Chromatic Polynomials

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ABSTRACT

In this thesis, we shall investigate chromatic polynomials of graphs, and some related polynomials. In Chapter 1, we study the chromatic polynomial written in a modified form, and use these results to characterise the chromatic polynomials of polygon trees. In Chapter 2, we consider the chromatic polynomial written as a sum of the chromatic polynomials of complete graphs; in particular, we determine for which graphs the coefficients are symmetrical, and show that the coefficients exhibit a skewed property. In Chapter 3, we dualise many results about chromatic polynomials to flow polynomials, including the results in Chapter 1, and a result about a zero-free interval. Finally, in Chapter 4, we investigate the zeros of the Tutte Polynomial; in particular their observed proximity to certain hyperbolæ in the xy–plane.

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Chapter 3 - The Flow Polynomial

Chapter 4 - Zeros of the Tutte Polynomial

CHAPTER 0

Introduction and General Definitions

0.0. Introduction.

The *chromatic polynomial* of a graph, originally introduced in the hope that it would help to prove the 4-colour theorem, has since been the subject of much study in its own right.

A lot of research has been carried out into problems about to what extent the chromatic polynomial determines its graph. In this thesis, Chapter 1 is largely devoted to problems of this sort, in particular, it is shown that the chromatic polynomial of a polygon tree is unique to polygon trees with the same number of polygons of a given size.

Chapter 2 is devoted to the study of coefficients of the chromatic polynomial when it is written as a sum of the chromatic polynomials of complete graphs (that is, falling factorials). We characterise the graphs for which these coefficients have symmetry about the centre. It has been conjectured that these coefficients form a strongly log-concave sequence, and we present partial results towards proving this.

The *flow polynomial* of a graph is related to the chromatic polynomial; in particular, for a planar graph, the flow polynomial is more or less the chromatic polynomial of its dual. In Chapter 3, we show that many of the results about chromatic polynomials hold in a dual form for flow polynomials (of both planar and non-planar graphs), including the results in Chapter 1, and a result about a zero-free interval. We also present some results about the dual of the complete graph basis for the chromatic polynomial, introduced in Chapter 2.

The *Tutte polynomial* is a generalisation of both the chromatic and flow polynomials of a graph (in the sense that these can be calculated from the Tutte polynomial of that graph). In Chapter 4, we consider the zeros of the Tutte polynomial of a graph; in particular, the curious proximity of the zeros of Tutte polynomials to certain hyperbolæ in the xy–plane.

0.1. General Definitions.

Throughout this thesis, G will denote a graph with vertex-set $V(G)$, edge-set $E(G)$, n vertices, m edges, c components and b blocks. For a graph G_i, n_i, m_i, c_i and so on will denote the number of vertices, edges, components and so on. For a vertex v of G, $d(v)$ will denote the degree of v.

A graph H is a *subcontraction* of G, denoted $H \preccurlyeq G$, if G has a subgraph which is contractible to H.

A *circuit* of length n, C_n $(n \geq 1)$ is a connected 2-regular graph with n vertices. It is called a circuit of G if it is a subgraph of G. A *wheel* W_n $(n \geq 2)$ consists of a circuit C_{n-1} together with a vertex adjacent to every other vertex. Thus $C_3 \cong K_3$ and $W_4 \cong K_4$. The *girth* of G is the length of the shortest circuit of G.

$\gamma = \gamma(G)$ will denote the *circuit rank* of G, that is $\gamma = m - n + c$, the minimum number of edges whose removal from G destroys every circuit in G. It follows from Euler's Theorem that for a plane graph G, $\gamma(G)$ is one less than the number of faces of G.

For a graph G and $X \subseteq E(G)$, $c_G(X)$ and $\gamma_G(X)$ will denote the number of components and the circuit rank, respectively, of the graph with vertex-set $V(G)$ and edge-set X (so that $\gamma_G(X) = |X| - n + c_G(X)$).

A *cutset* of k vertices (edges) of a graph G is a set of k vertices (edges) whose deletion increases the number of components of G. For $k \geq 2$, G is said to be *k-connected* (*k-edge-connected*) if it is connected and has no cutset of fewer than k vertices (edges). A cutset of one vertex (edge) is called a *cut-vertex* (*cut-edge*).

A *block* of a graph G is a maximally 2-connected subgraph of G or a cut-edge (together with its end-vertices). Note that isolated vertices are not blocks.

For $k \geq 1$, a *k-colouring* of G is an assignment of k colours to the vertices of G such that adjacent vertices of G receive different colours. The *chromatic number* $\chi = \chi(G)$ of G is the smallest integer k such that G has a k-colouring.

The *chromatic polynomial* $P(G, t)$ of G is (for an integer $t > 0$) the number of t-colourings of G. We shall see in the next section that it actually is a polynomial.

For an edge e of G, $G-e$ and G/e will denote the graphs obtained from G by deleting and contracting e, respectively. For distinct vertices u and v of G, $(G)_{u=v}$ and $G+uv$ will denote the graphs obtained from G by identifying u and v, and adding an edge between u and v, respectively (so that $(G)_{u=v} = (G+uv)/uv$ and, if uv is an edge of G, $G/uv = (G-uv)_{u=v}$).

Finally, if G is a simple graph, $\bar{G}$ will denote the complement of G (that is the graph with vertex-set $V(G)$ and vertices adjacent in $\bar{G}$ if and only if they are non-adjacent in G), and, if G is a plane graph, G^* will denote the dual of G.

0.2. Basic Results.

In this section we present (without proof) some of the basic results that are known about the chromatic polynomial.

Theorem 0.1.

(i) $P(\bar{K}_n, t) = t^n$.

(ii) $P(K_n, t) = t(t-1)(t-2)\cdots(t-n+1)$.

(iii) If G is a tree then $P(G,t) = t(t-1)^{n-1}$.

(iv) $P(C_n, t) = (t-1)^n + (-1)^n(t-1)$. □

Theorem 0.2.

(i) The *deletion-contraction formula.*

If e is an edge of G then

$$P(G,t) = P(G-e,t) - P(G/e,t).$$

(ii) The *addition-identification formula.*

If u and v are non-adjacent vertices of G then

$$P(G,t) = P(G+uv,t) + P((G)_{u=v},t). \quad \square$$

Theorem 0.3.

(i) If $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \varnothing$, then

$$P(G,t) = P(G_1,t)P(G_2,t).$$

(ii) If $G = G_1 \cup G_2$ where $G_1 \cap G_2 = K_r$, then

$$P(G,t) = \frac{P(G_1,t)P(G_2,t)}{P(K_r,t)}. \quad \square$$

Theorem 0.4. If G has a loop, then $P(G,t) = 0$ for all t. Otherwise,

$$P(G,t) = \sum_{i=c}^{n} (-1)^{n-i} a_i t^i$$

where the a_i are all positive integers, $a_n = 1$, and, if G is simple, $a_{n-1} = m$. $\square$

CHAPTER 1

The Quotient Polynomial

1.0. Introduction and Definitions.

Throughout this chapter graphs will be assumed to be simple.

We define the *quotient polynomial* $q(G,t)$ by

$$q(G,t) := \frac{P(G,t)}{(-1)^{n-b-c}t^c(t-1)^b}\,.$$

We shall see in Section 1.1 that $q(G,t)$ actually is a polynomial.

For each i, $a_i(G)$ is defined by $q(G,t) = \sum_i a_i(G)s^i$ where $s = 1-t$.

A *polygon* in a graph G is a chordless circuit (that is, a circuit that is also an induced subgraph of G). It is an *r-gon* if it has r edges. Thus a triangle is always a 3-gon, but a circuit of length 4 is not necessarily a 4-gon. A *generalised polygon tree* is a 2-connected graph that does not have K_4 as a subcontraction. A *polygon tree* is defined recursively by the rules:

(i) A polygon is a polygon tree with one polygon.

(ii) Any graph $G = H \cup C$, where H is a polygon tree with k polygons, C is a polygon and $H \cap C = K_2$, is a polygon tree with $k+1$ polygons.

Equivalently, a polygon tree is a generalised polygon tree in which the intersection of any two polygons is empty or K_2.

A polygon tree in which every polygon is an r-gon is called an *r-gon tree.*

An *outerplanar* graph is a planar graph which can be drawn so that all the vertices lie on the boundary of a single face. Thus a 2-connected outerplanar graph is a polygon tree (but not necessarily vice versa).

For example, in Figure 1.0.1, G_1 is a polygon tree, G_2 is a generalised polygon tree but not a polygon tree, and neither is outerplanar.

A *separating edge* of a graph G is an edge uv whose contraction increases the number of blocks of the component C in which it lies (see Figure 1.0.2 (i)). It *separates* G into two subgraphs G_1 and G_2 (which need not be unique) such that $G_1 \cup G_2 = G$, $V(G_1 \cap G_2) = \{u, v\}$ and $E(G_1 \cap G_2) = \{uv\}$.

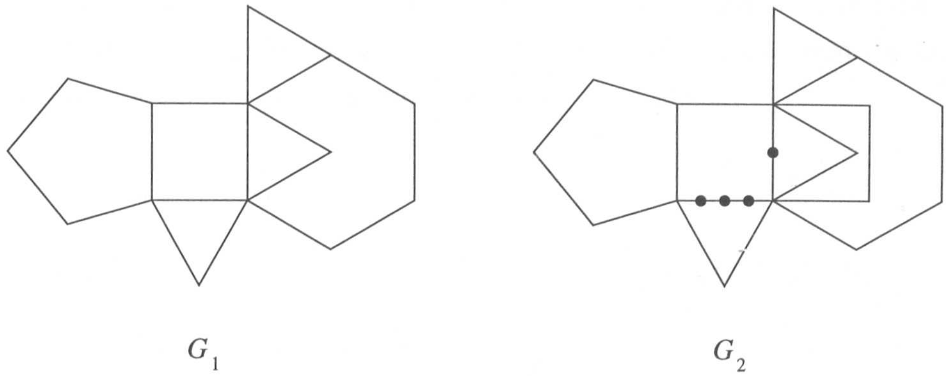


Figure 1.0.1

A connected subgraph H of a graph G is a *separating subgraph* if there exist connected subgraphs G_1 and G_2 and vertices u and v of G such that $G_1 \cup G_2 \cup H = C$, where C is the component of G containing H, $V(G_1 \cap G_2) = V(G_1 \cap H) = V(G_2 \cap H) = \{u, v\}$, H, G_1 and G_2 are edge-disjoint, and $uv \notin E(G)$ (see Figure 1.0.2 (ii)). It is a *separating path* if H is a path (see Figure 1.0.2 (iii)). Note that a separating edge is neither a separating subgraph nor a separating path.

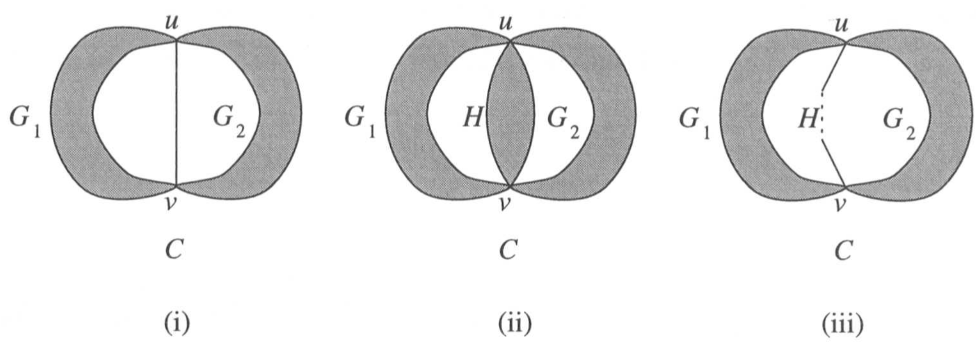
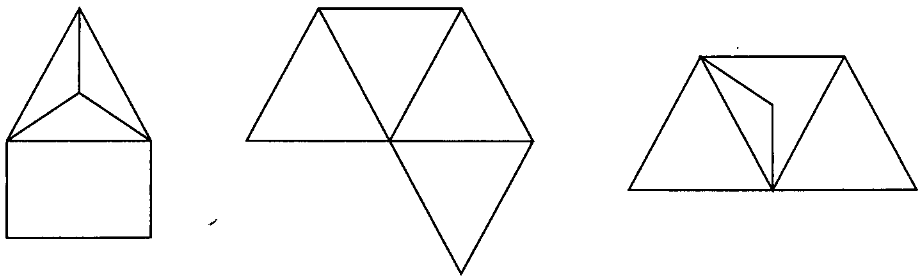


Figure 1.0.2

Chao and Li [1] claimed that it is possible to determine from the chromatic polynomial of a graph whether or not it is an r-gon tree with k r-gons. They showed that any graph with the same chromatic polynomial as an r-gon tree with k r-gons is a 2-connected planar graph with girth r and the right numbers of vertices, edges and r-gons; but unfortunately the rest of their proof is incorrect. The graphs in Figure 1.0.3 show that these properties are not enough to show that G is an r-gon tree. Each graph is 2-connected and planar, with girth 3, six vertices, nine edges and four 3-gons, but G_2 and G_3 are 3-gon trees while

G_1 is not. Also, G_2 is outerplanar, while G_3 is not, although they have the same chromatic polynomial, so it is impossible to determine from the chromatic polynomial of a graph whether or not it is outerplanar.



G_1 G_2 G_3

Figure 1.0.3

We shall present some basic results about the quotient form of the chromatic polynomial in Section 1.1, and apply these results in Section 1.2 to prove a stronger result than that claimed by Chao and Li. Finally, in Section 1.3, we shall present a result which evaluates some of the coefficients $a_i(G)$, and conjecture an improvement to Woodall's inequality [4]. Some of this work appears in a joint paper by D. R. Woodall and myself [3].

1.1. Basic Results.

Theorem 1.1.

(i) $q(G, t)$ is a polynomial in t.

(ii) $a_0(G) \geq a_{n-b-c}(G) = 1$, $a_i(G) \geq a_{n-b-c-1}(G) = \gamma(G)$ for $1 \leq i \leq n-b-c-2$, and $a_i(G) = 0$ for $i < 0$ or $i > n-b-c$.

(iii) If $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \varnothing$, K_1 or K_2, then $q(G, t) = q(G_1, t)q(G_2, t)$.

(iv) If T is a forest, then $q(T, t) = 1$, and if C_n is the circuit on n vertices, then $q(C_n, t) = 1 + s + s^2 + \cdots + s^{n-2}$.

(v) If $e \in E(G)$ is not a cut-edge of G, $G_1 = G - e$ and $G_2 = G/e$, then $q(G,t) = s^{b_1-b} q(G_1,t) + s^{b_2-b} q(G_2,t)$.

Proof. Parts (i) to (iv) are due to Woodall [4]; we prove (v). Note that $c_1 = c_2 = c$ and so, by the deletion-contraction formula and the definition of $q(G,t)$,

$$q(G,t) = \frac{P(G,t)}{(-1)^{n-b-c} t^c (t-1)^b}$$

$$= \frac{P(G_1,t)}{(-1)^{n-b_1-c_1} t^{c_1} (t-1)^{b_1} (-1)^{b_1-b} (t-1)^{b-b_1}}$$

$$- \frac{P(G_2,t)}{(-1)^{n-1-b_2-c_2} t^{c_2} (t-1)^{b_2} (-1)^{b_2-b+1} (t-1)^{b-b_2}}$$

$$= s^{b_1-b} q(G_1,t) + s^{b_2-b} q(G_2,t)$$

as required. □

It is worth noting that Theorem 1.1 can be used to show that $P(G,t)$ has zeros of multiplicity c at $t = 0$ and b at $t = 1$. Clearly $P(G,t)$ is zero at $t = 2$ if and only if G is non-bipartite. If G is non-bipartite but $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \emptyset$, K_1 or K_2, then by Theorem 0.3 $P(G,t)$ may have a zero of multiplicity 2 or more at $t = 2$. The graph G in Figure 1.1.1 is a non-bipartite graph such that $P(G,t)$ has a zero of multiplicity 2 at $t = 2$, but G is 3-connected.

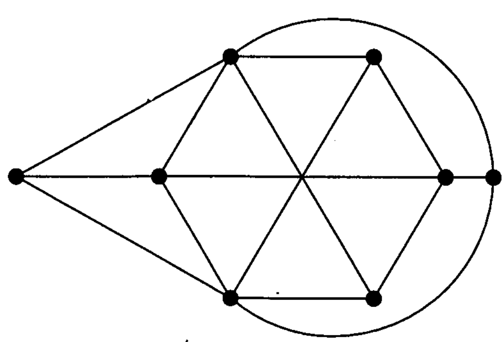


Figure 1.1.1

It is easy to see from Theorem 0.3 (ii), that by 'gluing' r copies of G together at a triangle, we can construct a 3-connected, non-bipartite graph G_r such that $P(G_r,t)$ has a zero of multiplicity $r+1$ at $t = 2$.

Returning to the quotient polynomial, the remainder of the results in this section place strong conditions on the graphs G for which $a_0(G)$ and $a_1(G)$ attain the minimum values permitted by Theorem 1.1 (ii).

Theorem 1.2. If H is a 2-connected subcontraction of G, then $a_i(G) \geq a_i(H)$ for each i.

Proof. We prove the result by induction on m. If $H = G$ then we are done, so suppose otherwise.

There are two cases to consider.

Case 1: $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \emptyset$, K_1 or K_2, and $G_1 \cap G_2$ is properly contained in G_1 and G_2. Then, by Theorem 1.1 (iii), $q(G,t) = q(G_1,t)q(G_2,t)$.

If $H \preccurlyeq G_1$ or $H \preccurlyeq G_2$, without loss of generality say $H \preccurlyeq G_1$, then

$$a_i(G) \geq a_0(G_2)a_i(G_1) \geq a_i(G_1) \geq a_i(H)$$

by Theorem 1.1 (ii) and the inductive hypothesis, as required. Note that this must happen if $G_1 \cap G_2 = \emptyset$ or K_1.

Now suppose otherwise. Then G has an edge e which separates G into G_1 and G_2. Moreover, $G - e$ is 2-connected. If $H \preccurlyeq G - e$, then by Theorem 1.1 (v) and the inductive hypothesis, $a_i(G) \geq a_i(G-e) \geq a_i(H)$, as required. Otherwise, $H = H_1 \cup H_2$ where $H_1 \cap H_2 = G_1 \cap G_2$, $H_1 \preccurlyeq G_1$ and $H_2 \preccurlyeq G_2$. Since H is 2-connected, and e is a separating edge of H (since H is not a subcontraction of G_1 or G_2), it follows that H_1 and H_2 are 2-connected. Then

$$a_i(G) = \textstyle\sum_r a_r(G_1)a_{i-r}(G_2) \geq \sum_r a_r(H_1)a_{i-r}(H_2) = a_i(H)$$

by the inductive hypothesis, as required.

Case 2: G is 2-connected with no separating edge. For $e \in E(G)$, let $G_1 = G - e$ and $G_2 = G/e$. Then G_2 is 2-connected (since G has no separating edge).

If $H \preccurlyeq G_2$ for some $e \in E(G)$, then by Theorem 1.1 (v) and the inductive hypothesis, $a_i(G) \geq a_i(G_2) \geq a_i(H)$ for each i, and we are done; so suppose otherwise. Then $V(H) = V(G)$ and, since H is not isomorphic to G, $E(G) \backslash E(H)$ is not empty. Since H is 2-connected, and $H \preccurlyeq G_1$ for any $e \in E(G) \backslash E(H)$, G_1 must be 2-connected, and so $b_1 = 1$ and $a_i(G) \geq a_i(G_1) \geq a_i(H)$ by the inductive hypothesis, as required. □

Corollary 1.2.1. (C.-Y. Chao, L.-C. Zhao [2]) If G has K_4 as a subcontraction, then $a_0(G) \geq 2$.

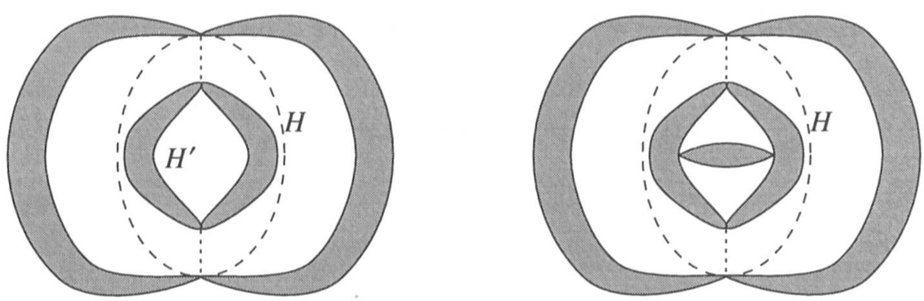
Proof. This follows from Theorem 1.2 and the fact that $q(K_4, t) = s^2 + 3s + 2$. $\square$

Lemma 1.3. Let G be a 2-connected graph without K_4 as a subcontraction, and suppose G has no separating edge. Then either G has a separating path or G is a circuit.

Proof. If G is a circuit then we are done, so suppose otherwise. Then it is not difficult to see, by considering any circuit in G, that G must have a separating subgraph.

Let H be a minimal separating subgraph of G. If H is a path then we are done, so suppose otherwise. Since H contains no separating edges of G, it is not difficult to see that either H has a proper subgraph H' which is also a separating subgraph of G (see Figure 1.1.2 (i)), contradicting the minimality of H, or $K_4 \preccurlyeq G$ (see Figure 1.1.2 (ii)), a contradiction.

The result now follows. $\square$



G (i) G (ii)

Figure 1.1.2

Corollary 1.3.1. A generalised polygon tree is a polygon tree if and only if it has no separating subgraph.

Proof. 'Only if' is obvious; we prove 'if'.

Let G be a minimal counterexample, that is a generalised polygon tree with no separating subgraphs that is not a polygon tree. Then G cannot be a circuit and so by Lemma 1.3, G has a separating edge e which separates G into G_1 and G_2, say. Then G_1 and G_2 are generalised polygon trees without separating subgraphs, and so by the minimality of G, G_1 and G_2 are polygon trees. But then G must be a polygon tree also, a contradiction. The result follows. □

Lemma 1.4. Let G be a 2-connected graph with a subgraph P which is either a separating path or a separating edge, and let l be the number of edges in P. Let G_1 be the graph obtained from G by removing all the edges and interior vertices of P and let G_2 be the graph obtained from G by contracting all but one of the edges of P. Note that G_1 and G_2 are 2-connected. Then

$$q(G,t) = q(G_1,t)(s^{l-1} + s^{l-2} + \cdots + s) + q(G_2,t)$$

Proof. We prove the result by induction on l.

If $l = 1$ then we are done since then $G_2 = G$, so suppose $l \geq 2$. Let e be an edge of P. Then $q(G-e,t) = q(G_1,t)$ by Theorem 1.1 (iii), since $G-e$ and G_1 differ only in $l-1$ blocks, each of which is K_2, and $q(K_2,t) = 1$. Thus, by Theorem 1.1 (v) and the inductive hypothesis,

$$\begin{aligned} q(G,t) &= s^{l-1}q(G-e,t) + q(G/e,t) \\ &= s^{l-1}q(G_1,t) + q(G_1,t)(s^{l-2} + s^{l-3} + \cdots + s) + q(G_2,t) \\ &= q(G_1,t)(s^{l-1} + s^{l-2} + \cdots + s) + q(G_2,t) \end{aligned}$$

as required. □

Corollary 1.4.1. (C.-Y. Chao, L.-C. Zhao [2]) Let G be a graph without K_4 as a subcontraction. Then $a_0(G) = 1$.

Proof. Let G be a minimal counterexample. By Theorem 1.1 (iii), G is 2-connected without separating edges. G cannot be a circuit, so by Lemma 1.3, G has a separating path P. By Lemma 1.4, $a_0(G) = a_0(G_2,t)$, where G_2 is defined as in Lemma 1.4. K_4 cannot be a subcontraction of G_2, so this is a contradiction of the minimality of G. Thus the statement must be true. □

Corollary 1.4.2. If G is a graph with a separating path P, then $a_1(G) > \gamma(G)$.

Proof. Let G be a minimal counterexample. Suppose there exist subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \varnothing$, K_1, or K_2. We may suppose without loss of generality that P is a separating path of G_1. By Theorem 1.1 (iii), $q(G,t) = q(G_1,t)q(G_2,t)$, and so, by the minimality of G and Theorem 1.1 (ii),

$$\begin{aligned} a_1(G) &= a_0(G_2)a_1(G_1) + a_0(G_1)a_1(G_2) \\ &\geq a_1(G_1) + a_1(G_2) \\ &> \gamma(G_1) + \gamma(G_2) \\ &= \gamma(G), \end{aligned}$$

a contradiction.

Thus G is 2-connected without separating edges. Let G_1 and G_2 be defined as in Lemma 1.4. Then $\gamma(G_2) = \gamma(G)$ and so by Theorem 1.1 (ii) and Lemma 1.4,

$$a_1(G) = a_0(G_1) + a_1(G_2) \geq 1 + \gamma(G_2) > \gamma(G),$$

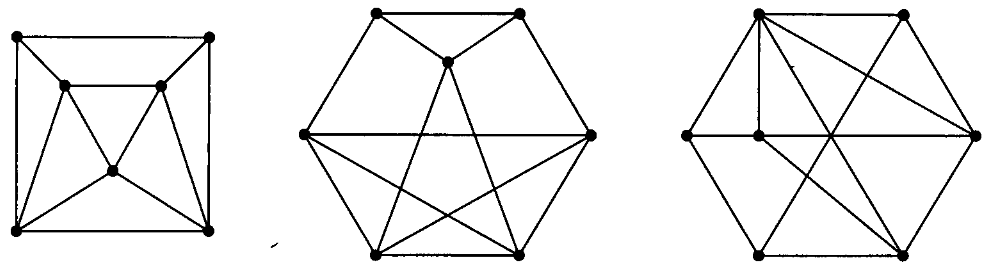
a contradiction. Thus the statement must be true. □

Corollary 1.4.3. Let G be a graph without K_4 as a subcontraction, and suppose that G has a separating subgraph H. Then $a_1(G) > \gamma(G)$.

Proof. Let G be a minimal counterexample. As in the proof of Corollary 1.4.2, G must be 2-connected without separating edges. G cannot be a circuit, and so by Lemma 1.3, G has a separating path. But then, by Corollary 1.4.2, $a_1(G) > \gamma(G)$, a contradiction. Thus the statement must be true. □

Corollary 1.2.1 and Corollary 1.4.1 together show that $a_0(G) = 1$ if and only if G does not have K_4 as a subcontraction. Thus it is possible to determine from the chromatic polynomial of a graph whether or not it has K_4 as a subcontraction. However, it is not possible to determine from the chromatic polynomial of a graph whether or not it has K_5 or $K_{3,3}$ as a subcontraction. For example, the graphs in Figure 1.1.3 all have the same chromatic polynomial, but G_1 is planar (and so cannot have K_5 or $K_{3,3}$ as a subcontraction by Kuratowski's

Theorem) whereas clearly G_2 has K_5 as a subcontraction and G_3 has $K_{3,3}$ as a subcontraction.



G_1 G_2 G_3

Figure 1.1.3

1.2. Polygon Trees.

In this section, we apply the results of Section 1.1 to polygon trees.

Theorem 1.5. Let G be a 2-connected graph. Then G is a polygon tree, with k_i i-gons for each i, if and only if

$$q(G,t) = \prod_{i=3}^{\infty}(1+s+s^2+\cdots+s^{i-2})^{k_i}.$$

Proof. 'Only if' follows inductively from Theorem 1.1 parts (iii) and (iv).

To prove 'if', suppose G is a graph with $q(G,t)$ as above. Then Theorem 1.1 (ii) gives $\gamma(G) = a_{n-b-c-1}(G) = \sum_{i=3}^{\infty} k_i = a_1(G)$ and $a_0(G) = 1$. By Corollary 1.2.1, G does not have K_4 as a subcontraction, and by Corollary 1.4.3, G has no separating subgraph. Thus by Corollary 1.3.1, G is a polygon tree (since it is 2-connected by hypothesis).

Suppose G has k_i' i-gons for each i. Then, by Theorem 1.1 (iii) and (iv),

$$q(G,t) = \prod_{i=3}^{\infty}(1+s+s^2+\cdots+s^{i-2})^{k_i'} = \prod_{i=3}^{\infty}(1+s+s^2+\cdots+s^{i-2})^{k_i}.$$

Multiplying through by $(1-s)^{\gamma(G)}$ gives

$$\prod_{i=3}^{\infty}(1-s^{i-1})^{k_i'} = \prod_{i=3}^{\infty}(1-s^{i-1})^{k_i}.$$

Equating coefficients of s^2 on each side, gives $k_3' = k_3$. Dividing through by $(1-s^2)^{k_3}$ and then equating coefficients of s^3, gives $k_4' = k_4$. Continuing in this way, we see that $k_i' = k_i$ for all $i \geq 3$. The result follows. □

Corollary 1.5.1. A graph G is a polygon tree, with k_i i-gons for each i, if and only if

$$P(G,t) = (-1)^n t(t-1)\prod_{i=3}^{\infty}(1+s+s^2+\cdots+s^{i-2})^{k_i},$$

where $s = 1-t$ and $n = 2+\sum_{i=3}^{\infty} k_i(i-2)$.

Proof. 'Only if' follows from Theorem 1.5 and the definition of $q(G,t)$.

