Two–Grid \( hp \)-DGFEM for Second Order Quasilinear Elliptic PDEs Based on an Incomplete Newton Iteration

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Abstract. In this paper we propose a class of so-called two-grid \( hp \)-version discontinuous Galerkin finite element methods for the numerical solution of a second-order quasilinear elliptic boundary value problem based on the application of a single step of a nonlinear Newton solver. We present both the \textit{a priori} and \textit{a posteriori} error analysis of this two-grid \( hp \)-version DGFEM as well as performing numerical experiments to validate the bounds.

1. Introduction

In our recent articles \([4, 5]\) we have considered a class of two-grid finite element methods for strongly monotone partial differential equations. Here, the underlying problem is first approximated on a coarse finite element space; the resulting coarse solution is then used to linearise the underlying problem on a finer finite element space, so that only a linear system of equations is solved on this richer space. In this paper we consider an alternative two-grid interior penalty (IP) discontinuous Galerkin finite element method (DGFEM), based on employing a single step of a Newton solver on the finer space, cf. \([1], [9], \text{Section 5.2}\), for the numerical solution of the following quasilinear elliptic boundary value problem:

\[
-\nabla \cdot (\mu(x, |\nabla u|)\nabla u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma, \tag{1.1}
\]

where \( \Omega \) is a bounded polygonal domain in \( \mathbb{R}^2 \), with boundary \( \Gamma \) and \( f \in L^2(\Omega) \).

We assume that \( \mu \in C^2(\bar{\Omega} \times [0, \infty)) \) satisfies the condition: there exists positive constants \( m_\mu \) and \( M_\mu \) such that the following monotonicity property is satisfied:

\[
m_\mu(t-s) \leq \mu(x, t)t - \mu(x, s)s \leq M_\mu(t-s), \quad t \geq s \geq 0, \quad x \in \bar{\Omega}. \tag{1.2}
\]

For ease of notation we write \( \mu(t) \) instead of \( \mu(x, t) \). The outline of this article is as follows. In Section 2 we state the proposed two-grid IP DGFEM. In Sections 3 and 4 we consider the \textit{a priori} and \textit{a posteriori} error analysis, respectively, of the two-grid IP DGFEM. Finally, in Section 5 we present some numerical results to validate the theoretical error bounds.

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2. Two-Grid $hp$-Version IP DGFEM

We consider shape-regular meshes $\mathcal{T}_h$ that partition $\Omega \subset \mathbb{R}^2$ into open disjoint elements $\kappa$ such that $\Omega = \bigcup_{\kappa \in \mathcal{T}_h} \kappa$. By $h_{\kappa}$ we denote the element diameter of $\kappa \in \mathcal{T}_h$, $h = \max_{\kappa \in \mathcal{T}_h} h_{\kappa}$, and $n_{\kappa}$ signifies the unit outward normal vector to $\kappa$. We allow the meshes $\mathcal{T}_h$ to be 1-irregular; further, we suppose that $\mathcal{T}_h$ is of bounded local variation, i.e., there exists a constant $\rho_1 \geq 1$, independent of the element sizes, such that $\rho_1^{-1} \leq h_{\kappa}/h_{\kappa'} \leq \rho_1$, for any pair of elements $\kappa, \kappa' \in \mathcal{T}_h$ which share a common edge $e = \partial \kappa \cap \partial \kappa'$. To each $\kappa \in \mathcal{T}_h$ we assign a polynomial degree $p_{\kappa} \geq 1$ and define the degree vector $\mathbf{p} = \{ p_{\kappa} : \kappa \in \mathcal{T}_h \}$. We suppose that $\mathbf{p}$ is also of bounded local variation, i.e., there exists a constant $\rho_2 \geq 1$, independent of the element sizes and $\mathbf{p}$, such that, for any pair of neighbouring elements $\kappa, \kappa' \in \mathcal{T}_h$, $\rho_2^{-1} \leq p_{\kappa}/p_{\kappa'} \leq \rho_2$.

