''VIBRATIONS OF THICK PLATES AND SHELLS''

by MICHAEL D J BRIDLE, B.Sc

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VIBRATIONS OF THICK PLATES AND SHELLS

M D J BRIDLE B.Sc.

ABSTRACT:

Using an asymptotic series approach, a thick shell theory is proposed for doubly curved shells with variable thickness. This theory includes the effects of transverse shear stresses and rotatory inertia. The displacement functions are designed to give non-zero transverse shear stresses internal to the shell, which satisfy the stress-free boundary conditions on the upper and lower surface. Use of the stress-free conditions makes the displacement functions, which vary through the thickness of the shell, dependent only on the middle surface displacements. This theory is applied to the twisted plate. A similar approach is applied to the cylindrical shell, but the effects of transverse normal stress are also included.

The theory is applied to the problem of free vibrations of shells clamped along one edge with the other three edges free. The results obtained are compared with practical and theoretical results of other researchers, and with those obtained from thin shell theory. The twisted plate results show the answers that are expected from a thick shell theory, in that it predicts lower frequencies than thin shell theory for modes in which the wavelength/thickness ratio is less than ten.

The results for the cylindrical shell show that the inclusion of transverse normal stress to the order assumed is not warranted.

The numerical techniques used for the solution of the free vibration problem are based on variational methods in which the Hamiltonian for the shell is minimised, subject to the constraints of the displacement boundary conditions.

CHAPTER 1. INTRODUCTION

1.1. Problem

The subject of this thesis is thick shell vibrations. The need for a theory of thick shells to predict natural frequencies is highlighted by the failures of turbine and compressor blading in such projects as the Queen Elizabeth II and the RB211 Jet engine. For certain blading, adequate results can be obtained using twisted beam or thin shell theory, dependent on the geometry of the blade. However, many modern blades are of such a shape that these theories are inadequate. A thick shell theory is required as a half-way stage between thin shell theory, and a full three dimensional analysis.

The major difference between thin and thick shell theories is the inclusion of transverse shear stresses in the latter case. These are neglected in thin shell theory, which therefore fails to give sufficiently accurate results as soon as these effects become important. In general the transverse shear modes do not appear by themselves, but are coupled with other modes. This coupling generally effects the frequencies of vibration for modes in which the wavelength to thickness ratio is less than ten.

1.2. Shell Theory

The basic property of shell theory is that one dimension, the thickness, is small compared with the other two. In shell theory this property is used to reduce the problem to two rather than three dimensions. To achieve this, certain approximations have to be made. The different approximations used, and the variation in the rigour applied in the analysis lead to the wide variety of shell theories to be found in the literature. Further references to and discussion of these theories can be found in Love (1), Nash (2), Flügge (3), Novozhilov (4), Goldenveiser (5), Naghdi (6), Koiter (7), and on vibrations of shells in Kalnins (8), Gros and Forsberg (9) and Hu (10).

In this section, an outline of the basic approximations is given, and the differences between thin and thick shell theories shown. Traditionally thin shell theories are based on the Kirchoff-Love assumptions:

a. Points which lie on one and the same normal, to the undeformed middle surface, also lie on one and the same normal to the deformed middle surface.

b. The effect of normal stress on surfaces parallel to the middle surface may be neglected in the stress-strain relation.

c. The displacements in the direction of the normal to the middle surface are approximately equal for all points on the same normal.

Assumptions a. and c. imply the displacement functions for the shell take the following form

$$
\begin{aligned}
u_1 &= u_1^0(\theta_1, \theta_2) + \theta_3 u_1'(\theta_1, \theta_2) \\
u_2 &= u_2^0(\theta_1, \theta_2) + \theta_3 u_2'(\theta_1, \theta_2) \\
u_3 &= u_3^0(\theta_1, \theta_2)
\end{aligned}
\qquad 1.2.1.
$$

where u_1' and u_2' are given in terms of the middle surface displacements u_1^0, u_2^0 and u_3^0 by the equations for the transverse shear strain implied by assumption a.

$$\gamma_{13} = \gamma_{23} = 0 \qquad 1.2.2.$$

Assumption b. affects the stress-strain relations; the fact that the transverse normal stress τ^{33} is zero is used to obtain an expression for the transverse normal strain γ_{33} which is substituted in the other stress-strain relations. This gives rise to one of the contradictions of thin shell theory in that assumption c. implies that $\gamma_{33} = 0$.

If assumption a. is relaxed, so that points on normals to the undeformed middle surface remain straight lines during deformation, but are not necessarily normal to the deformed middle surface, then the displacement functions are of the form 1.2.1. In this case, however, equations 1.2.2. do not apply, and so u_1' and u_2' are not dependent on the middle surface displacements. In this case substitution of these displacement functions in the strain-displacement and stress-strain relations gives non-zero transverse shear stresses and strains.

To include the effects of transverse normal stress, τ^{33}, the expansion for u_3 can be expanded to include terms of $O(\theta_3)$ or $O(\theta_3^2)$ i.e.

$$u_3 = u_3^0(\theta_1,\theta_2) + \theta_3 u_3^1(\theta_1,\theta_2) \qquad 1.2.3.$$

$$\text{or} \quad u_3 = u_3^0(\theta_1,\theta_2) + \theta_3 u_3^1(\theta_1,\theta_2) + \frac{\theta_3^2}{2} u_3^2(\theta_1,\theta_2)$$

The second of these cases has been considered by Reissner (11) and Naghdi (12).

In thick shell theory substitution of displacement functions 1.2.1. into the standard three dimensional strain-displacement and the stress-strain relations modified by the transverse normal stress assumption discussed earlier, gives transverse shear stresses which are constant through the thickness of the shell. For free vibration problems, the boundary conditions on the upper and lower surfaces of the shell imply that these stresses are zero on these boundaries. Hence direct application of this boundary condition would lead to thin shell theory. To surmount this difficulty changes have to be made to the standard three dimensional equations. This problem has been approached in two ways. In the first the stress-strain relations are modified by a shear constant K^2 so that

$$\tau^{\alpha 3} = K^2 G \gamma_{\alpha 3} \qquad (\alpha = 1,2) \qquad 1.2.4.$$

where G is the shear modulus of the material. This is the approach usually used in thick shell vibration problems. K^2 is evaluated so that the contribution of the transverse shear terms to the strain energy, matches the contribution of the true stress field. It is evaluated by matching frequencies obtained by the shell theory with those obtained by three dimensional theories. These methods are discussed by Mindlin (13) for plates and Herrmann and Mirsky (14) for shells. For the use of this technique in shell vibration problems see Herrmann and Mirsky (14), Tsui (15) and Zeinkiewicz et al (16, 17).

The other approach, which is often used in forming equations for shell theory, but has apparently not been applied to evaluate natural frequencies of shells is that presented by Naghdi (6). He assumes that the transverse stresses and strains are of the form:

$$\gamma_{\alpha 3} = \gamma^{0}_{\alpha 3}(\theta_1, \theta_2)\left(1 - \frac{\theta_3^2}{h^2}\right) \qquad 1.2.4.$$
$$\tau^{\alpha 3} = {}^{0}\tau^{\alpha 3}(\theta_1, \theta_2)\left(1 - \frac{\theta_3^2}{h^2}\right)$$

Higher order thick shell theories can be obtained by expanding the displacements u_1, u_2, u_3 to higher orders in θ_3. This approach has been taken by Martinez-Marquez (18) and Abe (19).

The discussion so far has only considered shell theories obtained by substituting assumed displacement functions in the three dimensional strain-displacement and stress-strain relations, with slight modifications to take account of stress assumptions made in the theories. The equations of motion or equilibrium can similarly be obtained by substituting the expressions for stresses, strains and displacements in the appropriate three dimensional equations, or by considering the forces acting on an element of the shell.

Another approach to obtain not only the equations of equilibrium but the stress-strain and the strain-displacement relations for the shell is by use of a variational theorem. This is done by Naghdi (6) who obtains these equations for a thick shell theory, and then shows the approximations introduced into various thin shell theories.

More recently, since the work of Johnson and Reissner (20) and Reiss (21) a great deal of interest has been shown in the use of asymptotic expansions for the solution of shell problems. In this case the displacements, strains and stresses are all expanded as series in a small parameter based on h/L where h is the thickness, and L some characteristic length of the shell. These expressions are substituted into the three dimensional equations, and a system of equations for varying orders of the parameter h/L extracted. This system can then be solved to give a solution to any order required. Here again as in the traditional approach to shell theory a variety of theories are obtained dependent on the asymptotic expansion used. The asymptotic technique was first applied by Johnson and Reissner (20) and Reiss (21) to static cylindrical shell problems.

Since then this has been extended by Green (22, 23) and Green and Naghdi (24) to general shells, and dynamic problems. Green and Naghdi (24) by this technique obtain equations of motion for thin shells.

The variational techniques used (as discussed in section 1.3 and Chapter 3), for the solution of the free vibration of shells, require that the kinetic and strain energy are expressed in terms of the displacement functions for the shell. For thin shell theories the displacements of the shell can be completely defined in terms of the three middle surface displacements, no matter which theory is applied. For the thick shell theories considered, five or more functions of the two middle surface coordinates are needed to define the displacements for the shell.

1.3. Vibrations of Shells

Only for shells with very special geometric properties and boundary conditions can the frequencies of natural vibration be found by exact methods. In order to obtain the solutions, in many cases, extra assumptions are made in the shell theory used. The most common of these are

a. The neglect of rotatory inertia.

b. The neglect of in-plane inertia effects. This is used most in the theory of shallow shells.

c. Bending stiffness of the shell is zero (Extensional vibrations).

d. Extensional stiffness of the shell is infinite (Inextensional vibrations).

The use of exact solution techniques and the effects of the above assumptions, as well as those in section 1.2, are discussed by Kalnins (8).

To obtain solutions for more general shell geometries and boundary conditions a large variety of approximate techniques have been developed. The most commonly used are those based on energy principles, such as the Rayleigh-Ritz method. In this method the displacements are expressed as a series of functions, each of which satisfies the displacement boundary conditions

A minimal energy condition is then used to find the values of the relative amplitudes of each member of the series, and the frequencies of vibration. The main drawback of this technique is that the displacement functions must each satisfy the displacement boundary conditions. This can be surmounted by finding stationary values of the Hamiltonian, subject to the constraints of the displacement boundary conditions. This technique is outlined in Chapter 3 and the basis of the method established in Appendix 1. For further details of the use of variational methods see Mikhlin (25) and Washizu (26).

For the application of these energy techniques, expressions are required for the strain and kinetic energy of the shell in terms of the displacement functions. For thin shell theory the displacements can be defined in terms of three functions of the two middle surface coordinates, but for the thick shell theories in use, five or more functions are involved. The energy techniques require sufficient terms in the approximating functions for convergence of the frequencies and mode shapes of vibration to be obtained. With the techniques available for solving the resultant eigenvalue problems, this convergence can be obtained for thin shell theory. For thick shell theory the number of functions involved does not allow adequate terms in each function to ensure convergence, except in special cases, such as axisymmetric shells where the problem is reduced to one dimension.

This problem has been overcome to some extent by the use of finite element techniques, Zeinkiewicz (27), which are very powerful in obtaining solutions to arbitrarily shaped shells, but in order to obtain solutions to thick shell problems assumptions must be introduced, usually involving the neglect of in-plane inertia terms. So although the results obtained using these techniques are very impressive, care must be taken over the effects of the other approximations introduced, especially for shells of high curvature.

1.4. Outline of Thesis

In this thesis a thick shell theory is proposed making use of the asymptotic series approach, and using the three dimensional strain-displacement and stress-strain equations. Displacement functions are proposed which satisfy the stress-free surface conditions on the shell, but which give non-zero transverse shear stresses in the interior. The use of the stress-free surface conditions make these displacement functions dependent only on the three middle surface displacements as opposed to the five or more functions normally encountered in thick shell theory. The resultant stress field satisfies the surface boundary conditions, and therefore no shear constant is required to modify the stress-strain relations. The stresses and strains are obtained by direct substitution of the displacements in the three dimensional equations, thus no extra assumptions have to be made as in equations 1.2.4. The only assumptions made in this theory are on the form of the displacement functions, and, in the resultant energy expression, the neglect of terms of $O(h^3)$. This theory is then applied to the problem of free vibrations of shells.

In Chapter 2 the thick shell theory is derived. The three-dimensional and shell equations in tensor form for classical infinitesimal elasticity are outlined in sections 2.2 and 2.3. The displacement functions are then derived in section 2.4. for a general variable thickness double curvature shell. This is for a theory where transverse shear stresses are included, but transverse normal stress neglected. The equations are also formed for the derivation of a theory including transverse normal stress, but these equations can only be solved in very special cases.

In sections 2.5 and 2.6 the equations are applied to the cylindrical shell. In this case transverse normal stress is included. In sections 2.7 and 2.8 the twisted plate is considered. This is a double curvature shell and so the theory in which transverse normal stress is neglected is applied.

In Chapter 3 the numerical methods employed in the solution of the free vibration problem are outlined. The method used was to minimise the Hamiltonian for the shell subject to the constraints of the displacement boundary conditions. The displacement functions are expanded as double power series. In order to obtain convergence of the frequencies and mode shapes of vibration the size of the system of equations had to be reduced. To achieve this two methods were applied, symmetry, and the use of the stress free edge conditions. The computer program developed to solve the problem is given, and the problems encountered in this discussed.

The results obtained for the two shells are presented in Chapter 4. The problem considered in each case was the shell clamped along one edge, the other three being stress free. The results obtained were compared with theoretical and experimental results obtained by other researchers, and with thin shell theory results obtained using Flügge's (3) equations.

CHAPTER 2 SHELL THEORY

2.1. Introduction

In this chapter displacement functions are proposed, which satisfy the stress free surface condition, but give non-zero interior transverse stresses. These displacement functions, which vary through the thickness of the shell, are made dependent on the middle surface displacements, by the use of the stress free surface condition. The functions are applied to two cases: the cylindrical shell and the twisted plate.

Use is made of the fundamental equations for the shell in tensor form obtained from Green and Zerna (28). These and the appropriate three dimensional equations and surface geometric properties are outlined in sections 2.2 and 2.3. Motivated by the works of Johnson and Reissner (20) and Reiss (21) for static shell problems, asymptotic series are introduced for the displacement function in section 2.4. The stress free surface conditions are then used to make these displacement functions dependent on the middle surface displacements This then gives for a particular shell only three functions to expand in series for the Rayleigh Ritz numerical solution (Chapter 3), as against the five normally encountered in thick shell theory.

These displacement functions are then applied in section 2.5 and 2.6 to the cylindrical shell, and in section 2.7 and 2.8 to the twisted plate. In each case giving expression for displacements, strains and stresses, to be used later in solving the free vibration problem.

For the twisted plate the theory makes the usual thick shell assumption that the transverse normal stress is zero. For the cylindrical shell the transverse normal stress is included. The effects of these assumptions are discussed in chapter 4. Also included in sections 2.6 and 2.8 are the relevant thin shell theory expressions for the cylinder and twisted plate respectively. These are used for comparison with the thick shell theory in the numerical work of chapter 4.

2.2. General Theory

In this section the relevant formulae for classical infinitesimal elasticity in general curvilinear coordinates Θ^i are summarised. The notation used is that adopted by Green and Zerna (28).

The covariant base vector $\underline{g}_i$ is defined by the equation

$$\underline{g}_i = \frac{\partial \underline{R}}{\partial \theta^i} \qquad 2.2.1$$

where $\underline{R}$ is the position vector of a point in the body

$$\underline{R} = \underline{R}(\theta^1, \theta^2, \theta^3) \qquad 2.2.2.$$

The covariant and contravariant metric tensors, and the contravariant base vector are then given by the equations.

$$g_{ij} = \underline{g}_i \cdot \underline{g}_j \qquad g^{ir} g_{rj} = \delta^i_j$$
$$\underline{g}^i = g^{ir} \underline{g}_r \qquad 2.2.3.$$

The displacement vector $\underline{u}$ of a point in space is written as

$$\underline{u} = u^i \underline{g}_i = u_j \underline{g}^j \qquad 2.2.4.$$

from which the covariant strain tensor γ_{ij} is obtained as

$$\gamma_{ij} = \frac{1}{2}\left\{ \underline{g}_i \cdot \frac{\partial \underline{u}}{\partial \theta^j} + \underline{g}_j \cdot \frac{\partial \underline{u}}{\partial \theta^i} \right\} \qquad 2.2.5.$$

For an isotropic material the stress strain relations are given by the equation

$$\tau^{ik} = \mu \left\{ g^{ir} g^{ks} + g^{is} g^{kr} + \frac{2\eta}{1-2\eta} g^{ik} g^{rs} \right\} \gamma_{rs} \qquad 2.2.6.$$

where τ^{ik} is the contravariant stress tensor, and μ and η are elastic constants for the material, the shear modulus and Poisson's ratio respectively.

The stress tensor τ^{ik} is related to a stress vector $\underline{T}_i$, acting on the surface θ^i = constant at any point, by the equation

$$\underline{T}_i = \tau^{ij} \underline{g}_j \sqrt{g} \qquad 2.2.7.$$

where $g = \det |g_{ij}|$

The physical components of displacement, strain and stress are given in terms of the tensor quantities by the following equations

$$V^i = u^j\sqrt{g^{jj}}$$
$$e_{ij} = \gamma_{lk}\sqrt{\frac{g^{kk}}{g^{ll}}} \qquad 2.2.8. \,*$$
$$\sigma_{ij} = \tau^{lk}\sqrt{\frac{g^{kk}}{g^{ll}}}$$

2.3. Fundamental Shell Equations

Points of the shell are defined by the position vector

$$\underline{R} = \underline{r}(\theta^1, \theta^2) + \theta_3\, \underline{a}_3(\theta^1, \theta^2) \qquad 2.3.1. \,***$$

where $\underline{r}$ is the position vector of points on the middle surface of the shell. This surface M is defined by a system of curvilinear coordinates θ^1, θ^2. $\underline{a}_3$ is a unit normal to the surface M at each point $\underline{r}$. The outer surfaces of the shell are given by

$$\theta_3 = \pm t \qquad 2.3.2.$$

where for a constant thickness shell t is a constant, but in general it is a function of θ^1, θ^2.

For the surface M, the covariant base vector is given by

$$\underline{a}_\alpha = \frac{\partial \underline{r}}{\partial \theta^\alpha} \qquad 2.3.3. \,**$$

The covariant and contravariant metric tensors, and the contravariant base vector then being given by the equations

$$a_{\alpha\beta} = \underline{a}_\alpha \cdot \underline{a}_\beta \qquad a^{\alpha\gamma} a_{\gamma\beta} = \delta^\alpha_\beta \qquad 2.3.4.$$
$$\underline{a}^\alpha = a^{\alpha\gamma}\underline{a}_\gamma$$

Other surface quantities required later are the second fundamental forms, and the Christoffel symbols. The second fundamental forms are defined by

* Note Indices repeated more than twice are not summed.

**Note From here on the middle surface coordinate system θ_1, θ_2 is denoted by Greek indices, the three dimensional system by latin indices.

*** Note $\theta^3 = \theta_3$

$$b_{\alpha\beta} = -\underline{a}_{\alpha}\cdot\underline{a}_{3,\beta} = -a_{3}\cdot\underline{a}_{\alpha,\beta}$$
$$b^{\alpha}_{\beta} = a^{\alpha\lambda}b_{\beta\lambda} = a_{\beta\lambda}b^{\alpha\lambda}$$

2.3.5.

where $,\beta$ denotes the partial derivative w.r.t. θ^{β}

The Christoffel symbols are given by

$$\Gamma^{\alpha}_{\beta\gamma} = \underline{a}^{\alpha}\cdot\frac{\partial \underline{a}_{\beta}}{\partial\theta^{\gamma}} = \frac{1}{2}a^{\alpha\lambda}\left[a_{\beta\lambda,\gamma} + a_{\lambda\gamma,\beta} - a_{\beta\gamma,\lambda}\right] \qquad 2.3.6.$$

Applying the equations of this section to those of section 2.2. gives the following equations for the shell base vectors and metric tensors.

$$\begin{aligned}
\underline{g}_{\alpha} &= \left(\delta^{\rho}_{\alpha} - \theta_{3}b^{\rho}_{\alpha}\right)\underline{a}_{\rho} \\
\underline{g}_{3} &= \underline{a}_{3} \\
g_{\alpha\beta} &= a_{\alpha\beta} - 2\theta_{3}b_{\alpha\beta} + \theta_{3}^{2}b^{\lambda}_{\beta}b_{\alpha\lambda} \\
g_{\alpha 3} &= g^{\alpha 3} = 0 \\
g_{33} &= g^{33} = 1 \\
g^{\alpha\beta} &= a^{\alpha\beta} + 2\theta_{3}b^{\alpha\beta} + 3\theta_{3}^{2}b^{\alpha\rho}b^{\beta}_{\rho} + O(\theta_{3}^{3})
\end{aligned} \qquad 2.3.7.$$

where $g^{\alpha\beta}$ is an infinite power series in θ_3, provided that t is sufficiently small.

For the shell the displacement vector is defined in terms of the middle surface base vector as

$$\underline{u} = u^{\alpha}\underline{a}_{\alpha} + u^{3}\underline{a}_{3} = u_{\alpha}\underline{a}^{\alpha} + u_{3}\underline{a}^{3}$$
$$\text{where} \quad u^{\alpha} = a^{\alpha\beta}u_{\beta} \qquad 2.3.8.$$

which gives expression for the strain tensor

$$2\gamma_{\alpha\beta} = u_{\alpha}|_{\beta} + u_{\beta}|_{\alpha} - 2b_{\alpha\beta}u_{3} - \theta_{3}\left[b_{\beta}^{\lambda}(u_{\lambda}|_{\alpha} - b_{\lambda\alpha}u_{3}) + b_{\alpha}^{\lambda}(u_{\lambda}|_{\beta} - b_{\lambda\beta}u_{3})\right] \qquad 2.3.9.$$

$$2\gamma_{\alpha 3} = u_{3,\alpha} + b_{\alpha}^{\lambda}u_{\lambda} + u_{\alpha,3} - \theta_{3}b_{\alpha}^{\lambda}u_{\lambda,3}$$

$$\gamma_{33} = u_{3,3}$$

where $u_{\alpha}|_{\beta}$ denotes the covariant derivative of u_{α} w.r.t. θ^{β}.
In terms of the partial derivatives

$$u_{\alpha}|_{\beta} = u_{\alpha,\beta} - \Gamma_{\alpha\beta}^{\lambda}u_{\lambda} \qquad 2.3.10.$$

$$u^{\alpha}|_{\beta} = u^{\alpha}_{,\beta} + \Gamma_{\lambda\beta}^{\alpha}u^{\lambda}$$

Substituting equation 2.3.7. into 2.2.6. gives for the shell stress-strain relations

$$\tau^{\alpha\beta} = \mu\left\{g^{\alpha\lambda}g^{\beta\rho} + g^{\alpha\rho}g^{\beta\lambda} + \frac{2\eta}{1-2\eta}g^{\alpha\beta}g^{\lambda\rho}\right\}\gamma_{\lambda\rho} + \frac{2\mu\eta}{1-2\eta}g^{\alpha\beta}\gamma_{33}$$ 2.3.11.

$$\tau^{\alpha 3} = 2\mu g^{\alpha\lambda}\gamma_{\lambda 3}$$

$$\tau^{33} = \frac{2\mu}{1-2\eta}\left\{(1-\eta)\gamma_{33} + \eta g^{\rho\lambda}\gamma_{\rho\lambda}\right\}$$

A shell stress tensor σ^{ij} is defined by

2.3.12. *

$$\underline{T}_i = \tau^{ij}\underline{g}_i\sqrt{g} = (\sigma^{i\lambda}\underline{a}_\lambda + \sigma^{i3}\underline{a}_3)\sqrt{a}$$

which implies that

$$\sigma^{i\lambda} = (\delta^\lambda_\mu - \theta_3 b^\lambda_\mu)\tau^{i\mu}\sqrt{\frac{g}{a}}$$ 2.3.13.

$$\sigma^{i3} = \tau^{i3}\sqrt{\frac{g}{a}}$$

Shell stress and couple resultants are defined in terms of σ^{ij} by

$$N^{\alpha\beta} = \int_{-t}^{t}\sigma^{\alpha\beta}d\theta_3$$ 2.3.14.

$$M^{\alpha\beta} = \int_{-t}^{t}\theta_3\sigma^{\alpha\beta}d\theta_3$$

$$Q^{\alpha} = \int_{-t}^{t}\sigma^{\alpha 3}d\theta_3$$

2.4. Displacement Functions

Using the general displacement-strain and stress-strain equations 2.3.9. and 2.3.11, displacement functions are proposed which satisfy the stress free conditions on the surface of the shell, but which give non zero transverse stresses interior to the shell.

Applying the stress free conditions to the stress-strain relations 2.3.8. implies that

$$\gamma_{13} = \gamma_{23} = 0 \qquad \theta_3 = \pm t$$ 2.4.1.

$$(1-\eta)\gamma_{33} + \eta g^{\rho\lambda}\gamma_{\rho\lambda} = 0$$

* Note a = $\det|a_{ij}|$

An asymptotic series for the displacement functions is to be used. Therefore a new normal coordinate ζ is defined, where

$$\zeta = \frac{\theta_3}{t} \qquad 2.4.2.$$

In general t is a function of the middle surface variables θ_1 and θ_2, this is now redefined by

$$t = hX(\theta^1, \theta^2) \qquad 2.4.3.$$

where h is a constant equal to half the maximum thickness of the shell, and X defines the variation in thickness.

Therefore $\theta_3 = hX(\theta^1, \theta^2)\zeta$

$$2.4.4.$$

where $-1 \leq \zeta \leq 1$

and $-1 \leq X(\theta^1, \theta^2) \leq 1$

With the adoption of the new normal coordinate ζ. derivatives w.r.t. the old coordinate system are given by

$$\frac{\partial V}{\partial \theta^\alpha} = \frac{\partial V}{\partial \theta^\alpha} - \left(\frac{\frac{\partial X}{\partial \theta^\alpha}}{X}\right)\zeta \frac{\partial V}{\partial \zeta}$$

$$\frac{\partial V}{\partial \theta^3} = \frac{1}{hX} \frac{\partial V}{\partial \zeta} \qquad 2.4.5.$$

The only condition on X is that it is a smooth function with continuous first derivatives of the same order of magnitude as the function itself.

The displacement functions U_i are now defined as asymptotic series in t.

$$U_i = \sum_{J=0}^{N} t^J \, {}^J u_i(\theta^1, \theta^2, \zeta) \qquad 2.4.6.$$

This series, unlike a Taylor's series expansion, can be truncated after any term, and still satisfy the stress free boundary conditions with non-zero transverse stresses. Therefore the series is truncated after the first term to give

$$u_i = {}^0u_i(\theta^1, \theta^2, \zeta) + t\,{}^1u_i(\theta^1, \theta^2, \zeta) \tag{2.4.7.}$$

Substituting 2.4.7. into the displacement-strain relations 2.3.9. and the stress-strain relations 2.3.11 gives rise to in-plane stresses of O(1) and transverse stresses of O $(\frac{1}{t})$. All stresses can be made of the same order, O(1) by making ${}^0u_i$ a function of the middle surface coordinates θ^1 and θ^2 only. With this assumption, and expanding ${}^1u_i$ as a polynomial in ζ gives

$$u_i = {}^0u_i(\theta^1, \theta^2) + t \sum_{J=0}^{M} {}^1_J u_i\, \zeta^J \tag{2.4.8.}$$

Substitution of these displacement functions 2.4.8. into the stress free boundary conditions 2.4.1. and the displacement-strain relations 2.3.8, neglecting terms of O(t^2), gives twelve equations in 3M + 6 unknowns. In order to give a set of equations to solve for ${}^1_J u_i$ in terms of the middle surface displacements ${}^0u_i$, M is taken as three, to leave twelve equations in fifteen unknowns. This gives displacements with terms up to and including ζ^3 , this being the least number of terms required to give non-zero transverse stresses, which satisfy the stress free boundary conditions The truncation of the polynomial after the ζ^3 term is made as a Rayleigh-Ritz approximation to the true solution, and it does not imply that higher order terms are negligible.

The twelve equations imply that

$${}^1_2 u_i = 0 \tag{2.4.9.}$$

i.e. no ζ^2 term. Also all the ${}^1_0 u_i$ terms are independent of all other terms but themselves, therefore they are incorporated into the $O(1)\,{}^0u_i$ term.

Taking account of these facts, the indicial notation is now dropped, and the displacement functions redefined as

$$\begin{aligned} u_1 &= A + hX\zeta(D + \zeta^2 E) \\ u_2 &= B + hX\zeta(F + \zeta^2 G) \\ u_3 &= C + hX\zeta(H + \zeta^2 J) \end{aligned} \tag{2.4.10.}$$

The six remaining stress free boundary conditions are now

$$
\begin{aligned}
&\frac{\partial C}{\partial \theta^1} + b_1^1 A + b_1^2 B + D + 3E = 0 \\
&\frac{\partial C}{\partial \theta^2} + b_2^1 A + b_2^2 B + F + 3G = 0 \\
&X\left(\frac{\partial H}{\partial \theta^1} + \frac{\partial J}{\partial \theta^1}\right) - 2\frac{\partial X}{\partial \theta^1} J - 2X\left(b_1^1 E + b_1^2 G\right) = 0 \\
&X\left(\frac{\partial H}{\partial \theta^2} + \frac{\partial J}{\partial \theta^2}\right) - 2\frac{\partial X}{\partial \theta^2} J - 2X\left(b_2^1 E + b_2^2 G\right) = 0 \\
&\eta P\left(A, \frac{\partial A}{\partial \theta^\alpha}, B, \frac{\partial B}{\partial \theta^\alpha}, C\right) + (1-\eta)(H + 3J) = 0 \\
&Q\left(A, \frac{\partial A}{\partial \theta^\alpha}, B, \frac{\partial B}{\partial \theta^\alpha}, C, D, \frac{\partial D}{\partial \theta^\alpha}, E, \frac{\partial E}{\partial \theta^\alpha}, F, \frac{\partial F}{\partial \theta^\alpha},\right. \\
&\qquad \left. G, \frac{\partial G}{\partial \theta^\alpha}, H, J : X, \frac{\partial X}{\partial \theta^\alpha}\right) = 0
\end{aligned}
\qquad 2.4.11.
$$

where P and Q are linear functions of the displacements and their derivatives. In the second case each displacement term being multiplied by X or one of its derivatives. These functions P and Q are dependent only on the shell middle surface geometry.

In general these equations are impossible to solve in terms of A, B and C, the middle surface displacements, but in certain special cases solution is possible, i.e. the cylindrical shell, section 2.6.

Now the more usual thick shell theory assumption is made, that the transverse normal stress is zero throughout the shell. This implies that J = 0. With this result, the first five equations of 2.4.11. are considered, the equation for the O(h) term in τ^{33} being neglected. The Rayleigh Ritz solution procedure will take care of this (see Chapter 3). Thus the equation 2.4.11. become

$$
\begin{aligned}
&\frac{\partial C}{\partial \theta^1} + b_1^1 A + b_1^2 B + D + 3E = 0 \\
&\frac{\partial C}{\partial \theta^2} + b_2^1 A + b_2^2 B + F + 3G = 0 \\
&\frac{\partial H}{\partial \theta^1} - 2\left(b_1^1 E + b_1^2 G\right) = 0 \\
&\frac{\partial H}{\partial \theta^2} - 2\left(b_2^1 E + b_2^2 G\right) = 0 \\
&\eta P\left(A, \frac{\partial A}{\partial \theta^\alpha}, B, \frac{\partial B}{\partial \theta^\alpha}, C\right) + (1-\eta) H = 0
\end{aligned}
\qquad 2.4.12.
$$

It should be noted that these equations are now independent of variable thickness, as all the derivative terms of X disappear with the assumption that J = 0. From the fifth equation H is obtained as a function of A, B, C and their derivatives. Substituting for H in the first four equations, then gives four linear equation for D, E, F and G in terms of the middle surface displacements.

The condition on these equations having a solution being that

$$K = b_1^1 b_2^2 - b_2^1 b_1^2 \neq 0 \qquad 2.4.13.$$

i.e. that the shell has double curvature.

For a single curvature shell it is possible to construct displacement functions to give a non zero transverse shear stress, dependent on the middle surface displacements, in the curved direction. In this case θ^1 and θ^2 must be principal coordinates. Assuming θ^2 to be the curved principal coordinate, gives, for a single curvature shell, b_2^2 as the only non zero mixed 2nd fundamental form. To obtain a non-zero transverse stress in the curved direction it is assumed that $\tau^{13} = 0$ throughout the shell, this implies that E = 0. Also as in the $\tau^{33} = 0$ assumption the $O(t)\tau^{13}$ equation is neglected, leaving the four equations

$$\frac{\partial C}{\partial \theta^1} + D = 0$$

$$\frac{\partial C}{\partial \theta^2} + b_2^2 B + F + 3G = 0 \qquad 2.4.14.$$

$$\frac{\partial H}{\partial \theta^2} - 2b_2^2 G = 0$$

$$\eta P\left(A, \frac{\partial A}{\partial \theta^\alpha}, B, \frac{\partial B}{\partial \theta^\alpha}, C\right) + (1-\eta)H = 0$$

Thus displacement functions can be obtained in terms of the middle surface displacement only, which give non-zero transverse shear stresses in any curved coordinate direction of the shell, and yet satisfy the stress free surface condition. Rotatory Inertia effects are taken account of by including $O(h^2)$ terms in the kinetic energy expression.

2.5. Cylindrical Shell

Here the equations of section 2.3 are applied to a cylindrical shell in order to obtain the displacement-strain and stress-strain relations. The shell is defined in terms of a middle surface M, taking the form of a cylinder of radius R. Thus in terms of cylindrical coordinates α and β , see fig 1, the position vector of the middle surface is given by

$$\underline{r} = (R\cos\alpha, R\sin\alpha, R\beta) \tag{2.5.1.}$$

where β and α correspond to the curvilinear coordinates of section 2.3 Θ^1 and Θ^2 respectively.

The length coordinates involved are scaled by the radius of the cylinder R. The normal to the surface is given by

$$\underline{a}^3 = (\cos\alpha, \sin\alpha, 0) \tag{2.5.2.}$$

Substituting equations 2.5.1 and 2.5.2 its equations 2.3.3., 2.3.4, and 2.3.5 gives the following expressions for the middle surface base vector, metric tensors and 2nd fundamental forms.

$$\underline{a}_1 = (0, 0, 1) \tag{2.5.3.}$$

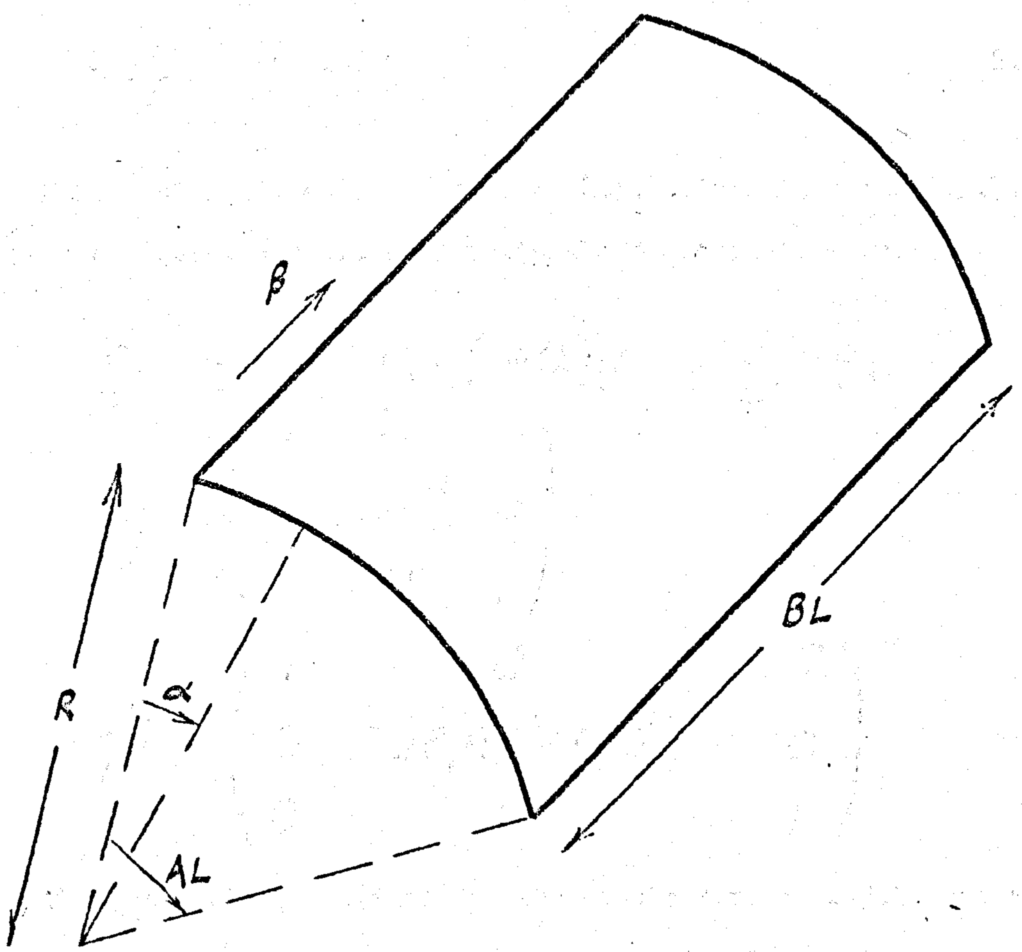
$$\underline{a}_2 = (-\sin\alpha, \cos\alpha, 0)$$

$$a_{\alpha\beta} = a^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$b_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$b_1^1 = b_1^2 = b_2^1 = 0 \qquad b_2^2 = -1$$

Fig 1. Cylindrical Surface



Since the $a_{\alpha\beta}$ terms are constant, all the Christoffel symbols are zero, therefore

$$u_\alpha|_\beta = u_{\alpha,\beta} \qquad 2.5.4.$$

Substituting equations 2.5.2 and 2.5.3. into equation 2.3.7. gives expressions for the shell base vectors and metric tensors.

$$\underline{g}_1 = \underline{a}_1 , \quad \underline{g}_2 = (1+\theta_3)\underline{a}_2 , \quad \underline{g}_3 = \underline{a}_3$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1+\theta_3)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad 2.5.5.$$

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-2\theta_3+3\theta_3^2+O(\theta_3^3) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The geometric quantities for the middle surface 2.5.4. are substituted into equations 2.3.9. to give the following displacement-strain relations.

$$\gamma_{11} = u_{1,1} \qquad 2.5.6.$$

$$\gamma_{12} = \frac{1}{2}\left\{u_{1,2} + u_{2,1}(1+\theta_3)\right\}$$

$$\gamma_{22} = (u_{2,2} + u_3)(1+\theta_3)$$

$$\gamma_{13} = \frac{1}{2}(u_{3,1} + u_{1,3})$$

$$\gamma_{23} = \frac{1}{2}\left\{u_{3,2} - u_2 + u_{2,3}(1+\theta_3)\right\}$$

$$\gamma_{33} = u_{3,3}$$

Substituting equations 2.5.5. into equations 2.3.11. gives stress-strain relations, ignoring terms of $O(\theta_3^3)$, of the form.

$$\tau^{11} = \frac{2\mu}{1-2\eta}\left\{(1-\eta)\gamma_{11} + \eta\left[(1-2\theta_3+3\theta_3^2)\gamma_{22} + \gamma_{33}\right]\right\} \tag{2.5.7.}$$

$$\tau^{12} = 2\mu(1-2\theta_3+3\theta_3^2)\gamma_{12}$$

$$\tau^{22} = \frac{2\mu}{1-2\eta}\left\{(1-\eta)(1-4\theta_3+10\theta_3^2)\gamma_{22} + \eta(1-2\theta_3+3\theta_3^2)(\gamma_{11}+\gamma_{33})\right\}$$

$$\tau^{13} = 2\mu\gamma_{13}$$

$$\tau^{23} = 2\mu(1-2\theta_3+3\theta_3^2)\gamma_{23}$$

$$\tau^{33} = \frac{2\mu}{1-2\eta}\left\{\eta\left[\gamma_{11} + (1-2\theta_3+3\theta_3^2)\gamma_{22}\right] + (1-\eta)\gamma_{33}\right\}$$

2.6. Cylindrical Shell - Displacement Functions

For the cylindrical shell three shell theories are considered. The thick shell theory using displacement functions 2.4.10 including non-zero transverse normal and shear stresses, secondly the thick shell theory with zero transverse normal stress τ^{33}, and thirdly the thin shell, Flügge theory.