To prove 'if', suppose G is a graph with $P(G,t)$ as above. Now neither $t-1=-s$ nor $t=1-s$ are factors of $p(t) = \prod_{i=3}^{\infty}(1+s+s^2+\cdots+s^{i-2})^{k_i}$, (since the result of substituting $s=0$ or $s=1$ is non-zero) and so $q(G,t) = p(t)$, and G is 2-connected. The result now follows by Theorem 1.5. □

Corollary 1.5.2. A polynomial $p(t)$ is the chromatic polynomial of an outerplanar graph if and only if

(i) $p(t) = t^n$ for some $n \geq 1$ or

(ii) $p(t) = (-1)^{n-b-c}t^c(t-1)^b\prod_{i=3}^{\infty}(1+s+s^2+\cdots+s^{i-2})^{k_i}$ for some integers $n, b, c \geq 1$, $k_i \geq 0$ for each i.

Proof. For 'only if', suppose G is an outerplanar graph. Then every block of G is either a polygon tree or an edge. If G has no edges, then $P(G,t) = t^n$; so suppose otherwise. Then by Theorem 1.5, Theorem 1.1 (iii) and the definition of $q(G,t)$, $P(G,t)$ has the required form, with b,c and k_i being the numbers of blocks, components and i-gons of G respectively.

For 'if', suppose $p(t)$ has the form given. If $p(t) = t^n$ then $P(\bar{K}_n, t) = p(t)$ and $\bar{K}_n$ is outerplanar; so suppose otherwise. Let G' be an outerplanar polygon tree with k_i i-gons for each i (note that this is easy to construct), and let G be a graph obtained from G' by adding $b-1$ pendant edges incident with a vertex of G' and $c-1$ isolated vertices. Then G is outerplanar, and $P(G,t)=p(t)$ by Theorem 1.5, Theorem 1.1 (iii) and the definition of $q(G,t)$. □

Theorem 1.6. Let G be a graph with K_4 as a subcontraction. Then either

(i) every circuit of G is contained in one block, which is isomorphic to K_4, or

(ii) $a_1(G) > \gamma(G)$.

Proof. We prove the result by induction on m. There are two cases to consider.

Case 1: $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \varnothing$, K_1 or K_2. Then either $K_4 \preccurlyeq G_1$ or $K_4 \preccurlyeq G_2$, say $K_4 \preccurlyeq G_1$. By Theorem 1.1 (iii), $q(G,t)=q(G_1,t)q(G_2,t)$, and so

$$a_1(G) = a_0(G_1)a_1(G_2) + a_1(G_1)a_0(G_2)$$

$$\geq 2a_1(G_2) + a_1(G_1)$$

$$\geq \gamma(G_2) + a_1(G_1) + a_1(G_2),$$

by Corollary 1.2.1 and Theorem 1.1 (ii).

If G_1 satisfies condition (i), then either G satisfies condition (i) also, or G_2 contains a circuit, in which case $\gamma(G_2) \geq 1$ and so

$$a_1(G) \geq 1 + \gamma(G_1) + \gamma(G_2) = \gamma(G) + 1 > \gamma(G),$$

as required.

Otherwise, $a_1(G_1) > \gamma(G_1)$ by the inductive hypothesis, and so

$$a_1(G) \geq a_1(G_1) + a_1(G_2) > \gamma(G_1) + \gamma(G_2) = \gamma(G),$$

as required.

Case 2: G is 2-connected, with no separating edge. If e is an edge of G, let $G_1 = G - e$ and $G_2 = G/e$ (with multiple edges removed). Then G_2 is 2-connected, and so by Theorem 1.1 (v), $q(G,t) = s^{b_1-1}q(G_1,t) + q(G_2,t)$. There are two subcases to consider.

Case 2a: e can be chosen so that it does not lie in a triangle. Then $\gamma(G_2) = \gamma(G)$.

If $K_4 \preccurlyeq G_2$, but G_2 is not isomorphic to K_4, then

$$a_1(G) \geq a_1(G_2) > \gamma(G_2) = \gamma(G)$$

by the inductive hypothesis, as required. If $G_2 = K_4$ then G is the graph obtained by subdividing an edge of K_4, which has $q(G,t) = 2 + 4s + 3s^2 + s^3$, and so $a_1(G) = 4 > 3 = \gamma(G)$, as required.

Thus we may suppose that K_4 is not a subcontraction of G_2. If G_1 has a cut-vertex then, since e does not lie in a triangle, it is easy to see that $K_4 \preccurlyeq G_2$ (since no circuits are destroyed in contracting e), a contradiction. Thus G_1 must be 2-connected, and so

$$a_1(G) = a_1(G_1) + a_1(G_2) \geq \gamma(G_1) + \gamma(G_2) = \gamma(G) - 1 + \gamma(G) > \gamma(G),$$

since $\gamma(G) \geq 2$, as required.

Case 2b: Every edge of G lies in a triangle. Then it is easy to see that e can be chosen in such a way as to make G_1 2-connected. If $K_4 \preccurlyeq G_1$ (note that G_1 is not isomorphic to K_4, since G is simple) then

$$a_1(G) = a_1(G_1) + a_1(G_2) > \gamma(G) - 1 + 1 = \gamma(G),$$

as required.

So suppose e cannot be chosen in such a way that G_1 is 2-connected and $K_4 \preccurlyeq G_1$. Let e be an edge of G such that G_1 is 2-connected. Then G_1 is a generalised polygon tree. If G_1 has a separating subgraph then, by Corollary 1.4.3,

$$a_1(G) \geq a_1(G_1) + a_1(G_2) > \gamma(G) - 1 + 1 = \gamma(G),$$

as required, so suppose otherwise.

Then G_1 is a polygon tree by Corollary 1.3.1. If G_1 contains a polygon adjacent to three or more others, then G must have a separating edge, a contradiction. If e is a chord of one of the polygons in G_1, then G is a polygon tree also, and so cannot have K_4 as a subcontraction, a contradiction. Since every edge of G lies in a triangle, it is easy to see that G_1 is a 3-gon tree with at least four 3-gons and (since G has no separating edges) e joins the two degree two vertices in G_1. But then G_1 has the triangulated pentagon as a proper subgraph, and if e is any edge of G_1 not in that subgraph, then $G - e$ is 2-connected and $K_4 \preccurlyeq G - e$, a contradiction. The result now follows. □

Corollary 1.6.1. If G is a graph such that $a_1(G) = \gamma(G)$, then either G has exactly one block containing a circuit, which is isomorphic to K_4, or every block of G containing a circuit is a polygon tree (and hence G has the same chromatic polynomial as some outerplanar graph).

Proof. By Theorem 1.6, either every circuit of G is contained in one block, which is isomorphic to K_4, in which case we are done, or G does not have K_4 as a subcontraction. Suppose the latter case holds. We prove the result by induction on the number of blocks of G.

If G is 2-connected, then G is a generalised polygon tree, and since G cannot have a separating subgraph by Corollary 1.4.3, G is in fact a polygon tree, as required. Now suppose G is not 2-connected. Then $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \varnothing$ or K_1, and then, by Theorem 1.1 parts (ii) and (iii) (since $a_0(G_1) = a_0(G_2) = 1$ by Corollary 1.4.1),

$$\gamma(G) = a_1(G) = a_1(G_1) + a_1(G_2) \geq \gamma(G_1) + \gamma(G_2) = \gamma(G),$$

and so equality must hold throughout, that is, $a_1(G_1) = \gamma(G_1)$ and $a_1(G_2) = \gamma(G_2)$. The result now follows by the inductive hypothesis. □

1.3. Identities for the Coefficients $a_i(G)$.

In this section, we derive explicit expressions for the last few coefficients $a_i(G)$. The first result is a very nice application of the Binomial Theorem to prove a combinatorial identity, and will be used in the main result of the section.

Lemma 1.7. Let α, β and r be non-negative integers with $\alpha \geq \beta$. Then

$$\binom{\alpha-\beta}{r} = \sum_{i=0}^{r} (-1)^{r-i} \binom{\alpha}{i} \binom{\beta+r-i-1}{r-i}.$$

Proof. The coefficient of x^r in $(1-x)^{\alpha-\beta}$ is $(-1)^r \binom{\alpha-\beta}{r}$. Now, $(1-x)^{\alpha-\beta} = (1-x)^{\alpha}(1-x)^{-\beta}$; the coefficient of x^i in $(1-x)^{\alpha}$ is $(-1)^i \binom{\alpha}{i}$, the coefficient of x^{r-i} in $(1-x)^{-\beta}$ is $\binom{\beta+r-i-1}{r-i}$, and so

$$(-1)^r\binom{\alpha-\beta}{r}=\sum_{i=0}^{r}(-1)^i\binom{\alpha}{i}\binom{\beta+r-i-1}{r-i},$$

whence the result. □

Lemma 1.8.

$$P(G,t)=\sum_{X\subseteq E}(-1)^{|X|}t^{c_G(X)}.$$

Proof. For a fixed positive integer t, let S be the set of all (improper) t-colourings of G, and, for each $i=1,2,\dots,m$, let S_i be the set of t-colourings in which the ith edge, e_i, is bad. Then $P(G,t)=|S\setminus\bigcup_i S_i|$. Also,

$$|S_{i_1}\cap S_{i_2}\cap\cdots\cap S_{i_r}|=t^{c_G(\{e_{i_1},e_{i_2},\dots,e_{i_r}\})}$$

for $i_j\in\{1,2,\dots,m\}$. By the inclusion-exclusion principle, $P(G,t)=\sum_{X\subseteq E}(-1)^{|X|}t^{c_G(X)}$, and since this holds for all positive integers t, it must hold for all t. □

The following result enumerates $a_{n-i}(G)$ for $0\le i\le g$, where g is the girth of G.

Theorem 1.9. Suppose G is a graph with girth g and k g-circuits. Then for $0\le r\le g-2$,

$$a_{n-b-c-r}(G)=\binom{\gamma(G)+r-1}{r}$$

and

$$a_{n-b-c-g+1}(G)=\binom{\gamma(G)+g-2}{g-1}-k.$$

Proof. First note that if $r=0$, the result follows by Theorem 1.1 (ii).

By the definition of $q(G,t)$ and Lemma 1.8,

$$q(G,t) = \frac{P(G,t)}{(-1)^{n-b-c}t^c(t-1)^b}$$

$$= \frac{\sum\limits_{X \subseteq E} (-1)^{|X|} t^{c_G(X)}}{(-1)^{n-c}t^c(1-t)^b}$$

$$= \frac{1}{(1-t)^b} \sum_{X \subseteq E} (-1)^{|X|-n+c_G(X)} (-t)^{c_G(X)-c}$$

$$= \frac{1}{s^b} \sum_{X \subseteq E} (-1)^{\gamma_G(X)} (s-1)^{c_G(X)-c}$$

where $s = 1 - t$.

Now, for $X \subseteq E$, if $|X| < g$, then $\gamma_G(X) = 0$ and $c_G(X) = n - |X|$, and if $|X| = g$, then $c_G(X) = n - g$ except for the k subsets X which form g-circuits, for which $\gamma_G(X) = 1$ and $c_G(X) = n - g + 1$. If $|X| > g$ then $c_G(X) \le n - g$. Thus,

$$q(G,t) = \frac{1}{s^b} \sum_{i=0}^{m} \sum_{|X|=i} (-1)^{\gamma_G(X)} (s-1)^{c_G(X)-c}$$

$$= \frac{1}{s^b} \left[\sum_{i=0}^{g} \sum_{|X|=i} (s-1)^{n-i-c} + \sum_{i=g+1}^{m} \sum_{|X|=i} (-1)^{\gamma_G(X)} (s-1)^{c_G(X)-c} \right.$$

$$\left. -k(s-1)^{n-g-c} - k(s-1)^{n-g+1-c} \right].$$

From this, for $1 \le r < g-1$, $a_{n-b-c-r}(G)$ is the coefficient of s^{n-c-r} in $\sum\limits_{i=0}^{r} \binom{m}{i} (s-1)^{n-i-c}$, and $a_{n-b-c-g+1}(G)$ is the coefficient of $s^{n-c-g+1}$ in $\sum\limits_{i=0}^{g-1} \binom{m}{i} (s-1)^{n-i-c} - k(s-1)^{n-g+1-c}$.

Thus $a_{n-b-c-r}(G) = \sum\limits_{i=0}^{r} (-1)^{r-i} \binom{m}{i} \binom{n-i-c}{r-i}$, which by Lemma 1.7, with $\alpha = m$ and $\beta = n - c - r + 1$ (note that $\gamma(G) \ge 0$ and so $\alpha - \beta = \gamma(G) + r - 1 \ge 0$), gives $a_{n-b-c-r}(G) = \binom{\gamma(G)+r-1}{r}$ as required.

Similarly,

$$a_{n-b-c-g+1}(G)=\sum_{i=0}^{g-1}(-1)^{g-1-i}\binom{m}{i}\binom{n-i-c}{g-1-i}-k=\binom{\gamma(G)+g-2}{g-1}-k,$$

as required. □

We finish this chapter with the conjecture of an improvement to Woodall's inequality.

Conjecture 1.9. $a_r(G)\geq\binom{\gamma(G)+r-1}{r}$ for $0\leq r\leq g-2$, and $a_r(G)\geq\binom{\gamma(G)+g-2}{g-1}-k$ for $g-1\leq r\leq n-b-c-g$.

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CHAPTER 2

The Chromatic Polynomial Relative to the Complete Graph Basis

2.0. Introduction and Definitions.

In this chapter, we present some results on the sequence of coefficients of the chromatic polynomial of a graph relative to the complete graph basis, that is, when it is expressed as the sum of the chromatic polynomials of complete graphs. We obtain necessary and sufficient conditions for this sequence to be symmetrical, and we prove that it is 'skewed' and decreasing beyond its mid-point.

Throughout this chapter, graphs will be assumed to be simple.

An *r-colouring* $\mathcal{C}$ of G is an assignment of r colours to the vertices of G such that adjacent vertices receive different colours. $\mathcal{C}$ is *non-degenerate* if all r colours are used. $\mathcal{C}$ determines a set of *colour classes*, each colour class consisting of all the vertices of a given colour. Thus, if $\mathcal{C}$ is non-degenerate, the colour classes correspond to a partition of $V(G)$ into r (non-empty) independent subsets. Two r-colourings of G are said to be *equivalent* if they have the same colour classes, that is, if one may be obtained from the other by permuting the colours. We say that $\mathcal{C}$ is the *unique* r-colouring of G if it is unique up to equivalence; clearly this implies $r = \chi(G)$.

A *non-edge* of G is a pair of non-adjacent vertices of G. A *non-edge* of $\mathcal{C}$ in G is a non-edge of G whose vertices are in different colour classes of $\mathcal{C}$. The *join* of two disjoint graphs G and H, denoted by $G+H$, is the graph formed by joining every vertex of G by an edge to every vertex of H. For a colouring $\mathcal{C}$ of G, let $\alpha(G,\mathcal{C})$ denote the number of non-edges of $\mathcal{C}$ in G. If $\alpha(G,\mathcal{C}) = 0$, then it is easy to see that G is a complete χ-partite graph, the join of χ null graphs whose vertex sets are the colour classes of $\mathcal{C}$, where $\chi = \chi(G)$, and $\mathcal{C}$ is the unique χ-colouring of G.

We denote by $k_i(G)$, the number of non-equivalent, non-degenerate i-colourings of G, that is, the number of partitions of the vertex set $V(G)$ of G into i independent (non-empty) subsets, and let $K(G,x)$ denote the polynomial $\sum_i k_i(G)x^i$. $\dfrac{K(G,x)}{x^\chi}$ is also known as the *σ-polynomial* of G, $\sigma(G,x)$.

Several results are known about σ-polynomials (see Brenti [1], Dhurandhar [3], Du [4], Korfhage [7,8] and Xu [14]).

Read [11] conjectured that the absolute values of the coefficients in $P(G,t)$ form a unimodal sequence. Hoggar [6] strengthened this conjecture to strong log-concavity (defined in Section 2.3). In [13] Read and Tutte mention the conjecture that the $k_i(G)$ form a strongly log-concave sequence.

It is well-known that if a polynomial has non-negative coefficients and only real zeros, then the coefficients form a strongly log-concave sequence. Brenti [1] has used this result to show that large classes of graphs have strongly log-concave $k_i(G)$:

Theorem 2.1. (Brenti [1])

(i) If G is a graph such that each odd circuit in its complement, $\bar{G}$, has at least one chord, then $K(G,x)$ has only real zeros.

(ii) If G is a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ such that $\bar{G}$ is triangle-free, and $H_1, H_2, \ldots, H_n$ are graphs such that each $K(H_i, x)$ has only real zeros, then the graph obtained from the disjoint union of the H_i by adding an edge from every vertex in H_i to every vertex in H_j if and only if $v_i v_j$ is an edge of G has only real zeros. In particular, $K(G,x)$ has only real zeros.

(iii) If G_1 and G_2 are graphs such that $K(G_1, x)$ and $K(G_2, x)$ have only real zeros, then $K(G_1 \cup G_2, x)$ has only real zeros.

(iv) If G is a complete multi-partite graph, then $K(G,x)$ has only real zeros.

(v) If $P(G,t)$ has zeros only in the half-open interval $[0, \chi)$ then $K(G,x)$ has only real zeros. In particular, if G is a chordal graph, (that is, a graph in which every circuit has a chord) then $K(G,x)$ has only real zeros. □

Chvátal [2] showed that $k_i(G) \geq k_{i-1}(G)$ whenever $(i+2)^{i-1} \leq 2^n$. In Section 2.5, we show that for any graph G, $k_{\chi+i}(G) \geq k_{n-i}(G)$ and $k_{n-i}(G) > k_{n-i+1}(G)$ for $i \leq \frac{1}{2}(n-\chi)$. In Section 2.2, we characterise the graphs for which $k_{\chi+i}(G) = k_{n-i}(G)$ for all i; such graphs are called *K-symmetrical.* Sections 2.3 and 2.4 are devoted to results about sequences and Stirling numbers that are needed in Section 2.5. In Section 2.1, we present some basic results and an improved method for computing the chromatic polynomial of a dense graph.

2.1. Basic Results.

The following result relates the coefficients $k_i(G)$ to the chromatic polynomial of G.

Lemma 2.2.

$$P(G,t) = \sum_{i=1}^{n} k_i(G)P(K_i,t) = \sum_{i=\chi}^{n} k_i(G)P(K_i,t) = \sum_{i=\chi}^{n} k_i(G)t(t-1)\cdots(t-i+1).$$

Proof. Each partition of $V(G)$ into i independent subsets corresponds to a non-degenerate i-colouring of G and vice versa. Thus for $i > n$ or $i < \chi$, $k_i(G) = 0$. Moreover, for each partition of $V(G)$ into i independent subsets there is a bijection between the t-colourings of G with colour classes corresponding to this partition, and the t-colourings of K_i. The result follows. □

Lemma 2.3. The *Addition-Identification Formula.* Suppose u and v are non-adjacent vertices in G. Let $G_1 := G + uv$ and $G_2 := (G)_{u=v}$. Then for each i, $k_i(G) = k_i(G_1) + k_i(G_2)$.

Proof. By the well-known formula for chromatic polynomials (Theorem 0.2 (i)) $P(G_1,t) = P(G,t) - P(G_2,t)$, and so $P(G,t) = P(G_1,t) + P(G_2,t)$ (Theorem 0.2 (ii)), from which the result follows by Lemma 2.2, since the $P(K_i,t)$ are linearly independent polynomials. □

The next result evaluates $k_i(G)$ for some values of i.

Lemma 2.4.

(i) $k_n(G) = 1$,

(ii) $k_{n-1}(G)$ is the number of non-edges in G, that is, $k_{n-1}(G) = \binom{n}{2} - m$,

(iii) $k_{n-2}(G)$ is the number of independent sets of three vertices plus the number of pairs of disjoint non-edges in G,

(iv) $k_{n-2}(G) = \binom{m}{2} - m\binom{n-1}{2} + \binom{n}{3}\frac{3n-5}{4} - t(G)$, where $t(G)$ is the number of triangles in G,

(v) $k_\chi(G)$ is the number of non-equivalent non-degenerate χ-colourings of G, and so $k_\chi(G) = 1$ if and only if G is uniquely χ-colourable.

Proof. All except (iv) is straightforward from the definitions; (iv) is proved in [1]. □

A good motivation for looking at the complete graph basis for the chromatic polynomial is given by the next rather simple result, proved by Zykov in [15]. It extends in an obvious way to the join of more than two graphs.

Lemma 2.5. $K(G+H,x) = K(G,x)K(H,x)$.

Proof. Each partition of $V(G+H)$ into i independent subsets corresponds to a partition of $V(G)$ into j independent subsets together with a partition of $V(H)$ into $i-j$ independent subsets, for some j. Thus

$$k_i(G+H) = \textstyle\sum_j k_j(G)k_{i-j}(H),$$

and so $K(G+H,x) = K(G,x)K(H,x)$, as required. □

Chromatic polynomials can be calculated by using the well-known deletion-contraction formula to express them in terms of the chromatic polynomials of null graphs. In their book [10], Nijenhuis and Wilf improved this algorithm by stopping the process when trees are reached (since all trees of a given order have the same chromatic polynomial, namely $t(t-1)^{n-1}$). This method works very well for sparse graphs, but becomes inefficient for dense graphs. For dense graphs, chromatic polynomials can be calculated by using the addition-identification formula (a rearrangement of the deletion-contraction formula) to express them in terms of the chromatic polynomials of complete graphs. Read [12], in an unpublished paper, provided a refinement of the Nijenhuis-Wilf algorithm, but commented that he had been unable to find a set of 'target graphs', analogous to trees in the Nijenhuis-Wilf algorithm, to aim for when calculating the chromatic polynomial of dense graphs.

Lemma 2.5, together with Lemma 2.3, yields an improved method for calculating the chromatic polynomial of dense graphs. Firstly, we find a colouring $\mathcal{C}$ of G (not necessarily optimal), perhaps using a greedy algorithm. Then we repeatedly apply Lemma 2.3 to the non-edges of $\mathcal{C}$ in G, until we are left with complete multipartite graphs, the chromatic polynomial of which can be calculated using Lemma 2.5. The following result shows that this process must stop.

Lemma 2.6. Let G be a graph which is not complete χ-partite. Let $\mathcal{C}$ be a colouring of G. Suppose uv is a non-edge of $\mathcal{C}$ in G, and let $G_2 = (G)_{u=v}$. Then there is a colouring $\mathcal{C}_2$ of G_2 such that $\alpha(G_2, \mathcal{C}_2) \leq \alpha(G, \mathcal{C}) - 1$.

Proof. Let $\mathcal{C}_2$ be the colouring of G_2 obtained by giving the amalgamated vertex uv a new colour, and the rest of the vertices the same colours as they have in $\mathcal{C}$. Then the number of non-edges of $\mathcal{C}_2$ not incident with uv is the same as the number of non-edges of $\mathcal{C}$ not incident with u or v. Any vertex non-adjacent to uv in G_2 must be non-adjacent to both u and v in G, and so for every non-edge of $\mathcal{C}_2$ incident with uv there must be at least one non-edge of $\mathcal{C}$ incident with u or v (since u and v are in different colour classes of $\mathcal{C}$). Thus $\alpha(G_2, \mathcal{C}_2) \leq \alpha(G, \mathcal{C}) - 1$ as required. □

2.2. K-Symmetry.

In this section, we study the conditions under which the coefficients $k_i(G)$ have symmetry about their centre. A graph G is said to be *r-K-symmetrical* if, for $0 \leq i \leq r$, $k_{n-i}(G) = k_{\chi+i}(G)$. Thus G is 0-K-symmetrical if and only if $k_\chi(G) = 1$, that is, G is uniquely χ-colourable (see, for example, Harary [5]). G is said to be *K-symmetrical* if it is s-K-symmetrical where $s = \left\lfloor \frac{1}{2}(n - \chi) \right\rfloor$, or, equivalently, if G is r-K-symmetrical for all r.

For example, if G_1 and G_2 are the graphs with complements $\bar{G}_1$ and $\bar{G}_2$ in Figure 2.2.1, and G_3 is the graph G_3 in Figure 2.2.1, then

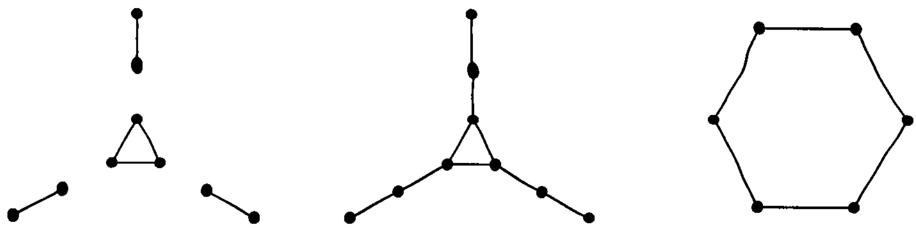
$$K(G_1, x) = x^9 + 6x^8 + 13x^7 + 13x^6 + 6x^5 + x^4,$$

$$K(G_{2,} x) = x^9 + 9x^8 + 25x^7 + 26x^8 + 9x^5 + x^4,$$

and

$$K(G_{3,} x) = x^6 + 9x^5 + 20x^4 + 10x^3 + x^2,$$

and so G_1 is K-symmetrical, G_2 is 1-K-symmetrical but not K-symmetrical, and G_3 is 0-K-symmetrical but not 1-K-symmetrical.



$\bar{G}_1$ $\bar{G}_2$ G_3

Figure 2.2.1

Lemma 2.7. The join of two r-K-symmetrical graphs is itself r-K-symmetrical.

Proof. Let G_1 and G_2 be r-K-symmetrical graphs, and let $G = G_1 + G_2$. Let $\chi_i = \chi(G_i)$ for each i. Then $n = n_1 + n_2$ and $\chi = \chi_1 + \chi_2$.

For $0 \le i \le r$, we have by Lemma 2.5,

$$k_{\chi+i}(G) = \sum_{j=0}^{i} k_{\chi_1+i-j}(G_1)k_{\chi_2+j}(G_2) = \sum_{j=0}^{i} k_{n_1-i+j}(G_1)k_{n_2-j}(G_2) = k_{n-i}(G)$$

since G_1 and G_2 are r-K-symmetrical, and so G is r-K-symmetrical as required. □

It follows that the join of two K-symmetrical graphs is itself K-symmetrical.

Lemma 2.8. Suppose a graph G has a unique χ-colouring $\mathcal{C}$, with colour classes of sizes $n_1, n_2, \ldots, n_\chi$. Then

$$k_{\chi+1}(G) \ge k_{n-1}(G) + \sum_{i=1}^{\chi}\left(S(n_i,2) - \binom{n_i}{2}\right) \qquad (2.2.1)$$

$$\ge k_{n-1}(G) \qquad (2.2.2)$$

where $S(t,i)$ denotes a Stirling number of the second kind (so that $S(t,2) = 2^{t-1} - 1$ is the number of ways of partitioning a set of t elements into two parts).

Proof. We shall show, by modifying $\mathcal{C}$, that the number of different non-degenerate $(\chi + 1)$-colourings of G is at least as large as the right-hand side of (2. 2. 1); this will prove the result, since $S(t,2) \ge \binom{t}{2}$ for all t.

By Lemma 2.4, $k_{n-1}(G)$ is the number of non-edges in G. Let $k_{n-1}(G) = k'_{n-1} + k''_{n-1}$ where k'_{n-1} is the number of non-edges of $\mathscr{C}$ in G and $k''_{n-1} = \sum_{i=1}^{\chi} \binom{n_i}{2}$ is the number of monochromatic non-edges of G. Then the right-hand side of (2.2.1) becomes $k'_{n-1} + \sum_{i=1}^{\chi} S(n_i, 2)$.

We can create k'_{n-1} different non-degenerate $(\chi + 1)$-colourings of G as follows. For each non-edge uv of $\mathscr{C}$, we can modify $\mathscr{C}$ by giving u and v the $(\chi + 1)$th colour; note that all $\chi + 1$ colours must be used, since $\chi = \chi(G)$ and $\mathscr{C}$ is the unique χ-colouring of G.

For each colour class C_i, there are $S(n_i, 2)$ $(\chi + 1)$-colourings that are obtained by dividing C_i into two non-empty colour classes, and these are all distinct, both from each other and from the k'_{n-1} $(\chi + 1)$-colourings previously mentioned. This completes the proof. □

This result can be used to give a characterisation of 1-K-symmetrical graphs:

Corollary 2.8.1. A graph G is 1-K-symmetrical if and only if

(i) G has a unique χ-colouring $\mathscr{C}$,

(ii) every colour class of $\mathscr{C}$ has at most three vertices,

and

(iii) the only non-degenerate $(\chi + 1)$-colourings of G are those described in the proof of Lemma 2.8.

Proof. Conditions (ii) and (iii) are the conditions that there is equality in (2.2.2) and (2.2.1) respectively, since $S(n_i, 2) = \binom{n_i}{2}$ if and only if $n_i \leq 3$. □

The next two results are easy results about 1-K-symmetry.

Lemma 2.9. If G is 1-K-symmetrical with unique χ-colouring $\mathscr{C}$, and C is a colour class of $\mathscr{C}$, then $G - C$ is 1-K-symmetrical.

Proof. $G - C$ must be uniquely $(\chi - 1)$-colourable (since any different $(\chi - 1)$-colouring of $G - C$ can be extended to a different χ-colouring of G). Every colour class of this $(\chi - 1)$-colouring is a colour class of $\mathscr{C}$ and so it has at most three vertices by Corollary 2.8.1. Every non-degenerate χ-colouring $\mathscr{C}'$ of $G - C$ must be described in the proof of Lemma 2.8, for otherwise the

$(\chi + 1)$-colouring of G formed from $\mathcal{C}'$ by adding the colour class C violates condition (iii) of Corollary 2.8.1. The result now follows by Corollary 2.8.1. □

Lemma 2.10. Suppose G is 1-K-symmetrical with unique χ-colouring $\mathcal{C}$. Then every independent set of three vertices in G is a colour class of $\mathcal{C}$.

