Theoretical studies of the two-dimensional interacting electron system in high magnetic field

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Abstract

This is a mathematical study of certain aspects of the interacting electron system in very high perpendicular magnetic field.

We analyse restrictions imposed upon the density correlation functions of this system and propose a set of sum rules which they must obey.

We study the possibility of building a bosonisation scheme for the projected density operators in the lowest Landau level. We suggest a second order bosonisation, along with an approximation scheme, which may be useful for carrying out calculations in the lowest Landau level.

We analyse the possible ground states of the system. We suggest a set of variational wavefunctions which can have lower energy than the Laughlin state for sufficiently soft interaction potentials.

We study the collective excitations of the system, paying particular attention to its symmetries. We suggest a set of variational excited states and discuss their applicability to finite as well as infinite systems.

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CHAPTER 1

Introduction

This is a mathematical study of certain aspects of the two dimensional interacting electron system with a high magnetic field applied perpendicularly to the plane of the electrons. We begin with a review of the model system under study and then go on to examine different aspects of the system within this model. This model is used to represent the 2des formed at a semiconductor heterojunction. The work is motivated by a current interest, within condensed matter physics, in the physical phenomena which can be observed in such systems (especially the integer [1–11] and fractional [12–44] quantum Hall effects). We are particularly interested in the system at certain filling factors, where the experimental evidence suggests that the electrons form an incompressible quantum fluid. An excellent review of these phenomena can be found in [45].

The work in this thesis has resulted in the following publications.

1. "An Investigation of the Collective Excitations of the FQHE using a New Boson Formalism", M.Brownlie and K.A. Benedict, Proceedings of 12th International Conference: High Magnetic Fields in the Physics of Semiconductors, vol II, p92 (1996) (Singapore: World Scientific).

2. "Three-point density correlation functions in the fractional quantum Hall regime", M Brownlie and Keith A Benedict, J. Phys A, **33**, 4283 (2000).

CHAPTER 2

The Model System

2.1 The 2D Plane

Obviously, the true physical system (*i.e.* the samples studied by the experiments investigating the phenomena that we are interested in) is extremely complicated and is represented by an enormous variety of parameters. Many of these variables have little or no qualitative effect on the nature of the physical processes under study and, as such, they serve mainly to obstruct the theoretical investigation of the system by throwing irrelevant complexity in the way of the idealized simplicity which will hopefully allow us an understanding of the physics of our system in terms of the clearest and most basic possible model.

We will adopt a very specific (and simplified) model with which to represent and study the electronic systems under consideration. We will assume that we have N electrons (where N may be allowed to tend to infinity) which are all confined to exist on a two dimensional plane (the x, y plane). We will also apply to this plane a completely uniform magnetic field in the negative z direction with magnitude B.

The phenomena which we wish to study occur at very high magnetic fields. In our theoretical model we will always assume that the magnetic field, B, can be considered sufficiently large to validate any approximation we may wish to make which depends on a large value of B (*i.e.* mathematically we will let $B \to \infty$). We expect that (for the effects that we are studying) having large but finite fields will cause only quantitative differences from the infinite field case and so using the infinite field

approximation should still allow us to study the basic physics going on in the system. This infinite magnetic field acts to freeze out the spin degrees of freedom of the electron system (so obviously we are not intending to study spin effects or effects due to the presence of spin excitations). This is inasmuch as that the infinite magnetic field forces the spins of all the electrons to be aligned in the same, most energetically favourable, direction. This is true even in the excited states as it is energetically infinitely unfavourable to be aligned against the B field when the magnetic field is infinite. This means that we can neglect the spin part of the electron Hamiltonian as it would merely contribute an additive constant to the energy of the system. As a consequence of the infinite magnetic field approximation we will ignore the spin of the electrons in what follows below (though we will of course continue to bear in mind the fermionic nature of the electron particles).

2.2   The Single Particle Case

2.2.1   The Non-Interacting Hamiltonian

We begin by considering the electrons in the system to be non-interacting *i.e.* we ignore the term in the Hamiltonian which corresponds to the potential energy of the electrons due to the presence and mutual repulsion of the other electrons in the system (via the electrostatic Coulomb force). The non-interacting many particle case can be readily extrapolated from the single particle case (by bearing the Pauli exclusion principle in mind and so constructing Slater determinant many particle eigenstates of the many particle Hamiltonian, which is simply the single electron Hamiltonian summed over the number of particles in the system). Therefore we will look at the single electron Hamiltonian for this system as a precursor to eventually studying the interacting many particle Hamiltonian for the system.

For our model system and ignoring the spin of the electrons (as described above) the Hamiltonian of a single electron confined to exist in the 2D x, y plane with an applied uniform magnetic field in the negative z direction and magnitude B is given

by

$$H = \frac{1}{2m} \left[\mathbf{p} - e\mathbf{A}(\mathbf{r}) \right]^2 + U(\mathbf{r}) \tag{2.1}$$

where $\mathbf{r} = (x, y)$ is the two dimensional position coordinate multiplication operator for the 2D plane and $\mathbf{p} = -i\hbar(\partial_x, \partial_y)$ is the canonical 2D momentum operator (*i.e.* which corresponds both classically and quantum mechanically to the observable which is conjugate to the position coordinate $\mathbf{r}$). The mass and charge of the electron are denoted by m and e, and $\mathbf{A}$ is the magnetic vector potential - where the magnetic field is given by the curl of $\mathbf{A}$. The $U(\mathbf{r})$ term is the external potential experienced by the electron which we will assume to be a constant.

This Hamiltonian is very simple and the corresponding Schrödinger equation can be solved exactly. The natural basis eigenstates for the Hamiltonian depend on the boundary conditions (external potential) and the vector potential used. We will look at two different gauges below for the vector potential - the Landau gauge and the symmetric gauge.

2.2.2 The Landau Gauge

There are many different vector potentials which will describe the same magnetic field. In order to find explicit solutions of the Schrödinger equation (*i.e.* eigenstates of the Hamiltonian) we need to adopt a particular gauge for the magnetic vector potential. One of the simplest to use is the Landau gauge where we have

$$\mathbf{A}(x, y, o) = (0, -Bx, 0). \tag{2.2}$$

This gauge allows us to find exact analytic solutions of the Hamiltonian eigenvalue-eigenfunction problem. In the Landau gauge the y-momentum, p_y, (which is the conjugate momentum operator to the y coordinate) and the Hamiltonian commute with one another (as the Hamiltonian in this gauge has no dependence on the y coordinate).This means that these two operators have a complete set of simultaneous eigenstates. These eigenstates are given by the (unnormalised) functions

$$\phi_{n,k}(x, y) = e^{iky}.\varphi_n\left(\frac{x - l_c^2 k}{l_c}\right) \tag{2.3}$$

where φ_n is the nth eigenfunction of the simple harmonic oscillator and

$$l_c = \sqrt{\frac{\hbar}{eB}} \tag{2.4}$$

is the magnetic (cyclotron) length. The corresponding eigenvalues are given by

$$E_{n,k} = \left(n + \frac{1}{2}\right)\hbar\omega_c \tag{2.5}$$

where ω_c is the cyclotron (angular) frequency and is given by

$$\omega_c = \frac{eB}{m}. \tag{2.6}$$

We see that the energy of these eigenstates, $E_{n,k}$, depends only on n so each energy level of the system contains many different eigenstates, as given by the k parameter (which has no effect on the energy). This means that the energy level occupied depends only on the harmonic oscillator part of the eigenfunction with each harmonic level corresponding to a different (highly degenerate) energy level in the system. We also note that, although the eigenfunctions are variables separable, the x part of the eigenfunctions are still coupled to the y part of the function via the k parameter. What we have is something where the x part of the function looks like a harmonic oscillator eigenfunction (centered on $x = l_c^2 k$) which is then multiplied by the y dependent part, e^{iky}, which causes the function to be rotated around the complex plane as a function of the y coordinate with a spatial frequency given by k. Hence the average x position of the electron determines the frequency in the y direction. Also interesting is that within one energy level (say the lowest, $n = 0$, level) there is an uncertainty between the x and y positions inasmuch as requiring the x coordinate of an electron to be contained within some narrow limits around a particular value (*i.e.* put the electron in one of the above eigenstates) causes the y coordinate to become completely undetermined.

2.2.3 The Symmetric Gauge

Another form of the magnetic vector potential which is useful in studying the Hamiltonian of this system is the symmetric gauge (and indeed for the investigations which

we carry out this is by far the most important). In the symmetric gauge the vector potential is given by

$$\mathbf{A}(x,y) = \left(\frac{1}{2}By, -\frac{1}{2}Bx, 0 \right) \tag{2.7}$$

This gauge, as before, allows us to find exact analytic solutions of the Hamiltonian eigenvalue/eigenfunction problem. This time, though, it is the (z component of the) angular momentum, $L_z = x.p_y - y.p_x$, which commutes with the Hamiltonian. This means that now it is the angular momentum which shares a complete set of eigenstates with the Hamiltonian. These eigenstates are given by the (unnormalised) functions - (using the polar coordinates r and θ) -

$$\phi_{m,n}(r,\theta) = e^{im\theta}.R_{m,n}\left(\frac{r}{l_c} \right) \tag{2.8}$$

where

$$R_{m,n}(u) = e^{-\frac{u^2}{4}}.u^{|m|}.L^{|m|}_{\alpha(m,n)}\left(\frac{u^2}{2} \right) \tag{2.9}$$

with

$$\alpha(m,n) = n + \frac{(m-|m|)}{2} \qquad \text{for} \quad -n \leq m \tag{2.10}$$

and $L^s_r(x)$ is the associated Laguerre polynomial.

The energy eigenvalues for these states are, of course, the same energies as given above for the case of the Landau gauge (an observable such as energy does not depend on which vector potential is used as long as it gives the correct magnetic field). These states are all circularly symmetric and they define a central point as the origin (this is due to choosing a particular angular momentum with a particular centre).

These eigenstates can be rewritten in the following way

$$\phi_{m,n}(x,y) = e^{\frac{1}{4}(x^2+y^2)}.\left(\partial_x + i\partial_y\right)^m.\left(\partial_x - i\partial_y\right)^n.e^{-\frac{1}{2}(x^2+y^2)} \tag{2.11}$$

where units have been adopted so that length is now measured in units of the magnetic length, so that now

$$l_c = \sqrt{\frac{\hbar}{eB}} = 1.$$

Remembering that the energy levels are given only by the n parameter *i.e.*

$$E_{m,n} = \left(n + \frac{1}{2} \right)\hbar\omega_c \tag{2.12}$$

we can see that the states in the lowest energy level (*i.e.* $n = 0$) take on a particularly simple form:

$$\phi_{m,0}(x,y) = e^{\frac{1}{4}(x^2+y^2)} \cdot (\partial_x + i\partial_y)^m \cdot e^{-\frac{1}{2}(x^2+y^2)} \tag{2.13}$$

which turns out to give

$$\phi_{m,0}(x,y) = (x+iy)^m \cdot e^{-\frac{1}{4}(x^2+y^2)}. \tag{2.14}$$

We immediately see that if we write the eigenfunctions in the lowest energy level in terms of a complex coordinate, $z = x + iy$, we obtain

$$\phi_{m,0}(z) = z^m \cdot e^{-\frac{1}{4}|z|^2}. \tag{2.15}$$

These states are the simultaneous eigenstates of the Hamiltonian and the angular momentum (z component) in the lowest energy level. This simplified form for these states will form the basis for much of the discussion in this work.

2.2.4 The Landau Levels

The energy levels of the single particle Hamiltonian, as we recall from above, are split into a series of discretely spaced levels each of which is enormously degenerate. The separation between the levels is given by $\hbar\omega_c$ and so is proportional to the magnetic field, B. These energy levels are known as Landau levels and we refer to the nth landau level as being the space spanned by all the eigenfunctions with energy

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c$$

and we often refer to the $n = 0$ eigenspace as simply the lowest Landau level.

Now, to calculate the degeneracy of the Landau levels we need to work out how many mutually orthogonal states there are in one Landau space. For example, use the Landau gauge of section 2.2.2 and assume the electron system to be confined to the rectangular area $0 \leq x \leq W$, $0 \leq y \leq L$ so that the total area is $\Omega = W.L$. Now, we know that as the energy depends solely on the x function in the eigenstates,

$$\phi_{n,k}(x,y) = e^{iky} \cdot \varphi_n\left(\frac{x - l_c^2 k}{l_c}\right),$$

then according to the formula for E_n, any eigenfunctions with different x functions are in different Landau levels. Therefore the degeneracy must come from the y functions. We can construct an orthonormal set of y functions quite easily *i.e.* the states

$$\epsilon_m = \frac{1}{\sqrt{L}}e^{ik_m y} \tag{2.16}$$

with

$$k_m = m\frac{2\pi}{L} \qquad m \in Z$$

obey the condition

$$\int_0^L dy.\epsilon_i^*.\epsilon_j = \delta_{i,j} \tag{2.17}$$

and we know from the theory of Fourier series that this set accounts for all possible y functions (over the interval $0 \leq y \leq L$). Now, we have an orthogonal set of degenerate eigenfunctions for the nth Landau level *i.e.*

$$\phi_{m,n}(x,y) = \epsilon_m(y).\varphi_n\left(\frac{x - l_c^2 k_m}{l_c}\right) \tag{2.18}$$

This result can alternatively be derived by simply applying periodic boundary conditions to the eigenstates in the y direction.

Because of the way that k_m determines where the x function is centered not all of these functions will fit into the interval $0 \leq x \leq W$. In order to consider only those functions that do fit into the rectangle we need m to satisfy the condition (not worrying about the boundary in too much detail)

$$0 \leq l_c^2 k_m \leq W \tag{2.19}$$

which gives us

$$0 \leq m \leq \frac{W.L}{2\pi l_c^2} = \frac{\Omega}{2\pi l_c^2} \tag{2.20}$$

showing that there are $\Omega/2\pi l_c^2$ orthogonal states in one Landau level.

Using the Landau gauge in this way is not, strictly speaking, rigorous as the eigenfunctions used (the simultaneous eigenstates of the Hamiltonian and y momentum) depend, on their validity, on ignoring any confining barrier in the Hamiltonian and by applying the boundary conditions to the edge of the sample in an inconsistent

manner. It may be thought that this shouldn't matter too much as the set of eigen-functions clearly form a complete set, but this is only over all Landau levels. If, for example, we were to use the same method but this time switch x and y directions we would find that the lowest Landau function spaces would be different for these two (supposedly) identical systems - most especially the filled Landau level states of the two cases would clearly be different from one another (although both cases would give the same degeneracy for the Landau level). This is caused by the inconsistent treatment given to the y boundary conditions compared to the x boundary conditions. We would expect, though, that a bulk effect such as the degeneracy of the level should not depend on the exact nature of the boundary conditions in the limit as the area, Ω, becomes very large. This is borne out by the fact that working in the symmetric gauge, as described above, gives the same answer for large areas. [The symmetric method uses a circularly symmetric geometry and treats the boundary conditions more consistently. It is also the case that adding a circularly symmetric barrier to the Hamiltonian does not affect the commutativity of the Hamiltonian with the angular momentum whereas adding a barrier to the Hamiltonian in the first method would affect the commutativity of the Hamiltonian with the y momentum.]

Now, we see that, as electrons are fermions and hence subject to the Pauli exclusion principle, only $1/2\pi l_c^2$ electrons per unit area can ever exist in any one Landau level at the same time. A useful unitless parameter to characterise the electron density of the system and how it compares with the magnetic field is the filling factor, ν, which is defined by

$$\nu = 2\pi l_c^2 n_e \tag{2.21}$$

where n_e is the electron density (number of electrons per unit area). So the filling factor is the number of electrons present divided by the number of states in one Landau level and is therefore a measure of how many Landau levels are filled at zero temperature by the electrons (strictly speaking for the non-interacting electrons).

2.3   The Many Particle Case

2.3.1   The Interacting Hamiltonian

The case of interacting particles is a much more complicated problem than the simple non-interacting case. The independent particle case is so simple because in the non-interacting system the total Hamiltonian is simply the summation of all the single particle Hamiltonians over the number of particles present. Then, as each of these single particle Hamiltonians is independent from all the others (*i.e.* they all commute with one another by virtue of their depending on different coordinates and momenta) then eigenstates of the total Hamiltonian can easily be constructed by forming the outer products of the single particle eigenstates, using one single particle state for each different particle coordinate. This gives a picture where the many body eigenstates can still be easily represented in terms of single particle states. This is, of course, not the case for the Hamiltonian of a system of interacting particles, which will contain terms that depend on two (or maybe more) different particle coordinates and so the interacting Hamiltonian cannot be deconstructed into mutually commuting single particle terms (at least not in terms of the original coordinates). The phenomena that we wish to study, in this system, all depend very strongly on the electron-electron interactions (*i.e.* those relevant to the fractional quantum Hall effect) and so we are forced to investigate the more complicated situation of a many particle interacting Hamiltonian.

