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STUDIES IN MULTIPLICATIVE NUMBER THEORY

BY

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ABSTRACT

This thesis gives some order estimates and asymptotic formulae associated with general classes of non-negative multiplicative functions as well as some particular multiplicative functions such as the divisor functions \( d_k(n) \).

In Chapter One we give a lower estimate for the number of distinct values assumed by the divisor function \( d(n) \) in \( 1 \leq n \leq x \). We also identify the smallest positive integer which is a product of triangular numbers and not equal to \( d_3(n) \) for \( 1 \leq n \leq x \).

In Chapter Two we show that if \( f(n) \) satisfies some conditions and if

\[
M = \max_{a \geq 1} \left( f(2^a) \right)^{1/a},
\]

then the maximum value of \( f(n) \) in \( 1 \leq n \leq x \) is about

\[
\frac{\log x}{M \log \log x}.
\]

We also show that a function which has a finite mean value cannot be large too often.

In Chapter Three we give an upper estimate to the average value of \( f(n) \) as \( n \) runs through a short interval in an arithmetic progression with a large modulus. As an application of our general theorem we show, for example, that if \( f(n) \) is the characteristic function of the set of integers which are the sum of two squares, then as \( x \to \infty \),
\[
\sum_{\substack{1 \leq n \leq x \\ n \equiv a \pmod{k}}} f(n) \ll \frac{1}{\phi(k)} \prod_{\substack{p \mid k \\ p \equiv 1 \pmod{4}}} \left( 1 - \frac{1}{p} \right) \frac{y}{\sqrt{\log x}}
\]

uniformly in \(a, k\) and \(y\) provided that

\[0 < a < k, \quad (a, k) = 1, \quad k < y^{1-\alpha}, \quad x^\beta < y \leq x.\]

where \(\alpha, \beta\) are positive constants.

We call a positive integer \(n\) a \(k\)-full integer if \(p^k\) divides \(n\) whenever \(p\) is a prime divisor of \(n\), and in Chapter Four we give an asymptotic formula for the number of \(k\)-full integers not exceeding \(x\). In Chapter Five we give an asymptotic formula for the number of \(2\)-full integers in an interval. We also study the problem of the distribution of the perfect squares among the sequence of \(2\)-full integers.

The materials in the first three chapters have been accepted for publication and will appear in [31], [22], [33] and [32].
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First of all I am greatly indebted to Professor H. Halberstam who gave an inspiring postgraduate course of lectures on analytic number theory and encouraged me to undertake research work in this area. As my supervisor he has given me much valuable advice, criticism and assistance throughout the course of this work, generously allowing me more of his time than is due to a part time student, and I wish to record my sincere thanks to him.

I would also like to thank Professor D.A. Burgess, Professor P. Erdős and Professor L.K. Hua for useful discussions. I also thank Dr. M. Nair for allowing me to include our joint work in the thesis.

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1.1 INTRODUCTION.

Throughout this chapter $c_1, c_2, \ldots$ denote absolute positive constants and $p_n$ denotes the $n$th prime.

Let $d(n)$ denote, as usual, the number of positive integer divisors of $n$. We partition the set of all positive integers into equivalence classes $H(k)$ where

$$H(k) = \{n; d(n) = k\}, \quad k = 1, 2, \ldots .$$

We call those integers which are least in their own equivalence classes the D-numbers and we denote by $D$ the set of all D-numbers. These D-numbers form a set of representatives for the equivalence classes and the divisor function maps $D$ bijectively onto the set of all positive integers. Since $m \in D$ implies $d(n) \neq d(m)$ for $0 < n < m$, it is clear that if we put

$$D(x) = \left|\{m; m \in D, m \leq x\}\right|$$

then $D(x)$ is the number of distinct values assumed by $d(n)$ in $1 \leq n \leq x$.

An integer $m$ is called highly composite if $d(n) < d(m)$ for $0 < n < m$. We see that our set $D$ contains all the highly composite integers.

In order to study $D(x)$, Erdös and Mirsky introduced in [6] another set of representatives from the equivalence classes $H(k)$, namely the set $B$ of B-numbers, defined as integers having the form...
where $q_1$ is prime, $q_1 \geq q_2 \geq \ldots \geq q_k$ and $k$ is arbitrary. It is easy to see that

$$
\frac{q_1^{-1}}{p_1} \ldots \frac{q_k^{-1}}{p_k}
$$

is the unique B-number in the equivalence class $H(q_1q_2\ldots q_k)$.

Putting

$$B(x) = \left\{ m; m \in B, m \leq x \right\},$$

Erdős and Mirsky proved in [6], among other things, that, as $x \to \infty$,

$$\log B(x) \sim \frac{2\sqrt{2} (\log x)^{1/2}}{\sqrt{3} \log \log x}, \quad (1.1)$$

and deduced that, as $x \to \infty$,

$$\log D(x) \sim \frac{2\sqrt{2} (\log x)^{1/2}}{\sqrt{3} \log \log x}. \quad (1.2)$$

They also proved that there is an absolute positive constant $c_1$ such that, for all sufficiently large values of $x$,

$$D(x) - B(x) > c_1 \log \log \log x, \quad (1.3)$$

and that

$$\frac{D(x)}{B(x)} = 1 + O \left\{ \frac{(\log x)^2}{(\log x)^{1/3}} \right\}. \quad (1.4)$$

There is a natural bijection from $D$ to $B$ via the equivalence classes and, because of the minimal property of D-numbers, it follows that $D(x) - B(x)$ counts precisely those D-numbers not exceeding $x$ whose corresponding B-numbers exceed $x$. Unless the
error term in (1.4) is exceedingly poor, (1.1) and (1.2) suggest that something better than (1.3) is true. In [31] I established the following improvement:

Let \( 0 < c_2 < 3^{4/3} \log 2 \). Then

\[
D(x) - B(x) > \exp \left\{ \frac{c_2 (\log \log x)^{1/3}}{\log \log \log x} \right\}
\]

(1.5)

for all sufficiently large values of \( x \). Later Dr. M. Nair and I elaborated on my method and proved:

**Theorem 1.** There exists \( c_3 > 0 \) such that

\[
D(x) - B(x) > \exp \left\{ \frac{c_3 (\log x)^{1/2}}{\log \log x} \right\}
\]

for all sufficiently large values of \( x \).

Compared with the upper bound for \( D(x) \) implied by (1.2) we see that, apart from the value of \( c_3 \), Theorem 1 gives the best possible result. We shall give the proof of (1.5) in section 1.2 and the proof of Theorem 1 in section 1.3. I would like to thank Dr. Nair for allowing me to include our joint work in my thesis.

In [6] Erdős and Mirsky also proved that, for \( x \geq 6 \), the least positive integer missed by \( d(n) \) in \( 1 \leq n \leq x \), that is, the least integer in the set

\[
M(x) = \left\{ m : d(n) \neq m, \quad 1 \leq n \leq x \right\},
\]

is the smallest prime \( p \) satisfying \( 2^{p-1} > x \). Here we give the generalisation of the missing value problem to \( d_3(n) \), the number of ways of writing \( n \) as a product of three factors. The function \( d_3(n) \) is multiplicative and, for each prime \( p \), we have
It follows that the range of $d_3(n)$ is the set $R_3$ of numbers which are products of triangular numbers. We now put, for $x \geq 1$,

$$M_3(x) = \left\{ m : m \in R_3, \, d_3(n) \neq m, \, 1 \leq n \leq x \right\},$$

and we shall identify the least integer in $M_3(x)$. The generalisation is non-trivial because there is no unique factorisation of numbers of $R_3$ into triangular numbers. A number $m \in R_3$ is said to be reducible, or irreducible, according to whether $m$ is, or is not, a product of smaller numbers in $R_3$. Every irreducible number must, of course, be triangular, but, as we shall see, there are some triangular numbers which are reducible. Even without unique factorisation these irreducible numbers do take the role of the primes in our missing value problem for $d_3(n)$ and in fact we have

Theorem 2. For $x \geq 1$ the least integer in $M_3(x)$ is the smallest irreducible number $\frac{1}{2} a(a + 1)$ such that $2^{a-1} > x$.

We shall give the proof of this in section 1.4.

1.2 A LOWER BOUND FOR $D(x) - B(x)$.

We first prove the following simple lemma.

Lemma 1. Let $1 < x < 3^{4/3}$, and for each sufficiently large value of $x$, define $r$ and $t$ by

$$r = \left[ x(\log \log x)^{1/3} \right],$$

(p. 1)

$$p_1 \cdots p_t < x < p_1 \cdots p_t p_{t+1}. \quad (2.2)$$
Then we have
\[ p_1^2 \ldots p_r^2 < p_t. \]  
(2.3)

Proof. We require the following consequences of the prime number theorem:

As \( x \to \infty \),
\[ \sum_{p \leq x} \log p \sim x, \]  
(2.4)
and
\[ \sum_{p \leq x} p^2 \log p \sim \frac{1}{3} x^3 \]  
(2.5)

and
\[ p_r \sim r \log r. \]  
(2.6)

From (2.1) we have
\[ (r \log r)^3 < (\alpha/3)^3 \log \log x. \]  
(2.7)

Choosing \( \alpha_1 \) and \( \alpha_2 \) so that \( \frac{1}{3} < \alpha_1 < \alpha_2 < (3/\alpha)^3 \), we derive at once from (2.5), (2.6) and (2.7), that
\[ \sum_{p \leq p_r} p^2 \log p < \alpha_1 p_r^3 < \alpha_2 (r \log r)^3 \]
\[ < \alpha_2 (\alpha/3)^3 \log \log x = \delta \log \log x, \]
say, where \( \delta < 1 \). Therefore
\[ p_1^2 \ldots p_r^2 = \exp \left( \sum_{p \leq p_r} p^2 \log p \right) < (\log x)^\delta. \]  
(2.8)

Now, in view of (2.4), it follows easily from (2.2) that
\[ p_t > \frac{1}{4} \log x, \]  
and the desired inequality (2.3) now follows from this and (2.8).

We can now prove (1.5). Given \( c_2 \), choose \( \alpha \) so that it satisfies
\[ c_2 < a \log 2 < 3^{4/3} \log 2, \] (2.9)

and let \( r \) and \( t \) be as defined by (2.1) and (2.2). The idea is to construct \( 2^{r-1} \) equivalence classes whose D-number representatives are less than or equal to \( x \) while the corresponding B-numbers exceed \( x \). Let

\[ u = (3, u_2, u_3, \ldots, u_r) \]

be a vector with each component \( u_i (2 \leq i \leq r) \) either 1 or 2. To each such vector \( u \) we assign the equivalence class

\[ H_u = H\left(2^{t-r+2} p_2^{u_2} p_3^{u_3} \cdots p_r^{u_r}\right), \]

and let \( d_u \) and \( b_u \) be respectively the D-number and B-number in \( H_u \).

We note immediately that \( b_u \) has exactly \( t - r + 2 + u_2 + u_3 + \cdots + u_r \) distinct prime factors. Since each \( u_i \geq 1 \), we have, by (2.2), that

\[ b_u \geq p_1 \cdots p_{t-r+2+r-1} = p_1 \cdots p_{t+1} > x. \]

Next, note that in order to prove \( d_u < x \), it suffices to construct a number \( a_u \in H_u \) such that \( a_u \leq x \). Now a suitable candidate is the number

\[ a_u = 2^7 p_2^{w_2} p_3^{w_3} \cdots p_r^{w_r} p_{r+1} \cdots p_{t-1}, \]

where \( w_i \) denotes \( \frac{u_i}{p_i - 1} \), for it is easy to verify that \( a_u \in H_u \) by computing \( d(a_u) \). Moreover, since each \( u_i \leq 2 \) we have, by (2.3) and (2.2), that

\[ a_u \leq p_2^2 p_3^2 \cdots p_r^2 p_{r+1} \cdots p_{t-1} < x. \]
We have therefore proved that $d_u < x < b_u$. Since the number of vectors $u$ considered is $2^{r-1}$, it now follows from (2.1) and (2.9) that

$$D(x) - B(x) \geq 2^{2-1} = \exp \left\{ (r-1) \log 2 \right\}$$

$$\geq \exp \left\{ \log 2 \left[ \frac{a(\log \log x)^{1/3}}{\log \log \log x} - 2 \right] \right\}$$

$$> \exp \left\{ \frac{c(\log \log x)^{1/3}}{\log \log \log x} \right\},$$

and (1.5) is proved.

1.3 PROOF OF THEOREM 1.

The idea in section 1.2 is to construct $2^{r-1}$ equivalence classes, with $r$ as large as the method allows, such that in each equivalence class the $D$-number does not exceed $x$, whereas the $B$-number exceeds $x$. Accordingly we now set

$$r = \left[ \frac{c(\log x)^{1/2}}{\log \log x} \right]. \quad (3.1)$$

We consider the vector

$$u = (u_1, u_2, \ldots, u_r)$$

where each component $u_i$ ($2 < i < r$) is either 0 or 1. The component $u_1$, which will be specified later, depends on the individual choices of $u_i$ ($2 < i < r$), so that the total number of such vectors we consider is $2^{r-1}$. To each such vector $u$ we assign the equivalence class

$$H_u = H\left\{ p_1^{u_1} p_2^{u_2} \ldots p_r^{u_r} \right\}.$$
Let \( d \) and \( b \) be the D-number and B-number in \( H_u \), and our task is to prove that

\[
d_u \leq x < b_u.
\]  

(3.2)

From the definition of a B-number, we must have

\[
b_u = g_r^{p_r-1} g_{r-1}^{p_{r-1}-1} \cdots g_2^{p_2-1} g_1 \]  

(3.3)

where

\[
g_i = p_r^{u_r} + u_{r-1}^{u_{r-1}} + \cdots + u_i^{u_i+1} \cdots p_2^{u_2} + u_1^{u_1+1}, \quad 1 \leq i \leq r,
\]

with the convention that an empty sum is 0 and an empty product is 1. Note that each \( g_i \) has exactly \( u_i \) prime factors so that

\[
d(g_i) = p_i^{u_i},
\]

and whence

\[
b_u \in H_u.
\]

Let us write

\[
G = g_r^{p_r-1} g_{r-1}^{p_{r-1}-1} \cdots g_2^{p_2-1}
\]  

(3.4)

and we note that

\[
G \leq (p_r^{r-1})^{p_r-1} < \exp (r p_r \log p_r)
\]

\[
< \exp (2r^2 \log^2 r) < x^{1/2}
\]

by (3.1). Moreover, since

\[
p_r^{u_r} + u_{r-1}^{u_{r-1}} + \cdots + u_2^{u_2+1} \leq p_r < (\log x)^{1/2},
\]

given any choice for \((u_2, u_3, \ldots, u_r)\) we can define \( u_1 \) in terms of this choice by
Indeed it is easy to show that $u_1$ satisfies

$$\frac{\log x}{10 \log \log x} < u_1 < \frac{10 \log x}{\log \log x}$$

uniformly with respect to $(u_2, u_3, \ldots, u_r)$.

From (3.3), (3.4) and (3.5) it follows at once that $b_u > x$. It remains to show that $d_u < x$, and, because of the minimal property of a D-number, it suffices to construct a number $a_u < x$ such that $a_u \in H_u$. Now a suitable candidate is the number

$$a_u = G(p_{u_r}+\ldots+u_2+1 \ldots p_{u_r}+\ldots+u_2+1)^3(p_{u_r}+\ldots+u_2+r+1 \ldots p_{u_r}+\ldots+u_2+r+1)$$

where $\ell$ satisfies $u_1 = 2r + \ell$. We note that, from (3.1) and (3.6),

$$\ell \sim u_1, \quad \text{as } x \to \infty.$$  \hfill (3.7)

Now

$$d(a_u) = d(G) 4^r 2^\ell = d(G) 2^{2r+\ell}$$

so that $a_u \in H$. Moreover, from (3.5) we see that in order to prove $a_u < x$ it suffices to show that

$$(p_{u_r}+\ldots+u_2+1 \ldots p_{u_r}+\ldots+u_2+r)^2 \leq p_{u_r}+\ldots+u_2+r+\ell+1 \ldots p_{u_r}+\ldots+u_2+2r+\ell-1 \cdot$$

(3.8)
Now the left hand side of (3.8) is at most

\[ p_{2r}^{2r} \leq \exp(2r \log r + 2r \log \log r + O(r)) \]

whereas the right hand side of (3.8) is at least

\[ p_{x}^{r-1} \geq \exp(r \log p_x + O(\log p_x)) \]

\[ \geq \exp(r \log x + r \log \log x + O(r) + O(\log p_x)) \]

\[ \geq \exp(r \log \log x + O(r)) \]

by (3.6) and (3.7). Thus (3.8) follows if

\[ 2 \log r + 2 \log \log r + O(1) \leq \log \log x , \]

and this certainly holds provided that the constant \( c_4 \) in (3.1) is sufficiently small to absorb the bounded term. Therefore (3.2) is proved, and so

\[ D(x) - B(x) = 2^{r-1} = \exp \{ (r-1) \log 2 \} , \]

and the theorem follows from (3.1) and this.

1.4 PROOF OF THEOREM 2.

It is convenient to write

\[ a = \frac{1}{2} a(a+1), \quad a = 1,2,\ldots . \]

Examples of reducible triangular numbers are:

\[ 8 = 3.3, \quad 9 = 2.5, \quad 20 = 4.6 . \]

An example of non-unique factorisation in \( R_3 \) is:

\[ 2.14 = 5.6 . \]
Indeed, even a triangular number itself may have two different sets of irreducible factors:

\[ 35 = 3 \cdot 14 = 2 \cdot 4 \cdot 6. \]

Let \( \bar{a} \) and \( \bar{b} \) be two successive irreducible numbers. There are \( b-a-1 \) reducible triangular numbers between them and we conjecture that this gap is bounded above by an absolute constant. However, the upper bound \( a/2 \) is quite sufficient for our purpose, and we now prove the following.

**Lemma 1.** Let \( 6 \leq \bar{a} < \bar{b} \). If \( \bar{a} \) and \( \bar{b} \) are successive irreducible numbers, then

\[ b-a < \frac{a}{2} \text{ or } 2b < 3a. \]

**Proof.** For \( 3 \leq a \leq 21 \), the lemma can be verified by computing \( b \). Assume therefore that \( a > 22 \). Now by a strong version of Bertrand's postulate, there is a prime \( p \) such that

\[ a < p-1 < \frac{3}{2} a. \]

Moreover the triangular number \( p-1 = \frac{1}{2} (p-1)p \) is the smallest triangular number divisible by \( p \) and is therefore necessarily irreducible. Since \( \bar{b} \) is the irreducible number immediately succeeding \( \bar{a} \) we now have

\[ a < b \leq p-1 < \frac{3}{2} a \]

and the required result follows.

**Theorem 2** can be verified by direct computation if \( x \leq 3072 \). We therefore assume that \( x > 3072 \).
Firstly, if $2^{a-1} > x$ and $\bar{a}$ is an irreducible number, then certainly $\bar{a} \in M_3(x)$. Let $m < \bar{a}$ and $m \in R_3$. We have to prove that $m \not\in M_3(x)$; that is we have to find a solution to
\[
d_3(n) = m \quad 1 \leq n \leq x. \quad (4.1)
\]
Let us denote by $\bar{b}$ the irreducible triangular number immediately preceeding $\bar{a}$ so that
\[
2^{b-1} \leq x. \quad (4.2)
\]
Now if $m$ itself is irreducible, then $m = \bar{a}_1$ where $a_1 \leq b$, and we see by (4.2) that
\[
n = 2^{a_1-1}
\]
is now a suitable solution to (4.1). We assume therefore that $m$ is a reducible number so that
\[
m = \bar{a}_1 \ldots \bar{a}_k, \quad a_1 \geq a_2 \geq \ldots \geq a_k
\]
where $k \geq 2$. We now put
\[
n = p_1^{a_1-1} \ldots p_k^{a_k-1}
\]
so that $d_3(n) = m$. In view of (4.2) we see that (4.1) will have a solution if $n \leq 2^{b-1}$; that is, if
\[
(a_1-1) \log p_1 + \ldots + (a_k-1) \log p_k \leq (b-1) \log 2.
\]
It suffices therefore to prove that
\[
(a_2-1)(\log p_2 + \ldots + \log p_k) \leq (b-a_1) \log 2,
\]
that is
\[
b-a_1 \geq \left\{\frac{\theta(p_k)}{\log 2} - 1\right\} (a_2-1) \quad (4.3)
\]
where, as usual, \( \theta(p_\ell) = \log p_1 + \ldots + \log p_\ell \).

Now from \( \bar{a}_1 \ldots \bar{a}_\ell = m < a \) we deduce that
\[
\frac{a - 1}{a_1 - 1} > \frac{a}{a_1} > \frac{a + 1}{a_1 + 1}
\]
whence
\[
\left( \frac{a - 1}{a_1 - 1} \right)^2 > \frac{a(a+1)}{a_1(a_1+1)} = \frac{\bar{a}}{\bar{a}_1} > \bar{a}_2 \ldots \bar{a}_\ell
\]
and so
\[
a - 1 > (\bar{a}_2 \ldots \bar{a}_\ell)^{1/2} (a_1 - 1).
\]

By Lemma 1 we also have \( 3b \geq 2a + 1 \) so that
\[
b - a_1 \geq \frac{2}{3} (a-1) - (a_1 - 1)
\]
and we arrive at
\[
b - a_1 > \left\{ \frac{2}{3} (\bar{a}_2 \ldots \bar{a}_\ell)^{1/2} - 1 \right\} (a_1 - 1). \quad (4.4)
\]
Here we note that (4.3) follows from (4.4) at once if
\[
\theta(p_\ell) \leq \frac{2}{3} \log 2 \left( \bar{a}_2 \ldots \bar{a}_\ell \right)^{1/2}. \quad (4.5)
\]
Since \( \bar{a}_2 \geq \ldots \geq \bar{a}_\ell \geq 3 \), we see that (4.5) certainly holds if \( \ell \geq 8 \).

