Developing the ARCH family model to evaluate the impact of inflation on the S&P 500 return volatility

by

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2014

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A Dissertation presented in part consideration for the Degree of Masters in Finance and Investment at the University of Nottingham

Acknowledgement

First and foremost, I would like to sincerely express deepest gratitude to my project supervisor Peter Oliver, for being a dedicated supervisor who has provided invaluable advice and encouragement throughout this period. Without your recommendations and guidance, I will not be able to complete the dissertation.

I would also like to thank my family and friends for providing motivational advice and giving me endless support. Especially my Mum and Dad, who funded my fantastic studies abroad and continuously supported and encourage me during the whole process of study.

Finally, I owe my deepest gratitude to Zhongyuan Gou, who helped me a lot in both life and studies.

Content

Abstract

This study investigates to apply the ARCH family model to assess the impact of inflation on S&P 500 stock return volatility using daily data from 2004 to 2014 in the U.S. it is found that no evidence shows that the inflation has the predictive power for stock return volatility in the U.S. this finding is consist with Davis and Kutan (2003) results for the U.S. market, but disagree with Schewert (1989) who find weak effect form inflation to the stock market volatility in U.S. In addition, this paper finds the GARCH model under the Generalized error distribution has more power when modeling the conditional volatility than the traditional normal distribution assumption. Moreover, the impact of asymmetric shocks exists in the S&P 500 conditional return volatility.

Key words: stock return volatility, ARMA, T-GARCH, GED, impact of inflation,

Chapter 1

Introduction

The majority of studies are to examine the effect of inflation to the stock returns based on the genesis of Fisher effect hypothesis financial theory, which suggests that the nominal stock returns should vary with the inflation. However, the realistic return series always tend to have a time-varying volatility feature found and provided evidence by Akgiray (1989), Bollerslev (1987), Chou (1988), and French et al. (1987) , therefore, it is also interesting to investigate and examine the impact of the inflation on the stock return volatility.

This paper is to investigate the possible predictive power of U.S inflation to S&P 500 stock return volatility by using daily data from 05^{th}/01/2004 to 30^{th}/06/2014. Where the approximate daily inflation data is obtained by linear interpolation method, and the S&P 500 stock return volatility is measured by using appropriate Autoregressive Conditional Heterosecaedastic (ARCH) family model.

The S&P 500 index is one of a most popular index in U.S. it is the market-value weighted index which are proxies for the market portfolio in the U.S. market rather than the share index which focus on some particular industries in the Country. Therefore, it is better to analyse the potential causes from the macro view, and the inflation is one of a key macro variables for a Country.

The rest of this paper shows the literature review of the topic; secondly the study for modeling daily S&P 500 index return volatility by ARCH family models and then finally investigates the study: does the inflation can predict and has effect on this volatility by adding the lag inflation value as the exiguous variable in the volatility time series model.

However, this paper does not separate the Methodology Part and Analysis Part. This because even based on reading the references for modeling the volatility, the empirical study should more obey the practical sample data. The most appropriate model for this sample return volatility should be identified according to the different problems and features in this sample data. The remainder of this paper is arranged as below:

Chapter 2: literature review

Chapter 3: obtain and analyse the S&P 500 return series;

Chapter4: according to the series correlation feature, investigate the ARMA model;

Chapter 5: modeling the volatility by simple ARCH;

Chapter 6: according to the problem of long lag order of ARCH model, implement the traditional GARCH model;

Chapter 7: according to the no feasible distribution assumption, implement the GARCH with GED assumption;

Chapter 8: test the possible leverage effect by TGARCH and try GARCH-M model.

Chapter 9: obtain the approximate daily inflation and use as the ex variable.

Chapter 10: conclusion

Chapter 2

Literature review

2.1 Stock return volatility

Return Volatility shows the return uncertainty, the risk when holding this stock. From the return time series picture, it means the fluctuations away from the average return. It is usually represented by the standard deviation (Std) of the return or the variance of the return. Stock return volatility refers to the variability of stock price changes a time period. It is a critical factor in options trading and it is regarded as a measure of risk by analysts, dealers, brokers, investors and regulators. (Karolyi et al, 2001) This paper uses the variance of the log return as the volatility. However, the return time-varying volatility exists, the higher volatility represents the higher distance that the return volatile away from the mean value. Therefore influence the wealth for the stock holders.

The analysis of the time varying stock return volatility involves two main filed: one is modeling volatility by using the high frequency data to obtain the low frequency volatility, using the time series model et al. or even forecast the volatility using these models. These are the univariate analysis.

The other research study field is to investigate the possible causes and potential explanations for this stock return volatility changes. Therefore, many articles study the influence from the view of macro-environment of the stock market, such as the relationship between macro variable volatility and the stock return volatility, or the macro variable for the stock market return volatility. The hypothesis that the macro-variables have effects on the stock market originally put forward to investigate the causes for the highly time varying stock return volatility. Because these will potentially contribute to the financial decision, which if successfully examine the association, therefore the investor would based on the macro variables changes as one of the indicator to make investment decision; besides, it would contribute to the financial risk management as well. The inflation is a key indicator of one of the macro-variable. Fama (1981) argues that the stock prices is the reflector of various macro variables such as inflation. Many other researchers also carried out study in this field: French and Roll (1986) and (Schwert, 1989) study the day-of-the-week effects, macroeconomic variables to the stock return volatility. Ederington and Lee (1993) and Berry and Howe (1994) study the release of scheduled macroeconomic news to the return volatility and the trading volume to the return volatility. Although Blair (2001) concludes some weaknesses of these possible explanations for time-varying stock return volatility, these articles still make a huge contribution for investigate the possible causes for volatility.

2.2 Inflation and importance of inflation

The Customer Price Index for All Urban (CPI-U) represents the changes in the prices paid by urban consumers for a representative basket of goods and services" (U.S. Department of Labor). The definition of inflation in U.S. given by U.S. Department of Labor is "the overall general upward price movement of goods and services in an economy." The CPI-U usually as the indicator represents the Inflation, or using the rate of return of CPI as the inflation rate. Therefore, the inflation represents the changes rate of the prices paid by urban consumers for the same basket of goods and services during a time duration.

Fisher (1930) puts forward the famous Fisher Effect Hypothesis that provides the normal interest rate and the real interest rate relationship:

$$(1 + \mathrm{R}_{\mathrm{nominal}}) = (1 + \mathrm{R}_{\mathrm{real}}) \cdot (1 + \mathrm{Inflation\ Rate})$$

$$\text{Or } \mathrm{R}_{\mathrm{nominal}} \approx \mathrm{R}_{\mathrm{real}} + \mathrm{Inflation\ rate}$$

Where: the $\mathrm{R}_{\mathrm{nominal}}$ is the nominal interest rate,

$\mathrm{R}_{\mathrm{real}}$ is the real interest rate.

This means the nominal interest rate is co-movements with its corresponding inflation rate.

A simple example to explain the importance of the inflation rate: if the nominal annual interest rate of return is 5% in one year in one Country, one investor invest 100 Pounds into bank, and after one year he will obtain 100 •(1+5%)=105 pounds which is the initial capital together with the interest. He loses or uses the opportunity cost for dong other invest during this period, instead he obtains the extra 5 pounds in return.

If further assumes that the inflation rate is 10% in this Country in this year, i.e. purchasing a set of basket of goods costs 100 Pounds at the beginning of this year, however at the end of this year, purchasing the same set of basket of goods costs 110 pounds. Therefore, in this example, the investor not only does not obtain the profit, but also even lose the value of the initial capital.

This is only an example for the importance of inflation to the bonds return based on the Fisher Effect. The same influence of inflation to the stock return based on this Effect that is the nominal stock return should co-movements with the inflation rate in one Country. And the real stock return should be immuse to the inflation.

There are a lots of papers empirically investigate this Fisher Hypothesis by examining the relationship between inflation and stock return, Firth (1979) and Gultekin (1983) give the positive relation between nominal stock returns and inflation in United Kingdom. Loannidis et al. (2004) examines the Finsher Effect in Greece from 1985 to 2003. Anari and Kolari (2001) and Luintel and Pandyal (2006) also give empirical evidences prove the Fisher effect, and some researchers find the Fisher effect does not hold in some Countries in some period. Linter (1975), Bodie (1976), Nelson (1976), Jaffe and Mandelker (1976), and Fama and Schwert (1977). Spyrou (2001) suggests the relation between stock return and inflation may change over time and may also depend on the period examined. Based on this strong evidence of the association between inflation and stock return, therefore, and also because the stock returns exhibit time vary volatility feature, this paper tends to study does the inflation can influence the stock return volatility.

2.3 Inflation and Stock return volatility

Schewert (1989) analyses and concludes the weak evidence of the relation of stock market volatility and the macroeconomic variables including inflation rate for USA from 1857 to 1987, using monthly data.

Palm (1996) motivates GARCH models of volatility into the Factor-GARCH models that are the return and conditional variance in GARCH-type models being interpreted by adding the new economic variables and financial data.

Hamilton and Lin (1996) analyses the U.S. market return and return conditional volatility by using S&P 500 index from 1965 to 1993, they find the evidence of the economic activity (including the inflation) drives stock return volatility in the economic recessions.

NICOLE DAVIS and ALI M. KUTAN (2003) use monthly data from 1957 to 1999 to investigate the impact of inflation as the exogenous variables on both the stock return and on the conditional volatility modeled by the standard GARCH (1,1) model and EGARCH (1,1) model among 13 industrial and developing countries. The results show that including the USA, there is no evidence that macroeconomic activity given by inflation has significant power to predict the stock return volatility in nine countries. The evidence shows in The Finland, Germany, Japan and the Netherlands, inflation plays a significant and negative impact on the conditional volatility. But the return value is sensitive to the inflation movement in the UAS.

Ali M. Kutan and Tansu Aksoy (2004) use the monthly data from 1996 to 2001 and expand the GARCH model to modeling the stock return volatility for The ISE composite (National-100) index in Turkey which has a relative high level of inflation rate. Then they use both CPI and first lag of CPI as the possible exogenous variables influencing the Turkey stock market, the results suggest that neither CPI nor lag-1 CPI is insignificantly influence the stock return volatility.

Generally, Engle and Rangel (2005) examine the impact of economy's overall health on unconditional market volatility. They provide that countries with high rates of inflation experience larger expected volatilities than those with more stable inflation rate.

After finding the strong time varying volatility in Toronto Stock Exchange (TSE) and Istanbul Stock Exchange (ISE) in Turkey with highly inflation rate, Saryal (2007), therefore, uses the same method, further discovers the inflation can determinant this return volatility. However, there is no evidence on Canada.

Bekaert and Engstrom (2009) study in the US show that high expected inflation has tended to coincide with periods of heightened uncertainty about real economic growth together with unusually high risk aversion, both in rationally raise equity yields.

Md. Arifur Rahman (2009) use the VAR method finds the significant association between the industrial-level stock returns volatility in Australia and the Australia inflation.

Sagarika Mishra and Harminder Singh (2010) investigate the relationship between macro-economic variables including inflation and stock volatility, using monthly Data from 1998 to 2008 from two major stock indices in India, although, no significance between return and inflation, but they find that the increased inflation affects one stock return volatility go up.

Shehu Usman Rano Aliyu (2012) use the monthly data and use the first lag of inflation (the difference of natural lgrithum CPI) as the variable to explain the time varying return volatility described by the simply GARCH(1,1) and QGARCH (1,1) model, the results show both two countries the Nigeria with Nigeria stock exchange (NSE) market and Ghana with GSE market show the significant evidence at 10% level that the inflation has an impact on the stock return volatility, the negative coefficient in the Nigeria and positive coefficient in the Ghana.

Engle, R. F. and Rangel, J. G. using the monthly data with five year interval for

50 countries observes that the higher return volatility is match the relative higher inflation, and for low return volatility countries, the inflation rate is relative low. Moreover, the volatility of inflation presents a positive impact to the return volatility.

Goabaone Otisitswe Boitumelo Moffat (2000) uses the quarterly data from 1999 to 2009 analysis the Botswana Stock Exchange market (BSE), using cointegration method. the results show firstly, there is a positive relationship between market volatility and market development, secondly there is a negative relationship between inflation rate and market development, therefore, indicating a negative association between market volatility and inflation rate.

Because after 1993, non study investigates the inflation and market return volatility in the U.S., also considering the timeliness, this paper investigates the effect of inflation on the market return for recent ten years from 2004 to 2014.

2.4 ARCH family model

However, the volatility for return series cannot be observed directly. Unless the higher frequency of the return data can be obtained, the standard way for calculating the daily standard deviation cannot compute the daily volatility. Therefore, the Autoregressive Conditionally Heteroscedastic (ARCH) family model, which models the conditional volatility of the stock price return conditional on current available information is popularly used in the financial analysis.

Engle (1982) firstly establishes the ARCH model, which can capture the heteroscedastic feature in the financial return series. Bollerslev (1986) improves the ARCH model into Generalised ARCH (GARCH) model, which adds the past value of variance(s) as the explanations, this releases the high constrains for the coefficients in the variance equation in the ARCH model and meanwhile reduces the number of parameters in the model and therefore minimises the standard errors when estimating the model. After using the GARCH model empirically to analyse the financial time series, Bollerslev, Ray and Kenneth (1992) concludes that the GARCH(1,1) model has the availability to estimate the conditional volatility for a wide range of financial series data. When studying the US stock market data, Akgiray (1989), Pagan and Schwert (1990), Brailsford and Faff (1996) and Brooks (1998) found that the GARCH model, especially GARCH (1,1) outperformed.

In order to capture the leverage effect, which means the asymmetrical effect from positive and negative news in error, one famous extension GARCH model is developed by Glosten, Jagannathan and Runkle (1993), namely Threshold GARCH (T-GARCH), it explains this asymmetrical effect more successfully than the simple GARCH. Chris Brooks (2008) represents an illustration for using T-GARCH to model monthly S&P 500 return series from December 1979 to June 1998, the result shows that the sample return volatility does have the asymmetric effect. The Exponential GARCH (E-GARCH) model by Nelson (1991) also can capture the asymmetric effect. Based on the financial theory that high risk should be reward high return, Engle, Lilen and Robins (1987) extend the GARCH-in-mean, which adds the conditional variance as the explanation in the return series.

However, for high frequency financial time series data, the conditional assumption of normality for the error term in the mean equation is often unrealistic due to the skewness and kurtosis of the distribution (Brooks, et al, 2000). Mandelbrot (1963) and Fama (1965) firstly observed the financial time series tend to be distributed as sharper peak and fatter tails than normal distribution. Therefore, researchers have begun to describe these non-normal distributions of return or other financial time series. Mandelbrot (1963) proposed the stable Paretian distribution. Bollerslev (1987), among others, implement the Student t distribution, and Nelson (1991) expands the use of the Generalized Error Distribution (GED), which has a thinner peak and thicker tails than the normal distribution when the sharp parameter less than 2.

Based on the previous studies, this paper will also try to model the return volatility under different distribution assumption, and also check the volatility asymmetric effect and try the GARCH-in-mean model in order to find the most appropriate model for modeling the S&P 500 stock return series.

Chapter 3

The S&P 500 Return Series

In order to analyses the causes of the stock index volatility, it is necessary to measure the daily stock return volatility. Before modeling the volatility by ARCH type model, it is necessary to obtain and analyse the stock return series.

3.1 S&P 500 index

This paper uses the recent ten years daily S&P 500 index data from 2nd January 2004 to 30th June 2014, totally 2641 observations. The S&P 500 index daily closing data is downloaded from *Yahoo finance* website. The main software used in the paper is the STATA software.

However, the daily stock price data obtained only includes the trading days during the chosen duration which does not contain all of the weekends and the bank holidays. This is because the stock market is not trading in the weekends and bank holidays. Therefore, when inputting the data into the STATA software, this time series data is with gaps avoiding the non-trading days.

3.2 Return series

This paper uses the log return as the daily S&P 500 index return (r_t) that is the changes in the natural logarithm stock index price. It is a daily rate of return shows the yield continuously compounded during the period from **t-1** to **t**, therefore, no any U.S. dollar or another unit describes this rate of return. This return is calculated as that the investor reinvests the profit obtained from the stock as the new capital, and the frequency of this compounded process is continuous during one period (*t-1* to *t*).

The daily S&P 500 return time series {$\mathbf{r_t}$} from the S&P 500 index is given by:

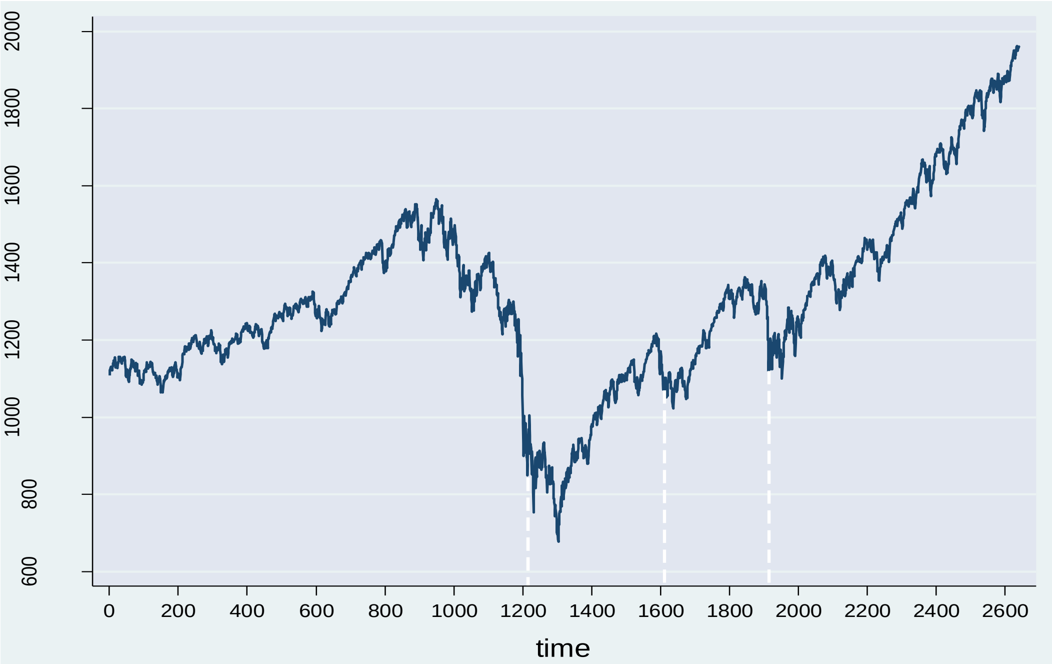
$$r_t = \ln_e\left(\frac{P_t}{P_{t-1}}\right) \tag{1}$$

Where: $\mathbf{r_t}$ is the log return at time t,
$\mathbf{P_t}$ is the stock price at time t,
t is the day number

Because the log return calculated based on two prices, the current day's ($\mathbf{P_t}$) and previous day's ($\mathbf{P_{t-1}}$), when obtain the daily log return series, there must at lease lose one value. Because the S&P 500 index price time series {$\mathbf{P_t}$} is with large volume of gaps, there must be lots of missing values for the log return, more specifically, one gap with ***n*** number of non-trading day(s) between the price data must lead to ***n+1*** lose values of the log return time series. Therefore, in order to compute a daily log return with no gaps, this paper only uses the trading calendar rather than the normal calendar to construct the daily price time series, and then obtain the log return time series with no gaps.

3.3 Data descriptive analysis

Figure 1
S&P 500 index time series from 02th/01/2004 to 30th/06/2014



Where: the time is the trading day count, totally 2641 observations.

The first observation in time series (i.e. t= 1) is 02nd/01/2004;

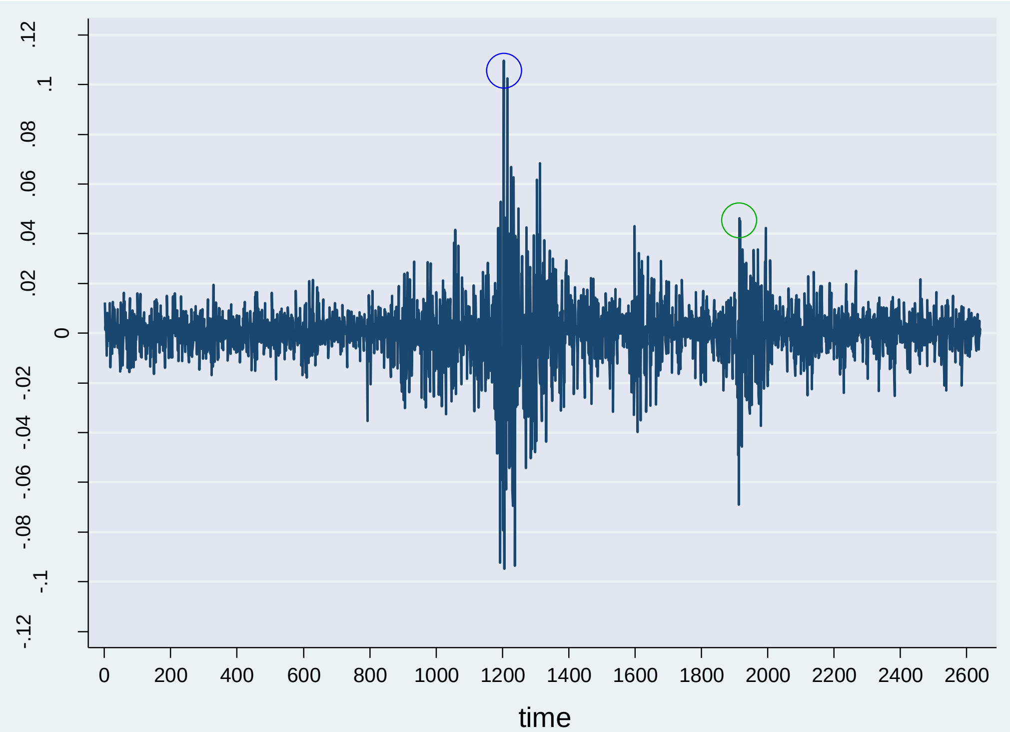
t= 600 is 19th/05/2006; t= 1200 is 07th/10/2008;

t= 1800 is 24th/02/2011; t= 2400 is 16th/07/2013.

The last observation in time series (i.e. t=2641) is 30th/06/2014

Figure 1 shows the daily S&P 500 index time series from 2nd January 2004 to 30th June 2014, totally 2641 observations. The horizontal line shows the daily time series, which is the trading day account; the vertical line shows the S&P 500 index price. Except the time interval from about t= 1000 to about t=1300, which is from December 2007 to March 2009, the S&P 500 index series is mainly increases with some big and small fluctuations during the sample duration.

Figure 2
S&P 500 log return time series from 02th/01/2004 to 30th/06/2014



Where: the time is the trading count, totally 2640 observations.

The first observation in time series (i.e. t= 1) is 05th/01/2004;

t= 600 is 22nd/05/2006; t= 1200 is 08th/10/2008;

t= 1800 is 25th/02/2011; t= 2400 is 17th/07/2013.