We will, as before, consider all the electrons to be confined to exist in the 2D x,y plane with a magnetic field,B, applied uniformly in the negative z direction over the whole plane and, again, we will neglect to consider the spin of the electrons. We will also take no account of the barrier potential and we will assume that the neutralising positively charged background averages over the plane, so contributing only a constant term to the energy which can be ignored. This gives us the many body Hamiltonian, describing our system, in the form

$$H = K + V = \sum_{i=1}^{N} K_i + \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^{N} v(\mathbf{r}_i - \mathbf{r}_j) \qquad (2.22)$$

with

$$K_i = \frac{1}{2m} \left[\mathbf{p}_i - e\mathbf{A}(\mathbf{r}) \right]^2 \tag{2.23}$$

where K_i is the single particle kinetic energy term for the ith electron. The $v(\mathbf{r}_i - \mathbf{r}_j)$ term is the mutually repulsive potential energy acting between the ith and j^{th} electrons at positions $\mathbf{r}_i$ and $\mathbf{r}_j$ respectively, and is usually represented as the pure Coulomb potential or some slightly softened version which takes into account sample effects.

The eigenstructure of the K_i operators is well known (as was discussed earlier) and so the kinetic part of the Hamiltonian, K, where

$$K = \sum_{i=1}^{N} K_i = \sum_{i=1}^{N} \frac{1}{2m} \left[\mathbf{p}_i - e\mathbf{A}(\mathbf{r}) \right]^2 \tag{2.24}$$

also has an easily understood eigenstructure (as it represents a non-interacting Hamiltonian term). This non-interacting term gives a discrete set of eigenvalues (whose eigenspaces are in general massively degenerate) separated from one another by $\hbar\omega_c = \hbar\frac{eB}{m}$. We can think of this structure as the set of Landau levels discussed earlier where every Landau level contains $\Omega/2\pi l_c^2$ single particle states each of which may or may not be filled by an electron. It remains, now, to examine the effect of the interaction potential term, V.

2.3.2 The Density Operators

The potential energy of the system (the interaction term) is due to the Coulombic repulsion acting between all the particles which depends, for its magnitude, on the separation between all the electrons - *i.e.* it depends on the density as a function of position and the mutual proximity of the electrons to one another. Therefore we might find it useful (in fact we shall do) to express the potential term in terms of the electron density.

In order to do this, we use the many particle density operator, $\rho(\mathbf{r})$, and its Fourier transform, $\rho_{\mathbf{k}}$, where

$$\rho(\mathbf{r}) = \sum_{i=1}^{N} \delta^2(\mathbf{r} - \mathbf{r}_i) \tag{2.25}$$

in the position representation, and its Fourier transform is given by

$$\rho_{\mathbf{k}} = \int d^2\mathbf{r}.\rho(\mathbf{r})e^{-i\mathbf{k}.\mathbf{r}}. \tag{2.26}$$

so that

$$\rho_{\mathbf{k}} = \sum_{j=1}^{N} e^{-i\mathbf{k}.\mathbf{r}_j} \tag{2.27}$$

where $\mathbf{r}_i$ is the multiplication operator for the position of the ith electron. These density operators have various important properties, as follows below.

The expectation of the number density at a point $\mathbf{r}$ in the state $|\psi\rangle$ is $\langle\psi|\rho(\mathbf{r})|\psi\rangle$ and similarly $\langle\psi|\rho_{\mathbf{k}}|\psi\rangle$ gives the Fourier components of the density.

$$\langle\psi|\rho_{\mathbf{k}}|\psi\rangle = \int d^2r.e^{-i\mathbf{k}.\mathbf{r}}\langle\psi|\rho(\mathbf{r})|\psi\rangle. \tag{2.28}$$

Also, it is worth noting that

$$\rho_{\mathbf{0}} = N \tag{2.29}$$

which is simply the dc component of the particle density.

Due to the position-momentum commutation relations $\rho_{\mathbf{k}}^{\dagger}$ is a shift operator for the momentum of the many particle system, *i.e.* if ψ is an eigenstate of the momentum operator:

$$\mathbf{P}\psi = \mathbf{p}\psi$$

then

$$\mathbf{P}(\rho_{\mathbf{k}}^{\dagger}\psi) = (\mathbf{p} + \hbar\mathbf{k})\rho_{\mathbf{k}}^{\dagger}\psi \tag{2.30}$$

so that $\rho_{\mathbf{k}}^{\dagger}$ raises the momentum of the system by $\hbar\mathbf{k}$.

The density operators obey the adjoint relation

$$\rho_{\mathbf{k}}^{\dagger} = \rho_{-\mathbf{k}} \tag{2.31}$$

which is the same as

$$\langle\psi|\rho_{\mathbf{k}}|\psi\rangle = \langle\psi|\rho_{-\mathbf{k}}^{\dagger}|\psi\rangle = \langle\psi|\rho_{-\mathbf{k}}|\psi\rangle^* \tag{2.32}$$

and, of course, this is simply the condition that the particle density, $\langle\psi|\rho(\mathbf{r})|\psi\rangle$, is real. We also readily notice that the density operators commute with one another as they only contain multiplying position operators, *i.e.*

$$[\rho_{\mathbf{k}}, \rho_{\mathbf{q}}] = 0. \tag{2.33}$$

2.3.3 The Interaction Potential

We now wish to rewrite the potential interaction term in the Hamiltonian as a function of the density operators. Consider the interaction term,

$$V = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^{N} v(\mathbf{r}_i - \mathbf{r}_j), \tag{2.34}$$

which, in order to complete the summation, we rewrite as

$$V = \frac{1}{2} \sum_{i,j}^{N} v(\mathbf{r}_i - \mathbf{r}_j) - \frac{1}{2} N.v(\mathbf{0}) \tag{2.35}$$

The second term in this equation is infinite but this is made up for by a correspondingly infinite term which is now included in the summation, so that the total energy is still finite. Now, we consider that all the electrons are contained within an area of size Ω (which can always be taken to the infinite limit when studying intrinsic quantities) and therefore there is some maximum distance which any two electrons can have from one another (*i.e.* for a circular area no two electrons can have a separation greater than the diameter of the circle). We are therefore able to replace $v(\mathbf{r})$ with any other function which has the same value as $v(\mathbf{r})$ over all the region of $\mathbf{r}$ corresponding to the electrons both being confined to the area. We therefore extend the interparticle potential term periodically and expand it as a Fourier series in a wavevector $\mathbf{q}$.

$$v(\mathbf{r}) = \frac{1}{\Omega} \sum_{\mathbf{q}} \tilde{v}_{\mathbf{q}} e^{i\mathbf{q}.\mathbf{r}_i} \tag{2.36}$$

where $\tilde{v}$ is the Fourier transform of the inter-electron potential

$$\tilde{v}_{\mathbf{q}} = \int_{\Omega} d^2 \mathbf{r}.v(\mathbf{r}) e^{-i\mathbf{q}.\mathbf{r}} \tag{2.37}$$

and the Fourier series expansion for $v(\mathbf{r})$ will become the inverse Fourier transform of $\tilde{v}_{\mathbf{q}}$ in the limit as Ω becomes infinitely large.

We now substitute the Fourier expansion for $v(\mathbf{r})$ into the expression for the energy to obtain

$$V = \frac{1}{2\Omega} \sum_{i,j}^{N} \sum_{\mathbf{q}} \tilde{v}_{\mathbf{q}} e^{i\mathbf{q}.\mathbf{r}_i - i\mathbf{q}.\mathbf{r}_j} - \frac{1}{2} N.v(\mathbf{0}) \tag{2.38}$$

which can be rearranged to give

$$V = \frac{1}{2\Omega} \sum_{\mathbf{q}} \tilde{v}_{\mathbf{q}} \left[\sum_i^N e^{i\mathbf{q}\cdot\mathbf{r}_i} \cdot \sum_j^N e^{-i\mathbf{q}\cdot\mathbf{r}_j} \right] - \frac{1}{2} N.v(\mathbf{0}) \tag{2.39}$$

and so allowing us to write the interaction potential energy in terms of the density operators

$$V = \frac{1}{2\Omega} \sum_{\mathbf{q}} \tilde{v}_{\mathbf{q}} \rho_{\mathbf{q}}^{\dagger} \rho_{\mathbf{q}} - \frac{1}{2} N.v(\mathbf{0}). \tag{2.40}$$

It is also interesting to express the potential energy term in a slightly different manner, using the fact that $\rho_{\mathbf{0}} = N$ (as mentioned earlier) we can write

$$V = \frac{N}{2\Omega} \sum_{\substack{\mathbf{q} \\ \mathbf{q} \neq \mathbf{0}}} \tilde{v}_{\mathbf{q}} \frac{1}{N} \rho_{\mathbf{q}}^{\dagger} \rho_{\mathbf{q}} + \frac{N}{2\Omega}.N.\tilde{v}_{\mathbf{0}} - \frac{N}{2\Omega} \sum_{\mathbf{q}} \tilde{v}_{\mathbf{q}} \tag{2.41}$$

where we have also used the definition of the Fourier series from above

$$v(\mathbf{0}) = \frac{1}{\Omega} \sum_{\mathbf{q}} \tilde{v}_{\mathbf{q}} \tag{2.42}$$

and $\tilde{v}_{\mathbf{0}}$ is the dc component of $v(\mathbf{r})$, *i.e.*

$$\tilde{v}_{\mathbf{0}} = \int d^2\mathbf{r}.v(\mathbf{r}). \tag{2.43}$$

Now, we define the static structure factor, $S(\mathbf{k})$ of a state, ψ, to be

$$S(\mathbf{k}) = \begin{cases} 0 & \text{if } \mathbf{k} = \mathbf{0} \\ \frac{1}{N} \langle \psi | \rho_{\mathbf{q}}^{\dagger} \rho_{\mathbf{q}} | \psi \rangle & \text{if } \mathbf{k} \neq \mathbf{0} \end{cases}. \tag{2.44}$$

The static structure factor is an intrinsic quantity of a system in the thermodynamic limit. The $\mathbf{k} = \mathbf{0}$ condition is to keep the structure factor free of the discontinuity which would otherwise occur due to the fact that $\rho_{\mathbf{0}}^{\dagger} = N$ (for states which have infinite area).

We can now rewrite the energy per particle of a particular electronic state, ψ, as

$$\frac{1}{N} \langle \psi | V | \psi \rangle = \frac{1}{2\Omega} \sum_{\mathbf{q}} \tilde{v}_{\mathbf{q}} \left[S(\mathbf{q}) - 1 \right] + \frac{1}{2} \frac{N}{\Omega}.\tilde{v}_{\mathbf{0}} \tag{2.45}$$

which, allowing Ω to tend to infinity and the summations to become integrals, gives

$$\frac{1}{N}\langle\psi|V|\psi\rangle = \frac{1}{2}\cdot\frac{1}{4\pi^2}\int d^2\mathbf{q}.\tilde{v}_\mathbf{q}\left[S(\mathbf{q})-1\right].+\frac{1}{2}\frac{N}{\Omega}.\tilde{v}_\mathbf{0}. \qquad (2.46)$$

This shows how the interaction energy of the system is made up of two terms; the first term depends on the explicit form of the wavefunction for the system via the density operators whereas the second term depends linearly on the average density and is independent of the actual form of the electron wavefunction. Therefore if we set the number of electrons and the area of the system (N and Ω) to be fixed (or for an infinite system take the average density N/Ω to be fixed) then part of the interaction term will simply behave like a constant. Therefore we can ignore it and write

$$V = \frac{1}{2\Omega}\sum_\mathbf{q}\tilde{v}_\mathbf{q}\rho_\mathbf{q}^\dagger\rho_\mathbf{q} = \frac{1}{2}\cdot\frac{1}{4\pi^2}\int d^2\mathbf{q}.\tilde{v}_\mathbf{q}\rho_\mathbf{q}^\dagger\rho_\mathbf{q} \qquad (2.47)$$

as long as we are clear that the density of the system is a constant and that the interparticle potential is fixed.

2.3.4 The Lowest Landau Level Approximation

Remember that the Hamiltonian of our model system is given by

$$H = K + V = \sum_{i=1}^{N}K_i + \frac{1}{2}\sum_{\substack{i,j \\ i\neq j}}^{N}v(\mathbf{r}_i - \mathbf{r}_j) \qquad (2.48)$$

with

$$K_i = \frac{1}{2m}\left[\mathbf{p}_i - e\mathbf{A}(\mathbf{r})\right]^2. \qquad (2.49)$$

We know the Landau level structure of the kinetic energy operator, K, where the Landau levels are separated by $\hbar\omega_B = \hbar\frac{eB}{m}$ and so the possible kinetic energies form a discrete set of levels with a separation which is proportional to the strength of the magnetic field.

Now, for very high magnetic fields the $\hbar\omega_B$ spacing is very large and therefore we can think of V as a perturbation to K. The eigenstates of the total hamiltonian will be formed from linear superpositions of the eigenstates of K, and so each eigenstate

$|k\rangle$ of K gives rise to an eigenstate of H

$$|\psi_k\rangle = |k\rangle + \sum_i \{L_i\} \tag{2.50}$$

where $\{L_i\}$ denotes the contribution to ψ_k from the ith Landau level. If $|k\rangle$ is from the jth Landau level then the scalar coefficients for each $\{L_i\} : i \neq j$ will have factors of $[(j-i)\hbar\omega_B]^{-1}$ and so for very large magnetic field ,B, we expect that the Landau level mixing is small and it is therefore a good approximation to classify the eigenstates of H according to Landau level.

In the infinite magnetic field limit we would then expect that all the low energy states of the system (such as the ground state) will be formed from superpositions of wavefunctions taken from the lowest Landau level (*i.e.* any state with a component in a higher Landau level will have such a high contribution of kinetic energy due to that component that it will have a higher energy than a state contained completely within the lowest Landau level and it will, therefore, be energetically unfavourable in comparison to the lowest Landau level states). This will only be true in the limit of infinitely large magnetic field strength (an approximation which we have already used to justify the neglecting of the electron spin term in the Hamiltonian) but we would expect that finite (though still large) fields would only lead to quantitative differences and we expect that the physical picture and general details for the finite B case should be well represented by the lowest Landau level (infinite magnetic field) approximation.

Therefore, in studying the low lying energy states of electron system, we shall only consider those states lying within the lowest Landau level. In the lowest Landau level, all the states have the same kinetic energy (by definition) and so the kinetic energy term may be regarded as a simple additive constant in the Hamiltonian when considering such wavefunctions. This means that for wavefunctions in the lowest Landau level the only important term in the Hamiltonian is the interaction potential

$$V = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^{N} v(\mathbf{r}_i - \mathbf{r}_j) \tag{2.51}$$

which was discussed in section 2.3.3.