Proof. Suppose otherwise. Then there is an independent set of three vertices from two or more colour classes of $\mathcal{C}$. We can obtain a $(\chi + 1)$-colouring of G by assigning the $(\chi + 1)$th colour to these three vertices, contrary to Corollary 2.8.1 (iii). □

Now we are ready to prove an inequality for 1-K-symmetrical graphs similar to that given in Lemma 2.8 for uniquely χ-colourable graphs.

Lemma 2.11. Suppose G is 1-K-symmetrical. Then $k_{\chi+2}(G) \geq k_{n-2}(G)$.

Proof. Let $\mathcal{C}$ be the unique χ-colouring of G. By Lemma 2.4, $k_{n-2}(G)$ is the number of independent sets of three vertices plus the number of pairs of disjoint non-edges in G. As in the proof of Lemma 2.8, we shall show, by modifying $\mathcal{C}$, that the number of non-degenerate $(\chi + 2)$-colourings of G is at least as large as this.

By Lemma 2.10, every independent set of three vertices is a colour class of $\mathcal{C}$, and so, for each independent set of three vertices, we obtain a non-degenerate $(\chi + 2)$-colouring of G by giving two of the vertices new (different) colours.

For each pair of disjoint non-edges in G, we obtain a non-degenerate $(\chi + 2)$-colouring of G as follows. For each of the non-edges that is a non-edge of $\mathcal{C}$ in G, move both of its vertices into a new colour class. For each other non-edge that is now a whole colour class, move one of its vertices into a new colour class. Any remaining non-edge must be properly contained in a single colour class: move both of its vertices into a new class. Note that all $\chi + 2$ colours must be used by Corollary 2.8.1 (iii). The result follows. □

Corollary 2.11.1. Suppose G is 2-K-symmetrical. Then the only non-degenerate $(\chi + 2)$-colourings of G are those described in the proof of Lemma 2.11. □

The next result places strong conditions on r-K-symmetrical graphs for $r \leq 2$.

Lemma 2.12. Suppose G is r-K-symmetrical, where $0 \leq r \leq 2$, with unique χ-colouring $\mathcal{C}$. Then every colour class C of $\mathcal{C}$ with $r+1$ or fewer vertices contains a vertex adjacent to every vertex not in that class. In particular, if G is 2-K-symmetrical, then every colour class of $\mathcal{C}$ contains such a vertex.

Proof. Every vertex that forms a singleton colour class of $\mathcal{C}$ must be adjacent to every other vertex, for otherwise G has another χ-colouring.

Suppose that $r \geq 1$ and that the result fails for a colour class C of two vertices, say $C = \{v_1, v_2\}$. Then there are vertices u_1, u_2 outside C, which are distinct by Lemma 2.10, such that u_1v_1 and u_2v_2 are non-edges. Make $\{u_1, v_1\}$ and $\{u_2, v_2\}$ into colour classes (so that C disappears). The resulting $(\chi+1)$-colouring contradicts Corollary 2.8.1 (iii).

Finally, suppose that $r = 2$ and that the result fails for a colour class C of three vertices, say $C = \{v_1, v_2, v_3\}$. Then there are vertices u_1, u_2 and u_3 outside C, which are distinct by Lemma 2.10, such that u_1v_1, u_2v_2 and u_3v_3 are non-edges. Make $\{u_1, v_1\}$, $\{u_2, v_2\}$ and $\{u_3, v_3\}$ into colour classes (so that C disappears). Note that all $\chi+2$ colours must be used by Corollary 2.8.1 (iii). But this $(\chi+2)$-colouring is not one of those constructed in the proof of Lemma 2.11, and so G cannot be 2-K-symmetrical, a contradiction. □

This next result will be used in the inductive proof of Theorem 2.15.

Lemma 2.13. Suppose G has a unique χ-colouring $\mathcal{C}$ with a non-edge uv of $\mathcal{C}$ in G. Let $G_1 := G + uv$ and $G_2 := (G)_{u=v}$. Then G_1 is uniquely χ-colourable and $\chi(G_2) = \chi + 1$.

Proof. First note that since u and v are in different colour classes of $\mathcal{C}$, $\mathcal{C}$ is also the unique χ-colouring of G_1, and so G_1 is uniquely χ-colourable.

Now G_2 has a $(\chi+1)$-colouring obtained from $\mathcal{C}$ by assigning the $(\chi+1)$th colour to the amalgamated vertex uv, and keeping the other colours the same as in $\mathcal{C}$. Thus $\chi(G_2) \leq \chi + 1$. But a χ-colouring of G_2 would yield a χ-colouring of G in which u and v have the same colour, which would contradict the hypothesis that G is uniquely χ-colourable. Thus $\chi(G_2) = \chi + 1$ as required. □

Lemma 2.14. Suppose G is a graph with a colouring $\mathcal{C}$ such that

(i) every colour class of $\mathcal{C}$ with one or two vertices contains a vertex adjacent to every vertex not in that class,

and

(ii) every independent set of three vertices is a colour class of $\mathcal{C}$.

Then G is uniquely χ-colourable, and $\mathcal{C}$ is the unique χ-colouring.

Proof. Let the colour classes of $\mathcal{C}$ be $C_1, \ldots, C_r$ where $C_1, \ldots, C_t$ are the classes of size 1 and 2, and let $\mathcal{C}'$ be another r-colouring with colour classes $C'_1, \ldots, C'_r$. For $1 \le i \le t$, let c_i be a vertex of C_i that is adjacent to every vertex of $G - C_i$, which exists by condition (i), and let C'_i be the colour class of $\mathcal{C}'$ that contains c_i. Then $C'_1, \ldots, C'_t$ are distinct since the vertices $c_1, \ldots, c_t$ induce a complete subgraph of G. In fact, $C'_i \subseteq C_i$ for $1 \le i \le t$, since if x is a vertex in C'_i different from c_i then x must be in C_i, for otherwise c_i would be adjacent to x by condition (i). By condition (ii), G contains no independent sets of four vertices, and $C_{t+1}, \ldots, C_\chi$ are the only independent sets of three vertices. Thus

$$3(r-t) = |C_{t+1} \cup \cdots \cup C_r| \le |C'_{t+1} \cup \cdots \cup C'_r| \le 3(r-t).$$

It follows that there is equality throughout, so $C'_i = C_i$ for $1 \le i \le t$, $|C'_i| = 3$ for $t+1 \le i \le r$, and $C'_{t+1}, \ldots, C'_r$ are the same as $C_{t+1}, \ldots, C_r$ in some order. Hence $\mathcal{C}'$ is equivalent to $\mathcal{C}$ and so G is uniquely r-colourable and $r = \chi$. □

We are now ready to prove the main theorems of the section, characterisations of K-symmetrical and 1-K-symmetrical graphs, together with the surprising result that all 2-K-symmetrical graphs are, in fact, K-symmetrical.

Theorem 2.15. A graph G is K-symmetrical if and only if, for some $r > 0$, G has an r-colouring $\mathcal{C}$ such that

(i) every colour class of $\mathcal{C}$ contains a vertex adjacent to every vertex not in that class,

and

(ii) every independent set of three vertices is a colour class of $\mathcal{C}$.

Moreover G is K-symmetrical if and only if it is 2-K-symmetrical.

Proof. First note that K-symmetrical graphs are 2-K-symmetrical by definition, and that conditions (i) and (ii) hold for the unique χ-colouring of a 2-K-symmetrical graph by Lemmas 2.12 and 2.10. It remains to prove that G is K-symmetrical if it has an r-colouring that satisfies conditions (i) and (ii).

Let $\mathcal{C}$ be an r-colouring of G that satisfies conditions (i) and (ii). If $\alpha(G,\mathcal{C}) = 0$, then G is a complete χ-partite graph, and by condition (ii), the colour classes of $\mathcal{C}$ are of size three or less. Thus G is the join of null graphs of order three or less, and it is easy to check that these null graphs are K-symmetrical. It follows by Lemma 2.7 that G is itself K-symmetrical.

So suppose $\alpha(G,\mathcal{C}) > 0$ and that the result holds for all graphs G' with a colouring $\mathcal{C}'$, satisfying conditions (i) and (ii), such that $\alpha(G',\mathcal{C}') < \alpha(G,\mathcal{C})$. Then G has a non-edge uv of $\mathcal{C}$. Let $G_1 := G + uv$ and $G_2 := (G)_{u=v}$.

Now, $\mathcal{C}$ is a colouring of G_1 which satisfies conditions (i) and (ii) and $\alpha(G_1,\mathcal{C}) = \alpha(G,\mathcal{C}) - 1$, and so, by the inductive hypothesis, G_1 is K-symmetrical. In G_2, the amalgamated vertex uv must be adjacent to every other vertex, for if there is a vertex, w say, which is not adjacent to uv, then u, v and w form an independent set in G which violates condition (ii). Let $G_3 := G_2 - uv = G - \{u, v\}$. Then, by Lemma 2.5, $K(G_2, x) = xK(G_3, x)$. By Lemma 2.14, G must be uniquely χ-colourable, and so by Lemma 2.13 $\chi(G_2) = \chi + 1$. Thus $\chi(G_3) = \chi$. Neither u nor v has the property of being adjacent to every vertex not in its class, so that $\mathcal{C}$, when restricted to G_3, still satisfies condition (i). It also satisfies condition (ii), since G_3 is an induced subgraph of G. Also, $\alpha(G_3,\mathcal{C}) < \alpha(G,\mathcal{C})$, and so, by the inductive hypothesis, G_3 is K-symmetrical, and so G_2 is also K-symmetrical. By Lemma 2.3, G is K-symmetrical as required. □

Theorem 2.16. A graph G is 1-K-symmetrical if and only if, for some $r > 0$, G has an r-colouring such that

(i) every colour class of $\mathcal{C}$ with one or two vertices contains a vertex adjacent to every vertex not in that class,

(ii) every independent set of three vertices is a colour class of $\mathcal{C}$,

and

(iii) $\mathcal{C}$ does not contain two colour classes whose union induces C_6, the circuit of order 6.

Proof. The unique χ-colouring of a 1-K-symmetrical graph satisfies conditions (i) and (ii) by Lemmas 2.12 and 2.10, and condition (iii), because the union of any two colour classes induces a 1-K-symmetrical subgraph by repeated application of Lemma 2.9, and it is easy to check that C_6 is not 1-K-symmetrical.

It remains to prove that if G has an r-colouring $\mathcal{C}$ satisfying conditions (i), (ii) and (iii), then it is 1-K-symmetrical. Let $\mathcal{C}$ be such an r-colouring. Then G is uniquely χ-colourable, and $\mathcal{C}$ is the unique χ-colouring, by Lemma 2.14. By Corollary 2.8.1, it suffices to show that every $(\chi + 1)$-colouring of G can be obtained by the constructions in the proof of Lemma 2.8.

Let $\mathcal{D}$ be a non-degenerate $(\chi + 1)$-colouring of G, with colour classes $D_1, \ldots, D_{\chi+1}$, and let $D_1, \ldots, D_s$ be the classes that straddle more than one class of $\mathcal{C}$. By condition (ii), $|D_1| = \cdots = |D_s| = 2$. If $s = 0$ then $\mathcal{D}$ is obtained from $\mathcal{C}$ by splitting a colour class into two, as in the proof of Lemma 2.8. If $s = 1$, then $D_1 = \{u, v\}$, say, and then uv is a non-edge of $\mathcal{C}$ and so $\mathcal{D}$ is again constructed from $\mathcal{C}$ as in Lemma 2.8. For $s \geq 2$, $D_1 \cup \cdots \cup D_s$ must contain the union of $s - 1$ colour classes of $\mathcal{C}$ (for otherwise $\mathcal{D}$ would have at least $\chi + 2$ classes, since every colour class of $\mathcal{C}$ not contained in $D_1 \cup \cdots \cup D_s$ is the union of colour classes of $\mathcal{D}$). But $D_1 \cup \cdots \cup D_s$ cannot contain a class of $\mathcal{C}$ with one or two vertices, by condition (i). The only possibility is that $s = 3$ and $D_1 \cup D_2 \cup D_3$ is the union of two classes of size 3, say C_i and C_j. Hence $C_i \cup C_j$ induces a subgraph of C_6. By condition (ii), $C_i \cup C_j$ actually induces C_6, which is impossible by condition (iii). This completes the proof. □

2.3. Sequences.

We shall denote the sequence $a_0, a_1, \ldots, a_n$ of non-negative terms by (a_i). It will be convenient to set $a_i = 0$ for $i > n$ or $i < 0$. A sequence (a_i) is said to be *log-concave* if, for each i, $a_i^2 \geq a_{i-1}a_{i+1}$, and *strongly log-concave* if this inequality is strict for $0 \leq i \leq n$. For two sequences (a_i) and (b_i), the *Cauchy product* is (c_k), where for each k, $c_k = \sum_i a_i b_{k-i}$. The following result is well known (see for example Hoggar [6]).

Theorem 2.17. The Cauchy product of two log-concave sequences is itself log-concave. Moreover, if the two sequences are, in fact, strongly log-concave, then the Cauchy product is strongly log-concave also.

In the rest of this section (only), we shall adopt the convention that for a sequence (a_i), a_i is defined for all $i \in \mathbb{R}$, but $a_i = 0$ whenever $i \notin \mathbb{Z}$. Let $p = \frac{1}{2}n$. A sequence $(a_i) = a_0, \ldots, a_n$ is said to be *skewed* if $a_i \geq a_{n-i}$ for $0 \leq i \leq p$, or, equivalently, if $a_{p-i} \geq a_{p+i}$ for all real $i \geq 0$.

The following result will be used later.

Lemma 2.18. Suppose $(a_i) = a_0, \ldots, a_n$ is a log-concave, skewed sequence. Let $p = \frac{1}{2}n$. Then $a_{p+i} \geq a_{p+i+1}$ for all real $i \geq 0$. Moreover, if (a_i) is strongly log-concave, then this inequality is strict for all i such that $p+i \in \mathbb{Z}$ and $0 \leq i \leq p$.

Proof. We prove the result by induction on $\lfloor i \rfloor$. If $p+i \notin \mathbb{Z}$ then $a_{p+i} = a_{p+i+1} = 0$ and the result holds, so suppose $p+i \in \mathbb{Z}$.

Suppose $0 \leq i < 1$. Then $a_{p+i}^2 \geq a_{p+i-1}a_{p+i+1}$ and $a_{p+i-1} \geq a_{n-p-i+1} = a_{p-i+1}$. If $i = 0$ then $a_{p-i+1} = a_{p+i+1}$ and if $i = \frac{1}{2}$ then $a_{p-i+1} = a_{p+i}$. In either case, the result holds.

Suppose now that $i \geq 1$. Then $a_{p+i}^2 \geq a_{p+i-1}a_{p+i+1} \geq a_{p+i}a_{p+i+1}$, by the inductive hypothesis. Thus $a_{p+i} \geq a_{p+i+1}$ as required.

The second part is proved in the same way. □

Corollary 2.18.1. Suppose (a_i) is a strongly log-concave, skewed sequence. Then for all real i and j such that $j > i \geq 0$, if $a_{p+i} = a_{p+j}$, then $a_{p+i} = 0$.

Proof. This is straightforward from Lemma 2.18. □

We are now ready to prove the main results of the section, which will be applied in Section 2.5 to the coefficients of the chromatic polynomial relative to the complete graph basis. The first result shows that, for log-concave sequences, the skewed property is preserved under the Cauchy product.

Theorem 2.19. The Cauchy product of two log-concave skewed sequences is itself log-concave and skewed.

Proof. Let $(a_i) = a_0, \ldots, a_n$ and $(b_i) = b_0, \ldots, b_m$ be log-concave skewed sequences, and let $(c_i) = c_0, \ldots, c_{n+m}$ be their Cauchy product. Let $p = \frac{1}{2}n$, $q = \frac{1}{2}m$ and $s = p + q = \frac{1}{2}(n+m)$. Then (c_i) is log-concave by Theorem 2.17; we must prove that it is skewed.

Suppose $k \geq 0$ (k real), and let $\gamma_k = c_{s-k} - c_{s+k}$. We show that $\gamma_k \geq 0$. If $s + k \notin \mathbb{Z}$ then $c_{s-k} = c_{s+k} = 0$ and the result holds, so suppose $s + k \in \mathbb{Z}$. In what follows, the summations are over all real values of i in the range specified (note that, in practice, only half integer values of i contribute to the sums, and so the summations are countable). If $k = 0$ then $\gamma_k = c_s - c_s = 0$, so suppose $k > 0$. It is not difficult to check that

$$\gamma_k = \sum_i a_{p+i} b_{q-k-i} - \sum_i a_{p+i} b_{q+k-i}$$

$$= \sum_{i<0} a_{p+i} b_{q-k-i} + \sum_{i\geq 0} a_{p+i} b_{q-k-i} - \sum_{i\leq 0} a_{p+i} b_{q+k-i} - \sum_{i>0} a_{p+i} b_{q+k-i}$$

$$= \sum_{i>0} a_{p-i} b_{q-k+i} + \sum_{i\geq 0} a_{p+i} b_{q-k-i} - \sum_{i\geq 0} a_{p-i} b_{q+k+i} - \sum_{i>0} a_{p+i} b_{q+k-i}$$

$$= \sum_{0<i\leq k} (a_{p-i} b_{q-(k-i)} - a_{p+i} b_{q+(k-i)})$$

$$+ \sum_{i>k} (a_{p-i} b_{q+(i-k)} - a_{p+i} b_{q-(i-k)}) + \sum_{i\geq 0} (a_{p+i} b_{q-(k+i)} - a_{p-i} b_{q+(k+i)}).$$

For each (real) $i \geq 0$, let $\alpha_i = a_{p-i} - a_{p+i}$ and $\beta_i = b_{q-i} - b_{q+i}$. Note that $\alpha_i, \beta_i \geq 0$. Then, substituting for a_{p-i} and b_{q-i},

$$\gamma_k = \sum_{0<i\leq k} ((\alpha_i + a_{p+i})(\beta_{k-i} + b_{q+k-i}) - a_{p+i} b_{q+k-i})$$

$$+ \sum_{i>k} ((\alpha_i + a_{p+i}) b_{q+i-k} - a_{p+i}(\beta_{i-k} + b_{q+i-k}))$$

$$+ \sum_{i\geq 0} (a_{p+i}(\beta_{k+i} + b_{q+k+i}) - (\alpha_i + a_{p+i}) b_{q+k+i})$$

$$= \sum_{0<i\le k} (a_{p+i}\beta_{k-i} + \alpha_i b_{q+k-i} + \alpha_i \beta_{k-i})$$

$$+ \sum_{i>k}(\alpha_i b_{q+i-k} - a_{p+i}\beta_{i-k}) + \sum_{i\ge 0}(a_{p+i}\beta_{k+i} - \alpha_i b_{q+k+i}).$$

Rearranging (noting that $\alpha_0 = \beta_0 = 0$),

$$\gamma_k = \sum_{0<i\le k} \alpha_i(b_{q+k-i} - b_{q+k+i}) + \sum_{i>k} \alpha_i(b_{q+i-k} - b_{q+i+k})$$

$$+ \sum_{0<i\le k} \beta_{k-i}a_{p+i} + \sum_{i\ge 0} \beta_{k+i}a_{p+i} - \sum_{i>k} \beta_{i-k}a_{p+i} + \sum_{0<i\le k} \alpha_i\beta_{k-i}.$$

Using suitable changes of variables for terms involving β_r,

$$\gamma_k = \sum_{0<i\le k} \alpha_i(b_{q+k-i} - b_{q+k+i}) + \sum_{i>k} \alpha_i(b_{q+i-k} - b_{q+i+k})$$

$$+ \sum_{0\le j<k} \beta_j a_{p+k-j} + \sum_{j\ge k} \beta_j a_{p+j-k} - \sum_{j>0} \beta_j a_{p+j+k} + \sum_{0<i\le k} \alpha_i\beta_{k-i}$$

$$= \sum_{0<i\le k} \alpha_i(b_{q+k-i} - b_{q+k+i}) + \sum_{i>k} \alpha_i(b_{q+i-k} - b_{q+i+k})$$

$$+ \sum_{0<i<k} \beta_i(a_{p+k-i} - a_{p+k+i}) + \sum_{i\ge k} \beta_i(a_{p+i-k} - a_{p+i+k}) + \sum_{0<i\le k} \alpha_i\beta_{k-i} \qquad (2.3.1)$$

which is non-negative, since $b_{q+k-i} \ge b_{q+k+i}$ and $a_{p+k-i} \ge a_{p+k+i}$ if $0 \le i \le k$, and $b_{q+i-k} \ge b_{q+i+k}$ and $a_{p+i-k} \ge a_{p+i+k}$ if $i > k$, by repeated application of Lemma 2.18, as required. □

Corollary 2.19.1 The Cauchy product of two strongly log-concave skewed sequences is itself strongly log-concave and skewed.

Proof. This follows from Theorem 2.17 and Theorem 2.19. □

Theorem 2.20. Let $(a_i) = a_0, \ldots, a_n$ and $(b_i) = b_0, \ldots, b_m$ be strongly log-concave skewed sequences, and let $(c_i) = c_0, \ldots, c_{n+m}$ be their Cauchy product. Let $p = \frac{1}{2}n$, $q = \frac{1}{2}m$ and $s = p + q = \frac{1}{2}(n+m)$. Suppose that whenever $i < p$ and $a_i = a_{n-i}$, then $a_{i-1} = a_{n-i+1}$ also, and similarly for (b_i). Then whenever $i < s$ and $c_i = c_{n+m-i}$, then $c_{i-1} = c_{n+m-i+1}$ also.

Proof. As in the proof of Theorem 2.19, let $\alpha_i = a_{p-i} - a_{p+i}$ and $\beta_i = b_{q-i} - b_{q+i}$ for each (real) $i \geq 0$. Then whenever $i > 0$ and $\alpha_i = 0$, then $\alpha_{i+1} = 0$ also, and similarly for the β_i. Suppose $k > 0$ and $\gamma_k = c_{s-k} - c_{s+k} = 0$. It remains to prove that $\gamma_{k+1} = 0$ also. Note that this is true if $s + k \notin \mathbb{Z}$ (since then $s - k = n + m - (s + k) \notin \mathbb{Z}$ also), so suppose $s + k \in \mathbb{Z}$. Note also that $2k \in \mathbb{Z}$, since $2s = n + m \in \mathbb{Z}$.

Every term in (2. 3. 1) in the proof of Theorem 2.19 must be zero; in particular, taking $i = k$ in the relevant sums, $\alpha_k(b_q - b_{q+2k}) = 0$ and $\beta_k(a_p - a_{p+2k}) = 0$. We shall show that $\alpha_k = \beta_k = 0$.

Suppose first that $s \in \mathbb{Z}$, so that $k \in \mathbb{Z}$ also. If $p \in \mathbb{Z}$, then $q = s - p \in \mathbb{Z}$ also; by Lemma 2.18 $b_q > b_{q+2k}$, and so $\alpha_k = 0$. If $p \notin \mathbb{Z}$, then $\alpha_k = 0$ anyway (since $k \in \mathbb{Z}$). Similarly $\beta_k = 0$.

Now suppose $s \notin \mathbb{Z}$, so that $k \notin \mathbb{Z}$ also. We may assume without loss of generality that $p \in \mathbb{Z}$ and $q \notin \mathbb{Z}$. Then $\alpha_k = 0$ (since $k \notin \mathbb{Z}$). Also, $a_p > a_{p+2k}$ by Lemma 2.18, and so $\beta_k = 0$.

Thus $\alpha_i = \beta_i = 0$ for $i \geq k$. Using this, and substituting $k + 1$ for k in (2. 3. 1),

$$\gamma_{k+1} = \sum_{0<i<k} \alpha_i(b_{q+k+1-i} - b_{q+k+1+i})$$

$$+ \sum_{0<i<k} \beta_i(a_{p+k+1-i} - a_{p+k+1+i}) + \sum_{1<i<k} \alpha_i \beta_{k+1-i}. \qquad (2.3.2)$$

Also, for $0 < i < k$, $\alpha_i(b_{q+k-i} - b_{q+k+i}) = 0$ and so either $\alpha_i = 0$ or $b_{q+k-i} = b_{q+k+i}$. The latter implies $b_{q+k+1-i} = b_{q+k+1+i} = 0$, by Corollary 2.18.1, and so the first sum in (2. 3. 2) is zero.

Similarly, for $0 < i < k$, either $\beta_i = 0$ or $a_{p+k-i} = a_{p+k+i}$, and again the latter implies $a_{p+k+1-i} = a_{p+k+1+i} = 0$, and so the second sum in (2. 3. 2) is zero also.

For $1 < i < k$, either $\alpha_i = 0$ or $\beta_{k-i} = 0$, and the latter implies $\beta_{k+1-i} = 0$, and so the final sum in (2. 3. 2) is zero. Thus $\gamma_{k+1} = 0$ as required. □

2.4. Stirling Numbers.

The Stirling number of the second kind $S(n,i)$ is the number of ways of partitioning a set of n elements into i non-empty subsets. Thus $k_i(\bar{K}_n) = S(n,i)$ for each i. With this is mind, in this section we show that the Stirling numbers form strongly log-concave skewed sequences. The Stirling numbers satisfy the basic recurrence $S(n,i) = S(n-1,i-1) + iS(n-1,i)$, with $S(0,0) = 1$ and $S(n,0) = 0$ for $n \geq 1$.

Theorem 2.21. (Lieb [9])

The Stirling numbers are strongly log-concave. □

Lemma 2.22. $S(n,n-i) \leq (i+1)S(n-1,n-1-i)$ for $i \leq \frac{1}{2}(n-2)$, $n \geq 2$.

Proof. We prove the result by induction on n. If $n = 2$ then either $i < 0$, in which case both sides of the equation are zero, or $i = 0$, in which case the result follows, since $S(2,2) = S(1,1) = 1$.

So suppose $n \geq 3$. If $i < \frac{1}{2}(n-2)$ then

$$\begin{aligned} S(n,n-i) &= S(n-1,n-1-i) + (n-i)S(n-1,n-i) \\ &\leq (i+1)S(n-2,n-2-i) + (n-i)iS(n-2,n-1-i) \\ &\qquad\qquad \text{by the inductive hypothesis} \\ &\leq (i+1)[S(n-2,n-2-i) + (n-1-i)S(n-2,n-1-i)] \\ &\qquad\qquad \text{since } i \leq \tfrac{1}{2}(n-1) \Rightarrow (n-i)i \leq (i+1)(n-1-i) \\ &= (i+1)S(n-1,n-1-i), \end{aligned}$$

as required.

If $i = \frac{1}{2}(n-2)$, that is, $n = 2i+2$, then, using the recurrence relation,

$$\begin{aligned} (i+1)S(n-1,n-1-i) &= iS(n-1,n-1-i) \\ &\quad + S(n-1,n-1-i) - S(n,n-i) + S(n,n-i) \end{aligned}$$

$$= i[S(n-2, n-2-i) + (n-1-i)S(n-2, n-1-i)]$$

$$+ S(n-1, n-1-i) - [S(n-1, n-1-i)$$

$$+(n-i)S(n-1, n-i)] + S(n, n-i)$$

$$\geq i[(n-2-i)S(n-3, n-2-i)$$

$$+ (n-1-i)S(n-2, n-1-i)]$$

$$-(n-i)S(n-1, n-i) + S(n, n-i)$$

$$= i^2 S(n-3, n-2-i) + i(i+1)S(n-2, n-1-i)$$

$$-(i+2)S(n-1, n-i) + S(n, n-i) \qquad \text{since } n = 2i+2$$

$$\geq iS(n-2, n-1-i) + i(i+1)S(n-2, n-1-i)$$

$$- i(i+2)S(n-2, n-1-i) + S(n, n-i)$$

by the inductive hypothesis

$$= S(n, n-i),$$

as required. □

Theorem 2.23. $S(n, i+1) \geq S(n, n-i)$ for $i \leq \frac{1}{2}(n-1)$, $n \geq 1$.

Proof. We prove the result by induction on n. If $n = 1$ then either $i < 0$, in which case both sides of the inequality are zero, or $i = 0$, in which case the result is trivial, as it is for all values of n if $i = \frac{1}{2}(n-1)$, when $i + 1 = n - i$.

So suppose $n \geq 2$ and $i \leq \frac{1}{2}(n-2)$. Then

$$S(n, i+1) = S(n-1, i) + (i+1)S(n-1, i+1)$$

$$\geq (i+1)S(n-1, i+1)$$

$$\geq (i+1)S(n-1, n-1-i) \qquad \text{by the inductive hypothesis}$$

$$\geq S(n, n-i) \qquad \text{by Lemma 2.22}$$

as required. □

Corollary 2.23.1. Let $s = \left\lfloor \frac{1}{2}(n-1) \right\rfloor$. Then $S(n, n-i) > S(n, n-i+1)$ for $0 \leq i \leq s$.

Proof. This follows immediately from Theorem 2.21, Theorem 2.23 and Lemma 2.18. □

Lemma 2.24. $S(n, i+1) > S(n, n-i)$ for $0 < i \leq \frac{1}{2}(n-2)$.

Proof. Suppose otherwise. Then, by Theorem 2.23, there is some value of i such that $S(n, n-i) = S(n, i+1)$. But then

$$S(n, n-i) = S(n, i+1)$$

$$= S(n-1, i) + (i+1)S(n-1, i+1)$$

$$\geq (i+1)S(n-1, i+1)$$

$$\geq (i+1)S(n-1, n-1-i) \qquad \text{by Theorem 2.23}$$

$$\geq S(n, n-i) \qquad \text{by Lemma 2.22}$$

and so equality must hold throughout. In particular, $S(n-1, i) = 0$. But since $1 \leq i \leq n-1$, $S(n-1, i) > 0$, a contradiction. □

2.5. Applications to Graphs.

We now apply the results of Sections 2.3 and 2.4 to the coefficients $k_i(G)$.