The last observation in time series (i.e. t=2640) is 30th/06/2014

Figure 2 shows the S&P 500 daily log return time series from 5th January 2004 to 30th June 2014, totally 2640 trading days. The horizontal line (i.e. the X-axis) shows the daily time series, the vertical line (i.e. the Y-axis) shows the log return values. This return series is non-linear with highly frequent fluctuations over whole sample interval. But the main trend of this series seems horizontal with the X-axis, which is near and just above the Y = 0 line.

Together with **Figure 1**, interestingly, the extremely big magnitudes of return variations become when the index significantly drop, such as in October 2008 (when about t=2000 in both figure 1 and Figure 2), May 2010 and August 2011. But these big variations of return do not last for a very long time. This together with the main horizontal trend (i.e. no trend) reflects the mean reversion feature in the return time series. That is the time-varying return volatility is not unlimited, and the return volatile around its mean values, and will back to its mean values.

- *Volatility Clustering*

There were large fluctuations around t=1200. i.e. in year 2008, this return series becomes more volatile, this heavy fluctuations did not decay immediately, but followed by some large fluctuations too. **Figure 2** shows a tendency that some large changes of fluctuations of return tend to be followed by large changes, like the blue circle in year 2008, and vice verse, like the green circle in year 2011 (around t=2000). This feature reflected by return series is called the volatility clustering.

Because of the feature of return volatile continuously with different magnitudes as the time varying, this daily return has a time varying volatility, and these volatile return, for instance, holding the stock on current day **t**, the return on next day $\mathbf{r_{t+1}}$ maybe unchanged, maybe dramatically increase or unacceptable decrease. These fluctuations of the return series in **Figure 2** represent risks for the stockholders, i.e the return volatility in the economic way. From a statistical view, these volatility is measured by the second moments i.e. the standard deviation (Std) or variance (the square of the Std).

Table 1
Descriptive statistics for return series

Mean	0.0002159
Minimum value	-0.094695
Maximum value	0.109572
Standard deviation	0.012657
Skewness	-0.336674
Kurtosis	14.35698

Table 1 shows the descriptive statistics for the return samples. The mean of the sample is above zero (0.0002159 or 0.02159%). However, comparing to the mean value, the minimum value and the maximum value in the sample are extremely small and extremely large representing the high fluctuations for the return values. This is showed by the standard deviation (0.012657) which is not a small figure, comparing with the mean value.

The skewness and kurtosis describe the sample distribution, therefore, these two figures cannot be compared with return value. The sample distribution is nearly symmetrical (Skewness= -0.3366737), but the Kurtosis of 14.35698 means that the considerable volume of returns around the mean value.

Figure 3

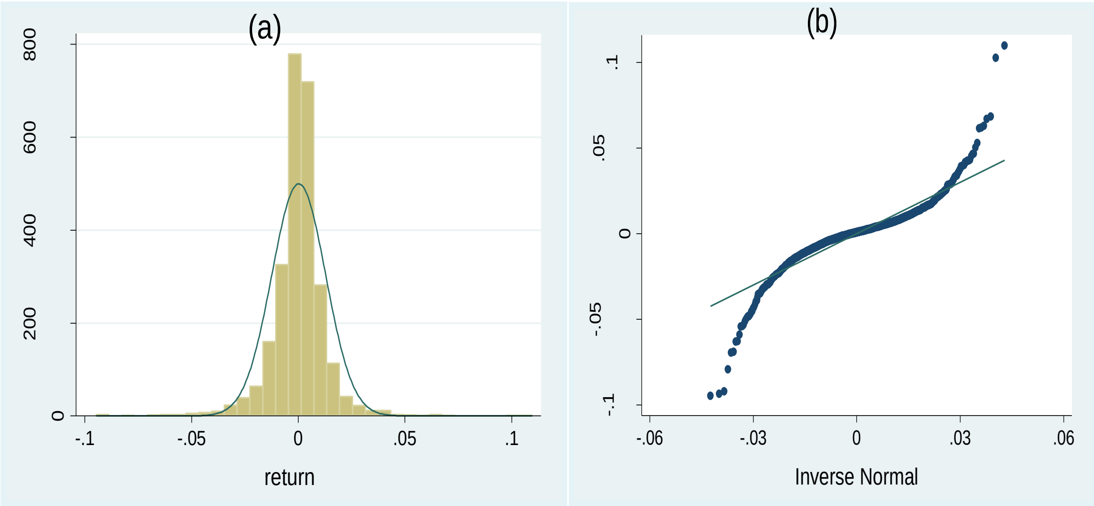


Figure 3 (a): Distribution of sample return,
Figure 3 (b): Q-Q Plot of sample return.

- *Distribution Leptokurtic*

Figure 3 (a) the gray area shows the totally 2640 return samples' distribution. The X-axis shows the return values, the Y-axis is the frequency of the return appearing in the 2640 observations. The blue bell curve together with the X-axis constitutes the normal distribution area which is the best fitted normal distribution to the samples. However, the sample distribution is higher peak and fatter tails than the corresponding normal distribution and this feature of return distribution is called *leptokurtic*. This means the considerable volume of return values around its mean value, but also not small number of return values are extremely large or small representing the very high risk for S&P 500 index in the sample period.

Figure 3 (b) is the Q-Q Plot of the return series which re-examines this leptokurtic feature more clearly. It compares the normal distribution with the return sample distribution. If the bold line is fitted with the thin straight line, it means the sample distribution is normality; otherwise, the sample distribution is not normality. The bold line deeply below the reference line in the left side and above the reference line in the right side means the higher peak and fatter tails of the sample distribution respectively.

Figure 4

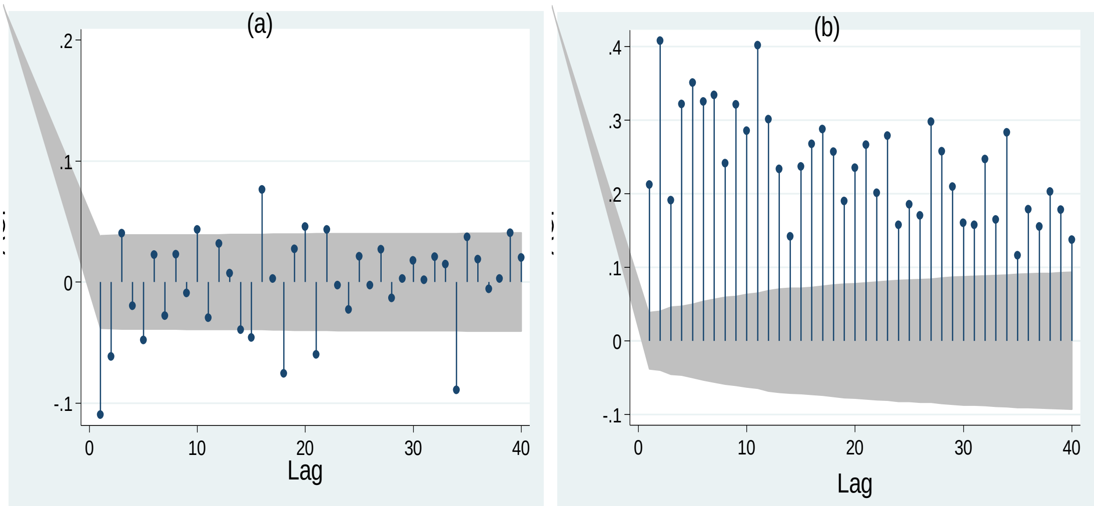


Figure 4 (a): ACF picture of return series,
Figure 4 (b): ACF picture of square of return series

Figure 4 (a) is the Autocorrelation function (ACF) of the return time series $\{r_t\}$. It shows the linear correlation coefficients between series $\{r_t\}$ and its lagged values. The X-axis shows the lag number and the Y-axis shows the auto-correlation coefficients values. If the series is serial correlation, the autocorrelation values of return series will significantly differ from zero in 95% confidence interval. Otherwise, all the plots will be in the gray area in ACF picture.

- *Serial Correlation*

The **Fig 4 (a)** ACF of S&P 500 return series shows that this time series firstly exhibits low levels of significant autocorrelations in first 5 lags; until the lag-15, lag-18, lag-21, the autocorrelations become significantly different from zero again. This is a small magnitude of the serial correlation in the return series. The Autoregressive (AR) model can describe this serial correlation characteristic in the return series or even using Autoregressive Moving Average (ARMA) model.

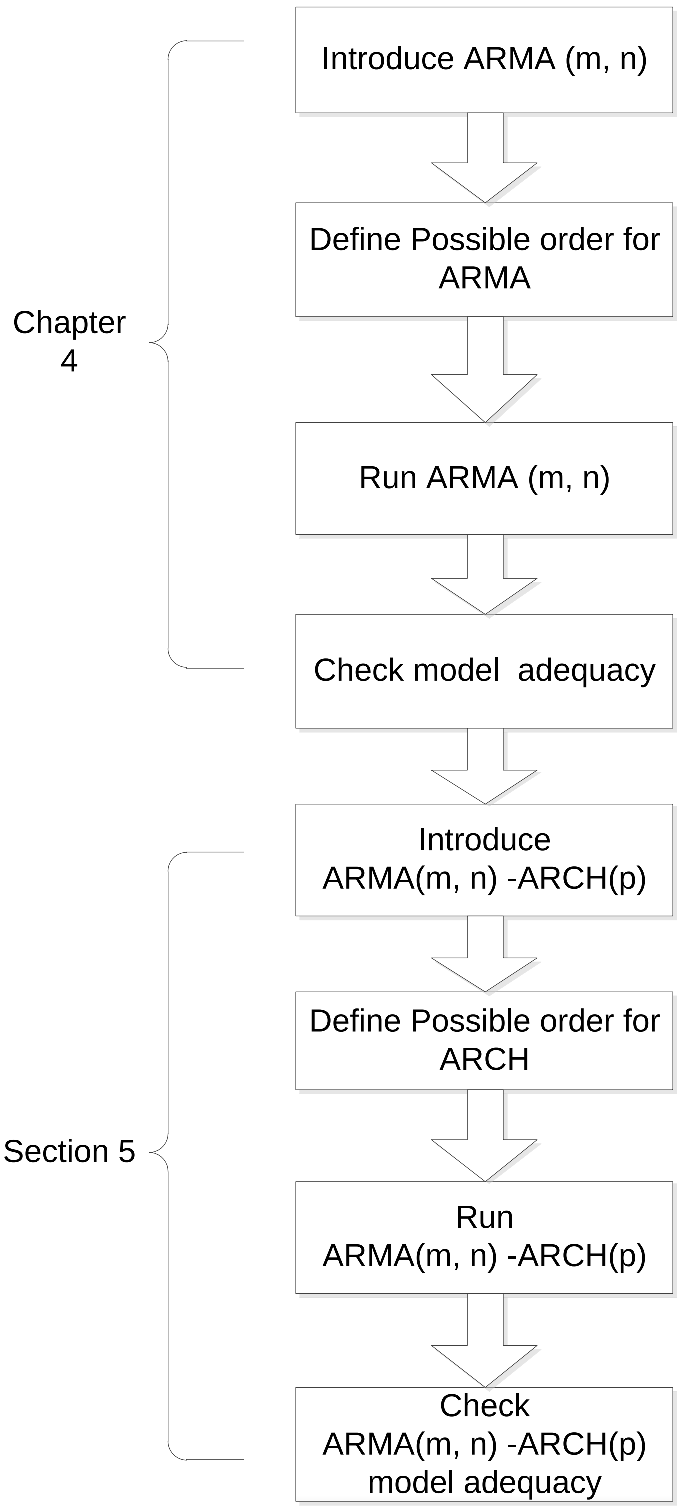
- *Serial Dependence*

However, the square of return series $\{\mathbf{r}_t^2\}$ has significantly much higher level of autocorrelations at least within 40-lags shown by **Figure 4 (b)**. Therefore, this return time series is low serial correlation but with highly non-linear dependencies in the second moment. The ARCH family model will try to capture this dependence in the return series.

3.4 Section summary

Therefore, after analysis this return series, it has four main characteristics which are the same findings in other stock returns time series no matter the high-frequency data or low-frequency data. These are mean reversion, volatility clustering, distribution leptokurtic, and series dependence. Based on the previous research for financial assets return series such as Hinich and Patterson (1985); Baillie and Bollerslev (1989); Brooks (1996), these characteristics can be best captured by an ARCH family model, which is modeling non-linear in variance (Chris Brooks,2008). However, the serial correlation feature in sample return cannot be described by ARCH-type model. Therefore, this paper will use ARCH-type model together with ARMA model (ARMA-ARCH) rather than a simple ARCH-type model for modeling return volatility.

The outline for next two Chapters is shown by the process chart below:


Introduce ARMA (m, n)
Define Possible order for ARMA
Run ARMA (m, n)
Check model adequacy
Chapter 4
Introduce ARMA(m, n) -ARCH(p)
Define Possible order for ARCH
Run ARMA(m, n) -ARCH(p)
Check ARMA(m, n) -ARCH(p) model adequacy
Section 5

Chapter 4

ARMA model for the mean equation

This S&P 500 return series has a small magnitude autocorrelation tested by ACF picture, which can be described by the autoregressive (AR) model, or even autoregressive moving average (ARMA) model (Box and Jenkins, 1976). Therefore, this paper is going to add ARMA model to describe the autocorrelation in mean equation in ARCH model.

However, the majority of the literatures, modeling for S&P 500 or other stock return volatility by using time series models, just simply regress the return on a constant and use ARCH–type model, without combining the AR or ARMA model. This may because their return series do not exhibit the serial correlation or just exhibit very slight magnitude autocorrelations, such as the low-frequency monthly data rather than the high frequency daily data used in this paper.

Because the main aim is to choose the best model for modeling sample volatility, the necessary for using AR or ARMA model just make sure the mean equation is adequacy.

The rest of this Chapter outline:

- Introducing ARMA model
- Stationary process --- *Dickey-Fuller* test
- ARMA order --- *AIC & BIC*
- Modeling adequacy checking --- *Portmanteau test* & ACF

For an **ARMA (m, n)** model:

$$\mathrm{r_t} = \left(\mathrm{c_0} + \sum_{i=1}^{\mathrm{m}} \mathrm{c}_i \cdot \mathrm{r}_{\mathrm{t}-i} + \sum_{j=1}^{\mathrm{n}} \mathrm{d}_j \cdot \varepsilon_{\mathrm{t}-j} \right) + \varepsilon_{\mathrm{t}} \qquad (2)$$

Where:

m is the lag order for AR part, which is a positive integral number;

n is the lag order for MA part, which is a positive integral number;

$\mathbf{c}_i$ is the coefficient parameter for lag return value, for i = {1, 2, 3, … , m};

$\mathbf{r}_{t-i}$ is the lagged return value, for i = {1, 2, 3, … , m};

$\mathbf{d}_j$ is the coefficient parameter for lag error term value, for j = {1, 2, 3, … , n};

$\boldsymbol{\varepsilon}_{t-j}$ is the lagged error term value, for j = {1, 2, 3, … , n}.

$\boldsymbol{\varepsilon}_t$ is the error term at time t.

The **Equation (2)** is called the mean equation, describing the return series. It shows the return $\mathbf{r}_t$ is a linear combination of its lagged values ($\mathbf{r}_{t-i}$) and lagged error terms ($\boldsymbol{\varepsilon}_{t-j}$). The second term in the bracket is the AR model part, capturing the *serial correlation* in S&P 500 return series {$\mathbf{r}_t$}. The third term in the bracket is the MA model part.

Because the square of sample series exhibits high autocorrelation, which means the sample series is with heteroscedastic, which the variance of error term is not constant over time, and also because the ARMA model cannot capture this heteroscedastic feature, therefore the error term in equation (2) here only is assumed with zero mean, and zero autocovariance over time.

This ARMA model (**equation 2**) is used as the mean equation in ARCH model.

4.2 Check the series Stationary

- *Stationary process*

Firstly examining return series is a stationary process or not from statistical way, because a weak stationary process has a mean reverting property. It means the effect from the previous value(s) become small and small, and decay as time passed, for a return series, it means even the return is volatile; it would not volatile unlimited, and will back to its mean values.

A weakly stationary process **{Y_t}**, but not a white noise*,* has a constant mean and constant variance over time, and a non-zero auto-covariance only depends on the number of lag k, not depends on the time t, i.e.:

$$\mathrm{E}(Y_t) = U, \text{ for all time t,} \tag{3}$$

$$\mathrm{Var}(Y_t) = \gamma_0, \text{ for all time t,} \tag{4}$$

$$\mathrm{Cov}(Y_t, Y_{t-k}) = \gamma_k, \text{ for all time t, } k \neq 0, \ \gamma_k \neq 0 \tag{5}$$

The *lag-k* autocorrelation $\boldsymbol{\rho_k}$ for a time series {$\mathbf{Y_t}$}:

$$\rho_k = \frac{\mathrm{Cov}(Y_t, Y_{t-k})}{\mathrm{Var}(Y_t)} = \frac{\gamma_k}{\gamma_0} \tag{6}$$

If the absolute values of auto-correlation of the series is less than 1, (**$|\rho_k|$ <1** for k≠0), this time series could be a *stationary process*, which the effect from the previous value(s) become small and small, and decay as time passed; if no auto-correlation feature (**ρ_k**=0, for k≠0), this could be a white noise; if the autocorrelation equal to 1 (**ρ_k**=1, for k≠0), it is a *unit root process* or called random walk, which the shocks from lag values do not decay and do not increase, and its ACF picture are firstly approaching 1 and be seen decay very slowly; if the autocorrelation of the series is above 1, (**ρ_k >1** for k≠0), it is a deterministic trend process, which can be obviously observed from the time series picture with observed main trend with fluctuations.

Because sample return series has an autocorrelation, and does not exhibit deterministic trend, therefore, the sample series either a stationary process or unit root process. Because the ACF for lag-1 in **Figure 3 (a)** seems between (-0.2, -0.1) not approach value 1, simply observed, the sample series is a stationary process. However sometimes it is not easy to distinct these two processes clearly from simply observed an ACF picture (Brooks, 2008), for insurance for the result, this paper implements *Dickey-Fuller test* for non-unit root (or stationary) test to re-examine the observed feature of mean reversion in sample return series.

- *Dickey-Fuller test*

The null hypothesis (H_0) is: series being a unit root process; reject the null here (H_1): series is a stationary process.

Step 1: For a series {$\mathbf{y_t}$},

$$y_t = c \cdot y_{t-1} + \varepsilon_t \qquad (7.1)$$

Where: c is the coefficient parameter for y_{t-1}, and ε_t is the error term, the null hypothesis becomes: $H_0\colon\ c = 1$

Step 2: Taking the first difference for **Equation (7.1)**:

$$\begin{aligned}\Delta y_t &= y_t - y_{t-1} && (7.2)\\ &= (c \cdot y_{t-1} + \varepsilon_t) - y_{t-1} \\ &= (c-1) \cdot y_{t-1} + \varepsilon_t && \text{or,}\end{aligned}$$

$$\Delta y_t = d \cdot y_{t-1} + \varepsilon_t, \qquad \text{for } d = c - 1 \qquad (7.3)$$

Therefore the null hypothesis becomes: $H_0\colon\ c - 1 = 0,\ \text{ or }\ \ d = 0$

Step 3: Under the null hypothesis, Test statistics does not follow a usual ***t***-distribution, but rather than following a non-standard distribution, where:

$$\text{Test statistics} = \frac{\hat{d}}{\widehat{SE(\hat{d})}} \tag{8}$$

Where: SE means standard error.

Accept the null, if the p-value of the test statistics is above the significant level. Reject the null, if the P-value for test statistics is at 5% significant level.

Table 2: Dickey-Fuller test for S&P 500 return

Test Statistic:	-57.3370
p-value:	0.0000

Table 2 shows the result of Dickey-Fuller test for sample return. The p-value of the test statistic (0.0000) is highly significantly at 1% level. Therefore, we strongly reject the null of unit root process, so the sample S&P 500 return series is indeed a stationary process.

4.3 Re-examining the Serial Correlation

- *Portmanteau Test*:

Secondly, this paper re-examines the return serial correlation by Portmanteau Test. This is the same function as ACF picture to test the series $\{\mathbf{r_t}\}$ is serial correlation or not, from the statistic way. This method is modified by *Ljung and Box* (1978). The null hypothesis (H_0) is the series being white noise from the view of no autocorrelation within at least *m* lags; then the series' autocorrelations are all jointly zero within m lags. Reject the null, if any of the autocorrelation is significantly different zero.

$$H_0: \rho_1 = \rho_2 = \cdots \rho_m = 0 \tag{9}$$

$$H_1: \rho_k \neq 0 \; for\ any\ one\ or\ some\ k\epsilon\{1, 2, \ldots, m\} \tag{10}$$

Where: $\boldsymbol{\rho_k}$ is the lag-k autocorrelation, **k** is the positive integer.

Under the null hypothesis, the general statistic of linear dependence Q (m) should be distributed as a Chi-square random variable with m degree of freedom (df), then the p-value of Q(m) should greater than the **α**, where the **α** is the significance level chose.

$$\mathrm{Q(m)} = \mathrm{T(T+2)}\sum_{i=1}^{11}\frac{\widehat{\rho_i^2}}{T-i} \sim \chi_\alpha^2(m) \qquad (11)$$

Where:

T is the sample size;

m is the maximum lag length;

χ^2_α is the 100•(1-α)th percentile of a Chi-squared distribution with *m* degree of freedom.

Ruey s. Tsay shows that the simulation studies result for $\mathrm{m} \approx \ln(\mathrm{T})$ provides better performance. Reject the null if $\mathrm{Q(m)} > \chi^2_\alpha$ with *m* degrees of freedom, then the provided p-value of Q (m) by software is less than or equal to ***α***, where ***α*** is the significance level.

Because the return sample observation is: $\mathrm{T} = 2640$

Then: $$\mathrm{m} \approx \ln(\mathrm{T}) = \ln(2640) \approx 7.8785342 \approx 8$$

Table 3

Portmanteau Test for sample S&P 500 return series

Portmanteau (Q) statistic :	116.3556
Prob > chi2(20) :	0.0000
Portmanteau (Q) statistic :	57.9747
Prob > chi2(8) :	0.0000

According to the Portmanteau test with m=20 and m=8 degree of freedom (df) that the Q (20) = 116.3556 and Q (8) = 57.9747. The p-values of these two statistics both are 0.0000 less than 5% significant level, then reject the null hypothesis of series no autocorrelation, together with the ACF of return picture, suggesting the sample return does have an autocorrelation feature.