2.4 The Projected Hilbert Space

2.4.1 The Complex Function Space

The advantage of the lowest Landau level approximation is that it allows us to restrict our attention to a particular sub-Hilbert space of wavefunctions (*i.e.* those states in the lowest Landau level) and enables us to more or less ignore most of the true Hilbert space of allowed wavefunctions. We follow closely the example of Girvin and Jach [46]. We will use the model Hamiltonian discussed earlier and we will take the vector potential to be in the symmetric gauge (so defining a centre point in the plane). In this gauge the (z component of) angular momentum commutes with the Hamiltonian (interacting and non-interacting) and the simultaneous eigenstates of angular momentum and the non-interacting Hamiltonian are given by (see section 2.2.3)

$$\phi_{m,n}(x,y) = e^{\frac{1}{4}(x^2+y^2)} . \left(\partial_x + i\partial_y\right)^m . \left(\partial_x - i\partial_y\right)^n . e^{-\frac{1}{2}(x^2+y^2)} \tag{2.52}$$

where units have been adopted so that length is now measured in units of the magnetic length, so that now

$$l_c = \sqrt{\frac{\hbar}{eB}} = 1.$$

The energy levels are given only by the n parameter *i.e.*

$$E_{m,n} = \left(n + \frac{1}{2}\right)\hbar\omega_c. \tag{2.53}$$

Now, we are interested in the states which occupy the lowest single particle energy level and the simultaneous eigenstates of the Hamiltonian and the angular momentum which span the lowest Landau level are

$$\phi_{m,0}(x,y) = (x+iy)^m . e^{-\frac{1}{4}(x^2+y^2)} \tag{2.54}$$

and if we write the eigenfunctions in the lowest energy level in terms of a complex coordinate, $z = x + iy$, then we find that these states can be expressed as complex functions of the z variable

$$\phi_{m,0}(z) = z^m . e^{-\frac{1}{4}|z|^2} \tag{2.55}$$

(where we note that, as it commutes with the kinetic energy, the angular momentum preserves Landau level and is as good a label for eigenstates in the lowest Landau level as it is in the whole Hilbert space). These states form a basis for the lowest Landau level Hilbert space, W_0, and any wavefunction in the lowest Landau level can therefore be written as a superposition of these states *i.e.*

$$\psi(z) = \sum_{m=0}^{\infty} \alpha_m . z^m . e^{-\frac{1}{4}|z|^2}. \tag{2.56}$$

The first thing that we notice about the wavefunctions in the lowest Landau level is that they all have the same Gaussian factor, $e^{-\frac{1}{4}|z|^2}$, which makes sure that they go to zero at infinity (*i.e.* every basis vector has such a factor, from equation 2.55, and so obviously any possible superposition will also have such a factor). We can therefore define a simple isomorphism from the lowest Landau level, W_0, of the original wavefunction space to a new Hilbert space, H_0, by applying the transformation

$$f(z)e^{-\frac{1}{4}|z|^2} \mapsto f(z) \tag{2.57}$$

to all the wavefunctions in the lowest Landau level. For example, the basis set formed from the simultaneous eigenstates of the Hamiltonian and angular momentum are transformed according to

$$z^m . e^{-\frac{1}{4}|z|^2} \mapsto z^m. \tag{2.58}$$

In the original wavefunction space the inner product is defined by (where F and G are wavefunctions in the lowest Landau level of the original space *i.e.* $F, G \in W_0$)

$$\langle F, G \rangle_{W_0} = \int dx dy . F^*(x, y) G(x, y). \tag{2.59}$$

in order to preserve the values of all the inner products between corresponding states we define the inner product in H_0 to be given by

$$\langle f, g \rangle_{H_0} = \int dx dy . e^{-\frac{1}{2}|z|^2} . f^*(x, y) g(x, y). \tag{2.60}$$

for any $f, g \in H_0$. This gives that whenever

$$F \leftrightarrow f \quad \text{and} \quad G \leftrightarrow g$$

that the corresponding inner products are equal *i.e.*

$$\langle F, G \rangle_{W_0} = \langle f, g \rangle_{H_0}. \tag{2.61}$$

As we saw above, the basis set in equation 2.58 for the wavefunction space is represented by the integer powers of z in the transformed Hilbert space H_0,

$$\left\{ z^m . e^{-\frac{1}{4}|z|^2} \right\}_{m=0,1,2,\dots} \qquad \leftrightarrow \qquad \{ z^m \}_{m=0,1,2,\dots} \qquad . \tag{2.62}$$

Therefore the z^m form a basis for this representation of the lowest Landau level and H_0 is simply the completion of their span with respect to the defined inner product. This then defines a Hilbert space, H_0, which represents all (and only) those states in the lowest Landau level. In H_0 we see that all of the statevectors are represented by functions of z only (there is no z^* dependence).

2.4.2 The Projected Operators

We are going to restrict our attention to only this lowest Landau level Hilbert space, $W_0 \leftrightarrow H_0$ and we want to generate state vectors which are completely contained within it. To do this we consider the lowest Landau level as a Hilbert space in its own right, W_0, and we define operators to act in/on this space. We define the projection onto W_0 of a wavefunction in W (the whole wavefunction Hilbert space) in the obvious manner via the projection operator

$$P_0 = \sum_\alpha |\alpha\rangle\langle\alpha| \tag{2.63}$$

where α is any set of basis vectors complete in W_0. We define the projection of any operator A, in W, into W_0 as

$$\ddot{A} = P_0 A P_0 \mid_{W_0} \tag{2.64}$$

This definition guarantees that, for any $F, G \in W_0$

$$\langle F|\ddot{A}|G \rangle_{W_0} = \langle F|A|G \rangle_W \tag{2.65}$$

and that

$$\ddot{A}\phi \in W_0 \qquad \forall \phi \in W_0 \tag{2.66}$$

We then transform the projected operator, $\ddot{A}$, which is defined on W_0, on to the Hilbert space H_0 and we denote this transformed projected operator by $\bar{A}$. The transformation is effected (in the only way that it can be done) by simply transforming the component ket bra pairs of the $\ddot{A}$ operator (in some basis for W_0, $i.e.$ F_m)

$$\ddot{A} = \sum_{m,n} |F_m\rangle a_{m,n} \langle F_n| \quad \longmapsto \quad \sum_{m,n} |f_m\rangle a_{m,n} \langle f_n| = \bar{A} \qquad (2.67)$$

where f_m is the transformation of F_m from W_0 into H_0

$$W_0: \quad F_m \leftrightarrow f_m \quad : H_0.$$

This ensures that for any states in the lowest Landau level that $\bar{A}$ has the same matrix elements as A, $i.e.$

$$\langle f|\bar{A}|g\rangle_{H_0} = \langle F|\ddot{A}|G\rangle_{W_0} = \langle F|A|G\rangle_W \qquad (2.68)$$

for any $W_0 \ni F, G \leftrightarrow f, g \in H_0$. It also ensures that

$$\bar{A}f \in H_0 \quad \forall f \in H_0 \quad (i.e. \ \forall F \in W_0). \qquad (2.69)$$

In this manner, it is readily found that the (transformed) projected operators are given by (see Girvin, Jach [46])

$$
\begin{aligned}
\bar{z} &= z \\
\overline{z^*} &= 2\partial \\
\overline{z^*z} = \overline{zz^*} &= 2\partial z \\
\overline{(z^*)^m z^n} = \overline{z^n (z^*)^m} &= (2\partial)^m z^n
\end{aligned}
$$

$$(2.70)$$

where z is the operator 'multiply by z' and ∂ is the operator defined by its action

$$
\begin{aligned}
\partial : \phi_0 &\mapsto 0 \\
\partial : \phi_n &\mapsto n.\phi_{n-1} \quad for \ n \geq 1
\end{aligned}
$$

$$(2.71)$$

where ϕ_n is the basis function z^n and ∂ therefore behaves like differentiate *w.r.t.* z when acting on analytic functions in H_0. The ∂ and z operators obey the commutation relation

$$[\partial, z] = 1. \tag{2.72}$$

One important consequence of this is that the process of projecting operators onto the lowest Landau level does not preserve commutation relations, for example

$$[\overline{z^*}, \overline{z}] = 1 \quad \neq \quad 0 = [z^*, z]. \tag{2.73}$$

All of these relations are easily extended to many electron statefunctions and they can be used to build up an algebra of operators in H_0.

We note that the transformation into H_0 always preserves the actional form of all multiplication operators (such as position) but that the projection onto W_0 from W does not always respect the actional form of such operators. The actional form of the 'operator multiply by z' is preserved by the projection and this is why its action is unchanged in H_0. The action of the operator 'multiply by z^*' is changed by the projection onto the lowest Landau level because the z^* operator does not preserve Landau level and this is why $\overline{z^*}$ has such a different actional form in H_0 from that of z^* in W.

2.4.3 The Projected Density Operators

Some of the most important operators for our system are the density operators and the projected density operators are, correspondingly, just as important in the lowest Landau level. Remember that the normal density operators are given by

$$\rho_{\mathbf{k}} = \int d^2 \mathbf{r} . \rho(\mathbf{r}) e^{-i\mathbf{k}.\mathbf{r}}.$$

$$\rho_{\mathbf{k}} = \sum_{j=1}^{N} e^{-i\mathbf{k}.\mathbf{r}_j}. \tag{2.74}$$

Now, as well as expressing the position vectors of electrons as a complex number we can do the same for any two dimensional vector

$$\mathbf{k} = (k_x, k_y) \mapsto k = k_x + ik_y$$

$$\mathbf{q} = (q_x, q_y) \mapsto q = q_x + iq_y$$

so that

$$\mathbf{k}_x = \frac{1}{2}\left(k + k^*\right)$$

$$\mathbf{k}_y = \frac{1}{2i}\left(k - k^*\right)$$

$$k^*q = (k_x q_x + k_y q_y) + i\left(k_x q_y - k_y q_x\right) = \mathbf{k}.\mathbf{q} + i\mathbf{k} \wedge \mathbf{q}$$

$$\mathbf{k}.\mathbf{q} = k_x q_x + k_y q_y = \frac{1}{2}\left(k^*q + kq^*\right)$$

$$\mathbf{k} \wedge \mathbf{q} = k_x q_y - k_y q_x = \frac{1}{2i}\left(k^*q - kq^*\right)$$

$$(2.75)$$

where the wedge product, $\wedge$, of a pair of two dimensional vectors is the same as the z component of the vector cross product which would be obtained from the vectors by adding a z component of 0 to both of them.

Rewriting the vectors and the dot products in the expression for the density operators (above) and then (transforming and) projecting onto the lowest Landau level, H_0, gives

$$\overline{\rho}_k = \sum_{j=1}^{N} e^{-ik\partial_j} e^{-ik^* \frac{1}{2} z_j} \tag{2.76}$$

where we see that, like the normal density operators, the projected density operators obey the conditions

$$\overline{\rho}_0 = N \tag{2.77}$$

and

$$\overline{\rho}_k^\dagger = \overline{\rho}_{-k}. \tag{2.78}$$

Using a similar method, we obtain

$$\overline{\rho_k \rho_q} = \overline{\rho}_k \overline{\rho}_q + (1 - e^{\frac{k^* q}{2}})\overline{\rho}_{k+q} \tag{2.79}$$

and this, along with the fact that all of the normal density operators commute with one another ($[\rho_k, \rho_q] = 0$) leads to the commutation relation for the projected density operators

$$[\overline{\rho}_k, \overline{\rho}_q] = \phi(k, q).\overline{\rho}_{k+q} \tag{2.80}$$

where

$$\begin{aligned}
\phi(k,q) &= e^{\frac{k^*q}{2}} - e^{\frac{q^*k}{2}} \\
&= 2i.e^{\frac{1}{2}\mathbf{k.q}}.\sin\left(\frac{\mathbf{k \wedge q}}{2}\right).
\end{aligned}$$

This further demonstrates the way in which the process of projection onto the lowest Landau level fails to preserve commutation relations, as we see that where the normal density operators all commute with one another the projected ones fail to do so.

The projected density operators also differ from the normal ones in other ways. For example, whereas the normal density operators are shift operators for the total momentum of a system we can show that the projected density operators are actually shift operators for position. They are the symmetrised version of an operator which moves one electron by a displacement $(k_y, -k_x)$. To see this, define the operators

$$x = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{2}(z_j + z_j^*) \tag{2.81}$$

and similarly

$$y = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{2i}(z_j - z_j^*). \tag{2.82}$$

The x and y are multiplication operators representing the x and the y coordinates of the centre of mass of the system. Then $\bar{x}, \bar{y}$ have the following commutation relations with $\bar{\rho}_k^\dagger$:

$$[\bar{x}, \bar{\rho}_k^\dagger] = \frac{1}{N}k_y.\bar{\rho}_k^\dagger$$

$$[\bar{y}, \bar{\rho}_k^\dagger] = \frac{1}{N}(-k_x).\bar{\rho}_k^\dagger$$

$$\text{and also}\quad [\bar{x}, \bar{y}] = -i$$

so that $\bar{x}, \bar{y}$ are conjugate operators in H_0. We see that $\bar{\rho}_k^\dagger$ is a shift operator for the $\bar{x}$ and $\bar{y}$ operators, moving the centre of mass of the system by $(k_y, -k_x)/N$ - this is the effect of $\rho_k^\dagger$ trying to increase the electron velocity in the direction k, but due to the magnetic field in the $-z$ direction this results in a movement in the direction $-\underline{v} \times \underline{B}$. This is more easily seen, as the projected density operator is

the symmetrised version of the single particle translation operator, by considering the single electron eigenfunctions found by solving in the Landau gauge where the position of an eigenfunction in one coordinate depends explicitly on the momentum in the perpendicular direction, so that by increasing the momentum in one direction we move the electron by a fixed amount in the orthogonal direction. We also see that $\overline{H}$ conserves k (*i.e.* it conserves centre of mass) and therefore in H_0, k is a good label for statefunctions.

2.4.4 The Projected Hamiltonian

Ultimately we want to investigate the energies of different states. In order to do this we use the projected Hamiltonian, $\bar{H}$, which will still give accurate values for any states in the lowest Landau level (as discussed in section 2.4.2)

$$\langle f|\bar{H}|f\rangle_{H_0} = \langle F|\ddot{H}|F\rangle_{W_0} = \langle F|H|F\rangle_W. \tag{2.83}$$

Now, we know that the kinetic energy is, by definition, the same for all states in the lowest Landau level, hence behaving just like a constant, and so it can be reasonably ignored. Therefore, we only need to calculate the projection of the interaction term in the Hamiltonian. This is readily carried out by using the relations discussed in sections 2.4.2 and 2.4.3 and the expression for the interaction potential in terms of the normal density operators from equation 2.40 in section 2.3.3.

$$V = \frac{1}{2\Omega} \sum_{\mathbf{q}} \tilde{v}_{\mathbf{q}} \rho^{\dagger}_{\mathbf{q}} \rho_{\mathbf{q}} - \frac{1}{2} N.v(\mathbf{0}). \tag{2.84}$$

Using equation 2.79 from the last section

$$\overline{\rho^{\dagger}_k \rho_k} = \bar{\rho}^{\dagger}_k \bar{\rho}_k + (1 - e^{-\frac{1}{2}|k|^2})N \tag{2.85}$$

and combining this with the expression for V gives

$$\overline{V} = \frac{1}{2\Omega} \sum_{q} \tilde{v}_q \left[\bar{\rho}^{\dagger}_q \bar{\rho}_q + N \left(1 - e^{-\frac{1}{2}|q|^2} \right) \right] - \frac{1}{2} N.v(0) \tag{2.86}$$

which expresses the projection of the interaction term in the Hamiltonian in terms of the projected density operators.

If we define a projected static structure factor, $\bar{S}(k)$, of a state, ψ, in analogy with the normal static structure factor from equation 2.44

$$\bar{S}(k) = \begin{cases} 0 & \text{if} \quad k = 0 \\ \frac{1}{N}\langle\psi|\bar{\rho}_k^\dagger\bar{\rho}_k|\psi\rangle & \text{if} \quad k \neq 0 \end{cases} \tag{2.87}$$

so that

$$S(k) = \bar{S}(k) + \left(1 - e^{-\frac{1}{2}|k|^2}\right) \tag{2.88}$$

then we can now rewrite the energy per particle of a particular electronic state, ψ, as

$$\frac{1}{N}\langle\psi|V|\psi\rangle = \frac{1}{2\Omega}\sum_q \tilde{v}_q\left[\bar{S}(q) - e^{-\frac{1}{2}|q|^2}\right] + \frac{1}{2}\frac{N}{\Omega}.\tilde{v}_0 \tag{2.89}$$

which, allowing Ω to tend to infinity and the summations to become integrals, gives

$$\frac{1}{N}\langle\psi|V|\psi\rangle = \frac{1}{2}.\frac{1}{4\pi^2}\int d^2q.\tilde{v}_q\bar{S}(q). - \frac{1}{2}.\frac{1}{4\pi^2}\int d^2q.\tilde{v}_qe^{-\frac{1}{2}|q|^2}. + \frac{1}{2}\frac{N}{\Omega}.\tilde{v}_0 \tag{2.90}$$

all in analogy to equations 2.45 and 2.46. This shows how the interaction energy of the system depends on three separate terms. The first term depends on the explicit form of the wavefunction via the projected static structure factor (the projected density operators). The second term is independent of the wavefunction and depends only on the interparticle potential ($\tilde{v}_q$) and the last term is simply proportional to the average density of the system.

Therefore if we set the number of electrons and the area of the system (N and Ω) to be fixed (or for an infinite system take the average density N/Ω to be fixed) then part of the interaction term will simply behave like a constant. Therefore we can ignore it along with the kinetic energy constant and write (in analogy with equation 2.47)

$$\bar{H} = \bar{V} + const \quad = \quad \frac{1}{2\Omega}\sum_q \tilde{v}_q\bar{\rho}_q^\dagger\bar{\rho}_q \tag{2.91}$$

$$= \quad \frac{1}{2}.\frac{1}{4\pi^2}\int d^2q.\tilde{v}_q\bar{\rho}_q^\dagger\bar{\rho}_q \tag{2.92}$$

as long as we are clear that the density of the system is a constant and that the interparticle potential is fixed. It is this expression for the projected Hamiltonian in the lowest Landau level that we will generally make use of.

2.4.5   The Projected Evolution Equation

We note that for $\psi \in \mathcal{H}_0$

$$\langle \psi | [H, P_0 A P_0] | \psi \rangle = \langle \psi | [\bar{H}, \bar{A}] | \psi \rangle \qquad (2.93)$$

and so the time evolution of any lowest Landau level operator, A, can be calculated using its commutation with the projected Hamiltonian.

CHAPTER 3

The Correlation Functions

3.1 Overview

We discuss some of the elementary restrictions which can be placed on the form of the projected third order correlation function, which is relevant to current phonon experiments. We also derive a sum rule for the third order correlation function. This provides a useful check on any approximations which may be adopted for the correlation function. We note that the method employed is readily extended to produce a hierarchy of sum rules for all of the nth order correlation functions.