Lemma 2.25. Suppose G is a complete χ-partite graph, with n vertices. Let $s = \frac{1}{2}(n-\chi)$. Then

(i) (Brenti [1]) $k_i(G)^2 > k_{i-1}(G)k_{i+1}(G)$ for $\chi \le i \le n$, that is, the $k_i(G)$ are strongly log-concave,

(ii) $k_{\chi+i}(G) \ge k_{n-i}(G)$ for $0 \le i \le s$, that is, the $k_i(G)$ are skewed, and

(iii) $k_{n-i}(G) > k_{n-i+1}(G)$ for $0 \le i \le s$.

Proof. G is the join of χ null graphs, whose coefficients relative to the complete graph basis are Stirling numbers. The result thus follows from Lemma 2.5, Lemma 2.18, Corollary 2.19.1, Theorem 2.21 and Theorem 2.23. □

Lemma 2.26. Let G be a complete χ-partite graph with n vertices. Suppose $k_{n-i}(G) = k_{\chi+i}(G)$ for some $i \le \frac{1}{2}(n-\chi-1)$. Then $k_{n-i+1}(G) = k_{\chi+i-1}(G)$.

Proof. We prove the result by induction on χ. Suppose first that $\chi = 1$, and that $k_{n-i}(G) = k_{i+1}(G)$. Then G is the null graph $\bar{K}_n$, and so

$$S(n, n-i) = k_{n-i}(G) = k_{i+1}(G) = S(n, i+1).$$

By Lemma 2.24, $S(n, n-i) < S(n, 1+i)$ for $0 < i \le \frac{1}{2}(n-2)$, and so $i \le 0$. But then $k_{n-i+1}(G) = k_i(G) = 0$, and so the result holds in this case.

Suppose now that $\chi > 1$, and that $k_{n-i}(G) = k_{\chi+i}(G)$. Let C be a colour class of G, so that $G = (G-C) + \bar{K}_{|C|}$. Then the coefficients $k_i(G-C)$ and $k_i(\bar{K}_{|C|})$ are strongly log-concave and skewed by Lemma 2.25. By the inductive hypothesis, if $k_{n-i}(G-C) = k_{\chi-1+i}(G-C)$ for some $i \le \frac{1}{2}(n-\chi)$ then $k_{n-i+1}(G-C) = k_{\chi+i-2}(G-C)$, and similarly for the $k_i(\bar{K}_{|C|})$. The result now follows by Theorem 2.20. □

Theorem 2.27. Let G be an arbitrary graph and let $s = \frac{1}{2}(n-\chi)$. Then

(i) $k_{\chi+i}(G) \ge k_{n-i}(G)$ for $0 \le i \le s$, that is, the $k_i(G)$ are skewed, and

(ii) $k_{n-i}(G) > k_{n-i+1}(G)$ for $0 \le i \le s$.

Proof. Let $\mathcal{C}$ be a χ-colouring of G. If $\alpha(G, \mathcal{C}) = 0$ then G is a complete χ-partite graph, and so the result follows by Lemma 2.25. So suppose that $\alpha(G, \mathcal{C}) > 0$ and that the result holds for all graphs G' with a colouring $\mathcal{C}'$ (not necessarily a $\chi(G')$-colouring) such that $\alpha(G', \mathcal{C}') < \alpha(G, \mathcal{C})$. Then G must have a pair of non-adjacent vertices, u and v, in different colour classes of $\mathcal{C}$.

Let $G_1 = G + uv$ and $G_2 = (G)_{u=v}$. Let $\mathcal{C}_2$ be the colouring of G_2 obtained by giving the amalgamated vertex uv a $(\chi + 1)$th colour, and keeping the other vertex colours the same as in $\mathcal{C}$. Then $\alpha(G_2, \mathcal{C}_2) \leq \alpha(G, \mathcal{C}) - 1$, by Lemma 2.6, and $\alpha(G_1, \mathcal{C}) = \alpha(G, \mathcal{C}) - 1$. Also $\chi(G_1) = \chi$ and $\chi \leq \chi(G_2) \leq \chi + 1$.

We prove property (ii) first. If $0 \leq i \leq s$ then $i - 1 \leq s - 1 \leq \frac{1}{2}(n - 1 - \chi(G_2))$ and so

$$k_{n-i}(G) = k_{n-i}(G_1) + k_{(n-1)-(i-1)}(G_2) \tag{2.5.1}$$

$$> k_{n-i+1}(G_1) + k_{(n-1)-(i-1)+1}(G_2)$$

$$= k_{n-i+1}(G)$$

by the inductive hypothesis and (if $i = 0$) the obvious fact that $k_n(G_2) = k_{n+1}(G_2) = 0$. This proves property (ii). To prove property (i), note that if $\chi(G_2) = \chi + 1$ then (2. 5. 1) gives

$$k_{n-i}(G) \leq k_{\chi+i}(G_1) + k_{\chi+1+i-1}(G_2) = k_{\chi+i}(G)$$

by the inductive hypothesis, as required. So suppose $\chi(G_2) = \chi$. If $0 \leq i \leq \frac{1}{2}(n - 1 - \chi)$ then (2. 5. 1) gives

$$k_{n-i}(G) < k_{n-i}(G_1) + k_{n-1-i}(G_2) \qquad \text{by (ii)}$$

$$\leq k_{\chi+i}(G_1) + k_{\chi+i}(G_2) \qquad \text{by the inductive hypothesis}$$

$$= k_{\chi+i}(G)$$

as required. Now, if $i > \frac{1}{2}(n - 1 - \chi)$ then $i = s = \frac{1}{2}(n - \chi)$ and $\chi + i = n - i$, so the result holds trivially. This completes the proof of property (i). □

Corollary 2.27.1. For a K-symmetrical graph G, the $k_i(G)$ are unimodal.

Proof. This follows from the definition of K-symmetry and Theorem 2.27 (ii). □

Theorem 2.28. Let G be an arbitrary graph. Suppose $k_{n-i}(G) = k_{\chi+i}(G)$ for some $i \leq \frac{1}{2}(n - \chi - 1)$. Then $k_{n-i+1}(G) = k_{\chi+i-1}(G)$.

Proof. Let $\mathcal{C}$ be a χ-colouring of G. If $\alpha(G,\mathcal{C})=0$ then G is a complete χ-partite graph, and the result follows by Lemma 2.26. So suppose that $\alpha(G,\mathcal{C})>0$ and that the result holds for every graph G' with a colouring $\mathcal{C}'$ such that $\alpha(G',\mathcal{C}')<\alpha(G,\mathcal{C})$. Define u, v, G_1 and G_2 as in the proof of Theorem 2.27.

Suppose $k_{n-i}(G)=k_{\chi+i}(G)$ for some $i\leq\left\lfloor \frac{1}{2}(n-\chi-1)\right\rfloor$. Then

$$k_{n-i}(G_1)+k_{n-i}(G_2)=k_{n-i}(G)=k_{\chi+i}(G)=k_{\chi+i}(G_1)+k_{\chi+i}(G_2). \quad (2.5.2)$$

Suppose first that $\chi(G_2)=\chi+1$. Then $k_{n-i}(G_1)\leq k_{\chi+i}(G_1)$ and $k_{n-i}(G_2)\leq k_{\chi+i}(G_2)$ by Theorem 2.27, and so equality holds in both. Thus by the inductive hypothesis, $k_{n-i+1}(G_1)=k_{\chi+i-1}(G_1)$ and $k_{n-i+1}(G_2)=k_{\chi+i-1}(G_2)$, and so

$$k_{n-i+1}(G)=k_{n-i+1}(G_1)+k_{n-i+1}(G_2)=k_{\chi+i-1}(G_1)+k_{\chi+i-1}(G_2)=k_{\chi+i-1}(G)$$

as required.

Suppose now that $\chi(G_2)=\chi$. Then $k_{n-i}(G_1)\leq k_{\chi+i}(G_1)$, and $k_{\chi+i}(G_2)\geq k_{n-1-i}(G_2)\geq k_{n-i}(G_2)$ by Theorem 2.27, and so, by (2. 5. 2), equality must hold throughout. In particular, $k_{n-1-i}(G_2)=k_{n-i}(G_2)$, and so $i<0$ and $k_{n-i+1}(G)=k_{\chi+i-1}(G)=0$ as required. □

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CHAPTER 3

The Flow Polynomial

3.0. Introduction and Definitions.

In this chapter, we allow graphs to have multiple edges and loops. We shall say that such a graph G is *2-connected* if and only if it has at least two vertices and every pair of vertices is contained in a circuit. A *block* of G is a maximal 2-connected subgraph of G, a loop (with its vertex) or a cut-edge (with its end vertices). Note that for a simple graph, these definitions are equivalent to the more usual ones.

Let G be a graph, and $\vec{G}$ be any orientation of G.

A function $f\colon E(\vec{G}) \to \{0, 1, \ldots, t-1\}$ is called a *(maybe zero) t-flow* of G if, for every vertex v of $\vec{G}$, $\sum_{\text{in}} f(e) - \sum_{\text{out}} f(e) \equiv 0 \bmod t$, where the sum $\sum_{\text{in}}$ is over all edges e of $\vec{G}$ that enter v and the sum $\sum_{\text{out}}$ is over all edges leaving v.

f is called a *nowhere-zero t-flow* or simply a *t-flow* if $f(e) \neq 0$ for every edge e. The number of distinct t-flows of $\vec{G}$ is called the *flow polynomial* of G, and is well known to be independent of the orientation of G. It is denoted by $F(G,t)$, and is a polynomial in t of order $\gamma = \gamma(G)$. $F(G,t) := 1$ if $E(G) = \emptyset$.

We define the *quotient flow polynomial* $q^*(G,t)$ by

$$q^*(G,t) := \frac{F(G,t)}{(-1)^{\gamma-b}(t-1)^b}\,.$$

We shall see in Section 3.1 that $q^*(G,t)$ actually is a polynomial.

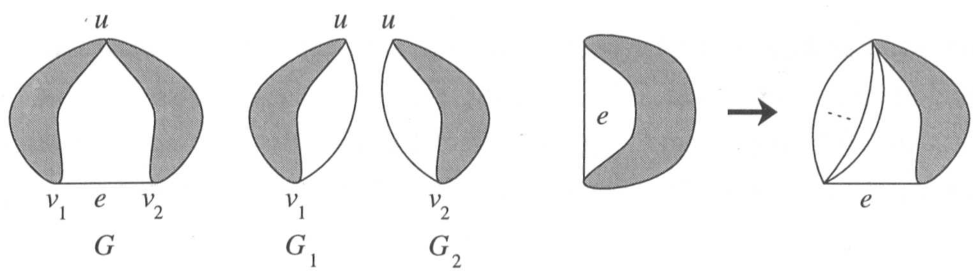
For each i, $a_i^*(G)$ is defined by $q^*(G,t) = \sum_i a_i^*(G)s^i$ where $s = 1-t$. It is well known that for a planar graph G,

$$F(G,t) = \frac{P(G^*,t)}{t}\,. \tag{3.0.1}$$

Thus, for a planar graph G, $q^*(G,t) = q(G^*,t)$ and $a_i^*(G) = a_i(G^*)$.

The chromatic polynomial of a graph G with multiple edges is the same as that of the simple graph obtained from G by removing all the multiple edges; that is, by repeatedly deleting one edge in any pair of parallel edges. In the case of flow polynomials, a graph G with a cutset of two edges and no cut-edge has the same flow polynomial as a graph G', in which every component is 3-edge-connected, obtained from G by repeatedly contracting one edge in any cutset of two edges. It is easy to see that if G is 2-connected, then G' is either 2-connected or a loop, since no circuits are destroyed.

Suppose a graph G has an edge $e = v_1v_2$ such that $G - e$ has a cut-vertex u which is not also a cut-vertex of G. Then there exist subgraphs H_1 and H_2 of G such that $G - e = H_1 \cup H_2$, $H_1 \cap H_2 = \{u\}$, $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$. Let $G_1 = H_1 + uv_1$ and $G_2 = H_2 + uv_2$. Then e is said to be a *cleaving edge* which *cleaves* G into G_1 and G_2 (see Figure 3.0.1 (i)). Then $n = n_1 + n_2 - 1$, $c = c_1 + c_2 - 1$, and so $n - c = n_1 - c_1 + n_2 - c_2$.



(i) (ii)

Figure 3.0.1

We define a *dual generalised polygon tree* to be a 2-connected, 3-edge-connected graph that does not have K_4 as a subcontraction. Clearly it is planar.

A *dual polygon tree* is defined recursively as follows:

(i) A sheaf of three or more parallel edges is a dual polygon tree.

(ii) Any graph formed from a dual polygon tree G by detaching one end of an edge e of G from its incident vertex, and adding a sheaf of two or more edges between them (see Figure 3.0.1 (ii)), is a dual polygon tree.

It is easy to see that a dual generalised polygon tree is the dual of some generalised polygon tree and a dual polygon tree is the dual of some polygon tree.

A 2-edge-connected subgraph H of a graph G is a *cleaving subgraph* if there exist 2-edge-connected subgraphs G_1, G_2 and vertices u, v and w of G such that $G_1 \cup G_2 \cup H = C$, where C is the component of G containing H, $V(G_1 \cap H) = \{u\}$, $V(G_2 \cap H) = \{v\}$, $V(G_1 \cap G_2) = \{w\}$, and H, G_1 and G_2

are edge-disjoint (see Figure 3.0.2 (i)). It is a *cleaving sheaf* if H is a sheaf (see Figure 3.0.2 (ii)).

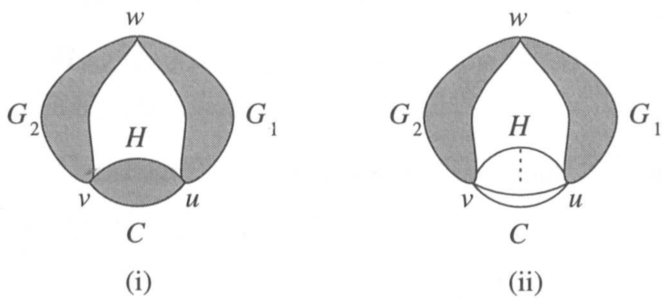


Figure 3.0.2

It is not difficult to see that if G is a plane graph with a separating edge, then G^* has a cleaving edge (see Figure 3.0.3 (i)), if G has a separating subgraph H, then G^* has a cleaving subgraph $H' = ((H)_{u=v})^*$ (see Figure 3.0.3 (ii)), and if G has a separating path H, then G^* has a cleaving sheaf $H' = ((H)_{u=v})^*$ (see Figure 3.0.3 (iii)).

Similarly, the converse holds; if G is a plane graph with a cleaving subgraph H', then G^* has a separating subgraph $H = ((H')_{a=b})^*$, and so on.

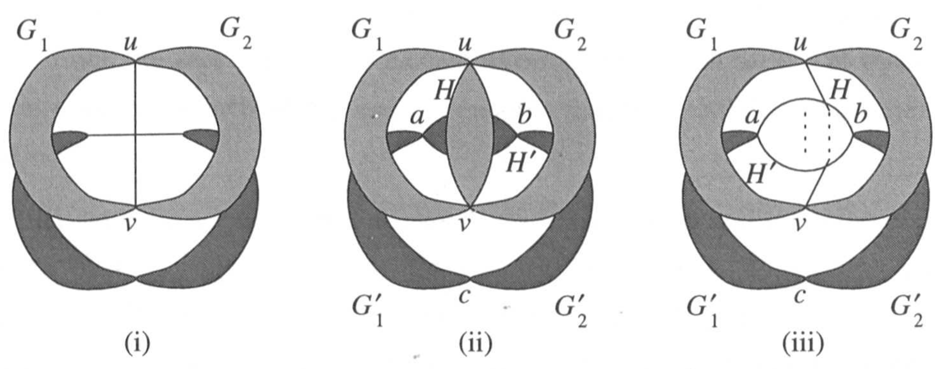


Figure 3.0.3

In this chapter, we shall see that many of the results about chromatic polynomials hold in a dual form for flow polynomials. For planar graphs, the dual forms hold by (3.0.1) and so the proofs often rely on reducing the problem to that of planar graphs.

3.1. Basic Results.

Theorem 3.1. Let G be a graph without cut-edges.

(i) If e is a non-loop edge of G, then $F(G,t) = F(G/e,t) - F(G-e,t)$, and if e is a loop, then $F(G,t) = (t-1)F(G-e,t)$.

(ii) If $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \varnothing$ or K_1, then $F(G,t) = F(G_1,t)F(G_2,t)$.

(iii) If G has an edge e which cleaves it into G_1 and G_2, then $F(G,t) = \dfrac{F(G_1,t)F(G_2,t)}{(t-1)}$.

(iv) If there exist subgraphs G_1, G_2 and vertices u and v of G such that $G_1 \cup G_2 = G$, $E(G_1 \cap G_2) = \varnothing$ and $V(G_1 \cap G_2) = \{u, v\}$ then

$$(t-1)F(G,t) = (t-1)F(G_1,t)F((G_2)_{u=v},t)$$

$$+ F(G_2 + uv, t)[F(G_1 + uv, t) - (t-1)F(G_1,t)].$$

Proof. For (i), a t-flow of G/e can be extended to a t-flow of G provided that the total flow in $G-e$ at each end of e is not zero, in which case it gives a t-flow of $G-e$. Also, the flow on a loop-edge of G has no effect on the rest of G. The result follows.

For (ii), a t-flow for G_1 together with a t-flow for G_2 yields a t-flow for G. Conversely, given a t-flow f for G, the restrictions of f to G_1 and G_2 are t-flows, since the fact that the vertex condition is satisfied at all but at most one of the vertices in each of G_1 and G_2 ensures that it is satisfied at all the vertices.

For (iii), let f be a p-flow of G for some prime p. Let v_1 and v_2 be the ends of e in G_1 and G_2 respectively, and let u be the cut-vertex of $G-e$ which is in both G_1 and G_2 (see Figure 3.0.1 (i)). Then f yields a p-flow f_1 of G_1 with $f_1(h) = f(h)$ for $h \neq uv_1$ and $f_1(uv_1) = f(e)$, and similarly f yields a p-flow f_2 of G_2. Since p is prime, exactly $\frac{1}{p-1}$ of the p-flows f_1 of G_1 have $f_1(uv_1) = f(uv_2)$. It follows that $F(G,p) = \dfrac{F(G_1,p)F(G_2,p)}{(p-1)}$. Since there are infinitely many primes, and $F(G,t)$ is a polynomial, the result now follows.

Finally, for (iv), let G' be the graph with an edge e such that $G'/e = G$ and $G'-e$ is the graph obtained by 'gluing' G_1 to G_2 at the vertex u (see

Figure 3.1.1). Then, by parts (i) and (ii), $F(G',t) = F(G,t) - F(G_1,t)F(G_2,t)$. Rearranging and applying part (iii) to $F(G',t)$, we have

$$F(G,t) = \frac{F(G_1+uv,t)F(G_2+uv,t)}{t-1} + F(G_1,t)F(G_2,t). \qquad (3.1.1)$$

Also, by part (i), $F(G_2+uv,t) = F((G_2)_{u=v},t) - F(G_2,t)$, and substituting this into (3.1.1) gives

$$F(G,t) = \frac{F(G_1+uv,t)F(G_2+uv,t)}{t-1} + F(G_1,t)[F((G_2)_{u=v},t) - F(G_2+uv,t)]$$

which can be rearranged to give the required result. □

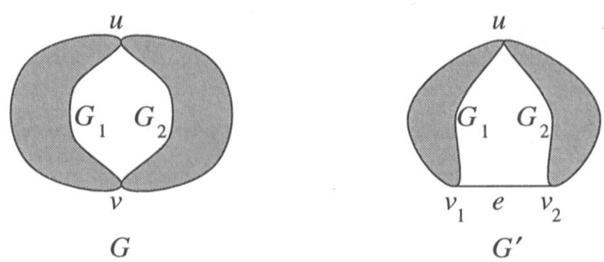


Figure 3.1.1

Corollary 3.1.1.

(i) If e is a non-loop edge of G, $G_1 = G-e$ and $G_2 = G/e$, then $q^*(G,t) = s^{b_1-b}q^*(G_1,t) + s^{b_2-b}q^*(G_2,t)$.

(ii) If $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \varnothing$ or K_1, then $q^*(G,t) = q^*(G_1,t)q^*(G_2,t)$.

(iii) If G has an edge which cleaves it into G_1 and G_2, then $q^*(G,t) = q^*(G_1,t)q^*(G_2,t)$.

Proof. Part (i) follows in a similar fashion to Theorem 1.1 (v). The rest follows from the definition of $q^*(G,t)$ and Theorem 3.1. □

Theorem 3.2.

Let G be a graph in which every component is 3-edge-connected. Then

(i) $q^*(G,t)$ is a polynomial in t,

(ii) $q^*(G,t) \gg_s 1+(n-c)s+(n-c)s^2+\cdots+(n-c)s^{\gamma-b-1}+s^{\gamma-b}$ where $s = 1-t$, and

(iii) $a^*_{\gamma-b}(G) = 1$ and $a^*_{\gamma-b-1}(G) = n-c$.

Proof. We prove the result by induction on $\gamma(G)$. There are three cases to consider.

Case 1: There exist graphs G_1 and G_2 such that $n_1 < n$ and $n_2 < n$ and either $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \varnothing$ or K_1, or G has an edge which cleaves it into G_1 and G_2. Then $q^*(G,t) = q^*(G_1,t)q^*(G_2,t)$ by Corollary 3.1.1 (ii) and (iii). Thus, by the inductive hypothesis, $q^*(G,t)$ is a polynomial in t and

$$q^*(G,t) \gg_s (1+(n_1-c_1)s+(n_1-c_1)s^2+\cdots+(n_1-c_1)s^{\gamma_1-b_1-1}+s^{\gamma_1-b_1})$$

$$\times(1+(n_2-c_2)s+(n_2-c_2)s^2+\cdots+(n_2-c_2)s^{\gamma_2-b_2-1}+s^{\gamma_2-b_2})$$

$$\gg_s 1+(n-c)s+(n-c)s^2+\cdots+(n-c)s^{\gamma-b-1}+s^{\gamma-b}$$

since $n_1-c_1+n_2-c_2 = n-c$ and $\gamma_1-b_1+\gamma_2-b_2 = \gamma-b$, in each case. Moreover, $a^*_{\gamma-b}(G) = a^*_{\gamma_1-b_1}(G_1)a^*_{\gamma_2-b_2}(G_2) = 1$ and

$$a^*_{\gamma-b-1}(G) = a^*_{\gamma_1-b_1}(G_1)a^*_{\gamma_2-b_2-1}(G_2) + a^*_{\gamma_1-b_1-1}(G_1)a^*_{\gamma_2-b_2}(G_2)$$

$$= a^*_{\gamma_1-b_1-1}(G_1) + a^*_{\gamma_2-b_2-1}(G_2)$$

$$= n_1-c_1+n_2-c_2 = n-c.$$

The result follows.

Case 2: G is K_1 or K_2^* or K_3^*.

Then $q^*(K_1,t) = q^*(K_2^*,t) = 1$ and $q^*(K_3^*,t) = 1+s$, from which the result clearly holds.

Case 3: Neither Case 1 nor Case 2 applies.

Then G is 2-connected, $\gamma(G) \geq 3$ and for each edge e of G, $G-e$ is 2-connected. Choose an edge e of G (note that e cannot be a loop), let G_1 be a graph obtained from $G-e$ by repeatedly contracting an edge in any cutset of two edges (so that G_1 is 2-connected, 3-edge-connected and $F(G_1,t) = F(G-e,t)$) and let $G_2 := G/e$. Then, by Corollary 3.1.1 (i),

$$q^*(G,t) = q^*(G_1,t) + s^{b_2-1}q^*(G_2,t)$$

and so $q^*(G,t)$ is a polynomial in t by the inductive hypothesis.

There are two subcases to consider.

Case 3a: e can be chosen so that it does not lie in a cutset of three edges in G.

Then $G-e$ is 3-edge-connected, and so $G_1 = G-e$ and, by the inductive hypothesis,

$$q^*(G_1,t) \gg_s 1+(n-c)s+(n-c)s^2+\cdots+(n-c)s^{\gamma-b-2}+s^{\gamma-b-1}$$

with $a^*_{\gamma-b-1}(G_1)=1$ and $a^*_{\gamma-b-2}(G_1)=n-c$. Also

$$s^{b_2-1}q^*(G_2,t) \gg_s s^{b_2-1}(1+(n-c-1)s+(n-c-1)s^2+$$

$$\cdots+(n-c-1)s^{\gamma_2-b_2-1}+s^{\gamma_2-b_2})$$

$$\gg_s (n-c-1)s^{\gamma-b-1}+s^{\gamma-b}$$

(since $\gamma_2=\gamma$ and $b=1$), with $a^*_{\gamma_2-b_2-1}(G_2)=n-c-1$ and $a^*_{\gamma_2-b_2}(G_2)=1$.

Thus

$$q^*(G\ ,t) \gg_s 1+(n-c)s+(n-c)s^2+\cdots+(n-c)s^{\gamma-b-1}+s^{\gamma-b}.$$

Moreover, $a^*_{\gamma-b}(G)=1$ and $a^*_{\gamma-b-1}(G)=n-c$, as required.

Case 3b: Every edge of G lies in a cutset of three edges.

Since $G \neq K_3^*$, it is not difficult to find an edge e such that $G_2 = G/e$ is 2-connected, for otherwise e lies in a double edge in G and the third edge in the cutset of three edges can then be contracted. Then $q^*(G,t)=q^*(G_1,t)+q^*(G_2,t)$. By the inductive hypothesis,

$$q^*(G_1,t) \gg_s 1+(n-c)s+(n-c)s^2+\cdots+(n-c)s^{\gamma_1-b_1-1}+s^{\gamma_1-b_1}$$

$$\gg_s 1+s+s^2+\cdots+s^{\gamma-b-2}+s^{\gamma-b-1}$$

since $n>c$. Also,

$$q^*(G_2,t) \gg_s 1+(n-c-1)s+(n-c-1)s^2+\cdots+(n-c-1)s^{\gamma-b-1}+s^{\gamma-b}.$$

Thus

$$q^*(G,t) \gg_s 1+(n-c)s+(n-c)s^2+\cdots+(n-c)s^{\gamma-b-1}+s^{\gamma-b}.$$

Moreover,

$$a^*_{\gamma-b}(G) = a^*_{\gamma_2-b_2}(G_2) = 1$$

and

$$a^*_{\gamma-b-1}(G) = a^*_{\gamma_2-b_2-1}(G_2) + a^*_{\gamma_1-b_1}(G_1) = n - c - 1 + 1 = n - c.$$

The result now follows. □

Corollary 3.2.1.

Let G be a graph without cut-edges. Then

(i) $q^*(G,t)$ is a polynomial in t,

(ii) $q^*(G,t) \gg_s 1 + ks + ks^2 + \cdots + ks^{\gamma-b-1} + s^{\gamma-b}$ where $k := a^*_{\gamma-b-1}(G) \geq 1$,

and

(iii) $a^*_{\gamma-b}(G) = 1$.

Proof. Let G' be a graph obtained from G by repeatedly contracting one edge of any cutset of two edges. Then $F(G,t) = F(G',t)$ and every component of G' is 3-edge-connected. The result now follows by Theorem 3.2. □

Corollary 3.2.2.

Let G be a graph in which every component is 3-edge-connected. Then

(i) if $t < 1$ then $F(G,t)$ is non-zero with the sign of $(-1)^\gamma$,

and

(ii) at 1, $F(G,t)$ has a zero of multiplicity b (hence a simple zero if G is 2-connected).

Proof. This follows immediately from Theorem 3.2 and the definition of $q^*(G,t)$. □

Theorem 3.3.

Let G be a graph in which every component is 3-edge-connected. If H is a 3-edge-connected, 2-connected sub-contraction of G, then $a_i^*(G) \geq a_i^*(H)$ for each i.

Proof. We prove the result by induction on m. If $H = G$ then we are done, so suppose otherwise. There are two cases to consider.

Case 1: There exist graphs G_1 and G_2 such that $n_1 < n$ and $n_2 < n$ and either $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \varnothing$ or K_1, or G has an edge which cleaves it into G_1 and G_2. Then $q^*(G,t) = q^*(G_1,t)q^*(G_2,t)$, by Corollary 3.1.1 (ii) and (iii).

If $H \preccurlyeq G_1$ or $H \preccurlyeq G_2$, without loss of generality say $H \preccurlyeq G_1$, then $a_i^*(G) \geq a_0^*(G_2)a_i^*(G_1) \geq a_i^*(G_1) \geq a_i^*(H)$ by Theorem 3.2 and the inductive hypothesis, as required. Note that this must happen if $G_1 \cap G_2 = \varnothing$ or K_1.

Now suppose otherwise. Then G has an edge e which cleaves G into G_1 and G_2. Moreover, G/e is 2-connected. If $H \preccurlyeq G/e$, then by Corollary 3.1.1 (i), $a_i(G) \geq a_i(G/e) \geq a_i(H)$, by the inductive hypothesis, as required. Otherwise e cleaves H into H_1 and H_2, where $H_1 \preccurlyeq G_1$ and $H_2 \preccurlyeq G_2$. Since H is 2-connected, it follows that H_1 and H_2 are 2-connected. Then

$$a_i^*(G) = \textstyle\sum_r a_r^*(G_1)a_{i-r}^*(G_2) \geq \sum_r a_r^*(H_1)a_{i-r}^*(H_2) = a_i^*(H)$$

by the inductive hypothesis, as required.

