

Hence
$$K \frac{\partial \phi^*}{\partial \theta_1^*}\bigg|_{\underline{\theta},\underline{\theta}=\underline{0}} = \tfrac{1}{2}c\left[(c-1)!\right]^b 2(c-1)c^{b-2}$$
$$= (1-\tfrac{1}{c})(\ c!\ )^b$$

giving
$$E(\ l_1^*\ ) = 1-\tfrac{1}{c}\ .$$

The expected value of L2 is now given by

$$E(\ L2\ ) = b + b(1-\tfrac{1}{c})$$
$$= b(2-\tfrac{1}{c})\ .$$

To calculate the variance of L2 we require the expected value of $(L2)^2$.

Now $E(\ (L2)^2\ ) = E(\ (L1)^2\ ) + E(\ L^{*\,2}\ ) + 2E(\ L1.L^*\ )$

where $E(\ L^{*\,2}\ ) = bE(\ l_1^{*\,2}\ ) + b(b-1)E(\ l_1^*.l_2^*\ )$

and $E(\ L1.L^*\ ) = bE(\ l_1 l_1^*\ ) + b(b-1)E(\ l_1 l_2^*\ )$.

Now
$$E(\ l_1^{*\,2}\ ) = K \frac{\partial^2 \phi^*}{\partial \theta_1^{*\,2}}\bigg|_{\underline{\theta},\underline{\theta}^*=\underline{0}} \bigg/ (\ c!\ )^b$$

where
$$\frac{\partial^2 \phi^*}{\partial\ \theta_1^{*\,2}} =$$

$$c(c-1)u^{*(c-2)}\left\{\sum_{r=1}^{c}\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c} R_r x_{1i_1}\ \cdots\ x_{bi_b}\delta_{r1}^{*} f\right\}^2$$
$$+ cu^{*(c-1)}\left\{\sum_{r=1}^{c}\sum_{i_1=1}^{c}\cdots\sum_{i_b=1}^{c} R_r x_{1i_1}\ \cdots\ x_{bi_b}\delta_{r1}^{*2} f\right\}$$

Thus $\left.\dfrac{\partial^2 \phi^*}{\partial \theta_1^*}\right|_{\underline{\theta},\underline{\theta}^* = \underline{0}}$ =

$$\tfrac{1}{4}c(c-1)u_o^{*(c-2)}\left\{ \sum_{r=1}^{c-1} \sum_{i_2=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1r+1} x_{2i_2} \cdots\cdots x_{bi_b} \right.$$

$$\left. + \sum_{r=2}^{c} \sum_{i_2=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1r-1} x_{2i_2} \cdots\cdots x_{bi_b} \right\}^2$$

$$\tfrac{1}{4}\, cu_o^{*(c-1)}\left\{ \sum_{r=1}^{c-1} \sum_{i_2=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1r+1} x_{2i_2} \cdots\cdots x_{bi_b} \right.$$

$$\left. + \sum_{r=2}^{c} \sum_{i_2=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1r-1} x_{2i_2} \cdots\cdots x_{bi_b} \right\}$$

giving
$$K \left.\frac{\partial^2 \phi^*}{\partial \theta_1^{*\,2}}\right|_{\underline{\theta},\underline{\theta}^* = \underline{0}} = \frac{(\,c!\,)^b\,(3c^2 - 9c + 8)}{2c(c-1)} .$$

Hence
$$E(\,l_1^{*\,2}\,) = \frac{3c^2 - 9c + 8}{2c(c-1)} .$$

Now
$$E(\,l_1^*.l_2^*\,) = K \left.\frac{\partial^2 \phi^*}{\partial \theta_1^* \partial \theta_2^*}\right|_{\underline{\theta},\underline{\theta}^* = \underline{0}} \Big/ (\,c!\,)^b ,$$

where $\dfrac{\partial^2 \phi^*}{\partial \theta_1^* \partial \theta_2^*}$ =

$$c(c-1)u^{*(c-2)}\left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1i_1} x_{2i_2} \cdots x_{bi_b} \delta^{*}_{r2} f \right\} \times$$

$$\left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1i_1} x_{2i_2} \cdots x_{bi_b} \delta^{*}_{r1} f \right\}$$

$$+ cu^{*(c-1)}\left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1i_1} x_{2\ 2} \cdots x_{bi_b} \delta^{*}_{r1} \delta^{*}_{r2} f \right\}$$

so that $\dfrac{\partial^2 \phi^*}{\partial \theta_1^* \partial \theta_2^*}\Big|_{\underline{\theta},\underline{\theta}^* = \underline{0}} =$

$$\tfrac{1}{4}c(c-1)u_o^{*(c-2)}\left\{ \sum_{r=1}^{c-1} \sum_{i_1=1}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1i_1} x_{2r+1} \cdots x_{bi_b} \right.$$

$$\left. + \sum_{r=2}^{c} \sum_{i_1=1}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1i_1} x_{2r-1} \cdots x_{bi_b} \right\} \times$$

$$\left\{ \sum_{r=1}^{c-1} \sum_{i_2=1}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1r+1} x_{2i_2} \cdots x_{bi_i} \right.$$

$$\left. + \sum_{r=2}^{c} \sum_{i_2=1}^{c} \sum_{i_3=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1r-1} x_{2i_2} \cdots x_{bi_b} \right\},$$

whence, after some simplification,

$$K \frac{\partial^2 \phi^*}{\partial \theta_1^* \partial \theta_2^*}\Big|_{\underline{\theta},\underline{\theta}^* = \underline{0}} = (c!)^b \frac{(c-1)^2}{c^2} .$$

Hence $\quad E(\, L^{*\,2}\,) \;=\; \dfrac{b(3c^2 - 9c + 8)}{2c(c-1)} \;+\; b(b-1)(1 - \frac{1}{c})^2 \;.$

Now $\quad E(\, l_1 . l_1^* \,) \;=\; K \dfrac{\partial^2 \phi^*}{\partial \theta_1^* \partial \theta_1} \Big|_{\underline{\theta}, \underline{\theta}^* = \underline{0}} \Big/ (\, c!\,)^b$

Thus $\quad \dfrac{\partial^2 \phi^*}{\partial \theta_1^* \partial \theta_1} \;=$

$$c(c-1)u^{*(c-2)} \left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1i_1} \cdots\cdots x_{bi_b} \delta^*_{r1} f \right\} \times$$

$$\left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1i_1} \cdots\cdots x_{bi_b} \delta_{r1} f \right\}$$

$$+ \; cu^{*(c-1)} \left\{ \sum_{r=1}^{c} \sum_{i_1=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1i_1} \cdots\cdots x_{bi_b} \delta^*_{r1} \delta_{r1} f \right\}$$

so that $\quad K \dfrac{\partial^2 \phi^*}{\partial \theta_1^* \partial \theta_1} \Big|_{\underline{\theta}, \underline{\theta}^* = \underline{0}} \;=$

$$c(c-1)u_o^{*(c-2)} \left\{ \sum_{r=1}^{c-1} \sum_{i_2=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1r+1} x_{2i_2} \cdots\cdots x_{bi_b} \right.$$

$$\left. + \sum_{r=2}^{c} \sum_{i_2=1}^{c} \cdots \sum_{i_b=1}^{c} R_r x_{1r-1} x_{2i_2} \cdots\cdots x_{bi_b} \right\}$$

Hence, after some simplification,

$$K \frac{\partial^2 \phi^*}{\partial \theta_1^* \partial \theta_1} \Big|_{\underline{\theta}, \underline{\theta}^* = \underline{0}} \;=\; (1 - \tfrac{2}{c})(\, c!\,)^b \qquad ,$$

which gives $E(l_1 . l_1^*) = 1 - \frac{2}{c}$.

Now $E(l_1 . l_2^*)$ is computed in a similar manner to $E(l_1^* . l_1)$ and so we simply quote the result

$$E(l_1 . l_2^*) = 1 - \frac{1}{c} .$$

Thus we have $E(L1 . L^*) = b(1 - \frac{2}{c}) + b(b - 1)(1 - \frac{1}{c})$.

Finally,

$$\begin{aligned} var(L2) &= E(L2^2) - (E(L2))^2 \\ &= 2b + b(b - 1) + \frac{b(3c^2 - 9c + 8)}{2c(c - 1)} \\ &= + b(b - 1)(1 - \frac{1}{c})^2 + 2b(1 - \frac{2}{c}) \\ &= - b^2(2 - \frac{1}{c})^2 \end{aligned}$$

$$\text{i.e. } var(L2) = \frac{b}{c} (\frac{3(c - 2)}{2} + \frac{1}{c(c - 1)} .$$

Since L2 is the sum of the b independent variables $l_i + l_i^*$ $(i = 1, 2, \ldots , b)$. we may invoke the central limit theorem. Thus as $b \to \infty$ the distribution of L2 tends to the normal distribution with mean $b(2 - \frac{1}{c})$ and variance $\frac{b}{c} (\frac{3(c - 2)}{2} + \frac{1}{c(c - 1)})$.

If c is large then the approximations 2b and 3b/2 for the mean and variance, respectively, may be more convenient to use.

7. Upper Tail Probabilities for the Null Distribution of L1

Once the distribution for 1 block had been calculated the distributions for higher numbers of blocks were derived by convolution.

The exact distributions of L1 are given for c = 3, b = 2 to 10; c = 4, b = 2 to 10; c = 5, b = 2 to 7; c = 6, b = 2 to 5; c = 7, b = 2 to 4. Unfortunately, integer overflow prevented us presenting b = 2 to 10 in all cases.

c = 3 b = 2

x	P(L1 ⩾ x)
0	1
1	.888889
2	.555556
3	.305555
4	.194444
6	.277778

c = 3 b = 3

x	P(L1 ⩾ x)
0	1
1	.962963
2	.796296
3	.546296
4	.365741
5	.199074
6	.074074
7	.046296
9	.004630

c = 3 b = 4

x	P(L1 ⩾ x)
0	1
1	.987654
2	.913580
3	.746914
4	.555556
5	.381944
6	.215278
7	.113426
8	.057870
9	.016204
10	.010031
12	.000772

c = 3 b = 5

x	P(L1 ⩾ x)
0	1
1	.995885
2	.965021
3	.872428
4	.723251
5	.557356
6	.387217
7	.238040
8	.139660
9	.070216
10	.030350
11	.014917
12	.003344
13	.002058
15	.000129

c = 3 b = 6

x	P(L1 ⩾ x)
0	1
1	.998628
2	.986283
3	.939986

x	$P(L1 \geq x)$
4	.843278
5	.708248
6	.553155
7	.393497
8	.258466
9	.157772
10	.084897
11	.043424
12	.020276
13	.007416
14	.003558
15	.000664
16	.000407
18	.000021

c = 3 b = 7

x	$P(L1 \geq x)$
0	1
1	.999543
2	.994742
3	.973137
4	.917524
5	.822102
6	.695173
7	.548290
8	.400945
9	.274016

x	$P(L1 \geq x)$
10	.172768
11	.100001
12	.054766
13	.026760
14	.012131
15	.005380
16	.001704
17	.000804
18	.000129
19	.000079
21	.000004

c = 3 b = 8

x	$P(L1 \geq x)$
0	1
1	.999848
2	.998019
3	.988416
4	.959000
5	.898586
6	.804965
7	.683270
8	.544810
9	.407674
10	.285979
11	.186542
12	.113925

x	$P(L1 \geq x)$
13	.065098
14	.034091
15	.016987
16	.007718
17	.003142
18	.001341
19	.000374
20	.000174
21	.000024
22	.000015
24	.000001

c = 3 b = 9

x	$P(L1 \geq x)$
0	1
1	.999949
2	.999263
3	.995148
4	.980516
5	.945365
6	.882351
7	.790073
8	.673003
9	.542457
10	.413042
11	.295972
12	.199192

x	P(L1 ≥ x)
13	.126325
14	.075003
15	.041695
16	.021916
17	.010588
18	.004849
19	.002082
20	.000769
21	.000319
22	.000080
23	.000037
24	.000005
25	.000003
27	.000000

c = 3 b = 10

x	P(L1 ≥ x)
0	1
1	.999983
2	.999729
3	.998014
4	.991071
5	.971924
6	.932658
77	.867952

x	P(L1 ≥ x)
8	.776932
9	.664379
10	.540588
11	.417345
12	.304793
13	.210545
14	.137492
15	.084598
16	.049207
17	.026988
18	.013860
19	.006771
20	.003062
21	.001299
22	.000533
23	.000180
24	.000073
25	.000016
26	.000008
27	.000001
28	.000001
30	.000000

c = 4 b = 2

x	P(L1 ≥ x)
0	1
1	.859375
2	.609375
3	.310764
4	.144097
5	.050347
6	.022569
8	.001736

c = 4 b = 3

x	P(L1 ≥ x)
0	1
1	.947266
2	.806641
3	.576172
4	.351635
5	.180411
6	.086661
7	.033709
8	.012762
9	.003111
10	.000137
12	.000072

c = 4 b = 4

x	P(L1 ⩾ x)
0	1
1	.980225
2	.909912
3	.763428
4	.567247
5	.368378
6	.214154
7	.110638
8	.052381
9	.021403
10	.008382
11	.002667
12	.000931
13	.000172
14	.000075
16	.000003

c = 4 b = 5

x	P(L1 ⩾ x)
0	1
1	.992584
2	.959625
3	.876312
4	.736338
5	.558924
6	.382825
7	.236778
8	.133696
9	.068495
10	.032296
11	.013755
12	.005517
13	.001934
14	.000668
15	.000180
16	.000060
17	.000009
18	.000004
20	.000001

c = 4 b = 6

x	P(L1 ⩾ x)
0	1
1	.997219
2	.982388
3	.938305
4	.849804
5	.715478
6	.553936
7	.392644
8	.255449
9	.152734
10	.084318
11	.042913
12	.020298
13	.008857
14	.003620
15	.001347
16	.000479
17	.000149
18	.000047
19	.000011
20	.000004
21	.000004
22	.000002
24	.000000

c = 4 b = 7

x	P(L1 ⩾ x)
0	1
1	.998957
2	.992468
3	.970298
4	.918708
5	.827699
6	.699603
7	.549853

x	P(L ⩾ x)
8	.400567
9	.270398
10	.169473
11	.098742
12	.053639
13	.027180
14	.012897
15	.005714
16	.002379
17	.000921
18	.000338
19	.000113
20	.000037
21	.000010
22	.000003
23	.000001
24	.000000
25	.000000
26	.000000
28	.000000

c = 4 b = 8

x	P(L1 ⩾ x)
0	1
1	.999609
2	.996828
3	.986091
4	.957732
5	.900821
6	.809281
7	.686750
8	.546677
9	.406872
10	.282977
11	.184029
12	.112087
13	.064024
14	.034368
15	.017351
16	.008256
17	.003700
18	.001566
19	.000624
20	.000235
21	.000083
22	.000028
23	.000009
24	.000003
25	.000001
26	.000000
27	.000000
28	.000000
29	.000000
30	.000000
32	.000000

c = 4 b = 9

x	P(L1 ⩾ x)
0	1
1	.999853
2	.998680
3	.993629
4	.978720
5	.945225
6	.884722
7	.793584
8	.676146
9	.544024
10	.412130
11	.293669
12	.196898
13	.124332
14	.074042
15	.041636
16	.022142
17	.011145
18	.005316
19	.002404

x	P(L1 ≥ x)
20	.001032
21	.000419
22	.000162
23	.000059
24	.000021
25	.000007
26	.000002
27	.000001
28	.000000
29	.000000
30	.000000
31	.000000
32	.000000
33	.000000
34	.000000
36	.000000

c = 4 b = 10

x	P(L1 ≥ x)
0	1
1	.999945
2	.999456
3	.997134
4	.989566
5	.970767
6	.933137

x	P(L1 ≥ x)
7	.870209
8	.780043
9	.667171
10	.541790
11	.416575
12	.302932
13	.208342
14	.135606
15	.083613
16	.048897
17	.027180
18	.014329
19	.007195
20	.003440
21	.001566
22	.000680
23	.000281
24	.000111
25	.000042
26	.000015
27	.000005
28	.000002
29	.000001
30	.000000
31	.000000
32	.000000

x	P(L1 ≥ x)
33	.000000
34	.000000
35	.000000
36	.000000
37	.000000
38	.000000
40	.000000

c = 5 b = 2

x	P(L1 ≥ x)
0	1
1	.865556
2	.590556
3	.327708
4	.141597
5	.051319
6	.017431
7	.004236
8	.001458
10	.000069

c = 5 b = 3

x	P(L1 ≥ x)
0	1
1	.950704

x	P(L1 ≥ x)
2	.799454
3	.577544
4	.353698
5	.184080
6	.083758
7	.033364
8	.012036
9	.003911
10	.001075
11	.000304
12	.000053
13	.000018
15	.000000

c = 5 b = 4

x	P(L1 ≥ x)
0	1
1	.981925
2	.907980
3	.761679
4	.567069
5	.371345
6	.214742
7	.110460
8	.051037
9	.021435

x	P(L1 ≥ x)
10	.008181
11	.002868
12	.000917
13	.000267
14	.000074
15	.000017
16	.000000
17	.000000
18	.000000
20	.000000

c = 5 b = 5

x	P(L1 ≥ x)
0	1
1	.993372
2	.959481
3	.875096
4	.735045
5	.559756
6	.384161
7	.237743
8	.133219
9	.068029
10	.031844
11	.013735
12	.005482

x	P(L1 ≥ x)
13	.002027
14	.000697
15	.000223
16	.000066
17	.000018
18	.000005
19	.000001
20	.000000
21	.000000
22	.000000
23	.000000
25	.000000

c = 5 b = 6

x	P(L1 ≥ x)
0	1
1	.997570
2	.982658
3	.937903
4	.848705
5	.715025
6	.554478
7	.393775
8	.255966
9	.152664
10	.083867

x	P(L1 ≥ x)
11	.042619
12	.020118
13	.008852
14	.003641
15	.001403
16	.000507
17	.000173
18	.000055
19	.000017
20	.000005
21	.000001
22	.000000
23	.000000
24	.000000
25	.000000
26	.000000
27	.000000
28	.000000
30	.000000

c = 5 b = 7

x	P(L1 ≥ x)
0	1
1	.999109
2	.992730
3	.970323
4	.918146
5	.826979
6	.699365
7	.550398
8	.401334
9	.270872
10	.169434
11	.098472
12	.053340
13	.027015
14	.012831
15	.005730
16	.002412
17	.000958
18	.000360
19	.000128
20	.000043
21	.000014
22	.000004
23	.000001
24	.000000
25	.000000
26	.000000
27	.000000
28	.000000
29	.000000
30	.000000
31	.000000
32	.000000
33	.000000
35	.000000

c = 6 b = 2

x	P(L1 ≥ x)
0	1
1	.864535
2	.594278
3	.322162
4	.143767
5	.052535
6	.016424
7	.004502
8	.001169
9	.000214
10	.000060
12	.000002

c = 6 b = 3

x	P(L1 ≥ x)
0	1
1	.950141

x	P(L1 ⩾ x)
2	.801130
3	.576482
4	.352785
5	.184891
6	.083939
7	.033429
8	.011900
9	.003806
10	.001112
11	.000294
12	.000072
13	.000015
14	.000003
15	.000000
16	.000000
18	.000000

<u>c = 6 b = 4</u>

x	P(L1 ⩾ x)
0	1
1	.981649
2	.908523
3	.761854
4	.566441
5	.371186
6	.214941

x	P(L1 ⩾ x)
7	.110677
8	.051110
9	.021348
10	.008132
11	.002844
12	.000919
13	.000275
14	.000076
15	.000020
16	.000005
17	.000001
18	.000000
19	.000000
20	.000000
21	.000000
22	.000000
24	.000000

<u>c = 6 b = 5</u>

x	P(L1 ⩾ x)
0	1
1	.993246
2	.959603
3	.875367
4	.734935
5	.559473

x	P(L1 ⩾ x)
6	.384061
7	.237848
8	.133378
9	.068082
10	.031817
11	.013692
12	.005455
13	.002021
14	.000699
15	.000227
16	.000069
17	.000020
18	.000005
19	.000001
20	.000000
21	.000000
22	.000000
23	.000000
24	.000000
25	.000000
26	.000000
27	.000000
28	.000000
30	.000000

c = 7 b = 2

x	P(L1 ≥ x)
0	1
1	.864681
2	.593897
3	.323552
4	.142616
5	.052779
6	.016573
7	.004508
8	.001098
9	.000238
10	.000049
11	.000007
12	.000002
14	.000000

c = 7 b = 3

x	P(L1 ≥ x)
0	1
1	.950222
2	.800807
3	.576887
4	.352725
5	.184717
6	.083940
7	.033515
8	.011900
9	.003799
10	.001103
11	.000293
12	.000072
13	.000016
14	.000003
15	.000001
16	.000000
17	.000000
18	.000000
19	.000000
20	.000000

c = 7 b = 4

x	P(L1 ≥ x)
0	1
1	.981689
2	.908404
3	.761915
4	.566536
5	.371147
6	.214866
7	.110681
8	.051138
9	.021362
10	.008130
11	.002839
12	.000915
13	.000274
14	.000076
15	.000020
16	.000005
17	.000001
18	.000000
19	.000000
20	.000000
21	.000000
22	.000000
23	.000000
24	.000000
25	.000000
26	.000000
28	.000000

8. Upper Tail Probabilities Of The Null Distribution of L2

The exact distributions of L2 have been derived using a convolution process and are given for c = 3, b = 2 to 10; c = 4, b = 2 to 10; c = 5, b = 2 to 7; c = 6, b = 2 to 5; c = 7, b = 2 to 4. The tables give the probabilities $P(L2 \geqslant x)$

c = 3 b = 2

x	$P(L2 \geqslant x)$
2	1
3	.750000
4	.416667
5	.138889
6	.027778

c = 3 b = 3

x	$P(L2 \geqslant x)$
3	1
4	.875000
5	.625000
6	.333333
7	.129630
8	.032407
9	.004630

c = 3 b = 4

x	$P(L2 \geqslant x)$
4	1
5	.937500
6	.770833
7	.520833
8	.280093
9	.114969
10	.034722
11	.006944
12	.000772

c = 3 b = 5

x	$P(L2 \geqslant x)$
5	1
6	.968750
7	.864583
8	.673611
9	.442130
10	.237654
11	.102366
12	.034208
13	.008488
14	.001415
15	.000129

c = 3 b = 6

x	$P(L2 \geqslant x)$
6	1
7	.984375
8	.921875
9	.786458
10	.589699
11	.378472
12	.204089
13	.090835
14	.032707
15	.009238
16	.001950

x	P(L2 ≥ x)
17	.000279
18	.000021

c = 3 b = 7

x	P(L2 ≥ x)
7	1
8	.992188
9	.955729
10	.864583
11	.710648
12	.516879
13	.326485
14	.176526
15	.080647
16	.030661
17	.009506
18	.002329
19	.000429
20	.000054
21	.000004

c = 3 b = 8

x	P(L2 ≥ x)
8	1
9	.996094
10	.975260
11	.916233
12	.802807
13	.639419
14	.453977
15	.283238
16	.153580
17	.071634
18	.028414
19	.009443
20	.002575
21	.000558
22	.000091
23	.000010
24	.000001

c = 3 b = 9

x	P(L2 ≥ x)
9	1
10	.998047
11	.986328
12	.949219
13	.869358
14	.740017
15	.573929
16	.399515
17	.246865
18	.134216
19	.063682
20	.026132
21	.009171
22	.002711
23	.000661
24	.000128
25	.000019
26	.000002
27	.000000

c = 3 b = 10

x	P(L2 ≥ x)
10	1
11	.999023
12	.992513
13	.969727
14	.915473
15	.817998
16	.678530
17	.514403
18	.352259
19	.215982
20	.117724
21	.056663

x	P(L2 ⩾ x)
22	.023910
23	.008768
24	.002763
25	.000736
26	.000162
27	.000029
28	.000004
29	.000000
30	.000000

c = 4 b = 2

x	P(L2 ⩾ x)
0.0	1
0.5	.998264
1.0	.9913194
1.5	.973958
2.0	.932292
2.5	.847222
3.0	.729167
3.5	.572917
4.0	.385417
4.5	.239583
5.0	.170139
5.5	.076389
6.0	.055556

x	P(L2 ⩾ x)
7.0	.012153
8.0	.001736

c = 4 b = 3

x	P(L2 ⩾ x)
0.0	1
0.5	.999928
1.0	.999494
1.5	.997975
2.0	.993490
2.5	.981988
3.0	.958116
3.5	.914714
4.0	.843099
4.5	.741753
5.0	.619358
5.5	.481120
6.0	.348307
6.5	.243996
7.0	.162833
7.5	.095775
8.0	.060619
8.5	.026982
9.0	.018736
9.5	.005715

x	P(L2 ⩾ x)
10.0	.004413
11.0	.000723
12.0	.000072

c = 4 b = 4

x	P(L2 ⩾ x)
0.0	1
0.5	.999997
1.0	.999973
1.5	.999864
2.0	.999479
2.5	.998303
3.0	.995265
3.5	.988393
4.0	.974383
4.5	.948773
5.0	.906889
5.5	.844389
6.0	.759657
6.5	.656678
7.0	.542697
7.5	.426342
8.0	.320511
8.5	.231234
9.0	.158173

x	P(L2≥ x)
9.5	.102798
10.0	.064821
10.5	.036929
11.0	.022678
11.5	.010525
12.0	.006619
12.5	.002276
13.0	.001600
13.5	.000370
14.0	.000298
15.0	.000039
16.0	.000003

c = 4   b = 5

x	P(L2≥ x)
0.0	1
0.5	.999999
1.0	.999999
1.5	.999992
2.0	.999962
2.5	.999861
3.0	.999553
3.5	.998734
4.0	.996772
4.5	.992513
5.0	.984103
5.5	.968862
6.0	.943420
6.5	.904340
7.0	.848954
7.5	.776231
8.0	.688069
8.5	.589285
9.0	.486152
9.5	.385878
10.0	.295174
10.5	.217066
11.0	.153225
11.5	.104084
12.0	.067953
12.5	.042426
13.0	.025967
13.5	.014496
14.0	.008554
14.5	.004086
15.0	.002440
15.5	.000929
16.0	.000590
16.5	.000165
17.0	.000118
17.5	.000022
18.0	.000018
19.0	.000002
20.0	.000000

c = 4   b = 6

x	P(L2≥ x)
0.0	1
0.5	.999999
1.0	.999999
1.5	.999999
2.0	.999998
2.5	.999990
3.0	.999963
3.5	.999881
4.0	.999657
4.5	.999102
5.0	.997840
5.5	.995195
6.0	.990056
6.5	.980773
7.0	.965157
7.5	.940619
8.0	.904578
8.5	.855070
9.0	.791346
9.5	.714424
10.0	.627330

x	P(L2 ≥ x)
10.5	.534553
11.0	.441272
11.5	.352662
12.0	.272755
12.5	.203891
13.0	.147361
13.5	.102966
14.0	.069520
14.5	.045384
15.0	.028670
15.5	.017363
16.0	.010357
16.5	.005724
17.0	.003290
17.5	.001603
18.0	.000917
18.5	.000375
19.0	.000223
19.5	.000072
20.0	.000046
20.5	.000011
21.0	.00008
21.5	.000001
22.0	.000001
23.0	.000000
24.0	.000000

c = 4 b = 7

x	P(L2 ≥ x)
0.0	1
0.5	.999999
1.0	.999999
1.5	.999999
2.0	.999999
2.5	.999999
3.0	.999997
3.5	.999990
4.0	.999968
4.5	.999906
5.0	.999750
5.5	.999381
6.0	.998572
6.5	.996915
7.0	.993737
7.5	.988014
8.0	.978314
8.5	.962829
9.0	.939508
9.5	.906344
10.0	.861782
10.5	.805156
11.0	.737024
11.5	.659336

x	P(L2 ≥ x)
12.0	.575270
12.5	.488741
13.0	.403871
13.5	.324386
14.0	.253086
14.5	.191715
15.0	.141009
15.5	.100668
16.0	.069766
16.5	.046934
17.0	.030636
17.5	.019397
18.0	.011947
18.5	.007084
19.0	.004142
19.5	.002271
20.0	.001278
20.5	.000633
21.0	.000351
21.5	.000151
22.0	.000085
22.5	.000030
23.0	.000018
23.5	.000005
24.0	.000003

x	P(L2 ≥ x)
24.5	.000001
25.0	.000001
25.5	.000000
26.0	.000000
27.0	.000000
28.0	.000000

c = 4 b = 8

x	P(L2 ≥ x)
0.0	1
0.5	.999999
1.0	.999999
1.5	.999999
2.0	.999999
2.5	.999999
3.0	.999999
3.5	.999999
4.0	.999997
4.5	.999991
5.0	.999974
5.5	.999930
6.0	.999823
6.5	.999580
7.0	.999061
7.5	.998017
8.0	.996035
8.5	.992470
9.0	.986398
9.5	.976582
10.0	.961511
10.5	.939516
11.0	.908979
11.5	.868617
12.0	.817794
12.5	.756773
13.0	.686843
13.5	.610270
14.0	.530039
14.5	.449473
15.0	.371825
15.5	.299875
16.0	.235666
16.5	.180422
17.0	.134542
17.5	.097706
18.0	.069104
18.5	.047591
19.0	.031914
19.5	.020839
20.0	.013250
20.5	.008194
21.0	.004945
21.5	.002885
22.0	.001660
22.5	.000905
23.0	.000501
23.5	.000251
24.0	.000136
24.5	.000061
25.0	.000033
25.5	.000013
26.0	.000007
26.5	.000002
27.0	.000001
27.5	.000000
28.0	.000000
28.5	.000000
29.0	.000000
29.5	.000000
30.0	.000000
31.0	.000000
32.0	.000000

c = 4 b = 9

x	P(L2 ≥ x)
0.0	1
0.5	.999999
1.0	.999999

x	P(L2 ⩾ x)	x	P(L2 ⩾ x)	x	P(L2 ⩾ x)
1.5	.999999	14.5	.710873	27.5	.000025
2.0	.999999	15.0	.640817	28.0	.000013
2.5	.999999	15.5	.566386	28.5	.000005
3.0	.999999	16.0	.490265	29.0	.000003
3.5	.999999	16.5	.415237	29.5	.000001
4.0	.999999	17.0	.343883	30.0	.000001
4.5	.999999	17.5	.278323	30.5	.000000
5.0	.999998	18.0	.220063	31.0	.000000
5.5	.999993	18.5	.169944	31.5	.000000
6.0	.999990	19.0	.128158	32.0	.000000
6.5	.999949	19.5	.094369	32.5	.000000
7.0	.999877	20.0	.067847	33.0	.000000
7.5	.999718	20.5	.047623	33.5	.000000
8.0	.999385	21.0	.032637	34.0	.000000
8.5	.998724	21.5	.021835	35.0	.000000
9.0	.997478	22.0	.014261	36.0	.000000
9.5	.995242	22.5	.009092		
10.0	.991416	23.0	.005659		
10.5	.985169	23.5	.003434		
11.0	.975433	24.0	.002039		
11.5	.960928	24.5	.001174		
12.0	.940266	25.0	.000667		
12.5	.912101	25.5	.000362		
13.0	.875343	26.0	.000198		
13.5	.829379	26.5	.000010		
14.0	.774272	27.0	.000053		

<u>c = 4   b = 10</u>

x	P(L2 ⩾ x)
0.0	1
0.5	.999999
1.0	.999999
1.5	.999999
2.0	.999999
2.5	.999999
3.0	.999999

x	P(L2 ≥ x)	x	P(L2 ≥ x)	x	P(L2 ≥ x)
3.5	.999999	16.5	.667794	29.5	.000040
4.0	.999999	17.0	.598674	30.0	.000021
4.5	.999999	17.5	.526937	30.5	.000010
5.0	.999999	18.0	.454943	31.0	.000005
5.5	.999999	18.5	.385011	31.5	.000002
6.0	.999998	19.0	.319202	32.0	.000001
6.5	.999994	19.5	.259149	32.5	.000000
7.0	.999986	20.0	.205964	33.0	.000000
7.5	.999964	20.5	.160214	33.5	.000000
8.0	.999916	21.0	.121957	34.0	.000000
8.5	.999812	21.5	.090840	34.5	.000000
9.0	.999598	22.0	.066204	35.0	.000000
9.5	.999178	22.5	.047207	35.5	.000000
10.0	.998390	23.0	.032935	36.0	.000000
10.5	.996979	23.5	.022480	36.5	.000000
11.0	.994555	24.0	.015012	37.5	.000000
11.5	.990568	24.5	.009807	38.0	.000000
12.0	.984274	25.0	.006268	39.0	.000000
12.5	.974742	25.5	.003918	40.0	.000000
13.0	.969879	26.0	.002397		
13.5	.941506	26.5	.001432		
14.0	.915479	27.0	.000839		
14.5	.881849	27.5	.000480		
15.0	.840026	28.0	.000268		
15.5	.789939	28.5	.000145		
16.0	.732133	29.0	.000079		

c = 5 b = 2

x	P(L2 ≥ x)
0.0	1.
0.5	.998889
1.0	.992222
1.5	.970000

x	P(L2 ≥ x)
2.0	.922222
2.5	.840833
3.0	.727500
3.5	.585278
4.0	.439167
4.5	.299722
5.0	.193611
5.5	.108889
6.0	.062778
6.5	.029652
7.0	.017986
7.5	.006041
8.0	.003819
9.0	.000625
10.0	.000069

<u>c = 5   b = 3</u>

x	P(L2 ≥ x)
0.0	1
0.5	.999963
1.0	.999630
1.5	.998019
2.0	.992796
2.5	.979880
3.0	.953907
3.5	.909106
4.0	.841523
4.5	.750481
5.0	.640907
5.5	.520706
6.0	.401567
6.5	.292264
7.0	.201965
7.5	.131288
8.0	.081994
8.5	.047782
9.0	.027602
9.5	.014357
10.0	.007815
10.5	.003420
11.0	.001843
11.5	.000637
12.0	.000380
12.5	.000089
13.0	.000061
14.0	.000008
15.0	.000001

<u>c = 5   b = 4</u>

x	P(L2 ≥ x)
0.0	1
0.5	.999999
1.0	.999984
1.5	.999890
2.0	.999488
2.5	.998176
3.0	.994715
3.5	.986988
4.0	.971953
4.5	.945920
5.0	.905293
5.5	.847536
6.0	.772242
6.5	.681605
7.0	.580551
7.5	.475636
8.0	.374147
8.5	.282069
9.0	.203912
9.5	.141183
10.0	.093963
10.5	.059906
11.0	.036908
11.5	.021711
12.0	.012458
12.5	.006729
13.0	.003622
13.5	.001774

x	P(L2 ⩾ x)
14.0	.000915
14.5	.000396
15.0	.000202
15.5	.000072
16.0	.000039
16.5	.000010
17.0	.000006
17.5	.000001
18.0	.000001
19.0	.000000
20.0	.000000

c = 5   b = 5

x	P(L2 ⩾ x)
0.0	1
0.5	.999999
1.0	.999999
1.5	.999994
2.0	.999969
2.5	.999865
3.0	.999526
3.5	.998592
4.0	.996363
4.5	.991649
5.0	.982672
5.5	.967103
6.0	.772242
6.5	.681605
7.0	.855596
7.5	.791374
8.0	.714244
8.5	.627120
9.0	.534343
9.5	.440963
10.0	.351983
10.5	.271501
11.0	.202323
11.5	.145629
12.0	.101319
12.5	.068124
13.0	.044341
13.5	.027901
14.0	.017032
14.5	.010040
15.0	.005762
15.5	.003179
16.0	.001722
16.5	.000886
17.0	.000456
17.5	.000216
18.0	.000107
18.5	.000046
19.0	.000022
19.5	.000008
20.0	.000004
20.5	.000001
21.0	.000001
21.5	.000000
22.0	.000000
22.5	.000000
23.0	.000000
24.0	.000000
25.0	.000000

c = 5   b = 6

x	P(L2 ⩾ x)
0.0	1
0.5	.999999
1.0	.999999
1.5	.999999
2.0	.999998
2.5	.999991
3.0	.999964
3.5	.999874
4.0	.999621
4.5	.998987
5.0	.997561
5.5	.994648

x	P(L2 ≥ x)
6.0	.989182
6.5	.979693
7.0	.964339
7.5	.941175
8.0	.908246
8.5	.864158
9.0	.808352
9.5	.741406
10.0	.665130
10.5	.582429
11.0	.496961
11.5	.412630
12.0	.333072
12.5	.261197
13.0	.198935
13.5	.147132
14.0	.105686
14.5	.073741
15.0	.050001
15.5	.032951
16.0	.021121
16.5	.013163
17.0	.007988
17.5	.004712
18.0	.002711
18.5	.001514
19.0	.000827
19.5	.000437
20.0	.000227
20.5	.000113
21.0	.000056
21.5	.000026
22.0	.000012
22.5	.000005
23.0	.000003
23.5	.000001
24.0	.000000
24.5	.000000
25.0	.000000
25.5	.000000
26.0	.000000
26.5	.000000
27.0	.000000
27.5	.000000
28.0	.000000
29.0	.000000
30.0	.000000

c = 5 b = 7

x	P(L2 ≥ x)
0.0	1
0.5	.999999
1.0	.999999
1.5	.999999
2.0	.999999
2.5	.999999
3.0	.999998
3.5	.999990
4.0	.999966
4.5	.999897
5.0	.999718
5.5	.999296
6.0	.998385
6.5	.996568
7.0	.993194
7.5	.987333
8.0	.977765
8.5	.963026
9.0	.941535
9.5	.911796
10.0	.872643
10.5	.823503
11.0	.764600
11.5	.697059
12.0	.622867
12.5	.544691
13.0	.465573
13.5	.388579

x	P(L2 ≥ x)
14.0	.316451
14.5	.251330
15.0	.194610
15.5	.146893
16.0	.108083
16.5	.077531
17.0	.054230
17.5	.036994
18.0	.024621
18.5	.015988
19.0	.010135
19.5	.006271
20.0	.003790
20.5	.002236
21.0	.001290
21.5	.000726
22.0	.000400
22.5	.000215
23.0	.000113
23.5	.000058
24.0	.000029
24.5	.000014
25.0	.000007
25.5	.000003
26.0	.000001
26.5	.000001

x	P(L2 ≥ x)
27.0	.000000
27.5	.000000
28.0	.000000
28.5	.000000
29.0	.000000
29.5	.000000
30.0	.000000
30.5	.000000
31.0	.000000
31.5	.000000
32.0	.000000
32.5	.000000
33.0	.000000
34.0	.000000
35.0	.000000

x	P(L2 ≥ x)
4.0	.454994
4.5	.324747
5.0	.216113
5.5	.133663
6.0	.077413
6.5	.041690
7.0	.021644
7.5	.010108
8.0	.004992
8.5	.001956
9.0	.001022
9.5	.000278
10.0	.000162
11.0	.000021
12.0	.000002

$c = 6 \quad b = 2$

x	P(L2 ≥ x)
0.0	1
0.5	.998378
1.0	.990770
1.5	.968648
2.0	.921356
2.5	.841202
3.0	.729288
3.5	.594863

$c = 6 \quad b = 3$

x	P(L2 ≥ x)
0.0	1
0.5	.999935
1.0	.999475
1.5	.997600
2.0	.992030
2.5	.978865
3.0	.952927
3.5	.908963

x	P(L2 ⩾ x)
4.0	.843572
4.5	.756817
5.0	.652919
5.5	.539466
6.0	.425674
6.5	.320149
7.0	.229318
7.5	.156372
8.0	.101596
8.5	.062927
9.0	.037243
9.5	.021077
10.0	.011462
10.5	.005967
11.0	.003012
11.5	.001439
12.0	.000683
12.5	.000294
13.0	.000134
13.5	.000049
14.0	.000023
14.5	.000007
15.0	.000004
15.5	.000001
16.0	.000000
17.0	.000000
18.0	.000000

c = 6 b = 4

x	P(L2 ⩾ x)
0.0	1
0.5	.999997
1.0	.999973
1.5	.999843
2.0	.999353
2.5	.997884
3.0	.994209
3.5	.986287
4.0	.971255
4.5	.945747
5.0	.906569
5.5	.851582
6.0	.780506
6.5	.695359
7.0	.600291
7.5	.500935
8.0	.403364
8.5	.313032
9.0	.233971
9.5	.168381
10.0	.116690
10.5	.077901
11.0	.050132
11.5	.031124
12.0	.018662
12.5	.010805

x	P(L2 ⩾ x)
13.0	.006066
13.5	.003293
14.0	.001734
14.5	.000884
15.0	.000438
15.5	.000209
16.0	.000098
16.5	.000044
17.0	.000020
17.5	.000008
18.0	.000003
18.5	.000001
19.0	.000001
19.5	.000000
20.0	.000000
20.5	.000000
21.0	.000000
21.5	.000000
22.0	.000000
23.0	.000000
24.0	.000000

c = 6 b = 5

x	P(L2 ⩾ x)
0.0	1
0.5	.999999
1.0	.999999

x	P(L2 ≥ x)
1.5	.999991
2.0	.999954
2.5	.999822
3.0	.999424
3.5	.998390
4.0	.996026
4.5	.991185
5.0	.982200
5.5	.966939
6.0	.943029
6.5	.908257
7.0	.861057
7.5	.800982
8.0	.729006
8.5	.647548
9.0	.560208
9.5	.471255
10.0	.385001
10.5	.305206
11.0	.234646
11.5	.174906
12.0	.126400
12.5	.088573
13.0	.060201
13.5	.039703
14.0	.025422

x	P(L2 ≥ x)
14.5	.015812
15.0	.009559
15.5	.005621
16.0	.003216
16.5	.001792
17.0	.000972
17.5	.000514
18.0	.000265
18.5	.000133
19.0	.000065
19.5	.000031
20.0	.000015
20.5	.000007
21.0	.000003
21.5	.000001
22.0	.000001
22.5	.000000
23.0	.000000
23.5	.000000
24.0	.000000
24.5	.000000
25.0	.000000
25.5	.000000
26.0	.000000
26.5	.000000
27.0	.000000

x	P(L2 ≥ x)
27.5	.000000
28.0	.000000
29.0	.000000
30.0	.000000

c = 7 b = 2

x	P(L2 ≥ x)
0.0	1
0.5	.998329
1.0	.990285
1.5	.967218
2.0	.919689
2.5	.841139
3.0	.732551
3.5	.602772
4.0	.466681
4.5	.339039
5.0	.231218
5.5	.147929
6.0	.089063
6.5	.050394
7.0	.026950
7.5	.013570
8.0	.006546
8.5	.002967
9.0	.001332

x	P(L2 ≥ x)
9.5	.000535
10.0	.000234
10.5	.000078
11.0	.000037
11.5	.000008
12.0	.000005
13.0	.000001
14.0	.000000

c = 7   b = 3

x	P(L2 ≥ x)
0.0	1
0.5	.999932
1.0	.999438
1.5	.997431
2.0	.991588
2.5	.978113
3.0	.952145
3.5	.908915
4.0	.845389
4.5	.761680
5.0	.661618
5.5	.552097
6.0	.441529
6.5	.337919
7.0	.247342
7.5	.173124
8.0	.115916
8.5	.074285
9.0	.045610
9.5	.026855
10.0	.015187
10.5	.008258
11.0	.004326
11.5	.002185
12.0	.001067
12.5	.000503
13.0	.000230
13.5	.000101
14.0	.000044
14.5	.000018
15.0	.000007
15.5	.000002
16.0	.000001
16.5	.000000
17.0	.000000
17.5	.000000
18.0	.000000
18.5	.000000
19.0	.000000
20.0	.000000
21.0	.000000

c = 7   b = 4

x	P(L2 ≥ x)
0.0	1
0.5	.999997
1.0	.999970
1.5	.999829
2.0	.999299
2.5	.997739
3.0	.993920
3.5	.985857
4.0	.970837
4.5	.945742
5.0	.907654
5.5	.854636
6.0	.786427
6.5	.704808
7.0	.613492
7.5	.517538
8.0	.422478
8.5	.333390
9.0	.254177
9.5	.187174
10.0	.133137
10.5	.091496
11.0	.060775
11.5	.039040
12.0	.024268

x	P(L2 ≥ x)	x	P(L2 ≥ x)
12.5	.014608	25.5	.000000
13.0	.008522	26.0	.000000
13.5	.004822	27.0	.000000
14.0	.002649	28.0	.000000
14.5	.001414		
15.0	.000734		
15.5	.000370		
16.0	.000182		
16.5	.000087		
17.0	.000041		
17.5	.000019		
18.0	.000008		
18.5	.000004		
19.0	.000002		
19.5	.000001		
20.0	.000000		
20.5	.000000		
21.0	.000000		
21.5	.000000		
22.0	.000000		
22.5	.000000		
23.0	.000000		
23.5	.000000		
24.0	.000000		
24.5	.000000		
25.0	.000000		

9. Asymptotic Critical Values of L1

As a consequence of the asymptotic ($c \to \infty$) distribution of L1 being Poisson with mean b, we are able to obtain approximate critical values which are independent of the number of treatments.

Comparison with the exact null distributions given in section 7 reveals that these approximate critical values agree with the known true best conservative critical values in all cases except $c = 3$, $b = 5$ and $c = 4$, $b = 4$.

A selection of best conservative critical values obtained from the Poisson approximation is given in the table below.

b	Significance Level 5 %	1 %	0.1 %
2	6	7	9
3	7	9	11
4	9	10	12
5	10	12	14
6	11	13	16
7	13	15	17
8	14	16	19
9	15	18	21
10	16	19	22
11	18	20	23
12	19	22	25
13	20	23	26
14	21	24	28

b	5 %	1 %	0.1 %
15	23	26	29
16	24	27	31
17	25	28	32
18	26	30	33
19	27	31	35
20	29	32	36
21	30	33	37
22	31	35	39
23	32	36	40
24	33	37	41
25	34	38	43

When b is also large we may employ the normal distribution to obtain approximate critical values. Using a normal distribution with mean and variance equal to b, we obtain the following table.

	Significance Level		
	5 %	1 %	0.1 %
Critical value	$1.65\sqrt{b} + b + \frac{1}{2}$	$2.33\sqrt{b} + b + \frac{1}{2}$	$3.09\sqrt{b} + b + \frac{1}{2}$

10. Asymptotic Critical Values of L2

In view of L2 being asymptotically normal ($b \to \infty$), approximate critical values may be obtained from a normal distribution with mean $b(2 - \frac{1}{c})$ and variance $\frac{b}{c}\left(\frac{3(c-2)}{2} + \frac{1}{c(c-1)} \right)$. A comparison of some true (best conservative) critical values with the appropriate approximation is given in the table below.

		Significance Level					
		5 %		1 %		0.1 %	
c	b	True	Approx.	True	Approx.	True	Approx.
4	3	8	8	9	9	10.5	10
	4	10	10	11.5	11.5	13	13
	5	12.5	12	14	13.5	15	15
	6	14.5	14.5	16	16	18	17.5
	7	16.5	16.5	18.5	18	20	20
	8	18.5	18.5	20.5	20	22.5	22
	9	20.5	20.5	22.5	22.5	24.5	24.5
	10	22.5	22.5	24.5	24.5	27	26.5
5	3	8	8.5	10	10	11	11
	4	10.5	10.5	12	12	14	13.5
	5	13	13	14.5	14.5	16.5	16
	6	15	15	17	16.5	19	18.5
	7	17	17	19	19	21.5	21

These results quite justify the use of the normal distribution in obtaining approximate critical values of L2.

Should c be sufficiently large then the mean and variance of L2 approximate to 2b and 3b/2 respectively. This simplifies the calculation of the approximate critical values by the use of $z_c\sqrt{3b/2} + 2b + \frac{1}{2}$, where z_c is the appropriate critical value from the standard normal distribution.

11. Exact Power Calculations for L1

Before analysing the computer simulations it is interesting to reflect on the validity of such results. Fortunately, it is a comparatively easy task to calculate the exact power of L1 for three treatments and four blocks. We shall use an exponential and then a uniform distribution.

For the purpose of the exact power calculations we reformulate our model. Let X_j ($j = 1, 2, 3$) denote independent random variables with a continuous distribution function given by

$$F_j(x - \alpha_j) = P(X_j \leqslant x) \quad ,$$

where α_j is a location parameter corresponding to the j^{th} treatment.

We test the null hypothesis

$$H_o : F_1 = F_2 = F_3$$

against the ordered alternative

$$H_1 :: F_1 < F_2 < F_3 \quad .$$

The probabilities of obtaining exactly 0, 1 and 3 matches between the predicted order and any particular block are denoted by $P_{[0]}$, $P_{[1]}$ and $P_{[3]}$ respectively.

We then have

$$P_{[1]} = P(X_1 < X_3 < X_2) + P(X_3 < X_2 < X_1) + P(X_2 < X_1 < X_3)$$

$$P_{[3]} = P(X_1 < X_2 < X_3)$$

and $P_{[0]} = 1 - P_{[1]} - P_{[3]}$

For the exponential distribution case we consider the distribution functions,

$$F_1(x_1) = 1 - e^{-x_1} \quad , \quad (x_1 \geqslant 0)$$

$$F_2(x_2) = 1 - e^{-x_2/a_1} = F_1(x_2/a_1) \quad , \quad (x_2 \geqslant 0)$$

$$F_3(x_3) = 1 - e^{-x_3/a_2} = F_1(x_3/a_2) \quad , \quad (x_3 \geqslant 0) .$$

Now $P_3 = P(X_1 \quad X_2 \quad X_3)$

$$= \int_{-\infty}^{\infty} dF_3(x_3) \int_{-\infty}^{x_3} dF_2(x_2) \int_{-\infty}^{x_2} dF(x_1)$$

$$= \int_0^{\infty} dF_3(x_3) \int_0^{x_3} (1 - e^{-x_2}) dF_2(x_2)$$

$$= \int_0^{\infty} \left\{ \frac{a_1}{1+a_1} - e^{-x_3/a_1} + \frac{1}{1+a} e^{-x_3(1+a_1)/a_1} \right\} dF_3(x_3)$$

$$= \frac{a_1 a_2^2}{(a_1 + a_2)(a_1 + a_2 + a_1 a_2)}$$

In a similar manner we calculate the components of $P_{[1]}$.

$$P(X_1 < X_3 < X_2) = \int_{-\infty}^{\infty} dF(x_2) \int_{-\infty}^{x_2} dF_3(x_3) \int_{-\infty}^{x_3} dF_1(x_1)$$

$$= \frac{a_1^2 a_2}{(a_1 + a_2)(a_1 + a_2 + a_1 a_2)}$$

$$P(X_3 < X_2 < X_1) = \int_{-\infty}^{\infty} dF_1(x_1) \int_{-\infty}^{x} dF_2(x_2) \int_{-\infty}^{x_2} dF_3(x_3)$$

$$= \frac{1}{(a_1 + 1)(a_1 + a_2 + a_1 a_2)}$$

$$P(X_2 < X_1 < X_3) = \int_{-\infty}^{\infty} dF_3(x_3) \int_{-\infty}^{x_3} dF_1(x_1) \int_{-\infty}^{x_1} dF_2(x_2)$$

$$= \frac{a_2^2}{(a_2 + 1)(a_1 + a_2 + a_1 a_2)} .$$

Hence $$P_{[1]} = \frac{1}{a_1 + a_2 + a_1 a_2} \left\{ \frac{a_1^2 a_2}{a_1 + a_2} + \frac{1}{a_1 + 1} + \frac{a_2^2}{a_2 + 1} \right\}$$

If we now let $a_1 = 1 + \theta$, $a_2 = 1 + 2\theta$ $(0 \leqslant \theta < \infty)$ so that when $\theta = 0$ H_o holds true, then we obtain the above probabilities in terms of θ.

$$P_{[1]} = \frac{1}{2\theta^2 + 6\theta + 3} \left\{ \frac{(1+\theta)^2(1+2\theta)}{(2+3\theta)} + \frac{1}{2+\theta} + \frac{(1+2\theta)^2}{2(1+\theta)} \right\}$$

$$P_{[3]} = \frac{(1+\theta)(1+2\theta)^2}{(2\theta^2 + 6\theta + 3)(2+3\theta)}$$

with $P_{[0]} = 1 - P_{[1]} - P_{[3]}$.

We now derive similar expressions for $P_{[0]}$, $P_{[1]}$ and $P_{[3]}$ for the uniform distribution.

The three distribution functions are now taken to be

$$F_1(x_1) = \begin{cases} 0 & (\, x < 0 \,) \\ x & (\, 0 \leqslant x \leqslant 1 \,) \\ 1 & (\, x > 1 \,) \end{cases}$$

$$F_2(x_2) = \begin{cases} 0 & (\, x < \theta \,) \\ x - \theta & (\, \theta \leqslant x \leqslant 1 + \theta \,) \\ 1 & (\, x > 1 + \theta \,) \end{cases}$$

$$F_3(x_3) = \begin{cases} 0 & (\, x < 2\theta \,) \\ x - 2\theta & (\, 2\theta \leqslant x \leqslant 1 + 2\theta \,) \\ 1 & (\, x > 1 + 2\theta \,) \end{cases} .$$

As before, for $P_{[1]}$ we require $P(X_2 < X_1 < X_3)$, $P(X_3 < X_2 < X_1)$ and $P(X_1 < X_3 < X_2)$.

Now

$$P(X_2 < X_1 < X_3) = \int_{-\infty}^{\infty} dF_3(x_3) \int_{-\infty}^{x_3} dF_1(x_1) \int_{-\infty}^{x} dF_2(x_2) \quad .$$

We let

$$I_1 = \int_{-\infty}^{x} dF_2(x_2) = \begin{cases} x_1 - \theta & (x_1 > \theta) \\ 0 & (0 \leq x_1 \leq \theta \leq 1) \end{cases}$$

so that

$$I_2 = \int_{-\infty}^{x_3} I_1 dF_1(x_1)$$

$$= \begin{cases} \tfrac{1}{2}x_3^2 + \tfrac{1}{2}\theta^2 - \theta x_3 & (0 \leq \theta < \tfrac{1}{2},\ x_3 < 1) \\ \tfrac{1}{2}(1 - \theta)^2 & (\tfrac{1}{2} \leq \theta < 1,\ x_3 \geq 1) \end{cases}$$

Hence

$$P(X_2 < X_1 < X_3) = \int_{-\infty}^{\infty} I_2 dF_3(x_3)$$

$$= \begin{cases} \frac{1}{6} + \frac{\theta}{2} - \frac{3\theta^2}{2} + \frac{2\theta^3}{3} & (0 \leq \theta < \tfrac{1}{2}) \\ \tfrac{1}{2}(1 - \theta)^2 & (\tfrac{1}{2} \leq \theta \leq 1) \quad . \end{cases}$$

In a similar manner we obtain

$$P(X_3 < X_2 < X_1) = \begin{cases} \frac{1}{6} - \theta + 2\theta^2 - \frac{4\theta^3}{3} & (0 \leq \theta < \tfrac{1}{2}) \\ 0 & (\tfrac{1}{2} \leq \theta \leq 1) \end{cases}$$

$$P(X_1 < X_3 < X_2) = \begin{cases} \frac{1}{6} + \frac{\theta}{2} - \frac{3\theta^2}{2} + \frac{2\theta^3}{3} & (0 \leq \theta < \tfrac{1}{2}) \\ \tfrac{1}{2}(1 - \theta)^2 & (\tfrac{1}{2} \leq \theta < 1) \quad . \end{cases}$$

Combining these probabilities we obtain

$$P_{[1]} = \begin{cases} \frac{1}{2} - \theta^2 & (0 \leqslant \theta < \frac{1}{2}) \\ (1 - \theta)^2 & (\frac{1}{2} \leqslant \theta \leqslant 1) \ . \end{cases}$$

Now

$$\begin{aligned} P_{[3]} &= P(X_1 < X_2 < X_3) \\ &= \int_{-\infty}^{\infty} dF_3(x_3) \int_{-\infty}^{x_3} dF_2(x_2) \int_{-\infty}^{x_2} dF_1(x_1) \\ &= \begin{cases} \frac{1}{6} + \theta + \theta^2 - \frac{4\theta^3}{3} & (0 \leqslant \theta \leqslant \frac{1}{2}) \\ \theta(2 - \theta) & (\frac{1}{2} \leqslant \theta \leqslant 1) \end{cases} \end{aligned}$$

Of course, as before, $P_{[0]} = 1 - P_{[1]} - P_{[3]}$.

From the probability distribution for L1 with three treatments and four blocks we see that $P(L1 \geqslant 8) = 0.0579$; it is this critical value we use in our comparision of the powers.

In terms of the above probabilities $P_{[0]}$, $P_{[1]}$ and $P_{[3]}$,

$$P(L1 \geqslant 8) = 4P_{[0]}P^3_{[3]} + 6P^2_{[1]}P^2_{[3]} + 4P_{[1]}P^3_{[3]} + P^4_{[3]} \ ,$$

and so by varying the value of θ from 0 upwards we may compare the exact and simulated powers. The results of these comparisons are given in the following tables.

Exponential Distribution

θ	0	.2	.4	.6	.8	1
Exact power	.058	.093	.130	.167	.203	.238
Simulated power	.050	.095	.132	.197	.220	.263

Uniform Distribution

θ	0	.2	.4	.6	.8	1
Exact power	.058	.374	.833	.986	1	1
Simulated power	.063	.371	.836	.984	.999	1

12. Comments and Results of the Simulations

As previously, the comments are in two sections, one for the linear case and the other for the non-linear case. The Inversion test to which we refer is our version of Jonckheere's test. We included the F-test simply to discover how well it would perform under ordered alternatives. The simulations are based on four treatments and four blocks.

(i) Results from the linear model $X_{ij} = M + A_i + B_j + z_{ij}$.

Normal Distribution. Although the F-test is not one of best performers it has certainly produced a creditable result. Of the nonparametric tests, there is little to choose between Page, Inversion and L2. Even L1, the simplest of all the tests, produced a good performance.

Uniform Distribution. Clearly, Page's and the Inversion tests are at the forefront in overall performance. However in the 5 % case L2 performs as well as these upto $\theta = 0.25$.

Double Exponential Distribution. Throughout the range L2, Page's and the Inversion tests produced excellent results. L1 also rendered a good result, achieving a maximum power of approximately 0.7 in the 5 % case.

Cauchy Distribution. In both the 5 % and 1 % cases L2, Page's and the Inversion tests produced indistinguishable results, attaining a maximum power of approximately 0.8 in the 5 % case. Somewhat predictably, the F-test exhibited non-robust features.