3.2 The Magnetoroton Scattering Term

In recent phonon scattering experiments [47] ballistic phonons are fired at the electron system in order to probe its collective excitations. The transition rate for the process where a phonon, of in plane wavevector k, causes a magnetoroton (see chapter 4) to scatter from a mode q into a mode $k + q$ is controlled by the matrix element [48, 49] $\langle \psi | \bar{\rho}_{k+q} \bar{\rho}_k^\dagger \bar{\rho}_q^\dagger | \psi \rangle$ via

$$\text{Transition rate} \propto \left| \langle \psi | \bar{\rho}_{k+q} \bar{\rho}_k^\dagger \bar{\rho}_q^\dagger | \psi \rangle \right|^2 . \tag{3.1}$$

Quite simply, this is the matrix element of the operator $\bar{\rho}_k^\dagger$ (which represents the phonon interacting with the electron system) taken between the initial sate on the right (a magnetoroton of mode q) and the final state on the left (a magnetoroton of mode $k+q$). The higher order scattering rates depend on matrix elements constructed

in a similar way. We are mainly interested in the second order scattering process and so we shall mostly restrict our attention to studying the projected third order correlation function, which is defined by

$$\bar{P}(k,q) = \begin{cases} 0 & \text{if} \quad k \text{ or } q \;=\; 0 \\ \frac{1}{N}\langle\psi|\bar{\rho}_{k+q}\bar{\rho}_k^{\dagger}\bar{\rho}_q^{\dagger}|\psi\rangle & \text{if} \quad k \text{ and } q \;\neq\; 0 \end{cases} \qquad (3.2)$$

where, as before with the structure factor (section 2.3.3), the zero wavevector condition is to avoid the singularities at $k = 0$ and $q = 0$ which occur for states, ψ, with infinite area.

3.3 The Conditions on the Correlation Function

3.3.1 The Restrictions on Possible Solutions

In order to be able to accurately calculate the transition rates for phonon assisted magnetoroton scattering we need to have an expression for the third order correlation function, $\bar{P}(k,q)$. Calculating the correlation function is a highly non trivial business and we are undoubtedly going to be forced into using some sort of approximation. The correlation function has to obey certain restrictions which any candidate solution or any approximation to $\bar{P}(k,q)$ should also satisfy. It is, of course, desirable to know and understand as many of these restrictions as possible, in order to limit the known set of functions, of which $\bar{P}(k,q)$ is a member, to be as small as possible (ideally we would have enough conditions to give a unique solution). Also, any conditions which we know that the correlation function has to satisfy can be used as a method to test any approximation we may have for $\bar{P}(k,q)$. Therefore, we investigate various conditions which the correlation function must satisfy.

3.3.2 The Symmetry Conditions

The first type of restrictions on the correlation function which we consider are the symmetry conditions. These provide fundamental conditions which can limit the possible solutions and, in general, the symmetry conditions are the restrictions which it is most important for any approximation to obey.

The Particle Hole Symmetry

One of the most obvious and important conditions is particle hole symmetry. This is due to the fact that the phonons will interact equally with electrons or the holes (unfilled electron states) so that a certain density of electrons should absorb the same phonons as the same density of holes would do (in the equivalent hole state). The scattering is actually proportional to the electron density as well as the modulus square of the correlation function so that we can say

$$\nu.\bar{P}_\nu(k,q) = (1-\nu).\bar{P}_{1-\nu}(k,q) \tag{3.3}$$

which is the same condition that the projected static structure factor has to obey, *i.e.*

$$\nu.\bar{S}_\nu(k) = (1-\nu).\bar{S}_{1-\nu}(k) \tag{3.4}$$

This is because the phonons should be equally affected whether the electron filling factor is ν or the hole filling factor is ν (which would give electron filling factor as $1-\nu$). This symmetry condition gives us some idea of how the correlation function depends on the electron density.

The Adjoint Symmetry

We recall from section 2.4.3 the equality

$$\bar{\rho}_k^\dagger = \bar{\rho}_{-k} \tag{3.5}$$

which is simply equivalent to the condition that the density of the electron system is real valued. By using this equation and the definition of the adjoint operation we arrive at another symmetry condition for the correlation function, namely

$$\langle\psi|\bar{\rho}_{k+q}\bar{\rho}_k^\dagger\bar{\rho}_q^\dagger|\psi\rangle = \langle\psi|\bar{\rho}_q\bar{\rho}_{-k}^\dagger\bar{\rho}_{k+q}^\dagger|\psi\rangle^*$$

which implies the adjoint symmetry condition (which is equivalent to the condition of micro-reversibility)

$$\bar{P}(k,q) = \bar{P}^*(-k,q+k) \tag{3.6}$$

This condition gives us information on the form of the correlation function's dependence on the variables k and q.

The Axial Symmetry

The whole electron system is axially symmetric about the z axis (assuming that the ground state, ψ is isotropic and homogeneous) and so we expect that the scattering term will share this axial symmetry. This means that if we use polar coordinates $|k|,|q|,\theta_k$ and θ_q then we would expect that the correlation function will depend only on the angle between k and q and that

$$|\bar{P}(|k|,|q|,\theta_q - \theta_k)|^2 = |\bar{P}(|k|,|q|,\theta_k - \theta_q)|^2 \tag{3.7}$$

so that the orientation (sign) of the angle should make no difference to the phonon interaction. This condition, like the adjoint symmetry, gives us information on the possible forms that the correlation function's dependence on the k,q variables may take *i.e.* we now know that $|\bar{P}|$ can be written as a function of the variables $|k|,|q|$ and $|\theta|$.

The Form of the Possible Solutions

As a result of the symmetries discussed above we can rewrite the functional dependence of the correlation function in a more restrictive form. First we separate the magnitude and phase of the correlation function, by writing

$$\bar{P}(k,q) = e^{i\phi(k,q)}.|\bar{P}(k,q)| \tag{3.8}$$

and it is the magnitude, the physically important part of the correlation function (*i.e.* it is the scattering term), which we are interested in calculating whereas $\phi(k,q)$ is the relatively unimportant phase. Now, as a result of the adjoint and axial symmetries we can write

$$|\bar{P}(k,q)| = f\left(|k \wedge q|, |k|, q.(k+q)\right) \tag{3.9}$$

(we have dropped the boldface vector notation - it is always obvious whether we are using the vector or complex representation) so that the symmetry conditions have allowed us to predict, to some degree, the functional form of the correlation function. This provides quite a useful limitation on the correlation function and it can be used to test the validity of approximations.

It is obvious that any function which depends only on the variables $|k|$, $|k \wedge q|$ and $q.(k+q)$ will satisfy the axial and adjoint symmetry conditions. In order to prove the result, though, we must show that <u>any</u> function which obeys the adjoint and axial symmetries can be expressed as a function of the $|k \wedge q|$,$|k|$ and $q.(k+q)$ variables. To do this, we need to show that we can write the original vector variables as functions of the new ones in such a form as does not restrict their range in any way.

We define the coordinate transformation

$$
\begin{aligned}
a^2 &= |k \wedge q|^2 \\
b &= |k| \\
c &= q.(k+q).
\end{aligned}
\tag{3.10}
$$

Now, we know, from the axial condition, that the correlation function depends only on the magnitudes of the vector variables and the angle between them, so we adopt polar coordinates for our original variables and the coordinate transformation becomes (using the polar coordinates K,Q and θ which can again be replaced with the coordinates K,$Q \cos \theta$ and $Q \sin \theta$)

$$
\begin{aligned}
a^2 &= |k \wedge q|^2 &= K^2 (Q \sin \theta)^2 \\
b &= |k| &= K \\
c &= q.(k+q) &= (Q \cos \theta)^2 + (Q \sin \theta)^2 + K(Q \cos \theta)
\end{aligned}
\tag{3.11}
$$

We need to show that this transformation is (sufficiently) invertible so that all of the K,Q,θ variables can be written as functions of a^2, b and c. This would allow us to substitute these functions into the expression for $|\bar{P}(k,q)|$, so proving that the correlation function can be expressed in terms of the a, b, c variables.

This is equivalent to asking whether, given a point in a, b, c space, can we find K, Q and θ unambiguously. So, to see this, we solve equations 3.11 for a point a, b, c and show that our solution is (sufficiently) unique. Writing our solutions as U, V and ϕ (*i.e.* K, Q and θ) we obtain the inverting formulae

$$
\begin{aligned}
U &= b \\
V \sin \phi &= \pm \frac{a}{b} \\
V \cos \phi &= -\frac{b}{2} \pm \frac{1}{2} \sqrt{b^2 - 4 \left(\frac{a^2}{b^2} - c \right)}
\end{aligned}
\tag{3.12}
$$

(where $a, b \geq 0$ and the term inside the square root is always positive over the allowed range of a, b, c)

These formulae do not give a unique solution but rather there are four possible points in the K, Q, θ coordinates for every one point in the a, b, c coordinates. This is exactly what we want, though, as what the transformation does is to map every group of four points in the k, q coordinates, for which $|\bar{P}(k, q)|$ has the same value (from the symmetries), onto one point in the a, b, c coordinates - which guarantees that any function of the a, b, c coordinates will obey the symmetries. This proves that any function, $|\bar{P}(k, q)|$, which obeys the adjoint and axial symmetries can be written as a function of a, b and c as defined in equation 3.11.

3.3.3 The Commutation Equations

Another simple condition on the third order correlation function can be easily derived from the commutation relations for the projected density operators (section 2.4.3). We take the matrix element of the commutation relation in-between any electron statefunction

$$\langle \psi | [\bar{\rho}_k, \bar{\rho}_q] | \psi \rangle = \phi(k, q).\langle \psi | \bar{\rho}_{k+q} | \psi \rangle \tag{3.13}$$

where

$$\phi(k, q) = 2i.e^{\frac{1}{2}\mathbf{k}.\mathbf{q}}.\sin\left(\frac{\mathbf{k} \wedge \mathbf{q}}{2}\right)$$

and this immediately leads to

$$
\begin{aligned}
\bar{P}(k, q) - \bar{P}(q, k) &= \phi(k, q).\bar{S}(k + q) \\
&= 2i.e^{\frac{1}{2}\mathbf{k}.\mathbf{q}}.\sin\left(\frac{\mathbf{k} \wedge \mathbf{q}}{2}\right).\bar{S}(k + q).
\end{aligned}
\tag{3.14}
$$

This equation has to be obeyed by the correlation function of any state, ψ. In a similar fashion the commutation relations lead to equations relating the fourth order correlation function to the third and the fifth to the fourth and so on relating the $(n + 1)$th order correlation function to the nth in the general case but we are mainly interested in the third order function and so will confine our attention to the above equation.

As the right hand side of equation 3.14 is purely imaginary this means that (where $\Re$ denotes the real part)

$$\Re\left[\bar{P}(k,q)\right] = \Re\left[\bar{P}(q,k)\right] \tag{3.15}$$

and it is only the imaginary part of the correlation function which contributes to the static structure factor (*i.e.* this shows that the energy of the system can be expressed in terms of the imaginary part of $\bar{P}(k,q)$ but that the real part of the correlation function has no effect on the energy of the state).

3.3.4 The Equation of Motion

Another approach is to study the equation of motion for the system (where A is any general operator)

$$\frac{d}{dt}\langle\psi(t)|A|\psi(t)\rangle = \frac{i}{\hbar}\langle\psi(t)|[H,A]|\psi(t)\rangle \tag{3.16}$$

Using the projected form of this equation (section 2.4.5) and if we assume a ground state, ψ which is an eigenstate of the projected Hamiltonian, $\bar{H}$, then ψ must be a stationary state of $\bar{H}$ and so the static structure factor of the ground state will be a constant through time, giving

$$\frac{d}{dt}\langle\psi|\bar{\rho}_q^\dagger\bar{\rho}_q|\psi\rangle = 0 = \frac{i}{\hbar}\langle\psi|[\bar{H},\bar{\rho}_q^\dagger\bar{\rho}_q]|\psi\rangle. \tag{3.17}$$

We have (from section 2.4.4)

$$\bar{H} = \frac{1}{2\Omega}\sum_k \tilde{v}_k \bar{\rho}_k^\dagger \bar{\rho}_k \tag{3.18}$$

along with the following result

$$\begin{aligned}\left[\bar{\rho}_k^\dagger\bar{\rho}_k, \bar{\rho}_q^\dagger\bar{\rho}_q\right] &= \phi(k,q).\left[\bar{\rho}_{-k}\bar{\rho}_{-q}\bar{\rho}_{k+q} + \bar{\rho}_{-k-q}\bar{\rho}_q\bar{\rho}_k\right]\\ &+\phi(-k,q).\left[\bar{\rho}_{-k}\bar{\rho}_{k-q}\bar{\rho}_q + \bar{\rho}_{-q}\bar{\rho}_{q-k}\bar{\rho}_k\right].\end{aligned} \tag{3.19}$$

Dividing through by N (the number of electrons) and making use of the symmetry properties, discussed above, leads to

$$\begin{aligned}\frac{1}{N}\left[\bar{\rho}_k^\dagger\bar{\rho}_k, \bar{\rho}_q^\dagger\bar{\rho}_q\right] &= \phi(k,q).\left[\bar{P}(q,k) + \bar{P}^*(q,k)\right]\\ &+\phi(-k,q).\left[\bar{P}(k-q,q) + \bar{P}^*(k-q,q)\right].\end{aligned} \tag{3.20}$$

Substituting into the equation of motion (and again making frequent use of the symmetry conditions) gives

$$\sum_k \tilde{v}_k . e^{\frac{1}{2}k.q} . \sin\left(\frac{k \wedge q}{2}\right) . \bar{P}_R(q,k) = 0 \tag{3.21}$$

and this sum rule holds for all q. This condition was derived using only (as well as the symmetries) the commutation relations of the projected density operators and the assumption that ψ is an eigenstate of the projected Hamiltonian.

3.4 The Quantised Angular Momentum Sum Rules

3.4.1 The Second Order Sum Rule

So far we have discussed several restrictions which the third order correlation function must satisfy. All of these conditions are a result of simply applying three initial assumptions:

1. We made frequent use of the commutation relations for the projected density operators (these are exact results and we know that they are true).

2. The symmetry conditions, which, were perhaps, used most of all (again we should have no worries about applying these as long as we have an isotropic ground state).

3. We also used the condition that the ground state, ψ, is not only contained completely within the lowest Landau level but that it is also an eigenstate of the projected Hamiltonian. This is an approximation and we should bear in mind that those conditions which depend on this assumption will only be approximately true.

Obviously, we would like to have as many restrictions upon the form of the third order correlation function as possible (those we have provide nothing close to uniqueness). In this section we will derive a new restriction on the correlation functions in the form of sum rules linking the nth order correlation function to all those of lower order. This is especially useful for the third order function as this gives a sum rule linking $\bar{P}(k,q)$ to the static structure factor (for which calculations have already been made [50]). Also the sum rules bring further restriction upon the correlation functions in addition to those which depend on the assumptions above as they make use of an

extra fact *i.e.* that the electrons are fermions. Indeed we shall see that the validity of the sum rules to a wavefunction, ψ, depends only on the initial assumptions that ψ is completely contained within the lowest Landau level and that it is a fermion wavefunction.

We begin by deriving the sum rule for the second order projected correlation function (the projected static structure factor - section 2.4.4) partly to demonstrate that the method agrees with other known results and partly because the second order sum rule will be useful in the next section.

First we will consider the normal static structure factor

$$\langle\psi|\rho_q\rho_q^\dagger|\psi\rangle = \int_\Omega d^2\tilde{r}.\psi^*(\tilde{r})\left(\sum_{i,j=1}^{N} e^{-iq.(r_i-r_j)}\right)\psi(\tilde{r}). \tag{3.22}$$

where r_i is the two dimensional coordinate vector of the ith electron, $\tilde{r}$ represents the whole set of all the electron coordinates ($i = 1 \ldots N$) and Ω represents the area in the two dimensional plane within which the electrons are confined (we can let this tend to infinity when we need to). We separate the summation into the two parts $i = j$ and $i \neq j$ to obtain

$$\langle\psi|\rho_q\rho_q^\dagger|\psi\rangle = \int_\Omega d^2\tilde{r}.\psi^*(\tilde{r})N\psi(\tilde{r}). + \int_\Omega d^2\tilde{r}.\psi^*(\tilde{r})\left(\sum_{\substack{i,j=1\\i\neq j}}^{N} e^{-iq.(r_i-r_j)}\right)\psi(\tilde{r}). \tag{3.23}$$

Now, we integrate both sides over a range, Q, in q space (where we will allow Q to become the whole plane in the infinite limit)

$$\int_Q d^2q.NS(q). = \int_Q d^2q.\int_\Omega d^2\tilde{r}.\psi^*(\tilde{r})N\psi(\tilde{r}).+\int_\Omega d^2\tilde{r}.\psi^*(\tilde{r})\left(\sum_{\substack{i,j=1\\i\neq j}}^{N}\int_Q d^2q.e^{-iq.(r_i-r_j)}.\right)\psi(\tilde{r}). \tag{3.24}$$

where we have swapped the order of integration in the second term on the right. Using the fact that ψ is normalised and the result ([51, page 52])

$$T \overset{\text{Limit}}{\to} \infty \frac{1}{2\pi}\int_{-T}^{T} dt.e^{-i\omega t}. = \delta(\omega) \tag{3.25}$$

and allowing the region Q to become the whole plane in the infinite limit gives us

$$N \int d^2q. \left(S(q) - 1\right) = \int_\Omega d^2\tilde{r}.\psi^*(\tilde{r}) \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{N} 4\pi^2 \delta^2(r_i - r_j) \right) \psi(\tilde{r}). \tag{3.26}$$

Now, we know that the wavefunction of a fermion system must go to zero whenever any two particle coordinates have the same value but it is exactly these values of the wavefunction, when two of its particles are in the same position, that the delta function on the right hand side picks out. This means that the integral on the right must vanish, leaving us with

$$N \int d^2q. \left(S(q) - 1\right) = 0. \tag{3.27}$$

Now, substituting in the relation - which is valid for any wavefunction, ψ, as long as it is completely contained within the lowest Landau level (section2.4.4)

$$S(k) = \bar{S}(k) + \left(1 - e^{-\frac{1}{2}|k|^2}\right) \tag{3.28}$$

gives us the second order sum rule

$$\int d^2q. \bar{S}(q). = \int d^2q. e^{-\frac{1}{2}|q|^2}. = 2\pi \tag{3.29}$$

This result agrees with a direct calculation of the integral and will be useful in the next section.