Case 2: G is 2-connected and $G - e$ is 2-connected for each edge e of G. For $e \in E(G)$, let $G_1 = G - e$ and $G_2 = G/e$.

If $H \preccurlyeq G_1$ for some $e \in E(G)$, then, by Corollary 3.1.1 (i), $a_i^*(G) \geq a_i^*(G_1) \geq a_i^*(H)$ for each i, by the inductive hypothesis, and we are done; so suppose otherwise. Then $\gamma(H) = \gamma(G)$. Since H is not isomorphic to G, and H is 2-connected, there is an edge e such that $H \preccurlyeq G_2 = G/e$ where G_2 is 2-connected, and so $b_2 = 1$ and $a_i^*(G) \geq a_i^*(G_2) \geq a_i^*(H)$ by the inductive hypothesis, as required. □

Corollary 3.3.1.

Let G be a graph without cut-edges. If G has K_4 as a subcontraction, then $a_0^*(G) \geq 2$.

Proof. Suppose G is a minimal counterexample. If G has a cutset of two edges, then contracting one of these edges yields a smaller counterexample, so suppose every component of G is 3-edge-connected. By Theorem 3.3 and the fact that $q^*(K_4,t) = s^2 + 3s + 2$, $a_0^*(G) \geq a_0^*(K_4) = 2$, a contradiction. Thus the statement must be true. □

Lemma 3.4. Let G be a graph without K_4 as a subcontraction. Then $a_0^*(G) = 1$.

Proof. Since K_4 is not a subcontraction of G, G is planar and so it has a dual G^*. G^* does not have K_4 as a subcontraction, and so by Corollary 1.4.1, $a_0^*(G) = a_0(G^*) = 1$, as required. □

Lemma 3.5. Let G be a 2-connected graph without K_4 as a subcontraction, and suppose G has no cleaving edge. Then either G has a cleaving sheaf or G is a sheaf.

Proof. Since K_4 is not a subcontraction of G, G is planar and so it has a dual G^*. G^* is also 2-connected, does not have K_4 as a subcontraction and has no separating edge, so by Lemma 1.3, G^* either has a separating path or is a circuit. But then G either has a cleaving sheaf or is a sheaf, as required. □

The next result also follows by duality from Corollary 1.3.1.

Corollary 3.5.1. A dual generalised polygon tree is a dual polygon tree if and only if it has no cleaving subgraph.

Proof. 'Only if' is obvious; we prove 'if'.

Let G be minimal counterexample, that is a dual generalised polygon tree with no cleaving subgraphs that is not a dual polygon tree. Then G cannot be a sheaf and so by Lemma 3.5, G has an edge which cleaves it into G_1 and G_2, say. Then G_1 and G_2 are dual generalised polygon trees without cleaving subgraphs, and so by the minimality of G, G_1 and G_2 are dual polygon trees. But then G must be a dual polygon tree also, a contradiction. The result follows. □

Lemma 3.6. Let G be a 2-connected graph with a subgraph P which is either a cleaving sheaf or a cleaving edge, and let l be the number of edges in P. Let G_1 be the graph obtained from G by removing all but one of the edges of P and let G_2 be the graph obtained from G by contracting an edge of P and removing the $l-1$ loops so formed. Note that G_1 and G_2 are 2-connected. Then

$$q^*(G,t) = q^*(G_1,t) + q^*(G_2,t)(s^{l-1} + s^{l-2} + \cdots + s)$$

Proof. We prove the result by induction on l.

If $l = 1$ then we are done since then $G_1 = G$, so suppose $l \geq 2$. Let e be an edge of P. Then $q^*(G/e, t) = q^*(G_2, t)$ by Corollary 3.1 (ii), since G/e and G_2 differ only in $l-1$ blocks, each of which is a loop K_2^*, and $q^*(K_2^*, t) = 1$. Thus, by Corollary 3.1.1 (i) and the inductive hypothesis,

$$q^*(G,t) = q^*(G-e,t) + s^{l-1}q^*(G/e,t)$$

$$= q^*(G_1,t) + q^*(G_2,t)(s^{l-2} + s^{l-3} + \cdots s) + s^{l-1}q^*(G_2,t)$$

$$= q^*(G_1,t) + q^*(G_2,t)(s^{l-1} + s^{l-2} + \cdots + s)$$

as required. □

Corollary 3.6.1. If G is a graph with a cleaving sheaf P, then $a_1^*(G) > n - c$.

Proof. Let G be a minimal counterexample. Suppose G has an edge which cleaves it into G_1 and G_2 or there exist subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \emptyset$ or K_1. We may suppose without loss of generality that P is a separating path of G_1. By Corollary 3.1.1 (ii) and (iii), $q(G,t) = q(G_1,t)q(G_2,t)$ and so, by the minimality of G,

$$a_1^*(G) = a_0^*(G_2)a_1^*(G_1) + a_0^*(G_1)a_1^*(G_2)$$

$$\geq a_1^*(G_1) + a_1^*(G_2)$$

$$> n_1 - c_1 + n_2 - c_2 = n - c,$$

a contradiction.

Thus G is 2-connected without cleaving edges. Let G_1 and G_2 be defined as in Lemma 3.6. Then $n_1 - c_1 = n - c$ and so by Lemma 3.6,

$$a_1^*(G) = a_1^*(G_1) + a_0^*(G_2) \geq n_1 - c_1 + 1 > n - c,$$

a contradiction. Thus the statement must be true. □

The next result also follows by duality from Corollary 1.4.3.

Corollary 3.6.2. Let G be a graph without K_4 as a subcontraction, and suppose that G has a cleaving subgraph H. Then $a_1^*(G) > n - c$.

Proof. Let G be a minimal counterexample. As in the proof of Corollary 3.6.1, G must be 2-connected without cleaving edges. G cannot be a sheaf, and so by Lemma 3.5, G has a cleaving sheaf. But then, by Corollary 3.6.1, $a_1^*(G) > n - c$, a contradiction. Thus the statement must be true. □

In a similar way to the chromatic polynomial case (see Section 1.1 of Chapter 1), Corollary 3.3.1 and Lemma 3.4 together show that $a_0^*(G) = 1$ if and only if G does not have K_4 as a subcontraction. Thus it is possible to determine from the flow polynomial of a graph whether or not it has K_4 as a subcontraction. However, also in a similar way to the chromatic polynomial case, it is not possible to determine from the flow polynomial of a graph whether or not it has K_5 or $K_{3,3}$ as a subcontraction. For example, the graphs in Figure 3.1.2 both have the same flow polynomial, but G_1 is planar (it is, in fact, the dual of the graph G_1 in Figure 1.1.3 in Chapter 1), whereas G_2 has both K_5 and $K_{3,3}$ as subcontractions.

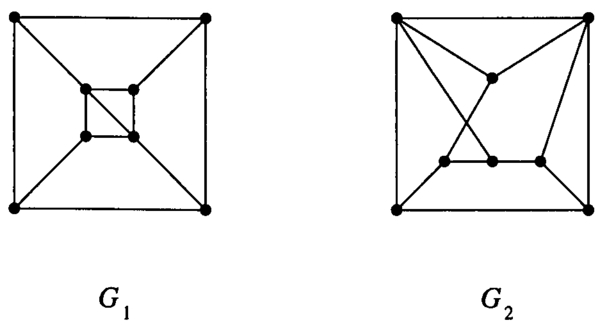


Figure 3.1.2

3.2. Dual Polygon Trees.

In this section, we apply the results of Section 3.1 and Chapter 1 to dual polygon trees.

Theorem 3.7. Let G be a 3-edge-connected, 2-connected graph. Then G is the dual of a polygon tree, with k_i i-gons for each i, if and only if

$$q^*(G,t) = \prod_{i=3}^{\infty} (1 + s + s^2 + \cdots + s^{i-2})^{k_i}.$$

Proof. To prove 'only if', let G be the dual of a polygon tree with k_i i-gons for each i. Then G is planar and G^* exists and is a polygon tree with k_i i-gons for each i, and so, by Theorem 1.5,

$$q^*(G,t) = q(G^*,t) = \prod_{i=3}^{\infty} (1+s+s^2+\cdots+s^{i-2})^{k_i}.$$

To prove 'if', suppose G is a graph with $q^*(G,t)$ as above. Then $a_0^*(G) = 1$, and so by Corollary 3.3.1, G cannot have K_4 as a subcontraction, and so is planar. Thus G^* exists, is simple (since G is 3-edge-connected) and $q(G^*,t) = q^*(G,t)$, and so, by Theorem 1.5, G^* is a polygon tree with k_i i-gons for each i. □

Corollary 3.7.1. A graph G in which every component is 3-edge-connected is the dual of a polygon tree, with k_i i-gons for each i, if and only if

$$F(G,t) = (-1)^{\gamma-1}(t-1)\prod_{i=3}^{\infty} (1+s+s^2+\cdots+s^{i-2})^{k_i},$$

where $s = 1-t$ and $\gamma = \sum_{i=3}^{\infty} k_i$.

Proof. 'Only if' follows from Theorem 3.7 and the definition of $q^*(G,t)$.

To prove 'if', suppose G is a graph with $F(G,t)$ as above. Now $t-1 = -s$ is not a factor of $p(t) = \prod_{i=3}^{\infty} (1+s+s^2+\cdots+s^{i-2})^{k_i}$, (since the result of substituting $s = 0$ is non-zero) and so $q^*(G,t) = p(t)$, and G is 2-connected. The result now follows by Theorem 3.7. □

Corollary 3.7.2. A polynomial $p(t)$ is the flow polynomial of the dual of an outerplanar graph (that is a planar graph with a vertex adjacent to every other non-isolated vertex of the graph) if and only if

(i) $p(t) = 1$

or

(ii) $p(t) = (-1)^{\gamma-b}(t-1)^b \prod_{i=3}^{\infty} (1+s+s^2+\cdots+s^{i-2})^{k_i}$ for some integers $\gamma, b, \geq 1$, $k_i \geq 0$ for each i.

Proof. For 'only if', suppose G is the dual of an outerplanar graph. If G has no edges then $F(G,t)=1$, so suppose otherwise. Then G^* is outerplanar and has at least one edge, so by Corollary 1.5.2,

$$F(G,t)=\frac{P(G^*,t)}{t}=(-1)^{\gamma-b}(t-1)^b\prod_{i=3}^{\infty}(1+s+s^2+\cdots+s^{i-2})^{k_i}.$$

as required.

For 'if', suppose $p(t)$ has the form given. If $p(t)=1$ then $F(K_1,t)=p(t)$ and K_1 is the dual of an outerplanar graph; so suppose otherwise. Then, by Corollary 1.5.2, there exists an outerplanar graph G for which $\dfrac{P(G,t)}{t}=p(t)=F(G^*,t)$, as required. □

Theorem 3.8. Let G be a graph in which every component is 3-edge-connected, and suppose G has K_4 as a subcontraction. Then either

(i) every non-loop edge of G is contained in one block, which is isomorphic to K_4,

or

(ii) $a_1^*(G)>n-c$.

Proof. We prove the result by induction on m. There are two cases to consider.

Case 1: There exist graphs G_1 and G_2 such that $n_1<n$ and $n_2<n$ and either $G=G_1\cup G_2$ where $G_1\cap G_2=\varnothing$ or K_1, or G has an edge which cleaves it into G_1 and G_2. Then either $K_4\preccurlyeq G_1$ or $K_4\preccurlyeq G_2$, say $K_4\preccurlyeq G_1$. By Corollary 3.1.1 (ii) and (iii), $q^*(G,t)=q^*(G_1,t)q^*(G_2,t)$, and so

$$a_1^*(G)=a_0^*(G_1)a_1^*(G_2)+a_1^*(G_1)a_0^*(G_2)$$

$$\geq 2a_1^*(G_2)+a_1^*(G_1)$$

$$\geq n_2-c_2+a_1^*(G_1)+a_1^*(G_2)$$

by Corollary 3.3.1 and Theorem 3.2.

If G_1 satisfies condition (i), then either G satisfies condition (i) also, or G_2 contains a non-loop edge, in which case $n_2-c_2\geq 1$ and so

$$a_1^*(G) \geq 1 + n_1 - c_1 + n_2 - c_2 > n - c,$$

as required. Otherwise, $a_1^*(G) > n_1 - c_1$, by the inductive hypothesis, and so

$$a_1^*(G) \geq a_1^*(G_1) + a_1^*(G_2) > n_1 - c_1 + n_2 - c_2 = n - c,$$

as required.

Case 2: G is 2-connected and $G - e$ is 2-connected for each edge e of G. Choose an edge e of G, let G_1 be a graph obtained from $G - e$ by repeatedly contracting an edge in any cutset of two edges and let $G_2 := G/e$. Then by Corollary 3.1.1 (i), and as in Case 3 of the proof of Theorem 3.2, $q^*(G,t) = q^*(G_1,t) + s^{b_2-1} q^*(G_2,t)$. Note that G_1 is 2-connected, and so $n_1 - c_1 \geq 2$. There are two subcases to consider.

Case 2a: e can be chosen so that it does not lie in a cutset of three edges in G. Then $G_1 = G - e$ and $n_1 = n$.

If $K_4 \preccurlyeq G_1$, but G_1 is not isomorphic to K_4, then

$$a_1^*(G) \geq a_1^*(G_1) > n_1 - c_1 = n - c,$$

by the inductive hypothesis, as required. If $G_1 = K_4$ then G is the graph obtained by doubling an edge of K_4, which has $q^*(G,t) = 2 + 4s + 3s^2 + s^3$, and so $a_1^*(G) = 4 > 3 = n - c$, as required.

Thus we may suppose that K_4 is not a subcontraction of G_1. If G_2 has a cut-vertex then e is a separating edge of G which separates G into two or more subgraphs, one of which must have K_4 as a subcontraction, and so $G_1 = G - e$ must have K_4 as a subcontraction, a contradiction. Thus G_2 must be 2-connected, and so

$$\begin{aligned} a_1^*(G) &= a_1^*(G_1) + a_1^*(G_2) \\ &\geq n_1 - c_1 + n_2 - c_2 \\ &= n - 1 + n - 2 \\ &> n - 1 \end{aligned}$$

since $n \geq 3$, as required.

Case 2b: Every edge of G lies in a cutset of three edges in G. Then it is easy to see that e can be chosen in such a way as to make G_2 2-connected. If $K_4 \preccurlyeq G_2$ (note that G_2 is not isomorphic to K_4, since G is 3-edge-connected) then

$$a_1^*(G) = a_1^*(G_1) + a_1^*(G_2) > 1 + n - 2 = n - c,$$

as required.

So suppose e cannot be chosen in such a way that G_2 is 2-connected and $K_4 \preccurlyeq G_2$. Let e be an edge of G such that G_2 is 2-connected. Then G_2 is a dual generalised polygon tree. If G_2 has a cleaving subgraph then, by Corollary 3.6.2,

$$a_1^*(G) = a_1^*(G_1) + a_1^*(G_2) > 1 + n - 2 = n - c,$$

as required, so suppose otherwise. Then by Corollary 3.5.1, G_2 is a dual polygon tree.

If $G_2^* = G^* - e^*$ has a polygon adjacent to three or more others, then G^* must have a separating edge, and so G must have a cleaving edge, a contradiction. If e^* is a chord of one of the polygons in G_2^*, then G^* is a polygon tree also, and so G^* (and hence G) cannot have K_4 as a subcontraction, a contradiction. Since every edge of G lies in a cutset of three edges, every edge of G^* lies in a triangle, and so, by a similar argument to that given in Case 2b of the proof of Theorem 1.6, G^* has an edge e^* such that $G^* - e^*$ (and hence G/e) is 2-connected, and $K_4 \preccurlyeq G^* - e^*$ (and hence $K_4 \preccurlyeq G/e$), a contradiction. The result now follows. □

Corollary 3.8.1. If G is a graph in which every component is 3-edge-connected and such that $a_1^*(G) = n - c$, then either G has exactly one block containing a non-loop edge, which is isomorphic to K_4, or every block of G containing a non-loop edge is the dual of a polygon tree (and hence G has the same flow polynomial as the dual of some outerplanar graph).

Proof. By Theorem 3.8, either every non-loop edge of G is contained in one block, which is isomorphic to K_4, in which case we are done, or G does not have K_4 as a subcontraction. Suppose the latter case holds. We prove the result by induction on the number of blocks of G.

If G is 2-connected, then G is a dual generalised polygon tree, and since G cannot have a cleaving subgraph by Corollary 3.6.2, G is in fact a dual polygon tree, as required. Now suppose G is not 2-connected. Then $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \varnothing$ or K_1, and then, by Corollary 3.1.1 part (ii) (since $a_0^*(G_1) = a_0^*(G_2) = 1$ by Lemma 3.4),

$$n - c = a_1^*(G) = a_1^*(G_1) + a_1^*(G_2) \geq n_1 - c_1 + n_2 - c_2 = n - c,$$

and so equality must hold throughout; that is, $a_1^*(G_1) = n_1 - c_1$ and $a_1^*(G_2) = n_2 - c_2$. The result now follows by the inductive hypothesis. □

3.3. Identities for the Coefficients $a_i^*(G)$.

In this section, we derive explicit expressions for the last few coefficients $a_i^*(G)$. The following result can be easily proved from a well-known result about the Tutte polynomial (see Chapter 4), but here we prove it directly.

Lemma 3.9.

$$F(G,t) = \sum_{X \subseteq E} (-1)^{|X|} t^{\gamma_G(E \setminus X)}.$$

Proof. Let S be the set of all (maybe zero) t-flows of G, and, for each $i = 1, 2, \ldots, m$, let S_i be the set of t-flows in which the ith edge, e_i, has zero flow. Then $F(G,t) = \left| S \setminus \bigcup_i S_i \right|$.

Let $\vec{G}$ be any orientation of G. Let T be a spanning forest of G, let $E(G) \setminus E(T) = \{e_1, e_2, \ldots, e_\gamma\}$ and let f be a function

$$f \colon E(G) \setminus E(T) \to \{0, 1, \ldots, t-1\}.$$

Then, for each i, $T + e_i$ contains a unique circuit C_i. For each edge e of T, let $f(e) = \sum_{i=1}^{\gamma} c(e,i)$, where

$$c(e,i)=\begin{cases} 0 & \text{if } e \notin E(C_i), \\ f(e_i) & \text{if } e \in E(C_i) \text{ and } e \text{ has the same orientation as } e_i \text{ in } C_i, \\ -f(e_i) & \text{if } e \in E(C_i) \text{ and } e \text{ has the opposite orientation to } e_i \text{ in } C_i. \end{cases}$$

Then f is a (maybe zero) t-flow of G. Now suppose f' is a (maybe zero) t-flow of G with $f'(e) = f(e)$ for each $e \in E(G) \backslash E(T)$. Then the function $f' - f$ is a (maybe zero) t-flow of G, and is zero on all the edges of $E(G) \backslash E(T)$ and it is easy to see that it must be zero on all the edges of T also. Thus $f' = f$, and so we have shown that there is a bijection between (maybe zero) t-flows on G and functions $f: E(G) \backslash E(T) \to \{0, 1, \ldots, t-1\}$. It follows that $|S| = t^{\gamma}$. It is easy to see that $S_{n_1} \cap S_{n_2} \cap \cdots \cap S_{n_r}$ is the set of all (maybe zero) t-flows of $G \backslash \{e_{n_1}, e_{n_2}, \cdots, e_{n_r}\}$, and so $|S_{n_1} \cap S_{n_2} \cap \cdots \cap S_{n_r}| = t^{\gamma_G(E \backslash \{e_{n_1}, e_{n_2}, \cdots, e_{n_r}\})}$, for $n_i \in \{1, 2, \ldots, m\}$. The result now follows by the inclusion-exclusion principle. □

Theorem 3.10. Let G be a graph in which every component is 3-edge-connec and let h be the largest number such that each component of G is h-edge-connected. Suppose G has k cutsets of h edges. Then

$$a^*_{\gamma-b-r}(G) = \binom{n-c+r-1}{r}$$

for $0 \le r \le h-2$ and

$$a^*_{\gamma-b-h+1}(G) = \binom{n-c+h-2}{h-1} - k.$$

Proof. First note that if $r = 0$, the result follows by Theorem 3.2, so suppose $r \ge 1$.

By the definition of $q^*(G,t)$ and Lemma 3.9,

$$q(G,t) = \frac{F(G,t)}{(-1)^{\gamma-b}(t-1)^b}$$

$$= \frac{\sum_{X \subseteq E} (-1)^{|X|} t^{\gamma_G(E \backslash X)}}{(-1)^{\gamma}(1-t)^b}$$

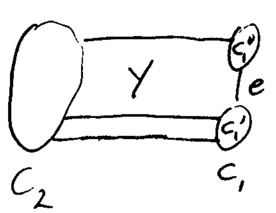
(*) Proof of inequality $\gamma_G(E\setminus X) \leq \gamma - h$ when $|X| > h \geq 3$:

Since the circuit rank $\gamma_G(E\setminus X)$ cannot be increased by removing edges, it suffices to prove the inequality when $|X| = h+1$.

We have,

$$\begin{aligned} & \gamma_G(E\setminus X) = \gamma - h \\ \Leftrightarrow \quad & |E\setminus X| - n + c_G(E\setminus X) \leq m - n + c - h \\ \Leftrightarrow \quad & c_G(E\setminus X) - c \leq |X| - h = 1 . \end{aligned}$$

So suppose $c_G(E\setminus X) - c \geq 2$. Clearly, X must contain a cutset, Y, of h edges, since if X is itself a cutset, then $c_G(E\setminus X) - c = 1$. Let C be the component of G containing Y, and let C_1 and C_2 be the components of $C\setminus Y$. Then one of C_1 and C_2, say C_1, contains a cut-edge e which separates it into two components C_1' and C_1''. Let C_1'' be the component of $C_1 - e$ incident with the fewest number of edges of Y. There can be at most $\lfloor \frac{h}{2} \rfloor$ of these edges, and so these edges, together with e, form a cutset of $\lfloor \frac{h}{2} \rfloor + 1$ edges, contradicting the minimality of h, since $h \geq 3 \Rightarrow \lfloor \frac{h}{2} \rfloor + 1 < h$.



C

$$= \frac{1}{(1-t)^b} \sum_{X \subseteq E} (-1)^{|X|+(|E\backslash X|-n+c_G(E\backslash X))-m+n-c} (-t)^{\gamma_G(E\backslash X)}$$

$$= \frac{1}{s^b} \sum_{X \subseteq E} (-1)^{c_G(E\backslash X)-c} (s-1)^{\gamma_G(E\backslash X)}$$

where $s = 1 - t$.

Now, for $X \subseteq E$, if $|X| < h$ then $c_G(E\backslash X) = c$ and

$$\gamma_G(E\backslash X) = |E\backslash X| - n + c = \gamma - |X|,$$

and if $|X| = h$ then $\gamma_G(E\backslash X) = \gamma - h$ except for the k subsets X which form cut-sets of h edges in G, for which $c_G(E\backslash X) = c + 1$ and $\gamma_G(E\backslash X) = \gamma - h + 1$. If $|X| > h$ then $\gamma_G(E\backslash X) \le \gamma - h$. Thus

$$q^*(G,t) = \frac{1}{s^b} \sum_{i=0}^{m} \sum_{|X|=i} (-1)^{c_G(E\backslash X)-c} (s-1)^{\gamma_G(E\backslash X)}$$

$$= \frac{1}{s^b} \left[\sum_{i=0}^{h} \sum_{|X|=i} (s-1)^{\gamma-i} + \sum_{i=h+1}^{m} \sum_{|X|=i} (-1)^{c_G(E\backslash X)-c} (s-1)^{\gamma_G(E\backslash X)} \right.$$

$$\left. - k(s-1)^{\gamma-h} - k(s-1)^{\gamma-h+1} \right].$$

From this, for $1 \le r \le h-2$, $a^*_{\gamma-b-r}(G)$ is the coefficient of $s^{\gamma-r}$ in $\sum_{i=0}^{r} \binom{m}{i} (s-1)^{\gamma-i}$, and $a^*_{\gamma-b-h+1}(G)$ is the coefficient of $s^{\gamma-h+1}$ in $\sum_{i=0}^{h-1} \binom{m}{i} (s-1)^{\gamma-i} - k(s-1)^{\gamma-h+1}$.

Thus $a^*_{\gamma-b-r}(G) = \sum_{i=0}^{r} (-1)^{r-i} \binom{m}{i} \binom{\gamma-i}{r-i}$, which by Lemma 1.7, with $\alpha = m$ and $\beta = \gamma - r + 1$ (note that $n \ge c$ and so $\alpha - \beta = n - c + r - 1 \ge 0$), gives $a^*_{\gamma-b-r}(G) = \binom{n-c+r-1}{r}$ as required. Similarly,

$$a^*_{\gamma-b-h+1}(G) = \sum_{i=0}^{h-1}(-1)^{h-1-i}\binom{m}{i}\binom{\gamma-i}{h-1-i} - k = \binom{n-c+h-2}{h-1} - k,$$

as required. □

We finish this section with the conjecture of an improvement to the inequality in Theorem 3.2.

Conjecture 3.11. $a^*_r(G) \geq \binom{n-c+r-1}{r}$ for $0 \leq r \leq h-2$, and $a^*_r(G) \geq \binom{n-c+h-2}{h-1} - k$ for $h-1 \leq r \leq \gamma - b - h$.

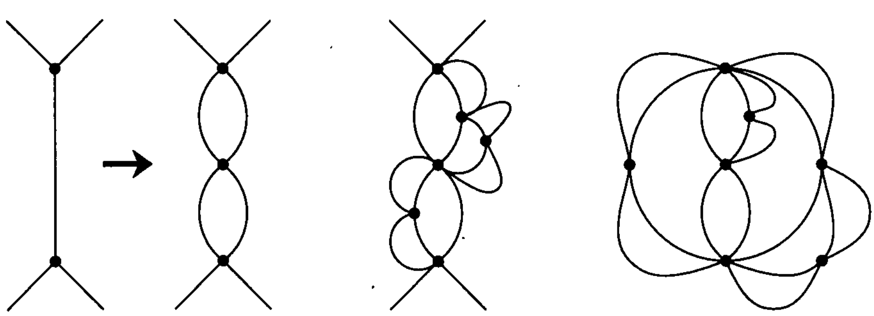
3.4. A Zero-Free Interval.

In this section, we present a dual result to Bill Jackson's zero-free interval for chromatic polynomials [1]. We begin with some definitions.

A *dual generalised edge* is defined recursively as follows.

(i) An edge (K_2) is a dual generalised edge.

(ii) A graph obtained from a dual generalised edge by replacing one edge by a double digon (see Figure 3.4.1 (i)), is a dual generalised edge also.



(i) (ii) (iii)

Figure 3.4.1

A *dual generalised triangle* is defined recursively as follows.

(i) A sheaf of three edges is a dual generalised triangle.

(ii) A graph obtained from a dual generalised triangle by replacing one edge by a dual generalised edge is a dual generalised triangle also.

Figure 3.4.1 (ii) and (iii) are examples of a dual generalised edge and a dual generalised triangle, respectively.

Theorem 3.10. Let G be a graph such that the following conditions hold:

(a) G is 2-connected and 3-edge-connected,

(b) for each edge e of G, G/e has exactly two blocks,

(c) if G' is a 3-edge-connected graph with an edge e such that $G'/e = G$, then $G' - e$ has an odd number of blocks,

(d) if G_1 and G_2 are subgraphs and u and v are vertices of G such that $G_1 \cup G_2 = G$, $E(G_1 \cap G_2) = \emptyset$, $V(G_1 \cap G_2) = \{u, v\}$ and G_1 is a dual generalised edge, then $(G_2)_{u=v}$ has exactly two blocks.

Then G is a dual generalised triangle.

Proof. Suppose otherwise, and let G be a minimal counterexample. If G has exactly two vertices then, by (a) and (b), G is a sheaf of three edges, and we are done; so suppose G has at least three vertices.

Note that G cannot be 3-connected, by (b), and so there exist subgraphs G_1 and G_2 and vertices u and v of G satisfying:

(i) $G_1 \cup G_2 = G$, $E(G_1 \cap G_2) = \emptyset$ and $V(G_1 \cap G_2) = \{u, v\}$,

(ii) $|V(G_1)| \geq 3$,

(iii) $uv \notin E(G_1)$,

(iv) G_1 is minimal subject to conditions (i) to (iii).

We shall show that G_1 is a dual generalised edge, with $|V(G_1)| = 3$. Suppose otherwise.

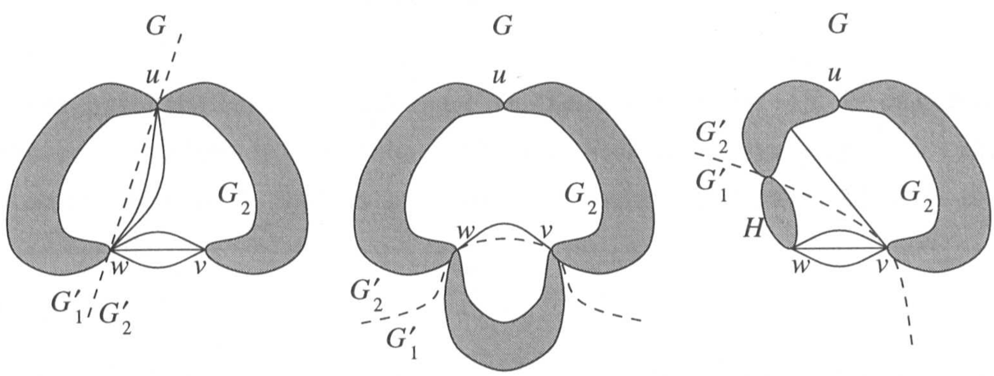
Suppose first that $|V(G_1)| = 3$, say $V(G_1) = \{u, v, w\}$. By conditions (b) and (iii), G_1 consists of two digons joined at w, and so G_1 is a dual generalised edge, a contradiction.