The first case uses displacement functions of the form

$$u_1 = A + hX\zeta(D + \zeta^2 E)$$

$$u_2 = B + hX\zeta(F + \zeta^2 G) \qquad 2.6.1.$$

$$u_3 = C + hX\zeta(H + \zeta^2 J)$$

Substituting these equations into the stress free surface condition 2.4.1. and with the use of the geometrical cylindrical shell relations of section 2.5. gives

$$D + 3E + \frac{\partial C}{\partial \beta} = 0 \qquad 2.6.2.$$

$$F + 3G + \frac{\partial C}{\partial \alpha} - B = 0$$

$$(1-\eta)(H + 3J) + \eta\left(\frac{\partial A}{\partial \beta} + \frac{\partial B}{\partial \alpha} + C\right) = 0$$

$$X\left(\frac{\partial H}{\partial \beta} + \frac{\partial J}{\partial \beta}\right) - 2\frac{\partial X}{\partial \beta} J = 0$$

$$X\left(\frac{\partial H}{\partial \alpha} + \frac{\partial J}{\partial \alpha}\right) - 2\frac{\partial X}{\partial \alpha} J + 2XG = 0$$

$$X\left(\frac{\partial D}{\partial \beta} - \frac{\partial E}{\partial \beta}\right) - 2\frac{\partial X}{\partial \beta} E + X\left(\frac{\partial F}{\partial \alpha} + \frac{\partial G}{\partial \alpha}\right) - 2\frac{\partial X}{\partial \alpha} G + X(H + J) - X\left(\frac{\partial B}{\partial \alpha} + C\right) = 0$$

For general X it is impossible to solve these equations to obtain the displacement functions in terms of the middle surface displacements A, B and C. However, in certain special cases solution is possible. Considered here is the case when X is a function of α only. The equation 2.6.2. then solve to give

$$u_1 = A + \tfrac{1}{2}h\chi\rho\Big\{(3-\rho^2)\Big(\bar{C} + \frac{\partial^2\bar{C}}{\partial\alpha^2} - \beta\Big(\frac{\eta}{1-\eta}\Big)\Big(T + \frac{d^2}{d\alpha^2}(\chi T)\Big)\Big) + \Big(\frac{\eta}{1-\eta}\Big)\frac{\partial}{\partial\alpha}\Big[\frac{d\chi}{d\alpha}\Big(A + \frac{\partial\bar{B}}{\partial\alpha} + \bar{C}\Big)\Big] + (1-\rho^2)\frac{\partial^2\bar{C}}{\partial\beta^2}\Big\} \qquad 2.6.3.$$

$$u_2 = \frac{\partial\bar{B}}{\partial\beta} + h\rho\chi\Big\{\frac{\partial\bar{B}}{\partial\beta} - \frac{\partial^2\bar{C}}{\partial\alpha\partial\beta} + \tfrac{1}{2}(3-\rho^2)\Big(\frac{\eta}{1-\eta}\Big)\Big(\frac{d}{d\alpha}(\chi T)\Big) + \frac{d\chi}{d\alpha}\Big(\frac{\partial A}{\partial\beta} + \frac{\partial^2\bar{B}}{\partial\alpha\partial\beta} + \frac{\partial\bar{C}}{\partial\beta}\Big)\Big\}$$

$$u_3 = \frac{\partial\bar{C}}{\partial\beta} + \tfrac{1}{2}h\rho\chi\Big(\frac{\eta}{1-\eta}\Big)\Big\{(1-\rho^2)\Big(\frac{\partial A}{\partial\beta} + \frac{\partial^2\bar{B}}{\partial\alpha\partial\beta} + \frac{\partial\bar{C}}{\partial\beta}\Big) + (3-\rho^2)T\Big\}$$

where $T = T(\alpha)$, an arbitrary function,

$$\frac{\partial\bar{B}}{\partial\beta} = B$$

and

$$\frac{\partial\bar{C}}{\partial\beta} = C$$

If the shell is of constant thickness then these equations reduce to the form

$$u_1 = A + \tfrac{1}{2}h\varsigma\left\{(3-\varsigma^2)\left(\bar{C} + \frac{\partial^2\bar{C}}{\partial\alpha^2} - \beta\left(\frac{\eta}{1-\eta}\right)\left(T + \frac{d^2T}{d\alpha^2}\right)\right.\right.$$
$$\left.+ (1-\varsigma^2)\frac{\partial^2\bar{C}}{\partial\beta^2}\right\} \qquad 2.6.4.$$

$$u_2 = B + h\varsigma\left\{B - \frac{\partial^2\bar{C}}{\partial\alpha\partial\beta} + \tfrac{1}{2}(3-\varsigma^2)\left(\frac{\eta}{1-\eta}\right)\frac{dT}{d\alpha}\right\}$$

$$u_3 = \frac{\partial\bar{C}}{\partial\beta} + \tfrac{1}{2}h\varsigma\left(\frac{\eta}{1-\eta}\right)\left\{(1-\varsigma^2)\left(\frac{\partial A}{\partial\beta} + \frac{\partial B}{\partial\alpha} + \frac{\partial\bar{C}}{\partial\beta}\right)\right.$$
$$\left. + (3-\varsigma^2)T\right\}$$

These equations for constant thickness give displacement strain relations of the form

$$\gamma_{11} = \frac{\partial A}{\partial\beta} + \tfrac{1}{2}h\varsigma\left\{(3-\varsigma^2)\left(\frac{\partial\bar{C}}{\partial\beta} + \frac{\partial^3\bar{C}}{\partial\alpha^2\partial\beta} - \left(\frac{\eta}{1-\eta}\right)\left(T + \frac{d^2T}{d\alpha^2}\right)\right)\right.$$
$$\left. + (1-\varsigma^2)\frac{\partial^3\bar{C}}{\partial\beta^3}\right\}$$

$$\gamma_{12} = \frac{1}{2}\left\{\frac{\partial A}{\partial\alpha} + \frac{\partial B}{\partial\beta} + h\varsigma\left[2\frac{\partial B}{\partial\beta} - \tfrac{1}{2}(1+\varsigma^2)\frac{\partial^3\bar{C}}{\partial\alpha\partial\beta^2}\right]\right.$$
$$+ \tfrac{1}{2}(3-\varsigma^2)\left[\frac{\partial\bar{C}}{\partial\alpha} + \frac{\partial^3\bar{C}}{\partial\alpha^3} - \beta\left(\frac{\eta}{1-\eta}\right)\left(\frac{dT}{d\alpha} + \frac{d^3T}{d\alpha^3}\right)\right] \qquad 2.6.5.$$
$$\left. + h^2\varsigma^2\left[\frac{\partial B}{\partial\beta} - \frac{\partial^3\bar{C}}{\partial\alpha\partial\beta^2}\right]\right\}$$

$$\gamma_{22} = \frac{\partial B}{\partial\alpha} + \frac{\partial\bar{C}}{\partial\beta} + h\varsigma\left\{2\frac{\partial B}{\partial\alpha} + \frac{\partial\bar{C}}{\partial\beta} - \frac{\partial^3\bar{C}}{\partial\alpha^2\partial\beta}\right.$$
$$\left. + \tfrac{1}{2}\left(\frac{\eta}{1-\eta}\right)\left[(3-\varsigma^2)\left(T + \frac{d^2T}{d\alpha^2}\right) + (1-\varsigma^2)\left(\frac{\partial A}{\partial\beta} + \frac{\partial B}{\partial\alpha} + \frac{\partial\bar{C}}{\partial\beta}\right)\right]\right\}$$

$$\gamma_{13} = \tfrac{1}{4}(1-\varsigma^2)\left\{3\left(\bar{C} + \frac{\partial^2\bar{C}}{\partial\alpha^2} + \frac{\partial^2\bar{C}}{\partial\beta^2} - \beta\left(\frac{\eta}{1-\eta}\right)\left(T + \frac{d^2T}{d\alpha^2}\right)\right)\right.$$
$$\left. + h\varsigma\left(\frac{\eta}{1-\eta}\right)\left(\frac{\partial^2 A}{\partial\beta^2} + \frac{\partial^2 B}{\partial\alpha\partial\beta} + \frac{\partial^2\bar{C}}{\partial\beta^2}\right)\right\}$$

$$\gamma_{23} = \frac{1}{4}\left(\frac{\eta}{1-\eta}\right)(1-\zeta^2)\left\{3\frac{dT}{d\alpha} + h\zeta\left[\frac{dT}{d\alpha} + \frac{\partial^2 A}{\partial\alpha\partial\beta} + \frac{\partial^2 B}{\partial\alpha^2} + \frac{\partial^2 \bar{C}}{\partial\alpha\partial\beta}\right]\right\}$$

$$\gamma_{33} = \frac{1}{2}\left(\frac{\eta}{1-\eta}\right)\left\{(1-3\zeta^2)\left(\frac{\partial A}{\partial\beta} + \frac{\partial B}{\partial\alpha} + \frac{\partial \bar{C}}{\partial\beta}\right) + 3(1-\zeta^2)T\right\} \qquad 2.6.5.$$

The displacement strain relations are used with the stress strain relations 2.5.7 to completely define the state of the shell in terms of the middle surface displacements and the arbitary function of α (T and the terms to make up $\bar{C}$).

It should be noted from these equations that the transverse shear terms are quadratics to the first order, but that the $h\zeta$ terms mean that these strains and hence the stresses are not symmetric about the middle surface. The in plane strains are all, to the first order, the same as in the thin shell theory presented later, but the inclusion of the J term in γ_{33} means that the in plane stresses, to the first order, are quadratics in ζ .

Secondly consider the thick shell theory with the assumption that to the first order $\tau^{33} = 0$, thus the displacements are of the form 2.6.1, but with J = 0.

The cylindrical shell being a single curvature shell means that, with the displacement functions considered, a non-zero transverse shear stress satisfying the surface conditions can only occur in the curved direction. The coordinates α, β considered for the cylindrical shell are principal co-ordinates, therefore assuming also that $\tau^{13} = 0$ equation 2.4.14 become

$$\begin{aligned}
&\frac{\partial C}{\partial\beta} + D = 0 \\
&\frac{\partial C}{\partial\alpha} - B + F + 3G = 0 \\
&\frac{\partial H}{\partial\alpha} - 2G = 0 \\
&\eta\left(\frac{\partial A}{\partial\beta} + \frac{\partial B}{\partial\alpha} + C\right) + (1-\eta)H = 0
\end{aligned} \qquad 2.6.6.$$

with solution

$$
\begin{aligned}
u_1 &= A - h\chi\zeta\frac{\partial C}{\partial \beta} \\
u_2 &= B + h\chi\zeta\left\{B - \frac{\partial C}{\partial \alpha} - \frac{1}{2}\left(\frac{\eta}{1-\eta}\right)(3-\zeta^2)\left(\frac{\partial^2 A}{\partial\alpha\partial\beta} + \frac{\partial^2 B}{\partial\alpha^2} + \frac{\partial C}{\partial\alpha}\right)\right\} \\
u_3 &= C - h\chi\zeta\left(\frac{\eta}{1-\eta}\right)\left(\frac{\partial A}{\partial\beta} + \frac{\partial B}{\partial\alpha} + C\right)
\end{aligned}
\qquad 2.6.7.
$$

Substituting in the displacement strain relations 2.5.6 gives

$$
\begin{aligned}
\gamma_{11} &= \frac{\partial A}{\partial\beta} - h\chi\zeta\frac{\partial^2 C}{\partial\beta^2} \\
\gamma_{12} &= \frac{1}{2}\Big\{\frac{\partial A}{\partial\alpha} + \frac{\partial B}{\partial\beta} + h\chi\zeta\Big[2\frac{\partial B}{\partial\beta} - 2\frac{\partial^2 C}{\partial\alpha\partial\beta} \\
&\quad - \frac{1}{2}\left(\frac{\eta}{1-\eta}\right)(3-\zeta^2)\left(\frac{\partial^3 A}{\partial\alpha\partial\beta^2} + \frac{\partial^3 B}{\partial\alpha^2\partial\beta} + \frac{\partial^2 C}{\partial\alpha\partial\beta}\right)\Big] \\
&\quad + h^2\chi^2\zeta^2\Big[\frac{\partial B}{\partial\beta} - \frac{\partial^2 C}{\partial\alpha\partial\beta} - \frac{1}{2}\left(\frac{\eta}{\eta-1}\right)(3-\zeta^2)\left(\frac{\partial^3 A}{\partial\alpha\partial\beta^2} + \frac{\partial^3 B}{\partial\alpha^2\partial\beta} + \frac{\partial^2 C}{\partial\alpha\partial\beta}\right)\Big] \\
&\quad - h\frac{\partial\chi}{\partial\beta}(1+h\chi\zeta)\zeta^3\left(\frac{\partial^2 A}{\partial\alpha\partial\beta} + \frac{\partial^2 B}{\partial\alpha^2} + \frac{\partial C}{\partial\alpha}\right)\Big\} \\
\gamma_{22} &= (1+h\chi\zeta)\Big\{\frac{\partial B}{\partial\alpha} + C + h\chi\zeta\Big[\frac{\partial B}{\partial\alpha} - \frac{\partial^2 C}{\partial\alpha^2} \\
&\quad - \left(\frac{\eta}{1-\eta}\right)\left(\frac{\partial A}{\partial\beta} + \frac{\partial B}{\partial\alpha} + C\right) - \frac{1}{2}\left(\frac{\eta}{1-\eta}\right)(3-\zeta^2)\left(\frac{\partial^3 A}{\partial\alpha^2\partial\beta} + \frac{\partial^3 B}{\partial\alpha^3} + \frac{\partial^2 C}{\partial\alpha^2}\right)\Big] \\
&\quad - h\frac{\partial\chi}{\partial\alpha}\zeta^3\left(\frac{\partial^2 A}{\partial\alpha\partial\beta} + \frac{\partial^2 B}{\partial\alpha^2} + \frac{\partial C}{\partial\alpha}\right)\Big\} \\
\gamma_{13} &= -\frac{1}{2}h\chi\zeta\left(\frac{\eta}{1-\eta}\right)\left(\frac{\partial A}{\partial\beta} + \frac{\partial B}{\partial\alpha} + C\right) \\
\gamma_{23} &= -\frac{1}{4}\left(\frac{\eta}{1-\eta}\right)(1-\zeta^2)\left\{(3-2h\chi\zeta)\left(\frac{\partial^2 A}{\partial\alpha\partial\beta} + \frac{\partial^2 B}{\partial\alpha^2} + \frac{\partial C}{\partial\alpha}\right)\right\} \\
\gamma_{33} &= -\left(\frac{\eta}{1-\eta}\right)\left(\frac{\partial A}{\partial\beta} + \frac{\partial B}{\partial\alpha} + C\right)
\end{aligned}
\qquad 2.6.8.
$$

The stress-strain relations to be used being 2.5.7.

Also included are the equations for Flugge's cylindrical shell theory. For this displacement functions are assumed of the form

$$\begin{aligned} U_1 &= A - \theta_3 \frac{\partial C}{\partial \beta} \\ U_2 &= B + \theta_3 \left(B - \frac{\partial C}{\partial \alpha} \right) \\ U_3 &= C \end{aligned} \qquad 2.6.9.$$

The displacement-strain relations are

$$\begin{aligned} \gamma_{11} &= \frac{\partial A}{\partial \beta} - \theta_3 \frac{\partial^2 C}{\partial \beta^2} \\ \gamma_{12} &= \frac{1}{2} \left\{ \frac{\partial A}{\partial \alpha} + \frac{\partial B}{\partial \beta} + 2\theta_3 \left(\frac{\partial B}{\partial \beta} - \frac{\partial^2 C}{\partial \alpha \partial \beta} \right) \right\} \\ \gamma_{22} &= \left\{ \frac{\partial B}{\partial \alpha} + C + \theta_3 \left(\frac{\partial B}{\partial \alpha} - \frac{\partial^2 C}{\partial \alpha^2} \right) \right\} (1+\theta_3) \\ \gamma_{13} &= \gamma_{23} = \gamma_{33} = 0. \end{aligned} \qquad 2.6.10.$$

and the stress-strain relations

$$\begin{aligned} \tau^{11} &= \frac{2\mu}{1-\eta} \left\{ \gamma_{11} + \eta (1 - 2\theta_3 + 3\theta_3^2) \gamma_{22} \right\} \\ \tau^{12} &= 2\mu (1 - 2\theta_3 + 3\theta_3^2) \gamma_{12} \\ \tau^{22} &= \frac{2\mu}{1-\eta} \left\{ \eta (1 - 2\theta_3 + 3\theta_3^2) \gamma_{11} + (1 - 4\theta_3 + 10\theta_3^2) \gamma_{22} \right\} \\ \tau^{13} &= \tau^{23} = \tau^{33} = 0 \end{aligned} \qquad 2.6.11.$$

2.7. Twisted Plate

The equations of section 2.3 are now applied to the geometry of a twisted plate. The middle surface of the twisted plate is defined by two curvilinear coordinates α and β , see fig 2. The characteristic length, a, is defined for the shell so that if β is the angle of twist at any point, then that point is a length $a\beta$ in the Z direction. α is defined as the angle such that any point on the surface is a length $a\tan\alpha$ from the axis, CC^1, of the shell.

The position vector of the middle surface is given by

$$\underline{r} = (a\tan\alpha\cos\beta,\ a\tan\alpha\sin\beta,\ a\beta) \qquad 2.7.1.$$

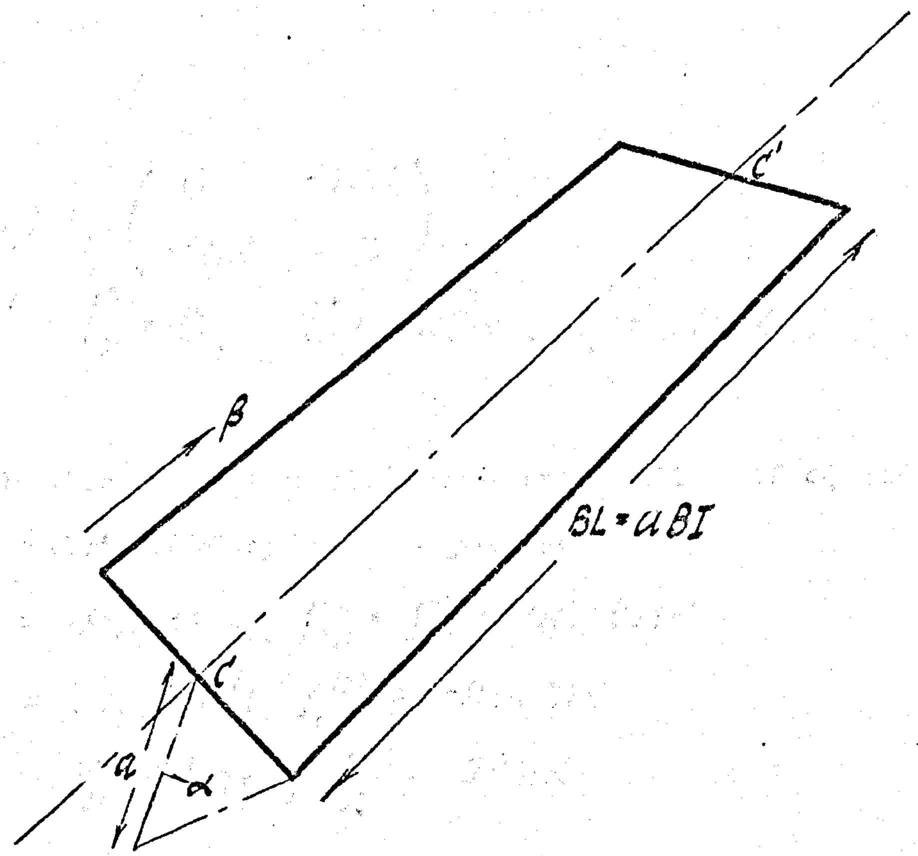
The length, a, is now scaled to unity and β and α are taken as the curvilinear coordinates θ^1 and θ^2 respectively. The normal to the middle surface is given by

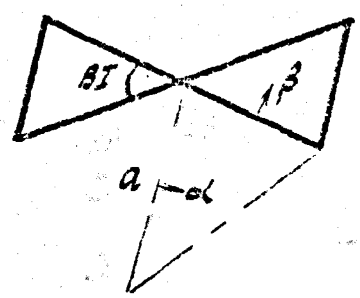
$$\underline{a}_3 = (\cos\alpha\sin\beta,\ -\cos\alpha\cos\beta,\ \sin\alpha) \qquad 2.7.2.$$

Substitution of equations 2.7.1 and 2.7.2 into 2.3.3., 2.3.5 and 2.3.6 gives for the middle surface base vectors, metric tensors and 2nd fundamental forms the following expressions.

$$\underline{a}_1 = (0, 0, 1)$$
$$\underline{a}_2 = (\sec^2\alpha\cos\beta,\ \sec^2\alpha\sin\beta,\ 0)$$
$$a_{\alpha\beta} = \begin{pmatrix} \sec^2\alpha & 0 \\ 0 & \sec^4\alpha \end{pmatrix}, \quad a^{\alpha\beta} = \begin{pmatrix} \cos^2\alpha & 0 \\ 0 & \cos^4\alpha \end{pmatrix} \qquad 2.7.3.$$

Fig 2. Twisted Plate Surface





Shell Viewed Along
Axis CC'

$$b_{\alpha\beta} = \begin{pmatrix} 0 & -\sec\alpha \\ -\sec\alpha & 0 \end{pmatrix} \qquad 2.7.3.$$

$$b_1^{\,1} = b_2^{\,2} = 0, \quad b_1^{\,2} = -\cos^3\alpha, \quad b_2^{\,1} = -\cos\alpha$$

In this case the metric tensors are functions of α and therefore the non-zero Christoffel symbols are given by

$$\Gamma_{112} = -\sec^2\alpha\tan\alpha, \quad \Gamma_{121} = \Gamma_{211} = \sec^2\alpha\tan\alpha$$
$$\Gamma_{222} = 2\tan\alpha\sec^4\alpha, \quad \Gamma_1^{\,22} = -\sin\alpha\cos\alpha \qquad 2.7.4.$$
$$\Gamma_{12}^{\,1} = \Gamma_{21}^{\,1} = \tan\alpha, \quad \Gamma_{22}^{\,2} = 2\tan\alpha$$

The shell base vectors and metric tensors are obtained by substituting equations 2.7.3 and 2.7.4 into 2.3.7 to give

$$\underline{g}_1 = \underline{a}_1 + \theta_3\cos^3\alpha\,\underline{a}_2, \quad \underline{g}_2 = \underline{a}_2 + \theta_3\cos\alpha\,\underline{a}_1, \quad \underline{g}_3 = \underline{a}_3$$

$$\underline{g}_{ij} = \begin{pmatrix} \sec^2\alpha(1+\cos^4\alpha\,\theta_3^2) & 2\theta_3\sec\alpha & 0 \\ 2\theta_3\sec\alpha & \sec^4\alpha(1+\cos^4\alpha\,\theta_3^2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad 2.7.5.$$

$$\underline{g}^{ij} = \begin{pmatrix} \cos^2\alpha(1+3\cos^4\alpha\,\theta_3^2) & -2\theta_3\cos^5\alpha & 0 \\ -2\theta_3\cos^5\alpha & \cos^4\alpha(1+3\cos^4\alpha\,\theta_3^2) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The geometric properties of the shell are applied to the displacement-strain equation 2.3.9, with the use of equation 2.3.8, to give the strains in terms of the contravariant displacement as

$$\begin{aligned}
\gamma_{11} &= \sec^2\alpha\, u^1_{,1} + \sec^2\alpha \tan\alpha\, u^2 + \theta_3(\sec\alpha\, u^2_{,1} - \sin\alpha\, u^1 + \cos^2\alpha\, u^3) \\
\gamma_{12} &= \tfrac{1}{2}\Big\{\sec^2\alpha\, u^1_{,2} + \sec^4\alpha\, u^2_{,1} + \sec\alpha\, u^3 \\
&\qquad + \theta_3(\sec\alpha\, u^1_{,1} + \sec\alpha\, u^2_{,2} + 3\sec\alpha\tan\alpha\, u^2)\Big\} \\
\gamma_{22} &= \sec^4\alpha\, u^2_{,2} + 2\sec^4\alpha\tan\alpha\, u^2 + \theta_3(\sec\alpha\, u^1_{,2} + \sec\alpha\tan\alpha\, u^1 + u^3) \\
\gamma_{13} &= \tfrac{1}{2}\Big\{u^3_{,1} - \sec\alpha\, u^2 + \sec^2\alpha\, u^1_{,3} + \theta_3\sec\alpha\, u^2_{,3}\Big\} \\
\gamma_{23} &= \tfrac{1}{2}\Big\{u^3_{,2} - \sec\alpha\, u^1 + \sec^4\alpha\, u^2_{,3} + \theta_3\sec\alpha\, u^1_{,3}\Big\} \\
\gamma_{33} &= u^3_{,3}
\end{aligned} \qquad 2.7.6.$$

These equations are given in terms of the contravariant displacement functions as they give simpler equations in later applications, see section 2.8.

Substitution of equations 2.7.5 into 2.3.11 gives the following stress-strain relations, terms of $O(\theta_3^3)$ being ignored.

$$\begin{aligned}
\tau^{11} &= \frac{2\mu}{1-2\eta}\Big\{(1-\eta)\cos^4\alpha(1+6\cos^4\alpha\,\theta_3^2)\gamma_{11} - 4(1-\eta)\cos^7\alpha\,\theta_3\gamma_{12} \\
&\qquad + \big[\eta\cos^6\alpha(1+6\cos^4\alpha\,\theta_3^2) + 4(1-2\eta)\cos^{10}\alpha\,\theta_3^2\big]\gamma_{22} + \eta\cos^2\alpha(1+3\cos^4\alpha\,\theta_3^2)\gamma_{33}\Big\}
\end{aligned} \qquad 2.7.7.$$

$$\begin{aligned}
\tau^{12} &= \frac{2\mu}{1-2\eta}\Big\{-4(1-\eta)\cos^7\alpha\,\theta_3\gamma_{11} + \big[(1-2\eta)\cos^6\alpha(1+6\cos^4\alpha\,\theta_3^2) + 8\eta\cos^{10}\alpha\,\theta_3^2\big]\gamma_{12} \\
&\qquad - 4(1-\eta)\cos^9\alpha\,\theta_3\gamma_{22} - 4\eta\cos^5\alpha\,\theta_3\gamma_{33}\Big\}
\end{aligned}$$

$$\begin{aligned}
\tau^{22} &= \frac{2\mu}{1-2\eta}\Big\{\big[\eta\cos^6\alpha(1+6\cos^4\alpha\,\theta_3^2) + 4(1-2\eta)\cos^{10}\alpha\,\theta_3^2\big]\gamma_{11} - 4(1-\eta)\cos^9\alpha\,\theta_3\gamma_{12} \\
&\qquad + (1-\eta)\cos^8\alpha(1+6\cos^4\alpha\,\theta_3^2)\gamma_{22} + \eta\cos^4\alpha(1+3\cos^4\alpha\,\theta_3^2)\gamma_{33}\Big\}
\end{aligned}$$

$$\tau^{13} = 2\mu\Big\{\cos^2\alpha(1+3\cos^4\alpha\,\theta_3^2)\gamma_{13} - 2\cos^5\alpha\,\theta_3\gamma_{23}\Big\}$$

$$\tau^{23} = 2\mu\left\{-2\cos^5\alpha\,\theta_3\,\gamma_{13} + \cos^4\alpha\left(1+3\cos^4\alpha\,\theta_3^2\right)\gamma_{23}\right\}$$

$$\tau^{33} = \frac{2\mu}{1-2\eta}\left\{\eta\cos^2\alpha\left(1+3\cos^4\alpha\,\theta_3^2\right)\gamma_{11} - 4\eta\cos^5\alpha\,\theta_3\,\gamma_{12} + \eta\cos^4\alpha\left(1+3\cos^4\alpha\,\theta_3^2\right)\gamma_{22} + (1-\eta)\gamma_{33}\right\} \qquad 2.7.7.$$

2.8. Twisted Plate - Displacement Functions

For the twisted plate only the thick shell theory with zero τ^{33} is considered. The six equations 2.4.11 for the theory with non-zero τ^{33} being impossible to solve for the displacement functions in terms of the middle surface displacement, even for special cases i.e. constant thickness.

Thus displacements functions are considered of the form

$$\begin{aligned} u^1 &= A + h\times\zeta(D+\zeta^2 E) \\ u^2 &= B + h\times\zeta(F+\zeta^2 G) \\ u^3 &= C + h\times\zeta H \end{aligned} \qquad 2.8.1.$$

Substituting these expressions into the stress free boundary equations 2.4.1., using the geometric properties of the shell from section 2.7, and the relations between contravariant and covariant displacements 2.3.8. gives the set of equations

$$\frac{\partial C}{\partial \beta} - \sec\alpha\,B + \sec^2\alpha(D+3E) = 0 \qquad 2.8.2.$$

$$\frac{\partial C}{\partial \alpha} - \sec\alpha\,A + \sec^4\alpha(F+3G) = 0$$

$$\frac{\partial H}{\partial \beta} + 2\sec\alpha\,G = 0$$

$$\frac{\partial H}{\partial \alpha} + 2\sec\alpha\,E = 0$$

$$\eta\left(\frac{\partial A}{\partial \beta} + \frac{\partial B}{\partial \alpha} + \tan\alpha\,B\right) + (1-\eta)H = 0$$

These equations solve to give displacements functions of the form

$$
\begin{aligned}
u^1 &= A + h\chi\varsigma\Big\{\cos\alpha\, B - \cos^2\alpha\,\frac{\partial C}{\partial\beta} \\
&\quad -\tfrac{1}{2}\left(\frac{\eta}{1-\eta}\right)(3-\varsigma^2)\left(\cos\alpha\,\frac{\partial^2 A}{\partial\alpha\partial\beta} + \cos\alpha\,\frac{\partial^2 B}{\partial\alpha^2} + 3\sin\alpha\,\frac{\partial B}{\partial\alpha} + 3\sec\alpha\, B\right)\Big\} \\
u^2 &= B + h\chi\varsigma\Big\{\cos^3\alpha\, A - \cos^4\alpha\,\frac{\partial C}{\partial\alpha} \\
&\quad -\tfrac{1}{2}\left(\frac{\eta}{1-\eta}\right)(3-\varsigma^2)\left(\cos\alpha\,\frac{\partial^2 A}{\partial\beta^2} + \cos\alpha\,\frac{\partial^2 B}{\partial\alpha\partial\beta} + 3\sin\alpha\,\frac{\partial B}{\partial\beta}\right)\Big\} \\
u^3 &= C - h\chi\varsigma\left(\frac{\eta}{1-\eta}\right)\left(\frac{\partial A}{\partial\beta} + \frac{\partial B}{\partial\alpha} + 3\tan\alpha\, B\right)
\end{aligned}
\qquad 2.8.3.
$$

Substitution into the displacement-strain relations 2.7.6. gives the equations

$$
\begin{aligned}
\gamma_{11} &= \sec^2\alpha\,\frac{\partial A}{\partial\beta} + \sec^2\alpha\tan\alpha\, B + h\chi\varsigma\Big\{2\sec\alpha\,\frac{\partial B}{\partial\beta} + \cos^2\alpha\, C - \frac{\partial^2 C}{\partial\beta^2} \\
&\quad - \sin\alpha\cos\alpha\,\frac{\partial C}{\partial\alpha} - \tfrac{1}{2}\left(\frac{\eta}{1-\eta}\right)(3-\varsigma^2)\Big[\sec\alpha\tan\alpha\,\frac{\partial^2 A}{\partial\beta^2} + \sec\alpha\,\frac{\partial^3 A}{\partial\alpha\partial\beta^2} \\
&\quad + 3\sec\alpha(2\sec^2\alpha - 1)\frac{\partial B}{\partial\beta} + 4\sec\alpha\tan\alpha\,\frac{\partial^2 B}{\partial\alpha\partial\beta} + \sec\alpha\,\frac{\partial^3 B}{\partial\alpha^2\partial\beta}\Big]\Big\} \\
&\quad + h^2\chi^2\varsigma^2\Big\{\cos^2\alpha\left(\frac{1-2\eta}{1-\eta}\right)\frac{\partial A}{\partial\beta} - \sin\alpha\cos\alpha\left(\frac{1+2\eta}{1-\eta}\right)B - \cos^2\alpha\left(\frac{\eta}{1-\eta}\right)\frac{\partial B}{\partial\alpha} \\
&\quad + \sin\alpha\cos^2\alpha\,\frac{\partial C}{\partial\beta} - \cos^3\alpha\,\frac{\partial^2 C}{\partial\alpha\partial\beta} - \tfrac{1}{2}\left(\frac{\eta}{1-\eta}\right)(3-\varsigma^2)\Big[\frac{\partial^3 A}{\partial\beta^3} - \sin\alpha\cos\alpha\,\frac{\partial^2 A}{\partial\alpha\partial\beta} \\
&\quad - 3\tan\alpha\, B + 3\tan\alpha\,\frac{\partial^2 B}{\partial\beta^2} - 3\sin^2\alpha\,\frac{\partial B}{\partial\alpha} - \sin\alpha\cos\alpha\,\frac{\partial^2 B}{\partial\alpha^2} + \frac{\partial^3 B}{\partial\alpha\partial\beta^2}\Big]\Big\} \\
&\quad - h\frac{\partial\chi}{\partial\beta}\varsigma^3\left(\frac{\eta}{1-\eta}\right)\Big\{\sec\alpha\,\frac{\partial^2 A}{\partial\alpha\partial\beta} + \sec\alpha\,\frac{\partial^2 B}{\partial\alpha^2} + 3\sec\alpha\tan\alpha\,\frac{\partial B}{\partial\alpha} + 3\sec^2\alpha\, B\Big\} \\
&\quad - h^2\chi\frac{\partial\chi}{\partial\beta}\varsigma^4\left(\frac{\eta}{1-\eta}\right)\left(\frac{\partial^2 A}{\partial\beta^2} + \frac{\partial^2 B}{\partial\alpha\partial\beta} + 3\tan\alpha\,\frac{\partial B}{\partial\beta}\right)
\end{aligned}
\qquad 2.8.4.
$$

$$\begin{aligned}\gamma_{12} = {} & \tfrac{1}{2}\left(\sec^2\alpha\frac{\partial A}{\partial\alpha} + \sec^4\alpha\frac{\partial B}{\partial\beta} + 2\sec\alpha\, C'\right) + h\chi\zeta\left\{\sec\alpha\left(\frac{1-2\eta}{1-\eta}\right)\frac{\partial A}{\partial\beta}\right. \\ & + \sec\alpha\tan\alpha\left(\frac{1-4\eta}{1-\eta}\right)B + \sec\alpha\left(\frac{1-2\eta}{1-\eta}\right)\frac{\partial B}{\partial\alpha} + \tan\alpha\frac{\partial C'}{\partial\beta} - \frac{\partial^2 C'}{\partial\alpha\partial\beta} \\ & - \tfrac{1}{4}(3-\zeta^2)\left(\frac{\eta}{1-\eta}\right)\left[\sec^3\alpha\frac{\partial^3 A}{\partial\beta^3} - \sec\alpha\tan\alpha\frac{\partial^2 A}{\partial\alpha\partial\beta} + \sec\alpha\frac{\partial^3 A}{\partial\alpha^2\partial\beta}\right. \\ & + 3\sec^3\alpha\tan\alpha\, B + 3\sec^3\alpha\tan\alpha\frac{\partial^2 B}{\partial\beta^2} + 3\sec\alpha(1+\sec^2\alpha)\frac{\partial B}{\partial\alpha} \\ & \left.\left. + 2\sec\alpha\tan\alpha\frac{\partial^2 B}{\partial\alpha^2} + \sec\alpha\frac{\partial^3 B}{\partial\alpha^3} + \sec^3\alpha\frac{\partial^3 B}{\partial\alpha\partial\beta^2}\right]\right\} \\ & + \tfrac{1}{2}h^2\chi^2\zeta^2\left\{\cos^2\alpha\frac{\partial A}{\partial\alpha} + \frac{\partial B}{\partial\beta} - \cos\alpha\frac{\partial^2 C'}{\partial\beta^2} + \sin\alpha\cos^2\alpha\frac{\partial C'}{\partial\alpha}\right. \\ & - \cos^3\alpha\frac{\partial^2 C'}{\partial\alpha^2} - (3-\zeta^2)\left(\frac{\eta}{1-\eta}\right)\left[\tan\alpha\frac{\partial^2 A}{\partial\beta^2} + \frac{\partial^3 A}{\partial\alpha\partial\beta^2} + 3(2\sec^2\alpha-1)\frac{\partial B}{\partial\beta}\right. \\ & \left.\left. + 4\tan\alpha\frac{\partial^2 B}{\partial\alpha\partial\beta} + \frac{\partial^3 B}{\partial\alpha^2\partial\beta}\right]\right\} - \tfrac{1}{2}h\zeta^3\left(\frac{\eta}{1-\eta}\right)\left\{\frac{\partial\chi}{\partial\alpha}\left[\sec\alpha\frac{\partial^2 A}{\partial\alpha\partial\beta}\right.\right. \\ & \left. + \sec\alpha\frac{\partial^2 B}{\partial\alpha^2} + 3\sec\alpha\tan\alpha\frac{\partial B}{\partial\alpha} + 3\sec^3\alpha\, B\right] + \frac{\partial\chi}{\partial\beta}\left[\sec^3\alpha\frac{\partial^2 A}{\partial\beta^2}\right. \\ & \left.\left. + \sec^2\alpha\frac{\partial^2 B}{\partial\alpha\partial\beta} + 3\sec^3\alpha\tan\alpha\frac{\partial B}{\partial\beta}\right]\right\} - \tfrac{1}{2}h^2\chi\zeta^4\left(\frac{\eta}{1-\eta}\right)\left\{\frac{\partial\chi}{\partial\alpha}\left[\frac{\partial^2 A}{\partial\beta^2}\right.\right. \\ & \left. + \frac{\partial^2 B}{\partial\alpha\partial\beta} + 3\tan\alpha\frac{\partial B}{\partial\beta}\right] + \left.\frac{\partial\chi}{\partial\beta}\left[\frac{\partial^2 A}{\partial\alpha\partial\beta} + \frac{\partial^2 B}{\partial\alpha^2} + 3\tan\alpha\frac{\partial B}{\partial\alpha} + 3\sec^2\alpha\, B\right]\right\}\end{aligned}$$

$$\begin{aligned}\gamma_{22} = {} & 2\sec^4\alpha\tan\alpha\, B + \sec^4\alpha\frac{\partial B}{\partial\alpha} + h\chi\zeta\left\{2\sec\alpha\frac{\partial A}{\partial\alpha} + C + 2\tan\alpha\frac{\partial C}{\partial\alpha} - \frac{\partial^2 C}{\partial\alpha^2}\right. \\ & - \tfrac{1}{2}(3-\zeta^2)\left(\frac{\eta}{1-\eta}\right)\left[\sec^3\alpha\tan\alpha\frac{\partial^2 A}{\partial\beta^2} + \sec^3\alpha\frac{\partial^3 A}{\partial\alpha\partial\beta^2} + 3\sec^2\alpha(2\sec^2\alpha-1)\frac{\partial B}{\partial\beta}\right. \\ & \left.\left. + 4\sec^3\alpha\tan\alpha\frac{\partial^2 B}{\partial\alpha\partial\beta} + \sec^3\alpha\frac{\partial^3 B}{\partial\alpha^2\partial\beta}\right]\right\} - h^2\chi^2\zeta^2\left\{\left(\frac{\eta}{1-\eta}\right)\frac{\partial A}{\partial\beta}\right. \\ & + 3\left(\frac{\eta}{1-\eta}\right)\tan\alpha\, B - \left(\frac{1-2\eta}{1-\eta}\right)\frac{\partial B}{\partial\alpha} - \sin\alpha\frac{\partial C}{\partial\beta} + \cos\alpha\frac{\partial^2 C}{\partial\alpha\partial\beta} \\ & + \tfrac{1}{2}\left(\frac{\eta}{1-\eta}\right)(3-\zeta^2)\left[\frac{\partial^3 A}{\partial\alpha^2\partial\beta} + 6\sec^2\alpha\tan\alpha\, B + 6\sec^2\alpha\frac{\partial B}{\partial\alpha}\right. \\ & \left.\left. + 3\tan\alpha\frac{\partial^2 B}{\partial\alpha^2} + \frac{\partial^3 B}{\partial\alpha^3}\right]\right\} - h\zeta^3\frac{\partial\chi}{\partial\alpha}\left(\frac{\eta}{1-\eta}\right)\left[\sec^3\alpha\frac{\partial^2 A}{\partial\beta^2} + \sec^3\alpha\frac{\partial^2 B}{\partial\alpha\partial\beta}\right. \\ & \left. + 3\sec^3\alpha\tan\alpha\frac{\partial B}{\partial\beta}\right] - h^2\chi\frac{\partial\chi}{\partial\alpha}\zeta^4\left(\frac{\eta}{1-\eta}\right)\left[\frac{\partial^2 A}{\partial\alpha\partial\beta} + \frac{\partial^2 B}{\partial\alpha^2}\right. \\ & \left. + 3\tan\alpha\frac{\partial B}{\partial\alpha} + 3\sec^2\alpha\, B\right]\end{aligned} \qquad 2.8.4.$$

$$\begin{aligned}\gamma_{13} = {} & -\tfrac{1}{2}\left(\frac{\eta}{1-\eta}\right)(1-\zeta^2)\left\{\tfrac{3}{2}\sec\alpha\frac{\partial^2 A}{\partial\alpha\partial\beta} + \tfrac{9}{2}\sec^3\alpha\, B + \tfrac{9}{2}\sec\alpha\tan\alpha\frac{\partial B}{\partial\alpha} + \tfrac{3}{2}\sec\alpha\frac{\partial^2 B}{\partial\alpha^2}\right. \\ & \left. + h\chi\zeta\left[\frac{\partial^2 A}{\partial\beta^2} + 3\tan\alpha\frac{\partial B}{\partial\beta} + \frac{\partial^2 B}{\partial\alpha\partial\beta}\right]\right\}\end{aligned}$$

$$\begin{aligned}\gamma_{23} = {} & -\tfrac{1}{2}\left(\frac{\eta}{1-\eta}\right)(1-\zeta^2)\left\{\tfrac{3}{2}\sec^3\alpha\frac{\partial^2 A}{\partial\beta^2} + \tfrac{9}{2}\sec^3\alpha\tan\alpha\frac{\partial B}{\partial\beta} + \tfrac{3}{2}\sec^3\alpha\frac{\partial^2 B}{\partial\alpha\partial\beta}\right. \\ & \left. + h\chi\zeta\left[\frac{\partial^2 A}{\partial\alpha\partial\beta} + 3\sec^2\alpha\, B + 3\tan\alpha\frac{\partial B}{\partial\alpha} + \frac{\partial^2 B}{\partial\alpha^2}\right]\right\}\end{aligned}$$

$$\gamma_{33} = -\left(\frac{\eta}{1-\eta}\right)\left[\frac{\partial A}{\partial\beta} + 3\tan\alpha\, B + \frac{\partial B}{\partial\alpha}\right]$$

The stress strain relations for this theory are those in equation 2.7.7.

The equations for the thin shell theory for the twisted plate, equivalent to the Flügge Theory equation for the cylindrical shell, are also given here.