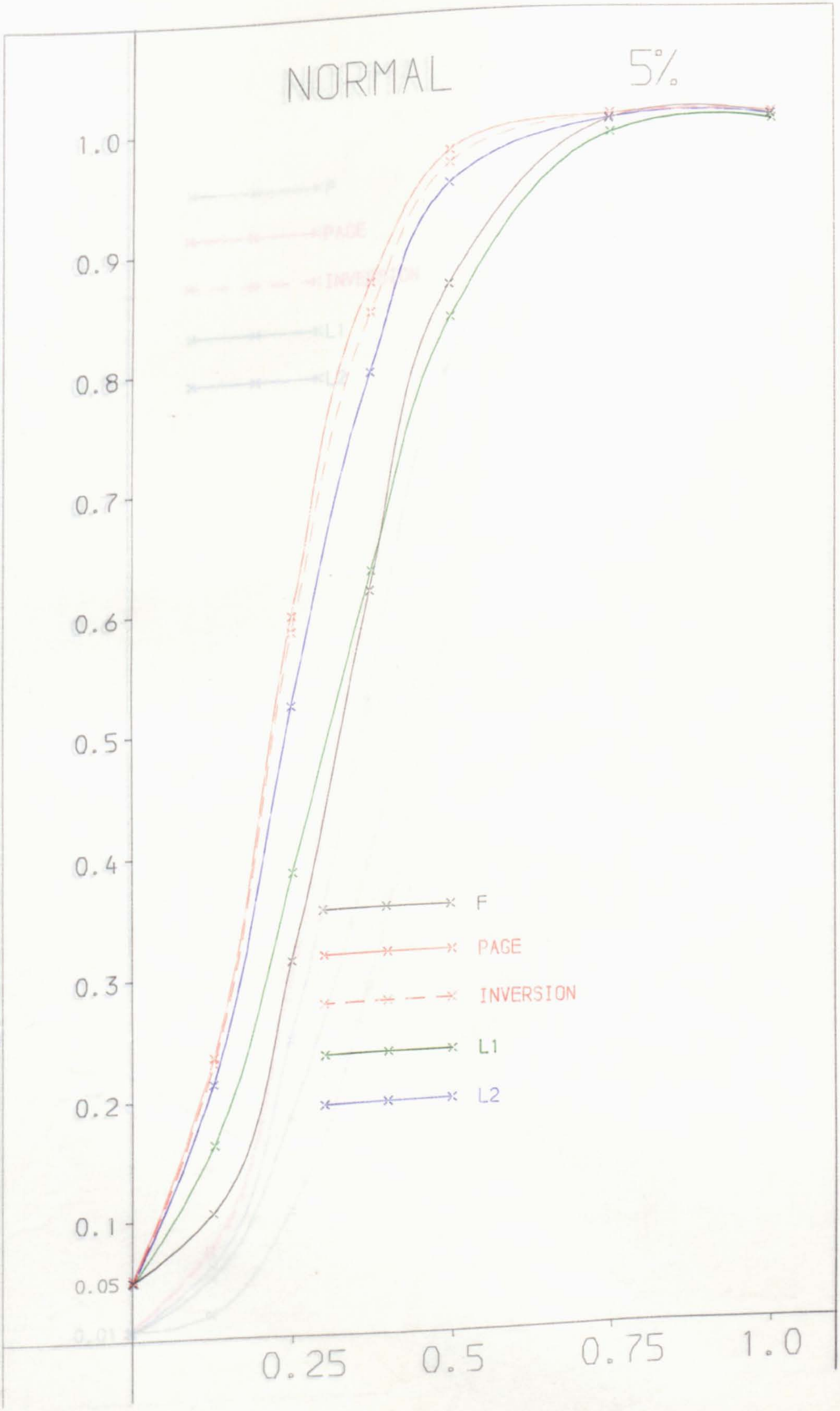
Exponential Distribution. All tests have produced a greater maximum power than in the corresponding general alternatives case, being in excess of 0.8 in the 5 % case for the top three tests.

(ii) Results from the non-linear model $X_{ij} = M + A_i + B_j z_{ij}$.

Normal Distribution. Once again, L2, Page's and the Inversion tests have produced virtually identical results. However the maximum achieved is only approximately 0.4 as compared to 1 in the linear model. Note the non-robust behaviour of the F-test.

Uniform Distribution. The Inversion and Page's tests have produced almost identical results with L2 following. A reasonable maximum power is achieved by the nonparametric tests.

<u>Exponential Distribution</u>. The most notable feature is the poor performance by all the tests; the maximum power in the 5 % case being only approximately 0.3 and 0.1 in the 1 % case.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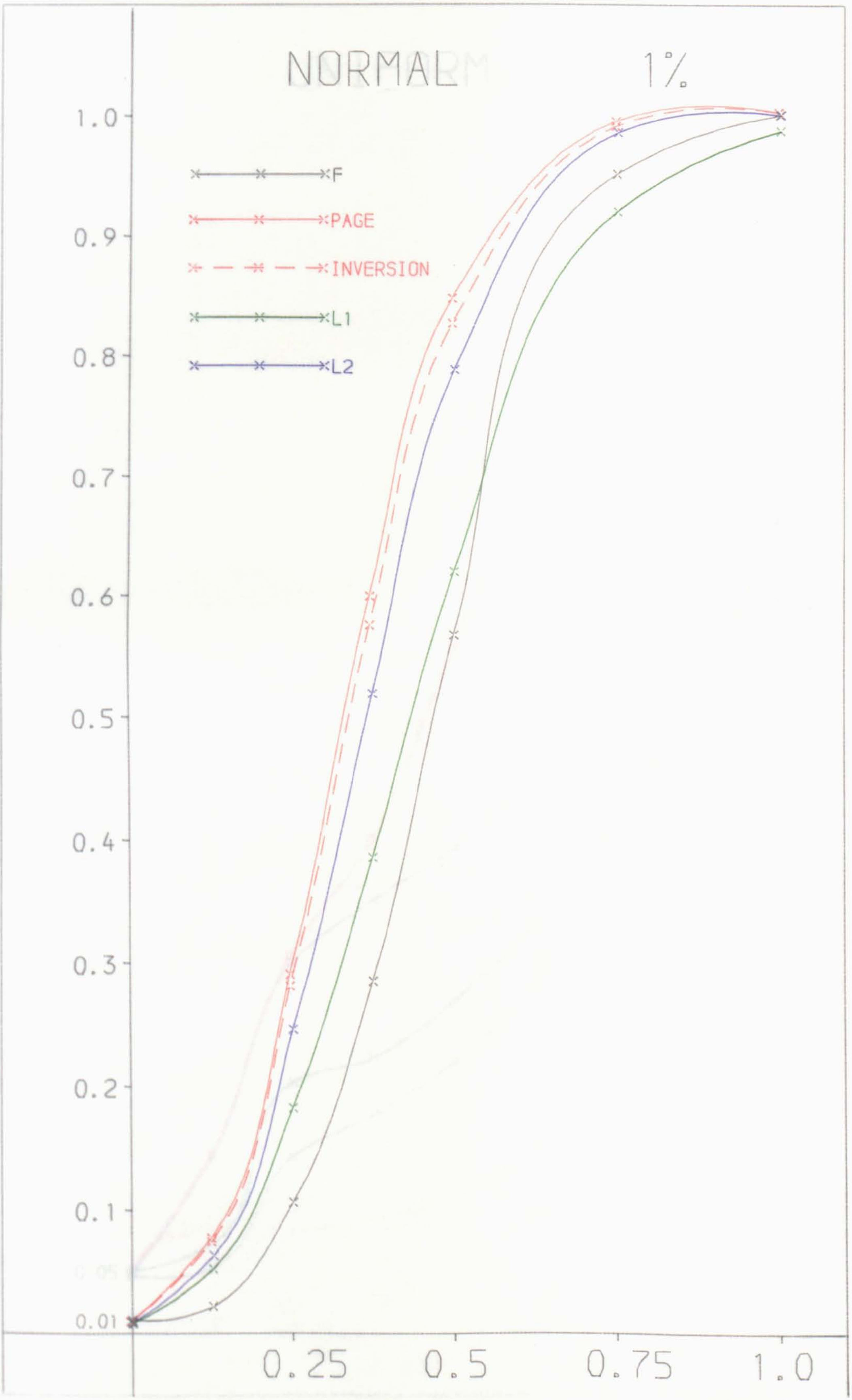

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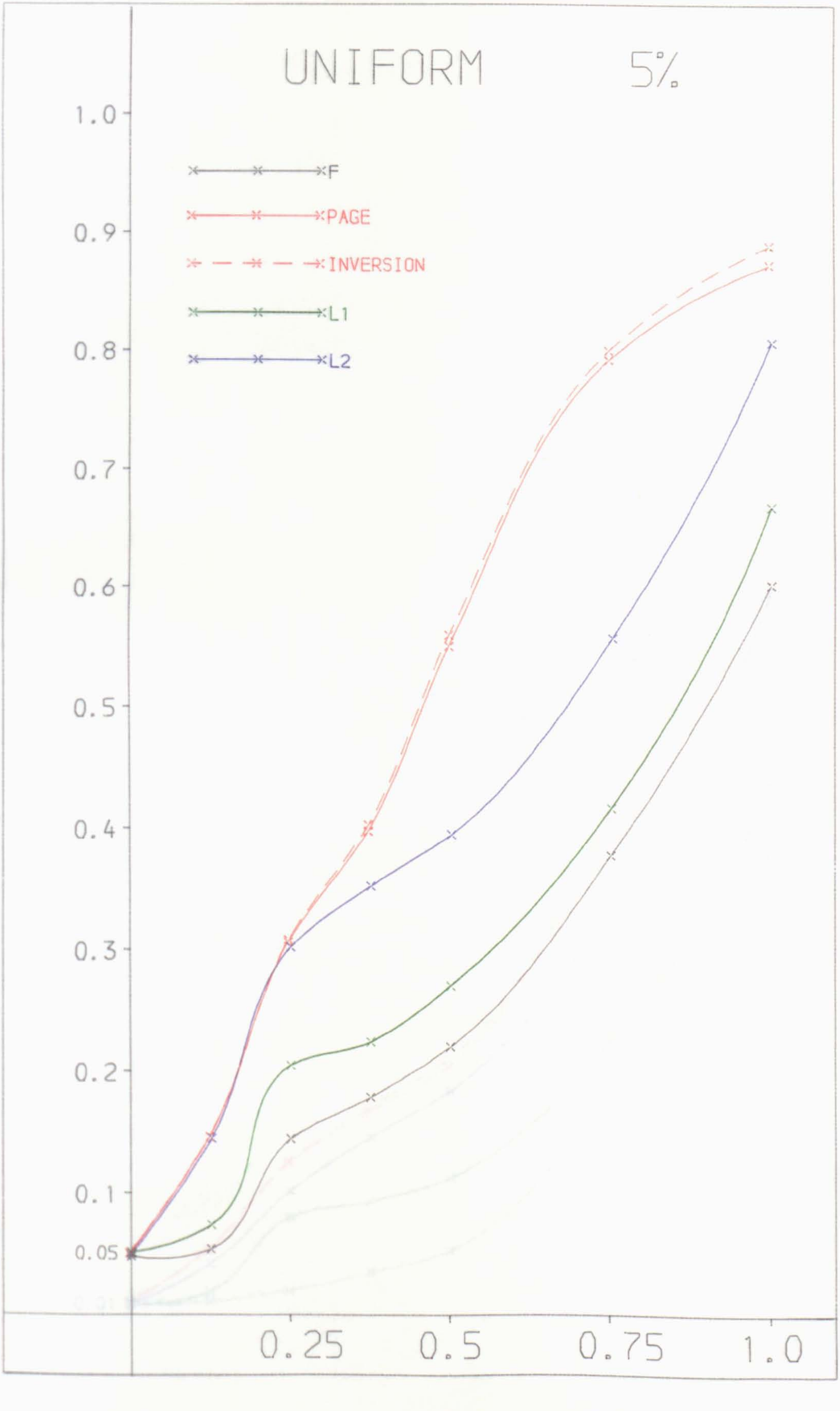
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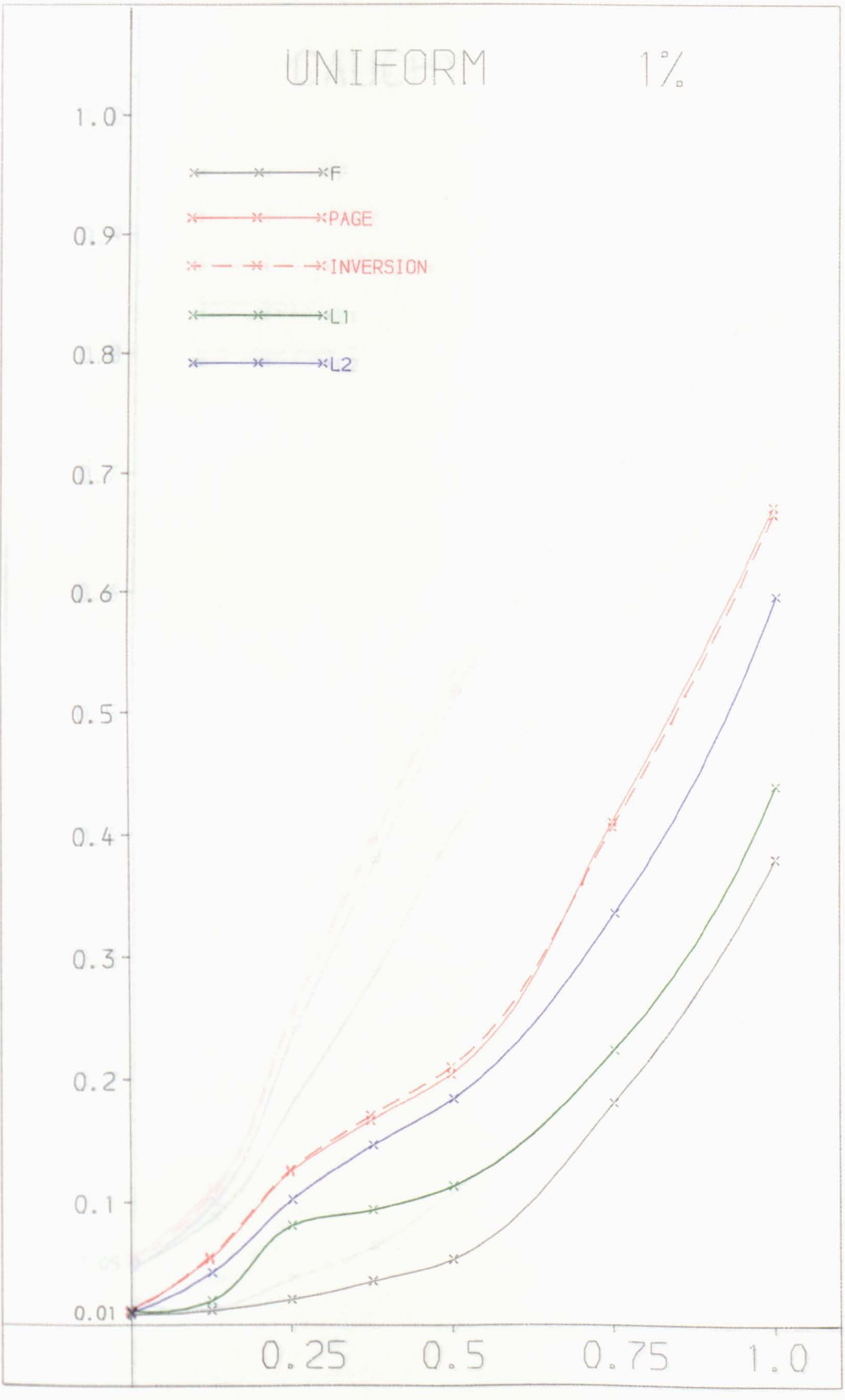

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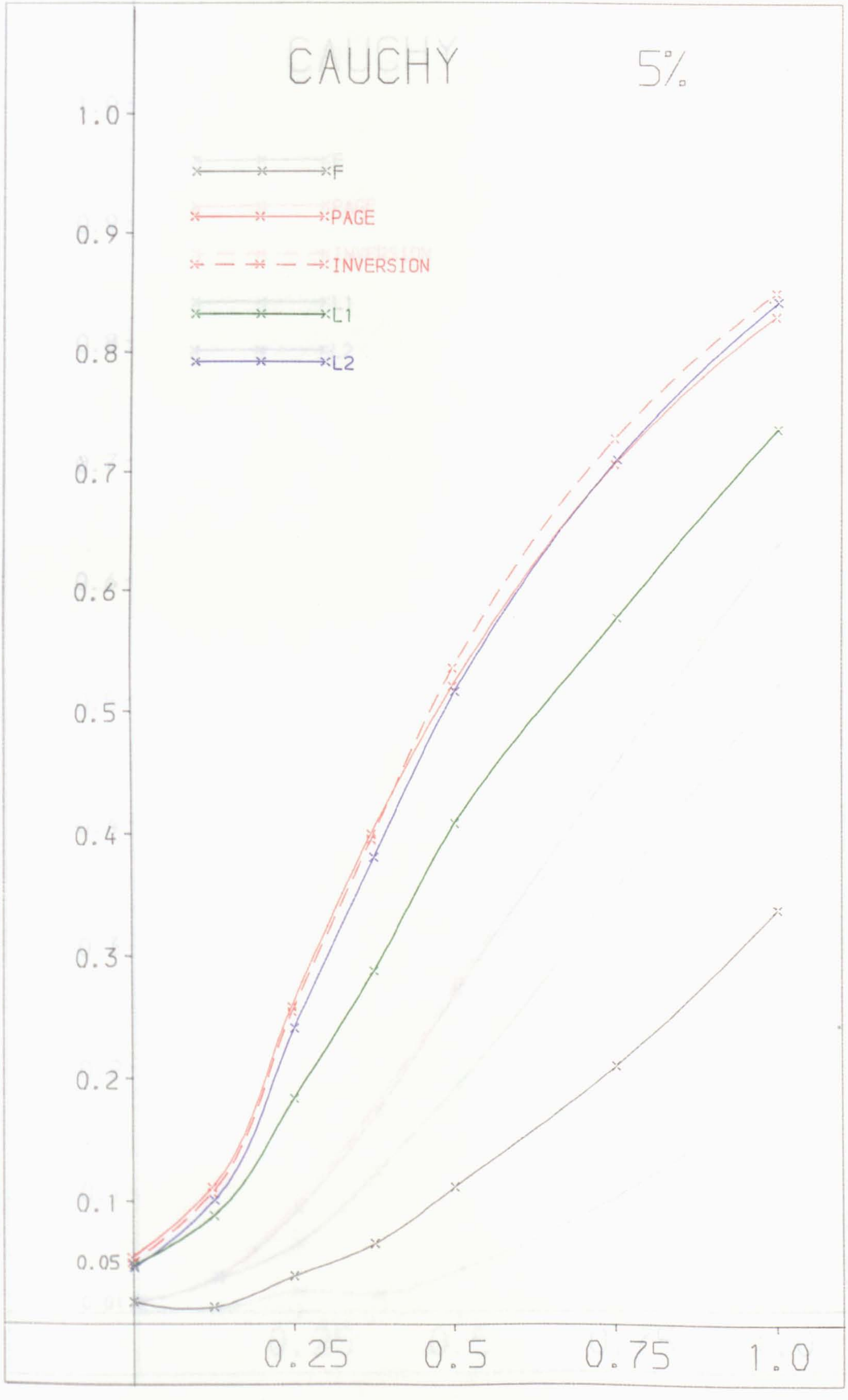
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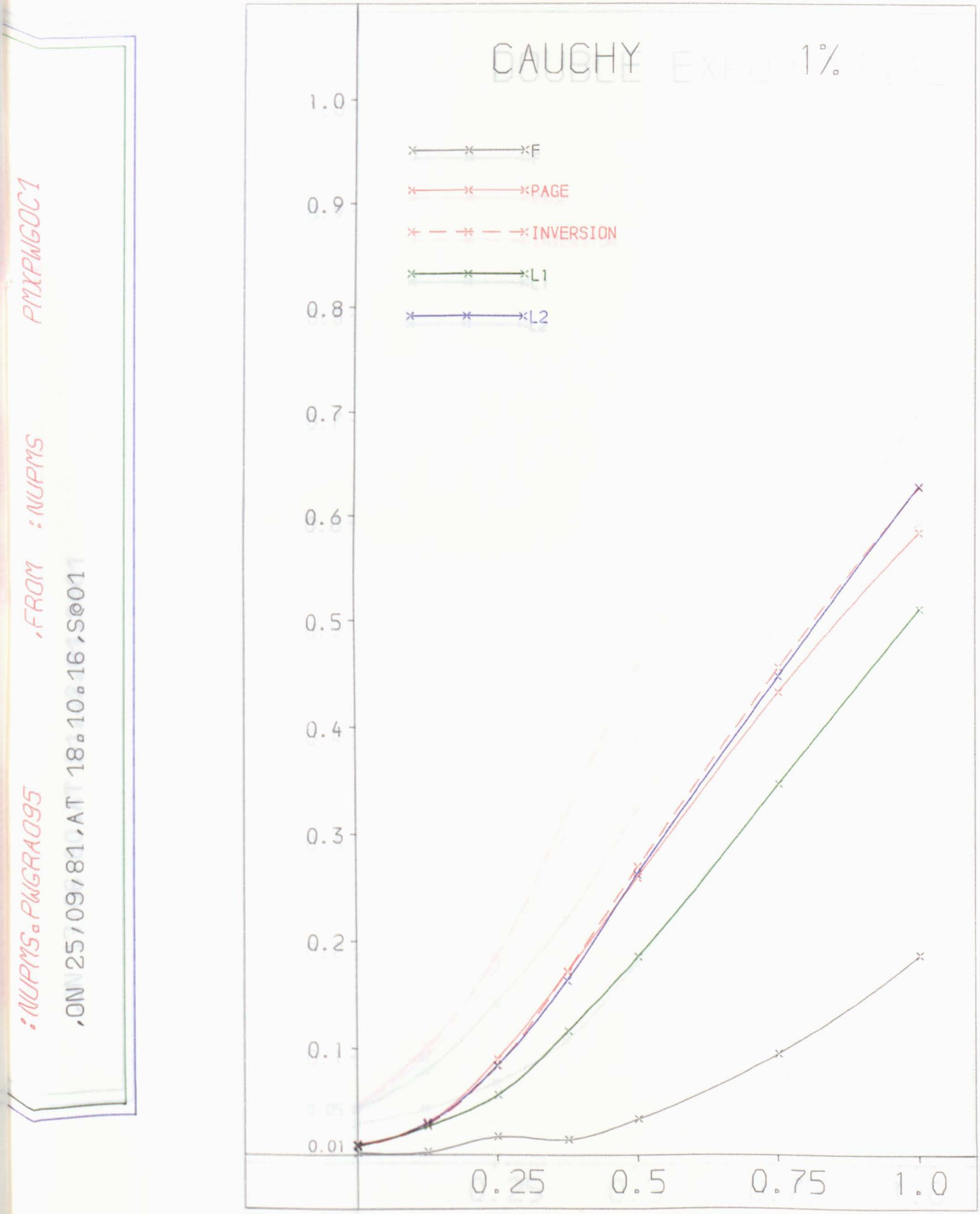
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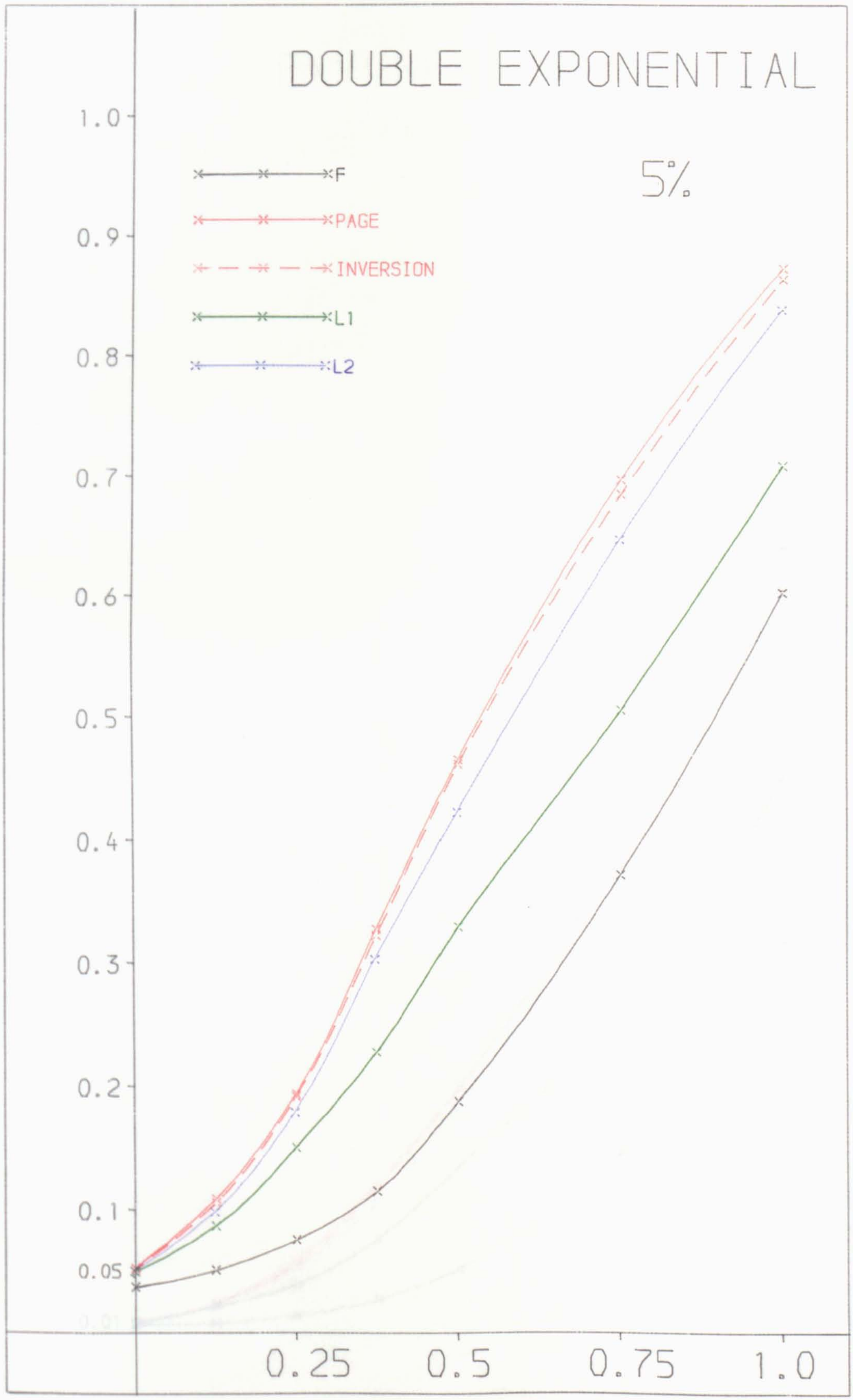

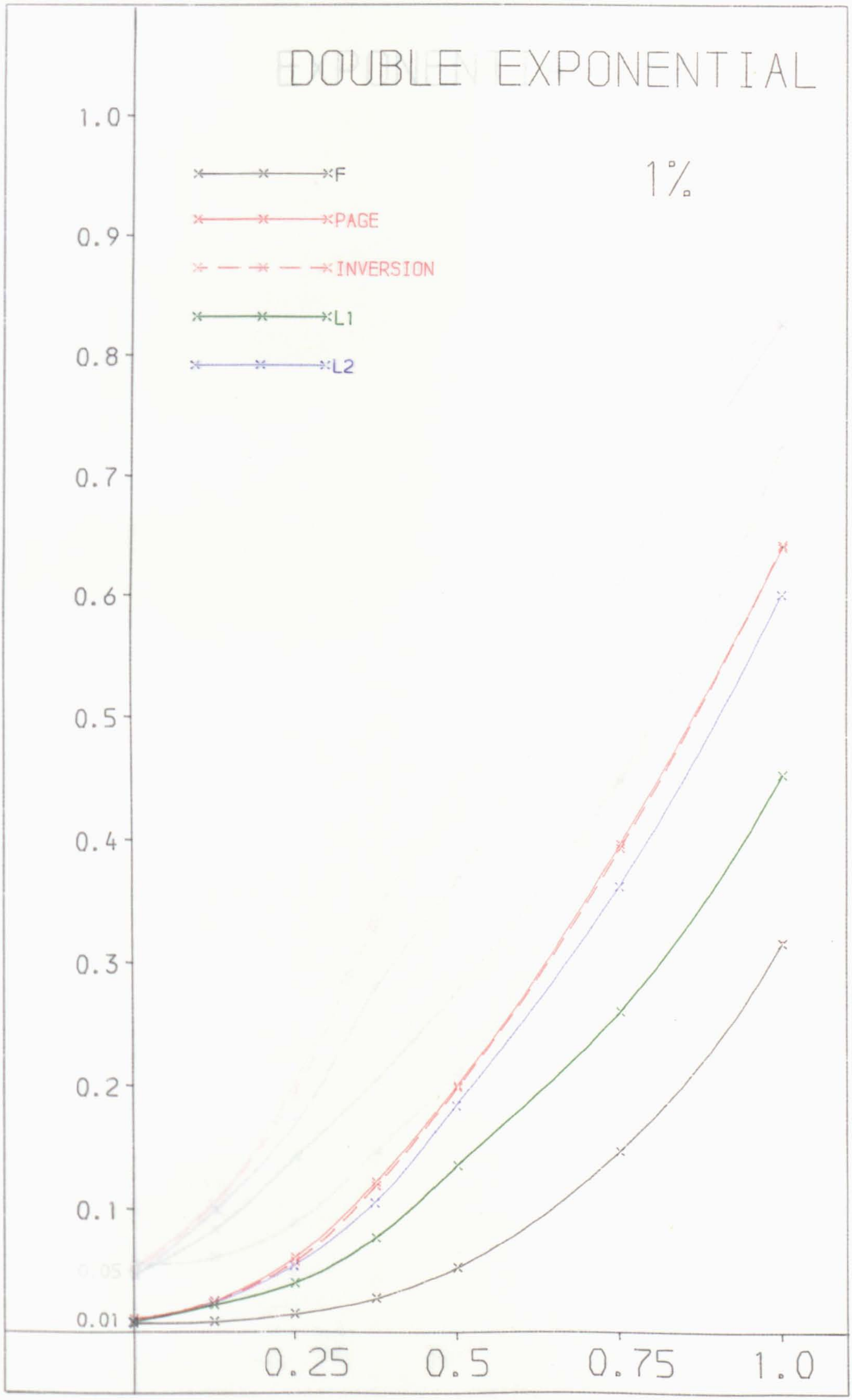
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INVERSION
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L2
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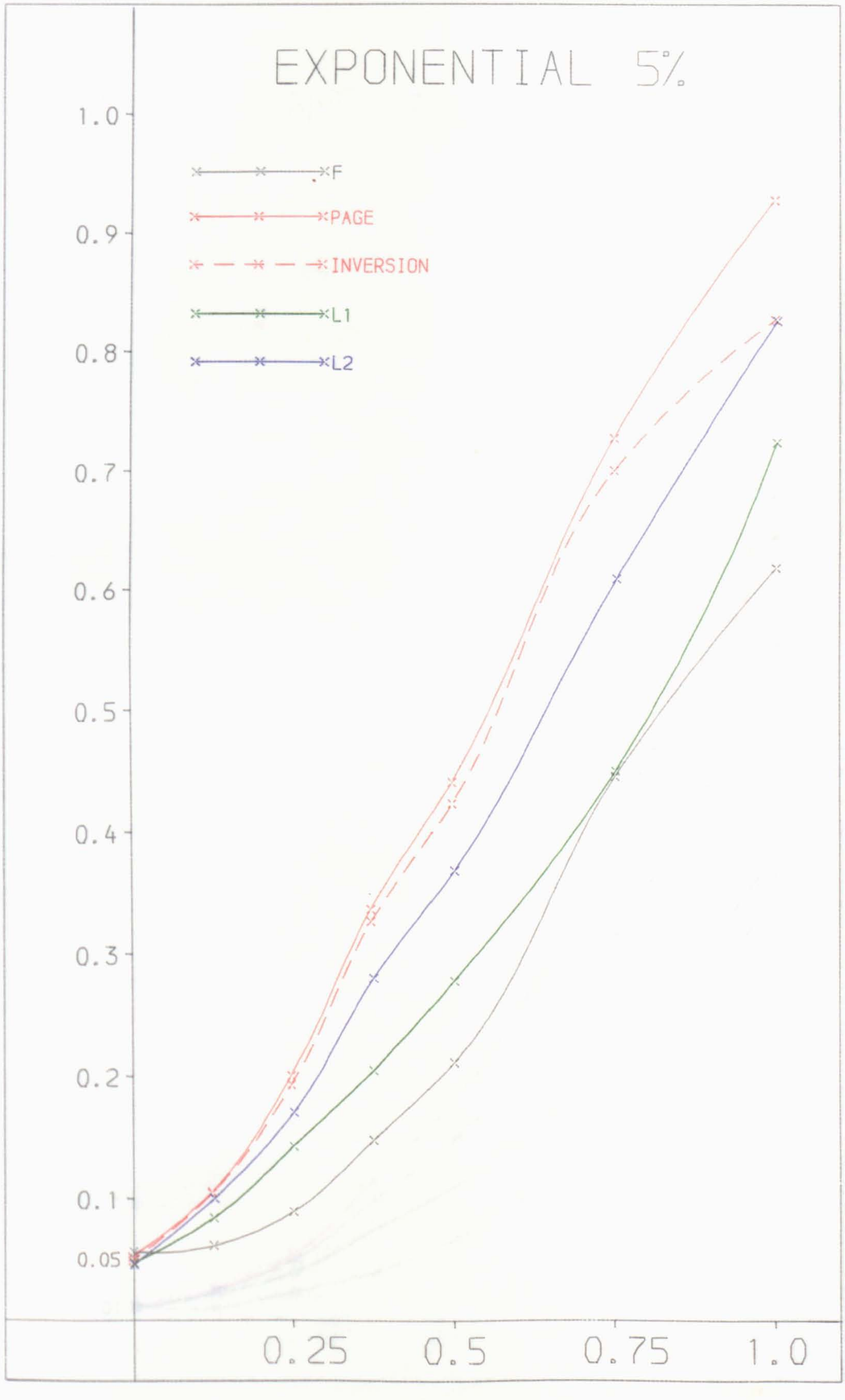
EXPONENTIAL 5%
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0.05
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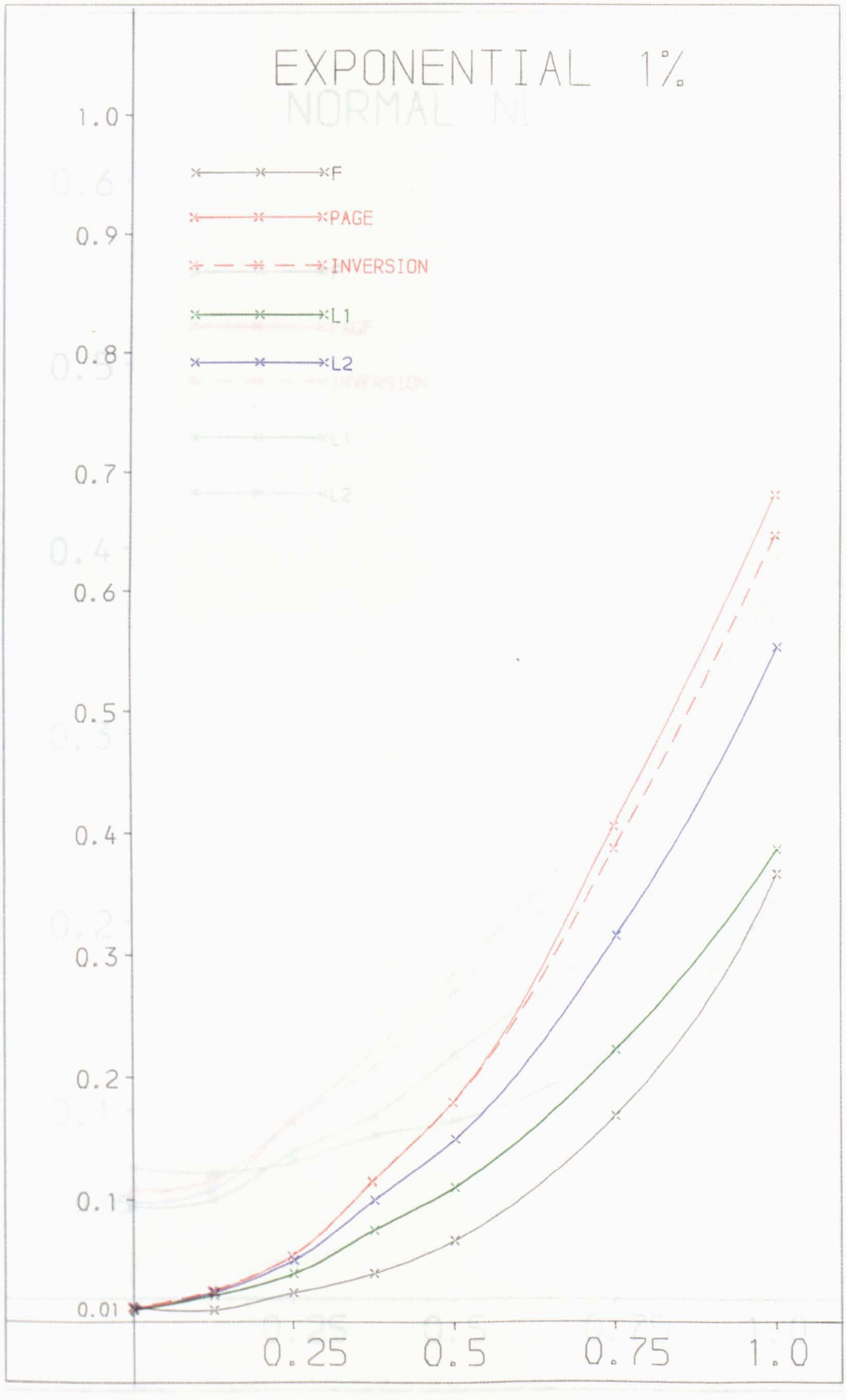
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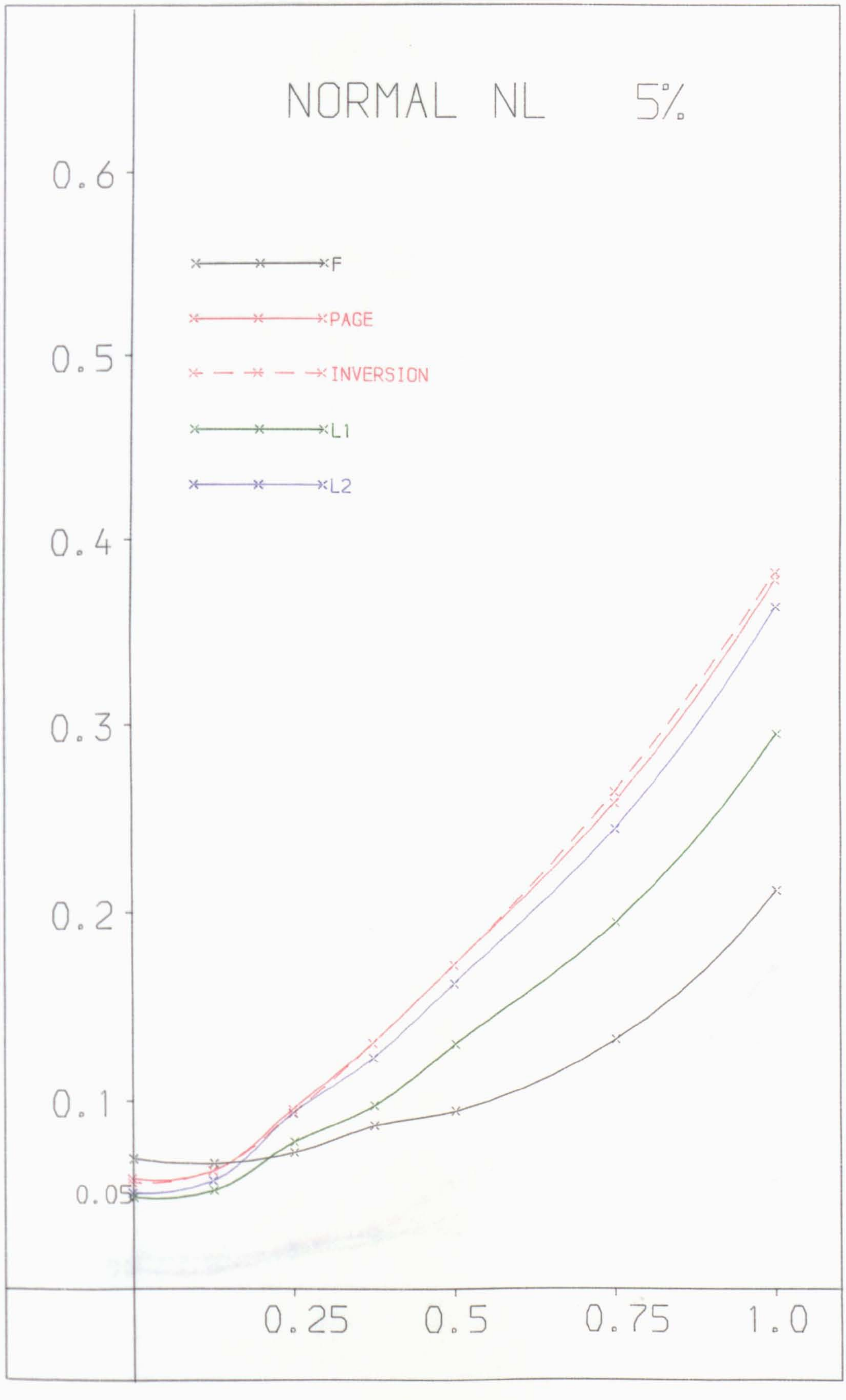
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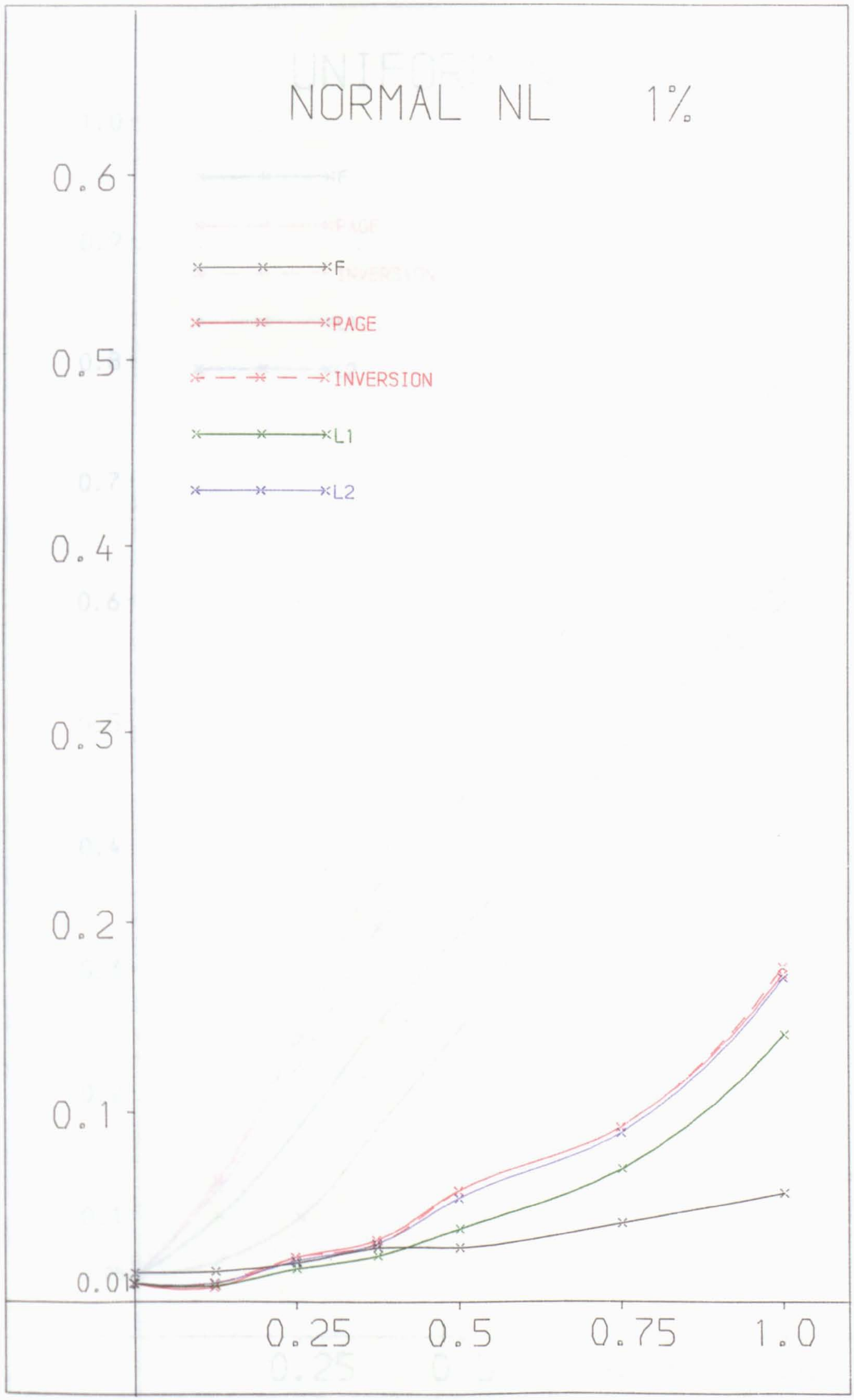
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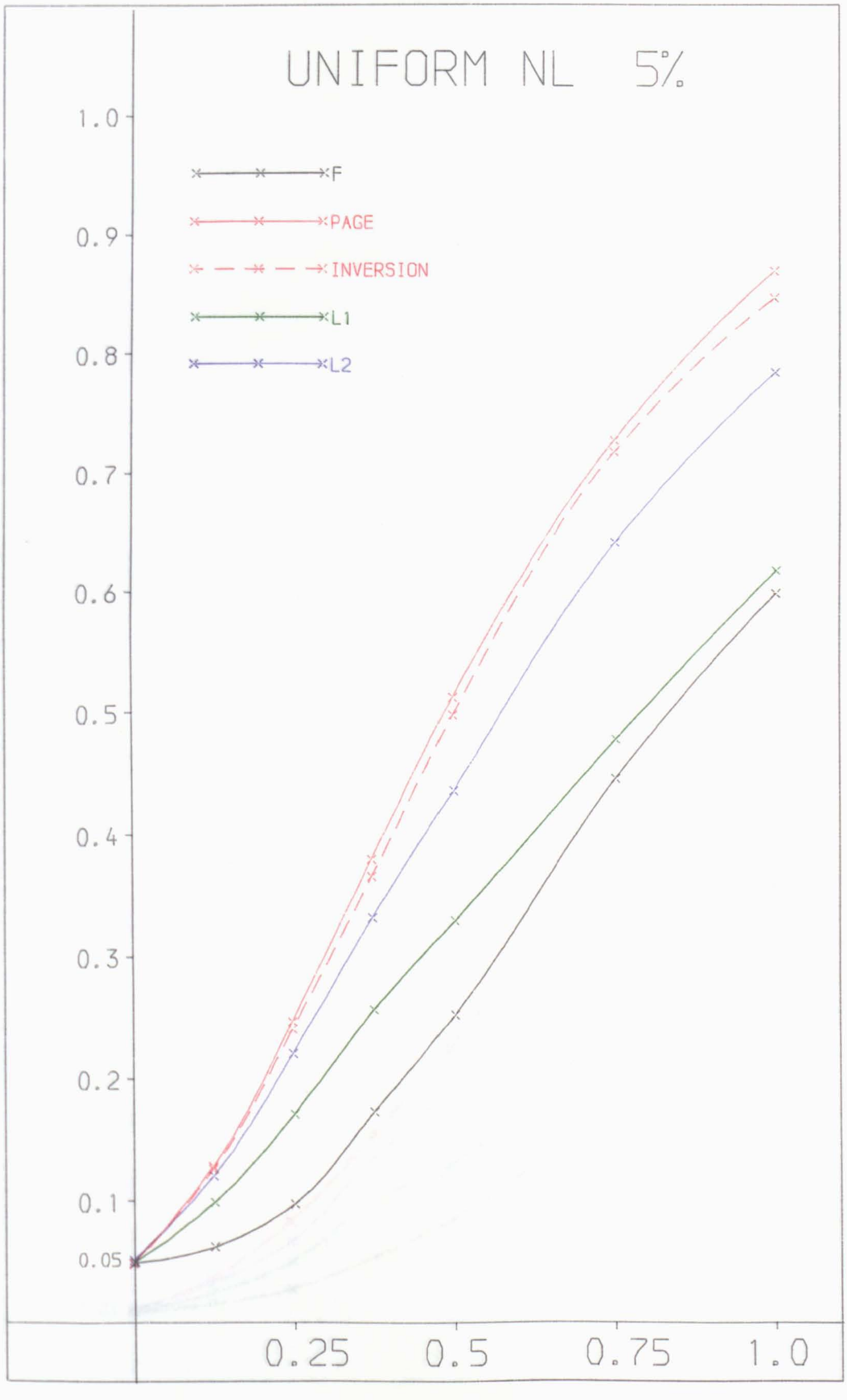
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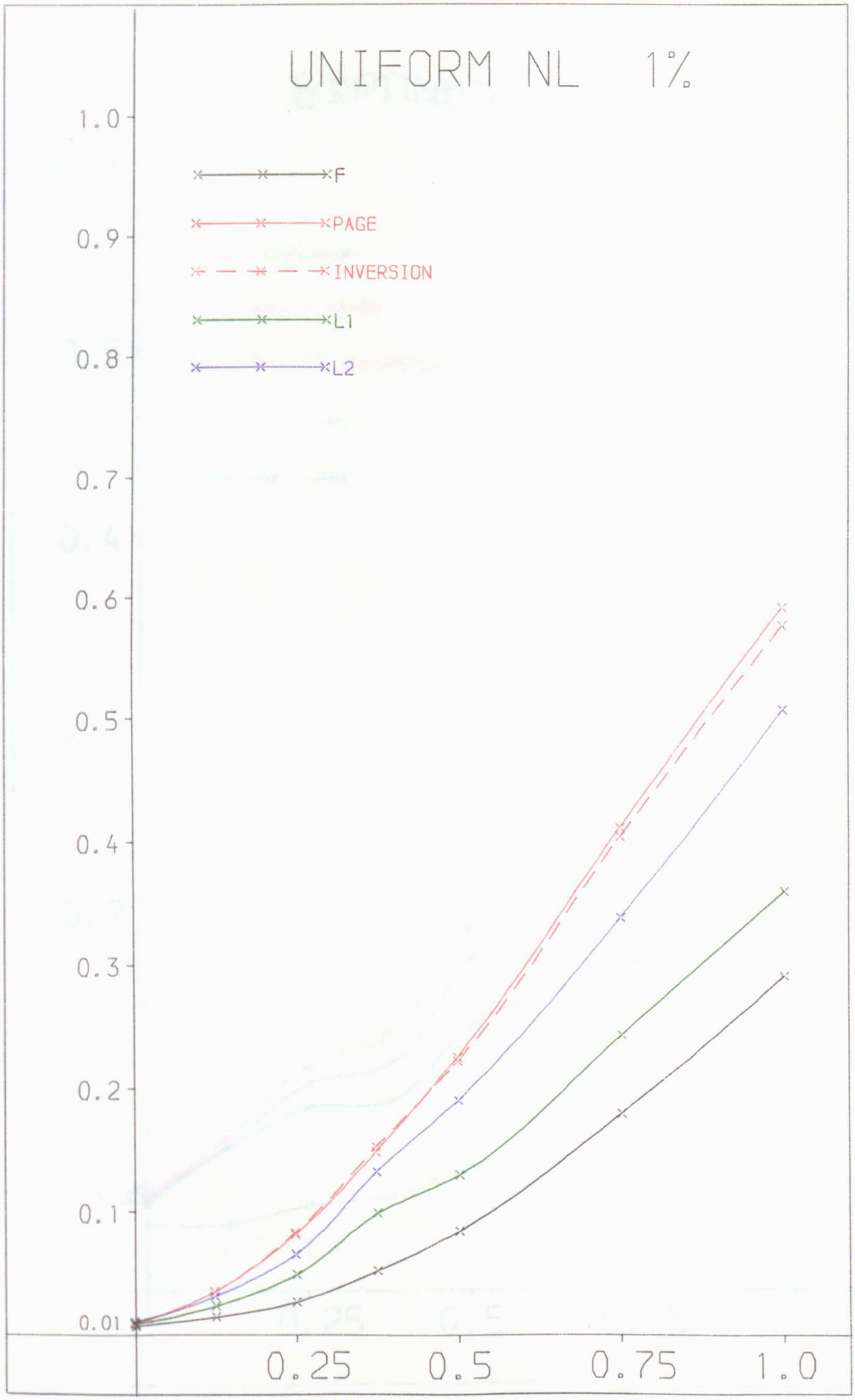

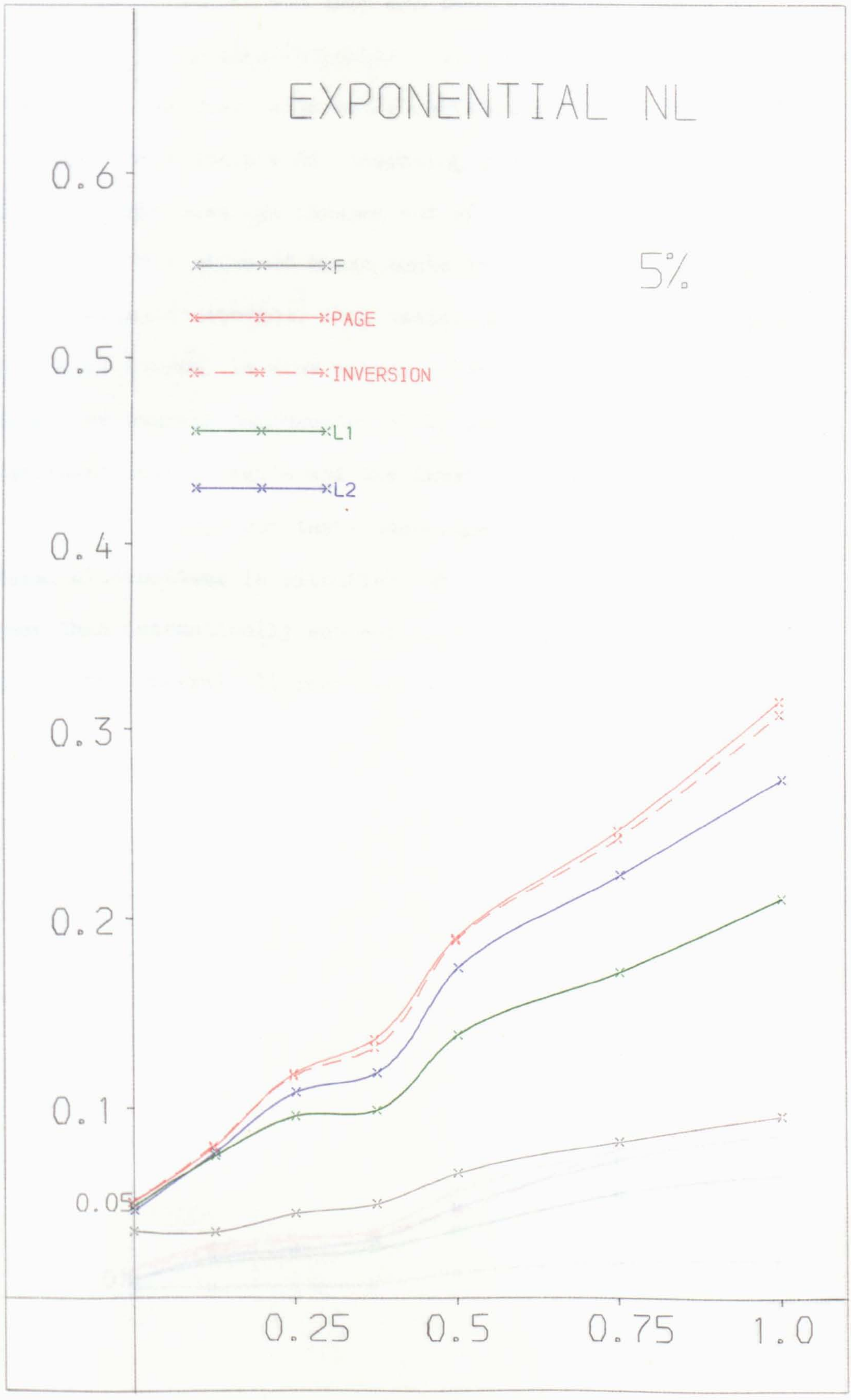
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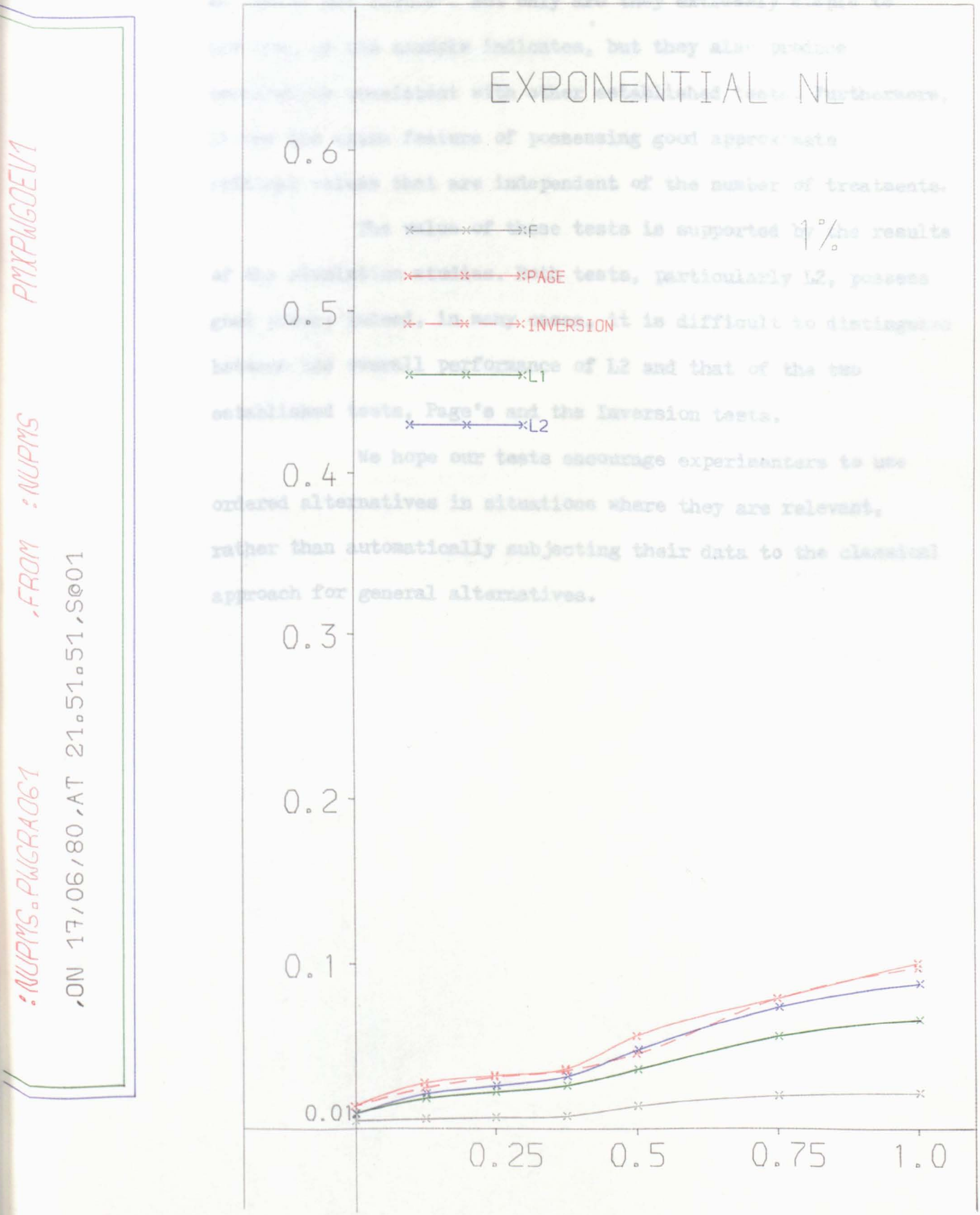
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13. Conclusion.

It is clear that both L1 and L2 may be classified as "quick and simple". Not only are they extremely simple to use but, as the example indicates, but they also produce conclusions consistent with other established tests. Furthermore, L1 has the extra feature of possessing good approximate critical values that are independent of the number of treatments.

The value of these tests is supported by the results of the simulation studies. Both tests, particularly L2, possess good power; indeed, in many cases, it is difficult to distinguish between the overall performance of L2 and that of the two established tests, Page's and the Inversion tests.

We hope our tests encourage experimenters to use ordered alternatives in situations where they are relevant, rather than automatically subjecting their data to the classical approach for general alternatives.

CHAPTER 5

INTERACTION IN TWO-WAY ANALYSIS OF VARIANCE

1. Introduction.

Wilcoxon was the first to produce a nonparametric test for interaction in two-way analysis of variance. This appeared in his rather concise yet informative booklet "Some Rapid Approximate Statistical Procedures" in 1949.