With this notation, we introduce the finite element space

$$V(\mathcal{T}_h, \mathbf{p}) = \{ v \in L^2(\Omega) : v|_{\kappa} \in S_{p_{\kappa}}(\kappa) \forall \kappa \in \mathcal{T}_h \},$$

where $S_{p_{\kappa}}(\kappa) = P_{p_{\kappa}}(\kappa)$ if $\kappa$ is a triangle and $S_{p_{\kappa}}(\kappa) = Q_{p_{\kappa}}(\kappa)$ if $\kappa$ is a parallelogram. Here, for $p \geq 0$, $P_{p}(\kappa)$ denotes the space of polynomials of degree at most $p$ on $\kappa$, while $Q_{p}(\kappa)$ is the space of polynomials of degree at most $p$ in each variable on $\kappa$.

For the mesh $\mathcal{T}_h$, we write $\mathcal{E}_h^B$ to denote the set of all interior edges of the partition $\mathcal{T}_h$ of $\Omega$, $\mathcal{E}_h^B$ the set of all boundary edges of $\mathcal{T}_h$, and set $\mathcal{E}_h = \mathcal{E}_h^B \cup \mathcal{E}_h^I$. Let $v$ and $q$ be scalar- and vector-valued functions, respectively, which are sufficiently smooth inside each element $\kappa \in \mathcal{T}_h$. Given two adjacent elements, $\kappa^+, \kappa^- \in \mathcal{T}_h$ which share a common edge $e \in \mathcal{E}_h^I$, i.e., $e = \partial \kappa^+ \cap \partial \kappa^-$, we write $v^\pm$ and $q^\pm$ to denote the traces of the functions $v$ and $q$, respectively, on the edge $e$, taken from the interior of $\kappa^\pm$; respectively. With this notation, the averages of $v$ and $q$ at $x \in e$ are given by $\| v \| = \| v^+ + v^- \|$ and $\| q \| = \| q^+ + q^- \|$, respectively.

Similarly, the jumps of $v$ and $q$ at $x \in e$ are given by $[ v ] = v^+ n_{\kappa^+} + v^- n_{\kappa^-}$ and $[ q ] = q^+ n_{\kappa^+} + q^- n_{\kappa^-}$, respectively, where $n_{\kappa^\pm}$ denotes the unit outward normal vector on $\partial \kappa^\pm$, respectively. On a boundary edge $e \in \mathcal{E}_h^B$, we set $\| v \| = | v |$, $\| q \| = | q |$, $[ v ] = v n$ and $[ q ] = q n$, with $n$ denoting the unit outward normal vector on the boundary $\Gamma$. For $e \in \mathcal{E}_h^I$, we define $h_e$ to be the length of the edge; moreover, we set $p_e = \max(p_{\kappa}, p_{\kappa'})$, if $e = \partial \kappa \cap \partial \kappa' \in \mathcal{E}_h^I$, and $p_e = p_{\kappa}$, if $e = \partial \kappa \cap \Gamma \in \mathcal{E}_h^B$.

2.1. Standard IP DGFEM discretisation. Given a fine mesh partition $\mathcal{T}_h$ of $\Omega$, with the corresponding polynomial degree vector $\mathbf{p}$, the standard IP DGFEM is defined as follows: find $u_{h,p} \in V(\mathcal{T}_h, \mathbf{p})$ such that

$$A_{h,p}(u_{h,p}, v_{h,p}) = F_{h,p}(v_{h,p})$$

for all $v_{h,p} \in V(\mathcal{T}_h, \mathbf{p})$, where $F_{h,p}(v) = \int_{\Omega} f u \, dx$ and

$$A_{h,p}(u, v) = \int_{\Omega} \mu(|\nabla_h u|) \nabla_h u \cdot \nabla_h v \, dx - \sum_{e \in \mathcal{E}_h} \int_{e} \mu(|\nabla_h u|) \nabla_h u : [ v ] \, ds$$

$$+ \theta \sum_{e \in \mathcal{E}_h} \int_{e} \mu(h_e^{-1} | u |) \nabla_h u \cdot [ u ] \, ds + \sum_{e \in \mathcal{E}_h} \int_{e} \sigma_{h,p} | u | \cdot [ v ] \, ds.$$