To see this we note that, for \( \ell \geq 9 \),
\[
\theta(p_\ell) \leq (2 \log 2) p_\ell, \quad p_\ell \leq 3^{\frac{\ell-3}{2}}
\]
whereas the right hand side of (4.4) is
\[
\theta(p_\ell) \leq \left( \frac{2}{3} \log 2 \right) \left( \frac{\ell-1}{3^2} \right)^{\frac{\ell-1}{2}}
\]
so that we have
\[
\theta(p_\ell) \leq \left( \frac{2}{3} \log 2 \right)^{\frac{\ell-1}{2}}
\]
The theorem is proved.
and it is easy to check that this holds even for \( k = 8 \). We are now left to deal with the cases \( 2 \leq k \leq 7 \) with the additional assumption that (4.5) does not hold; that is

\[
\overline{a}_2 \ldots \overline{a}_k < \left( \frac{3\theta(p_k)}{2 \log 2} \right)^2.
\]

But this implies that

\[
\overline{a}_2 < 3^{\frac{k-2}{k}} \left( \frac{\theta(p_k)}{2 \log 2} \right)^2
\]

and we see that \( \overline{a}_2 \) is bounded above by 15, 15, 10, 10, 6 and 3 for \( k = 2, 3, 4, 5, 6 \) and 7 respectively. The corresponding upper bound for \( a_2 \) itself is thus 5, 5, 4, 4, 3 and 2 respectively. Since \( x > 3072 \) we can take \( a_1 \) large enough so that (4.4) implies (4.3). For example, we take the worst case when \( k = 2 \) and \( a_2 = 2 \).

Since \( x > 3072 = 2^{11-1} 3^{2-1} \), we see that if \( m = \overline{a}_1 \overline{a}_2 \) where \( a_1 < 11 \) and \( a_2 = 2 \), then certainly \( m \in M_3(x) \), and therefore we can safely assume that \( a_1 \geq 12 \), and now the right hand side of (4.4) is

\[
\left( \frac{2\sqrt{3}}{3} - 1 \right)(12 - 1) > 1.7
\]

whereas the right hand side of (4.3) is

\[
\log 3 \log 2 < 1.6.
\]

The theorem is proved.
1.5 **THE LEAST REDUCIBLE NUMBER IN \( M_3(x) \)**

Returning to the original missing value problem in \( M(x) \) considered by Erdős and Mirsky, we remark that, for large \( x \), one would expect not just the first number, but the initial block of numbers in \( M(x) \) to be primes. This is indeed the case; for it can be proved that, for \( x \geq 36 \), the first composite number in \( M(x) \) is \( 2p \) where \( p \) is the least prime satisfying \( 3 \cdot 2^{p-1} > x \).

The problem of identifying the first pair of consecutive numbers in \( M(x) \) seems very difficult. We conjecture that, for large \( x \), they are numbers of the form \( 2p-1, 2p \) or \( 2p, 2p+1 \) where in either case, \( p, 2p \pm 1 \) are primes. It is not known if there are infinitely many primes of the form \( 2p \pm 1 \).

One might guess that the least reducible number in \( M_3(x) \) is \( 3\overline{a} \) where \( \overline{a} \) is the smallest irreducible number satisfying \( 3 \cdot 2^{a-1} > x \); but this is not always the case. We recall that there is no unique factorisation in \( R_3 \), and therefore if \( \overline{a} \) is an irreducible number, \( 3\overline{a} \) may still be factorised into other triangular numbers. If this is so, then \( 3\overline{a} \) may not be a missing value after all. For example, \( \overline{14} \) is an irreducible number but

\[
3 \cdot \overline{14} = 5.6
\]

and so the least solution to \( d_3(n) = 3.\overline{14} \) is actually \( 2^5.3^4 \) and not \( 2^{13}.3 \). However we do have the following:

**Theorem 3.** For \( x \geq 1 \), the least reducible number in \( M_3(x) \) is the number \( 3\overline{a} \) where \( \overline{a} \) is the smallest integer satisfying
(i) $3.2^{a-1} > x$

(ii) $3^a$ has no other factorisation in $R_3$.

We shall omit the proof of this theorem since it is very similar to the proof of Theorem 2.
CHAPTER TWO

THE MAXIMUM ORDERS OF MUTIPLICATIVE FUNCTIONS

2.1 INTRODUCTION.

Let \( d(n) \) be the divisor function. It is well known (see, for example, Hardy and Wright [11], Theorem 317, p. 262) that

\[
\limsup_{n \to \infty} \frac{\log d(n) \log \log n}{\log n} = \log 2. \tag{1.1}
\]

Let \( a(n) \) denote the number of non-isomorphic Abelian groups of order \( n \). Kendall and Rankin [14] proved that

\[
\frac{1}{2} \log 2 \leq \limsup_{n \to \infty} \frac{\log a(n) \log \log n}{\log n} \leq \frac{2\pi}{\sqrt{6}},
\]

and later Kratzel [16] proved that actually

\[
\limsup_{n \to \infty} \frac{\log a(n) \log \log n}{\log n} = \frac{1}{4} \log 5. \tag{1.2}
\]

We call a positive integer \( m \) square-full if \( m \) is a product of squares and cubes. Let \( \beta(n) \) denote the number of square-full divisors of \( n \). Knopfmacher [15] proved that

\[
\limsup_{n \to \infty} \frac{\log \beta(n) \log \log n}{\log n} = \frac{1}{3} \log 3. \tag{1.3}
\]

The arguments used in the proofs of (1.2) and (1.3) are parallel to that of (1.1) given in Hardy and Wright, and the
The main purpose of this chapter is to show that the method is applicable to a general class of multiplicative functions. We shall prove:

**Theorem 1.** Let \( f(n) \) be a multiplicative function satisfying the following conditions.

(i) There exist constants \( A \) and \( \theta \) \((0 < \theta < 1)\) such that
\[
f(2^a) \leq \exp(A \theta^a), \quad a \geq 1.
\]

(ii) For all primes \( p \), and all \( a \geq 1 \),
\[
f(p^a) = f(2^a) \geq 1.
\]

Then we have
\[
\limsup_{n \to \infty} \frac{\log f(n) \log \log n}{\log n} = \log M
\]
where
\[
M = \max_{a \geq 1} (f(2^a))^{1/a}.
\]

It is also well known (Hardy and Wright [11], Theorem 432, p.359) that if \( \varepsilon > 0 \), then
\[
|\log d(n) - \log 2 \log \log n| < \varepsilon \log \log n \quad (1.4)
\]
for almost all \( n \). In [14] and [15] respectively, it is proved that
\[
\log \alpha(n) < \left\{ \frac{2\pi}{\sqrt{6}} + \varepsilon \right\} \log \log n, \quad (1.5)
\]
and
\[
\log \beta(n) < \left\{ \frac{1}{3} \log 3 + \varepsilon \right\} \log \log n, \quad (1.6)
\]
for almost all \( n \). Actually (1.5) and (1.6) are rather misleading in that they have the following drastic improvement: given any unbounded increasing function \( g(n) \), we have \( a(n) < g(n) \) and \( \beta(n) < g(n) \) for almost all \( n \). This follows from the following simple theorem.

**Theorem 2.** Let \( f(n) \) be any non-negative function satisfying the condition \( \sum_{n \leq x} f(n) = O(x) \) as \( x \to \infty \). Then, for any unbounded increasing function \( g(n) \), we have \( f(n) < g(n) \) for almost all \( n \).

### 2.2 PROOF OF THEOREM 1.

That \( M \) exists follows from

\[
1 \leq (f(2^a))^{1/a} \leq \exp(Aa^{0-1}) + 1 \text{ as } a \to \infty.
\]

We first show that

\[
\limsup_{n \to \infty} \frac{\log f(n) \log \log n}{\log n} \geq \log M. \tag{2.1}
\]

We can choose \( b \) so that \( M^b = f(2^b) \). Let \( p_1, p_2, \ldots, p_r \) be the first \( r \) primes and set

\[ n = (p_1 p_2 \cdots p_r)^b \]

so that

\[ f(n) = (f(2^b))^r = M^{br} = M^{b \pi(p_r)}. \]

From the prime number theorem we have, as \( r \to \infty \),

\[ \pi(p_r) \sim \frac{p_r}{\log p_r}, \quad \sum_{p \leq p_r} \log p \sim p_r. \]
so that

$$\log n = b \sum_{p \leq p_r} \log p \sim b p_r,$$

and

$$\log \log n = \log p_r + o(1).$$

Consequently we have, as $r \to \infty$, that

$$\log f(n) = b \log M \cdot \pi(p_r)$$

and so (2.1) is proved.

In order to prove that

$$\limsup_{n \to \infty} \frac{\log f(n) \log \log n}{\log \log n} \leq \log M$$

we shall require the following lemma.

**Lemma.** Let $A > 0$ and $0 < \theta < 1$. Then there exist constants $B$ and $\lambda$ such that

$$\frac{B}{\delta^\lambda} + a \delta \log 2 \geq A a^\theta$$

for all $a \geq 1$ and all $\delta > 0$.

**Proof.** We set

$$\lambda = \frac{\theta}{1 - \theta}, \quad B = \left(\frac{A \delta}{\lambda}\right)^\theta \left(\frac{\lambda}{\log 2}\right)^\lambda,$$

$$x = \frac{B}{\delta^\lambda}, \quad y = \frac{a \delta \log 2}{\lambda}.$$
The required result follows at once from the inequality

\[(1-\theta)x + \theta y \geq x^{1-\theta}y^\theta.\]

Returning to the proof of (2.2), let \(n = \Pi p^a\) be the factorisation of \(n\) into prime powers, and \(\delta > 0\). We have

\[
\frac{f(n)}{\delta} = \Pi \frac{f(p^a)}{\delta} = \Pi \frac{f(2^a)}{\delta}.
\]

(2.3)

From condition (i) we have, for all \(p\) and \(a\), that

\[
\frac{f(p^a)}{\delta} \leq \frac{f(2^a)}{\delta} \leq \exp(Aa^\theta - a\delta \log 2) \leq \exp \left(\frac{B}{\delta^\lambda}\right)
\]

by our lemma. For \(p \geq M^{1/\delta}\) we also have that

\[
\frac{f(2^a)}{\delta} \leq \frac{f(2^a)}{M^a} \leq 1.
\]

It now follows from (2.3) that

\[
\frac{f(n)}{\delta} \leq \Pi \text{exp} \left(\frac{B}{\delta^\lambda}\right) \leq \exp \left(\frac{BM^{1/\delta}}{\delta^\lambda}\right)
\]

so that

\[
\log f(n) \leq \delta \log n + \frac{BM^{1/\delta}}{\delta^\lambda}.
\]

Now let \(\epsilon > 0\) and set

\[
\delta = \left(1 + \frac{\epsilon}{2}\right) \frac{\log M}{\log \log n}
\]

so that

\[
\log f(n) \leq \left(1 + \frac{\epsilon}{2}\right) \frac{\log M \log n}{\log \log n} + \frac{B(\log n)}{2} \frac{1 + \frac{\epsilon}{2}}{(\log \log n)^\lambda} \frac{\log \lambda}{\log M}.
\]

\[
\leq \left(1 + \frac{\epsilon}{2}\right) \frac{\log M \log n}{\log \log n},
\]
provided that \( n \) is sufficiently large. This proves (2.2) and so completes the proof of Theorem 1.

2.3 PROOF OF THEOREM 2.

Let \( S = \{ n : f(n) \geq g(n) \} \) and suppose, if possible, that \( S \) has positive upper asymptotic density \( \delta \). Then there exists arbitrarily large \( x \) such that

\[
\sum_{n \leq x} f(n) \geq \sum_{n \in S} g(n) \geq \sum_{n \leq \delta x} g(n),
\]

since there must be at least \( \frac{1}{4} \delta x \) positive integers belonging to \( S \) in the interval \( 1 \leq n \leq x \), and \( g(n) \) is an increasing function.

We now have that

\[
\sum_{n \leq x} f(n) \geq \sum_{\frac{1}{4} \delta x \leq n \leq \delta x} g(n) \geq \frac{1}{4} \delta x g\left( \left\lceil \frac{\delta x}{4} \right\rceil \right).
\]

Since \( g(n) \) is also unbounded, this contradicts \( \sum_{n \leq x} f(n) = O(x) \), \( \sum_{n \leq x} g(n) \) as \( x \to \infty \). Therefore \( S \) must have zero asymptotic density and the theorem is proved.

2.4 REMARKS.

1. We have \( d(2^a) = a + 1 \), \( \beta(2^a) = a \) and \( \alpha(2^a) = P(a) \) where \( P(a) \) is the number of partitions of \( a \) into positive parts. It is well known (see, for example, Apostol [1], Theorem 14.7, p.316) that \( P(a) < \exp(A\sqrt{a}) \) with \( A = 2\pi/\sqrt{6} \), so that our Theorem 1 is applicable with \( \theta = \frac{1}{2} \). We see that (1.1), (1.2) and (1.3) follow from the fact that...
(a+1)^{1/a} , (P(a))^{1/a} and a^{1/a}

have maximum values at a = 1, 4 and 3 respectively.

2. It is clear from the proof of Theorem 1 that if f(n) satisfies
only the condition

\[ f(p^a) \leq \exp(Aa^a), \ p \text{ prime, } a \geq 1, \]

then we can still deduce that

\[ \lim \sup \frac{\log f(n) \log \log n}{\log n} \leq \log M^* \]

where

\[ M^* = \sup_p \left\{ \max_{a \geq 1} (f(p^a))^{1/a} \right\}. \]

3. It is shown in [14] and [15] respectively that

\[ \sum_{n \leq x} \alpha(n) \sim \prod_{r=2}^{\infty} \xi(r), \ x = (2.29...)x, \ \text{as } x \to \infty, \]

\[ \sum_{n \leq x} \beta(n) \sim \frac{\xi(2)\xi(3)}{\xi(6)} \ x = (1.94...)x, \ \text{as } x \to \infty, \]

whereas

\[ \sum_{n \leq x} d(n) \sim x \log x, \ \text{as } x \to \infty, \]

so that \( \alpha(n) \) and \( \beta(n) \) cannot be compared with \( d(n) \) in relation to

(1.4). We saw that, for \( a > 4 \), \( \alpha(p^a) \) is much larger than \( d(p^a) \),

but \( \alpha(p) = \beta(p) = 1 \) whereas \( d(p) = 2 \). Thus, as is to be expected,

the behaviour of a multiplicative function depends most heavily on

its values at the primes, and less on its values at prime powers.
CHAPTER THREE

A BRUN-TITCHMARSH THEOREM FOR

MULTIPLICATIVE FUNCTIONS

3.1 INTRODUCTION.

Let \( d(n) \) be the divisor function, and let \( a \) and \( k \) be integers satisfying

\[
0 < a < k, \quad (a,k) = 1 .
\]  (1.1)

In 1957 Linnik and Vinogradov [20] proved that if \( 0 < a < \frac{1}{2} \),

then, as \( x \to \infty \),

\[
\sum_{\substack{n \leq x \\text{n} \equiv a \pmod{k}}} d(n) \ll \frac{\phi(k)}{k^2} x \log x
\]  (1.2)

uniformly in \( a \) and \( k \), provided only that \( k < x^{1-\alpha} \). Here \( \phi(k) \) is Euler's function and the implied constant depends only on \( a \).

Their proof reduces to estimating the number \( \psi(x,y) \) of positive integers not exceeding \( x \) having no prime factor greater than \( y \).

Unfortunately they made use of a uniform upper estimate of \( \psi(x,y) \)

by A.I. Vinogradov which is incorrect (see Norton [23]). Later

Linnik [19] stated the generalisation of the problem to the functions

\[
d_r^\ell (n) = (d_r (n))^\ell , \quad r = 2, 3, \ldots , \quad \ell = 1, 2, \ldots
\]

where \( d_r (n) \) is the number of ways of writing \( n \) as a product of \( r \)
factors, taking account of ordering. The expected result is that,

as \( x \to \infty \),
\[
\sum_{\substack{n \leq x \\ n \equiv a \pmod{k}}} d_r(n) \ll \frac{x}{k} \left( \frac{\phi(k)}{k} \log x \right)^{r-1}
\]  

(1.3)

uniformly in \( k < x^{1-\alpha} \) with the implied constant depending on \( r, \ell \) and \( \alpha \). This result has been used by others (see, for example [21]) to tackle a variety of problems.

The merit of these results centres on the range of uniformity for the modulus \( k \) being large. Indeed, in the shorter range \( k < x^{2/3-\alpha} \), Selberg (unpublished manuscript) has obtained even an asymptotic formula for the sum in (1.2) (see also Hooley [13] and Heath-Brown [12]). The situation is similar to that in estimating \( \pi(x;k,a) \), the number of primes \( p \leq x \) such that \( p \equiv a \pmod{k} \); the Brun-Titchmarsh inequality

\[
\pi(x;k,a) \ll \frac{x}{\phi(k) \log \frac{x}{k}}
\]

valid uniformly in \( k < x \), is often used to supplement the asymptotic formula for \( \pi(x;k,a) \) given by the Siegel-Walfisz theorem, which is known to be valid only for a much shorter range of \( k \). Indeed, the Brun-Titchmarsh inequality is sometimes used to supplement even Bombieri's theorem; for example, in the solution to the Titchmarsh-Linnik divisor problem (see [10]).

We remark however that, unlike the Brun-Titchmarsh inequality, (1.2) cannot be extended all the way to \( k < x \). To see this, we choose \( k \) to be the largest prime less than \( x \); then the right hand side of (1.2) is of order \( \log x \), whereas we can choose \( a < k \) such that \( d(a) > \log^2 x \).
In 1971 Wolke [39] applied the method of Erdös [5] to study the sum \( \sum_{n \leq x} f(a_n) \) where \( f \) is a non-negative multiplicative function satisfying the condition that there exist positive constants \( c_1 \) and \( c_2 \) such that for all primes \( p \) and all \( l \geq 1 \),

\[
f(p^l) \leq c_1 ^l c_2 , \tag{1.4}
\]

and \( (a_n) \) is a strictly increasing sequence of positive integers. However, the results they obtained are not uniform with respect to any given class of sequences \( (a_n) \), such as the class of arithmetic progressions. Nevertheless, using their method, we can now give the generalisation of the Brun-Titchmarsh problem for a class of non-negative multiplicative functions \( f \) which satisfy conditions weaker than (1.4), and in a short interval \( x - y < n \leq x \).

In section 3.6 we apply our main theorem to give a proof of (1.3), and to other results. In section 3.7 we use the method to give a new proof, and a generalisation, of a famous result of Turan, (see [37]) namely that, as \( x \to \infty \),

\[
\sum_{n \leq x} w^2(n) = x(\log \log x)^2 + O(x \log \log x) \tag{2.1}
\]

Here \( w(n) \) denotes the number of distinct prime factors of \( n \).
3.2 THE MAIN THEOREM.

We shall consider the class $M$ of arithmetic functions of which are non-negative, multiplicative and satisfy the following two conditions.

(i) There exists a positive constant $A_1$ such that

$$f(p^k) ≤ A_1^k, \quad p \text{ prime}, \quad k ≥ 1.$$ 

(ii) For every $\varepsilon > 0$, there exists a positive constant $A_2 = A_2(\varepsilon)$ such that

$$f(n) ≤ A_2 n^\varepsilon, \quad n ≥ 1.$$ 

The following is our main theorem.

**Theorem 1.** Let $f \in M$, $0 < \alpha < \frac{1}{2}$, $0 < \beta < \frac{1}{2}$ and let $a, k$ be integers satisfying

$$0 < a < k, \quad (a,k) = 1.$$ 

Then, as $x \to \infty$,

$$\sum_{\substack{x-y<n≤x \\ n≡a(\text{mod } k)}} f(n) ≤ \frac{y}{\phi(k)} \frac{1}{\log x} \exp \left( \sum_{p≤x} \frac{f(p)}{p} \right), \quad (2.1)$$

uniformly in $a, k$ and $y$ provided that

$$k < y^{\frac{1-\alpha}{\beta}}, \quad x^\beta < y ≤ x. \quad (2.2)$$

We make the following remarks on our conditions (i) and (ii).