The displacement functions are

$$
\begin{aligned}
u^1 &= A + \theta_3\left(\cos\alpha\, B - \cos^2\alpha \frac{\partial C}{\partial \beta}\right)\\
u^2 &= B + \theta_3\left(\cos^3\alpha\, A - \cos^4\alpha \frac{\partial C}{\partial \alpha}\right)\\
u^3 &= C
\end{aligned}
\qquad 2.8.5.
$$

the displacement strain relations are

$$
\begin{aligned}
\gamma_{11} = {} & \sec^2\alpha \frac{\partial A}{\partial \beta} + \sec^2\alpha \tan\alpha\, B + \theta_3\Big(2\sec\alpha \frac{\partial B}{\partial \beta} + \cos^2\alpha\, C\\
& - \frac{\partial^2 C}{\partial \beta^2} - \sin\alpha\cos\alpha \frac{\partial C}{\partial \alpha}\Big) + \theta_3^2\Big(\cos^2\alpha \frac{\partial A}{\partial \beta} - \sin\alpha\cos\alpha\, B\\
& + \sin\alpha\cos^2\alpha \frac{\partial C}{\partial \beta} - \cos^3\alpha \frac{\partial^2 C}{\partial\alpha\partial\beta}\Big)
\end{aligned}
\qquad 2.8.6.
$$

$$
\begin{aligned}
\gamma_{12} = \frac{1}{2}\Big\{ & \sec^2\alpha \frac{\partial A}{\partial \beta} + \sec^4\alpha \frac{\partial B}{\partial \beta} + 2\sec^2\alpha\, C + \theta_3\Big(\sec\alpha \frac{\partial A}{\partial \beta} + \sec\alpha\tan\alpha\, B\\
& + \sec\alpha \frac{\partial B}{\partial \alpha} + \tan\alpha \frac{\partial C}{\partial \beta} - \frac{\partial^2 C}{\partial\alpha\partial\beta}\Big) + \frac{1}{2}\theta_3^2\Big(\cos^2\alpha \frac{\partial A}{\partial \alpha} + \frac{\partial B}{\partial \beta}\\
& - \cos\alpha \frac{\partial^2 C}{\partial \beta^2} + \sin\alpha\cos^2\alpha \frac{\partial C}{\partial \alpha} - \cos^3\alpha \frac{\partial^2 C}{\partial \alpha^2}\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{22} = {} & 2\sec^4\alpha\tan\alpha\, B + \sec^4\alpha \frac{\partial B}{\partial \alpha} + \theta_3\Big(2\sec\alpha \frac{\partial A}{\partial \alpha} + C + 2\tan\alpha \frac{\partial C}{\partial \alpha}\\
& - \frac{\partial^2 C}{\partial \alpha^2}\Big) + \theta_3^2\Big(\frac{\partial B}{\partial \alpha} + \sin\alpha \frac{\partial C}{\partial \beta} - \cos\alpha \frac{\partial^2 C}{\partial\alpha\partial\beta}\Big)
\end{aligned}
$$

$$\gamma_{13} = \gamma_{23} = \gamma_{33} = 0$$

and finally the stress-strain relations are

$$\begin{aligned}
\tau^{11} = \frac{2\mu}{1-\eta}\Big\{ & \cos^4\alpha\,(1+6\cos^4\alpha\,\theta_3^2)\,\gamma_{11} - 4\cos^7\alpha\,\gamma_{12} \\
& + \left[\eta\cos^6\alpha\,(1+6\cos^4\alpha\,\theta_3^2) + 4\cos^{10}\alpha\,\theta_3^2\right]\gamma_{22}\Big\} \\
\tau^{12} = \frac{2\mu}{1-\eta}\Big\{ & -2\theta_3\cos^7\alpha\,\gamma_{11} + \left[\cos^6\alpha\,(1+10\cos^4\alpha\,\theta_3^2) \right. \\
& \left. + 8\eta\cos^{10}\alpha\,\theta_3^2\right]\gamma_{12} - 2\theta_3\cos^9\alpha\,\gamma_{22}\Big\} \qquad 2.8.7. \\
\tau^{22} = \frac{2\mu}{1-\eta}\Big\{ & \left[\eta\cos^6\alpha\,(1+6\cos^4\alpha\,\theta_3^2) + 4\cos^{10}\alpha\,\theta_3^2\right]\gamma_{11} \\
& - 4\cos^9\alpha\,\theta_3\gamma_{12} + \cos^8\alpha\,(1+6\cos^4\alpha\,\theta_3^2)\,\gamma_{22}\Big\} \\
\tau^{13} = \tau^{23} = & \ \tau^{33} = 0
\end{aligned}$$

CHAPTER 3 NUMERICAL ANALYSIS AND COMPUTING:

3.1. Introduction

The numerical techniques and computing methods used in this investigation for the solution of shell free vibration problems are presented in this chapter. The computer programs have been applied to the twisted plate and cylindrical shell, to both thin and thick theories for comparison, in every case to the problem of the shell clamped along one edge. The results obtained are presented in Chapter 4. The programs have been made as general as possible, so that they can be used with ease for any shell geometry and boundary condition.

The techniques used have been restricted due to the limitations of computing facilities available. The computer used was a KDF9, which is a 32K, 48 bit word machine with extensive disc and magnetic tape facilities. The machine was in heavy use, so the time available for running programs was rather restricted. Therefore the programs had to be made as efficient as possible. They were written in Egdon Algol (a specially modified version of Algol for use on the KDF9) and, where possible, standard library routines were used. The efficiency could have been improved by the use of a lower level language, but the use of Algol makes the program machine independent. The core store available for use was 26K. By careful chaining of the program, 22K was left for data space, this being the limit on the size of the system of equations that could be considered.

The numerical techniques applied to the problem are approximate solution techniques based on the Rayleigh-Ritz method. For this, strain and kinetic energy matrices are set up in terms of coefficients in the expansions of the middle surface displacements. Boundary conditions are then applied to reduce the size of these equations, so that all the displacement boundary conditions are satisfied. The resulting eigenvalue problem is then solved to give the frequencies and mode shapes for the shell. Other techniques used to reduce the size of the eigenvalue problem are symmetry, and the use of the stress free edge conditions.

The numerical techniques used are outlined in section 3.2, and the computer program in section 3.3. The particular sections of the program are then discussed in more detail in sections 3.4 to 3.6, and a general discussion of the program is presented in section 3.7.

3.2. Numerical Analysis

Exact solutions of free vibration problems for general shaped shells, with arbitrary boundary conditions, are not possible. For this problem a wide variety of approximate solutions can be used, most of these being relevant only to particular geometries or boundary conditions. The best general method, in the sense that it can be applied to any shell geometry and boundary conditions, is one using an energy principle. Essentially this is very similar to the Rayleigh-Ritz method, in that an approximation to the energy of the structure is minimised. In the Rayleigh-Ritz method the displacements of the shell are defined as a series of functions, each of which satisfy the displacement boundary conditions. The main drawback of the Rayleigh-Ritz method is that the displacement functions must be chosen to satisfy the displacement boundary conditions.

The solution technique used here is to find stationary values of the Hamiltonian:

$$H = \int_{t_0}^{t_1} (V - T)\, dt. \qquad 3.2.1.$$

subject to the constraints imposed by the displacement boundary conditions. The functions used in the approximations to the displacements can now be general, and are chosen as double power series in the middle surface coordinates α and β . The basis of this variational method is given in Appendix 1.

For a general shell problem, the potential energy V and the kinetic energy T are defined as

$$V = \frac{1}{2}\int_V \gamma_{ij} \tau^{ij} dV - \int_V F_i u^i - \int_S T_i u^i dS \qquad 3.2.2.$$

$$T = \frac{1}{2}\int_V \rho \dot{u}_i \dot{u}^i dV. \qquad 3.2.2.$$

where $\dot{u}_i$ and $\dot{u}^i$ are the covariant and contravariant displacements, the dot representing the time derivatives. γ_{ij} is the covariant strain tensor, τ^{ij} the contravariant stress tensor, F_i the body forces, and $\dot{T}_i$ the surface tractions. The first integral in the potential energy equation is known as the strain energy. For a free vibration problem the body force and surface traction integrals are both zero. For this reason only the displacement boundary conditions have to be satisfied; the stress free conditions are satisfied by the solution, which would otherwise give non-zero surface tractions T_i .

The strain and stress tensors are known linear functions of the displacements u_i and their derivatives.

$$\tau^{ij} = D\gamma_{ij} \qquad 3.2.3.$$

$$\gamma_{ij} = \gamma_{ij}(u_i)$$

where D is the stress strain matrix.

The displacement functions u_i are in turn defined in terms of the middle surface displacements A, B, C and their derivatives (see Chapter 2).

$$u_i = u_i(A, B, C) \qquad 3.2.4.$$

$$u^i = a^{ij}u_j$$

The middle surface displacements are defined as double power series in the scaled middle surface coordinates of the shell.

$$A = \sum_{I=0}^{M-1}\sum_{J=0}^{N-1} A_{IJ}\left(\frac{\alpha}{AL}\right)^I\left(\frac{\beta}{BL}\right)^J e^{-i\omega t}$$

$$B = \sum_{I=0}^{M-1}\sum_{J=0}^{N-1} B_{IJ}\left(\frac{\alpha}{AL}\right)^I\left(\frac{\beta}{BL}\right)^J e^{-i\omega t}$$

$$C = \sum_{I=0}^{M-1}\sum_{J=0}^{N} C_{IJ}\left(\frac{\alpha}{AL}\right)^I\left(\frac{\beta}{BL}\right)^J e^{-i\omega t} \qquad 3.2.5.$$

Substituting equations 3.2.5, 3.2.4., 3.2.3 and 3.2.2. into 3.2.1. gives a quadratic in the coefficients A_{IJ}, B_{IJ}, C_{IJ} . To find the stationary values of H, it is differentiated by each of the coefficients to give a system of linear equations in the form of an eigenvalue problem

$$(K - \omega^2 M)\underline{x} = 0 \qquad 3.2.6.$$

where $\underline{x}$ is a vector of the coefficients A_{IJ}, B_{IJ}, C_{IJ}.
These equations are subject to the constraints implied by the displacement boundary conditions. The standard technique for this is to introduce Lagrange multipliers for each of the constraint equations. This, however, increases the number of equations to be solved, whereas the need is to decrease it as much as possible. The alternative method of achieving this is outlined below.

The constraint equations are set up in the form

$$G\underline{x} = 0 \qquad 3.2.7.$$

where, if there are R constraints, G is a R x 3MN matrix. These equations are used to reduce out R of the coefficients of $\underline{x}$.

Partitioning 3.2.7. gives

$$G_1\underline{q} + G_2\underline{p} = 0 \qquad 3.2.8.$$

where $\underline{p}$ is a vector of the R coefficients to be reduced out and $\underline{q}$ the remaining 3MN - R coefficients.

Therefore,

$$\underline{p} = -G_2^{-1} G_1 \underline{q} \qquad 3.2.9.$$

From this a transformation matrix Q can be formed such that

$$\underline{x} = Q\underline{q} \qquad 3.2.10.$$

The strain and kinetic energy matrices are modified to become

$$\begin{aligned} \bar{K} &= Q^T K Q \\ \bar{M} &= Q^T M Q \end{aligned} \qquad 3.2.11.$$

leaving the eigenvalue problem

$$(\bar{K} - \omega^2 \bar{M})\underline{q} = 0 \qquad 3.2.12.$$

This technique has been successfully applied, by Webster (29), to the problem of a thin cylindrical panel clamped on four sides.

For the problems of shells clamped along one edge $\beta = 0$, considered later, the displacement boundary conditions lead to equations of the form

$$\sum_{I=0}^{M-1} A_{I0} \left(\frac{\alpha}{AL}\right)^I = 0 \qquad 3.2.13.$$

$$\Rightarrow A_{I0} = 0$$

Thus, this boundary condition leads to certain coefficients being zero. For this case, it is not necessary to apply the boundary condition reduction above, as the conditions can simply be satisfied by leaving the appropriate terms out of the displacement functions. In this particular case, the displacement functions used then satisfy all the displacement boundary conditions: and therefore the Rayleigh-Ritz technique is being applied. However, it is emphasised that this is true only for certain boundary conditions: in general, the Hamiltonian approach with reduction is required.

The eigenvalue problem is solved employing standard library routines, which can only be used to solve problems with up to 75 unknowns. For a general shell problem this restricts the (M, N) values used in the displacement functions to (5, 6) or (6,5), which are insufficient to ensure convergence of the frequencies.

Two techniques have been utilised to give larger (M, N) values, and thus better convergence: symmetry and stress-free boundary conditions.

Symmetry along one axis of a shell doubles the number of terms in the displacement functions, by solving two 75 x 75 problems. This increases the (M, N) values to (7, 8), (8, 7), which are high enough to give good convergence (Section 4.2). However, the use of symmetry restricts the shells and boundary conditions for which solutions can be found.

For the Hamiltonian solution it is only necessary that the displacement boundary conditions are satisfied, stress-free conditions are satisfied automatically. By utilising these stress-free conditions the number of terms in the displacement functions can be increased, whilst the size of the eigenvalue problem remains the same. For example, consider the problem of a shell clamped along one edge. Putting the resultant forces and moments zero on the other three edges gives possible (M, N) values of (8, 9), (9, 8). Application of the force-conditions alone gives values (6, 8), (7, 7), (8, 6) similar to those which ensure adequate convergence in the symmetry problem. It should be noted that as the stress-free conditions are not necessary for application of the solution technique, only sufficient conditions for convergence need to be applied.

Standard library routines are used to solve the eigenvalue problem

$$(Z - \lambda I)\underline{y} = 0 \qquad 3.2.14.$$

to which equation 3.2.12 can be transformed.

Two solution techniques were used. The first should be sufficient to solve the free vibration problem. However, due to the build up of errors the second approach was also applied.

The first technique is to solve equation 3.2.16 by Householder's method, Wilkinson (30). In this Z is first transformed to tri-diagonal form by a sequence of transformations.

$$Z_r = P_r Z_{r-1} P_r \quad , \qquad Z_0 = Z \tag{3.2.15.}$$

$$P_r = I - 2\omega_r \omega_r' , \qquad \omega_r' \omega_r = 1$$

The elements of ω_r are chosen so that Z_r will have zeros in all except the tri-diagonal positions of a particular column. Each transformation leaves unchanged the elements of the previous columns, so that for a system of equations of order J, J-2 transformations are required to reduce Z to its tri-diagonal form Z_T . P_r has the property that it is orthogonal, therefore the eigenvalues of Z_T are the same as those of Z.

The eigenvalues of Z_T are then found by the method of Sturm sequences. Given a tri-diagonal eigenvalue problem

$$\begin{bmatrix} z_1-\lambda & y_2 & & & 0 \\ y_2 & z_2-\lambda & y_3 & & \\ & - & - & - & - \\ & & & & \\ 0 & & y_{n-1} & z_{n-1}-\lambda & y_n \\ & & & y_n & z_n-\lambda \end{bmatrix} \tag{3.2.16.}$$

let $f_r(\lambda)$ be the determinant of the matrix formed from the first r rows and columns. It can then be proved that

$$f_{r+1}(\lambda) = (z_{r+1}-\lambda) f_r(\lambda) - y_{r+1}^2 f_{r-1}(\lambda) \qquad 3.2.17.$$

For a particular value of λ, say $\bar{\lambda}$, the sequence $f_r(\bar{\lambda})$ is a Sturm sequence, that is, the number of changes of sign in the sequence is equal to the number of roots of $f_r(\lambda)$ less than $\bar{\lambda}$ in algebraic value. This result can be used to compute all of the roots, or any particular one required, simply by bisecting the real axis and applying Newton's method to obtain the root to the accuracy required.

The eigenvectors are then evaluated by the method of inverse iteration. It can be shown that for the root nearest to S the eigenvector of the eigenvalue problem is given by the iteration.

$$\underline{y}^{(r)} = (Z - I_S)^{-1} \underline{y}^{(r-1)} = ----- = (Z - I_S)^{-r} \underline{y}^{(0)} \qquad 3.2.18.$$

This iteration is performed by successively solving the system of linear equations with the same matrix on the left, for varying right hand sides. This is performed by Gaussian elimination, the first step forms the upper triangular matrix, which is then used in all successive iterations.

When convergence is obtained the ratio of the components of $y^{(r)}/y^{(r+1)} = \varepsilon$ gives a correction to the eigenvalue S, so that the better approximation to the eigenvalue is given by

$$\lambda_r = S + \varepsilon \qquad 3.2.19.$$

This technique depends on Z being symmetric. This can be achieved by the transformation

$$Z = L^{-1} \bar{K} L^{-T} \qquad y = L^{T} q \tag{3.2.20.}$$

where $\bar{M} = LL^{T}$

and $\lambda = \omega^2$

This transformation can only be applied if $\bar{M}$ is symmetric and positive definite. From the definition of $\bar{M}$ this must be so, but as the number of terms in the displacement function are increased, error terms make the evaluation of L impossible. To evaluate the effects of these errors the second technique was used. For this the following transformation is applied.

$$Z = M^{-1} K \qquad y = q \qquad \lambda = \omega^2 \tag{3.2.21.}$$

This Z is not symmetric, so Householder's method cannot be applied. Therefore the technique referred to as the H.Q.R. method is applied Wilkinson (30), Frances (31). Z is first reduced to Hessenberg form (an upper triangle matrix, with the first lower diagonal non-zero also). This is achieved by a sequence of transformations

$$Z_r = M_r^{-1} Z_{r-1} M_r \; , \quad Z_0 = Z \tag{3.2.22.}$$

The M_r's are chosen so that they reduce to zero all the terms in one column that need to be zero in the final Hessenberg form. In addition, the form of the M_r's is such that they do not affect previously zeroed columns. The order in which the columns are reduced is dependent on the largest term in the lower triangle, thus reducing the effects of error. By this method the matrix is reduced to Hessenberg form by J-2 transformations.

The Q-R Algorithm is then applied to the resulting Z_H so that

$$Z_{H1} = Q_1 R_1 , \quad Z_{H2} = R_1 Q_1 = Q_2 R_2$$
$$Z_{H3} = R_2 Q_2 = Q_3 R_3 \qquad 3.2.23.$$

such that

$$Z_{H2} = R_1 Z_{H1} R_1^{-1} , \quad Z_{H3} = R_2 Z_{H2} R_2^{-1} = (R_2 R_1) Z_{H1} (R_2 R_1)^{-1} \qquad 3.2.24.$$

where Qr is an orthogonal matrix, and R_r is an upper triangular matrix. This sequence converges to a matrix Z_Q. In the case of all real eigenvalues Z_Q is an upper triangular matrix, the diagonal elements of which are the eigenvalues of Z. For complex roots Z_Q has elements just below the diagonal. The solution of the 2 x 2 determinant of this term with the adjacent diagonal and upper triangular terms give a complex pair of roots of Z.

The eigenvectors are then formed by inverse iteration as in Householder's method.

The HQR method, in general gives complex eigenvalues and eigenvectors, and is much slower than Householder's method. The use of the two techniques is discussed in section 3.7.

3.3. Computer Program. General Outline

The program has been made as general as possible, so that only certain blocks have to be changed dependent on the shell geometry, the boundary conditions applied, and the type of solution required. The flow diagram for the program is given in fig 3.

The program has been chained so that the maximum amount of core store is available for data storage. All the blocks in fig 3 are separate chains, except BCVAR which is performed manually, the program being split in two at this point. The first chain is very straightforward, involving only the setting up of control variables required later in the program. The other chains are discussed in more detail in the following sections. The computer program for the constant thickness thick twisted plate is given in Appendix 2. This program does not involve any boundary condition reduction. The appropriate parts of the program for boundary condition reduction applied to the thin cylindrical shell are given in Appendix 4. In Appendix 3 the data required to define the energy expressions for the constant thickness thick twisted plate is presented.

The approximate times for the various parts of the program are shown in table 1; first for the solution using symmetry and Householder's eigenvalue solution; second for a problem without symmetry, using the stress free boundary condition reduction and the HQR eigenvalue solution. The times given are for the maximum size problem by each method, so as to give the greatest accuracy possible.

Fig 3. Program Flow Diagram

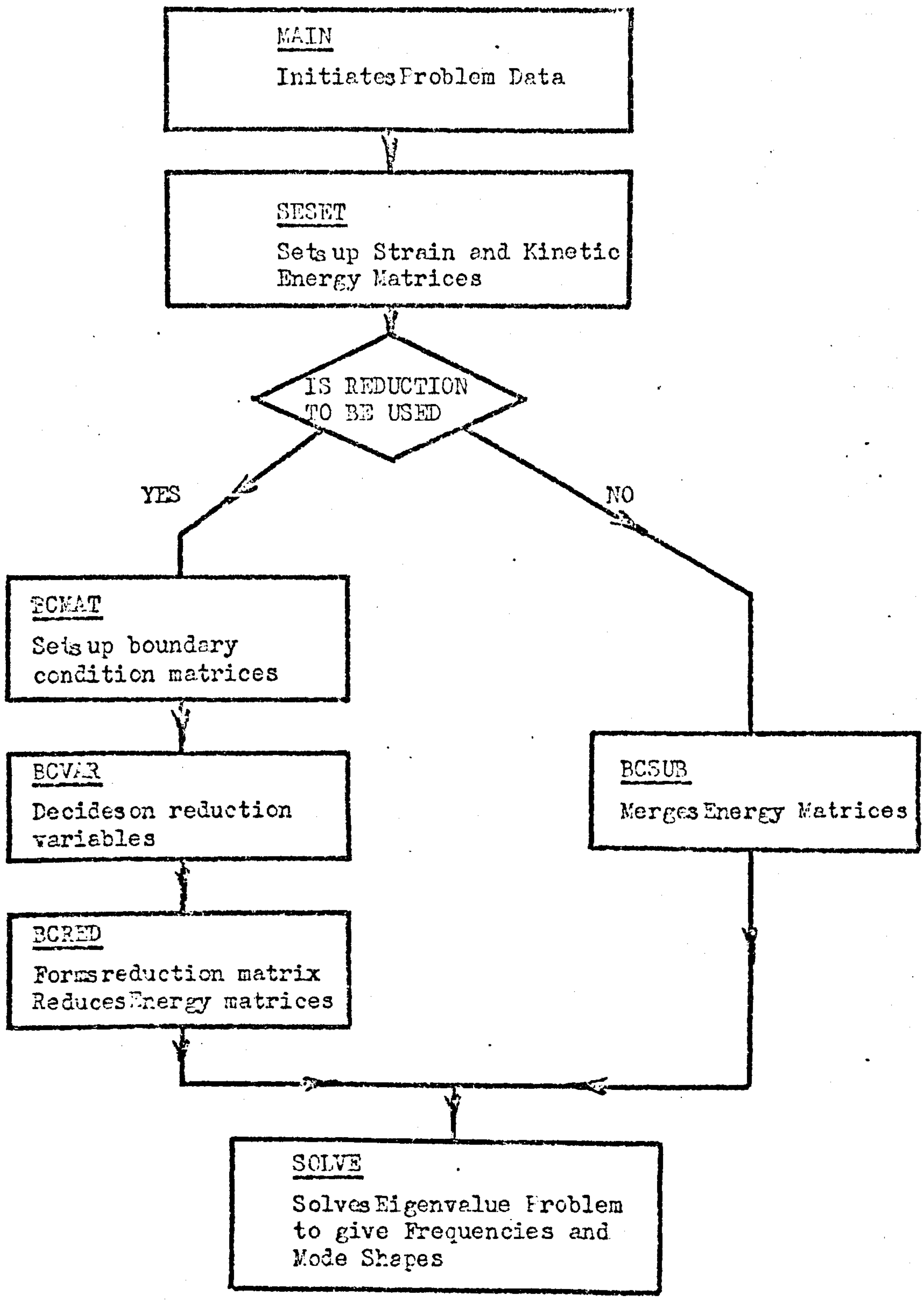


Table 1 Computer Times

	Computer Core Time (sec)			
	Thin Theory Twisted Plate	Thick Theory Twisted Plate	Thin Theory Cylinder	Thick Theory Cylinder
(a) Symmetry & Householder				
MAIN	10	10	10	10
SESET	330	1030	100	250
BCSUB	10	10	10	10
SOLVE	350	350	350	350
TOTAL	700	1400	470	620
(b) Boundary Condition Reduction and HQR				
MAIN	10	10	10	10
SESET	650	2050	190	490
BCMAT	40	40	40	40
BCRED	180	180	180	180
SOLVE	300	300	300	300
TOTAL	1180	2580	720	1020

3.4. Strain and Kinetic Energy Matrices

Applying equations 3.2.1 to 3.2.7 gives rise to a set of linear equations in the coefficients of the displacement function. The matrices, K and M, of these terms are referred to as the strain energy and kinetic energy matrices (or in the engineering literature as the "stiffness" and "mass" matrices respectively).

The energy matrices are set up separately, both being symmetric matrices of the form below -

$$\left[\begin{array}{c|c|c} AA & AB & AC \\ \hline BA & BB & BC \\ \hline CA & CB & CC \end{array}\right] \qquad 3.4.1.$$

where AB refers to coefficients arising from $A_{IJ}\ B_{KL}$ terms, similarly for the other partitions. To keep the core requirements to a minimum the matrices are partitioned as above, each partition being worked out separately. Only six matrices have to be evaluated, because of symmetry.

The flow diagram for this part of the program, referred to as SESET is given in fig 4.

For this section of the program, data is required to define the energy expressions. For the thick shell theories these can be very complicated, in fact for the variable thickness twisted plate theory there are over 600 terms, each being of the form

$$\iint_S CE(IC)\,X\sin^{JS}\alpha\cos^{JC}\alpha\ \frac{\partial^{ID+JD}F}{\partial\alpha^{ID}\partial\beta^{JD}}\ \frac{\partial^{KD+HD}G}{\partial\alpha^{KD}\partial\beta^{HD}}\ \bar{X}\left(X,\frac{\partial X}{\partial\alpha},\frac{\partial X}{\partial\beta}\right)d\alpha\, d\beta \qquad 3.4.2.$$

where F and G can be either A, B or C; CE(IC) is a constant, dependent on shell geometry, and material properties; and $\overline{X}$ is a quadratic function of X and its first derivatives (X defines the thickness variation of the shell. See section 2.4). In this expression the integration over the thickness has already been performed. Thus, for this shell, every term in the energy expressions can be defined in terms of a set of numbers

IS IA ID JD HD KD JS JC IC JX 3.4.4.

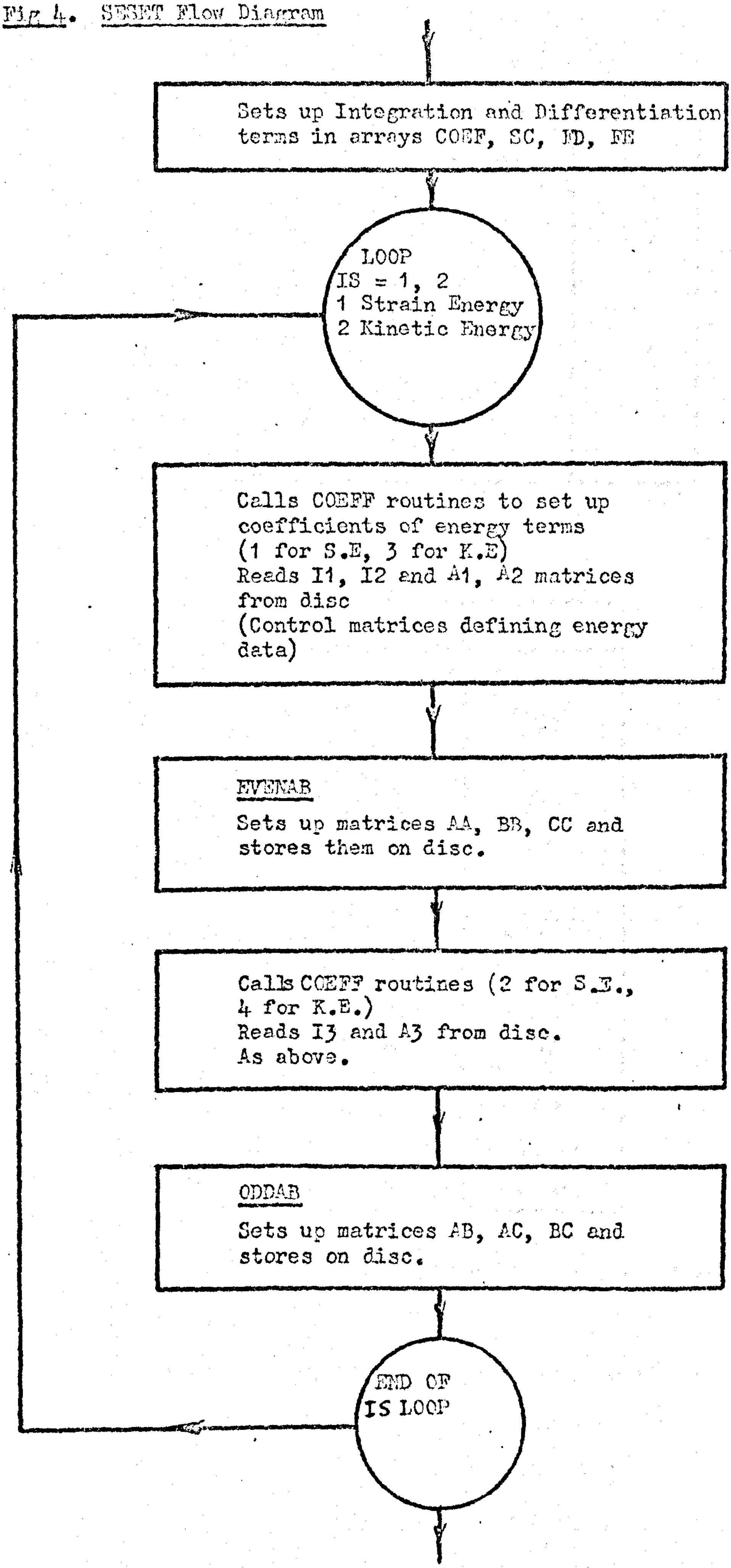
and a set of expressions defining CE(IC)

IS defines whether the term is strain or kinetic energy,

$$IS = \begin{cases} 1 \text{ Strain Energy} \\ 2 \text{ Kinetic Energy} \end{cases} \qquad 3.4.5.$$

IA defines F and G in terms of A and B.

Fig 4. SESET Flow Diagram



IA	F	G
1	A	A
2	B	B
3	C	C
4	A	B
5	A	C
6	B	C

3.4.6.

and JX defines the quadratic $\overline{X}$.

JX	$\overline{X}$
1	1
2	X^2
3	$\left(\frac{\partial X}{\partial \alpha}\right)^2$
4	$\left(\frac{\partial X}{\partial \beta}\right)^2$
5	$X \frac{\partial X}{\partial \alpha}$
6	$X \frac{\partial X}{\partial \beta}$
7	$\frac{\partial X}{\partial \alpha} \cdot \frac{\partial X}{\partial \beta}$

3.4.7.

For a constant thickness twisted plate JX is not required. For a cylinder, JS and JC are not required.

The number of terms in the energy expressions make it inefficient to store all these values at the same time. Thus, the terms are divided into sub-sets for IS and IA, only certain sub-sets being used at any one time. Also for IA = 1, 2 and 3, the numbers are divided into further sub-sets so those where ID = KD and JD = HD are separate from the remaining terms.

The symmetry of these terms means that the evaluation and storage of their resultant energy terms can be performed twice as efficiently as for the general terms.

Arrays of the set of numbers 3.4.4, excepting IS and IA, are set up on disc for each value of IS, for three distinct cases.

(a) IA = 1, 2, 3   ID = JD and JD = KD

(b) IA = 1, 2, 3   Terms not included in (a)

(c) IA = 4, 5, 6

Reference is made to these arrays in the program as A1, A2, A3. For each case, arrays I1, I2, I3 are formed, which define the position of the terms in A1, A2 and A3 respectively, for the different values of IA. This division of the arrays makes the program more efficient as different operations are performed for each of the cases (a), (b), (c) above, and in addition saves storage space as the arrays A1, A2, A3 and CE need only be declared so as to contain the largest sub-set of values. The coefficients CE are set up using four routines, which define CE for the separate cases.

(i) COEFF 1   AA, BB, CC   Strain Energy terms

(ii) COEFF 2   AA, BB, CC   Kinetic Energy terms

(iii) COEFF 3   AB, AC, BC   Strain Energy terms

(iv) COEFF 4   AB, AC, BC   Kinetic Energy terms

The remainder of this section will be devoted to discussion of the constant thickness twisted plate, with comments on the changes to be made for other shells. When referring in general to the arrays and routines defined above, AN will be used to refer to A1, A2 and A3, IN for I1, I2 and I3 and COEFFN for COEFF 1, 2, 3 and 4.

All the information is now readily available for the evaluation of the general energy expression 3.4.2. After taking out all constant terms and substituting the power series for the displacements the only terms left to be evaluated are of the form

$$\int_0^{BL}\int_0^{AL}\left(\frac{\alpha}{AL}\right)^{R}\left(\frac{\beta}{BL}\right)^{S}\sin^{JS}\alpha\,\cos^{JC}\alpha\;d\alpha\,d\beta \qquad 3.4.8.$$

where AL and BL define the boundary values of α and β.

For constant AL and BL, the α and β terms in this integral are independent, and so the integrals with respect to each can be evaluated separately. The integral with respect to β can be carried out analytically, but the α integral has to be evaluated numerically. This is done by Simpson's rule in the procedure INTEG. In the program, both integrations are carried out for all terms to be encountered in the energy expressions and the results stored in arrays. This means that each integration has only to be carried out once. Two arrays are used to store this data.

$$SC(I) = \int_0^{BL} \left(\frac{\beta}{BL}\right)^I d\beta \qquad 3.4.9.$$

$$COEF(I,J,K) = \int_0^{AL} \sin^I\alpha \cos^J\alpha \left(\frac{\alpha}{AL}\right)^K d\alpha$$

Tests were made on the accuracy of the numerical integration, and finally an accuracy to the sixth figure was used. This had no noticeable effect on the resulting frequencies.

For the constant thickness cylinder, the α integrals can also be carried out analytically. When variable thickness is included the arrays must be expanded to include all possibilities. In this case the α and β integrals may not be independent.

Also set up initially are arrays FD and FE which give the coefficients obtained as the result of differentiation with reference to α and β

$$FD(IJ) = \text{Coefficient}\left[\frac{d^J\left(\frac{\alpha}{AL}\right)^I}{d\alpha^J}\right] = I(I-1)\cdots\left(I-(J-1)\right)/AL^J \qquad 3.4.10.$$

$$FE(IJ) = \text{Coefficient}\left[\frac{d^J\left(\frac{\beta}{BL}\right)^I}{d\alpha^J}\right] = I(I-1)\cdots\left(I-(J-1)\right)/BL^J$$

From this initial information the contribution from every energy term can be evaluated as a simple product for each term in the power series expansion of the middle surface displacements.

For example refer back to equation 3.4.2, and consider the constant thickness case, so that the X and $\overline{X}$ terms are not included. Evaluation of the contribution to the energy expression of the term arising from the general cross term

$$\left[\left(\frac{\alpha}{AL}\right)^{I}\left(\frac{\beta}{BL}\right)^{J}\right]\left[\left(\frac{\alpha}{AL}\right)^{K}\left(\frac{\beta}{BL}\right)^{H}\right] \qquad 3.4.11.$$

can be written down for each term in the energy expression as

$$\begin{array}{l}CE(IC)*\ COEF(JS,\ JC,\ I + K)*\ SC(J + H)* \\ FD(I,\ ID)*\ FD(K,\ KD)*\ FE(J,\ JD)*\ FE(H,\ HD)\end{array} \qquad 3.4.12.$$

and its position in the appropriate partition of the energy array is defined by I, J, K, H, ID, JD, KD and HD.

The program SESET organises the setting up of the energy matrices. Having formed the initial matrices SC, COEF, FD and FE, it then sets up each of the partitions of the energy matrices. The IS loop controls the strain energy and kinetic energy calculations. For each in turn it calls the appropriate COEFFN routines, reads AN and IN from disc and evaluates each of the six partitions. The evaluations for AA, BB and CC are carried out in EVENAB, and those for AB, AC and BC in ODDAB. The flow diagram for EVENAB is given in fig 5; ODDAB is very similar. The routines given here are for shells symmetric about $\alpha = 0$, the array AS and AT storing the symmetric and asymmetric parts of each partition. Without symmetry the routines become much simpler.

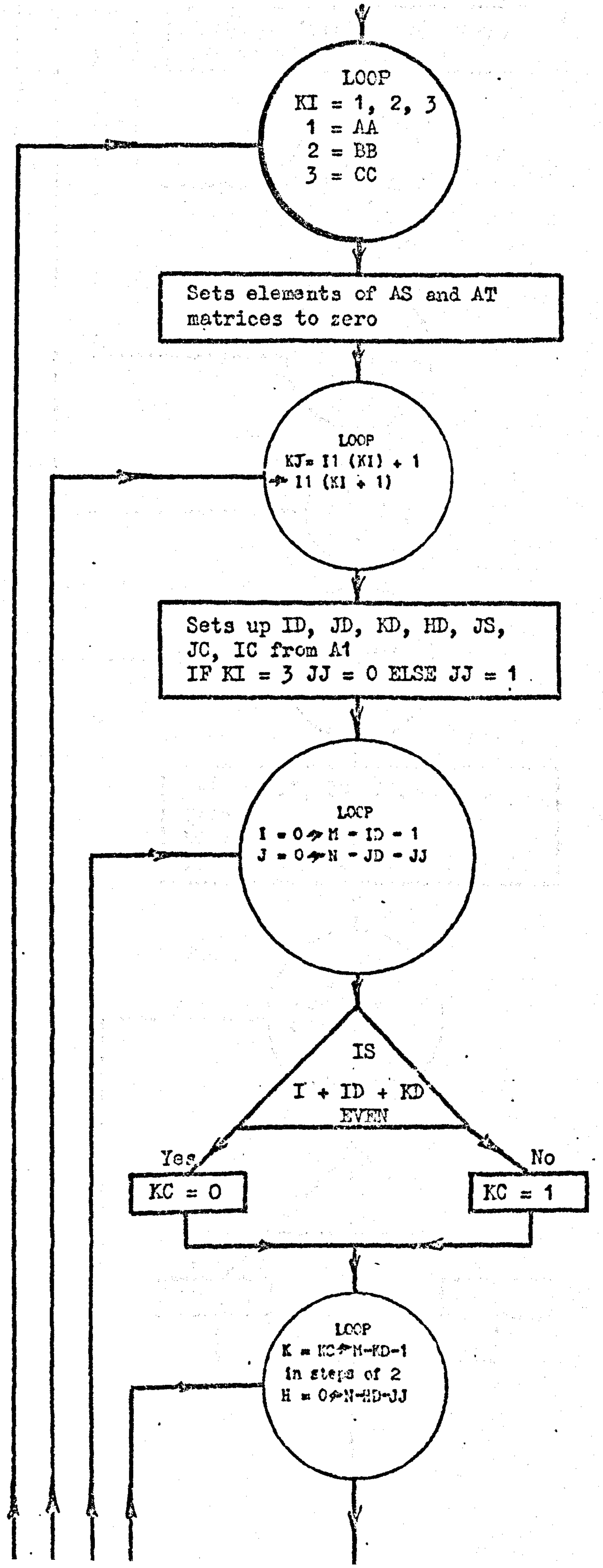
The routines EVENAB and ODDAB set up loops, one for the control information stored in AN for the particular partition under consideration, the others for four integers I, J, K, H, which define a cross term in the energy expression of the form 3.4.11. In these latter loops the coefficient of the term is evaluated and stored in the appropriate position in either the AS or AT matrix.

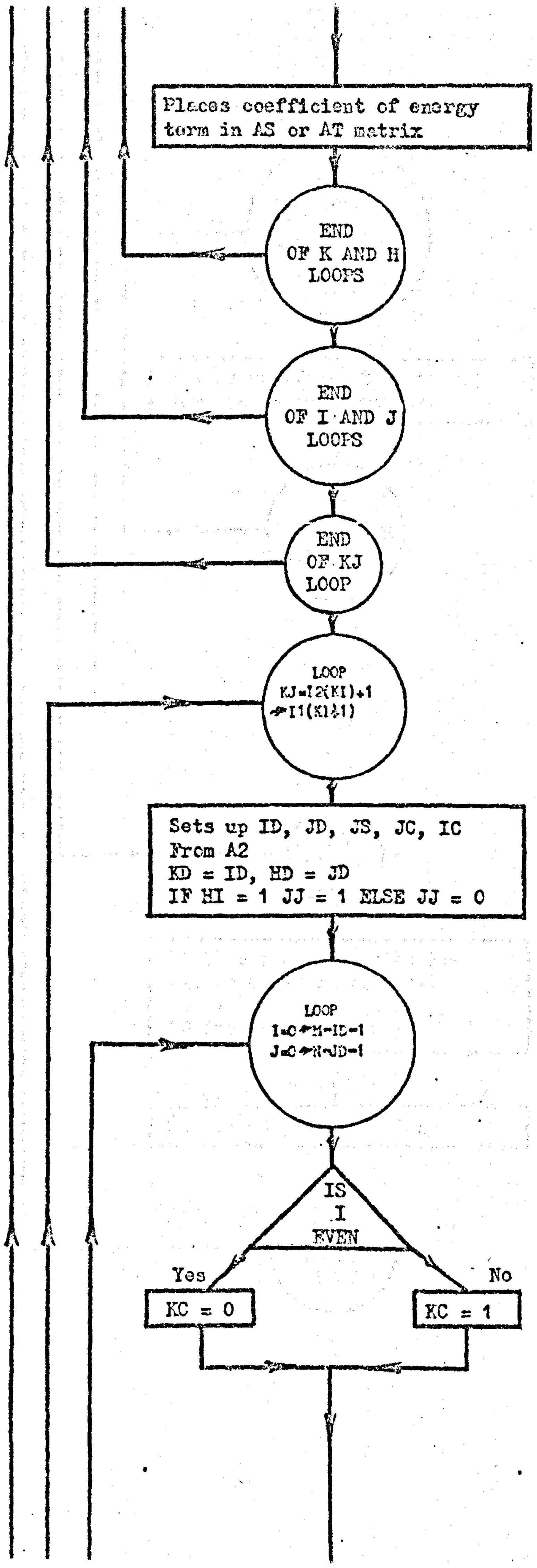
The resulting AS and AT matrices are stored in predetermined position on disc. In EVENAB, scale factors are also stored, equal to the diagonal terms in the strain energy. These are used later, in the boundary condition reduction (section 3.5).