Here, $\theta \in [-1, 1]$, $\nabla_h$ is the element-wise gradient operator and $\sigma_{h,p} = \gamma p_e^2/h_e$, where $\gamma > 0$ is a sufficiently large constant. We define the energy norm on $V(\mathcal{T}_h, \mathbf{p})$:

$$\| v \|_{DG}^2 = \| \nabla_h v \|_{L^2(\Omega)}^2 + \sum_{e \in \mathcal{E}_h} \int_{e} \sigma_{h,p} | v |^2 \, ds.$$
Lemma 2.1 (See [6]). The semilinear form \( A_{h,p}(\cdot, \cdot) \) is strongly monotone in the sense that, there exists \( \gamma_{\min} > 0 \), such that for any \( \gamma \geq \gamma_{\min} \)
\begin{equation}
A_{h,p}(w_1, w_1 - w_2) - A_{h,p}(w_2, w_1 - w_2) \geq C_m \|w_1 - w_2\|^2_{D^2 G} \quad \forall w_1, w_2 \in V(T_h, p),
\end{equation}
where \( C_m \) is a positive constant, independent of the discretisation parameters.

2.2. Two-grid IP DGFEM discretisation. We now introduce a two-grid IP DGFEM based on employing a single step of the Newton iteration on the fine mesh. To this end, we consider two partitions \( T_h \) and \( T_H \) of \( \Omega \), with granularity \( h \) and \( H \), respectively. We assume that \( T_h \) and \( T_H \) are nested in that sense that for any element \( \kappa_h \in T_h \) there exists an element \( \kappa_H \in T_H \) such that \( \Gamma_h \subseteq \Gamma_H \). Moreover for each mesh, \( T_h \) and \( T_H \), we have a corresponding polynomial degree vector \( p = \{p_\kappa : \kappa \in T_h\} \) and \( P = \{p_\kappa : \kappa \in T_H\} \), respectively, where given an element \( \kappa_h \in T_h \) and an element \( \kappa_H \in T_H \), such that \( \Gamma_h \subseteq \Gamma_H \), the polynomial degree vectors satisfy the condition that \( p_{\kappa_h} \geq p_{\kappa_H} \). Thereby, the finite element spaces \( V(T_h, p) \) and \( V(T_H, P) \) satisfy the following the condition: \( V(T_H, P) \subseteq V(T_h, p) \).

Using this notation we introduce the \( hp \)-version two-grid IP DGFEM discretisation of (1.1) based on a single Newton iteration step, cf. [1], [9, Section 5.2]:

1. Compute the coarse grid approximation \( u_{h,p} \in V(T_h, p) \) such that
\begin{equation}
A_{H,p}(u_{h,p}, v_{h,p}) = F_{H,p}(v_{h,p}) \quad \text{for all } v_{h,p} \in V(T_h, p).
\end{equation}

2. Determine the fine grid solution \( u_{2G} \in V(T_h, p) \) such that
\begin{equation}
A'_{h,p}[u_{h,p}(u_{2G}, v_{h,p}) = A'_{h,p}[u_{h,p}(u_{h,p}, v_{h,p}) - A_{h,p}(u_{h,p}, v_{h,p}) + F_{h,p}(v_{h,p}) \quad \text{for all } v_{h,p} \in V(T_h, p).
\end{equation}

Here, \( A'_{h,p}[u](\phi, v) \) denotes the Fréchet derivative of \( u \rightarrow A_{h,p}(u, v) \), for fixed \( v \), evaluated at \( u \); thereby, given \( \phi \) we have \( A'_{h,p}[u](\phi, v) = \lim_{t \to 0} \frac{A_{h,p}(u + tv, v) - A_{h,p}(u, v)}{t} \).