4.4 Decide using ARMA for return series

Figure 5

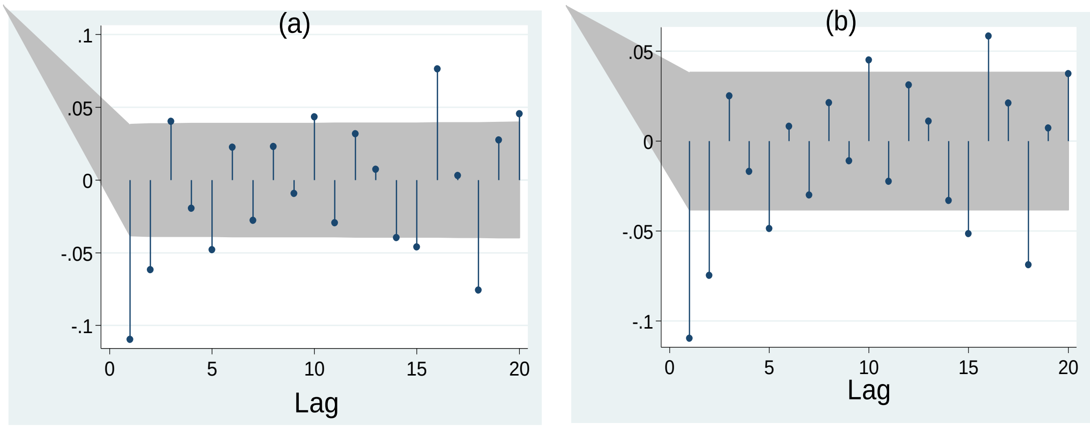


Figure 5 (a): ACF of the sample S&P 500 log returns from 5th/01/2004 to 30th/06/2014;
Figure 5 (b): PACF of the sample S&P 500 log returns in same time interval.

- *PACF picture*

The partial autocorrelation function (PACF) $\boldsymbol{\tau_k}$ measures the partial autocorrelation between current value and lag-k value after controlling for the effects of observations at all lags<k. For example: for a stationary AR (1) model for $\{\mathbf{y_t}\}$ time series:

$$y_t = c_1 \cdot y_{t-1} + \varepsilon_t \qquad (12.1)$$

Where: the $\mathbf{c_1}$ is the correlation coefficient for $\mathbf{y_{t-1}}$,

$\boldsymbol{\varepsilon_t}$ is the error term which assumption should be a white noise process, that is zero mean, constant variance all time and zero autocorrelations for all the time.

Then: $$y_{t-1} = c_1 \cdot y_{t-2} + \varepsilon_{t-1} \qquad (12.2)$$

Then $\mathbf{y_t}$ could be written as: $y_t = c_1 \cdot y_{t-1} + \varepsilon_t$ (12.3)

$$= c_1 \cdot (c_1 \cdot y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \qquad (12.4)$$

$$= c_1^2 \cdot y_{t-2} + c_1 \cdot \varepsilon_{t-1} + \varepsilon_t \qquad (12.5)$$

Therefore the correlation coefficient between $\mathbf{y_t}$ and $\mathbf{y_{t-2}}$ is $\mathbf{c_1^2}$. However, because $\mathbf{y_t}$ is an AR (1) model that only depends on lag-1 value, hence, after controlling the effect of $\mathbf{y_{t-1}}$ to the $\mathbf{y_t}$, $\mathbf{y_{t-2}}$ cannot provide additional information for $\mathbf{y_t}$. The effect of $\mathbf{y_{t-2}}$ to the $\mathbf{y_t}$ is zero. i.e. The partial autocorrelation for *lag-2* is zero. The same reason for PACF of *lag-2, lag-4*, … … are all zero. Therefore, for a stationary AR (1) series, this series' PACF is cut off at lag-1. For a pure AR (*m*) time series, it is PACF is cut off at *lag-m*.

Figure 5 (a) and **(b)** shows the ACF and PACF of sample return series within 20 lags respectively. The ACF picture shows the sample series has the significantly small magnitude serial autocorrelation the same result with *Portmanteau Test*, and this can be captured by the AR model. However, the PACF firstly significantly differ from zero within first five lags, and re-significantly differ from zero at lag-10, lag-15, and lag-18, which means the PACF of this sample series does not exactly significantly cut off at some lag order at least within 20 lags. Therefore, **Fig. 5 (a)** and **(b)** show that it is necessary to implementing an AR model, but only a pure AR model might not be adequate to describe the mean equation for sample S&P 500 return series. Therefore, this paper tries to implement the ARMA model, which explains the series using both the lagged return and lagged error term in the mean equation.

4.5 Define the orders for ARMA model

Fourthly, define the possible orders for ARMA model for the/a mean equation by using the Akaike's (1974) information criterion (AIC) and Schwarz's (1978) Bayesian information criterion (BIC):

$$\mathrm{AIC} = -2 \cdot \ln\mathrm{L} + 2 \cdot \mathrm{K} \qquad (13)$$

$$\mathrm{BIC} = -2 \cdot \ln\mathrm{L} + \mathrm{K} \cdot \ln\mathrm{T} \qquad (14)$$

Where:

L is the maximum likelihood estimates,

T is the sample size,

K is the number of parameters in the mean equation,

The first term in Equation AIC measures the goodness of fit of the ARMA model to the data, the bigger the ***lnL*** is, the better the goodness of fit of the model is. Whereas the second term in Equation **(13)** is the penalty function of the criterion. The less the parameters used, the better the model. Or consider, if two models both can explain a series well in the same level, the less the parameters the better the model. Therefore, the less the AIC value the better the model fit to the series.

The ***K*lnT*** is another penalty function resulting in another information criteria (i.e. BIC). The less the BIC value the better the model fit to the series. Therefore choose the orders for ARMA model that has the minimum AIC value and the minimum BIC value. According to these two information criteria, these two 'best' orders may be different or may be the same orders for ARMA model.

- *Implementing AIC & BIC*:

Because the autocorrelation coefficients after lag-5 decays and insignificantly differ from zero until lag-10, although in lag-10 and lag-15, the autocorrelation coefficients become significantly differ from zero, the ACF of return series is approximately cut off at lag 5.

From the PACF picture, the partial autocorrelation coefficients are significantly different from zero for lag-1, lag-2, lag-5. Lag-5 and re-significantly differ from zero until lag-18. Therefore, this paper estimates the ARMA(p,q) for p={0,1,2,3,4,5} and q={0,1,2,3,4,5}, and choose the one which minimizes the AIC and the one minimizes the BIC value.

Table 4

AIC & BIC values for ARMA model for return series

(m,n)	AIC	BIC	(m,n)	AIC	BIC
(0,0)	-15576.34	-15564.58	(3,0)	-15618.63	-15589.24
(0,1)	-15610.93	-15593.29	(3,1)	-15616.79	-15581.52
(0,2)	-15617.24	-15593.73	(3,2)	-15627.29	-15586.14
(0,3)	-15617.42	-15588.03	(3,3)	-15639.82	-15592.79
(0,4)	-15617.28	-15582.01	(3,4)	-15637.84	-15584.93
(0,5)	-15621.15	-15580	(3,5)	-15625.01	-15566.23
(1,0)	-15606.22	-15588.58	(4,0)	-15617.39	-15582.12
(1,1)	-15614.76	-15591.25	(4,1)	-15618.19	-15577.04
(1,2)	-15617.34	-15587.95	(4,2)	-15639.86	-15592.84
(1,3)	-15615.83	-15580.55	(4,3)	-15637.83	-15584.92
(1,4)	-15617.6	-15576.45	(4,4)	-15636	-15577.21
(1,5)	-15628.63	-15581.6	(4,5)	-15631.8	-15567.13
(2,0)	-15618.96	-15595.45	(5,0)	-15621.64	-15580.49
(2,1)	-15618.56	-15589.17	(5,1)	-15627.02	-15579.99
(2,2)	-15617.61	-15582.34	(5,2)	-15626.13	-15573.23
(2,3)	-15626.58	-15585.43	(5,3)	-15624.21	-15565.42
(2,4)	-15639.99	-15592.96	(5,4)	-15634.4	-15579.73
(2,5)	-15626.97	-15574.06	(5,5)	-15636.88	-15566.34

Where: the (m,n) means the ARMA order number, m for AR order, n for MA order.

Table 4 shows the AIC and BIC values output by STATA. The minimizing value of AIC (-15639.99) is for ARMA (2,4) and the minimizing value for BIC is

-15595.45 for ARMA (2,0). Thus, the next step is to implement ARMA (2,0) and ARMA(2,4) for S&P 500 return series respectively and to check which model is more suitable.

- The ARMA (2,0) model for sample return series is shown below:

$$\widehat{r_t} = 0.0\widehat{002}151 - 0.1\widehat{177}303 \cdot r_{t-1} - 0.\widehat{746}346 \cdot r_{t-2} \qquad (15.1)$$

The standard errors for estimated parameters are 0.0002158; 0.0108606; and 0.0082858 respectively, all are very small. They measure the accuracy for the estimated parameters, which means the values of parameters' coefficients provide good performance. Except the constant has p-value larger than 10% significant level, other two lagged return values are both significantly at 1% level. Therefore, equation (15.1) could be re-written as:

$$\widehat{r_t} = -0.1\widehat{177}303 \cdot r_{t-1} - 0.\widehat{746}346 \cdot r_{t-2} \qquad (15.2)$$

- *ARMA (2,0) adequacy:*

If an AR or ARMA model is adequate, then the estimated residual $\widehat{\varepsilon_t}$, which estimates for the value of error term ε_t in the mean equation, should be a white noise process with zero mean, constant variance and zero autocorrelation all the time. However, the sample return series has a time varying variance, thus the time-varying variance is for error term, because this heteroscedastic feature will be described by ARCH –type model, here just test the ARMA(2,0) model's adequacy from the view of no-autocorrelation in the estimated residual by observing the ACF picture of estimated residual (shown in Figure 6) and *Portmanteau Test*

Figure 6

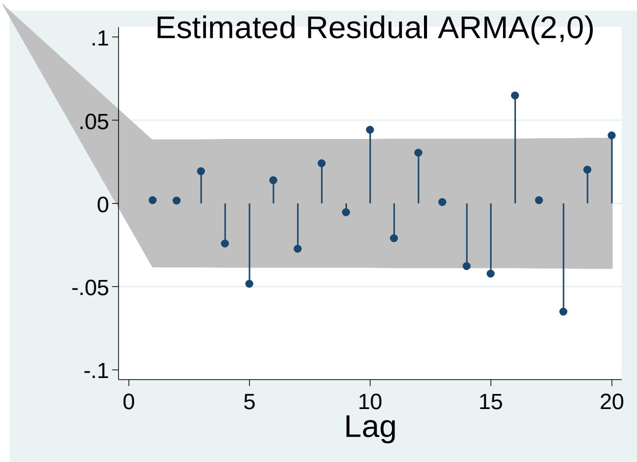


Figure 6 shows the autocorrelation function of estimated residual, when using ARMA (2,0) model to sample S&P 500 log return series. Because majority ACFs within 20 lags are in the gray area, which are not significantly different from zero, only two ACF values are just above 0.05 and just under -0.05 respectively, which suggests the estimated residual $\hat{\varepsilon}_t$ is not significantly serial correlation.

Table 5

Portmanteau Test for estimated residual in ARMA (2,0)

Portmanteau Q(10) statistic :	18.0729
Prob > $chi^2(10)$:	0.0537
Portmanteau Q(8) statistic :	12.8091
Prob > chi2(8) :	0.1186

Table 5 shows the Portmanteau Test for estimated residual in ARMA (2,0) model. because Q(10)=18.0729 with p-value =0.0537 is slightly bigger than 5% significant level, and Q(8)=12.8091 with p-value is bigger than the 5% significant level (11.86% > 5%), then accept the null hypothesis of estimated residual in ARMA(2,0) model is not serial correlation, therefore together with the ACF picture examined, the ARMA (2,0) model for return series in the mean equation is adequate.

The ARMA(2,4) for sample series as the mean equation is shown below:

$$\widehat{r_t} = 0.0\widehat{002}152 + -0.\widehat{720}8521 \cdot r_{t-1} + -0.\widehat{986}8351 \cdot r_{t-2} + 0.6\widehat{087}11 \cdot \varepsilon_{t-1}$$
$$+ 0.8\widehat{535}012 \cdot \varepsilon_{t-2} - 0.1\widehat{303}663 \cdot \varepsilon_{t-3} - 0.0\widehat{490}067 \cdot \varepsilon_{t-4}$$

Except the p-value of constant is not significant within 5% level, other p-values for explainers all are significantly in 5% level. All of the standard errors for estimated coefficients are less than 0.013, which means the estimated coefficients are estimated well. The ARMA(2,4) equation could be re-written as:

$$\widehat{r_t} = -0.\widehat{720}8521 \cdot r_{t-1} - 0.\widehat{986}8351 \cdot r_{t-2} + 0.6\widehat{087}11 \cdot \varepsilon_{t-1} + 0.8\widehat{535}012 \cdot \varepsilon_{t-2}$$
$$- 0.1\widehat{303}663 \cdot \varepsilon_{t-3} - 0.0\widehat{490}067 \cdot \varepsilon_{t-4}$$

It is noticed that the coefficient of r_{t-1} (-0.7208521) and coefficient of ε_{t-1} (0.6808711) tend to be offset, and the same situation to the coefficient of r_{t-2} (-0.9868351) and the coefficient of ε_{t-2} (0.8535012). Because the software run the ARMA(2,4) for sample return series, if the ARMA(2,4) is the suitable model for sample return, then output results with significant p-value and small standard error for coefficients would be obtained, however, if the ARMA(2,4) model is not suitable for sample return series, then the software will impose this model to sample series. Simply observed the output for the offsetting estimated coefficients, it is conjectured that the ARMA(2,4) is not suitable for sample series, and the ARMA(2,4) model may contain too much over lagged values for describing return series.

- Checking ARMA (2,4) model adequacy:

Table 6

Portmanteau Test for estimated residual in ARMA (2,4)

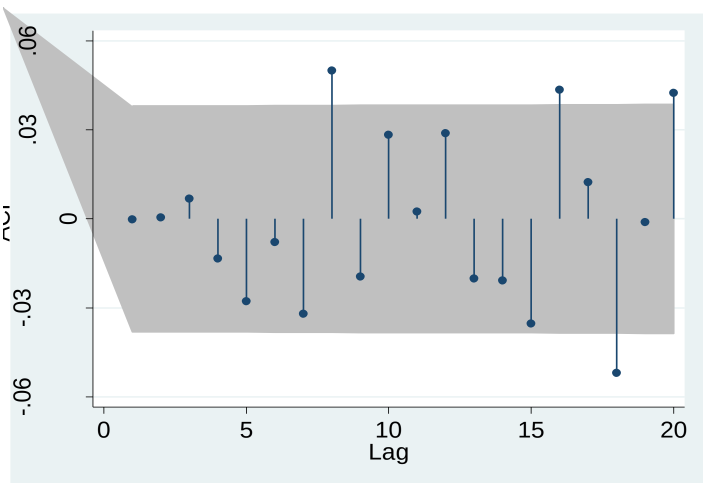
Portmanteau (Q=20) statistic	=	40.4684
Prob > chi2(20)	=	0.0044
Portmanteau (Q=8) statistic	=	15.2945
Prob > chi2(8)	=	0.1217

Table 6 shows the portmanteau test for estimated residual in ARMA (2,4) model. even though the p-value of Q(8) is insignificant at 5% level (0.1217 >5%), the p-value of Q(20) significantly at 5% level means the residual series in ARMA(2,4) model exhibits low autocorrelations within 20 lags.

The **Figure 7** of ACF picture of estimated residual in ARMA (2,4) shows the this residual series has low magnitude autocorrelations with lag-8, lag-16 and lag-18., which is consistent with the results in table 5. Therefore, due to the autocorrelation feature in residual in ARMA(2,4), the ARMA(2,4) cannot capture all the autocorrelation features in the sample return series and it is not adequate for using in the mean equation.

Figure 7

ACF picture: estimated residual in ARMA (2,4)



Therefore, after examining the two models for the mean equation, only ARMA (2,0) is adequate for describing mean equation.

Table 7
Descriptive statistics for residual in ARMA (2,0)

Mean	0.0000008970
Standrd deviation	0.0125454
Variance	0.0001574
Skewness	-0.5478653
Kurtosis	13.44373

Table 7 shows the descriptive statistics for residual obtained from ARMA (2,0) model. The mean value is extremely near zero. However, the 13.44373 kurtosis shows the distribution of this residual series is still excess taller than a normal distribution with kurtosis equal to 3. This also examined in the **Figure 8 (a)** and **(b)**. The inverse 'S' shape of the bold line in the Q-Q plot **Fig.(b)** shows the distribution of the residual series is distribution leptokurtic as the sample return series.

Figure 8

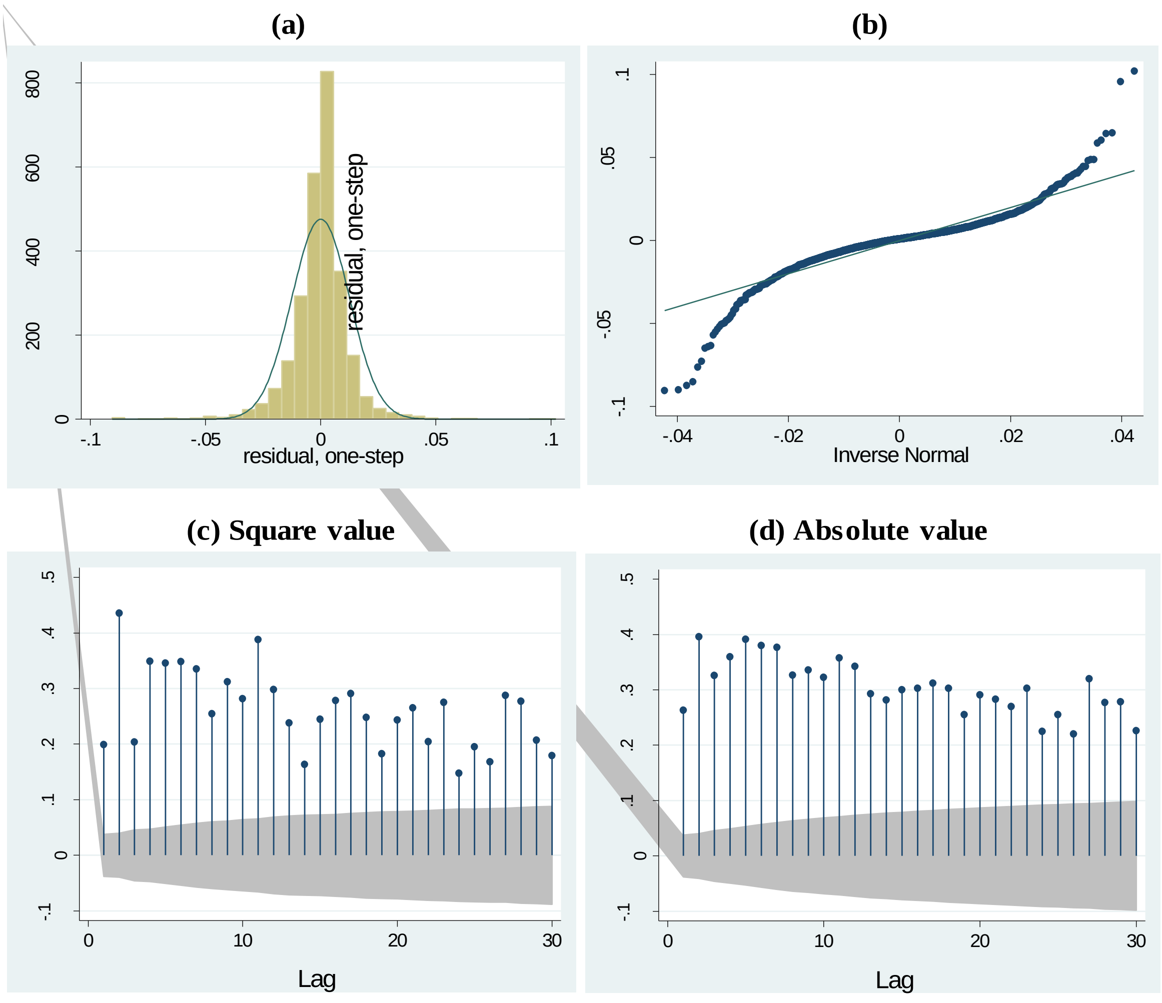


Figure 8 (a): distribution of the residual in ARMA (2,0)

(b): Q-Q Plot of residual in ARMA (2,0)

(c): ACF of square of residual in ARMA (2,0)

(d): ACF of absolute residual in ARMA (2,0)

Figure 8 (c) and **(d)** show the high autocorrelations in both the square of residual and absolute residual in ARMA (2,0) model, which means the dependence feature in the residual series, therefore it can be described by ARCH type model, i. e, implementing ARCH type model is meaningful. Therefore, the next section is to combine the ARCH model with ARMA(2,0).

Chapter 5

Modeling Volatility: ARMA-ARCH model

An ARMA (m, n) - Autoregressive Conditionally Heteroscedastic ARCH (p) model is consistent with the following three equations (Engle, 1982):

$$r_t = \left(c_0 + \sum_{i=1}^{m} c_i \cdot r_{t-i} + \sum_{j=1}^{n} d_j \cdot \varepsilon_{t-j} \right) + \varepsilon_t \qquad \text{recall (2)}$$

$$\varepsilon_t \mid I_{t-1} \sim N\,(0, h_t) \qquad (17.1)$$

$$h_t = a_0 + \textstyle\sum_{k=1}^{p} a_k \cdot \varepsilon_{t-k}^2 \qquad (18)$$

Where: **t** is the positive integral number, stand for the dates during the chosen duration

r_t is the log daily return at time t,

ε_t is the error term or residual at time t,

I_{t-1} is the all of the information set at time t-1, it also includes the information before t-1.

h_t is the conditional variance of ε_t,

a_0 is a non-negative constant,

a_k is the parameter of the **ε^2_{t-k}**, for k = { 1, 2, 3, … , p}, and a_k <1, otherwise the h_t will continuously increase over time.

p represents the lagged order number, which is a positive integral number,

All the boundaries of the parameters in equation (18) are making sure the h_t is non-negative.

Equation (2) represents the mean equation, it describes that the return series at time t is explained by the ARMA model and the rest residual par at time t **ε_t.**

Thus, this error term represents the unexpected news or innovations on the return at time t.

Equation (17) means that conditional on all of the information set in time **t-1**, the error term $\boldsymbol{\varepsilon_t}$ at time **t** is normally distributed with mean zero and variance $\mathbf{h_t}$ at time t. Therefore, this allows the error term variance to be a time-varying variance and this feature is called the heteroscedastic in the equation (2), rather than the homoscedastic, which the variance of the error term is constant all the time.

Based on Engle (1982) said the ARCH model can capture the leptokurtic feature in error term, therefore, Here the assumption for error term is the conditional normality assumption, which cannot directly be observed from the data, rather than an unconditional distribution shown by **Figure 8 (a)**.

Equation (17.1) can also be alternatively written down as (17.2), (17.3):

$$\varepsilon_t = \epsilon_t \cdot \sqrt{h_t} \mid I_{t-1} \qquad (17.2)$$

$$\epsilon_t \sim N\,(0,1) \qquad (17.3)$$

Where: ϵ_t is the standardized error term, which is an unconditionally identical independent distribution (i.i.d). Based on the assumption for error term conditional normality distribution, the standardized residual ϵ_t, therefore, is unconditionally normally distributed with zero mean and unit variance. Therefore, in order to check this normality assumption right or not, it is easy to check the standardized residual unconditional distribution is normal or not.