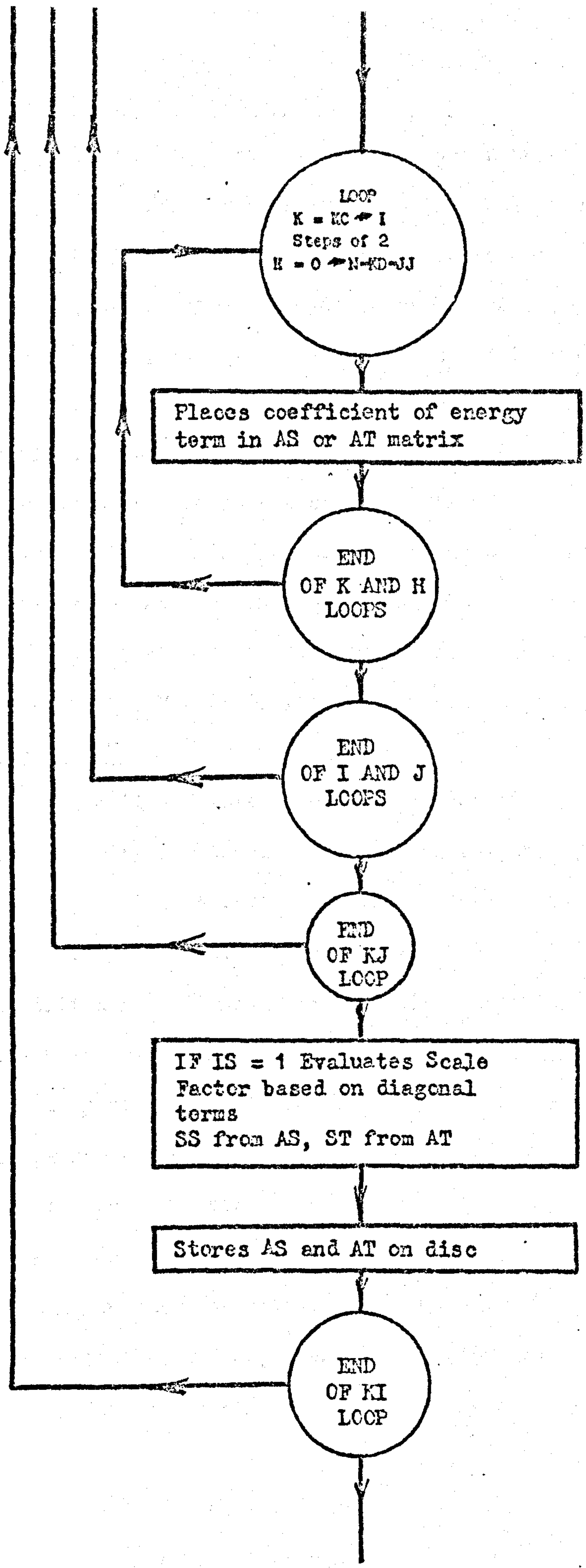
Thus, at the end of SESET all the energy matrices have been set up and stored on disc. The program for the constant thickness thick twisted plate theory is presented in Appendix 2, and the energy control information, required for this shell, in Appendix 3. The programs for the thin twisted plate and the thin cylindrical shell require a few changes in the dimensions of arrays, and in the case of the cylinder a different array COEF. The main differences are that the routines COEFFN must be set up and the control information IN and AN stored on disc for the particular theory.

For the thick cylinder theory, other changes have to be made to take account of the arbitrary function $T(\alpha)$ introduced in section 2.6.

Fig 5. EVENAB Flow Diagram




Places coefficient of energy term in AS or AT matrix
END OF K AND H LOOPS
END OF I AND J LOOPS
END OF KJ LOOP
LOOP
KJ=I2(KI)+1 → I1(KI+1)
Sets up ID, JD, JS, JC, IC
From A2
KD = ID, HD = JD
IF HI = 1 JJ = 1 ELSE JJ = 0
LOOP
I=0 → M-ID-1
J=0 → N-JD-1
IS I EVEN
Yes
No
KC = 0
KC = 1


LOOP
K = KC → I
Steps of 2
H = 0 → N-KD-JJ
Places coefficient of energy term in AS or AT matrix
END OF K AND H LOOPS
END OF I AND J LOOPS
END OF KJ LOOP
IF IS = 1 Evaluates Scale Factor based on diagonal terms
SS from AS, ST from AT
Stores AS and AT on disc
END OF KI LOOP

3.5. Boundary Conditions

This part of the program divides naturally into three parts:-

(i) Forming boundary condition matrices.

(ii) Choosing variables to reduce out.

(iii) Forming reduction matrix, and operating on energy matrices.

These three parts are referred to as BCMAT, BCVAR and BCRED respectively. For problems where this boundary condition reduction does not have to be applied, a substitute routine BCSUB has been written to merge all the partitions of the energy matrices, in preparation for the solution phase. This routine is straightforward and is not discussed further here; the program is included in the overall program, given in Appendix 2. The reduction parts of the program are now discussed in more detail. The program is given in Appendix 4 for application to the thin cylinder problem.

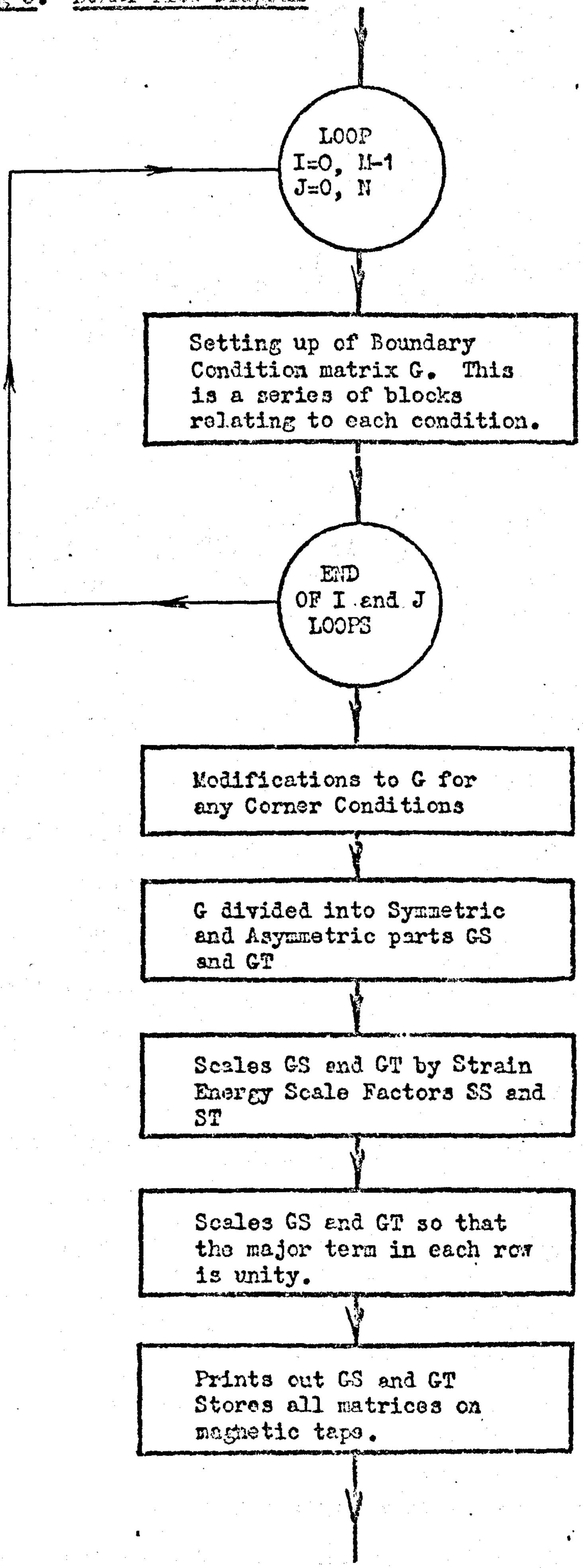
(i) BCMAT

The flow diagram for this part of the program is given in fig 6. This is for symmetry problems. Without symmetry, the program becomes much simpler. The whole of this section of the program is written for symmetry problems, so that comparisons can be made with results obtained without the boundary condition reductions. Most of this program is common for all shells. It has been written so that small blocks have to be added for particular boundary conditions to be applied. These blocks are included inside two loops for I and J which refer to terms $\left(\frac{\alpha}{AL}\right)^{I}\left(\frac{\beta}{BL}\right)^{J}$ in each displacement function. Referring to the program in Appendix 4 the lines between each updating of the counter CC are for one boundary condition. These are displacement or stress-free boundary conditions. The stress-free conditions are applied to the major terms in the resultant forces and couples, on the edges, defined by equations 2.3.14. As an example consider the condition

$$N^{12} = 0 \quad \text{on} \quad \alpha = AL \text{ and } \beta = BL \qquad 3.5.1.$$

for the thin cylindrical shell.

Fig 6. BCMAT Flow Diagram



These imply that

$$\frac{\partial A}{\partial \alpha} + \frac{\partial B}{\partial \beta} = 0 \quad \text{on } \alpha = AL, \beta = BL \qquad 3.5.2.$$

Substituting the displacement function 3.2.5. into these equations gives

$$\sum_{I=0}^{M-1} \frac{I}{AL} A_{I(J+1)} + \frac{(J+1)}{BL} B_{IJ} = 0 \qquad J=1, N-1 \qquad 3.5.3.$$

on α = AL

and $$\sum_{J=0}^{N-1} \frac{(I+1)}{AL} A_{IJ} + \frac{I}{BL} B_{(I+1)J} = 0 \qquad I=1, M-1 \qquad 3.5.4.$$

on β = BL

where in both cases the suffices of A and B cannot exceed M-1 or N-1 respectively. The parts of the computer program relating to these conditions are shown in Appendix 4 between lines 42 to 47 and lines 56 to 60 respectively. All other boundary conditions are set up in a similar manner. For the cylindrical shell the equations are straightforward, but for the twisted plate difficulties occur for conditions on lines of constant β, because of the trignometric terms involved. In this case, instead of separating out orthogonal terms, the equations should be set up at M equally spaced points along the edge.

It should be noted that in applying equations 3.5.3 and 3.5.4, one equation overdetermines the system, as the condition is applied twice at one point α = AL, β = BL. This is referred to as a corner condition. Care must be taken to eliminate such equations as they give rise to a singular matrix G_2.

The scaling applied to the equations at this stage is necessary because of the problem of errors later in the program. For this purpose, scale factors were evaluated in SESET, in terms of the diagonal terms in the strain energy matrix. Thus,

$$SS(I) = 1 / \sqrt{K(I, I)} \qquad 3.5.5.$$

This scaling is applied to each term in the energy matrices so that

$$K(I, J) = K(I, J)* SS(I)* SS(J) \qquad 3.5.6.$$

which makes each diagonal term unity, and all terms in the energy matrices of the same order of magnitude. The scaling is applied to the boundary condition matrix so that

$$G(I, J) = G(I, J)* SS(J) \qquad 3.5.7.$$

The boundary condition matrix is then scaled again so that the major term in each row is unity. If this scaling is not applied, the order of terms in each equation can vary greatly, leading to an ill-conditioned reduction matrix.

(b) BCVAR

All the matrices referred to in this section are defined in section 3.2, equations 3.2.7 to 3.2.12. The problem is to choose reduction variables so that the resulting matrix G_2 is non-singular, and the resulting energy matrices are well conditioned. The boundary condition matrix, G, is, in general, sparse, and has a definite pattern to it. These properties mean that the choice of reduction variables is better made by manual rather than automatic methods. For this purpose, at the end of BCMAT, all the information required later in the program is stored on magnetic tape, and the boundary condition matrices printed out. The reduction variables are then chosen and input to the next section of the program BCRED.

The basic criteria applied in the choice of the reduction variables is to make the major terms in each row and column of G_2 as large as possible, whilst ensuring that the matrix is not obviously singular. The terms are made large so that the coefficients of the matrix $G_2^{-1}G_1$ are as small as possible. The matrix Q, which is used to reduce the energy matrices, is made up of a unit matrix and the matrix $G_2^{-1}G_1$

$$Q = \left[\begin{array}{c|c} \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 \\ \hline 0 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} \\ \hline \multicolumn{2}{c}{-G_2^{-1}G_1} \end{array}\right] \qquad 3.5.8.$$

Therefore, the reduced energy matrices $\bar{M}$, $\bar{K}$ are made up of original matrices for the unreduced variables, plus terms from the rest of the energy matrix times $G_2^{-1}G_1$ terms. Thus, if the terms of $G_2^{-1}G_1$ are kept as small as possible, the effect on the conditioning of the resultant energy matrices is minimised.

For displacement boundary conditions this section of the program can be easily automated; but, for the stress free conditions the automatic methods tried, either gave rise to a singular G_2, extreme ill-conditioning of the resultant energy matrices, or became time consuming.

(c) <u>BCRED</u>

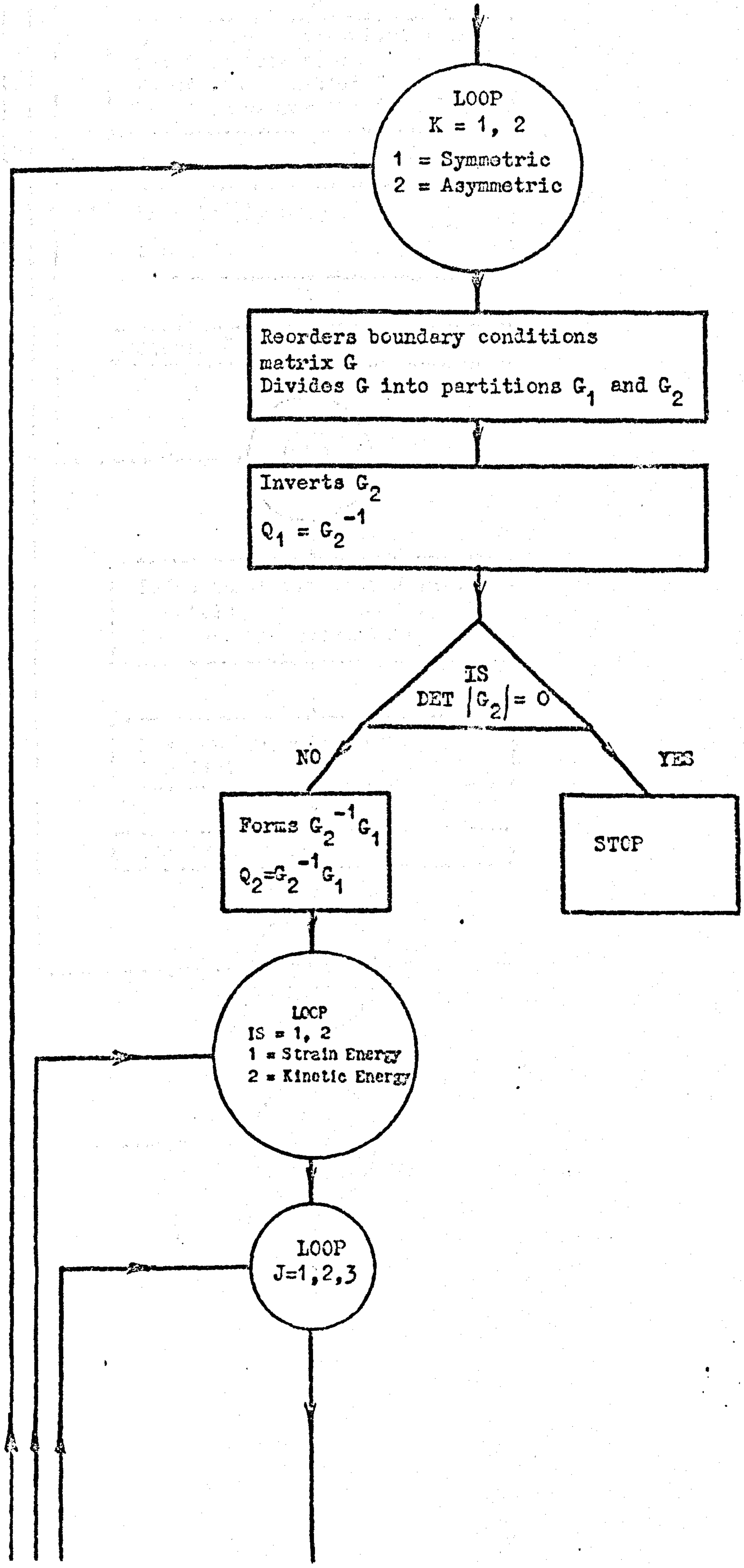
The only problem in this section of the program is with core store on the computer, as the arrays involved can be very large. This means the use of a complex block structure. The flow diagram is given in fig 7, and the program in Appendix 4.

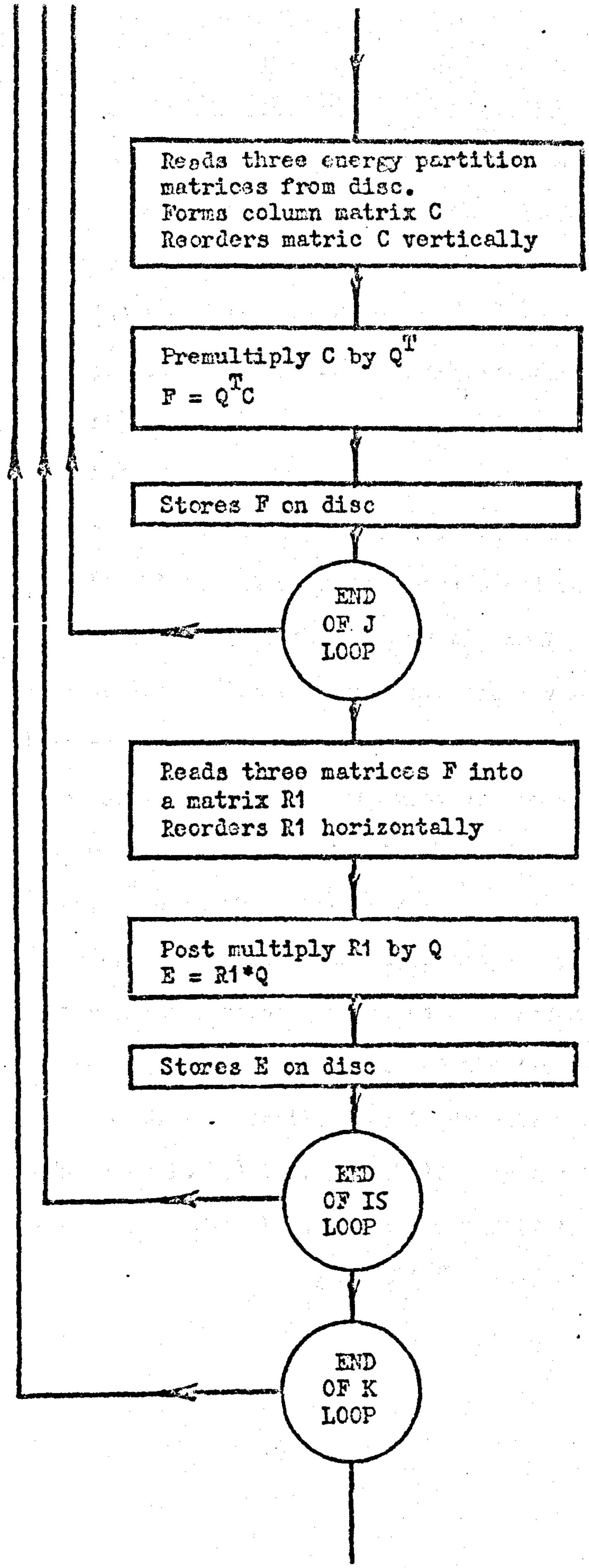
First the program reorders the boundary condition matrix G into G_1 and G_2, this being dependent on the reduction variables chosen in BCVAR. From these, the reduction matrix Q is formed. Only the partition $G_2^{-1}G_1$ of this is stored as the other partition is the unit matrix (see 3.5.8.)

The energy equations must now be pre- and post-multiplied by Q and its transpose. Here the problem arises with core store requirements, because due to the size of equations to be considered it becomes impossible to store the reduction matrix, and the whole energy matrix, at one time. Therefore, only three partitions of the energy matrix are in core at one time. These are brought down from disc to form complete columns. For example in 3.4.1.

partitions AA, BA and CA are formed in one large partition. The columns of this matrix are then reordered vertically to correspond with the Q matrix, and the premultiplication is carried out. The resultant matrix is stored on disc, and the operation repeated for all the other columns. These matrices are then brought down to form complete rows. The rows are reordered horizontally, and the post-multiplication performed. The resultant matrix is the energy matrix required by the eigenvalue solution procedure. The process is repeated for all the energy matrices.

Fig 7. BCRED Flow Diagram




Reads three energy partition matrices from disc.
Forms column matrix C
Reorders matric C vertically
Premultiply C by Q^T
F = Q^TC
Stores F on disc
END OF J LOOP
Reads three matrices F into a matrix R1
Reorders R1 horizontally
Post multiply R1 by Q
E = R1*Q
Stores E on disc
END OF IS LOOP
END OF K LOOP

3.6. Solution of the Eigenvalue Problem

The flow diagram for this section of the program is given in fig 8 and the program in Appendix 2. This section can be divided into four distinct sections.

(a) Operation on matrices $\overline{M}$, $\overline{K}$ of equation 3.2.15 to bring the problem into the form of the standard eigenvalue equation 3.2.16.

(b) The eigenvalue solution routine.

(c) Operation on eigenvectors to convert them into coefficients of the original displacement functions.

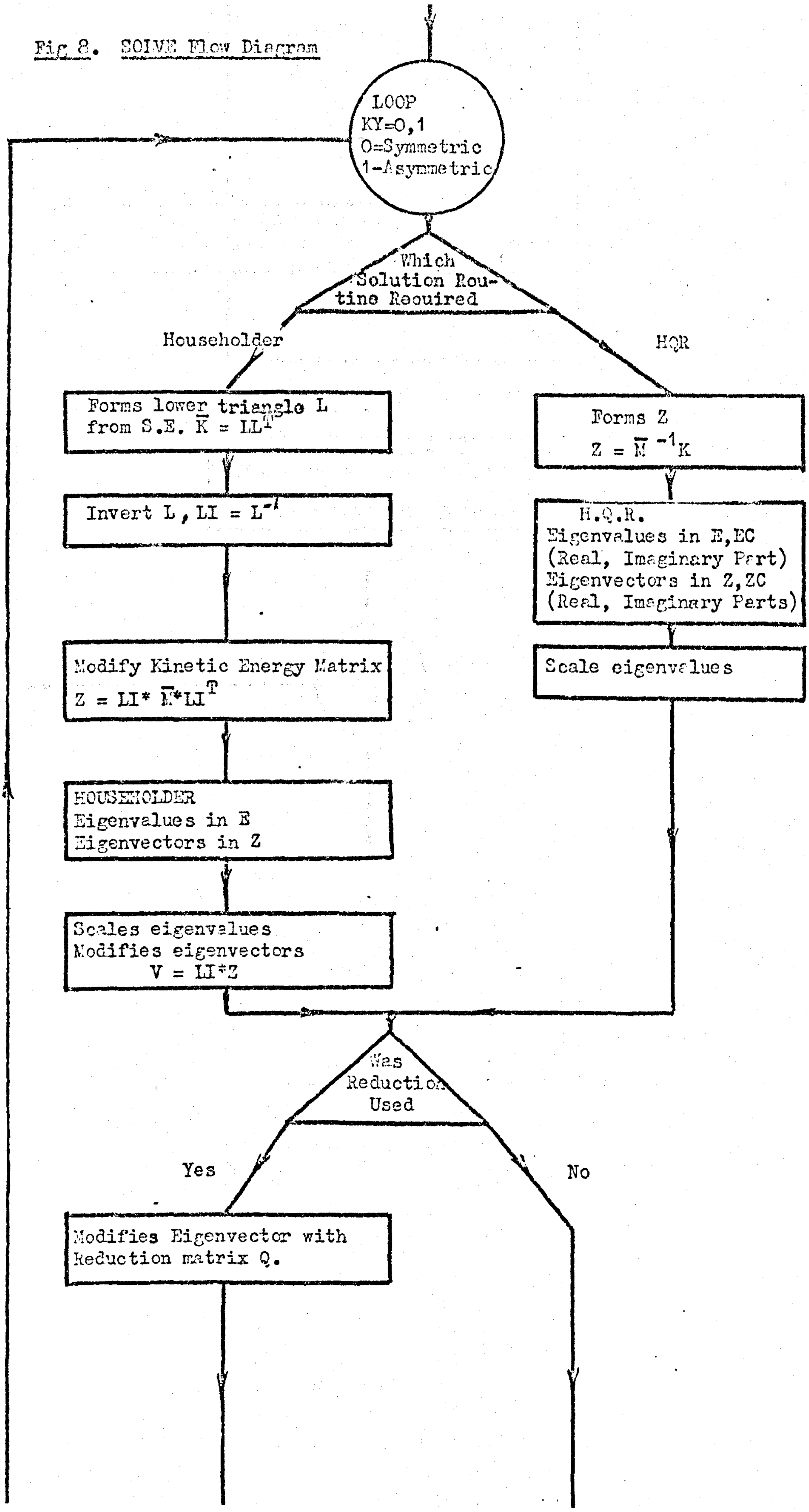
(d) Evaluation of frequencies and mode shapes.

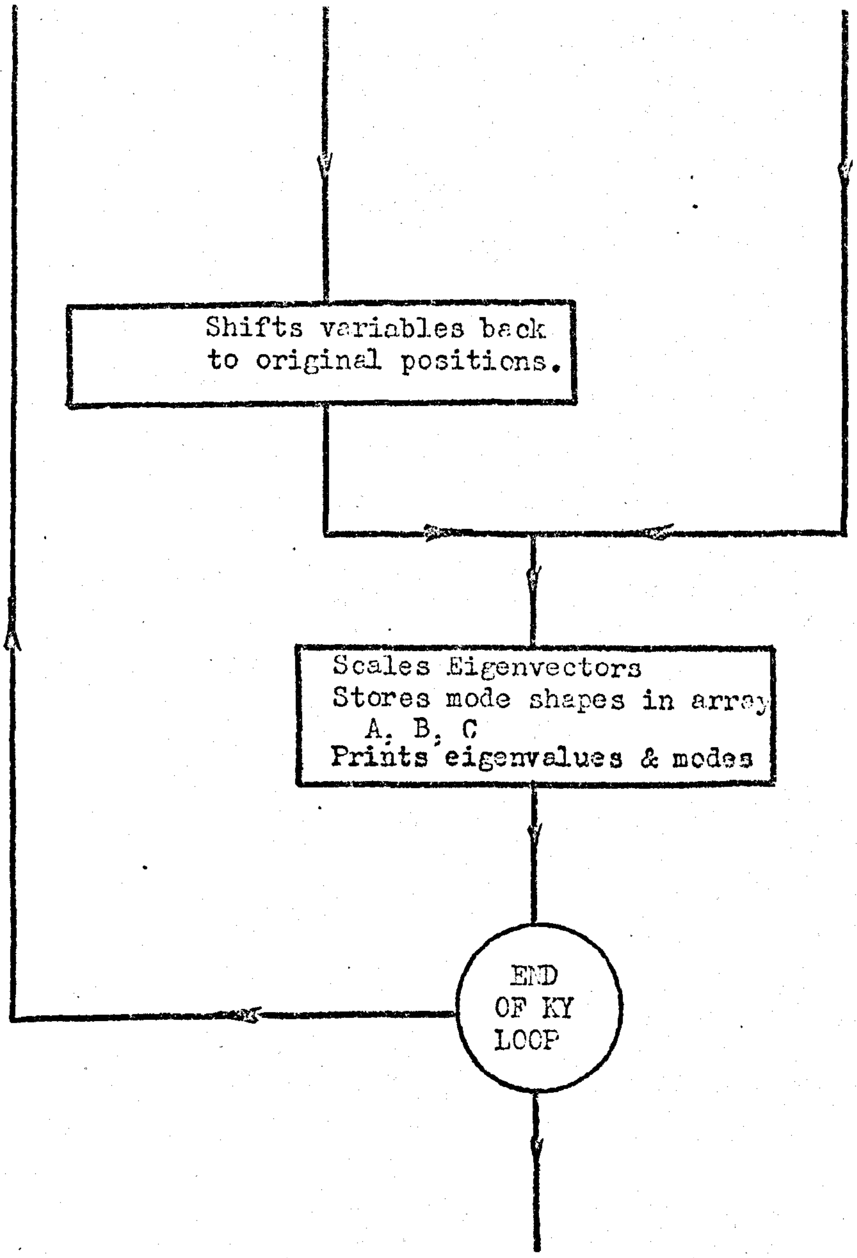
In the program the inverse eigenvalue problem is solved. This is because the smallest frequencies are the most important. By solving the inverse eigenvalue problem, these correspond to the largest eigenvalues, which are the ones that are most accurately determined. Unless all rigid body motions are reduced out, this technique cannot be used as $\overline{K}$ would be singular, $\overline{M}$ on the other hand is positive definite. For a singular matrix $\overline{K}$ the operations 3.6.1 or 3.6.2 are not possible. Thus if a completely free vibration problem is to be considered, the matrices $\overline{M}$ and $\overline{K}$ must be interchanged in the program, and the true eigenvalue problem solved.

The first operation (a) is dependent on the eigenvalue solution routine to be used in (b) i.e. Householder or the Hessenberg Q-R (HQR). In the latter case, the library routine MATDIV is used which forms

$$Z = \overline{K}^{-1}\,\overline{M} \qquad 3.6.1.$$

Fig 8. SOLVE Flow Diagram




Shifts variables back to original positions.
Scales Eigenvectors
Stores mode shapes in array
A, B, C
Prints eigenvalues & modes
END OF KY LOOP

In Householder's method the lower triangular matrix L must be formed so that

$$\bar{K} = LL^T \tag{3.6.2.}$$

The matrix L is formed first, then the matrix Z is formed by operating on the mass matrix

$$Z = L^{-1}\,\bar{M}\,\,L^{-T} \tag{3.6.3.}$$

Special routines, INVLOW, MATML and TRAML have been written to do this as opposed to using the standard library routines, because with L being a lower triangle matrix and Z and $\bar{M}$ being symmetric this makes the program more efficient.

Part (b) the eigenvalue solution is carried out using standard library routines HOUSEH for the Householder solution and NSEIGB for the HQR method.

For the Householder solution, the eigenvectors obtained have to be operated on to reconvert them to the eigenvectors of the original problem. For this,

$$\underline{q} = L^{-T}\underline{y} \tag{3.6.4.}$$

where $\underline{y}$ are the eigenvectors obtained in HOUSEH.

The above parts of the program have been written as two routines SOLUTION, one for each technique. The appropriate one is substituted as required.

If reduction has been used, then the eigenvectors must now be operated on again to obtain the original coefficients of the displacement functions. For this, the operation

$$\underline{x} = \underline{Qq} \tag{3.6.5.}$$

is carried out. The resulting matrix $\underline{x}$ is reordered to return it to its original form. The eigenvectors are then rescaled to take account of the scaling introduced into the energy expressions. For the reduction operations, two routines TRANSF are provided; one to perform the operations outlined above, the other a dummy to be used when no reduction has been applied.

The operations on the eigenvectors are only performed for those mode shapes required.

The frequencies are found from the eigenvalues as

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{\lambda}} \qquad 3.6.6.$$

The mode shapes for the shell are output in the form of three matrices, one for each of the middle surface displacements A, B and C. Each element of the matrix refers to the displacements at a mesh point in the shell, and this matrix then gives a complete picture of the mode involved. It should be noted that the displacements obtained directly from the eigenvectors are tensor displacements, and these have to be converted to physical displacements by the use of equation 2.2.8.

3.7. Discussion

The program outlined in this chapter has been used to obtain the results presented in Chapter 4. All these results have been obtained for symmetric shells, without the use of the stress-free boundary conditions. It is shown (section 4.2) that good convergence is obtained for the lower frequencies of such shells. The accuracies of the shell theories are discussed in Chapter 4.

The one major difficulty in the numerical part of this project has been with the reduction technique used on the stress-free boundary conditions. The technique works for displacement boundary conditions, as has been shown by Webster (29). The program was tested on Webster's problem of a cylindrical panel clamped on all four sides, and gave identical results.

Testing of the stress-free boundary conditions was carried out on the thin cylindrical shell clamped at one end. Symmetry about the middle axis $\alpha = 0$ was used, so that direct comparison could be made with well converged results obtained without the stress free conditions.

With Householder's eigenvalue solution, break down of the program occurs when the build up of errors makes it impossible to form the lower triangular matrix L from the strain energy matrix $\overline{K}$. This is caused by the ill-conditioning of the matrix $\overline{K}$. This ill-conditioning can occur without the use of a boundary condition reduction, because of ill-conditioning of high order polynomial expressions used in the displacement function. However, with the use of boundary condition reduction it occurs for lower order polynomials.

The form of the matrices $\bar{M}$ and $\bar{K}$ means that they are both symmetric and positive definite. For matrices with this property the eigenvalue problem 3.2.15. gives real positive eigenvalues and real eigenvectors. As extra terms are included in the displacement functions, the resulting low frequencies of vibration converge, but at a certain stage error terms give rise to negative eigenvalues. Additional terms then cause the break down in the formation of the lower triangular matrix L. However, until this break down occurs the major real eigenvalues are still well converged. To investigate the effects of the additional error terms on the major real eigenvalues, the HQR eigenvalue solution was used. This gave results similar to those of the Householder method, up to the break down point. At this stage the HQR method gave complex eigenvalues and eigenvectors, but still well converged real eigenvalues. The addition of further terms gave an increase in number and magnitude of the complex eigenvalues. Eventually these were as large as the major real eigenvalues. However, the real eigenvalues were still well converged. The problem now became one of time, as with this number of terms included, very few runs could be made. In fact, insufficient computer time was available to carry on with this work.

From the results obtained, however, it seems that the HQR method coupled with the stree-free boundary conditions is capable of solving general shell problems. The number of terms that had been used were sufficient to give good convergence for the problem without symmetry, provided the convergence criteria is the same as for the problem with symmetry. But, errors involved in this technique require further study.

The ill-conditioning of the energy matrices $\bar{M}$ and $\bar{K}$ is caused by the use of high order polynomials. It could have been overcome to some extent by using better conditioned series. The simple power series expansions were used here because of the ease of applying the boundary conditions.

The other reduction technique, mentioned earlier, but not applied, is that using the finite element technique, referred to as mass condensation or Guyan reduction, Zeinkiewitz (27). In this, degrees of freedom for which the inertia effects are negligible are reduced out. In the problems considered here this could be applied for a flap type mode to the in-plane displacements A and B. However, as the purpose of the shell theories presented here is to investigate small order effects, it is dangerous to reduce out of the solution any minor terms. For this reason the technique was not used.

CHAPTER 4

RESULTS AND CONCLUSIONS

4.1. Introduction

In this chapter are presented the results of the application of the numerical techniques, outlined in Chapter 3, to the thick shell theories proposed in Chapter 2. The results are compared with those for the relevent thin shell theories, and with the experimental and theoretical results obtained by other researchers.

The thick shell theories considered are:

1) For the twisted plate the theory presented in section 2.8, which includes transverse shear stresses of the same order as the in-plane stresses.

2) For the cylindrical shell the theory presented in section 2.6, which includes both transverse shear and normal stresses to the same order as the in-plane stresses.

All the shells considered are of constant thickness, and all have the following material constants

$E = 3 \times 10^7$ p.s.i.

$\eta = 0.3$

$\rho = 0.284$ lbs/cu. in.

the values for steel. In each case, the shell is clamped along one edge, the other three being free. Also, the shells have an axis of symmetry, along the middle axis perpendicular to the clamped edge. This facilitates the use of the numerical techniques outlined in Chapter 3, without use being made of the stress-free boundary conditions. Thus, the numerical method is an application of the Rayleigh-Ritz technique.

In section 4.2, the convergence of the frequencies and mode shapes for varying number of terms in the power series expansions, of the middle surface displacements, is considered. This gives a measure of the accuracy of the numerical methods. The accuracy of the theories as compared with experimental and theoretical results obtained by other researchers is analysed in section 4.3.

The effects of the thin and thick shell theories on the natural frequencies and mode shapes is then shown in section 4.4 for the twisted plate, and section 4.5 for the cylindrical shell. In both cases the effects of the two theories for various shapes and thicknesses are considered, and in the case of the twisted plate the effects of different angles of twist.

Conclusions on the use of the thick shell theories, and the finite element solutions to the problem are then drawn in section 4.6.

4.2. Convergence

The numerical techniques used here in the solution of the free vibration problem are approximate methods, the accuracy of which depend on how well the series expansions for the middle surface displacements approximate to the true solution. For convenience, in setting up energy expressions and in applying boundary conditions a double power series in each of the middle surface displacements has been used. The test of the numerical accuracy of the computing methods employed, is how well the frequencies and mode shapes converge, as the number of terms in the power series is increased.

The limitations on the number of terms that can be included in the series depend on two factors: the eigenvalue solution routines, and the conditioning of the system of equations obtained. In the results considered here, the limit has been set by the eigenvalue routines which are only effective for systems of equations of order 75.

In practice, this necessitates limiting the terms in the power series to those of order $\alpha^5\beta^7$ or $\alpha^7\beta^5$, where α and β are the middle surface coordinates of the shell. For the convergence tests, series including terms up to $\alpha^5\beta^7$, $\alpha^5\beta^6$, $\alpha^5\beta^7$ and $\alpha^7\beta^5$ were used.

Convergence tests were carried out for both the cylindrical shell and the twisted plate, for thick and thin shell theories, and for different shapes. In all cases, the results obtained were similar, so here two cases are presented.

1) The square cylindrical shell, referred to later as the Lindberg-Olsen fan blade, using Flügge's thin shell theory.
2) The 5:1 Length/Width ratio twisted plate with angle of twist of 30° using the thick shell theory.

For the cylindrical shell case, the blade considered is shown in fig 9. It is a square cylindrical panel 12" x 12" with radius of curvature 24" and thickness 0.12". The frequencies of the blade for varying number of terms in the power series are shown in table 2. In fig 10 the mode shapes are plotted, in each case for the lowest and highest frequency presented in table 2; the modes for the other values are intermediate to these two cases. The full line shows the lower frequency value, the better converged, and the dotted line the higher value. Where no dotted line is shown the two lines coincide. The tables and figures show good convergence for the lower frequencies, to within 1%. The fact that the frequencies have converged to the correct values isshown later in section 4.3.

For the twisted plate, the blade considered has dimensions 24" x 4.8", an angle of twist of 30° and a thickness of 0.24". The frequencies and mode shapes for this blade are given in table 3 and fig 11 respectively, in the same way as for the cylindrical shell. The results show the same degree of accuracy as for the cylinder.

Based on the results here, all results presented later in this chapter use the series with terms up to the order $\alpha^5\beta^7$ except for shells with a length/width ratio of 1:1 where the $\alpha^7\beta^5$ series has been used. This does not, for every mode, give the most closely converged result, but the accuracy is sufficient to show the effects of the various theories, and of varying thickness and angles of twist. If more accurate results are required, it is necessary to run the computer program with the $\alpha^5\beta^7$ and the $\alpha^7\beta^5$ series, choosing the lowest value of frequency for each mode.

4.3. Comparison of Theoretical with Experimental Results

To investigate the validity of the theories used, comparisons are made with the practical and theoretical results obtained by other researchers.

The cylindrical shell results are compared with those of Lindberg and Olson (32,33) for a cylindrical fan blade. They present experimental and theoretical results. The shell is that considered in the convergence tests of section 4.2, and it is shown in fig 9. The blade was welded to a thick steel block, and the vibrational modes excited using a sinusoidal magnetic force.

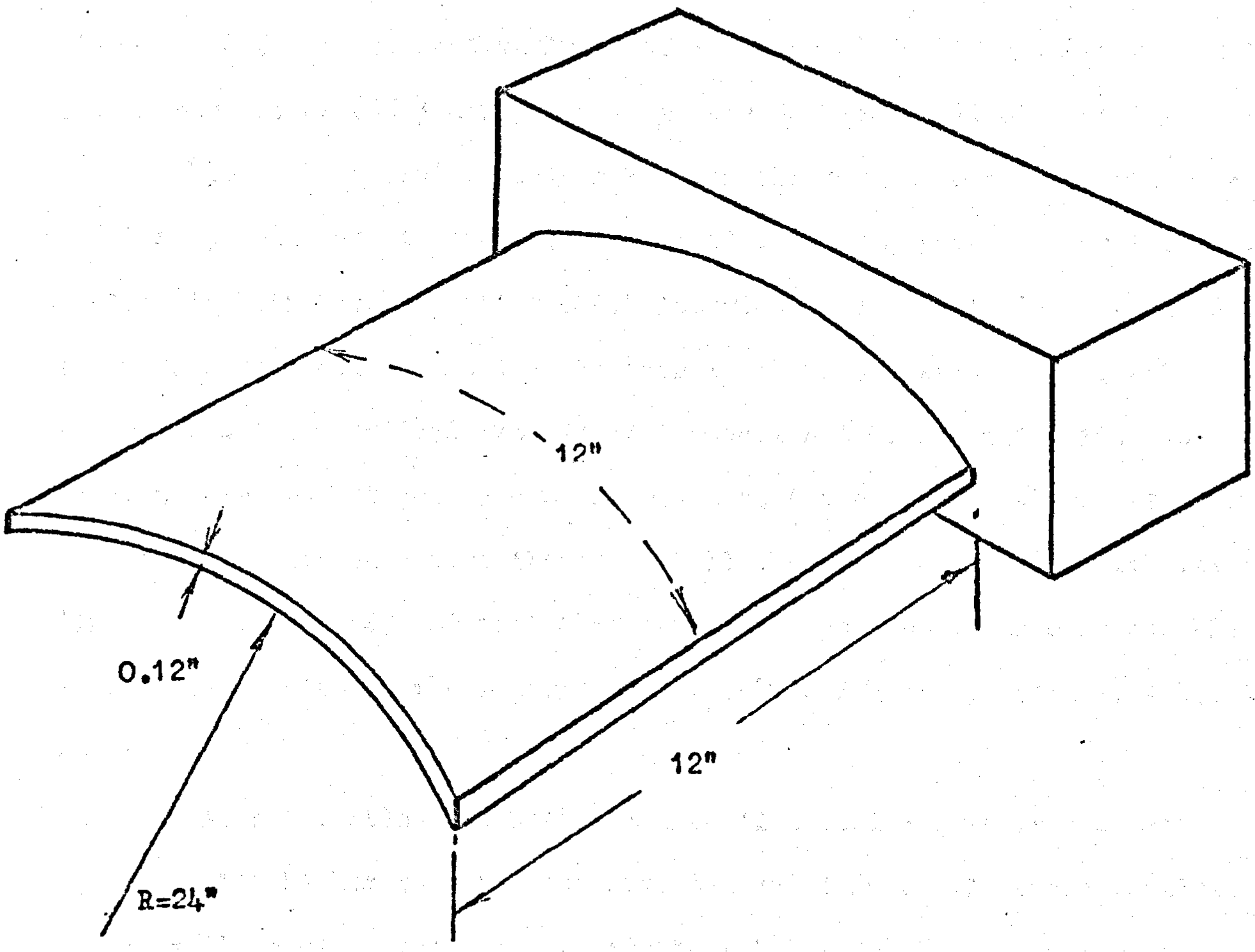


Fig 9. Lindberg-Olsen Fan Blade

This work was carried out twice by Lindberg and Olsen, and was, in each case, compared with a different finite element theory. First, a non-conforming cylindrical shell element was used, and second, a doubly curved triangular shell element. A finite element analysis of the same blade has been carried out by Zeinkiewicz et al (17) using a doubly curved thick shell element.