Now suppose $|V(G_1)| > 3$. Let w be a neighbour of v in G_1. If v has exactly one neighbour in G_1 then $G_1' = G_1 \backslash \{v\} \backslash \{\text{edges } uw\}$ and $G_2' = G_2 \cup \{v\} \cup \{\text{edges } vw\} \cup \{\text{edges } uw\}$ (see Figure 3.4.2 (i)) satisfy (i) to (iii) and so contradict (iv). Thus v must have at least two neighbours in G_1.

If there is exactly one edge vw, then by (b) there exist subgraphs G_1' and G_2' with $G_1' \cup G_2' = G$, $E(G_1' \cap G_2') = \varnothing$, $V(G_1' \cap G_2') = \{v, w\}$, $vw \notin G_1'$, $|V(G_1')| \geq 3$ and G_1' a subgraph of G_1 (see Figure 3.4.2 (ii)), contradicting (iv). Thus there are at least two edges vw.

If $G \backslash \{\text{edges } vw\}$ is 2-connected then there exists a graph G' with an edge e such that $G'/e = G$ and $G' - e$ has $G \backslash \{\text{edges } vw\}$ and $\{\text{edges } vw\}$ as blocks, which violates condition (c), so let H be the block of $G \backslash \{\text{edges } vw\}$ containing the vertex w. Then $G_1' = H \cup \{\text{edges } vw\}$ and $G_2' = G \backslash H \backslash \{\text{edges } vw\}$ (see Figure 3.4.2 (iii)) satisfy conditions (i) to (iii) and so contradict condition (iv) (since G_1' is a proper subgraph of G_1).

Thus G_1 is a dual generalised edge with $|V(G_1)| = 3$; that is, it is a double digon.



(i) (ii) (iii)

Figure 3.4.2

Let $H = G_2 + uv$. Then H satisfies (a) since G does. Let e be an edge of H. If $e \in E(G_2)$ then H/e has exactly two blocks since G/e has by (b), otherwise $e = uv$ and then by (d) H/e has exactly two blocks. Thus H satisfies (b).

Now let H' be a 3-edge-connected graph with an edge e such that $H'/e = H$. Let G' be the graph formed from H' by replacing uv by a double digon. Then G' is 3-edge-connected and $G'/e = G$ and so $G' - e$ has an odd number of blocks by (c). But $H' - e$ must have the same number of blocks as $G' - e$, for otherwise the edge uv is a block of $H' - e$ and then $(G_2)_{u=v} = H/uv$ is 2-connected, contrary to (d). Thus H satisfies (c).

Finally, let H_1 and H_2 be subgraphs and w and z be vertices of H such that $H_1 \cup H_2 = H$, $E(H_1 \cap H_2) = \varnothing$, $V(H_1 \cap H_2) = \{w, z\}$ and H_1 is a dual generalised edge. If $uv \in E(H_1)$ then the graph G_1' formed from H_1 by replacing uv

by a double digon is a dual generalised edge, with $G = H_2 \cup G_1'$, and so by (d), $(H_2)_{w=z}$ has exactly two blocks. If $uv \in E(H_2)$ then $(G_2')_{w=z}$ has exactly two blocks by (d), where G_2' is obtained from H_2 by replacing uv by a double digon, and so $(H_2)_{w=z}$ has exactly two blocks also.

Thus H satisfies conditions (a) to (d), contradicting the minimality of G. It follows that G is in fact a dual generalised triangle. □

Theorem 3.11. Let G be a graph without cut-edges. Then $F(G,t)$ is non-zero with sign $(-1)^{\gamma-b}$ for all $t \in (1, \frac{32}{27}]$.

Proof. Suppose otherwise. Let G be a minimal counterexample. Then every component of G must be 3-edge-connected, for if G has a cutset of two edges, a graph G' obtained from G by contracting one of the edges in the cutset has the same flow polynomial, circuit rank, and number of blocks as G, contradicting the minimality of G.

There must exist $t \in (1, \frac{32}{27}]$ such that $F(G,t)$ either has sign $(-1)^{\gamma-b+1}$ or is zero. However, it follows from Corollary 3.2.2 that $F(G,t)$ has sign $(-1)^{\gamma-b}$ for t sufficiently close to 1, and so by continuity, there exists $t \in (1, \frac{32}{27}]$ such that $F(G,t) = 0$.

Claim 1. G is 2-connected.

Proof. Suppose otherwise. Then $G = G_1 \cup G_2$ where $G_1 \cap G_2$ is K_1 or $\emptyset$. Then by Theorem 3.1 (ii), $F(G,t) = F(G_1,t)F(G_2,t)$, which by the minimality of G is non-zero with sign $(-1)^{\gamma_1-b_1+\gamma_2-b_2} = (-1)^{\gamma-b}$, a contradiction.

Claim 2. For each edge e of G, $G-e$ is 2-connected.

Proof. Suppose otherwise, and let e be an edge which cleaves G into G_1 and G_2. Then by Theorem 3.1 (iii), $F(G,t) = \dfrac{F(G_1,t)F(G_2,t)}{(t-1)}$, which by the minimality of G is non-zero with sign $(-1)^{\gamma_1-1+\gamma_2-1} = (-1)^{\gamma-1}$, a contradiction.

Claim 3. Suppose G' is a 3-edge-connected graph with an edge e such that $G'/e = G$. Let r be the number of blocks of $G'-e$. Then r is odd.

Proof. If $r = 1$ then we are done, so suppose $r \geq 2$ (so that e cleaves G'). Let $G_1, G_2, \ldots, G_r$ be the blocks of $G'-e$ and $G_1', G_2', \ldots, G_r'$ be the corresponding graphs into which e cleaves G', so that $G_i' = G_i \cup e_i$ for some e_i, (see Figure 3.4.3). Note that none of the G_i can consist of a single edge, since G' is 3-edge-connected, so each G_i is 2-edge-connected.

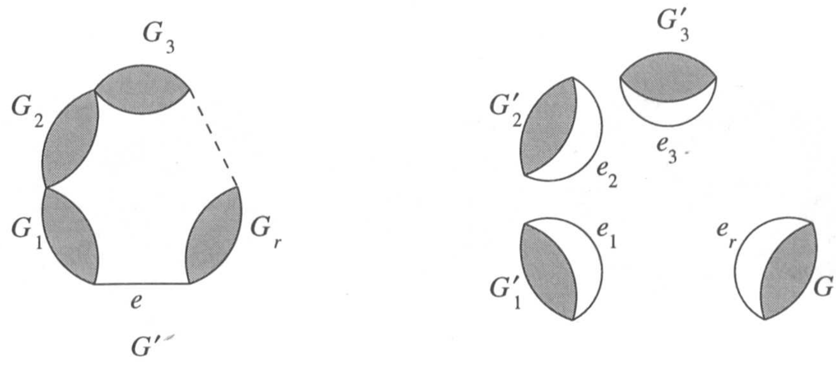


Figure 3.4.3

Then $F(G'-e,t) = \prod_{i=1}^{r} F(G_i,t)$ and $F(G',t) = \dfrac{\prod_{i=1}^{r} F(G'_i,t)}{(t-1)^{r-1}}$.

Now $F(G,t) = F(G',t) + F(G'-e,t)$ by Theorem 3.1 (i). By the minimality of G, $F(G',t)$ is non-zero with sign

$$(-1)^{(\sum_{i=1}^{r}\gamma(G'_i)-1)} = (-1)^{(\sum_{i=1}^{r}\gamma_i)} = (-1)^{\gamma(G')-1} = (-1)^{\gamma-1},$$

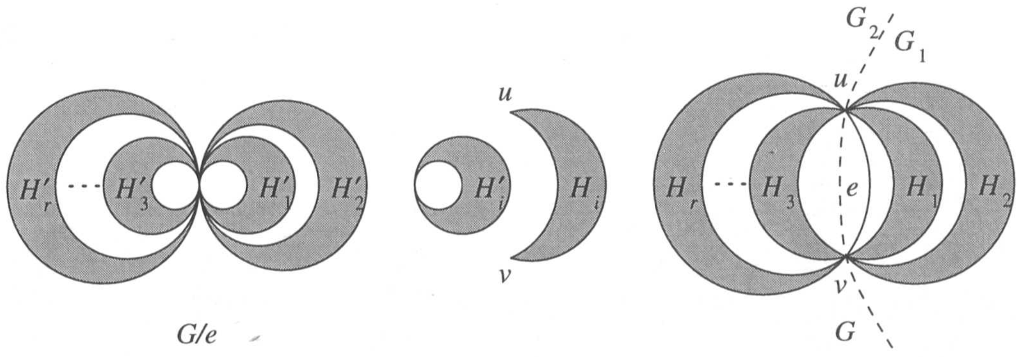
and $F(G'-e,t)$ is non-zero with sign $(-1)^{(\sum_{i=1}^{r}\gamma_i-1)} = (-1)^{\gamma-r-1}$. Thus if r is even, $F(G,t)$ is non-zero with sign $(-1)^{\gamma-1}$, a contradiction.

Claim 4. For each edge e of G, G/e has exactly two blocks.

Proof. By the minimality of G, $F(G-e,t)$ is non-zero with sign $(-1)^{\gamma-2}$ (since $\gamma(G-e) = \gamma-1$ and $G-e$ is 2-connected by Claim 2). Suppose G/e has r blocks. If r is odd, then $F(G/e,t)$ is non-zero with sign $(-1)^{\gamma-r} = (-1)^{\gamma-1}$, and so by Theorem 3.1 (i), $F(G,t)$ is non-zero with sign $(-1)^{\gamma-1}$, a contradiction. Thus r is even.

Suppose now that $r \geq 4$, and let u and v be the ends of e. Let $H'_1, H'_2, \ldots, H'_r$ be the blocks of G/e (see Figure 3.4.4 (i)) and let H_i be the subgraph of G corresponding to H'_i (so that $H'_i = (H_i)_{u=v}$) for $i = 1, 2, \ldots, r$ (see Figure 3.4.4 (ii)). Then there exists a graph G' with an edge f such that $G'/f = G$ and $G'-f$ has $G_1 = H_1 \cup H_2 \cup e$ and $G_2 = H_3 \cup H_4 \cup \cdots H_r$ as blocks (see Figure 3.4.4 (iii)), contradicting Claim 3. Thus $r = 2$ as required.

Claim 5. If G_1 and G_2 are subgraphs and u and v are vertices of G such that $G_1 \cup G_2 = G$, $E(G_1 \cap G_2) = \varnothing$, $V(G_1 \cap G_2) = \{u, v\}$ and G_1 is a dual generalised edge, then $(G_2)_{u=v}$ has exactly two blocks.



(i) (ii) (iii)

Figure 3.4.4

Proof. If $G_1 = K_2$ then $G_2 = G - uv$ and $(G_2)_{u=v} = G/uv$, which has exactly two blocks by Claim 4. So suppose $G_1 \neq K_2$. Then γ_1 must be even. Let H be the graph obtained from G_1 by adding two edges uv (see Figure 3.4.5 (i)). By the minimality of G, $F(H,t)$ is non-zero with sign $(-1)^{(\gamma_1+2)-1} = -1$ (since γ_1 is even). By Theorem 3.1 (iv), (with $H = G_1 \cup$ {two edges uv})

$$(t-1)F(H,t) = (t-1)F(G_1,t)(t-1)^2$$

$$+ (t-1)(t-2)[F(G_1+uv,t) - (t-1)F(G_1,t)].$$

Now $F(G_1,t)$ is non-zero with sign $(-1)^{\gamma_1 - b_1} = 1$ (since G_1 is a dual generalised edge and $G_1 \neq K_2$). This gives $F(G_1+uv,t) - (t-1)F(G_1,t) > 0$ (since otherwise $F(H,t) > 0$). By Theorem 3.1 (iv),

$$(t-1)F(G,t) = (t-1)F(G_1,t)F((G_2)_{u=v},t)$$

$$+ F(G_2+uv,t)[F(G_1+uv,t) - (t-1)F(G_1,t)].$$

Now, by the minimality of G, $F(G_2+uv,t)$ is non-zero with sign $(-1)^{(\gamma_2+1)-1}$ and $F((G_2)_{u=v},t)$ is non-zero with sign $(-1)^{(\gamma_2+1)-r}$, where r is the number of blocks in $(G_2)_{u=v}$. Moreover, $\gamma = \gamma_1 + \gamma_2 + 1$, and so $\gamma - \gamma_2$ is odd.

If r is odd then $F(G,t)$ is non-zero with sign $(-1)^{\gamma_2} = (-1)^{\gamma-1}$, a contradiction. Thus r is even. Suppose $r \geq 4$. Let $H'_1, H'_2, \ldots, H'_r$ be the blocks of $(G_2)_{u=v}$ and let H_i be the subgraph of G corresponding to H'_i (so that $H'_i = (H_i)_{u=v}$) for $i = 1, 2, \ldots, r$ (see Figure 3.4.5 (ii)). Then there exists a graph G' with an edge f such that $G'/f = G$ and $G' - f$ has $G'_1 = G_1 \cup H_1 \cup H_2$ and

$G_2' = H_3 \cup H_4 \cup \cdots H_r$ as blocks (see Figure 3.4.5 (iii)), contradicting Claim 3. Thus $r = 2$ as required.

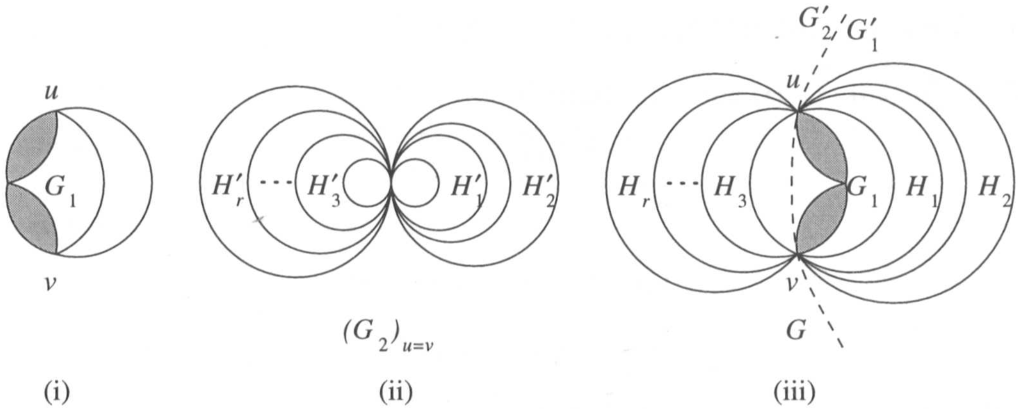


Figure 3.4.5.

By Theorem 3.10, G is a dual generalised triangle, and in particular G is planar. Thus, by Bill Jackson's result [1], $F(G,t) = \dfrac{P(G^*,t)}{t}$ is non-zero, a contradiction. The result now follows. □

The next theorem, which follows by duality with a result in Bill Jackson's paper [1], shows that this interval is 'best possible'.

Theorem 3.12. There exist graphs G in which $F(G,t)$ is zero arbitrarily close to $\frac{32}{27}$.

Proof. In his paper [1], Bill Jackson presents a series of planar graphs G in which $P(G,t)$ is zero arbitrarily close to $\frac{32}{27}$. Then $F(G^*,t)$ is zero arbitrarily close to $\frac{32}{27}$. □

3.5. The Flow Polynomial in Terms of Falling Factorials.

In this section, we consider the dual of the complete graph basis for chromatic polynomials (see Chapter 2), that is, writing a modified flow polynomial as a linear combination of the falling factorials $(t)_i := t(t-1)(t-2)\cdots(t-i+1)$. Note that since every graph without cut-edges is known to have a 6-flow (see

Seymour [2]), and is strongly conjectured to have a 5-flow (see Tutte [3]), there is no analogue for flow polynomials to complete graphs for chromatic polynomials. We begin with some definitions and some basic results.

For each $i \geq 0$, let $k_i^*(G)$ be defined by $tF(G,t) = \sum_{i \geq 0} k_i^*(G)(t)_i$, and for $i < 0$, let $k_i^*(G) = 0$. Note that $k_i^*(G)$ is well-defined, since the $(t)_i$ are linearly independent polynomials. Let $K^*(G,x) := \sum_{i \geq 0} k_i^*(G)x^i$.

Lemma 3.13. Let G be a graph.

(i) If G has a cut-edge, then $k_i^*(G) = 0$ for each i.

(ii) $k_i^*(G) = 0$ for $i = 0$ and for each $i > \gamma(G) + 1$.

(iii) If G has no r-flow for some $r > 0$, then $k_t^*(G) = 0$ for each non-negative integer $t \leq r$. In particular, if G has an edge, then $k_1^*(G) = 0$, and if G is non-Eulerian, then $k_2^*(G) = 0$.

(iv) For a non-loop edge e of G, $k_i^*(G) = k_i^*(G/e) - k_i^*(G-e)$ for each i.

(v) For a planar graph G, $k_i(G^*) = k_i^*(G)$ for each i, and $K(G^*,x) = K^*(G,x)$.

Proof. Parts (i) and (iv) follow directly from the definitions, Theorem 3.1 (i) and the fact that if G has a cut-edge, then $F(G,t) = 0$.

Since $F(G,t)$ has degree $\gamma(G)$ providing G has no cut-edges, it follows that $k_i^*(G) = 0$ for $i > \gamma(G) + 1$. Also, $k_0^*(G) = 0$ since $tF(G,t) = 0$ when $t = 0$. This proves part (ii).

If G has no r-flow, then $F(G,t) = 0$ for each non-negative integer $t \leq r$, and so $k_t^*(G) = 0$. The remainder of part (iii) follows from the fact that G has a 1-flow if and only if it is a null graph, and a 2-flow if and only if it is Eulerian (that is, every vertex of G has even degree).

Finally, for part (v), $P(G^*,t) = tF(G,t) = \sum_{i \geq 0} k_i^*(G)(t)_i$, and the result follows from Lemma 2.2. □

In view of Lemma 3.13 (v), we make the following conjecture.

Conjecture 3.14.

(i) For each i, $k_i^*(G)$ is a non-negative integer.

(ii) If G has an r-flow but no $(r-1)$-flow, then $k_i^*(G)$ is a positive integer for $r \le i \le \gamma(G)+1$.

(iii) The $k_i^*(G)$ form a log-concave sequence, that is, $k_i^*(G)^2 \ge k_{i-1}^*(G)k_{i+1}^*(G)$ for each i.

Since, for each i, $k_i(G)$ is the number of partitions of $V(G)$ into i independent (non-empty) subsets, it is trivial that $k_i(G)$ is a non-negative integer. However, Conjecture 3.14 (i) seems to be far from trivial. By Lemma 3.13 (ii) and (iii), Conjecture 3.14 (ii) implies Conjecture 3.14 (i). Conjecture 3.14 (iii) cannot be strengthened to strong log-concavity, since $K^*(K_{3,3}, x) = x^5 + x^4 + x^3$.

In the remainder of this section, we show that Conjecture 3.14 (i) holds in general if it holds for simple, 2-connected, cubic graphs of girth at least 5. We will make extensive use of the following result.

Lemma 3.15.

(i) For integers i and r, $(t-r)(t)_i = (t)_{i+1} + (i-r)(t)_i$.

(ii) If G' and G are graphs such that $F(G',t) = (t-r)F(G,t)$ for some integer r, then $k_i^*(G') = k_{i-1}^*(G) + (i-r)k_i^*(G)$ for each i. In particular, if r and i are integers such that $r \le i$ and $k_{i-1}^*(G)$ and $k_i^*(G)$ are non-negative integers, then $k_i^*(G')$ is a non-negative integer.

Proof. Part (i) follows immediately since $t-r = (t-i)+(i-r)$. Let G' and G be graphs such that $F(G',t) = (t-r)F(G,t)$ for some integer r. Then

$$
\begin{aligned}
tF(G',t) &= t(t-r)F(G,t) \\
&= (t-r)\textstyle\sum_i k_i^*(G)(t)_i \\
&= \textstyle\sum_i k_i^*(G)(t)_{i+1} + \sum_i (i-r)k_i^*(G)(t)_i && \text{by part (i)} \\
&= \textstyle\sum_i k_{i-1}^*(G)(t)_i + \sum_i (i-r)k_i^*(G)(t)_i,
\end{aligned}
$$

from which the result follows. □

Lemma 3.16. Let G be a graph.

(i) $F(G,t)$ can be expressed as the sum of the flow polynomials of graphs of maximum degree 3 or less.

(ii) $F(G,t)$ can be expressed as the sum of the flow polynomials of graphs in which every component is cubic or a loop.

Proof.

(i) Let $\alpha(G) = \sum_{v \in V(G)} \max(d(v)-3,0)$. We prove the result by induction on $\alpha(G)$.

If $\alpha(G)=0$ then G has maximum degree 3 or less, and we are done, so suppose $\alpha(G)>0$. Let v be a vertex of degree at least 4. Let G_1 be a graph obtained from G by 'splitting' v into v_1 and v_2 and adding the edge v_1v_2, in such a way that v_2 has degree 3 in G_1 (so that $G = G_1/v_1v_2$ and $d(v_1)$ in G_1 is equal to $d(v)-1$ in G, see Figure 3.5.1). Let G_2 be the graph obtained from G_1 by deleting the edge v_1v_2 and contracting one of the remaining edges incident with v_2. Then it is easy to see that $\alpha(G_1) = \alpha(G)-1$ and $\alpha(G_2) \le \alpha(G)-1$.

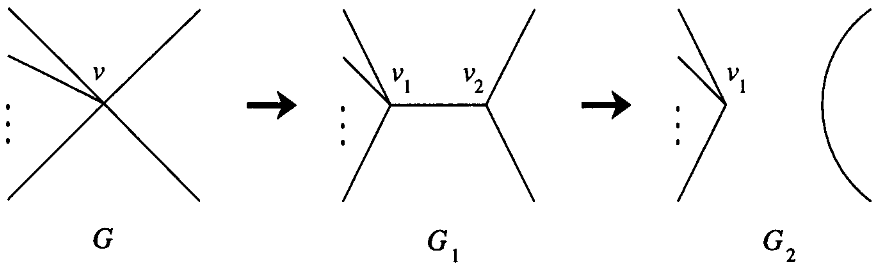


Figure 3.5.1

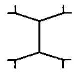
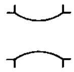
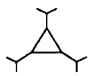
Thus, by the inductive hypothesis, $F(G_1,t)$ and $F(G_2,t)$ can be expressed as the sum of the flow polynomials of graphs of maximum degree 3 or less, and the result follows by Theorem 3.1 (i) (since $F(G_2,t) = F(G_1 - v_1v_2, t)$).

(ii) If G has a cut-edge, then $F(G,t)=0$ and we are done, so suppose otherwise. By part (i), we need only consider the case where every vertex of G is of degree 2 or 3. Let G' be the graph obtained from G by repeatedly contracting one edge incident with any degree 2 vertex that is not the vertex of a loop component. Then $F(G,t) = F(G',t)$, and every component of G' is cubic or a loop. The result follows. □

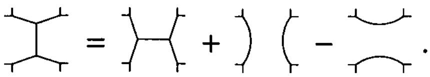
Corollary 3.16.1. Conjecture 3.14 (i) holds in general if it holds for loopless cubic graphs.

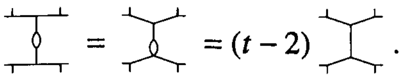
Proof. Let G be any graph, and suppose that the conjecture holds for loopless cubic graphs. Then by Lemma 3.16 (ii), $F(G,t)$ can be expressed as the sum of the flow polynomials of graphs in which every component is cubic or a loop. By the definition of $k_i^*(G)$, it is only necessary to show that the conjecture holds for such graphs.

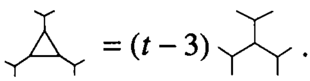
Let H be such a graph. If H has a cut-edge, then $F(H,t)=0$, and we are done, so suppose otherwise. Then the only loops of H are components of H. If H has no such components, then H is loopless and cubic and we are done, so suppose otherwise. Let r be the number of components of H which are loops, and let H' be the graph consisting of the non-loop components of H. Then $F(H,t)=(t-1)^r F(H',t)$ and H' is loopless and cubic. Since by Lemma 3.13 (ii), $k_0^*(H')=0$, the result follows by r applications of Lemma 3.15 (ii). □

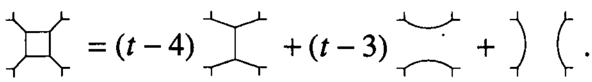
In what follows, we shall use symbols such as ,  and  as shorthand for the flow polynomials of graphs containing the given configuration.

Lemma 3.17.

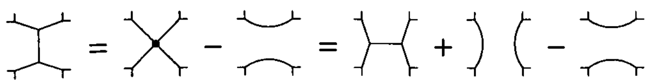
(i) 

(ii) 

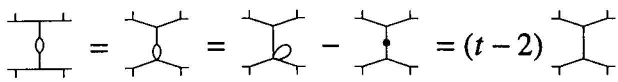
(iii) 

(iv) 

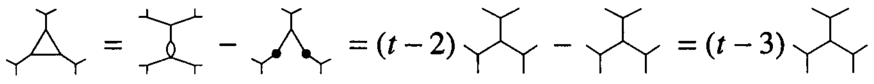
Proof. Using Theorem 3.1 (i), we have:

(i) 

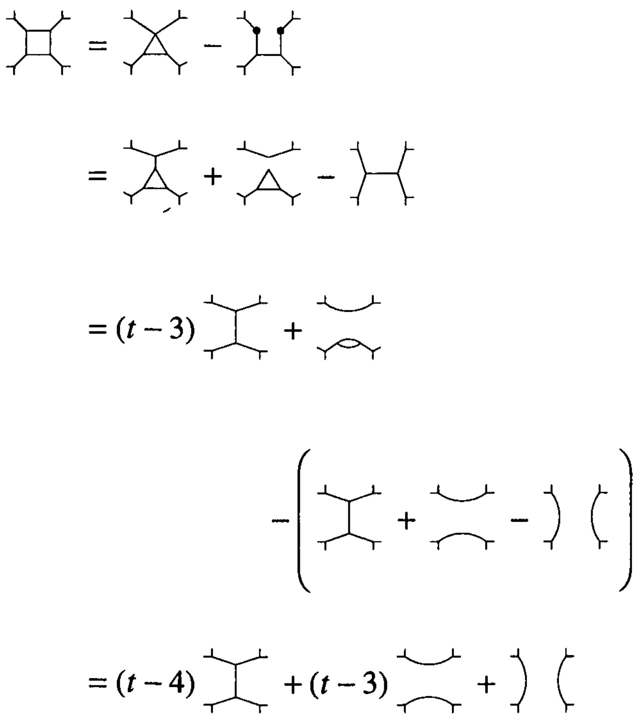
, as required.

(ii) 

, as required.

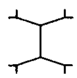
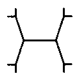
(iii)  by

part (ii), as required.

(iv) 

by parts (i) and (iii)

by part (ii), as required. □

Lemma 3.18. Let G be a cubic graph containing the configuration  and let G' be the graph obtained from G by replacing this configuration by .

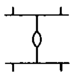
Then either G or G' has no 3-flow. In particular, either $k_3^*(G) = 0$ or $k_3^*(G') = 0$.

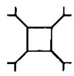
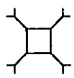
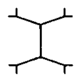
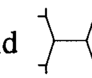
Proof. It is not difficult to see that a cubic graph has a 3-flow if and only if it has no cut-edges and is bipartite (since the two possible flows on each edge must alternate around a circuit). If G has no 3-flow, then we are done; if G has a 3-flow, then it is bipartite, and it is easy to see that G' in non-bipartite, and hence that it has no 3-flow. The rest of the result follows from Lemma 3.13 (iii). □

We are now ready to prove the main results of this section.

Theorem 3.19. Conjecture 3.14 (i) holds in general if it holds for simple cubic graphs of girth at least 5.

Proof. Let G be any graph, and suppose that the conjecture holds for simple cubic graphs of girth at least 5. By Corollary 3.16.1, we need only consider the case where G is cubic and loopless. We prove the result by induction on the number of circuits of length 4 or less in G.

By Lemma 3.13 (iii), $k_1^*(G) = k_2^*(G) = 0$. Suppose some component C of G is isomorphic to K_3^*. Then, since $F(K_3^*, t) = (t-1)(t-2)$, $F(G,t) = (t-1)(t-2)F(G-C,t)$ by Theorem 3.1 (ii), and the result follows by Lemma 3.15 (ii) and the inductive hypothesis. Otherwise, any digon in G must occur in a configuration . If G has such a digon, then the result follows by Lemma 3.15 (ii), Lemma 3.17 (ii) and the inductive hypothesis. Similarly, if G has a triangle, the result follows as above, using Lemma 3.17 (iii) instead of (ii).

Suppose G contains the configuration . Let G_1 and G_2 be the graphs obtained from G by replacing the configuration  by  and , respectively. If $k_3^*(G_1) = 0$, then the result follows as above by Lemma 3.15 (ii), Lemma 3.17 (iv) and the inductive hypothesis. Otherwise, $k_3^*(G_2) = 0$ by Lemma 3.18, and again the result follows as above, with the configurations in Lemma 3.17 (iv) rotated through a right-angle. □

Lemma 3.20. Let G be a graph and G_1 and G_2 be subgraphs of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \varnothing$ or K_1. Then if Conjecture 3.14 (i) holds for G_1 and G_2, it also holds for G.