First, the condition (1.4) clearly implies (i) and we now show that it also implies (ii). Since $f(n)$ is multiplicative we only need to consider the case $n = p$. Choose $c_1, c_2$ accordingly and
let $\epsilon > 0$. There exists $A_2$ such that, for all $\ell$,
\[ c_1 \ell^c_2 \leq A_2 (2^\epsilon \ell) \]

It follows that
\[ f(n) = f(p^\ell) \leq c_1 \ell^c_2 \leq A_2 (2^\epsilon \ell) \]
\[ \leq A_2 \ell^{\epsilon \ell} = A_2 n^\epsilon . \]

We next show that there exists $f \in M$ such that (1.4) does not hold. Consider the function
\[ f(n) = \begin{cases} \exp \left( \frac{\log n}{\log \log n} \right) & \text{if } n = 3^\ell, \quad \ell \geq 3, \\ 1 & \text{otherwise.} \end{cases} \]
We see that $f(n)$ is multiplicative and
\[ f(3^\ell) = \exp \left( \frac{\ell \log 3}{\log \ell + \log \log 3} \right), \quad \ell \geq 3. \]
With $A = e^5$ we have
\[ f(3^\ell) \leq \exp \left( 5\ell \right) = A^\ell . \]
Thus condition (i) is satisfied, and clearly (ii) also holds, so that $f \in M$. But, for any fixed $c_1, c_2$ we have
\[ f(3^\ell) \geq \exp \left( \sqrt{\ell} \right) > c_1 \ell^c_2 \]
for sufficiently large $\ell$. Therefore (1.4) cannot hold.

Next we note that if $f \in M$ and $\delta > 0$, then there exists $A_3 = A_3(\delta)$ such that
\[ \sum_{p} \sum_{\ell=2}^{\infty} \frac{f(p^\ell)}{p^{\ell \delta}} \leq A_3 . \]

(2.3)
Finally we point out that the rather strong condition (ii) is actually vital to the truth of the theorem, and, in particular, we cannot replace it by (2.3). For suppose that (ii) does not hold; then there exist \( \epsilon_0 > 0 \) and a strictly increasing sequence of positive integers \( (n_r) \) such that \( f(n_r) > n_r^{\epsilon_0} \). We can now show that (2.1) does not hold by setting \( \alpha = \frac{1}{10} \min (\epsilon_0, 1) \). We define \( a, k, x \) and \( y \) as follows: For large \( r \), we put

\[
a = n_r,
\]

\[
k = \text{the least prime in the interval } (a, 2a),
\]

\[
x = y = k^{1-\alpha} + 1.
\]

Then the left hand side of (2.1) is at least \( f(a) > a^{\epsilon_0} \), whereas the right hand side is at most

\[
\frac{y}{\phi(k)} y^{1+\alpha} \leq \frac{1+\alpha}{k-1} \leq \frac{1}{k-1} - 1 \leq \frac{3\alpha}{k} \leq \frac{3\alpha}{a} \leq \frac{3\epsilon_0}{10}.
\]

We note, in particular, that Theorem 1 cannot be extended to cover the case when \( f(n) = 2^{\Omega(n)} \), where \( \Omega(n) \) denotes the total number of prime factors of \( n \).

**Lemma 1.** For all sufficiently large \( x \), we have

\[29\]

3.3 **NOTATIONS.**

The letters \( a, b, d, h, i, j, k, t, m, n, r \) and \( s \) are used to denote positive integers and we shall assume that (1.1) always holds. The letters \( p \) and \( q \) are reserved for prime numbers. We let \( p(n) \) and \( q(n) \) denote the greatest and the least prime factors of \( n \) respectively. We let \( \omega(n) \) denote the number of distinct prime factors of \( n \) while \( \Omega(n) \) is the total number of prime factors.
of \( n \), taking account of multiplicity. We also define \( p(l), q(l), w(l), \Omega(l) \) to be 1. As usual \( r(n) \) is the number of ways of writing \( n \) as a sum of two squares. We call a positive integer \( s \) a square full integer if \( s \) is a product of squares and cubes, and we let \( \delta(n) \) be the number of square full divisors of \( n \).

The letters \( x, y, z, \alpha, \beta, \delta \) and \( \epsilon \) denote positive real numbers and \( \lambda \) is a real number. We put

\[
\psi(x,z) = \sum_{n \leq x \atop p(n) \leq z} 1
\]

and

\[
\phi(x,y,z;k,a) = \sum_{x-y<n \leq x \atop n \equiv a (\text{mod } k) \atop q(n) > z} 1.
\]

All the constants implied by the \( \ll \) and \( O \)-notations depend at most on \( \alpha, \beta \) and \( \epsilon \).

### 3.4 Preliminary Lemmas.

We shall require the following lemmas.

**Lemma 1.** For all sufficiently large \( x \), we have

\[
\psi(x, \log x \log \log x) \ll \exp \left( \frac{3 \log x}{(\log \log x)^{1/2}} \right).
\]

**Proof.** As a consequence of the prime number theorem, we have, for all large \( y \), that

\[
\sum_{p \leq y} \frac{1}{\log p} \ll \frac{2y}{\log y}.
\]

(4.1)
We now use Rankin's method (see [25]). Let $\delta > 0$. We have, for large $y$,

$$\psi(x, y) = \sum_{n \leq x} \frac{1}{\sqrt{n}} \sum_{\mathfrak{p} \leq y} \frac{1}{\mathfrak{p}^\delta} \ll x^\delta \prod_{\mathfrak{p} \leq y} \left(1 + \frac{1}{\mathfrak{p}^\delta - 1}\right) \ll \exp \left(\delta \log x + \sum_{\mathfrak{p} \leq y} \frac{1}{\log \mathfrak{p}}\right) \ll \exp \left(\delta \log x + \frac{2y}{\delta \log 2}\right)$$

by (4.1). Now put $y = \log x \log \log x$ and set $\delta = (\log \log x)^{-\frac{1}{2}}$ and we see that the result follows.

**Lemma 2.** Let $a$ and $k$ satisfy (1.1) and suppose that $k < y \leq x$ and $z \geq 2$. Then

$$\Phi(x, y, z; k, a) \ll \frac{y}{\Phi(k) \log z + z^2}.$$

This can be proved using the simplest Selberg upper bound sieve method; see [10], p.104.

**Lemma 3.** Let $f \in \mathbb{M}$. Then as $x \to \infty$,

$$\sum_{n \leq x} \frac{f(n)}{n} \ll \exp \left(\sum_{\mathfrak{p} \leq x} \frac{f(p)}{p}\right),$$

uniformly in $k$. 

**Proof.** As in (4.3) and (4.1). We have, 

$$\sum_{n \leq x} \frac{f(n)}{n} \ll \exp \left(\sum_{\mathfrak{p} \leq x} \frac{f(p)}{p}\right),$$

uniformly in $k$. 

This lemma corresponds to Lemma 3 in [10] except that we generalize to the non-integer situation $(n,k)=1$, and replace the constants involved are independent of $k$, and cut class $M$ is
Proof. Since $f \in M$, we have, by (2.3), that
\[
\sum_{n \leq x, (n,k)=1} \frac{f(n)}{n} \ll \prod_{p \leq x, p \nmid k} \left( 1 + \frac{f(p)}{p} + \sum_{l=2}^{\infty} \frac{f(p^l)}{p^l} \right)
\]
\[
\ll \exp \left( \sum_{p \leq x, p \nmid k} \frac{f(p)}{p} + \sum_{l=2}^{\infty} \frac{f(p^l)}{p^l} \right) \quad \text{and, by condition (ii).}
\]
\[
= \exp \left( \sum_{p \leq x, p \nmid k} \frac{f(p)}{p} + O(1) \right) \ll \exp \left( \sum_{p \leq x, p \nmid k} \frac{f(p)}{p} \right).
\]

Lemma 4. Let $f \in M$. Then, as $z \to \infty$
\[
\sum_{n \leq z, p(n) \leq z^{1/r}, (n,k)=1} \frac{f(n)}{n} \ll \exp \left( \sum_{p \leq z, p \nmid k} \frac{f(p)}{p} - \frac{1}{10} r \log r \right)
\]
uniformly in $k$ and $r$, provided that $1 \leq r \leq \frac{\log z}{\log \log z}$.

This lemma corresponds to Lemma 3 in [39] except that we generalise it to incorporate the condition $(n,k) = 1$, and replace the implicit constant for the coefficient of $r \log r$ by an absolute constant. Since it is crucial for our application that the constants involved are independent of $k$, and our class $M$ is larger than Wolke's, we shall give the detailed proof here.

Proof. Again we use Rankin's method. Let $\frac{3}{4} \leq \delta \leq 1$. We have,
by (2.3), that

\[ \sum_{n \gg x}^{\infty} \frac{f(n)}{p} \leq x^{\delta-1} \sum_{n \gg x}^{\infty} \frac{f(n)}{n^\delta} \leq x^{\delta-1} \sum_{n \gg 1}^{\infty} \frac{f(n)}{n^\delta} \]

\[ = x^{\delta-1} \prod_{p \leq h} \left(1 + \frac{f(p)}{p} + \sum_{\ell=2}^{\infty} \frac{f(p^\ell)}{p^\ell} \right) \]

\[ \leq \exp\left\{ (\delta-1)\log x + \sum_{p \leq h} \frac{f(p)}{p} + O(1) \right\} . \]

Now \( \frac{f(p)}{p^\delta} = \frac{f(p)}{p} + \frac{f(p)}{p} (p^{1-\delta} - 1) \), and, by condition (i),

\[ \sum_{p \leq h} \frac{f(p)}{p} (p^{1-\delta} - 1) \leq \sum_{p \leq h} \frac{A_1}{p} \sum_{n=1}^{\infty} \frac{((1-\delta) \log p)^n}{n!} \]

\[ \leq A_1 \sum_{n=1}^{\infty} \frac{((1-\delta) \log p)^n}{n!} \sum_{p \leq h} \frac{\log p}{p} \]

and the required

\[ \leq 2A_1 \sum_{n=1}^{\infty} \frac{((1-\delta) \log y)^n}{n!} \leq 2A_1 \exp\left\{ (1-\delta) \log y \right\} \]
Therefore we have

\[ \sum_{n \geq x} \frac{f(n)}{n} \ll \exp\left( (\delta - 1) \log x + \sum_{p \leq y} \frac{f(p)}{p} + 2A \frac{1}{y^{1-\delta}} \right). \]

Now put \( x = \frac{z}{2} \), \( y = z^r \) and set \( \delta = 1 - \frac{r \log r}{4 \log z} \).

Note that if \( 1 \leq r \leq \log z / \log \log z \), then \( r \log r < \log z \) and so \( \frac{3}{4} < \delta \leq 1 \). Moreover, we now have

\[ (1-\delta) \log x = \frac{r \log r}{4 \log z} \cdot \frac{1}{2} \log z = \frac{1}{8} r \log r \]

and

\[ y^{1-\delta} = z^r, \quad \frac{1-\delta}{r} = \frac{\log r}{4 \log z} = \frac{1}{r^4}. \]

We now have

\[ \sum_{n \geq x} \frac{f(n)}{n} \ll \exp\left( \sum_{p \leq z^{1/r}} \frac{f(p)}{p} - \frac{1}{8} r \log r + 2A_1 \frac{1}{r^4} \right) \]

and the required result follows at once.

3.5 PROOF OF THE MAIN THEOREM.

Let \( k \) and \( y \) satisfy (2.2) and put

\[ z = y^{10}. \]  

For each \( n \) satisfying \( x - y < n < x \), \( n \equiv a \pmod{k} \) we express \( n \) in the form

\[ n = p_1^{s_1} \cdots p_j^{s_j} p_{j+1}^{s_{j+1}} \cdots p_k^{s_k} = b_n d, \quad (p_1 < p_2 < \cdots < p_k), \]
where \( b_n \) is chosen so that
\[
    b_n \leq z < b_{n+1}.
\]  

We divide the set of such integers \( n \) into the following classes:

I. \( q(d_n) > z^2 \);

II. \( q(d_n) \leq z^2, \ b_n \leq z^2 \);

III. \( q(d_n) \leq \log x \log \log x, \ b_n > z^2 \);

IV. \( \log x \log \log x < q(d_n) \leq z^2, \ b_n > z^2 \).

First we have
\[
\sum_{n \in I} f(n) = \sum_{n \in I} f(b_n) f(d_n) \leq \sum_{b \leq z} f(b) \sum_{x-y \leq n \leq x} f(\frac{n}{b})
\]
from Legendre's identity we arrive at
\[
= \sum_{b \leq z} f(b) \sum_{(b,k)=1} {x - \frac{y}{b} < d \leq \frac{x}{b}} f(d) \quad \text{where} \quad a' \equiv a \ (\text{mod} \ k), \quad \frac{a}{\alpha} = \frac{20}{20} \quad \text{and hence, by condition (i)},
\]

so that
\[
\Omega(d) \leq \frac{\log x}{\log q(d)} < \frac{20}{a\beta},
\]
and hence, by condition (i),
\[
f(d) \leq \Omega(d) \leq A_1 \frac{20}{20}.
\]
We have therefore

\[ \sum_{n \in I} f(n) \ll \sum_{b \leq z} f(b) \phi \left( \frac{x}{b}, \frac{y}{b}, \frac{z^2}{k}, a^t \right). \]

Since \( k < y^{1-a} \) and \( b \leq z < y^a \), so that \( kb < y \), it follows from Lemma 2 that

\[ \phi \left( \frac{x}{b}, \frac{y}{b}, \frac{z^2}{k}, a^t \right) \ll \frac{2y}{\phi(k)b \log z + z}, \]

and therefore

\[ \sum_{n \in I} f(n) \ll \left\{ \frac{y}{\phi(k) \log z + z} + z^2 \right\} \sum_{b \leq z} \frac{f(b)}{b}. \]

From Lemma 3 we arrive at

\[ \sum_{n \in I} f(n) \ll \left\{ \frac{y}{\phi(k) \log z + z} + z^2 \right\} \exp \left( \sum_{p \leq z} \frac{f(p)}{p} \right). \quad (5.3) \]

Next, to each \( n \in I \), there correspond \( p \) and \( s \) such that

\[ p^s \mid n, \ p \leq z^2 \] and by (5.2) \( p^s > z^2 \). Let \( s_p \) denote the least positive integer \( s \) satisfying \( p^s > z^2 \) so that \( s_p \geq 2 \) and hence

\[ -s_p \leq \min(z^{1/2}, p^{-2}); \] thus

\[ \sum_{p \leq z^{1/2}} \frac{1}{s_p} \leq \sum_{p \leq z^{1/4}} z^{-1/2} + \sum_{p > z^{1/4}} p^{-2} \ll z^{-1/4}. \]
It now follows that

\[ \sum_{n \in \mathbb{II}} 1 \leq \sum_{p \leq z^{1/2}} \sum_{x-y \leq n \leq x} \frac{1}{n^{\alpha} \mod k} \]

\[ n \equiv 0 \mod p \]

\[ = \sum_{p \leq z^{1/2}} \left\{ \frac{y}{k} + O(1) \right\} \ll \frac{y}{k} z - \frac{1}{4} + \frac{1}{2}. \quad (5.4) \]

Suppose next that \( n \in \mathbb{III} \). Then there exists \( b \) such that

\[ \frac{1}{b} \mid n, \quad z^2 < b \leq z, \]

and

\[ p(b) < \log x \log \log x. \]

Consequently we have

\[ \sum_{n \in \mathbb{III}} 1 \leq \sum_{z^{1/2} < b \leq z} \sum_{x-y \leq n \leq x} \frac{1}{n^{\alpha} \mod k} \]

\[ n \equiv 0 \mod b \]

\[ = \sum_{z^{1/2} < b \leq z} \left\{ \frac{y}{kb} + O(1) \right\} \ll \frac{y}{z} \cdot \frac{\log x \log \log x}{z} + O(z) \ll \frac{y}{k} z - \frac{1}{4} + z. \quad (5.5) \]

by Lemma 1. Since \( k < y^{1-\alpha} \) we have, by (5.1),

\[ z < y^{\alpha} z^4 < \frac{y}{k} z^4, \]

so that \( \log x \log \log x \) is small and that \( \frac{y}{k} \) is

\[ \frac{z^4}{z^{1/2}} \]

\[ \frac{y}{k} \]
and so, from (5.4) and (5.5) we have

\[ \sum_{n \in \text{II}} 1 + \sum_{n \in \text{III}} 1 \ll \frac{y}{k} z^{-\frac{1}{4}}. \]

For \( y > x^\beta \) we have, by condition (ii) and (5.1), that

\[ f(n) \ll n \ll x < y \ll z^\frac{1}{8} \]

so that we arrive at

\[ \sum_{n \in \text{II}} f(n) + \sum_{n \in \text{III}} f(n) \ll \frac{y}{k} z^{-\frac{1}{8}}. \quad (5.6) \]

Lastly we deal with the class IV. We have

\[ \sum_{n \in \text{IV}} f(n) = \sum_{n \in \text{IV}} f(b_n) f(d_n) \ll \sum_{n \in \text{IV}} f(b_n) \sum_{n/a \equiv b \pmod{k}} f(d_n). \]

Let us put

\[ r_0 = \left[ \frac{\log z}{\log(\log x \log \log x)} \right], \quad (5.7) \]

so that \( \log x \log \log x > z^{r_0} \). Let \( 2 \leq r \leq r_0 \) and consider those

\[ n \text{ for which } z^{r+1} < q(d_n) \leq z^r. \]

For such \( n \), we have \( p(b_n) = p(b) < q(d_n) < z^r \) and moreover, as before,
\[
\Omega(d_n) \leq \frac{\log x}{\log q(d_n)} \leq \frac{(r+1)\log x}{\log x} < \frac{10(r+1)}{a\beta} < 20r/a\beta
\]

so that \( f(d_n) \leq \Omega(d_n) \leq A_1^r \), \( A_5 = A_1^{a\beta} \).

It follows that

\[
\sum_{n \in \mathbb{N}} f(n) \leq \sum_{2 \leq r \leq r_0} A_5^r \sum_{z^{1/2} < b \leq z} \sum_{x-y < n < x} f(b) \sum_{n=a (\text{mod } k)}^{x-y+n < x} 1
\]

and (3.3) and

\[
\sum_{b \geq 2} \sum_{b \leq z} f(b) \phi \left( \frac{x}{b}, \frac{y}{b}, \frac{z^{r+1}}{k}, a' \right)
\]

where \( a' \equiv ab \), \( b^b \equiv 1 (\text{mod } k) \). From Lemma 2 we have

\[
\phi \left( \frac{x}{b}, \frac{y}{b}, \frac{z^{r+1}}{k}, a' \right) \leq \frac{y(r+1)}{\phi(b) \log z + z^{r+1}}
\]

and therefore

\[
\sum_{n \in \mathbb{N}} f(n) \leq \left( \frac{y}{\phi(k) \log z + z^{r+1}} \right) \left( \frac{r+1}{2}\right) \sum_{2 \leq r \leq r_0} A_5^r \sum_{z^{1/2} < b \leq z} \sum_{p(b) < z^{1/r}} f(b) \cdot
\]

From (5.7) we see that we can apply Lemma 4 to the inner sum here giving
\[
\sum_{n \in \mathbb{IV}} f(n) \ll \left( \frac{y}{\phi(k) \log z} + z^2 \right) \exp \left( \sum_{p \leq z} \frac{f(p)}{p} \right) \sum_{p \mid k} r \Lambda_p \exp \left( -\frac{1}{10} r \log r \right).
\]

For \( k < y^{1-\alpha} \) we have, by (5.1),

\[
z^2 < \frac{k}{\phi(k) \log z} < \frac{y}{\phi(k) \log z} < \frac{y}{\phi(k) \log z}.
\]

From (5.3) and (5.8) we now have

\[
\sum_{n \in \mathbb{I}} f(n) + \sum_{n \in \mathbb{IV}} f(n) \ll \frac{y}{\phi(k) \log z} \exp \left( \sum_{p \leq z} \frac{f(p)}{p} \right)
\]

and this, together with (5.6), gives the desired result (2.1).

### 3.6 APPLICATIONS OF THE MAIN THEOREM.

It is easy to see that the expected result (1.3) is an immediate consequence of our main theorem; indeed we have:

**Theorem 2.** Let \( \alpha, \beta, \lambda \) be real numbers and let \( a, k, r \) be integers.

Suppose that

\[
0 < \alpha < \frac{1}{2}, \quad 0 < \beta < \frac{1}{2}, \quad r \geq 2, \quad 0 < a < k, \quad (a,k) = 1.
\]

We have, as \( x \to \infty \):
\[
\sum_{\substack{1 \leq y < n \leq x \\ n \equiv a \pmod{k}}} d_r^\lambda(n) \ll \frac{y}{k} \left( \frac{k}{\log x} \right)^{r^\lambda - 1}
\]

uniformly in \(a, k\) and \(y\) provided that

\[k < y^{1-\alpha}, \quad x^\beta < y \leq x.\]

**Proof.** We put

\[f(n) = d_r^\lambda(n)\]

so that \(f(n)\) is multiplicative. Also, given any fixed \(r\) there exists \(c = c(r)\) such that

\[d_r(p^\ell) \leq c \ell^{r^\lambda - 1}, \quad p \text{ prime}, \quad \ell \leq 1,
\]

so that (1.4) holds, and hence \(f \in M\). Next we have

\[f(p) = d_r^\lambda(p) = r^\lambda
\]

so that, for \(k < y^{1-\alpha}\),

\[
\sum_{p \leq x \atop p \nmid k} \sum_{p \leq x \atop p \nmid k} \frac{f(p)}{p} = r^\lambda \left\{ \sum_{p \leq x} \frac{1}{p} - \sum_{p \mid k} \frac{1}{p} \right\}
\]

where the implied constant is absolute. Since

\[
\exp \left( - r^\lambda \sum_{p \mid k} \frac{1}{p} \right) = \prod_{p \mid k} \exp \left( - \frac{r^\lambda}{p} \right)
\]

uniformly in \(a, k\) and \(y\) provided that

\[
\sum_{\substack{1 \leq y < n \leq x \\ n \equiv a \pmod{k}}} d_r^\lambda(n) \ll \frac{y}{k} \left( \frac{k}{\log x} \right)^{r^\lambda - 1}
\]
\[
<< \prod_{p \mid k} \left(1 - \frac{\lambda}{p}\right) \ll \prod_{p \mid k} \left(1 - \frac{1}{p}\right)^{R^\lambda} = \left(\frac{\phi(k)}{k}\right)^{R^\lambda}
\]

we see that

\[
\exp\left(\sum_{p \leq x} f(p) \frac{\lambda}{p} \right) \ll \left(\frac{\phi(k)}{k} \log x\right)^{R^\lambda}
\]

where the implied constant depends on \( r \) and \( \lambda \). The required result therefore follows from Theorem 1.