Since then, of course, other nonparametric tests for interactions have been developed. However, all these methods suffer from one or more problems such as being only asymptotically distribution-free, being computationally difficult or having no exact distribution available even for small size experiments.

In this chapter we propose two tests for interaction in two-way experiments, both tests being based on the matching principle. Before presenting these tests it is profitable to consider some features of the earlier methods.

Tests for interaction can be classified into two categories; namely, those tests dealing with the ordinary two-way factorial experiment (the univariate case), and those tests dealing with the less common experiments in which the observations within each cell can be ordered so that the k^{th} observation in one cell can be "paired" with the k^{th} observation in another cell (the multivariate case). It is interesting to note that while discussing this latter case Lin and Crump (1974) recommended that " if there is no natural pairing, the observations can be randomly paired, although then, unfortunately, the values of the test statistic depend upon the particular pairings chosen, " ; this seems rather an understatement. From time to time various authors have either adopted this random approach or simply pretended that their experiment does in fact exhibit natural pairing; for examples of this see Koch (1970) or Wilcoxon (1949).

Weber's test makes use of normal scores and, at best, it is suitable only for large samples since exact critical values are not calculable. Indeed for large samples the statistic is only approximately χ^2 distributed.

Bhapkar and Gore's test is based on Hoeffding's (1948) U-statistics. Unfortunately, it is only asymptotically distribution-free and, furthermore, an extra problem is introduced by the necessity to estimate a "nuisance parameter" $\psi(F)$ whose value depends on the continuous distribution F of the random variables z_{ijk} in the model $X_{ijk} = M + A_i + B_j + (AB)_{ij} + z_{ijk}$. Another feature of this test is the extraordinary amount of computation required even for quite small experiments, e.g. just one part of the calculation for a 2 x 3 x 3 experiment requires $3^5 = 243$ computations. The dependence of their test statistic on $\psi(F)$ means that no exact tables of critical values are possible and so critical values are obtained from a χ^2 approximation.

Lin and Crump's test is in fact a modification of a test proposed by Patel and Hoel (1973) which they discovered to be adversely affected by the presence of strong first-order effects. Their modification consists of replacing the actual observations X_{ijk} by the aligned observations given by $Y_{ijk} = X_{ijk} - \bar{X}_{.jk} - \bar{X}_{i.k} + \bar{X}_{...}$, and then performing Patel and Hoel's procedure which is based on the quantity

$P(X_{12k} \leq X_{11k}) - P(X_{21k} \leq X_{21k})$, estimates for the probabilities being derived from Wilcoxon-Mann-Whitney statistics. Both tests were in fact designed for only 2 x 2 experiments, although the authors do say that the procedures may be extended to larger experiments. Both Lin and Crump's and Patel and Hoel's test statistics are asymptotically normally distributed and, because of their reliance on estimates, no exact tables of critical values are possible.

With regard to the multivariate analysis of interaction the main contributors have been Wilcoxon (1949), Puri and Sen (1966), Mehra and Sen (1969) and Mehra and Smith (1970).

Wilcoxon applied Friedman's test to the differences between the pairings, so that in an experiment with three treatments A_1, A_2 and A_3 the test statistic is the sum of two components; one component is obtained by tabulating $A_1 - A_2$ for the different blocks and the other by tabulating $A_1 + A_2 - 2A_3$ for the different blocks. The statistic is asymptotically distributed as χ^2 and requires only a moderate amount of computation. However, because of the non-symmetric way in which the components are derived it is quite possible that contradictory conclusions can be obtained by re-arranging the order of the treatments.

Puri and Sen's test, which is a derivative of Wilcoxon's idea but employing the Kruskal-Wallis statistic, requires quite sophisticated mathematics and involved computations. Furthermore it suffers from being only asymptotically distribution-free.

Mehra and Sen extended the theory of permutation rank-order tests for main effects to provide a test for interaction. Its major drawback, apart from the nonfeasibility of exact tables, is the great computational effort required which makes the test virtually impractical; even microcomputers would have storage problems in analysing just small size experiments.

The great failing of Mehra and Smith's test is its reliance on the use of scores which are directed towards specific, but arbitrary, distributions. It is also only asymptotically distribution-free.

All the above mentioned tests suffer to a greater or lesser extent from computational troubles. The tests we now introduce for univariate analysis of interaction are free from such worries. The presentation of tests for multivariate analysis of interaction will be deferred to the chapter dealing with second-order interaction. There we shall see that multivariate analysis is easily accommodated.

2. Definition of the Test Statistics.

The model upon which our considerations are based is one where the observations X_{ijk} may be modelled as

$$X_{ijk} = M + A_i + B_j + (AB)_{ij} + z_{ijk} ,$$

$$i = 1, 2, \ldots, b$$
$$j = 1, 2, \ldots, c$$
$$k = 1, 2, \ldots, n_{ij}$$

where M represents the overall mean,

A_i represents the effect of the i^{th} level of factor A with $\sum_{i=1}^{c} A_i = 0$,

B_j represents the effect of the j^{th} level of factor B with $\sum_{j=1}^{c} B_j = 0$,

$(AB)_{ij}$ represents an interaction effect between the i^{th} and j^{th} levels of factors A and B respectively with $\sum_{i=1}^{c} (AB)_{ij} = \sum_{j=1}^{c} (AB)_{ij} = 0$,

z_{ijk}'s are independent random variables possessing some continuous distribution with $E(z_{ijk}) = 0$

and n_{ij} is the number of replications in the i^{th} and j^{th} levels of factors A and B respectively; unlike classical analysis of variance we do not exclude the possibility of $n_{ij} = 1$ for all i and j.

We seek to test the null hypothesis

$$H_0 : \quad (AB)_{ij} = 0 \quad \text{for all } i \text{ and } j$$

against the alternative hypothesis

$$H_1 : \quad (AB)_{ij} \neq 0 \quad \text{for some } i \text{ and } j.$$

For our procedure we first replace each cell of observations by their mean $\bar{X}_{ij}$, which of course alleviates any problems due to unequal replication sizes although naturally some information is lost. The aligned observations $\bar{X}_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..}$ are then formed where $\bar{X}_{i.}$ is the mean of the i^{th} level of factor A, $\bar{X}_{.j}$ is the mean of the j^{th} level of factor B and $\bar{X}_{..}$ is the overall sample mean. These aligned observations are then ranked, either by column (factor A) or by row (factor B), and the match statistic, M1 or M2, is calculated.

Because of the unpredictable nature of interactions, we expect the presence of interaction to yield few matches and near-matches and the opposite to happen for no interaction effects. Hence the null hypothesis is rejected whenever M1 or M2 $\leqslant$ critical value, where as we comment below, the critical value is an approximation from the relevant null distribution of M1 or M2.

3. <u>Comment on the Effect of Alignment</u>.

Aligning the observations in the above manner causes a restriction in the possible arrangment of ranks and so the distributions of the interaction match statistics is only approximately equal to the null distributions of M1 and M2.

To gain some idea of the extent of this restriction we simulated the null distributions for interaction of M1, M2 and Friedman's statistics, the latter being included as a potential rival to the match statistics. The simulations were based on a 4 x 4 experiment, the observations being taken from (a) the uniform distribution U(0,1), (b) the standard normal distribution. The results below give the observed frequencies out of a total of 30,000 together with the respective expected frequencies derived from the null distributions of M1, M2 and Friedman's statistics.

Simulated Distribution of M1

M1	Expected Frequency	Observed Frequency	
		Uniform	Normal
0	52	74	77
2	1250	1308	1334
3	2083	2127	1770
4	6094	6406	6326
5	4062	4237	4336
6	5938	6510	6614
7	3124	2578	3693
8	3535	4076	3656
9	1458	1024	1348
10	1093	815	394
12	833	667	294
13	312	129	116
15	69	49	42
16	39	-	-
18	52	-	-
24	2	-	-

Simulated Distribution of M2

M2	Expected Frequency	Observed Frequency Uniform	Observed Frequency Normal
6	52	72	77
7	729	829	760
7.5	312	367	258
8	3227	4019	4183
8.5	2448	2223	2209
9	3177	3275	3455
9.5	2500	4037	4303
10	3450	4103	4204
10.5	1615	1265	1313
11	2943	2109	2106
11.5	2031	1674	1998
12	1812	1735	1528
12.5	1042	1176	1059
13	1588	1408	1030
13.5	694	529	416
14	742	589	559
14.5	260	113	140
15	547	390	306
15.5	156	87	96
16	273	-	-
16.5	52	-	-
17	156	-	-
18	147	-	-

M2	Expected Frequency	Observed Frequency Uniform	Normal
20	19	-	-
21	26	-	-
24	2	-	-

Simulated Distribution of Friedman's Statistic

χ_r^2	Expected Frequency	Observed Frequency Uniform	Normal
0	228	2373	2816
.3	1862	13014	10546
.6	964	3710	3512
.9	3112	5559	7500
1.2	1232	1794	2051
1.5	2203	1671	1989
1.8	868	264	412
2.1	3694	1116	764
2.4	445	117	174
2.7	2444	234	177
3.0	1235	148	59
3.3	1155	-	-
3.6	942	-	-
3.9	2498	-	-
4.5	1150	-	-

χ_r^2	Expected Frequency	Observed Frequency Uniform	Normal
4.8	317	-	-
5.1	892	-	-
5.4	516	-	-
5.7	1036	-	-
6.0	380	-	-
6.3	486	-	-
6.6	243	-	-
6.9	334	-	-
7.2	109	-	-
7.5	540	-	-
7.8	50	-	-
8.1	434	-	-
8.4	204	-	-
8.7	65	-	-
9.3	169	-	-
9.6	11	-	-
9.9	95	-	-
10.2	13	-	-
10.8	20	-	-
11.1	26	-	-
12.0	2	-	-

Clearly all the distributions have been affected by the process of alignment. However the changes in the distributions of the match statistics is not too severe, particularly in the lower tails which are of course the critical regions for the interaction test. The greatest change has occurred in Friedman's distribution where the restriction in values is quite dramatic.

The results indicate that, in practice the match statistics, when used with critical values from the null distributions for general alternatives, are likely to give valid conclusions. The same cannot be said of Friedman's test which in similar circumstances would tend to reject the null hypothesis of no interaction too readily. These comments on the behaviour of the tests are certainly borne out in the examples that follow.

4. Examples.

Example 1 (Johnson and Leone, 1964).

Four laboratories are invited to participate in an experiment to test the chemical content of four different specimens. Each laboratory is given two samples of each. The data below give the percentage by weight of a basic ingredient.

Specimens	Laboratory I	II	III	IV
1	8, 11	10, 8	7, 10	9, 12
2	14, 19	11, 15	13, 11	10, 13
3	20, 16	21, 18	21, 20	22, 25
4	19, 13	11, 12	17, 15	19, 17

The hypotheses of interest are :

H_0 : there is no interaction between types of specimen and laboratory.

H_1 : there is some interaction between types of specimen and laboratory.

Tests (i) - the match tests

The approximate critical values are obtained from the null distributions given in Chapter 3.

For the M1 test, the null hypothesis is rejected at the 5 % and 1 % levels of significance if $M1 \leqslant 2$ and $M1 = 0$ respectively, while for the M2 rejection occurs at the same levels of significance if $M2 \leqslant 7.5$ and $M2 \leqslant 6$ respectively.

The table of aligned mean observations is given below.

Aligned Mean Observations

-0.28125	0.96875	-0.53125	-0.15625
2.84375	1.09375	-0.90625	-3.03125
-2.78125	0.46875	0.46975	1.84375
0.21875	-2.53125	0.96875	1.34750

Ranking these observations horizontally produces the following table of ranks.

	3	4	1	2
	4	3	2	1
	1	2	3	4
	2	1	3	4
Rank sums	9	10	9	12

Hence $M1 = 1 + 0 + 2 = 3$

and $M2 = 3 + \frac{1}{2}(5 + 3 + 2) = 8$.

Clearly both tests produce no evidence to support the alternative hypothesis.

An alternative analysis may be obtained by ranking the aligned mean observations vertically. Doing so produces the following table of ranks.

				Rank sums
2	3	2	2	9
4	4	1	1	10
1	2	3	4	10
3	1	4	3	11

Hence $M1 = 4 + 0 + 2 = 6$

and $M2 = 6 + \frac{1}{2}(3 + 3 + 2) = 10$.

Again there is no evidence to support the alternative hypothesis.

Test (ii) - Friedman's test

The values of Friedman's statistic from the horizontal and vertical ranks are 0.1 and 0.3 respectively. Both of these results would appear to be significant when compared to the critical values from Friedman's null distribution. However the simulation results make one rather cautious about such a conclusion.

Test (iii) - the classical F-test

The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $F > 2.54$ and $F > 3.78$ respectively, there being (9,16) degrees of freedom.

Performing the usual analysis of variance calculations produces the value $F = 1.784$ which clearly provides no support for the alternative hypothesis.

Example 2

In this example we use artificial data which has been constructed so as to indicate the presence of interaction.

	Factor A			
	1.44, 1.96	2.39, 2.81	3.18, 3.01	1.59, 1.66
	2.26, 2.87	1.97, 1.86	2.99, 3.22	3.44, 3.53
Factor B	3.70, 3.96	4.21, 3.87	2.72, 3.07	2.68, 2.55
	4.90, 4.03	3.08, 3.98	3.25, 2.63	3.83, 4.42

The hypotheses of interest are :

H_0 : there is no interaction between factor A and factor B

H_1 : there is some interaction between factor A and factor B.

Tests (i) - the match tests

For the M1 test, the null hypothesis is rejected at the 5 % and 1 % levels of significance if $M1 \leq 2$ and $M1 = 0$ respectively, while for the M2 test rejection occurs at the same levels of significance if $M2 \leq 7.5$ and $M2 \leq 6$ respectively.

The table of aligned mean observations is given below.

Aligned Mean Observations

-0.66187	0.35687	0.86437	-0.55937
-0.30937	-0.84062	0.36187	0.78812
0.37812	0.70687	-0.42562	-0.65937
0.59312	-0.22312	-0.80062	0.43062

Ranking these observations horizontally and vertically gives, respectively,

	1	3	4	2
	2	1	3	4
	3	4	2	1
	4	2	1	3
Rank sum	10	10	10	10

				Rank sum
1	3	4	2	10
2	1	3	4	10
3	4	2	1	10
4	2	1	3	10

Both sets of rankings produce $M1 = 0$ and $M2 = 6$. Clearly there is strong evidence to support the hypothesis.

Test (ii) - Friedman's test

Friedman's test, for both the horizontal and vertical rankings, returns a value of 0. This is the smallest possible value and so is, at least, not inconsistent with the alternative claim.

Test (iii) - the classical F-test

The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $F \geqslant 2.54$ and $F \geqslant 3.78$ respectively, there being (9,16) degrees of freedom.

Performing the usual analysis of variance calculations produces the value $F = 11.35$ clearly a highly significant result.

5. Comments and Results of the Simulations.

In the simulations for interaction in two-way experiments we have used three tests namely, the classical F-test, the M1 and the M2 tests. No other nonparametric tests such as Weber's normal scores tests were used. It was felt that the necessity to use asymptotic approximations for the critical values reduces the value of these tests in comparative study.

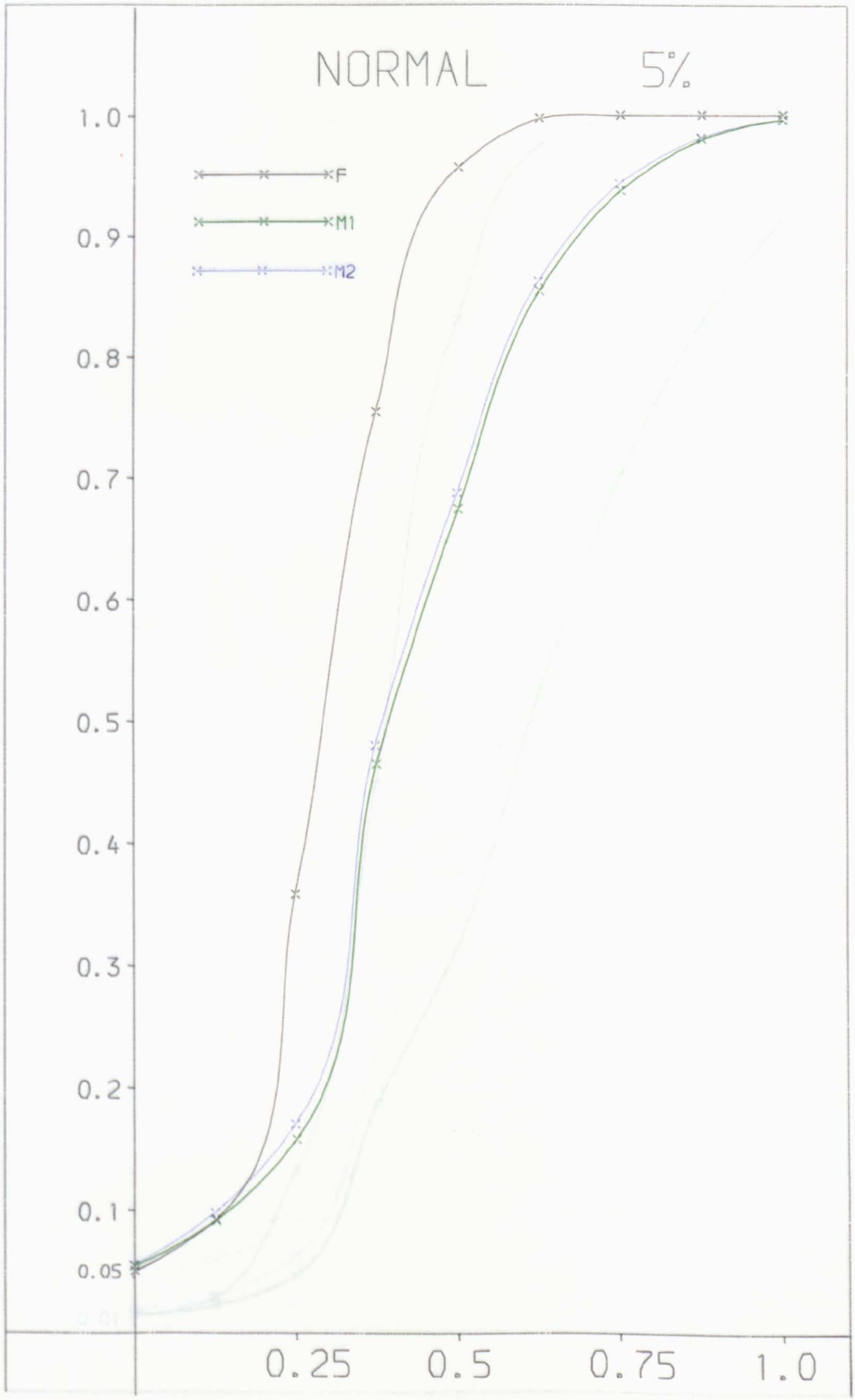
Normal Distribution. As expected the ~~normal~~ F-test ~~distribution~~ reigned supreme. However M1 and M2 performed well and produced similar results.

Uniform Distribution. The notable feature in this case is the superior performance of M1 and M2 until θ reaches about 0.5 .

Double Exponential Distribution. The performance of all tests is very similar to their performance with the uniform distribution.

Cauchy Distribution. The Cauchy distribution has certainly confused all the tests. They all have low power, this being a maximum of 0.1 in the 5 % case. The F-test has particularly poor robustness. Throughout the range both M1 and M2 are superior to the F-test.

Exponential Distribution. Low power is the characteristic feature with this distribution. M1 and M2 are both reasonable performers throughout the range.



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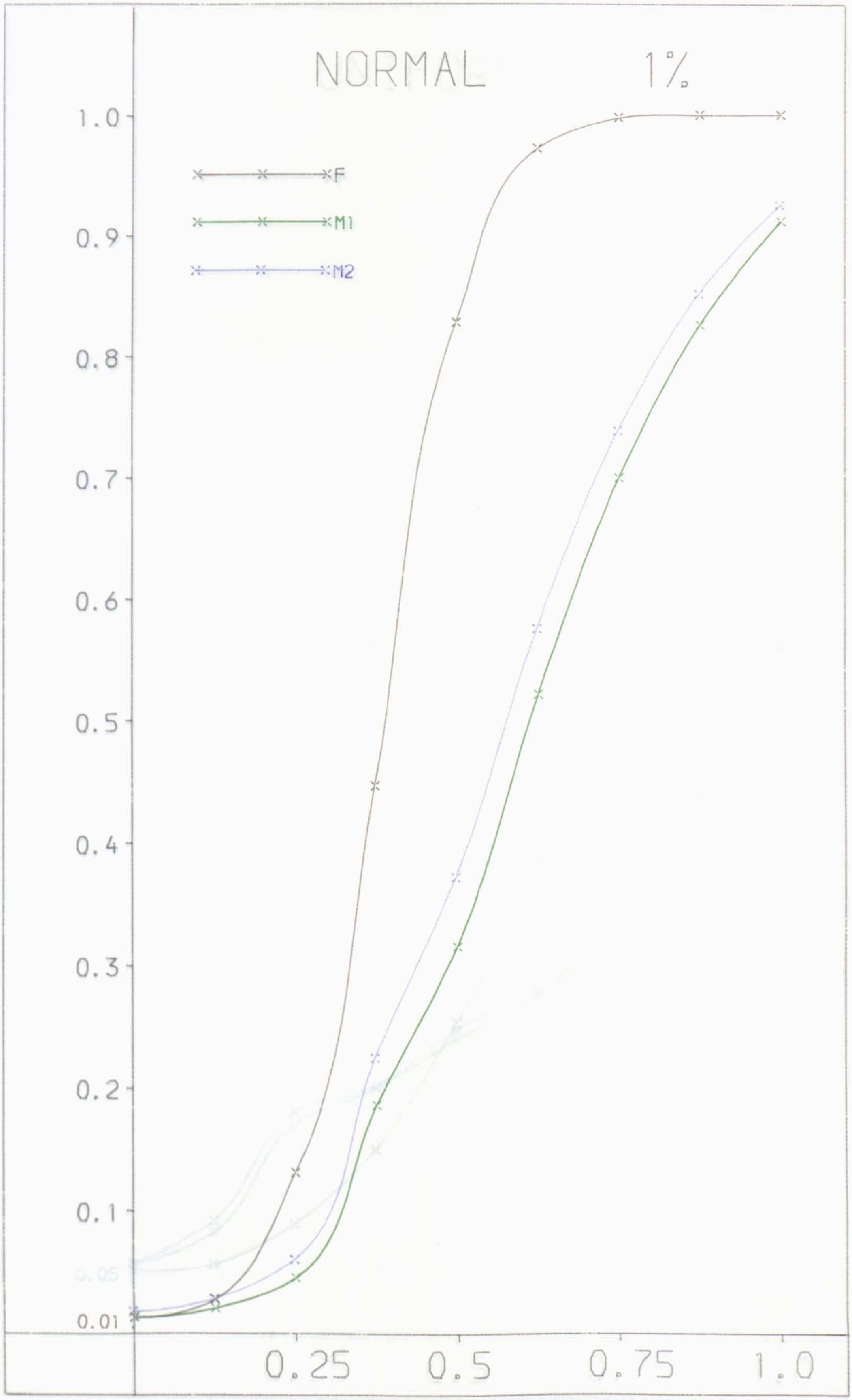

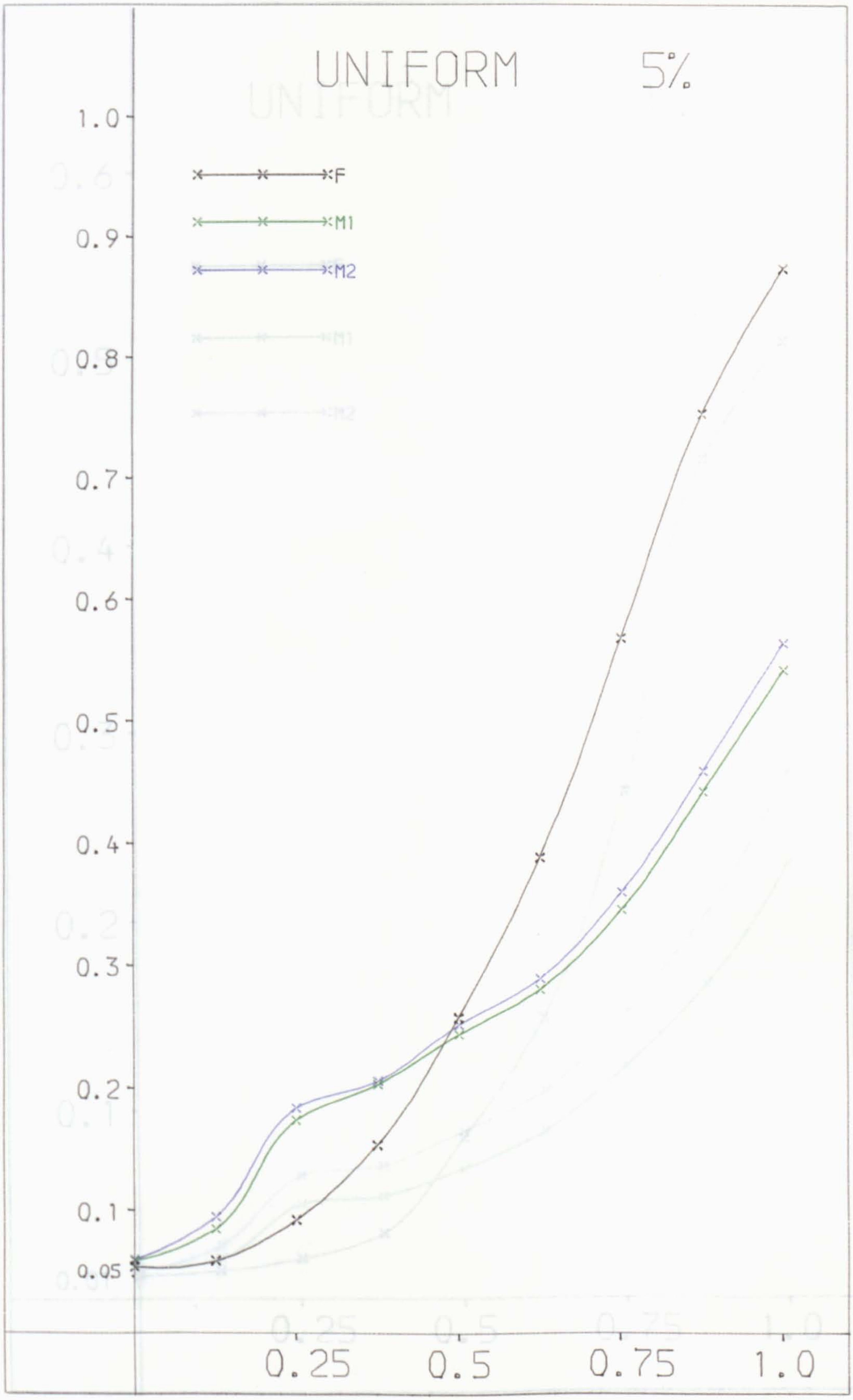
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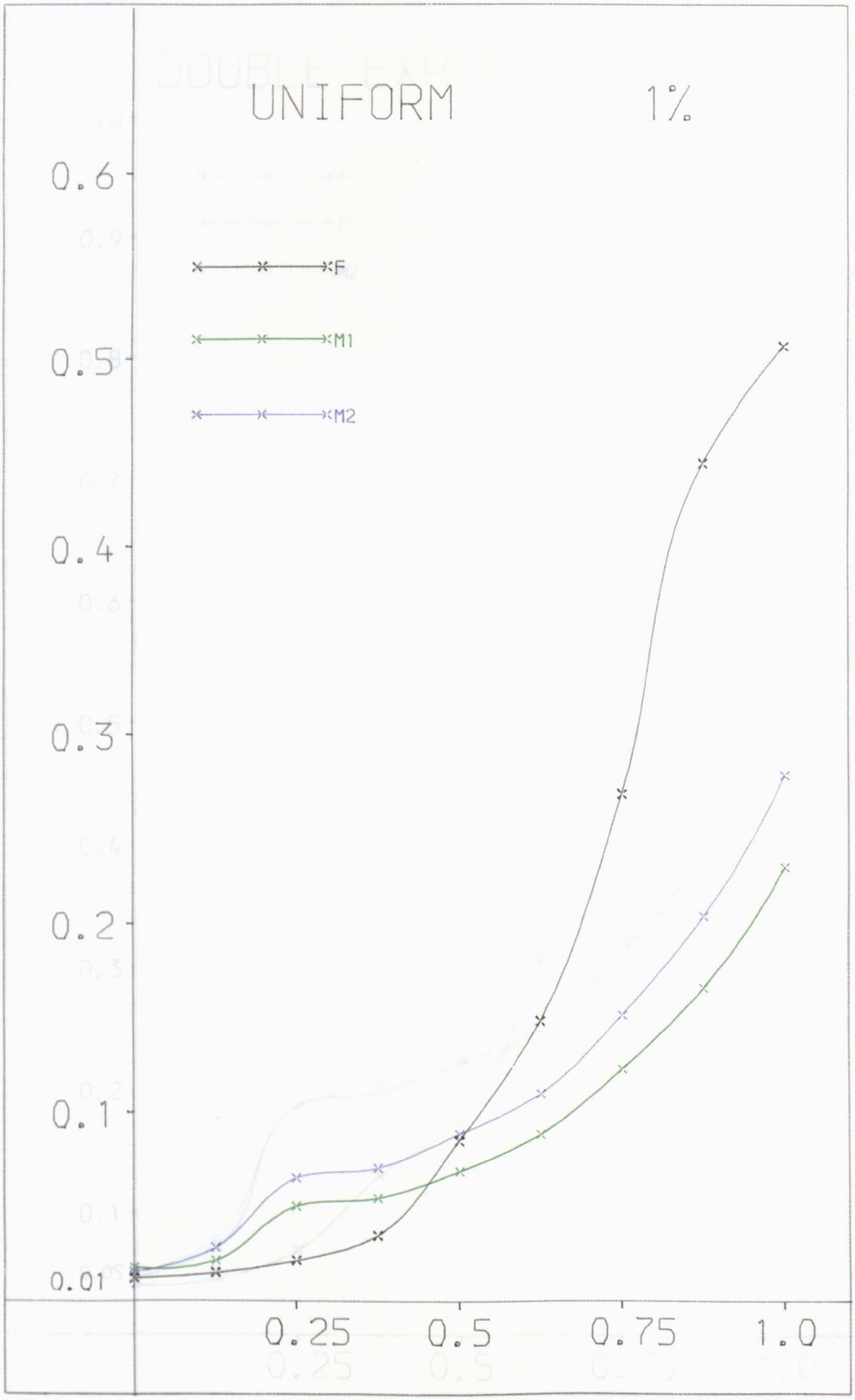

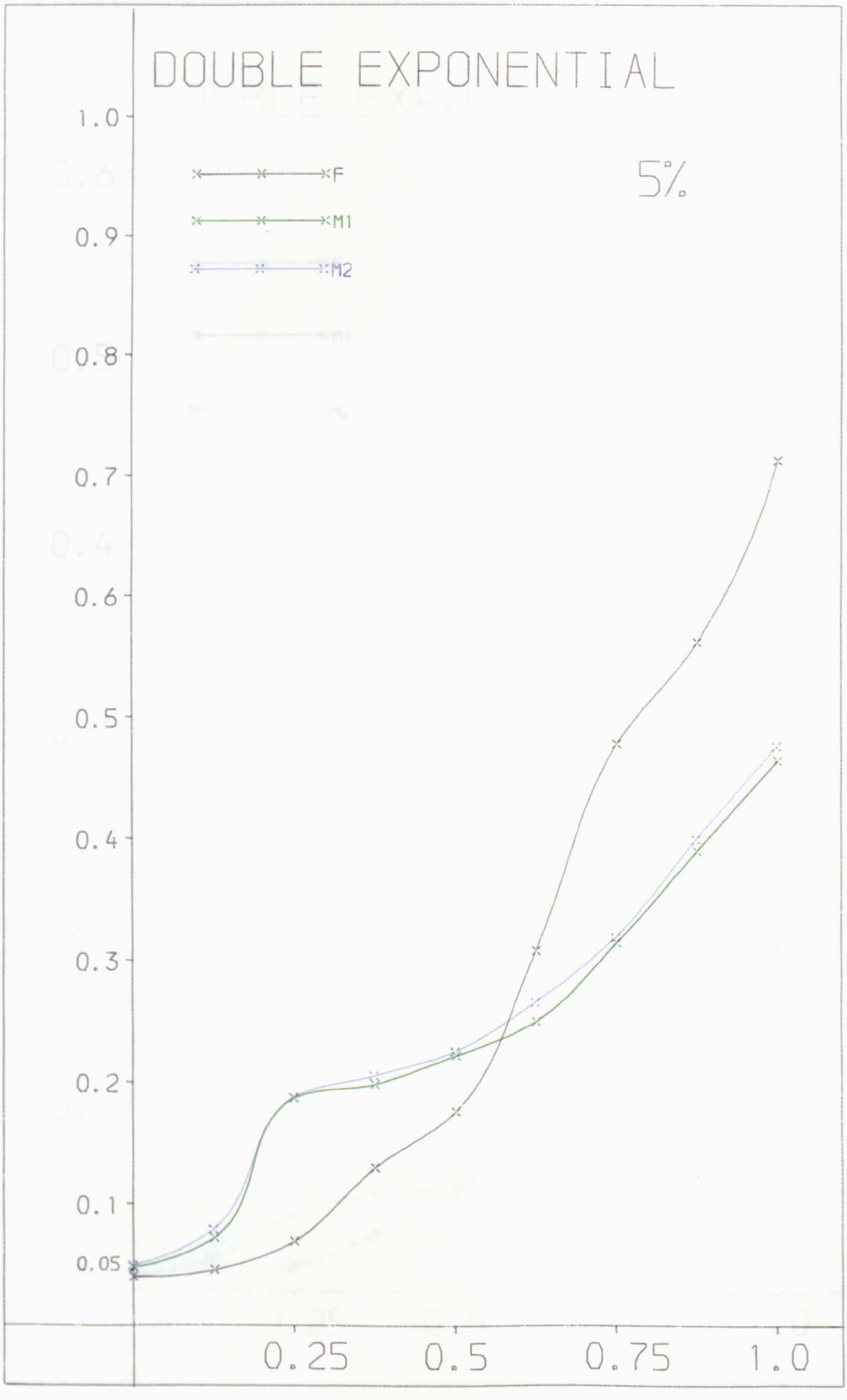
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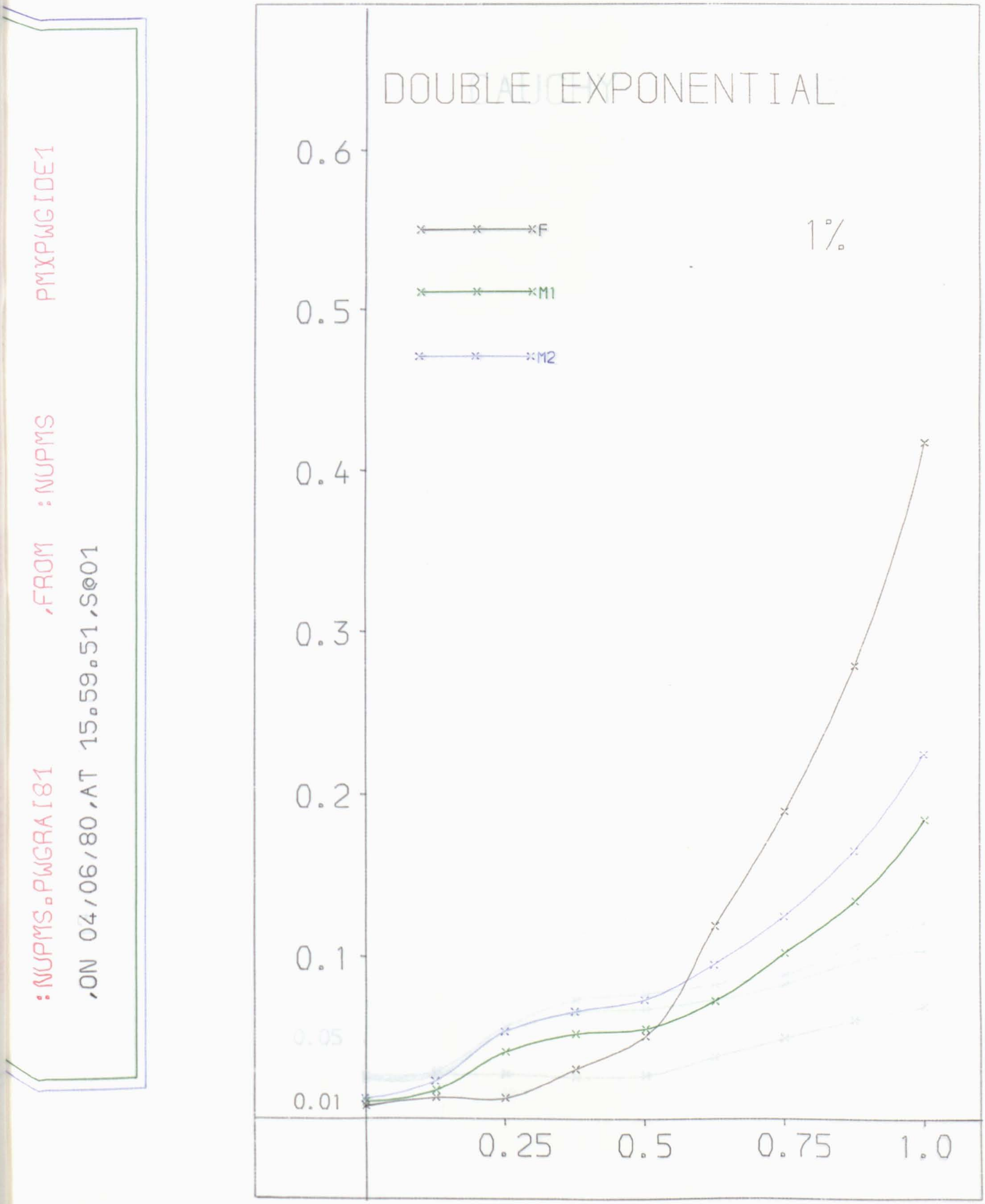
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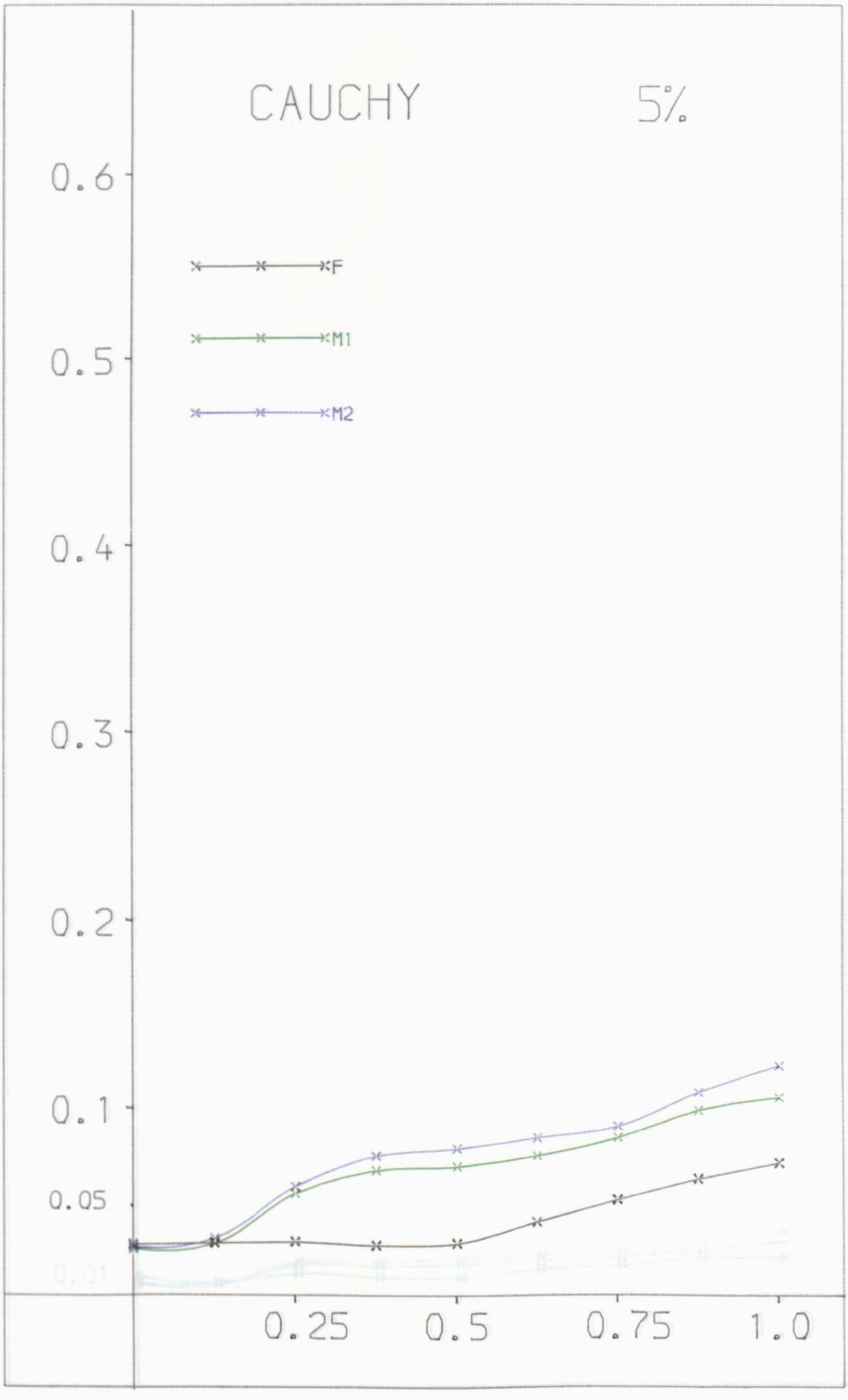
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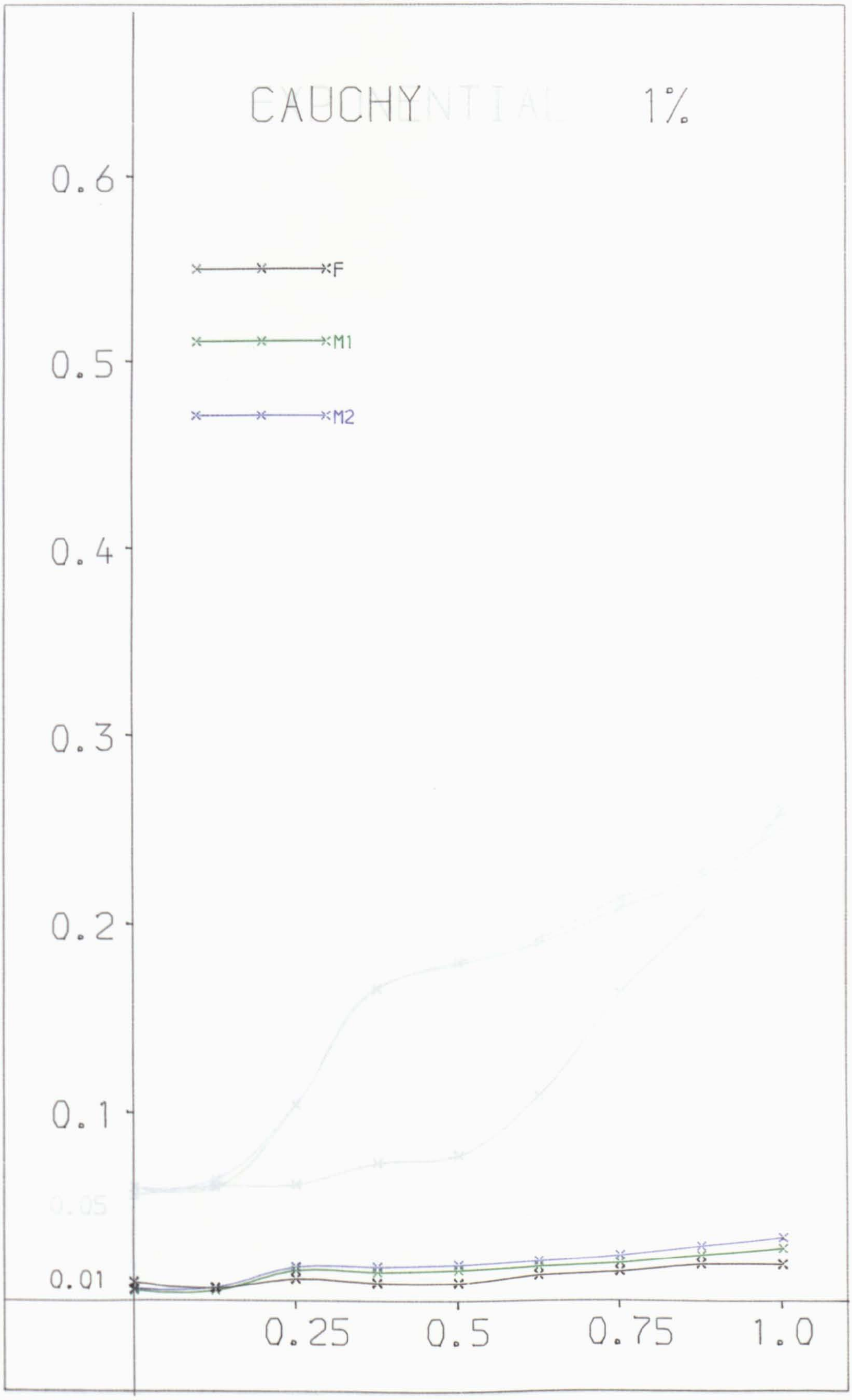

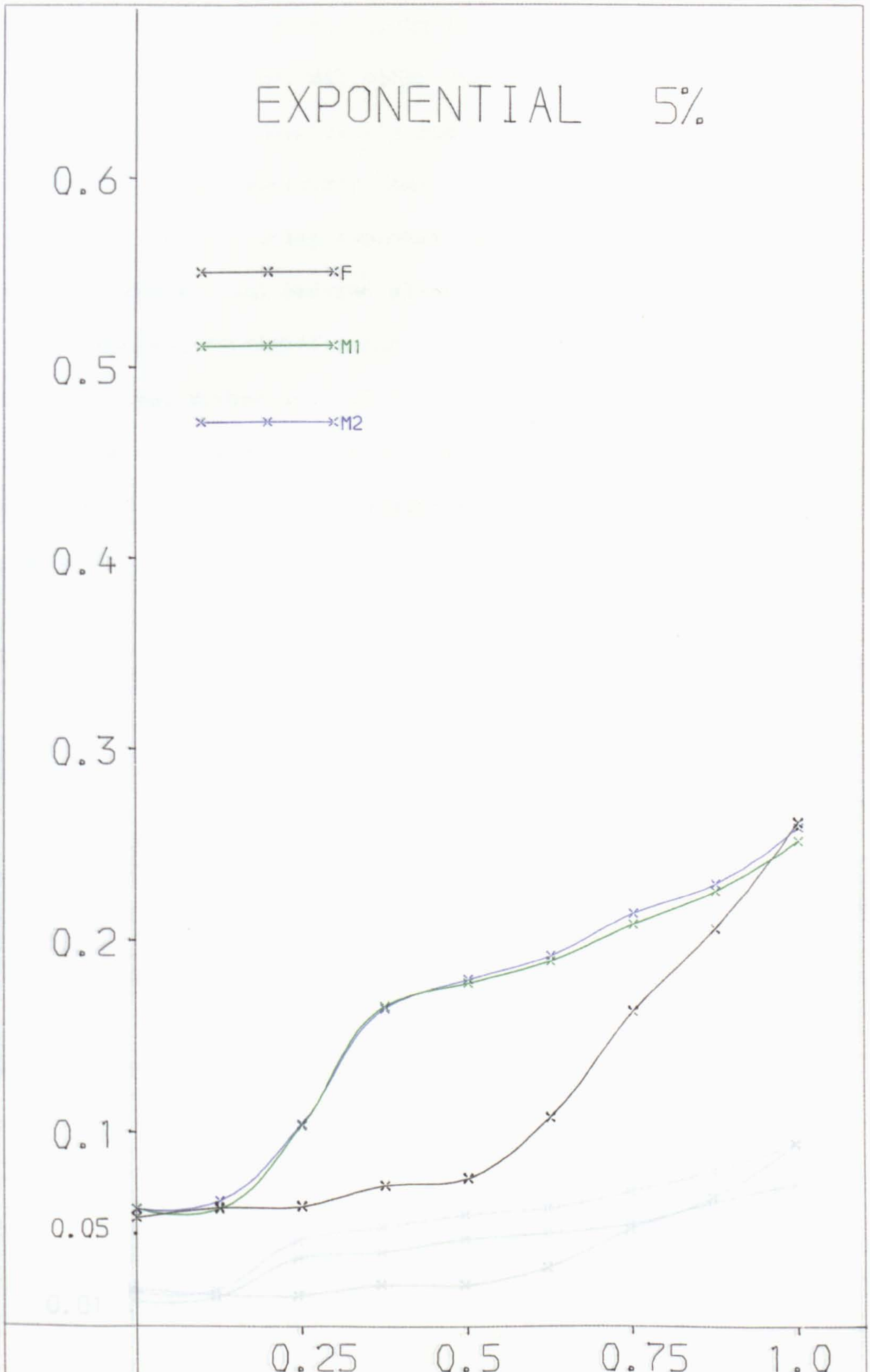
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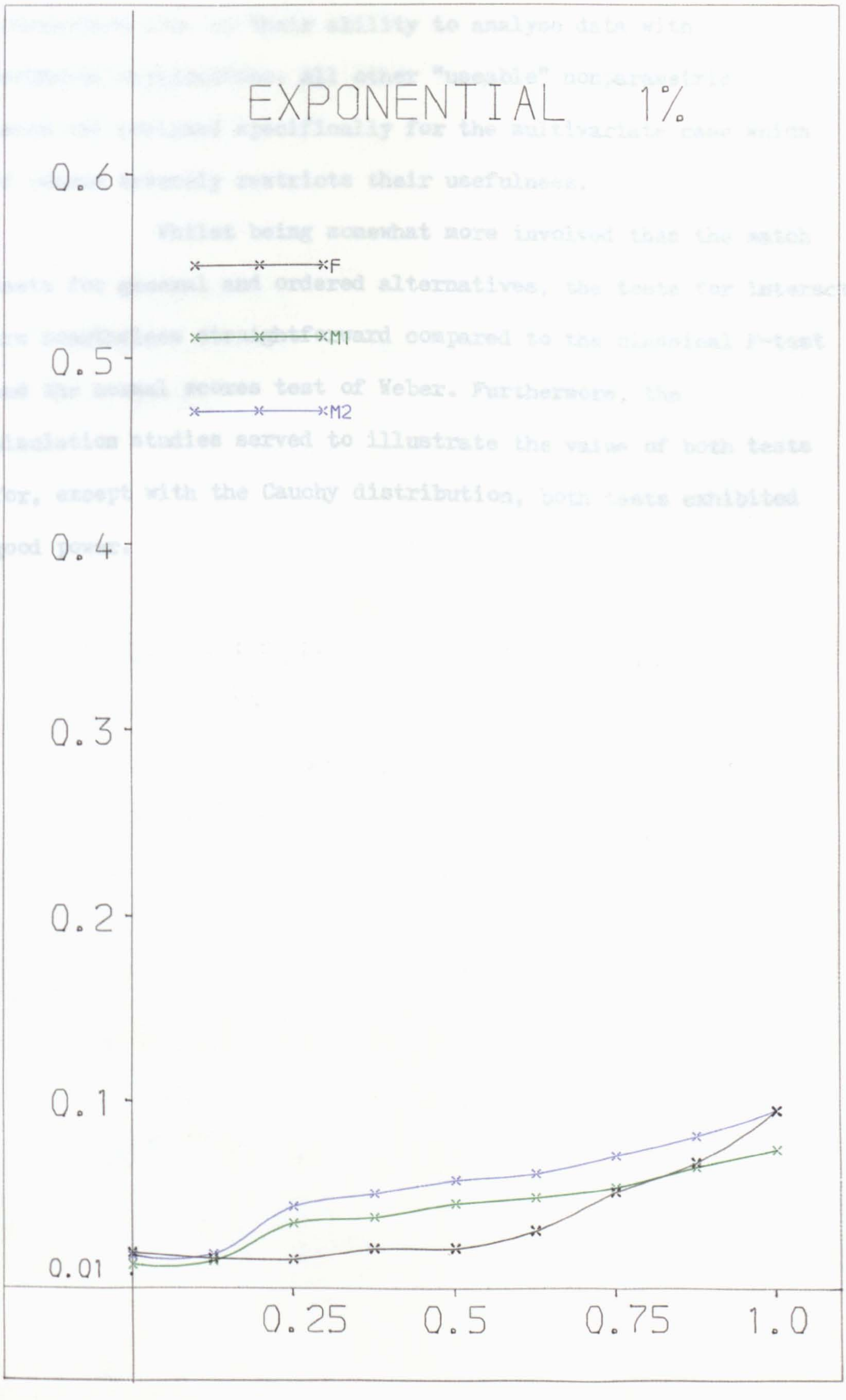
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0.6
0.5
0.4
0.3
0.2
0.1
0.01
0.25
0.5
0.75
1.0

6. Conclusion.

The value of our match tests for first-order interaction lies in their ability to analyse data with unordered replications. All other "useable" nonparametric tests are designed specifically for the multivariate case which of course severely restricts their usefulness.

Whilst being somewhat more involved than the match tests for general and ordered alternatives, the tests for interaction are nonetheless straightforward compared to the classical F-test and the normal scores test of Weber. Furthermore, the simulation studies served to illustrate the value of both tests for, except with the Cauchy distribution, both tests exhibited good power.

CHAPTER 6

SECOND-ORDER INTERACTION IN THREE-WAY ANALYSIS OF VARIANCE

1. Introduction.

It is frequently necessary to consider the existence of more than two factors in an experimental design. Certainly this is so if there is any likelihood that additional factors may corrupt the results. In such higher-order designs not only do we need to allow for first-order interaction but also for possible second-order interactions.

In the classical analysis one considers a model of the type

$$x_{ijkl} = M + A_i + B_j + C_k + (AB)_{ij} + (AC)_{ik} + (BC)_{jk} + (ABC)_{ijk} + z_{ijkl} ,$$

$$i = 1, 2, \ldots, c$$
$$j = 1, 2, \ldots, b$$
$$k = 1, 2, \ldots, v$$
$$l = 1, 2, \ldots, n_1 ,$$

where A_i, B_j and C_k represent the i^{th}, j^{th} and k^{th} levels of the main effects A, B and C, with $\sum_{i=1}^{c} A_i = \sum_{j=1}^{b} B_j = \sum_{k=1}^{v} C_k = 0$,

$(AB)_{ij}$, $(AC)_{ik}$ and $(BC)_{jk}$ represent the first-order interactions with $\sum_{i=1}^{c} (AB)_{ij} = \sum_{j=1}^{b} (AB)_{ij} = \sum_{i=1}^{c} (AC)_{ik}$ $= \sum_{k=1}^{v} (AC)_{ik} = \sum_{j=1}^{b} (BC)_{jk} = \sum_{k=1}^{v} (BC)_{jk} = 0$,

$(ABC)_{ijk}$ represents the second-order interaction with $\sum_{i=1}^{c} (ABC)_{ijk} = \sum_{j=1}^{b} (ABC)_{ijk} = \sum_{k=1}^{v} (ABC)_{ijk} = 0$;

z_{ijkl}'s are independent random variables possessing a normal distribution with $E(z_{ijkl}) = 0$,

and n_l is the number of replications in the i^{th}, j^{th} and k^{th} cell.

Hypotheses concerning the main effects and interactions are then tested using the F-ratios, with the assumption that the underlying distributions are normal with equal variances.

However there are many practical situations where the normality assumptions may not hold true. So once again we have a situation where the validity of results is questionable because of ignorance regarding the assumptions.

In other experimental designs, such as one-way analysis of variance and randomised blocks, there are highly satisfactory nonparametric tests serving as alternatives to the classical analyses which overcome the dilemma of the normality assumptions. However in the case of three-way analysis of variance, particularly with second-order interactions, there has been little alternative to the classical analysis.

In 1979 Bradley published a method for analysing interactions of any order. Unfortunately, his method is simply a modification of Wilcoxon's (1949) test for first-order interactions, which suffers from requiring a natural ordering of the observations. Indeed, Bradley admits that "the test statistic is somewhat influenced by (a) the assignment of independent observations to rows within a cell, (b) the particular sequence in which the levels of a variable are presented in the data table.". He supplies no satisfactory remedy for this fault, although he does warn against the temptation of reversing an unwelcome decision by redoing the test under a different permutation of columns, blocks or different arrangement of observations within cells.

Our tests for second-order interactions, based on the matching principle, suffer from none of the above faults. They also have the added bonus of being "quick and easy" tests.

2. Definition of the Test Statistics.

The linear model on which our considerations are based has been introduced in the previous section. Now the z_{ijkl}'s represent independent random variables possessing some continuous distribution.

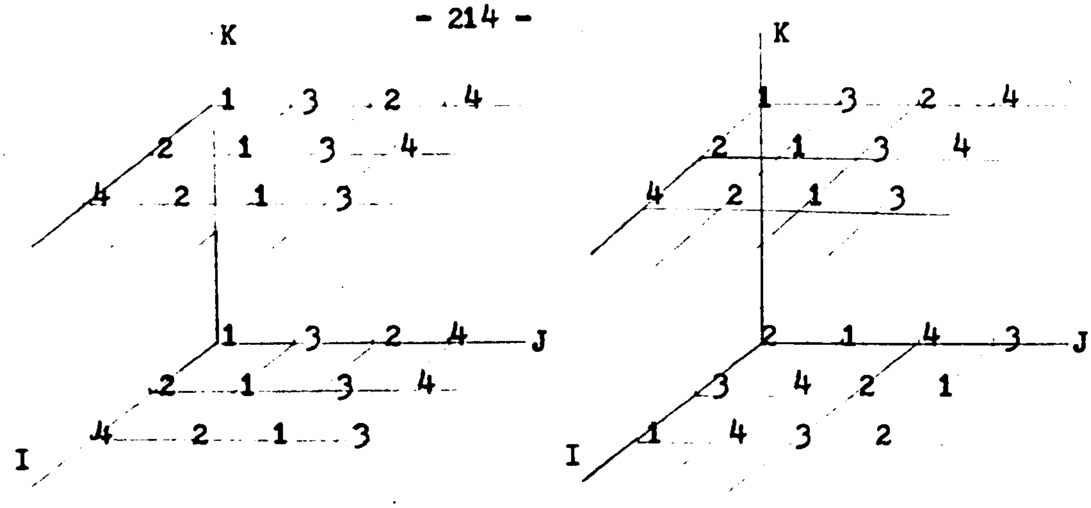
We seek to test the null hypothesis

$$H_0 : \quad (ABC)_{ijk} = 0 \quad \text{for all } i, j \text{ and } k$$

against the alternative hypothesis

$$H_1 : \quad (ABC)_{ijk} \neq 0 \quad \text{for some } i, j, k.$$

The idea and the procedure of the tests is best explained in conjunction with the following diagrams where the ranks are those of aligned mean observations and indicate in (a) no second-order interaction, (b) possible second-order interaction.



First of all we replace each cell of observations by their mean $\bar{X}_{ijk}$. We then consider each horizontal plane in turn and form on each plane the mean aligned observations $\bar{X}_{ijk} - \bar{X}_{i.k} - \bar{X}_{.jk} + \bar{X}_{..k}$, where, in the k^{th} plane, the means of the i^{th} row and j^{th} column are $\bar{X}_{i.k}$ and $\bar{X}_{.jk}$ respectively and the overall mean is $\bar{X}_{..k}$. So for each horizontal plane the row and column effects have been eliminated leaving the (AB) interaction. These values are now ranked (in either direction), typical values are shown in the diagrams.