Proof. By Theorem 3.1 (ii), $F(G,t) = F(G_1,t)F(G_2,t)$, and so

$$tF(G,t) = tF(G_1,t)F(G_2,t)$$

$$= \frac{1}{t} \sum_{i \geq 0} k_i^*(G_1)(t)_i \sum_{j \geq 0} k_j^*(G_2)(t)_j$$

$$= \sum_{i\geq 0} \sum_{0\leq j\leq i} k_i^*(G_1)k_j^*(G_2)(t-1)(t-2)\cdots(t-j+1)(t)_i$$

$$+ \sum_{i\geq 0} \sum_{j\geq i+1} k_i^*(G_1)k_j^*(G_2)(t-1)(t-2)\cdots(t-i+1)(t)_j,$$

and since, for each i, $k_i^*(G)$ is the coefficient of $(t)_i$ in $tF(G,t)$, it follows by repeated applications of Lemma 3.15 (i) that $k_i^*(G)$ is non-negative. The result follows. □

Theorem 3.21. Conjecture 3.14 (i) holds in general if it holds for simple, 2-connected cubic graphs of girth at least 5.

Proof. Suppose the conjecture holds for simple, 2-connected, cubic graphs of girth at least 5. Let G be a simple cubic graph of girth at least 5. Then the conjecture holds for every block of G (since it is simple, 2-connected and cubic, of girth at least 5), and so by repeated application of Lemma 3.20, it holds for G. The result now follows from Theorem 3.19. □

It is known that the conjecture that every graph without cut-edges has a 5-flow is true if it holds for snarks, that is, cyclically-4-edge-connected (and hence, simple, 2-connected and 3-edge-connected) cubic graphs, of girth at least 5, that have no 4-flow.

Since Conjecture 3.14 (i) holds for planar graphs by Lemma 3.13 (v), we have shown that it holds in general if it holds for non-planar, simple, 2-connected, cubic graphs of girth at least 5, and although these need not be snarks, it seems possible that this conjecture is as hard as the aforementioned 5-flow conjecture.

References

[1] B. Jackson, A zero-free interval for chromatic polynomials of graphs, *Combinatorics, Probability and Computing*, **2** (1993), 325–336.

[2] P. D. Seymour, Nowhere-zero 6-flows, *J. Combinatorial Theory (B)*, **30** (1981), 130–135.

[3] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Can. J. Math.*, **6** (1954), 80–91.

CHAPTER 4

Zeros of the Tutte Polynomial

4.0. Introduction and Definitions.

In this chapter, we allow graphs to have multiple edges and loops.

The *Tutte polynomial*, $T(G, x, y)$ of a graph G, introduced in [3], is defined by

$$T(G, x, y) := \sum_{X \subseteq E(G)} (x-1)^{c_G(X)-c}(y-1)^{\gamma_G(X)}.$$

It is a bivariate polynomial with non-negative coefficients.

A *plane near-triangulation* is a plane graph in which every inside face is a triangle. For example, the wheels W_n are plane near-triangulations. A *plane triangulation* is a plane graph in which every face is a triangle. A *separating polygon* in a plane graph is a circuit which has at least one vertex inside it and at least one vertex outside it.

When the points at which $T(G, x, y) = 0$ are plotted in the xy–plane, there often seem to be lines of zeros close to the hyperbolæ H_1, H_2 and $H_{\tau+1}$, where H_α is the hyperbola $(x-1)(y-1) = \alpha$ and $\tau = \frac{1}{2}(\sqrt{5}+1)$, the golden ratio. Figure 4.0.1 shows the zeros of the Tutte polynomial of some plane triangulations with eight or nine vertices, together with the hyperbolæ H_1, H_2 and $H_{\tau+1}$.

In Section 4.1 we present some basic results about the Tutte polynomial, and show that $T(G, x, y) \to 0$ as $n \to \infty$ on part of the hyperbola H_1. In Section 4.2 we present some partial results towards showing that a similar result holds for plane triangulations on H_2.

4.1. Basic Results.

In this section, we present some basic results about the Tutte Polynomial. The first two results are well known.

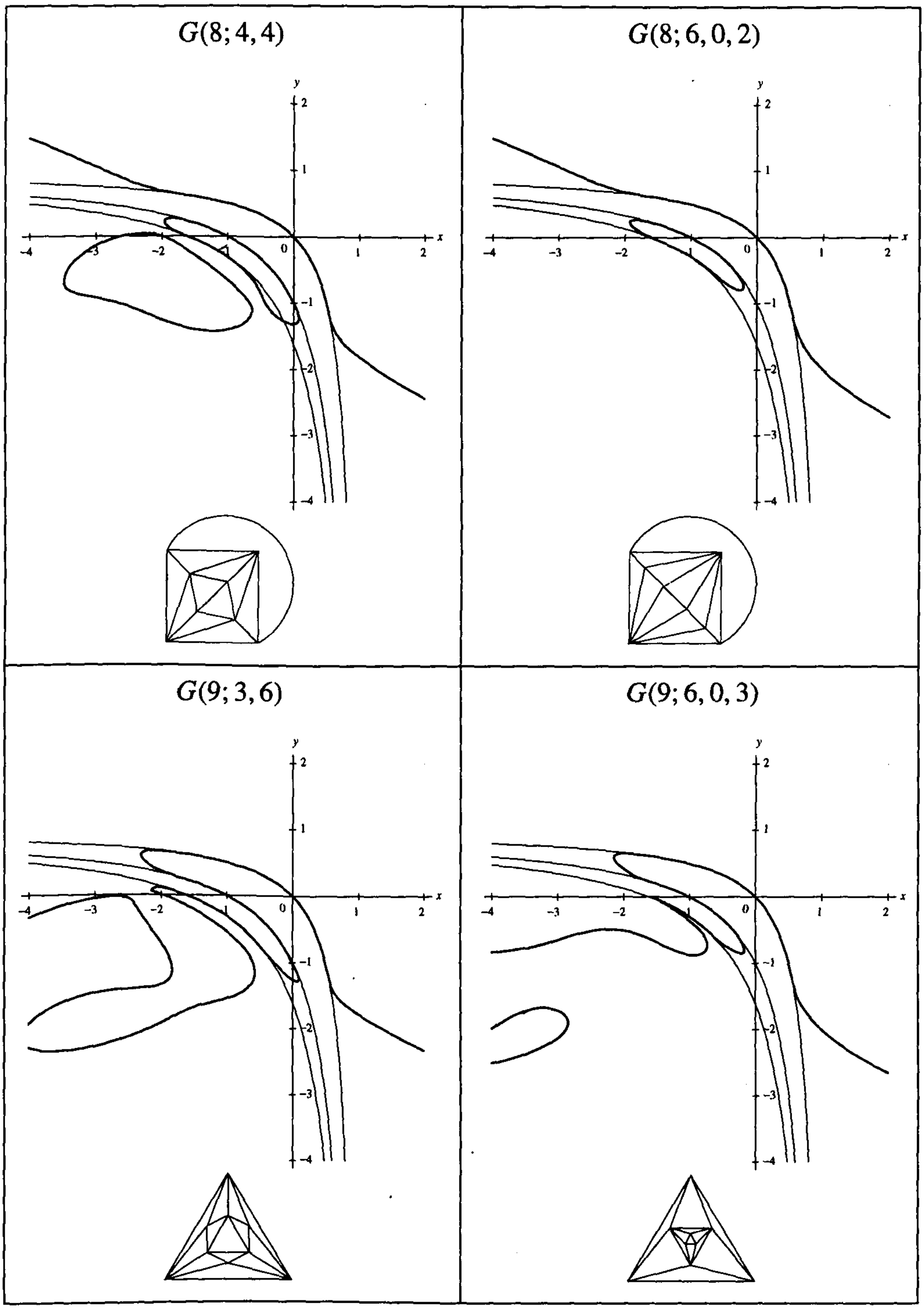


Figure 4.0.1

Theorem 4.1. Let G be a graph.

(i) If G has no edges, then $T(G, x, y) = 1$.

(ii) If e is an edge of G which is not a cut-edge or loop, then $T(G, x, y) = T(G - e, x, y) + T(G/e, x, y)$.

(iii) If e is a cut-edge of G then $T(G, x, y) = xT(G/e, x, y)$.

(iv) If e is a loop of G then $T(G, x, y) = yT(G - e, x, y)$.

(v) If there exist subgraphs G_1 and G_2 of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \varnothing$ or K_1, then $T(G, x, y) = T(G_1, x, y)T(G_2, x, y)$. □

Theorem 4.2. Let G be a graph.

(i) The chromatic polynomial of G, $P(G, t) = (-1)^{n-c} t^c T(G, 1 - t, 0)$.

(ii) The flow polynomial of G, $F(G, t) = (-1)^{m-n+c} T(G, 0, 1 - t)$.

(iii) If G is planar, then $T(G^*, x, y) = T(G, y, x)$. □

Corollary 4.2.1. Let G be a graph.

(i) If G is simple, then $q(G, t) = \dfrac{1}{s^b} T(G, s, 0)$, where $s = 1 - t$.

(ii) If every component of G is 3-edge-connected, then $q^*(G, t) = \dfrac{1}{s^b} T(G, 0, s)$.

(iii) If G is planar, then $F(G, t) = \dfrac{P(G^*, t)}{t}$ and $q^*(G, t) = q(G^*, t)$.

Proof. This follows from Theorem 4.2 and the definitions of $q(G, t)$ and $q^*(G, t)$ (see Chapters 1 and 3). □

The next result can be used to show that $T(G, x, y) \to 0$ as $n \to \infty$ on part of the hyperbola H_1. Note that $y = \frac{x}{x-1}$ on H_1.

Lemma 4.3. $T(G, x, \frac{x}{x-1}) = \dfrac{x^m}{(x - 1)^{m-n+c}}$.

Proof. $$T(G, x, \tfrac{x}{x-1}) = \sum_{X \subseteq E(G)} (x - 1)^{c_G(X) - c} \left(\frac{x}{x - 1} - 1 \right)^{\gamma_G(X)} \quad \text{by definition}$$

$$= \sum_{X\subseteq E(G)} (x-1)^{c_G(X)-c}\left(\frac{1}{x-1}\right)^{|X|-n+c_G(X)}$$

$$= (x-1)^{n-c} \sum_{X\subseteq E(G)} \left(\frac{1}{x-1}\right)^{|X|}$$

$$= (x-1)^{n-c} \sum_{i=0}^{m} \binom{m}{i}\left(\frac{1}{x-1}\right)^{i}$$

$$= (x-1)^{n-c}\left(\frac{1}{x-1}+1\right)^{m}$$

$$= (x-1)^{n-c}\left(\frac{x}{x-1}\right)^{m}$$

$$= \frac{x^m}{(x-1)^{m-n+c}},$$

as required. □

Corollary 4.3.1.

(i) For connected graphs, $T(G, x, \frac{x}{x-1}) \to 0$ as $n \to \infty$ for $-1 < x < \frac{1}{2}$.

(ii) For fixed $n - c$, $T(G, x, \frac{x}{x-1}) \to 0$ as $m \to \infty$ for $x < \frac{1}{2}$.

(iii) For fixed $\gamma(G)$, $T(G, x, \frac{x}{x-1}) \to 0$ as $m \to \infty$ for $-1 < x < 1$.

(iv) For plane triangulations, $T(G, x, \frac{x}{x-1}) \to 0$ as $n \to \infty$ for $-2.147\ldots < x < 0.569\ldots$, where $-2.147\ldots$ and $0.569\ldots$ are the real roots of $x^3 + x^2 - 2x + 1$ and $x^3 - x^2 + 2x - 1$, respectively.

Proof. $\left|T(G, x, \frac{x}{x-1})\right| = \left|\frac{x}{x-1}\right|^{m-n+c} |x|^{n-c}$ by Lemma 4.3, and $\left|\frac{x}{x-1}\right| < 1$ if and only if $x < \frac{1}{2}$. Parts (i) to (iii) follow.

For (iv), $m = 3(n-2)$ and $c = 1$, and so $\left|T(G, x, \frac{x}{x-1})\right| = \left|\frac{x^3}{(x-1)^2}\right|^{n} \frac{|x-1|^5}{|x|^6}$, which tends to zero as $n \to \infty$ if and only if $\left|\frac{x^3}{(x-1)^2}\right| < 1$, from which the result follows. □

The zeros of chromatic polynomials, particularly of plane triangulations, have been much studied. The following theorem summarises what is known, and the corollary interprets these results in terms of Tutte polynomials.

Theorem 4.4. Let G be a loopless graph.

(i) (Tutte [5], Jackson [2]) $P(G,t)$ is non-zero for $t<0$, $0<t<1$, $1<t\le \frac{32}{27}$.

(ii) (Birkhoff and Lewis [1]) If G is a plane near-triangulation then $P(G,t)$ is non-zero for $1<t<2$.

(iii) (Woodall [6]) If G is a plane triangulation then $P(G,t)$ is non-zero for $2<t<2.546\ldots$ where $2.546\ldots$ is the real zero of $t^3-9t^2+29t-32$ (a factor of the chromatic polynomial of the octahedron).

(iv) (Tutte [4]) If G is a plane triangulation then $|P(G,\tau+1)|\le \tau^{5-n}$ which tends to 0 as $n\to\infty$.

(v) (Corollary 3.2.2 and Theorem 3.11) If G has no cut-edges (but may have loops) then $F(G,t)$ is non-zero for $t<1$, $1<t\le \frac{32}{27}$. □

Corollary 4.4.1. Let G be a loopless graph.

(i) $T(G,x,0)$ is non-zero for $-\frac{5}{27}\le x<0$, $x>0$.

(ii) If G is a plane near-triangulation, then $T(G,x,0)$ is non-zero for $-1<x<0$.

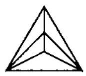
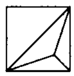
(iii) If G is a plane triangulation then $T(G,x,0)$ is non-zero for $-1.546\ldots<x<-1$, and $|T(G,-\tau,0)|\le\tau^{3-n}$.

(iv) If G has no cut-edges (but may have loops) then $T(G,0,y)$ is non-zero for $-\frac{5}{27}\le y<0$, $y>0$.

Proof. This follows from Theorem 4.4 and Theorem 4.2. □

Theorem 4.4 (iv) has been used as partial justification for the observation that chromatic polynomials of plane triangulations usually have a zero near to $1+\tau$. Since $(-\tau,0)$ lies on the hyperbola $H_{\tau+1}$, any similar result to Corollary 4.3.1 for $H_{\tau+1}$ is a generalisation of Tutte's result.

4.2. The Hyperbola H_2.

In this section, we present some partial results about the zeros of the Tutte polynomial, evaluated on the hyperbola H_2, that is, the hyperbola $xy = x + y + 1$, which can be parameterised as $x = 1 - t$, $y = \dfrac{t-2}{t}$. In what follows, we shall use symbols such as ,  and  as shorthand for the Tutte polynomials of graphs containing the given configurations, where the outer polygon of each symbol represents a separating polygon in the graph. We shall use '$=_H$' to denote equality between Tutte polynomials evaluated on the hyperbola H_2.

For $0 \le z < 1$, $c > 0$ and $I \subseteq \mathbb{R}$, let $\mathscr{S}[z, c, I]$ denote the class of graphs for which $|T(G, 1-t, \frac{t-2}{t})| \le cz^n$ for all $t \in I$. For $0 < z \le 1$, $c > 0$ and $I \subseteq \mathbb{R}$, let $\mathscr{S}(z, c, I)$ denote the class of graphs for which there exists a constant $d \in [0, z)$ such that $|T(G, 1-t, \frac{t-2}{t})| \le cd^n$ for all $t \in I$. Thus if $0 < z < 1$ then $\mathscr{S}(z, c, I) \subseteq \mathscr{S}[z, c, I]$.

We start with two conjectures.

Conjecture 4.5.

(i) For simple plane triangulations G, $|T(G, 1-t, \frac{t-2}{t})| \le f(n, t)$ for some function f such that $f(n, t) \to 0$ as $n \to \infty$ for $\frac{3}{2} \le t \le \frac{5}{2}$.

(ii) There exists $c > 0$ such that all simple plane triangulations belong to $\mathscr{S}(1, c, [\frac{3}{2}, \frac{5}{2}])$.

Note that (ii) implies (i). The conjectures cannot be extended to all connected graphs. For example, if C_n denotes the circuit on n vertices, then $T(C_n, x, y) = y + x + x^2 + x^3 + \cdots + x^{n-1}$ and so $\lim_{n\to\infty} T(C_n, x, y) = \dfrac{xy - x - y}{x - 1}$ for $-1 < x < 1$, which is non-zero ($\frac{1}{x-1}$) on H_2. However, it may be possible to extend the conjecture to all 3-connected non-bipartite graphs.

As a step to proving Conjecture 4.5 (ii), we shall present several relations between the Tutte polynomials of graphs containing certain configurations. We shall then optimise these relations to show that several classes of graphs satisfy Conjecture 4.5 (ii). We begin with a lemma which will be used to simplify this optimisation.

Lemma 4.6. Let G be a graph, and suppose there exist graphs $G_1, G_2, \ldots, G_r$ with $n_1 \leq n_2 \leq \cdots \leq n_r$ and functions $f_1, f_2, \ldots, f_r$ of t such that

$$T(G, 1-t, \tfrac{t-2}{t}) = \sum_{i=1}^{r} f_i(t) T(G_i, 1-t, \tfrac{t-2}{t}). \tag{4.2.1}$$

(i) If $G_1, G_2, \ldots, G_r \in \mathcal{S}[z, c, I]$ for some $z \in [0, 1)$, $c > 0$ and $I \subseteq \mathbb{R}$, and $\sum_{i=1}^{r} |f_i(t)| \leq z^{n-n_1}$ for all $t \in I$, then $G \in \mathcal{S}[z, c, I]$.

(ii) If $G_1, G_2, \ldots, G_r \in \mathcal{S}(z, c, I)$ for some $z \in (0, 1]$, $c > 0$ and $I \subseteq \mathbb{R}$, and $\sum_{i=1}^{r} |f_i(t)| < z^{n-n_1}$ for all $t \in I$, then $G \in \mathcal{S}(z, c, I)$.

Proof.

(i) For $t \in I$ and $i = 1, 2, \ldots, r$, $|T(G_i, 1-t, \tfrac{t-2}{t})| \leq cz^{n_i}$. Thus

$$|T(G, 1-t, \tfrac{t-2}{t})| \leq \sum_{i=1}^{r} |f_i(t)| cz^{n_i} \leq cz^{n_1} \sum_{i=1}^{r} |f_i(t)| \leq cz^{n_1} z^{n-n_1} = cz^n,$$

and so $G \in \mathcal{S}[z, c, I]$, as required.

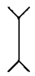
(ii) There exists a constant $d \in [0, z)$, arbitrarily close to z, such that $|T(G_i, 1-t, \tfrac{t-2}{t})| \leq cd^{n_i}$ for $t \in I$ and $i = 1, 2, \ldots, r$. Then

$$|T(G, 1-t, \tfrac{t-2}{t})| \leq \sum_{i=1}^{r} |f_i(t)| cd^{n_i} \leq cd^{n_1} \sum_{i=1}^{r} |f_i(t)| < cd^{n_1} z^{n-n_1},$$

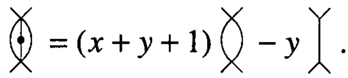
and so, since d can be chosen arbitrarily close to z, $|T(G, 1-t, \tfrac{t-2}{t})| \leq cd^n$ and $G \in \mathcal{S}(z, c, I)$, as required. □

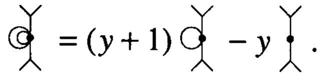
With this in mind, we now present several relations of the form (4.2.1).

Lemma 4.7.

(i)  $= (y+1)$  $- y$  providing  is not a cut-edge.

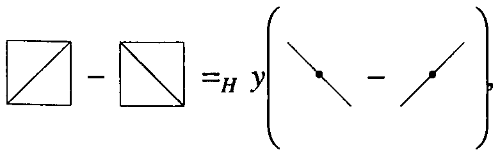
(ii)  $= (y+1)$  $- y$  .

(iii) 

(iv) 

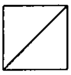
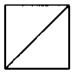
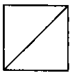
Proof. Suppose is not a cut-edge. Then $=$ $+$ $=$ $+ y$, by Theorem 4.1 (ii) and (iv), and $=$ $+$, by Theorem 4.1 (ii), and so $= (y+1)$ $- y$, as required. This proves (i). (ii) follows immediately; (iii) follows from (ii) since $= x$ $+$ by Theorem 4.1 (ii) and (iii); (iv) follows from Theorem 4.1 (iv). □

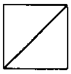
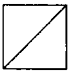
The next result is crucial to the other relations given in this section.

Lemma 4.8.  where represents the graph obtained from by removing two 'outside edges' incident with the 'diagonal edge' and then contracting the 'diagonal edge' (so that is a plane triangulation if and only if is).

Proof. Let $G = G_0$ be a graph with the configuration . We prove the result by induction on m. Let G_1, G_2 and G_3 be the graphs obtained from G by substituting , and respectively for .

If G has an edge e which is a loop, then by Theorem 4.1 (iv) $T(G_i, x, y) = yT(G_i - e, x, y)$ for each i, $G - e$ contains the configuration , and since, by the inductive hypothesis, the result holds for $G - e$, it also holds for G. Similarly, if G has a cut-edge e, then by Theorem 4.1 (iii) $T(G_i, x, y) = xT(G_i/e, x, y)$ for each i, and the result follows from the inductive hypothesis. Thus we may suppose that G has no cut-edges or loops.

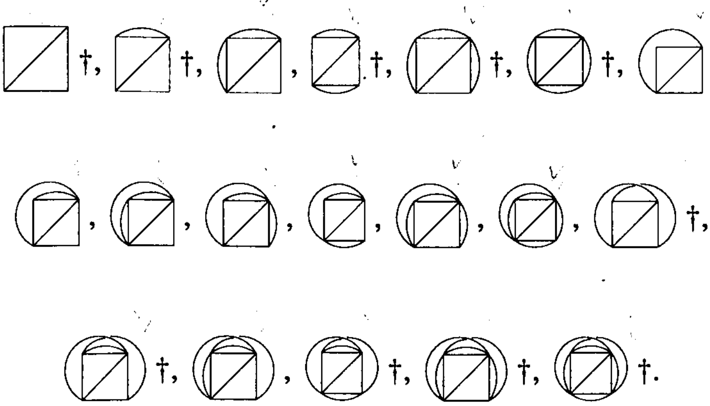
If G contains an edge e which does not join vertices in the configuration , then by Theorem 4.1 (ii) $T(G_i, x, y) = T(G_i - e, x, y) + T(G_i/e, x, y)$ for each i, $G - e$ and G/e both contain the configuration , and since, by the inductive hypothesis, the result holds for $G - e$ and G/e, it also holds for G. Thus we may suppose that G has exactly four vertices, and no cut-edges or loops (that is, $G =$ , possibly with extra edges).

Suppose G has an edge of multiplicity three or more, joining vertices on an 'outside edge' of , and let e_1 and e_2 be two of the edges. Then by Lemma 4.7 (ii), $T(G_i, x, y) = (y+1)T(G_i - e_1, x, y) - yT(G_i - \{e_1, e_2\}, x, y)$ for each i, and the result follows from the inductive hypothesis. Suppose now that the rest of G has an edge of multiplicity two or more, joining vertices at 'opposite corners' of , and let e_1 and e_2 be two of the edges. Then by Lemma 4.7 (i) and (iv),

$$T(G_i, x, y) = (y+1)T(G_i - e_{1,}\, x, y) - yT(G_i - \{e_1, e_2\}, x, y)$$

for each i, and again the result follows from the inductive hypothesis.

Thus we may suppose that either G or G_1 is one of the following graphs:



The result holds trivially for the marked graphs (since then $G \cong G_1$ and $G_2 \cong G_3$). The others are checked in Table 4.2.1. □

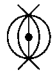
Lemma 4.9.

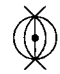
(i)  $=_H y(x+y)$  $- y^2 x$  .

(ii)  $=_H y(x+y)$  $- y^2(x+1)$  .

(iii)  $=_H y(2x+y-1)$  $- y^2(x^2+1)$  .

Proof. We prove (i) directly. By Theorem 4.1 (ii) and (v),

 $= (x^2+x+xy+y+y^2)$  $+$  (4.2.2)

since  $= x^2+x+xy+y+y^2$. Repeatedly applying Lemma 4.7 (i) to  ,

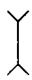
 $= (y+1)$  $- y$ 

$= (y+1)\Big((y+1)$  $- y$  $\Big) - y$ 

$= (y^2+y+1)$  $- xy(y+1)$  $- y(y+1)$  ,

by Theorem 4.1 (ii) and (iii). Substituting this into (4.2.2),

 $= (y^2+y+1)$  $- xy(y+1)$  $+ x(x+y+1)$ 

$= (y^2+y+1)$  $- xy(y+1)$  $+ x\Big($  $+ y$  $\Big)$ by Lemma 4.7 (iii)

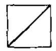
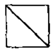
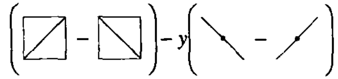
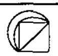
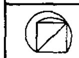
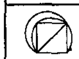
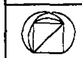
G	 $T(G,x,y)$	 $T(G_1,x,y)$	 $T(G_2,x,y)$	 $T(G_3,x,y)$	
	$x^3+2x^2+2x^2y+x+4xy$ $+2xy^2+xy^3+y+3y^2$ $+2y^3+y^4$	$x^3+2x^2+2x^2y+x+4xy$ $+3xy^2+y+3y^2+3y^3+y^4$	$x^2+xy+xy^2$	x^2y	0
	$x^3+2x^2+x^2y+x+2xy$ $+2xy^2+y+y^2+y^3$	$x^3+3x^2+2x+4xy$ $+2y+3y^2+y^3$	x^2y	x^2+x+y	$-(xy-x-y)(xy-x-y-1)$
	$x^3+2x^2+2x^2y+x+3xy$ $+3xy^2+xy^3+y+2y^2$ $+2y^3+y^4$	$x^3+3x^2+x^2y+2x+5xy$ $+2xy^2+2y+4y^2$ $+3y^3+y^4$	x^2y+xy^2	$x^2+x+xy+y+y^2$	$-(xy-x-y)(xy-x-y-1)$
	$x^3+2x^2+3x^2y+x+4xy$ $+4xy^2+2xy^3+xy^4+y$ $+3y^2+3y^3+2y^4+y^5$	$x^3+3x^2+2x^2y+2x+6xy$ $+4xy^2+xy^3+2y+5y^2$ $+5y^3+3y^4+y^5$	$x^2y+xy^2+xy^3$	$x^2+x+2xy+y+2y^2+y^3$	$-(xy-x-y)(xy-x-y-1)$
	$x^3+2x^2+3x^2y+x+4xy$ $+5xy^2+2xy^3+y+3y^2$ $+4y^3+3y^4+y^5$	$x^3+3x^2+2x^2y+2x+6xy$ $+4xy^2+xy^3+2y+5y^2$ $+5y^3+3y^4+y^5$	$x^2y+2xy^2+y^3$	$x^2+x+xy+xy^2$ $+y+y^2+y^3$	$-(xy-x-y)(xy-x-y-1)$
	$x^3+2x^2+3x^2y+x+4xy$ $+5xy^2+2xy^3+y+3y^2$ $+4y^3+3y^4+y^5$	$x^3+3x^2+2x^2y+2x+6xy$ $+5xy^2+2y+5y^2+6y^3$ $+3y^4+y^5$	$x^2y+2xy^2+y^3$	$x^2+x+2xy+y+2y^2+y^3$	$-(xy-x-y)(xy-x-y-1)$
	$x^3+2x^2+4x^2y+x+5xy$ $+7xy^2+3xy^3+xy^4+y+4y^2$ $+6y^3+5y^4+3y^5+y^6$	$x^3+3x^2+3x^2y+2x+7xy$ $+7xy^2+2xy^3+2y+6y^2$ $+8y^3+6y^4+3y^5+y^6$	$x^2y+2xy^2+xy^3+y^3+y^4$	$x^2+x+2xy+xy^2+y$ $+2y^2+2y^3+y^4$	$-(xy-x-y)(xy-x-y-1)$
	$x^3+2x^2+5x^2y+x+6xy$ $+10xy^2+4xy^3+2xy^4+y+5y^2$ $+9y^3+8y^4+6y^5+3y^6+y^7$	$x^3+3x^2+4x^2y+2x+8xy$ $+10xy^2+4xy^3+2y+7y^2+11y^3$ $+10y^4+6y^5+3y^6+y^7$	$x^2y+2xy^2+2xy^3+y^3$ $+2y^4+y^5$	$x^2+x+2xy+2xy^2+y$ $+2y^2+3y^3+2y^4+y^5$	$-(xy-x-y)(xy-x-y-1)$
	$x^3+3x^2+3x^2y+2x+7xy$ $+6xy^2+2xy^3+xy^4+2y+6y^2$ $+7y^3+5y^4+3y^5+y^6$	$x^3+3x^2+3x^2y+2x+7xy$ $+6xy^2+3xy^3+2y+6y^2$ $+7y^3+6y^4+3y^5+y^6$	$x^2y+xy+xy^2+xy^3$ $+y^2+y^3+y^4$	$x^2y+xy+2xy^2+y^2$ $+2y^3+y^4$	0

Table 4.2.1

$= (y^2 + x + y + 1)$  $- xy^2$ 

$=_H \; y(x+y)$  $- xy^2$  ,

as required.