Remark 2.2. For simplicity of presentation, throughout the rest of this article we shall only consider the incomplete IP variation of the DGFEM, i.e., when \( \theta = 0 \).

Lemma 2.3. Under the assumptions on \( \mu \), the following inequality holds:
\begin{equation}
A'_{h,p}[u](v, v) \geq C_m \|v\|^2_{D^2 G} \quad \forall u, v \in V(T_h, p).
\end{equation}

Proof. Setting \( w_1 = u + tv \) and \( w_2 = u \) in Lemma 2.1, \( u, v \in V(T_h, p) \), \( t > 0 \):
\begin{equation}
\frac{A_{h,p}(u + tv, v) - A_{h,p}(u, v)}{t} \geq C_m \|v\|^2_{D^2 G}.
\end{equation}
Taking the limit as \( t \to 0 \), we deduce the statement of the Lemma.

3. A Priori Error Analysis

For simplicity of presentation, in this section we assume that the mesh is quasiuniform with mesh size \( h \) and that \( p \) is uniform over the mesh, i.e., \( p \equiv p \).

Theorem 3.1. Assuming that \( u \in C^1(\Omega) \) and \( u \in H^k(\Omega), \allowbreak k \geq 2 \), the solution of \( u_{2G} \in V(T_h, p) \) of the two-grid IP DGFEM satisfies
\begin{align}
\|u_{h,p} - u_{2G}\|_{DG} & \leq C \frac{p^{7/2}}{h} \frac{H^{2S - 2}}{P^{3k - 3}} \|u\|^2_{H^k(\Omega)}, \\
\|u - u_{2G}\|_{DG} & \leq C \frac{h^{k - 1}}{P^{3k - 3}} \|u\|^2_{H^k(\Omega)} + C \frac{h^{7/2}}{h} \frac{H^{2S - 2}}{P^{3k - 3}} \|u\|^2_{H^k(\Omega)},
\end{align}
with $1 \leq s \leq \min \{ p + 1, k \}$, $p \geq 1$ and $1 \leq S \leq \min \{ P + 1, k \}$, $P \geq 1$, where $C > 0$ is independent of the discretisation parameters.

3.1. Auxiliary Results. We first state the following auxiliary results.

LEMMA 3.2. For a function $v \in V(\mathcal{T}_h, p)$ we have the inverse inequality
\[
\|v\|_{L^4(\Omega)} \leq C \eta^{-1/2} \|v\|_{L^2(\Omega)},
\]
where $C$ is a positive constant, independent of the discretisation parameters.

PROOF. Given $\kappa \in \mathcal{T}_h$, employing standard inverse inequalities, see [8], gives
\[
\int_{\kappa} |v|^4 \, dx \leq \|v\|_{L^\infty(\kappa)}^2 \|v\|_{L^2(\kappa)}^2 \leq C p^4 h^{-2} \|v\|_{L^2(\kappa)}^2 \|v\|_{L^2(\kappa)} = C p^4 h^{-2} \|v\|_{L^2(\kappa)}^4.
\]
Summing over $\kappa \in \mathcal{T}_h$, employing the inequality $\sum_{i=1}^n a_i \leq (\sum_{i=1}^n \sqrt{a_i})^2$, $a_i \geq 0$, $i = 1, \ldots, n$, and taking the fourth root of both sides, completes the proof.