If there is no second-order interaction we expect the same array of ranks on each horizontal plane (diagram (a)) whilst the presence of second-order interaction would tend to produce different arrays (diagram (b)).

The test statistics, C1 and C2, are based on M1 and M2, the statistics used in the general alternatives situation. M1 and M2 are calculated for each vertical layer, then C1 and C2 are given by

C1 = sum of all the M1's

C2 = sum of all the M2's .

The presence of second-order interaction will tend to yield low values of C1 and C2 while the absence of such interaction will tend to give higher values. Thus the null hypothesis of no second-order interaction will be rejected if C1 and C2 $\leqslant$ a critical value obtained from the appropriate table in sections 6 and 7 respectively. For the reasons outlined in Chapter 5 the critical values are approximate.

Given a set of data the user may select any of the three factors to be the 'vertical' layer, etc. However with small sized experiments, in order to avoid a limited range of critical values it is advisable to choose the vertical layer to be given by the factor with the smallest number of levels.

3. Example. (Miller and Freud, 1965)

A warm sulphuric pickling bath is used to remove oxides from the surface of a metal prior to plating. It is desired to determine what factors, in addition to the concentration of the sulphuric acid, might affect the electrical conductivity of the bath. As it is felt that the salt concentration and the bath temperature might also affect the conductivity, an experiment is planned to determine the individual and joint efffects of these three variables on the electrical conductivity of the bath. The three factors, acid concentration (A), salt concentration (S) and bath temperature (B), were at 4, 3 and 2 levels respectively, there being 2 replicates at each level combination. The results are given in the table below.

		B_1				B_2		
		A				A		
		1	2	3	4	1	2	3	4
	1	0.99	1.00	1.24	1.24	1.15	1.12	1.12	1.32
		0.93	1.17	1.22	1.20	0.99	1.13	1.15	1.24
	2	0.97	0.99	1.15	1.14	0.87	0.96	1.11	1.20
S		0.91	1.04	0.95	1.10	0.86	0.98	0.95	1.19
	3	0.95	0.97	1.03	1.02	0.91	0.94	1.12	1.02
		0.86	0.95	1.01	1.01	0.85	0.99	0.96	1.00

The hypotheses of particular interest to us are :

H_0 : there are no second-order interaction effects

H_1 : there exist some second-order interaction effects.

<u>Tests (i) - the match test</u>

Using the tables for $c = 4$, $b = 3$ and $v = 2$ given in sections 6 and 7 we obtain the following decision rules.

For the C1 test, the null hypothesis is rejected at the 5 % and 1 % levels of significance if $C1 \leqslant 2$ and $C1 \leqslant 1$ respectively, while for the C2 test rejection occurs at the same levels if $C2 \leqslant 7$ and $C2 \leqslant 6$ respectively.

From the above data we obtain two 'vertical' layers where the observations in each cell have been replaced by their mean.

Vertical layer 1	0.960	1.085	1.230	1.220	a_1
	0.940	1.025	1.050	1.120	b_1
	0.905	0.960	1.020	1.015	c_1
Vertical layer 2	1.070	1.125	1.135	1.280	a_2
	0.865	0.970	1.030	1.195	b_2
	0.880	0.965	1.040	1.010	c_2

Thus the three horizontal layers are a_1a_2, b_1b_2 and c_1c_2 . We now align the observations on each of these layers to obtain :

Vertical layer 1	0.041	-0.006	0.619	-0.016
	0.028	0.018	0.001	-0.469
	0.012	-0.003	-0.011	0.002
Vertical layer 2	-0.041	0.006	-0.619	0.016
	-0.028	-0.018	-0.001	0.469
	-0.012	0.003	0.011	-0.002

Hence after ranking each horizontal layer we obtain :

Vertical layer 1	1	3	4	2
	4	3	2	1
	4	2	1	3
Vertical layer 2	4	2	1	3
	1	2	3	4
	1	3	4	2

So, $C1 = (1 + 1) + (1 + 1) = 4$

and $C2 = 4 + \frac{1}{2}((3 + 2) + (3 + 2)) = 9$

Clearly neither of these results supports the alternative hypothesis. In fact, under H_0, $P(C1 \leqslant 4) = 0.2820$ and $P(C2 \leqslant 9) = 0.2397$.

Test (ii) - the classical F-test

The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $F > 2.53$ and $F > 3.71$ respectively, the values being obtained from the F-distribution with (6,23) degrees of freedom.

Performing the usual analysis of variance calculations produces $F = 1.47$. Clearly this result is quite consistent with the other tests in not supporting the alternative hypothesis.

4. First-order Interaction with Ordered Replicates.

Without any modification we can apply our match tests for second-order interaction to analysing interactions in two-way experiments where the replicates are ordered (the multivariate case).

To illustrate the procedure we shall analyse the problem presented in Mehra and Smith's paper. For our purposes the replicates correspond to the elements of the vertical layers in three-factor analysis.

We shall compare the results from the match tests with those from Mehra and Smith's, Wicoxon's and the classical F tests.

An experiment was conducted involving three varieties of sugar cane V_i $(i = 1, 2, 3)$ and three different levels of nitrogen N_j $(j = 1, 2, 3)$. Four replications R_k $(k = 1, .., 4)$

were taken. The yields in tons per acre are given in the table below.

	R_1			R_2		
	V_1	V_2	V_3	V_1	V_2	V_3
N_1	70.5	58.6	65.8	67.5	65.2	68.3
N_2	67.3	64.3	64.1	75.9	48.3	64.8
N_3	79.9	64.4	56.3	72.8	67.3	54.7

	R_3			R_4		
	V_1	V_2	V_3	V_1	V_2	V_3
N_1	63.9	70.2	72.7	64.2	51.8	67.6
N_2	72.2	74.0	70.9	60.5	63.6	58.3
N_3	64.8	78.0	66.2	86.3	72.0	54.4

The hypotheses under investigation are :

H_0 : there is no interaction between varieties of sugar cane and levels of nitrogen.

H_1 : there exists interaction between varieties of sugar cane and levels of nitrogen.

<u>Tests (i) - the match tests</u>.

Using the tables for $c = 3$, $b = 3$ and $v = 4$ given in sections 6 and 7 we obtain the following decision rules.

For the C1 test, the null hypothesis is rejected at the 5 % and 1 % levels of significance if $C1 \leq 6$ and $C1 \leq 5$ respectively, while for the C2 test, rejection at the same levels occurs if $C2 \leq 16$ and $C2 \leq 14$ respectively.

Regarding the replicates as vertical layers, we obtain three horizontal planes of data. Alongside each we show the aligned data.

Plane 1

	V_1	V_2	V_3			
	70.5	58.6	65.8	4.533	-2.292	-2.242
N_1	67.5	65.2	68.3	-0.500	2.275	-2.242
	63.9	70.2	72.7	-6.033	5.342	0.692
	64.2	51.8	67.6	2.000	-5.325	3.325

Plane 2

	V_1	V_2	V_3			
	67.3	64.3	64.1	-1.558	1.867	-0.308
N_2	75.9	48.3	64.8	9.275	-11.90	2.625
	72.2	74.0	70.9	-3.792	4.433	-0.642
	60.5	63.6	58.3	-3.925	5.600	-1675

Plane 3

	V_1	V_2	V_3			
	79.9	64.4	56.3	5.183	-4.792	-0.392
N_3	72.8	67.3	54.7	0.017	0.042	-0.0583
	64.8	78.0	66.2	-12.72	6.008	6.708
	86.3	72.0	54.5	7.517	-1.258	-6.258

We now obtain for the ranks within the vertical layers :

layer 1 :	3	1	2	layer 2 :	2	3	1
	1	3	2		3	1	2
	3	1	2		2	3	1
layer 3 :	1	2	3	layer 4 :	3	2	1
	1	3	2		1	3	2
	1	3	2		2	1	3

So $C1 = 4 + 3 + 5 + 0 = 12$

and $C2 = 4 + 4 + 7 + 2 = 17$

On consulting the decision rules we see that neither C1 nor C2 support the alternative hypothesis.

<u>Test (ii) - Wilcoxon's test</u>

For this test we follow the procedure outlined in Wilcoxon's (1949) booklet.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if $\chi_r^2 > 9.488$ and $\chi_r^2 > 13.28$ respectively, these critical values being approximate values based on the χ^2 - distribution with 4 degrees of freedom.

The test value is the sum of two χ_r^2 values. One component is obtained from the tabulation of $N_1 - N_2$ for the different V's; the other component is obtained from the tabulation of $N_1 + N_2 - 2N_3$ for the different V's. Details of the calculation are given below.

The $N_1 - N_2$ component

	V_1	Rank	V_2	Rank	V_3	Rank
	3.2	3	-5.7	1	1.7	2
	-8.4	1	16.9	3	3.5	2
	-8.3	1	-3.8	2	1.8	3
	2.7	2	-11.8	1	9.3	3
Rank sum		7		7		10

Hence $\chi_r^2 = \frac{12}{48}(49 + 49 + 100) - 48 = 1.5$

The $N_1 + N_2 - 2\,N_3$ component

	V_1	Rank	V_2	Rank	V_3	Rank
	-22.0	1	-5.9	2	17.3	3
	-2.2	2	-21.1	1	23.7	3
	6.5	2	-11.8	1	11.2	3
	-47.9	1	-28.6	2	16.9	3
Rank sum		6		6		12

Hence $\chi_r^2 = \frac{12}{48}(36 + 36 + 144) - 48 = 6$

So the test value is equal to $1.5 + 6 = 7.5 < 9.488$, the 5 % critical value thereby indicating the lack of evidence to support the alternative hypothesis.

Test (iii) - the Mehra and Smith test.

Because of the extremely lengthy computation involved with this test, we omit the calculations. In their paper they show that their statistic, χ_o^2, is asymptotically distributed as a χ^2 distribution with $(r - 1)(c - 1)$ degrees of freedom. Accordingly then the null hypothesis is rejected at the 5 % and 1 % levels of significance if $\chi_o^2 \geqslant 9.488$ and $\chi_o^2 \geqslant 13.28$ respectively, these critical values being from the χ^2 distribution with 4 degrees of freedom.

After much computation, Mehra and Smith obtain the value $\chi_o^2 = 9.12$, a result which is not significant at the 5 % level.

Test (iv) - the classical F-test.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if $F > 2.76$ and $F > 4.18$ respectively, the critical values being obtained from the F distribution with (4,27) degrees of freedom.

Performing the usual analysis of variance calculations produces $F = 3.01 > 2.76$, a result which is significant at the 5 % level.

It is interesting to note that the four nonparametric tests agree in not rejecting the null hypothesis at the 5 % level of significance.

5. A Note on the Distributions of C1 and C2.

Because of the large number of combinations of treatments, blocks and vertical layers we only present a selection of null distributions of C1 and C2. Furthermore, the length of these distributions has forced us to only present values whose cumulative probability is no greater than 0.3 .

The distributions of C1 and C2 were obtained by convolution using the distributions of M1 and M2 respectively.

6. Lower Tail Probabilities for the Null Distribution of C1

Below we give the approximate probabilities (see Chapter 5) $P(C1 \leqslant x)$ for c = 3, b = 3, v = 2 to 6; c = 3, b = 4, v = 2 to 6; c = 4, b = 3, v = 2 to 6; c = 4, b = 4, v = 2 to 6.

c = 3 b = 3

v = 2

x	P(C1 ≤ x)
0	.003086
2	.058642
3	.077160

v = 3

x	P(C1 ≤ x)
0	.000171
2	.004801
3	.006344
4	.048011
5	.078104
6	.207733

c = 3 b = 3

v = 4

x	P(C1 ≤ x)
0	.000009
2	.000352
3	.000467
4	.005096
5	.008354
6	.036646
7	.069054
8	.142356
9	.268404

v = 5

x	P(C1 ≤ x)
0	.000000
2	.000024
3	.000032
4	.000461
5	.000759
6	.004664
7	.008951
8	.027741
9	.056820
10	.103610
11	.193678
12	.266246

v = 6

x	P(C1 ≤ x)
0	.000000
2	.000002
3	.000002
4	.000038
5	.000062

x	P(C1 ≤ x)
6	.000495
7	.000960
8	.004008
9	.008525
10	.021308
11	.044960
12	.078305
13	.141562
14	.207573
15	.287025

c = 3 b = 4

v = 2

x	P(C1 ≤ x)
6	.027778
7	.083333
8	.111111
9	.231481

v = 3

x	P(C1 ≤ x)
9	.004630
10	.018519
11	.032407
12	.067130
13	.141204
14	.199074
15	.281250

v = 4

x	P(C1 ≤ x)
12	.000772
13	.003858
14	.008488
15	.018261
16	.042181
17	.071502
18	.109868
19	.179312
20	.250171

v = 5

x	P(C1 ≤ x)
15	.000129
16	.000772
17	.002058
18	.004737
19	.0011596
20	.022655
21	.038266
22	.066647
23	.104906
24	.147588
25	.208614
26	.279261

v = 6

x	P(C1 ≤ x)
18	.000021
19	.000150
20	.000472
21	.001179
22	.003022
23	.006580
24	.012212
25	.022496
26	.038814
27	.059945
28	.090482
29	.131740
30	.177517
31	.232846
32	.298246

c = 4 b = 3

v = 2

x	P(C1 ≤ x)
0	.001736
1	.012153
2	.048611
3	.136574
4	.281973

v = 3

x	P(C1 ≤ x)
0	.000072
1	.000723
2	.003979
3	.015336
4	.044822
5	.104772
6	.202028

v = 4

x	P(C1 ≤ x)
0	.000003
1	.000039
2	.000274
3	.001339
4	.005006
5	.015083
6	.037810
7	.080615
8	.148971
9	.242702

v = 5

x	P(C1 ≤ x)
0	.000000
1	.000002
2	.000017
3	.000101
4	.000457
5	.001676
6	.005148
7	.013526
8	.030915
9	.062302
10	.112056
11	.182017
12	.270210

v = 6

x	P(C1 ≤ x)
0	.000000
1	.000000
2	.000001
3	.000007
4	.000037
5	.000158
6	.000571
7	.001776
8	.004827
9	.011607
10	.024950
11	.048386
12	.085396
13	.138330
14	.207423
15	.290394

c = 4 b = 4

v = 2

x	P(C1 ≤ x)
0	.000003
2	.000147
3	.000389
4	.002830
5	.009087
6	.031524

x	P(Cl ≤ x)
7	.071383
8	.148352
9	.239703

v = 3

x	P(Cl ≤ x)
0	.000000
2	.000000
3	.000001
4	.000012
5	.000043
6	.000230
7	.000799
8	.002860
9	.007902
10	.020021
11	.042037
12	.080158
13	.133798
14	.207056
15	.292361

v = 4

x	P(Cl ≤ x)
0	.000000
2	.000000

x	P(Cl ≤ x)
3	.000000
4	.000000
5	.000000
6	.000001
7	.000004
8	.000020
9	.000074
10	.000265
11	.000804
12	.002263
13	.005554
14	.012434
15	.024868
16	.045740
17	.076945
18	.120836
19	.177036
20	.245477

v = 5

x	P(Cl ≤ x)
0	.000000
2	.000000
3	.000000
4	.000000
5	.000000

x	P(Cl ≤ x)
6	.000000
7	.000000
8	.000000
9	.000000
10	.000002
11	.000007
12	.000025
13	.000080
14	.000240
15	.000648
16	.001614
17	.003664
18	.007688
19	.014870
20	.026818
21	.045164
22	.071660
23	.107415
24	.153187
25	.208469
26	.272287

v = 6

x	P(Cl ≤ x)
0	.000000
2	.000000

x	$P(C1 \leqslant x)$	x	$P(C1 \leqslant x)$
3	.000000	29	.133127
4	.000000	30	.179021
5	.000000	31	.232401
6	.000000	32	.292196
7	.000000		
8	.000000		
9	.000000		
10	.000000		
11	.000000		
12	.000000		
13	.000001		
14	.000002		
15	.000008		
16	.000025		
17	.000071		
18	.000191		
19	.000474		
20	.001096		
21	.002359		
22	.004752		
23	.008977		
24	.015984		
25	.026900		
26	.042994		
27	.065454		
28	.095310		

Lower Tail Probabilities For The Null Distribution Of C2

Below we give the probabilities P(C2 ≤ x) for c = 3, b = 3, v = 2 to 6; c = 3 b = 4 v = 2 to 6; c = 4 b = 3 v = 2 to 6; c = 4 b = 4 v = 2 to 6. Probabilities exceeding 0.3 are not recorded.

c = 3 b = 3

v = 2

x	P(C2 ≤ x)
6	.003086
7	.040123
8	.197531

v = 3

x	P(C2 ≤ x)
9	.000171
10	.003258
11	.025634
12	.108968
13	.278335

v = 4

x	P(C2 ≤ x)
12	.000010
13	.000238
14	.002582
15	.015956
16	.062495
17	.164861

v = 5

x	P(C2 ≤ x)
15	.000001
16	.000016
17	.000227
18	.001846
19	.009868
20	.036679
21	.098736
22	.201005

v = 6

x	P(C2 ≤ x)
18	.000000
19	.000001
20	.000018
21	.000185
22	.001258
23	.006099
24	.021861
25	.059670
26	.128060
27	.225374

c = 3 b = 4

v = 2

x	P(C2 ≤ x)
16	.049383
17	.148148

v = 3

x	P(C2 ≤ x)
24	.010974
25	.043896
26	.115912

x	P(C2 ≤ x)
27	.221536

v = 4

32	.002439
33	.012193
34	.038409
35	.087791
36	.165123
37	.264613

v = 5

x	P(C2 ≤ x)
40	.000542
41	.003252
42	.011888
43	.031533
44	.067965
45	.124287
46	.200488
47	.291709

v = 6

x	P(C2 ≤ x)
48	.000120
49	.000843
50	.003507
51	.010567
52	.025669
53	.052726
54	.094947
55	.153341
56	.226538

c = 4 b = 3

v = 2

6.0	.006944
6.5	.020833
7.0	.044271
7.5	.089699
8.0	.144052
8.5	.214871

v = 3

x	P(C2 ≤ x)
9.0	.000579
9.5	.002315
10.0	.006113
10.5	.014431
11.0	.028128
11.5	.049294
12.0	.080612
12.5	.120962
13.0	.172337
13.5	.233098

v = 4

x	P(C2 ≤ x)
12.0	.000048
12.5	.000241
13.0	.000760
13.5	.002042
14.0	.004608
14.5	.009230
15.0	.017044
15.5	.028982
16.0	.046316
16.5	.070118
17.0	.100941
17.5	.139496
18.0	.185382
18.5	.238178
19.0	.296909

v = 5

x	P(C2 ≤ x)
15.0	.000004
15.5	.000024
16.0	.000088

x	P(C2 ≤ x)
16.5	.000266
17.0	.000676
17.5	.001516
18.0	.003107
18.5	.005866
19.0	.010358
19.5	.017274
20.0	.027344
20.5	.041392
21.0	.060151
21.5	.084247
22.0	.114160
22.5	.150006
23.0	.191726
23.5	.238852
24.0	.290605

v = 6

x	P(C2 ≤ x)
18.0	.000000
18.5	.000002
19.0	.000010
19.5	.000033
20.0	.000092
20.5	.000227
21.0	.000510

x	P(C2 ≤ x)
21.5	.001054
22.0	.002031
22.5	.003687
23.0	.006344
23.5	.010414
24.0	.016383
24.5	.024797
25.0	.036242
25.5	.051287
26.0	.070457
26.5	.094177
27.0	.122714
27.5	.156172
28.0	.194426
28.5	.237146
29.0	.283797

c = 4 b = 4

v = 2

x	P(C2 ≤ x)
12.0	.000003
13.0	.000087
13.5	.000124
14.0	.001087
14.5	.001877
15.0	.007575

x	P(C2 ≤ x)
15.5	.014069
16.0	.032856
16.5	.056831
17.0	.093909
17.5	.134343
18.0	.189968
18.5	.243407

v = 3

x	P(C2 ≤ x)
18.0	.000000
19.0	.000000
19.5	.000000
20.0	.000004
20.5	.000008
21.0	.000051
21.5	.000102
22.0	,000397
22.5	.000831
23.0	.002211
23.5	.004395
24.0	.008981
24.5	.015866
25.0	.027023
25.5	.041688
26.0	.062349

x	P(C2 ≤ x)
26.5	.087229
27.0	.118668
27.5	.154497
28.0	.196657
28.5	.241827
29.0	.292186

v = 4

x	P(C2 ≤ x)
24.0	.000000
25.0	.000000
25.5	.000000
26.0	.000000
26.5	.000000
27.0	.000000
27.5	.000000
28.0	.000003
28.5	.000006
29.0	.000021
29.5	.000048
30.0	.000136
30.5	.000295
31.0	.000674
31.5	.001341
32.0	.002614
32.5	.004664
33.0	.008019
33.5	.012891
34.0	.019990
34.5	.029496
35.0	.042166
35.5	.058048
36.0	.077864
36.5	.101351
37.0	.129085
37.5	.160494
38.0	.195891
38.5	.234375
39.0	.276089

v = 5

x	P(C2 ≤ x)
30.0	.000000
31.0	.000000
31.5	.000000
32.0	.000000
32.5	.000000
33.0	.000000
33.5	.000000
34.0	.000000
34.5	.000000
35.0	.000000
35.5	.000000
36.0	.000001
36.5	.000003
37.0	.000008
37.5	.000019
38.0	.000046
38.5	.000098
39.0	.000209
39.5	.000412
40.0	.000785
40.5	.001408
41.0	.002436
41.5	.004009
42.0	.006375
42.5	.009739
43.0	.014429
43.5	.020692
44.0	.028904
44.5	.039299
45.0	.052241
45.5	.067891
46.0	.086534
46.5	.108191
47.0	.133018
47.5	.160856
48.0	.191702

x	P(C2 ≤ x)
48.5	.225215
49.0	.261240
49.5	.299281

<u>v = 6</u>

x	P(C2 ≤ x)
36.0	.000000
37.0	.000000
37.5	.000000
38.0	.000000
38.5	.000000
39.0	.000000
39.5	.000000
40.0	.000000
40.5	.000000
41.0	.000000
41.5	.000000
42.0	.000000
42.5	.000000
43.0	.000000
43.5	.000000
44.0	.000000
44.5	.000001
45.0	.000003
45.5	.000007
46.0	.000015

x	P(C2 ≤ x)
46.5	.000032
47.0	.000065
47.5	.000128
48.0	.000241
48.5	.000433
49.0	.000752
49.5	.001256
50.0	.002033
50.5	.003183
51.0	.004843
51.5	.007159
52.0	.010322
52.5	.014519
53.0	.019981
53.5	.026914
54.0	.035562
54.5	.046125
55.0	.058818
55.5	.073793
56.0	.091204
56.5	.111114
57.0	.133582
57.5	.158559
58.0	.185995
58.5	.215721

8. Comments and Results of the Simulations.

In the simulations for second-order interaction we used the two match tests, C1 and C2, and the F-test. Bradley's test was excluded because of its reliance on ordered replications.

The simulations are based on four treatments, four blocks, two vertical layers and two replications. As before, the parameter θ varies from 0 to 1 and allows the effect of increasing the magnitude of the second -order interaction to be observed.

Normal Distribution. In both the 5 % and 1 % cases, all the tests achieved the maximum power of 1. It is encouraging to see C2 matching the performance of the F-test over part of the range.

Uniform Distribution. Both the match tests are superior to the F-test until θ reaches 0.5. All the tests have attained good overall power.

Double Exponential Distribution. Again, upto $\theta = 0.5$ both the match tests are superior to the F-test.

Cauchy Distribution. All the tests performed poorly, the maximum power in the 5 % case is only approximately 0.3. The F-test also exhibited poor robustness features.

Exponential Distribution. All the tests performed erratically and achieved low power. The match tests performed better than the F-test.

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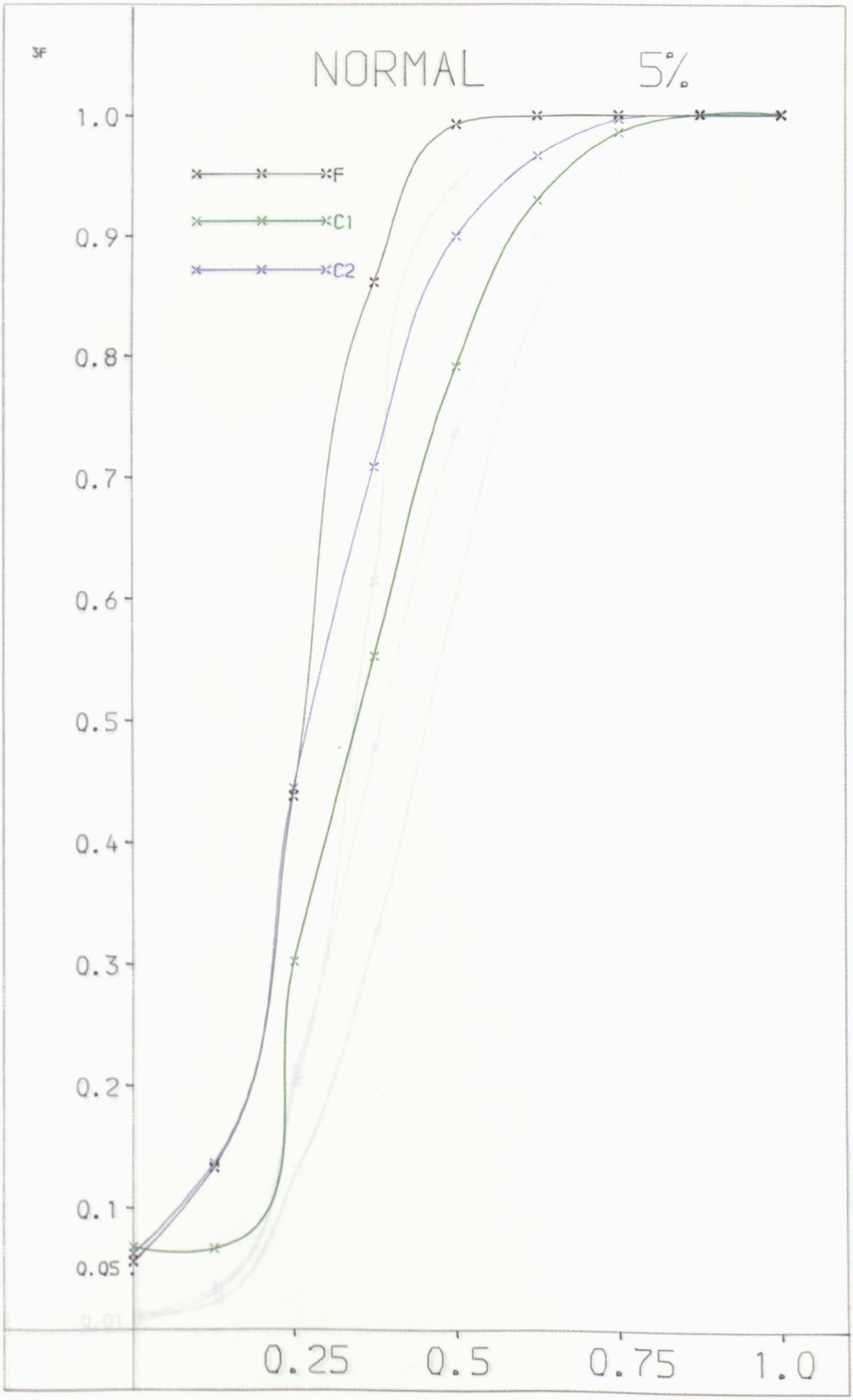

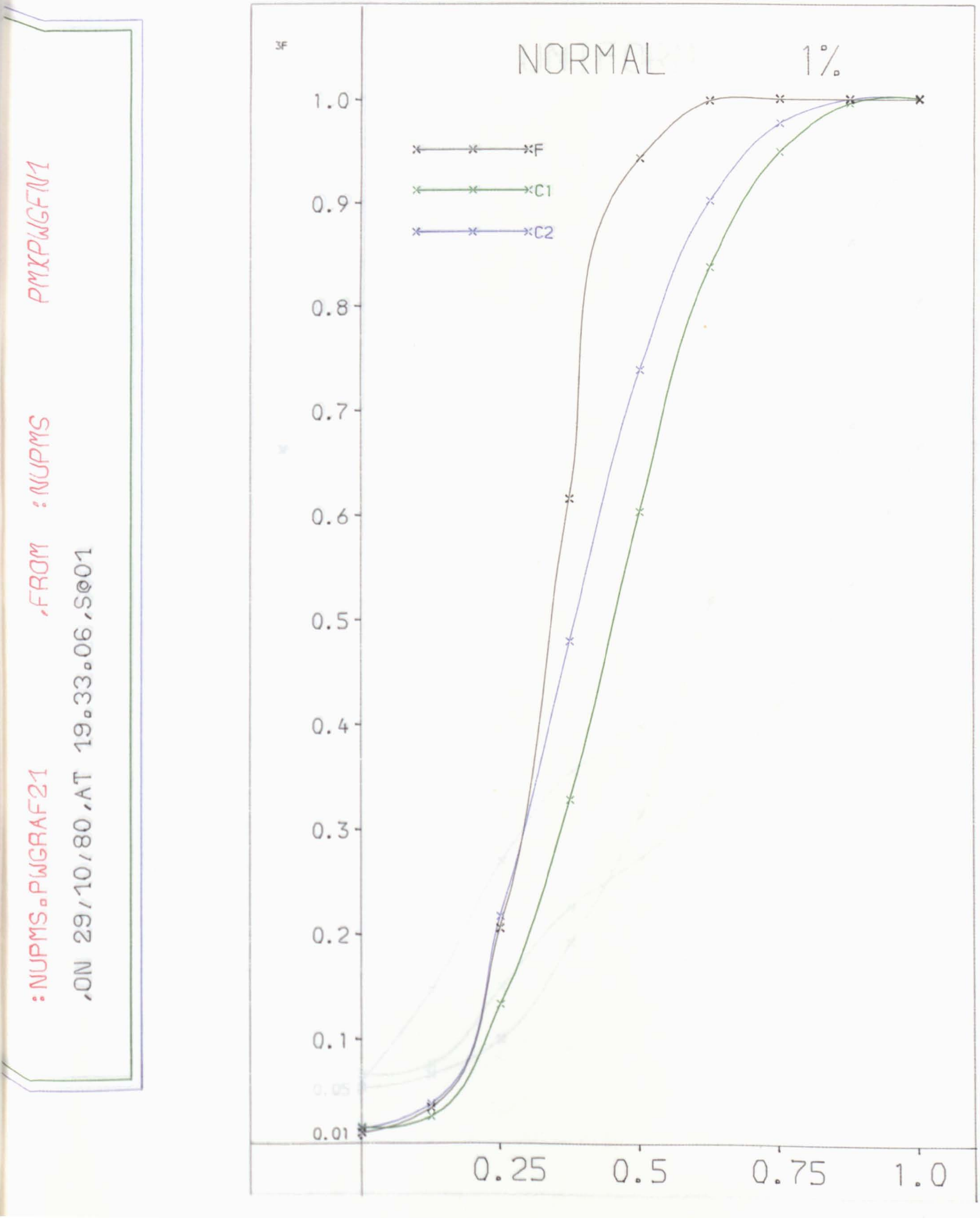
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3F
NORMAL 1%
F
C1
C2
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.01
0.25
0.5
0.75
1.0

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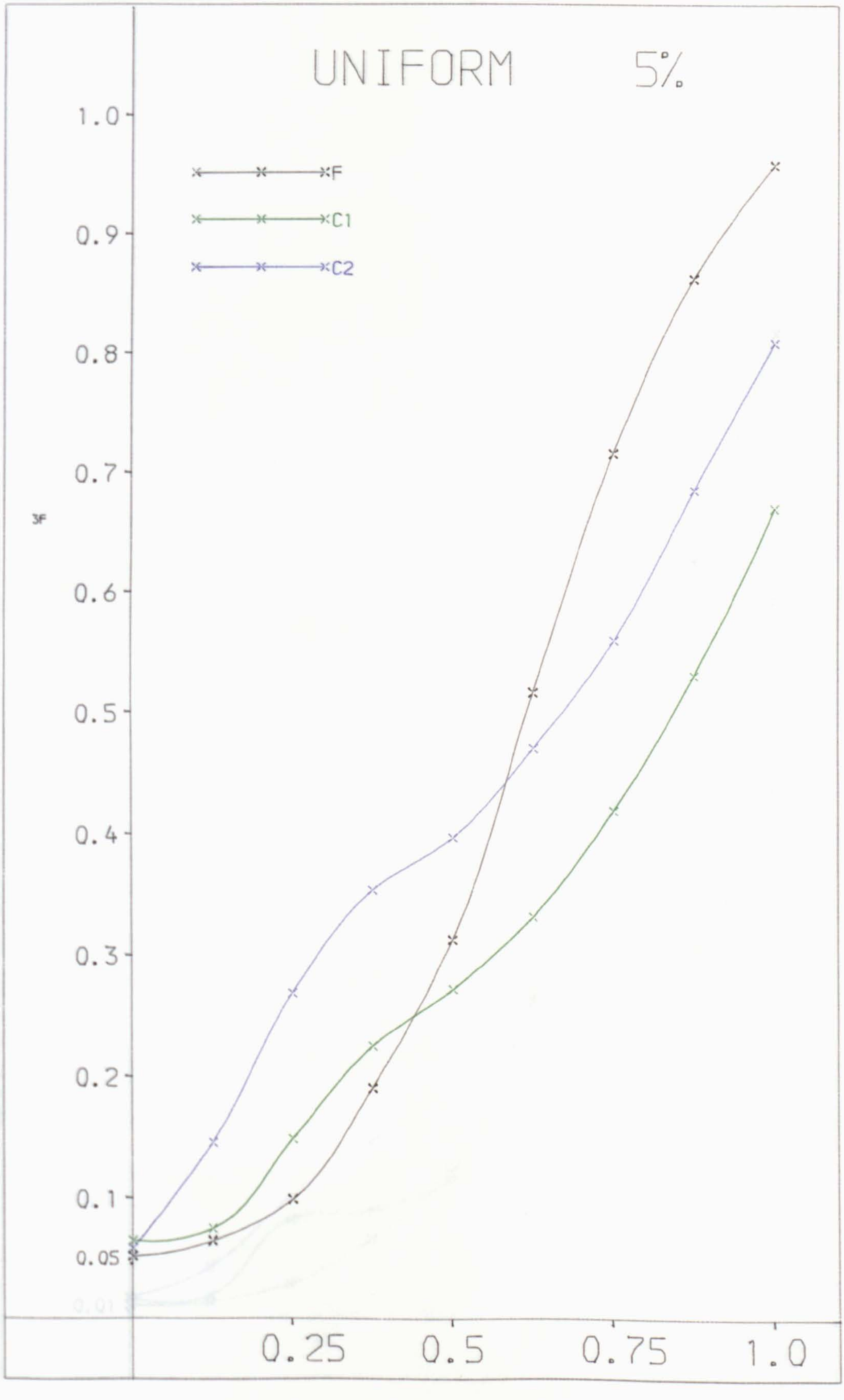

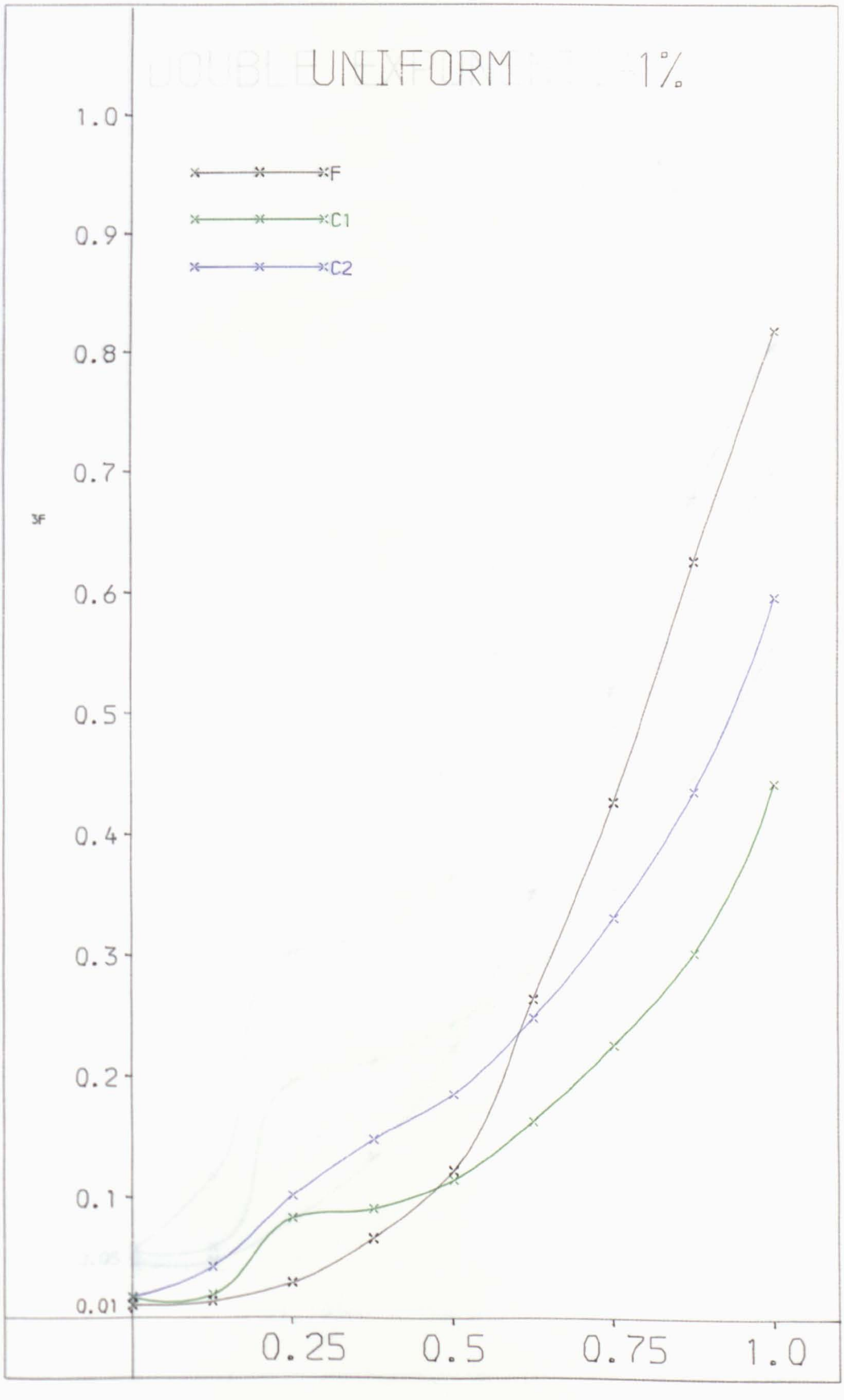
UNIFORM 1%
F
C1
C2
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.01
3F
0.25
0.5
0.75
1.0

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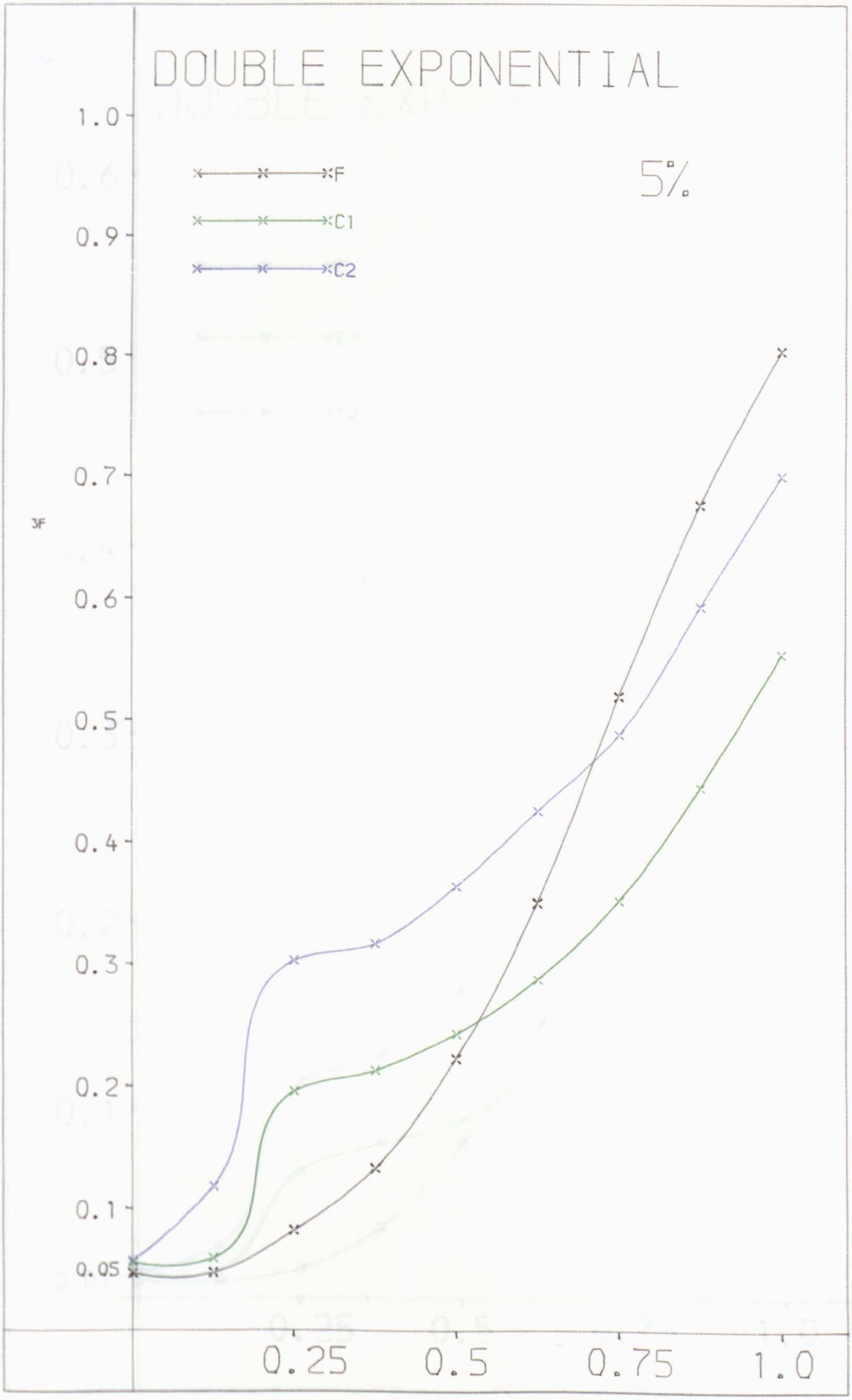
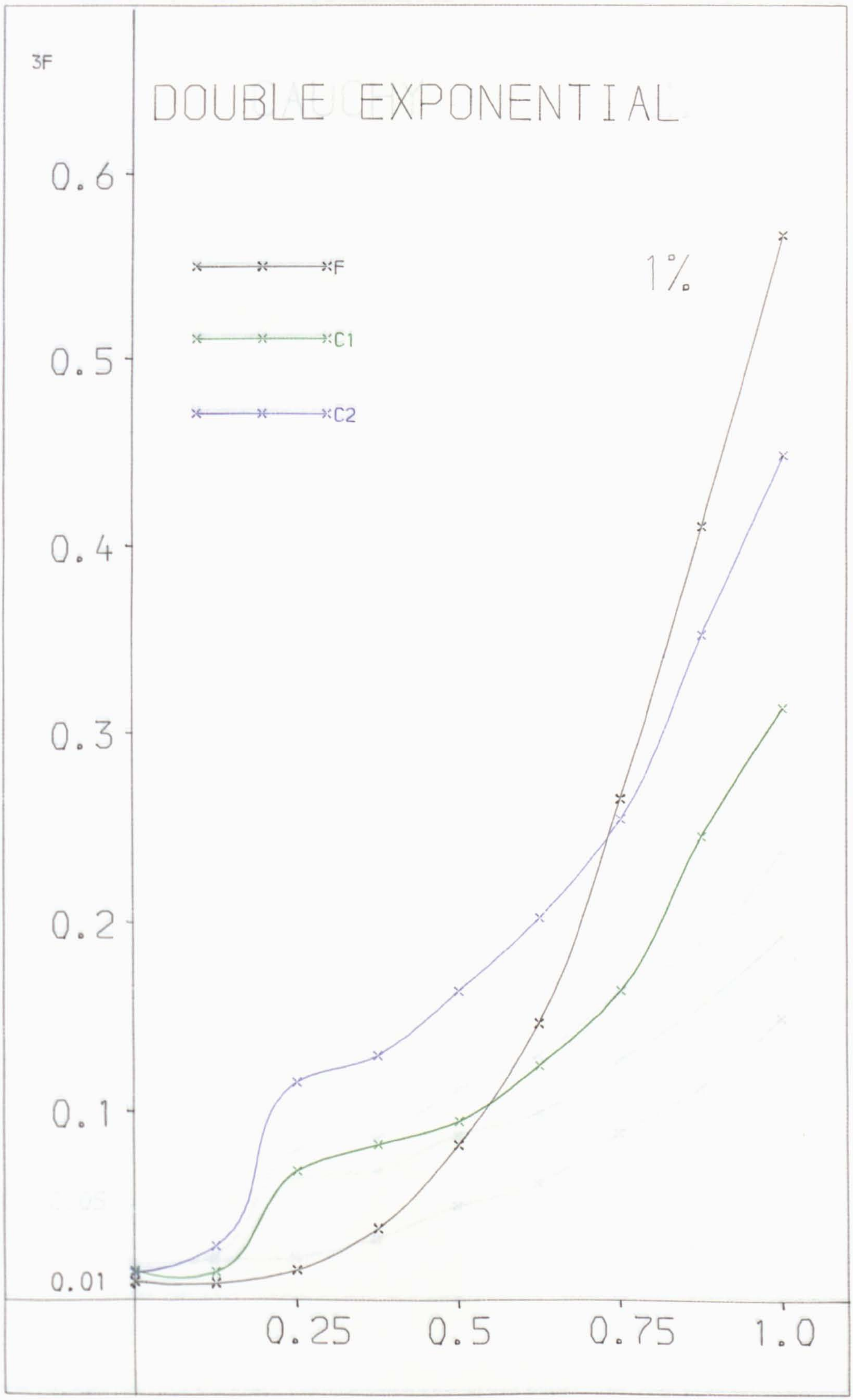
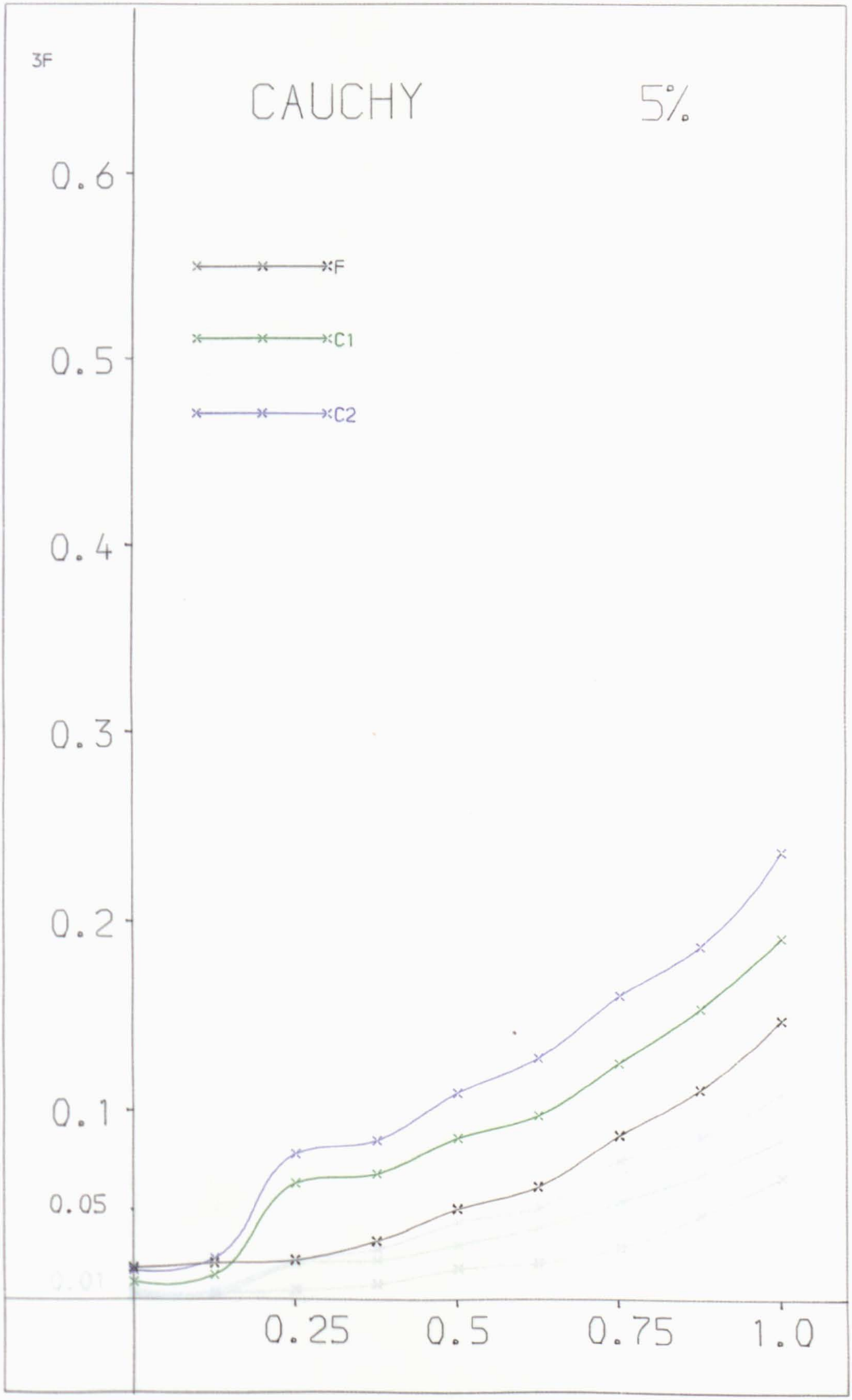
,ON 29/10/80,AT 19.46.35,S@01




3F
DOUBLE EXPONENTIAL
1%
F
C1
C2
0.6
0.5
0.4
0.3
0.2
0.1
0.01
0.25
0.5
0.75
1.0


3F
CAUCHY
5%
F
C1
C2
0.6
0.5
0.4
0.3
0.2
0.1
0.05
0.25
0.5
0.75
1.0

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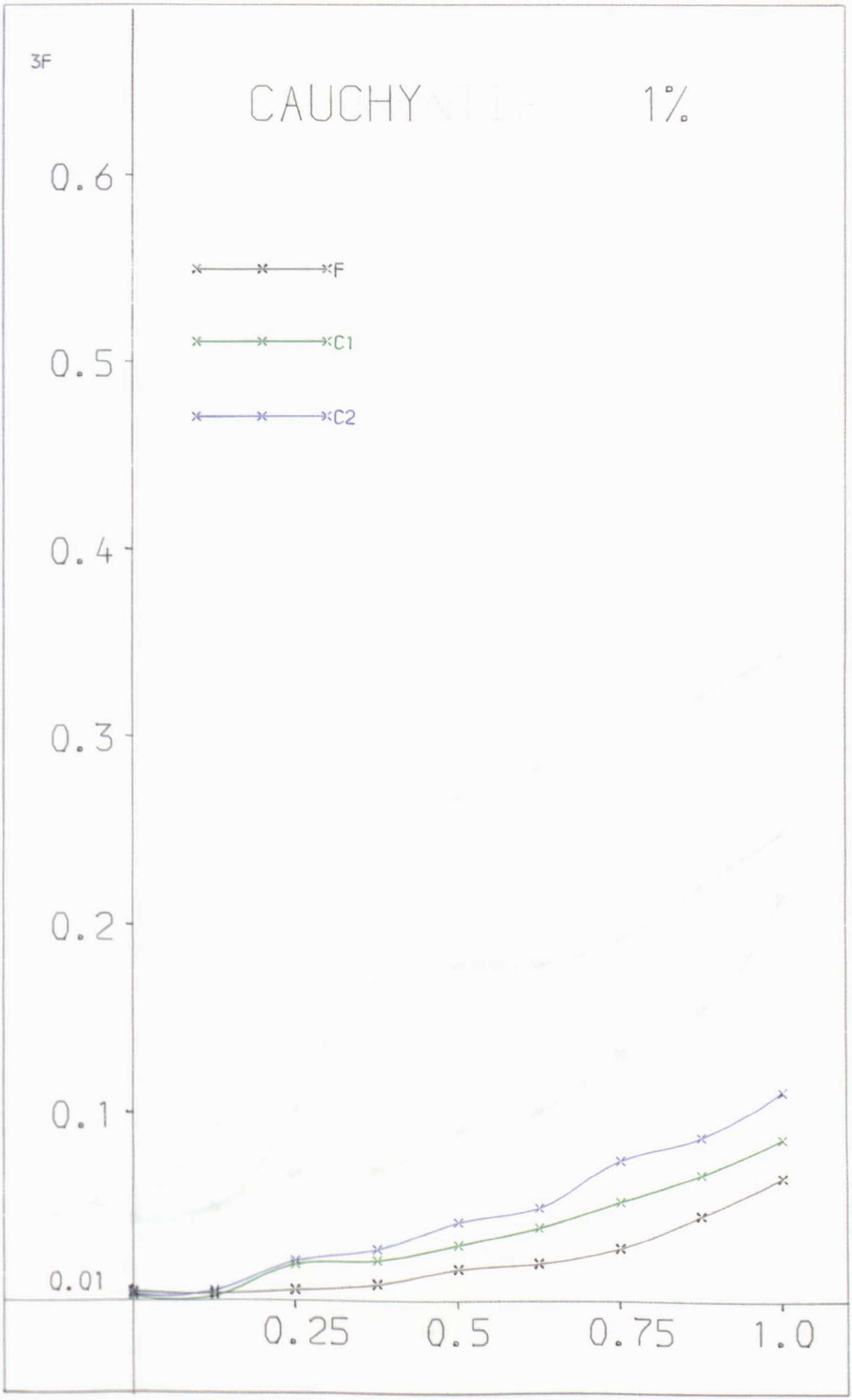

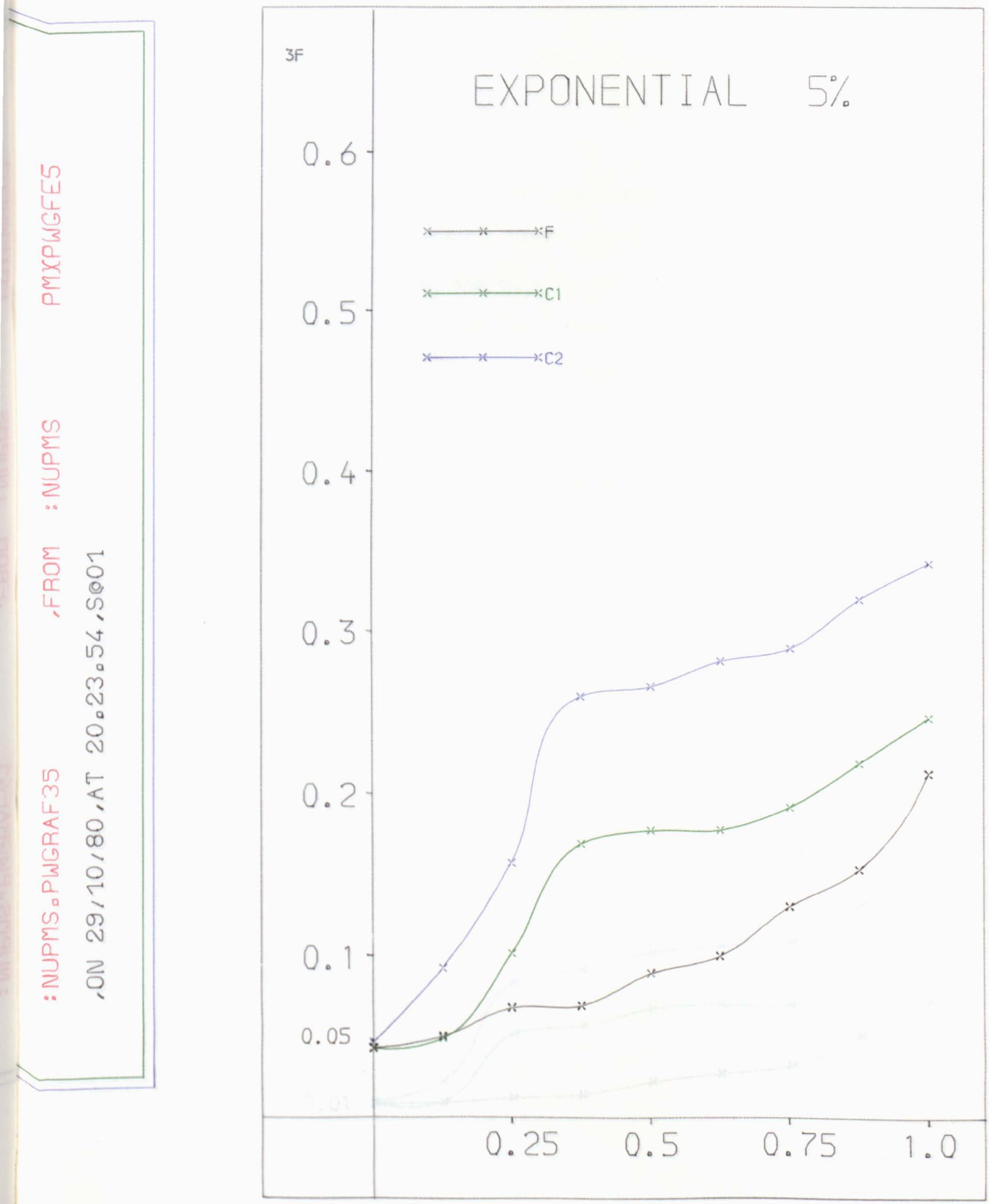
:NUPMS.PWGRAF35 ,FROM :NUPMS PMXPWGFE5
,ON 29/10/80,AT 20.23.54,S@01
3F
EXPONENTIAL 5%
F
C1
C2
0.6
0.5
0.4
0.3
0.2
0.1
0.05
0.25
0.5
0.75
1.0

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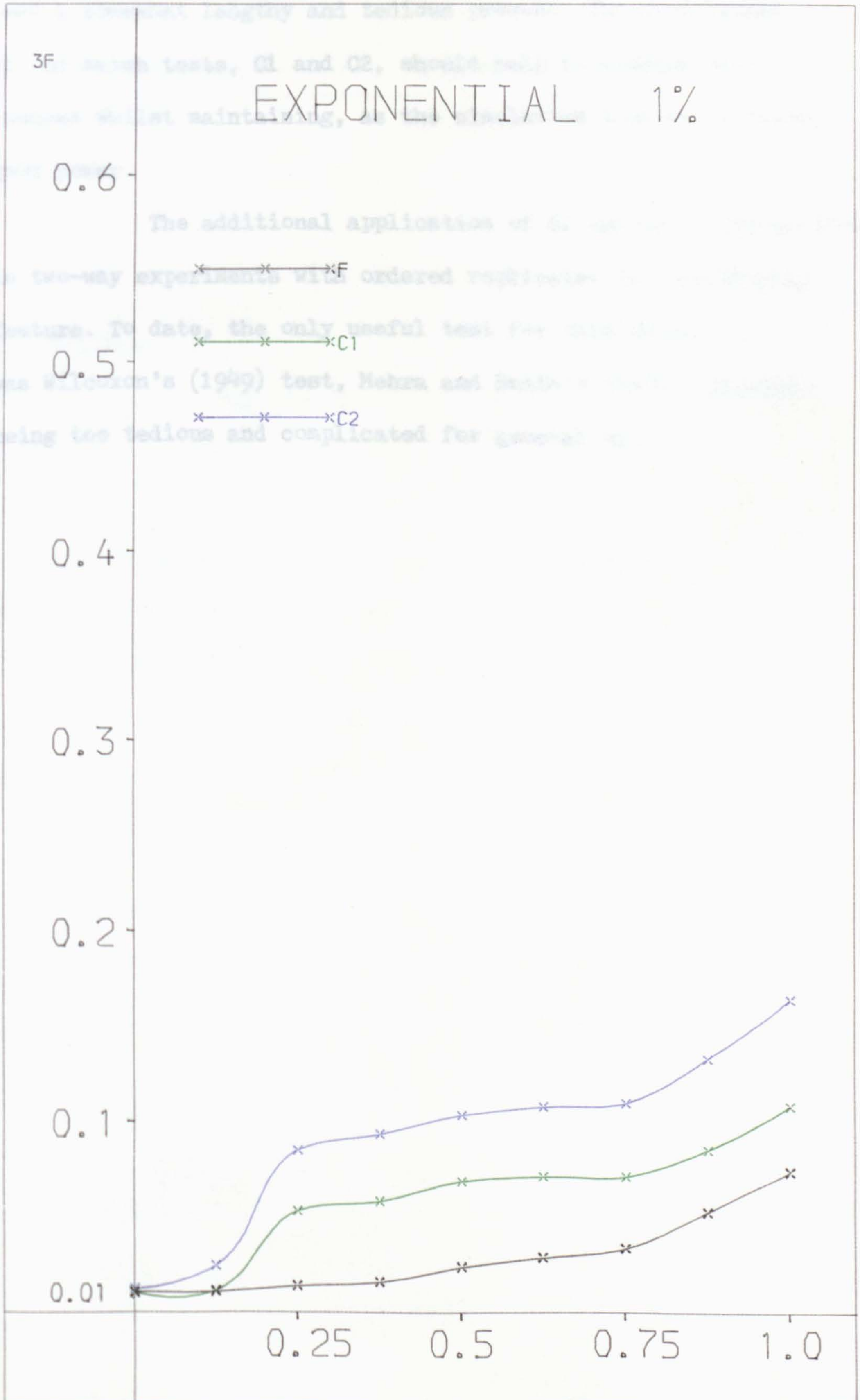
,ON 29/10/80,AT 20.25.52,S@01



9. Conclusion.

The analysis of second-order interaction has always been a somewhat lengthy and tedious process. The development of our match tests, C1 and C2, should help to shorten this process whilst maintaining, as the simulation studies indicate, good power.