For (ii) and (iii) (and for completeness, (i)), by a similar argument to that given in the proof of Lemma 4.8, it is only necessary to check two cases for each relation (see Table 4.2.2). □

Corollary 4.9.1. Let G be a graph containing the configuration  .

(i) If there exists a constant $c > 0$ such that  ,  $\in \mathcal{G}(1, c, (1, 3.205\ldots))$, where $3.205\ldots$ is the real root of $t^3 - 5t^2 + 7t - 4$, then $G \in \mathcal{G}(1, c, (1, 3.205\ldots))$.

(ii) If there exists a constant $c > 0$ such that  ,  $\in \mathcal{G}[0.577\ldots, c, [\frac{3}{2}, \frac{5}{2}]]$ then $G \in \mathcal{G}[0.577\ldots, c, [\frac{3}{2}, \frac{5}{2}]]$, where $0.577\ldots$ is $\frac{1}{\sqrt{3}}$

Proof. Let $x = 1 - t$, $y = \frac{t-2}{t}$ and $f(t) = |y(x+y)| + |xy^2|$. Then

$$f(t) = \frac{|t-2||t^2 - 2t + 2| + |t-1||t-2|^2}{|t|^2}$$

$$= \frac{|t-2|(t^2 - 2t + 2) + |t-1|(t-2)^2}{t^2}$$

since $t^2 - 2t + 2 = (t-1)^2 + 1 > 0$. Thus

$$f(t) = \begin{cases} \dfrac{2t^3 - 9t^2 + 14t - 8}{t^2} & \text{if } t > 2, \\ \dfrac{2-t}{t} & \text{if } 1 < t \le 2, \\ -\dfrac{(2t^3 - 9t^2 + 14t - 8)}{t^2} & \text{if } t \le 1. \end{cases} \qquad (4.2.3)$$

G / G'	$T(G,x,y)$ / $T(G',x,y)$	$f(x,y)$	$g(x,y)$	$T(G,x,y)-f(x,y)$ $-\,g(x,y)$ / $T(G',x,y)-f(x,y)$ $-\,g(x,y)$
	$x^3+2x^2+2x^2y+x+3xy+3xy^2+xy^3+y+2y^2+2y^3+y^4$	$y(x+y)$	$-y^2x$	$-(xy-x-y-1)(x^2+x+xy+y+y^2)$
	$x^3+2x^2+2x^2y+x^2y^2+x+3xy+4xy^2+2xy^3+xy^4+y+2y^2+3y^3+2y^4+y^5$			$-(xy-x-y-1)(x^2+x+xy+xy^2+y+y^2+y^3)$
	$x^3+3x^2+x^2y+2x+5xy+2xy^2+2y+4y^2+3y^3+y^4$	$y(x+y)$	$-y^2(x+1)$	$-(xy-x-y-1)(x^2+2x+xy+2y+2y^2)$
	$x^3+3x^2+x^2y+x^2y^2+2x+5xy+3xy^2+2xy^3+2y+4y^2+4y^3+3y^4+y^5$			$-(xy-x-y-1)(x^2+xy^2+2x+xy+2y+2y^2+2y^3)$
	$x^3+2x^2+x^2y+x^2y^2+x+2xy+2xy^2+2xy^3+y+y^2+y^3+y^4$	$y(2x+y-1)$	$-y^2(x^2+1)$	$-(xy-x-y-1)(x^2-x^2y+x+xy+y+y^2)$
	$x^3+2x^2+x^2y+x^2y^2+x^2y^3+x+2xy+2xy^2+2xy^3+2xy^4+y+y^2+y^3+y^4+y^5$			$-(xy-x-y-1)(x^2-x^2y+x+xy+y+y^2+y^3)$

Table 4.2.2

(i) We solve $f(t) < 1$, that is, $f(t) - 1 < 0$.

If $t > 2$ then $f(t) - 1 = \dfrac{2(t^3 - 5t^2 + 7t - 4)}{t^2}$, and so $f(t) < 1$ if and only if $t < 3.205\ldots$.

If $1 < t \le 2$ then $f(t) - 1 = \dfrac{2(1-t)}{t} < 0$.

If $t \le 1$ then $f(t) - 1 = -\dfrac{2(t-1)(t^2 - 3t + 4)}{t^2} > 0$

Thus $f(t) < 1$ if and only if $1 < t < 3.205\ldots$. The result now follows by Lemma 4.6 (ii) and Lemma 4.9 (i).

(ii) We now maximise $f(t)$ on the interval $[\frac{3}{2}, \frac{5}{2}]$.

If $2 < t \le \frac{5}{2}$ then by (4.2.3)

$$f'(t) = \frac{2(t^3 - 7t + 8)}{t^3} > \frac{4t^2 - 14t + 16}{t^3} = \frac{(2t - \frac{7}{2})^2 + \frac{15}{4}}{t^3} > 0.$$

If $\frac{3}{2} \le t \le 2$ then by (4.2.3) $f'(t) = -\dfrac{2}{t^2} < 0$.

Thus $f(t)$ has no turning points in $[\frac{3}{2}, \frac{5}{2}]$ and so $f(t) \le \max(f(\frac{3}{2}), f(\frac{5}{2})) = \frac{1}{3}$ for $\frac{3}{2} \le t \le \frac{5}{2}$. The result now follows by Lemma 4.6 (i) and Lemma 4.9 (i). □

As a special case of this corollary, we have the following theorem.

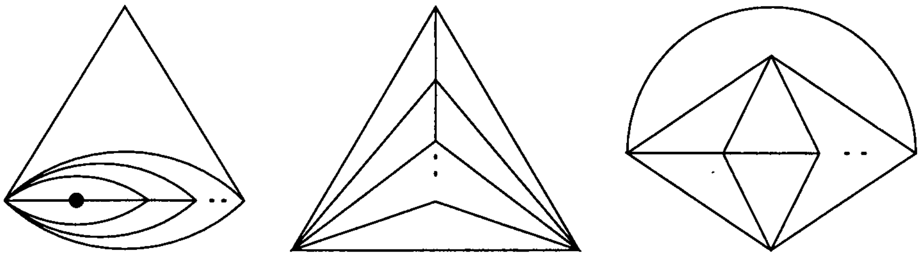
Theorem 4.10. There exist constants $c_1, c_2 > 0$ such that the family of graphs D_n (see Figure 4.2.1) is a subset of $\mathcal{G}(1, c_1, (1, 3.205\ldots))$, where $3.205\ldots$ is the real root of $t^3 - 5t^2 + 7t - 4$, and of $\mathcal{G}[0.577\ldots, c_2, [\frac{3}{2}, \frac{5}{2}]]$, where $0.577\ldots$ is $\frac{1}{\sqrt{3}}$. In particular, the D_n satisfy Conjecture 4.5 (ii).

Proof. Let $G_1 = D_3$ and $G_2 = D_4$ (that is, the members of D_n with 3 and 4 vertices, respectively). Let

$$c_1 = \sup\,\{|T(G_i, 1-t, \tfrac{t-2}{t})| : i = 1,2 \text{ and } t \in (1, 3.205\ldots)\},$$

$$c_2 = \frac{\sup\,\{|T(G_i, 1-t, \frac{t-2}{t})| : i = 1,2 \text{ and } t \in [\frac{3}{2}, \frac{5}{2}]\}}{(0.577\ldots)^4}.$$

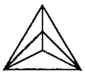
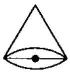
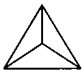
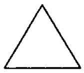
The result follows from the definitions of $\mathcal{G}(z, c, I)$ and $\mathcal{G}[z, c, I]$ and Corollary 4.9.1. □

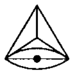
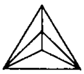
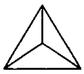
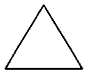


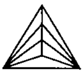
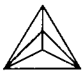
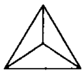
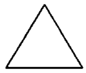
D_n S_n B_n

Figure 4.2.1

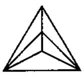
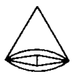
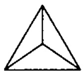
Lemma 4.11.

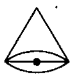
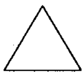
(i)  $=_H y$  $+ y(x+y)$  $- y^2(2x+y)$  .

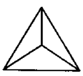
(ii)  $=_H x$  $- y(x^2+1)$  $+ x^2y^2$  .

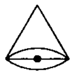
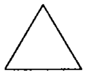
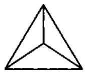
(iii)  $=_H y(2x+y)$  $- x^2y^3$  $+ x^2y^3$  .

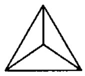
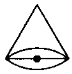
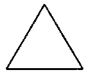
Proof.

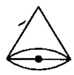
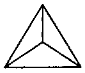
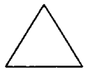
 $=_H$  $+ y\Big($  $-$  $\Big)$ by Lemma 4.8

$=_H \Big(y(x+y)$  $- y^2(x+1)$  $\Big) - y\Big(y(x+y-1)$  $\Big)$

$+ y$  since  $\doteq_H xy(x+y-1)$ and by Lemma 4.9 (ii)

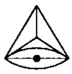
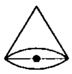
$=_H y(x+y)$  $-y^2(x+1)$  $+y$ 

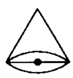
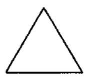
$+y(x+y-1)\Big($  $-$  $-y$  $\Big)$ by Lemma 4.8

$=_H y$  $+y(x+y)$  $-y^2(2x+y)$  ,

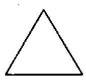
as required. This proves (i).

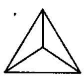
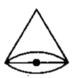
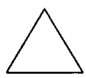
We now prove (ii).

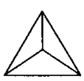
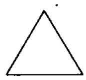
 $=_H$  $+y\Big($  $-$  $\Big)$ by Lemma 4.8

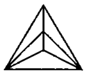
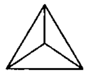
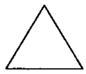
$=_H \Big(y(2x+y-1)$  $-y^2(x^2+1)$  $\Big)+y$ 

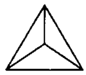
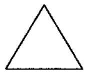
$-y^2(x+y)$  by Lemma 4.9 (iii) and Theorem 4.1

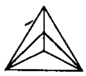
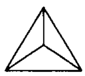
$=_H y(2x+y)$  $-y^2(x^2+1)$ 

$+y(x+y)\Big($  $-$  $-y$  $\Big)$ by Lemma 4.8

$=_H xy$  $+y(x+y)$  $-xy^2(x+y)$ 

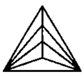
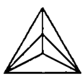
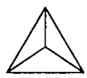
$$=_H x\left(\right.$$  $$- y(x+y)$$  $$+ y^2(2x+y)$$  $$\left. \right)$$

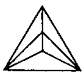
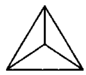
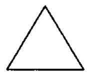
$$+ y(x+y)$$  $$- xy^2(x+y)$$  by part (i)

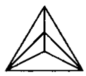
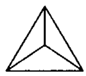
$$=_H x$$  $$- y(x^2+1)$$  $$+ x^2y^2$$  ,

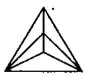
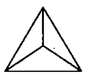
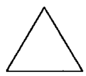
as required.

Finally, we prove (iii).

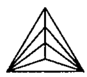
 $$=_H y$$  $$+ y(x+y)$$  $$- y^2(2x+y)$$  by part (i)

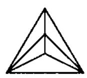
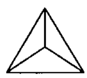
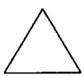
$$=_H y\left(x \right.$$  $$- y(x^2+1)$$  $$+ x^2y^2$$  $$\left. \right)$$

$$+ y(x+y)$$  $$- y^2(2x+y)$$  by part (ii)

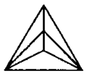
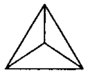
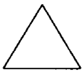
$$=_H y(2x+y)$$  $$- x^2y^3$$  $$+ x^2y^3$$  ,

as required. □

Corollary 4.11.1. Let G be a graph containing the configuration  .

(i) If there exists a constant $c > 0$ such that  ,  ,  $\in \mathcal{S}(1, c, (1, 2.747\ldots))$, where $2.747\ldots$ is a root of

$t^5 - 7t^4 + 21t^3 - 34t^2 + 26t - 8$, then $G \in \mathscr{CP}(1, c, (1, 2.747\ldots))$.

(ii) If there exists a constant $c > 0$ such that , ,  $\in \mathscr{CP}[0.841\ldots, c, [\frac{3}{2}, \frac{5}{2}]]$ then $G \in \mathscr{CP}[0.841\ldots, c, [\frac{3}{2}, \frac{5}{2}]]$, where $0.841\ldots$ is the cube root of 0.596.

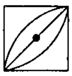
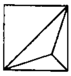
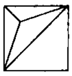
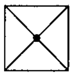
Proof. This can be proved from Lemma 4.11 (iii) by a similar method to the proof of Corollary 4.9.1. □

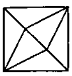
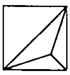
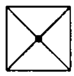
As a special case, we have the following theorem.

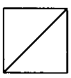
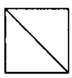
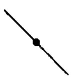
Theorem 4.12. There exist constants $c_1, c_2 > 0$ such that the family of 'stack polyhedra' S_n (see Figure 4.2.1) is a subset of $\mathscr{CP}(1, c_1, (1, 2.747\ldots))$, where $2.747\ldots$ is a root of $t^5 - 7t^4 + 21t^3 - 34t^2 + 26t - 8$, and of $\mathscr{CP}[0.841\ldots, c_2, [\frac{3}{2}, \frac{5}{2}]]$, where $0.841\ldots$ is the cube root of 0.596. In particular, the 'stack polyhedra' satisfy Conjecture 4.5 (ii).

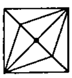
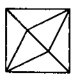
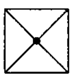
Proof. This is proved in an exactly similar way to Theorem 4.10. □

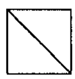
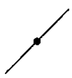
Lemma 4.13.

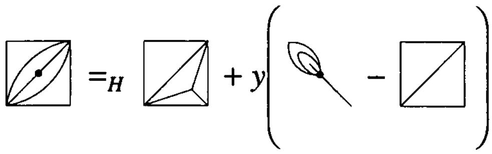
(i)  $=_H$  $+$  $-$  $+ y^2$  .

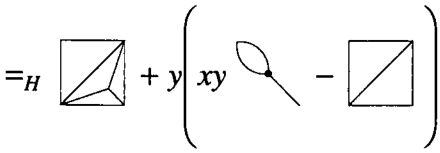
(ii)  $=_H y$ $\Big($  $+$  $\Big)$ $+ y(x+y)$ 

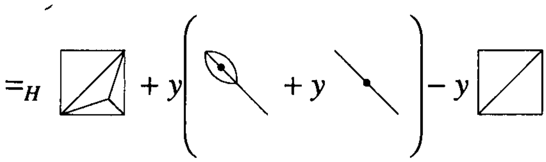
$- xy^2(y+1)$  $- y^2(x+y-1)$  $+ y^3(x+1)$  .

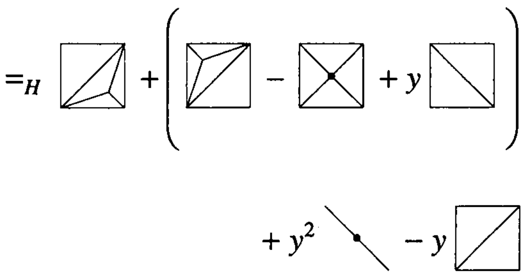
(iii)  $=_H y(2x+y+1)$  $- y^2(x^2+4x+1+2y)$ 

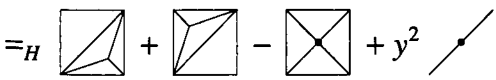
$+ x^2y^3(y+1)$  $- x^2y^4$  .

Proof. 

 by Theorem 4.1 (iii) and (iv)

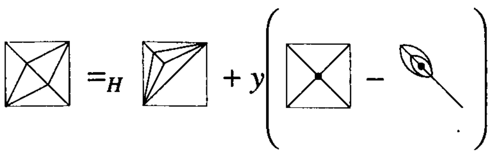
 by Lemma 4.7 (iii)

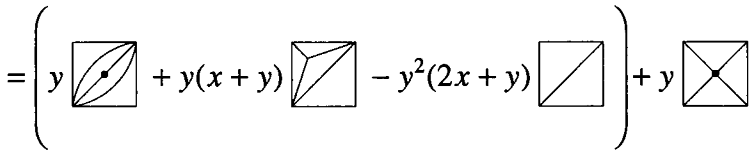
 by Lemma 4.8

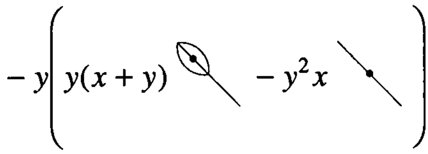
 by Lemma 4.8

as required. This proves (i).

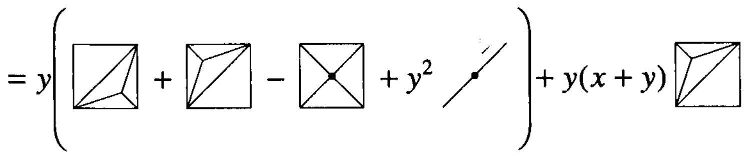
We now prove (ii).

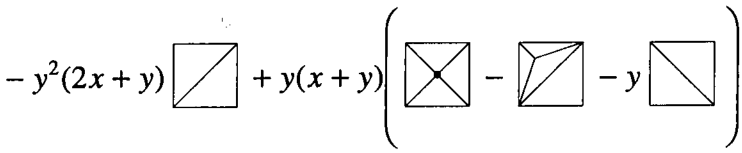
 by Lemma 4.8

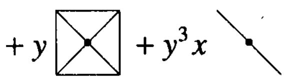




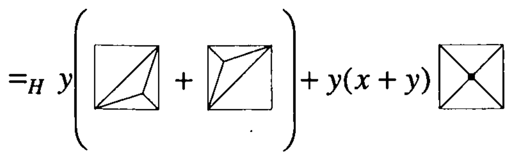
by Lemma 4.9 (i) and Lemma 4.11 (i)

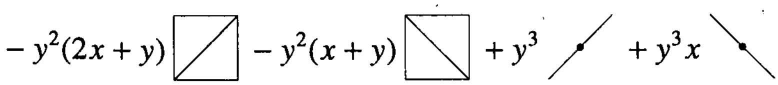


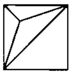
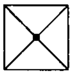
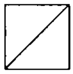


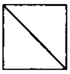


by part (i) and Lemma 4.8



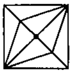
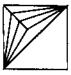
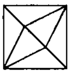


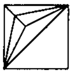
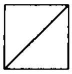
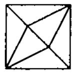
$=_H y\Big($  $+$  $\Big) + y(x+y)$  $- xy^2(y+1)$ 

$- y^2(x+y-1)$  $+ y^3(x+1)$  by Lemma 4.8

as required.

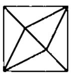
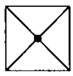
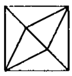
Finally, we prove (iii).

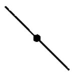
 $=_H$  $+ y\Big($  $-$  $\Big)$ by Lemma 4.8

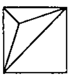
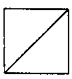
$=_H \Big(y(2x+y)$  $- x^2y^3$  $+ x^2y^3$  $\Big) + y$ 

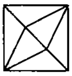
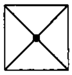
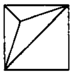
$- y\Big(y(x+y)$  $- y^2x$  $\Big)$

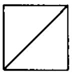
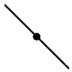
by Lemma 4.11 (iii) and Lemma 4.9 (i)

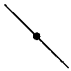
$=_H y(2x+y)\Big($  $+ y$  $- y$  $\Big) + y$ 

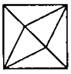
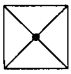
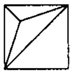
$- y\Big(y^2(x+y)^2$  $- y^3x(x+y)$  $- y^2x$  $\Big)$

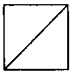
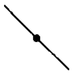
$- x^2y^3$  $+ x^2y^3$  by Lemma 4.8 and Lemma 4.9 (i)

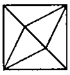
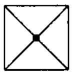
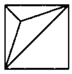
$=_H\ y(2x+y+1)$  $-\,y^2(2x+y)$  $-\,x^2y^3$ 

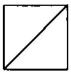
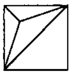
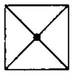
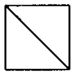
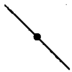
$+\,x^2y^3$  $+\,y^2(2x+y)\Big($ $y(x+y)$  $-\,y^2x$  $\Big)$

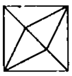
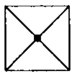
$-\,y^3((x+y)^2-x)$  $+\,y^4x(x+y)$  by Lemma 4.9 (i)

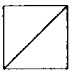
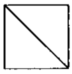
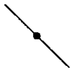
$=_H\ y(2x+y+1)$  $-\,y^2(2x+y)$  $-\,x^2y^3$ 

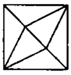
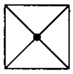
$+\,x^2y^3$  $+\,y^3(x^2+xy+x)$  $-\,x^2y^4$ 

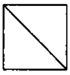
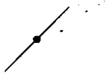
$=_H\ y(2x+y+1)$  $-\,y^2(2x+y)$  $-\,x^2y^3$ 

$+\,x^2y^3$  $+\,x^2y^3\Big($  $-$  $+\,y$  $\Big)-x^2y^4$ 

$=_H\ y(2x+y+1)$  $-\,y^2(2x+y+x^2y)$ 

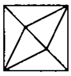
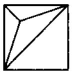
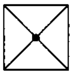
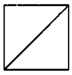
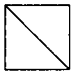
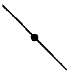
$+\,x^2y^3$  $+\,x^2y^4$  $-\,x^2y^4$ 

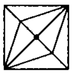
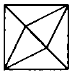
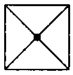
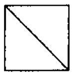
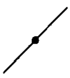
$=_H$ $y(2x+y+1)$  $-\, y^2(x^2+4x+1+2y)$ 

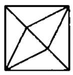
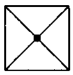
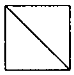
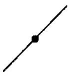
$+\, x^2y^3(y+1)$  $-\, x^2y^4$  by Lemma 4.8

as required. □

Corollary 4.13.1.

(i) Let G be a graph containing the configuration . If there exists a constant $c > 0$ such that , , , , ,  $\in \mathscr{A}(1, c, (1.557\ldots, 2.591\ldots))$, where $1.557\ldots$ and $2.591\ldots$ are roots of $3t^4 - 24t^3 + 58t^2 - 64t + 32$ and $5t^4 - 28t^3 + 62t^2 - 72t + 32$, respectively, then $G \in \mathscr{A}(1, c, (1.557\ldots, 2.591\ldots))$.

(ii) Now let G be a graph containing the configuration . If there exists a constant $c > 0$ such that , , ,  $\in \mathscr{A}(1, c, (1, 2.812\ldots))$, where $2.812\ldots$ is a root of $t^5 - 7t^4 + 24t^3 - 48t^2 + 44t - 16$, then $G \in \mathscr{A}(1, c, (1, 2.812\ldots))$.

If there exists a constant $c > 0$ such that , , ,  $\in \mathscr{A}[0.832\ldots, c, [\frac{3}{2}, \frac{5}{2}]]$ then $G \in \mathscr{A}[0.832\ldots, c, [\frac{3}{2}, \frac{5}{2}]]$, where $0.832\ldots$ is the fourth root of 0.4792.

Proof. This can be proved from Lemma 4.13 (ii) and (iii) by a similar method to the proof of Corollary 4.9.1. □

As special cases, we have the following theorem, part (i) of which is a slightly stronger version of Theorem 4.12.

Theorem 4.14.

(i) There exist constants $c_1, c_2 > 0$ such that the family of 'stack polyhedra' S_n is a subset of $\mathcal{G}(1, c_1, (1, 2.812\ldots))$, where $2.812\ldots$ is a root of $t^5 - 7t^4 + 24t^3 - 48t^2 + 44t - 16$, and of $\mathcal{G}[0.832\ldots, c_2, [\frac{3}{2}, \frac{5}{2}]]$, where $0.832\ldots$ is the fourth root of 0.4792.

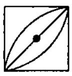
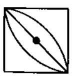
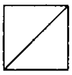
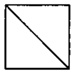
(ii) There exist constants $c_1, c_2 > 0$ such that the family of bipyramids B_n (see Figure 4.2.1) is a subset of $\mathcal{G}(1, c_1, (1, 2.812\ldots))$ and of $\mathcal{G}[0.832\ldots, c_2, [\frac{3}{2}, \frac{5}{2}]]$.

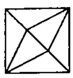
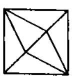
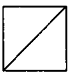
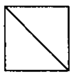
In particular, the bipyramids satisfy Conjecture 4.5 (ii).

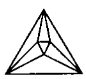
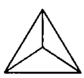
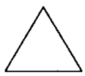
Proof. This is proved in a similar way to Theorem 4.10. □

To conclude this chapter, we present an example of a relation of the sort that may be required to prove Conjecture 4.5 (ii) for general (simple) plane triangulations. Such a proof will probably rely on showing that every plane triangulation contains a configuration which can be 'reduced' in this way.

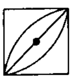
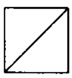
Lemma 4.15.

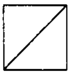
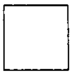
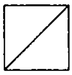
(i)  $-$  $=_H y(2x+y)\left(\right.$  $-$  $\left.\right).$

(ii)  $-$  $=_H y^2(2x+y)\left(\right.$  $-$  $\left.\right).$

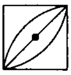
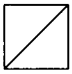
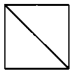
(iii)  $=_H xy^3(x+y)$  $- y^3(x+y)(3x+y)$  .

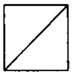
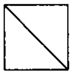
Proof.

 $= (x+y+1)$  $- y$  by Lemma 4.7 (iii)

$= (x+y+1)\Big($ $(y+1)$  $- y$  $\Big) - y$  by Lemma 4.7 (i)

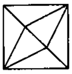
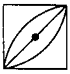
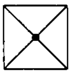
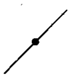
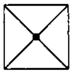
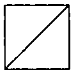
and so

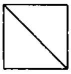
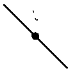
 $-$  $= ((x+y+1)(y+1) - y)\Big($  $-$  $\Big)$

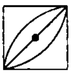
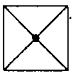
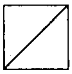
$=_H\ y(2x+y)\Big($  $-$  $\Big)$

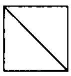
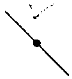
as required. This proves (i).

We now prove (ii).

 $=_H\ y\Big($  $+$  $- y^2$  $\Big) + y(x+y)$  $- xy^2(y+1)$ 

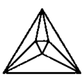
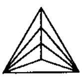
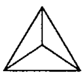
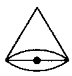
$- y^2(x+y-1)$  $+ y^3(x+1)$  by Lemma 4.13 (i) and (ii)

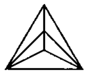
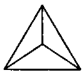
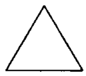
$=_H\ y$  $+ xy^2$  $- y^2(2x+y)$ 

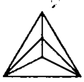
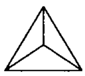
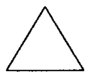
$- y^2(x+y)$  $+ xy^3$  by Lemma 4.8

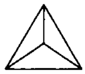
from which the result follows by part (i) and Lemma 4.8.

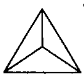
Finally, we prove (iii).

 $=_H$  $+ y^2(2x+y)\Big($  $-$  $\Big)$ by part (ii)

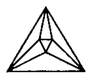
$=_H \Bigg($ $y(2x+y)$  $-\,x^2y^3$  $+\,x^2y^3$  $\Bigg)$

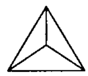
$-\,y(2x+y)\Bigg($  $-\,y(x+y)$  $+\,y^2(2x+y)$  $\Bigg)$

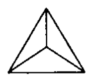
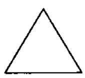
$+\,y^2(2x+y)$  by Lemma 4.11 (iii) and (i)

$=_H\ xy^3(x+y)$  $-\,y^3(x+y)(3x+y)$ 

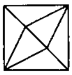
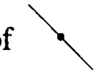
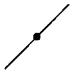
as required. □

Corollary 4.15.1. Let G be a graph containing the configuration  .

(i) If there exists a constant $c > 0$ such that  ,  $\in \mathscr{Z}(1, c, (1, 3.288\ldots))$, where $3.288\ldots$ is a root of $4t^7 - 37t^6 + 145t^5 - 322t^4 + 432t^3 - 352t^2 + 160t - 32$, then $G \in \mathscr{Z}(1, c, (1, 3.288\ldots))$.

(ii) If there exists a constant $c > 0$ such that  ,  $\in \mathscr{Z}[0.416\ldots, c, [\frac{3}{2}, \frac{5}{2}]]$ then $G \in \mathscr{Z}[0.416\ldots, c, [\frac{3}{2}, \frac{5}{2}]]$, where $0.416\ldots$ is the cube root of $\frac{35}{486}$.

Proof. This can be proved from Lemma 4.15 (iii) by a similar method to the proof of Corollary 4.9.1. □

Corollary 4.15.1 shows that any plane triangulation containing a triangle with vertex-degrees $4, 4, 4$ can be 'reduced'. It may be possible to show that every simple plane triangulation contains a triangle which can be 'reduced' in this way. It is known that every simple plane triangulation without vertices of degree four has a triangle whose vertex-degrees sum to at most 29, and that every simple plane triangulation with minimum degree 5 has a triangle whose vertex-degrees sum to at most 17 (both of these are 'best possible'). Thus, providing we can deal with degree four vertices, the number of triangles it is necessary to 'reduce' is finite. The problem with degree four vertices is that triangulations like the bipyramids B_n and the 'stack polyhedra' S_n (see Figure 4.2.1) have triangles whose vertex-degrees may sum to an arbitrarily high number ($n+6$ in the case of the B_n). However, Corollary 4.13.1 (i), together with Lemma 4.8 and Lemma 4.13 (ii), offers some hope of dealing with this problem (note that if G is a simple planar graph containing the configuration , then one of  and  must be simple).

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