Smith [34] has obtained an asymptotic formula for the sum

\[
\sum_{n \leq x} r(n) \quad n \equiv a (\text{mod } k)
\]

valid for \( k < x^{2/3-\alpha} \). Here we have:

**Theorem 3.** Let \( \alpha, \beta, \lambda \) be real numbers and let \( a, k \) be integers.

Suppose that

\[
0 < \alpha < \frac{1}{2}, \quad 0 < \beta < \frac{1}{2}, \quad 0 < a < k, \quad (a,k) = 1.
\]

We have, as \( x \to \infty \),

\[
\sum_{x-y < n \leq x \atop n \equiv a (\text{mod } k) \atop r(n) > 0} r^\lambda(n) \ll \frac{\lambda}{\phi(k)} \prod_{p \mid k} \left(1 - \frac{2^\lambda}{p}\right) (\log x)^{2^{\lambda-1}-1} \quad (6.1)
\]

uniformly in \( a,k \) and \( y \) provided that
Note that if we put $\lambda = 0$, then (6.1) becomes

$$\sum_{x-y<n<x; \ n=a \ (mod \ k); \ r(n)>0} 1 \ll \frac{1}{\phi(k)} \prod_{p|k, \ p \equiv 1 \ (mod \ 4)} \left( 1 - \frac{1}{p} \right) \cdot \frac{y}{\sqrt{\log x}}.$$ 

Proof of Theorem 3. Here we let

$$f(n) = \begin{cases} \left( \frac{r(n)}{4} \right) & \text{if } r(n) > 0, \\ 0 & \text{if } r(n) = 0, \end{cases}$$

so that $f(n)$ is multiplicative and it is clear that $f \in M$.

We have

$$f(p) = \begin{cases} 2^\lambda & \text{if } p \equiv 1 \ (mod \ 4), \\ 0 & \text{if } p \equiv 3 \ (mod \ 4), \\ 1 & \text{if } p = 2. \end{cases}$$

Also, by Merten's theorem,

$$\sum_{p \leq x; \ p \equiv 1 \ (mod \ 4)} \frac{1}{p} = \frac{1}{2} \log \log x + O(1), \quad x \to \infty,$$

so that we have, for $k < y^{1-\alpha}$,

$$\sum_{p \leq x; \ p \equiv 1 \ (mod \ 4)} \frac{f(p)}{p} = 2^\lambda \left( \frac{1}{2} \log \log x - \sum_{p|k, \ p \equiv 1 \ (mod \ 4)} \frac{1}{p} + O(1) \right).$$
Now
\[
\exp\left\{ -\sum_{\substack{p|k \atop p \equiv 1 \pmod{4}}} \frac{2\lambda}{p} \right\} = \prod_{\substack{p|k \atop p \equiv 1 \pmod{4}}} \exp\left\{ -\frac{2\lambda}{p} \right\} \quad \ll \prod_{\substack{p|k \atop p \equiv 1 \pmod{4}}} \left( 1 - \frac{2\lambda}{p} \right)
\]
\quad \text{when } p > 2^\lambda
\]
and the required result follows from Theorem 1.

Let \( \delta(n) \) denote the number of square full divisors of \( n \).

Knopfmacher \([15]\) has obtained an asymptotic formula for the sum
\[
\sum_{\substack{n \leq x \atop n \equiv a \pmod{k}}} \delta(n)
\]
valid for \( k < x^{7/36-\alpha} \). As our last application of our main theorem we have:

**Theorem 4.** Let \( \alpha, \beta, \lambda \) be real numbers and \( a, k \) be integers.

Suppose that
\[
0 < \alpha < \frac{1}{2}, \quad 0 < \beta < \frac{1}{2}, \quad 0 < a < k, \quad (a,k) = 1.
\]
We have, as \( x \to \infty \),

(i) The implied constant depends on \( \beta \) and \( \alpha \).

(ii) If \( k \) varies we may only include the first term in (7.1) in only \( \Theta(x \log x) \). This is clear from the proof.

\[
\sum_{\substack{x-y<n\leq x \atop n \equiv a \pmod{k}}} \delta^\lambda(n) \ll \frac{\sqrt{y}}{k}
\]
uniformly in \( a,k \) and \( y \) provided that

**Proof.** Consider first the case \( a \neq 0 \). For \( n \neq x \) we have

\[
k < y^{1-\alpha}, \quad x^\beta < y \leq x.
\]
Proof. Let \( f(n) = \delta_x(n) \). Then clearly \( f \in M \) and \( f(p) = 1 \) for all \( p \). As in the proof of Theorem 2, we have

\[
\exp \left( \sum_{\substack{p \leq x \in M \backslash \{y\} \mid p \mid k}} \frac{1}{p} \right) \ll \frac{\phi(k)}{k} \log x, \quad k < y^{1-\alpha}
\]

and so the required result follows from Theorem 1.

3.7 A GENERALISATION OF TURAN'S LEMMA.

We shall prove the following generalisation of Turan's lemma.

**Theorem 5.** Let \( 0 < \alpha < \frac{1}{2} \) and \( a, k, \ell \) be integers satisfying

\[
0 < a < k, \quad (a,k) = 1, \quad \ell > 0.
\]

Let \( f(n) \) denote either \( w(n) \) or \( \Omega(n) \). Then, as \( x \to \infty \),

\[
\sum_{\substack{n \leq x \equiv \ell \pmod{k} \atop n \in A \pmod{k}}} f(n) = \frac{x}{k} (\log \log x)^\ell + O \left( \frac{x}{k} (\log \log x)^{\ell-1} \log \log \log x \right), \quad (7.1)
\]

uniformly in \( a \) and \( k \) provided that \( k < x^{1-\alpha} \).

**Remarks.**

(i) The implied constant depends on \( \ell \) and \( \alpha \).

(ii) If \( k \) varies boundedly the error term in (7.1) is only \( O(x(\log \log x)^{\ell-1}) \). This is clear from the proof.

**Proof.** Consider first the case \( \ell = 1 \). For \( n \leq x \) we have

\[
\delta(n) = \delta(n_2) + O(1)
\]
\[ w(n) = \sum_{\substack{p | n \leq x \leq \alpha \atop p \nmid x}} 1 = \sum_{\substack{p | n \leq x \leq \alpha \atop p \nmid x}} 1 + O(1) \]

since
\[ \sum_{\substack{p | n \leq x \leq \alpha \atop p \nmid x}} 1 < \frac{1}{\alpha} . \]

It follows that
\[ w(n) = \sum_{\substack{n \leq x \leq \alpha \atop n \equiv a \pmod{k}}} \left\{ \sum_{\substack{p | n \leq x \leq \alpha \atop p \nmid x}} 1 + O(1) \right\} \]
\[ = \sum_{\substack{p | x \leq \alpha \atop n \equiv a \pmod{k}}} 1 + O\left(\frac{x}{k}\right) \]
\[ = \sum_{\substack{p | x \leq \alpha \atop p \nmid x}} 1 + O\left(\frac{x}{k}\right) \]
\[ = \sum_{\substack{p | x \leq \alpha \atop p \nmid x}} 1 + O\left(\frac{x}{k}\right) \]
where \( a' \equiv ab \pmod{k} \). Since \( x^{\alpha} < x/k \) we now have
\[ \sum_{\substack{n \leq x \leq \alpha \atop n \equiv a \pmod{k}}} w(n) = \sum_{\substack{p | x \leq \alpha \atop p \nmid k}} \left\{ \sum_{\substack{p' | x \leq \alpha \atop p' \nmid p}} \frac{1}{p' + O(1)} \right\} + O\left(\frac{x}{k}\right) \]
\[ = \sum_{\substack{p | x \leq \alpha \atop p \nmid k}} \frac{1}{p} + O\left(\frac{x}{k}\right) . \hspace{1cm} (7.2) \]

By writing \( n = n_1 n_2 \) where \( p(n_2) \leq x^{\alpha} < q(n_1) \) we see that
\[ \Omega(n) = \Omega(n_2) + O(1) \]
so that
\[
\sum_{n \equiv a \pmod{k}} \Omega(n) = \sum_{n \equiv a \pmod{k}} \Omega(n_2) + 0\left(\frac{x}{k}\right)
\]

\[
= \sum_{n \equiv a \pmod{k}} \left\{ \sum_{m \leq x} \sum_{p \leq x} \alpha \right\} + 0\left(\frac{x}{k}\right)
\]

\[
= \sum_{m \leq \frac{\log x}{\log 2}} \sum_{p \leq x} \sum_{n \equiv a \pmod{k}} \left\{ \frac{x}{m} + 0(1) \right\} + 0\left(\frac{x}{k}\right).
\]

Now
\[
\sum_{2 \leq m \leq \frac{\log x}{\log 2}} \sum_{p \leq x} \left\{ \frac{x}{m} + 0(1) \right\} << \frac{x}{k} \sum_{p} \sum_{m \geq 2} \frac{1}{m} + x(\alpha) \log x
\]

since \(\omega(x) = x \log \log x\). The required result (7.1) therefore follows from (7.3).

We next proceed to prove the general case by induction on \(k\).

We use as induction hypothesis that (7.1) holds uniformly in \(a\) and \(k\) provided \(\omega_k = \omega_k\log \log x\) and we consider

\[
\sum_{n \equiv a \pmod{k}} \Omega(n) = \frac{x}{k} \sum_{p \leq x} \frac{1}{p} + 0\left(\frac{x}{k}\right)
\]

and, together with (7.2) we have
\[
\sum_{n \leq x} f(n) = \frac{x}{k} \sum_{\substack{p \leq x^a \backslash \mathfrak{p} \mid k}} \frac{1}{p} + O\left(\frac{x}{k}\right). \tag{7.3}
\]

Now
\[
\sum_{p \leq x} \frac{1}{p} = \sum_{p \leq x \backslash \mathfrak{p} \mid k} \frac{1}{p} + O(1)
\]
\[
= \sum_{p \leq x} \frac{1}{p} + O\left(\sum_{p \mid k} \frac{1}{p}\right) + O(1),
\]

and
\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + O(1),
\]
and
\[
\sum_{p \mid k} \frac{1}{p} \leq \sum_{p \leq \log x} \frac{1}{p} + \sum_{p \mid k} \frac{1}{p} \quad p > \log x
\]

\[
<< \log \log \log x + \frac{w(k)}{\log x}
\]

<< \log \log \log x,
since \(w(k) \ll \log k < \log x\). The required result (7.1) therefore follows from (7.3).

We next proceed to prove the general case by induction on \(k\).

We use as induction hypothesis that (7.1) holds uniformly in \(a\) and \(k\) provided that \(k < x^{1-\alpha/2}\). Suppose now that \(k < x^{1-\alpha}\) and we consider
\[
\sum_{n \leq x} (w(n))^x = \sum_{n \leq x} w(n)\left\{\sum_{p \mid n} 1 + O(1)\right\}
\]

\[
\sum_{n \leq x} (w(n))^x = \sum_{n \leq x} \alpha/2
\]
\[
\sum_{\substack{p \leq x^{1/2} \\
\text{p divides } k}} \sum_{\substack{m \leq x/p \\ m \equiv \alpha (\text{mod } k) \\ m \equiv 0 (\text{mod } p)}} w^\ell(n) + O\left(\frac{x}{k} (\log \log x)^\ell\right)
\]

\[
= \sum_{\substack{p \leq x^{1/2} \\
\text{p divides } k}} \sum_{\substack{m \leq x/p \\ m \equiv \alpha' \text{ (mod } k) \text{ (mod } p)}} w^\ell(mp) + O\left(\frac{x}{k} (\log \log x)^\ell\right),
\]

(7.4)

where \(\alpha' \equiv \alpha b, \ bp \equiv 1 \pmod{k}\). Now

\[
w^\ell(mp) = w^\ell(m) + O\left((\omega(m))^{\ell-1}\right),
\]

(7.5)

and if \(p \leq x^{1/2}, \ k < x^{1-\alpha}\), then \(k < (x/p)^{1-\alpha/2}\) so that, by the induction hypothesis, we have

\[
\sum_{\substack{m \leq x/p \\ m \equiv \alpha' \text{ (mod } k) \text{ (mod } p)}} w^\ell(m) = \frac{x}{pk} \left(\log \log \frac{x}{p}\right)^\ell + O\left(\frac{x}{pk} (\log \log x)^{\ell-1} \log \log \log x\right)
\]

\[
= \frac{x}{pk} \left(\log \log x\right)^\ell + O\left(\frac{x}{pk} (\log \log x)^{\ell-1} \log \log \log x\right).
\]

(7.6)

Also

\[
\sum_{\substack{m \leq x/p \\ m \equiv \alpha' \text{ (mod } k) \text{ (mod } p)}} w^{\ell-1}(m) \ll \frac{x}{pk} (\log \log x)^{\ell-1}
\]

(7.7)

uniformly in \(k < x^{1-\alpha}, \ p \leq x^{1/2}\). As before we have

\[
\sum_{\substack{p \leq x^\alpha \\ p \text{ divides } k}} \frac{1}{p} = \log \log x + O(\log \log \log x)
\]
uniformly in \( k < x \), so that from (7.4), (7.5), (7.6) and (7.7) we have

\[
\sum_{\substack{n \leq x \\ n \equiv a \mod k}} (w(n))^{\ell+1} = \frac{x}{k} (\log \log x)^{\ell+1} + O\left(\frac{x}{k} (\log \log x)^{\ell} \log \log \log x\right)
\]

uniformly in \( k < x^{1-\alpha} \).

The same result is obtained for \( \Omega(n) \) by applying the method used for \( \ell = 1 \). The inductive step is complete and so the theorem is proved.

where each \( a_i \) is a positive integer; moreover, this representation is unique if we stipulate that \( a_1 \ldots a_p \) is square free.

For \( \ell \geq 1 \) we denote by \( \Omega_\ell(x) \) the number of \( k \)-full integers not exceeding \( x \) so that

\[
\Omega_\ell(x) = \sum_{\substack{a_1, \ldots, a_k \leq x \\ a_1 \ldots a_k \neq 0}} \mu^2(a_1 \ldots a_k)
\]

where \( \mu(n) \) is the Möbius function. We shall see that \( \Omega_\ell(x) \) is related to the corresponding unweighted sum

\[
S_\ell(x) = \sum_{\substack{a_1, \ldots, a_k \leq x \\ a_1 \ldots a_k = 0}} 1
\]

which, following standard procedures, satisfies

\[
S_\ell(x) = \sum_{\ell \leq 2k} \Lambda^k x^\ell = \Theta_k(x)
\]
CHAPTER FOUR

THE DISTRIBUTION OF POWER FULL INTEGERS

4.1 INTRODUCTION.

Let $k$ be an integer greater than 1. We call a positive integer $n$ a $k$-full integer if $p^k$ divides $n$ whenever $p$ is a prime divisor of $n$. It is clear that each $k$-full integer can be written in the form

$$a_1^k a_2^{k+1} \ldots a_k^{2k-1}$$

where each $a_i$ is a positive integer; moreover, this representation is unique if we stipulate that $a_2 \ldots a_k$ is square free.

For $x \geq 1$ we denote by $Q_k(x)$ the number of $k$-full integers not exceeding $x$ so that

$$Q_k(x) = \sum \mu^2(a_2 \ldots a_k)$$

where $\mu(n)$ is the Möbius function. We shall see that $Q_k(x)$ is related to the corresponding unweighted sum

$$S_k(x) = \sum 1$$

where $l(x)$ is an error term. We define $\psi(x)$ to be the influence of all $p$ satisfying $p \leq 1/\psi(x)$.

which, following standard procedures, satisfies

$$S_k(x) = \sum_{k \leq r < 2k} A^*_r x^r + A^*_k(x)$$

(1.3)
where

$$A_{kr} = \prod_{\substack{n \leq n < 2k \atop n \neq r}} \zeta \left( \frac{n}{r} \right), \quad k \leq r < 2k,$$

and $\Delta_k(x)$ is an error term. Let us define $\rho_k^*$ to be the infimum of all $\rho$ satisfying

$$\Delta_k^*(x) \ll x^\rho, \quad x \to \infty.$$

From the application of a special case of the colossal lattice point theorem due to Landau [17], [18], we can show that

$$\frac{k-1}{k(3k-1)} \leq \rho_k^* \leq \frac{1}{k+2}, \quad k \geq 2.$$

If we write

$$A_{kr} = A_k^* J_k \left( \frac{1}{r} \right), \quad k \leq r < 2k$$

where $J_k(s)$ is a function defined later in section 4.2, then we can write

$$Q_k(x) = \sum_{k \leq r < 2k} \frac{1}{A_{kr} x^r + \Delta_k(x)}$$

where $\Delta_k(x)$ is an error term. We define $\rho_k$ to be the infimum of all $\rho$ satisfying

$$\Delta_k(x) \ll x^\rho, \quad x \to \infty.$$

In 1934 Erdős and Szekeres [7] proved in an elementary way that

$$\rho_k \leq \frac{1}{k+1}, \quad k \geq 2,$$
and in 1958 Bateman and Grosswald \( [2] \) improved this to

\[
\rho_k \leq \frac{1}{k+2} , \quad k \geq 2.
\]

They proved also that \( \rho_2 \leq \frac{1}{6} \) and

\[
\rho_3 \leq \frac{7}{46} = 0.157 ... \tag{1.8}
\]

and, by relating \( Q_k(x) \) to the sum

\[
\sum_{a_1 \cdots a_{k+r+1} \leq x} 1 , \quad r = \left\lfloor \sqrt[2k]{x} \right\rfloor
\]

and then appealing to Landau's theorem, they proved that

\[
\rho_k \leq \max \left\{ \frac{r}{k(r+2)} , \frac{1}{k+r+1} \right\} , \quad r = \left\lfloor \sqrt[2k]{x} \right\rfloor , \quad k \geq 4 . \tag{1.9}
\]

Here we shall relate \( Q_k(x) \) to \( S_k(x) \) and prove:

**Theorem 1.** We have

\[
\rho_k \leq \max \left\{ \rho_k^* , \frac{1}{2k+2} \right\} , \quad k \geq 2 . \tag{1.10}
\]

Moreover, if \( \rho \) and \( \lambda \) are constants satisfying

\[
\rho \geq \frac{1}{2k+2} , \quad \lambda > 0
\]

and

\[
\Delta_k^*(x) \ll x^\rho \log^\lambda x , \quad x \to \infty ,
\]

then

\[
\Delta_k(x) \ll x^\rho \log^\lambda x , \quad x \to \infty
\]
where
\[
\lambda' = \begin{cases} 
\lambda & \text{if } \rho > \frac{1}{2k+2}, \\
\lambda+1 & \text{if } \rho = \frac{1}{2k+2}.
\end{cases}
\]

From (1.10) we see that the inequality
\[
\rho_k^* < \frac{1}{2k+2}
\]
is of crucial relevance to our upper bound for \( \rho_k \). As a particular case of a general result, Richert [27] has proved that
\[
\rho_2^* \leq \frac{2}{15}
\]
which establishes (1.11) when \( k = 2 \), the only case settled up to the present. Indeed, from this Bateman and Grosswald proved that
\[
\Delta_2(x) \ll x^\frac{3}{5} \exp(-c w(x)), \quad x \to \infty
\]
where
\[
w(x) = (\log x)^{\frac{1}{5}} (\log \log x)^{-\frac{1}{5}}
\]
and \( c \) is a positive constant. They also pointed out that \( \rho_2 < \frac{1}{6} \) if and only if the supremum of the real parts of the zeros of the Riemann zeta function is less than 1. It will be clear from the proof of Theorem 1 that if (1.11) holds for any particular \( k \), then we can apply the method of Bateman and Grosswald to prove that
\[
\Delta_k(x) \ll x^{\frac{1}{2k+2}} \exp(-c w(x)), \quad x \to \infty.
\]
In section 4.4 we shall apply Theorem 1 together with some results of Schmidt [29] and Srinivasan [35] to prove:

**Theorem 2.** We have, as \( x \to \infty \),

\[
\Delta_3(x) \ll x^{2052} \log^2 x
\]

so that

\[
\rho_3 \leq \frac{263}{2052} = 0.128 \ldots
\]

Our upper bound for \( \rho_3 \) is an improvement on (1.8), and indeed is quite close to \( \frac{1}{8} \), the critical value on the right hand side of (1.11) when \( k = 3 \).

In section 4.5 we shall apply Theorem 1 to give the following improvement on (1.9).