The additional application of C1 and C2 to interaction in two-way experiments with ordered replicates is a worthwhile feature. To date, the only useful test for this situation was Wilcoxon's (1949) test, Mehra and Smith's (1970) procedure being too tedious and complicated for general use.

CHAPTER 7

THIRD-ORDER INTERACTION IN FOUR-WAY ANALYSIS OF VARIANCE

1. Introduction.

Our final tests are designed to detect the presence of third-order interaction in four factor experiments. Traditionally this analysis is accomplished by the classical F-test while the only non-traditional contender has been a test due to Bradley (1979) who presented a nonparametric procedure for interactions of any order in multivariate experiments.

By the very nature of the complexity of four factor experiments, any test for third-order interaction is likely to involve considerable computation. This may be appreciated simply by considering the usual parametric model for four factor experiments, namely

$$\begin{aligned} X_{ijklt} = {} & M + A_i + B_j + C_k + D_l + (AB)_{ij} + (AC)_{ik} + (AD)_{il} \\ & + (BC)_{jk} + (BD)_{jl} + (CD)_{kl} + (ABC)_{ijk} + (ABD)_{ijl} \\ & + (ACD)_{ikl} + (BCD)_{jkl} + (ABCD)_{ijkl} + z_{ijklt} \ ; \end{aligned}$$

$$\begin{aligned} \text{for } i &= 1, 2, \ldots\ldots, r \\ j &= 1, 2, \ldots\ldots, c \\ k &= 1, 2, \ldots\ldots, p \\ l &= 1, 2, \ldots\ldots, q \\ t &= 1, 2, \ldots\ldots, n_{ijkl} \ , \end{aligned}$$

where

M represents the overall mean,

A_i, B_j, C_k, D_l represent the main effects with

$$\sum_{i=1}^{r} A_i = \sum_{j=1}^{c} B_j = \sum_{k=1}^{p} C_k = \sum_{l=1}^{q} D_l = 0 \ ,$$

$(AB)_{ij}$, $(AC)_{ik}$, $(AD)_{il}$, $(BC)_{jk}$, $(BD)_{jl}$, $(CD)_{kl}$ represent first-order interactions where, as above, there are the usual restrictions on their sums,

$(ABC)_{ijk}$, $(ABD)_{ijl}$, $(ACD)_{ikl}$, $(BCD)_{jkl}$ represent the second-order interactions with the usual restrictions on their sums,

$(ABCD)_{ijkl}$ represents the third-order interaction with sum-to zero restrictions,

z_{ijklt}'s are random variables having a normal distribution with a zero location parameter,

and n_{ijkl} is the replications per cell.

The tests we propose for third-order interaction involve substantial, but not unreasonable amounts of computation. Furthermore, when the classical assumption of normality is not known to be true then our tests will provide valid alternative procedures.

Before presenting the tests it is necessary to define rank vectors and their related match functions. This will enable us to present the tests in a much more concise manner than would otherwise be possible using our previous notation.

2. Matches between Rank Vectors.

By a rank vector $\underline{a}$ we shall mean the n-tuple $\underline{a} = (a_1, a_2, \ldots\ldots\ldots, a_n)$ where the a_i's $(i = 1, 2, \ldots., n)$ are the ranks of n observations.

Given two rank vectors, $\underline{a} = (a_1, a_2, \ldots, a_n)$ and $\underline{b} = (b_1, b_2, \ldots, b_n)$ of equal length, we define the match function $m(\underline{a} ; \underline{b})$ of $\underline{a}$ and $\underline{b}$ by

$$m(\underline{a} ; \underline{b}) = p/n ,$$

where p is the number of matches between the ranks $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ respectively. Thus we have a perfect match between $\underline{a}$ and $\underline{b}$ if and only if $m(\underline{a} ; \underline{b}) = 1$.

As an example of this matching process consider the rank vectors $\underline{a} = (4, 1, 3, 2)$ and $\underline{b} = (1, 4, 3, 2)$. A simple comparison reveals that $m(\underline{a} ; \underline{b}) = 2/4$.

Just as we previously extended the concept of matches to near-matches which resulted in more powerful tests, so too we can extend the above matching idea to produce the modified match function $m'(\underline{a} ; \underline{b})$ of $\underline{a}$ and $\underline{b}$ by

$$m'(\underline{a} ; \underline{b}) = (p + p')/n ,$$

where p' is half the number of near-matches between the ranks $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. So, for example, if $\underline{a} = (1, 3, 2, 4)$ and $\underline{b} = (1, 2, 3, 4)$ then $m' = (2 + \frac{1}{2}.2)/4 = 3/4$.

We are now in a position to describe our tests for third-order interaction.

3. Definition of the Tests.

Our procedure is best explained by considering an experiment of a specific size, such as 4 x 4 x 4 x 3. Thus the data may be considered to be in three "cubes", D_1, D_2 and D_3 (corresponding to the three levels of factor D), each of size 4 x 4 x 4 with n_{ijkl} (i, j, k = 1, 2, 3, 4 and l = 1, 2, 3) replications in each cell. The decision to split the data in this manner is quite arbitrary; the data could equally well have been arranged in four "cubes" C_1, C_2, C_3 and C_4 (corresponding to the four levels of factor C) each of size 4 x 4 x 3.

The observations in each of the 3.4^3 cells are replaced by their mean $\bar{X}_{ijkl}$; thus although some information is lost by this process, we are able to deal with unequal replication sizes. Each mean is now replaced by the appropriate aligned mean observation given by

$$\bar{X}_{ijkl} - \bar{X}_{ij.l} - \bar{X}_{i.kl} - \bar{X}_{.jkl} + \bar{X}_{i..l} + \bar{X}_{.j.l} + \bar{X}_{..kl} + \bar{X}_{..}$$

where, for a given cube l,

$\bar{X}_{ij.l}$, $\bar{X}_{i.kl}$, $\bar{X}_{.jkl}$ are the means over the planes (specified by directions i, j, etc.) that pass through the $(i, j, k)^{th}$ mean observation,

$\bar{X}_{i..l}$, $\bar{X}_{.j.l}$, $\bar{X}_{..kl}$ are the means over the lines (specified by the directions of i, j and l respectively) that pass through the $(i, j, k)^{th}$ mean observation.

Thus each cube is transformed to data representing second-order interactions.

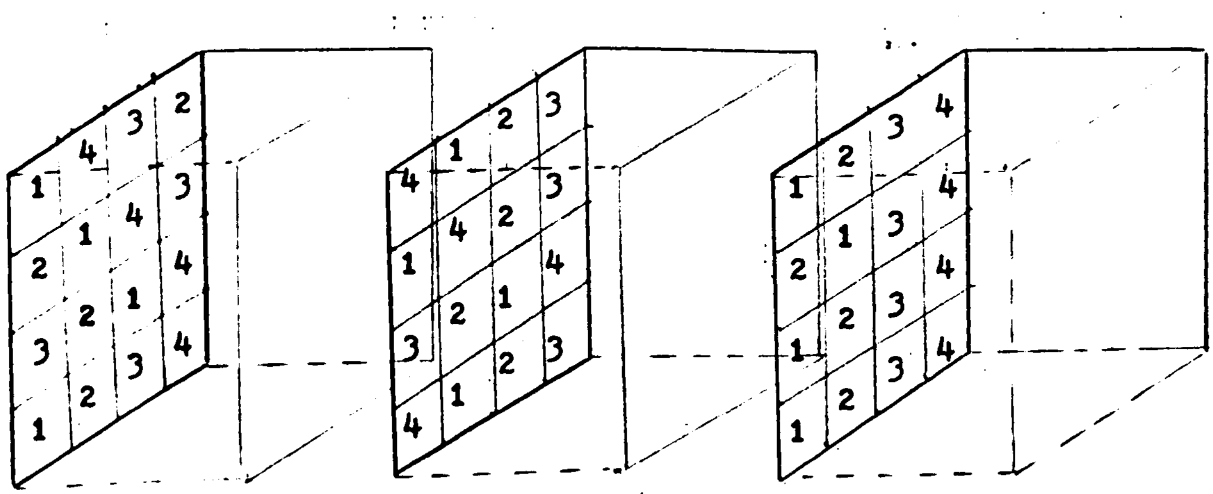
In each cube, the mean aligned observations in each i-k plane (the direction being quite arbitrary) are ranked in, for example, the i^{th} direction. Thus each cube will consist of four planes of ranks.

Suppose now that the ranks for the first such plane in each cube, D_1, D_2 and D_3, in terms of rank vectors are as follows.

<u>Cube</u>

D_1	D_2	D_3
$\underline{a}_{11} = (1, 4, 3, 2)$	$\underline{a}_{21} = (4, 1, 2, 3)$	$\underline{a}_{31} = (1, 2, 3, 4)$
$\underline{a}_{12} = (2, 1, 4, 3)$	$\underline{a}_{22} = (1, 4, 2, 3)$	$\underline{a}_{32} = (2, 1, 3, 4)$
$\underline{a}_{13} = (3, 2, 1, 4)$	$\underline{a}_{23} = (3, 2, 1, 4)$	$\underline{a}_{33} = (1, 2, 3, 4)$
$\underline{a}_{14} = (1, 2, 3, 4)$	$\underline{a}_{24} = (4, 1, 2, 3)$	$\underline{a}_{34} = (1, 2, 3, 4)$

These ranks are shown in the diagram below.



From these we calculate $m(\underline{a}_{1i} : \underline{a}_{2i})$, $m(\underline{a}_{1i} : \underline{a}_{3i})$ and $m(\underline{a}_{2i} : \underline{a}_{3i})$ for $i = 1, 2, 3, 4$ to give

	$m(\underline{a}_{1i} : \underline{a}_{2i})$	$m(\underline{a}_{1i} : \underline{a}_{3i})$	$m(\underline{a}_{2i} : \underline{a}_{3i})$	Total
$i = 1$	0	2/4	0	1/2
2	1/4	2/4	0	3/4
3	1	2/4	2/4	2
4	0	1	0	1

The sum $V_1 = \sum_{i=1}^{4} \left(m(\underline{a}_{1i} : \underline{a}_{2i}) + m(\underline{a}_{1i} : \underline{a}_{3i}) + m(\underline{a}_{2i} : \underline{a}_{3i}) \right)$

is then calculated. In the above example this gives $V_1 = 4\frac{1}{4}$.

Similar calculations are performed for the remaining three planes to produce V_2, V_3 and V_4 . The test statistic is then given by

$$V1 = \sum_{k=1}^{4} V_k \ .$$

The presence of third-order interactions will produce different second-order interactions from cube to cube. This will cause the cubes to have different rank structures which will result in a small value of V1. Conversely, the absence of third-order interaction will tend to preserve the rank structure of the aligned observations thereby resulting in a high value of V1. Thus the null hypothesis of no third-order interaction will be rejected if $V1 \leq$ a critical value obtained from the appropriate table in section 7. For the reasons outlined in Chapter 5, the critical values are approximate.

In general, in a r x c x p x q experiment the statistic V1 becomes

$$V1 = \sum_{j=1}^{q} V_k ,$$

where

$$V_k = \sum_{i=1}^{r} \sum_{j=1}^{p-1} \sum_{j'=j+1}^{p} m(\underline{a}_{ij} ; \underline{a}_{ij'})$$

with $\underline{a}_{ij}$ being the i^{th} rank vector in the j^{th} cube.

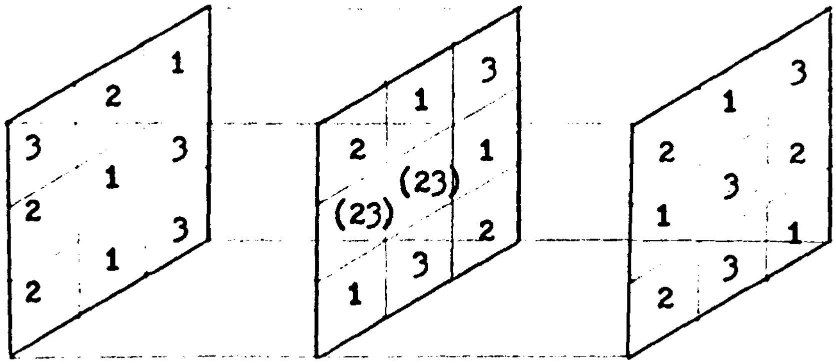
In a similar way as we extended the general alternatives test M1 to the more powerful version M2, so here by using $m'(\underline{a}_{ij} ; \underline{a}_{ij'})$ in place of $m(\underline{a}_{ij} ; \underline{a}_{ij'})$ we obtain a test statistic V2, that incorporates more information regarding the nearness of matches. Clearly V2 is calculated in a similar manner to V1, approximate critical values for V2 being given in section 8.

4. Examples.

In order to economise on space, we reproduce only the mean aligned observations. The data are constructed to form a 3 x 3 x 3 x 3 experiment with two replications per cell.

Example 1.

The mean aligned observations are given below where the diagram illustrates the ranked data for the first cube.



	Cube 1			Ranks		
	-0.0457	0.1991	-0.1534	3	2	1
Plane 1	0.0046	-0.0370	0.0324	2	1	3
	0.0411	-0.1620	0.1209	2	1	3
	0.0046	-0.0370	0.0324	2	1	3
Plane 2	0.0185	0.0185	-0.0370	(23)	(23)	1
	-0.0231	0.0185	0.0046	1	3	2
	0.0411	-0.1620	0.1209	2	1	3
Plane 3	-0.0231	0.0185	0.0046	1	3	2
	-0.0179	0.1435	-0.1256	2	3	1

	Cube 2					
	-0.0041	0.0324	-0.0284	2	3	1
Plane 1	0.0602	0.0185	-0.0787	3	2	1
	-0.0561	-0.0509	0.1071	1	2	3
	-0.1065	-0.1481	0.2546	2	1	3
Plane 2	0.1296	-0.0370	-0.0926	3	2	1
	-0.0231	0.1852	-0.1620	2	3	1

	0.1105	0.1157	-0.2263	2	3	1
Plane 3	-0.1898	0.0185	0.1713	1	2	3
	0.0793	-0.1343	0.0550	3	1	2

Cube 3

	0.2043	-0.0787	-0.1256	3	2	1
Plane 1	-0.1065	-0.0926	0.1991	1	2	3
	-0.0978	0.1713	-0.0735	1	3	2
	0.0602	-0.0926	0.0324	3	1	2
Plane 2	-0.0370	0.1852	-0.1481	2	3	1
	-0.0231	-0.0926	0.1157	2	1	3
	-0.2645	0.1713	0.0932	1	3	2
Plane 3	0.1435	-0.0926	-0.0509	3	1	2
	0.1209	-0.0787	-0.0422	3	1	2

The hypotheses of interest are

H_0 : there is no third-order interaction.

H_1 : there is some third-order interaction.

Tests (i) - the match tests

The approximate critical values are obtained from the tables in sections 6 and 7.

For the V1 test, the null hypothesis is rejected at the 5 % and 1 % levels of significance if $V1 \leq 6$ and $V1 \leq 5$ respectively, while for the V2 test rejection occurs at the same levels if $V2 \leq 12.67$ and $V2 \leq 12$ respectively.

Values of $m(a_{ij} : a_{ij'})$

		$m(a_{11} : a_{12})$	$m(a_{11} : a_{13})$	$m(a_{12} : a_{13})$
	$i = 1$	$\frac{1}{3}$	1	$\frac{1}{3}$
Plane 1	2	0	$\frac{1}{3}$	$\frac{1}{3}$
	3	$\frac{1}{3}$	0	$\frac{1}{3}$
	$i = 1$	1	$\frac{1}{3}$	$\frac{1}{3}$
Plane 2	2	2/3	2/3	$\frac{1}{3}$
	3	$\frac{1}{3}$	0	$\frac{1}{3}$
	$i = 1$	$\frac{1}{3}$	0	$\frac{1}{3}$
Plane 3	2	$\frac{1}{3}$	$\frac{1}{3}$	0
	3	0	0	1

Hence $V_1 = 3$, $V_2 = 4$ and $V_3 = 2\frac{1}{3}$ giving $V1 = 9\frac{1}{3}$.

Values of $m'(a_{ij} : a_{ij'})$

		$m'(a_{11} : a_{12})$	$m'(a_{11} : a_{13})$	$m'(a_{12} : a_{13})$
	$i = 1$	2/3	1	2/3
Plane 1	2	$\frac{1}{3}$	2/3	$\frac{1}{3}$
	3	2/3	$\frac{1}{3}$	2/3
	$i = 1$	1	2/3	2/3
Plane 2	2	5/6	5/6	2/3
	3	2/3	$\frac{1}{3}$	$\frac{1}{3}$
	$i = 1$	$\frac{1}{3}$	$\frac{1}{3}$	2/3
Plane 3	2	2/3	$\frac{1}{3}$	$\frac{1}{3}$

Hence $V_1' = 5\frac{1}{3}$, $V_2' = 6$ and $V_3' = 4\frac{1}{3}$ giving $V2 = 15\frac{2}{3}$.

For reasons of space, only the range method was used for ties. Clearly, neither V1 nor V2 provides evidence to support the alternative hypothesis.

Test (ii) - the classical F-test.

The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $F > 1.79$ and $F > 2.27$ respectively, there being (16,81) degrees of freedom.

Performing the usual analysis of variance calculations produces the value $F = 1.437$ which clearly provides no support for the alternative hypothesis.

Example 2.

The mean aligned observations for this example are given below.

	Cube 1			Ranks		
	-0.0561	-0.2176	0.2737	2	1	3
Plane 1	-0.1343	0.2407	-0.1065	1	3	2
	0.1904	-0.0231	-0.1672	3	2	1
	-0.0231	0.0185	0.0046	1	3	2
Plane 2	0.0185	0.0185	-0.0370	(23)	(23)	1
	0.0046	-0.0370	0.0324	2	1	3
	0.0793	0.1991	-0.2784	2	3	1
Plane 3	0.1157	-0.2593	0.1435	2	1	3
	-0.1950	0.0602	0.1348	1	2	3

Cube 2

	-0.1534	0.1713	-0.0179	1	3	2
Plane 1	0.1435	-0.2037	0.0602	3	1	2
	0.0098	0.0324	-0.0422	2	3	1
	0.0880	-0.0926	0.0046	3	1	2
Plane 2	-0.2037	0.2407	-0.0370	1	3	2
	0.1157	-0.1481	0.0324	3	1	2
	0.0654	-0.0787	0.0133	3	1	2
Plane 3	0.0602	-0.0370	-0.0231	3	1	2
	-0.1256	0.1157	0.0098	1	3	2

Cube 3

	0.1383	0.0602	-0.1985	3	2	1
Plane 1	0.0880	-0.0370	-0.0509	3	2	1
	-0.2263	-0.0231	0.2494	1	2	3
	-0.1343	0.0741	0.0602	1	3	2
Plane 2	0.0185	-0.1481	0.1296	2	1	3
	0.1157	0.0741	-0.1898	3	2	1
	-0.0041	-0.1343	0.1383	2	1	3
Plane 3	-0.1065	0.1852	-0.0787	1	3	2
	0.1105	-0.0509	-0.0596	3	2	1

The hypotheses of interest are

H_0 : there is no third-order interaction.

H_1 : there is some third-order interaction.

Tests (i) - the match tests.

For the V1 test, the null hypothesis is rejected at the 5 % and 1 % levels of significance if $V1 \leqslant 6$ and $V1 \leqslant 5$ respectively, while for the V2 test rejection occurs at the same levels if $V2 \leqslant 12.67$ and $V2 \leqslant 12$ respectively.

Values of $m(a_{ij} : a_{ij'})$

		$m(a_{i1} : a_{i2})$	$m(a_{i1} : a_{i3})$	$m(a_{i2} : a_{i3})$
	$i = 1$	0	0	0
Plane 1	2	$\frac{1}{3}$	0	$\frac{1}{3}$
	3	$\frac{1}{3}$	$\frac{1}{3}$	0
	$i = 1$	$\frac{1}{3}$	1	$\frac{1}{3}$
Plane 2	2	1/6	1/6	0
	3	$\frac{1}{3}$	0	$\frac{1}{3}$
	$i = 1$	0	$\frac{1}{3}$	$\frac{1}{3}$
Plane 3	2	$\frac{1}{3}$	0	$\frac{1}{3}$
	3	$\frac{1}{3}$	$\frac{1}{3}$	0

Hence $V_1 = 1\frac{1}{3}$, $V_2 = 2\frac{2}{3}$ and $V_3 = 2$ giving $V1 = 6$.

Values of $m'(a_{ij} : a_{ij'})$

		$m'(a_{i1} : a_{i2})$	$m'(a_{i1} : a_{i3})$	$m'(a_{i2} : a_{i3})$
	$i = 1$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
Plane 1	2	$\frac{1}{3}$	$\frac{1}{3}$	2/3
	3	2/3	$\frac{1}{3}$	$\frac{1}{3}$

		$m'(a_{i1} : a_{i2})$	$m'(a_{i1} : a_{i3})$	$m'(a_{i2} : a_{i3})$
	$i = 1$	⅓	1	⅓
Plane 2	2	2/3	⅓	⅓
	3	1	⅓	2/3
	$i = 1$	⅓	⅓	2/3
Plane 3	2	2/3	⅓	⅓
	3	2/3	⅓	⅓

Hence $V_1' = 3\frac{2}{3}$, $V_2' = 5$ and $V_3' = 4$ giving $V2 = 12\frac{2}{3}$.

Again, the range method was used for ties. From the above values of V1 and V2 we see that both tests provide evidence to support the alternative hypothesis at the 5 % level of significance but not at the 1 % level.

Test (ii) - the classical F-test.

The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $F > 1.79$ and $F > 2.27$ respectively, there being (16,81) degrees of freedom.

Performing the usual analysis of variance calculations produces the value $F = 2.22$ which is significant at the 5 % but not the 1 % level of significance.

5. Example of the Analysis of a Four-Factor Experiment.

In this example we analyse a 4 x 4 x 2 x 3 experiment with two 2 replications per cell. We shall investigate main effects, first, second and third order interactions. The situation is based on the four-factor model given in section 1 with factors A, B, C and D at 4, 4, 2 and 3 levels respectively.

Since our aim is to simply illustrate the various procedures we only investigate a selection of the possible hypotheses, namely

(I) $H_0 : A_i = 0$ for all i, $(i = 1, 2, 3, 4)$

$H_1 : A_i \neq 0$ for some i

(II) $H_0 : (AB)_{ij} = 0$ for all i and j, $(i, j = 1, 2, 3, 4)$

$H_1 : (AB)_{ij} \neq 0$ for some i and j

(III) $H_0 : (ABC)_{ijk} = 0$ for all i, j and k, $(i, j = 1, 2, 3, 4$
$k = 1, 2)$

$H_1 : (ABC)_{ijk} \neq 0$ for some i, j and k

(IV) $H_0 : (ABCD)_{ijkl} = 0$ for all i, j, k and l
$(i, j = 1, 2, 3, 4$
$k = 1, 2$ and $l = 1, 2, 3)$

$H_1 : (ABCD)_{ijkl} \neq 0$ for some i, j, k and l.

Accordingly we give only the data relevant to each set of hypotheses.

Hypotheses (I).

The relevant data are as follows.

						Ranks	
4.12	3.19	3.01	3.31	4	2	1	3
3.84	3.35	2.61	4.34	3	2	1	4
2.81	3.76	2.95	2.63	2	4	3	1
4.03	2.42	3.55	3.16	4	1	3	2

Tests (i) - the match tests.

The critical values are obtained from the exact null distributions given in Chapter 3 and are the best conservative values.

For the M1 test, the null hypothesis is rejected at the 5 % and 1 % levels of significance if $M1 \geq 12$ and $M1 \geq 15$ respectively, while for the M2 test rejection occurs at the same levels of significance if $M2 \geq 15$ and $M2 \geq 18$ respectively.

Performing the usual comparison of ranks produces $M1 = 4$ and $M2 = 8$ with neither value supporting the alternative hypothesis.

Test (ii) - Friedman's test.

The critical values are obtained from the exact null distribution for $c = 4$ and $b = 4$ and are the best conservative values.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if $\chi_r^2 \geq 7.8$ and $\chi_r^2 \geq 9.6$ respectively.

Performing the usual calculations produces $\chi_r^2 = 2.1$ which clearly is a result that does not support the alternative hypothesis.

Test (iii) - the classical F-test.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if $F > 2.70$ and $F > 3.98$ respectively, the values being obtained from the F - distribution with (3,96) degrees of freedom.

Performing the usual analysis of variance calculations produces $F = 1.68$, again a result which does not support the alternative hypothesis.

Hypotheses (II).

The relevant mean aligned data are as follows.

				Ranks			
0.055	-0.013	-0.018	-0.023	4	3	2	1
-0.013	-0.008	-0.107	0.128	2	3	1	4
-0.102	0.143	0.034	-0.076	1	4	3	2
0.060	-0.122	0.091	-0.029	3	1	4	2

Tests (i) - the match tests.

For the M1 test the null hypothesis is rejected at the 5 % and 1 % levels of significance if $M1 \leqslant 2$ and $M1 = 0$ respectively, while for the M2 test rejection occurs at the same levels of significance if $M2 \leqslant 7.5$ and $M2 \leqslant 6$ respectively.

Performing the usual comparison of ranks produces $M1 = 2$ and $M2 = 7$, results which are significant at the 5 % level of significance.

Test (ii) - the classical F-test.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if F > 1.97 and F > 2.59 respectively, the values being obtained from the F - distribution with (9,96) degrees of freedom.

Performing the usual analysis of variance calculations gives F = 1.98, a result significant at the 5 % level.

Hypotheses (III).

The relevant mean aligned data are as follows.

Vertical Layer 1

				Ranks			
0.050	-0.039	0.039	-0.050	4	2	3	1
-0.081	0.029	-0.195	0.247	2	3	1	4
-0.086	0.055	0.029	0.003	1	4	3	2
0.117	-0.044	0.128	-0.201	3	2	4	1

Vertical Layer 2

0.060	0.013	-0.076	0.003	4	3	1	2
0.055	-0.044	-0.018	0.008	4	1	2	3
-0.118	0.232	0.039	-0.154	2	4	3	1
0.003	-0.201	0.055	0.143	2	1	3	4

Tests (i) - the match tests.

For the C1 test the null hypothesis is rejected at the 5 % and 1 % levels of significance if C1 $\leq$ 6 and C1 $\leq$ 5 respectively, while for the C2 test rejection occurs at the same levels if C2 $\leq$ 16 and C2 $\leq$ 15 respectively.

Performing the usual comparisons of ranks in each vertical layer produces C1 = 7 and C2 = $15\frac{1}{2}$, the result for the C2 test being significant at the 5 % level.

Test (ii) - the classical F-test.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if $F > 1.97$ and $F > 2.59$ respectively, the values being obtained from the F - distribution with (9,96) degrees of freedom.

Performing the usual analysis of variance calculations gives F = 1.78 which is not significant at the 5 % level.

Hypotheses (IV).

The relevant mean aligned data are as follows.

Cube 1

					Ranks			
	-0.031	-0.031	0.094	-0.031	(1-3)	(1-3)	4	(1-3)
Plane 1	-0.156	-0.031	-0.031	0.219	1	(2-3)	(2-3)	4
	0.094	-0.156	-0.031	0.094	(3-4)	1	2	(3-4)
	0.094	0.219	-0.031	-0.281	3	4	2	1
	0.031	0.031	-0.094	0.031	(2-4)	(2-4)	1	(2-4)
Plane 2	0.156	0.031	0.031	-0.219	4	(2-3)	(2-3)	1
	-0.094	0.156	0.031	0.094	(1-2)	4	3	(1-2)
	-0.094	-0.219	0.031	0.281	2	1	3	4

Cube 2

	0.109	0.109	-0.141	-0.078	(3-4)	(3-4)	1	2
Plane 1	0.109	-0.016	-0.141	0.047	4	2	1	3
	-0.141	-0.141	0.109	0.172	(1-2)	(1-2)	3	4
	-0.078	0.047	0.172	-0.141	2	3	4	1
	-0.109	-0.109	0.141	0.078	(1-2)	(1-2)	4	3
Plane 2	-0.109	0.016	0.141	-0.047	1	3	4	2
	0.141	0.141	-0.109	-0.172	(3-4)	(3-4)	2	1
	0.078	-0.047	-0.172	0.141	3	2	1	4

Cube 3

	-0.094	-0.156	0.219	0.031	2	1	4	3
Plane 1	-0.156	0.156	-0.094	0.094	1	4	2	3
	0.094	0.031	-0.094	-.031	4	3	1	2
	0.156	-0.031	-0.031	-0.094	4	(2-3)	(2-3)	1
	0.094	0.156	-0.219	-0.031	3	4	1	2
Plane 2	0.156	-0.156	0.094	-0.094	4	1	3	2
	-0.094	-0.031	0.094	0.031	1	2	4	3
	-0.156	0.031	0.031	0.094	1	(2-3)	(2-3)	4

In order to economise only the range method has been used for ties.

Tests (i) - the match tests.

For the V1 test the null hypothesis is rejected at the 5 % and 1 % levels of significance if V1 $\leqslant$ 6.50 and V1 $\leqslant$ 5.75 respectively, while for the V2 test rejection occurs at the same levels if V2 $\leqslant$ 13.625 and V2 $\leqslant$ 12.875 respectively. These values are obtained from the tables in sections 6 and 7.

Performing the various comparisons of ranks between the cubes produces V1 = 5.58 and V2 = 10.563 both of which are significant at the 1 % level.

Test (ii) - the classical F-test.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if F > 1.68 and F > 2.07 respectively, the values being obtained from the F - distribution with (18,96) degrees of freedom.

Performing the usual analysis of variance calculations gives F = 2.46 which is significant at the 1 % level of significance.

6. A Note on the Distributions of V1 and V2.

Because of the large number of combinations of treatments, blocks, vertical layers and cubes we present only a selection of null distributions of V1 and V2. Furthermore, the length of these distributions has forced us to only present values whose cumulative probability is no greater than 0.3 .

The distributions of V1 and V2 were obtained by convolution using the distributions of C1 and C2 respectively.

7. Lower Tail Probabilities for the Null Distribution of V1.

Below we give the probabilities $P(V1 \leq x)$ for $c = 3$, $b = 3$, vertical layers $v = 2$, number of cubes $n = 2$ to 4, $v = 3$, $n = 3$, 4 and $v = 4$, $n = 4$; $c = 4$, $b = 4$, $v = 2$ and $n = 2$ to 4.

c = 3 b = 3

v = 2 n = 2

x	P(V1 ≤ x)
0	.000000
0.67	.000352
1	.000467
1.33	.005096
1.67	.008354
2	.036646
2.33	.069054
2.67	.142356
3	.268404

v = 2 n = 3

x	P(V1 ≤ x)
0	.000000
0.67	.000002
1	.000002
1.33	.000038
1.67	.000062
2	.000495
2.33	.000960
2.67	.004007
3	.008524
3.33	.021308
3.67	.044960
4	.078305
4.33	.141562
4.67	.207573
5	.287025

v = 2 n = 4

x	P(V1 ≤ x)
0	.000000
0.67	.000000
1	.000000
1.33	.000000
1.67	.000000

x	P(V1 ⩽ x)
2	.000004
2.33	.000008
2.67	.000051
3	.000112
3.33	.000444
3.67	.001034
4	.002804
4.33	.006396
4.67	.013203
5	.026909
5.33	.046796
5.67	.079269
6	.124176
6.33	.176377
6.67	.248051

c = 3 b = 3

v = 3 n = 3

x	P(V1 ⩽ x)
0	.000000
.67	.000000
1	.000000
1.33	.000000
1.67	.000000
2	.000000

x	P(V1 ⩽ x)
2.33	.000001
2.67	.000005
3	.000011
3.33	.000051
3.67	.000123
4	.000398
4.33	.000959
4.67	.002336
5	.005271
5.33	.010526
5.67	.020761
6	.036384
6.33	.060546
6.67	.095836
7	.138675
7.33	.195646
7.67	.262112

v = 3 n = 4

x	P(V1 ⩽ x)
0	.000000
.67	.000000
1	.000000
1.33	.000000
1.67	.000000

x	P(V1 ⩽ x)
2	.000000
2.33	.000000
2.67	.000000
3	.000000
3.33	.000000
3.67	.000000
4	.000001
4.33	.000002
4.67	.000006
5	.000015
5.33	.000043
5.67	.000120
6	.000267
6.33	.000622
6.67	.001330
7	.002764
7.33	.005331
7.67	.009795
8	.017153
8.33	.028228
8.67	.044716
9	.067359
9.33	.096918
9.67	.135073
10	.180199
10.33	.232634

c = 3 b = 3

v = 4 n = 4

x	P(V1 ≤ x)
0	.000000
.67	.000000
1	.000000
1.33	.000000
1.67	.000000
2	.000000
2.33	.000000
2.67	.000000
3	.000000
3.33	.000000
3.67	.000000
4	.000000
4.33	.000000
4.67	.000000
5	.000000
5.33	.000000
5.67	.000000
6	.000000
6.33	.000000
6.67	.000001
7	.000002
7.33	.000005
7.67	.000011
8	.000027
8.33	.000063
8.67	.000138
9	.000292
9.33	.000589
9.67	.001138
10	.002115
10.33	.003763
10.67	.006448
11	.010632
11.33	.016871
11.67	.025886
12	.038341
12.33	.054985
12.67	.076557
13	.103390
13.33	.135967
13.67	.174335
14	.217947

c = 4 b = 4

v = 2 n = 2

x	P(V1 ≤ x)
0	.000000
0.5	.000000
0.75	.000000
1	.000000
1.25	.000000
1.5	.000001
1.75	.000004
2	.000020
2.25	.000074
2.5	.000265
2.75	.000804
3	.002262
3.25	.005554
3.5	.012434
3.75	.024868
4	.045740
4.25	.076945
4.5	.120836
4.75	.177036
5	.245477

v = 2 n = 3

x	P(V1 ≤ x)
0	.000000
0.5	.000000
0.75	.000000
1	.000000
1.25	.000000
1.5	.000000
1.75	.000000
2	.000000
2.25	.000000

x	P(V1 ≤ x)
2.5	.000000
2.75	.000000
3	.000000
3.25	.000001
3.5	.000002
3.75	.000008
4	.000025
4.25	.000071
4.5	.000191
4.75	.000474
5	.001096
5.25	.002359
5.5	.004752
5.75	.008977
6	.015984
6.25	.026899
6.5	.042994
6.75	.065454
7	.095310
7.25	.133127
7.5	.179021
7.75	.232401
8	.292196

v = 2 n = 4

x	P(V1 ≤ x)
0	.000000
0.5	.000000
0.75	.000000
1	.000000
1.25	.000000
1.5	.000000
1.75	.000000
2	.000000
2.25	.000000
2.5	.000000
2.75	.000000
3	.000000
3.25	.000000
3.5	.000000
3.75	.000000
4	.000000
4.25	.000000
4.5	.000000
4.75	.000000
5	.000000
5.25	.000001
5.5	.000002
5.75	.000007
6	.000017
6.25	.000043

x	P(V1 ≤ x)
6.5	.000100
6.75	.000223
7	.000473
7.25	.000951
7.5	.001825
7.75	.003342
8	.005854
8.25	.009831
8.5	.015859
8.75	.024627
9	.036891
9.25	.053409
9.5	.074879
9.75	.101855
10	.134638
10.25	.173297
10.5	.217550
10.75	.266803

8. <u>Lower Tail Probabilities for the Null Distribution of V2.</u>

Below we give the probabilities P(V2 ≤ x) for c = 3, b = 3, vertical layers v = 2, number of cubes n = 2 to 4, v = 3, n = 3, 4 and v = 4, n = 4; c = 4, b = 4, v = 2 and n = 2 to 4.

c = 3 b = 3

v = 2 n = 2

x	P(V2 ≤ x)
4	.000000
4.33	.000238
4.67	.002582
5	.015956
5.33	.062495
5.67	.164861

v = 2 n = 3

x	P(V2 ≤ x)
6	.000000
6.33	.000001
6.67	.000018
7	.000185
7.33	.001258
7.67	.006099
8	.021861
8.33	.059670
8.67	.128060
9	.225374

v = 2 n = 4

x	P(V2 ≤ x)
8	.000000
8.33	.000000
8.67	.000000
9	.000001
9.33	.000014
9.67	.000100
10	.000546
10.33	.002341
10.67	.008005
11	.022242
11.33	.051197
11.67	.099960
12	.170164
12.33	.259613

c = 3 b = 3

v = 3 n = 3

x	P(V2 ≤ x)
9	.000000
9.33	.000000
9.67	.000000
10	.000000
10.33	.000001
10.67	.000011
11	.000069
11.33	.000354
11.67	.001455
12	.004896
12.33	.013685
12.67	.032279
13	.065455
13.33	.116566
13.67	.186359
14	.272683

v = 3 n = 4

x	P(V2 ≤ x)
12	.000000
12.33	.000000
12.67	.000000
13	.000000
13.33	.000000
13.67	.000000
14	.000000
14.33	.000001
14.67	.000004
15	.000021
15.33	.000094
15.67	.000354
16	.001152
16.33	.003258
16.67	.008083
17	.017770
17.33	.035016
17.67	.062607
18	.102783
18.33	.156640
18.67	.223716

c = 3 b = 3

v = 4 n = 4

x	P(V2 ≤ x)
16	.000000
16.33	.000000
16.67	.000000
17	.000000
17.33	.000000
17.67	.000000
18	.000000
18.33	.000000
18.67	.000000
19	.000000
19.33	.000000
19.67	.000000
20	.000001
20.33	.000004
20.67	.000015
21	.000055
21.33	.000174
21.67	.000498
22	.001284
22.33	.003014
22.67	.006471
23	.012797
23.33	.023466
23.67	.040165

x	P(V2 ≤ x)
24	.064576
24.33	.098098
24.67	.141564
25	.195014
25.33	.257597

c = 4 b = 4

v = 2 n = 2

x	P(V2 ≤ x)
6	.000000
6.25	.000000
6.375	.000000
6.5	.000000
6.625	.000000
6.75	.000000
6.875	.000000
7	.000003
7.125	.000006
7.25	.000021
7.375	.000048
7.5	.000136
7.625	.000295
7.75	.000674
7.875	.001341
8	.002614
8.125	.004664

x	$P(V2 \leq x)$
8.25	.008019
8.375	.012891
8.5	.019991
8.625	.029496
8.75	.042166
8.875	.058048
9	.077864
9.125	.101351
9.25	.129085
9.375	.160494
9.5	.195890
9.625	.234375
9.75	.276088

v = 2 n = 3

x	$P(V2 \leq x)$
9	.000000
9.25	.000000
9.375	.000000
9.5	.000000
9.625	.000000
9.75	.000000
9.875	.000000
10	.000000
10.125	.000000
10.25	.000000
10.375	.000000
10.5	.000000
10.625	.000000
10.75	.000000
10.875	.000000
11	.000000
11.125	.000001
11.25	.000003
11.375	.000007
11.5	.000015
11.625	.000032
11.75	.000066
11.875	.000128
12	.000241
12.125	.000433
12.25	.000752
12.375	.001256
12.5	.002033
12.625	.003183
12.75	.004843
12.875	.007160
13	.010322
13.125	.014519
13.25	.019980
13.375	.026914
13.5	.035562
13.625	.046124
13.75	.058819
13.875	.073794
14	.091205
14.125	.111114
14.25	.133582
14.375	.158559
14.5	.185995
14.625	.215721
14.75	.247575

v = 2 n = 4

x	$P(V2 \leq x)$
12	.000000
12.25	.000000
12.375	.000000
12.5	.000000
12.625	.000000
12.75	.000000
12.825	.000000
13	.000000
13.125	.000000
13.25	.000000
13.375	.000000
13.5	.000000
13.625	.000000
13.75	.000000
13.875	.000000
14	.000000

x	P(V2 ≤ x)	x	P(V2 ≤ x)
14.125	.000000	17.375	.003753
14.25	.000000	17.5	.005241
14.375	.000000	17.625	.007193
14.5	.000000	17.75	.009716
14.625	.000000	17.875	.012923
14.75	.000000	18	.016941
14.875	.000000	18.125	.021899
15	.000000	18.25	.027936
15.125	.000000	18.375	.035188
15.25	.000000	18.5	.043792
15.375	.000001	18.625	.053872
15.5	.000002	18.75	.065548
15.625	.000003	18.875	.078916
15.75	.000007	19	.094059
15.875	.000013	19.125	.111031
16	.000023	19.25	.129862
16.125	.000042	19.375	.150551
16.25	.000074	19.5	.173068
16.375	.000126	19.625	.197349
16.5	.000208	19.75	.223301
16.625	.000336	19.875	.250800
16.75	.000531		
16.875	.000818		
17	.001234		
17.125	.001823		
17.25	.002641		

9. Conclusion.

As we remarked in the introduction, any test for third-order interaction is likely to involve much computation. The match tests are no exception to this statement. However, in their favour we observe that they involve only light arithmetic unlike, for example, the classical F-test. Indeed, once the data have been split into "cubes" and the mean aligned observations obtained there remains only the simple tasks of ranking and matching.

The examples in section 4 have illustrated the procedure for experiments of size 3 x 3 x 3 x 3. Clearly the analysis of an r x c x p x q experiment would be performed in a similar manner; the division of the data into cubes being decided by the availability of suitable tables.

The final example illustrated the use of the match tests to analyse not only interactions of different orders in a four factor experiment but also the main effects. In fact this example served as a summary of our match tests.

CHAPTER 8

LATIN SQUARE DESIGNS

1. Introduction.

A most interesting application of our ideas is to the analysis of Latin square designs. There appears to be no nonparametric procedure specifically catering for these designs, though, as we see, undoubtedly it is possible to modify an existing procedure to cope with the analysis. This is rather surprising since Latin square designs are popular in view of their ability to analyse three factors in the same experiment but using relatively few observations.

The applicability of the matching principle to Latin squares does mean that not only is there available a nonparametric test but also one that is "quick - and - easy". Should a more powerful nonparametric test be required then our procedure for Latin squares is equally applicable to Friedman's test.

A typical 4 x 4 Latin square design is illustrated below.

		Factor A			
		1	2	3	4
Factor B	1	C_1	C_2	C_3	C_4
	2	C_4	C_1	C_2	C_3
	3	C_3	C_4	C_1	C_2
	4	C_2	C_3	C_4	C_1

Two of the factors (A and B) are represented by the columns and rows of the square arrangement; each column or row corresponds to one level of the appropriate factor. The levels of the third factor C are indicated by the suffices of C within the square.

With an n x n design there are n^2 different factor level combinations as compared to n^3 possible arrangements. This substantial saving in the experimental effort is paid for by the assumption of no interaction between the factors. Nevertheless, we shall see that some information concerning interactions may be forthcoming.

2. The Test Procedure.

Our model for the Latin square design is

$$X_{ijk} = M + A_i + B_j + C_k + z_{ijk} ,$$

$$i, j, k = 1, 2, \ldots , n ,$$

where M represents the overall mean,

A_i, B_j and C_k represent the i^{th}, j^{th} and k^{th} levels of factors A, B and C respectively,

and z_{ijk}'s are independent random variables having some continuous distribution.

We have three sets of hypotheses to investigate :

(I) $H_0 : A_i = 0$ for all i

$H_1 : A_i \neq 0$ for some i

(II) $H_0 : B_i = 0$ for all i

$H_1 : B_i \neq 0$ for some i

(III) $H_0 : C_i = 0$ for all i

$H_1 : C_i \neq 0$ for some i .

We extract from the Latin square design three tables, one for each of the possible pairs of factors. Then the combination of factors A and B may be employed to investigate hypotheses (I) and (II), the combination of factors A and C for hypotheses (I) and (III) and the combination of factors B and C for hypotheses (II) and (III). It is clear that each set of hypotheses may be investigated by using either of two combinations. This choice has the advantage of being able to infer from inconsistent conclusions the possible existence of interactions, hitherto assumed not to exist.

Using the matching principle the actual analysis of the hypotheses is undertaken by calculating either of the statistics M1 or M2. The null hypothesis is rejected for M1, M2 $\geq$ critical value.

3. Examples.

Our first example is taken from Johnson and Leone (1964) while the next two examples consist of data constructed to illustrate the effects of interaction.

Example 1

A particular missile alternator design is made up of three separate power generating sections, considered mutually independent. The alternator is driven by a turbine which is powered by hot gas supplied from a solid grain gas generator. The parasitic section of the alternator supplies power to a dummy electrical load as required in order to maintain alternator speed at a constant value of 24,000 rpm. The parasitic section is comprised of a 4-pole stator, 6-pole rotor and a shaft. The rotor turns concentrically within the stator bore while the stator is held fixed within the housing. The stator is wound with both DC and AC turns of fixed wire size. The AC output voltage is a function of DC input current and AC turns. The rotor is stacked from individual laminations punched from 0.004in thick stock. The laminations are coated for insulation purposes.

The purpose of the experiment was to determine which factors were most closely associated with performance and what levels of these factors gave the best performances. A 5 x 5 Latin square experiment was designed with the factors and levels as follows.,

a. The number of AC turns for the stators. The levels were at 145, 150, 155, 160 and 165 AC turns.

b. The number of laminations per stack for the rotors. The levels were 230, 240, 250, 260 and 270.

c. The quality (visual) of lamination coatings. The five levels were on an arbitrary scale with A the best and E the worst.

A conventional alternator was built for test purposes. The unit was assembled and disassembled as necessary to test components and follow the Latin square design. A random testing order was established. The background of the test conditions was controlled as rigidly as possible. The feature observed was the maximum parasitic AC output voltage. The data are given in the table below.

Output Voltage of Missile Alternators

	Stators				
Rotors	145	150	155	160	165
230	310C	312B	320A	306D	300E
240	309D	310C	324B	300E	305A
250	312B	303E	325C	307A	302D
260	316A	306D	318E	304C	294B
270	314E	308A	323D	309B	303C

We have three sets of hypotheses to investigate, namely

(I) H_0 : there is no difference between the stators.

H_1 : there is some difference between the stators.

(II) H_0 : there is no difference between the rotors.

H_1 : there is some difference between the rotors.

(III) H_0 : performance is not affected by the coating quality.

H_1 : performance is affected by the coating quality.

Tests (i) - the match tests.

The critical values for M1 and M2 are obtained from the approximate distributions given in Chapter 3.

For the M1 test, the null hypothesis is rejected at the 5 % and 1 % levels of significance if M1 $\geqslant$ 16 and M1 $\geqslant$ 19 respectively, while for the M2 test rejection at the same levels occurs if M2 $\geqslant$ 23 and M2 $\geqslant$ 25 respectively.

Before ranking the observations we construct three tables, one for each of the combinations rotors x stators, rotors x quality and quality x stators. These tables are given below.

Table 1

	Stators				
Rotors	145	150	155	160	165
230	310	312	320	306	300
240	309	310	324	300	305
250	312	303	325	307	302
260	316	306	318	304	294
270	314	308	323	309	303

Table 2

	Quality				
Rotors	A	B	C	D	E
230	320	312	310	306	300
240	305	324	310	309	300
250	307	312	325	302	303
260	316	294	304	306	318
270	308	309	303	323	314

Table 3

Quality	Stators 145	150	155	160	165
A	316	308	320	307	305
B	312	312	324	309	294
C	310	310	325	304	303
D	309	306	323	306	302
E	314	303	318	300	300

Hypotheses (I).

We may use either table 1 or table 3 to test these hypotheses. Using table 1 we obtain the following table of ranks.

	3	4	5	2	1
	3	4	5	1	2
	4	2	5	3	1
	4	3	5	2	1
	4	2	5	3	1
Rank sums	18	15	25	11	6

From this table we obtain the values M1 = 24 and M2 = 34.

On the other hand, using table 3 we obtain the following table.

	4	3	5	2	1
	(3-4)	(3-4)	5	2	1
	(3-4)	(3-4)	5	2	1
	4	(2-3)	5	(2-3)	1
	4	3	5	(1-2)	(1-2)
Rank sums	19	15.5	25	10	5.5

Using the average rank and the range methods for ties gives M1 = 36 and M1 = 36.75 respectively, while the two average rank methods and the range method give M2 = 40.25, M2 = 36.5 and M2 = 42.25 respectively.

Clearly, in each case both tests produce evidence strongly supporting the alternative hypothesis.

<u>Hypotheses (II)</u>.

We may use either table 1 or table 2 in order to investigate these hypotheses. Using table 1 produces the following table of ranks.

					Rank sums
2	5	2	3	2	14
1	4	4	1	5	15
3	1	5	4	3	16
5	2	1	2	1	11
4	3	3	5	4	19

From this table we obtain M1 = 10 and M2 = 19.

On the other hand, table 2 produces the following rank table.

					Rank sums
5	(3-4)	(3-4)	(2-3)	(1-2)	16
1	5	(3-4)	4	(1-2)	15
2	(3-4)	5	1	3	14.5
4	1	2	(2-3)	5	14.5
3	2	1	5	4	15

Using the average rank and the range methods for ties gives $M1 = 3$ and $M1 = 3\frac{1}{4}$ respectively, while the two average rank methods and the range method give $M2 = 11\frac{1}{4}$, $M2 = 12\frac{1}{2}$ and $M2 = 12.125$ respectively.

Clearly, in each case both tests produce no evidence to support the alternative hypothesis.

Hypotheses (III).

We may use either table 2 or table 3 to investigate these hypotheses. Using table 2 produces the following table of ranks.

	5	4	3	2	1
	2	5	4	3	1
	3	4	5	1	2
	4	1	2	3	5
	2	3	1	5	4
Rank sums	16	17	15	14	13

From this table we obtain $M1 = 4$ and $M2 = 13$ ..

On the other hand, table 3 produces the following table of ranks.

					Rank sums
5	3	2	4	5	19
3	5	4	5	1	18
2	4	5	2	4	17
1	2	3	3	3	12
4	1	1	1	2	9

From this table we obtain M1 = 10 and M2 = $18\frac{1}{2}$.

Clearly, in each case both provide no evidence to support the alternative hypothesis.

Test (ii) - Friedman's test.

The critical values are obtained from the exact null distribution. The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $\chi_r^2 \geqslant 8.96$ and $\chi_r^2 \geqslant 11.68$ respectively.

Hypotheses (I).

Table 1 gives the value $\chi_r^2 = 16.48$ while table 3 gives $\chi_r^2 = 18.52$.

Both cases produce results strongly supporting the alternative hypothesis.

Hypotheses (II).

Table 1 gives the value $\chi_r^2 = 2.72$ while table 2 gives $\chi_r^2 = 0.12$.

Both cases produce results that do not support the alternative hypothesis.

Hypotheses (III).

Table 2 gives the value $\chi_r^2 = 0.80$ while table 3 gives $\chi_r^2 = 5.92$.

Both cases produce results that do not support the alternative hypothesis.

<u>Test (iii) - the classical F-test.</u>

The null hypothesis will be rejected at the 5 % and 1 % levels of significance if $F > 3.26$ and $F > 5.41$ respectively, the values being obtained from the F - distribution with (4,12) degrees of freedom.

Performing the usual analysis of variance calculations produces :

<u>Hypotheses (I)</u>. $F = 27.07$ which strongly supports the validity of the alternative hypothesis.

<u>Hypotheses (II)</u>. $F = 0.76$ which clearly provides no evidence to support the alternative hypothesis.

<u>Hypotheses (III)</u>. $F = 1.09$ which provides no evidence to support the alternative hypothesis.

It is reassuring that the nonparametric tests produce conclusions consistent with the classical F-test.

<u>Example 2</u>.

The model from which the data are derived is

$$X_{ijk} = M + A_i + B_j + C_k + (AB)_{ij} + z_{ijk} ,$$

where, apart from the interaction term $(AB)_{ij}$, the model is the same as that in section 2. The factors A and B were contrived to have some effect, C being the only main effect not contributing to the observations and the only factor not affected by the interaction.

The data are given below.

	Factor A			
Factor B	1	2	3	4
1	1.79C_1	1.30C_2	2.45C_3	2.55C_4
2	1.04C_2	2.71C_1	1.58C_4	3.68C_3
3	1.67C_3	2.99C_4	2.88C_1	3.78C_2
4	2.91C_4	3.64C_3	3.36C_2	4.36C_1

The hypotheses under investigation are :

(I) $H_0 : A_i = 0$ for all i

$H_1 : A_i \neq 0$ for some i

(II) $H_0 : B_j = 0$ for all j

$H_1 : B_j \neq 0$ for some j

(III) $H_0 : C_k = 0$ for all k

$H_1 : C_k \neq 0$ for some k.

Tests (i) - the match tests.

The critical values are obtained from the exact null distributions given in Chapter 3 and are the best conservative values.

For the M1 test, the null hypothesis is rejected at the 5 % and 1 % levels of significance if M1 $\geqslant$ 12 and M1 $\geqslant$ 15 respectively, while for the M2 test, rejection occurs at the same levels of significance if M2 $\geqslant$ 15 and M2 $\geqslant$ 18 respectively.

Proceeding as in the previous example gives the following results.

<u>Hypotheses (I)</u>. Using the combination A with B gives M1 = 15 and M2 = 18, while the combination A with C gives M1 = 6 and M2 = 12.

<u>Hypotheses (II)</u>. Using the combination A with B gives M1 = 12 and M2 = 17, while the combination B with C gives M1 = 8 and M2 = 13.

<u>Hypotheses (III)</u>. Using the combination B with C gives M1 = 3 and M2 = $7\frac{1}{2}$, while the combination A with C gives M1 = 4 and M2 = 8.

All these results are consistent with the conditions under which the data were obtained.

<u>Test (ii) - Friedman's test</u>.

The critical values are obtained from the exact null distribution for c = 4 and b = 4, and are the best conservative values.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if $\chi_r^2 \geqslant 7.8$ and $\chi_r^2 \geqslant 9.6$ respectively.

Proceeding as in the previous example gives the following results.

Hypotheses (I). Using the combination A with B gives $\chi_r^2 = 9.3$, while the combination A with C gives $\chi_r^2 = 5.7$.

Hypotheses (II). Using the combination A with B gives $\chi_r^2 = 9.3$, while the combination B with C gives $\chi_r^2 = 3.9$.

Hypotheses (III). Using the combination B with C gives $\chi_r^2 = 0.899$, while the combination A with C gives $\chi_r^2 = 0.599$.

Test (iii) - the classical F-test.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if $F > 4.76$ and $F > 9.78$ respectively, the values being obtained from the F - distribution with (3,6) degrees of freedom.

Performing the usual analysis of variance calculations produces :

Hypotheses (I). $F = 7.74$, a result which is significant at the 5 % level but not the 1 % level of significance.

Hypotheses (II). $F = 7.21$, a result which is significant at the 5 % level but not the 1 % level of significance.

Hypotheses (III). $F = 1.12$, a result which is not significant the 5 % level.

The above results certainly seem to be consistent with the model; the nonparametric tests revealing the presence of interaction between A and B.

Example 3.

In order to see the effect of omitting the interaction term we have obtained another set of data though this time based on the ordinary Latin squares model given in section 2. This time only factor B contributes to the observations. The data are given in the following table.

Factor A

Factor B	1	2	3	4
1	$0.56C_1$	$1.07C_2$	$1.29C_3$	$0.69C_4$
2	$1.28C_2$	$2.15C_1$	$1.30C_4$	$1.39C_3$
3	$3.01C_3$	$2.70C_4$	$3.23C_1$	$3.04C_2$
4	$3.37C_4$	$2.80C_3$	$2.22C_2$	$2.86C_1$

The hypotheses under investigation are :

(I) $H_0 : A_i = 0$ for all i

$H_1 : A_i \neq 0$ for some i

(II) $H_0 : B_j = 0$ for all j

$H_1 : B_j \neq 0$ for some j

(III) $H_0 : C_k = 0$ for all k

$H_1 : C_k \neq 0$ for some k.

Tests (i) - the match tests.

The critical values are obtained from the exact null distributions given in Chapter 3 and are the best conservative values.

For the M1 test, the null hypothesis is rejected at the 5 % and 1 % levels of significance if $M1 \geqslant 12$ and $M1 \geqslant 15$ respectively, while for the M2 test rejection occurs at the same levels of significance if $M2 \geqslant 15$ and $M2 \geqslant 18$ respectively.

Proceeding as before gives the following results.

Hypotheses (I). Using the combination A with B gives $M1 = 5$ and $M2 = 8\frac{1}{2}$, while the combination A with C gives $M1 = 2$ and $M2 = 7$.

Hypotheses (II). Using the combination A with B gives $M1 = 16$ and $M2 = 20$, while the combination B with C gives $M1 = 18$ and $M2 = 21$.

Hypotheses (III). Using the combination B with C gives $M1 = 5$ and $M2 = 8$, while the combination A with C gives $M1 = 0$ and $M2 = 6$.

Test (ii) - Friedman's test.

The critical values are obtained from the exact null distribution for $c = 4$ and $b = 4$, and are the best conservative values.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if $\chi_r^2 \geqslant 7.8$ and $\chi_r^2 \geqslant 9.6$ respectively.

Proceeding as before gives the following results.

Hypotheses (I). Using the combination A with B gives $\chi_r^2 = 0.899$, while the combination A with C gives $\chi_r^2 = 0.3$.

Hypotheses (II). Using the combination A with B gives $\chi_r^2 = 10.8$, while the combination B with C gives $\chi_r^2 = 11.1$.

Hypotheses (III). Using the combination B with C gives $\chi_r^2 = 1.5$, while the combination A with C gives $\chi_r^2 = 0$.

Test (iii) - the classical F-test.

The null hypothesis is rejected at the 5 % and 1 % levels of significance if $F > 4.76$ and $F > 9.78$ respectively, the values being obtained from the F - distribution with (3,6) degrees of freedom.

Performing the usual analysis of variance calculations produces :

Hypotheses (I). $F = 0.12$, clearly a result that is not significant.

Hypotheses (II). $F = 17.58$, a highly significant result.

Hypotheses (III). $F = 0.29$, not a significant result.

Once again we have results that are consistent with the conditions of the model.