LEMMA 3.3. For any $v, w, \phi \in V(\mathcal{T}_h, p)$,
\[
(3.3) \quad A_{h,p}(w, \phi) = A_{h,p}(v, \phi) + A_{h,p}[v](w - v, \phi) + Q(v, w, \phi),
\]
where the remainder $Q$ satisfies
\[
|Q(v, w, \phi)| \leq C p^2 h^{-1} (1 + \|\nabla w\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)}) \|\nabla (w - v)\|_{DG}^2 \|\nabla \phi\|_{DG},
\]
and $C$ is a positive constant, independent of the discretisation parameters.

PROOF. We follow the proof outlined by [9, Lemma 3.1]; to this end, setting $\xi(t) = v + t(w - v)$ and $\eta(t) = A_{h,p}[\xi(t), \phi]$, we note that the first equation follows from the identity
\[
\eta(1) = \eta(0) + \int_0^1 \eta''(t)(1 - t) \, dt,
\]
where $Q(v, w, \phi) = \int_0^1 \eta''(t)(1 - t) \, dt$ and $\eta''(t) = A_{h,p}[\xi(t)](w - v, w - v, \phi)$. Thereby,
\[
Q(v, w, \phi) = 2 \int_0^1 \int_\Omega \mu'_{\nabla u}((\nabla \xi(t))) \cdot \nabla(w - v) \nabla(w - v) \cdot \nabla \phi \, dx(1 - t) \, dt
\]
\[
+ \int_0^1 \int_\Omega \mu''_{\nabla u}((\nabla \xi(t))) |\nabla(w - v)|^2 \nabla \xi(t) \cdot \nabla \phi \, dx(1 - t) \, dt
\]
\[
- 2 \int_0^1 \sum_{e \in \mathcal{E}_h} \int_e \|\mu'_{\nabla u}((\nabla \xi(t))) \cdot \nabla(w - v) \| \cdot \|\phi\| \, ds(1 - t) \, dt
\]
\[
- \int_0^1 \sum_{e \in \mathcal{E}_h} \int_e \|\mu''_{\nabla u}((\nabla \xi(t))) |\nabla(w - v)|^2 \nabla \xi(t) \| \cdot \|\phi\| \, ds(1 - t) \, dt
\]
\[
= T_1 + T_2 + T_3 + T_4.
\]
Here, $\mu'_{\nabla u}(|\cdot|)$ and $\mu''_{\nabla u}(|\cdot|)$ denote the first and second derivatives of $\mu(|\cdot|)$, respectively. First consider $T_1$: given that $\mu \in C^2(\bar{\Omega} \times [0, \infty))$, Lemma 3.2 gives
\[
T_1 \leq C \|\nabla(w - v)\|_{L^4(\Omega)}^2 \|\nabla \phi\|_{L^2(\Omega)} \leq C p^2 h^{-1} \|\nabla(w - v)\|_{L^2(\Omega)}^2 \|\nabla \phi\|_{L^2(\Omega)}.
\]
Secondly, term $T_2$ is bounded in an analogous fashion as follows:
\[
T_2 \leq C \left( \|\nabla w\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} \right) \|\nabla(w - v)\|_{L^4(\Omega)} \|\nabla \phi\|_{L^2(\Omega)}
\]
\[
\leq C \left( \|\nabla w\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} \right) p^2 h^{-1} \|\nabla(w - v)\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)}.
\]
Term $T_3$ is bounded via the inverse trace inequality, see [8], and Lemma 3.2:

$$T_3 \leq C \left\{ \sum_{e \in E_h} h_e p_e^{-2} \| \nabla (w - v) \|^2_{L^2(e)} \right\}^{1/2} \left\{ \sum_{e \in E_h} \int_{e \in E_h} p_e^{-2} h_e^{-1} \| \phi \|^2 \, ds \right\}^{1/2} \leq C \| \nabla (w - v) \|^2_{L^2(\Omega)} \| \phi \|_{DG} \leq C p^2 h^{-1} \| \nabla (w - v) \|^2_{L^2(\Omega)} \| \phi \|_{DG}.$$