**Theorem 3.** For \( 4 \leq k \leq 12 \) we have

\[
\Delta_k(x) \ll x^{\frac{1}{2k}} \log^\frac{3}{2} x , \quad x \to \infty ,
\]

so that

\[
\rho_k \leq \frac{1}{2k} , \quad 4 \leq k \leq 12.
\]

We also have

\[
\rho_k \leq \frac{1}{k+u} , \quad k \geq 13,
\]

where

\[
u = \sqrt{6k + \frac{25}{4}} + \frac{5}{2}.
\]
Furthermore we have

\[ \beta_k \leq \frac{1}{2k}, \quad k \geq 13 \]

on the assumption of the Lindelöf hypothesis.

4.2 THE DIRICHLET SERIES ASSOCIATED WITH \( \mathcal{Q}_k(x) \).

Let \( k \) be fixed and let \( \alpha_k(n) \) be the characteristic function of the set of \( k \)-full integers. It is clear that \( \alpha_k(n) \) is multiplicative and that, for each prime \( p \),

\[ \alpha_k(p^a) = \begin{cases} 1 & a = 0, k, k+1, \ldots, \\ 0 & a = 1, 2, \ldots, k-1. \end{cases} \]

For \( s = \sigma + it, \quad \sigma > 1/k \) we have that

\[ \sum_{n=1}^{\infty} \frac{\alpha_k(n)}{n^s} = \prod_p \left( 1 + \sum_{a=k}^{\infty} \frac{p^{-as}}{a} \right) = \prod_p \left( 1 + \frac{p^{-ks}}{1-p^{-s}} \right). \]

(2.1)

Lemma 2.1. Let \( k \geq 2 \) and \( K = \frac{1}{2} (3k^2 + k - 2) \). Then there are constants \( a_{kr} \) \( (2k + 2 < r \leq k) \) such that

\[ \left( 1 + \frac{v^k}{1-v} \right)(1-v^{k+1})(1-v^{2k-1}) = 1 - v^{2k+2} + \sum_{r=2k+3}^{K} a_{kr} v^r \]

(2.2)

holds for all \( v \neq 1 \).

Proof. The product of the first two factors on the left hand side of (2.2) is

\[ (1-v+v^k)(1+v+v^2+\ldots+v^{k+1}) = 1 + v^{k+1} + v^{k+2} + \ldots + v^{2k-1}. \]
It follows that the left hand side of (2.2) is a polynomial in $v$ of degree

$$(2k-1) + (k+1) + (k+2) + \ldots + (2k-1) = K.$$ 

Moreover, when the product is multiplied out, the terms $v^r$ $(1 \leq r < 2k+2)$ cancel, leaving $-v^{2k+2}$ as the first non-zero term. The lemma is proved.

We now define, for $\sigma > 1/k$,

$$H_k(s) = \prod_{k \leq n < 2k} \zeta(ns)$$

and, for $\sigma > 1/(2k+2)$,

$$J_k(s) = \prod_p \left( 1 - \frac{(2k+2)s}{p} + \sum_{r=2k+3}^{K} \frac{a_{kr} p^{-rs}}{r} \right)$$

so that from (2.1) and (2.2) we have that

$$\sum_{n=1}^{\infty} \frac{a_k(n)}{n^s} = J_k(s) H_k(s).$$

We note that if $k = 2$, then $K = 6$ and $2k+3 = 7$ so that the sum in the Euler product for $J_2(s)$ is empty, giving $J_2(s) = 1/\zeta(6s)$.

We remark that, for $k \geq 3$, the line $\sigma = 0$ is a natural boundary for $J_k(s)$. To see this we need a lemma of Esterman [8] which states that, for small $x$,

$$1 - x + x^k = \prod_{n=1}^{\infty} \left( 1 - x^n \right)^{\ell_k(n)}.$$ 

Here $\ell_k(n)$ is an integer given by
\( \lambda_k(n) = \frac{1}{n} \sum_{ab=n} \mu(a) \prod_{r=1}^{k} \lambda_r^b \)

where \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are the roots of the equation

\[ \lambda^k - \lambda^{k-1} + 1 = 0. \]

From (2.1) and (2.2) we have that

\[ J_k(s) = \frac{\zeta(s)}{H_k(s)} \prod \left( 1 - \frac{s}{p} + \frac{p^{-ks}}{p^s} \right) \]

so that from (2.6) we see that \( J_k(s) \) can be written as an infinite product of the Riemann zeta functions. For example, we have

\( (1, 0, -1, -1, 0, 0, 1, 1, 1, 1, 0, 0, -1, -1, 0, 0, 1, 1, 1, 0, -1, -2, -2, -1, 1, 3, \ldots) \)

for the sequence \( (\lambda_3(n)) \) so that

\[ J_3(s) = \frac{\zeta(13s)\zeta(14s)\zeta(21s)\zeta^2(22s)\zeta^2(23s)\zeta(24s)}{\zeta(8s)\zeta(9s)\zeta(10s)\zeta(17s)\zeta(18s)\zeta(19s)\zeta(25s)\zeta^3(26s)} \ldots \]

If we assume the Riemann hypothesis we can deduce easily that the zeros of \( J_k(s) \) \((k \geq 3)\) are dense in the line \( \sigma = 0 \).

If we follow the proof of the main theorem in Estermann's paper we can give an unconditional proof using only simple zero density estimates for the Riemann zeta function. We shall not require this result in our proofs of the theorems and shall therefore leave it to section 4.6.

We now define \( \beta_k(n) \) and \( \tau_k(n) \) by
\[ J_k(s) = \sum_{n=1}^{\infty} \frac{\beta_k(n)}{n^s}, \]

and

\[ H_k(s) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s}, \]

so that, from (2.5), we have

\[ \alpha_k = \beta_k \cdot \tau_k. \]  

\text{(2.7)}

We also note that, by (2.3),

\[ \tau_k(n) = \sum_{a_1 \cdots a_k = n} 1 \]

so that

\[ S_k(x) = \sum_{n \leq x} \tau_k(n). \]

Lemma 2.2. We have, as \( x \to \infty \),

\[ \sum_{n \leq x} |\beta_k(n)| \ll x^{2k+2}, \quad k = 3, 4, \ldots. \]

Proof. We see from (2.4) that there are constants \( a'_{k,r} \) \((2k+2 < r \leq K + 2k+3)\) such that

\[ J_k(s) = \frac{1}{\zeta((2k+2)s)} \left\{ 1 + \sum_{r=2k+3}^{k+2k+3} a'_{k,r} \frac{-rs}{p} \right\} \left\{ \frac{1}{1 - \sum_{r=2k+3}^{k+2k+3} a'_{k,r} \frac{-(4k+4)s}{p}} \right\} . \]

It follows that we can write
\[
\sum_{n=1}^{\infty} \frac{b_k(n)}{n^s} = \sum_{n=1}^{\infty} \frac{h_1(n)}{n^s} \times \sum_{n=1}^{\infty} \frac{h_2(n)}{n^s}
\]

where

\[
h_1(n) = \begin{cases} 
\mu(m) & \text{if } n = m^{2k+2} \\
0 & \text{otherwise}
\end{cases}
\]

and the series \( \sum h_2(n)/n^s \) converges absolutely in \( \sigma > 1/(2k+3) \).

Now

\[
b_k = h_1 \ast h_2
\]

and, as \( y \to \infty \)

\[
\sum_{n \leq y} |h_1(n)| = \sum_{m \leq y} \frac{\mu(m)}{2^{k+2}} << y^{2k+2}.
\]

It follows that, as \( x \to \infty \),

\[
\sum_{n \leq x} |b_k(n)| = \sum_{a \leq x} \left| h_1(a) h_2(b) \right| = \sum_{b \leq x} \left| h_2(b) \right| \sum_{a \leq x/b} |h_1(a)| << \sum_{b \leq x} \left| h_2(b) \right| \left( \frac{x}{b} \right)^{2k+2} << x^{2k+2},
\]

since

\[
\sum_{b=1}^{\infty} \frac{|h_2(b)|}{b^{2k+2}} < \infty.
\]

4.3 \textbf{PROOF OF THEOREM 1.}

We have, by (2.7) and (2.8), that

\[
Q_k(x) = \sum_{n \leq x} a_k(n) = \sum_{m \leq x} \beta_k(m) r_k(n) = \sum_{m \leq x} \beta_k(m) s_k \left( \frac{x}{m} \right).
\]
From (1.3) we now have that

\[ Q_k(x) = \sum_{m \leq x} A_k^* \frac{1}{m} \sum_{m \leq x} \frac{\beta_k(m)}{m^{1/r}} + \sum_{m \leq x} \beta_k(m) \Delta_k^*(\frac{x}{m}). \] (3.1)

From Lemma 2.2 we have, by partial summations, that

\[ \sum_{m>x} \frac{|\beta_k(m)|}{m^{1/r}} << x^{\frac{1}{2k+2}} - \frac{1}{r}, \quad (k < r < 2k), \]

so that

\[ \sum_{k \leq r < 2k} A_k^* x^r \sum_{m \leq x} \frac{\beta_k(m)}{m^{1/r}} = \sum_{k \leq r < 2k} A_k^* x^r \left\{ J_k(\frac{1}{r}) + 0\left( x^{\frac{1}{2k+2}} - \frac{1}{r} \right) \right\} \]

\[ = \sum_{k \leq r < 2k} A_k^* x^r + 0\left( x^{\frac{1}{2k+2}} \right) \]

by (1.6). It now follows from (1.7) and (3.1) that

\[ \Delta_k(x) = \sum_{m \leq x} \beta_k(m) \Delta^*_k\left(\frac{x}{m}\right) + 0\left( x^{\frac{1}{2k+2}} \right). \] (3.2)

Suppose first that \( \rho_k^* < 1/(2k+2) \). We can then choose \( \rho \) so that \( \rho_k^* < \rho < 1/(2k+2) \). From Lemma 2.2 and partial summations, we have

\[ \sum_{m \leq x} \frac{|\beta_k(m)|}{m^{\rho}} << x^{\frac{1}{2k+2}} - \rho \]

so that
\[ \sum_{m \leq x} \beta_k(m) \Delta_k^* \left( \frac{x}{m} \right) \ll \sum_{m \leq x} |\beta_k(m)| \left( \frac{x}{m} \right)^\rho \ll x^{1/(2k+2)} \]

and hence, from (3.2)

\[ \Delta_k(x) \ll x^{1/(2k+2)} \]

This proves that \( \rho_k \leq 1/(2k+2) \) provided that \( \rho_k^* < 1/(2k+2) \).

Suppose next that \( \rho_k^* > 1/(2k+2) \) and let \( \rho \) and \( \lambda \) be such that \( \rho > 1/(2k+2) \), \( \lambda > 0 \) and

\[ \Delta_k^*(x) \ll x^\rho \log^\lambda x , \quad x \to \infty . \]

Then

\[ \sum_{m \leq x} \beta_k(m) \Delta_k^* \left( \frac{x}{m} \right) \ll x^\rho \log^\lambda x \sum_{m \leq x} \frac{|\beta_k(m)|}{m^\rho} \ll x^\rho \log^\lambda x , \]

since the Dirichlet series for \( J_k(s) \) converges absolutely in \( \sigma > 1/(2k+2) \). From (3.2) we now have

\[ \Delta_k(x) \ll x^\rho \log^\lambda x , \quad x \to \infty , \]

and in particular we have

\[ \rho_k < \rho_k^* \quad \text{if} \quad \rho_k^* > \frac{1}{2k+2} . \]

It is also clear from the above that if

\[ \Delta_k^*(x) \ll x^{1/(2k+2)} \log^\lambda x , \]

then
The theorem is proved.

Cohen and Davis [4] pointed out that the uniqueness theorem for Dirichlet series is used in the Batemann-Grosswald proof of \( \rho_k \leq 1/(k+2) \), and in [4] they gave another elementary proof by means of some special divisor and totient functions to avoid the use of the uniqueness theorem. Here we give a sketch of a direct and elementary proof of Theorem 1 itself without the use of special functions or the uniqueness theorem. We confine ourselves to the case \( k = 3 \). We have

\[
Q_3(x) = \sum_{a \leq x} \mu^2(bc) = \sum_{a \leq x} \mu^2(b) \mu^2(c) \sum_{\ell | b, c} \mu(\ell)
\]

\[
= \sum_{a \leq x} \mu^2(m) \mu^2(n) \mu(\ell)
\]

\[
= \sum_{a \leq x} \mu(b) \mu(c) \mu(\ell)
\]

\[
= \sum_{b \leq x} \mu(b) \mu(c) \mu(\ell) S'_3 \left( \frac{x}{b, 8, 9, 10, c}, \ell \right)
\]

say, where

\[
S'_3(x, \ell) = \sum_{a \leq x} \mu(m) S'_3(x, m)
\]

\[
= \sum_{m | \ell} \mu(m) S'_3(x, m)
\]
say, where

$$S''_3(x, m) = \sum_{h \mid m} \sum_{a \mid 3, b \mid 4, c \mid 5} \sum_{\substack{\ell \mid b \mid m_1 \mid \lambda \mid m_1}} \frac{1}{a b c \xi x \ell}$$

$$bc \equiv 0 \pmod{m}$$

$$b, m) = h$$

$$= \sum_{m_1, h = m} S''_3 \left( \frac{x}{h m_1} \right)$$

say, where

$$S''_3(x, m_1) = \sum_{a \mid 3, b \mid 4, c \mid 5} \sum_{\substack{\ell \mid b \mid m_1 \mid \lambda \mid m_1}} \mu(\ell)$$

$$= \sum_{\ell \mid m_1} \mu(\ell) S_3 \left( \frac{x}{\ell} \right)$$

where $S_3(x)$ is defined by (1.2). Substituting (1.3) into here and working back to $Q_3(x)$ we find that

$$Q_3(x) = \sum_{r=3}^{5} A^*_3 r \frac{1}{x^r} Q_3(x, \frac{1}{r}) + \Delta_3^+(x)$$

(3.3)

where

$$Q_3(x, \frac{1}{r}) = \sum_{b \mid 8, g \mid 10, \xi x} \frac{\mu(b)\mu(c)\mu(\xi)}{(b, g \cdot c \cdot 10)^{1/r}}$$

$$\sum_{m \mid \ell} \mu(m) g(m, \frac{1}{r})$$

$$g(m, \frac{1}{r}) = \sum_{m_1, h = m} \sum_{\substack{\ell \mid b \mid m_1 \mid \lambda \mid m_1 \mid \ell \mid \xi \mid m_1}} \frac{\mu(\ell)}{(h m_1 \ell)^{1/r}}$$

$$\Delta_3^+(x) = \sum_{b \mid 8, g \mid 10, \xi x} \mu(b)\mu(c)\mu(\xi) \Delta_3^+ \left( \frac{x}{b, g \cdot c \cdot 10}, \xi \right)$$

(3.4)
\[ \Delta_3'(x, \ell) = \sum_{m \mid \ell} \sum_{n \mid m} \sum_{v \mid n} \mu(m) \mu(v) \Delta_3^x \left( \frac{x}{v} \right). \] (3.5)

If we write
\[ f(\ell, \frac{1}{r}) = \frac{\mu(\ell)}{\zeta(8/r)} \sum_{m \mid \ell} \mu(m) g(m, \frac{1}{r}), \]
and
\[ F(\ell, \frac{1}{r}) = f(\ell, \frac{1}{r}) \prod_{p \mid \ell} (1 - p^{-8/r})^{-1} (1 - p^{-10/r})^{-1}, \]
then we will arrive at
\[ Q_3(x, \frac{1}{r}) = \frac{1}{\zeta(8/r) \zeta(10/r)} \sum_{\ell=1}^{\infty} F(\ell, \frac{1}{r}) + O\left( \frac{1}{x} \right). \]

It can be verified that
\[ F(p^a, \frac{1}{r}) = 0, \quad a = 2, 3, \ldots \]
and
\[ F(p, \frac{1}{r}) = \frac{-p^{-9/r} + p^{-13/r} + p^{-14/r} - p^{-18/r}}{(1 - p^{-8/r})(1 - p^{-10/r})}, \]
so that
\[ \sum_{\ell=1}^{\infty} F(\ell, \frac{1}{r}) = \prod_p \left( 1 + F(p, \frac{1}{r}) \right) = \zeta(8/r) \zeta(10/r) J_3(\frac{1}{r}). \]

From (1.6), (1.7) and (3.3) we now have that
\[ \Delta_3(x) = \Delta_3^+(x) + O(x^3), \]
and the required result can be derived from (3.4) and (3.5).
4.4 **PROOF OF THEOREM 2.**

In [7] Erdős and Szekeres also gave the asymptotic formula associated with the number of Abelian groups of a given order mentioned in Chapter Two, section 2.4. This asymptotic formula was rediscovered by Kendall and Rankin [14] who gave a superior estimate for the error term. This then led to a succession of improvements by Richert [27], Schwarz [30], Schmidt [29], and, more recently, by Srinivasan [35].

We define

\[ \psi(x) = x - \left\lfloor x \right\rfloor - \frac{1}{2}, \]

(4.1)

and, for positive constants \( \alpha, \beta, \gamma \) we write, following Richert [27],

\[ R(x; \alpha, \beta, \gamma) = \sum_{n \leq x} \psi(x^{\beta}) \]

(4.2)

and, following Schmidt [29],

\[ S_{\alpha, \beta, \gamma}(x) = \sum_{m^{\alpha} \leq x} \sum_{n^{\beta} \leq x} \psi\left(\frac{x}{mn^{\gamma}}\right) \]

(4.3)

Richert [27] showed that, for positive constants \( u, v \) the error term associated with the sum

\[ \sum_{a^{u}b^{v} \leq x} 1 \]

(4.4)

can be expressed in terms of \( R(x; \alpha, \beta, \gamma) \) with \( \alpha, \beta, \gamma \) depending on \( u \) and \( v \). In particular the error term \( \Delta_{2}^{*}(x) \) associated with our sum \( S_{2}(x) \) is given by
\[ \Delta_2^* = - R \left( x; \frac{1}{5}, \frac{1}{2}, \frac{3}{2} \right) - R \left( x; \frac{1}{5}, \frac{1}{3}, \frac{2}{3} \right) + O(1), \quad (4.4) \]

a formula that we shall use later in section 5.2. In order to improve on Richert's result Schmidt [29] had to consider the error term \( \Delta_3^{(1)}(x) \) associated with the sum

\[ \sum_{ab^2c^3 \leq x} \frac{1}{ab^2c^3} \]

and he proved that

\[ \Delta_3^{(1)}(x) = - \sum_{(\alpha, \beta, \gamma)} S_{\alpha, \beta, \gamma}(x) + O(x^6) \quad (4.5) \]

where the summation is over the six permutations of \((1, 2, 3)\).

For our present problem of the error term \( \Delta_3^*(a) \) associated with the sum \( S_3(x) \) we have

\[ \Delta_3^*(x) = - \sum_{(\alpha, \beta, \gamma)} S_{\alpha, \beta, \gamma}(x) + O(x^{12}) \quad (4.6) \]

where the summation is now over the six permutations of \((3, 4, 5)\).

We shall omit the proof of (4.6) since it is, of course, the same as that of (4.5).

Next, in order to estimate \( S_{\alpha, \beta, \gamma}(x) \) we shall apply the following result due to Srinivasan [35].

**Lemma 4.1.** (Srinivasan [35], Main Theorem).

Let \( \rho, \sigma > 0 \) and \((\lambda_0, \lambda_1)\) be any two dimensional exponent pair. Let \( z, M, N, F \) satisfy

\[ F = z^{-\rho} N^{-\sigma} , \quad 1 \ll M \ll F , \quad 1 \ll N \ll F. \]
Then, for any region \( D \) in the rectangle \( M < m \leq 2M, \)
\( N < n \leq 2N, \) we have

\[
\sum_{(m,n) \in D} \psi \left( \frac{z}{m,n} \right) < \left( \frac{1}{F^2 + \lambda_0 - \lambda_1} \right) \left( \frac{1}{\frac{3}{2} + \lambda_0 - \lambda_1} \right) \frac{1}{4} \frac{1}{4} + \frac{F}{M^2} N + \frac{F}{M^2} M N.
\]

(4.7)

The definition of a two dimensional exponent pair is given
in \( [35] \) where it is also shown that

\[
\left[ \begin{array}{c} 23 \\ 250 \\ 45 \\ 250 \\ 250 \end{array} \right]
\]

(4.8)
is such a pair. With this pair the first term on the right hand
side becomes

\[
\left[ \begin{array}{c} F^{92} M^{171} N^{263} \end{array} \right]^{1/342}
\]

(4.9)

Now let us write

\[
S_{\alpha,\beta,\gamma}(x; m, N) = \sum_{M < m \leq 2M} \sum_{N < n \leq 2N} \sum_{a+b \leq x} \sum_{m > n} \psi \left( \left( \frac{x}{m,n,\gamma} \right)^{1/\alpha} \right).
\]

With \( z = x^{1/\alpha}, \ \rho = \beta/\alpha, \ \sigma = \gamma/\alpha \) we have, from Lemma 4.1
with the exponent pair (4.8), that

\[
S_{\alpha,\beta,\gamma}(x; M, N) < \left( \frac{1}{F^{92} M^{171} N^{263}} \right)^{1/342} + \frac{1}{4} \frac{1}{4} + \frac{F}{M^2} N + \frac{F}{M^2} M N.
\]
where
\[ F = (x^{1/\alpha} M^{-\beta} N^{-\gamma})^{1/\alpha}. \]

Let \((\alpha, \beta, \gamma)\) be any permutation of \((3, 4, 5)\). Then

\[ M^4 N^8 << (MN)^6 << M^{a+\beta} N^\gamma << x \]

and so

\[
\begin{align*}
\frac{1}{F^4} \frac{1}{M^4 N} &= \left\{ x(M^4 N^8)^{1/2} (M^{a+\beta} N^\gamma)^{-1}\right\} \ll x^{1/8}, \\
\frac{1}{F^2} \frac{1}{M N} &= \left\{ x^{-1}(M^4 N^8)^{1/4} M^{a+\beta} N^\gamma\right\} \ll x^{1/8}
\end{align*}
\]

and

\[
\frac{1}{(F^{92} M^{171} N^{263})^{342}} = \left\{ x^{92}(MN)^{263a} (M^{a+\beta} N^\gamma)^{-92}\right\} \ll x^{263} \frac{1}{2052}.
\]

Therefore

\[ S_{\alpha, \beta, \gamma}(x; M, N) \ll x^{\frac{263}{2052}} \]

and so, from (4.3),

\[ S_{\alpha, \beta, \gamma}(x) \ll x^{\frac{263}{2052}} \log^2 x. \]

The required result now follows from (4.6) and Theorem 1.