4. Comments and Results of the Simulations.

For the simulations the three treatments were taken at four levels. We took the model

$$X_{ijk} = M + A_i\theta + B_j + C_k + z_{ijk}$$

where θ varied from 0 to 1 and the rest of the parameters are as in section 2.

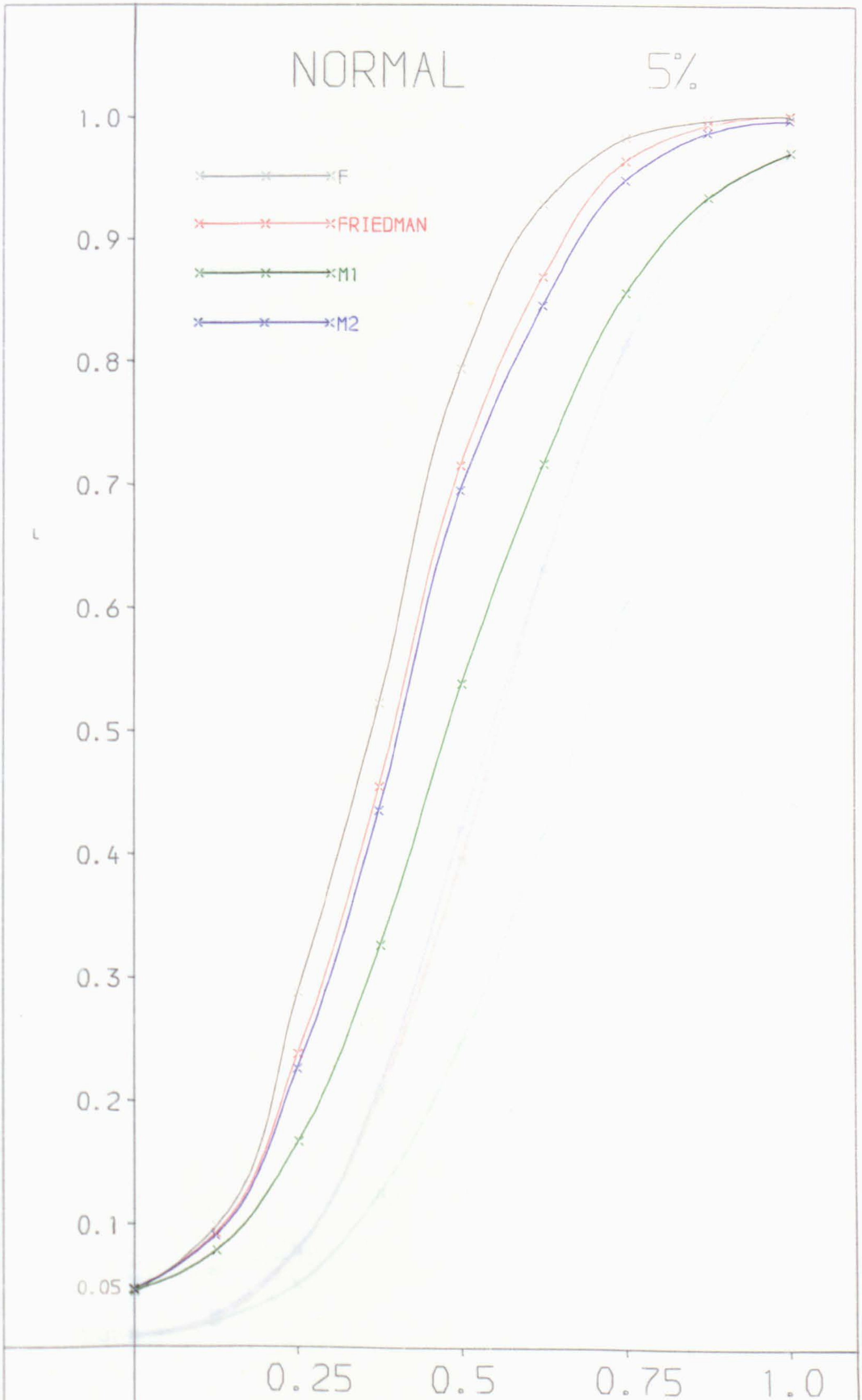
Normal Distribution. All the tests achieved good overall power with all but the M1 test reaching the maximum of 1. It is encouraging to see Friedman's and the M2 tests matching the performance of the F-test in the 1 % case.

Uniform Distribution. The overall power performance is only moderate, the F-test achieving a maximum of 0.6 in the 5 % case. Once again, Friedman's and the M2 tests match the performance of the F-test in the 1 % case.

Double Exponential Distribution. In both the 5 % and 1 % cases, Friedman's and the M2 tests are similar in performance to the F-test. The performance of the M1 test is also very creditable.

Exponential Distribution. Overall the tests achieved low power, the maximum in the 5 % case being only 0.28. Once again the nonparametric tests produced the superior results with the F-test suffering from non-robustness.

Cauchy Distribution. The nonparametric tests are certainly the superior tests with this distribution. The F-test suffers from non-robustness and low power.



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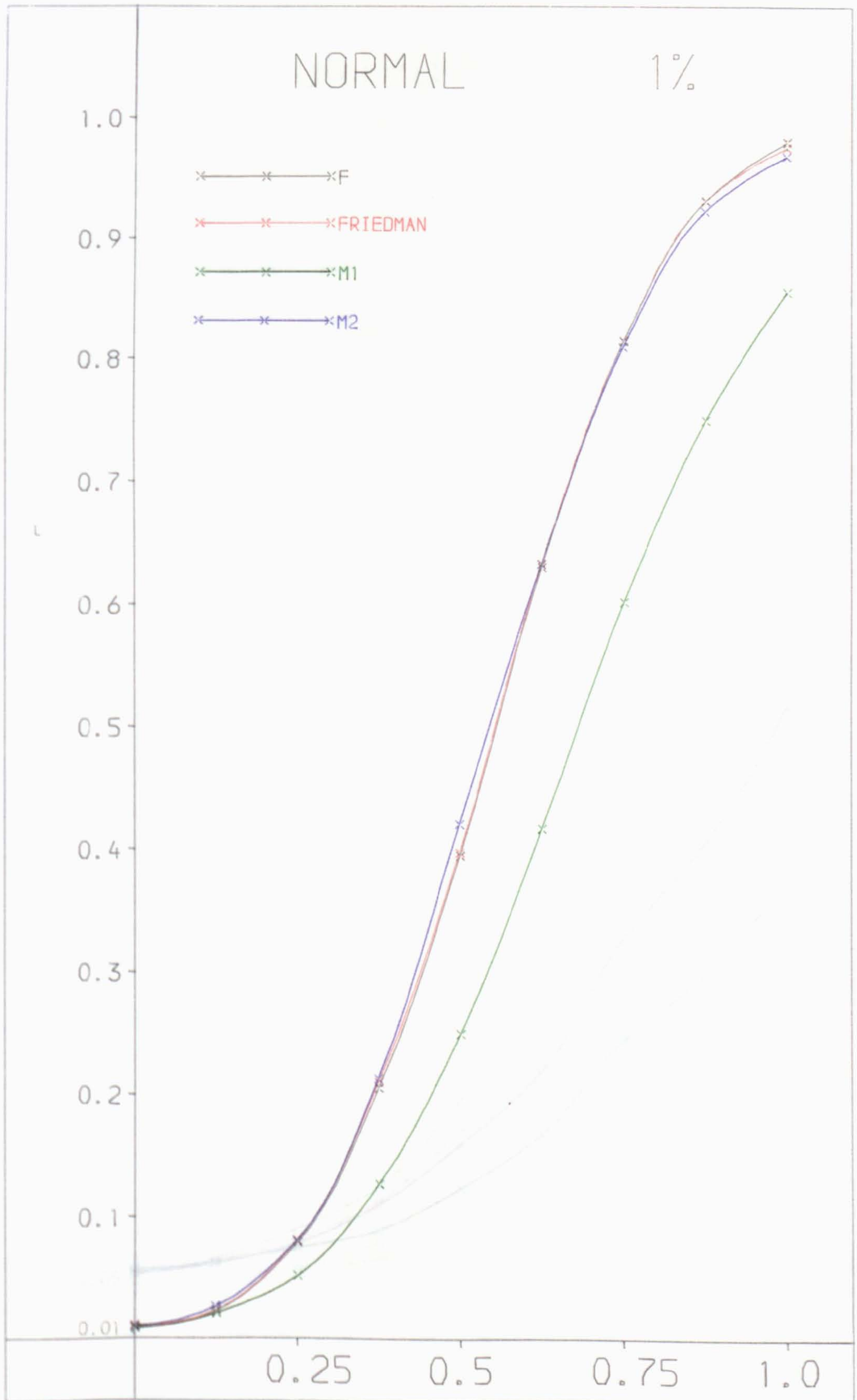

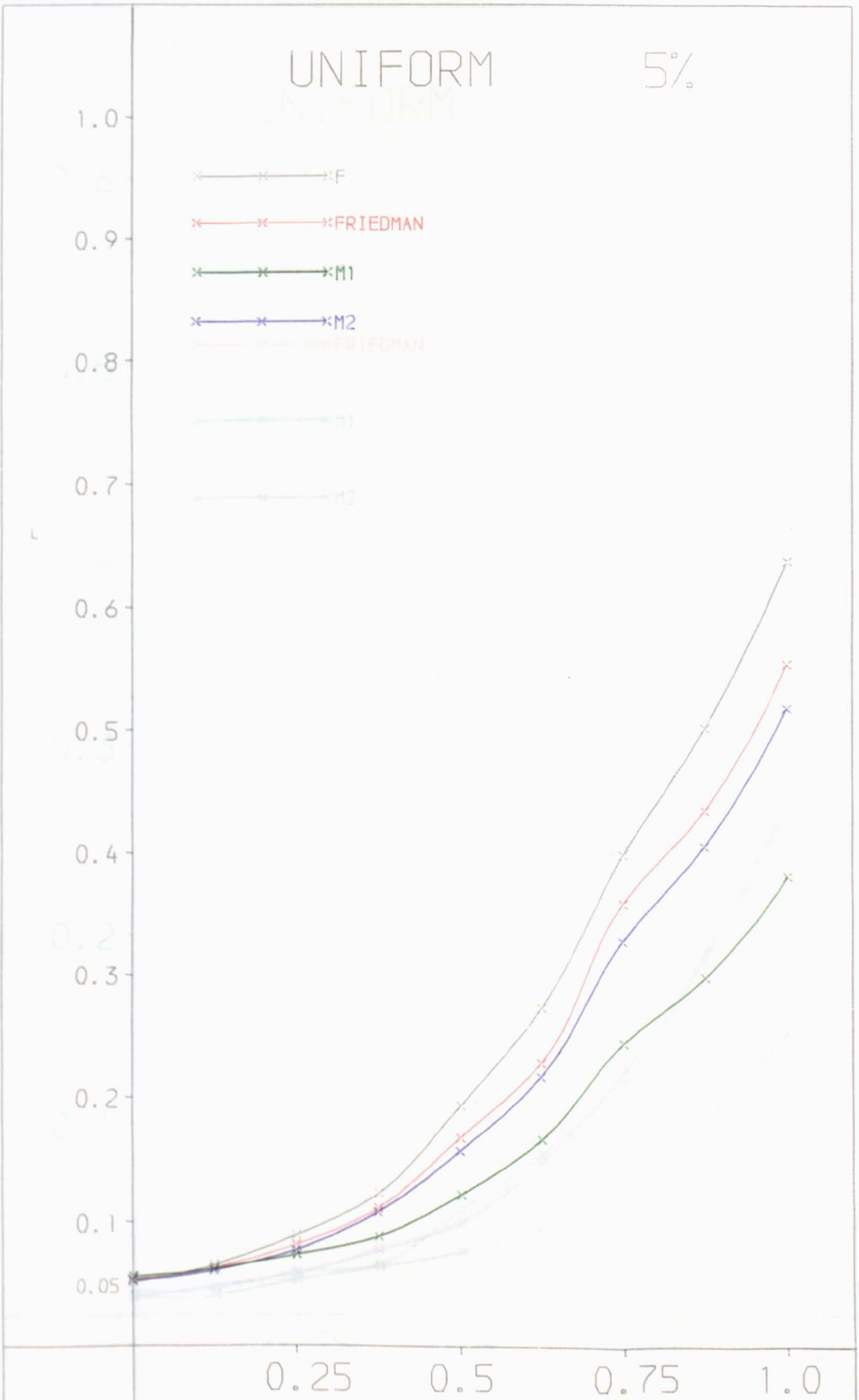
UNIFORM 5%
F
FRIEDMAN
M1
M2
1.0
0.9
0.8
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0.5
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1.0

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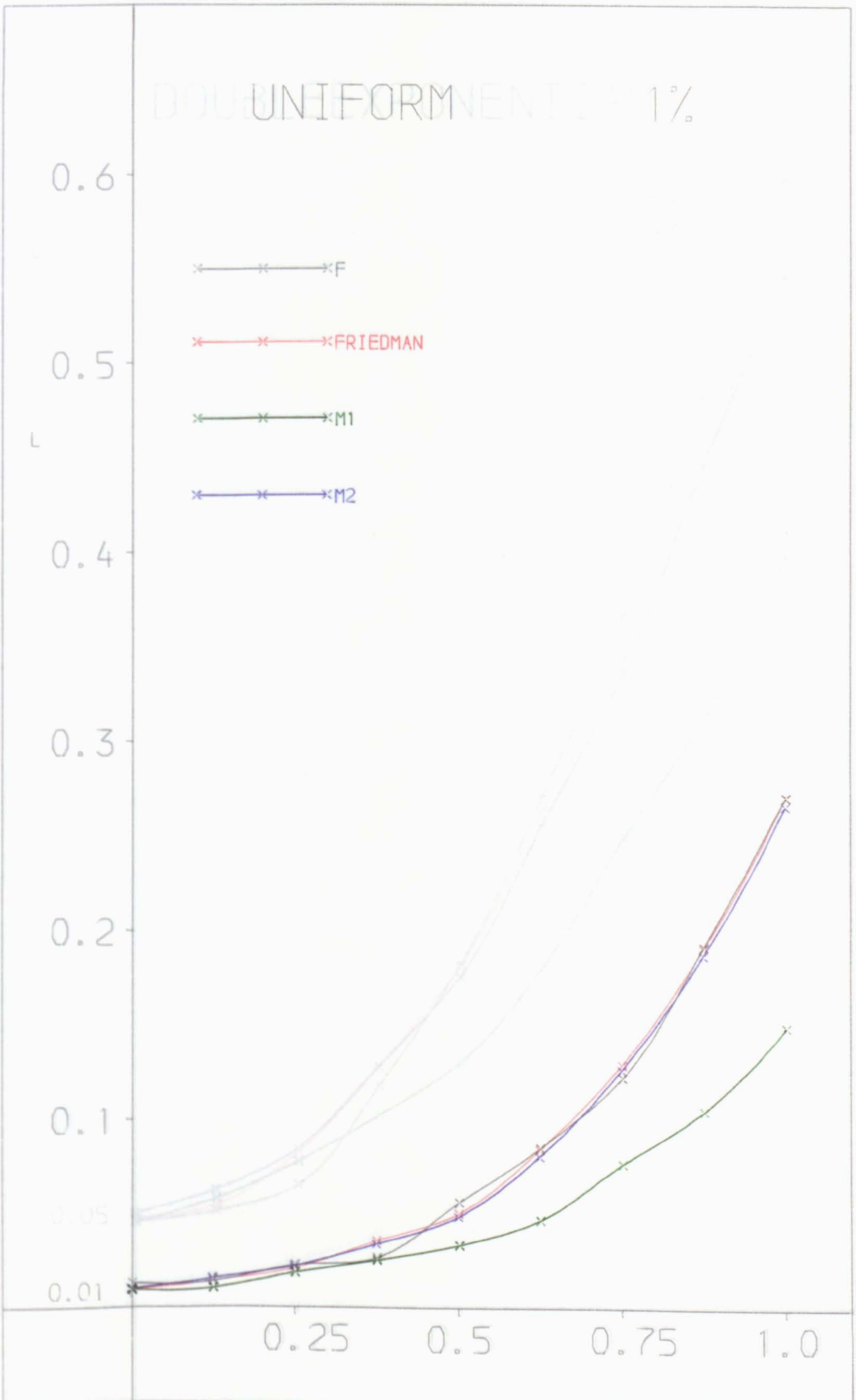

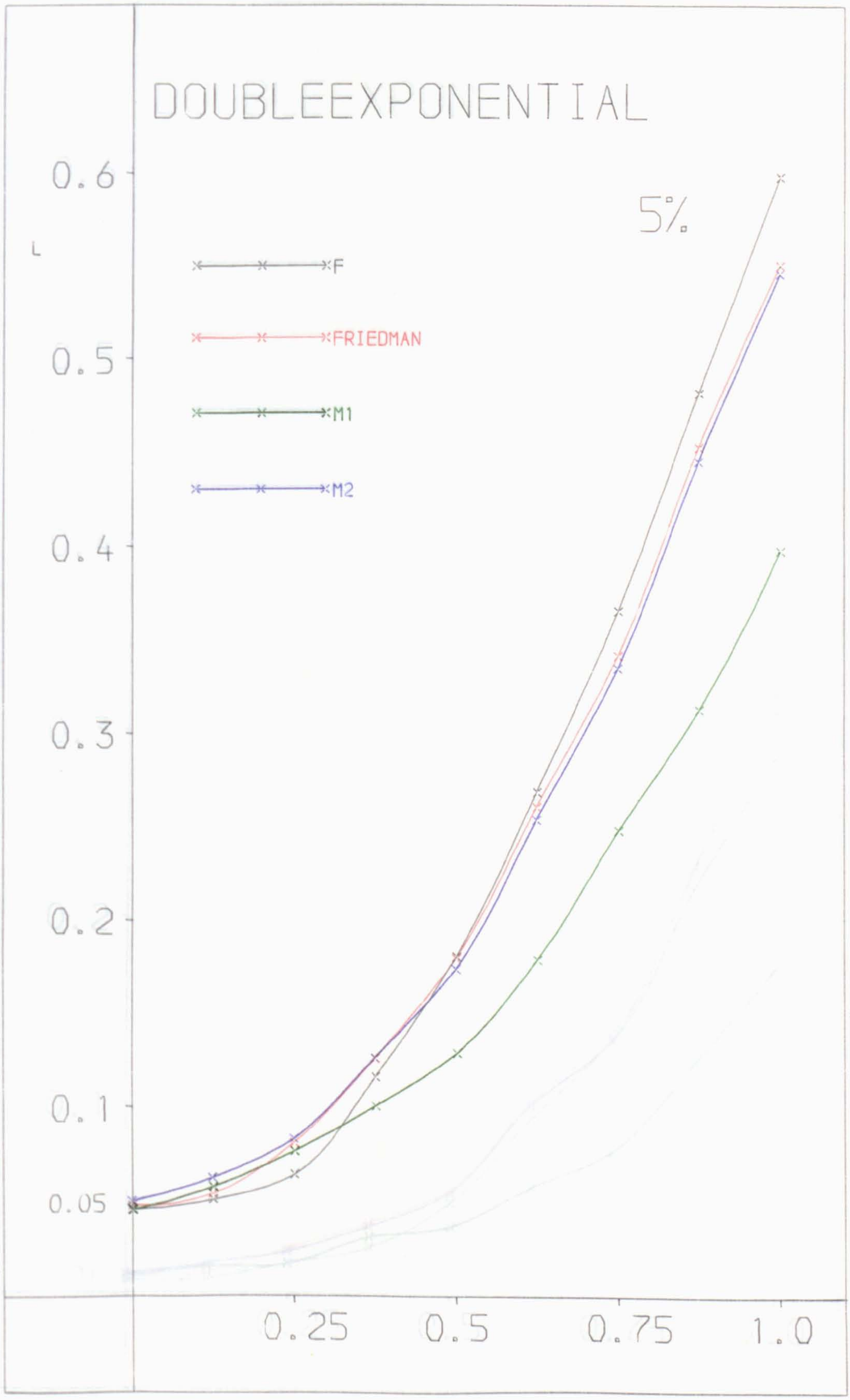
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L
F
FRIEDMAN
M1
M2
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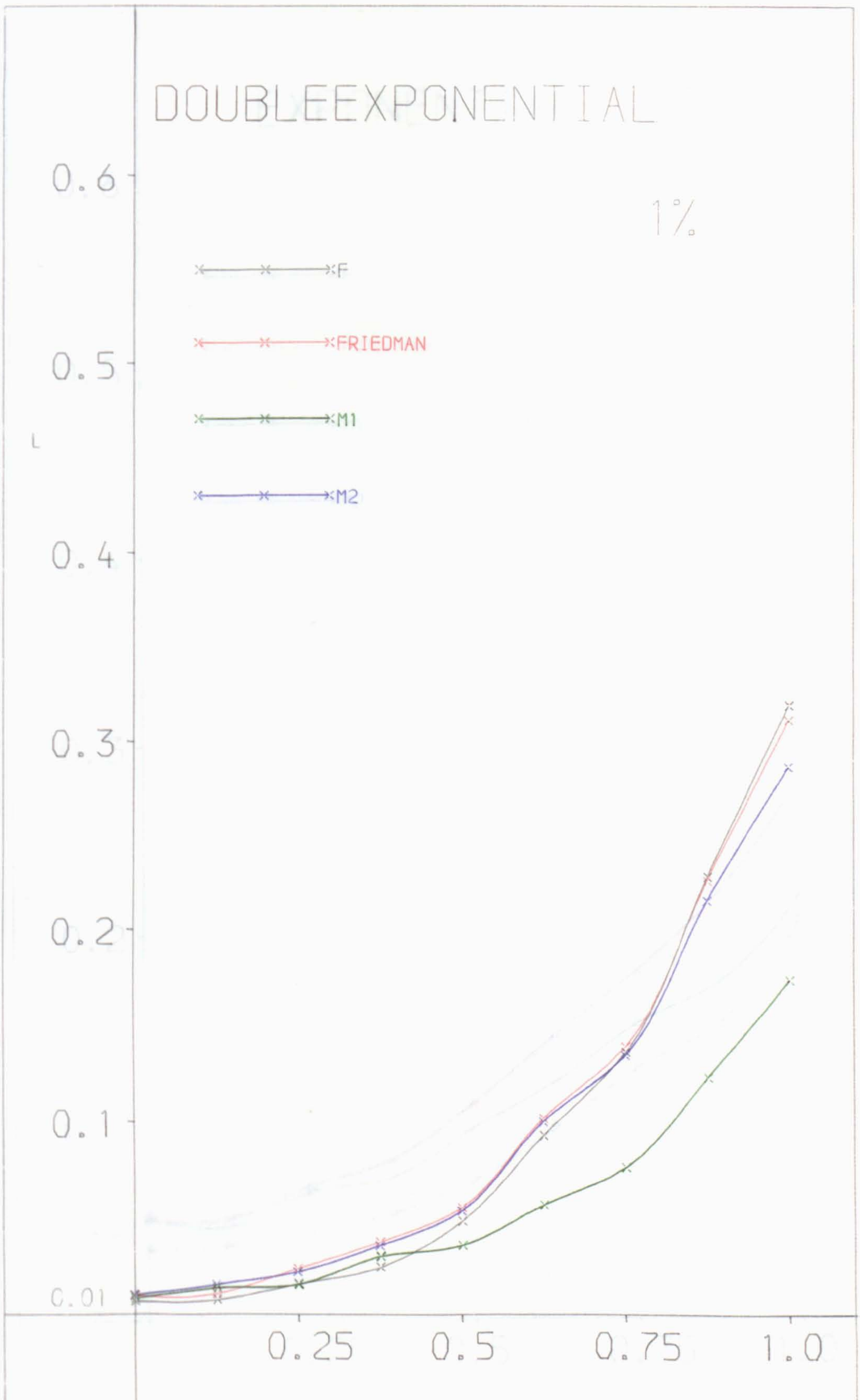

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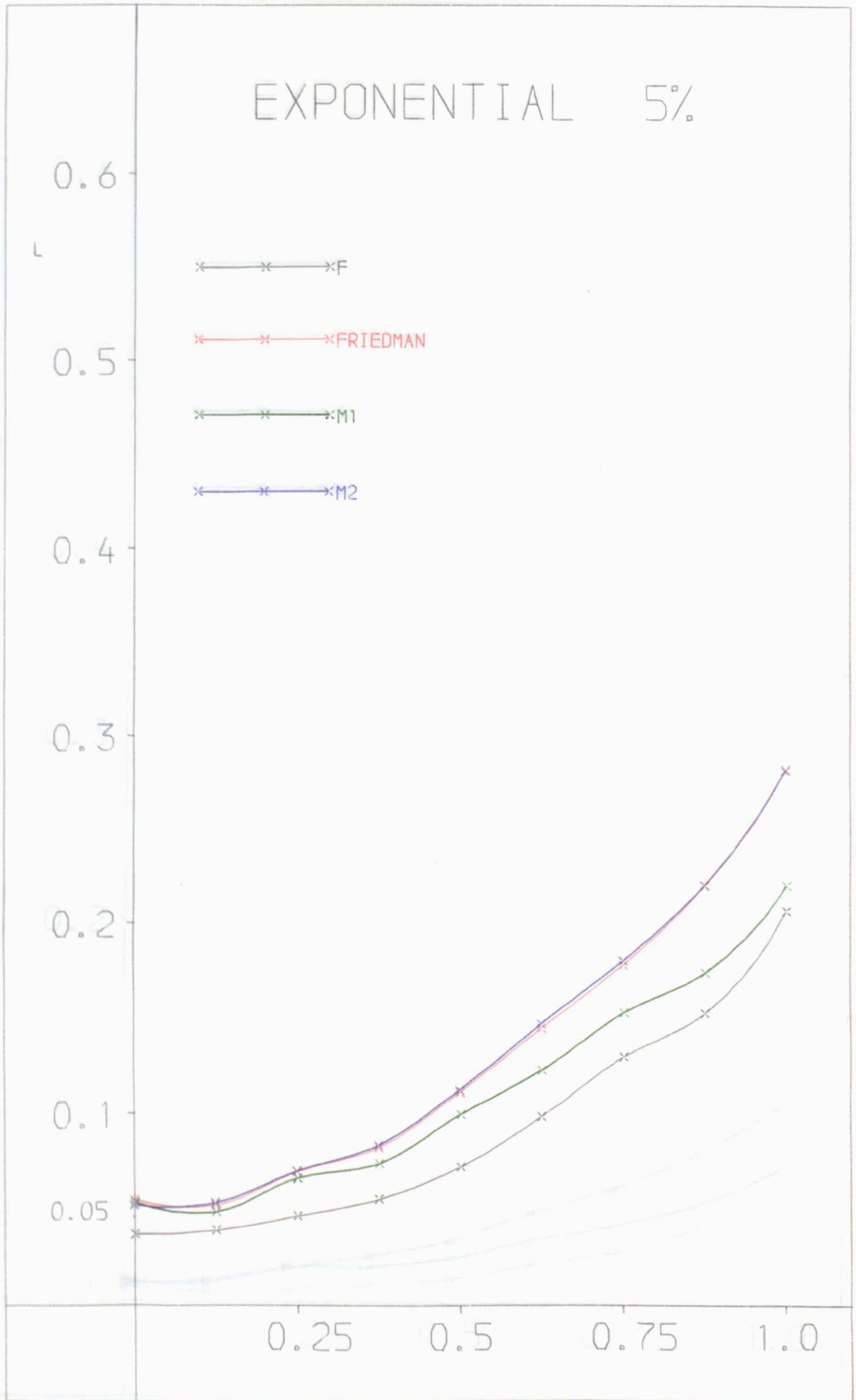
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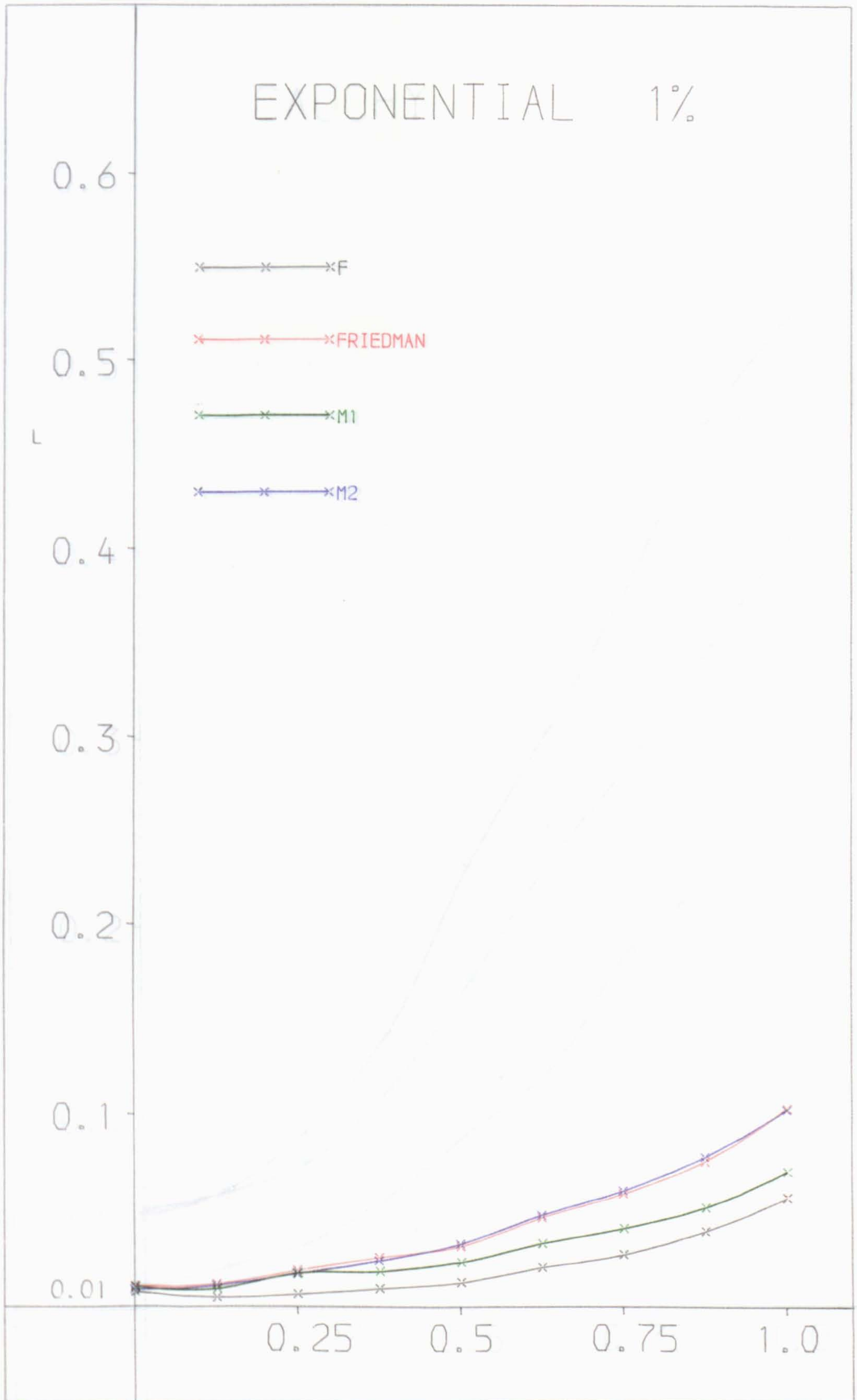
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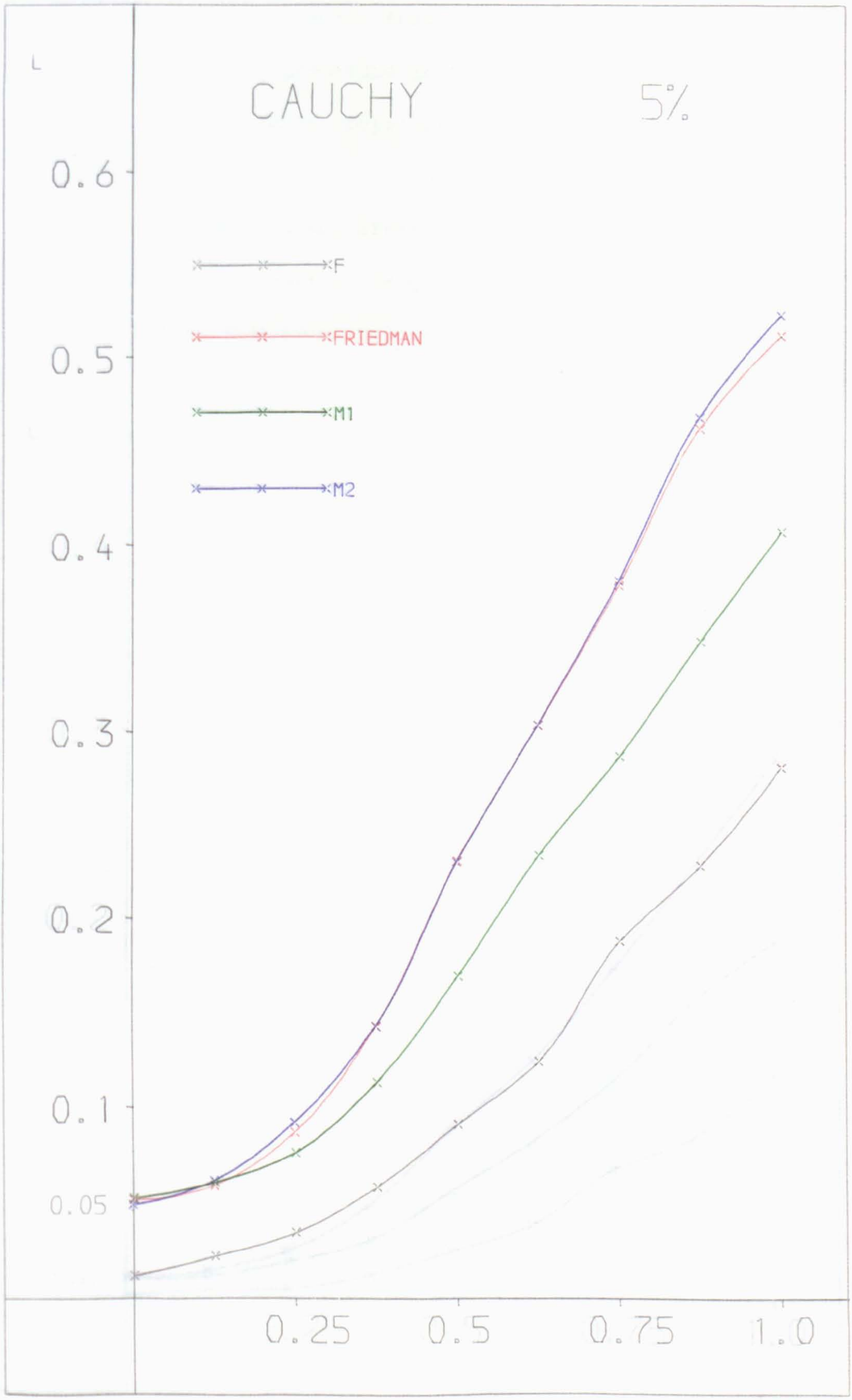

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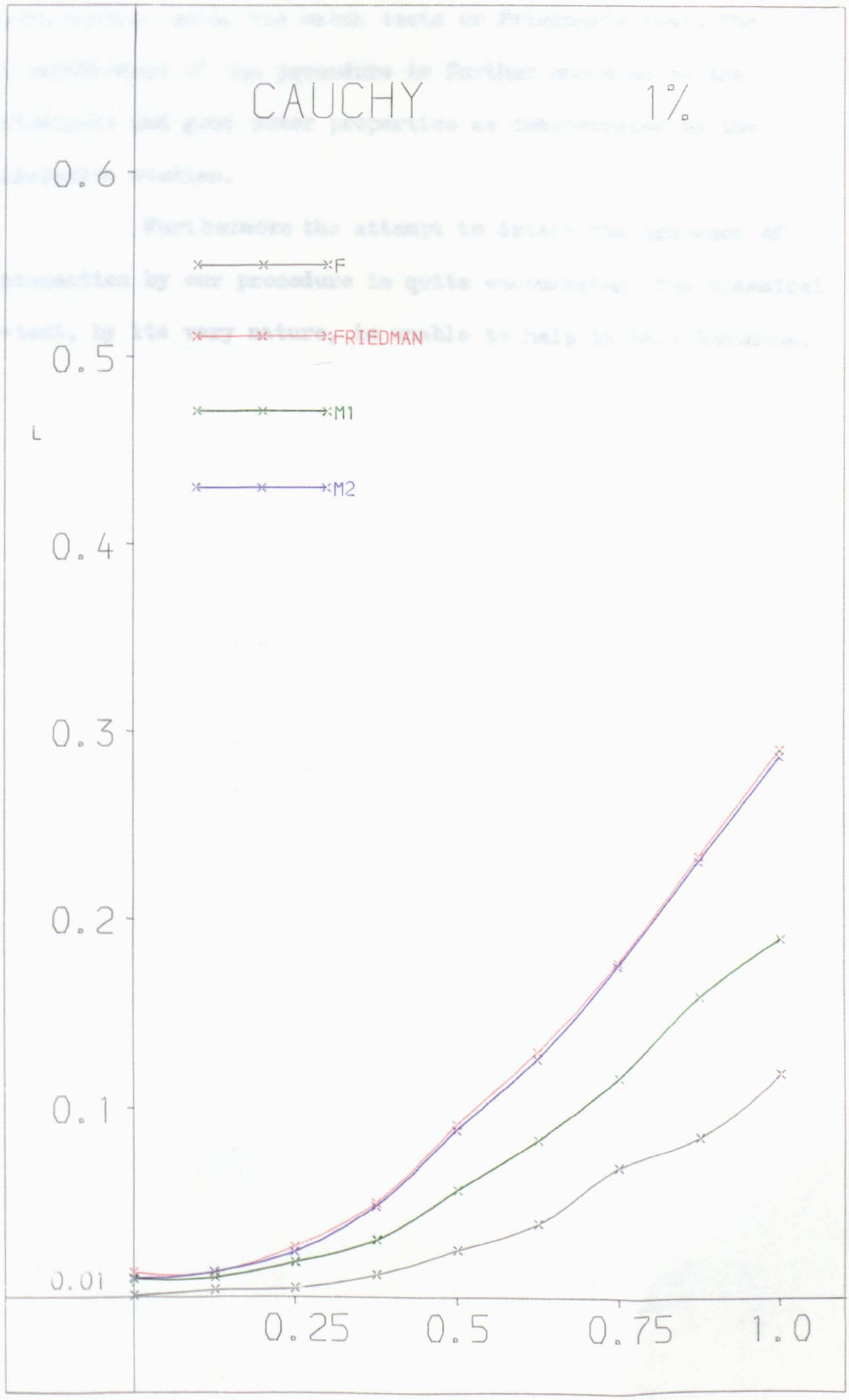


5. Conclusion.

Our procedure for Latin square designs is easy to apply whether using the match tests or Friedman's test. The attractiveness of the procedure is further enhanced by the robustness and good power properties as demonstrated by the simulation studies.

Furthermore the attempt to detect the presence of interaction by our procedure is quite encouraging. The classical F-test, by its very nature, is unable to help in this instance.

CHAPTER 9

FUTURE DEVELOPMENTS

1. <u>Introduction</u>.

Our analysis of common experimental designs has hardly been extensive. We have discussed, in varying degrees of depth, some of the more common designs. Unfortunately, the circumstances of a particular experiment may prevent it being analysed by such straightforward designs. Thus the experimenter must always be prepared to search for a more specialised or unusual design.

In this chapter we take a look at areas where further explorations might be profitable. These are discussed under the following titles.

(i) Specialised Experimental Designs.

(ii) Interaction Patterns.

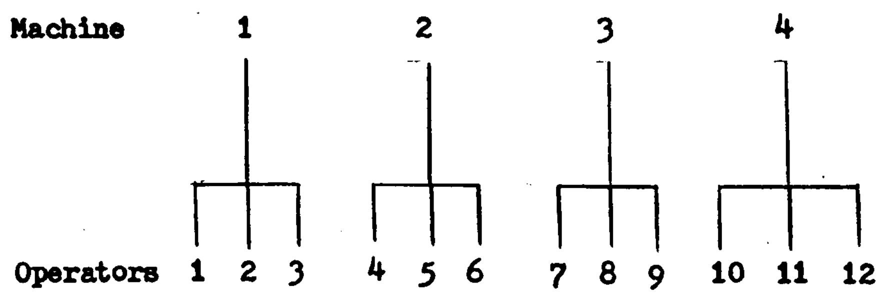
(iii) Optimum Contribution from a Near-match.

2. <u>Specialised Experimental Designs</u>.

Out of the many specialised experimental designs there are two in particular that seem suited to analysis by nonparametric methods. These are the nested (or hierarchal) and split-plot designs.

(a) <u>Nested Designs</u>. We have already discussed in Chapters 3 and 4 various aspects of cross-classified experiments in two-way layouts. A second type of relationship involving two factors is the nested design. The essential difference between them is that in the former each level of one factor is combined with all levels of the second factor. However with the nested design each level of one factor (the <u>main group</u> factor) is associated with a different set of levels of the second factor (the <u>subgroup</u> factor).

A fairly typical nested design is illustrated in the diagram below.



In this experiment, samples of the work of 3 operators on each of 4 machines (12 operators in all) are recorded. So, for example, operators 1, 2 and 3 are excluded from machines 2, 3 and 4 this would not be so in a cross-classified experiment.

To test for differences between the machines a procedure of the Kruskal-Wallis type where the combined sample is ranked seems appropriate. For the other feature of interest, namely differences between the operators, it seems that each machine must be considered separately; differences between operators on that machine being tested by a Kruskal-Wallis type procedure.

(b) <u>Split-Plot Designs</u>. Some work on applying nonparametric procedures to split-plot designs has already been carried out by Koch (1970). However, although they are essentially straightforward crossed designs, each design generally has its own peculiar characteristics that call for special ways of grouping the factor level combinations. This makes it difficult to recommend a universally applicable procedure for split-plot designs. The basic idea of a split-plot design is to confound a main effect factor thereby sacrificing its accuracy in order to gain accuracy in other, more important factors or interactions.

The following example of a split-plot experiment, taken from Johnson and Leone (1964), will serve to illustrate the possible use of the match tests in these designs.

In a study of the strength properties of polymers five different polymers were chosen. The polymers were applied to test papers which were subsequently dried. Two drying times were chosen, namely 4 minutes and 10 minutes. The specimens were then placed in steel cylindrical containers, each container having 10 small steel balls, a fixed amount of water and detergent. One specimen from each of the polymers was placed in each of 5 containers for the 4 minute group and similarly for the 10 minute group. The containers were then rotated for 60 minutes, after which time the specimens were removed and examined.

In this split-plot experiment, the 10 cylinders are the "main plots". Each cylinder is split into 5 "subplots", one for each polymer. The main features of interest are differences in the polymers and interaction between polymers and time, differences between the cylinders being of no interest. The diagrammatical representation of the experiment is shown below.

		Time 1 4 min					Time 2 10 min				
Cylinders :		C1	C2	C3	C4	C5	C6	C7	C8	C9	C10
	P1	x	x	x	x	x	x	x	x	x	x
	P2	x	x	x	x	x	x	x	x	x	x
Polymers	P3	x	x	x	x	x	x	x	x	x	x
	P4	x	x	x	x	x	x	x	x	x	x
	P5	x	x	x	x	x	x	x	x	x	x

To test for diferences between the polmers a test based on the general alternatives match tests is quite possible. For the interaction between polymers and time a test based on the ideas used in the second-order interaction tests should be possible.

3. Interaction Patterns.

In our investigation of interaction effects in two-way layouts we concentrated on situations where a general alternatives hypothesis was appropriate. However Hirostu (1978) has produced parametric tests designed to detect interaction effects in situations where an ordered alternative hypohesis is appropriate. In fact he investigated seven interaction patterns based on the relative values of μ_{ij}, the expected response under an ordered alternative hypothesis in the $(ij)^{th}$ cell.

This is certainly an interesting development to explore with the match tests L1 and L2, applying a similar idea to our interaction tests in general alternatives experiments. The possibility of detecting interaction patterns in general experiments is also worth investigating.

4. Optimum Contribution from a Near-match.

Our basic match tests M1 and L1 for general and ordered alternatives respectively were made more powerful by incorporating the idea of a near-match. Whenever a difference in ranks was 1 we contributed $\frac{1}{2}$ to the value of the test statistic; the $\frac{1}{2}$ being not only midway between 0 (no contribution) and 1 (the contribution for a match) but also convenient to apply.

It is pertinent to enquire whether the contribution of $\frac{1}{2}$ gives rise to a test with optimum power or whether some other contribution, say for example $\frac{1}{4}$, would give a more powerful test.

Suppose a near-match contributed α ($0 < \alpha \leqslant 1$) giving rise to tests $M2^{*}$ and $L2^{*}$ for general and ordered alternatives respectively. The mean and variance of these statistics can be found in terms of α by using the methods of Chapters 3 and 4 respectively. However this information is of limited use in power considerations.

A series of computer simulation studies using various values of α would undoubtedly reveal useful information concerning the optimum value of α, though of course, for each value of α and size of experiment the null distribution of each statistic would be required.

PART II

AN ASYMPTOTIC EXPANSION OF THE NULL DISTRIBUTIONS OF KRUSKAL-WALLIS'S AND FRIEDMAN'S STATISTICS

CHAPTER 9

THE METHOD OF STEEPEST DESCENTS

1. Introduction.

During the preparation of the simulation studies we became aware of the limitations in availability of known exact null distributions for various nonparametric statistics. Our attention was drawn initially to Friedman's statistic and then later to the Kruskal-Wallis statistic. For both of these the exact null distributions are difficult to derive for even quite small total sample sizes N; in fact $N > 18$, say, involves considerable computational problems for the Kruskal-Wallis distributions. One of the most extensive collections of critical values for these statistics is in Neave (1978) where selected values are given for Friedman's test for $c = 3$, $b = 2$ to 50; $c = 4$, $b = 2$ to 22; $c = 5$, $b = 2$ to 9; $c = 6$, $b = 2$ to 4 and for the Kruskal-Wallis test for $c = 3$, $\max n = 6$; $c = 4$, $\max n = 4$; $c = 5$, $\max n = 3$.

Clearly the availability of good approximations for both distributions is desirable. It is unfortunate that both have a chi-square asymptotic distribution as this excludes the use of an Edgeworth-type expansion which requires the limiting distribution to be normal.

Using the chi-square distribution as an approximation produces somewhat conservative critical values. Other approximations have been derived by Wallace (1959), Alexander and Quade (1968) for the Kruskal-Wallis test and by Iman and Davenport (1980) for Friedman's test. All these methods stem from Kruskal and Wallis's (1952) Beta approximation and have been obtained by varying the number of degrees of freedom.

In 1954, Daniels applied the method of steepest descents to obtain an approximation to the probability density function of a sample mean. Prior to this, only Jeffreys (1948) seems to have applied this method in Statistics. We have adapted the method of steepest descents to obtain an asymptotic expansion of the probability function of the Kruskal-Wallis and Friedman statistics. In order to derive the expansion we required the first four moments of these statistics.

In section 2 we outline the method of steepest descents and then apply it to our situations in section 3.

2. Outline of the Method of Steepest Descents.

A full account of the development of the method is given in Jeffreys and Jeffreys (1966) and so it is sufficient for us to present just a brief summary.

The method of steepest descents, introduced by Debye in 1909 for Bessel functions of large order, produces an approximate evaluation of integrals of the form

$$I(t) = \int_A^B e^{tF(z)}\,dz \quad ,$$

where t is large, real and positive,

and $F(z)$ is analytic with $F(z) = \phi + i\psi$, ϕ and ψ both satisfying Laplace's equation.

Consider a path from A to B where, as often happens, there are points such that ϕ is greater than ϕ_A and ϕ_B. Thus ϕ has a maximum at an interior point z_0 of the path. Suppose that the section of the path passing through z_0 is one of constant ψ (it cannot be one of constant ϕ). If ds and dn are elements of length along and normal to the path respectively then, at this maximum point, $\partial\phi/\partial s = 0$ and $\partial\psi/\partial n = 0$ (since ψ is constant); thus by the Cauchy-Riemann relations $\partial\psi/\partial n = 0$ and $\partial\phi/\partial n = 0$ giving $F'(z_0) = 0$. The point z_0 is called a saddle-point since there F(z) is neither a true maximum nor a true minimum.

Now lines of constant ψ are called lines of steepest descent as the direction of any point on them is such that $|\partial\phi/\partial s|$ is a maximum. This we can see by considering

$$\frac{\partial\phi}{\partial s} = \cos\theta.\frac{\partial\phi}{\partial x} + \sin\theta.\frac{\partial\phi}{\partial y} ,$$

where θ is the inclination of the path to the x-axis. For extreme values of $\partial\phi/\partial s$, for variations in θ, we require $\partial^2\phi/\partial s^2 = 0$. This gives

$$\begin{aligned} 0 &= -\sin\theta.\frac{\partial\phi}{\partial x} + \cos\theta.\frac{\partial\phi}{\partial y} \\ &= -\sin\theta.\frac{\partial\psi}{\partial y} - \cos\theta.\frac{\partial\psi}{\partial x} \\ &= -\frac{\partial\psi}{\partial s} , \end{aligned}$$

which is satisfied on a path of constant ψ.

So the path of integration is chosen so that part of it consists of a line of steepest descent through a saddle-point so that the larger values of ϕ are concentrated in as short an interval of the path as possible.

Now given that z_o is a saddle-point of $F(z)$ and presuming $F''(z_o) \neq 0$, then $F(z)$ can be expanded in the form

$$F(z) = F(z_o) + \tfrac{1}{2}(z - z_o)^2 F''(z_o) + \ldots\ldots ,$$

where the direction of the path will be such that $(z - z_o)^2 F''(z_o)$ is real and negative.

If we now let $F(z) - F(z_o) = -u^2$ and change the variable to u then the integral $I(t)$ becomes

$$I(t) = e^{tF(z_o)} \int_{-\infty}^{\infty} e^{-tu^2} \frac{dt}{du}\, du ,$$

a form that is similar to that considered in Watson's lemma (see Jeffreys and Jeffreys). This lemma ensures the existence of constants c_0, c_1, c_2, ... such that

$$\frac{dt}{du} = c_0 + c_1 u + c_2 u^2 + \ldots\ldots .$$

Substituting this series into $I(t)$ and performing the integrations produces

$$I(t) \approx \sqrt{\frac{\pi}{t}}\; e^{tF(z_o)} \left\{ c_0 + \frac{1}{2}\frac{c_2}{t} + \frac{1.3}{2^2}\frac{c_4}{t^2} + \frac{1.3.5}{2^3}\frac{c_6}{t^3} + \ldots \right\}$$

It is this expansion for $I(t)$ that enables us to derive approximate probability functions for the Kruskal-Wallis and Friedman statistics. Of course, except for small N, this is feasible since the possible values in the discrete distribution are so close together that an approximation by a density function provides a good fit.

3. Derivation of an Approximating Density Function.

We now suppose that a random variable T has finite moments μ_1', μ_2', μ_3' and μ_4' where $\mu_i' = E(X^i)$. Then if the characteristic function of T is $\phi(t)$ setting $k = it$ gives

$$\phi(-ik) \equiv \Omega(k) = 1 + \mu_1' k + \mu_2' \frac{k^2}{2!} + \mu_3' \frac{k^3}{3!} + \mu_4' \frac{k^4}{4!} \quad .$$

The usual inversion theorem, which in terms of k can be written

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Omega(k)\, e^{-kx}\, dk \quad ,$$

is now employed to obtain

$$f_T(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \exp(\, Ak + Bk^2 + Ck^3 + Dk^4\,)\, dk \qquad \ldots\ldots (1)$$

where the coefficients A, B, C and D are obtained by equating

$$\left(1 + \mu_1' k + \mu_2' \frac{k^2}{2!} + \mu_3' \frac{k^3}{3!} + \mu_4' \frac{k^4}{4!}\right) e^{-kx} \quad \text{to}$$

$\exp(\, Ak + Bk^2 + Ck^3 + Dk^4\,)$. Their values are

$$A = \mu_1' - x \,,$$

$$B = \tfrac{1}{2}\mu_2' - \tfrac{1}{2}\mu_1'^{\,2} \,,$$

$$C = \frac{1}{6}\mu'_3 - \frac{1}{2}\mu'_1\mu'_2 + \frac{1}{3}\mu'^{3}_1 ,$$

$$D = \frac{1}{24}\mu'_4 - \frac{1}{8}\mu'^{2}_2 - \frac{1}{6}\mu'_1\mu'_3 + \frac{1}{2}\mu'^{2}_1\mu'_2 - \frac{1}{4}\mu'^{4}_1$$

If we now define $F(k)$ by

$$F(k) = \frac{A}{x}k + \frac{B}{x}k^2 + \frac{C}{x}k^3 + \frac{D}{x}k^4$$

then (1) becomes

$$f_T(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{xF(k)}\, dk$$

which is of the required format for applying the method of steepest descents.

Suppose a stationary point of $F(k)$ occurs at k_o so that $F'(k_o) = 0$, then, as before, we define u by $F(k) - F(k_o) = u^2$. Hence the expansion of $F(k)$ about k_o is

$$F(k) = F(k_o) + \frac{1}{2}F''(k_o)(k-k_o)^2 + \frac{1}{6}F'''(k_o)(k-k_o)^3$$
$$+ \frac{1}{24}F^{iv}(k_o)(k-k_o)^4 + \ldots\ldots \qquad ..(2)$$

We now denote $k - k_o$ by r, $\frac{1}{2}F''(k_o)$ by a_2, $\frac{1}{6}F'''(k_o)$ by a_3 and $\frac{1}{24}F^{iv}(k_o)$ by a_4 so that (2) becomes

$$-u^2 = a_2r^2 + a_3r^3 + a_4r^4 .$$

Now let $\quad r = b_1u + b_2u^2 + b_3u^3 + b_4u^4 + \ldots\ldots$

Then

$$-u^2 = a_2(b_1u + b_2u^2 + b_3u^3 + b_4u^4 + \;....)^2$$
$$+ \; a_3(b_1u + b_2u^2 + b_3u^3 + b_4u^4 + \;....)^3$$
$$+ \; a_4(b_1u + b_2u^2 + b_3u^3 + b_4u^4 + \;....)^4 \quad .$$

On equating coefficients we obtain, after setting

$$\alpha_1 = a_3 / a_2 \;, \quad \alpha_2 = a_4 / a_2 \;,$$

the following values for the b's

$$b_1 = i / a_2^{\frac{1}{2}}$$
$$b_2 = -\tfrac{1}{2}\alpha_1 \, b_1^2$$
$$b_3 = \left(\tfrac{5}{8}\alpha_1^2 - \tfrac{1}{2}\alpha_2\right) b_1^3$$
$$b_4 = \left(-\alpha_1^3 + \tfrac{3}{2}\alpha_1\alpha_2\right) b_1^4$$
$$b_5 = \left(\tfrac{231}{128}\alpha_1^4 + \tfrac{7}{8}\alpha_2^2 - \tfrac{63}{16}\alpha_2\alpha_1^2\right) b_1^5$$
$$b_6 = \left(-\tfrac{7}{2}\alpha_1^5 + 10\,\alpha_2\alpha_1^3 - 5\,\alpha_2^2\alpha_1\right) b_1^6$$
$$b_7 = \left(\tfrac{7293}{1024}\alpha_1^6 + \tfrac{1287}{64}\alpha_2^2\alpha_1^2 - \tfrac{33}{16}\alpha_2^3 - \tfrac{6435}{256}\alpha_2\alpha_1^4\right) b_1^7$$

From $r = k - k_o = b_1u + b_2u^2 + b_3u^3 + b_4u^4 + \ldots$

we obtain

$$\frac{dk}{du} = b_1 + 2b_2u + 3b_3u^2 + 4b_4u^3 + 5b_5u^4 + \dots$$

which in conjunction with (2) produces

$$f_T(x) \approx \frac{1}{2\pi i}\sqrt{\frac{\pi}{x}}\, e^{xF(k_o)}\left\{ b_1 + \frac{3b_3}{2x} + \frac{1.3\ 5b_5}{2^2\ x^2} + \frac{1.3.5\ 7b_7}{2^3\ x^3} + \dots \right\}$$

$$\text{i.e. } f_T(x) \approx \frac{e^{xF(k_o)}}{2\sqrt{\pi\beta}}\left\{ 1 - \frac{3}{2\beta}\left(\frac{5}{8}\alpha_1^2 - \frac{1}{2}\alpha_2\right)\right.$$

$$+ \frac{5}{4\beta^2}\left(\frac{231}{128}\alpha_1^4 + \frac{7}{8}\alpha_2^2 - \frac{63}{16}\alpha_2\alpha_1^2\right)$$

$$\left. - \frac{105}{8\beta^3}\left(\frac{7293}{1024}\alpha_1^6 + \frac{287}{64}\alpha_2^2\alpha_1^2 - \frac{33}{16}\alpha_2^3 - \frac{6438}{256}\alpha_2\alpha_1^4\right)\right\}$$

where $\beta = a_2x = \frac{1}{2}xF''(k_o)$.

The value of k_o is obtained by solving the cubic equation $F'(k) = 0$, that is

$$A + 2B + 3Ck^2 + 4Dk^3 = 0 .$$

This may be solved by first solving the reduced equation

$$y^3 + 3\Delta_1 y + \Delta_2 = 0$$

where

$$y = k + \frac{C}{4D} ,$$

$$\Delta_1 = \frac{8BD - 3C^2}{48D^2}$$

$$\Delta_2 = \frac{C^3 - 4BCD + 8AD^2}{32D^3}$$

Now Jackson (1964) has shown that if $\Delta_2^2 + 4\Delta_1^3 > 0$ then the cubic has only one real solution. This would indicate that the function F(k) has a unique saddle-point, which is clearly desirable. However, should more than one saddle-point exist then the path of integration with the steepest descent is selected by considering the behaviour of the respective arguments of the saddle-points.

For the Kruskal-Wallis statistic, computer calculations have shown that for $c \geqslant 3$ and $N > 9$, $\Delta_2^2 + 4\Delta_1^3 > 0$ and thus F(k) has a unique saddle-point. Similar calculations for Friedman's statistic indicate that a unique saddle-point exists whenever $b > 3$. These conditions adequately cover the range of sample sizes we have considered.

CHAPTER 10

COMPARISON OF RESULTS

1. Comparison of Results for the Kruskal-Wallis Distribution.

As previously mentioned there have been several approximations proposed for the exact null distribution of the Kruskal-Wallis statistic. Before comparing performances with that of the steepest descent approximation we shall first briefly describe some of these approximations.

(a) The Chi-square Approximation.

Kruskal (1952) showed that under the null hypothesis H is asymptotically (as all sample sizes $\rightarrow \infty$) distributed as a chi-square distribution with $c - 1$ degrees of freedom.

(b) The Beta (B_1) Approximation.

In their paper, Kruskal and Wallis (1952) proposed an approximation that matches the distribution of H / M, where M is the maximum possible value of H, to a Beta distribution whose parameters are chosen so that the means and variances of the two distributions are equal. They employed the F - distribution, a form of the incomplete Beta distribution, and set

$$F = \frac{H(M - \mu_H)}{\mu_H(M - H)}$$

where $\mu_H = c - 1$, the mean of H,

and F has degrees of freedom (not necessarily integral) given by

$$f_1 = \frac{\mu_H(\mu_H(M - \mu_H) - V)}{\frac{1}{2}MV} ,$$

$$f_2 = \frac{(M - \mu_H) f_1}{\mu_H}$$

with V being the variance of H and M being given by

$$M = \frac{(N^3 - \sum_{i=1}^{c} n_i^3)}{N(N + 1)} \quad .$$

(c) <u>Wallace's B_2 - III Approximation.</u>

In 1959, Wallace gave an approximation in which the usual analysis of variance calculations are performed on ranks. This results in the test statistic

$$F = \frac{(N - c) H}{(c - 1)(N - 1 - H)}$$

with (c - 1, N - c) degrees of freedom. Clearly this is a fairly simple statistic to compute and test which appear to be its main attributes.