Therefore, the conditional variance of the return at time t, conditional on information set at time **t-1**, is equal to this conditional variance of the error term at time t (**ε_t**)

Equation (18) is the variance equation to capture the *dependence* in the return series, i.e. the serial correlation in the square of return series. **Equation (18)** shows this conditional volatility of the error term at time t (**h_t**) depends on the lagged of the square of error term values (**ε^2_t**). Therefore, this variance equation can capture the changes in the conditional variance of the error term over time, hence captures the changes in the conditional variance of the return series over time.

Furthermore, variance equation explains the return *volatility clustering* feature Engle (1982).

Moreover, In the variance **equation (18)**: all of the boundaries for the parameters firstly make sure the ht is a non-negative value, and the secondly the effect from lagged value ε^2_t should decay, such as: $a_1 + \ldots + a_q < 1$, and $a_k < 1$, for k={1, 2, …, q}.

5.2 Implement ARCH model with ARMA (2,0)

Because the appropriate order for ARMA for return series in the mean equation is ARMA(2,0), this paper next is to combine this model AR(2) together with ARCH-type model.

5.2.1 Possible order determination for ARCH

Firstly this paper defines the possible lagged order for ARCH model for sampling series. There is no accurate method to determine the order of ARCH model, Ruey S. Tsay (2005) uses the Partial autocorrelation function (PACF) of the square of error term series $\{\varepsilon^2_t\}$ obtained in mean equation (2) to define the possible order.

The PACF measures the partial autocorrelation for the series. If the PACF of series $\{\varepsilon^2_t\}$ cut off at lagged q, controlling for the effect from ε^2_{t-1}, …, ε^2_{t-q}, the partial autocorrelation between ε^2_t and ε^2_{t-k} is zero, for k>q. Any k lagged values cannot provide additional information. ε^2_{t-1}, ε^2_{t-2}, …, ε^2_{t-q} exhausts all information about the past values of the variable ε^2_t. Therefore the series $\{\varepsilon^2_t\}$ is autoregressive within lagged q:

$$\varepsilon_t^2 = c_0 + \sum_{k=1}^{p} c_k \cdot \varepsilon_{t-k}^2 + e_t \qquad (25)$$

Where: c_0 is constant;
C_k are parameters for ε_{t-k}^2, for k = {1, 2, 3, …, p};
e_t is the error term at time t.

If the PACF of ε^2_t cut off at lagged *p*, Ruey S. Tsay (2005) says then the possible order for ARCH model for return series $\{r_t\}$ is *p* order, i.e. ARCH(*p*).

Because of equation (19):

$$h_t|I_{t-1} = E[(\varepsilon_t - 0)^2|I_{t-1}]$$

Therefore:

$$h_t|I_{t-1} = E[(\varepsilon_t - 0)^2|I_{t-1}]$$

$$=E[\varepsilon^2|I_{t-1}]$$

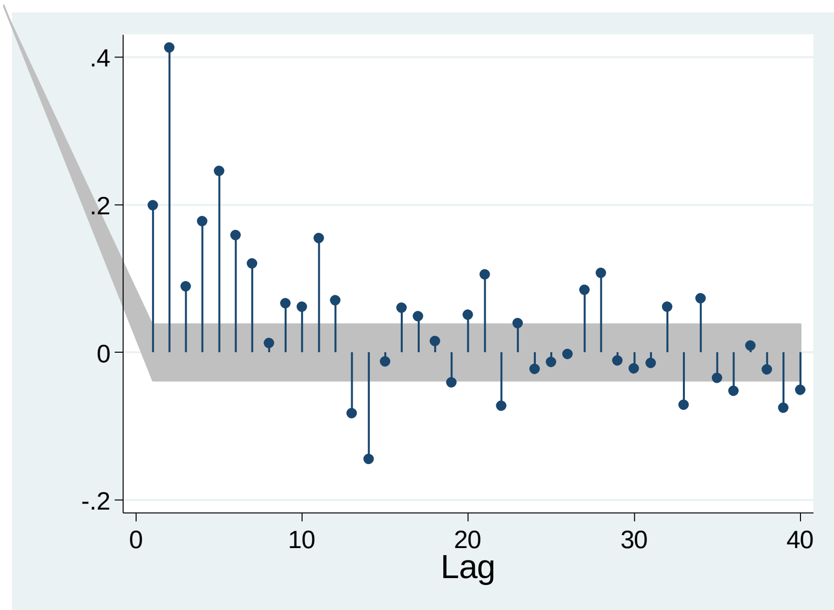
$$= E\left[c_0 + \sum_{k=1}^{p} c_k \cdot \varepsilon_{t-k}^2 + e_t|I_{t-1}\right]$$

$$= c_0 + \sum_{k=1}^{p} c_k \cdot \varepsilon_{t-k}^2$$

However, this method only provides the possible order, it is necessary to test this ARCH (*p*) effect in the next step.

Figure 9

PACF of square of estimated error in AR (2)



Regress the S&P 500 log return series $\{r_t\}$ on the appropriate ARMA model, i.e. AR (2), to predict the residual $\boldsymbol{\varepsilon_t}$ series, and then obtain the estimated $\{\varepsilon^2_t\}$ series. **Figure 9** shows the PACF of the $\{\varepsilon^2_t\}$ time series after regressing sample return on AR (2) model. Because from the first lagged order to 14th lagged order, except the *lag-8*, all partial autocorrelation are statistically significance at 95% confidence interval, while almost partial autocorrelation of lag k values are nearly insignificant, for $k > 14$. This suggests the possible ARCH order for this return series is 14.

5.2.2 Test the ARCH (p) effect --- *Lagrange multiplier test*

Before applying the possible AR(2)-ARCH(p) model, we should examine whether this return series presents an ARCH (*p*) effect or not by using the *Lagrange multiplier test*.

Step 1, regress the return series on propitiate ARMA model and obtain the estimated residuals $\{\hat{\varepsilon}_t\}$ and then obtain its squares value $\{\widehat{\varepsilon_t^2}\}$.

Step 2, based on the possible ***p*** order obtained before, regress estimated $\boldsymbol{\varepsilon^2_t}$ on its lags $\boldsymbol{\varepsilon^2_{t-1}, \ldots, \varepsilon^2_{t-p}}$, i.e. the equation (25), obtain $\mathbf{R^2}$ of this regression.

Step 3, the joint null hypothesis (H_0) of no ARCH (q) effect is that all parameters of ***p*** lags of the squared residuals ($\mathbf{c_1, c_2, \ldots, c_q}$) are all significantly zero, which means the calculated value $(T\text{-}q)\bullet R^2$ should be distributional with critical Chi-square value at the ***p*** degree of freedom with 5% significant level, where the ***T*** is the total sample return population:

$$H_0\text{: no ARCH(p)effect,} \qquad (T-p)\cdot R^2 \sim \chi^2_p \qquad (26)$$

$$H_0\text{: has ARCH(p)effect,} \qquad (T-p)\cdot R^2 > \chi^2_p \qquad (27)$$

Reject the null (H_1), if the calculated value $(T\text{-}q)\bullet R^2$ is greater that it's critical value. It means at least one of this coefficient parameter (c_1, c_2, …, c_p) is significantly not zero.

Test the AR(2)-ARCH(14) for S&P 500 return series, the R^2 in step 2 is 0.3524, the total number of observations for return is 2640, the Chi-square with 14 degrees of freedom at 5% significant level is 23.68, therefore:

$$(T\text{-}q)\bullet R^2 = 0.3524 \bullet (2640 - 14) = 925.4024 > 23.68$$

Where: All the critical chi-square values used in this paper are obtained from: http://passel.unl.edu/pages/informationmodule.php?idinformationmodule=1130447119&topicorder=8&maxto=15&minto=1
Then null hypothesis of no ARCH (14) on return series is rejected. i.e. the ARCH effect does exist in the sample return series.

4.2.3 Modeling the ARCH (p) model

After running the ARMA (2,0)-ARCH(14) model for S&P 500 return time series, However, the p-values for ARCH part lag-14, lag-13, lag-12, and lag 1 of square of residuals are not significantly within 5% level, therefore, it could be a ARCH (11) model.

Test the ARCH (11) effect for S&P 500 return series using *Lagrange multiplier test* using STATA software: $(T\text{-}q)\bullet R^2 = (2640 - 11)\bullet 0.3306 = 869.1474 > \chi^2_{11} = 19.68$, at 5% significant level, then reject the null-hypothesis of no ARCH (11) effect. Use STATA to run ARCH (11). Except the lag-1 in ARCH model part and lag-2 in AR model part, all the p-values for explainers are significantly at 5% level.

5.3 Check the adequacies of the ARMA(2,0)-ARCH(11) model

Firstly, check the mean equation adequacy: if the mean equation is adequate, the estimated standardized residual ($\widehat{\epsilon_t}$) should be a white noise process, rather than serial correlation, where the standardized residual is equal to:

$$\varepsilon_t = \epsilon_t \cdot \sqrt{h_t} \mid I_{t-1} \qquad \text{recall (6)}$$

A white noise process $\{y_t\}$ has a constant mean and constant variance for all time, and zero auto-covariance. These three conditions make the white noise series is independent and identical distribution:

$$E(y_t) = 0, \text{ for all time t,} \qquad (28)$$

$$Var(y_t) = \sigma^2, \text{ for all t,} \qquad (29)$$

$$Cov(y_t, y_s) = 0, \text{ fot time t} \neq \text{ time s} \qquad (30)$$

The white noise is checked by the *Portmanteau test* and also checked by the *ACF picture* of the series. These are checked from the view of no serial correlation.

Secondly, check the Variance equation adequacy: the variance equation is adequate, only if the square of standardized residual $\{\epsilon_t^2\}$ is white noise rather than serially correlated.

Thirdly, check that the standardized residual term (ϵ_t) that should be normally distributed. This is checked from the assumption of equation (7):

$$\epsilon_t \sim N\ (0{,}1) \qquad \text{recall (7)}$$

(1) The Mean equation checking:

Figure 10

ACF of standardized residual ARMA(2,0)-ARCH(11)

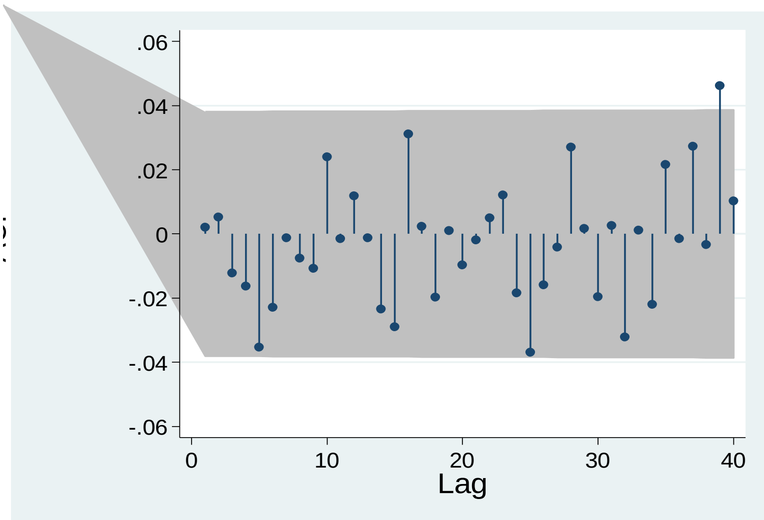


Figure 10 shows the autocorrelation function of standardized residual, when using ARMA(2,0) - ARCH (11) model for S&P 500 log return series. However, according to this graph, this standardized residual is not significantly auto-correlated with its past lag values, together with **Table 8**, the p-value for Q(20) and for Q(8) both are above 5% significant level, therefore accept the null hypothesis of no serial autocorrelation for standardized residual; therefore the mean equation is adequate.

Table 8

Portmanteau Test for standardized residual in AR(2)-ARCH(11)

Portmanteau Q(20) statistic :	15.8130
Prob > $chi^2(20)$:	0.7282
Portmanteau Q(8) statistic :	6.0260
Prob > chi2(8) :	0.6443

(2) Variance equation checking:

Figure 11

ACF of square of standardized residual AR(2)-ARCH(11)

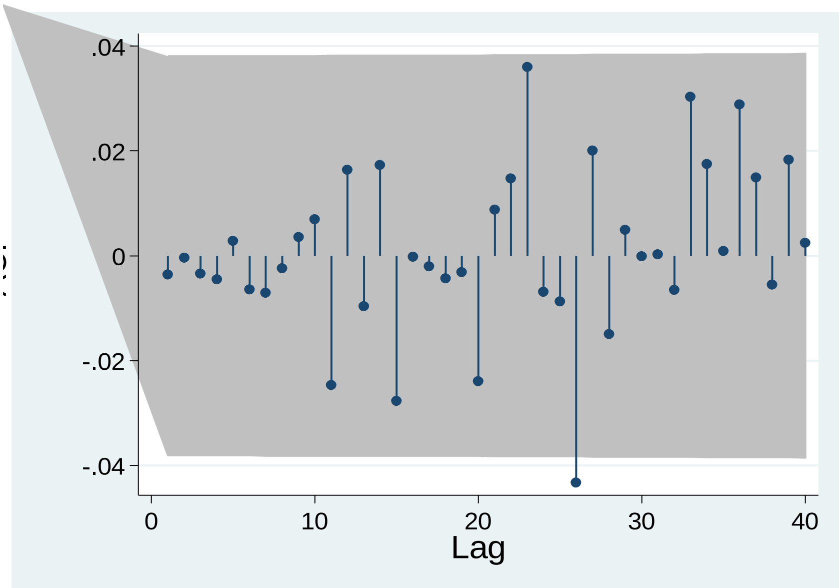


Table 9

Portmanteau Test for square of standardized residual in AR(2)-ARCH(11)

Portmanteau Q(20) statistic :	7.6022
Prob > $chi^2(20)$:	0.9942
Portmanteau Q(8) statistic :	0.4015
Prob > chi2(8) :	0.9999

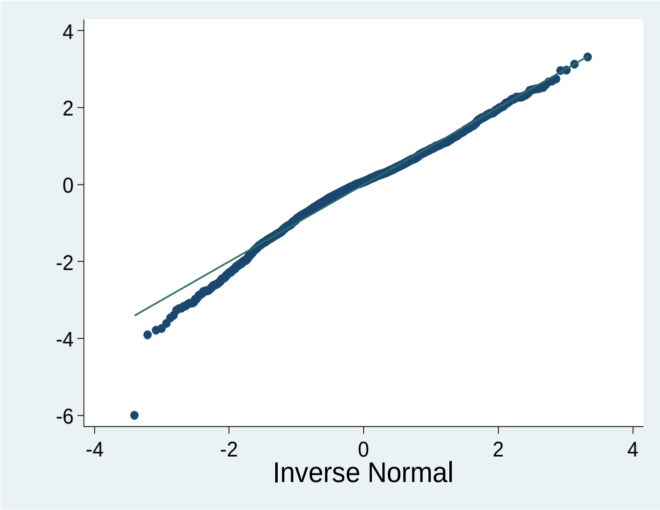
Because the ACF of the square of standardized residual is not significantly different from zero in Figure 9, and Portmanteau test for the square of standardized residual shows the p-value for Q(20) and Q(8) both are above 5% significant level. Therefore, the square of standardized residual is no autocorrelation, hence, the variance equation for AR(2)-ARCH(11) is adequate.

(3) normality checking:

Table 10
Descriptive statistics for Standardized residual
AR(2)-ARCH(11) Distribution

Mean	-0.034907
Std. Dev.	0.9995707
Skewness	-0.4466433
Kurtosis	4.137793

Figure 12
Q-Q Plot: standardized residual ARMA(2,0)-ARCH(11)



The Q-Q plot for the square of standardized residual in ARMA (2,0)-ARCH(11) shows that the bold line is nearly fit the reference thin straight line, which means the distribution of the standardized residual ARMA (2,0)-ARCH(11) is nearly normality distributed but not exactly normality distributed. This may because the sample size is not big enough, so the distribution is not perfectly normality. This may because the model is not adequate enough for the sample S&P500 return, which cannot meet this initial assumption in the ARCH model that the standardized residual is normality distributed.

Table 7 analyses this standardized residual in ARMA (2,0)-ARCH(11) in detail. Comparing with the critical value for normal distribution, The -0.4466433

Skewness is nearly zero, and Kurtosis is a little bit bigger than the critical value (4.137793 > 3). Consequently, it can be concluded that the standardized residual in AR(2)-ARCH(11) is nearly normality distributed, considering the simple size is 2640, we accept that the normality distribution is adequate. Thus, the AR(2)-ARCH(11) model is adequate for modeling the sample S&P 500 return series variance, then the volatility.

However, the basic assumption $a_1 + \ldots + a_q < 1$ for the variance equation (4) in ARCH model gives the bounders of the ARCH model, and the more lagged the ARCH model, the more complex the bounders will be. Moreover, the more the parameters needed to estimate, the more likely the error occurs in estimating the parameters.

As for the sample return, the appropriate ARCH order is 11 which contains many lagged values, Therefore it is needed to introduce and use the Generalized ARCH (GARCH) model, which can express the variance equation using smaller explanatory, by simply adding the lagged variance value as the explanatory in variance equation. This is because the ht can be explained by lagged square of error terms, then use the lagged variance as the explanatory can replace may lagged square of error terms. The next section (4.4) describes the application of the GARCH model with ARMA (2,0) to model the volatility of sample return series.

Chapter 6 GARCH model

The variance equation in a GARCH (p, q) model (Bollerslev, 1986) is:

$$h_t = a_0 + \sum_{k=1}^{p} a_k \cdot \varepsilon_{t-k}^2 + \sum_{j=1}^{q} b_q \cdot h_{t-j} \qquad (5)$$

Where:

ε_t is the error term at time t,

I_{t-1} is the all of the information before and including the t-1 time,

h_t is the conditional variance of ε_t,

p is the lag square error term order number; which is a positive integral number,

q is the lag conditional variance order number; which is a positive integral number,

c is a constant,

a_0 is a non-negative constant,

a_p is the parameter of the ε^2_{p-1},

b_q is the parameter of the h_{q-1},

all the boundaries of the parameters in equation (5) are making sure the h_t is a non-negative.

Therefore, the conditional variance in GARCH model, which conditional on the information set at time t-1, is consistent by the square of lagged error terms and the lagged conditional variance term.

The necessary for implementing ARCH model is that if series can be described by low order ARCH model, for example ARCH (1), then it might be the ARCH (1) model is enough. Secondly, it is necessary to test the ARCH effect, even if we directly use GARCH, otherwise either ARCH or GARCH model cannot be used.

Because the small order for p, and q, in GARCH can have the same effect as the large order for ARCH model, then try the three most popular GARCH model and contain small orders: GARCH(1,1); GARCH(1,2); GARCH(2,1) for modeling variance for sample return series.

Bollerslev, Ray and Kenneth (1992) state that the GARCH(1,1) has superiors for modeling the majority time series volatility and except the GARCH (1,2) and GARCH (2,1) no higher order implementing in the GARCH model. Consequently, this paper tries to implement the ARMA (2,0) with GARCH(1,1), GARCH (1,2) and GARCH (2,1).

Table 11.1

	ARMA(2,0)-GARCH(1,1)			ARMA(2,0)-GARCH(1,2)			ARMA(2,0)-GARCH(2,1)		
	coefficient	std.error	p-value	coefficient	std.error	p-value	coefficient	std.error	p-value
Distribution assumption	Normal			Normal			Normal		
Mean equation									
Constant	0.000546	0.000	0.000	0.0005439	0.000	0.000	0.000537	0.000	0.000
r_{t-1}	-0.0605861	0.023	0.007	-0.0608913	0.021	0.004	-0.0622873	0.018	0.001
r_{t-2}	-0.0370281	0.021	0.073	-0.0380397	0.020	0.062	-0.0401465	0.022	0.063
Variance equation									
constant	1.68E-06	2.58E-07	0.000	1.00E-06	1.96E-07	0.000	2.59E-06	3.79E-07	0.000
ε_{t-1}^2	0.0899301	0.008	0.000	0.0457729	0.007	0.000	-0.0068691	0.012	0.557
ε_{t-2}^2	/	/	/	/	/	/	0.1228789	0.016	0.000
h_{t-1}	0.894596	0.0094706	0.000	1.553264	0.073	0.000	0.8591149	0.013	0.000
h_{t-2}	/	/	/	-0.6081486	0.065	0.000	/	/	/
sum of coefficients	0.9845261			0.9908883			0.9751247		
log-likelihood	8578.54			8587.675			8593.281		
AIC	-17145.09			-17161.35			-17172.56		
BIC	-17109.82			-17120.2			-17131.41		

Where: the std. error is the standard errors for estimated coefficients.

Table 11.1 shows the ARMA(2,0) model with GARCH(1,1); GARCH(1,2) and with GARCH(2,1) respectively. For model ARMA (2,0)-GARCH(1,1), all the standard errors for estimated coefficients are less than 0.023. The smaller the standard errors are, the better for the estimated coefficients, then the better for the model. Excepting the p-value for r_{t-2} (0.073), all the p-values of parameters are at 5% significant level.

For the model ARMA(2,0)-GARCH(1,2), all the standard errors for estimated coefficients are less than 0.073. Excepting the p-value for $\mathbf{r_{t-2}}$ (0.062), all the p-values of parameters are at 5% significant level. It is noticed that the coefficient for h_{t-1} is very large (1.553264). The total sum of the coefficients in the variance equation is extremely near 1 (0.9908883), which means if this model is appropriate for modeling the returns variance, then the past values of variance and the square of error term would dominantly influence the current variance, and this effect would decay very slowly.