Table 4 shows the comparison between the results obtained by Lindberg and Olsen, Zeinkiewicz, and those obtained using the power series technique applied to both thin and thick shell theories presented in Chapter 2. Also, the mode shapes obtained using the thin shell theory are compared with the theoretical and practical results of Lindberg and Olsen in fig 12. The thick shell theory mode shapes bore no relation to the experimental results.

The results show that the thin shell theory is comparable with any of the finite elements used, and that they have converged to within a reasonable accuracy of the frequencies of the experimental model, which are of course, subject to errors.

It is noticeable that both thick shell theories give frequencies significantly higher than the experimental and thin shell theory results. This effect will be discussed later in sections 4.4 and 4.5.

The twisted plate theories have been compared with results for twisted beams presented by Carnegie (34) and Dawson (35). A great deal of work has been published for twisted turbine and compressor blades, but no other results were found for constant thickness twisted plates clamped at one end.

The blade considered by Carnegie was 6" long, 1" wide and 0.0635" thick, the angles of twist were varied in steps of 15° from 0° to 90°. Carnegie compares his experimental results with theoretical values for the fundamental bending-bending frequency and the first five torsional frequencies, these being obtained by application of the Rayleigh method to twisted beams. For twisted beams, the two independent flexural modes of a straight beam become coupled. These are referred to, by Carnegie, as bending-bending modes.

Dawson's beam was 12" long, 1" wide and 0.25" thick. He gives results for angles of twist of 30°, 60° and 90°. He only presents results for bending-bending frequencies. His theoretical results were obtained using the Rayleigh-Ritz technique, expanding the displacements as power series in the length variabl

The comparison of Carnegie's results with both thick and thin shell theories is shown for frequencies in table 5, and for mode shapes in fig 13. These comparisons show that the thin shell theory gives the more accurate results for the bending-bending modes, whilst the thick shell theory is better for the torsional ones. In each case, the frequencies are predicted to within 10%, except in the thin shell theory for angles of twist of 90°. The mode shapes compare very well, the effect of twist on these being negligible. The break down of the thin shell theory for 90° angles of twist is shown later, in section 4.4, to occur for other shapes as well as the "beams" considered here. Discussion of this effect is left until then.

For the Dawson blade, the frequency comparisons are given in table 6, and the mode shapes in fig 14. Once again the thin shell theory gives more accurate results for the bending-bending modes, the frequencies being predicted to within 5% for all except the 60° fundamental mode, and all the 90° twist cases. For the torsional modes the thick theory gives the lower frequencies. But here, there are no results from Dawson to make comparisons. The comparisons in the tables are with Dawson's theoretical results. He only presents experimental results for mode shapes, these being indistinguishable from the mode shapes presented in fig 14.

4.4. Twisted Plate Results

Two further twisted plate shapes are considered here, these have length/width (L/W) ratios of 5:1 and 1:1. The effects of varying the amount of twist, and the thickness of the shells, for both thin and thick shell theories are given. The dimensions of the two shells are 24" x 4.8" and 12" x 12" respectively.

The effects of angle of twist on the frequencies of the two shells is shown in tables 7 and 8. The results given in these tables for the flat plate case are obtained from Plunkett (36), who presents tables of frequency parameters for clamped cantilever plates with various L/W ratios. These tables of Plunkett are based on Rayleigh-Ritz results obtained by Barton (37) and Young and Felgar (38), and on experimental work by Plunkett himself and Grinstead (39). The flat plate values are only given for the flexural and torsional modes, as only these are considered by Plunkett.

The mode shapes for the 5:1 L/W ratio shell are shown in fig 15, these mode shapes are for both shell theories, there being no noticeable difference between them for this particular shell. Also, the angle of twist has a negligible effect on these modes, the 30° case being plotted. For the 1:1 L/W ratio shell the mode shapes are shown in figs 16 and 17 for the thin and thick shell theories respectively. In this case the effects of angle of twist are shown. These figures highlight the differences between the two shell theories, especially for the higher angles of twist.

The effect of thickness on the frequencies of the two shells, for the 30° angle of twist case, is shown in tables 9 and 10. For the 1:1 L/W ratio shell the mode shapes are given for the two theories in figs 18 and 19. For the 5:1 L/W ratio, shell thickness had very little effect on the modes shapes.

The thin shell theory applied to the two shells considered here, shows the same effect as with the Carnegie and Dawson blades; of very high frequencies for the 90° twist case. This effect, relative to the lower angles of twist, is not shown by the thick shell theory, even when it is applied to "thin shells" *. Considering the comparison with Carnegie's results, for the symmetric modes, it can be seen that the thick shell theory is predicting the lower frequency for all the 90° twist cases, except the fundamental. However, the frequencies are much higher than Carnegie's experimental results. The thick theory is giving the correct trend as the angle of twist is increased, but incorrect values.

Cohen (40) in his static study of the twisted plates, showed that O(t) terms in the in-plane stresses and strains (neglected by Love (1)) are important to the analysis of this shell. He calls these "strain-curvature" terms. These terms are included in the thin shell theory used here, but the theory has proved inadequate to deal with shells with high angles of twist.

*Note: "Thin shells" are those for which the ratio of wavelength/thickness is greater than 10.

It appears that for "thin shells", with high angles of twist, a theory is required which includes transverse shear stresses of O(t) compared to the in-plane stresses, rather than the O(1) ones included in the thick shell theory.

The major point of interest here is the difference between the thin and thick shell theories. The general "rule of thumb" approach to applying thin shell theory, is that results should be accurate for wavelength/thickness (λ/t) ratios greater than 10. Thin shell theory then breaks down, giving too high frequencies. This is because it overestimates the stiffness of the shell, by failing to take account of the transverse shear effects. Thick shell theories, on the other hand, include these transverse shear terms, and so for λ/t ratios less than 10 tend to give lower frequencies than thin shell theory. However, when thick shell theory is applied to "thin shells", ie with λ/t ratios greater than 10, the inclusion of the transverse shear terms stiffens the shell and gives high frequencies. This result is shown by Zeinkiewicz's application of thick shell theory to the Lindberg-Olsen fan blade, table 4.

In the results of table 10 for the 1:1 L/W ratio shell the thick shell theory is predicting lower frequencies than the thin theory for λ/t ratio less than 10. For the 5:1 L/W ratio shell, and the Carnegie and Dawson blades, all symmetric modes have λ/t ratios greater than 10, and all the thin shell results, apart from those for 90° of twist, give lower frequencies. For the asymmetric modes, however, the thick shell theory is predicting lower frequencies for λ/t ratios of up to 80:1. This is because these blades can be considered as beams rather than shells, and for torsional vibrations of beams both shear terms in the plane of the cross-section are important. The thin shell theory assumes that one of these, the transverse shear term, is zero, and thus stiffens the blade too much.

Thus, the thick shell theory in predicting lower frequencies than the thin theory for all cases where λ/t is less than 10. These are the results that should be obtained from such a theory. The accuracy of the results obtained requires further confirmation from experiments carried out on shells equivalent to the 1:1 L/W ratio shell considered here.

4.5. Cylindrical Shell Results

For the cylindrical shell, as with the twisted plate, two shapes are considered with 5:1 and 1:1 L/W ratios. The thin and thick shell theories are applied for shells of various thicknesses. The dimensions of the blades are 24" x 4.8" and 12" x 12", both with a radius of curvature of 24". The second one is equivalent to the Lindberg-Olsen fan blade considered earlier.

Tables 11 and 12 show the effects of varying the thickness on the frequencies predicted by the two shell theories. The mode shapes are given in fig 20 for the 5:1 L/W ratio shell, and in fig 21 for the 1:1 L/W ratio shell. For the 5:1 L/W shell the thickness had no noticeable effect on the mode shapes for the thin shell theory, but they do vary slightly for the thick theory. Fig 20, therefore, shows one representative thin mode shape, and the two extremes of the thick theory. For the 1:1 L/W shell, the thick theory gave extremely distorted mode shapes for the thinner shells, so results are given in this case for only the thickest shell considered, 1.2" thick. Fig 21, gives the comparisons between the two theories for this blade. The effect of thickness on the mode shapes for the thin shell theory is shown in fig 22.

For the cylindrical shell, the thick shell theory applied contains transverse normal stress, as well as the transverse shear stresses. It can be seen in tables 11 and 12 that the thick shell theory is predicting higher frequencies than the thin theory, even for λ/t ratios much smaller than 10:1, well after the thin shell theory should have broken down. To confirm this, experimental results are required, but even so, the thick shell theory is obviously not going to predict frequencies more effectively than the far simpler thin theory.

The effect of thick shell theory predicting high frequencies for thin shells is shown in table 4 for the Lindberg-Olsen fan blade. The finite element thick theory, which does not include transverse normal stress, predicts high frequencies, although not to the extent that the thick shell theory presented here does. The reasons for this have been discussed in section 4.4. The inclusion of transverse normal stress has increased the stiffness still further, thus giving higher frequencies.

It would be of interest to see what results the thick theory presented in section 2.6, where transverse shear stress in the curved coordinate direction only is included, would give. However, this was discovered too late for the computer runs to be made.

4.6. Conclusions

The results presented for the twisted plate in section 4.4 show that the thick shell theory predicts lower frequencies than the thin shell theory for λ/t ratios less than 10, this being the result that is expected from a thick shell theory. The accuracy of the results need to be evaluated by experimental means, however, before this theory can be applied with confidence. If experimental confirmation is obtained then the general thick shell theory presented in section 2.5, for shell with double curvature, including transverse shear stresses of the same order as the in-plane stresses, can be used as a half-way stage between thin-shell and full three-dimensional analysis. This theory could then be applied to more complicated shell structures, by the construction of a finite element based on it.

Over the last three years curved thick shell elements have been developed based on the usual thick shell theory discussed in Chapter 1, with five functions defining the displacement field, and constant transverse shear stresses across the element. This kind of element has been applied with reasonable success by Zeinkiewicz to Compressor and Turbine blades. Similar thin shell elements have also been developed. It would be interesting to see what results these give when applied to the twisted plate with a 90° twist angle.

The break down of the thin shell theory for twisted plates with high angles of twist is a topic that requires further study. This appears to necessitate a thin shell theory which includes transverse shear stresses of $O(t)$ compared to the in plane stresses.

For the thick shell theories, presented it can be seen that the energy expressions, and hence the solutions, become extremely complicated, when compared to the far simpler thin shell theories. Therefore, its use can only be justified when all other techniques have broken down. Finite element techniques based on traditional thin and thick shell theories would appear to be the best approaches to solving shell vibration problems. However the break down of thin shell theories for the twisted plate with high angle of twist shows the need for other approaches to be investigated.

The results for the cylindrical shell show that a thick shell theory, including transverse normal stresses in this way is not worthwhile. A three-dimensional theory would be required before the theory could give accurate results.

TABLE 2. THIN CYLINDER CONVERGENCE TESTS

Dimensions 12" x 12" x 0.12". 24" Radius

$\alpha^I \beta^J$ / Mode	Frequency c/s 5 x 6	5 x 7	5 x 8	7 x 6
1	85.9	85.9	85.9	85.8
2	139	139	139	138
3	248	248	248	247
4	345	345	344	342
5	393	392	392	387
6	576	576	576	529
7	738	735	734	734
8	750	741	738	736
9	790	781	780	780
10	848	845	844	807
11	-	-	-	1064
12	1242	1233	1226	1240
13	1291	1268	1238	1254
14	1409	1318	1275	1299
15	1402	1318	1297	1389
16	1764	1762	1761	1763

TABLE 3. THICK TWISTED PLATE CONVERGENCE TESTS

Dimensions: 24" x 4.8" x 0.24", 30° Twist

	Frequency c/s			
Mode \ $\alpha^I \beta^J$	5 x 6	5 x 7	5 x 8	7 x 6
1	15.7	15.7	15.6	15.7
2	85	85	85	85
3	154	154	154	154
4	283	248	247	283
5	395	386	379	395
6	468	467	467	467
7	579	569	510	578
8	811	798	796	808
9	1320	1192	1170	1316
10	-	1781	1701	-

TABLE 4. LINDBERG - OLSEN CYLINDER COMPARISON

Mode	Frequency c/s						
	Experimental Results		Lindberg Olsen Finite Elements		Thin Cylinder Theory	Zeinkiewicz Thick Shell Theory	Thick Cylinder Theory
	1971	1967	Triangular Curved(1971)	Cylinder (1967)			
1	85.6	86.6	86.6	93.5	85.8	113	183.5
2	134.5	135.5	139.2	147.6	138	147	413.2
3	259	258.9	251.3	255.1	247	296	543
4	351	350.6	348.6	393.1	342	440	758
5	395	395.2	393.4	423.5	387	475	1200
6	531	531.1	533.4	534.3	529		
7	751	751.2	746.4	781.5	736		
8	743	743.2	752.1	792.2	734		
9	790.	792.1	790.1	863.2	780		
10	809	809.2	813.8	862.4	807		
11	997	996.8	1009	1002	1064		
12	1216	1215	1232	1175	1233		
13	1252		1246		1299		
14	1241		1266		1254		
15	1281		1286		1318		
16	1310		1303		1318		
17	1706		1652		1682		
18	1625		1653		1762		
19	1643		1678		1811		
20	1841		1838		1881		
21	1830		1885		2128		
22	1867		1926		2006		
23	1890		1941		2746		
24	2257		2224		2151		
25	2207		2274		2378		

TABLE 5. CARNEGIE TWISTED BEAM COMPARISON

Bending Mode	Angle (Degrees)	Frequency c/s			
		Carnegie		Thin Twisted Plate	Thick Twisted Plate
		Experimental	Theoretical		
1	30	57	62	64	71
1	60	59	63	77	132
1	90	61	65	174	231
2	30	320		349	380
2	60	265		284	334
2	90	210		637	338
3	30	1000		952	1116
3	60	900		940	1056
3	90	800		1100	1020
4	30	1000		1124	1344
4	60	1250		1324	1668
4	90	1500		2900	2044
5	30	1950		2396	2524
5	60	1980		2640	2760
5	90	2000		5516	3064
Torsional Mode					
1	30	730	725	896	770
1	60	800	790	948	864
1	90	900	910	1556	992
2	30	2200	2150	2712	2340
2	60	2500	2450	2860	2612
2	90	2800	2650	4696	3000
3	30	3800	3750	4596	4000
3	60	4200	4100	4824	4408
3	90	4700	4600	7920	5004
4	30	5500	5500	6668	5924
4	60	5900	5950	7012	6500
4	90	6600	6600	11652	7308

TABLE 6. DAWSON TWISTED BEAM COMPARISON

Mode	Angle of Twist (Degrees)	Frequency (C/S)		
Symmetric		Dawson	Thin Twisted Plate	Thick Twisted Plate
1	30	28.5	29.3	31.8
1	60	28.8	35.5	56.9
1	90	29.3	79.9	98.7
2	30	107	106	134
2	60	92.8	94.3	117
2	90	80.8	211	116
3	30	192	191	216
3	60	221	218	272
3	90	256	478	283
4	30	476	478	507
4	60	430	435	483
4	90	385	995	451
5	30	749	725	933
5	60	821	798	1021
5	90	849	1821	1079
Asymmetric				
1	30		775	602
1	60		775	603
1	90		1060	604
2	30		2104	2175
2	60		2104	2151
2	90		3041	2140
3	30		2329	1808
3	60		2330	1815
3	90		3752	1820
4	30		3895	3029
4	60		3895	3039
4	90		5553	3047
5	30		5487	4284
5	60		5494	4309
5	90		8007	4332

TABLE 7. EFFECTS OF TWIST ON TWISTED PLATE FREQUENCIES

5:1 L/W Ratio Shell

Dimensions 24" x 4.8" x 0.48"

	Frequency C/S								
	Plunkett Flat Plate	Thin Twisted Plate Theory				Thick Twisted Plate Theory			
Mode \ Angle of Twist (Degrees)	0	15	30	60	90	15	30	60	90
Symmetric									
1	28.0	27.2	28.2	34.1	76.5	29.2	31.0	57.9	101.3
2	171	167	151	122	273	178	165	143	143
3	-	276	303	360	786	355	382	432	443
4	481	483	474	463	1031	511	508	539	631
5	934	1014	1018	1040	2240	1072	1073	1098	1142
6	-	1490	1515	1596	3188	1756	1758	1779	1817
Asymmetric									
1	260	325	328	339	545	272	277	300	330
2	795	986	996	1028	1658	832	848	914	1010
3	1370	1687	1700	1749	2845	1441	1463	1557	1698
4	-	2106	2105	2101	3144	2632	2176	2148	2125
5	2011	2481	2498	2562	4265	2169	2195	2315	2492
6	-	3637	3662	3760	6272	3211	3252	3433	3704

TABLE 8. EFFECTS OF TWIST ON TWISTED PLATE FREQUENCIES

1:1 L/W Ratio Shell

Dimensions 12" x 12" x 0.48"

	Frequency c/s								
	Plunkett Flat Plate	Thin Twisted Plate Theory				Thick Twisted Plate Theory			
Mode \ Angle of Twist (Degrees)	0	15	30	60	90	15	30	60	90
Symmetric									
1	113	110	113	126	256	121	128	223	347
2	695	677	619	485	1005	715	674	571	552
3	891	936	952	1009	1476	959	986	1092	1176
4	1982	1859	1792	1586	2958	2019	1921	1756	1573
5	-	1959	1997	2133	3335	1863	1992	2239	2362
6	-	2098	2270	2658	4272	2234	2393	2776	3031
Asymmetric									
1	277	397	561	857	1257	413	640	1008	1200
2	1011	1155	1297	1589	2078	1119	1315	1700	1899
3	-	2175	2182	2220	2794	2220	2240	2296	2346
4	2497	2414	2447	2515	3854	2487	2528	2649	2692
5	-	3354	3385	3550	4832	3236	3301	3558	3718
6	4458	4217	4166	3969	5527	4539	4335	4071	3854

TABLE 9. EFFECT OF THICKNESS ON TWISTED PLATE FREQUENCIES

5:1 L/W Ratio Shell

Dimensions 24" x 4.8" 30° Twist

	Frequency (C/S)					
	Thin Twisted Plate Theory			Thick Twisted Plate Theory		
Mode \ Thickness/Length	0.005	0.01	0.02	0.005	0.01	0.02
Symmetric						
1	6.69	14.1	28.2	7.83	15.6	31.1
2	40.3	77.8	151	42.7	84.6	165
3	119	229	474	126	248	508
4	285	541	1018	315	569	1073
5	328	304	303	388	386	382
6	535	925	1799	538	971	1758
Asymmetric						
1	98.2	171	328	97.6	154	277
2	296	520	996	293	467	848
3	498	884	1700	487	798	1463
4	722	1295	2498	709	1192	2195
5	1066	1904	3662	1072	1781	3252
6	2105	2105	2105	2528	2176	2176

TABLE 10. EFFECT OF THICKNESS ON TWISTED PLATE FREQUENCIES

1:1 L/W Ratio Shell

Dimensions 12" x 12", 30° Twist

	Frequency c/s							
	Thin Twisted Plate				Thick Twisted Plate			
Thickness/Length — Mode	0.01	0.02	0.04	0.1	0.01	0.02	0.04	0.1
Symmetric								
1	26.9	53.6	113	281	32.9	64.8	128	311
2	161	321	619	1424	175	346	674	1512
3	407	564	952	2038	476	623	986	2038
4	478	944	1792	4468	530	1028	1921	4096
5	920	1255	1997	4736	1069	1335	1992	4746
6	1086	1932	3519	5509	1153	2005	3381	5484
Asymmetric								
1	409	480	561	836	477	571	640	864
2	639	883	1297	2706	693	943	1315	2473
3	789	1189	2182	5115	906	1270	2240	4953
4	941	1415	2447	5658	1043	1499	2528	5491
5	1291	1890	3385	7888	1408	1946	3301	6940
6	1361	2330	4397	10220	1463	2431	4335	9350

TABLE 11. EFFECT OF THICKNESS ON CLYINDRICAL SHELL FREQUENCIES

5:1 L/W Ratio Shell

Dimensions 24" x 4.8" Radius of Curvature 24"

	Frequency c/s					
	Thin Cylinder Theory			Thick Cylinder Theory		
Thickness/Length Mode	0.005	0.01	0.02	0.005	0.01	0.02
Symmetric						
1	9.82	15.4	28.4	16.9	25.6	44.2
2	60.5	96.0	177	107	167	292
3	168	269	497	371	578	921
4	333.	550	1031	1084	1362	2010
5	792	1108	1882	1942	3269	5387
6	2106	2106	2106	2129	2129	2129
Asymmetric						
1	68.9	137.3	261	83.2	161	279
2	211	264	276	264	280	308
3	264	420	838	280	506	957
4	366	727	1425	579	1014	1493
5	564	1102	2172	1209	1495	1779
6	970	1428	1448	1495	1819	3082

Table 12. Effect of Thickness on Cylindrical Shell Frequencies

1:1 L/W Ratio Shell

Dimensions 12" x 12", Radius of Curvature 24"

Mode \ Thickness/Length	Frequency C/S — Thin Cylinder Theory 0.01	0.02	0.04	0.1	Thick Cylinder Theory 0.01	0.02	0.04	0.1
Symmetric								
1	139	162	192	319	413	462	535	836
2	248	441	824	1773	543	821	1497	2608
3	392	632	948	2170	1735	1971	3011	4351
4	735	1005	1808	4301	4123	4404	6197	8060
5	781	1279	2159	4877	7896	8415	8594	8748
6	1233	1762	3229	7563	8164	8777	9918	11632
16	4224	4226	4225	4226	4286	4284	4284	4284
Asymmetric								
1	85.9	147	280	685	183	278	501	908
2	345	557	1027	2469	758	1106	1423	3318
3	576	1090	2063	5066	1199	2009	3107	6684
4	741	1252	2377	5577	2521	3234	5434	9236
5	845	1579	2989	7312	5350	6308	8507	11180
6	1263	2302	4421	10350	11180	12970	14010	15950
8	1765	1764	1762	1763	1825	1823	1824	1825
15	4747	4743	4743	4742	4878	4878	4879	4880

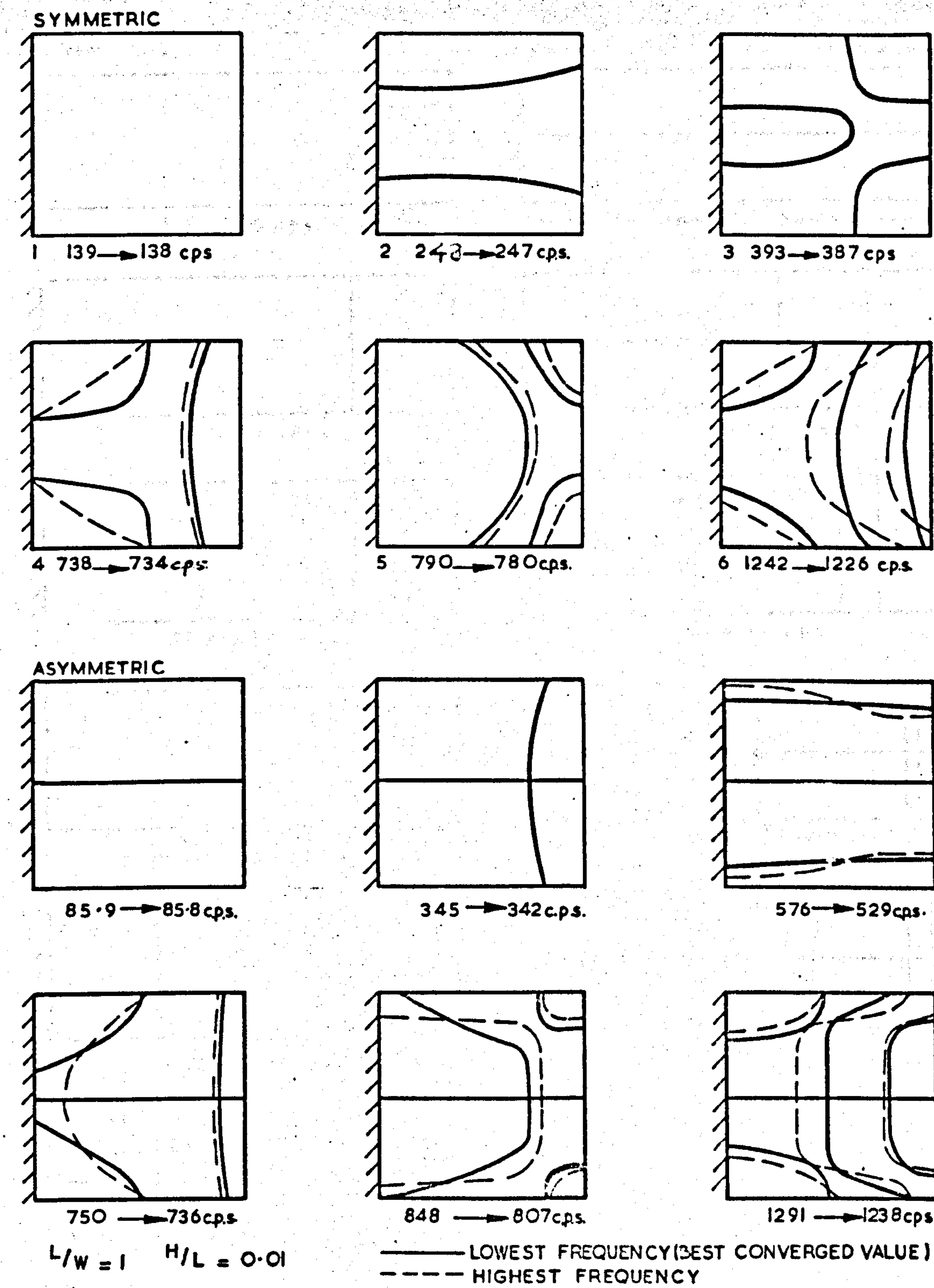


FIG. 10. CYLINDRICAL SHELL CONVERGENCE TESTS, THIN SHELL THEORY.

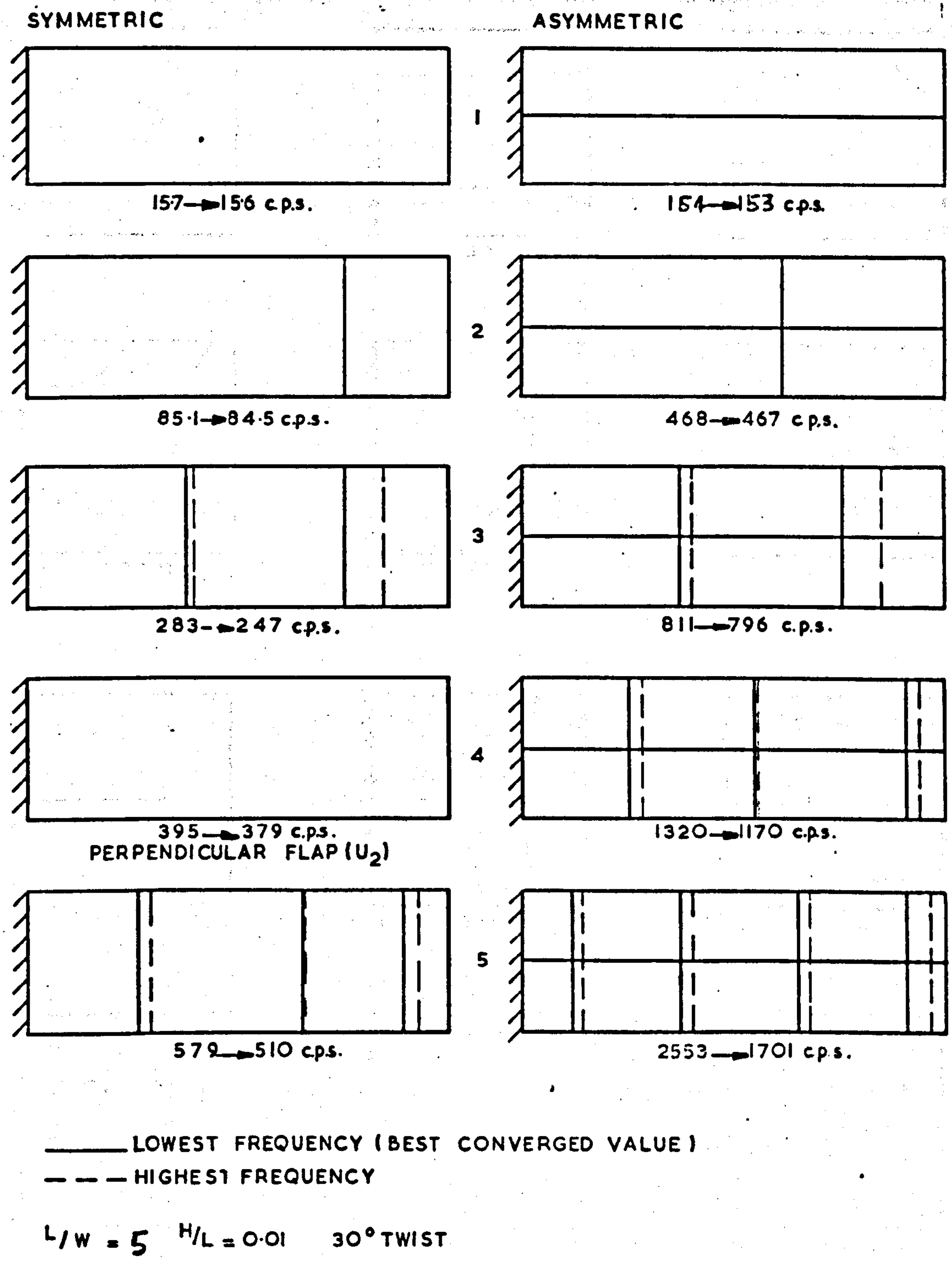


FIG.11. TWISTED PLATE CONVERGENCE TESTS, THICK SHELL THEORY.

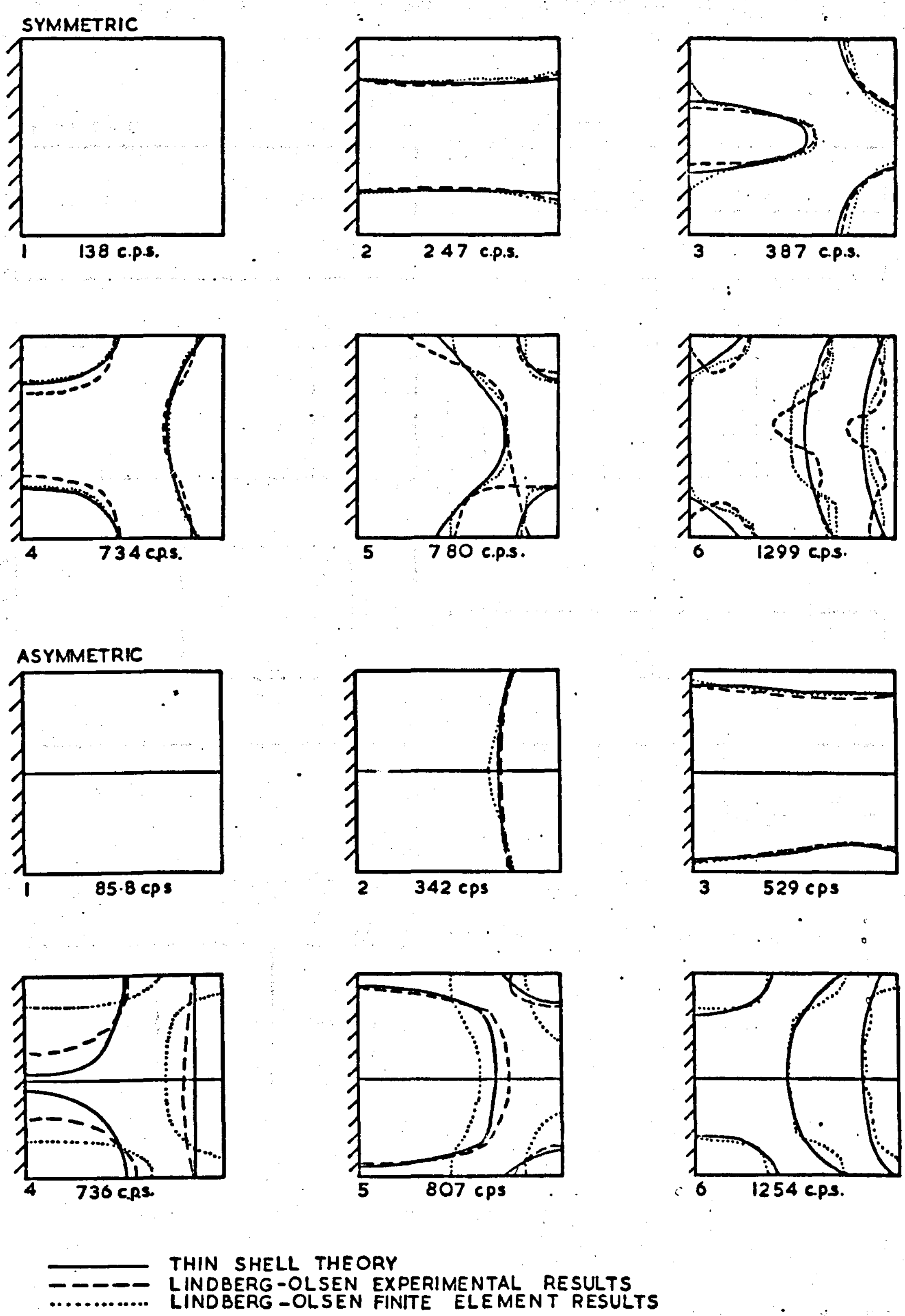


FIG.12. LINDBERG-OLSEN FAN BLADE MODES

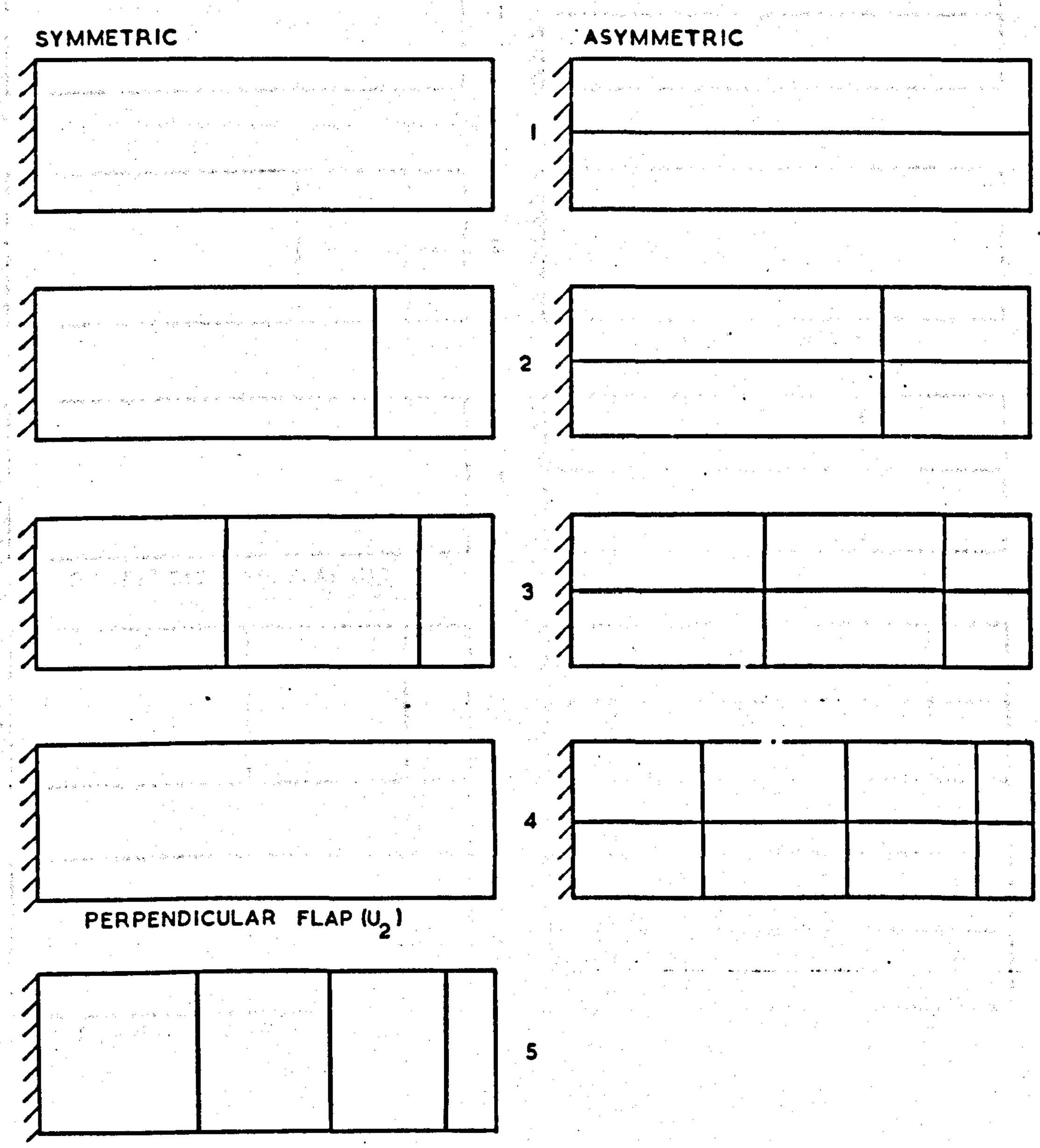


FIG.13. CARNEGIE BEAM MODES.