We can bound $T_4$ in an analogous manner as follows:

$$T_4 \leq C \left\{ \sum_{e \in E_h} h_e p_e^{-2} \| \nabla (w - v) \|^2 |\nabla w| \|^2_{L^2(F)} \right\}^{1/2} \left\{ \sum_{e \in E_h} \int_{F} p_e^{-2} h_e^{-1} \| \phi \|^2 \, ds \right\}^{1/2} + C \left\{ \sum_{e \in E_h} h_e p_e^{-2} \| \nabla (w - v) \|^2 |\nabla v| \|^2_{L^2(F)} \right\}^{1/2} \left\{ \sum_{e \in E_h} \int_{F} p_e^{-2} h_e^{-1} \| \phi \|^2 \, ds \right\}^{1/2} \leq C \{ \| \nabla (w - v) \|^2 |\nabla w| \|_{L^2(\Omega)} + \| \nabla (w - v) \|^2 |\nabla v| \|_{L^2(\Omega)} \} \| \phi \|_{DG} \leq C p^2 h^{-1} \{ \| \nabla w \|_{L^2(\Omega)} + \| \nabla v \|_{L^2(\Omega)} \} \| \nabla (w - v) \|_{L^2(\Omega)} \| \phi \|_{DG}.$$

Combining these bounds for terms $T_1$, $T_2$, $T_3$ and $T_4$ completes the proof. 

**Lemma 3.4.** Let $u \in H^2(\Omega)$ be the analytical solution of (1.1), such that $\nabla u \in [L^\infty(\Omega)]^2$, and $u_{h,p} \in V(T_h, p)$ be the IP DGFEM defined by (2.1), we have that

$$\| \nabla u_{h,p} \|_{L^\infty(\Omega)} \leq C p^{3/2},$$

where $C$ is a positive constant, independent of the discretisation parameters.

**Proof.** Writing $P_u$ to denote the projection of $u$ onto the finite element space $V(T_h, p)$ defined in [2], we have that $\| u - P_u \|_{H^0(\Omega)} \leq C h^{1/2-k} \| u \|_{H^2(\Omega)}$ and $\| \nabla (u - P_u) \|_{L^\infty(\Omega)} \leq C \| u \|_{H^2(\Omega)}$ for all $q \leq 2$. Exploiting these bounds, standard inverse inequalities, [8], and the a priori bound for the IP DGFEM, [6], gives

$$\| \nabla u_{h,p} \|_{L^\infty(\Omega)} \leq \| \nabla (u_{h,p} - P_u) \|_{L^\infty(\Omega)} + \| \nabla P_u \|_{L^\infty(\Omega)} \leq C p^2 h^{-1} \| \nabla (u_{h,p} - P_u) \|_{L^2(\Omega)} + \| \nabla (u - P_u) \|_{L^\infty(\Omega)} + \| \nabla u \|_{L^\infty(\Omega)} \leq C p^{3/2} \{ \| u \|_{H^2(\Omega)} + \| \nabla u \|_{L^\infty(\Omega)} \}.$$

Since $u \in H^2(\Omega)$ and $\nabla u \in [L^\infty(\Omega)]^2$, the quantities $\| u \|_{H^2(\Omega)}$ and $\| \nabla u \|_{L^\infty(\Omega)}$ are both bounded uniformly by a constant; this then completes the proof. 

**3.2. Proof of Theorem 3.1.** We now exploit the above results to prove Theorem 3.1. For the first bound (3.1), we employ Lemma 2.3, (2.1), (2.4) and (3.3); thereby, with $\phi = u_{h,p} - u_{2G}$, we deduce that