For the model ARMA(2,0)-GARCH(2,1), all the standard errors for estimated coefficients are less than 0.022, which is the smallest standard errors in these three models, shows that the model estimates the coefficients very well. The p-value for $\mathbf{r_{t-2}}$ is also insignificant at 5% level (0.063> 5%). Besides, however, the p-value for $\boldsymbol{\varepsilon^2_{t-1}}$ is 0.557 even more than 10% significant level, therefore, this model can be re-written as in **Table 11.2** below:

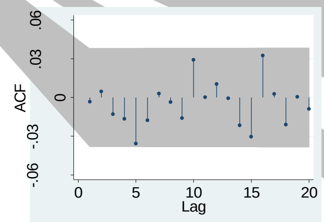
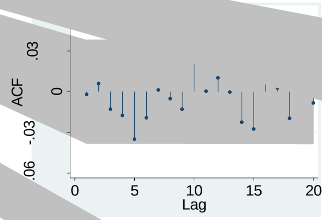
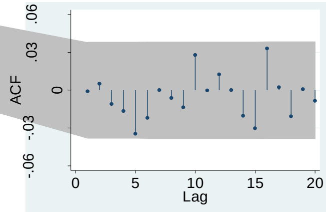
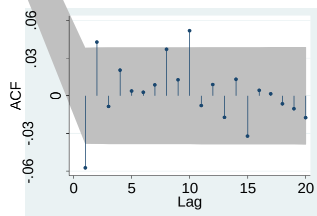
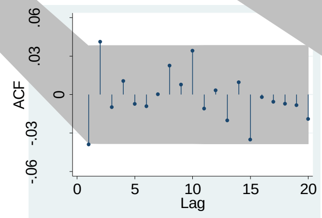
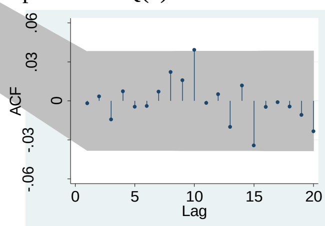
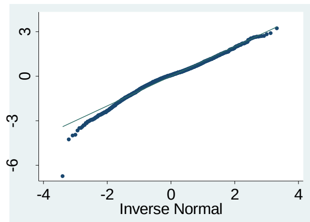
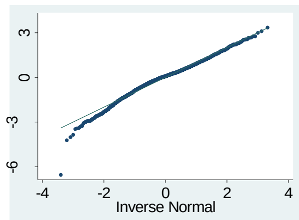
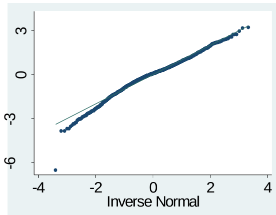
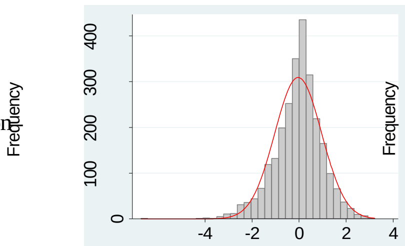
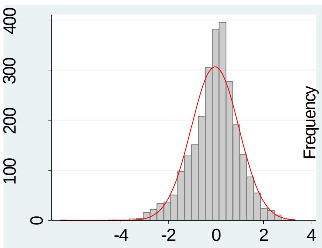
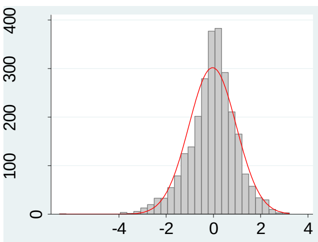
Table 11.2

	ARMA(2,0)-GARCH(1,1)	**ARMA(2,0)-GARCH(1,2)**	**ARMA(2,0)-GARCH(2,1)**
	coefficient	coefficient	coefficient
Mean equation			
Constant	0.000546	0.0005439	0.000537
r_{t-1}	-0.0605861	-0.0608913	-0.0622873
r_{t-2}	-0.0370281	-0.0380397	-0.0401465
Variance equation			
constant	1.68E-06	1.00E-06	2.59E-06
ε_{t-1}^2	0.0899301	0.0457729	/
ε_{t-2}^2	/	/	0.1228789
h_{t-1}	0.894596	1.553264	0.8591149
h_{t-2}	/	-0.6081486	/

Table 11.1 also shows the AIC and BIC information criteria for three models respectively. The minimum value for AIC among these three models and the minimum value for BIC among these three models both are the third model ARMA (2,0)-GARCH(2,1) under normal distribution assumption. Therefore, the third model provides a better describe for sample return volatility.

Table 11.3

Checking adequacies for ARMA-GARCH models under Normal distribution

	ARMA(2,0)-GARCH(1,1)	ARMA(2,0)-GARCH(1,2)	ARMA(2,0)-GARCH(2,1)
Adequ[illegible]			
Por[illegible]tman[illegible] [illegible]est	p-va[illegible] [illegible] p[illegible]lue of [illegible] [illegible]>5%	p-value of Q(20)= 0.7392 >5% [illegible]Q(8)= 0.6851 >5%	p-value of Q(20)= 0.6925 >5% p-value of Q(8)= 0.6855 >5%
ACF pictu[illegible]	 (a[illegible]	 (b)	 (c)
Adequacy 2			
Portmanteau test	[illegible]value of Q(20)= 0.0440 <[illegible] p[illegible]lue of Q(8)= 0.0163 <5%	p-value of Q(20[illegible] [illegible]5% [illegible]-value of Q(8)= 0.[illegible]	p-value of Q(20)= 0.8537 >5% p-value of Q(8)= 0.9720 >5%
ACF picture	 (d)	 (e)	 (f)
Adequacy 3			
Kurtosis	4.453757	4.365114	4.323114
Skewness	-0.5042864	-0.4866329	-0.4867017
Q-Q Plot	 (g)	 (h)	 (i)
Distribution picture	 (j)	 (k)	 (l)

Where:

- the ***adequacy 1*** is checking the **Standardized Residual** for mean equation checked by Portmanteau test and ACF picture;
- The ***adequacy 2*** is checking the **Square of Standardized Residual** for variance equation checked by Portmanteau test and ACF picture;
- The ***adequacy 3*** is checking the **standardized residual distribution normality** checked by the Q-Q plot and the standardized residual distribution kurtosis and skewness.

Table 11.3 shows the checking adequacies for the three models ARMA (2,0)-GARCH(1,1); ARMA(2,0)-GARCH(1,2) and ARMA(2,0)-GARCH(2,1).

Adequacy 1 means the test for the mean equation adequacy, if it is adequate, i.e. no serial autocorrelations in the standardized residual series, then the p-values of the Portmanteau test for standardized residual should be bigger than the 5% significant level. And the ACF picture of standardized residual should not be significantly different from zero. From the tables 8.3, all the p-value of Q(20) and p-value of Q(8) for standardized residuals in three models are bigger than 5%, and ACF pictures for three standardized residuals **Table.8.3 (a), (b), (c)** are not significantly different from zero, therefore all three models' mean equation are adequate.

Adequacy 2 means the test for the variance equation adequacy, if it is adequate, i.e. no serial correlations in the square of standardized residual series. Then the p-values of the Portmanteau test for square of standardized residual should be bigger than the 5% significant level. And the ACF picture of square of standardized residual should not exhibit significantly difference from zero too. However, the p-value of Q(20) and p-value of Q(8) for the square of standardized residual in ARMA(2,0)-GARCH(1,1) model both are less than the 5% significant level, which means the square of standardized residual in this model has serial correlation. The ACF picture of this square of standardized residual **Table 11.3 (d)** also exhibits the serial correlation feature in lag-1, lag-2, and lag-10, therefore, the variance equation for ARMA(2,0)-GARCH(1,1) model is not adequate. Therefore, the ARMA(2,0)-GARCH(1,1) model is not adequate for modeling the sample return variance.

Both the model ARMA(2,0)-GARCH(1,2) and ARMA(2,0)-GARCH(2,1) are adequate in the variance equation. Especially the third model

ARMA(2,0)-GARCH(2,1) shows more adequacy than the second model ARMA(2,0)-GARCH(1,2), because the p-value of Q(20) and Q(8) are extremely larger than 5% for the square of standardized residual in the third model than in the second model (0.8537 > 0.4350 > 5%, and 0.9720 > 0.2151 >5% respectively).

Adequacy 3 means the test for the standardized residual normality distributed. There is no need to check the first model due to the first model is not adequate in variance equation.

Because no one model's standardized error is exactly normality distributed, especially due to the litter bit higher peaked in the distribution than normal distribution. Then **Table 11.3** adds the standardized error distribution kurtosis figures and skewness figures separately compared with the normal distribution kurtosis critical value **3** and skewness critical value **0**. Both ARMA(2,0)-GARCH(1,2) and ARMA(2,0)-GARCH(2,1) models' standardized residual distributions are near the normal distribution, but with higher peak and fat tails than the normal distribution. Due to the third model's standardized residual distribution has a lower kurtosis (4.323114) which has smaller difference from normal distribution kurtosis (4.323114 – 3= 1.323114 < 4.365114 – 3= 1.365114), then together with the ***Adequacy 1*** and ***Adequacy 2*** checking, the ARMA(2,0)-GARCH(2,1) is and more appropriate for sample return series.

For the non-exactly normal distribution for the standardized residual in ARMA(2,0)-GARCH(2,1), considering the total 2640 observation in the sample series, it may because the sample size is not big enough, thus the distribution for standardized residual is near normal distribution but not exactly. Secondly it may be that the standardized residual is actually not a normal distribution, not as initially assumed, therefore, in these circumstance, the ARMA(2,0)-GARCH(2,1) model with standardized error normal distribution would be not appropriate for modeling the return variance. Therefore, it is necessary to try to change the assumption for residual being conditional normal distribution.

Chapter 7

Generalized error distribution (GED)

Although the ARCH-type model can capture the leptokurtic feature in the residual series $\{\varepsilon_t\}$ from the mean equation (Giorgio Canarella, 2008), Considering the standardized residual $\left\{\frac{\varepsilon_t}{\sqrt{h_t}}\right\}$ unconditional distribution cannot meet the normal assumption, which tends to be excess peak and fat tails and has a little skewness comparing with the corresponding normal distribution, therefore, this conditional normal distribution assumption for residual $\{\varepsilon_t\}$ may be not the most appropriate assumption for our sample study, which is consistent with the Bollerslev (1987) result that the leptokurtosis captured by ARCH –type model is not large enough to explain the extremely excess kurtosis.

Furthermore, based on the unconditional distribution for error term in ARMA (2,0) model shows a highly peak, fatter tails and little skewness features, which means the returns series tend to exhibit some more risky extremely large or extremely small values, therefore, when making the conditional assumption for the residual distribution, it is more likely to build a distribution which could have a high peak, fatter tails and skewness features than the normal distribution.

Bollerslev (1987), among others, proposes the conditional Student-t distribution and in 1991, Nelson suggests the Generalized Error Distribution (GED) as the conditional distribution assumption. No matter the changing into the Student-t or GED or keeping the normality assumption for the error term {εt}, the aim for this is to make sure the unconditional variance of error to be infinite. Therefore, this section will change the conditional distribution assumption into GED distribution, which has a thinner peak and fatter tails than the normal distribution when the shape value equal to 2. Therefore, this paper compares the results with the ARMA (2,0)-GARCH (2,1) under the normality assumption and under the GED assumption, in order to choose the most appropriate model for the sample return series.

The probability density function of **Generalized Error Distribution** (GED) is:

$$f(t) = \nu \exp\left(-\frac{1}{2}\cdot\left|\frac{t}{\lambda\cdot\sigma_t}\right|\right)\cdot\left[2^{\frac{\nu+1}{\nu}}\cdot\Gamma\left(\frac{1}{\nu}\right)\cdot\lambda\cdot\sigma_t\right]^{-1} \qquad \text{(x)}$$

Where:

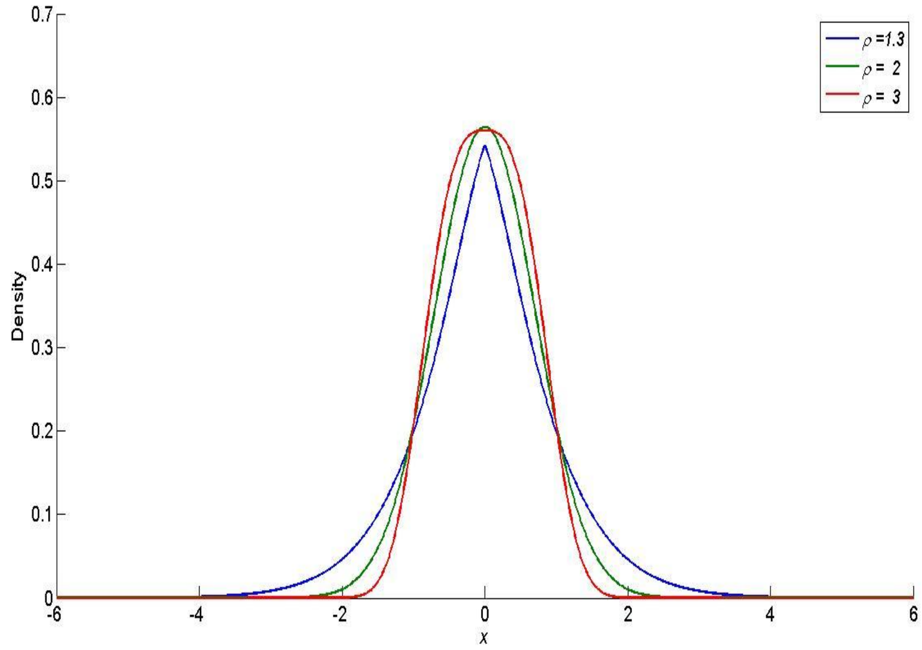
$$\lambda = \left[\frac{\Gamma\left(1/\nu\right)}{2^{(2/\nu)}\cdot\Gamma\left(3/\nu\right)}\right]^{(1/2)}$$

v is a shape parameter determining the high or low peak and fat or thin tails, which is estimated during the Maximum log-likelihood estimation by STATA software when estimates the models. Figure below shows the GED with different shape values. When $v = 2$ the GED becomes the normal distribution; when $v = 1$ the GED becomes the Laplace distribution; when $v < 2$ then GED becomes the high and thin peak with thicker tails than the corresponding normal distribution; when $v > 2$ then GED becomes the thinner tails than the corresponding normal distribution; however when $v < 1$ he unconditional variance for error term ε_t does not exist.

$\Gamma(\cdot)$ is the Gamma function.

σ_t is the distribution's standard deviation.

GED distribution with different shape values



Where: the ρ is the GED shape parameters.

This paper uses the maximum log-likelihood function estimation to estimates the parameters in the ARCH-type models. Essentially, this method is to find the most likely values of the parameters to produce the actual data. And all the works are to search over the parameter-space until the parameter values maximize the log-likelihood function, and then choose these set of parameter values as the coefficient values in a defined order ARCH-type models (or together with the ARMA models).

Giorgio Canarella and Stephen K Pollard (2008) simply uses the maximum log-likelihood values as a kind of indicators together with the AIC and BIC when comparing three GARCH models with exactly same orders but under different conditional distribution assumptions. Therefore, this paper also uses the log-likelihood values as one of the indicators to compare GARCH models under different distribution assumptions.

Table12.1
ARMA-GARCH model under GED assumption

	ARMA(2,0)-GARCH(1,1)			ARMA(2,0)-GARCH(1,2)			ARMA(2,0)-GARCH(2,1)		
	coefficient	std.error	p-value	coefficient	std.error	p-value	coefficient	std.error	p-value
Distribution	GED			GED			GED		
shape *v*	1.281734	0.048		1.290955	0.049		1.294779	0.049	
Mean equation									
Constant	0.0008283	0.000	0.000	0.0008152	0.000	0.000	0.0008205	0.000	0.000
r_{t-1}	-0.0593772	0.020	0.002	-0.05987	0.019	0.001	-0.0603098	0.017	0.000
r_{t-2}	-0.0295614	0.019	0.114	-0.029806	0.019	0.109	-0.0310191	0.019	0.111
Variance equation									
constant	1.46E-06	0.000	0.001	9.04E-07	0.000	0.001	2.39E-06	0.000	0.000
ε_{t-1}^2	0.0906167	0.013	0.000	0.0468828	0.012	0.000	-0.0096535	0.017	0.580
ε_{t-2}^2	/	/	/	/	/	/	0.1301933	0.025	0.000
h_{t-1}	0.8974576	0.014	0.000	1.549168	0.113	0.000	0.8586905	0.019	0.000
h_{t-2}	/	/	/	-0.603448	0.101	0.000	/	/	/
sum of coefficients	0.988076			0.9926024			0.9792303		
Log-likelihood	8641.608			8648.425			8652.938		
AIC	-17269.22			-17280.85			-17289.88		
BIC	-17228.07			-17233.82			-17242.85		

Where:

The distribution is the distribution assumption for conditional error term $\{\varepsilon_t\}$ in mean equation. Therefore, in table 10.1, all the models are assumed as the GED distribution. i.e.:

$$\varepsilon_t = \epsilon_t \cdot \sqrt{h_t} \mid I_{t-1} \qquad \text{recall (17.2)}$$

$$\epsilon_t \sim \mathrm{GED}\,(0,1,\nu) \qquad (17.5)$$

The ν is the shape parameter for distribution estimated by the maximum log-likelihood function.

Log-likelihood is maximized log-likelihood values output by STATA.

For model ARMA(2,0)-GARCH(1,1)-GED, excepting the lag-2 value of return in the mean equation with p-value of 0.114, all the p-values for the parameters are at the 5% significant levels, and the standard error for all parameters are less than 0.02.

For model ARMA(2,0)-GARCH(1,2)-GED, excepting the lag-2 value of return in the mean equation with p-value of 0.109, all the p-values for the parameters are highly significant at the 5% level. Excepting the lag-1 and lag-2 values of variance in the variance equation with a standard error of 0.113 and 0.101 respectively, the standard errors for rest parameters are less than 0.019.

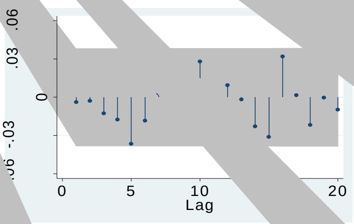
For model ARMA (2,0)-GARCH(2,1)-GED, the p-value for lag-2 return value is still insignificant with 0.111, however, the p-value for ε_{t-1}^{2} in the variance equation is highly insignificant (0.580), the p-value for rest parameters are highly significant at 1% level. The standard errors for all the parameters are all less than 0.025. Therefore, the lag-2 return values in all three models here under GED assumption exhibit insignificance with p-values all more than the 10% significant level. This agrees Bera (1995) that the adding GARCH model for describing the return variance feature will minimizes the effect of the return autocorrelation feature. Because of highly insignificance p-value in the third model, the third model ARMA (2,0)-GARCH(2,1)-GED should be modified without ε_{t-1}^{2} value.

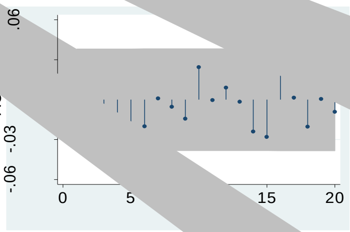
Table 12.2

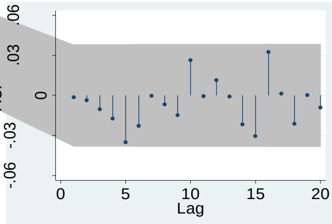
Residual Diagnostics for ARMA-GARCH model under GED

	ARMA(2,0)-GARCH(1,1)		ARMA(2,0)-GARCH(1,2)		ARMA(2,0)-GARCH(2,1)	
	statistic value	P-value	statistic value	P-value	statistic value	P-value
Distribution assumption	GED		GED		GED	
Q(20) for ϵ_t	[illegible]0981	[illegible]	15.938	0.7205	16.6504	0.6755
Q(8) for ϵ_t	[illegible]	[illegible]	5.8694	0.6619	5.9018	0.6582
Q(20) for ϵ_t^2	29.7484	[illegible]074	[illegible]7175	0.4757	13.2948	0.8644
Q(8) for ϵ_t^2	[illegible]6473	[illegible]	[illegible]	[illegible]2907	2.0726	0.9787

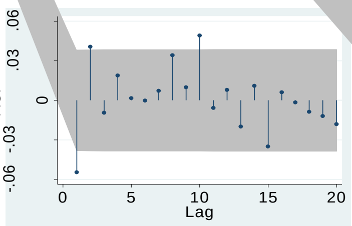
ACF for ϵ_t

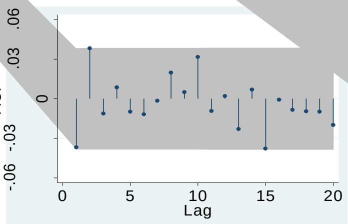

(a)

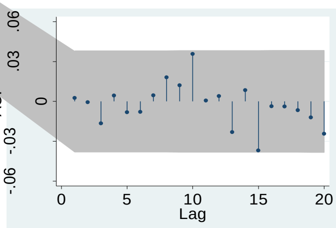

(b)


(c)

ACF for ϵ_t^2

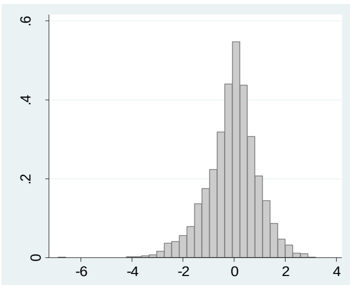

(d)

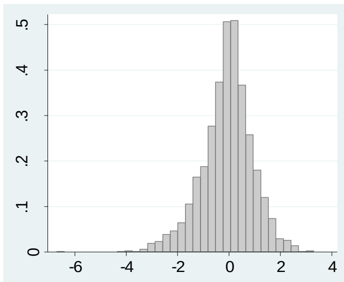

(e)

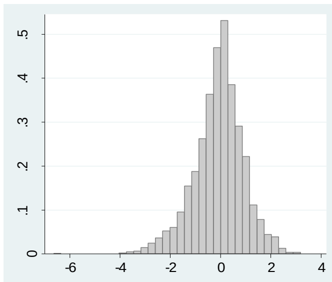

(f)

Distribution for ϵ_t^2

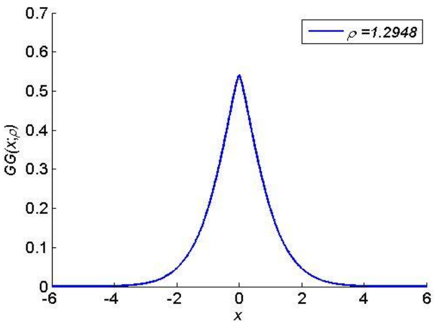
Density


(g)


(h)


(i)

	ARMA(2,0)-GARCH(1,1)		ARMA(2,0)-GARCH(1,2)		ARMA(2,0)-GARCH(2,1)	
Kurtosis	4.503198		4.403601		4.360299	
Skewness	-0.5089808		-0.4890501		-0.4887276	
Bera-Jaeque		0.000		0.000		0.000



(j)

Where: the Q(m) is the Ljung-Box Portmanteau Test for lag m.

ϵ_t is the standardized residual: $\epsilon_t = \varepsilon_t/\sqrt{h_t}$

the Bera-jaeque is the kurtosis and skewness normality test.

(j) is the standard GED distribution ,with mean zero, unit variance, and shape equal to 1.294779, the same shape with the GED in ARMA(2,0)-GARCH(2,1)-GED model

Table 10.2 shows the residual Diagnostics for ARMA-GARCH model under GED assumption. Portmanteau Test for standardized residual Q(20) and Q(8) with p-value 0.647 and 0.672 respectively, both more than the 5% significant level show that the standardized residual for model ARMA-GARCH(1,1)-GED is serial uncorrelated. The ACF picture of the standardized residual in **Figure (a)** also examines this result. Therefore the mean equation in this model is adequate for describing the autocorrelation feature in return series. However, the p-values in the portmanteau test for the square of standardized residual Q(20) and Q(8) are both more than 5% significant level (0.074 and 0.034 respectively), which means $\{\epsilon_t^2\}$ series has the serial correlation, therefore, the variance equation for describing return variance is not enough adequate for describing the return series dependence feature. Therefore, the ARMA (2,0)-GARCH(1,1) is not enough adequate for modeling the sample return variance.