3.4.2 The Third Order Sum Rule

We're now ready to work out the third order sum rule. We proceed in much the same way as in the last section, *i.e.* by first studying the normal third order correlation function

$$\langle \psi | \rho_{q+h} \rho_q^\dagger \rho_h^\dagger | \psi \rangle = \int_\Omega d^2\tilde{r}.\psi^*(\tilde{r}) \left(\sum_{i,j,s=1}^{N} e^{-i(q+h).r_i} e^{iq.r_j} e^{ih.r_s} \right) \psi(\tilde{r}). \tag{3.30}$$

and as before we separate the summation into the parts $i = j$ and $i \neq j$ and then integrate over the region Q in q space

$$\int_Q d^2q. \langle \psi | \rho_{q+h} \rho_q^\dagger \rho_h^\dagger | \psi \rangle = \int_Q d^2q. \int_\Omega d^2\tilde{r}.\psi^*(\tilde{r}) \left(\sum_{i,s=1}^{N} e^{ih.(r_s - r_i)} \right) \psi(\tilde{r})..$$

$$+ \int_Q d^2q. \int_\Omega d^2\tilde{r}.\psi^*(\tilde{r}) \left(\sum_{\substack{i,j,s=1 \\ i \neq j}}^N e^{ih.(r_s - r_i)} e^{iq.(r_j - r_i)} \right) \psi(\tilde{r}). \qquad . \tag{3.31}$$

Now, we use the result

$$\int d^2q. e^{iq.(r_j - r_i)}. = 4\pi^2 \delta^2(r_j - r_i) \tag{3.32}$$

and, as before, swap the order of the integrals in the second term on the right and allow Q to become the whole plane in the infinite limit to obtain

$$\int d^2q.\langle\psi|\rho_{q+h}\rho_q^\dagger\rho_h^\dagger|\psi\rangle = \int d^2q. \int_\Omega d^2\tilde{r}.\psi^*(\tilde{r}) \left(\sum_{i,s=1}^N e^{ih.(r_s - r_i)} \right) \psi(\tilde{r})..$$

$$+ \int_\Omega d^2\tilde{r}.\psi^*(\tilde{r}) \left(\sum_{\substack{i,j,s=1 \\ i \neq j}}^N e^{ih.(r_s - r_i)} 4\pi^2 \delta(r_j - r_i) \right) \psi(\tilde{r}).. \tag{3.33}$$

Again, making use of the fermion argument that $\psi(\tilde{r}; r_i = r_j)$ must always be zero, the second integral on the right hand side vanishes and noticing that the term underneath the remaining q integral is simply the static structure factor we can write

$$\int d^2q.\langle\psi|\rho_{q+h}\rho_q^\dagger\rho_h^\dagger|\psi\rangle = \int d^2q.\langle\psi|\rho_h\rho_h^\dagger|\psi\rangle. \tag{3.34}$$

and this is the unprojected form of the third order sum rule. In this form it tells us very little, other than that the integral of the normal third order correlation function diverges rather badly, but if we make the additional assumption (additional to the fermion assumption) that the wavefunction, ψ, is contained completely within the lowest Landau level then we find that the sum rule gives up more information.

We use the result (derived from the trivial but tedious repeated application of the commutation relation $[\partial, z] = 1$) that

$$\overline{\rho_{q+h}\rho_q^\dagger\rho_h^\dagger} - \left\{ \bar{\rho}_h\bar{\rho}_h^\dagger + \left(1 - e^{-\frac{1}{2}|h|^2}\right) \right\} = \bar{\rho}_{q+h}\bar{\rho}_q^\dagger\bar{\rho}_h^\dagger + \left(1 - e^{\frac{1}{2}q^*h}\right) \bar{\rho}_{q+h}^\dagger\bar{\rho}_{q+h}$$

$$+ \left(1 - e^{-\frac{1}{2}|h|^2} e^{-\frac{1}{2}q^*h}\right) \bar{\rho}_q^\dagger\bar{\rho}_q - e^{-\frac{1}{2}|q|^2} e^{-\frac{1}{2}h^*q}\bar{\rho}_h^\dagger\bar{\rho}_h$$

$$\left\{ e^{-\frac{1}{2}|q+h|^2}. \left(e^{\frac{1}{2}h^*q} + e^{\frac{1}{2}q^*h}\right) - e^{-\frac{1}{2}|q|^2} - e^{-\frac{1}{2}|q+h|^2} \right\} \bar{\rho}_0.$$

$$\tag{3.35}$$

Making use of the fact that $\langle\psi|A|\psi\rangle = \langle\psi|\bar{A}|\psi\rangle$ whenever ψ is completely contained within the lowest Landau level and substituting the last equation into the unprojected sum rule gives (where we have also divided by N)

$$\int d^2q.\bar{P}(q,h). = \bar{S}(h)\int d^2q.e^{-\frac{1}{2}|q|^2}e^{-\frac{1}{2}h^*q}.$$

$$-\int d^2q.2\bar{S}(q)\left(1 - e^{-\frac{1}{2}|h|^2}e^{\frac{1}{2}q^*h}\right).$$

$$+\int d^2q.\left\{e^{-\frac{1}{2}|q+h|^2}\left(e^{\frac{1}{2}h^*q} + e^{\frac{1}{2}q^*h}\right) - 2e^{-\frac{1}{2}|q|^2}\right\}.$$
$$(3.36)$$

This is the projected version of the normal sum rule given in equation 3.34. As we can see this has the potential to be much more informative than its unprojected counterpart. We can simplify the integrals significantly by applying the second order sum rule from last section (equation 3.29) and so obtain

$$\int d^2q.\bar{P}(q,h). = \bar{S}(h)\int d^2q.e^{-\frac{1}{2}|q|^2}e^{-\frac{1}{2}h^*q}.$$

$$+ 2e^{-\frac{1}{2}|h|^2}\int d^2q.e^{\frac{1}{2}q^*h}\bar{S}(q).$$

$$- e^{-\frac{1}{2}|h|^2}\int d^2q.e^{-\frac{1}{2}|q|^2}\left(e^{\frac{1}{2}h^*q} + e^{\frac{1}{2}q^*h}\right).$$
$$(3.37)$$

and this is the third order sum rule.

We would not normally expect such a modest assumption such as the particles being indistinguishable fermions to lead to a condition giving much significant information on the spatial distribution of all the possible wavefunctions (remember that these density operators are related to the electron density as a function of position) and this is indeed the case as we saw from the unprojected sum rules. The fact that all the possible wavefunctions in the lowest Landau level do obey the restriction of the sum rule above is a result of the spatial quantisation that all the single particle states undergo (section 2.2). It is this angular momentum quantisation ,apparent in the lowest Landau level, enforced by the fermionic nature of the electrons (which ensures that no more than one particle can occupy any one of the spatially quantised single particle states at any one time, so forcing the electrons to maintain a sufficient

distance from one another that they can exist in different single particle states) which restricts the relative spatial distribution of the particles, and it is this effect which leads directly to the sum rules derived in this section.

We also note that using the methods of the previous derivations it is possible to derive a hierarchy of sum rules where the nth order sum rule will relate the integral of the nth order projected correlation function to an integral involving all of the lower order correlation functions. We do not discuss the details here for ecological reasons.

3.5   Summary

We derived a sum rule which related the third order correlation function to the second order one. We note that the same method can be readily applied to derive a similar sum rule relating any n^{th} order correlation function to all those of lower order.

CHAPTER 4

Bosonisation in the Lowest Landau Level

4.1 Overview

We briefly summarise the results of Girvin, MacDonald and Platzman [50] on magnetoroton excitations in the lowest Landau level. We then discuss a first order bosonisation scheme for the magnetorotons and study some of the apparent problems of the scheme. We introduce a second order bosonisation scheme for the projected density operator which we then discuss. We also suggest a random phase approximation, within this scheme, in order to make calculations of the correlation functions computationally tractable.

4.2 Magnetorotons in the 2D Electron System

The Feynman theory [50, 52] of the low lying collective excitations in liquid helium used a variational excited state of the form

$$\phi_k = \rho_k \psi \tag{4.1}$$

where ψ is the ground state of the system. ϕ_k is orthogonal to ψ for any value of k for which the Fourier transform of the density distribution of the ground state is zero. The energy of this state is given by

$$\Delta(k) = \frac{\langle \psi | \rho_k^\dagger [H, \rho_k] | \psi \rangle}{\langle \psi | \rho_k^\dagger \rho_k | \psi \rangle} \tag{4.2}$$

which, because the density operator commutes with the potential energy term in the Hamiltonian but not with the kinetic term, gives

$$\Delta(k) = \frac{\hbar^2 |k|^2}{2ms(k)} \tag{4.3}$$

where m is the mass of a single particle and these quasiparticle excitations were called rotons (we have ignored backflow corrections).

Girvin, MacDonald and Platzman [50] suggested a variational form for the low lying collective excitations in the Lowest Landau level,

$$\phi_k = \bar{\rho}_k \psi, \tag{4.4}$$

which was based on the roton theory of Feynman and in analogy these states were called magnetorotons. The projection onto the lowest Landau level is to maintain the low energy nature of the wavefunction and ψ is assumed to be contained within the lowest Landau level. The energy of this state is again given by

$$\Delta(k) = \frac{\langle \psi | \bar{\rho}_k^\dagger [H, \bar{\rho}_k] | \psi \rangle}{\langle \psi | \bar{\rho}_k^\dagger \bar{\rho}_k | \psi \rangle} \tag{4.5}$$

which, if we assume that ψ is an eigenstate of $\bar{H}$, is the same as

$$\Delta(k) = \frac{1}{2} \frac{\langle \psi | \left[\bar{\rho}_k^\dagger, [\bar{H}, \bar{\rho}_k] \right] | \psi \rangle}{\langle \psi | \bar{\rho}_k^\dagger \bar{\rho}_k | \psi \rangle}. \tag{4.6}$$

¿From chapter 2 and particularly section 2.4.4 we have

$$\bar{H} = \frac{1}{2\Omega} \sum_q v_q \bar{\rho}_q^\dagger \bar{\rho}_q \tag{4.7}$$

$$= \frac{1}{2} \frac{1}{4\pi^2} \int d^2q . v(q) \bar{\rho}_q^\dagger \bar{\rho}_q. \tag{4.8}$$

The projected density operators do not commute with the potential energy term in the Hamiltonian (though, of course, they do commute with the kinetic term - which is just a constant in the lowest Landau level). Using the commutation relations

$$[\bar{\rho}_k, \bar{\rho}_q] = \phi(k, q) \bar{\rho}_{k+q} \tag{4.9}$$

where

$$\phi(k,q) \quad = \quad e^{\frac{k^*q}{2}} - e^{\frac{q^*k}{2}} \tag{4.10}$$

$$= \quad 2ie^{\frac{1}{2}k.q} \sin\left(\frac{k \wedge q}{2}\right) \tag{4.11}$$

it is straightforward to show that the magnetoroton energy (the amount by which the average energy of the state $\bar{\rho}_k$ is larger than the ground state energy) is a functional of the projected static structure factor of the ground state,

$$\Delta(k) = \frac{1}{4.\bar{S}(k)} \sum_q v_q.(e^{\frac{q^*k}{2}} - e^{\frac{k^*q}{2}}).f(k,q) \tag{4.12}$$

where

$$f(k,q) = \bar{S}(q).e^{-\frac{|k|^2}{2}}.(e^{\frac{-k^*q}{2}} - e^{\frac{-q^*k}{2}}) + \bar{S}(k+q).(e^{\frac{k^*q}{2}} - e^{\frac{q^*k}{2}}) \tag{4.13}$$

i.e.

$$\Delta(k) = \frac{1}{4.\bar{S}(k)} \sum_q v_q.2ie^{\frac{1}{2}k.q} \sin\left(\frac{q \wedge k}{2}\right).f(k,q) \tag{4.14}$$

where

$$f(k,q) = \bar{S}(q).e^{-\frac{|k|^2}{2}}.2ie^{-\frac{1}{2}k.q} \sin\left(\frac{q \wedge k}{2}\right) + \bar{S}(k+q).2ie^{\frac{1}{2}k.q} \sin\left(\frac{k \wedge q}{2}\right). \tag{4.15}$$

Girvin, MacDonald and Platzman [46] calculated the projected static structure factor for the first few Laughlin states using numerical techniques (the Laughlin states are described in some detail in chapter 5 as well as in [45]). By then approximating the true ground state of the projected Hamiltonian as the Laughlin wavefunction they obtained the energy of the magnetoroton from equation 4.12. The energy varies as a function of k. It begins at a finite value when $k = 0$, falls to a minimum (at around the value at which $\bar{S}(k)$ has its peak) and then rises again as k is increased.

4.3 First Order Bosonisation for Magnetorotons

4.3.1 Roton Bosonisation in Liquid Helium

In the liquid helium system, where we have a Hamiltonian of the form

$$H = \frac{1}{2\Omega} \sum_q v_q \rho_q^\dagger \rho_q + E_K, \tag{4.16}$$

the density operators behave somewhat like generalised coordinates for the system and the Hamiltonian (in this coordinate system) looks something like a collection of harmonic oscillators (the generalised momentum terms are contained within the kinetic energy part of the Hamiltonian). In analogy to coordinate operators we can write

$$\rho_k^\dagger = A(k) \left[a_k^\dagger + a_{-k} \right] \tag{4.17}$$

and this obeys the adjoint condition

$$\rho_k^\dagger = \rho_{-k} \tag{4.18}$$

and the commutation relation

$$[\rho_k, \rho_q] = 0 \tag{4.19}$$

and the vacuum state $|0\rangle$ is a good approximation to the ground state of the Hamiltonian - or we could consider the vacuum state to be the ground state of the Hamiltonian and expect that equation 4.17 is a good approximation for the density operators in terms of the quasiparticle creation and annihilation operators.

4.3.2 First Order Bosonisation of the Projected Density Operators

Introduction

It would be useful if we could bosonise the collective excitations of the lowest Landau level in a similar fashion to liquid helium. This would be a useful analytical tool for studying the electron system and its interactions with external stimuli. This is especially true for interactions with other boson fields such as photons and phonons. For example, in the phonon case the interaction Hamiltonian between the electron system and the lattice vibrations has been calculated [48, 49] to be

$$H_{e\phi} = \frac{1}{\sqrt{V}} \sum_{\lambda, Q} M(\lambda, Q) \left(c_{\lambda, Q}^\dagger + c_{\lambda, -Q} \right) \rho_q \tag{4.20}$$

where $Q = (q, q_z)$ (q is the projection of Q onto the x, y plane) and λ is a label of polarisation. $c_{\lambda, Q}^\dagger$ and $c_{\lambda, Q}$ are the raising and lowering operators for the phonon quasiparticle modes labeled by λ and Q. We are interested in transitions between electron

states in the lowest landau level and hence the relevant interaction Hamiltonian is

$$\bar{H}_{e\phi} = \frac{1}{\sqrt{V}} \sum_{\lambda,Q} M(\lambda,Q)\left(c_{\lambda,Q}^\dagger + c_{\lambda,-Q}\right)\bar{\rho}_q \qquad (4.21)$$

and so we see that a bosonisation for the projected density operators would allow us to study the phonon-electron interaction in terms of the creation-annihilation processes of Feynman diagrams.