$$C_m \| u_{h,p} - u_{2G} \|_{DG}^2 \leq A_{h,p}[u_{H,P}](u_{h,p} - u_{2G}, \phi) = A_{h,p}[u_{H,P}](u_{h,p} - u_{H,P}, \phi) + A_{h,p}[u_{H,P}](u_{H,P} - u_{2G}, \phi) = A_{h,p}[u_{H,P}](u_{h,p} - u_{H,P}, \phi) + A_{h,p}(u_{H,P}, \phi) - F_{h,p}(\phi) = A_{h,p}[u_{H,P}](u_{h,p} - u_{H,P}, \phi) + A_{h,p}(u_{H,P}, \phi) - A_{h,p}(u_{h,p}, \phi) = -Q(u_{H,P}, u_{h,p}, \phi).$$
Hence, from Lemma 3.3 we get that
\[ \|u_{h,p} - u_{2G}\|_{DG} \leq C p^2 h^{-1} \left( 1 + \|\nabla u_{h,p}\|_{L^\infty(\Omega)} + \|\nabla u_{H,P}\|_{L^\infty(\Omega)} \right) \|u_{h,p} - u_{H,P}\|_{DG}. \]

Applying Lemma 3.4, noting that \( p^{3/2} \geq P^{3/2} \geq 1 \), and the \textit{a priori} bound for the standard IP DGFEM, cf. [6, Theorem 3.3], gives
\[ \|u_{h,p} - u_{2G}\|_{DG} \leq C p^{7/2} h^{-1} \left( 1 + p^{3/2} + P^{3/2} \right) \left\{ \|u - u_{h,p}\|_{DG} + \|u - u_{H,P}\|_{DG} \right\} \]
\[ \leq C p^{7/2} h^{-1} \left( \frac{h^{2s-2}}{p^{2k-3}} \|u\|_{H^s(\Omega)} + \frac{H^{2s-2}}{p^{2k-3}} \|u\|_{H^s(\Omega)} \right). \]

Noting that \( h \leq H \) and \( p \geq P \) completes the proof of the first bound (3.1). To prove the second inequality (3.2), we first employ the triangle inequality
\[ \|u - u_{2G}\|_{DG} \leq \|u - u_{h,p}\|_{DG} + \|u_{h,p} - u_{2G}\|_{DG}. \]

Thereby, applying the \textit{a priori} error bound for the standard IP DGFEM, together with the bound (3.1), completes the proof of Theorem 3.1.

4. \textbf{A Posteriori Error Analysis}

Here, we state an \textit{a posteriori} error bound for the two-grid IP DGFEM.

**Theorem 4.1.** Let \( u \in H^1_0(\Omega) \) be the analytical solution of (1.1), \( u_{H,P} \in V(\mathcal{T}_H, \mathbf{P}) \) and \( u_{2G} \in V(\mathcal{T}_h, \mathbf{p}) \) the numerical approximations obtained from (2.3) and (2.4), respectively; then the following \( hp \)-a posteriori error bound holds
\[ \|u - u_{2G}\|^2_{DG} \leq C \sum_{\kappa \in \mathcal{T}_h} \left( \eta^2_k + \xi^2_k \right), \]
with a constant \( C > 0 \), which is independent of \( h, H, p \) and \( \mathbf{P} \). Here, for \( \kappa \in \mathcal{T}_h \),
\[ \eta^2_k = h^2 \|\Pi_{\kappa,p_c} f + \nabla \cdot \{ \mu(\nabla u_{H,P}) \nabla u_{2G} \} \|_{L^2(\kappa)}^2 \]
\[ + h \|\mu(\nabla u_{H,P}) \nu u_{2G} \|_{L^2(\partial\kappa)}^2 \]
\[ + \gamma h^{-1} \|\mu(\nabla u_{H,P}) \nu u_{2G} \|_{L^2(\partial\kappa)}^2 \]
\[ + \|\mu(\nabla u_{H,P}) \cdot (\nabla u_{2G} - \nabla u_{H,P}) \|_{L^2(\kappa)}^2 \]
\[ + \|\mu(\nabla u_{H,P}) \cdot (\nabla u_{2G} - \nabla u_{H,P}) \|_{L^2(\partial\kappa)}^2 \]
\[ + \|\mu(\nabla u_{H,P}) \cdot (\nabla u_{2G} - \nabla u_{H,P}) \|_{L^2(\partial\kappa)}^2 \],
\[ \xi^2_k = \|\mu(\nabla u_{H,P}) - \mu(\nabla u_{2G})\|_{L^2(\kappa)}^2 \]
\[ + \|\mu(\nabla u_{H,P}) - \mu(\nabla u_{2G})\|_{L^2(\partial\kappa)}^2 \]
\[ + h \|\mu(\nabla u_{H,P}) - \mu(\nabla u_{2G})\|_{L^2(\partial\kappa)}^2 \]
\[ + \|\mu(\nabla u_{H,P}) - \mu(\nabla u_{2G})\|_{L^2(\partial\kappa)}^2 \],
and \( \Pi_{\kappa,p_c} \) denotes the (elementwise) \( L^2 \)-projection onto \( V(\mathcal{T}_h, \mathbf{p}) \).