4.5 PROOF OF THEOREM 3.

We see from Theorem 1 that we need deal only with \(A_k^*(x)\).

Suppose first that \(4 \leq k \leq 12\). Let \(x\) be half an odd positive
integer and $\varepsilon > 0$. We put

$$c = \frac{1}{k} + \varepsilon, \quad T = x^M \quad (M > 1 + c). \quad (5.1)$$

From (2.3) and (2.8) and the usual method (see, for example, Titchmarsh [36], p. 53) of finding the partial sum of the coefficients of a Dirichlet series we have that

$$S_k(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{H_k(s) x^s}{s} \, ds + R(x, T) \quad (5.2)$$

where

$$|R(x, T)| \leq \frac{1}{\pi} \sum_{n=1}^{\infty} \tau_k(n) \left(\frac{x}{n}\right)^c \min \left(1, \frac{1}{T|\log \frac{x}{n}|}\right)$$

$$<< \frac{x^{c+1}}{T} << 1, \quad (5.3)$$

by (5.1). We move the line of integration from $\sigma = c$ to $\sigma = 1/2k$ passing through the $k$ simple poles of the integrand at

$$s = \frac{1}{k}, \frac{1}{k+1}, \ldots, \frac{1}{2k-1}.$$  

The sum of the residues at these poles is

$$\sum_{k \leq r < 2k} A_{kr}^* \frac{1}{x^r} \quad (5.5)$$

where $A_{kr}^*$ is defined in (1.4). From the definition of $\Delta_k^*(x)$ together with (5.2) and (5.3) we now have, from the residue theorem, that

$$\Delta_k^*(x) = \frac{1}{2\pi i} \left[I_1 + I_2 + I_3\right] + O(1) \quad (5.4)$$
where \( I_1, I_2, I_3 \) are the integrals of \( H_k(s) x^s / s \) along the lines joining the points

\[
\frac{1}{2k} - iT, \quad \frac{1}{2k} + iT, \quad c + iT
\]

in the respective order.

We first estimate \( I_3 \). Here we have

\[
s = \sigma + iT, \quad \frac{1}{2k} < \sigma < c.
\]

From the well known estimates (see, for example, Titchmarsh [36], p. 99.):

\[
\zeta(\frac{1}{2} + it) \ll t^\frac{1}{6}, \quad \zeta(1 + it) \ll t^\varepsilon, \quad (t > 1)
\]

we have that

\[
\zeta(ns) \ll T^3(1-n\sigma+\varepsilon), \quad \frac{1}{2n} \leq \sigma \leq \frac{1}{n}.
\]

(5.5)

Let us put

\[
I_3(n) = \int_{\frac{1}{n+1} + iT}^{\frac{1}{n} + iT} \frac{H_k(s) x^s}{s} \, ds, \quad k \leq n < 2k
\]

so that

\[
I_3 = \sum_{k \leq n < 2k} I_3(n) + O(1), \quad \text{(5.6)}
\]

since

\[
\int_{\frac{1}{k} + iT}^{c + iT} \frac{H_k(s) x^s}{s} \, ds \ll \frac{x}{T} < < 1.
\]
Let

$$\beta_n = \frac{1}{3} \sum_{r=k}^{n} \left( 1 - \frac{r}{n} \right) = \frac{1}{6n} (n-k)(n-k+1), \quad (5.7)$$

and we shall prove that, for $k \leq n < 2k$,

$$I_3(n) \ll \frac{1}{n} + \varepsilon + \frac{1}{n+1} + \varepsilon \quad (5.8)$$

For $r$ satisfying $n < r < 2k$ we have

$$\zeta(rs) \ll T^\varepsilon, \quad \frac{1}{n+1} < \sigma < \frac{1}{n}$$

and therefore, from (5.5),

$$H_k(s) \ll T \sum_{r=k}^{n} (1-r\sigma+k\varepsilon)$$

It follows that

$$I_3(n) \ll \int \frac{1}{n} + \frac{1}{n+1} \sum_{r=k}^{n} (1-r\sigma+k\varepsilon-1) \frac{x^\sigma}{\sigma} dx$$

since the integrand is maximum at $\sigma = 1/n$ or $\sigma = 1/(n+1)$, and

$$\frac{1}{n} \sum_{r=k}^{n} \left( 1 - \frac{r}{n+1} \right) = \beta_{n+1}.$$
Since \( T^k = x^{Mk} \) by (5.1) we see that (5.8) is proved if we replace \( \epsilon \) by \( \epsilon/Mk \).

From (5.6) and (5.8) we now have that

\[
I_3 \ll \sum_{k \leq n < 2k} \frac{1}{n} + \epsilon \cdot \frac{x}{T}^{1-\beta} + 1.
\]

Similarly we have

\[
I_1 \ll \sum_{k \leq n < 2k} \frac{1}{n} + \epsilon \cdot \frac{x}{T}^{1-\beta} + 1.
\]

We next deal with \( I_2 \). Let us write

\[
f(s) = \prod_{k<n<2k} \zeta(ns)
\]

so that

\[
I_2 = \int_{\frac{1}{2k} - iT}^{\frac{1}{2k} + iT} \frac{\zeta(ks)f(s)x^s}{s} \, ds \ll \frac{1}{2k} \int_{1}^{T} \frac{1}{t} \left| \zeta\left(\frac{1}{2} +ikt\right)f\left(\frac{1}{2k} + it\right)\right| \, dt + \frac{1}{x^2}
\]

since the integral along \( \frac{1}{2k} - i \) and \( \frac{1}{2k} + i \) is bounded. Applying the Cauchy-Schwarz inequality to the integral we have that

\[
I_2 \ll \frac{1}{x^2} \left\{ \int_{1}^{T} \frac{\left| \zeta\left(\frac{1}{2} + ikt\right)\right|^2}{t} \, dt \int_{1}^{T} \frac{\left| f\left(\frac{1}{2k} + it\right)\right|^2}{t} \, dt \right\}^{\frac{1}{2}} + \frac{1}{2k}
\]

From Titchmarsh [36], Theorems 7.3 and 7.1 we have
\[ \int_{0}^{T} |\zeta(\frac{1}{2} + it)|^2 \, dt \ll T \log T , \]

and

\[ \int_{0}^{T'} |f(\frac{1}{2k} + it)|^2 \, dt \ll T . \]

Let \( R \) be the integer satisfying \( 2^{R-1} < T \leq 2^R \). Then

\[ \int_{1}^{T} \frac{|\zeta(\frac{1}{2} + ikt)|^2}{t} \, dt \ll \sum_{r=1}^{R} 2^{1-r} \int_{2^{r-1}}^{2^r} |\zeta(\frac{1}{2} + ikt)|^2 \, dt \ll \sum_{r=1}^{R} 2^{1-r} \, 2^{r} \log 2^r \ll \sum_{r=1}^{R} \frac{r}{2^r} \]

\[ \ll R^2 \ll \log^2 T , \]

and similarly

\[ \int_{1}^{T} \frac{|f(\frac{1}{2k} + it)|^2}{t} \, dt \ll \log T . \]

Therefore we arrive at

\[ I_2 \ll \frac{1}{2k} \frac{3}{\log \frac{2}{T}} . \]

Returning to (5.4) we now have that

\[ \Delta_k^*(x) \ll x^{2k} \log^2 T + \sum_{k \leq n < 2k} \frac{1}{T^{1-\beta_n}} + 1 . \]

Since \( 4 \leq k \leq 12 \) we see from (5.7) that \( \beta_n < 1 \) for \( k \leq n < 2k \).
With $M$ sufficiently large in (5.1), the sum on the right hand side is bounded and therefore

$$\Delta^*_k(x) \ll \frac{1}{2k} \frac{3}{\log^2 x}, \quad 4 \leq k \leq 12.$$  

Suppose now that $k \geq 13$. From (4.7) we see that $\beta_n < 1$ provided that $k \leq n < k+u$ where $u$ is given in the theorem. We now consider the integral in (5.2) and move the line of integration from $\sigma = c$ to $\sigma = 1/(k+u)$ only. The same argument can be applied, except that we now have that

$$I_1 + I_3 \ll \sum_{k \leq n < k+u} \frac{1}{n} + \frac{1}{k+u} + \epsilon$$

and

$$I_2 = \int_{k+u}^{1} \frac{1}{s} \frac{H_k(s) x^s}{s} ds$$

and

$$\ll \frac{1}{k+u} \int_{1}^{T} |H_k(\frac{1}{k+u} + it)| dt + \frac{1}{k+u}$$

4.6 A PROBLEM OF ANALYTIC CONTINUATION.

For $k \geq 3$ the line of the natural boundary to the function $\zeta(s)$ is at $\sigma = 1$. The proof requires only a small modification of the proof of the main theorem in Estermann's paper [8].

From these estimates we deduce that

$$\Delta^*_k(x) \ll \frac{1}{k+u} + \epsilon,$$  

and hence

$$\rho_k^* \ll \frac{1}{k+u}, \quad k \geq 13.$$  

If we apply deeper theorems concerning the order of $\zeta(s)$ on the critical line $\sigma = \frac{1}{2}$, we can improve on our result for $k \geq 13$ slightly. In fact, if $0 < \lambda < \frac{1}{6}$ and

$$\zeta\left(\frac{1}{2} + it\right) \ll t^\lambda, \quad t \to \infty,$$

then we can take

$$\beta_n = 2\lambda \sum_{r=k}^{n} \left(1 - \frac{r}{n}\right)$$

and now the condition $\beta_n < 1$ will hold for $n < k + u'$ where $u' > u$. In particular, assuming the Lindelöf hypothesis,

namely that

$$\zeta\left(\frac{1}{2} + it\right) \ll t^\varepsilon, \quad t \to \infty,$$

for all $\varepsilon > 0$, we have that

$$\Lambda_k^*(x) \ll x^{\frac{1}{2k}} \log^2 x, \quad k \geq 13.$$

The theorem now follows from these results and Theorem 1.

### 4.6 A PROBLEM OF ANALYTIC CONTINUATION.

For $k \geq 3$ the line $\sigma = 0$ is a natural boundary to the function $J_k(s)$ defined in (2.4). The proof requires only a small modification of the proof of the main theorem in Estermann's paper [8]. We shall restrict ourselves to the case $k = 3$.

Let us write, for $\sigma > 1$,

$$g(s) = \prod_p \left(1 - p^{-s} + p^{-3s}\right) \quad (6.1)$$
so that, by (2.1), (2.3) and (2.5),

\[ J_3(s) = \frac{\zeta(s)g(s)}{\zeta(3s)\zeta(4s)\zeta(5s)}. \]

Since \( \zeta(s) \), \( \zeta(3s) \), \( \zeta(4s) \) and \( \zeta(5s) \) are meromorphic in the whole \( s \)-plane it remains to show that \( \sigma = 0 \) is a natural boundary to \( g(s) \).

Let \( \lambda_1, \lambda_2, \lambda_3 \) be the roots of the equation

\[ \lambda^3 - \lambda^2 + 1 = 0 \]

and let

\[ \alpha = \max(|\lambda_1|, |\lambda_2|, |\lambda_3|) > 1. \]

We recall that Estermann proved that

\[ \ell(n) = \frac{1}{n} \sum_{ab=n} \mu(a) \left( \lambda_1^b + \lambda_2^b + \lambda_3^b \right) \]

is an integer which satisfies

\[ 1 - x + x^3 = \prod_{n=1}^{\infty} \left( 1 - x^n \right) \ell(n), \quad |x| < \frac{1}{\alpha}, \quad (6.2) \]

and it is clear that

\[ |\ell(n)| \leq 3a^n, \quad n = 1, 2, 3, \ldots. \]

Let \( 0 < \delta < 1, \ M > \alpha^{1/\delta} \). For \( p > M \) and \( \sigma > \delta \) we have

\[ \left| \frac{\ell(n)}{p^{n\delta}} \right| \leq 3 \left( \frac{\alpha}{\sigma} \right)^n \]

and since
\[
\frac{\alpha}{\sigma} < \frac{\alpha}{M} < 1
\]

it follows that

\[
\sum_{n=1}^{\infty} \left| \frac{\ell(n)}{p^n} \right| < \frac{3\alpha}{\sigma} \cdot \frac{1}{1 - \alpha p^{-\delta}} < \frac{3\alpha}{1 - \alpha M^{-\delta}} \cdot \frac{1}{\sigma} .
\]

Therefore

\[
\sum_{p>M} \sum_{n=1}^{\infty} \left| \frac{\ell(n)}{p^n s} \right| < \infty , \quad \sigma > 1
\]

and so

\[
\prod_{p>M} \prod_{n=1}^{\infty} (1 - p^{-ns}) \ell(n) = \prod_{n=1}^{\infty} \left\{ \prod_{p>M} (1 - p^{-ns}) \right\} \ell(n) , \quad \sigma > 1 .
\]

From (6.2) we now have, for \( \sigma > 1 \),

\[
\prod_{p>M} (1 - p^{-s} + p^{-3s}) = \prod_{n=1}^{\infty} \left\{ \zeta_M(ns) \right\}^{-\ell(n)}
\]

where

\[
\zeta_M(ns) = \prod_{p>M} (1 - p^{-ns})^{-1} = \zeta(ns) \prod_{p \leq M} (1 - p^{-ns}) .
\]

From (6.1) we now have, for \( \sigma > 1 \),

\[
g(s) = \prod_{p \leq M} (1 - p^{-s} + p^{-3s}) \prod_{n=1}^{\infty} \left\{ \zeta_M(ns) \right\}^{-\ell(n)}
\]

\[
= g_1(s) \cdot g_2(s) \cdot g_3(s) ,
\]
say, where

\[ g_1(s) = \prod_{\nu \in M} (1 - p^{-s} + p^{-3s}) \]

and

\[ g_2(s) = \prod_{n \geq \left\lceil \frac{1}{5} \right\rceil} \left( \zeta_M(ns) \right)^{-\tau(n)} \]

and

\[ g_3(s) = \prod_{n > \left\lceil \frac{1}{5} \right\rceil} \left( \zeta_M(ns) \right)^{-\tau(n)} \]

where we can assume that \( a = 0 \). We shall also assume that

Now \( g_1(s) \) is an entire function, and \( g_2(s) \) is meromorphic in the whole \( s \)-plane. We next show that the product for \( g_3(s) \) converges uniformly in \( \sigma > \delta \). In fact, for

\[ n \geq \left\lceil \frac{1}{\delta} \right\rceil + 1 \quad \text{and} \quad \sigma > \delta \]

we have

\[ |\zeta_M(ns) - 1| \leq \sum_{m=M+1}^{\infty} m^{-\delta} < \frac{1}{n^\delta - 1} \cdot \frac{M}{M^{\delta}} \]

and

\[ \frac{|\tau(n)|}{M^{\delta}} < 3 \left( \frac{\alpha}{M^{\delta}} \right) \]

so that

\[ \sum_{n > \left\lceil \frac{1}{\delta} \right\rceil} |\tau(n)| M^{\delta} < \infty \]

Therefore the series

\[ \sum_{n > \left\lceil \frac{1}{\delta} \right\rceil} |\tau(n)| |\zeta_M(n^\delta) - 1| \]

converges uniformly in \( \sigma > \delta \), and hence the product for \( g_3(s) \)
converges uniformly in $\sigma \geq \delta$. Therefore $g_3(s)$ is meromorphic in $\sigma > \delta$ and since $\delta$ is arbitrary it follows that $g(s)$ has a meromorphic continuation into $\sigma > 0$.

We next show that the zeros of $g(s)$ are dense in the line $\sigma = 0$. Let $\epsilon > 0$. We show that $g(s)$ has a zero in the square

$$0 < \sigma < \epsilon, \quad u < t < u + \epsilon$$

where we can assume that $u > 0$. We shall also assume that

$$\alpha = |\lambda_1|$$

so that we can write

$$\lambda_1 = e^{\beta + i\gamma}, \quad \alpha = e^\beta, \quad \beta > 0.$$

For $x > 0$ we shall denote by $\overline{w}(x)$ the number of primes not exceeding $x$, and by $N(x)$ the number of zeros of $\zeta(s)$ with

$$0 < t < x.$$ We can choose $V$ so large that

$$0 < \frac{1}{V} < \epsilon, \quad \frac{2\pi}{VB} < \epsilon,$$

$$\overline{w}(e^{2VB}) - \overline{w}(e^{VB}) > e^{VB}$$

and

$$e^{VB} > 2V N(2V(u + \epsilon)).$$

The last two inequalities being possible because, as $x \to \infty$, we have the well known estimates

$$\overline{w}(x) \sim \frac{x}{\log x}, \quad N(x) \ll x \log x.$$ We note that the rectangle

$$\frac{1}{2V} \leq \sigma < \frac{1}{V}, \quad u < t < u + \epsilon$$
is contained in the square (6.3). Now take
\[ \delta = \frac{1}{2V} , \quad M = \left[ \frac{1}{\delta} \right] + 1 = \left[ e^{2VB} \right] + 1. \]

Then \( g_3(s) \) has no poles in the rectangle (6.6). Let \( Z \) denote the number of distinct zeros of \( g_1(s) \) and let \( P \) denote the number of distinct poles of \( g_2(s) \) in the rectangle (6.6), so that it suffices to prove that
\[ Z > P. \tag{6.7} \]

Now \( 1 - p^{-s} + p^{-3s} = 0 \) if \( p^K = \lambda_1 \), which is the case if
\[ s \log p = \log \lambda_1 = \beta + i(\gamma + 2\pi m) \]
where \( m \) is any integer. Thus, if \( s = \sigma + it \) where
\[ \sigma = \frac{\beta}{\log p} , \quad t = \frac{\gamma + 2\pi m}{\log p} , \quad p \leq M \tag{6.8} \]
then \( s \) is a zero of \( g_1(s) \). If, moreover,
\[ e^{VB} < p \leq e^{2VB} \]
so that
\[ \frac{1}{2V} \leq \frac{\beta}{\log p} < \frac{1}{V} , \quad \frac{2\pi}{\log p} < c \]
which means that, for each such prime \( p \), at least one of the numbers \( s \) given in (6.8) is in the rectangle (6.6) and so
\[ Z \geq \sum \left( e^{2VB} \right) - \sum \left( e^{VB} \right). \tag{6.9} \]

On the other hand, each pole of \( g_2(s) \) must be a zero of \( \zeta_M(2s), \zeta_M(2s), \ldots, \zeta_M(2Vs) \) which have the same zeros in \( \sigma > 0 \).
as $\zeta(s), \zeta(2s), \ldots, \zeta(2V_s)$. Hence

$$P \leq \sum_{n=1}^{2V} P_n$$

where $P_n$ is the number of zeros of $\zeta(ns)$ in the rectangle (6.6), and this is the same as the number of zeros of $\zeta(s)$ in the rectangle

$$\frac{n}{2V} \leq \sigma < \frac{n}{V}, \quad \text{un} < t < (u+\varepsilon)n$$

and so

$$P_n \leq N((u+\varepsilon)n) \leq N(2V(u+\varepsilon))$$

and hence

$$P \leq 2V N(2V(n+\varepsilon)) \quad (6.10)$$

The required result (6.7) now follows from (6.4), (6.5), (6.9) and (6.10).
CHAPTER FIVE

THE DISTRIBUTION OF SQUARE-FULL INTEGERS

5.1 INTRODUCTION.

In this chapter we study the finer distribution of the 2-full integers, or the square-full integers. We shall write $S(x)$ and $Q(s)$ for $S_2(x)$ and $Q_2(x)$ respectively. We recall from (4.1.3) and (4.1.7) that

$$S(x) = A_{22}^* x^2 + A_{23}^* x^3 + \Delta_2^*(x),$$

(1.1)

and

$$Q(x) = A_{22}^* x^2 + A_{23}^* x^3 + \Delta_2^*(x),$$

(1.2)

and that $\rho_2^*$ and $\rho_2$ are the infima of the sets of exponents associated with the error terms $\Delta_2^*(x)$ and $\Delta_2(x)$ respectively.