An $a^\dagger$,a Representation for Magnetorotons

We assume that a set of quasiparticle modes exist which can be labeled by a vector, k, which we expect to be a good quantum number [45, page 359] - *i.e.* we use the same vector label as appears in the expansion of the Hamiltonian so that k is conserved by the Hamiltonian. This means that we assume the existence of a set of quasiparticle raising and lowering operators $a_k^\dagger$ and a_k such that

$$\left[a_k, a_q^\dagger\right] = \delta_{k,q} \qquad (4.22)$$

$$\left[a_k, a_q\right] = 0 \qquad (4.23)$$

$$\left[a_k^\dagger, a_q^\dagger\right] = 0. \qquad (4.24)$$

We now expand the projected density operator as a series in these magnetoroton raising and lowering operators (this idea has also been expressed by other authors, for example [53])

$$\begin{aligned}
\bar{\rho}_k^\dagger &= A(k)(a_k^\dagger + a_{-k}) + \sum_h B(k,h).a_{h+k}^\dagger a_h \\
&\quad + \sum_h \left\{C(k,h)a_{k+h}^\dagger a_{-h}^\dagger + C^*(-k,-h)a_h a_{-k-h}\right\} + ... \qquad (4.25)\\
&\qquad\qquad\qquad\qquad \text{for } k \neq 0
\end{aligned}$$

where the coefficients in the expansion have to fit the projected density commutation relations and the adjoint condition $\bar{\rho}_k^\dagger = \bar{\rho}_{-k}$. We take the vacuum state, $|0\rangle$, as representing the ground state of the system. We expect that

$$\bar{\rho}_k^\dagger \psi \approx A(k)a_k^\dagger|0\rangle \qquad (4.26)$$

in the region where $\bar{\rho}_k^\dagger \psi$ is a good approximation to a single magnetoroton quasiparticle.

4.3.3 Problems with this Picture

The Magnetoroton Energy

Using the bosonisation above we can rewrite the Hamiltonian in terms of the magnetoroton raising and lowering operators and hence calculate the energy of a single excitation,

$$a_q^\dagger |0\rangle \approx \frac{\bar{\rho}_q^\dagger \psi}{\sqrt{\langle \psi | \bar{\rho}_q^\dagger \bar{\rho}_q | \psi \rangle}}. \tag{4.27}$$

Using

$$
\begin{aligned}
\langle n | \bar{\rho}_k \bar{\rho}_k^\dagger | n \rangle &= |A(k)|^2 . n_k + |A(k)|^2 . (1 + n_{-k}) \\
&\quad + \sum_h |B(k,h)|^2 . n_h (1 + n_{h+k}) \\
&\quad + \sum_h |C(k,h)|^2 . (1 + n_{k+h})(1 + n_{-h}) \\
&\quad + \sum_h |C(-k,-h)|^2 . n_h n_{-k-h}
\end{aligned} \tag{4.28}
$$

we obtain

$$
\begin{aligned}
\Delta(q) &= \frac{\langle \psi | \bar{\rho}_q^\dagger [\bar{H}, \bar{\rho}_q] | \psi \rangle}{\langle \psi | \bar{\rho}_q^\dagger \bar{\rho}_q | \psi \rangle} \tag{4.29} \\
&= \frac{\langle \psi | \bar{\rho}_q \bar{H} \bar{\rho}_q^\dagger | \psi \rangle}{\langle \psi | \bar{\rho}_q^\dagger \bar{\rho}_q | \psi \rangle} - \langle \psi | \bar{H} | \psi \rangle \tag{4.30} \\
&\approx E(q) \tag{4.31} \\
&= \langle 0 | a_q \bar{H} a_q^\dagger | 0 \rangle - \langle 0 | \bar{H} | 0 \rangle \tag{4.32} \\
&= \frac{1}{2\Omega} v_q |A(q)|^2 + \frac{1}{2\Omega} v_{-q} |A(-q)|^2 \\
&\quad + \frac{1}{2\Omega} \sum_k v_k |B(k,q)|^2 \\
&\quad + \frac{1}{2\Omega} \sum_k v_k |C(k, q-k)|^2 + \frac{1}{2\Omega} \sum_k v_k |C(k, -q)|^2 \\
&\quad + \dots \quad \dots \quad \dots \tag{4.33} \\
&= n_e v_q \frac{1}{N} |A(q)|^2 + \quad higher\ order\ terms \tag{4.34} \\
&\approx n_e v_q \bar{S}(q) + higher\ order\ terms. \tag{4.35}
\end{aligned}
$$

The first term is the direct coupling of the magnetoroton to the ground state through v_q and $\bar{S}(q)$. The other terms are self interaction terms representing the quasiparticle's coupling to other modes (as it undergoes transitions to those modes and then returns to its original mode).

This is not a surprising result. The energy of an excitation at wavevector k depends on the Fourier component of the interaction potential at k and the projected static structure factor at k and it is proportional to the density of the system.

The first thing that we notice about the above expression is that the first term, $n_e v_q \bar{S}(q)$, starts at zero for $|q| \to 0$ (because $\bar{S}(q) \sim |q|^4$ as $|q| \to 0$ - see [50]) rises to a maximum and then goes to zero again as $|q|$ increases to infinity. Moreover, the maximum is caused by the peak in the projected static structure factor and therefore occurs at approximately the value of q where we would actually expect the magnetoroton minimum to occur. It is possible that the higher order terms contribute in such a way as to turn this maximum into a minimum - which would require that the higher order terms are significant in comparison to the first order terms. We also see that, since all the terms are positive, the first term should provide a minimum value for the energy. We also note that this expression for the energy goes to zero as $|q|$ goes to zero - as the higher order terms equal zero when $q = 0$ (if we want the coefficients to be well behaved and equal zero for a_0 and $a_0^\dagger$ terms) and cannot help the first term to rise above its zero value. This contrasts with the energy from equation 4.12 which has a finite value as $|q| \to 0$ and has a minimum near the peak of the projected static structure factor - *i.e.* it is the exact opposite of what the above expression might suggest.

In fact, we see that the expression for the energy of the magnetoroton state,

$$\Delta(q) = \frac{1}{4.\bar{S}(q)} \sum_k v_k . 2ie^{\frac{1}{2}q.k} \sin\left(\frac{k \wedge q}{2}\right) . f(q,k) \qquad (4.36)$$

where

$$f(q,k) = \bar{S}(k).e^{-\frac{|q|^2}{2}}.2ie^{-\frac{1}{2}q.k} \sin\left(\frac{k \wedge q}{2}\right) + \bar{S}(q+k).2ie^{\frac{1}{2}q.k} \sin\left(\frac{q \wedge k}{2}\right), \quad (4.37)$$

does not contain any terms in v_q or $\bar{S}(q)$ whatsoever although it does contain evaluations of these functions at almost every other wavevector value. This is surprising;

it is as if $\Delta(q)$ contains only interaction terms.

The reason for these inconsistent results is very simple; the vacuum state, $|0\rangle$, is not an eigenstate of the Hamiltonian. Indeed, we see, even from an inspection of the leading order terms,

$$\bar{\rho}_k^\dagger \bar{\rho}_k = |A(k)|^2.(a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger + a_{-k} a_{-k}^\dagger + a_{-k} a_k) + ..., \qquad (4.38)$$

that it is very unlikely that the vacuum state can even be used as a reasonable approximation to an eigenstate of the Hamiltonian (even if it is possible for any of the higher order terms to cancel the offending leading terms they would have to be of comparable size compared to these terms and hence the pure creation terms that they would generate in the Hamiltonian would be as equally problematic as the original leading order terms). We also see that (regardless of any higher order terms) the Hamiltonian in this picture does not commute with the total quasiparticle number operator,

$$\left[\bar{H}, \sum_k a_k^\dagger a_k\right] \neq 0, \qquad (4.39)$$

and so we have a system where the Hamiltonian does not (even approximately) conserve the total quasiparticle number - *i.e.* in this picture the total number of magnetorotons is not a good quantum number and is not conserved through time by the electron Hamiltonian. It would seem, then, that this first order bosonisation of the projected density operators suffers from some problems and we still have to explain why the magnetoroton energy, $\Delta(q)$, does not depend on v_q or $\bar{S}(q)$ but does depend on almost every other wavevector.

This did not happen in the case of liquid helium because the density operators behaved like generalised coordinates in a Hamiltonian which contained squared coordinate terms (in the interparticle potential) and squared momentum terms (in the kinetic energy). This meant that the Hamiltonian could be considered as a collection of independent harmonic oscillators with additional (small) perturbation terms so that the vacuum state could be taken as a good approximation to the ground state of the system. This is not the case in the lowest Landau level. Inspection of the Hamiltonian shows that if we are to interpret the projected density operators as

collective coordinates then we have a Hamiltonian which is made up of coordinate terms only - there are no momentum terms in the Hamiltonian for the lowest Landau level. This explains why we cannot both regard the projected density operators as collective coordinates and simultaneously expect the Hamiltonian to behave like a collection of harmonic oscillators in these coordinates to any sensible approximation.

A physical analogy can be made with a phonon system. In a crystal lattice system, if we linearise the forces acting on each atom, we can express the Hamiltonian as a collection of independent harmonic oscillators which has the vacuum state (*i.e.* no phonons) as its ground state. Of course, in reality the forces acting on the atoms are not so simple and so the Hamiltonian also contains higher order terms (expressible in the raising and lowering operators of the phonon modes) which represent phonon interactions. As long as these interaction terms are comparatively small then it is still a good approximation to treat the ground state of the system as the vacuum state of phonons. But, if these interaction terms increase to be much larger than the linear terms then the vacuum state will cease to be a good approximation to the ground state of the system. We suggest that the first order bosonisation scheme fails in the lowest Landau level because it is a very strongly interacting system, with no kinetic term to compensate, so that the vacuum state of the bosonisation scheme is not a good approximation to the ground state of the system.

4.4 Quasiparticle Transformations

Conditions on Feasible Bosonisations

We have seen in the last section that the first order bosonisation of the projected density operators using magnetoroton creation and annihilation operators ran into some difficulty. We would still like to pursue the idea of a boson representation for the projected density operators as a formalism is likely to facilitate calculations of interesting quantities for the lowest Landau level states.

Mathematically, there are an infinite number of different possible bosonisations for the system. Most of these will not be amenable to simple physical interpretation

or of any in use in doing calculations. We want to restrict the number of possible bosonisations to only those which will have a sensible physical interpretation. Therefore, before we attempt to formulate such a bosonisation scheme, we shall consider some of the restrictions which we might like to place on such a representation.

Natural Representations

Just from the fact that any Hilbert spaces with the same dimension are isomorphic to one another we know that any system will have more than one (usually infinitely many) bosonisations which are mathematically possible. We are only interested in those schemes which might turn out to be useful as a means to study the physical system. We would, therefore, like to have some method of discriminating between useful and non-useful bosonisations.

Consider the bosonisation schemes for the one dimensional harmonic oscillator in figures 4.1 and 4.2, where ψ_n is the nth eigenstate of the oscillator Hamiltonian, $a^\dagger$ is a second quantised boson creation operator, $\hat{S}$ is the boson symmetrising operator and $\phi(q_i)$ is the single particle state of the ith particle which is created by the creation operator (q_i is the ith particle's coordinate).

$$
\begin{aligned}
\psi_n &\longleftrightarrow \left(a^\dagger\right)^n |0\rangle \leftrightarrow \hat{S}\phi(q_1)\phi(q_2)\ldots\phi(q_n) \\
&\vdots \qquad\quad \vdots \\
\psi_3 &\longleftrightarrow a^\dagger a^\dagger a^\dagger |0\rangle \leftrightarrow \hat{S}\phi(q_1)\phi(q_2)\phi(q_3) \\
\psi_2 &\longleftrightarrow a^\dagger a^\dagger |0\rangle \leftrightarrow \hat{S}\phi(q_1)\phi(q_2) \\
\psi_1 &\longleftrightarrow a^\dagger |0\rangle \leftrightarrow \hat{S}\phi(q_1) \\
\psi_0 &\longleftrightarrow |0\rangle \leftrightarrow 0
\end{aligned}
$$

FIGURE 4.1: A Natural Representation

Both of these representations are equally valid in a mathematical sense as they simply identify one basis element in the original space with one unique basis ele-

$$\psi_n \quad \longleftrightarrow \quad \left(a^\dagger\right)^n |0\rangle \leftrightarrow \hat{S}\phi(q_1)\phi(q_2)\ldots\phi(q_n)$$

$$\vdots \qquad \vdots$$

$$\psi_3 \quad \longleftrightarrow \quad a^\dagger a^\dagger a^\dagger |0\rangle \leftrightarrow \hat{S}\phi(q_1)\phi(q_2)\phi(q_3)$$

$$\psi_2 \quad \longleftrightarrow \quad a^\dagger |0\rangle \leftrightarrow \hat{S}\phi(q_1)$$

$$\psi_1 \quad \longleftrightarrow \quad a^\dagger a^\dagger |0\rangle \leftrightarrow \hat{S}\phi(q_1)\phi(q_2)$$

$$\psi_0 \quad \longleftrightarrow \quad |0\rangle \leftrightarrow 0$$

FIGURE 4.2: An Unnatural Representation

ment in the boson space. Obviously, though, the first one is a much more natural representation in a (human) physically intuitive way.

What is the difference between these representations ? We suggest that the important difference is the connection between the operators in the two spaces. In the first one the harmonic oscillator raising and lowering operators, $b^\dagger$ and b, have the simple representation

$$b^\dagger \quad \longleftrightarrow \quad a^\dagger \tag{4.40}$$

$$b \quad \longleftrightarrow \quad a \tag{4.41}$$

in terms of the boson particle creation and annihilation operators. On the other hand, the second picture leads to a representation of the original harmonic oscillator operators, in terms of the boson operators, which is badly divergent if we attempt to expand it in powers of the a,$a^\dagger$'s, *i.e.*

$$b^\dagger \longleftrightarrow a^\dagger \quad -\sqrt{1}|1\rangle\langle 0| - \sqrt{2}|2\rangle\langle 1| - \sqrt{3}|3\rangle\langle 2| \tag{4.42}$$

$$+\sqrt{1}|2\rangle\langle 0| + \sqrt{2}|1\rangle\langle 2| + \sqrt{3}|3\rangle\langle 1| \tag{4.43}$$

where

$$|T\rangle\langle M| = \frac{1}{\sqrt{(T+1)!M!}} \left(a^\dagger\right)^T a^M \hat{A}_M \tag{4.44}$$

and

$$\hat{A}_M = \prod_{\substack{j=0 \\ j \neq M}}^{\infty} \frac{j - \hat{n}}{j - M} = \prod_{\substack{j=0 \\ j \neq M}}^{\infty} \frac{j - a^\dagger a}{j - M} \qquad (4.45)$$

so that

$$\hat{A}_M|n\rangle = \begin{cases} |M\rangle & \text{if } n = M \\ 0 & \text{if } n \neq M \end{cases} . \qquad (4.46)$$

We see from the form of the $\hat{A}_M$ that the representation of the harmonic oscillator operators cannot be represented as a sensible series expansion in the $a^\dagger, a$'s.

It seems likely that representations like figure 4.1 are more natural and hence will be more useful than representations such as figure 4.2. It appears, then, that the boson representation of the important operators is a good guide to the usefulness of a particular bosonisation scheme. We define a natural representation as being one in which the physically relevant operators (*i.e.* the Hamiltonian and projected density operators in the lowest Landau level) have a sensible series expansion in terms of the boson creation and annihilation operators.

The point of the preceding example was to demonstrate that almost any bosonisation is mathematically possible but that most of them will be quite unnatural in a physical sense. For example, we could easily think of a representation where

$$\bar{\rho}_k^\dagger |\psi_0\rangle \longleftrightarrow a_k^\dagger |0\rangle \qquad (4.47)$$

but such a representation, while mathematically possible, may still break sensible physical restrictions (e.g as discussed in the last section). It was also to demonstrate that the assumption that the projected density *operator* can be expanded in a power series of raising and lowering *operators* and the vacuum state can be used as the ground state of the projected Hamiltonian is an assumption that does restrict the range of possible bosonisations.

Unless we can solve for the system exactly, we will probably need to make series and perturbation type approximations and so we will only look for bosonisation schemes which are natural representations. This means that we will restrict our attention to only those bosonisation schemes which allow the projected density operator

to be expanded as a series in powers of the raising and lowering operators (*i.e.* natural representations). We would also expect a natural representation to allow a more physical interpretation of the projected density operators.

Conservation Rules

In order to allow a sensible physical interpretation of the bosonisation scheme and to ensure that some good quantum numbers are easily accessible (for the purpose of building up statevectors which are suitable starting points for perturbation expansions) we would like the representation to respect certain conservation rules. The first one is that we would like the interparticle Hamiltonian to conserve the total k and for the projected density operators to be shift operators for this quantity. We will therefore only look for bosonisation schemes in which the modes are labeled by this same k vector. For example, the representation

$$\bar{\rho}_k = \sum_q e^{-\frac{ik}{\sqrt{2}}a_q} e^{-\frac{ik^*}{\sqrt{2}}a_q^\dagger} \tag{4.48}$$

is a possible bosonisation for the projected density operator but it is not a bosonisation in terms of k labeled modes such that $\bar{\rho}_k$ is a shift operator for this label.