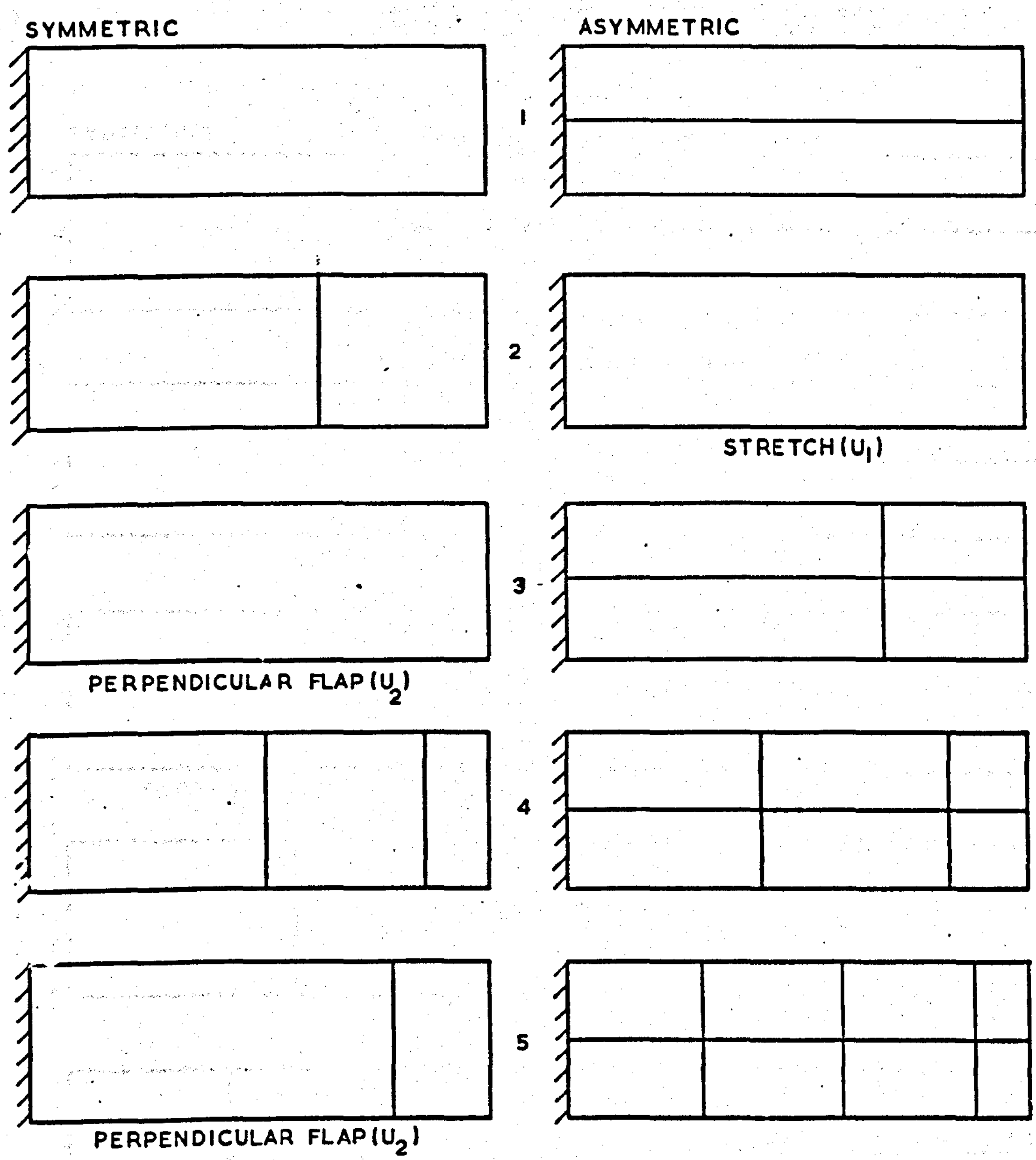


FIG.14. DAWSON BEAM MODES

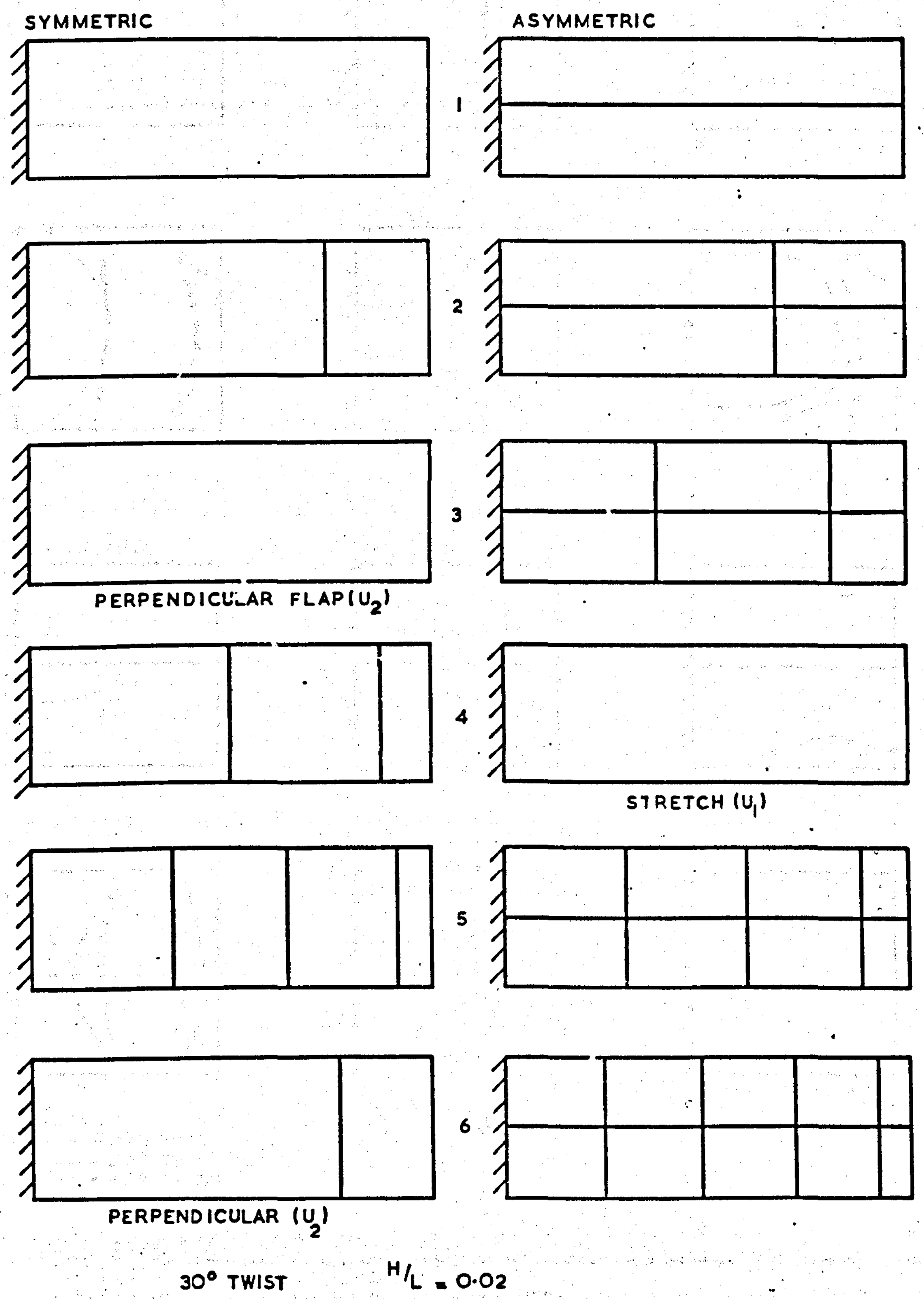


FIG.15. MODES FOR 5:1 L/W RATIO TWISTED PLATE.

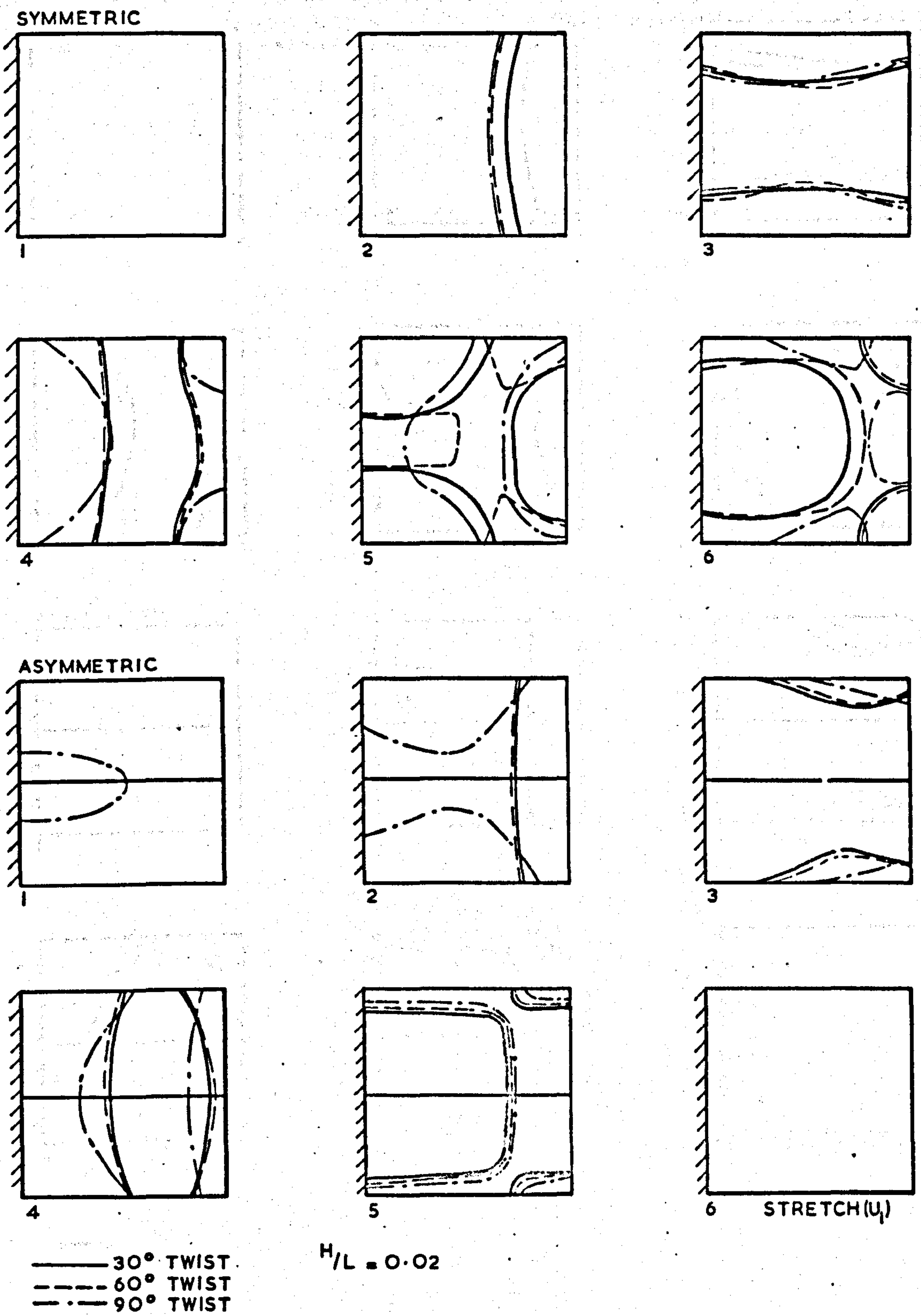


FIG.16. EFFECT OF TWIST ON MODES OF 1:1 L/W RATIO TWISTED PLATE, THIN SHELL THEORY.

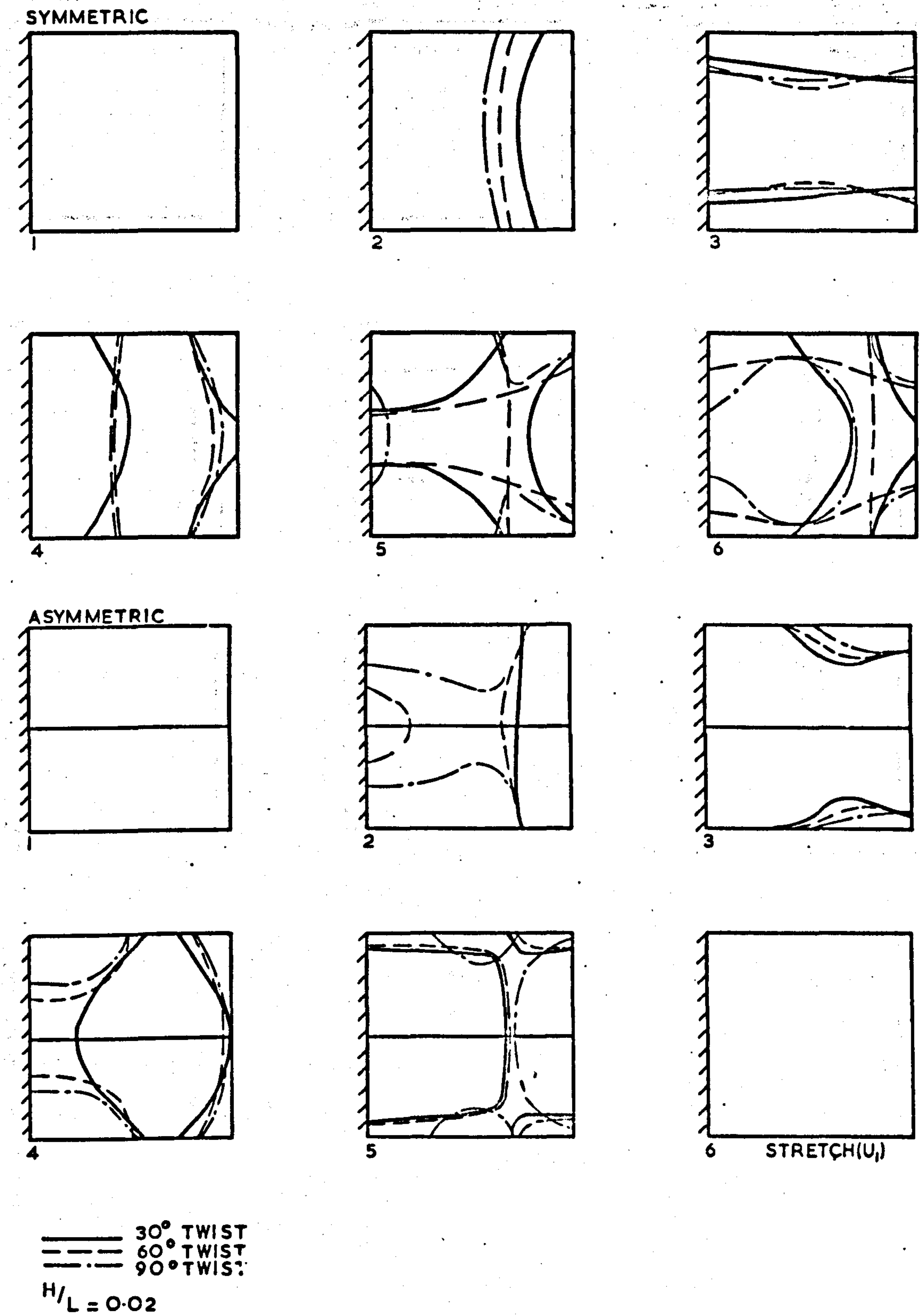


FIG.17. EFFECT OF TWIST ON MODES OF 1:1 L/W RATIO TWISTED PLATE, THICK SHELL THEORY.

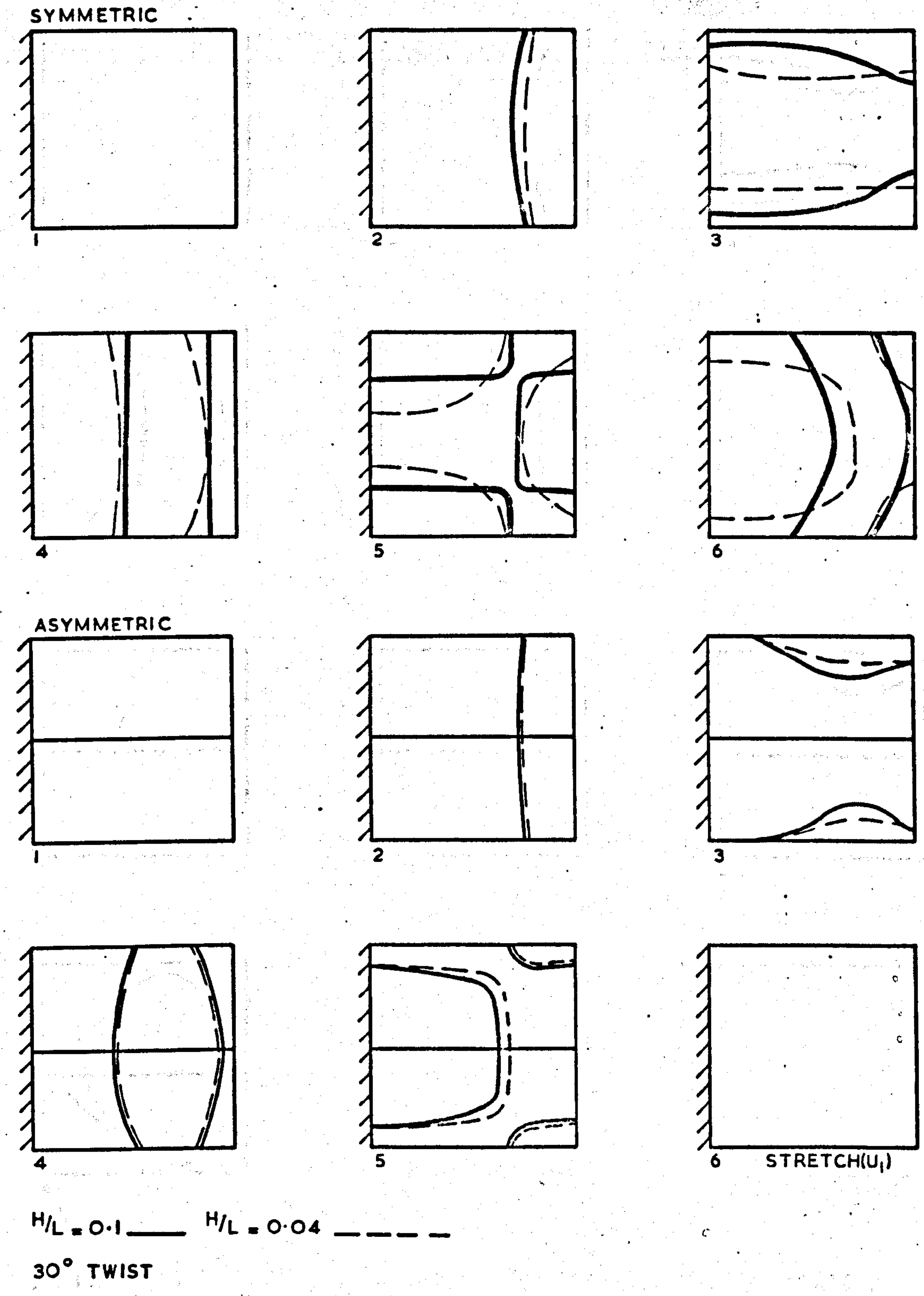
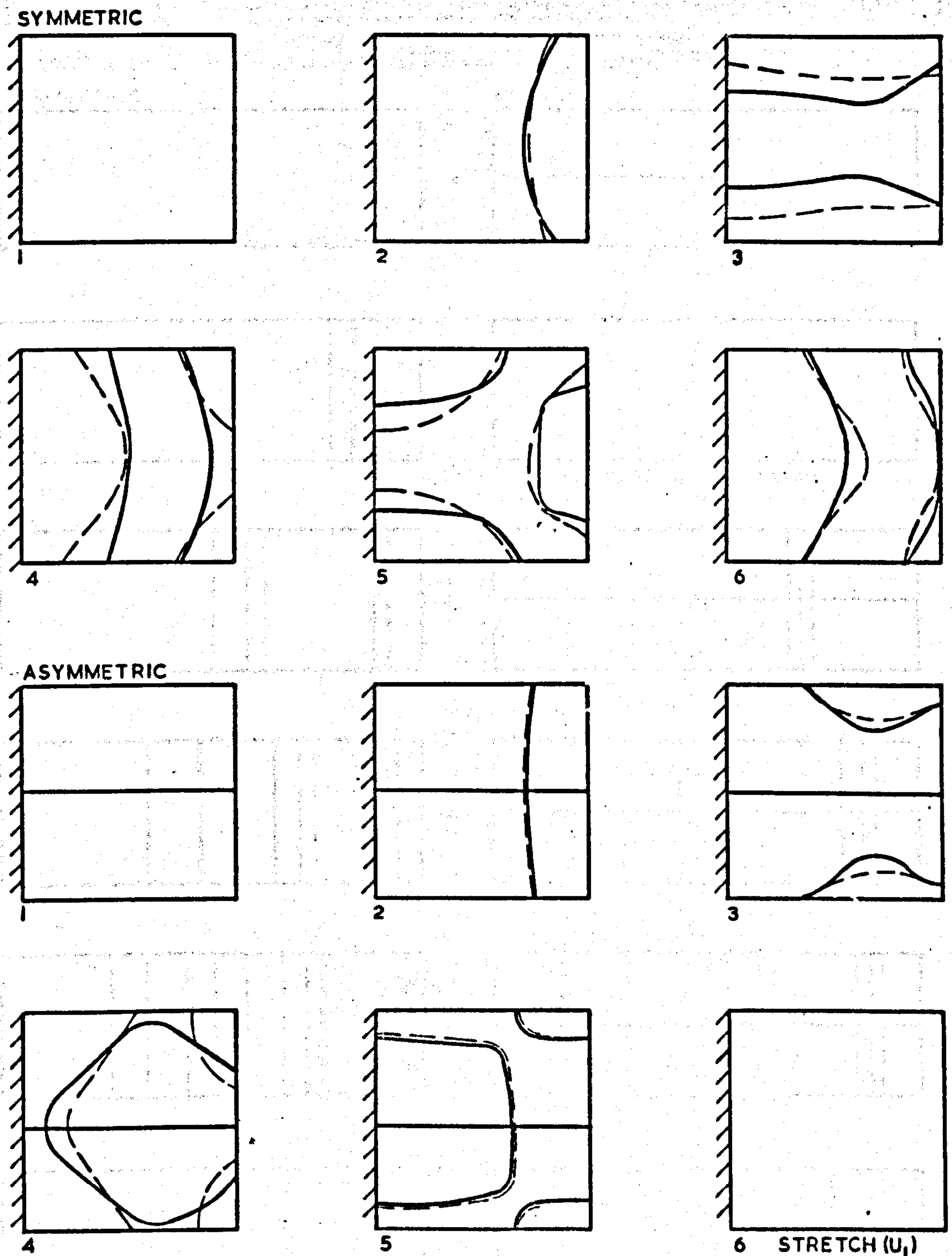


FIG.18. EFFECT OF THICKNESS ON MODES OF 1:1 L/W RATIO TWISTED PLATE, THIN SHELL THEORY.



H/L = 0·1 ______ H/L = 0·04 _ _ _ _ 30° TWIST

FIG 19. EFFECT OF THICKNESS ON MODES OF 1:1 L/W RATIO TWISTED PLATE, THICK SHELL THEORY.

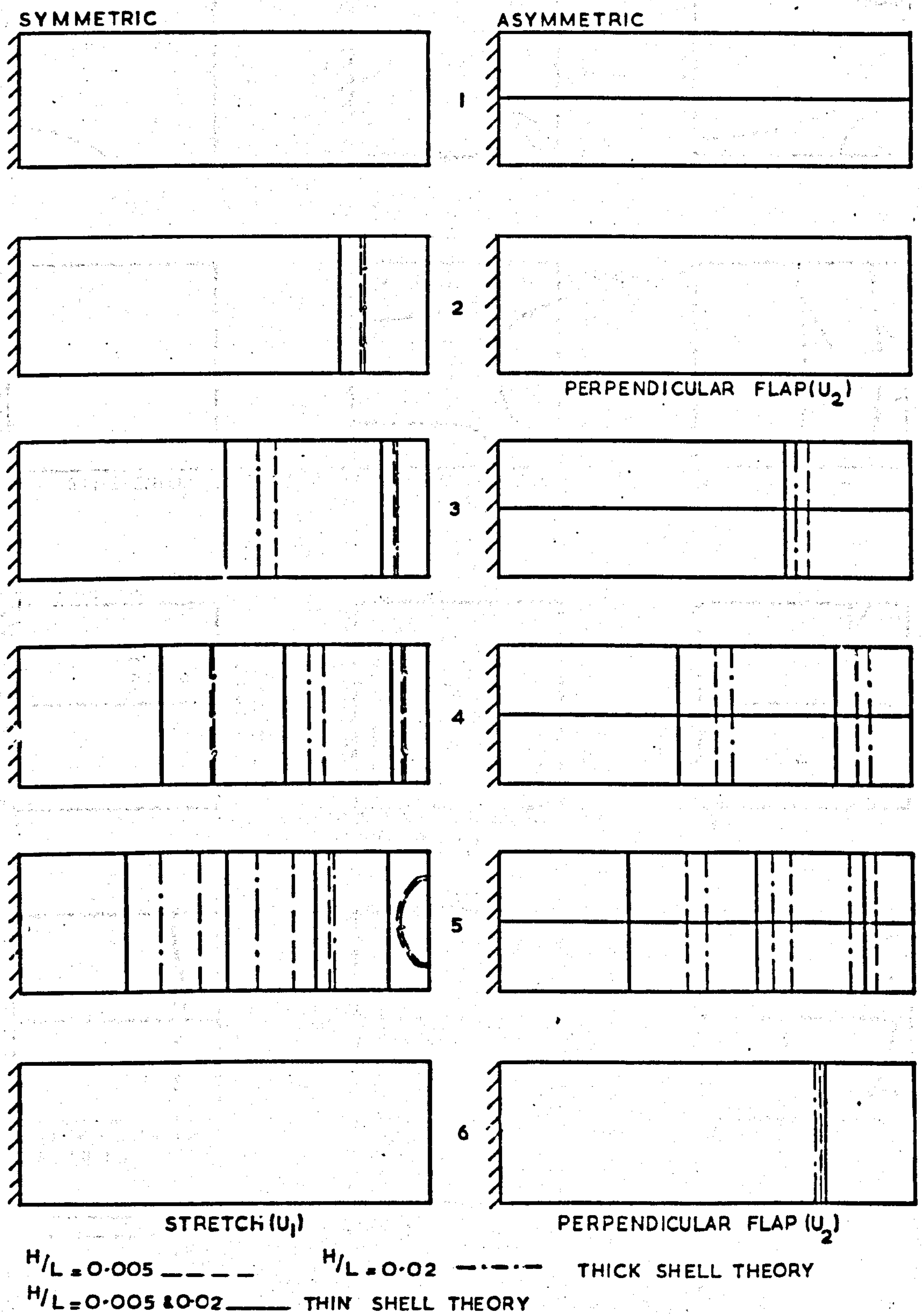


FIG.20. COMPARISON OF THIN AND THICK CYLINDRICAL SHELL THEORIES EFFECT OF THICKNESS ON MODES FOR 5:1 L/W RATIO SHELL.

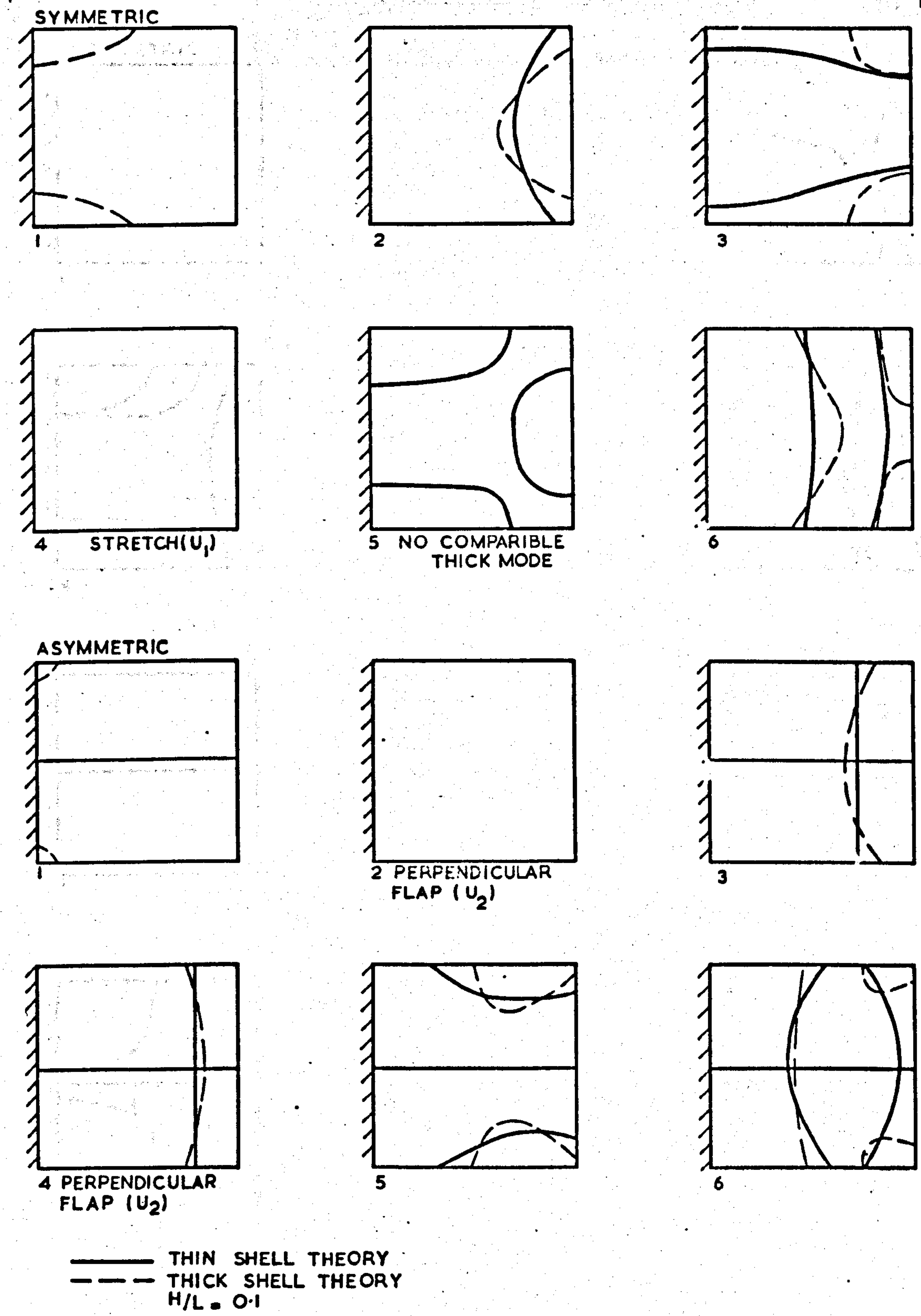


FIG.21. COMPARISON BETWEEN THIN AND THICK CYLINDRICAL SHELL THEORIES, MODES FOR 1:1 L/W RATIO SHELL.

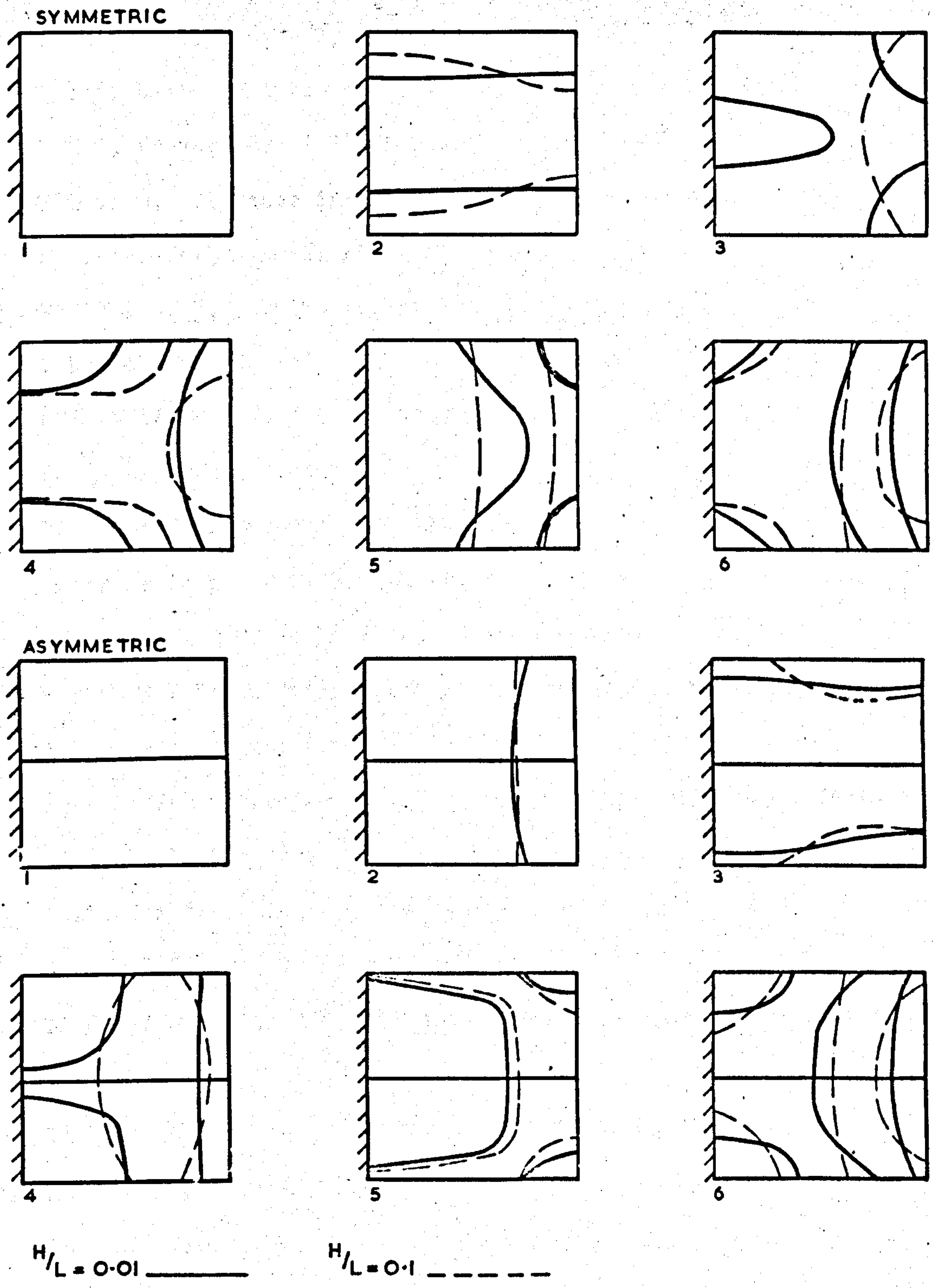


FIG.22. EFFECT OF THICKNESS ON MODES FOR 1:1 L/W RATIO CYLINDRICAL SHELL THIN SHELL THEORY.

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APPENDIX 1. BASIS OF THE VARIATIONAL METHOD

Consider the vibrations of an elastic body subject to the conditions

$$u_i = 0 \text{ on } S_u \text{ for all } t$$
$$t_{ij}n_j = 0 \text{ on } S_T \text{ for all } t \qquad \text{A.1}$$

Assume there exist solutions

$$u_i = U_i^{\alpha} e^{i\Omega_\alpha t} \qquad \alpha = 1,2,\dots\dots \qquad \text{A.2}$$

so that if t_{ij}^{α} is the stress arising from U_i^{α}

$$\frac{\partial t_{ij}^{\alpha}}{\partial x_j} = -\rho \Omega_\alpha^2 U_i^{\alpha} \qquad \text{A.3.}$$

Multiplying this equation by U_i^{β} and the similar equation for t_{ij}^{β} and U_i^{β} by U_i^{α} and subtracting gives

$$U_i^{\beta} \frac{\partial t_{ij}^{\alpha}}{\partial x_j} - U_i^{\alpha} \frac{\partial t_{ij}^{\beta}}{\partial x_j} = \rho(\Omega_\beta^2 - \Omega_\alpha^2) U_i^{\alpha} U_i^{\beta} \qquad \text{A.4.}$$

Integrating over the body gives

$$\int_V \frac{\partial}{\partial x_j}\left[U_i^{\beta} t_{ij}^{\alpha} - U_i^{\alpha} t_{ij}^{\beta}\right] dV \qquad \text{A.5.}$$
$$- \int_V \left[t_{ij}^{\alpha} \frac{\partial U_i^{\beta}}{\partial x_j} - t_{ij}^{\beta} \frac{\partial U_i^{\alpha}}{\partial x_j} \right] dV$$
$$= \rho(\Omega_\beta^2 - \Omega_\alpha^2) \int_V U_i^{\alpha} U_i^{\beta} dV$$

Using the divergence theorem and the boundary condition makes the first integral on the L.H.S. of A.5 vanish, and the second also vanishes since

$$t_{ij}^{\alpha} \frac{\partial U_i^{\beta}}{\partial x_j} = t_{ij}^{\alpha} e_{ij}^{\beta} = t_{ij}^{\beta} e_{ij}^{\alpha} \qquad \text{A.6.}$$

Hence since $\Omega_\beta^2 \neq \Omega_\alpha^2$

$$\int_V U_i^{\alpha} U_i^{\beta} dV = 0 \qquad \text{A.7.}$$

Assume that the solutions U_i^{α} form a complete set, and consider the Hamiltonian, H, associated with an arbitrary displacement function

$$w_i = W_i e^{i\omega t} \qquad \text{A.8.}$$

satisfying

$$W_i = 0 \quad \text{on} \quad S_u \qquad \text{A.9.}$$

$$H = V - T = \frac{1}{2}\int_V t_{ij}^{*} e_{ij}^{*} dV - \frac{1}{2}\int_V \rho \dot{w}_i \dot{w}_i dV \qquad \text{A.10}$$

where t_{ij}^{*} and e_{ij}^{*} are the stresses and strain arising from w_i. Writing these as $\hat{t}_{ij} e^{i\omega t}$ and $\hat{e}_{ij} e^{i\omega t}$ and dropping the time dependent term

$$\hat{H} = \frac{1}{2}\int_V \hat{t}_{ij} \hat{e}_{ij} dV + \frac{1}{2}\omega^2 \int \rho W_i W_i dV \qquad \text{A.11.}$$

Now since u_i^α are a complete set

$$W_i = \sum_{\alpha=1}^{\infty} a^\alpha u_i^\alpha \qquad \text{A.12.}$$

$$\hat{e}_{ij} = \sum_{\alpha=1}^{\infty} a^\alpha e_{ij}^\alpha \qquad \hat{t}_{ij} = \sum_{\alpha=1}^{\infty} a^\alpha t_{ij}$$

and

$$\hat{H} = \frac{1}{2}\sum_{\alpha=1}^{\infty}\sum_{\beta=1}^{\infty}\int_V a^\alpha a^\beta t_{ij}^\alpha e_{ij}^\beta \, dV + \frac{1}{2}\omega^2 \sum_{\alpha=1}^{\infty}\int_V \rho a_\alpha^2 u_i^\alpha u_i^\alpha \, dV \qquad \text{A.13}$$

Also since

$$\int_V u_i^\alpha u_i^\beta = 0 \qquad \alpha \neq \beta \qquad \text{A.14.}$$

$$-\frac{1}{\rho\Omega_\alpha^2}\int_V \frac{\partial t_{ij}^\alpha}{\partial x_j} u_i^\beta = 0$$

or $$-\frac{1}{\rho\Omega_\alpha^2}\int_V \frac{\partial}{\partial x_j}(t_{ij}^\alpha u_i^\beta)\,dV + \frac{1}{\rho\Omega_\alpha^2}\int_V t_{ij}^\alpha e_{ij}^\beta \, dV = 0$$

Using the divergence theorem and the boundary condition, the first integral vanishes giving that

$$\int_V t_{ij}^\alpha e_{ij}^\beta \, dV = 0 \qquad \alpha \neq \beta \qquad \text{A.15.}$$

Hence

$$\hat{H} = \frac{1}{2}\sum_{\alpha=1}^{\infty} a_\alpha^2 \int_V (t_{ij}^\alpha e_{ij}^\alpha + \rho\omega^2 u_i^\alpha u_i^\alpha)\,dV \qquad \text{A.16.}$$

Also

$$\int_V \frac{\partial t_{ij}^{\alpha}}{\partial x_j} u_i^{\alpha} dv = \rho \Omega_{\alpha}^2 \int_V u_i^{\alpha} u_i^{\alpha} dv \tag{A.17.}$$

or

$$\int_V \frac{\partial}{\partial x_j} (t_{ij}^{\alpha} u_i^{\alpha}) dv - \int_V t_{ij}^{\alpha} e_{ij}^{\alpha} dv = \rho \Omega_{\alpha}^2 \int_V u_i^{\alpha} u_i^{\alpha} dv \tag{A.18.}$$

Divergence theorem and boundary conditions, imply the first integral vanishes, therefore

$$\int_V t_{ij}^{\alpha} e_{ij}^{\alpha} dv = -\rho \Omega_{\alpha}^2 \int_V u_i^{\alpha} u_i^{\alpha} dv \tag{A.19.}$$

Thus

$$\hat{H} = \frac{1}{2} \sum_{\alpha=1}^{\infty} a_{\alpha}^2 \rho (\omega^2 - \Omega_{\alpha}^2) \int_V u_i^{\alpha} u_i^{\alpha} dv \tag{A.20.}$$

and

$$\frac{\partial \hat{H}}{\partial a_{\alpha}} = a_{\alpha} \rho (\omega^2 - \Omega_{\alpha}^2) \int_V u_i^{\alpha} u_i^{\alpha} dv = 0$$

$$\text{if } a_{\alpha} = 0 \quad \text{or} \quad \omega^2 = \Omega_{\alpha}^2 \tag{A.21.}$$

Thus the stationary values of the Hamiltonian, $\hat{H}$, give the frequencies of free vibration $\omega^2 = \Omega_\alpha^2 \quad \alpha = 1, 2, \ldots$

This is the basis of the method.