For the model ARMA(2,0)-GARCH(1,2)-GED and the model ARMA(2,0) -GARCH(2,1)-GED, both of them use the ARMA(2,0) to describe the return series, the p-values for portmanteau test within 20 lags Q(20) and Q(8) for standardized residual are 0.7205 and 0.6619 in ARMA(2,0)-GARCH(1,2)-GED model and 0.6755 and 0.6582 in ARMA(2,0)-GARCH(2,1)-GED model, which all are bigger than 5% significance level, this means both models' standardized residuals are not serial correlation, therefore the ARMA(2,0) in the two models is enough to describe the return series.

The portmanteau test of the square of standardized residuals give Q(20) with p-value 0.4757 and Q(8) with p-value 0.2907 confirming that the variance equation in ARMA(2,0)-GARCH(1,2)-GED is adequate at the 5% significant level. The variance equation in ARMA(2,0)-GARCH(2,1)-GED is also adequate at 5% level with p-value of Q(20) and Q(8) bigger than 5% (0.8644>5%; 0.9787>5%). Comparing these two models, however, the p-values for portmanteau test Q(20) and Q(8) for GARCH (2,1)-GED both are bigger than corresponding p-values in GARCH(1,2)-GED, (0.8644 > 0.4757 > 5%; and 0.9787 > 0.2907 > 5%), therefore, the ARMA(2,0)-GARCH(1,2)-GED is more appropriate than ARMA(2,0)-GARCH(2,1)-GED. The ACF pictures under these two models can also confirm this conclusion (see **figure (e)** and **figure (f)**).

- *Bera-Jarque test*

The Bera Jarque (1981) test is for the distribution normality test from view of skewness and kurtosis of the distribution. The null hypothesis is the distribution is a normal distribution with symmetric and mesocratic (kurtosis equal to 3).

$$\mathrm{H_0}\text{: distribution is a normal distribution}$$
$$\mathrm{H_1}\text{: distribution is not a normal distribution}$$

The test statistic **W** is:

$$\mathbf{w} = \mathrm{T} \cdot \left[\frac{\mathrm{b}_1^2}{6} + \frac{(\mathrm{b}_2 - 3)^2}{24}\right]$$

Where: the b_1 and b_2 are the coefficients of skewness and kurtosis:

$$b_1 = \frac{E[\mu^3]}{(\sigma^2)^{3/2}}$$

$$b_2 = \frac{E[\mu^4]}{(\sigma^2)^2}$$

Where the $\boldsymbol{\mu}$ is the mean of the sample distribution, and the σ^2 is the variance of the sample distribution, the T is the sample size. Under the null hypothesis, the test statistics **W** should follow a Chi-Square with 2 degree of freedom; then the test statistic would not be significant at the 5% level. Reject the null, if the test statistics is above the critical value, then the test statistic would be statistically significant.

From **table 10.2** the Bera-Jarque test for ARMA(2,0)-GARCH(1,2)-GED and ARMA(2,0)-GARCH92,1)-GED both are statistically significant at the 1% level, therefore, rejecting the null hypothesis of standardized residual being normal distribution. As a result, under the both models, the standardized residual distributions are not normal. The distribution pictures for two model standardized residual shown in **(h)** and **(i)** both show the distributions are with higher peak and fatter tails than the corresponding normal distributions. Because the shape parameters $\boldsymbol{\nu}$ estimated by the maximum log-likelihood function for these two models are 1.295 and 1.291 (see **table 10.1**) respectively, which are in the (1,2) interval. Therefore the estimated GED distributions for these two models both have a high peak and fatter tails than the corresponding normal distribution, which correspond with the results by the distribution pictures in **Table 12.2 (h)** and **(i)**. Therefore, the conditional GED distribution assumption for error term $\boldsymbol{\varepsilon_t}$ with shape parameters between (1,2) interval cannot be rejected. Therefore, two models are adequate for GED assumption. Considering the GARCH(2,1) shows better in the variance equation, therefore, the ARMA(2,0)-GARCH(2,1)-GED is the most appropriate models under the GED assumption.

In summary, under the normal distribution assumption, the ARMA (2,0)-GARCH(2,1)-N is the most appropriate model, but the distribution of standardized error still does not meet the assumption and the p-value for r_{t-2} and ε_{t-1}^2 are insignificant at the 5% level. Under the GED assumption, the ARMA(2,0)-GARCH(2,1)-GED is adequate and the most appropriate model, also with p-value for r_{t-2} and ε_{t-1}^2 are insignificant at the 5% level, therefore these two models should be modified without the r_{t-2} and ε_{t-1}^2. **Table 13** shows the Log-likelihood value; AIC and BIC values for two models. Because the log-likelihood value of second model is bigger than the value in first model and the minimum value for AIC and minimum value for BIC both are the second model, therefore the second model ARMA(2,0)-GARCH(2,1)-GED provides a better performance for modeling the return volatility.

Table 13

	ARMA(2,0)-GARCH(2,1)-Normal	ARMA(2,0)-GARCH(2,1)-GED
Log-likelihood	8652.938	8593.281
AIC	-17289.88	-17172.56
BIC	-17242.85	-17131.41

Chapter 8

Extension of GARCH

- T-GARCH

However, the GARCH model assumes the lagged positive shocks (i.e. the positive error term) and lagged negative shocks (negative error term) have the same effect to the current variance. However, in reality, people tend to have more panic when the return decrease than the return increase, therefore, Glosten, Jagannathan, and Runkle (1993) develop the GARCH model into a Threshold-GARCH model, which aims to capture this leverage effect in the financial series, that is a volatility reacts differently to a increase in the return (or to a good news coming) than to a decrease in the return or bad news coming. In Threshold GARCH (p,q) model, the variance equation changes into:

$$h_t = a_0 + \sum_{k=1}^{p} (a_k + \gamma_k \cdot D_{t-k}) \cdot \varepsilon_{t-k}^2 + \sum_{j=1}^{q} \beta_j \cdot h_{t-j} \qquad \text{(x)}$$

- Where **D_{t-k}** is a dummy variable:

$$D_{t-k} = \begin{cases} 1 \text{if } \varepsilon_{t-k} < 0, \\ 0 \text{ if} \varepsilon_{t-k} \geq 0, \end{cases} \qquad \text{(y)}$$

- K is a positive number, for k = {1, 2, 3, …, p};
- Y_k is the coefficient for $D_{t-k,}$

Therefore, when the shock is positive, the effect from $\varepsilon_{t-k}^2$to the h_t is a_k, when the shock is negative, the effect from the ε_{t-k}^2 is $(a_k + \gamma_k)$. When the $\gamma_k > 0$, the effect from negative shocks (negative lag ε) for the variance is bigger than the effect from positive shocks, and vice verse. When the $\gamma_k = 0$, the effect from negative news and positive news are the same to the return variance. This is allows testing whether the downward movements in the sample return treated as the bad news has the same magnitude variance of upward movements (i.e. the good news) to the current variance.

Because ARMA(2,0)-GARCH(2,1) with GED distribution assumption are adequacies for sample return series, therefore, implementing ARMA(2,0)-TGARCH(2,1) with GED distribution.

- **GARCH-in-mean**

Based on the capital asset pricing model (CAPM) in the financial theory that encountering with higher risk tends to have a higher return, Engle, Lilen and Robins (1987) extend the GARCH-in-mean model (GARCH-M), that is adding the current variance obtained in the variance equation as the variable to explain the current return in the mean equation: therefore comparing with the GARCH model, the GARCH-M model changes the mean equation into:

$$r_t = c \cdot h_t + \left(c_0 + \sum_{i=1}^{m} c_i \cdot r_{t-i} + \sum_{j=1}^{n} d_j \cdot \varepsilon_{t-j} \right) + \varepsilon_t \qquad (10.2)$$

Where: the c is the coefficient parameter for h_t. It means variance increases return by a factor of c. This paper also implements the ARMA (2,0)-GARCH(2,1)-M in the conditionally GED distribution assumption for error term ε_t.

Table 14.1

	ARMA(2,0)-T-GARCH(2,1)			ARMA(2,0)-GARCH(2,1)-M		
	Coefficient	**Std.error**	**P-value**	**Coefficient**	**Std.error**	**P-value**
Mean equation						
h_t	/	/	/	1.527031	1.592	0.338
Constant	0.0005992	0.000	0.000	0.0007227	0.000	0.000
r_{t-1}	-0.054579	0.017	0.001	-0.0613843	0.017	0.000
r_{t-2}	-0.0300582	0.019	0.116	-0.0323712	0.020	0.097
Variance equation						
constant	1.71E-06	3.30E-07	0.000	2.39E-06	6.05E-07	0.000
ε_{t-1}^2	0.0479151	0.034	0.158	-0.0084632	0.018	0.633
ε_{t-2}^2	0.1087425	0.037	0.003	0.1289647	0.025	0.000
$D_{t-1} \bullet \varepsilon_{t-1}^2$	-0.1256137	0.034	0.000	/	/	/
$D_{t-2} \bullet \varepsilon_{t-2}^2$	-0.0497472	0.038	0.192	/	/	/
h_{t-1}	0.9052297	0.014	0.000	0.8587674	0.019	0.000
sum of coefficients	0.8865264			0.9792689		
Distribution	**GED**			**GED**		
Shape	1.370588	0.050		1.294538	0.049	
log-likelihood	8693.484			8653.233		
AIC	-17366.97			-17288.47		
BIC	-17308.18			-17235.56		

Table 14.1 shows the results by running three ARMA(2,0)-GARCH(2,1) models. The first model is **ARMA(2,0)-T-GARCH(2,1)** model based on the conditional GED distribution assumption. The shape parameter estimated by the log likelihood function for the GED distribution is 1.370588 which is less than 2 with the standard error of 0.050, which means the conditional distribution is higher and thinner peak and thinker tails than the corresponding normal distribution. All the standard errors of the parameters are less than 0.038. However the p-value of the parameters suggests that the r_{t-2}; ε_{t-1}^2; and $D_{t-1} \cdot \varepsilon_{t-1}^2$are insignificant at the 5% level. Therefore the model should be refined by dropping these three parameters:

$$r_t = 0.0005992 - 0.054579 \cdot r_{t-1} - 0.0300582 \cdot r_{t-1} + \varepsilon_t$$

$$\varepsilon_t | I_{t-1} = \epsilon_t \cdot \sqrt{h_t} | I_{t-1} \qquad \epsilon_t \sim GED(0, 1, 1.370558)$$

$$h_t = -0.1256137 \cdot D_{t-1} \cdot \varepsilon_{t-1}^2 + 0.1087425 \cdot \varepsilon_{t-2}^2 + 0.9052297 \cdot h_{t-1}$$

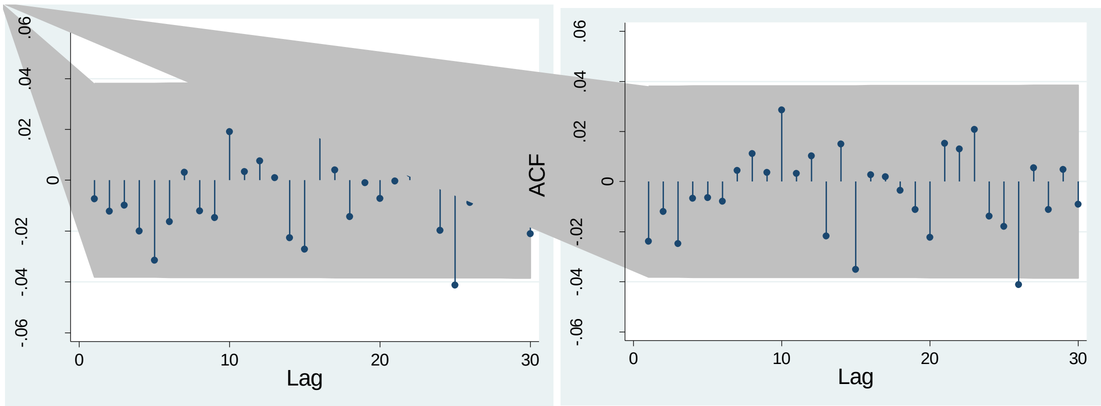
The non-zero parameter (-0.1256137) for $D_{t-1} \cdot \varepsilon_{t-1}^2$ statistically significantly captures the different effect from the negative shock and positive shocks, that is when the shock ε_{t-1} is negative, the effect from ε_{t-1}^2 to h_t is -0.1256137, whereas, when the shock ε_{t-1} is positive, the effect from ε_{t-1}^2 to h_t is zero. This means when the past innovation is negative to the return r_t, i.e. the negative ε_t, the return volatile is less than the good news coming, but this difference is at the small level. Therefore, the leverage effect indeed exists and is successfully captured by the Threshold GARCH model.

The sum of the coefficients in the variance equation becomes: 1.0139722>1 when the past shock is positive. This means that if the shocks continuously being positive, then the return variance would continuously increase. The sum of the coefficients in the variance equation becomes 0.8883585 <1 when the past shocks is negative. This means that if the shocks continuously being negative, the past effect from past square errors and past variance would decay slowly. This shows that the return variance has a mean reversion feature that is the sample return variance would back to its mean value.

The second model is the **ARMA (2,0)-GARCH(2,1)-in-mean** model under the GED distribution assumption. However, the coefficient for h_t in the mean equation as the unique feature in the GARCH-in-mean model has the p-value insignificantly even bigger than 40%, it means this variable has no significantly power to explain the dependent variable return series. Together with the high standard error for estimating the parameter for h_t (1.736), which cannot be ignored, this means that this model should be re-fine by deleting this h_t variable in the mean equation. Therefore, both p-value and standard error for $\mathbf{h_t}$ indicate that the GARCH-in-mean is not suitable for describing sample return series.

Therefore, there is no need to further implement Threshold-GARCH together with GARCH-in-mean model, and it will be sufficient by only examines the model adequacies for ARMA(2,0)-T-GARCH(2,1) model.

Figure 13



(a) (b)

(a): the ACF for standardized residual in ARMA(2,0)-T-GARCH(2,1)
(b): the ACF for Square of standardized residual in ARMA(2,0)-T-GARCH(2,1)

The ACF of sample standardized residual in **Figure (a)** fails to suggest any significant serial correlations in the standardized residual series $\{\epsilon_t\}$. The ACF of the square of standardized residual in **Figure (b)** fails to suggest any significant serial correlations in the square of standardized residual series $\{\epsilon_t^2\}$, which suggests no conditional heteroscedasticity in the standardized residual series $\{\epsilon_t\}$. More specifically, in Table 13.2 the portmanteau test for standardized residual Q(20) with p-value 0.8217 and Q(8) with p-value 0.6909 both are higher than the 5% significant level, and portmanteau test for the square of standardized residual Q(20) with p-value of test statistic 0.8507 and Q(8) with p-value 0.8310 both are more than 5%, therefore, the model

ARMA(2,0)-T-GARCH(2,1) appears to be adequate in describing the linear dependence in return series and return variance series.

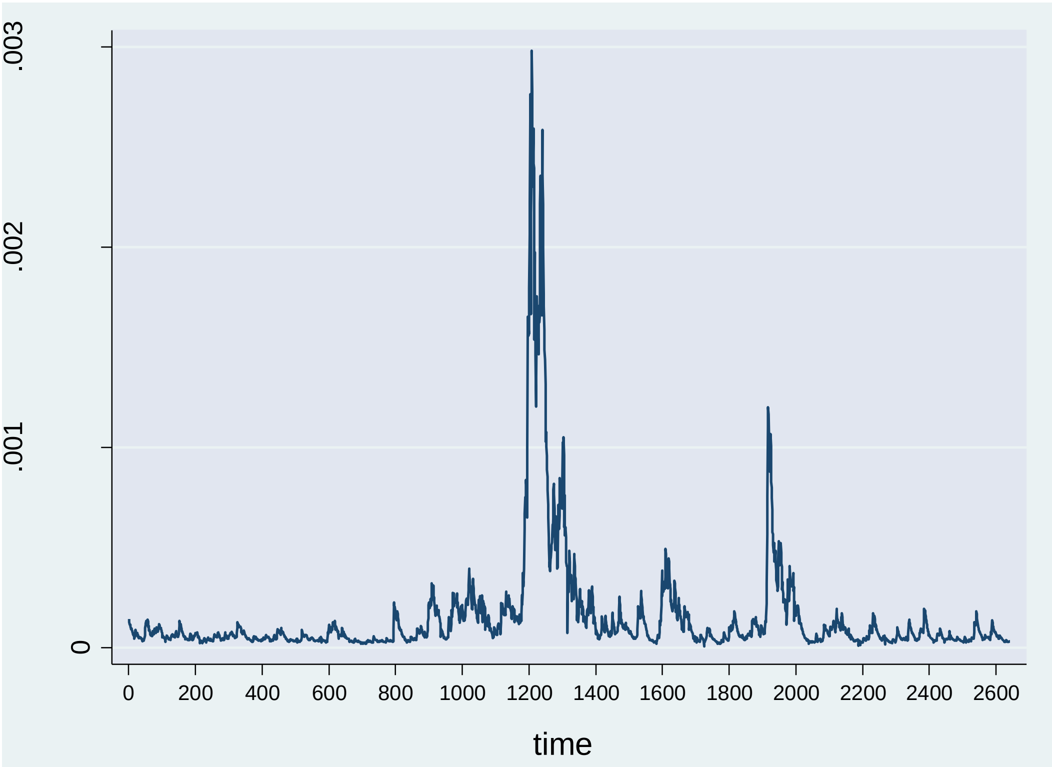
Table 14.2
Checking adequacies for ARMA (2,0)-T-GARCH (2,1)

	Statistic value	P-value
Q(20) for ϵ_t	14.1714	0.8217
Q(8) for ϵ_t	5.6089	0.6909
Q(20) for ϵ_t^2	13.5889	0.8507
Q(8) for ϵ_t^2	4.2798	0.8310

The distribution of standardized residual is near a normal distribution represented by Q-Q plot **(j)** which shows that the bold line is near the straight reference line. However, compared with the normal distribution, this distribution of standardized residual is with taller peak and little fat tail on the left side, and not exactly symmetric with -0.491 skewness.

Therefore, the ARMA(2,0)-T-GARCH(2,1)-GED model is adequate for modeling the sample return volatility, and because the p-values for $D_{t-1} \cdot \varepsilon_{t-1}^2$ and shown in Table 10.1 are significant with small standard error, which means the symmetric effect does exist in the sample return volatility, therefore, the T-GARCH model rather than the traditional GARCH can better explain the return volatility. Graph X shows the predicted conditional variance of the sample return from ARMA (2,0)-**T**-GARCH(2,1)-GED model.

Figure 14



Where: The first observation in time series (i.e. t= 1) is 05th/01/2004;
t= 600 is 18th/05/2006; t= 1200 is 06th/10/2008;
t= 1800 is 23rd/02/2011; t= 2400 is 15th/07/2013.

In summary, the S&P 500 return series has the time-varying variance, and this variance can be captured by the ARCH-type models. Because the small orders in GARCH can capture the longer orders in ARCH model, therefore the GARCH model is superior than the ARCH model, especially in the return series exhibits long lag serial dependence. However, the traditional conditional normal assumption for the error term obtained in the mean equation is not suitable for daily S&P 500 log return series as the distribution of the sample return series and the error term series in the mean equation both exhibit higher and thinner peak and thinker tails than the corresponding normal distribution, which suggests that the conditional distribution for the error term series might be thinner peak and thinker tails. This paper empirically examines this assumption changing into a generalized error distribution (GED). The results show that the conditional assumption for the

GED distribution is more appropriate than the normal distribution assumption, and can provide a better performance confirmed by the AIC, BIC information criteria and log-likelihood values. Furthermore, the Threshold-GARCH model proves the return variance has the leverage effect, and therefore has a better performance than the simple GARCH model. However, this paper fails to use the GARCH- in-mean model, which suggests that the current variance has no direct influence on the return at the same time. Therefore, the most appropriate model for modeling the daily S&P 500 return series is ARMA(2,0)-T-GARCH under GED assumption.

However, this paper does not deal with the small magnitude sknewness in the standardized error distribution.

Chapter 9

Inflation

This paper uses the first difference of the natural logarithm of the daily U.S. Customer Price Index (CPI) as the daily inflation rate $\{f_t\}$. "The CPI (CPI-U) represents the changes in the prices paid by urban consumers for a representative basket of goods and services" (U.S. Department of Labor, 时间). Therefore, the log inflation rate f_t will reflect the changing rate of the prices paid by urban consumers from time t-1 to time t. therefore, the inflation rate is one of and is the most important rate indicating the whole macro-environment in a Country. The daily inflation rate is obtained as below:

$$f_t = \ln_e\left(\frac{CPI_t}{CPI_{t-1}}\right)$$

Where: the **t** is the day number;

CPI_t is the daily customer price index at time t.

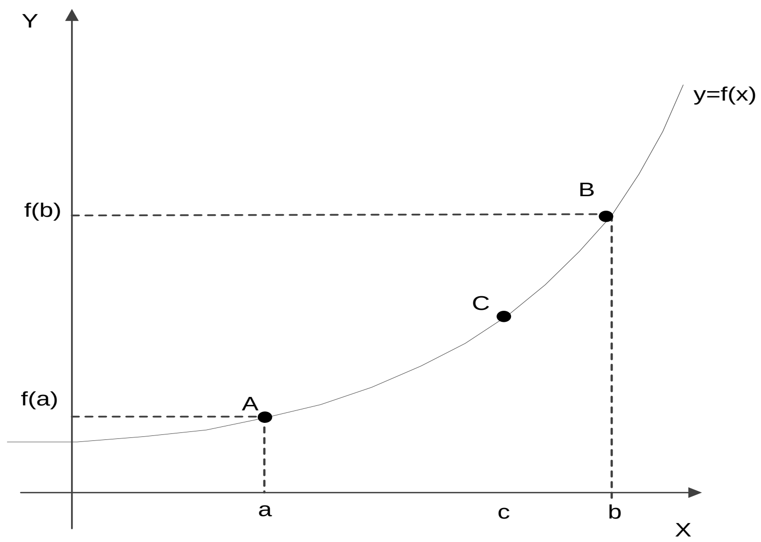
However, there is no published daily CPI available. The most frequent data for CPI is monthly data. This paper obtained the U.S. Customer price index for all urban (CPI-U) provided by the U.S. Department of Labor Bureau of Labor Statistic obtained from the *Inflation.data.com* as the monthly CPI data, which is from December 2003 to July 2014, ten years duration. This CPI data is based on the average of the year from 1982 to 1984 as the 100 value, and then based this value, the CPI in the other time periods is calculated (U.S. Department of Labor).

In order to obtain the daily CPI data, this paper uses the *Linear Interpolation* method to obtain the approximate daily data from the monthly data.