We would also like the Hamiltonian to conserve total boson particle number so that quasiparticle number is a good quantum number (in the first order bosonisation, from last section, the Hamiltonian failed to do this and this was the source of the problems we then encountered with that scheme).

So, defining the operators

$$\hat{N} = \sum_k a_k^\dagger a_k = \sum_k \hat{n}_k \tag{4.49}$$

and

$$\hat{K} = \sum_k k\hat{n}_k, \tag{4.50}$$

we wish to look for bosonisation schemes which have the properties

$$[\bar{H}, \hat{N}] = 0 \tag{4.51}$$

and

$$[\bar{H}, \hat{K}] = 0. \qquad (4.52)$$

We also expect, from symmetry grounds, to find a representation such that the ground state of the system is an eigenstate of K with eigenvalue **0**.

Finite Reliability

In order to carry out calculations in the boson scheme it will probably be necessary to describe wavefunctions in terms of some set of basis states. The most natural basis set in a bosonisation scheme is the number states (especially since, from above, we are looking for a representation where $\hat{N}$ and $\hat{K}$ are conserved). It is possible, though, that for some representations physical quantities (such as energy) may diverge for some number states. There can still be, of course, other states (superpositions of different number states) in the boson representation which do represent physical states. Because we wish to work in the number state basis we would like to avoid representations in which the number states represent such unphysical states and so avoid having to cancel out various diverging terms with one another. We therefore define those representations in which the number states have finite values for physical quantities as being finitely reliable and we will look only for those bosonisation schemes which have this property. We note that the finite reliability condition is an unphysical one in the sense that it is basis dependent but that if we expect to be able to form a useful physical interpretation of the boson scheme in terms of number states it will be a very important condition. We also note that finite reliability is an important condition if the boson number states are to have a physical interpretation as it requires them all to have finite energy.

4.5   Second Order Bosonisation of the Projected Density Operators

4.5.1   An Exact Solution

We will now look for a bosonisation scheme for the projected density operators which is a natural transformation, finitely reliable and which obeys the conservation rules mentioned in the last section.

¿From the square coordinate nature of the projected Hamiltonian (and the fact that $v(q)$ is an even function) we see that in order for the total boson number to be conserved and the k labeling condition to be satisfied it is necessary that the expansion (which we are assuming exists) for the projected density operators must not contain any pure creation terms (or any pure annihilation terms). This immediately removes the possibility of a first order bosonisation scheme of the form used in section 4.3. It might also be suspected that the relation $\bar{\rho}_k^\dagger = \bar{\rho}_{-k}$ would enable us to conclude that the expansion for the projected density operators would only contain terms with equal numbers of creation and annihilation operators (*i.e.* that $\bar{\rho}_k$ itself would conserve boson number) but this is complicated by the different possible combinations of higher order terms which might appear in the Hamiltonian and, so, we do not consider the details here. It might be possible simply to require the projected density operators to conserve boson number but we shall not do so as we will only really be concerned with the second order terms in the $\bar{\rho}_k$ expansion for which such a requirement (or result) makes no difference.

We also wish the projected density operator to be represented as a shift operator for $\hat{K}$. From the assumptions of a natural representation and the conservation rules discussed earlier we can write our bosonisation scheme as

$$\bar{\rho}_k^\dagger = \sum_h B(k,h)a_{h+k}^\dagger a_h + higher \ \ order \ \ terms \qquad (4.53)$$

where the higher order terms are ones with at least three raising/lowering operators. The expansion must obey the relations

$$\bar{\rho}_k^\dagger = \bar{\rho}_{-k} \qquad (4.54)$$

and

$$[\bar{\rho}_k, \bar{\rho}_q] = \phi(k, q)\bar{\rho}_{k+q} \tag{4.55}$$

where

$$\phi(k, q) = e^{\frac{k^*q}{2}} - e^{\frac{q^*k}{2}} \tag{4.56}$$

$$= 2ie^{\frac{1}{2}k.q}\sin\left(\frac{k \wedge q}{2}\right). \tag{4.57}$$

These relations translate into a set of simultaneous equations for all of the coefficients in the boson expansion from equation 4.53. We can see from inspection, however, that the first set of such equations are equations in the $B(k, h)$ coefficients only. These are the only coefficients for which this occurs; the higher order equations contain combinations of the coefficient sets - including the second order ones. This is interesting as it means that either an exact solution exists in second order terms only or no solution exists at all (The second order coefficients identically equal to zero solution is easily shown to imply that all the coefficients are zero). This, of course, does not preclude the possibility that solutions exist which do possess higher order terms.

The relations governing the second order coefficients are readily found to be

$$B(k, h) = B^*(-k, h + k) \tag{4.58}$$

and

$$B(q, h).B(k, q + h) - B(k, h).B(q, h + k) = \phi(k, q).B(k + q, h). \tag{4.59}$$

Within the assumptions already mentioned, we find one exact solution to these equations - appendix N. We therefore suggest the second order bosonisation scheme [54]

$$\bar{\rho}_k^\dagger = \sum_h B(k, h).a_{h+k}^\dagger a_h$$

$$B(k, h) = 2i.e^{-\frac{1}{4}|k|^2}.\sin\left(\frac{k \wedge h}{2}\right) \tag{4.60}$$

for $k \neq 0$.

We see that in this representation the projected density operator does not create a boson quasiparticle at wavevector k but displaces ones which are already there. This is consistent with the form of the magnetoroton energy, $\Delta(k)$, discussed earlier.

Using the result that [50]

$$\bar{S}(k) \to (1-\nu)e^{-\frac{1}{2}|k|^2} \quad \text{as} \quad |k| \to \infty \tag{4.61}$$

and

$$\bar{\rho}_k^\dagger.\bar{\rho}_k = \sum_h |B(k,h)|^2.(1+n_{h+k}).n_h + \sum_{\substack{h,s \\ h\neq s}} B(k,h).B^*(k,s).a_{h+k}^\dagger a_s^\dagger a_h a_{s+k} \tag{4.62}$$

we can derive the result that

$$n_g = \frac{\nu(1-\nu)}{4\pi} \tag{4.63}$$

$$\frac{n_e}{n_g} = \frac{2}{1-\nu} \tag{4.64}$$

where we have assumed that n_g is finite (which is a necessary condition for the static structure factor to be finite valued) and

$$n_g = \frac{N_g}{\Omega} \tag{4.65}$$

$$= \frac{1}{\Omega}\sum_k n_k \tag{4.66}$$

$$= \frac{1}{4\pi^2}\int d^2s.n(s) \tag{4.67}$$

is the number of bosons per unit area.

In this picture (except for $\nu = 0$ and $\nu = 1$) the ground state is not a vacuum state but rather it is an interacting sea of bosons and the $\bar{\rho}_k$ operator acts to redistribute these bosons.

4.5.2   The Random Phase Approximation

Using the bosonisation scheme introduced in the last section we can rewrite the projected Hamiltonian as

$$
\begin{aligned}
\bar{H} &= \frac{1}{2\pi} \sum_k v_k . \bar{\rho}_k^\dagger . \bar{\rho}_k \\
\bar{\rho}_k^\dagger . \bar{\rho}_k &= \sum_h |B(k,h)|^2 . (1 + n_{h+k}) . n_h + \\
&\quad \sum_{\substack{h,s \\ h \neq s}} B(k,h) . B^*(k,s) . a_{h+k}^\dagger a_s^\dagger a_h a_{s+k} .
\end{aligned}
\tag{4.68}
$$

We see that the Hamiltonian contains a number operator term (a highly nonlinear one) which is always positive and another term which oscillates between positive and negative. This suggests that a zeroth order approximation for the ground state may be found via a random phase argument. Therefore we suppose that, for the projected static structure factor, the oscillations of the second term approximately cancel out to zero. This leaves the only contribution to come from the (always positive) number term. This is equivalent to considering the oscillating term as a perturbation to the (dominant) number term and it is easy to see that the first order energy correction to a number state caused by the oscillating term is zero, suggesting that the RPA might be quite useful.

Using the R.P.A. in this way we can obtain an integral equation relating the projected static structure factor to the average boson distribution:

$$
2\pi\nu\bar{S}(k) = \int d^2s . |B(k,s)|^2 . n(s)[1 + n(s+k)] .
\tag{4.69}
$$

This is a quite complicated nonlinear integral equation which defines the relationship between the projected static structure factor and the boson number distribution. Calculations have been made for the projected static structure factor [50] for several possible ground states. It would be interesting, then, to be able to calculate the number distribution, $n(s)$, from the above formula, using these previously calculated values for $\bar{S}(k)$.

We suggest an iteration scheme for solving equation 4.69 which is based on the

simplifying fact that

$$k \wedge q = k_x q_y - k_y q_x = k'.q \tag{4.70}$$

where $k' = (-k_y, k_x) = ik$.

We have from equation 4.69 and assuming $n(s) = n(|s|)$ that

$$
\begin{aligned}
2\pi\nu e^{\frac{1}{2}|k|^2} \bar{S}(k) &= 2 \int d^2s.n(s) - 2 \int d^2s.\cos(k \wedge s)\, n(s) \\
&\quad + 4 \int d^2s.\sin^2\left(\frac{k \wedge s}{2}\right) n(s)n(s+k)
\end{aligned}
\tag{4.71}
$$

$$
\begin{aligned}
\Rightarrow \int d^2s.\cos(k.s)\, n(s) &= \pi\nu(1-\nu) - \pi\nu e^{\frac{1}{2}|k|^2} \bar{S}(k) \\
&\quad + 2 \int d^2s.\sin^2\left(\frac{k \wedge s}{2}\right) n(s)n(s+k).
\end{aligned}
\tag{4.72}
$$

So, we have

$$\mathcal{F}_c[n(s)] = F(k) + G(k; n) \tag{4.73}$$

where

$$F(k) = \pi\nu(1-\nu) - \pi\nu e^{\frac{1}{2}|k|^2} \bar{S}(k) \tag{4.74}$$

and

$$G(k; n) = 2 \int d^2s.\sin^2\left(\frac{k \wedge s}{2}\right) n(s)n(s+k). \tag{4.75}$$

We therefore suggest the iterative scheme

$$n_{i+1}(s) = \mathcal{F}_c^{-1}\left[F(k) + G(k; n_i)\right] \tag{4.76}$$

to solve equation 4.69 for the number distribution, $n(s)$. We expect that the Fourier transform nature of the iteration should make the scheme computationally tractable. Computational calculations, though, are not considered here.

4.5.3 The Third Order Correlation Function

If we have a known expression for $\bar{S}(k)$ (*i.e.* the calculation for the Laughlin states performed in [50]) then we can find $n(s)$ by solving the integral equation 4.69. Once $n(s)$ is known it is quite simple to calculate any of the correlation functions (within the R.P.A.) - they will just be integrals involving $n(s)$. We would not expect the

R.P.A. to respect phase properties but we might expect that it may be a reasonable approximation for energy type calculations - *i.e.* for calculating the Hermitian correlation functions or the imaginary parts of the odd correlation functions (such as the three point one). Within the R.P.A. the third order correlation function is given by the integral

$$2\pi\nu.P(k,q) =$$

$$\int d^2h.B(-k-q,h)B(k,h)B(q,h-k).n(h-k-q)\left[1+n(h-k)\right]\left[1+n(h)\right].$$

$$+ \int d^2h.B(-k-q,h)B(k,h-q)B(q,h).n(h-k-q)n(h-q)\left[1+n(h)\right]$$
(4.77)

The higher order correlation functions can, of course, be calculated in a similar fashion using the R.P.A. and doing so will always result in an integral over the distribution $n(s)$. This calculation scheme may be useful for more than simply direct calculations of correlation functions. It may be useful as a means of studying variational ground states of the form suggested by Girvin [45, page 387]

$$\Psi = \left(\sum_q a_q \bar{\rho}_q^\dagger \bar{\rho}_q + \sum_{k,q} b(k,q)\bar{\rho}_{k+q}^\dagger \bar{\rho}_q \bar{\rho}_k + \dots \quad \dots\right).\psi. \qquad (4.78)$$

It might also be interesting to analyse the ground state of the bosonised Hamiltonian using perturbation techniques, using for a starting point the R.P.A. number distribution - but as a number state. Such work would likely involve a significant numerical component and we do not consider it here.

4.6 Summary

We discussed a first order bosonisation scheme for the projected density operators and identified problems with this picture. We then suggested a second order bosonisation scheme which was an exact solution of the commutation relations. We discussed an approximation scheme which led to an integral transform method for calculating the correlation functions of the system. We comment that the bosonisation scheme

employed here leads to a significantly different physical interpretation than the first order schemes discussed at the beginning of the chapter.

CHAPTER 5

Rotations, Translations and the Infinite Limit in the Lowest Landau Level

5.1 Overview

We describe the Laughlin wavefunction [55] which is the ground state for an infinitely hard interparticle potential. We then discuss what sort of ground state we would expect for softer interactions. We analyse the translationally generated wavefunctions suggested by Musaelin and Joynt. We identify some problems with these states. We then go on to suggest a set of rotationally generated states, which overcome these problems, as good variational groundstate wavefunctions.

We also put forward an argument which suggests that the class of rotationally generated wavefunctions will always contain the exact ground state of the (idealised) system for the Laughlin fractions. We note that this argument also predicts (from first principles) that the Laughlin fraction ground states are incompressible - *i.e.* that there is an energy gap involved in moving ν away from these fractions.

We suggest that the results of this chapter will be especially relevant to systems possessing circular symmetry such as quantum dots.

5.2 Ground States of the 2D Electron System

5.2.1 The Laughlin Wavefunction

In this section we intend to study the nature of the possible ground states of the 2D electron system in high magnetic field. The starting point of our discussion is the Laughlin wavefunction [45, chapter 7] [55], which is given by (unnormalised)

$$\psi_L = \prod_{i<j}(z_i - z_j)^m . e^{-\frac{1}{4}\sum_{k=1}^{N}|z_k|^2} \qquad , \tag{5.1}$$

or in the Lowest Landau level notation it is simply

$$\psi_L = \prod_{i<j}(z_i - z_j)^m. \tag{5.2}$$

This wavefunction is, at the moment, the approximation most widely used to represent the ground state of the FQHE system for filling factors

$$\nu = \frac{1}{m} \qquad , m = 1, 3, 5, 7, \dots \quad . \tag{5.3}$$

We will concentrate on the $\nu = 1/3$ case for most of this discussion as it is the simplest non-trivial case and the other $\nu = 1/m$ filling factor wavefunctions can be obtained as a simple extension of the $\nu = 1/3$ case. The Laughlin wavefunction is antisymmetric as it must be in order to represent a system of fermionic particles. It is completely contained within the lowest Landau level, ensuring that it has a low energy (*i.e.* the lowest possible kinetic energy) so that it is a good candidate for the ground state of the system. It is an eigenstate of angular momentum which we know a non-degenerate ground state must be because of the fact that angular momentum commutes with the Hamiltonian (or if there are degenerate ground states then there must exist at least one ground state which is an eigenstate of angular momentum). The Laughlin state is also an eigenstate of the projected angular momentum operator (projected onto the lowest Landau level) which is a result of the fact that the angular momentum operator conserves Landau level. The wavefunction represents a system with a uniform density (at least in the infinite limit) which, again, we would expect our true ground state to do. One of the Laughlin wavefunction's most important properties is that, due to the

power dependency on the particle separation, the wavefunction keeps the electrons apart from one another, *i.e.* there is only a very small probability of any two electrons being close to one another. This mutual repulsion between the electrons, described by the Laughlin wavefunction, is what gives the wavefunction such a low energy when measured against a Hamiltonian with a strongly repulsive interaction potential.

5.2.2 The Classical Plasma Description

A useful way of working with the Laughlin wavefunction is via an analogy with a thermodynamical classical system, often called the classical plasma description. To construct this picture we use the fact that for any multiplication only operator, A, (such as an operator that depends only on position *i.e.* $v(r)$) we have

$$\langle\psi|A(\tilde{r})|\psi\rangle = \int d^2\tilde{r}.A(\tilde{r})|\psi(\tilde{r})|^2. \qquad (5.4)$$

which we can rewrite as

$$\langle\psi|A(\tilde{r})|\psi\rangle = \int d^2\tilde{r}.A(\tilde{r})e^{\ln|\psi(\tilde{r})|^2}. \qquad (5.5)$$

and again as

$$\langle\psi|A(\tilde{r})|\psi\rangle = \int d^2\tilde{r}.A(\tilde{r})e^{-\beta E(\tilde{r})}. \qquad (5.6)$$

where we have defined

$$E(\tilde{r}) = -\frac{1}{\beta}\ln|\psi(\tilde{r})|^2 \qquad (5.7)$$

and β is an arbitrary constant. Now, if we choose to interpret β in terms of some imaginary temperature, T, according to

$$\beta = \frac{1}{kT}, \qquad (5.8)$$

where k is Boltzmanns constant, we see from equation 5.6 that the expectation value of any multiplication operator, $A(\tilde{r})$, in the state ψ is absolutely equivalent to the classical thermodynamical Boltzmann average for $A(\tilde{r})$ over a system at temperature T, energy of configuration $E(\tilde{r})$ and density of states $\rho(\tilde{r}) = constant$ (noting that the configuration energy depends only on position).