APPENDIX 2 THICK TWISTED PLATE COMPUTER PROGRAM

MAIN

```
        PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
        PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
        PUBLIC CODP(1),CNDP(1),AA
        PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
        PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
        PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
        PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
        REAL KS
        INTEGER RN
        INTEGER RM,RX,FA,GA
        INTEGER R
'BEGIN'
'REAL'AX;
'INTEGER'I,J,K,H;
START'.'ININT(M);'IF'M'EQ'99'THEN'STOP;
SUPDV1(CO,CODP,15);SUPDV1(CN,CNDP,5);
SUPDV1(SS,SSDP,200);SUPDV1(SB,SBDP,200);
SUPDV1(JF,JFDP,3);SUPDV2(FA,FADP,2,30);
SUPDV2(GA,GADP,2,30);SUPDV1(MA,MADP,2);
SUPDV2(IX,IXDP,3,3);
  SUPDV1(NMA,NMDP,3);SUPDV1(NMB,NBDP,3);
SUPDW3(COEF,0,4,-8,5,0,4*M);
ININT(N);
NA=NB=NC=M*N-M;
NAB=NA+NB;R=NB;
NN=NA+NB+NC;
RN=NN;
NMA(1)=NA;NMA(2)=NB;NMA(3)=NC;
NMB(1)=0;NMB(2)=NA;NMB(3)=NA+NB;
INREAL(BI,AL,BL,ETA,HT,KS);
LINES(4);
SPACES(2);TEXT('('M'('5S')'N'('8S')'BI'('14S')'AL'('14S')'BL'('14S
')'ETA'('14S')'HT'('14S')'KS')');
LINES(2);IPRINT(M,2);SPACES(3);IPRINT(N,2);SPACES(3);
EPRINT(BI,4);SPACES(3);
EPRINT(AL,4);SPACES(3);EPRINT(BL,4);SPACES(3);EPRINT(ETA,4);
SPACES(3);EPRINT(HT,4);SPACES(3);EPRINT(KS,4);LINES(4);
AL=ATAN(0.5*AL*BI/BL);
HT=0.5*HT*BI/BL; AA=BL/BI;
BL=BI;
AI=1/AL;AL=1;BI=1/BL;BL=1;
HT2=HT*HT;ETA2=ETA*ETA;
ACC=1.0'E'-6;
CN(1)=1/3;CN(2)=1/105;CN(3)=1/35;
CO(1)=1-ETA;CO(2)=1/CO(1);CO(3)=ETA2*CO(2);CO(4)=1-2*ETA;
CO(5)=CO(2)*CO(2);CO(6)=ETA2*CO(5);CO(7)=CO(4)*CO(4);
CO(8)=ETA*CO(6);CO(9)=CO(6)*CO(4);CO(10)=ETA*CO(7)*CO(5);
CO(11)=CO(3)+CO(8);CO(12)=1+2*ETA;CO(13)=1-4*ETA;CO(14)=ETA*CO(2);
IX(1,1)=0;IX(2,2)=1;IX(3,3)=2;IX(1,2)=3;IX(1,3)=4;IX(2,3)=5;
'FOR'I=1,2'DO''FOR'J=I+1'STEP'1'UNTIL'3'DO'IX(J,I)=IX(I,J);
CLEAR;
M=2*M;
JF(1)=JF(2)=M;JF(3)=2*M;
CHAIN(2);
'END'
```

SESET

```
    PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
    PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
    PUBLIC CODP(1),CNDP(1),AA
    PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
    PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
    PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
    PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
    REAL KS
    INTEGER RN
    INTEGER RM,RX,FA,GA
    INTEGER R
'BEGIN'
'REAL'AX;
'REAL'CW;'INTEGER'I,J,K,H,Z,Y,IP,IC;
'INTEGER' KJ,W,JJ,KC;
'ARRAY'A,B(1'.'R,1'.'R), CE(1'.'112);
'ARRAY' SC(1'.'2*N+1), FD(0'.'M-1,0'.'3), FE(0'.'N;0'.'3);
'INTEGERARRAY' I1, I2, I3  (1'.'4);
'PROCEDURE' EVENAB (A1, A2, HI);
'INTEGERARRAY' A1, A2;
'INTEGER' HI;
'BEGIN'
'INTEGER' KI;
'PROCEDURE'RESET(P,Q);'INTEGER'P,Q;
'BEGIN'
'IF'0.5*P-P'/'2'GT'0.3'THEN''GOTO'SYM;
P=P'/'2;Q=Q'/'2;A(P,Q)=A(P,Q)+CW;'GOTO'REST;
SYM'.'P=P'/'2+1;Q=Q'/'2+1;B(P,Q)=B(P,Q)+CW;
REST'.''END';
'FOR'KI=1'STEP'1'UNTIL'3'DO''BEGIN'
NULL(A,R,R);
NULL(B,R,R);
'FOR' KJ = I1(KI) + 1 'STEP' 1 'UNTIL' I1 (KI + 1) 'DO'
'BEGIN'
ID = A1 (KJ,1); JD = A1(KJ,2); KD = A1(KJ,3); HD = A1(KJ,4);
JS = A1(KJ,5); JC = A1(KJ,6); IC = A1(KJ,7);
'IF' KI 'EQ' 3 'THEN' JJ = Ø 'ELSE' JJ = 1;
'FOR' I = Ø 'STEP' 1 'UNTIL' M - 1 - ID 'DO'
'FOR' J = Ø 'STEP' 1 'UNTIL' N - JD - JJ 'DO' 'BEGIN'
IE = I + ID + KD;
'IF 0-5* IE - IE'/'2 'GT' 0.3 'THEN' KC = 1 'ELSE' KC = Ø;
'FOR' K = KC 'STEP' 2 'UNTIL'  M - 1 - KD 'DO'
'FOR' H = Ø 'STEP' 1 'UNTIL' N - JD - JJ 'DO' BEGIN'
Z = I + K;  Y = J + H +1;
P=(J+JD)*M+I+ID+1-JF(KI);Q=(H+HD)*M+K+KD+1-JF(KI);
'IF'P'LT'1'OR'Q'LT'1'THEN''GOTO'FINE;
CW=CE(IC)*COEF(JS,JC,Z)*SC(Y)*FD(I,ID)*FD(K,KD)*FE(J,JD)*FE(H,HD);
'IF'P'EQ'Q'THEN'CW=2*CW;
'IF'P'GT'Q'THEN''BEGIN'IP=P;P=Q;Q=IP;'END';
RESET(P,Q);
FINE'.''END';
'END'; 'END',
'FOR' KJ = I2(KI) +1 'STEP' 1 'UNTIL' I2(KI+1) 'DO'
'BEGIN'
ID = A2 (KJ,1);JD = A2(KJ,2);
JS = A2(KJ,3); JC = A2(KJ,4); IC = A2(KJ,5);
'IF' KI 'EQ'3 'THEN' JJ = Ø 'ELSE' JJ = 1;
'FOR' I = Ø 'STEP' 1 'UNTIL' M - 1 - ID 'DO'
'FOR' J = Ø 'STEP' 1 'UNTIL' N - JD - JC 'DO' 'BEGIN'
'IF' 0.5* I - I '/' 2 'GT' 0.3 'THEN' KC = 1 'ELSE' KC = Ø;
'FOR' K = KC 'STEP' 2 'UNTIL' I 'DO'
'FOR' H = Ø 'STEP' 1 'UNTIL' N - JD - JJ 'DO' 'BEGIN'
Z = I + K;  Y= J + H + 1;
KD=ID;HD=JD;
P=(J+JD)*M+I+ID+1-JF(KI);Q=(H+HD)*M+K+KD+1-JF(KI);
'IF'P'LT'1'OR'Q'LT'1'THEN''GOTO'FINEE;
CW=2*CE(IC)*COEF(JS,JC,Z)*SC(Y)*FD(I,ID)*FD(K,KD)*FE(J,JD)*FE(H,HD);
RESET(P,Q);
FINEE'.''END';
'END'; 'END';
```

```
'IF'HI'EQ'0'THEN''FOR'I=1'STEP'1'UNTIL'NMA(KI)'DO''BEGIN'
'IF'KI'NE'1'THEN''BEGIN'
SS(I+NMB(KI))=1/SQRT(ABS(B(I,I)));SB(I+NMB(KI))=1/SQRT(ABS(A(I,I)));
'END';
'IF'KI'EQ'1'THEN''BEGIN'
SS(I+NMB(KI))=1/SQRT(ABS(A(I,I)));SB(I+NMB(KI))=1/SQRT(ABS(B(I,I)));
'END';'END';
'IF'KI'NE'1'THEN'WATD(B,1,6*HI+KI-1)'ELSE'WATD(B,1,11+6*HI+KI);
'IF'KI'NE'1'THEN'WATD(A,1,11+6*HI+KI)'ELSE'WATD(A,1,6*HI+KI-1);
'END';'END';
'PROCEDURE' ODDAB (A3,HI);
'INTEGERARRAY' A3;'INTEGER' HI;
'BEGIN' 'INTEGER KI;
'FOR'KI=1'STEP'1'UNTIL'3'DO''BEGIN'
NULL(A,R,R);
NULL(B,R,R);
'FOR KJ = I3 (KI) +1 'STEP' 1 'UNTIL' I3 (KI+1) 'DO'
'BEGIN'
ID = A3(KJ,1); JD = A3(KJ,2); KD = A3(KJ,3); HD = A3(KJ,4);
JS = A3(KJ,5); JC = A3(KJ,6); IC = A3(KJ,7);
'IF' KI 'EQ' 1 'THEN' JJ = 1 'ELSE' JJ = Ø;
'FOR' I = Ø 'STEP' 1 'UNTIL' M - 1 - ID 'DQ'
'FOR' J = Ø 'STEP' 1 'UNTIL' J - 1 - JD 'DO' 'BEGIN'
IE = I + ID + ID;
'IF 0.5* IE - IE '/' 2 'GT' 0.3 'THEN' KC = 1 'ELSE' KC = Ø;
'FOR' K = KC 'STEP' 2 'UNTIL' M - 1 - KD 'DO'
'FOR' H = Ø 'STEP' 1 'UNTIL' N - JD - JJ 'DO' 'BEGIN'
Z = I + K;  Y = J + H + 1;
P=2;Q=3;'IF'KI'NE'3'THEN'P=1;'IF'KI'EQ'1'THEN'Q=2;
P=(J+JD)*M+I+ID+1-JF(P);Q=(H+HD)*M+K+KD+1-JF(Q);
'IF'P'LT'1'OR'Q'LT'1'THEN''GOTO'FINI;
CW = CE(IC)* COEF(JS,JC,Z)* SC(Y)*
FD(I,ID)* FD(K,KD)* FE(J,JD)* FD(H,HD);
'IF'KI'NE'3'THEN''BEGIN'
'IF'0.5*P-P'/'2'GT'0.3'THEN''GOTO'S1;
P=P'/'2;Q=Q'/'2+1;A(P,Q)=A(P,Q)+CW;'GOTO'FINI;
S1'.'P=P'/'2+1;Q=Q'/'2;B(P,Q)=B(P,Q)+CW;'GOTO'FINI;
'END';
'IF'KI'EQ'3'THEN''BEGIN'
'IF'0.5*P-P'/'2'GT'0.3'THEN''GOTO'S2;
P=P'/'2;Q=Q'/'2;A(P,Q)=A(P,Q)+CW;'GOTO'FINI;
S2'.'P=P'/'2+1;Q=Q'/'2+1;B(P,Q)=B(P,Q)+CW;'END';
FINI'.''END';
'END'; 'END';
M = M '/' 2;
INTEG;
 M=2*M;
JA=JB=JC=M;
'FOR'I=1'STEP'1'UNTIL'2*N+1'DO'SC(I)=BL**I/I;
'FOR'J=0'STEP'1'UNTIL'M-1'DO'FD(J,0)=1;
'FOR'J=0'STEP'1'UNTIL'N'DO'FE(J,0)=1;
'FOR'I=1'STEP'1'UNTIL'3'DO''BEGIN'
'FOR'J=0'STEP'1'UNTIL'M=1'DO''BEGIN'
W=1;'FOR'K=1'STEP'1'UNTIL'I'DO'W=W*(J+K);FD(J,I)=W*AI**I;
'END';
'FOR'J=0'STEP'1'UNTIL'N'DO''BEGIN'
W=1;'FOR'K=1'STEP'1'UNTIL'I'DO'W=W*(J+K);FE(J,I)=W*BI**I;
'END';
'END';

 JM=JA;JN=JB;
 MAXARR(2500);
 COEFF1(CE);
```

```
I1 (1) = I2 (1) = I3 (1) = 0 ;
'BEGIN'
'INTEGERARRAY' A1 (1'.'53,1'.'7), A2 (1'.'47,1'.'5);
MAXARR (800);
RAFD (A1,4,Ø); RAFD (A2,4,1);
MAXARR (2500);
I1(2) = 9; I1(3) = 46; I1(4) = 53;
I2(2) = 13; I2(3) = 40; I2(4) = 47;
EVENAB (A1, A2, Ø);
'END';
COEFF2 (CE);
'BEGIN'
'INTEGERARRAY' A3(1'.' 109, 1'.'7);
MAXARR (800);
RAFD (A3,4,3);
MAXARR (2500);
I3(2) = 56; I3(3) = 76; I3(4) = 109;
ODDAB (A3, Ø);
'END';
COEFF3 (CE);
'BEGIN'
'INTEGERARRAY' A1(1'.'7, 1'.'7), A2(1'.'16, 1'.'5);
MAXARR (800);
RAFD (A1,4,4); RAFD (A2,4,5);
MAXARR (2500);
I1(2) = 1; I1(3) = 7; I1(4) = 7;
I2(2) = 4; I2(3) = 12; I2(4) = 16;
EVENAB (A1,A2,1);
'END';
COEFF4 (CE);
'BEGIN'
'INTEGERARRAY' A3(1'.'19, 1'.'7);
MAXARR (800);
RAFD (A3,4,6);
MAXARR (2500);
I3(2) = 10; I3(3) = 13; I3(4) = 19;
ODDAB (A3,1);
'END';
M=M '/' 2;
CHAIN (3);
'END'
```

INTEG

```
      PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
      PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
      PUBLIC CODP(1),CNDP(1),AA
      PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
      PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
      PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
      PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
      REAL KS
      INTEGER RN
      INTEGER RM,RX,FA,GA
      INTEGER R
'PROCEDURE'INTEG;
'BEGIN'
'REAL'AC;
'INTEGER'I,J,K;
'PROCEDURE'SIMP(J);'INTEGER'J;
'BEGIN'
'REALPROCEDURE'EVAL(XX);'VALUE'XX;'REAL'XX;
'BEGIN'
'REAL'XI,XJ,XK;
'IF'I'EQ'0'THEN'XI=1'ELSE'XI=SIN(XX/AI)**I;
'IF'J'EQ'0'THEN'XJ=1'ELSE''IF'J'LT'0'THEN'XJ=1/(COS(XX/AI)**ABS(J))
'ELSE'XJ=COS(XX/AI)**J;
'IF'K'EQ'0'THEN'XK=1'ELSE'XK=XX**K;
EVAL=XI*XJ*XK;
'END';
    'REAL PROCEDURE'SIMPIN(FNX,AA,BB,EE );
    'VALUE'AA,BB,EE;
    'REAL'AA,BB,EE;
    'REAL PROCEDURE' FNX;
    'COMMENT' (X)'.' 'VALUE' X'.' 'REAL' X;
     'BEGIN'
     'REAL' OLD,NEW,ENDS,EVENS,ODDS,H;
    'REAL'X,A,B,E;
    A=AA;B=BB;E=EE;
     H= B-A;
     'IF' H'EQ'0 'THEN'
      'BEGIN'
      NEW= 0;
      'GOTO' END;
      'END';
    ENDS= FNX(A); ENDS=ENDS+FNX(B);
    ODDS=(A+B)/2;
    ODDS= FNX(ODDS);
     NEW= ENDS+2*ODDS;
     EVENS= 0;
AGAIN'.'
     OLD= NEW;
     EVENS= EVENS+ODDS;
     H=0.5*H;
     ODDS= 0;
    'FOR' X=A+0.5*H 'STEP' H 'UNTIL' B 'DO'
     ODDS= ODDS+FNX(X);
     NEW= (ENDS+2*EVENS+4*ODDS)*H*(1/6);
     'IF' ABS(NEW-OLD)'GT'E 'THEN GOTO' AGAIN;
END'.'
    SIMPIN=NEW;
'END';
AC=0.5*ACC*AL*EVAL(AL);
COEF(I,J,K)=SIMPIN(EVAL,0.0,AL,AC);
'END';
M=2*M;
'FOR'I=0'STEP'1'UNTIL'4'DO''FOR'J=-8'STEP'1'UNTIL'5'DO'
'FOR'K=0'STEP'1'UNTIL'2*M'DO'COEF(I,J,K)=0;
'FOR'K=0'STEP'1'UNTIL'2*M'DO''BEGIN'
I=0;
'FOR'J=-7'STEP'1'UNTIL'5'DO'SIMP(J);
I=2;
'FOR'J=-7,-5,-3,-2,-1,0,1,3'DO'SIMP(J);
I=4;SIMP(-5);
I=1;
'FOR'J=-6'STEP'1'UNTIL'4'DO'SIMP(J);
I=3;
'FOR'J=-4,-2,-1'DO'SIMP(J);
'END';
M=M'/'2;
'END'
```

COEFFI

```
      PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
      PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
      PUBLIC CODP(1),CNDP(1),AA
      PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
      PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
      PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
      PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
      REAL KS
      INTEGER RN
      INTEGER RM,RX,FA,GA
      INTEGER R
'PROCEDURE'COEFF1(CE);'ARRAY'CE;
'BEGIN'
CE(1)=CO(1)-CO(3);
CE(2)=CN(1)*(5*CO(1)-6*CO(4)-7*CO(3)+6*CO(9)+2*CO(5)*CO(4)*CO(7)+
2*CO(8))*HT2;CE(3)=0.8*(ETA-CO(8)-CO(10))*HT2;CE(4)=-CE(3);
CE(5)=0.8*(2*ETA-CO(3)-CO(8)-CO(10))*HT2;CE(6)=0.6*CO(9)*KS;
CE(7)=-2*CN(2)*CO(9)*KS*HT2;CE(8)=34*CN(3)*CO(11)*HT2;
CE(9)=0.8*(2*ETA-2*CO(3)-ETA*CO(2)*CO(4))*HT2;CE(10)=2*CE(8);
CE(11)=8.5*CN(3)*CO(9)*HT2;CE(12)=-2*CE(11);CE(13)=-CE(12);
CE(14)=0.5*CO(4);CE(15)=CN(1)*(5.5*CO(4)-4*CO(1)+4*ETA)*HT2;
CE(16)=0.8*ETA*CO(2)*CO(4)*HT2;CE(17)=CE(6);CE(18)=CE(7);
CE(19)=CE(11);CE(20)=CE(12);CE(21)=CE(8);CE(22)=CE(11);
CE(23)=5*CO(1)+4*ETA-9*CO(3);
CE(24)=5.4*CO(9)*KS;
CE(25)=CN(1)*(-2*CO(12)+25*CO(1)-24*CO(13)+8*ETA+16*CO(4)-51*CO(3)-
4*ETA*CO(2)*CO(12)+6*CO(6)*CO(12)+2*CO(4)*CO(13)*CO(13)*CO(5)+
24*CO(6)*CO(13)+18*CO(8))*HT2;
CE(26)=0.8*(9*ETA-9*CO(8)-3*ETA*CO(4)*CO(13)*CO(5))*HT2;
CE(27)=76.5*CN(3)*CO(9)*HT2;CE(28)=-6*CN(3)*CO(9)*HT2*KS;
CE(29)=0.8*(15*ETA-6*CO(3)-9*CO(8)-3*ETA*CO(4)*CO(13)*CO(5))*HT2;
CE(30)=153*CN(3)*CO(9)*HT2;
CE(31)=2*ETA+4*CO(1)-6*CO(3);CE(32)=10.8*CO(9)*KS;
CE(36)=0.8*(18*ETA-18*CO(8)-3*ETA*CO(4)*CO(13)*CO(5))*HT2;
CE(33)=CE(36)+CN(1)*(2*ETA-12*CO(4)-42*CO(3)+2*CO(5)*CO(12)+20*CO(1)+
12*CO(8)+4*CO(5)*CO(7)*CO(13)-8*CO(13)+18*CO(9)+3*CO(6)*CO(13))*HT2-
2.4*CO(3)*HT2;
CE(34)=0.8*(3*ETA-9*CO(8))*HT2;
CE(36)=CE(36)+0.8*(-12*ETA+15*CO(8)-3*ETA*CO(7)*CO(5))*HT2+CE(30)+
(-36*CN(2)*CO(9)*KS*HT2);CE(37)=CE(30);
CE(38)=3.6*CO(9)*KS;CE(39)=-12*CN(2)*CO(9)*KS*HT2;
CE(40)=0.8*(7*ETA-CO(3)-6*CO(8)-2*ETA*CO(4)*CO(13)*CO(5))*HT2;
CE(41)=102*CN(3)*CO(9)*HT2;
CE(42)=0.8*(4*ETA-CO(3)-ETA*CO(4)*CO(13)*CO(5)-3*CO(8))*HT2;
CE(43)=51*CN(3)*CO(9)*HT2;
CE(44)=0.8*(5*ETA-2*CO(3)-3*CO(8)-ETA*CO(4)*CO(13)*CO(5))*HT2;
CE(45)=CE(43);CE(46)=0.5*CE(32);CE(47)=0.5*CO(4);
CE(50)=0.8*(6*ETA-6*CO(3)-3*ETA*CO(4)*CO(2))*HT2;
CE(51)=306*CN(3)*CO(11)*HT2;
CE(48)=-18*CN(2)*CO(9)*KS*HT2+CE(50)+4*CE(51);
CE(49)=CN(1)*(-4*CO(1)+5.5*CO(4)+4*ETA)*HT2+CE(51);
CE(50)=CE(50)+CE(51);
CE(51)=4*CE(51);CE(52)=CE(38);
CE(54)=24.68*CN(3)*CO(11)*HT2;
CE(53)=-12*CN(2)*CO(9)*KS*HT2+0.8*(8*ETA-8*CO(3)-4*ETA*CO(4)*CO(2))*
HT2+0.5*CE(54);
CE(56)=408*CN(3)*CO(11)*HT2;
CE(55)=0.8*(2*ETA-2*CO(3)-CO(4)*ETA*CO(2))*HT2+0.5*CE(56);
CE(57)=153*CN(3)*CO(9)*HT2;
CE(58)=0.8*(6*ETA-3*CO(3)-3*CO(8)-3*ETA*CO(7)*CO(5))*HT2;
CE(58)=CE(58)+CE(57);
CE(59)=CE(57);
CE(60)=2*CN(1)*CE(57);CE(61)=0.5*CE(60);CE(62)=CE(61);
CE(57)=0.5*CE(57);
CE(63)=CE(38);
CE(64)=CE(60)-12*CN(2)*CO(9)*KS*HT2+0.8*(ETA+CO(3)-2*CO(8)-
2*ETA*CO(7)*CO(5))*HT2;
CE(65)=CE(60);CE(66)=CE(61)+0.8*(ETA-CO(8)-ETA*CO(7)*CO(5))*HT2;
CE(67)=CE(61);CE(68)=CE(61)+0.8*(2*ETA-CO(3)-CO(8)-ETA*CO(7)*CO(5))*
HT2;CE(69)=CE(61);CE(70)=CO(1)-CO(3);CE(71)=CE(24);
CE(72)=2.4*(CO(3)-CO(8))*HT2-18*CN(2)*CO(9)*KS*HT2;
CE(74)=0.8*(6*ETA-6*CO(8)-3*ETA*CO(7)*CO(5))*HT2;
CE(75)=0.5*CE(30);
CE(70)=CE(70)+0.5*CE(30);
CE(73)=CN(1)*(5*CO(1)-6*CO(4)-7*CO(3)+2*CO(8)+2*CO(7)*CO(4)*CO(5)+
6*CO(9))*HT2+CE(74)+CE(75);
CE(74)=CE(74)-4.8*HT2*(ETA-CO(8))+2*CE(75);
CE(76)=0.6*CO(9)*KS;CE(77)=-2*CN(2)*CO(9)*KS*HT2;
CE(78)=34*CN(3)*CO(9)*HT2;CE(79)=CE(80)=CE(78);
CE(81)=0.25*CE(78);CE(82)=0.5*CE(78);CE(83)=CE(76);
CE(84)=CE(77);CE(85)=544*CN(3)*CO(11)*HT2;CE(86)=0.5*CE(85);
CE(87)=CE(81);CE(88)=34*CN(3)*CO(11)*HT2;CE(89)=2*CO(4);
CE(90)=CN(1)*HT2*(18*ETA-14*CO(1)+18*CO(4));
CE(91)=CN(1)*HT2*(6*CO(1)-2*ETA-2*CO(4));
CE(92)=-CE(91);CE(93)=CE(91);CE(94)=2*CN(1)*CO(4)*HT2;
CE(95)=-2*CE(94);CE(96)=CN(1)*CO(1)*HT2;
CE(97)=CN(1)*HT2*(2*CO(1)-4*ETA);CE(98)=2*CN(1)*ETA*HT2;
CE(100)=CN(1)*HT2*(5*CO(1)-4*ETA);CE(101)=CN(1)*HT2*(2*ETA-4*CO(1));
CE(102)=CE(96);CE(103)=CE(94);
'END'
```

COEFF2

```
        PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
        PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
        PUBLIC CODP(1),CNDP(1),AA
        PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
        PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
        PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
        PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
        REAL KS
        INTEGER RN
        INTEGER RM,RX,FA,GA
        INTEGER R
'PROCEDURE'COEFF2(COD);'ARRAY'COD;
'BEGIN'
COD(1)=2*CO(1)+4*ETA-6*CO(3);
COD(2)=CN(1)*HT2*(10*CO(1)-2*CO(12)-6*CO(4)-8*CO(13)+16*ETA-38*CO(3)+
4*ETA*CO(4)*CO(2)+18*CO(9)+2*CO(6)*CO(12)+4*CO(7)*CO(13)*CO(5)+
8*CO(6)*CO(13)+12*CO(8));
COD(3)=0.8*HT2*(9*ETA-6*CO(3)-3*CO(8)-3*ETA*CO(7)*CO(5));
COD(5)=2*ETA-2*CO(3);
COD(4)=0.8*HT2*(3*ETA-3*CO(8)-3*ETA*CO(7)*CO(5));
COD(6)=CN(1)*HT2*(6*ETA-8*CO(4)+4*ETA*CO(4)*CO(2)-10*CO(3)+4*CO(8)+
12*CO(9)+4*CO(7)*CO(4)*CO(5))+0.8*HT2*(6*ETA-3*CO(7)*ETA*CO(5)-
6*CO(8));
COD(7)=2.4*HT2*(ETA-CO(8));
COD(8)=0.8*HT2*(6*ETA-6*CO(3)-3*ETA*CO(7)*CO(5));
COD(9)=0.8*HT2*(5*ETA-3*CO(3)-2*CO(8)-2*ETA*CO(7)*CO(5));
COD(10)=0.8*HT2*(2*ETA-CO(3)-CO(8)-ETA*CO(7)*CO(5));
COD(11)=0.8*HT2*(ETA-CO(8)-ETA*CO(7)*CO(5));
COD(12)=3.6*KS*CO(9);COD(13)=-4*CN(3)*CO(9)*KS*HT2;
COD(15)=408*CN(3)*CO(11)*HT2;
COD(14)=0.5*COD(15)+0.8*HT2*(2*ETA-2*CO(3)-ETA*CO(4)*CO(2));
COD(16)=CN(1)*COD(12);COD(17)=CN(1)*COD(13);
COD(18)=272*CN(3)*CO(11)*HT2;COD(19)=0.25*COD(18);
COD(20)=0.8*HT2*(5*ETA-2*CO(3)-3*CO(8)-ETA*CO(4)*CO(13)*CO(5));
COD(21)=51*CN(3)*CO(9)*HT2;COD(22)=COD(21);
COD(23)=0.8*HT2*(2*ETA-CO(11)-ETA*CO(7)*CO(5))+COD(21);
COD(24)=COD(21);COD(25)=2*CN(1)*COD(21);COD(26)=0.5*COD(25);
COD(27)=COD(26);COD(28)=CO(4);
COD(30)=1.6*HT2*(6*ETA-6*CO(3)-3*ETA*CO(4)*CO(2));
COD(29)=0.5*COD(30)+CN(1)*HT2*(-16*CO(1)+16*ETA+11*CO(4));
COD(31)=3.2*HT2*(2*ETA-2*CO(3)-ETA*CO(4)*CO(2));
COD(32)=0.25*COD(31);COD(33)=COD(12);COD(34)=COD(13);
COD(35)=0.8*HT2*(-5*ETA+2*CO(3)+3*CO(8)+ETA*CO(4)*CO(2)*CO(2)*CO(13));
COD(36)=-51*CN(3)*CO(9)*HT2;COD(37)=COD(36);COD(38)=COD(12);
COD(39)=COD(13);COD(40)=0.8*HT2*(-2*ETA+CO(11)+ETA*CO(7)*CO(5))+
COD(36);
COD(39)=COD(39)+COD(40);
COD(41)=COD(36);COD(42)=CN(1)*COD(12);COD(43)=CN(1)*COD(13);
COD(44)=2*CN(1)*COD(36);COD(45)=0.5*COD(44);COD(46)=COD(45);
COD(48)=400*CN(3)*CO(11)*HT2;
COD(47)=0.8*HT2*(2*ETA-2*CO(3)-ETA*CO(4)*CO(2))+0.5*COD(48);
COD(49)=272*CN(3)*CO(11)*HT2;COD(50)=0.25*COD(49);
COD(51)=0.8*HT2*(4*ETA-CO(3)-ETA*CO(4)*CO(13)*CO(5)-3*CO(8));
COD(52)=-COD(36);COD(53)=COD(52);
COD(54)=0.8*HT2*(ETA-CO(8)-ETA*CO(7)*CO(5))+COD(52);
COD(55)=COD(52);COD(56)=-COD(44);COD(57)=0.5*COD(56);
COD(58)=COD(57);
COD(59)=CN(1)*HT2*(-6*CO(1)+2*ETA+4*CO(3)+4*CO(7)*CO(2));
COD(60)=-COD(59);
COD(61)=0.8*HT2*(6*ETA-2*CO(3)-2*ETA*CO(4)*CO(2));
COD(62)=0.8*HT2*(ETA+CO(3));COD(63)=-COD(62);COD(64)=COD(62);
COD(65)=-0.8*HT2*ETA*CO(4)*CO(2);COD(66)=-COD(65);COD(67)=2*CO(4);
COD(69)=CN(1)*HT2*(4*CO(1)-4*ETA-CO(4));
COD(70)=2*CN(1)*HT2*CO(4)+COD(69);COD(71)=-CN(1)*HT2*CO(4);
COD(72)=COD(66);COD(73)=COD(65);
COD(74)=0.8*HT2*(6*ETA-2*CO(3)-2*ETA*CO(2)*CO(4));
COD(75)=0.8*HT2*(ETA+CO(3));COD(76)=-COD(75);COD(77)=COD(75);
COD(78)=COD(65);COD(79)=COD(66);
COD(80)=CN(1)*HT2*(-18*CO(1)+6*ETA+12*CO(3)+4*CO(4)*CO(13)*CO(2));
COD(81)=3*COD(65);COD(82)=-COD(80);COD(83)=-COD(81);
COD(84)=2*CO(4);COD(86)=12*CN(1)*HT2*(3*ETA-CO(3)-ETA*CO(4)*CO(2));
COD(85)=0.5*COD(86);COD(88)=4.8*HT2*(ETA+CO(3));
COD(90)=-COD(88);COD(87)=-CN(1)*HT2*CO(4)+0.5*COD(88);
COD(89)=CN(1)*HT2*(8*ETA+CO(4)-8*CO(1))+0.5*COD(90);
COD(91)=CN(1)*HT2*(4*CO(1)-CO(4)-4*ETA)+0.5*COD(88);
COD(92)=COD(88);COD(93)=COD(81);COD(94)=COD(83);
COD(95)=COD(81)+2*CN(1)*HT2*(ETA+2*CO(3)+2*CO(7)*CO(2)-3*CO(1));
COD(96)=COD(81);COD(97)=-COD(95);COD(98)=COD(83);COD(99)=2*COD(65);
COD(100)=-COD(99);COD(101)=COD(65);COD(102)=COD(66);
COD(103)=6.4*HT2*(3*ETA-CO(4)*CO(14)-CO(3));
COD(104)=3.2*HT2*(ETA+CO(3));COD(105)=-COD(104);COD(106)=COD(104);
COD(107)=COD(65);COD(108)=COD(66);COD(109)=0.25*COD(103);
COD(110)=0.25*COD(104);COD(111)=-COD(110);COD(112)=COD(110);
COD(13)=COD(13)+COD(14);
'END'
```

COEFF3

```
      PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
      PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
      PUBLIC CODP(1),CNDP(1),AA
      PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
      PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
      PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
      PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
      REAL KS
      INTEGER RN
      INTEGER RM,RX,FA,GA
      INTEGER R
'PROCEDURE'COEFF3(CE);'ARRAY'CE;
'BEGIN'
CE(1)=1.0;CE(2)=-0.8*HT2*CO(14);CE(3)=CN(1)*HT2*CO(6);
CE(4)=17*CN(3)*HT2*CO(6);CE(5)=CE(4);CE(6)=1+153*CN(3)*HT2*CO(6);
CE(7)=3*HT2*CO(6);CE(8)=-2.4*HT2*CO(14);
CE(9)=2*HT2*CO(6)-2.4*HT2*CO(14);CE(10)=306*CN(3)*HT2*CO(6);
CE(11)=CE(2);CE(12)=102*CN(3)*HT2*CO(6);
CE(13)=0.5*CE(10);CE(14)=CE(12);
CE(15)=CE(3);CE(16)=CE(13);CE(17)=CE(12);CE(18)=CE(4);CE(19)=CE(4);
CE(20)=1.0;CE(21)=-CN(1)*HT2;CE(22)=CE(23)=-CE(2);
'END'
```

COEFF4

```
      PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
      PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
      PUBLIC CODP(1),CNDP(1),AA
      PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
      PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
      PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
      PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
      REAL KS
      INTEGER RN
      INTEGER RM,RX,FA,GA
      INTEGER R
'PROCEDURE'COEFF4(COD);
'ARRAY'COD;
'BEGIN'
COD(1)=-2.4*HT2*CO(14);COD(2)=CN(1)*COD(1);COD(3)=2*HT2*CO(6);
COD(4)=CN(1)*COD(3);COD(5)=102*CN(3)*HT2*CO(6);
COD(6)=CN(1)*COD(5);COD(7)=COD(2);COD(8)=COD(5);COD(9)=COD(5);
COD(10)=COD(6);COD(11)=-COD(2);COD(12)=COD(11);COD(13)=-2*CN(1)*HT2;
COD(14)=COD(13);COD(15)=-COD(1);COD(16)=COD(15);COD(17)=COD(15);

COD(18)=COD(19)=COD(11);
'END'
```

BCSUB

```
    PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
    PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
    PUBLIC CODP(1),CNDP(1),AA
    PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
    PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
    PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
    PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
    REAL KS
    INTEGER RN
    INTEGER RM,RX,FA,GA
    INTEGER R
'BEGIN''INTEGER'I,J,K,H,HI;
'INTEGER'KY;
'ARRAY'A(1':'NN,1':'NN),B(1':'R,1':'R);
'PROCEDURE'FULL(EF,R);'ARRAY'EF;'INTEGER'R;
'BEGIN''INTEGER'I,J;
'FOR'I=1'STEP'1'UNTIL'R-1'DO''FOR'J=I'STEP'1'UNTIL'R'DO'
EF(J,I)=EF(I,J);
'END';
MAXARR(2500);
'FOR'KY=0,2'DO''BEGIN'
NMB(1)=0;NMB(2)=NA;NMB(3)=NA+NB;
'FOR'HI=0'STEP'1'UNTIL'1'DO''BEGIN'
'FOR'I=1'STEP'1'UNTIL'3'DO'
'FOR'J=I'STEP'1'UNTIL'3'DO''BEGIN'
'IF'I'EQ'J'THEN'K=I-1'ELSE'K=I+J;
RAFD(B,1,6*KY+6*HI+K);
'FOR'K=1'STEP'1'UNTIL'NMA(I)'DO''FOR'H=1'STEP'1'UNTIL'NMA(J)'DO'
A(K+NMB(I),H+NMB(J))=B(K,H);
'END';
FULL(A,NN);
MAXARR(5000);
WATD(A,2,KY+HI+2);
MAXARR(2500);
'END';
'END';CHAIN(5);
'END'
```

SOLVE

```
      PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
      PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
      PUBLIC CODP(1),CNDP(1), AA
      PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
      PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
      PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
      PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
      REAL KS
      INTEGER RN
      INTEGER RM,RX,FA,GA
      INTEGER R
'BEGIN'
'REAL' SC,E,RO,PI,G;
'INTEGER' NS,KY,RD;
'ARRAY' EV(1'.'RN), A(1'.'RN,1'.'RN);
NS = N
INREAL (E,RO);
ININT (RD);
PI = 3.141592654;
G  = 386.4;
SC = 1.0/(2*PI*AA)* SQRT (E*G/ (RO* (1-2*ETA)* (1+ETA)));
'FOR' KY = 0,2 'DO' 'BEGIN'
SOLUTION (A,EV,SC,NS,KY);
'BEGIN''REAL'SUM;'INTEGER'I,J,K,H,NX,NY;
'ARRAY'D(1'.'RD,1'.'NN);
'PROCEDURE'EVALU(A,X);'ARRAY'A;'INTEGER'X;
'BEGIN''ARRAY'A1(1'.'NX+1,1'.'NY+1);
'REAL'AX,BX;
'INTEGER'K;
'FOR'I=0'STEP'1'UNTIL'NX'DO''FOR'J=0'STEP'1'UNTIL'NY'DO''BEGIN'
AX=I*AL/NX;BX=J*BL/NY;
SUM=0;
'IF'KY'EQ'0'AND'X'NE'1'OR'KY'EQ'2'AND'X'EQ'1'THEN''BEGIN'
'FOR'H=1'STEP'1'UNTIL'NS'DO'SUM=SUM+A(1,H)*BX**H;
'FOR'K=2'STEP'1'UNTIL'M'DO''FOR'H=1'STEP'1'UNTIL'NS'DO'
SUM=SUM+A(K,H)*AX**(2*K-2)*BX**H;
'END';
'IF'KY'EQ'2'AND'X'NE'1'OR'KY'EQ'0'AND'X'EQ'1'THEN''BEGIN'
'FOR'K=1'STEP'1'UNTIL'M'DO''FOR'H=1'STEP'1'UNTIL'NS'DO'
SUM=SUM+A(K,H)*AX**(2*K-1)*BX**H;
'END';
A1(I+1,J+1)=SUM;
'END';
LINES(4);EMATPR(A1,NX+1,NY+1,4);
'END';
'PROCEDURE'SCALE(D);'ARRAY'D;
'BEGIN''REAL'SUM,AB;
'FOR'I=1'STEP'1'UNTIL'RD'DO''BEGIN'
SUM=0;
'FOR'J=1'STEP'1'UNTIL'NN'DO''BEGIN'
AB=ABS(D(I,J));
'IF'AB'GT'SUM'THEN'SUM=AB;
'END';
SUM=1/SUM;
'FOR'J=1'STEP'1'UNTIL'NN'DO'D(I,J)=D(I,J)*SUM;
'END';'END';
TRANSF (A,D,RD,NN,KY);
SCALE(D);
ININT(NX,NY);
LINES(1);
'IF'KY'EQ'0'THEN'TEXT('('SYM')')'ELSE'TEXT('('ANTISYM')');
SPACES(1);TEXT('('FREQ'('S')'AND'('S')'MODES')');
'BEGIN''ARRAY'A,B,C(1'.'M,1'.'NS);
'FOR'K=1'STEP'1'UNTIL'RD'DO''BEGIN'
'FOR'I=1'STEP'1'UNTIL'M'DO''FOR'J=1'STEP'1'UNTIL'NS'DO''BEGIN'
'IF'J'GE'NS'THEN''GOTO'U3;
A(I,J)=D(K,(J-1)*M+I);
B(I,J)=D(K,NMB(2)+(J-1)*M+I);
U3'.'
C(I,J)=D(K,NMB(3)+(J-2)*M+I);
'END';
'FOR'I=1'STEP'1'UNTIL'M'DO'A(I,NS)=B(I,NS)=0;
```

```
'FOR'I=1'STEP'1'UNTIL'M'DO'C(I,1)=0;
LINES(1);
TEXT('('FREQ'('S')'NUMBER')');SPACES(1);PRINT(K,2);
SPACES(4);EPRINT(EV(K),4);
EVALU(A,1);
EVALU(B,2);
EVALU(C,3);
LINES(6);
'END';
'END';'END';
'END';
CHAIN(1);
'END'
```

INVLOW

```
'PROCEDURE'INVLOW(A,L,N);'ARRAY'A,L;'INTEGER'N;
'BEGIN''REAL'SUM;'INTEGER'I,J,K;NULL(A,N,N);
'FOR'I=1'STEP'1'UNTIL'N'DO'A(I,I)=1/L(I,I);
'FOR'I=2'STEP'1'UNTIL'N'DO''FOR'J=1'STEP'1'UNTIL'I-1'DO''BEGIN'
SUM=0;'FOR'K=J'STEP'1'UNTIL'I-1'DO'SUM=SUM+L(I,K)*A(K,J);
A(I,J)=-SUM*A(I,I);
'END';
'END'
```

MATML

```
'PROCEDURE'MATML(L,A,N);'ARRAY'L,A;'INTEGER'N;
'BEGIN''INTEGER'I,J,K;'ARRAY'EE(1':'N);
'FOR'I=1'STEP'1'UNTIL'N'DO''BEGIN'
'FOR'J=1'STEP'1'UNTIL'N'DO''BEGIN'
EE(J)=0;'FOR'K=1'STEP'1'UNTIL'I'DO'EE(J)=EE(J)+A(I,K)*L(K,J);'END';
'FOR'J=1'STEP'1'UNTIL'N'DO'A(I,J)=EE(J);'END';
'END'
```

TRAML

```
'PROCEDURE'TRAML(L,A,N);'ARRAY'L,A;'INTEGER'N;
'BEGIN''INTEGER'I,J,K;'ARRAY'EE(1':'N);
'FOR'I=1'STEP'1'UNTIL'N'DO''BEGIN'
'FOR'J=1'STEP'1'UNTIL'N'DO''BEGIN'
EE(J)=0;'FOR'K=1'STEP'1'UNTIL'J'DO'EE(J)=EE(J)+L(I,K)*A(J,K);'END';
'FOR'J=1'STEP'1'UNTIL'N'DO'L(I,J)=EE(J);
'END';
'END'
```

HOUSEHOLDER SOLUTION

```
      PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
      PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
      PUBLIC CODP(1),CNDP(1)
      PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
      PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
      PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
      PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
      REAL KS
      INTEGER RN
      INTEGER RM,RX,FA,GA
      INTEGER R
'PROCEDURE' SOLUTION (A,EV,SC,NS,KY.);
'ARRAY' A,EV; 'REAL' SC; 'INTEGER' NS,KY;
'BEGIN'
'INTEGER' N,I,J,K;
'ARRAY' L(1'.'RN, 1'.'RN);
N = RN
NULL(L,N,N);
RAFD(A,2,KY+2);
'FOR'I=1'STEP'1'UNTIL'N'DO'JS(I)=SQRT(ABS(1/A(I,I)));
'FOR'I=1'STEP'1'UNTIL'N'DO''FOR'J=1'STEP'1'UNTIL'N'DO'
A(I,J)=A(I,J)*JS(I)*JS(J);
L(1,1)=SQRT(A(1,1));'FOR'I=2'STEP'1'UNTIL'N'DO'L(I,1)=A(I,1)/L(1,1);
'FOR'I=2'STEP'1'UNTIL'N'DO''FOR'J=1'STEP'1'UNTIL'N'DO''BEGIN'
'IF'I'NE'J'THEN''GOTO'OUT;'FOR'K=1'STEP'1'UNTIL'I-1'DO'
L(I,I)=L(I,I)+L(I,K)*L(I,K);L(I,I)=SQRT(A(I,I)-L(I,I));
'GOTO'EXIT;
OUT'.''FOR'K=1'STEP'1'UNTIL'I-1'DO'L(J,I)=L(J,I)+L(I,K)*L(J,K);
L(J,I)=(A(J,I)-L(J,I))/L(I,I);
EXIT'.''END';
INVLOW(A,L,N);
RAFD(L,2,KY+3);
'FOR'I=1'STEP'1'UNTIL'N'DO''FOR'J=1'STEP'1'UNTIL'N'DO'
L(I,J)=L(I,J)*JS(I)*JS(J);
TRAML(L,A,N);WATD(A,2,2);MATML(L,A,N);
'FOR'I=1'STEP'1'UNTIL'N-1'DO''FOR'J=I+1'STEP'1'UNTIL'N'DO'
A(J,I)=A(I,J)=0.5*(A(I,J)+A(J,I));
'END';
MONITO;
ENJP'.''BEGIN''INTEGER'I,J,K,H;'ARRAY'EE,EN(1'.'N);
K=1;
HOUSEH(A,EE,N,K);
'IF'K'EQ'9"'THEN'TEXT('(''('N')'JACOBI')');
'FOR'K=1'STEP'1'UNTIL'N'DO'EE(K)=1/EE(K);
LINES(5);EVECPR(EE,N,4);LINES(5);
'FOR'I=1'STEP'1'UNTIL'N'DO'
'IF'EE(I)'GT'0'THEN'EV(I)=SC*SQRT(EE(I));
'BEGIN'
'REAL'SUM;'ARRAY'B(1'.'N,1'.'N);
RAFD(B,2,2);
'FOR'I=1'STEP'1'UNTIL'N'DO''BEGIN''FOR'J=1'STEP'1'UNTIL'N'DO'
EE(J)=A(I,J);
'FOR'K=1'STEP'1'UNTIL'N'DO''BEGIN'
SUM=0;'FOR'H=K'STEP'1'UNTIL'N'DO'SUM=SUM+B(H,K)*EE(H);
EN(K)=SUM;
'END';
'FOR'J=1'STEP'1'UNTIL'N'DO'A(I,J)=EN(J);'END';
'END';
'FOR'I=1'STEP'1'UNTIL'RD'DO''FOR'J=1'STEP'1'UNTIL'RN'DO'
A(I,J)=JS(J)*A(I,J);
'END';
'END'
```