(d) <u>The Quade Approximation</u>.

This is similar to Wallace's B_2 - III approximation with the difference that the number of degrees of freedom in the denominator is decreased by one. This results in an approximation that, at least for equal n_i, is identical to Wallace's B_2 - I.

We compare these approximations by calculating the difference Δ = (true probability) - (approximate probability) in various cases. The following table shows values of Δ at the 1 %, 2 %, 5 % and 10 % conservative critical values. The number of comparisons is restricted by the availibility of exact distributions, thus we only have comparisons for c = 3 from n = 5 to 8 and c = 4 for n = 4.

c	n	H	True Prob	Steepest Descent	χ^2	B_1	B_2-III	Quade
3	5	8	.0095	-.0002	-.0088	.0026	.0032	.0022
		7.22	.0194	.0018	-.0077	.0044	.0065	.0050
		5.78	.0488	.0016	-.0068	.0028	.0084	.0057
		4.56	.0995	.0006	-.0028	-.0017	.0067	.0031
	6	8.22	.0099	.0001	-.0065	.0018	.0028	.0021
		7.24	.0198	.0003	-.0070	.0022	.0043	.0054
		5.80	.0491	.0001	-.0060	.0016	.0061	.0060
		4.64	.0987	.0003	.0006	.0022	.0086	.0023
	7	8.38	.0099	.0001	-.0053	.0014	.0023	.0018
		7.33	.0197	.0002	-.0059	.0016	.0034	.0027
		5.82	.0491	.0000	-.0054	.0008	.0046	.0033
		4.59	.0993	-.0001	-.0013	.0001	.0053	.0037
	8	8.47	.0099	.0000	-.0046	.0010	.0019	.0014
		7.36	.0199	-.0005	-.0054	.0009	.0026	.0020
		5.81	.0497	-.0015	-.0052	.0001	.0034	.0025
		4.61	.0985	-.0001	-.0015	-.0002	.0043	.0031
4	4	9.29	.0100	.0000	-.0157	-.0012	.0025	.0012
		8.52	.0199	.0000	-.0165	-.0013	.0047	.0023
		7.24	.0492	.0000	-.0156	-.0033	.0074	.0041
		6.09	.0990	-.0006	-.0084	-.0045	.0095	.0051

Even from these limited comparisons we see that the steepest descent method provides considerable improvement over the previous approximations. Admittedly this is at the expense of computational ease; the steepest descent method can hardly be described as computationally straightforward. However we feel the effort is justified, particularly as the calculations are performed once and for all when establishing a set of critical values which can then be tabulated for future use.

2. Comparison of Results for Friedman's Distribution.

Until recently the only approximation to the null distribution of Friedman's χ_r^2 - statistic was the chi-square approximation proposed by Friedman (1937). In 1980 Iman and Davenport presented approximate critical values based on the B_1 approximation. In our comparison we shall investigate suitably modified versions of the B_1, B_2 - III and Quade's approximations.

(a) The Chi-square Approximation.

Friedman (1937) showed that under the null hypothesis χ_r^2 is asymptotically distributed as the chi-square distribution with $c - 1$ degrees of freedom.

(b) The Beta (B_1) Approximation.

This is derived from the approximation proposed by Kruskal and Wallis (1952) for their H - statistic. Using the same idea for Friedman's χ_r^2 - statistic produces an F - ratio

$$F = \frac{\chi_r^2 (b - 1)}{b(c - 1) - \chi_r^2}$$

with degrees of freedom

$$f_1 = \frac{b(c - 1) - 2}{b}$$

$$f_2 = (b - 1)f_1 \quad .$$

(c) Wallace's B_2 - III Approximation.

The F - ratio in Wallace's approximation is obtained by performing the usual analysis of variance calculations on the ranks. For Friedman's statistic the F - ratio is in fact identical to the B_1 approximation though with degrees of freedom given by $f_1 = c - 1$ and $f_2 = (b - 1)f_1$.

(d) Quade's Approximation.

Quade's approximation uses the same F - ratio as Wallace's. Quade simply takes $f_2 = (b - 1)f_1 - 1$ in an attempt to achieve a better approximation.

Comparisons are again effected by examining the difference Δ = (true probability) - (approximate probability). We have chosen the 1 %, 2 %, 5 % and 10 % conservative critical values for c = 3, b = 8 to 15, c = 4, b = 7 to 12 and c = 5, b = 5 to 6.

c	n	χ_r^2	Prob	Steepest Descent	χ^2	B_1	B_2-III	Quade
3	8	9	.0099	.0009	-.0012	.0050	.0067	.0063
		7.75	.0179	-.0020	-.0029	.0045	.0082	.0073
		6.25	.0469	.0000	.0030	.0092	.0161	.0143
		5.25	.0789	.0000	.0065	.0092	.0181	.0157
	9	9.56	.0060	-.0002	-.0024	.0024	.0036	.0033
		8	.0189	.0017	.0006	.0064	.0098	.0092
		6.22	.0475	-.0008	.0029	.0082	.0144	.0130
		5.56	.0689	.0000	.0067	.0103	.0176	.0159
	10	9.60	.0075	.0007	-.0008	.0035	.0046	.0044
		7.80	.0179	-.0009	-.0023	.0031	.0063	.0056
		6.20	.0456	.0005	.0006	.0052	.0112	.0096
		5	.0924	.0018	.0103	.0117	.0186	.0171
	11	9.46	.0065	.0003	-.0023	.0015	.0028	.0025
		7.82	.0187	-.0002	-.0014	.0035	.0064	.0058
		6.55	.0435	.0033	.0056	.0102	.0147	.0139
		5.09	.0867	-.0008	.0083	.0100	.0161	.0148
	12	9.50	.0074	.0004	-.0013	.0022	.0034	.0032
		8	.0197	.0008	.0014	.0057	.0082	.0078
		6.50	.0381	-.0026	-.0007	.0035	.0077	.0069
		5.17	.0796	-.0040	.0041	.0060	.0114	.0104

c	b	χ_r^2	True Prob	Steepest Descent	χ^2	B_1	B_2-III	Quade
3	12	9.50	.0074	.0004	-.0013	.0022	.0034	.0032
		8	.0197	.0008	.0014	.0057	.0082	.0078
		6.50	.0381	-.0026	-.0007	.0035	.0077	.0069
		5.17	.0796	-.0040	.0041	.0060	.0114	.0104
	13	9.39	.0087	.0005	-.0005	.0028	.0040	.0038
		8	.0161	-.0002	-.0022	.0017	.0041	.0037
		6.62	.0371	.0000	.0005	.0044	.0082	.0075
		4.77	.0979	-.0013	.0058	.0065	.0116	.0107
	14	9.14	.0077	-.0008	-.0026	.0004	.0018	.0016
		8.14	.0167	.0007	-.0004	.0032	.0053	.0050
		6.14	.0480	-.0019	.0016	.0050	.0088	.0082
		5.14	.0896	.0040	.0132	.0150	.0195	.0188
	15	8.93	.0097	-.0002	-.0018	.0012	.0026	.0024
		8.13	.0179	.0009	.0008	.0041	.0060	.0057
		6.40	.0468	.0000	.0061	.0094	.0127	.0123
		4.93	.0958	.0003	.0109	.0122	.0165	.0159
4	7	10.54	.0091	.0001	-.0054	.0023	.0041	.0036
		9.17	.0196	-.0003	-.0075	.0019	.0055	.0046
		7.80	.0413	-.0005	-.0090	.0002	.0061	.0049
		6.43	.0929	.0016	.0004	.0057	.0135	.0115
	8	10.50	.0094	-.0001	-.0054	.0012	.0031	.0027
		9.45	.0188	.0005	-.0051	.0027	.0056	.0050
		7.65	.0488	-.0004	-.0050	.0026	.0080	.0068
		6.30	.0999	.0012	.0020	.0061	.0127	.0112

c	b	χ_r^2	True Prob	Steepest Descent	χ^2	B_1	B_2-III	Quade
4	9	10.75	.0094	.0003	-.0039	.0016	.0032	.0029
		9.40	.0194	-.0001	-.0050	.0018	.0045	.0040
		7.67	.0488	.0001	-.0046	.0021	.0067	.0058
		6.20	.0978	-.0005	-.0045	-.0012	.0046	.0035
	10	10.68	.0099	.0003	-.0037	.0012	.0028	.0025
		9.48	.0194	.0003	-.0041	.0018	.0042	.0038
		7.68	.0471	-.0013	-.0060	.0010	.0047	.0039
		6.36	.0948	.0003	-.0006	.0030	.0080	.0071
	11	10.75	.0099	.0001	-.0033	.0011	.0025	.0022
		9.66	.0180	-.0001	-.0037	.0015	.0036	.0033
		7.69	.0492	-.0006	-.0036	.0018	.0055	.0049
		6.27	.0979	.0008	-.0012	.0018	.0064	.0057
	12	10.80	.0098	.0002	-.0031	.0009	.0021	.0020
		9.50	.0198	.0000	-.0035	.0013	.0034	.0031
		7.70	.0483	-.0009	-.0043	.0006	.0041	.0036
		6.30	.0988	.0008	.0010	.0038	.0079	.0072
5	5	11.68	.0094	.0001	-.0105	.0021	.0042	.0036
		10.56	.0190	-.0001	-.0130	.0022	.0061	.0049
		8.96	.0488	.0003	-.0133	.0052	.0121	.0100
		7.68	.0944	-.0002	-.0096	.0013	.0105	.0076
	6	11.87	.0099	.0001	-.0085	.0014	.0033	.0028
		10.80	.0193	.0001	-.0096	.0021	.0053	.0045
		9.07	.0491	-.0003	-.0103	.0021	.0078	.0063
		7.73	.0951	.0000	-.0068	.0020	.0093	.0074

The above comparisons of approximations for Friedman's distribution confirm our previous thoughts regarding the steepest descent approximation. It certainly appears to be consistent in giving good approximations and, once again, justifies the great computation involved.

APPENDIX 1

THE THIRD AND FOURTH MOMENTS OF THE KRUSKAL-WALLIS DISTRIBUTION

1. Introduction.

The third and fourth moments of the Kruskal-Wallis H-statistic have been derived using the method employed by Kruskal (1952) to calculate the first two moments of H. Our results have been verified by checking with moments calculated from exact null distributions.

The first two moments of H are given by

$$E(H) = c - 1$$

$$\operatorname{var}(H) = 2(c-1) - \frac{2(3c^2 - 6c + N(2c^2 - 6c + 1))}{5N(N+1)} - \frac{6}{5}\Sigma \frac{1}{n_i}$$

In the following calculations we use the notation :

$$H = \frac{12}{N(N+1)} \sum_{j=1}^{c} \frac{R_j^2}{n_j} - 3(N+1) \qquad \dots (1)$$

$X_j^{(i)}$ is the rank in the overall sample of the i^{th} observation from the j^{th} sample.

R_j is the sum of the ranks from the j^{th} sample.

2. Calculation of the Third Moment.

Directly from (1) we have

$$E(H^3) = \frac{12^3}{N^3(N+1)^3} E\left\{\sum_{j=1}^{c}\left(\frac{R_j^2}{n_j}\right)^3\right\} - \frac{1296}{N^2(N+1)} E\left\{\sum_{j=1}^{c}\left(\frac{R_j^2}{n_j}\right)^2\right\}$$

$$+ \frac{324(N+1)}{N} E\left\{\sum_{j=1}^{c}\frac{R_j^2}{n_j}\right\} - 27(N+1)^3 \quad \ldots.(2)$$

We now consider separately the three expectations in (2).

First,

$$E\left\{\sum_{j=1}^{c}\left(\frac{R_j^2}{n_j}\right)^3\right\} = \sum_{j=1}^{c} E\left(\frac{R_j^6}{n_j^3}\right) + 3\sum_{\substack{j=1\\ j\neq k}}^{c}\sum_{k=1}^{c} E\left(\frac{R_j^4 R_k^2}{n_j^2 n_k}\right)$$

$$+ \sum_{\substack{j=1\\ j\neq k}}^{c}\sum_{\substack{k=1\\ j\neq l}}^{c}\sum_{\substack{l=1\\ k\neq l}}^{c} E\left(\frac{R_j^2 R_k^2 R_l^2}{n_j n_k n_l}\right) \quad \ldots(3)$$

Now,

$$E(R_j^6) = \sum_{i_1=1}^{n_j}\sum_{i_2=1}^{n_j}\sum_{i_3=1}^{n_j}\sum_{i_4=1}^{n_j}\sum_{i_5=1}^{n_j}\sum_{i_6=1}^{n_j} E(X_{i_1}^{(j)} X_{i_2}^{(j)} X_{i_3}^{(j)} X_{i_4}^{(j)} X_{i_5}^{(j)} X_{i_6}^{(j)})$$

which by symmetry becomes

$$E(R_j^6) =$$

$$n_j(n_j - 1)(n_j - 2)(n_j - 3)(n_j - 4)(n_j - 5)E(X_{i_1}^{(j)}X_{i_2}^{(j)}X_{i_3}^{(j)}X_{i_4}^{(j)}X_{i_5}^{(j)}X_{i_6}^{(j)})$$

$$+ 15n_j(n_j - 1)(n_j - 2)(n_j - 3)(n_j - 4)\ E(X_1^{(j)}X_2^{(j)}X_3^{(j)}X_4^{(j)}X_5^{(j)^2})$$

$$+ 20n_j(n_j - 1)(n_j - 2)(n_j - 3)\ E(X_1^{(j)}X_2^{(j)}X_3^{(j)}X_4^{(j)^3})$$

$$+ 15n_j(n_j - 1)(n_j - 2)\ E(X_1^{(j)}X_2^{(j)}X_3^{(j)^4})$$

$$+ 6n_j(n_j - 1)\ E(X_1^{(j)}X_2^{(j)^5}) \quad + \quad n_j\ E(X_1^{(j)^6})$$

$$+ 45n_j(n_j - 1)(n_j - 2)(n_j - 3)\ E(X_1^{(j)}X_2^{(j)}X_3^{(j)^2}X_4^{(j)^2})$$

$$+ 60n_j(n_j - 1)(n_j - 2)\ E(X_1^{(j)}X_2^{(j)^3}X_3^{(j)^2})$$

$$+ 15n_j(n_j - 1)\ E(X_1^{(j)^4}X_2^{(j)^2}) \quad + \quad 10n_j(n_j - 1)\ E(X_1^{(j)^3}X_2^{(j)^3})$$

$$+ 15n_j(n_j - 1)(n_j - 2)\ E(X_1^{(j)^2}X_2^{(j)^2}X_3^{(j)^2})$$

$$= \frac{n_j(n_j - 1)(n_j - 2)(n_j - 3)(n_j - 4)(n_j - 5)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma\ p_1p_2p_3p_4p_5p_6$$

$$+ \frac{15n_j(n_j - 1)(n_j - 2)(n_j - 3)(n_j - 4)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma\ p_1^2p_2p_3p_4p_5$$

$$+ \frac{20n_j(n_j-1)(n_j-2)(n_j-3)}{N(N-1)(N-2)(N-3)} \Sigma\, p_1^3 p_2 p_3 p_4$$

$$+ \frac{15n_j(n_j-1)(n_j-2)}{N(N-1)(N-2)} \Sigma\, p_1^4 p_2 p_3 + \frac{6n_j(n_j-1)}{N(N-1)} \Sigma\, p_1^5 p_2$$

$$+ \frac{n_j}{N} \Sigma\, p_1^6 + \frac{45n_j(n_j-1)(n_j-2)(n_j-3)}{N(N-1)(N-2)(N-3)} \Sigma\, p_1^2 p_2^2 p_3 p_4$$

$$+ \frac{60n_j(n_j-1)(n_j-2)}{N(N-1)(N-2)} \Sigma\, p_1^3 p_2^2 p_3 + \frac{5n_j(n_j-1)}{N(N-1)} \Sigma\, p_1^4 p_2$$

$$+ \frac{15n_j(n_j-1)(n_j-2)}{N(N-1)(N-2)} \Sigma\, p_1^2 p_2^2 p_3^2 + \frac{10n_j(n_j-1)}{N(N-1)} \Sigma\, p_1^3 p_2^3 ,$$

where the p_i's run from 1 to N and within any term of a summation no two are equal.

Summing over j we obtain, after some algebraic effort,

$$\sum_{j=1}^{c} E\left\{R_j^6 \,/\, n_j^3\right\} =$$

$$\frac{(N+1)}{4032}(63N^5 - 315N^3 - 224N^2 + 140N + 96)\left(\Sigma n_j^3 - 15\, \Sigma n_j^2 + 85N - 225c + 274\, \Sigma \frac{1}{n_j} - 120\, \Sigma \frac{1}{n_j^2} \right)$$

$$+ \frac{(N+1)}{672}(210N^5 + 105N^4 - 812N^3 - 693N^2 + 302N + 240)$$
$$(\Sigma n_j^2 - 10N + 35c - 50\,\Sigma \frac{1}{n_j} + 24\,\Sigma \frac{1}{n_j^2})$$

$$+ \frac{(N+1)}{168}(105N^5 + 126N^4 - 231N^3 - 276N^2 + 76N + 80)$$
$$(N - 6c + 11\,\Sigma \frac{1}{n_j} - 6\,\Sigma \frac{1}{n_j^2})$$

$$+ \frac{(N+1)}{168}(126N^5 + 231N^4 - 78N^3 - 226N^2 + 37N + 60)$$
$$(c - 3\,\Sigma \frac{1}{n_j} + 2\,\Sigma \frac{1}{n_j^2})$$

$$+ \frac{(N+1)}{28}(14N^5 + 32N^4 + 7N^3 - 17N^2 + 4)(\Sigma \frac{1}{n_j} - \Sigma \frac{1}{n_j^2})$$

$$+ \frac{(N+1)}{42}(6N^5 + 15N^4 + 6N^3 - 6N^2 - N + 1)\ \Sigma \frac{1}{n_j^2}$$

$$+ \frac{(N+1)}{336}(420N^5 + 364N^4 - 1267N^3 - 1291N^2 + 370N + 360)$$
$$(N - 6c + 11\,\Sigma \frac{1}{n_j} - 6\,\Sigma \frac{1}{n_j^2})$$

$$+ \frac{(N+1)}{42}(105N^5 + 147N^4 - 183N^3 - 268N^2 + 37N + 60)$$
$$(c - 3\,\Sigma \frac{1}{n_j} + 2\,\Sigma \frac{1}{n_j^2})$$

$$+ \frac{(N+1)}{84}(84N^5 + 156N^4 - 49N^3 - 159N^2 + 7N + 30)(\Sigma \frac{1}{n_j} - \Sigma \frac{1}{n_j^2})$$

$$+ \frac{(N+1)}{504}(280N^5 + 308N^4 - 682N^3 - 797N^2 + 153N + 180)\left(c - 3\Sigma\frac{1}{n_j} + 2\Sigma\frac{1}{n_j^2}\right)$$

$$+ \frac{10(N+1)}{336}(21N^5 + 36N^4 - 21N^3 - 48N^2 + 8)\left(\Sigma\frac{1}{n_j} - \Sigma\frac{1}{n_j^2}\right)$$

To obtain $\sum_{\substack{i=1\\i\neq j}}^{c} \sum_{j=1}^{c} E\left\{\frac{R_i^4 R_j^2}{n_i^2 n_j}\right\}$ we consider $E(R_i^4 R_j^2)$.

Now $E(R_i^4 R_j^2)$ =

$$\sum_{i_1=1}^{n_i}\sum_{i_2=1}^{n_i}\sum_{i_3=1}^{n_i}\sum_{i_4=1}^{n_i}\sum_{j_1=1}^{n_j}\sum_{j_2=1}^{n_j} E(X_{i_1}^{(i)}X_{i_2}^{(i)}X_{i_3}^{(i)}X_{i_4}^{(i)}X_{j_1}^{(j)}X_{j_2}^{(j)})$$

$$= n_i(n_i-1)(n_i-2)(n_i-3)n_j(n_j-1)E(X_1^{(i)}X_2^{(i)}X_3^{(i)}X_4^{(i)}X_1^{(j)}X_2^{(j)})$$

$$+ n_i(n_i-1)(n_i-2)(n_i-3)n_jE(X_1^{(i)}X_2^{(i)}X_3^{(i)}X_4^{(i)}X_1^{(j)^2})$$

$$+ 6n_i(n_i-1)(n_i-2)n_j(n_j-1)E(X_1^{(i)}X_2^{(i)}X_3^{(i)^2}X_1^{(j)}X_2^{(j)})$$

$$+ 6n_i(n_i-1)(n_i-2)n_jE(X_1^{(i)}X_2^{(i)}X_3^{(i)^2}X_1^{(j)^2})$$

$$+ 4n_i(n_i-1)n_j(n_j-1)E(X_1^{(i)}X_2^{(i)^3}X_1^{(j)}X_2^{(j)})$$

$$+ 4n_i(n_i-1)n_jE(X_1^{(i)}X_2^{(i)^3}X_1^{(j)^2})$$

$$+ n_in_j(n_j-1)E(X_1^{(i)^4}X_1^{(j)}X_2^{(j)}) + n_in_jE(X_1^{(i)^4}X_1^{(j)^2})$$

$$+ 3n_i(n_i - 1)n_j(n_j - 1)E(X_1^{(i)^2} X_2^{(i)^2} X_1^{(j)} X_2^{(j)})$$

$$+ 3n_i(n_i - 1)n_j E(X_1^{(i)^2} X_2^{(i)^2} X_1^{(j)^2})$$

$$= \frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma p_1 p_2 p_3 p_4 p_5 p_6$$

$$+ \frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma p_1^2 p_2 p_3 p_4 p_5$$

$$+ \frac{6n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma p_1^2 p_2 p_3 p_4 p_5$$

$$+ \frac{6n_i(n_i - 1)(n_i - 2)n_j}{N(N - 1)(N - 2)(N - 3)} \Sigma p_1^2 p_2^2 p_3 p_4$$

$$+ \frac{4n_i(n_i - 1)n_j(n_j - \ )}{N(N - 1)(N - 2)(N - 3)} \Sigma p_1^3 p_2 p_3 p_4$$

$$+ \frac{4n_i(n_i - 1)n_j}{N(N - 1)(N - 2)} \Sigma p_1^2 p_2^3 p_3 + \frac{n_i n_j(n_j - 1)}{N(N - 1)(N - 2)} \Sigma p_1^4 p_2 p_3$$

$$+ \frac{n_i n_j}{N(N - 1)} \Sigma p_1^2 p_2^4 + \frac{3n_i(n_i - 1)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)} \Sigma p_1^2 p_2^2 p_3 p_4$$

$$+ \frac{3n_i(n_i - 1)n_j}{N(N - 1)(N - 2)} \Sigma p_1^2 p_2^2 p_3^2$$

Summing over i and j $(i \neq j)$ we obtain after some simplification,

$$\sum_{i=1}^{c} \sum_{\substack{j=1 \\ i \neq j}}^{c} E\left\{\frac{R_i^4 R_j^2}{n_i^2 n_j}\right\} =$$

$$\frac{(N+1)}{4032}(63N^5 - 315N^3 - 224N^2 + 140N + 96)\Big((N-c)(\Sigma n_i^2 - 6N + 11c - 6\,\Sigma \frac{1}{n_i}) - \Sigma n_i^3 + 7\,\Sigma n_i^2 - 17N + 17c - 6\,\Sigma \frac{1}{n_i}\Big)$$

$$+ \frac{(N+1)}{10080}(210N^5 + 105N^4 - 812N^3 - 693N^2 + 302N + 240)\Big((c-1)(\Sigma n_i^2 - 2N + 29c - 18\,\Sigma \frac{1}{n_i}) - 6\,\Sigma n_i^2 + 18N - 2c + 6N(N - 3c + 2\,\Sigma \frac{1}{n_i})\Big)$$

$$+ \frac{(N+1)}{2520}(420N^5 + 364N^4 - 1267N^3 - 1291N^2 + 370N + 360)(c-1)(N - 3c + 2\,\Sigma \frac{1}{n_i})$$

$$+ \frac{(N+1)}{840}(105N^5 + 126N^4 - 231N^3 - 276N^2 + 76N + 80)\Big(c - N + (c - \Sigma \frac{1}{n_i})(N - c + 1)\Big)$$

$$+ \frac{(N+1)}{630}(105N^5 + 147N^4 - 183N^3 - 268N^2 + 37N + 60)(c-1)(c - \Sigma \frac{1}{n_i})$$

$$+ \frac{(N+1)}{2520}(126N^5 + 231N^4 - 78N^3 - 226N^2 + 37N + 60)\Big((N - c + 1)\Sigma \frac{1}{n_i} - c\Big)$$

$$+ \frac{(N+1)}{1260}(84N^5 + 156N^4 - 49N^3 - 159N^2 + 7N + 30)(c-1)\Sigma \frac{1}{n_i}$$

$$+ \frac{(N+1)}{5040}(420N^5 + 364N^4 - 1267N^3 - 1291N^2 + 370N + 360)$$

$$(c - N + (N - c + 1)(c - \Sigma \frac{1}{n_i}))$$

$$+ \frac{(N+1)}{2520}(280N^5 + 308N^4 - 682N^3 - 797N^2 + 153N + 180)(c-1)(c - \Sigma \frac{1}{n_i})$$

To obtain $\sum_{\substack{i=1 \\ i \neq j}}^{c} \sum_{\substack{j=1 \\ i \neq k}}^{c} \sum_{\substack{k=1 \\ j \neq k}}^{c} E\left\{\frac{R_i^2 R_j^2 R_k^2}{n_i n_j n_k}\right\}$ we consider $E(R_i^2 R_j^2 R_k^2)$.

Now $E(R_i^2 R_j^2 R_k^2) =$

$$\sum_{i_1=1}^{n_i} \sum_{i_2=1}^{n_i} \sum_{j_1=1}^{n_j} \sum_{j_2=1}^{n_j} \sum_{k_1=1}^{n_k} \sum_{k_2=1}^{n_k} E(X_{i_1}^{(i)} X_{i_2}^{(i)} X_{j_1}^{(j)} X_{j_2}^{(j)} X_{k_1}^{(k)} X_{k_2}^{(k)})$$

$$= n_i(n_i - 1)n_j(n_j - 1)n_k(n_k - 1)E(X_1^{(i)} X_2^{(i)} X_1^{(j)} X_2^{(j)} X_1^{(k)} X_2^{(k)})$$

$$+ 3n_i n_j(n_j - 1)n_k(n_k - 1)E({X_1^{(i)}}^2 X_1^{(j)} X_2^{(j)} X_1^{(k)} X_2^{(k)})$$

$$+ 3n_i n_j n_k(n_k - 1)E({X_1^{(i)}}^2 {X_1^{(j)}}^2 X_1^{(k)} X_2^{(k)})$$

$$+ n_i n_j n_k E({X_1^{(i)}}^2 {X_1^{(j)}}^2 {X_1^{(k)}}^2)$$

$$= \frac{n_i(n_i - 1)n_j(n_j - 1)n_k(n_k - 1)}{N(N-1)(N-2)(N-3)(N-4)(N-5)} \Sigma\, p_1p_2p_3p_4p_5p_6$$

$$+ \frac{3n_in_j(n_j - 1)n_k(n_k - 1)}{N(N-1)(N-2)(N-3)(N-4)} \Sigma\, p_1^2p_2p_3p_4p_5$$

$$+ \frac{3n_in_jn_k(n_k - 1)}{N(N-1)(N-2)(N-3)} \Sigma\, p_1^2p_2^2p_3p_4$$

$$+ \frac{n_in_jn_k}{N(N-1)(N-2)} \Sigma\, p_1^2p_2^2p_3^2$$

Summing over i, j and k (with no two equal) we obtain

$$\sum_{\substack{i=1\\ i\neq j}}^{c} \sum_{\substack{j=1\\ i\neq k}}^{c} \sum_{\substack{k=1\\ j\neq k}}^{c} E\left\{\frac{R_i^2\; R_j^2\; R_k^2}{n_i\; n_j\; n_k}\right\} =$$

$$\frac{(N+1)}{4032}(63N^5 - 315N^4 - 224N^2 + 140N + 96)(2(\Sigma n_i^3 - \Sigma n_i^2\,)$$
$$-(\Sigma n_i^2 - N)(3N - 3c + 4) + (N - c)(N - c + 1)(N - c+2))$$

$$+ \frac{(N+1)}{10080}(210N^5 + 105N^4 - 812N^3 - 693N^2 + 302N + 240)$$
$$(N(2Nc - 6N - 2(c-2)(c-3)\,) - 2\,\Sigma n_i^2\,(c - 3)$$
$$+ c(N^2 - \Sigma n_i^2 - 4N(c - 2) + 3(c - 1)(c - 2)\quad)$$

$$+ \frac{(N+1)}{1540}(420N^5 + 364N^4 - 1267N^3 - 1291N^2 + 370N + 360)$$
$$(c((c-1)(N-c+2) - N) - N(c-2))$$

$$+ \frac{(N+1)}{7560}(280N^5 + 308N^4 - 682N^3 - 797N^2 + 153N + 180)c(c-1)(c-2)$$

To produce $E\left\{\sum_{j=1}^{c}\left(\frac{R_j^2}{n_j}\right)^3\right\}$ we combine all the above terms according to relation (3). So,

$$E\left\{\sum_{j=1}^{c}\left(\frac{R_j^2}{n_j}\right)^3\right\} =$$

$$(N+1)\Big\{(63N^5 - 315N^3 - 224N^2 + 140N + 96)(N^3 - 12N^2 + 42Nc$$
$$+ 40N - 3N^2c + 3Nc^2 - c^3 - 30c^2 - 176c - 18N\Sigma\frac{1}{n_i}$$
$$+ 8c\Sigma\frac{1}{n_i} + 256\Sigma\frac{1}{n_i} - 120\Sigma\frac{1}{n_i^2}) / 4032$$

$$+ (210N^5 + 105N^4 - 812N^3 - 693N^2 + 302N + 240)$$
$$(12N^2 + 3N^2c - 6Nc^2 - 72Nc - 72N + 3c^3 + 78c^2 + 408c$$
$$- 696\Sigma\frac{1}{n_i} - 54c\Sigma\frac{1}{n_i} + 36N\Sigma\frac{1}{n_i} + 360\Sigma\frac{1}{n_i^2}) / 10080$$

$$+ (420N^5 + 364N^4 - 1267N^3 - 1291N^2 + 370N + 360)$$
$$(3Nc^2 + 18Nc + 24N - 3c^3 - 54c^2 - 204c + 45c\Sigma\frac{1}{n_i}$$
$$+ 450\Sigma\frac{1}{n_i} - 9N\Sigma\frac{1}{n_i} - 270\Sigma\frac{1}{n_i^2}) / 15120$$

$$+ (280N^5 + 308N^4 - 682N^3 - 797N^2 + 153N + 180)$$

$$(c^3 + 6c^2 + 8c - 9c\,\Sigma \frac{1}{n_i} - 36\,\Sigma \frac{1}{n_i} + 30\,\Sigma \frac{1}{n_i^2}) / 7560$$

$$+ (105N^5 + 126N^4 - 231N^3 - 276N^2 + 76N + 80)$$

$$(8N - 96c + 12Nc - 12c^2 - 12N\,\Sigma \frac{1}{n_i} + 12c\,\Sigma \frac{1}{n_i}$$

$$+ 208\,\Sigma \frac{1}{n_i} - 120\,\Sigma \frac{1}{n_i^2}) / 3360$$

$$+ (105N^5 + 147N^4 - 183N^3 - 268N^2 + 37N + 60)$$

$$(12c^2 - 12c\,\Sigma \frac{1}{n_i} + 48c - 168\,\Sigma \frac{1}{n_i} + 120\,\Sigma \frac{1}{n_i^2}) / 2520$$

$$+ (126N^5 + 231N^4 - 78N^3 - 226N^2 + 37N + 60)$$

$$(3N\,\Sigma \frac{1}{n_i} - 3c\,\Sigma \frac{1}{n_i} - 42\,\Sigma \frac{1}{n_i} + 12c + 30\,\Sigma \frac{1}{n_i^2}) / 2520$$

$$+ (84N^5 + 156N^4 - 49N^3 - 159N^2 + 7N + 30)$$

$$(3c\,\Sigma \frac{1}{n_i} + 12\,\Sigma \frac{1}{n_i} - 15\,\Sigma \frac{1}{n_i^2}) / 1260$$

$$+ (14N^5 + 32N^4 + 7N^3 - 17N^2 + 4)(\Sigma \frac{1}{n_i} - \Sigma \frac{1}{n_i^2}) / 28$$

$$+ (6N^5 + 15N^4 + 6N^3 - 6N^2 - N + 1)\Sigma \frac{1}{n_i^2} / 420$$

$$+ 10(21N^5 + 36N^4 - 21N^3 - 48N^2 + 8)(\Sigma \frac{1}{n_i} - \Sigma \frac{1}{n_i^2}) / 336 \Big\}$$

The expectation in the second term of (2) is easily obtained using the results of Kruskal (1952).

$$E\left\{\sum_{j=1}^{c}\left(\frac{R_j^2}{n_j}\right)^2\right\} = \sum_{j=1}^{c} E\left(\frac{R_j^4}{n_j^2}\right) + \sum_{\substack{j=1\\ j\neq k}}^{c}\sum_{k=1}^{c} E\left(\frac{R_j^2\, R_k^2}{n_j\, n_k}\right)$$

$$= \frac{N^2(N+1)^2}{144}\left\{2(c-1) - \frac{2(3c^2 - 6c + N(2c^2 - 6c + 1))}{5N(N+1)}\right.$$

$$\left. - \frac{6}{5}\Sigma\frac{1}{n_i} + (c-1)^2 - 9(N+1)^2 + 6(N+1)(3N+c+2)\right\} .$$

The third expectation is also obtained from Kruskal, and is given by

$$E\left\{\sum_{j=1}^{c}\frac{R_j^2}{n_j}\right\} = \sum_{j=1}^{c} E\left(\frac{R_j^2}{n_j}\right)$$

$$= N(N+1)(3N+2+c)/\,12 .$$

Combining all the above results finally produces the following expression for $E(\,H^3\,)$

$$
\begin{aligned}
E(H^3) = \Big\{ & -105N^4 - 336N^3 - 279N^2 \\
& + c(-35N^4 + 644N^3 + 1547N^2 + 484N - 480) \\
& + c^2(105N^4 + 210N^3 - 69N^2 - 246N) \\
& + c^3(35N^4 - 14N^3 - 143N^2 + 2N + 120) \\
& - \Sigma \frac{1}{n_i}(378N^4 + 1332N^3 + 1170N^2 - 240N - 480) \\
& + \Sigma \frac{1}{n_i^2}(240N^4 + 480N^3 + 120N^2 - 120N) \\
& + c\, \Sigma \frac{1}{n_i}(-126N^4 + 36N^3 + 450N^2 - 360) \Big\} \Big/ 35N^2(N+1)^2
\end{aligned}
$$

As a matter of interest, we see that as each $n_i \to \infty$, and thus $N \to \infty$,

$$E(H^3) \to c^3 + 3c^2 - c - 3$$

which is the third moment of the chi-square distribution with $c - 1$ degrees of freedom.

3. <u>Calculation of the Fourth Moment.</u>

Directly from (1) we have

$$
\begin{aligned}
E(H^4) = & \frac{20736}{N^4(N+1)^4} E\left\{ \sum_{i=1}^{c} \left(\frac{R_i^2}{n_i} \right)^4 \right\} - \frac{20736}{N^3(N+1)^2} E\left\{ \sum_{i=1}^{c} \left(\frac{R_i^2}{n_i} \right)^3 \right\} \\
& + \frac{7776}{N^2} E\left\{ \sum_{i=1}^{c} \left(\frac{R_i^2}{n_i} \right)^2 \right\} - \frac{296}{N}(N+1)^2 E\left\{ \sum_{i=1}^{c} \frac{R_i^2}{n_i} \right\} + 81(N+1)^4
\end{aligned}
$$

Of the terms in this expression only the first is unknown; we now proceed to obtain its value.

$$E\left\{ \sum_{i=1}^{c}\left(\frac{R_i^2}{n_i}\right)^4 \right\} = \sum_{i=1}^{c} E\left(\frac{R_i^8}{n_i^4}\right) + 4\sum_{\substack{i=1\\i\neq j}}^{c}\sum_{j=1}^{c} E\left(\frac{R_i^6 R_j^2}{n_i^3 n_j}\right)$$

$$+ 6\sum_{\substack{i=1\\i\neq j}}^{c}\sum_{\substack{j=1\\i\neq k}}^{c}\sum_{\substack{k=1\\j\neq k}}^{c} E\left(\frac{R_i^4 R_j^2 R_k^2}{n_i^2 n_j n_k}\right) + 3\sum_{\substack{i=1\\i\neq j}}^{c}\sum_{j=1}^{c} E\left(\frac{R_i^4 R_j^4}{n_i^2 n_j^2}\right)$$

$$+ \underset{\text{no two equal}}{\sum_{i=1}^{c}\sum_{j=1}^{c}\sum_{k=1}^{c}\sum_{l=1}^{c}} E\left(\frac{R_i^2 R_j^2 R_k^2 R_l^2}{n_i n_j n_k n_l}\right) \qquad \ldots\ldots\ (4)$$

To obtain $\sum_{i=1}^{c} E\left(\frac{R_i^8}{n_i^4}\right)$ we first consider $E\left(\frac{R_i^8}{n_i^4}\right)$.

$$E(R_i^8) = \sum_{i_1=1}^{n_i}\sum_{i_2=1}^{n_i}\sum_{i_3=1}^{n_i}\sum_{i_4=1}^{n_i}\sum_{i_5=1}^{n_i}\sum_{i_6=1}^{n_i}\sum_{i_7=1}^{n_i}\sum_{i_8=1}^{n_i}$$

$$E(X_{i_1}^{(i)}X_{i_2}^{(i)}X_{i_3}^{(i)}X_{i_4}^{(i)}X_{i_5}^{(i)}X_{i_6}^{(i)}X_{i_7}^{(i)}X_{i_8}^{(i)})$$

$$= n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)(n_i - 6)(n_i - 7)$$

$$E(X_1^{(i)}X_2^{(i)}X_3^{(i)}X_4^{(i)}X_5^{(i)}X_6^{(i)}X_7^{(i)}X_8^{(i)})$$

$$+ 28n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)(n_i - 6)$$

$$E({X_1^{(i)}}^2X_2^{(i)}X_3^{(i)}X_4^{(i)}X_5^{(i)}X_6^{(i)}X_7^{(i)})$$

$$+ \ 56n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)$$

$$E(X_1^{(i)^3} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_5^{(i)} X_6^{(i)})$$

$$+ \ 70n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)E(X_1^{(i)^4} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_5^{(i)})$$

$$+ \ 56n_i(n_i - 1)(n_i - 2)(n_i - 3)E(X_1^{(i)^5} X_2^{(i)} X_3^{(i)} X_4^{(i)})$$

$$+ \ 28n_i(n_i - 1)(n_i - 2)E(X_1^{(i)^6} X_2^{(i)} X_3^{(i)})$$

$$+ \ 8n_i(n_i - 1)E(X_1^{(i)^7} X_2^{(i)}) \ + \ n_i E(X_1^{(i)^8})$$

$$+ \ 210n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)$$

$$E(X_1^{(i)^2} X_2^{(i)^2} X_3^{(i)} X_4^{(i)} X_5^{(i)} X_6^{(i)})$$

$$+ \ 420n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)$$

$$E(X_1^{(i)^2} X_2^{(i)^2} X_3^{(i)^2} X_4^{(i)} X_5^{(i)})$$

$$+ \ 105n_i(n_i - 1)(n_i - 2)(n_i - 3)E(X_1^{(i)^2} X_2^{(i)^2} X_3^{(i)^2} X_4^{(i)^2})$$

$$+ \ 840n_i(n_i - 1)(n_i - 2)(n_i - 3)E(X_1^{(i)^3} X_2^{(i)^2} X_3^{(i)^2} X_4^{(i)})$$

$$+ \ 210n_i(n_i - 1)(n_i - 2)E(X_1^{(i)^4} X_2^{(i)^2} X_3^{(i)^2})$$

$$+ \; 560n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)$$

$$E(X_1^{(i)^3} X_2^{(i)^2} X_3^{(i)} X_4^{(i)} X_5^{(i)})$$

$$+ \; 280n_i(n_i - 1)(n_i - 2)(n_i - 3)E(X_1^{(i)^3} X_2^{(i)^3} X_3^{(i)} X_4^{(i)})$$

$$+ \; 280n_i(n_i - 1)(n_i - 2)E(X_1^{(i)^3} X_2^{(i)^3} X_3^{(i)^2})$$

$$+ \; 280n_i(n_i - 1)(n_i - 2)E(X_1^{(i)^4} X_2^{(i)^3} X_3^{(i)})$$

$$+ \; 35n_i(n_i - 1)E(X_1^{(i)^4} X_2^{(i)^4})$$

$$+ \; 420n_i(n_i - 1)(n_i - 2)(n_i - 3)E(X^{(i)^4} X^{(i)^3} X^{(i)} X^{(i)})$$

$$+ \; 168n_i(n_i - 1)(n_i - 2)E(X_1^{(i)^5} X_2^{(i)^2} X_3^{(i)})$$

$$+ \; 56n_i(n_i - 1)E(X_1^{(i)^5} X_2^{(i)^3}) \; + \; 28n_i(n_i - 1)E(X_1^{(i)^6} X_2^{(i)^2}) \; .$$

So $E(R_i^8) \;\; =$

$$\frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)(n_i - 6)(n_i - 7)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)(N - 7)} \quad x$$

$$\Sigma \; P_1P_2P_3P_4P_5P_6P_7P_8$$

$$+ \frac{28n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)(n_i - 6)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)} \times$$

$$\Sigma\, p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \frac{56n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \quad \Sigma\, p_1^3 p_2 p_3 p_4 p_5 p_6$$

$$+ \frac{70n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \quad \Sigma\, p_1^4 p_2 p_3 p_4 p_5$$

$$+ \frac{56n_i(n_i - 1)(n_i - 2)(n_i - 3)}{N(N - 1)(N - 2)(N - 3)} \quad \Sigma\, p_1^5 p_2 p_3 p_4$$

$$+ \frac{28n_i(n_i - 1)(n_i - 2)}{N(N - 1)(N - 2)} \quad \Sigma\, p_1^6 p_2 p_3$$

$$+ \frac{8n_i(n_i - 1)}{N(N - 1)} \quad \Sigma\, p_1^7 p_2 \qquad + \frac{n_i}{N} \quad \Sigma\, p_1^8$$

$$+ \frac{210n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \quad \Sigma\, p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{420n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \quad \Sigma\, p_1^2 p_2^2 p_3^2 p_4 p_5$$

$$+ \frac{105 n_i(n_i - 1)(n_i - 2)(n_i - 3)}{N(N - 1)(N - 2)(N - 3)} \quad \Sigma\, p_1^2 p_2^2 p_3^2 p_4^2$$

$$+ \frac{840 n_i(n_i - 1)(n_i - 2)(n_i - 3)}{N(N - 1)(N - 2)(N - 3)} \quad \Sigma\, p_1^3 p_2^2 p_3^2 p_4$$

$$+ \frac{210 n_i(n_i - 1)(n_i - 2)}{N(N - 1)(N - 2)} \quad \Sigma\, p_1^2 p_2^2 p_3^4$$

$$+ \frac{560 n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \quad \Sigma\, p_1^3 p_2^2 p_3 p_4 p_5$$

$$+ \frac{280 n_i(n_i - 1)(n_i - 2)(n_i - 3)}{N(N - 1)(N - 2)(N - 3)} \quad \Sigma\, p_1^3 p_2^3 p_3 p_4$$

$$+ \frac{280 n_i(n_i - 1)(n_i - 2)}{N(N - 1)(N - 2)} \quad \Sigma\, p_1^3 p_2^3 p_3^2$$

$$+ \frac{280 n_i(n_i - 1)(n_i - 2)}{N(N - 1)(N - 2)} \quad \Sigma\, p_1^4 p_2^3 p_3$$

$$+ \frac{35 n_i(n_i - 1)}{N(N - 1)} \quad \Sigma\, p_1^4 p_2^4$$

$$+ \frac{420 n_i(n_i - 1)(n_i - 2)(n_i - 3)}{N(N - 1)(N - 2)(N - 3)} \quad \Sigma\, p_1^4 p_2^2 p_3 p_4$$

$$+ \frac{168n_i(n_i - 1)(n_i - 2)}{N(N-1)(N-2)} \quad \Sigma\, p_1^5 p_2^2 p_3$$

$$+ \frac{56n_i(n_i - 1)}{N(N-1)} \quad \Sigma\, p_1^5 p_2^3 \quad + \quad \frac{28n_i(n_i - 1)}{N(N-1)} \quad \Sigma\, p_1^6 p_2^2 \quad ,$$

where, as before, the p_i's run from 1 to N and within any term of a summation no two are equal. Summing over i we obtain, after some simplification,

$$\sum_{i=1}^{c} E\left(\frac{R_i^8}{n_i^4}\right) =$$

$$P_a(N)(\Sigma n_i^4 - 28\,\Sigma n_i^3 + 322\,\Sigma n_i^2 - 1960N + 6769c - 13132\,\Sigma \frac{1}{n_i}$$
$$+ 13068\,\Sigma \frac{1}{n_i^2} - 5040\,\Sigma \frac{1}{n_i^3} \quad)$$

$$+ 28P_b(N)(\Sigma n_i^3 - 21\,\Sigma n_i^2 + 175N - 735c + 1624\,\Sigma \frac{1}{n_i} - 1764\,\Sigma \frac{1}{n_i^2}$$
$$+ 720\,\Sigma \frac{1}{n_i^3} \quad)$$

$$+ (56P_c(N) + 210P_i(N)\ )(\Sigma n_i^2 - 15N + 85c - 225\,\Sigma \frac{1}{n_i} + 274\,\Sigma \frac{1}{n_i^2}$$
$$- 120\,\Sigma \frac{1}{n_i^3} \quad)$$

$$+ 70(P_d(N) + 6P_j(N) + 8P_n(N)\ )(N - 10c + 35\,\Sigma \frac{1}{n_i} - 50\,\Sigma \frac{1}{n_i^2} + 24\,\Sigma \frac{1}{n_i^3}\ )$$

$$+ (56P_e(N) + 105P_k(N) + 840P_l(N) + 280P_p(N) + 420P_t(N)) \times$$

$$(c - 6\Sigma \frac{1}{n_i} + \quad \Sigma \frac{1}{n_i^2} - 6\Sigma \frac{1}{n_i^3})$$

$$+ (28P_f(N) + 210P_m(N) + 280P_q(N) + 280P_r(N) + 168P_u(N)) \times$$

$$(\Sigma \frac{1}{n_i} - 3\Sigma \frac{1}{n_i^2} + 2\Sigma \frac{1}{n_i^3})$$

$$+ (8P_g(N) + 35P_s(N) + 56P_v(N) + 28P_w(N))(\Sigma \frac{1}{n_i^2} - \Sigma \frac{1}{n_i^3})$$

$$+ \; P_h(N)\,\Sigma \frac{1}{n_i^3}$$

The polynomials P(N) are defined below.

$$P_a(N) = \frac{(N+1)}{34560}(135N^7 - 315N^6 - 1575N^5 + 735N^4 + 5320N^3 + 2820N^2 - 1936N - 1152)$$

$$P_b(N) = \frac{(N+1)}{120960}(630N^7 - 945N^6 - 6615N^5 + 595N^4 + 18665N^3 + 11438N^2 - 6200N - 4032)$$

$$P_c(N) = \frac{(N+1)}{40320}(315N^7 - 105N^6 - 2625N^5 - 1035N^4 + 5694N^3 + 4276N^2 - 1816N - 1344)$$

$$P_d(N) = \frac{(N+1)}{50400}(630N^7 + 525N^6 - 3495N^5 - 3216N^4 + 5891N^3 + 5799N^2 - 1934N - 1680)$$

$$P_e(N) = \frac{(N+1)}{10080}(210N^7 + 390N^6 - 555N^5 - 1020N^4 + 855N^3 + 1260N^2 - 300N - 336)$$

$$P_f(N) = \frac{(N+1)}{2520}(90N^7 + 240N^6 - 10N^5 - 335N^4 + 100N^3 + 340N^2 - 47N - 84)$$

$$P_g(N) = \frac{(N+1)}{720}(45N^7 + 145N^6 + 75N^5 - 125N^4 - 30N^3 + 106N^2 - 24)$$

$$P_h(N) = \frac{(N+1)}{90}(10N^7 + 35N^6 + 25N^5 - 25N^4 - 17N^3 + 17N^2 + 3N - 3)$$

$$P_i(N) = \frac{(N+1)}{60480}(420N^7 - 336N^6 - 3913N^5 - 872N^4 + 9337N^3 + 6632N^2 - 2772N - 2016)$$

$$P_j(N) = \frac{(N+1)}{453600}(4200N^7 - 980N^6 - 34590N^5 - 17129N^4 + 70104N^3 + 57961N^2 - 18006N - 15120)$$

$$P_k(N) = \frac{(N+1)}{226800}(2800N^7 + 560N^6 - 20504N^5 - 14920N^4 + 35455N^3 + 33935N^2 - 7626N - 7560)$$

$$P_l(N) = \frac{(N+1)}{151200}(2100N^7 + 1260N^6 - 13347N^5 - 12230N^4 + 20825N^3 + 22220N^2 - 4628N - 5040)$$

$$P_m(N) = \frac{(N+1)}{37800}(840N^7 + 1188N^6 - 3514N^5 - 5225N^4 + 4030N^3 + 6272N^2 - 801N - 1260)$$

$$P_n(N) = \frac{(N+1)}{100800}(1050N^7 + 210N^6 - 7590N^5 - 5052N^4 + 14032N^3 + 12498N^2 - 3868N - 3360)$$

$$P_p(N) = \frac{(N+1)}{100800}(1575N^7 + 1545N^6 - 8451N^5 - 9525N^4 + 11880N^3 + 14280N^2 - 2904N - 3360)$$

$$P_q(N) = \frac{(N+1)}{10080}(210N^7 + 255N^6 - 1015N^5 - 1385N^4 + 1165N^3 + 1750N^2 - 188N - 336)$$

$$P_r(N) = \frac{(N+1)}{3600}(90N^7 + 156N^6 - 298N^5 - 560N^4 + 280N^3 + 574N^2 - 62N - 120)$$

$$P_s(N) = \frac{(N+1)}{900}(36N^7 + 80N^6 - 69N^5 - 220N^4 + 19N^3 + 170N^2 - N - 30)$$

$$P_t(N) = \frac{(N+1)}{75600}(1260N^7 + 1500N^6 - 5937N^5 - 7270N^4 + 8350N^3 + 10315N^2 - 2278N - 2520)$$

$$P_u(N) = \frac{(N+1)}{10080}(280N^7 + 570N^6 - 630N^5 - 1490N^4 + 710N^3 + 1540N^2 - 188N - 336)$$

$$P_v(N) = \frac{(N+1)}{720}(30N^7 + 70N^6 - 45N^5 - 170N^4 + 15N^3 + 136N^2 - 24)$$

$$P_w(N) = \frac{(N+1)}{1260}(60N^7 + 160N^6 - 15N^5 - 260N^4 + 20N^3 + 223N^2 - 5N - 42)$$

To obtain the second expectation in (4) we first consider $E\left(\frac{R_i^6 \, R_j^2}{n_i^3 \, n_j}\right)$.

Now, $$E(R_i^6 R_j^2) = \sum_{i_1=1}^{n_i} \sum_{i_2=1}^{n_i} \sum_{i_3=1}^{n_i} \sum_{i_4=1}^{n_i} \sum_{i_5=1}^{n_i} \sum_{i_6=1}^{n_i} \sum_{j_1=1}^{n_j} \sum_{j_2=1}^{n_j} E(X_{i_1}^{(i)} X_{i_2}^{(i)} X_{i_3}^{(i)} X_{i_4}^{(i)} X_{i_5}^{(i)} X_{i_6}^{(i)} X_{j_1}^{(j)} X_{j_2}^{(j)})$$

$$= n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)n_j(n_j - 1) \, E(X_1^{(i)} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_5^{(i)} X_6^{(i)} X_1^{(j)} X_2^{(j)})$$

$$+ \; n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)n_j \, E(X_1^{(i)} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_5^{(i)} X_6^{(i)} X_1^{(j)^2})$$

$$+ \; 15n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)n_j(n_j - 1) \, E(X_1^{(i)^2} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_5^{(i)} X_1^{(j)} X_2^{(j)})$$

$$+ \ 15n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)n_j \\ E(X_1^{(i)^2} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_5^{(i)} X_1^{(j)^2})$$

$$+ \ 20n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1) \\ E(X_1^{(i)^3} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_1^{(j)} X_2^{(j)})$$

$$+ \ 20n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j \ E(X_1^{(i)^3} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_1^{(j)^2})$$

$$+ \ 15n_i(n_i - 1)(n_i - 2)n_j(n_j - 1) \ E(X_1^{(i)^4} X_2^{(i)} X_3^{(i)} X_1^{(j)} X_2^{(j)})$$

$$+ 15n_i(n_i - 1)(n_i - 2)n_j \ E(X_1^{(i)^4} X_2^{(i)} X_3^{(i)} X_1^{(j)^2})$$

$$+ 6n_i(n_i - 1)n_j(n_j - 1) \ E(X_1^{(i)^5} X_2^{(i)} X_1^{(j)} X_2^{(j)})$$

$$+ \ 6n_i(n_i - 1)n_j \ E(X_1^{(i)^5} X_2^{(i)} X_1^{(j)^2})$$

$$+ \ n_i n_j(n_j - 1) \ E(X_1^{(i)^6} X_1^{(j)} X_2^{(j)}) \ + \ n_i n_j \ E(X_1^{(i)^6} X_1^{(j)^2})$$

$$+ \ 45n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1) \ E(X_1^{(i)^2} X_2^{(i)^2} X_3^{(i)} X_4^{(i)} X_1^{(j)} X_2^{(j)}$$

$$+ \ 45n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j \ E(X_1^{(i)^2} X_2^{(i)^2} X_3^{(i)} X_4^{(i)} X_1^{(j)^2})$$

$$+ \ 15n_i(n_i - 1)(n_i - 2)n_j(n_j - 1) \ E(X_1^{(i)^2} X_2^{(i)^2} X_3^{(i)^2} X_1^{(j)} X_2^{(j)})$$

$$+ \ 15n_i(n_i - 1)(n_i - 2)n_j \ E(X_1^{(i)^2} X_2^{(i)^2} X_3^{(i)^2} X_1^{(j)^2})$$

$$+ \ 60n_i(n_i - 1)(n_i - 2)n_j(n_j - 1) \ E(X_1^{(i)^3} X_2^{(i)^2} X_3^{(i)} X_1^{(j)} X_2^{(j)})$$

$$+ \ 60n_i(n_i - 1)(n_i - 2)n_j \ E(X_1^{(i)^3} X_2^{(i)^2} X_3^{(i)} X_1^{(j)^2})$$

$$+ \ 15n_i(n_i - 1)n_j(n_j - 1) \ E(X_1^{(i)^4} X_2^{(i)^2} X_1^{(j)} X_2^{(j)})$$

$$+ \ 15n_i(n_i - 1)n_j \ E(X_1^{(i)^4} X_2^{(i)^2} X_1^{(j)^2})$$

$$+ \ 10n_i(n_i - 1)n_j(n_j - 1) \ E(X_1^{(i)^3} X_2^{(i)^3} X_1^{(j)} X_2^{(j)})$$

$$+ \ 10n_i(n_i - 1)n_j \ E(X_1^{(i)^3} X_2^{(i)^3} X_1^{(j)^2}) \quad .$$

So $E(R_i^6 \, R_j^2) \quad =$

$$\frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)(N - 7)} \ \times$$

$$\Sigma \ p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8$$

$$+ \ \frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)(n_i - 5)n_j}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)} \ \Sigma \ p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \ \frac{15n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)} \ \Sigma \ p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \frac{15n_i(n_i - 1)(n_i - 2)(n_i - 3)(n_i - 4)n_j}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma p_1^2p_2^2p_3p_4p_5p_6$$

$$+ \frac{20n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma p_1^3p_2p_3p_4p_5p_6$$

$$+ \frac{20\ n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma p_1^3p_2^2p_3p_4p_5$$

$$+ \frac{15n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma p_1^4p_2p_3p_4p_5$$

$$+ \frac{15n_i(n_i - 1)(n_i - 2)n_j}{N(N - 1)(N - 2)(N - 3)} \Sigma p_1^4p_2^2p_3p_4$$

$$+ \frac{6n_i(n_i - 1)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)} \Sigma p_1^5p_2p_3p_4$$

$$+ \frac{6n_i(n_i - 1)n_j}{N(N - 1)(N - 2)} \Sigma p_1^5p_2^2p_3 \quad + \frac{n_in_j(n_j - 1)}{N(N - 1)(N - 2)} \Sigma p_1^6p_2p_3$$

$$+ \frac{45n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma p_1^2p_2^2p_3p_4p_5p_6$$

$$+ \frac{45n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j}{N(N - 1)(N - 2)(N - 3)(N - 4)} \; \Sigma \; p_1^2p_2^2p_3^2p_4p_5$$

$$+ \frac{15n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \; \Sigma \; p_1^2p_2^2p_3^2p_4p_5$$

$$+ \frac{15n_i(n_i - 1)(n_i - 2)n_j}{N(N - 1)(N - 2)(N - 3)} \; \Sigma \; p_1^2p_2^2p_3^2p_4^2 \quad + \frac{n_in_j}{N(N - 1)} \; \Sigma \; p_1^6p_2^2$$

$$+ \frac{60n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \; \Sigma \; p_1^3p_2^2p_3p_4p_5$$

$$+ \frac{60n_i(n_i - 1)(n_i - 2)n_j}{N(N - 1)(N - 2)(N - 3)} \; \Sigma \; p_1^3p_2^2p_3^2p_4$$

$$+ \frac{15n_i(n_i - 1)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)} \; \Sigma \; p_1^4p_2^2p_3p_4$$

$$+ \frac{15n_i(n_i - 1)n_j}{N(N - 1)(N - 2)} \; \Sigma \; p_1^4p_2^2p_3^2 \quad + \frac{10n_i(n_i - 1)n_j}{N(N - 1)(N - 2)} \; \Sigma \; p_1^3p_2^3p_3^2$$

$$+ \frac{10n_i(n_i - 1)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)} \; \Sigma \; p_1^3p_2^3p_3p_4 \quad .$$

Hence, after performing the summations, we obtain

$$\sum_{i=1}^{c} \sum_{\substack{j=1 \\ i \neq j}}^{c} E\left(\frac{R_i^6 \, R_j^2}{n_i^3 \, n_j}\right) =$$

$$P_a(N)\Big((N + c + 1)(\Sigma n_i^3 - 15\,\Sigma n_i^2 + 85N - 225c + 274\,\Sigma \frac{1}{n_i} - 120\,\Sigma \frac{1}{n_i^2})$$

$$- (\Sigma n_i^4 - 15\,\Sigma n_i^3 + 85\,\Sigma n_i^2 - 225N + 274c - 20\,\Sigma \frac{1}{n_i}\,)\Big)$$

$$+ P_b(N)(c - 1)(\Sigma n_i^3 - 15\Sigma n_i^2 + 85N - 225c + 274\,\Sigma \frac{1}{n_i} - 20\,\Sigma \frac{1}{n_i^2})$$

$$+ 15P_b(N)\ (N - c + 1)(\Sigma n_i^2 - 10N + 35c - 50\,\Sigma \frac{1}{n_i} + 24\,\Sigma \frac{1}{n_i^2}$$

$$- (\Sigma n_i^3 - 10\,\Sigma n_i^2 + 35N - 50c + 24\,\Sigma \frac{1}{n_i})$$

$$+ 15P_i(N)(c - 1)(\Sigma n_i^2 - 10N + 35c - 50\,\Sigma \frac{1}{n_i} + 24\,\Sigma \frac{1}{n_i^2}$$

$$+ (20P_c(N) + 45P_i(N)\,)\Big((N - c + 1)(N - 6c + 11\,\Sigma \frac{1}{n_i} - 6\,\Sigma \frac{1}{n_i^2}\,)$$

$$- (\Sigma n_i^2 - 6N + 11c - 6\,\Sigma \frac{1}{n_i}\,)\Big)$$

$$+ (20P_n(N) + 45P_j(N)\,)(c - 1)(N - 6c + 11\,\Sigma \frac{1}{n_i}\ 6\,\Sigma \frac{1}{n_i^2})$$

$$+ 15(P_d(N) + P_j(N) + 4P_n(N)\,)\Big((N - c + 1)(c - 3\,\Sigma \frac{1}{n_i} + 2\,\Sigma \frac{1}{n_i^2})$$

$$- (N - 3c + 2\,\Sigma \frac{1}{n_i}\,)\Big)$$

$$+ \ 15(P_t(N) + P_k(N) + 4P_l(N))(c-1)(c - 3\Sigma \frac{1}{n_i} + 2\Sigma \frac{1}{n_i^2})$$

$$+ \ (6P_e(N) + 10P_p(N) + 15P_t(N))((N - c + 1)(\Sigma \frac{1}{n_i} - \Sigma \frac{1}{n_i^2}) - c + \Sigma \frac{1}{n_i})$$

$$+ \ (6P_u(N) + 10P_q(N) + 15P_m(N))(c-1)(\Sigma \frac{1}{n_i} - \Sigma \frac{1}{n_i^2})$$

$$+ \ P_f(N)((N - c + 1)\Sigma \frac{1}{n_i^2} - \Sigma \frac{1}{n_i}) \quad + \quad P_w(N)(c-1)\Sigma \frac{1}{n_i^2} \quad .$$

To obtain the third expectation in (4) we first consider $E(R_i^4 R_j^2 R_k^2)$.