- *Linear interpolation:*

For example: In a two-dimensional coordinate, the horizontal line is X-axis, the vertical line is Y-axis. For a function $y = f(x)$ shown in the **Figure (15.1)** with only two known values **A** $(a,\ f(a))$ and **B** $(b,\ f(b))$ in this two-dimensional coordinate, where the point **A** with x value equals to a, y value equals to $f(a)$, and the point **B** with x value equals to b, y value equals to $f(b)$ shown in the **Figure (15.1)**, the linear interpolation method can calculate the approximate function's value for a point **C** with x value equaling to c, under the $y = f(x)$ function, where the c is in between the a and b:

Figure (15.1)



(drawn by)

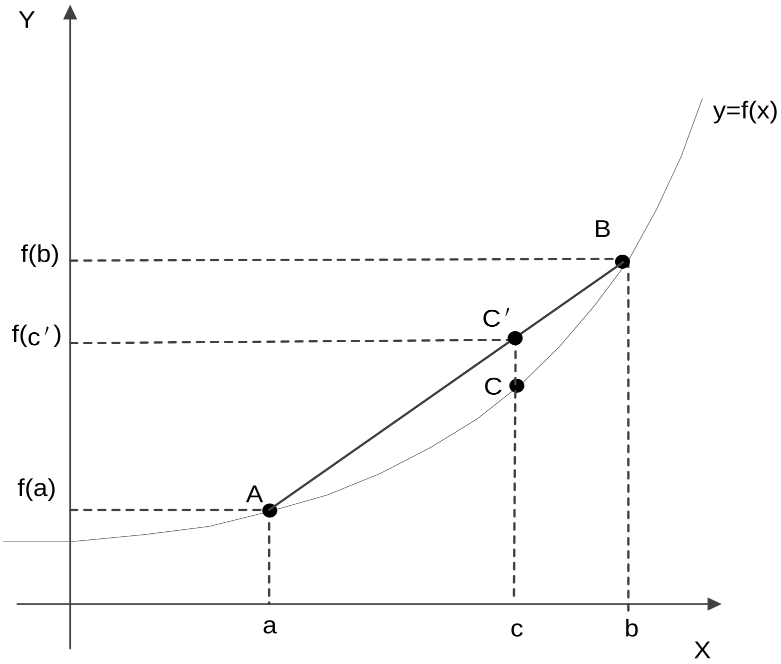
(Source: Peter Oliver, Lecture notes)

The approximate value for $f(c)$ calculated by using linear interpolation method is:

$$f(c) \approx f(a) + \frac{c - a}{b - a} \cdot [f(b) - f(a)]$$

The approximate value for $f(c)$ is represented as $f(c')$ which is shown in the **Figure (15.2)**. The point **C'** is the approximate point estimated by the linear interpolation method.

Figure (15.2)



(drawn by)

- Calculating the approximate daily CPI

This paper supposes that the monthly CPI data is treated as the daily data in the middle of the date in the corresponding month, i.e. the 15th date in the corresponding month, and uses the linear interpolation method to obtain the approximate daily data in this month.

Taking the January 2014 for an example, the monthly CPI index for January 2014 is 233.916, and the monthly CPI index for February 2014 is 234.781. Therefore, the daily CPI for 15th January 2014 is treated as 233.916, and the daily CPI for 15th February 2014 is treated as 234.781. Because the first month in year 2014 has 31 calendar days, therefore from day 15th January 2014 (treated as the day zero) to the day 15th February 2014 (treated as the day 31st) totally is 31 days, which means this paper then needs to interpolate total 30 days values between 15th/01/2014 to 15th/02/2014. Therefore:

The CPI of the first day (i.e. the day 16th/01/2014) is approximately:

$$\mathrm{CPI}_1 \approx \mathrm{CPI}_0 + \frac{(1-0)}{(31-0)} \cdot (\mathrm{CPI}_{31} - \mathrm{CPI}_0)$$

$$= 233.916 + \frac{(1-0)}{(31-0)} \cdot (234.781 - 233.916)$$

$$= 233.9439032$$

The Nth day daily CPI is approximately:

$$\mathrm{CPI}_\mathrm{N} \approx \mathrm{CPI}_0 + \frac{(\mathrm{N}-0)}{(31-0)} \cdot (\mathrm{CPI}_{31} - \mathrm{CPI}_0)$$

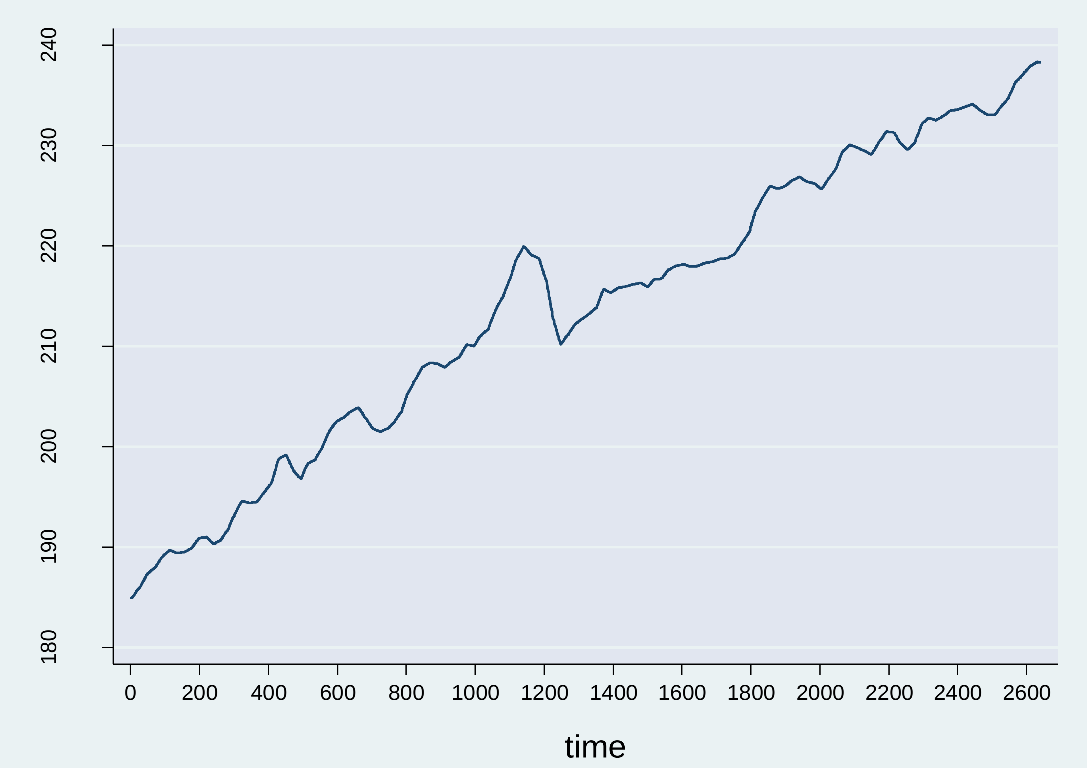
Table 15
The approximate daily CPI using linear interpolation method

Calender day	Day's number N	Approx. daily CPI
2014/1/15	0	$CPI_0 = 233.916$
2014/1/16	1	233.944
2014/1/17	2	233.972
2014/1/18	3	234.000
2014/1/19	4	234.028
2014/1/20	5	234.056
2014/1/21	6	234.083
2014/1/22	7	234.111
2014/1/23	8	234.139
2014/1/24	9	234.167
2014/1/25	10	234.195
2014/1/26	11	234.223
2014/1/27	12	234.251
2014/1/28	13	234.279
2014/1/29	14	234.307
2014/1/30	15	234.335
2014/1/31	16	234.362
2014/2/1	17	234.390
2014/2/2	18	234.418
2014/2/3	19	234.446
2014/2/4	20	234.474
2014/2/5	21	234.502
2014/2/6	22	234.530
2014/2/7	23	234.558
2014/2/8	24	234.586
2014/2/9	25	234.614
2014/2/10	26	234.641
2014/2/11	27	234.669
2014/2/12	28	234.697
2014/2/13	29	234.725
2014/2/14	30	234.753
2014/2/15	31	$CPI_{31} = 234.781$

The **Table 15** shows the approximate daily U.S. CPI values from 15^{th}/01/2014 to 15^{th}/02/2014 using linear interpolation method. Therefore, this daily approximate CPI index for these 31 days constitute an arithmetic sequence, which means that the increased CPI part during these 31 days is assumed to be equally weighted increased by these 31 days. Therefore, all the approximate daily U.S. CPI values from 02/01/2004 to 30^{th}/06/2014 are obtained by using the same method.

However, this daily CPI series is based on the calendar days rather than trading days, thus, when inputting the CPI data into the STATA software, all of the non-trading day's values are deleted.

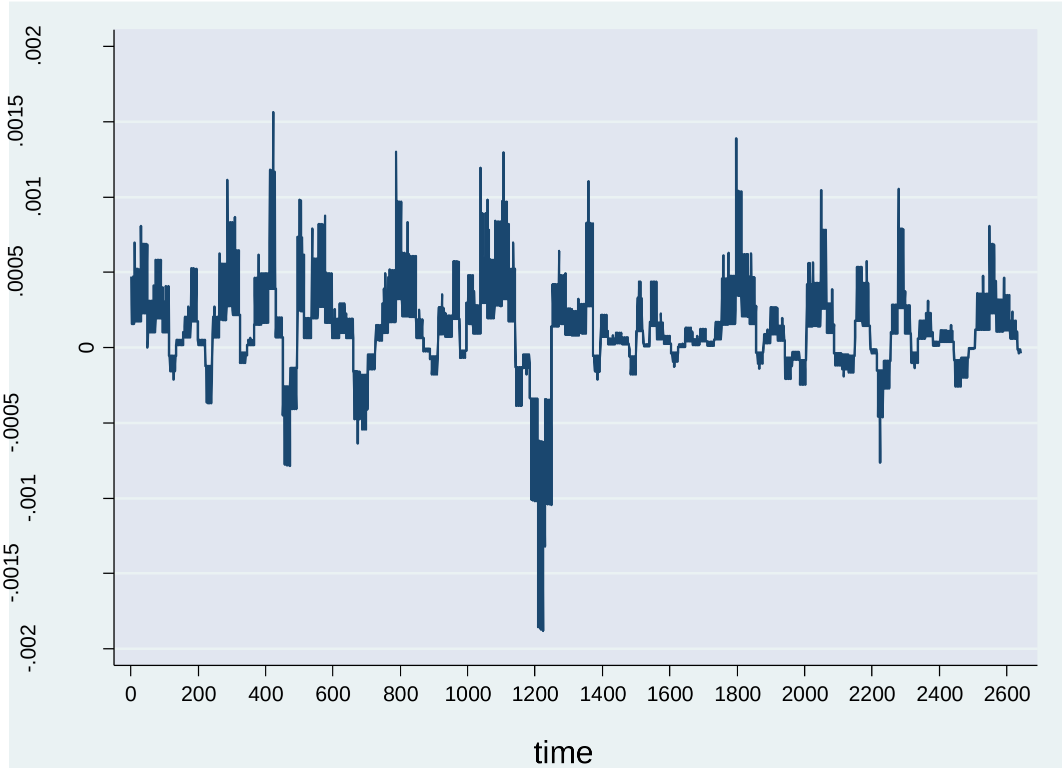
Figure 16
U.S. Daily approximate CPI index from 02/01/2004 to 30/06/2014



Where: the time is the trading day count, totally 2641 observations.
The first observation in time series (i.e. t= 1) is 02nd/01/2004;
t= 600 is 19th/05/2006; t= 1200 is 07th/10/2008;
t= 1800 is 24th/02/2011; t= 2400 is 16th/07/2013.
The last observation in time series (i.e. t=2641) is 30th/06/2014

Figure 16 shows the U.S. daily approximate CPI index (trading days account) using the linear interpolation method, from 2nd/01/2004 to 30th/06/2014 totally 2641 observations. The approximate U.S. daily CPI mainly increases with some fluctuations. The non-highly frequent fluctuations is mainly because the actual data obtained is low monthly frequent data, the approximate daily data by the linear interpolation method only show the main trend between two actual values.

Figure 17
U.S. Daily approximate Inflation from 05/01/2004 to 30/06/2014



Where: the time is the trading day count, totally 2640 observations.
The first observation in time series (i.e. t= 1) is 05th/01/2004;
t= 600 is 22nd/05/2006; t= 1200 is 08th/10/2008;
t= 1800 is 25th/02/2011; t= 2400 is 17th/07/2013.
The last observation in time series (i.e. t=2640) is 30th/06/2014

The U.S. daily approximate log inflation rate is obtained and shown by **Figure 17**, which is from 5th/01/2004 to 30th/06/2014, totally 2640 observations. This inflation rate mainly fluctuates between zero and value 0.0005. It is noticed that in August 2008 (about t=2000), this inflation rate is sharply down to its bottom value about -0.0018 during the ten year duration, and then increase back to its mean value. This significant drop in the August 2008 is corresponding to the extremely significant S&P 500 return volatiles around August 2008 in **Figure 2**, and therefore also corresponding to the huge sharply increased conditional daily S&P 500 return variance obtained by ARMA (2,0)-T-GARCH(2,1)-GED model in **Figure 14**, where the daily conditional variance reaches its highest peak value around August 2008 (about t=2000 in **Figure 14**). Moreover, the main ten fluctuations for this inflation series in the ten-year duration reflect the possible seasonal influence on the log inflation rate.

Figure 18
Distribution of the U.S. Daily approximate log Inflation

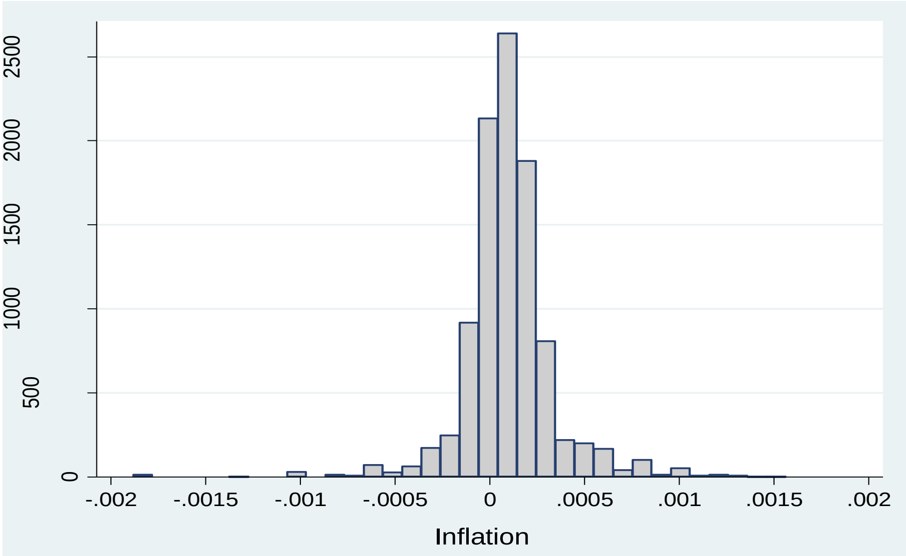


Figure 18 shows the distribution of the sample approximate U.S. daily log inflation rate. This distribution exhibits highly kurtosis, together with **Table 16**, the variance of this distribution is very small (6.17×10^{-8}), therefore, the approximate daily inflation rate is mainly around its mean value (9.63×10^{-5}). However, the sample inflation series also contains extreme values, which the minimum value is down to -1.882×10^{-3}, and the maximum value reaches to 1.5644×10^{-3}. The skewness (-0.336188) is not obvious in the distribution picture.

Table 16
Descriptive statistics for sample U.S. log inflation

Minimum value	-0.0018820
Maximum value	0.0015644
Mean	0.0000963
Variance	0.0000000617
Skewness	-0.3361884
Kurtosis	13.3033400

Once the appropriate model for modeling daily S&P 500 stock return volatility from 05th/01/2004 to 30th/06/2014 is determined i.e. the ARMA(2,0)-T-GARCH(2,1) model under the GED assumption for error term, the next step is to investigate the predictive power of inflation on the conditional variance. To do so, this paper follows the method adapted by Lin and Hamilton (1996), who use the conditional variance of return series as the function of the past square errors, the past conditional variance(s) and the possible past variable value(s) to study the influence from the inflation to the conditional variance in the U.S. Palm (1996) motivates this kind of model as the Factor-GARCH model.

In the U.S. market, the Schewert (1989) firstly studies the stock market volatility and the macro-variables including Inflation from 1859 to 1987 using monthly data. Until 2003, Nicole Davis and M.Kutan used the Factor-GARCH model to examine the power of inflation for U.S. stock market volatility from 1957 to 1999 monthly data. The appropriate model they used is the EGARCH (1,1) under the normal assumption for error term. Based on the likelihood method, they decide to add lag-1, lag-2 and lag-3 of inflation values as the exogenous variables in the EGARCH variance equation. Because the value of monthly inflation rate is small compared with the value of conditional variance, the inflation data used by them is computed based on the log-differenced of CPI multiplied by 100.

Considering the previous studies, this paper uses the Factor-ARMA-T-GARCH-GED model with the lag inflation value(s) as the exogenous variables in the conditional variance equation to investigate the possible predictive power from daily inflation to the S&P 500 return volatility which the S&P 500 stock index is treated as the U.S. stock market index.

Because the mean of the approximate daily inflation data is 9.63×10^{-5} with 6.17×10^{-8} small variance, while the mean for the conditional variance obtained from ARMA (2,0)-T-GARCH(2,1)-GED is relative big 1.542×10^{-4} with relative big variance 8.83×10^{-8} shown in **Table 16**, therefore, this paper decides to use the inflation multiply 100 as the exogenous variables adding in the Factor-GARCH model, which is as the same method as Nicole Davis and M.Kutan's. The model is as follow:

$$r_t = c_0 + c_1 \cdot r_{t-1} + c_2 \cdot r_{t-2} + \varepsilon_t$$

$$\varepsilon_t | I_{t-1} = \epsilon_t \cdot \sqrt{h_t} | I_{t-1}$$

$$\epsilon_t \sim GED(0,1,v)$$

$$h_t = a_0 + \sum_{k=1}^{p} \left(a_k + \gamma_k \cdot D_{t-k}\right) \cdot \varepsilon_{t-k}^2 + \sum_{j=1}^{q} \beta_j \cdot h_{t-j} + \sum_{m=1}^{M} \theta_m \cdot f_{t-m} \cdot 100$$

The same explanations as before, however, the new explanation θ_m is the coefficient of the parameter $f_{t-m} \cdot 100$. The m is the positive integer number for $m = \{1, 2, \dots, M\}$. f_{t-m} is the lag-m approximate daily inflation value.

When deciding how many lag values of inflation need in the conditional variance equation, this paper follow the Nicole Davis and M.Kutan's method by using the likelihood ration tests.

- *Likelihood ratio (LR) test*:

The LR test is implemented to consider whether the model should add one or some new parameter(s). It involves estimation under the two models. One is the model with restrictions of adding new parameter(s) in the model, the other model is called unconstrained model, which allows the researcher to adds m new parameter(s) in the model. The null hypothesis is the first constrained model is more sufficient than the unconstrained model. Under the null hypothesis, the LR statistics should follow the Chi-Square with m degree of freedom.

$$\mathrm{LR} = -2 \cdot (\mathrm{LLF}_{\mathrm{constrained}} - \mathrm{LLF}_{\mathrm{unconstrained}}) \sim \chi^2(m)$$

LLF is the log-likelihood function. Here the LLF denotes the after the model uses the log-likelihood function method to estimates the model parameters' coefficients, Therefore, $\mathrm{LLF}_{\mathrm{constrained}}$ means the maximum log-likelihood number for the first constrained model; $\mathrm{LLF}_{\mathrm{unconstrained}}$ means the maximum log-likelihood value for the unconstrained model. The *m* is the number of new parameters added in the unconstrained model.

Reject the null hypothesis, if the LR is above the corresponding Chi-square critical values. Then the unconstrained model with new adding parameters is more sufficient than the constrained model.

Therefore, this paper is to decide how many lag values of the approximate inflation should be added into the model ARMA(2,0)-T-GARCH(2,1)-GED, here called original model, by comparing the original model with the model adding lag-1 of inflation, and comparing the this original model as constrained model with the model adding lag-1 and lag-2 of inflation values, and so on.

This test is used in the situation when adding the new parameters into the model, the new model should have a bigger maximum log-likelihood value than that of constrained model. However, if the model with more parameter(s) has a maximum log-likelihood value which is smaller than the original model, then the new model will not to be considered.

Table 17
Maximum log-likelihood value for models

	Model	Model's variance equation	Maximum log-likelihood value
1st	ARMA(2,0)-T-GARCH(2,1)-GED	original model	8693.484
2nd	ARMA(2,0)-T-GARCH(2,1)-GED	with lag-1 inflation value	8703.585
3rd	ARMA(2,0)-T-GARCH(2,1)-GED	with lag-2 inflation value	8699.500
4th	ARMA(2,0)-T-GARCH(2,1)-GED	with lag-1, lag-2 inflation values	8699.663
5th	ARMA(2,0)-T-GARCH(2,1)-GED	with lag-1, lag-2, lag-3 inflation values	8696.015

Table 17 shows the maximum log-likelihood values for different models when using log-likelihood function to estimate the model's parameters. Beside the original model, **Table 17** only lists four Factor GARCH models, because when adding more lag inflation values into the original model, the maximum log-likelihood values are even less than the original model's. Therefore, there is no need to list and compare those kinds of models. Because the second model in the table exhibits the highest value of the maximum log-likelihood value in these five models, therefore, if the second model is sufficient than the original model after testing the LR test for original model and this second model, then there is no need to implement other rest models into LR test.
LR test for comparing the original model with the second model is shown in Table 17:

$$\mathrm{LR} = -2 \cdot (8693.484 - 8703.585) = 20.202$$
$$\chi^2(1) = 3.84 \text{ at } 5\% \text{ significant level}$$

Because LR test is above the corresponding critical Chi-square value with 1 degree of freedom at the 5% significant level, the unconstrained model, i.e. the model with lag-1 inflation value is preferable.

Table 18 shows the result of ARMA (2,0)-T-GARCH (2,1)-GED with lag-1 of inflation as variable in the variance equation. The f_{t-1}•100 i.e. the lag-1 approximate daily inflation multiplied by 100 is as the possible variable to explain the S&P 500 return conditional volatility. If this past inflation value is appropriate to describe the conditional variance, it means the inflation can predict the conditional variance. The inflation movements can have effect on the conditional variance. However, the p-value of f_{t-1}•100 is 0.297, it is insignificant even at the 10% significant level. Furthermore, the coefficient of f_{t-1}•100 is 5.03430 however with big standard error of 4.830, together with the insignificant p-value, therefore, the lag-1 value of approximate daily inflation cannot explain the conditional variance of S&P 500 return. i.e. the inflation does not have the predictive power and no effect for this conditional variance.