HQR SOLUTION

```
      PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
      PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
      PUBLIC CODP(1),CNDP(1),AA
      PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
      PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
      PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
      PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
      REAL KS
      INTEGER RN
      INTEGER RM,RX,FA,GA
      INTEGER R
'PROCEDURE' SOLUTION (A,EV,SC,NS,KY);
'ARRAY' A,EV; 'REAL' SC; 'INTEGER' NS,KY;
'BEGIN'
'INTEGER' HX,I,J,K,H,N;
'ARRAY' SA(1'.'200), B(1'.'RN,1'.'RN), ER, EI(1'.'RN);
'PROCEDURE'VECCOP(SA,SS,X);'ARRAY'SA,SS;'INTEGER'X;
'BEGIN''INTEGER'I;
'FOR'I=1'STEP'1'UNTIL'X'DO'SA(I)=SS(I);'END';
HX = KY '/'2;
'IF' HX 'EQ' Ø 'THEN' VECCOP (SA,SS,NN)'ELSE' VECCOP (SA,SB,NN);
N=RN;
RAFD(A,2.2+2*HX);RAFD(B,2,3+2*HX);
  MATDIV(B,A,RN,RN);
WATD(B,2,2);
NSEICB(B,A,2,2,RN,ER,EI);
EVECPR(ER,RN,4);EVECPR(EI,RN,4);
EMATPR(B,RN,RN,4);EMATPR(A,RN,RN,4);
MATCOP(A,B,RN,RN);
'FOR'I=I'STEP'1'UNTIL'RN'DO'
'IF'ER(I)'GT'O'THEN'
EV(I)=SC/SQRT(ER(I));
'END'
```

TRANSF No Reduction

```
'PROCEDURE' TRANSF (A,D,RD,NN,KY);
'ARRAY' A,D; 'INTEGER' RD,NN,KY;
'BEGIN' 'INTEGER' I,J;
'FOR' I = 1 'STEP' 1 'UNTIL' RD 'DO'
'FOR' J = 1 'STEP' 1 'UNTIL' NN 'DO'
D(I,J) = A(I,J);
'END'
```

TRANSF With Reduction

```
    PUBLIC AL,BL,AI,BI,ETA,ETA2,HT,HT2,ACC,KS,M,N,NA,NB,NC,NN
    PUBLIC JA,JB,JC,R,JM,JN,NAB,CO(15),CN(5),COEF(5,14,21)
    PUBLIC CODP(1),CNDP(1)
    PUBLIC SS(200),SB(200),SSDP(1),SBDP(1)
    PUBLIC JF(3),JFDP(1),RM,RX,FA(2,30),GA(2,30),FADP(2),GADP(2)
    PUBLIC MA(2),MADP(1),IX(3,3),IXDP(2)
    PUBLIC RN,NMA(3),NMB(3),NMDP(1),NBDP(1)
    REAL KS
    INTEGER RN
    INTEGER RM,RX,FA,GA
    INTEGER R
'PROCEDURE' TRANSF ( A,D,RD,NN,KY);
'ARRAY' A,D; 'INTEGER' RD,NN,KY;
'BEGIN' 'INTEGER' HX,I,J,K; 'REAL'SUM;
'ARRAY' B(1'.'RN,1'.'RN);
'PROCEDURE'SHIFTBK(D);'ARRAY'D;
'BEGIN''INTEGER'I,J,K,H,CF,CG,CH,CJ,CK;
'ARRAY'E(1'.'RD,1'.'RM),F(1'.'RD,1'.'RN);
'FOR'I=1'STEP'1'UNTIL'RD'DO''BEGIN'
'FOR'J=1'STEP'1'UNTIL'RM'DO'E(I,J)=D(I,RN+J);
'FOR'J=1'STEP'1'UNTIL'RN'DO'F(I,J)=D(I,J);
CF=CG=CH=CJ=CK=0;
'FOR'K=1'STEP'1'UNTIL'MA(HX+1)'DO''BEGIN'
CH=CF+FA(HX+1,K) ;CJ=CG+G A(HX+1,K);CK=CH+GA(HX+1,K);
'FOR'J=CF+1'STEP'1'UNTIL'CH'DO'D(I,J)=F(I,J-CG);
'FOR'J=CH+1'STEP'1'UNTIL'CK'DO'D(I,J)=E(I,J-CH+CG);
CF=CK;CG=CJ;
'END';'END';
'END';
 MAXARR(2500);
 RAFD(B,2,H X);
 'FOR'I=1'STEP'1'UNTIL'RD'DO''FOR'J=1'STEP'1'UNTIL'RN'DO'D(I,J)=A(I,J);
 'FOR'K=1'STEP'1'UNTIL'RD'DO''BEGIN'
 'FOR'I=RN+1'STEP'1'UNTIL'NN'DO''BEGIN'
 SUM=0;'FOR'J=1'STEP'1'UNTIL'RN'DO'SUM=SUM+B(I-RN,J)*A(K,J);'END';
 D(K,I)=SUM;
 'END';
 'END';
 SHIFTBK(D); -
 'FOR'J=1'STEP'1'UNTIL' RD'DO''FOR'I=1'STEP'1'UNTIL'NN'DO'
 D(J,I)=D(J,I)*SA(I);
 'END'
```

APPENDIX 3 THICK TWISTED PLATE ENERGY DATA

The sets of numbers below define terms in the energy expressions for the shell of the form:

$$\iint CE(IC) \sin^{JS}\alpha \cos^{JC}\alpha \frac{\partial^{ID+JD} F}{\partial\alpha^{ID}\partial\beta^{JD}} \frac{\partial^{KD+HD} G}{\partial\alpha^{KD}\partial\beta^{HD}} \, d\alpha \, d\beta$$

Where F and G are A,B or C, and the CE(IC) terms are defined in the routines COEFF1,2,3,4 as follows:

(a) COEFF1 AA,BB,CC Strain Energy Terms

(b) COEFF2 AB,AC,BC Strain Energy Terms

(c) COEFF3 AA,BB,CC Kinetic Energy Terms

(d) COEFF4 AB,AC,BC Kinetic Energy Terms

and by the CO and CN arrays defined in MAIN.

The true energy terms can be obtained by multiplying these terms by the following scale factors:

(a) $\dfrac{Ea}{2(1-2\eta)(1+\eta)}$ for the strain energy

(b) $\dfrac{\rho a^3}{2g}$ for the kinetic energy

Where E is Young's modulus, η Poisson's ratio, ρ the density and a the length parameter defined such that the length of the shell is a times the total twist angle BI.

1. AA STRAIN ENERGY TERMS

(a) TERMS FOR WHICH ID = KD AND JD = HD

ID	JD	JS	JC	IC
0	1	0	-3	1
0	1	0	1	2
0	2	0	-5	6
0	2	0	-1	7
0	2	2	-3	8
0	3	0	-3	11
1	0	0	-1	14
1	0	0	3	15
1	1	0	-3	17
1	1	0	1	18
1	1	2	-1	19
1	2	0	-1	21
2	1	0	1	22

(b) OTHER TERMS

ID	JD	KD	HD	JS	JC	IC
0	1	1	1	1	0	4
0	2	1	0	1	0	9
0	2	1	2	1	-2	10
0	3	1	1	1	-2	12
1	1	2	1	1	0	20
0	1	0	3	0	-1	3
0	1	2	1	0	1	5
0	3	2	1	0	-1	13
1	0	1	2	0	1	16

2. BB STRAIN ENERGY TERMS

(a) TERMS FOR WHICH ID = KD AND JD = HD

ID	JD	JS	JC	IC
0	0	2	-5	23
0	0	0	-7	24
0	0	2	-1	25
0	0	2	-3	26
0	0	2	-5	27
0	0	0	-3	28
0	1	2	-7	46
0	1	0	-5	47
0	1	2	-3	48
0	1	0	-1	49
0	1	0	-3	50
0	1	4	-5	51
0	2	2	-5	57
1	0	0	-3	70
1	0	2	-5	71
1	0	2	-1	72
1	0	0	1	73
1	0	0	-1	74
2	0	0	-3	76
2	0	0	1	77

ID	JD	JS	JC	IC
2	0	2	-1	78
3	0	0	1	81
1	1	0	-5	83
1	1	0	-1	84
1	1	2	-3	85
1	2	0	-3	87
2	1	0	-1	88

(b) OTHER TERMS

ID	JD	KD	HD	JS	JC	IC
0	0	1	0	1	-4	31
0	0	1	0	1	-6	32
0	0	1	0	1	0	33
0	0	1	0	3	-2	34
0	0	1	0	1	-2	36
0	0	1	0	1	-4	37
0	0	3	0	1	0	42
0	0	3	0	1	-2	43
0	0	1	2	1	-2	44
0	0	1	2	1	-4	45
0	1	1	1	1	-6	52
0	1	1	1	1	-2	53
0	1	1	1	3	-4	54
0	2	1	0	1	-2	58
0	2	1	0	1	-4	59
0	2	3	0	1	-2	61
0	2	1	2	1	-4	62
1	0	2	0	1	-4	63
1	0	2	0	1	0	64
1	0	2	0	1	-2	65
2	0	3	0	1	0	79
2	0	1	2	1	-2	80
1	1	2	1	1	-2	86
0	0	0	2	2	-3	29
0	0	0	2	2	-5	30
0	0	2	0	0	-5	38
0	0	2	0	0	-1	39
0	0	2	0	2	-1	40
0	0	2	0	2	-3	41
0	1	2	1	0	-1	55
0	1	2	1	2	-3	56
0	2	2	0	2	-3	60
1	0	3	0	0	1	66
1	0	3	0	0	-1	67
1	0	1	2	0	-1	68
1	0	1	2	0	-3	69
3	0	1	2	0	-1	82

3. CC STRAIN ENERGY TERMS

(a) TERMS FOR WHICH ID = KD AND JD = HD

ID	JD	JS	JC	IC
0	0	0	1	89
0	0	0	5	90
0	1	2	1	94
0	2	0	1	96
1	0	2	3	100
2	0	0	5	102
1	1	0	3	103

(b) OTHER TERMS

ID	JD	KD	HD	JS	JC	IC
0	0	1	0	1	4	92
0	1	1	1	1	2	95
0	2	1	0	1	2	97
1	0	2	0	1	4	101
0	0	2	0	0	3	91
0	0	2	0	0	5	93
0	2	2	0	0	3	98

4. AB STRAIN ENERGY TERMS

ID	JD	KD	HD	JS	JC	IC
0	1	1	0	0	-3	5
0	1	1	0	0	1	6
0	1	1	0	2	-1	7
0	1	1	0	0	-1	8
0	1	3	0	0	1	10
0	1	1	2	0	-1	11
0	2	1	1	0	-5	16
0	2	1	1	0	-1	17
0	2	1	1	2	-3	18
0	3	1	0	0	-1	23
0	3	1	0	0	-3	24
0	3	3	0	0	-1	26
0	3	1	2	0	-3	27
1	0	0	1	0	-3	28
1	0	0	1	0	1	29
1	0	0	1	2	-1	30
1	0	2	1	0	1	32
1	1	0	0	0	-5	33
1	1	0	0	0	-1	34
1	1	0	0	2	-1	35
1	1	0	0	2	-3	36
1	1	0	2	2	-3	37
1	1	2	0	0	-3	42
1	1	2	0	0	1	43
1	1	2	0	2	-1	44
1	2	0	1	0	-1	47
1	2	0	1	2	-3	48
1	2	2	1	0	-1	50

ID	JD	KD	HD	JS	JC	IC
2	1	1	0	0	1	54
2	1	1	0	0	-1	55
2	1	3	0	0	1	57
2	1	1	2	0	-1	58
0	1	0	0	1	-4	1
0	1	0	0	1	0	2
0	1	0	0	1	-2	3
0	1	0	2	1	-2	4
0	1	2	0	1	0	9
0	2	0	1	1	-6	12
0	2	0	1	1	-2	13
0	2	0	1	3	-4	15
0	2	2	1	1	-2	19
0	3	0	0	1	-2	20
0	3	0	0	1	-4	21
0	3	0	2	1	-4	22
0	3	2	0	1	-2	25
1	0	1	1	1	0	31
1	1	1	0	1	-4	38
1	1	1	0	1	0	39
1	1	1	0	1	-2	41
1	1	3	0	1	0	45
1	1	1	2	1	-2	46
1	2	1	1	1	-2	49
2	1	0	0	1	0	51
2	1	0	0	1	-2	52
2	1	0	2	1	-2	53
2	1	2	0	1	0	56

5. AC STRAIN ENERGY TERMS

ID	JD	KD	HD	JS	JC	IC
0	1	1	1	0	2	60
0	2	1	0	2	0	63
0	3	1	1	0	0	66
1	0	0	0	0	0	67
1	0	0	2	0	2	69
1	0	2	0	0	4	71
1	1	0	1	2	0	72
1	2	0	0	0	2	74
1	2	0	2	0	0	75
1	2	2	0	0	2	77
2	1	1	1	0	2	79
0	1	0	1	1	1	59
0	2	0	0	1	1	61
0	2	0	2	1	-1	62
0	2	2	0	1	1	64
0	3	0	1	1	-1	65
1	0	1	0	1	3	70
1	1	1	1	1	1	73
1	2	1	0	1	1	76
2	1	0	1	1	1	78

6. BC STRAIN ENERGY TERMS

ID	JD	KD	HD	JS	JC	IC
0	0	0	1	2	0	80
0	0	0	1	2	-2	81
0	1	0	0	0	-2	84
0	1	0	0	0	2	85
0	1	0	0	2	0	86
0	1	0	2	0	0	87
0	1	0	2	2	-2	88
0	1	2	0	0	2	91
0	1	2	0	2	0	92
0	2	0	1	2	-2	93
1	0	1	1	0	2	97
1	0	1	1	0	0	98
2	0	0	1	2	0	99
3	0	1	1	0	2	102
1	1	1	0	2	0	105
1	2	1	1	0	0	108
2	1	0	0	0	2	109
2	1	0	2	0	0	110
2	1	2	0	0	2	112
0	0	1	1	1	1	82
0	0	1	1	1	-1	83
0	1	1	0	1	1	89
0	1	1	0	3	-1	90
0	2	1	1	1	-1	94
1	0	0	1	1	1	95
1	0	0	1	1	-1	96
2	0	1	1	1	1	100
3	0	0	1	1	1	101
1	1	0	0	1	1	103
1	1	0	2	1	-1	104
1	1	2	0	1	1	106
1	2	0	1	1	-1	107
2	1	1	0	1	1	111

7. AA KINETIC ENERGY TERMS

(a) TERMS FOR WHICH ID = KD AND JD = HD

ID	JD	JS	JC	IC
0	0	0	-5	1
0	1	0	-3	3
0	2	0	-5	4
1	1	0	-3	5

(b) OTHER TERMS

ID	JD	KD	HD	JS	JC	IC
0	0	0	2	0	-3	2

8. BB KINETIC ENERGY TERMS

(a) TERMS FOR WHICH ID = KD AND JD = HD

ID	JD	JS	JC	IC
0	0	0	-7	6
0	0	2	-5	7
0	0	0	-5	8
0	1	2	-7	13
1	0	0	-3	15
1	0	2	-5	16
2	0	0	-3	18
1	1	0	-5	19

(b) OTHER TERMS

ID	JD	KD	HD	JS	JC	IC
0	0	1	0	1	-4	9
0	0	1	0	1	-6	10
0	1	1	1	1	-6	14
1	0	2	0	1	-4	17
0	0	2	0	0	-3	11
0	0	2	0	0	-5	12

9. CC KINETIC ENERGY TERMS

(a) TERMS FOR WHICH ID = KD AND JD = HD

ID	JD	JS	JC	IC
0	0	0	-3	20
0	0	0	1	21
0	1	0	-1	22
1	0	0	1	23

10. AB KINETIC ENERGY TERMS

ID	JD	KD	HD	JS	JC	IC
0	0	1	1	0	-3	2
0	1	1	0	0	-3	4
0	2	1	1	0	-5	6
1	1	0	0	0	-3	7
1	1	0	0	0	-5	8
1	1	2	0	0	-3	10
0	0	0	1	1	-4	1
0	1	0	0	1	-4	3
0	2	0	1	1	-6	5
1	1	1	0	1	-4	9

11. AC KINETIC ENERGY TERMS

ID	JD	KD	HD	JS	JC	IC
1	1	0	1	0	-2	11
0	2	1	0	0	-2	12
0	0	1	0	0	0	13

12. BC KINETIC ENERGY TERMS

ID	JD	KD	HD	JS	JC	IC
0	0	0	1	0	-2	14
0	0	0	1	0	-2	15
2	0	0	1	0	-4	18
1	1	1	0	0	-2	19
0	1	1	0	1	-3	16
1	0	0	1	1	-3	17

APPENDIX 4 REDUCTION PARTS OF THIN CYLINDER COMPUTER PROGRAM

BCMAT

```
      PUBLIC AL,BL,AI,BI,KS,HT2,ETA,ETA2,ETA3,HT,M,N,JM,JN,JA,JB,JC
      PUBLIC NA,NB,NC,ND,NN,R,CN(6),CO(6),CNDP(1),CODP(1)
      PUBLIC RM,RN,RX,AA
      PUBLIC SB(200),SS(200),SBDP(1),SSDP(1)
      PUBLIC IX(4,4),FA(40),FB(30),GA(40),GB(30),NMA(4),GBDP(1)
      PUBLIC NMB(4),NBDP(1)
      PUBLIC IXDP(2),FADP(1),FBDP(1),GADP(1),NMDP(1)
      REAL KS
      INTEGER R,RM,RN,RX,FA,FB,GA
      INTEGER GB
'PROCEDURE'CHAIN(KK);'INTEGER'KK;
'BEGIN'
'REAL'SUM;
'REAL'AX,BX;
'INTEGER'NAB,NABC;
'INTEGER'I,J,K,H,CC,MX,KX,HX;
MAXARR(2500);
M=M'/'2;
RX=RM=2*N+2*M;
WRITEA(85,SS,'('SSFL')');
WRITEA(85,SB,'('SSFL')');
'BEGIN'
'ARRAY'A(1':'RX,1':'2*NN);
NAB=NA+NB;NABC=NAB+NC;
MONITO;
NAB=NAB-2*M;
NAB=2*NAB;NA=2*NA;M=2*M;
NULL(A,RX,2*NN);
'FOR'I=1'STEP'1'UNTIL'M'DO''FOR'J=1'STEP'1'UNTIL'N+1'DO''BEGIN'
AX=AL**(I-2);
CC=0;
BX=BL**(J-1);MX=(J-1)*N+I;
K=(I+1)'/'2;
'IF'J'EQ'N'THEN''GOTO'B4;
'IF'J'EQ'N+1'THEN''GOTO'C3;
A(CC+J,MX)=ETA*J *AX*AL*B I;
B4':''IF'J'EQ'1'OR'I'EQ'1'THEN''GOTO'C3;
A(CC+J,NA+MX-M)=(I-1)*AX*AI;
C3':''IF'J'LE'2'THEN''GOTO'E5;
A(CC+J,NAB+MX)=AX*AL;
E5':'CC=CC+N+1;
'IF'J'EQ'N+1'THEN''GOTO'A5;
'IF'I'EQ'1'OR'J'EQ'1'THEN''GOTO'B2;
A(CC+J,MX-M)=(I-1)*AX*AI;
B2':''IF'J'EQ'N'THEN''GOTO'A5;
A(CC+J,NA+MX)=J*AX*AL*BI;
A5':'CC=CC+N;
'IF'J'GE'N'THEN''GOTO'C1;
A(CC+K,MX)=J*BX*BI;
'IF'I'EQ'1'THEN''GOTO'C1;
A(CC+I'/'2,NA+MX)=ETA*(I-1)*BX*BL*AI;
C1':''IF'J'LE'2'THEN''GOTO'A1;
A(CC+K,NAB+MX)=ETA*BX;
A1':'CC=CC+M'/'2;
CC=CC-1;
'IF'J'GE'N'OR'I'LE'2'THEN''GOTO'A2;
'IF'I'EQ'3'THEN''GOTO'B1;
A(CC+I'/'2,MX)=(I-1)*BX*BL*AI;
B1':'A(CC+K,NA+MX)=J*BX*BL;
A2':''END';
'BEGIN'
'REAL'SUM;
'ARRAY'C,D(1':'RX,1':'NN);
'PROCEDURE'SCALE(A,RI);'ARRAY'A;'INTEGER'RI;
'BEGIN''INTEGER'I,J,K;'ARRAY'JS(1':'RI);
'FOR'I=1'STEP'1'UNTIL'RI'DO''BEGIN'
SUM=0;
'FOR'J=1'STEP'1'UNTIL'NN'DO''BEGIN'
AX=ABS(A(I,J));
'IF'AX'GT'SUM'THEN'SUM=AX;
'END';
JS(I)=1/SUM;'END';
'FOR'I=1'STEP'1'UNTIL'RI'DO''FOR'J=1'STEP'1'UNTIL'NN'DO'
A(I,J)=JS(I)*A(I,J);
'END';
NAB=NAB+2*M;NN=2*NN;M=M'/'2;
'FOR'I=1'STEP'1'UNTIL'NN'DO''BEGIN'
SUM=0;
'FOR'J=1'STEP'1'UNTIL'N+1'DO'SUM=SUM+A(J,I)*BL**(J-1);
A(N+1,I)=SUM;'END';
```

```
'FOR'I=NA+1'STEF'1'UNTIL'NN'DO''BEGIN'
A(2*N+M+1,I)=A(N+1,I);A(N+1,I)=0;'END';
'FOR'I=1'STEP'1'UNTIL'NA'DO'A(2*N+M+1,I)=0;
'FOR'J=1'STEP'1'UNTIL'RX'DO''BEGIN'
'FOR'I=1'STEP'2'UNTIL'NA'DO''BEGIN'
K=I'/'2+1;C(J,K)=A(J,I);D(J,K)=A(J,I+1);'END';
'FOR'I=NA+1'STEP'2'UNTIL'NAB'DO''BEGIN'
K=I'/'2+1;C(J,K)=A(J,I+1);D(J,K)=A(J,I);'END';
'FOR'I=NAB+1'STEP'2'UNTIL'NN'DO''BEGIN'
K=I'/'2+1;C(J,K)=A(J,I);D(J,K)=A(J,I+1);'END';
'END';
NN=NN'/'2;
'FOR'I=1'STEP'1'UNTIL'RX'DO''FOR'J=1'STEP'1'UNTIL'NN'DO'
'BEGIN'C(I,J)=C(I,J)*SS(J);D(I,J)=D(I,J)*SB(J);'END';
SCALE(C,RX);SCALE(D,RX);
EMATPR(C,RX,NN,4);EMATPR(D,RX,NN,4);
WRITEA(85,C,'('FLBC')');WRITEA(85,D,'('FLBC')');
'END';
CLOSMT(85);
STOP;
'END';
'END'
```

MAIN For use with BCRED

```
        PUBLIC AL,BL,AI,BI,KS,HT2,ETA,ETA2,ETA3,HT,M,N,JM,JN,JA,JB,JC
        PUBLIC NA,NB,NC,ND,NN,R,CN(6),CO(6),CNDP(1),CODP(1)
        PUBLIC RM,RN,RX,AA
        PUBLIC SB(200),SS(200),SBDP(1),SSDP(1)
        PUBLIC IX(4,4),FA(2,40),GA(2,40),NMA(4),FADP(2),IXDP(2),GADP(2)
        PUBLIC MA(2),MADP(1)
        PUBLIC NMDP(1)
        PUBLIC NMB(4),NBDP(1)
        REAL KS
        INTEGER R,RM,RN,RX,FA,GA
'BEGIN'
'INTEGER'I,J,K;
SUPDV1(CN,CNDP,6);SUPDV1(CO,CODP,6);
SUPDV1(NMB,NBDP,4);
SUPDV1(SB,SBDP,200);SUPDV1(SS,SSDP,200);
SUPDV1(MA,MADP,2);
SUPDV2(IX,IXDP,4,4);SUPDV2(FA,FADP,2,40);
SUPDV2(GA,GADP,2,40);SUPDV1(NMA,NMDP,4);
START'.'ININT(M);'IF'M'EQ'99'THEN'STOP;
ININT(N);
NA=NB=NC=M*N-M;
NN=3*M*N-3*M;
R=NA;
RX=RM=2*M+2*N;
RN=NN-RM;
INREAL(AL,BL,ETA,HT,KS);
LINES(4);IPRINT(M,2);SPACES(3);IPRINT(N,2);SPACES(3);
EPRINT(AL,4);SPACES(3);EPRINT(BL,4);SPACES(3);EPRINT(ETA,4);
```

```
SPACES(3);EPRINT(HT,4);LINES(4);
AL=0.5*AL;
AI=1/AL;AL=1;BI=1/BL;BL=1;
HT=0.5*HT;
ETA2=ETA*ETA;ETA3=ETA*ETA2;HT2=HT*HT;
CO(1)=1-ETA;CN(1)=1/3;
NMA(1)=NA;NMA(2)=NB;NMA(3)=NC;NMA(4)=ND;
NMB(1)=0;NMB(2)=NMA(1);NMB(3)=NMA(1)+NMA(2);
IX(1,1)=0;IX(2,2)=1;IX(3,3)=2;IX(1,2)=3;IX(1,3)=4;IX(2,3)=5;
'FOR'I=1,2'DO''FOR'J=I+1'STEP'1'UNTIL'3'DO'IX(J,I)=IX(I,J);
M=2*M;
JA=JB=JC=M;
ININT(MA (1));
'FOR'I=1'STEP'1'UNTIL'MA( 1)'DO'ININT(FA(1,I));
'FOR'I=1'STEP'1'UNTIL'MA( 1)'DO'ININT(GA(1,I));
ININT(MA(2));
' FOR'I= 1'STEP'1'UNTIL'MA(2)'DO'I NINT(FA(2,I));
'FOR'I= 1'STEP'1'UNTIL'MA (2)'DO'ININT(GA(2,I));
LINES(4);
IPRINT(MA(1),3);
LINES(1);
'FOR'I=1'STEP'1'UNTIL'MA(1)'DO'IPRINT(FA(1,I),3);
LINES(1);
'FOR'I=1'STEP'1'UNTIL'MA(1)'DO'IPRINT(GA(1,I),3);
LINES(1);
IPRINT(MA(2),3);
LINES(1);
'FOR'I=1'STEP'1'UNTIL'MA(2)'DO'IPRINT(FA(2,I),3);
LINES(1);
'FOR'I=1'STEP'1'UNTIL'MA(2)'DO'IPRINT(GA(2,I),3);
LINES(4);
CLEAR;
CHAIN(2);
'END'
```

BCRED

```
    PUBLIC AL,BL,AI,BI,KS,HT2,ETA,ETA2,ETA3,HT,M,N,JM,JN,JA,JB,JC
    PUBLIC NA,NB,NC,ND,NN,R,CN(6),CO(6),CNDP(1),CODP(1)
    PUBLIC RM,RN,RX
    PUBLIC SB(200),SS(200),SBDP(1),SSDP(1)
    PUBLIC IX(4,4),FA(2,40),GA(2,40),NMA(4),FADP(2),IXDP(2),GADP(2)
    PUBLIC MA(2),MADP(1)
    PUBLIC NMDP(1)
    PUBLIC NMB(4),NBDP(1)
    REAL KS
    INTEGER R,RM,RN,RX,FA,GA
'BEGIN'
'INTEGER'NX;
'INTEGER'KX,HX;
'INTEGER'NAB;
'ARRAY'SA(1':'200);
MAXARR(2500);
'BEGIN'
'REAL'AX,BX;
'INTEGER'I,J,K,H ,C C,MX;
'PROCEDURE'SHIFTH(B);'ARRAY'B;
'BEGIN'
'INTEGER'I,J,K,H,CF,CG,CH,CJ,CK;
'ARRAY'A(1':'RX,1':'NN);
RAFD(A,3,HX);
CF=0;CG=0;CH=0;CJ=0;CK=0;
'FOR'K=1'STEP'1'UNTIL'MA(HX+1)'DO''BEGIN'
CH=CF+FA(HX+1,K) ;CJ=CG+G A(HX+1,K);CK=CH+GA(HX+1,K);
'FOR'I=CF+1'STEP'1'UNTIL'CH'DO'
'FOR'J=1'STEP'1'UNTIL'RX'DO'A(J,I-CG)=A(J,I);
'FOR'I=CH+1'STEP'1'UNTIL'CK'DO'
'FOR'J=1'STEP'1'UNTIL'RX'DO'B(J,I-CH+CG)=A(J,I);
CF=CK;CG=CJ;
'END';
WATD(A,3,3);
'END';
```

```
NAB=NA+NB;
'BEGIN''INTEGER'I;'ARRAY'A(1'.'R,1'.'R);
'FOR'I=0,12,13,1,2,14,3,15,4,16,17,5,6,18,19,7,8,20,9,21,10,22,23,11
'DO''BEGIN'
READAR(85,A,'('FLAR')');
WATD(A,1,I);
'END';'END';
READAR(85,SS,'('SSFL')');
READAR(85,SB,'('SSFL')');
'BEGIN''ARRAY'A,B(1'.'RX,1'.'NN);
READAR(85,A,'('FLBC')');
READAR(85,B,'('FLBC')');
EMATPR(A,RM,NN,4);EMATPR(B,RM,NN,4);
WATD(A,3,0);
WATD(B,3,1);
'END';
'FOR'HX=0,1'DO''BEGIN'
'BEGIN'
'ARRAY'B(1'.'RM,1'.'RM);
SHIFTH(B);
'BEGIN''ARRAY'A(1'.'RM,1'.'RM);
'ARRAY'C(1'.'RM,1'.'RM);
MATCOP(C,B,RM,RM);
MATINV(A,B,RM);MATCOP(B,A,RM,RM);
MATMUL(A,B,C,RM,RM,RM);EMATPR(A,RM,RM,4);
'END';
WATD(B,3,4);
'END';
'BEGIN''ARRAY'B(1'.'RM,1'.'NN);
RAFD(B,3,3);
'BEGIN''ARRAY'A(1'.'RX,1'.'RM),C(1'.'RX,1'.'RN);
RAFD(A,3,4);
'FOR'I=1'STEP'1'UNTIL'RX'DO''FOR'J=1'STEP'1'UNTIL'RN'DO''BEGIN'
C(I,J)=0;'FOR'K=1'STEP'1'UNTIL'RM'DO'
C(I,J)=C(I,J)-A(I,K)*B(K,J);
'END';WATD(C,2,HX);
'END';'END';
'END';
'END';
'BEGIN'
'PROCEDURE'FULL(EF,R);'ARRAY'EF;'INTEGER'R;
'BEGIN''INTEGER'I,J;
'FOR'I=1'STEP'1'UNTIL'R-1'DO''FOR'J=I+1'STEP'1'UNTIL'R'DO'
EF(J,I)=EF(I,J);
'END';
'PROCEDURE'VECCOP(SA,SS,X);'ARRAY'SA,SS;'INTEGER'X;
'BEGIN''INTEGER'I;
'FOR'I=1'STEP'1'UNTIL'X'DO'SA(I)=SS(I);'END';
'PROCEDURE'SHIFTH(A,RX,NN,RM,RN);'ARRAY'A;'INTEGER'RX,NN,RM,RN;
'BEGIN'
'INTEGER'I,J,K,H,CF,CG,CH,CJ,CK;
'ARRAY'B(1'.'RN,1'.'RM);
CF=0;CG=0;CH=0;CJ=0;CK=0;
'FOR'K=1'STEP'1'UNTIL'MA(HX+1)'DO''BEGIN'
CH=CF+FA(HX+1,K) ;CJ=CG+G A(HX+1,K);CK=CH+GA(HX+1,K);
'FOR'I=CF+1'STEP'1'UNTIL'CH'DO'
'FOR'J=1'STEP'1'UNTIL'RX'DO'A(J,I-CG)=A(J,I);
'FOR'I=CH+1'STEP'1'UNTIL'CK'DO'
'FOR'J=1'STEP'1'UNTIL'RX'DO'B(J,I-CH+CG)=A(J,I);
CF=CK;CG=CJ;
'END';
'FOR'I=RN+1'STEP'1'UNTIL'NN'DO'
'FOR'J=1'STEP'1'UNTIL'RX'DO'A(J,I)=B(J,I-RN);
'END';
'PROCEDURE'SHIFTV(A,H);'ARRAY'A;'INTEGER'H;
'BEGIN'
'INTEGER'I,J,K,CF,CG,CH,CJ,CK;
'ARRAY'B(1'.'RM,1'.'R);
CF=0;CG=0;CH=0;CJ=0;CK=0;
'FOR'K=1'STEP'1'UNTIL'MA(HX+1)'DO''BEGIN'
CH=CF+FA(HX+1,K) ;CJ=CG+G A(HX+1,K);CK=CH+GA(HX+1,K);
'FOR'I=CF+1'STEP'1'UNTIL'CH'DO'
'FOR'J=1'STEP'1'UNTI_'NMA(H)'DO'A(I-CG,J)=A(I,J);
'FOR'I=CH+1'STEP'1'UNTIL'CK'DO'
'FOR'J=1'STEP'1'UNTIL'NMA(H)'DO'B(I-CH+CG,J)=A(I,J);
CF=CK;CG=CJ;
'END';
'FOR'I=RN+1'STEP'1'UNTIL'NN'DO'
'FOR'J=1'STEP'1'UNTIL'NMA(H)'DO'A(I,J)=B(I-RN,J);
'END';
MONITO;
'FOR'HX=0,1'DO''BEGIN'
'IF'HX'EQ'0'THEN'VECCOP(SA,SS,NN)'ELSE'VECCOP(SA,SB,NN);
'BEGIN'
'INTEGER'KX,K,L;'ARRAY'C(1'.'NN,1'.'R);
'FOR'KX=0,3'DO''BEGIN'
'FOR'K=1,2,3'DO''BEGIN'
'FOR'L=1,2,3'DO''BEGIN'
'INTEGER'I,J;'ARRAY'A,B(1'.'R,1'.'R);
```

```
RAFD(A,1,12*HX+2*KX+IX(K,L));
'IF'K'EQ'L'THEN'FULL(A,R)'ELSE''IF'L'GT'K'THEN'
'BEGIN'TRANS(B,A,R,R);MATCOP(A,B,R,R);'END';
'FOR'I=1'STEP'1'UNTIL'NMA(L)'DO''FOR'J=1'STEP'1'UNTIL'NMA(K)'DO'
C(I+NMB(L),J)=A(I,J)*SA(I+NMB(L))*SA(J+NMB(K));
'END';
SHIFTV(C,K);
'BEGIN''REAL'SUM;'INTEGER'I,J;
'ARRAY'A(1':'RN,1':'R),B(1':'RM,1':'RN);
RAFD(B,2,HX);
'FOR'I=1'STEP'1'UNTIL'RN'DO''FOR'J=1'STEP'1'UNTIL'NMA(K)'DO''BEGIN'
SUM=C(I,J);'FOR'L=RN+1'STEP'1'UNTIL'NN'DO'
SUM=SUM+B(L-RN,I)*C(L,J);A(I,J)=SUM;'END';
WATD(A,3,KX+K-1);
'END';
'END';
'END';
'END';
'BEGIN'
'REAL'SUM;
'INTEGER'KI;
'INTEGER'I,J,K;'ARRAY'A(1':'RN,1':'NN);
'FOR'KI=0'STEP'1'UNTIL'1'DO''BEGIN'
'BEGIN''ARRAY'C(1':'RN,1':'R);
'FOR'K=0,1,2'DO''BEGIN'
RAFD(C,3,3*KI+K);
'FOR'I=1'STEP'1'UNTIL'RN'DO''FOR'J=1'STEP'1'UNTIL'NMA(K+1)'DO'
A(I,J+NMB(K+1))=C(I,J);
'END';'END';
SHIFTH(A,RN,NN,RM,RN);
'BEGIN''ARRAY'B(1':'RM,1':'RN),C(1':'RN,1':'RN);
RAFD(B,2,HX);
'FOR'I=1'STEP'1'UNTIL'RN'DO''FOR'J=1'STEP'1'UNTIL'RN'DO''BEGIN'
SUM=A(I,J);'FOR'K=RN+1'STEP' 1'UNTIL'NN'DO'
SUM=SUM+A(I,K)*B(K-RN,J);C(I,J)=SUM;'END';
'FOR'I=1'STEP'1'UNTIL'RN-1'DO''FOR'J=I+1'STEP'1'UNTIL'RN'DO'
C(J,I)=C(I,J);
MAXARR(5000);
WATD(C,2,2+2*HX+KI);
MAXARR(2500);
'END';
'END';
'END';
'END';
'END';
MONITO;
CHAIN(5);
'END'
```

APPENDIX 5. INPUT DATA FOR COMPUTER PROGRAMS

The following input data is required for the computer programs described in this thesis.

1. M = Order of α terms in displacements. Must be an odd number for symmetry solutions.
2. N = Order of β terms in displacements.
3. BI = Total twist in radians. (Not required for Cylindrical Shell)
4. AL = Width of Shell
5. BL = Length of Shell
6. ETA = Poisson's Ratio
7. HT = Thickness of Shell
8. KS = Transverse Shear Factor = 1.0 in all cases
9. E = Young's Modulus
10. RO = Density
11. RD = Number of Modes Required
12. NX = Number of sections in α direction required in mode shapes print out.
13. NY = Number of sections in β direction required in mode shapes print out.

APPENDIX 6. LIBRARY PROCEDURES USED IN COMPUTER PROGRAM

1. INREAL (X, Y, Z, ...)

Reads numbers from data into real locations X, Y, Z, etc.

2. ININT (I, J, K .....)

Reads numbers from data into integer locations I, J, K, etc.

3. EPRINT (X, I)

Prints real value X to I decimal places.

4. IPRINT (M, I)

Prints integer value M to I significant figures.

5. LINES (I)

Outputs I line feeds.

6. SPACES (I)

Output I space characters.

7. TEXT ('('STRING')')

Outputs characters STRING

8. EMATPR (A, M, N, I)

Outputs real array A(1:M, 1:N) row by row to I decimal places

9. EVECPR (A, M, I)

Outputs real array A(1:M) to I decimal places.

10. WATD (A, I, J)

Writes array A to position J on disc unit I

11. RAFD (A, I, J)

Reads array A from position J on disc unit I.

12. MAXARR (M)

Defines maximum size of arrays to be used in WATD and RAFD in order to define size of each area J on the disc.

13. WRITEA (I, A, '('NAME')')

Writes array A to present write position of magnetic tape unit I, and gives it the identifier NAME.

14. READAR (I, A, '('NAME')')

Reads next array with identifier NAME on magnetic tape unit I into array A.

15. CLOSMT (I)

Reallocates magnetic tape unit I

16. MONITO

Outputs computer processor time used by program, and real time.

17. STOP

Clears output buffers and stops execution of the program.

18. CLEAR

Clears output buffers

19. CHAIN (I)

Jumps from present chain to CHAIN I.

As the program is written in Algol, but the intermodular transfer of information is by way of Fortran PUBLIC variables, extra information is required to define the lower bounds of all arrays. This is performed in the program by the following routines.

20. SUPDV1 (A, ADOPE, M)

Sets up dope vector ADOPE (1:1) for array A(1:M)

21. SUPDV2 (A, ADOPE, M, N)

Sets up dope vector ADOPE (1:2) for array A(1:M, 1:N)

22. SUPDW3 (A, M, N, P, Q, I, J)

Sets up information in last three locations of array A(M:N, P:Q, I:J). Thus in the PUBLIC declaration the array A must be declared 3 elements larger than actually required.

These are the procedures used for this purpose in the program. Routines exist to perform the same operations for arrays of dimensions one to six.

In the following procedures the notation A(M, N) refers to an array of size A(1:M, 1:N)

23. NULL (A, M, N)

Nulls matrix A(M,N)

24. TRANS (A, B, M, N)

$A(M, N) = B^T(N, M)$

25. MATMUL (A, B, C, M, N, P)

$A(M, P) = B(M, N)* C(N,P)$

26. MATINV (A, B, N)

A(N, N) = B^{-1} (N, N)

27. MATCOP (A, B, M, N)

A(M, N) = B(M, N)

28. MATDIV (A, B, M, N)

A(M, N) = B^{-1}(M, M)* A(M, N)

29. HOUSEH (A, E, M, K)

Calculates eigenvalues of symmetric array A(M, M) by Householder's method. Stores eigenvalues in array E(M). If K $\neq$ 2 it stores eigenvectors in rows of A, calculating these by inverse iteration.

30. NSEIGB (A, B, K, L, M, ER, EI)

Calculates eigenvalues and eigenvectors of array A(M ,M) by HQR method and inverse iteration. It stores real and imaginary parts of eigenvalues in ER(M) and EI(M) respectively, and real and imaginary parts of eigenvectors in rows of A(M, M) and B(M, M) respectively. The integers K and L define normalisation of real and complex eigenvectors respectively as follows

1 The Manhattan Norm

2 The Euclidean Norm

3 The Maximum Norm