We also have, from (4.1.12), that

$$\rho_2^* \leq \frac{2}{15}$$

(1.3)

and, by Theorem 4.1,

$$\rho_2 \leq \frac{1}{6}.$$  

(1.4)

As we remarked in Chapter Four, (1.3) is due to Richert and it is the consequence of a particular case of his general result on the estimation of the sum

$$R(x; \alpha, \beta, \gamma) = \sum_{n \leq x} \psi \left( \frac{\beta}{n^\gamma} \right),$$

using the method of exponent pairs. In section 2 we shall choose
a more suitable exponent pair for our particular problem, and prove that

\[ \rho_2 < \frac{12}{91}. \] (1.5)

If \( \rho_2 < \theta < \frac{1}{2} \), then it is easy to deduce from (1.2) that,

as \( x \to \infty \),

\[ Q\left(x + x^\theta\right) - Q(x) \sim \frac{1}{2} A_{22} x^\theta. \] (1.6)

The question is whether there exists a \( \theta < \rho_2 \) such that (1.6) is true. Let \( \theta_0 \) be the infimum of all such \( \theta \) so that, by (1.4),

\[ \theta_0 \leq \frac{1}{6} = 0.166 \ldots \] (1.7)

In section 3 we shall prove:

**Theorem 1.** We have

\[ \theta_0 \leq \frac{1 + \rho_2^*}{9 - 12\rho_2^*} \cdot \]

From (1.5) and Theorem 1 we deduce that

\[ \theta_0 \leq \frac{103}{675} = 0.1529 \ldots , \]

which is an improvement on (1.7).

Let \( (q_n) \) denote the sequence of square-full integers. This sequence contains all the perfect squares together with square-full integers which are not squares, these being numbers of the form

\[ q = a^2 b^3, \quad \mu(b) = 1, \quad b > 1. \]

There are \( \left\lfloor x^{1/2} \right\rfloor \) squares not exceeding \( x \), and since
we see from (1.2) that the squares form a minority in the sequence \( \{q_n\} \). As a consequence of this the problem of whether there are infinitely many pairs of consecutive squares in the sequence \( \{q_n\} \) is not trivial. Here we prove much more.

**Theorem 2.** Let \( f(n) \) denote the number of square-full integers in the interval \( n^2 < q < (n+1)^2 \), and let

\[
F_m = \{n : f(n) = m\}, \quad m = 0, 1, 2, \ldots .
\]

Then each \( F_m \) has positive asymptotic density \( d_m \) given by

\[
d_m = \sum_{k=0}^{\infty} \frac{(-1)^k (m+k)!}{m! \cdot k! \cdot c_{m+k}}
\]

where

\[
c_0 = 1, \quad c_r = \frac{\mu^2(b_1) \cdots \mu^2(b_r)}{(b_1 \cdots b_r)^{3/2}}, \quad r = 1, 2, \ldots .
\]

In particular, since

\[
d_0 = \sum_{k=0}^{\infty} (-1)^k c_k > 0
\]

(see the calculations in section 5.5), it follows that \( \{q_n\} \) does indeed contain infinitely many pairs of consecutive terms that are both squares.

Let \( a(n) \) denote the number of non-isomorphic Abelian groups of order \( n \). Kendall and Rankin [14] showed that each of the sets \( \{n : a(n) = m\} \), \( m = 0, 1, 2, \ldots \), has asymptotic density \( P_m \), and
they call the sequence \((P_m)\) the asymptotic frequency distribution of \(a(n)\). They remarked that \(a(n)\) is a rare example of an integer-valued function with finite mean value for which the asymptotic frequency distribution \((P_m)\) can be calculated explicitly and satisfies

\[
\sum_{m=0}^{\infty} P_m = 1, \quad \sum_{m=0}^{\infty} m P_m = \lim_{x \to \infty} x \frac{1}{n} \sum_{n \leq x} a(n).
\]

We remark that the function \(f(n)\) in Theorem 2 is another such example. (Note that \(d_m > 0\) for all \(m\) whereas \(P_2 = P_{13} = 0\).)

Since the equation \(x^2 = 8y^2 + 1\) has infinitely many solutions in integers we see at once that

\[
\lim_{n \to \infty} \inf (q_{n+1} - q_n) = 1.
\]

In our final section we deduce from (1.2) and \(d_0 > 0\) that

\[
\lim_{n \to \infty} \sup \frac{q_{n+1} - q_n}{2n} = \frac{1}{A_{22}}.
\]

5.2 PROOF OF (1.5).

First we have

\[
\Delta_2^*(x) = -R(x; \frac{1}{5}, \frac{1}{2}, \frac{3}{2}) - R(x; \frac{1}{5}, \frac{1}{3}, \frac{2}{3}) + o(1)
\]

where, for positive \(\alpha, \beta, \gamma\),

\[
R(x; \alpha, \beta, \gamma) = \sum_{n \leq x} \psi \left(\frac{x}{n}\right) \cdot \psi \left(\frac{x}{n}\right).
\]
As we pointed out in Chapter Four this is a particular case of a general result due to Richert \[27\]. We see from (2.1) and (2.2) that trivially \( \Lambda_2^*(x) \ll x^{1/5} \). Since

\[
\psi(x) = x - \left\lfloor x \right\rfloor - \frac{1}{2} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{n}
\]

whenever \( x \) is not an integer, the sum in (2.2) can be transformed into a trigonometric sum. If we apply Vinogradov's method of estimating trigonometric sums (see, for example, Gelfond and Linnik \[9\] p. ) we can prove that \( \Lambda_2^*(x) \ll x^{1/7} \), giving \( \rho_2^* \leq 1/7 \). Although this is inferior to (1.3), it is nevertheless a significant result in the sense that it establishes the inequality (4.1.11) when \( k = 2 \).

Van der Corput's method of estimating trigonometric sums has been developed into a delicate theory of exponent pairs due to van der Corput \[38\], Phillips \[24\], and Rankin \[26\]. The rather complicated definition of an exponent pair is given in \[24\] and \[26\]. By means of this theory Richert \[27\] has proved the following:

**Lemma 2.1.** (Richert \[27\], Lemma 8).

Let \( \alpha, \beta, \gamma \) be positive constants and let \((k, \ell)\) be an exponent pair with \( k > 0 \). Then, as \( x \to \infty \),

\[
R(x; \alpha, \beta, \gamma) \ll x^{\frac{\alpha \ell + (\beta - \alpha \gamma)k}{k+1}} + \begin{cases} \frac{\beta k}{x^{k+1} \log x} & \text{if } \ell = \gamma k, \\ \frac{\beta k}{x^{1+(\gamma+1)k-\ell}} & \text{if } \ell < \gamma k. \end{cases}
\]
Let \((k, \ell)\) be an exponent pair such that \(2\ell = 3k\). We see from Lemma 2.1 that

\[
R(x; \frac{1}{5}, \frac{1}{2}, \frac{3}{2}) \ll x^{10} + x^{\frac{k}{2(k+1)}} \log x,
\]
and

\[
R(x; \frac{1}{5}, \frac{1}{3}, \frac{2}{3}) \ll x^{10} + x^{\frac{k}{2(k+1)}}.
\]

By the definition of an exponent pair we must have \(\ell \geq \frac{1}{2}\) so that \(k \geq 1/3\) and hence \(k/2(k+1) \geq 1/8\). It now follows from (2.1) that

\[
\Delta^*_2(x) \ll x^{\frac{k}{2(k+1)}} \log x
\]

giving

\[
\rho_2 \leq \frac{k}{2(k+1)}.
\]

(2.3)

By an application of the A-process Phillips [24] showed that if \((k, \ell)\) is an exponent pair, then so is \((k_1, \ell_1) = A(k, \ell)\) where

\[
k_1 = \frac{k}{2(k+1)}, \quad \ell_1 = \frac{1}{2} + \frac{\ell}{2(k+1)}.
\]

By applying the B-process he showed that if \((k, \ell)\) is an exponent pair, then so is \((k_1, \ell_1) = B(k, \ell)\) where

\[
k_1 = \ell - \frac{1}{2}, \quad \ell_1 = k + \frac{1}{2}.
\]

Rankin [26] showed that there is also a convexity process which asserts that if \((k_1, \ell_1)\) and \((k_2, \ell_2)\) are exponent pairs, then so is

\[
(k, \ell) = C\{(k_1, \ell_1), (k_2, \ell_2); t\}
\]

where
\[ k = t k_1 + (1-t)k_2, \quad \ell = t \ell_1 + (1-t)\ell_2, \quad (0 \leq t \leq 1). \]

Now let \((n, \frac{1}{2} + \eta)\) be an exponent pair, and consider the two exponent pairs

\[
(k_1, \ell_1) = BA(n, \frac{1}{2} + \eta) = \left( \frac{n + \frac{1}{2}}{2n + 2}, \frac{2n + 1}{2n + 2} \right),
\]

\[
(k_2, \ell_2) = BA^2(n, \frac{1}{2} + \eta) = \left( \frac{2n + \frac{3}{2}}{6n + 4}, \frac{4n + 2}{6n + 4} \right).
\]

We can apply the convexity process \(C\) to these two pairs to give an exponent pair \((k, \ell)\) such that \(2\ell = 3k\). Specifically, let

\[ t = \frac{1 - 3n - 4\eta^2}{3 + 4\eta + 2n^2}. \]

Then we have the exponent pair

\[
C\left\{ (k_1, \ell_1), (k_2, \ell_2); t \right\} = \left[ \frac{2n + 1}{2n + 4n + 3}, \frac{3}{2} \times \frac{2n + 1}{2n + 4n + 3} \right].
\]

It now follows from (2.3) that

\[ \frac{3}{2} \leq \frac{2n + 1}{4(n+1)(n+2)} \quad (2.4) \]

where \(\eta\) is any number such that \((n, \frac{1}{2} + \eta)\) is an exponent pair.

Now, as an immediate consequence of the definition of an exponent pair, \((0, 1)\) is an exponent pair so that we have the exponent pair

\[ \left[ \frac{1}{6}, \frac{2}{3} \right] = AB(0,1) \]

and so we can take \(\eta = 1/6\) in (2.4) giving our required result (1.5).
With more tedious calculations we can improve on (1.5) slightly. For example, we can consider

\[
\left[ \frac{11}{82}, \frac{57}{82} \right] = \text{ABA} \left[ \frac{1}{6}, \frac{2}{3} \right]
\]

\[
\left[ \frac{16}{82}, \frac{52}{82} \right] = \text{B} \left[ \frac{11}{82}, \frac{57}{82} \right]
\]

\[
\left[ \frac{27}{164}, \frac{109}{164} \right] = \text{C} \left[ \left[ \frac{11}{82}, \frac{57}{82} \right], \left[ \frac{16}{82}, \frac{52}{82} \right], \frac{1}{2} \right]
\]

so that we can take \( \eta = \frac{27}{164} \) giving

\[
\rho_2^* \leq \frac{8938}{67805} < \frac{12}{91}.
\]

Let \( \eta_0 \) be the infimum of all \( \eta \) such that \((\eta, \frac{1}{2} + \eta)\) is an exponent pair so that we have

\[
\rho_2^* \leq \frac{2\eta_0 + 1}{4(\eta_0 + 1)(\eta_0 + 2)}.
\]

From Rankin's calculations [26] we have

\[
\eta_0 = 0.1645106784 \ldots
\]

giving

\[
\rho_2^* \leq 0.13181619 \ldots .
\]

5.3 PROOF OF THEOREM 1.

Let \( \rho \) and \( \theta \) satisfy

\[
\rho_2^* < \rho < \frac{1}{6}, \quad 0 < \theta < \frac{1}{2}.
\]
and for $x > 1$ we let
\[ h = \frac{1}{x^2} \cdot \theta, \quad \log x < t < \frac{1}{x}. \]  

(3.2)

First we have, from (4.1.1),
\[
Q(x) = \sum_{q \leq x} \sum_{a \cdot b = q} \mu^2(b) = \sum_{q \leq x} \sum_{a \cdot b = q} \mu(c) = \sum_{a \cdot b \cdot c \leq x} \mu(c),
\]
so that
\[
Q(x+h) - Q(x) = \sum_{q \leq x} \sum_{a \cdot b \cdot c \leq x+h} \mu(c)
\]
\[
= \sum_{q \leq x} \sum_{a \cdot b \cdot c \leq x+h} \mu(c) + \sum_{c > t} \sum_{q \leq x+h} \mu(c)
\]
\[
= \sum_{1} + \sum_{2}
\]

(3.3)
say. We have at once
\[
\sum_{1} = \sum_{c \leq t} \mu(c) \left\{ s \left( \frac{x+h}{c} \right) - s \left( \frac{x}{c} \right) \right\}.
\]

Now, as $x \to \infty$, we have
\[ (x+h)^\frac{1}{2} - x^\frac{1}{2} = \theta + 0 \left( x \cdot \frac{2\theta - 1}{2} \right), \]  

(3.4)

We conclude then
\[ (x+h)^\frac{1}{3} - x^\frac{1}{3} \ll x \]  

(3.5)

and
so that, from (1.1) and (4.1.6), we arrive at

\[ \sum_{c \leq t} \left( \frac{x}{c^6} \right)^\rho \ll x^\rho t^{1-6\rho} \]

Next we put

\[ T(x,t) = \sum_{a^2 b^3 c^6 \leq x \atop c > t} 1 \]

so that

\[ |\sum_{t} | \leq T(x+h,t) - T(x,t) \]

and it now follows from (3.3) and (3.6) that

\[ |Q(x+h) - Q(x) - \left( \frac{1}{2} A_{22} + o(1) \right) t^\theta| \leq T(x+h,t) - T(x,t) + 0\left( x^\rho t^{1-6\rho} \right) \].

We first use the following crude method to estimate

\[ T(x+h,t) - T(x,t). \] Let \( u \) be a number satisfying

\[ ut^5 > h. \]

Corresponding to each pair of numbers \( a, b \) there are at most \( u \)
numbers \( c \) satisfying \( c > t \) and \( x < a^2 b^3 c^6 \leq x+h \), since

\[ a^2 b^3 (c+u)^6 \geq a^2 b^3 c^6 + uc^5 \geq x + ut^5 > x + h. \]

We conclude that

\[ T(x+h,t) - T(x,t) \leq \sum_{a^2 b^3 \leq 2xt^{-6}} u = u S(2xt^{-6}) \ll ux^2 t^{-3}. \]
We now set
\[ t = x \log x, \quad u = x - \frac{1}{8} \]
so that (3.9) holds by (3.2), and that
\[ T(x+h, t) - T(x, t) = 0 \left( x^\theta (\log x)^{-3} \right) = o(x^\theta). \]

From (3.8) we now have that
\[ Q(x+h) - Q(x) = \left( \frac{1}{2} A_{22} + o(1) \right) x^\theta + O \left( x^{-\frac{8}{8}} (\log x)^{1-6\rho} \right). \]

Consequently if, in (3.1), \( \theta \) further satisfies
\[ \frac{1 + 2\rho}{8} < \theta < \frac{1}{2}, \]
then the asymptotic formula (1.6) holds. This proves that
\[ \theta_0 \leq \frac{1 + 2\rho^*}{8} \]
and on applying (1.5) we arrive at
\[ \theta_0 \leq \frac{115}{728} = 0.1579... \] (3.10)

which is already an improvement on (1.7).

We next make a more careful estimate for \( T(x+h), t) - T(x, t) \).

We recall from section 4.2 that
\[ \tau_2(n) = \sum_{\substack{a^2 + b^3 = n}} 1 \]
so that, from (3.7),
\[ T(x, t) = \sum_{n \in \mathbb{Z}} \tau_2(n) = \sum_{n \leq x} \tau_2(n) \sum_{t \leq c \leq (xn)^{-1} \frac{1}{6}} 1 \]

\[ = \sum_{n \leq x} \tau_2(n) \left\{ \left\lfloor \frac{x}{n} \right\rfloor - \left\lfloor \frac{x}{n} \right\rfloor \right\} \]

\[ = \sum_{n \leq x} \tau_2(n) \left\{ \frac{1}{6} \left\lfloor \frac{x}{n} \right\rfloor - \psi \left( \frac{1}{6} \left\lfloor \frac{x}{n} \right\rfloor \right) \right\} - tS(xt^{-6}), \quad (3.11) \]

provided that \( \psi(t) = 0 \), and this we may assume by taking \( t \) to be half an odd integer. Now, by (1.1),

\[ tS(xt^{-6}) = A_{22}^* \left( \frac{1}{2} \right) t^{-2} + A_{23}^* \left( \frac{1}{3} \right) t^{-1} + t\Delta_2^*(xt^{-6}). \]

Similarly to the derivation of (1.1) itself we have that

\[ \sum_{n \leq x} \tau_2(n) \frac{1}{6} \left\lfloor \frac{x}{n} \right\rfloor = x^2 \sum_{a \in \mathbb{Z}, b \leq x} \frac{1}{a^2 b^3} \]

we arrive at

\[ = \frac{3}{2} A_{22}^* \left( \frac{1}{2} \right) t^{-2} + 2A_{23}^* \left( \frac{1}{3} \right) t^{-1} + \psi \left( \frac{1}{3} \right) \psi \left( \frac{1}{2} \right) + t\Delta_2^*(xt^{-6}) \]

Let us write

\[ U(x, t) = \sum_{n \leq x} \tau_2(n) \psi \left( \frac{1}{n} \right) \]

\[ = \sum_{a \in \mathbb{Z}, b \leq x} \psi \left( \frac{1}{a^2 b^3} \right) \quad (3.12) \]
In view of (3.4) and (3.5), it now follows from (3.11) that

\[ T(x+h,t) - T(x,t) = -U(x+h,t) + U(x,t) + O(x^\rho t^{1-6\rho}) + o(x^\theta), \]

and so, from (3.8), we now have

\[ \left| Q(x+h) - Q(x) - \left( \frac{1}{2} A_{22} + o(1) \right)x^6 \right| \leq \left| U(x+h,t) - U(x,t) \right| + O(x^\rho t^{1-6\rho}). \]  

(3.13)

Using the trivial estimate in (3.12) we have

\[ \left| U(x,t) \right| \leq \sum_{a^2b^3 \leq xt^{-6}} \frac{1}{a^b} = S(xt^{-6}) \ll \frac{1}{x^2 t^{-3}}. \]

On setting \( t = x^{1-12\rho} \)

so that \( \frac{1}{x^2 t^{-3}} = x^\rho t^{1-6\rho} = \frac{1}{x^{8-12\rho}} \)

we arrive at

\[ Q(x+h) - Q(x) = \left( \frac{1}{2} A_{22} + o(1) \right)x^6 + O\left( \frac{1}{x^{8-12\rho}} \right). \]

Consequently if, in (3.1), \( \theta \) satisfies also

\[ \frac{1}{8-12\rho} < \theta < \frac{1}{2}, \]

then the asymptotic formula (1.6) holds. This proves that

\[ \theta_0 \leq \frac{1}{8-12\rho_2}, \]

and on applying (1.5) we arrive at
which is an improvement on (3.10).

Our final improvement comes from a non-trivial estimate for $U(x, t)$ given by the following:

**Lemma 3.1.** We have, as $x \to \infty$,

$$U(x, t) \ll \left( \frac{1}{2} - \frac{7}{2} \frac{217}{855} t - \frac{1072}{855} \right) \log^2 x$$

uniformly in $t \leq x^\alpha$.

From the lemma we see that, as $x \to \infty$,

$$U(x, t) \ll \frac{1}{2} - \frac{7}{2} \log^2 x, \quad t \leq x^\beta,$$  \quad (3.14)

where

$$\beta = \frac{421}{3841} = 0.1096 \ldots.$$  

We finally set

$$t = x^{1-2\rho}$$

so that

$$\frac{1}{x^2} \frac{7}{2} = x^\rho t^{1-6\rho} = x^{\frac{1+\rho}{9-12\rho}}.$$  

Since $\rho > \rho_2 \geq \frac{1}{10}$ by (3.1) and (4.1.5) we see that the exponent of $t$ is at most

$$\frac{4}{39} = 0.10256 \ldots < \beta$$

so that (3.14) is valid, and so from (3.13) we have that

$$Q(x+h) - Q(x) = \left( \frac{1}{2} A_{22} + o(1) \right) x^\theta + o \left( x^{\frac{1+\rho}{9-12\rho}} \log^2 x \right).$$
Consequently if $\theta$ satisfies
\[ \frac{1+\rho}{91-2\rho} < \theta < \frac{1}{2}, \]
then the asymptotic formula (1.6) holds. This proves our required result that
\[ \theta_0 \leq \frac{1+\rho^*}{9-12\rho^*} \leq \frac{103}{675} = 0.1529 \ldots \]
by (1.5), subject to the proof of Lemma 3.1.

Let us write
\[ U_1(x,t) = \sum_{\substack{m \leq n \leq 2L \atop m > n}} \psi \left( \frac{\left(\frac{x}{2}\right)}{m \cdot n} \right) \]
and
\[ U_2(x,t) = \sum_{\substack{m \leq n \leq 2L \atop n > m}} \psi \left( \frac{\left(\frac{x}{2}\right)}{m \cdot n} \right) \]
so that, from (3.12),
\[ U(x,t) = U_1(x,t) + U_2(x,t) + O \left( \left(\frac{x^2}{t^6} \right)^{\frac{1}{5}} \right). \tag{3.15} \]

We shall apply Srinivasan's theorem, that is our Lemma 4.4.1, with the two dimensional exponent pair (4.4.8), to estimate $U_1(x,t)$ and $U_2(x,t)$.