Table 18

	ARMA(2,0)-T-GARCH(2,1) with lag-1 infation		
	Coefficient	**Std.error**	**P-value**
Mean equation			
Constant	0.0006073	0.000	0.000
r_{t-1}	-0.0533896	0.016	0.001
r_{t-2}	-0.0238756	0.019	0.199
Variance equation			
constant	-13.09523	0.188	0.000
f_{t-1}•100	5.03430	4.830	0.297
ε_{t-1}^2	0.0381544	0.034	0.265
ε_{t-2}^2	0.1420804	0.037	0.000
D_{t-1}•ε_{t-1}^2	-0.2283373	0.033	0.000
D_{t-2}•ε_{t-2}^2	0.0375888	0.034	0.270
h_{t-1}	0.9180229	0.011	0.000
sum of coefficients	0.9075092		
Distribution	**GED**		
Shape	1.37549	0.054	
log-likelihood	8703.585		
AIC	-17385.17		
BIC	-17320.51		

This result is consistent with the Davis and Kutan (2003) result, that in the 13 Countries, only four Countries show that the lag values of inflation have negative effect on the Countries' stock market return volatility at the 10% significant level. Other nine Countries including U.S. do not show any evidence that the lag values of influence have impact on the conditional variance. The proper lag length for the inflation variables they used are lag-1, lag-2 and lag-3 in the variance equation.

However, this result disagrees with results from Schewert (1989) and Hamilton and Lin (1996). Hamilton and Lin (1996) results treat S&P 500 stock index as the Market index, and study the market return volatility and the economic recessions, they use the output and inflation as the exogenous variables both on stock returns and its conditional volatility. The results shows together with other 13 Countries, the economic activities is an importance source of stock market return volatility, the U.S. inflation does have predictive power on both the return and return volatility at the 10% significant level after Wold War II from 1965 to 1993. The fact that this paper's result is differing from that of Hamilton and Lin (1996) might be attributed to the different study period. They study the period after the great economic recessions, and comparing the period from 1965 to 1993 and the period from 2004 to 2014, although there is a Financial Crisis during the 2008, however, the main economic development now is better than during the 1965 to 1993.

The possible predictive power from inflation to the stock return volatility is a hypothesis, there is no any financial theory supporting this point. The only direct relation between inflation and the stock return rather the stock return volatility is the famous Fisher Effect, which suggests that the stock return should fluctuate with inflation rate. This is the genius debate for the return and inflation relationship. Other journals show the relationship between stock return and inflation is based on the Country's monetary policy such as the countercyclical financial policies and procyclical policy. This second point is developed and examined by lots of studies such as Kaul (1987) study U.S. Country for this topic, whose time period of research is 1953 to 1983, that is slightly different with that of Hamilton and Lin (1996). Kaul (1987) finds that the link between inflation and stock return and this is related with the Counter-cyclical policy.

These previous results give a possible hypothesis that: does the relationship between inflation and stock return volatility is based on the relationship between inflation and stock return? However, from the journals Davis and Kutan (2003), which both investigates the inflation to return and inflation to return volatility, the Israel, Netherlands and U.S. all has the relation between inflation and Country's stock market return, however, these Countries in the same period do not exhibit the predictive power from inflation to stock return volatility. Therefore, this hypothesis is rejected.

After comparing the results with previous studies, secondly, the approximate data obtained by the linear interpolation method only captures the main trend of the CPI values, therefore, captures the main tend of the inflation movements. More specifically, under the linear interpolation method, the CPI within one month, from the middle of one month to next middle of month, constitute an arithmetic sequence, therefore, the log–differenced value of CPI i.e. the log inflation data within one month become a smooth relative steady line. However, return volatility fluctuates more frequently than the approximate log inflation, which means that only capturing the main trend by the approximate inflation may be not adequate enough to explain this high fluctuations data (return volatility). Therefore, even if the inflation had effect on the S&P 500 return series volatility, this effect could not be reflected clearly.

Therefore, this paper also investigates: does the inflation have effect on the return volatility by using monthly data during the same time interval from January 2004 to June 2014. Based on the previous studies, this paper uses the traditional GARCH(1,1) to model the monthly S&P 500 return volatility.

Table 19.1 shows the GARCH (1,1) with Normal distribution assumption for S&P 500 return series from January 2004 to June 2014. Except the p-value of constant in variance equation is 0.348 insignificance, the significance of the other parameters with p-values are all at the 5% level. The standard errors for all coefficient parameters are under 0.098.

Table 19.1
GARCH (1,1) for monthly S&P 500 return series

	GARCH(1,1)		
	Coefficient	**Std.error**	**P-value**
Mean equation			
Constant	0.0069293	0.003	0.023
Variance equation			
constant	0.00007	0.000	0.348
ε_{t-1}^2	0.2589835	0.087	0.003
h_{t-1}	0.7174895	0.096	0.000
sum of coefficients	0.976473		
Distribution	**Normal**		
log-likelihood	237.8328		
AIC	-467.6655		
BIC	-456.3523		

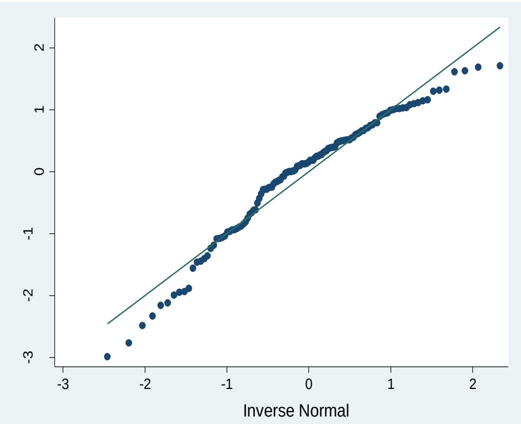
Table 19.2 shows the test for the standardized error in the GARCH (1,1) model. the Portmanteau Test for standardized residual gives Q(20)=21.949 with p-value 0.343, and Q(5)=3.027 with p-value 0.696>5% significant level. The Portmanteau Test for square of standardized residual Q(20)=15.930 with p-value 0.721 and Q(5)=5.870 with p-value 0.319 > 5% significant level,

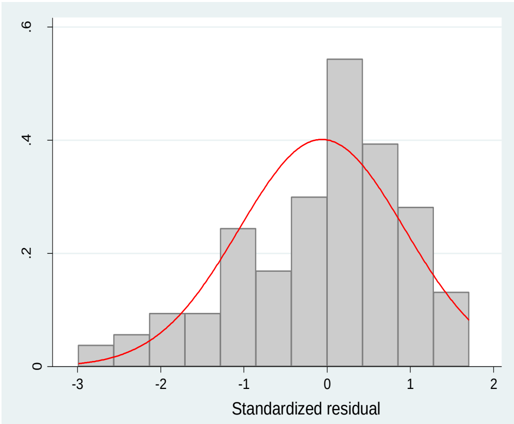
therefore, both standardized residual series and square of standardized residual are not serial correlation, therefore the GARCH(1,1) model for monthly S&P 500 return series appears to be adequate in describing the linear dependence in the return and conditional volatility series.

Table 19.2

Diagnostic Test for GARCH(1,1)		
	Statistic value	**P-value**
Q(20) for ϵ_t	21.949	0.343
Q(5) for ϵ_t	3.027	0.696
Q(20) for ϵ_t^2	15.930	0.721
Q(5) for ϵ_t^2	5.870	0.319
Bera-Jaeque		0.010
mean	-0.0626363	
variance	0.9872674	
Kurtosis	3.168346	
Skewness	-0.7228757	

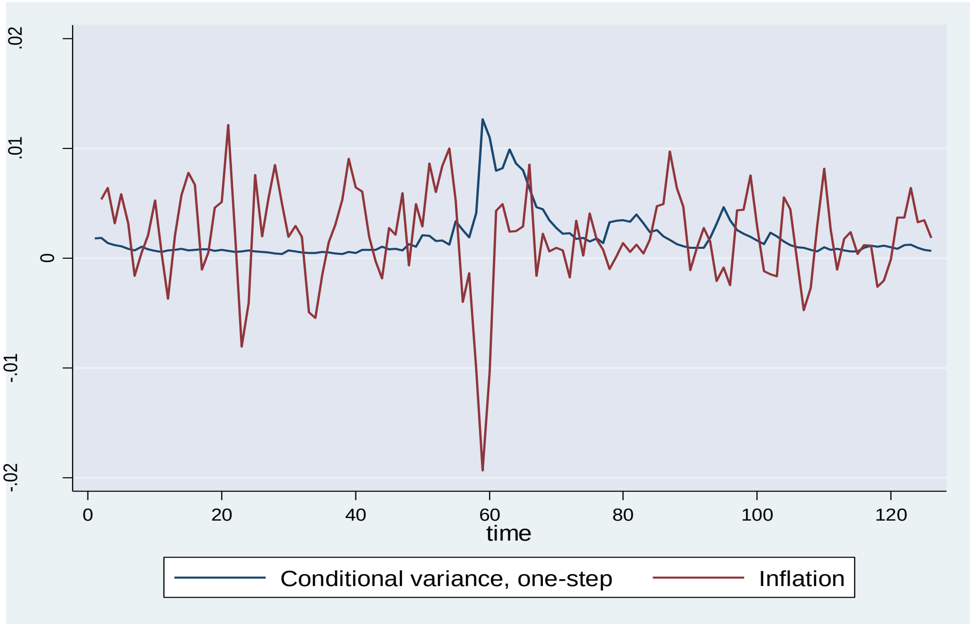
Figure 13





However, the Bera- Jaeque with p-value of 0.01 <5% shows that the standardized residual distribution is not normal. The Q-Q plot and Distribution picture of the standardized residal in GARCH(1,1) for monthly data also shows the non-normal distribution. This may because the sample size for monthly data is very small from January 2004 to June 2014, total 125 observations, therefore, this paper accepts this standardized residual is near normal, but with skewness. **Figure 14** shows the condtional variance by GARCH(1,1) and the monthly inflation series.

Figure 14



After checking the GARCH(1,1) model is adequate for modeling the monthly S&P 500 return volatility, next step is to add the monthly past inflation value as the possbile variable(s) in the variance equation. However, when try to adding the past inflation value(s) as the exogenous variables, such as adding lag-1, or adding lag-1 and lag-2, or adding lag-1, lag-2 and lag-3 of the inflation, non of the Factor model provides a higher maximum log-likelihood value than the original GARCH(1,1) model in Table 19 with maximum log-likelihood value of 237.8328, and the original GARCH(1,1) without any past inflation value(s) provides the lowest value of AIC and BIC.

The most approperiate model with past inflation value(s) is GARCH(1,1) with lag-1 inflation value shown in **Table 20**. However, f_{t-1}•100 is with large standard error of 3.038 and insignificant P-value of 0.583, which means the lag-1 monthly inflation has no effect to the monthly S&P 500 return conditional variance.

Therefore, this paper concludes that neither in daily data or using monthly data, the inflation has no predictive power for S&P 500 return conditional variance.

Table 20
GARCH(1,1) with lag-1 inflation for monthly S&P 500 return

	GARCH(1,1) with lag-1 infation		
	Coefficient	**Std.error**	**P-value**
Mean equation			
Constant	0.0071556	0.003	0.015
Variance equation			
constant	-10.17993	2.222	0.000
$f_{t-1} \bullet 100$	1.66910	3.038	0.583
ε_{t-1}^{2}	0.2567971	0.088	0.003
h_{t-1}	0.7198926	0.094	0.000
sum of coefficients	0.9766897		
Distribution	**Normal**		
log-likelihood	235.665		
AIC	-461.3301		
BIC	-447.2287		

Chapter 10

Conclusion

This paper investagates the possiable predictive power of inflation for S&P 500 stock return volatility by using high frequent daily data of ten-years period from 5th/01/2004 to 30th/06/2014. The method is using the past inflation value as the exogenous variable into the conditional variance equation. The approximate daily inflation data is obtained by using the linear interpolation method. The time varying return volatility is obtained by using ARCH famliy model.

The paper concludes that there is no evidence of the relationship between U.S. inflation and S&P 500 stock return volatility for both daily data and monthly data, the inflation has no predictive power for S&P 500 stock return volatility from 2004 to 2014, and it not a underlying determinant for stock market volatility in U.S. this result is cosistent with the results from Davis and Kutan (2003), who argue that the inflation cannot influence the market stock return volatility from 1957 to 1999. But this result disagrees with the Schewert (1989) using monthly data from 1857 to 1987.

the ARMA(2,0) model is applied to deal with the small magnitude serial correlation in the S&P 500 return series. Moreover, the possiable ARCH model for S&P 500 return series is ARCH(11), the GARCH model is used as well to minimise the standard errors when estimating the model. However, it is confirmed that the superious GARCH(1,1) model fail to model S&P 500 return series from 2004 to 2014,while the GARCH(2,1) is adequate to describe this return serial dependence feature.

It is found that the Generalised error distribution GED with shape parameter less than 2 has a thinner peak and fatter tails than the normal distribution, so it has more power and advance than the conditional normal assumption for the error term in the return mean equation, which confirms the previous studies. Furthermore, the daily S&P 500 stok return volatility does exist the aymmetric effect, which are successfully captured by the Thresholder GARCH model, and the negative shocks has slight effect on the conditional variance compared with the the positive shocks. However, the GARCH-in-mean model fails to model the sample series.

However, this paper ignores the slight skewness of the distribution in the return and error term when modeling the conditional variance. Secondly the approximate daily inflation data is obtained by using the linear interpolation method not the real data, which will influence the result’s accuracy.

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Appendix

Appendix 1: STATA software Code:

```
gen mydaily=daily(dailydate,"YMD")
gen time=_n
tsset time
gen price=adjclose
gen return= ln(price/l.price)


*analysis return
tsline price
tsline return
su return, d
summarize return
histogram return, freq normal
qnorm return
ac return
gen sreturn = abs(return)
gen sqreturn=return^2
ac sqreturn

*mean equation**********************
dfuller return,
wntestq return, lag(20)
wntestq return, lag(8)
ac return, lags (20)
pac return, lags (20)

forvalues p=0/4 {
forvalues q=0/4 {
arima return, arima (`p',0,`q')nolog
estat ic
}
}
*try ARMA(2,4)
arima return, arima(2,0,4) nolog
```

```
predict e24, residual
ac e24, lags(20)
wntestq e24, lags(20)
wntestq e24, lags(8)

*try ARMA(2,0):
arima return, arima(2,0,0) nolog
predict e20, residual
ac e20, lags(20)
wntestq e20, lags(20)
wntestq e20, lags(8)
*analysis e2
histogram e20, freq normal
qnorm e20
ac e20, lags(30)
gen se20 = e20^2
ac se20, lags (30)
summarize e20
su e20, d
gen abe20= abs(e20)
ac abe20, lags (30)

*possible arch order*****************
gen sqe2=e2^2
pac sqe2
*arch effect
regress sqe2 l(1/14).sqe2
arch return, arch(1/14) arima(2,0,0) nolog
regress sqe2 l(1/11).sqe2
arch return, arch(1/11) arima(2,0,0) nolog
*model adequacy check arma(2,0)-arch(11)
predict h211, variance
predict e211, residual
gen v211=e211 / h211^(1/2)
ac v211, lags
wntestq v211, lags(20)
wntestq v211, lags(8)
*check variance equation
gen sqv211= v211^2
```

```
ac sqv211
wntestq sqv211, lags(8)
wntestq sqv211, lags(20)
histogram v211, freq normal
qnorm v211
su v211, d
sktest v211

*arma (2,0)-GARCH (1,1)**********************************
arch return, arch(1/1) garch(1/1) arima(2,0,0)    nolog
estat ic
predict h2011, variance
predict e2011, residual
gen v2011=e2011 /h2011^(1/2)
ac v2011, lags (20)
wntestq v2011, lags(20)
wntestq v2011, lags(8)
*variance equation checking
gen sqv2011 = v2011^2
ac sqv2011, lags (20)
wntestq sqv2011, lags(20)
wntestq sqv2011, lags(8)
histogram v2011, freq normal
qnorm v2011
su v2011, d
sktest v2011

*ARMA(2,0)-GARCH(1,2)
arch return, arch(1/1) garch(1/2) arima(2,0,0)    nolog
estat ic
predict h2012, variance
predict e2012, residual
gen v2012=e2012 /h2012^(1/2)
ac v2012, lags (20)
wntestq v2012, lags(20)
wntestq v2012, lags(8)
*variance equation checking
gen sv2012 = v2012^2
ac sv2012, lags (20)
```

```
wntestq sv2012, lags(20)
wntestq sv2012, lags(8)
histogram v2012, freq normal
qnorm v2012
su v2012, d
sktest v2012

*ARMA(2,0)-GARCH(2,1)OKK!!!!
arch return, arch(1/2) garch(1/1) arima(2,0,0)    nolog
estat ic
predict h2021, variance
predict e2021, residual
gen v2021=e2021 /h2021^(1/2)
ac v2021, lags (20)
wntestq v2021, lags(20)
wntestq v2021, lags(8)
*variance equation checking
gen sv2021 = v2021^2
ac sv2021, lags (20)
wntestq sv2021, lags(20)
wntestq sv2021, lags(8)
histogram v2021, freq normal
qnorm v2021
su v2021, d
sktest v2021


*ARMA(1,0)-GARCH(2,1)OKK!!!
arch return, arch(1/2) garch(1/1) arima(1,0,0)    nolog
estat ic
predict h1021, variance
predict e1021, residual
gen v1021=e1021 /h1021^(1/2)
ac v1021, lags (20)
ac v1021
wntestq v1021, lags(20)
wntestq v1021, lags(8)
*variance equation checking
gen sv1021 = v1021^2
```

```
ac sv1021, lags (20)
wntestq sv1021, lags(20)
wntestq sv1021, lags(8)
histogram v1021, freq normal
qnorm v1021
su v1021, d
sktest v1021

*GED*******************************************************************
*
*arma(2,0) garch(1,1)GED
arch return, arch(1/1) garch(1/1) arima(2,0,0) distribution (ged)
estat ic
predict e2011e, residual
predict h2011e, variance
gen v2011e = e2011e/ h2011e^(1/2)
ac v2011e, lags(20)
wntestq v2011e, lags(20)
wntestq v2011e, lags(8)
gen sv2011e = v2011e^2
wntestq sv2011e, lags(20)
wntestq sv2011e, lags(8)
ac sv2011e, lags(20)
ac sv2011e
histogram v2011e
qnorm v2011e
su v2011e, d
sktest v2011e

*arma(2,0)-garch(1,2)GED
arch return, arch(1/1) garch(1/2) arima(2,0,0) distribution (ged)
estat ic
predict e2012e, residual
predict h2012e, variance
gen v2012e = e2012e/ h2012e^(1/2)
ac v2012e, lags(20)
wntestq v2012e, lags(20)
wntestq v2012e, lags(8)
gen sv2012e = v2012e^2
```

```
ac sv2012e, lags(20)
wntestq sv2012e, lags(20)
wntestq sv2012e, lags(8)
histogram v2012e, freq normal
qnorm v2012e
su v2012e, d
sktest v2012e


*GED arma(2,0)-GARCH(2,1)
arch return, arch(1/2) garch(1/1) arima(2,0,0) distribution (ged)
estat ic
predict e2021e, residual
predict h2021e, variance
gen v2021e = e2021e / h2021e^(1/2)
ac v2021e, lags(20)
wntestq v2021e, lags(20)
wntestq v2021e, lags(8)
gen sv2021e = v2021e^2
ac sv2021e, lags(20)
ac sv2021e
wntestq sv2021e, lags(20)
wntestq sv2021e, lags(8)
histogram v2021e, freq normal
qnorm v2021e
su v2021e, d
sktest v2021e

*tarch************************************************************
arch return, arch(1/2) tarch(1/2) garch(1/1) arima(2,0,0) nolog
predict ht, variance
tsline ht
predict et, residual
gen vt = et / ht^(1/2)
wntestq vt, lags(40)
wntestq vt, lags (20)
wntestq vt, lags (8)
ac vt, lags (60)
ac vt, lags (30)
```

```
gen svt=vt^2
wntestq svt, lags (40)
wntestq svt, lags (20)
wntestq svt, lags (8)
ac svt, lags (30)
histogram vt, freq normal
su vt, d
qnorm vt

predict htt, variance
predict returntt, xb
tsline htt

*
sort time,
gen t=_n
sum t
gen shock5 = (t-77)/15
sum shock5
predict h5, variance at (shock5 1)
twoway (spike h5 shock5, sort) if shock5>= -4 & shock5<= 4

***T+GED*****************************************************************
***
arch return, arch(1/2) tarch(1/2) garch(1/1) arima(2,0,0) distribution (ged) nolog
estat ic
predict htg, variance
su htg, d
tsline htg
predict etg, residual
gen vtg = etg / htg^(1/2)
wntestq vtg, lags(40)
wntestq vtg, lags (20)
wntestq vtg, lags (8)
ac vtg, lags (60)
ac vtg, lags (30)
gen svtg=vtg^2
wntestq svtg, lags (40)
wntestq svtg, lags (20)
```

```
wntestq svtg, lags (8)
ac svtg, lags (30)
histogram vtg,
su vtg, d
qnorm vtg
histogram vtg, normal kdensity
********************************************************************
* in mean
arch return, arch(1/2) garch(1/1) archm arima(2,0,0)    distribution (ged)
estat ic
predict hm, variance
predict em, residual
gen vm = em / hm^(1/2)
wntestq vm, lags(20)
wntestq vm, lags(8)
ac vm, lags (60)
ac vm, lags (30)
gen svm = vm^2
wntestq svm, lags(20)
wntestq svm, lags(8)
ac svm, lags (30)
su vm, d
histogram vm, normal
qnorm vm

gen logf100=logf1*100
arch return, abarch(1/2) atarch(1/2) sdgarch(1/1) arima(2,0,0)distribution(ged) het(logf100)
nolog
estat ic
* simple rate
gen simplef=(cpi-l.cpi)/l.cpi
gen simplef1=l.simplef
gen simplef2 = l.simplef1
* log rate
gen logf=ln(cpi/l.cpi)
gen logf1=l.logf
gen logf2=l.logf1
gen logf3=l.logf2
tsline cpi, from
```

```
tsline logf
histogram logf
su logf ,d
sum logf
gen cpi1=l.cpi

arch return, arch(1/1) garch(1/1) het(logf1) nolog
```

Appendix 2: Matlab software Code:

```
x = -6:0.01:6;
rho = [1.294779];
p = [];
for k = 1:length(rho)
    p = [p exp(-abs(x').^rho(k)) / (2*gamma(1+1/rho(k)))];
end
figure, hold on; set(gca,'fontsize',14);
plot(x,p,'linewidth',2);
str = num2str(rho');
clear str2;
for k = 1:length(rho)
    str2(k,:) = ['\it\rho =' str(k,:)];
end
legend(str2); xlabel('\itx'); ylabel('\itGG(x\rm;\it\rho)'); xlim([-6 6])
```

```
x = -6:0.01:6;
rho = [1.3 2 3];
p = [];
for k = 1:length(rho)
    p = [p exp(-abs(x').^rho(k)) / (2*gamma(1+1/rho(k)))];
end
figure, hold on; set(gca,'fontsize',14);
plot(x,p,'linewidth',2);
str = num2str(rho');
clear str2;
for k = 1:length(rho)
    str2(k,:) = ['\it\rho =' str(k,:)];
end
legend(str2); xlabel('\itx'); ylabel('\itGG(x\rm;\it\rho)'); xlim([-6 6])
```