**Proof.** The proof of this error bound follows in an analogous manner to the \textit{a posteriori} proof presented in [5], cf. also [7]. For details, we refer to [3].

5. \textbf{Numerical Experiments}

In this section we perform numerical experiments to validate the \textit{a priori} error bound, Theorem 3.1 and demonstrate the performance of the \textit{a posteriori} error bound, Theorem 4.1; here, we set \( \gamma = 10 \) and \( \theta = 0 \). Throughout this section, we let \( \Omega \) be the unit square \((0,1)^2 \subset \mathbb{R}^2 \) and define the nonlinear coefficient as \( \mu(x, |\nabla u|) = 2 + (1 + |\nabla u|)^{-1} \). We select the right-hand forcing function \( f \) so that the analytical solution to (1.1) is given by \( u(x, y) = x(1-x)y(1-y)(1-2y)e^{-20(2x-1)^2} \).
5.1. Validation of Theorem 3.1. We first validate the bound given in Theorem 3.1; to this end we first solve the standard IP DGFEM on a $256 \times 256$ uniform mesh of quadrilaterals to compute $u_{h,p}$ for a fixed constant polynomial degree $p = 1, 2, 3$. We then compute the solution $u_{2G}$ to (2.3)–(2.4), for $p = 1, 2, 3$, on a fixed fine $256 \times 256$ mesh, while performing uniform $h$-refinement of the coarse mesh, starting from a $4 \times 4$ mesh with polynomial degree $P = p$. Figure 1 shows the convergence rate of the error between $u_{h,p}$ and $u_{2G}$, measured in the DG norm, compared to the size of the coarse mesh. Here, we observe that $\|u_{h,p} - u_{2G}\|_{DG}$ tends to zero at the optimal rate $O(H^{2P})$, for each fixed $P$, cf. Theorem 3.1.

5.2. Adaptive Refinement using Theorem 4.1. For this experiment we use the two-grid mesh adaptation algorithm from [5], with the local error indicators $\eta_k$ and local two-grid error indicators $\xi_k$ from Theorem 4.1, to automatically refine the coarse and fine meshes employing both $h$- and $hp$-adaptive mesh refinement. Figure 2 shows $\|u - u_{2G}\|_{DG}$ compared to the third root of the degrees of freedom, as well as the effectivity indices of the error estimator. As can be seen for both $h$- and $hp$-adaptive refinement, the effectivity indices are roughly constant, indicating that the error bound overestimates the error by a roughly constant factor. For reference purposes, we also calculate the standard IP DGFEM solution $u_{h,p}$, using
both $h$– and $hp$–adaptive refinement; cf. Figure 2(a). Finally, in Figure 3 we compare the error in the standard and two-grid IP DGFEMs against the cumulative CPU time when both $h$– and $hp$–adaptive refinement are employed; here, we observe that the two-grid IP DGFEM is more efficient than the standard IP DGFEM.

References


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