Let $z = x^{1/6}$ and $(\rho, \sigma)$ be either $\left( \frac{1}{2}, \frac{1}{3} \right)$ or $\left( \frac{1}{3}, \frac{1}{2} \right)$, and put
\[ S_{\rho, \sigma}(x,t; M,N) = \sum_{\substack{M \leq m \leq 2M \atop N < n \leq 2N \atop m \cdot n \leq zt \atop m > n}} \psi \left( \frac{z}{m \cdot n} \right). \]
We have, from Lemma 4.4.1, that

\[ S_{\rho, \sigma}(z, t; M, N) \ll \left( F^{92} M^{171} N^{263} \right)^{\frac{1}{342}} + F^{\frac{1}{4}} M^{\frac{1}{4}} N + F \frac{1}{2} MN , \]

where

\[ F = z M^{\rho} N^{-\sigma} . \]

Now

\[ \frac{5}{N^6} \ll M^{\frac{1}{3}} N^{\frac{2}{3}} \ll M^{\rho} N^{\sigma} \ll z^{-1} . \]

Therefore

\[ \frac{1}{F^4} M^{\frac{1}{4}} N = \frac{1}{z^4} (M^\rho N^\sigma) - \frac{1}{4} \frac{1}{M^4} N \]

\[ = \frac{1}{z^4} (M^\rho N^\sigma) - \frac{1}{4} \left( \frac{1}{3} \frac{1}{N^2} \right) \left( \frac{3}{6} \right) \]

\[ \ll \frac{1}{z^4} (M^\rho N^\sigma)^4 \ll z^{-2} t^{-1/4} = x^4 t^{-1} . \]

\[ \frac{1}{2} \frac{1}{MN} = \frac{1}{z^2} (M^\rho N^\sigma)^2 MN \ll z^{-2} (M^\rho N^\sigma)^2 (M^\rho N^\sigma)^2 \]

\[ \ll z^{-2} (z t^{-1})^2 = x^2 t^{-2} . \]

and

\[ \left( F^{92} M^{171} N^{263} \right)^{\frac{1}{342}} = \left\{ z^{92} \left( M^\rho N^\sigma \right)^{-92} \frac{1}{3} \frac{1}{N^2} \frac{513}{5} \frac{5}{6} \frac{39}{5} \right\}^{\frac{1}{342}} \]

\[ \ll \left\{ z^{92} \left( z t^{-1} \right)^{\frac{1}{5}} \right\} \]

\[ = \frac{2144}{5} , \]

\[ \frac{217}{855} - \frac{1072}{855} . \]

Since \( x^4 t^{-1/4} \ll x^4 t^{-2} \) if and only if \( t \leq x \) it follows that
and therefore we have

\[ U_1(x,t) + U_2(x,t) \ll \left( \frac{1}{2} x^2 + \frac{7}{2} \frac{x}{855} + \frac{217}{855} - \frac{1072}{855} \right) \log^2 x. \]

Lemma 3.1 now follows from (3.15).

5.4 PROOF OF THEOREM 2.

We first prove a lemma which requires the notion of uniform distributions in r-dimensions.

Lemma 4.1. Let \( 1 < \beta_1 < \ldots < \beta_r \) be such that 1, \( 1/\beta_1, \ldots, 1/\beta_r \) are linearly independent over \( \mathbb{Q} \). Let \( N > 1 \), and

\[ A_{\beta_i} = \left\{ \left\lfloor a \beta_i \right\rfloor : a = 1, 2, \ldots, \left\lfloor \frac{N+1}{\beta_i} \right\rfloor \right\}, \quad 1 \leq i \leq r. \]

Then, as \( N \to \infty \),

\[ |A_{\beta_1} \cap \ldots \cap A_{\beta_r}| \sim \frac{N}{\beta_1 \ldots \beta_r}. \]

Moreover, if \( r = 1 \), then as \( N \to \infty \),

\[ |A_{\beta_1}| = \frac{N}{\beta_1} + O(1) \]

uniformly in \( \beta_1 \).

Proof. Suppose that \( n \in A_{\beta_i} \) (1 \( \leq i \leq r \)). Then there exists an integer \( a_i \) such that \( \left\lfloor a_i \beta_i \right\rfloor = n \), or \( o < a_i \beta_i - n < 1 \), or \( o < a_i - n/\beta_i < 1/\beta_i \), or

\[ 1 - \frac{1}{\beta_i} < \frac{n}{\beta_i} - (a_i - 1) < 1. \]
This means that \( n \in A_{\beta_1} \cap \ldots \cap A_{\beta_r} \) if and only if the \( r \)-dimensional point

\[
P_n = \left\{ \left( \frac{n}{\beta_1}, \ldots, \frac{n}{\beta_r} \right) \right\},
\]

where \((N/\beta_i)\) denotes the fractional part of \( n/\beta_i \), lies inside the \( r \)-dimensional interval

\[
\left\{ (x_1, \ldots, x_r) : 1 - \frac{1}{\beta_i} < x_i < 1, \quad i = 1, 2, \ldots, r \right\} \quad (4.1)
\]

which has \( r \)-dimensional Lebesgue measure \((\beta_1 \cdots \beta_r)^{-1}\).

Since \( 1, 1/\beta_1, \ldots, 1/\beta_r \) are linearly independent over \( \mathbb{Q} \) if follows from a well known theorem on uniform distributions (see, for example, Cassels [3] Theorem 1, p.64) that the sequence of points \((P_n)\) are uniformly distributed in the \( r \)-dimensional unit cube. Therefore the number of points \( P_n \) with \( n \leq N \) which lie inside the interval \((4.1)\) is asymptotic to \( N/\beta_1 \cdots \beta_r \) as \( N \to \infty \). This proves that

Following \( |A_{\beta_1} \cap \ldots \cap A_{\beta_r}| \approx \frac{N}{\beta_1 \cdots \beta_r} \), \( N \to \infty \).

Finally, if \( r = 1 \), then

\[
|A_{\beta_1}| = \sum_{1 \leq n \leq N} 1 = \sum_{a < \frac{N+1}{\beta_1}} 1
\]

\[
= \left\lfloor \frac{N+1}{\beta_1} \right\rfloor = \frac{N}{\beta_1} + O(1).
\]

This completes the proof of the lemma.

Now let \( N \geq 1 \) and write, for each square-free \( b > 1 \),

\[
A_b = \left\{ \left[ a \frac{b^{3/2}}{b^{3/2}} \right] : a = 1, 2, \ldots, \left\lfloor \frac{N+1}{b^{3/2}} \right\rfloor \right\}.
\]
For \( r = 0, 1, 2, \ldots \), we define \( M_r \) by

\[
M_0 = N, \quad M_r = \sum_{1 < b_1 < \cdots < b_r} |A_{b_1} \cap \cdots \cap A_{b_r}|,
\]

the summation is over all square-free \( b_i \). We note that \( A_{b_r} \) is empty if \( b_r^{3/2} > N + 1 \) so that, corresponding to and fixed \( N \), there are only finitely many positive \( M_r \). Let \( B_0 \) be the set of positive integers \( n \leq N \) which are not in any of the sets \( A_b \). We note that \( n \in B_0 \) if and only if \( n \leq N \) and \( f(n) = 0 \), where \( f(n) \) is defined in our theorem, so that

\[
|B_0| = |\{ n : n \leq N, \ n \in F \}|. \tag{4.2}
\]

We also have, from Sylvester's inclusion-exclusion principle, that

\[
|B_0| = M_0 - M_1 + M_2 - M_3 + \cdots. \tag{4.2}
\]

Following Brun's observation in the sieve of Eratosthenes we now use a similar device to obtain a quantitative result from this formula. Let \( H \) be a positive constant which we shall specify later, and write

\[
M_r(H) = \sum_{1 < b_1 < \cdots < b_r \leq H} |A_{b_1} \cdots A_{b_r}|, \quad r = 1, 2, \ldots.
\]

For any even positive integer \( \ell \) we have that

\[
|B_0| \leq M_0 - M_1(H) + M_2(H) - \cdots + M_\ell(H) \tag{4.3}
\]

and

\[
|B_0| \geq M_0 - M_1 + M_2(H) - \cdots - M_{\ell+1}(H). \tag{4.4}
\]

Where the implied constant is absolute.
From Lemma 4.1 with $\beta_1 = \beta_{1,3/2}$ we have that

$$|A_{b_1} \cap \ldots \cap A_{b_r}| \sim \frac{N}{(b_1 \ldots b_r)^{3/2}}, \quad N \to \infty,$$

and so

$$M_{r}(H) = N \sum_{1 < b_1 < \ldots < b_r \leq H} \frac{\mu^2(b_1) \ldots \mu^2(b_r)}{(b_1 \ldots b_r)^{3/2}} + o(N)$$

$$= N \left( c_r - \delta_r(H) \right) + o(N)$$

where $c_r$ is defined in the theorem,

$$\delta_r(H) = \sum_{1 < b_1 < \ldots < b_r \leq H} \frac{\mu^2(b_1) \ldots \mu^2(b_r)}{(b_1 \ldots b_r)^{3/2}}$$

and the implied constant depends on $r$ and $H$. We remark that, for $r \geq 1$,

$$0 \leq \delta_r(H) \leq \left( \sum_{b = 2}^{\infty} b^{-3/2} \right)^{r-1} \sum_{b > H} b^{-3/2} \leq 2 \left( \zeta\left( -\frac{3}{2} \right) - 1 \right)^{r-1} \frac{1}{H^{1/2}}.$$

We also have, from Lemma 4.1 together with the observation that $A_b$ is empty if $b > N^{2/3} + 1$, that

$$M_1 = \sum_{1 < b_1 \leq N^{2/3}} \mu^2(b_1) \left\{ \frac{N}{b_1^{3/2}} + o(1) \right\}$$

$$\quad = N \left\{ c_1 + 0 \left( \sum_{b > N^{2/3}} b^{-3/2} \right) \right\} + o(N^{2/3})$$

$$\quad = c_1 N + o(N^{2/3})$$

where the implied constant is absolute.
From (4.3) and (4.4) we now have that

\[ \frac{|B_0|}{N} \leq c_0 - c_1 + \ldots + c_\ell + \delta_1(H) + \delta_3(H) + \ldots + \delta_{\ell-1}(H) + o(1), \quad (4.5) \]

and

\[ \frac{|B_0|}{N} \geq c_0 - c_1 + \ldots - c_{\ell+1} - \delta_2(H) - \delta_4(H) - \ldots - \delta_{\ell}(H) + o(1), \quad (4.6) \]

where the implied constants depend on \( \ell \) and \( H \).

As we shall see in the next section, \( c_r \) satisfies

\[ 0 < c_r < \frac{1}{r} c_1 c_{r-1}, \quad r = 2, 3, \ldots \]

so that the series \( c_0 - c_1 + c_2 - \ldots \) converges to \( d_o \). Let \( \epsilon > 0 \).

We can choose \( \ell \) so that

\[ d_o - \frac{\epsilon}{4} < c_0 - c_1 + \ldots - c_{\ell+1} < c_0 - c_1 + \ldots + c_\ell < d_o + \frac{\epsilon}{4}, \]

and then choose \( H \) so that

\[ \delta_1(H) + \delta_2(H) + \ldots + \delta_\ell(H) < \frac{\epsilon}{4}. \]

It follows from (4.5) and (4.6) that for all sufficiently large \( N \), we have

\[ d_o - \epsilon < \frac{|B_0|}{N} < d_o + \epsilon. \]

Proof. For convenience we write \( m \equiv 3 \ell + 1 \) and we denote by \( \mathbb{N}_m \) and \( \mathbb{N}_m^\ast \) the submonoids with respect to \( m \)-divisibility and \( m \)-smoothness, respectively, by \( \mathbb{N}_m \), and we denote by \( \mathbb{N}_m^\ast \) the set of \( m \)-smooth integers \( \mathbb{N}_m^\ast \) satisfying \( 1 < \ldots < b \). The case \( F_{m}^\ast \quad (m \geq 1) \) can be proved similarly by using the generalised inclusion-exclusion principle (see, for example, [28] Theorem 1, p. 198.), namely

\[ |B_m| = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{(m+\ell)!}{m! \ell!} \cdot M^{m+\ell}. \]
where $B_m$ is the set of those positive integers $n \leq N$ which lie in exactly $m$ of the sets $A_b$.

5.5 NUMERICAL VALUES FOR $d_m$.

We first show that each of the constants $c_r$ in Theorem 2 can be expressed in terms of $\zeta(3a/2)$, $a = 1, 2, \ldots, 2r$. For $r = 1, 2, \ldots$ and $a = 0, 1, \ldots$, we define

$$g(r, a) = \sum_{1 < b_1 < \ldots < b_r} \frac{\mu^2(b_1) \ldots \mu^2(b_r)}{b_1 \ldots b_r^{3/2}} \prod_{\ell=1}^{r} \frac{1}{b_\ell^{3a/2}}.$$ 

We note that

$$g(r, 0) = r c_r, \quad r = 1, 2, \ldots, (5.1)$$

and

$$g(1, a) = \sum_{b=2}^{\infty} \frac{\mu^2(b)}{b^{(3a+3)/2}}$$

$$= \frac{\zeta(3a+3)}{2} \frac{1}{\zeta(3a+3)} - 1, \quad a = 0, 1, \ldots. (5.2)$$

**Lemma 5.1.** We have, for $r \geq 1$ and $a \geq 0$,

$$g(r+1, a) = g(1, a)c_r - g(r, a+1).$$

**Proof.** For convenience we write $s = 3/2$ and we denote by $\sum_{(r)}$ the summation with respect to square-free integers $b_1, b_2, \ldots, b_r$ satisfying $1 < b_1 < \ldots < b_r$.

We consider

$$\sum_{(r+1)} \frac{1}{(b_1 \ldots b_{r+1})^s} \cdot \frac{1}{b_{r+1}} = \sum_{(r)} \frac{1}{(b_1 \ldots b_r)^s} \sum_{b > b_r} \frac{\mu^2(b)}{b^{(a+1)s}}.$$
With $b_0 = 1$ we see that
\[
\sum_{b} \frac{\mu^2(b)}{b(b+1)s} = \sum_{b=2}^{\infty} \frac{\mu^2(b)}{b(b+1)s} - \sum_{\ell=1}^{r} \sum_{b \leq \ell - 1}^{\ell} \frac{\mu^2(b)}{b(b+1)s} = g(1,a) - \sum_{\ell=1}^{r} \sum_{b \leq \ell - 1}^{\ell} \frac{\mu^2(b)}{b(b+1)s} - \sum_{\ell=1}^{r} \frac{1}{b(b+1)s} \, ,
\]
and so
\[
\sum_{(r+1)} \frac{1}{(b_1 \cdots b_{r+1})^s} \cdot \frac{1}{b_{r+1}} \sum_{(r)} \frac{1}{(b_1 \cdots b_{r})^s} \sum_{\ell=1}^{r} \sum_{b \leq \ell - 1}^{\ell} \frac{\mu^2(b)}{b(b+1)s} = \sum_{(r)} \frac{1}{(b_1 \cdots b_{r})^s} \left\{ g(1,a) - \sum_{\ell=1}^{r} \frac{1}{b_{\ell}^{(a+1)s}} \right\} \, ,
\]
that is
\[
\sum_{(r+1)} \frac{1}{(b_1 \cdots b_{r+1})^s} \left\{ \frac{1}{b_{r+1}^{as}} + \sum_{\ell=1}^{r} \frac{1}{b_{\ell}^{as}} \right\} = c_r \, g(1,a) - g(r,a+1) \, ,
\]
which is the required result.

From the reduction formula in Lemma 5.1 we can express each $g(r,a)$ in terms of $g(1,a')$ which, by (5.2), can be calculated from a table of values for the Riemann zeta function. We give the following table of values for $g(r,a)$ truncated to 4 decimal places.
TABLE 1.

From (5.1) we can now calculate \( c_r \) giving

\[
\begin{align*}
    c_1 &= 1.1732 \ldots , \\
    c_2 &= 0.5974 \ldots , \\
    c_3 &= 0.1801 \ldots , \\
    c_4 &= 0.0368 \ldots , \\
    c_5 &= 0.0055 \ldots , \\
    c_6 &= 0.0006 \ldots .
\end{align*}
\]

From the formula

\[
d_m = \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{(m+\ell)!}{m! \ell! \cdot c_{m+\ell}}
\]

we can now calculate \( d_m \), giving

\[
\begin{align*}
    d_0 &= 0.275965 \ldots , \\
    d_1 &= 0.395565 \ldots , \\
    d_2 &= 0.231299 \ldots , \\
    d_3 &= 0.077074 \ldots , \\
    d_4 &= 0.017015 \ldots , \\
    d_5 &= 0.002714 \ldots , \\
    d_6 &= 0.000331 \ldots , \\
    d_7 &= 0.000031 \ldots , \\
    d_8 &= 0.000002 \ldots .
\end{align*}
\]
We remark that, in order to give allowance for the coefficient of $c_{m+1}$ in the formula for $d_m$, we need more accurate results for $c_r$ than (5.3) to arrive at (5.4); we actually calculated $g(r,a)$ to 8 decimal places. From (5.4) we see that

$$0 < 1 - \sum_{m=0}^{8} d_m < 0.000004,$$

and

$$0 < c_1 - \sum_{m=0}^{8} m d_m < 0.000002.$$

In order to verify (5.4) we use a computer to find the following empirical frequencies for $f(n)$ in $1 \leq n \leq N$ when $N = 5000$.

<table>
<thead>
<tr>
<th>m</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_m$</td>
<td>1485</td>
<td>2049</td>
<td>1087</td>
<td>313</td>
<td>58</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>$d'_m$</td>
<td>0.2970</td>
<td>0.4098</td>
<td>0.2174</td>
<td>0.0626</td>
<td>0.0116</td>
<td>0.0014</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

**TABLE 2.**

The table here shows some small agreement with out predicted frequencies and we note in particular that

$$d'_o > d'_o, \quad d'_m < d_m \quad m = 2, \ldots.$$

$R(x) = Q(x) - \left\lfloor \sqrt{x} \right\rfloor$,
the ratio of the number of square-full integers which are not squares to the number of squares. We have

$$\lim_{x \to \infty} R(x) = A_{22} - 1 = c_1 = 1.1732...,$$

by the asymptotic formula (1.2). From Table 2 we see that

$$Q(25 \times 10^6) = 10435$$
giving

$$R(25 \times 10^6) = 1.087$$

which is substantially smaller than its limiting value $c_1$. This means that, up to 25 million, we are still in the initial block of the sequence $(q_n)$ where the squares show up more frequently than it should asymptotically. That is the second dominating term, namely $A_{23} x^{1/3}$ where $A_{23} < 0$, still interferes with the result; indeed up to $10^5$, the squares actually form a majority in the sense that $R(10^5) < 1$. We therefore expect $d'_0$ to decrease while $d'_m$ $(m \geq 1)$ to increase to our theoretical values as $N$ increases.

We remark that our computation also show that

$$|A_2(x)| \leq 3 \quad \text{for} \quad 1 \leq x \leq 25 \times 10^6.$$  

We also mention that the least solution to $f(n) = 6$ is $n = 3611$, and the 6 square-full integers in the interval $3611^2 < q < 3612^2$ are

$$13,041,125 = 323^2 \cdot 5^3$$
$$13,041,675 = 695^2 \cdot 3^3$$
$$13,042,575 = 195^2 \cdot 7^3$$
$$13,043,800 = 35^2 \cdot 22^3$$
$$13,045,131 = 99^2 \cdot 11^3$$
$$13,048,832 = 1277^2 \cdot 2^3.$$
5.6 PROOF OF (1.8).

We first show that, as \( n \to \infty \),

\[
q_n = a n^2 + b n^{5/3} + o(n^{4/3}) \quad (6.1)
\]

where

\[
a = \frac{1}{A_{22}^2}, \quad b = \frac{2A_{23}}{A_{22}^{2/3}}. \quad (6.2)
\]

We shall use the result

\[
\Delta_2(x) \ll x^6, \quad x \to \infty
\]

together with the observation that \( Q(q_n) = n \). From (1.2) we have, as \( n \to \infty \),

\[
n = A_{22} q_n^2 + A_{23} q_n^3 + o(q_n^6)
\]

and so

\[
n^2 = A_{22} q_n^2 + 2A_{22} A_{23} q_n^3 + o(q_n^3) \quad (6.3)
\]

Since \( q_n \ll n^2 \) we now have, from (6.2) and (6.3),

\[
q_n = a n^2 + o(n^3)
\]

and so

\[
q_n^2 = a n^3 + o(n^3) \quad .
\]

The required result (6.1) now follows from (6.3) and (6.2).

Now given any \( n \), there exists a unique positive integer \( m \) such that

\[
m^2 \leq q_n < q_{n+1} \leq (m+1)^2
\]
From (6.1) we see that, as \( n \to \infty \),
\[
m^2 = a n^2 + O(n^{\frac{5}{3}})
\]
and so
\[
m = \sqrt{a} n + O(n^{\frac{2}{3}}).
\]
From (6.4) we now have
\[
\frac{q_{n+1} - q_n}{2n} \leq \sqrt{a} + O(n^{-\frac{1}{3}})
\]
and hence
\[
\limsup_{n \to \infty} \frac{q_{n+1} - q_n}{2n} \leq \frac{1}{A_{22}}
\]
by (6.2). Finally, since \( d_0 > 0 \), there are infinitely many \( n \) such that \( q_n = m^2 \) and \( q_{n+1} = (m+1)^2 \), so that (6.4) holds with equality infinitely often, and so the required result (1.8) follows.
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