Now $E(R_i^4 R_j^2 R_k^2) = \sum_{i_1=1}^{n_i} \sum_{i_2=1}^{n_i} \sum_{i_3=1}^{n_i} \sum_{i_4=1}^{n_i} \sum_{j_1=1}^{n_j} \sum_{j_2=1}^{n_j} \sum_{k_1=1}^{n_k} \sum_{k_2=1}^{n_k}$

$$E(X_{i_1}^{(i)} X_{i_2}^{(i)} X_{i_3}^{(i)} X_{i_4}^{(i)} X_{j_1}^{(j)} X_{j_2}^{(j)} X_{k_1}^{(k)} X_{k_2}^{(k)})$$

$$= \ n_i(n_i-1)(n_i-2)(n_i-3)n_j(n_j-1)n_k(n_k-1)$$

$$E(X_1^{(i)} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_1^{(j)} X_2^{(j)} X_1^{(k)} X_2^{(k)})$$

$$+ \ n_i(n_i-1)(n_i-2)(n_i-3)n_j(n_j-1)n_k$$

$$E(X_1^{(i)} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_1^{(j)} X_2^{(j)} X_1^{(k)^2})$$

$$+ \ n_i(n_i-1)(n_i-2)(n_i-3)n_j n_k(n_k-1)$$

$$E(X_1^{(i)} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_1^{(j)^2} X_1^{(k)} X_2^{(k)})$$

$$+ \; n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j n_k \; E(X_1^{(i)} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_1^{(j)^2} X_1^{(k)^2})$$

$$+ \; 6n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)n_k(n_k - 1)$$

$$E(X_1^{(i)^2} X_2^{(i)} X_3^{(i)} X_1^{(j)} X_2^{(j)} X_1^{(k)} X_2^{(k)})$$

$$+ \; 6n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)n_k \; E(X_1^{(i)^2} X_2^{(i)} X_3^{(i)} X_1^{(j)} X_2^{(j)} X_1^{(k)^2})$$

$$+ \; 6n_i(n_i - 1)(n_i - 2)n_j n_k(n_k - 1) \; E(X_1^{(i)^2} X_2^{(i)} X_3^{(i)} X_1^{(j)^2} X_1^{(k)} X_2^{(k)})$$

$$+ \; 6n_i(n_i - 1)(n_i - 2)n_j n_k \; E(X_1^{(i)^2} X_2^{(i)} X_3^{(i)} X_1^{(j)^2} X_1^{(k)^2})$$

$$+ \; 4n_i(n_i - 1)n_j(n_j - 1)n_k(n_k - 1) \; E(X_1^{(i)^3} X_2^{(i)} X_1^{(j)} X_2^{(j)} X_1^{(k)} X_2^{(k)})$$

$$+ \; 4n_i(n_i - 1)n_j(n_j - 1)n_k \; E(X_1^{(i)^3} X_2^{(i)} X_1^{(j)} X_2^{(j)} X_1^{(k)^2})$$

$$+ \; 4n_i(n_i - 1)n_j n_k(n_k - 1) \; E(X_1^{(i)^3} X_2^{(i)} X_1^{(j)^2} X_1^{(k)} X_2^{(k)})$$

$$+ \; 4n_i(n_i - 1)n_j n_k \; E(X_1^{(i)^3} X_2^{(i)} X_1^{(j)^2} X_1^{(k)^2})$$

$$+ \; n_i n_j(n_j - 1)n_k(n_k - 1) \; E(X_1^{(i)^4} X_1^{(j)} X_2^{(j)} X_1^{(k)} X_2^{(k)})$$

$$+ \; n_i n_j(n_j - 1)n_k \; E(X_1^{(i)^4} X_1^{(j)} X_2^{(j)} X_1^{(k)^2})$$

$$+ \; n_i n_j n_k(n_k - 1) \; E(X^{(i)^4} X^{(j)^2} X^{(k)} X^{(k)})$$

$$+ \; n_i n_j n_k \, E(X_1^{(i)^4} X_1^{(j)^2} X_1^{(k)^2})$$

$$+ \; 3n_i(n_i - 1)n_j(n_j - 1)n_k(n_k - 1) \; E(X_1^{(i)^2} X_2^{(i)^2} X_1^{(j)} X_2^{(j)} X_1^{(k)} X_2^{(k)})$$

$$+ \; 3n_i(n_i - 1)n_j(n_j - 1)n_k \; E(X_1^{(i)^2} X_2^{(i)^2} X_1^{(j)} X_2^{(j)} X_1^{(k)^2})$$

$$+ \; 3n_i(n_i - 1)n_j n_k(n_k - 1) \; E(X_1^{(i)^2} X_2^{(i)^2} X_1^{(j)^2} X_1^{(k)} X_2^{(k)})$$

$$+ \; 3n_i(n_i - 1)n_j n_k \; E(X_1^{(i)^2} X_2^{(i)^2} X_1^{(j)^2} X_1^{(k)^2}) \quad .$$

So $E(\,R_i^4 \, R_j^2 \, R_k^2\,) \quad =$

$$\frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)n_k(n_k - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)(N - 7)} \quad \times$$

$$\Sigma \; p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8$$

$$+ \; \frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)n_k}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)} \; \Sigma \; p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \; \frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j n_k(n_k - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)} \; \Sigma \; p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \; \frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j n_k}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \; \Sigma \; p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{6n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)n_k(n_k - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)} \Sigma\ p_1^2p_2p_3p_4p_5p_6p_7$$

$$+ \frac{6n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)n_k}{N(N- 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma\ p_1^2p_2^2p_3p_4p_5p_6$$

$$+ \frac{6n_i(n_i - 1)(n_i - 2)n_jn_k(n_k - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma\ p_1^2p_2^2p_3p_4p_5p_6$$

$$+ \frac{6n_i(n_i - 1)(n_i - 2)n_jn_k}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma\ p_1^2p_2^2p_3^2p_4p_5$$

$$+ \frac{4n_i(n_i - 1)n_j(n_j - 1)n_k(n_k - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma\ p_1^3p_2p_3p_4p_5p_6$$

$$+ \frac{4n_i(n_i - 1)n_j(n_j - 1)n_k}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma\ p_1^3p_2^2p_3p_4p_5$$

$$+ \frac{4n_i(n_i - 1)n_jn_k(n_k - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma\ p_1^3p_2^2p_3p_4p_5$$

$$+ \frac{4n_i(n_i - 1)n_jn_k}{N(N - 1)(N - 2)(N - 3)} \Sigma\ p_1^3p_2^2p_3^2p_4$$

$$+ \frac{n_i n_j (n_j - 1) n_k (n_k - 1)}{N(N-1)(N-2)(N-3)(N-4)} \Sigma\, p_1^4 p_2 p_3 p_4 p_5$$

$$+ \frac{n_i n_j (n_j - 1) n_k}{N(N-1)(N-2)(N-3)} \Sigma\, p_1^4 p_2^2 p_3 p_4$$

$$+ \frac{n_i n_j n_k (n_k - 1)}{N(N-1)(N-2)(N-3)} \Sigma\, p_1^4 p_2^2 p_3 p_4 \quad + \frac{n_i n_j n_k}{N(N-1)(N-2)} \Sigma\, p_1^4 p_2^2 p_3^2$$

$$+ \frac{3 n_i (n_i - 1) n_j (n_j - 1) n_k (n_k - 1)}{N(N-1)(N-2)(N-3)(N-4)(N-5)} \Sigma\, p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{3 n_i (n_i - 1) n_j (n_j - 1) n_k}{N(N-1)(N-2)(N-3)(N-4)} \Sigma\, p_1^2 p_2^2 p_3^2 p_4 p_5$$

$$+ \frac{3 n_i (n_i - 1) n_j n_k (n_k - 1)}{N(N-1)(N-2)(N-3)(N-4)} \Sigma\, p_1^2 p_2^2 p_3^2 p_4 p_5$$

$$+ \frac{3 n_i (n_i - 1) n_j n_k}{N(N-1)(N-2)(N-3)} \Sigma\, p_1^2 p_2^2 p_3^2 p_4^2 \quad .$$

Performing the summations produces

$$\sum_{\substack{i=1\\ i\neq j}}^{c} \sum_{\substack{j=1\\ i\neq j}}^{c} \sum_{\substack{k=1\\ j\neq k}}^{c} E\left(\frac{R_i^4 R_j^2 R_k^2}{n_i^2 n_j n_k}\right) =$$

$$P_a(N)\Big[(\Sigma n_i^2 - 6N + 11c - 6\,\Sigma \frac{1}{n_i})(N - c + 1)(N - c + 2)$$

$$+ 2(\Sigma n_i^4 - 6\,\Sigma n_i^3 + 11\,\Sigma n_i^2 - 6N)$$

$$- 2(\Sigma n_i^3 - 6\,\Sigma n_i^2 + 11N - 6c)(N - c + 2)$$

$$- (\Sigma n_i^2 - 6N + 11c - 6\,\Sigma \frac{1}{n_i})(\Sigma n^2 - N)\Big]$$

$$+ 2P_b(N)(c - 2)\Big((N - c + 1)(\Sigma n_i^2 - 6N + 11c - 6\,\Sigma \frac{1}{n_i})$$

$$- (\Sigma n_i^3 - 6\,\Sigma n_i^2 + 11N - 6c)\Big)$$

$$+ P_i(N)(c - 1)(c - 2)(\Sigma n_i^2 - 6N + 11c - 6\,\Sigma \frac{1}{n_i})$$

$$+ 6P_b(N)\Big((N - c + 1)(N - c + 2)(N - 3c + 2\,\Sigma \frac{1}{n_i})$$

$$+ 2(\Sigma n_i^3 - 3\,\Sigma n_i^2 + 2N) - 2(\Sigma n_i^2 - 3N + 2c)(N - c + 2)$$

$$- (N - 3c + 2\,\Sigma \frac{1}{n_i})(\Sigma n_i^2 - N)\Big)$$

$$+ 12P_i(N)(c - 2)\Big((N - c + 1)(N - 3c + 2\,\Sigma \frac{1}{n_i}) - (\Sigma n_i^2 - 3N + 2c)\Big)$$

$$+ 6P_j(N)(c - 1)(c - 2)(N - 3c + 2\,\Sigma \frac{1}{n_i})$$

$$+ \; 4P_c(N)\Big((c - \Sigma \tfrac{1}{n_i})(N - c + 1)(N - c + 2) + 2(\Sigma n_i^2 - N)$$

$$- 2(N - c)(N - c + 2) - (c - \Sigma \tfrac{1}{n_i})(\Sigma n_i^2 - N)\Big)$$

$$+ \; 8P_n(N) \; (N - c + 1)(c - \Sigma \tfrac{1}{n_i}) - N + c)$$

$$+ \; 4P_l(N)(c - 1)(c - 2)(c - \Sigma \tfrac{1}{n_i})$$

$$+ \; P_d(N)\Big(2N + \Sigma \tfrac{1}{n_i}(N - c + 1)(N - c + 2) - 2c(N - c + 2) - \Sigma \tfrac{1}{n_i}(\Sigma n_i^2 - N)\Big)$$

$$+ \; 2P_t(N)(\Sigma \tfrac{1}{n_i}(N - c + 1) - c)$$

$$+ \; P_m(N)(c - 1)(c - 2)\Sigma \tfrac{1}{n_i}$$

$$+ \; 3P_i(N)\Big((c - \Sigma \tfrac{1}{n_i})(N - c + 1)(N - c + 2) + 2(\Sigma n_i^2 - N)$$

$$- 2(N - c)(N - c + 2) - (c - \Sigma \tfrac{1}{n_i})(\Sigma n_i^2 - N)\Big)$$

$$+ \; 6P_j(N)(c - 2)\Big((c - \Sigma \tfrac{1}{n_i})(N - c + 1) - N + c\Big)$$

$$+ \; 3P_k(N)(c - 1)(c - 2)(c - \Sigma \tfrac{1}{n_i}) \quad .$$

To obtain the fourth expectation in (4) we first consider $E(R_i^4 R_j^4)$.

$$\text{Now} \quad E(R_i^4 R_j^4) = \sum_{i_1=1}^{n_i} \sum_{i_2=1}^{n_i} \sum_{i_3=1}^{n_i} \sum_{i_4=1}^{n_i} \sum_{j_1=1}^{n_j} \sum_{j_2=1}^{n_j} \sum_{j_3=1}^{n_j} \sum_{j_4=1}^{n_j}$$

$$E(X_{i_1}^{(i)} X_{i_2}^{(i)} X_{i_3}^{(i)} X_{i_4}^{(i)} X_{j_1}^{(j)} X_{j_2}^{(j)} X_{j_3}^{(j)} X_{j_4}^{(j)})$$

By symmetry we obtain $E(R_i^4 R_j^4) =$

$$n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)(n_j - 2)(n_j - 3)$$
$$E(X_1^{(i)}X_2^{(i)}X_3^{(i)}X_4^{(i)}X_1^{(j)}X_2^{(j)}X_3^{(j)}X_4^{(j)})$$

$$+ \ 6n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)(n_j - 2)$$
$$E(X_1^{(i)}X_2^{(i)}X_3^{(i)}X_4^{(i)}X_1^{(j)^2}X_2^{(j)}X_3^{(j)})$$

$$+ \ 4n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1) \ E(X_1^{(i)}X_2^{(i)}X_3^{(i)}X_4^{(i)}X_1^{(j)^3}X_2^{(j)})$$

$$+ \ n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j \ E(X_1^{(i)}X_2^{(i)}X_3^{(i)}X_4^{(i)}X_1^{(j)^4})$$

$$+ \ 6n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)(n_j - 2)(n_j - 3)$$
$$E(X_1^{(i)^2}X_2^{(i)}X_3^{(i)}X_1^{(j)}X_2^{(j)}X_3^{(j)}X_4^{(j)})$$

$$+ \ 36n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)(n_j - 2) \ E(X_1^{(i)^2}X_2^{(i)}X_3^{(i)}X_1^{(j)^2}X_2^{(j)}X_3^{(j)})$$

$$+ \ 24n_i(n_i - 1)(n_i - 2)n_j(n_j - 1) \ E(X_1^{(i)^2}X_2^{(i)}X_3^{(i)}X_1^{(j)^3}X_2^{(j)})$$

$$+ \ 6n_i(n_i - 1)(n_i - 2)n_j \ E(X_1^{(i)^2}X_2^{(i)}X_3^{(i)}X_1^{(j)^4})$$

$$+ \ 4n_i(n_i - 1)n_j(n_j - 1)(n_j - 2)(n_j - 3) \ E(X_1^{(i)^3}X_2^{(i)}X_1^{(j)}X_2^{(j)}X_3^{(j)}X_4^{(j)})$$

$$+ \ 24n_i(n_i - 1)n_j(n_j - 1)(n_j - 2) \ E(X_1^{(i)^3}X_2^{(i)}X_1^{(j)^2}X_2^{(j)}X_3^{(j)})$$

$$+ \; 16n_i(n_i - 1)n_j(n_j - 1) \; E(X_1^{(i)^3} X_2^{(i)} X_1^{(j)^3} X_2^{(j)})$$

$$+ \; 4n_i(n_i - 1)n_j \; E(X_1^{(i)^3} X_2^{(i)} X_1^{(j)^4})$$

$$+ \; n_i n_j(n_j - 1)(n_j - 2)(n_j - 3) \; E(X_1^{(i)^4} X_1^{(j)} X_2^{(j)} X_3^{(j)} X_4^{(j)})$$

$$+ \; 6n_i n_j(n_j - 1)(n_j - 2) \; E(X_1^{(i)^4} X_1^{(j)^2} X_2^{(j)} X_3^{(j)})$$

$$+ \; 4n_i n_j(n_j - 1) \; E(X_1^{(i)^4} X_1^{(j)^3} X_2^{(j)}) \; + \; n_i n_j \; E(X^{(i)^4} X^{(j)^4})$$

$$+ \; 3n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1) \; E(X_1^{(i)} X_2^{(i)} X_3^{(i)} X_4^{(i)} X_1^{(j)^2} X_2^{(j)^2})$$

$$+ \; 36n_i(n_i - 1)(n_i - 2)n_j(n_j - 1) \; E(X_1^{(i)^2} X_2^{(i)} X_3^{(i)} X_1^{(j)^2} X_2^{(j)^2})$$

$$+ \; 24n_i(n_i - 1)n_j(n_j - 1) \; E(X_1^{(i)^3} X_2^{(i)} X_1^{(j)^2} X_2^{(j)})$$

$$+ \; 6n_i n_j(n_j - 1) \; E(X_1^{(i)^4} X_1^{(j)^2} X_2^{(j)^2})$$

$$+ \; 3n_i(n_i - 1)n_j(n_j - 1)(n_j - 2)(n_j - 3) \; E(X_1^{(i)^2} X_2^{(i)^2} X_1^{(j)} X_2^{(j)} X_3^{(j)} X_4^{(j)})$$

$$+ \; 9n_i(n_i - 1)n_j(n_j - 1) \; E(X_1^{(i)^2} X_2^{(i)^2} X_1^{(j)^2} X_2^{(j)^2}) \quad .$$

Hence $E(R_i^4 R_j^4) \quad =$

$$\frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)(n_j - 2)(n_j - 3)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)(N - 7)} \quad x$$

$$\Sigma \; p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8$$

$$+ \frac{6n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)(n_j - 2)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)} \Sigma p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \frac{4n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma p_1^3 p_2 p_3 p_4 p_5 p_6$$

$$+ \frac{n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma p_1^4 p_2 p_3 p_4 p_5$$

$$+ \frac{6n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)(n_j - 2)(n_j - 3)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)(N - 6)} \Sigma p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \frac{36n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)(n_j - 2)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{24n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma p_1^3 p_2^2 p_3 p_4 p_5$$

$$+ \frac{6n_i(n_i - 1)(n_i - 2)n_j}{N(N - 1)(N - 2)(N - 3)} \Sigma p_1^4 p_2^2 p_3 p_4$$

$$+ \frac{4n_i(n_i - 1)n_j(n_j - 1)(n_j - 2)(n_j - 3)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma p_1^3 p_2 p_3 p_4 p_5 p_6$$

$$+ \frac{24n_i(n_i - 1)n_j(n_j - 1)(n_j - 2)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma p_1^3 p_2^2 p_3 p_4 p_5$$

$$+ \frac{16n_i(n_i - 1)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)} \Sigma\, p_1^3 p_2^3 p_3 p_4$$

$$+ \frac{4n_i(n_i - 1)n_j}{N(N - 1)(N - 2)} \Sigma\, p_1^4 p_2^3 p_3$$

$$+ \frac{n_i n_j(n_j - 1)(n_j - 2)(n_j - 3)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma\, p_1^4 p_2 p_3 p_4 p_5$$

$$+ \frac{6n_i n_j(n_j - 1)(n_j - 2)}{N(N - 1)(N - 2)(N - 3)} \Sigma\, p_1^4 p_2^2 p_3 p_4$$

$$+ \frac{4n_i n_j(n_j - 1)}{N(N - 1)(N - 2)} \Sigma\, p_1^4 p_2^3 p_3 \quad + \frac{n_i n_j}{N(N - 1)} \Sigma\, p_1^4 p_2^4$$

$$+ \frac{3n_i(n_i - 1)(n_i - 2)(n_i - 3)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \Sigma\, p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{36n_i(n_i - 1)(n_i - 2)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)(N - 4)} \Sigma\, p_1^2 p_2^2 p_3^2 p_4 p_5$$

$$+ \frac{24n_i(n_i - 1)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)} \Sigma\, p_1^3 p_2^2 p_3^2 p_4$$

$$+ \frac{6n_i n_j(n_j - 1)}{N(N - 1)(N - 2)} \Sigma\, p_1^4 p_2^2 p_3^2$$

$$+ \frac{3n_i(n_i - 1)n_j(n_j - 1)(n_j - 2)(n_j - 3)}{N(N - 1)(N - 2)(N - 3)(N - 4)(N - 5)} \quad \Sigma\, p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{9n_i(n_i - 1)n_j(n_j - 1)}{N(N - 1)(N - 2)(N - 3)} \quad \Sigma\, p_1^2 p_2^2 p_3^2 p_4^2 \quad .$$

After performing the various summations we obtain

$$\sum_{\substack{i=1 \\ i \neq j}}^{c} \sum_{j=1}^{c} E\left(\frac{R_i^4 R_j^4}{n_i^2 n_j^2}\right) =$$

$$P_a(N)\Big((\Sigma n_i^2 - 6N + 11c - 6\,\Sigma \frac{1}{n_i})(\Sigma n_i^2 - 6N + 11(c - 1) - 6\,\Sigma \frac{1}{n_i})$$
$$- (\Sigma n_i^4 + 47\,\Sigma n_i^2 + 36\,\Sigma \frac{1}{n_i^2} - 125\,\Sigma n_i^3 - 78N - 66\,\Sigma \frac{1}{n_i} + 72c\,)\Big)$$

$$+ 12P_b(N)\Big(\Sigma n_i^2(9 + N - 3c + 2\,\Sigma \frac{1}{n_i}) - \Sigma n_i^3 - N(6N - 18c + 3 + 12\,\Sigma \frac{1}{n_i})$$
$$+ c(8 + 11N - 33(c - 1) + 22\,\Sigma \frac{1}{n_i})$$
$$- \Sigma \frac{1}{n_i}(40 + 6N - 18c + 12\,\Sigma \frac{1}{n_i}) + 12\,\Sigma \frac{1}{n_i^2}\Big)$$

$$+ (8P_c(N) + 6P_1(N))\Big((\Sigma n_i^2 - 6N + 11c - 6\,\Sigma \frac{1}{n_i})(c - 1 - \Sigma \frac{1}{n_i})$$
$$+ N - 6c + 11\,\Sigma \frac{1}{n_i} - 6\,\Sigma \frac{1}{n_i^2}\Big)$$

$$+ 2P_d(N)(\Sigma n_i^2 \Sigma \frac{1}{n_i} - N(1 + 6\,\Sigma \frac{1}{n_i}) + c(6 + 11\,\Sigma \frac{1}{n_i}) - \Sigma \frac{1}{n_i}(6\,\Sigma \frac{1}{n_i} + 11)$$
$$+ 6\,\Sigma \frac{1}{n_i^2})$$

$$+ \ 36P_i(N)\Big((N - 3c + 2\,\Sigma \frac{1}{n_i})(N - 3c + 3 + 2\,\Sigma \frac{1}{n_i})$$

$$- 2(c - 3\,\Sigma \frac{1}{n_i} + 2\,\Sigma \frac{1}{n_i^2}) - \Sigma n^2 + 3N - 2c\Big)$$

$$+ \ 48P_n(N)\Big(N(c - 1 - \Sigma \frac{1}{n_i}) + c(4 - 3c + 3\,\Sigma \frac{1}{n_i})$$

$$+ \Sigma \frac{1}{n_i}(2c - 5 - 2\,\Sigma \frac{1}{n_i}) + 2\,\Sigma \frac{1}{n_i^2}\Big)$$

$$+ 12P_t(N)\Big(N\,\Sigma \frac{1}{n_i} - c(1 + 3\,\Sigma \frac{1}{n_i}) + \Sigma \frac{1}{n_i}(3 + 2\,\Sigma \frac{1}{n_i}) - 2\,\Sigma \frac{1}{n_i^2}\Big)$$

$$+ \ 16P_p(N)\Big(c(c - 1 - \Sigma \frac{1}{n_i}) + \Sigma \frac{1}{n_i}(2 - c + \Sigma \frac{1}{n_i}) - \Sigma \frac{1}{n_i^2}\Big)$$

$$+ \ (8P_r(N) + 6P_m(N))\Big(\Sigma \frac{1}{n_i}(c - 1 - \Sigma \frac{1}{n_i}) + \Sigma \frac{1}{n_i^2}\Big)$$

$$+ \ P_s(N)\Big((\Sigma \frac{1}{n_i})^2 - \Sigma \frac{1}{n_i^2}\Big)$$

$$+ \ 36P_j(N)\Big((N - 3c + 2\,\Sigma \frac{1}{n_i})(c - 1 - \Sigma \frac{1}{n_i}) + c - 3\,\Sigma \frac{1}{n_i} + 2\,\Sigma \frac{1}{n_i^2}\Big)$$

$$+ \ (24P_l(N) + 9P_k(N))\Big(\Sigma \frac{1}{n_i} - \Sigma \frac{1}{n_i^2} + (c - \Sigma \frac{1}{n_i})(c - 1 - \Sigma \frac{1}{n_i})\Big)$$

The final term of (4) is obtained by first considering $E(R_i^2\,R_j^2\,R_k^2\,R_l^2)$.

$$\text{Now}\quad E(R_i^2\,R_j^2\,R_k^2\,R_l^2) = \sum_{i_1=1}^{n_i}\sum_{i_2=1}^{n_i}\sum_{j_1=1}^{n_j}\sum_{j_2=1}^{n_j}\sum_{k_1=1}^{n_k}\sum_{k_2=1}^{n_k}\sum_{l_1=1}^{n_l}\sum_{l_2=1}^{n_l}$$

$$E(X_{i_1}^{(i)}X_{i_2}^{(i)}X_{j_1}^{(j)}X_{j_2}^{(j)}X_{k_1}^{(k)}X_{k_2}^{(k)}X_{l_1}^{(l)}X_{l_2}^{(l)})$$

So $\quad E(R_i^2 \, R_j^2 \, R_k^2 \, R_l^2) \quad =$

$$n_i(n_i - 1)n_j(n_j - 1)n_k(n_k - 1)n_l(n_l - 1)$$

$$E(X_1^{(i)}X_2^{(i)}X_1^{(j)}X_2^{(j)}X_1^{(k)}X_2^{(k)}X_1^{(l)}X_2^{(l)})$$

$$+ \; n_i(n_i - 1)n_j(n_j - 1)n_k(n_k - 1)n_l \; E(X_1^{(i)}X_2^{(i)}X_1^{(j)}X_2^{(j)}X_1^{(k)}X_2^{(k)}X_1^{(l)^2})$$

$$+ \; n_i(n_i - 1)n_j(n_j - 1)n_k n_l(n_l - 1) \; E(X_1^{(i)}X_2^{(i)}X_1^{(j)}X_2^{(j)}X_1^{(k)^2}X_1^{(l)}X_2^{(l)})$$

$$+ \; n_i(n_i - 1)n_j n_k(n_k - 1)n_l(n_l - 1) \; E(X_1^{(i)}X_2^{(i)}X_1^{(j)^2}X_1^{(k)}X_2^{(k)}X_1^{(l)}X_2^{(l)})$$

$$+ \; n_i n_j(n_j - 1)n_k(n_k - 1)n_l(n_l - 1) \; E(X_1^{(i)^2}X_1^{(j)}X_2^{(j)}X_1^{(k)}X_2^{(k)}X_1^{(l)}X_2^{(l)})$$

$$+ \; n_i(n_i - 1)n_j(n_j - 1)n_k n_l \; E(X_1^{(i)}X_2^{(i)}X_1^{(j)}X_2^{(j)}X_1^{(k)^2}X_1^{(l)^2})$$

$$+ \; n_i(n_i - 1)n_j n_k(n_k - 1)n_l \; E(X_1^{(i)}X_2^{(i)}X_1^{(j)^2}X_1^{(k)}X_2^{(k)}X_1^{(l)^2})$$

$$+ \; n_i(n_i - 1)n_j n_k n_l(n_l - 1) \; E(X_1^{(i)}X_2^{(i)}X_1^{(j)^2}X_1^{(k)^2}X_1^{(l)}X_2^{(l)})$$

$$+ \; n_i(n_i - 1)n_j n_k n_l \; E(X_1^{(i)}X_2^{(i)}X_1^{(j)^2}X_1^{(k)^2}X_1^{(l)^2})$$

$$+ \; n_i n_j(n_j - 1)n_k(n_k - 1)n_l \; E(X_1^{(i)^2}X_1^{(j)}X_2^{(j)}X_1^{(k)}X_2^{(k)}X_1^{(l)^2})$$

$$+ \; n_i n_j(n_j - 1)n_k n_l(n_l - 1) \; E(X^{(i)^2}X^{(j)}X^{(j)}X^{(k)^2}X^{(l)}X^{(l)})$$

$$+ \ n_i n_j (n_j - 1) n_k n_l \ E(X_1^{(i)^2} X_1^{(j)} X_2^{(j)} X_1^{(k)^2} X_1^{(l)^2})$$

$$+ \ n_i n_j n_k (n_k - 1) n_l (n_l - 1) \ E(X_1^{(i)^2} X_1^{(j)^2} X_1^{(k)} X_2^{(k)} X_1^{(l)} X_2^{(l)})$$

$$+ \ n_i n_j n_k (n_k - 1) n_l \ E(X_1^{(i)^2} X_1^{(j)^2} X_1^{(k)} X_2^{(k)} X_1^{(l)^2})$$

$$+ \ n_i n_j n_k n_l (n_l - 1) \ E(X_1^{(i)^2} X_1^{(j)^2} X_1^{(k)^2} X_1^{(l)} X_2^{(l)})$$

$$+ \ n_i n_j n_k n_l \ E(X_1^{(i)^2} X_1^{(j)^2} X_1^{(k)^2} X_1^{(l)^2})$$

Hence $E(R_i^2 R_j^2 R_k^2 R_l^2) =$

$$\frac{n_i(n_i - 1)n_j(n_j - 1)n_k(n_k - 1)n_l(n_l - 1)}{N(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)(N-7)} \times$$

$$\Sigma \ p_1 p_2 p_3 p_4 p_5 p_6 p_7 p_8$$

$$+ \frac{n_i(n_i - 1)n_j(n_j - 1)n_k(n_k - 1)n_l}{N(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)} \ \Sigma \ p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \frac{n_i(n_i - 1)n_j(n_j - 1)n_k n_l(n_l - 1)}{N(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)} \ \Sigma \ p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \frac{n_i(n_i - 1)n_j n_k(n_k - 1)n_l(n_l - 1)}{N(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)} \ \Sigma \ p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \frac{n_i n_j(n_j - 1)n_k(n_k - 1)n_l(n_l - 1)}{N(N-1)(N-2)(N-3)(N-4)(N-5)(N-6)} \Sigma\, p_1^2 p_2 p_3 p_4 p_5 p_6 p_7$$

$$+ \frac{n_i(n_i - 1)n_j(n_j - 1)n_k n_l}{N(N-1)(N-2)(N-3)(N-4)(N-5)} \Sigma\, p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{n_i(n_i - 1)n_j n_k(n_k - 1)n_l}{N(N-1)(N-2)(N-3)(N-4)(N-5)} \Sigma\, p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{n_i(n_i - 1)n_j n_k n_l(n_l - 1)}{N(N-1)(N-2)(N-3)(N-4)(N-5)} \Sigma\, p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{n_i n_j(n_j - 1)n_k(n_k - 1)n_l}{N(N-1)(N-2)(N-3)(N-4)(N-5)} \Sigma\, p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{n_i n_j(n_j - 1)n_k n_l(n_l - 1)}{N(N-1)(N-2)(N-3)(N-4)(N-5)} \Sigma\, p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{n_i n_j n_k(n_k - 1)n_l(n_l - 1)}{N(N-1)(N-2)(N-3)(N-4)(N-5)} \Sigma\, p_1^2 p_2^2 p_3 p_4 p_5 p_6$$

$$+ \frac{n_i(n_i - 1)n_j n_k n_l}{N(N-1)(N-2)(N-3)(N-4)} \Sigma\, p_1^2 p_2^2 p_3^2 p_4 p_5$$

$$+ \frac{n_i n_j(n_j - 1)n_k n_l}{N(N-1)(N-2)(N-3)(N-4)} \Sigma\, p_1^2 p_2^2 p_3^2 p_4 p_5$$

$$+ \frac{n_i n_j n_k (n_k - 1) n_l}{N(N-1)(N-2)(N-3)(N-4)} \quad \Sigma\, p_1^2 p_2^2 p_3^2 p_4 p_5$$

$$+ \frac{n_i n_j n_k n_l (n_l - 1)}{N(N-1)(N-2)(N-3)(N-4)} \quad \Sigma\, p_1^2 p_2^2 p_3^2 p_4 p_5$$

$$+ \frac{n_i n_j n_k n_l}{N(N-1)(N-2)(N-3)} \quad \Sigma\, p_1^2 p_2^2 p_3^2 p_4^2$$

Performing the summations as before, we have

$$\underset{\text{no two equal}}{\sum_{i=1}^{c} \sum_{j=1}^{c} \sum_{k=1}^{c} \sum_{l=1}^{c}} E\left(\frac{R_i^2\ R_j^2\ R_k^2\ R_l^2}{n_i\ n_j\ n_k\ n_l}\right) =$$

$$\begin{aligned} P_a(N) \Big\{ & 2(N-c)(\Sigma n_i^3 - \Sigma n_i^2) - 5(\Sigma n_i^4 - 2\,\Sigma n_i^3 + \Sigma n_i^2) \\ & - (N-c)(\Sigma n_i^2 - N)(3N - 3c + 7) + 3(\Sigma n_i^2 - N)^2 \\ & + (\Sigma n_i^3 - 2\,\Sigma n_i^2 + N)(3N - 3c + 7) \\ & + (N-c)(N-c+1)(N-c+2)(N-c+3) \\ & - (\Sigma n_i^2 - N)((N-c+1)(N-c+2) + (N-c+1)(N-c+3) \\ & + (N-c+2)(N-c+3)\ ) + 3(\Sigma n_i^3 - \Sigma n_i^2)(N-c+2) \\ & - (\Sigma n_i^4 - \Sigma n_i^3) \Big\} \end{aligned}$$

$$+ \; 4P_b(N)(c-3)\big((N-c)(N-c+1)(N-c+2) + 2(\Sigma n_i^3 - \Sigma n_i^2)$$
$$- 2(\Sigma n^2 - N)(N-c+2) - (N-c)(\Sigma n_i^2 - N)\big)$$

$$+ \; 6P_i(N)(c-2)(c-3)\big((N-c+1)(N-c) - \Sigma n_i^2 + N\big)$$

$$+ \; 4P_j(N)(c-1)(c-2)(c-3)(N-c)$$

$$+ \; P_k(N)c(c-1)(c-2)(c-3)$$

On combining the above expectations we obtain the following expression for $E(\left[\Sigma R_i^2 / n_i\right]^4)$:

$$E\left\{\sum_{i=1}^{c} \frac{R_i^2}{n_i}\right\}^4 =$$

$$\frac{(n+1)}{3628800}\Big\{14175N^{11} + 80325N^{10} + 174825N^9 + 155295N^8 - 26880N^7$$
$$- 180540N^6 - 156720N^5 - 51840N^4$$

$$+ c^4(175N^7 - 315N^6 - 1439N^5 + 1263N^4 + 4072N^3 - 732N^2 - 3024N)$$

$$+ c^3(2100N^8 + 4760N^7 - 8160N^6 - 26728N^5 - 2916N^4 + 32336N^3$$
$$+ 10704N^2 - 12096N)$$

$$+ c^2(9450N^9 + 45990N^8 + 80570N^7 + 45630N^6 - 42484N^5$$
$$- 93252N^4 - 60928N^3 + 2928N^2 + 12096N)$$

$$+ c(18900N^{10} + 113400N^9 + 304080N^8 + 524440N^7 + 708540N^6$$

$$+ 708592N^5 + 352224N^4 - 40064N^3 - 42816N^2 + 48384N)$$

$$- \Sigma \frac{1}{n_i}(11340N^9 + 79380N^8 + 257580N^7 + 496044N^6 + 490536N^5$$

$$+ 37920N^4 - 259200N^3 + 1536N^2 + 129024N)$$

$$+ c \Sigma \frac{1}{n_i}(-7560N^8 - 23040N^7 + 15840N^6 + 94320N^5 + 12648N^4$$

$$- 127728N^3 - 28992N^2 + 64512N)$$

$$+ c^2 \Sigma \frac{1}{n_i}(-1260N^7 + 2988N^6 + 10692N^5 - 12204N^4 - 31896N^3$$

$$+ 7488N^2 + 24192N)$$

$$+ \Sigma \frac{1}{n_i^2} (14400N^8 + 89664N^7 + 215136N^6 + 182400N^5 - 73440N^4$$

$$- 142464N^3 + 48384N^2 + 80640N)$$

$$+ c \Sigma \frac{1}{n_i^2} (4800N^7 - 2880N^6 - 31200N^5 + 5760N^4 + 63840N^3 - 40320N)$$

$$+ (\Sigma \frac{1}{n_i})^2 (756N^7 - 2916N^6 - 3780N^5 + 23220N^4 + 31104N^3$$

$$- 18144N^2 - 30240N)$$

$$+ \Sigma \frac{1}{n_i^3} (-15120N^7 - 45360N^6 - 25200N^5 + 25200N^4 + 10080N^3$$

$$- 10080N^2) \Big\}$$

Finally, we obtain the following expression for $E(H^4)$,

$$E(H^4) = \Big\{ -2625N^6 - 14175N^5 - 26535N^4 - 17145N^3$$

$$+ c^4(175N^6 - 315N^5 - 1439N^4 + 1263N^3 + 4072N^2 - 732N - 3024)$$

$$+ c^3(1400N^6 - 8848N^4 - 1776N^3 + 17816N^2 + 3504N - 12096)$$

$$+ c^2(2450N^6 + 14490N^5 + 18986N^4 - 16122N^3 - 34828N^2 + 2928N + 12096)$$

$$+ c(-1400N^6 + 30240N^5 + 115072N^4 + 97824N^3 - 34184N^2 - 14016N + 48384)$$

$$- \Sigma \frac{1}{n_i}(18900N^6 + 129924N^5 + 227916N^4 + 13980N^3 - 187200N^2$$

$$+ 30336N + 129024)$$

$$+ c\,\Sigma \frac{1}{n_i}(-10080N^6 - 7920N^5 + 38160N^4 + 7248N^3 - 84528N^2$$

$$- 7392N + 64512)$$

$$+ c^2\,\Sigma \frac{1}{n_i}(-1260N^6 + 2988N^5 + 10692N^4 - 12204N^3 - 31896N^2$$

$$+ 7488N + 24192)$$

$$+ \Sigma \frac{1}{n_i^2}(32064N^6 + 135936N^5 + 146400N^4 - 66240N^3 - 135264N^2$$

$$+ 48384N + 80640)$$

$$+ c\,\Sigma \frac{1}{n_i^2}(4800N^6 - 2880N^5 - 31200N^4 + 5760N^3 + 63840N^2 - 40320)$$

$$+ \left(\Sigma \frac{1}{n_i}\right)(756N^6 - 2916N^5 - 3780N^4 + 23220N^3 + 31104N^2$$

$$- 18144N - 30240)$$

$$\left. + \Sigma \frac{1}{n_i^3}(-15120N^6 - 45360N^5 - 25200N^4 + 25200N^3 + 10080N^2 - 10080N)\right\}$$

$$\Big/ 175N^3(N + 1)^3$$

As each $n_i \rightarrow \infty$, and thus $N \rightarrow \infty$, we see that $E(H^4) \longrightarrow c^4 + 8c^3 + 14c^2 - 8c - 15$, the fourth moment of the chi-square distribution with $c - 1$ degrees of freedom.

APPENDIX 2

THE FOURTH MOMENT OF FRIEDMAN'S DISTRIBUTION

1. Introduction.

We have obtained the fourth moment of Friedman's χ_r^2 - statistic by using the direct method employed by Friedman (1937) to obtain the first three moments. These moments are quoted below.

$$E(\chi_r^2) = c - 1$$

$$\mathrm{var}(\chi_r^2) = 2(b-1)(c-1) / b$$

$$E(\chi_r^2 - \mu_{\chi^2}) = 8(b-1)(b-2)(c-1) / b^2$$

In the following derivation of the fourth moment we use the notation :

r_{ij} = the rank of the observation in the i^{th} row and j^{th} column $(i = 1, 2, \ldots, b;\ j = 1, 2, \ldots, c)$

$$r'_{ij} = r_{ij} - \tfrac{1}{2}(c+1)$$

$$\bar{r}'_{ij} = \frac{1}{b}\sum_{i=1}^{b} r'_{ij}$$

$$\chi_r^2 = \frac{12b}{c(c+1)}\sum_{j=1}^{c} \bar{r}'^{\,2}_{ij}$$

$$= \frac{12}{bc(c+1)}\sum_{j=1}^{c}\left(\sum_{i=1}^{b} r'_{ij}\right)^2 \qquad \ldots. (1)$$

2. Calculation of the Fourth Moment.

Friedman shows that

$$\chi_r^2 - E(\chi_r^2) = \frac{24}{bc(c+1)} \sum_{i=1}^{b-1} \sum_{i_1=1}^{b} \sum_{j=1}^{c} r'_{ij} r'_{i_1 j}$$

so that for the fourth moment we are concerned with

$$\left\{ \sum_{i=1}^{b-1} \sum_{i_1=i+1}^{b} \sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right\}^4 .$$

Now,

$$\left\{ \sum_{i=1}^{b-1} \sum_{i_1=i+1}^{b} \sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right\}^4$$

$$= \sum_{i=1}^{b-1} \sum_{i_1=i+1}^{b} \left(\sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right)^4$$

$$+ 4 \sum_{i=1}^{b-1} \sum_{i_1=i+1}^{b} \sum_{i_2=1}^{b} \left(\sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right)^3 \left(\sum_{j=1}^{c} r'_{i_2 j} r'_{i_1 j} \right)$$

$$+ 6 \sum_{i=1}^{b-2} \sum_{i_1=i+1}^{b} \sum_{i_2=1}^{b-1} \sum_{i_3=i_2+1}^{b} \left(\sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right)^2 \left(\sum_{j=1}^{c} r'_{i_2 j} r'_{i_3 j} \right)^2$$

$$+ 12 \sum_{i=1}^{b-2} \sum_{i_1=i+1}^{b-1} \sum_{i_2=i_1+1}^{b} \left(\sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right)^2 \left(\sum_{j=1}^{c} r'_{ij} r'_{i_2 j} \right) \left(\sum_{j=1}^{c} r'_{i_1 j} r'_{i_2 j} \right)$$

$$+ 24 \sum_{i=1}^{b-3} \sum_{i_1=i+1}^{b-2} \sum_{i_2=i_1+1}^{b-1} \sum_{i_3=i_2+1}^{b} \left(\sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right) \left(\sum_{j=1}^{c} r'_{ij} r'_{i_2 j} \right) \left(\sum_{j=1}^{c} r'_{i_1 j} r'_{i_3 j} \right) \left(\sum_{j=1}^{c} r'_{i_2 j} r'_{i_3 j} \right)$$

We now consider the expectation, under H_o, of each of these terms.

$$E\left(\sum_{j=1}^{c} r'_{ij}\, r'_{i_1 j}\right)^4 =$$

$$\sum_{j=1}^{c} E(r'_{ij})^4\, E(r'_{i_1 j})^4 + 4\sum_{\substack{j=1 \\ j\neq j_1}}^{c}\ \sum_{j_1=1}^{c} E(r'^{3}_{ij}\, r'_{ij_1})\, E(r'^{3}_{i_1 j}\, r'_{i_1 j_1})$$

$$+\ 6\sum_{j=1}^{c-1}\ \sum_{j_1=j+1}^{c} E(r'_{ij}\, r'_{ij_1})^2\, E(r'_{i_1 j}\, r'_{i_1 j_1})^2$$

$$+\ 12\sum_{\substack{j=1 \\ j_2\neq j}}^{c}\ \sum_{j_1=1}^{c-1}\ \sum_{j_2=j_1+1}^{c} E(r'_{ij}\, r'_{ij_1}\, r'_{ij_2})\, E(r'_{i_1 j}\, r'_{i_1 j_1}\, r'_{i_1 j_2})$$

$$+\ 24\sum_{j=1}^{c-3}\ \sum_{j_1=j+1}^{c-2}\ \sum_{j_2=j_1+1}^{c-1}\ \sum_{j_3=j_2+1}^{c} E(r'_{ij}\, r'_{ij_1}\, r'_{ij_2}\, r'_{ij_3})$$

$$E(r'_{i_1 j}\, r'_{i_1 j_1}\, r'_{i_1 j_2}\, r'_{i_1 j_3})$$

By performing the appropriate summations we derive

$$E(r'_{ij}) = 0$$

$$E(r'^{2}_{ij}) = \frac{c^2 - 1}{12}$$

$$E(r'_{ij}\, r'_{ij_1}) = -\frac{c + 1}{12}$$

$$E(r'^{3}_{ij}\, r'_{ij_1}) = -\frac{(c + 1)(3c^2 - 7)}{240}$$

$$E(\,r'_{ij}\, r'_{ij_1}\,)^2 = \frac{c(c^2 - 1)(c + 1)}{144} - \frac{(c + 1)(3c^2 - 7)}{240}$$

$$E(\,r'^2_{ij}\, r'_{ij_1}\, r'_{ij_2}\,) = \frac{(c + 1)(3c^2 - 7)}{120(c - 2)} - \frac{c(c + 1)(c^2 - 1)}{144(c - 2)}$$

$$E(\,r'_{ij}\, r'_{ij_1}\, r'_{ij_2}\, r'_{ij_3}\,) = \frac{c(c + 1)(c^2 - 1)}{48(c - 2)(c - 3)} - \frac{(c + 1)(3c^2 - 7)}{40(c - 2)(c - 3)}$$

$$E(\,r'^4_{ij}\,) = \frac{(c^2 - 1)(3c^2 - 7)}{240}$$

Hence $\displaystyle\sum_{i=1}^{b-1} \sum_{i_1=i+1}^{b} E\left[\left(\sum_{j=1}^{c} r'_{ij}\, r'_{i_1 j} \right)^4 \right] =$

$$\frac{b(b - 1)}{2} \left\{ \frac{c(c^2 - 1)^2(3c^2 - 7)^2}{57600} + \frac{4c(c - 1)(c + 1)^2(3c^2 - 7)^2}{57600} \right.$$

$$+ 3c(c - 1) \left[\frac{c(c^2 - 1)(c + 1)}{144} - \frac{(c + 1)(3c^2 - 7)}{240} \right]^2$$

$$+ 6c(c - 1)(c - 2) \left[\frac{(c + 1)(3c^2 - 7)}{120(c - 2)} - \frac{c(c + 1)(c^2 - 1)}{144(c - 2)} \right]^2$$

$$+ c(c - 1)(c - 2)(c - 3) \left[\frac{c(c + 1)(c^2 - 1)}{48(c - 2)(c - 3)} - \frac{(c + 1)(3c^2 - 7)}{40(c - 2)(c - 3)} \right]^2$$

Also, $E \left(\sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right)^3 \left(\sum_{j=1}^{c} r'_{i_2 j} r'_{i_1 j} \right) = 0.$

Now Friedman derived

$$E \left\{ \left(\sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right)^2 \right\} = \sum_{j=1}^{c} E(r'^2_{ij}) E(r'^2_{i_1 j})$$

$$+ \; 2 \sum_{j=1}^{c-1} \sum_{j_1=j+1}^{c} E(r'_{ij} r'_{ij_1}) E(r'_{i_1 j} r'_{i_1 j_1})$$

so that

$$6 \sum_{i=1}^{b-2} \sum_{i_1=i+1}^{b} \sum_{i_2=1}^{b-1} \sum_{i_3=i_2+1}^{b} \left(\sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right)^2 \left(\sum_{j=1}^{c} r'_{i_2 j} r'_{i_3 j} \right)^2$$

$$= \frac{3b(b^2 - 1)(b - 2)c^4(c + 1)(c - 1)^2}{82944}$$

Now,

$$E \left\{ \left(\sum_{j=1}^{c} r'_{ij} r'_{i_1 j} \right)\left(\sum_{j=1}^{c} r'_{ij} r'_{i_2 j} \right)\left(\sum_{j=1}^{c} r'_{i_1 j} r'_{i_3 j} \right)\left(\sum_{j=1}^{c} r'_{i_2 j} r'_{i_3 j} \right) \right\}$$

$$= \sum_{j=1}^{c} E(r'^2_{ij} r'^2_{i_1 j} r'^2_{i_2 j} r'^2_{i_3 j})$$

$$+ \; 4 \sum_{j=1}^{c-1} \sum_{j_1=j+1}^{c} E(r'^2_{ij}) E(r'^2_{i_1 j}) E(r'_{i_2 j} r'_{i_3 j_1}) E(r'_{i_3 j} r'_{i_2 j_1})$$

$$+ \; 6 \sum_{j=1}^{c-1} \sum_{j_1=j+1}^{c} E(r'^2_{ij}) \, E(r'^2_{i_2 j}) \, E(r'_{i_2 j} \, r'_{i_2 j_1}) \, E(r'_{i_1 j} \, r'_{i_1 j_1})$$

$$+ \; 12 \sum_{j=1}^{c-2} \sum_{j_1=j+1}^{c-1} \sum_{j_2=j_1+1}^{c} E(r'^2_{ij}) \, E(r'_{i_1 j} \, r'_{i_2 j}) \, E(r'_{i_1 j_1} \, r'_{i_2 j_1})$$

$$E(r'_{i_2 j_2} \, r'_{i_3 j_3})$$

$$+ \; 24 \sum_{j=1}^{c-3} \sum_{j_1=j+1}^{c-2} \sum_{j_2=j_1+1}^{c-1} \sum_{j_3=j_2+1}^{c} E(r'_{ij} \, r'_{i_1 j}) \, E(r'_{i j_1} \, r'_{i_2 j_1})$$

$$E(r'_{i_1 j_2} \, r'_{i_2 j_2}) \, E(r'_{i_2 j_3} \, r'_{i_3 j_3})$$

Hence

$$24 \sum_{i=1}^{b-3} \sum_{i_1=i+1}^{b-2} \sum_{i_2=i_1+1}^{b-1} \sum_{i_3=i_2+1}^{b} \left(\sum_{j=1}^{c} r'_{ij} \, r'_{i_1 j} \right) \left(\sum_{j=1}^{c} r'_{ij} \, r'_{i_2 j} \right)$$

$$\left(\sum_{j=1}^{c} r'_{i_1 j} \, r'_{i_3 j} \right) \left(\sum_{j=1}^{c} r'_{i_2 j} \, r'_{i_3 j} \right)$$

$$= \frac{3b(b-1)(b-2)(b-3)c^4(c+1)^4(c-1)}{20736}$$

Now $E\left(\sum_{j=1}^{c} r'_{ij}\, r'_{i_1 j}\right)^2\left(\sum_{j=1}^{c} r'_{ij}\, r'_{i_2 j}\right)\left(\sum_{j=1}^{c} r'_{i_1 j}\, r'_{i_2 j}\right) = 0.$

Combining and simplifying the above results produces

$$E(\chi_r^2 - \mu_{\chi_r^2})^4 =$$

$$\frac{24(b-1)(c-1)(25c^3 - 38c^2 - 35c + 72)}{25b^3 c(c+1)}$$

$$+ \frac{12(b^2-1)(b-2)(c-1)^2}{b^3} + \frac{48(b-1)(b-2)(b-3)(c-1)}{b^3}$$

from which

$$E(\chi_r^2) =$$

$$\frac{24(b-1)(c-1)(25c^3 - 38c^2 - 35c + 72)}{25b^3 c(c+1)}$$

$$+ \frac{12(b-1)(b-2)(c-1)}{b^3}\,(b+1)(c-1) + 4(b-3)$$

$$+ \frac{32(b-1)(b-2)(c-1)^2}{b^2} + \frac{12(b-1)(c-1)^3}{b} + (c-1)^4 .$$

As $b \to \infty$ we see that

$E(\chi_r^2) \to c^4 + 8c^3 + 14c^2 - 8c - 15$, the fourth moment of the chi-square distribution with $c - 1$ degrees of freedom.

APPENDIX 3

APPROXIMATE CRITICAL VALUES FOR THE KRUSKAL-WALLIS AND FRIEDMAN STATISTICS BASED ON THE STEEPEST DESCENT METHOD

1. Approximate Critical Values for the Kruskal-Wallis Test.

The approximate critical values for the 10 %, 5 %, 2 % and 1 % significance levels are tabulated for $c = 3$, $n_i = 8$ to 25, $c = 4, 5, 6$ $n_i = 4$ to 25.

		Significance Level			
c	n_i	10 %	5 %	2 %	1 %
3	8	4.595	5.805	7.355	8.465
	9	4.586	5.831	7.418	8.529
	10	4.581	5.853	7.453	8.607
	11	4.587	5.885	7.489	8.648
	12	4.578	5.872	7.523	8.712
	13	4.601	5.901	7.551	8.735
	14	4.592	5.896	7.566	8.754
	15	4.591	5.902	7.582	8.821
	16	4.595	5.909	7.596	8.822
	17	4.593	5.915	7.609	8.856
	18	4.596	5.932	7.622	8.865
	19	4.598	5.923	7.634	8.887
	20	4.594	5.926	7.641	8.905
	21	4.597	5.930	7.652	8.918
	22	4.597	5.932	7.657	8.928
	23	4.598	5.937	7.664	8.947
	24	4.598	5.936	7.670	8.964
	25	4.599	5.942	7.682	8.975

c	n_i	10 %	5 %	2 %	1 %
		Significance Level			
4	4	6.088	7.235	8.515	9.287
	5	6.120	7.377	8.863	9.789
	6	6.127	7.453	9.027	10.09
	7	6.141	7.501	9.152	10.25
	8	6.148	7.534	9.250	10.42
	9	6.161	7.557	9.316	10.53
	10	6.167	7.586	9.376	10.62
	11	6.163	7.623	9.422	10.69
	12	6.185	7.629	9.458	10.75
	13	6.191	7.645	9.481	10.80
	14	6.198	7.658	9.508	10.84
	15	6.201	7.676	9.531	10.87
	16	6.205	7.678	9.550	10.90
	17	6.206	7.682	9.568	10.92
	18	6.212	7.698	9.583	10.95
	19	6.212	7.701	9.595	10.98
	20	6.216	7.703	9.606	10.98
	21	6.218	7.709	9.623	11.01
	22	6.215	7.714	9.629	11.03
	23	6.220	7.719	9.640	11.03
	24	6.221	7.724	9.652	11.06
	25	6.222	7.727	9.659	11.07

c	n_i	Significance Level 10 %	5 %	2 %	1 %
5	4	7.457	8.686	10.13	11.07
	5	7.532	8.876	10.47	11.57
	6	7.557	9.002	10.72	11.91
	7	7.600	9.080	10.87	12.14
	8	7.624	9.126	10.99	12.29
	9	7.637	9.166	11.06	12.41
	10	7.650	9.220	11.13	12.50
	11	7.660	9.242	11.19	12.58
	12	7.675	9.274	11.22	12.63
	13	7.685	9.303	11.27	12.69
	14	7.695	9.307	11.29	12.74
	15	7.701	9.302	11.32	12.77
	16	7.705	9.313	11.34	12.79
	17	7.709	9.325	11.36	12.83
	18	7.714	9.334	11.38	12.85
	19	7.717	9.342	11.40	12.87
	20	7.719	9.353	11.41	12.91
	21	7.723	9.356	11.43	12.92
	22	7.724	9.360	11.43	12.92
	23	7.727	9.368	11.44	12.94
	24	7.729	9.375	11.45	12.96
	25	7.730	9.377	11.46	12.96

c	n_i	Significance Level 10 %	5 %	2 %	1 %
6	4	8.800	10.14	11.71	12.72
	5	8.902	10.36	12.07	13.26
	6	8.958	10.50	12.33	13.60
	7	8.992	10.59	12.50	13.84
	8	9.037	10.66	12.62	13.99
	9	9.057	10.71	12.71	14.13
	10	9.078	10.75	12.78	14.24
	11	9.093	10.76	12.74	14.32
	12	9.105	10.79	12.90	14.38
	13	9.115	10.83	12.93	14.44
	14	9.125	10.84	12.98	14.49
	15	9.133	10.86	13.01	14.53
	16	9.140	10.88	13.03	14.56
	17	9.144	10.88	13.04	14.60
	18	9.419	10.89	13.06	14.63
	19	9.156	10.90	13.07	14.64
	20	9.159	10.92	13.09	14.67
	21	9.164	10.93	13.11	14.70
	22	9.168	10.94	13.12	14.72
	23	9.171	10.93	13.13	14.74
	24	9.170	10.93	13.14	14.74
	25	9.177	10.94	13.15	14.77

2. Approximate Critical Values for Friedman's Test.

The approximate critical values for the 10 %, 5 %, 2 % and 1 % significance levels are tabulated for c = 5, b = 11 to 25 and c = 6, b= 5 to 25.

c	b	Significance Level 10 %	5 %	2 %	1 %
5	11	7.782	9.309	11.20	12.58
	12	7.733	9.333	11.27	12.60
	13	7.754	9.354	11.32	12.68
	14	7.771	9.371	11.37	12.74
	15	7.787	9.387	11.36	12.80
	16	7.750	9.400	11.40	12.80
	17	7.765	9.412	11.44	12.85
	18	7.778	9.422	11.47	12.89
	19	7.789	9.432	11.45	12.88
	20	7.600	9.400	11.48	12.92
	21	7.771	9.448	11.50	12.91
	22	7.782	9.418	11.49	12.95
	23	7.791	9.426	11.51	12.97
	24	7.767	9.433	11.50	13.00
	25	7.776	9.440	11.52	12.99

c	b	Significance Level 10 %	5 %	2 %	1 %
6	5	9.000	10.49	12.09	13.23
	6	9.048	10.57	12.38	13.62
	7	9.122	10.67	12.55	13.86
	8	9.071	10.71	12.64	14.00
	9	9.127	10.78	12.75	14.14
	10	9.143	10.80	12.80	14.23
	11	9.130	10.84	12.92	14.32
	12	9.143	10.86	12.95	14.38
	13	9.176	10.89	13.00	14.45
	14	9.184	10.90	13.02	14.49
	15	9.210	10.92	13.06	14.54
	16	9.214	10.96	13.07	14.57
	17	9.202	10.95	13.10	14.61
	18	9.206	10.95	13.11	14.63
	19	9.196	11.00	13.14	14.67
	20	9.200	11.00	13.11	14.66
	21	9.218	10.99	13.14	14.69
	22	9.221	10.96	13.14	14.73
	23	9.236	11.00	13.19	14.73
	24	9.238	10.95	13.19	14.74
	25	9.229	10.99	13.21	14.74

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