Using this classical thermodynamical picture to describe the Laughlin wavefunction gives an energy of configuration

$$E(\tilde{r}) = \frac{1}{\beta}\left(-2m\sum_{i<j}\ln|r_i - r_j| + \frac{1}{2}\sum_{k=1}^{N}|r_k|^2\right) \tag{5.9}$$

or, in the complex notation

$$E(\tilde{z}) = \frac{1}{\beta}\left(-2m\sum_{i<j}\ln|z_i - z_j| + \frac{1}{2}\sum_{k=1}^{N}|z_k|^2\right) \tag{5.10}$$

and if we set β to $1/m$ then we have

$$E(\tilde{z}) = -2m^2\sum_{i<j}\ln|z_i - z_j| + m\frac{1}{2}\sum_{k=1}^{N}|z_k|^2. \tag{5.11}$$

This describes the potential energy of a two dimensional plasma with temperature $T = m/k$. Particles with charge m mutually repel one another via a logarithmic interaction potential. At the same time, though, they are also being pulled towards the centre by a potential, identical to what would be caused by an equivalent logarithmic attraction to a uniform, neutralising positive background. Because such a plasma would be electrically neutral everywhere, we see that the Laughlin states are of uniform density in their large scale features. In fact, Laughlin [45, chapter 7] was able to conclude, based on numerical studies, that the temperature/potential energy scale ratio of the plasma was high enough to guarantee a liquid state for the relatively small values of m we are interested in. This means, in turn, that the Laughlin states are of uniform density in their small scale features as well.

5.2.3 Ground States for Softer Interaction Potentials

The Laughlin wavefunction is known to provide a good variational ground state for the 2D system from numerical calculations on small numbers of particles using the coulomb potential [45, chapter 7]. In fact it is also known that the Laughlin wavefunction becomes an exact eigenstate of the Hamiltonian in the limit as the interaction potential becomes infinitely hard *i.e.* $v(r_i - r_j) \to \delta^2(r_i - r_j)$ [56]. Now, obviously

the exact nature of the ground state of the system will depend on the form of the interaction potential in the Hamiltonian,

$$H \;\; = \;\; \sum_q \tilde{v}_q . \rho_q^\dagger \rho_q \tag{5.12}$$

$$= \;\; \sum_{i,j=1}^{N} v(r_i - r_j), \tag{5.13}$$

and it is an interesting question to ask in what way does the ground state of the system depend on the interaction potential. To discuss this we will concentrate mainly on the hardness/softness of the interaction and the different ground states that we would expect for interactions of varying degrees of softness/hardness. For example the effective interaction potential can be softened from the Coulomb form by finite thickness effects and/or by the presence of image charges in metallic gates on top of experimental samples.

Consider the radial distribution function of a system where the interaction potential is very hard (*i.e.* short range) in figure 5.1. We see that where the potential is high there is a small probability of finding another electron and this probability increases as the repulsive potential decreases. Thus, for a short range potential we see that there will be a narrow, deep correlation hole surrounding each electron. This means that an electron excludes other particles from a small area around its own position but has little effect on particles further away. This is what we would expect for a short range interaction.

Now, consider the case where the interaction potential is softer (*i.e.* longer ranged) in figure 5.2. Again, the probability of finding an electron near our reference particle is inversely related to the potential at that distance. Now, though, because the potential decreases less rapidly, the radial distribution function increases more slowly. This creates a wider, shallower correlation hole around each electron than we would have in the previous case. This means that an electron does not exclude the other particles from the small area around its own position to the same extent as in the last case but, now, it does have a stronger repulsive effect at larger distances. This is, of course what we would expect in a system with a longer range interaction.
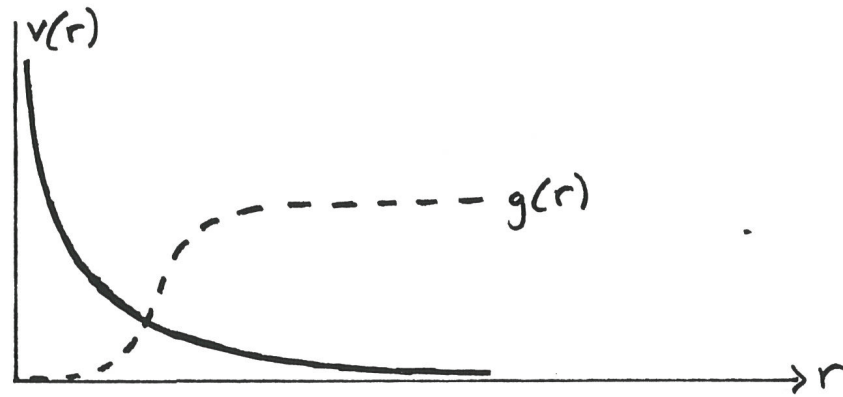
FIGURE 5.1: Short Range (Hard) Potential
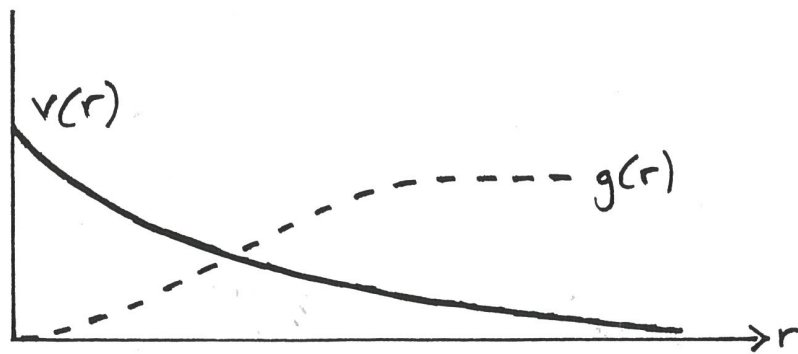
FIGURE 5.2: Long Range (Soft) Potential

So, we see that if we have a system with a hard, short range potential and we then soften that interaction, the effect on the wavefunction will be to change the ground state to one with a wider, shallower correlation hole around each electron.

5.3 Translationally Generated Variational Ground States

5.3.1 The Translationally Generated Wavefunctions

The Laughlin state has a very good variational energy for strongly repulsive potentials and is the exact ground state for the infinitely hard potential (*i.e.* a billiard ball interaction). It seems likely, then, that for softer potentials the Laughlin wavefunction will not be the best variational ground state. Such softer potentials can be caused by finite thickness effects in the two dimensional electron system and so we might expect that there will be some transition thickness above which the ground state departs from the Laughlin form and some better approximation is then possible.

A recent suggestion by Musaelian and Joynt [57] considered a generalisation of the Laughlin form which would be suitable for representing the ground state wavefunction as the inter-electron repulsion was softened, by, say, finite thickness effects. They were motivated to search for states with broken rotational symmetry in the FQHE system by the existence of such states in the analogous classical plasma.

They considered (unnormalised) wavefunctions of the form

$$\psi_\alpha = \prod_{i<j}(z_i - z_j)(z_i - z_j - \alpha l_c)(z_i - z_j + \alpha l_c).e^{-\frac{1}{4l_c^2}\sum_{k=1}^N |z_k|^2} \tag{5.14}$$

where $l_c = \sqrt{\hbar/eB}$ is the magnetic length, α is a complex constant and therefore αl_c is a position displacement vector. Rewriting in our usual notation, we have

$$\psi_\alpha = \prod_{i<j}(z_i - z_j)(z_i - z_j - \alpha)(z_i - z_j + \alpha). \tag{5.15}$$

Where we have concentrated on the filling factor 1/3 case and we will continue do so for most of what follows; the generalisation to other values of filling factor ($\nu = 1/m$) is usually obvious. This wavefunction is antisymmetric, is completely contained within the lowest Landau level and it reduces to the Laughlin wavefunction when $\alpha = 0$. It

is also of uniform density (far from the edges of the disc), translationally invariant over large distances and has broken rotational symmetry.

The principal difference between the Laughlin state and these translationally generated wavefunctions is the fact that in the Laughlin wavefunction an electron is kept apart from every other electron's position by a $|z_i - z_j|^6$ law whereas in the translationally generated states each electron is only kept apart from the other electron's precise positions by a $|z_i - z_j|^2$ law but is also kept away from the positions $z_j - \alpha$ and $z_j + \alpha$ by a similar power two law. This means that whereas the Laughlin wavefunction has a deep and narrow correlation hole in its radial distribution function, as we would expect to be the case for a hard interaction potential, the translationally generated wavefunctions tend to have wider and shallower correlation holes about each electron. Thus, we would therefore expect that these states will have lower energy for sufficiently soft interaction potentials. This suggests that at some degree of softness, in the inter-electron repulsion, the Laughlin wavefunction ceases to be the most energetically favoured state and becomes unstable to more energetically favourable translationally generated wavefunctions.

5.3.2 The Classical Plasma Picture

The translationally generated wavefunctions of the last section can be described in terms of a classical thermodynamical system in a very similar manner as the Laughlin wavefunction was in section 5.2.2. In order to best illuminate the comparison between the Laughlin and the translationally generated wavefunctions it is helpful to first, slightly, alter our perception of the Laughlin plasma picture. In section 5.2.2 we wrote the energy of the plasma as

$$E(\tilde{r}) = \frac{1}{\beta} \left(-2m \sum_{i<j} \ln |r_i - r_j| + \frac{1}{2} \sum_{k=1}^{N} |r_k|^2 \right) \tag{5.16}$$

or, in the complex notation with $m = 3$, *i.e.* $\nu = 1/3$,

$$E(\tilde{z}) = \frac{1}{\beta} \left(-3 \times 2 \sum_{i<j} \ln |z_i - z_j| + \frac{1}{2} \sum_{k=1}^{N} |z_k|^2 \right). \tag{5.17}$$

We interpreted this energy as representing a system of particles where each particle is pulled inwards towards the origin by the second term while the first term represents the mutual repulsion between every pair of particles. For future comparison we will find it beneficial to rewrite the energy term in the form

$$E(\tilde{z}) = \frac{1}{\beta}\left(-2\sum_{i<j}\{\ln|z_i - z_j| + \ln|z_i - z_j| + \ln|z_i - z_j|\} + \frac{1}{2}\sum_{k=1}^{N}|z_k|^2\right). \quad (5.18)$$

We interpret this energy in the following manner: each individual electron experiences an attraction towards the origin as before but, now, rather than simply be repelled by the other electrons directly, it is repelled by a collection of virtual particles where each other electron generates three, *i.e. m*, of these virtual particles. It just so happens, in the Laughlin case at least, that each of the three virtual particles, belonging to each electron, occupies the same position. The Boltzmann probability is maximised when the potential energy is minimised which in turn occupies that region in phase space where the classical force acting on each electron is zero. We see then that the configuration most likely to be taken by the classical plasma system is one where the attractive force towards the origin acting on each particle is exactly cancelled out by an equal and opposite force repelling each particle away from the centre, which is caused by the repulsive interaction that each particle suffers from all the other particles.

The benefits of the virtual particle picture become readily apparent when we consider the more general translationally generated wavefunctions of section 5.3.1. These wavefunctions can also be easily transformed to represent a classical thermodynamical system, in a similar fashion to the Laughlin state in section 5.2.2. The energy of the system corresponding to the wavefunction

$$\psi_\alpha = \prod_{i<j}(z_i - z_j)(z_i - z_j - \alpha)(z_i - z_j + \alpha).e^{-\frac{1}{4}\sum_{k=1}^{N}|z_k|^2} \quad (5.19)$$

is

$$E_\alpha(\tilde{z}) = \frac{1}{\beta}\left(-2\sum_{i<j}\{\ln|z_i - z_j| + \ln|z_i - z_j - \alpha| + \ln|z_i - z_j + \alpha|\} + \frac{1}{2}\sum_{k=1}^{N}|z_k|^2\right).$$
$$(5.20)$$

We see that now, according to the virtual particle interpretation, that each electron is still pulled towards the origin, as before, but that the particle-particle interaction has changed. Each electron suffers from the repulsive effects of the same collection of virtual particles as in the Laughlin case (all the other electrons each generating three virtual particles as previously) but, now, each of the virtual particles generated by the same electron have different positions, *i.e.* two out of every set of three virtual particles having been translated outwards, away from the electron by α and $-\alpha$ (see figure 5.3).

The Laughlin wavefunction led to a potential energy which caused an electron to be excluded from an area around any other electron position by virtue of the three repelling virtual particles at each electron position. In the translationally generated states, though, because two of the virtual particles belonging to each electron have been translated outwards from the electron position we see that an electron is now excluded from a larger area around each other electron but by less force. This leads to the wider, shallower correlation hole around each particle that we would expect to find in the ground state of a system with a softened interaction potential (*i.e.* a potential softer than the one represented by the Laughlin wavefunction).

5.3.3 Calculations and Results

Musaelian and Joynt performed numerical calculations to compare the energies of the translationally generated ground states with the energy of the Laughlin wavefunction. They used Monte Carlo simulations of the equivalent classical plasma and allowed for the softening effect of finite thickness on the effective interparticle potential by assuming the form

$$v(r) = \frac{e^2}{\epsilon\sqrt{r^2 + \lambda^2}} \tag{5.21}$$

for the inter-electron potential v. This form was suggested by Zhang and Das Sarma [29] and shown to be a reasonable approximation for a square shaped quantum well. The parameter λ is related to the layer thickness of the well, t, by $\lambda \approx 0.2t$.
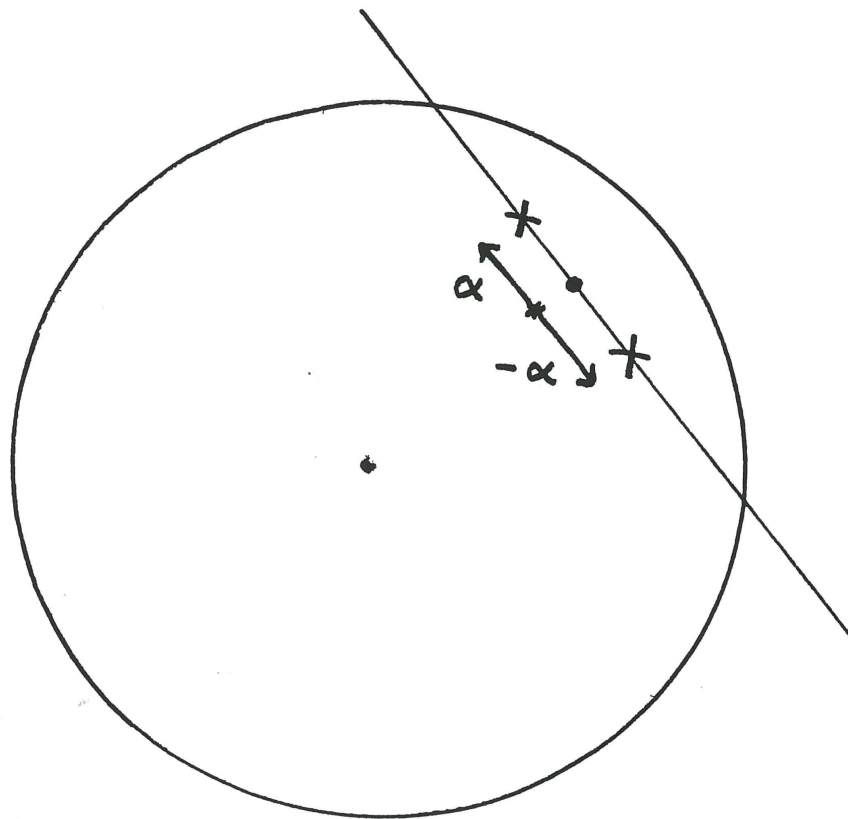
FIGURE 5.3: Virtual Particle Plasma for ψ_α

This gave them a formula for the total energy per particle of the system, U_α,

$$U_\alpha = \frac{n_e e^2}{2\epsilon} \int d^2r. \frac{1}{\sqrt{r^2 + \lambda^2}} \left[g_\alpha(\mathbf{r}) - 1 \right].$$

(5.22)

where n_e is the electron number density, $g_\alpha(\mathbf{r})$ is the radial distribution function for the wavefunction ψ_α and we have remembered that $g_\alpha(\mathbf{r})$ depends on the direction of $\mathbf{r}$ as well as its magnitude when α is non zero.

They calculated that the Laughlin wavefunction ($\alpha = 0$) was indeed the most energetically favourable one for the Coulomb potential ($\lambda = 0$) as may have been expected for such a hard potential. They did find, though, that for sufficiently soft potentials the Laughlin state could become unstable to translationally generated wavefunctions. This corresponds to the electron system undergoing a transition from the Laughlin state to a translationally generated state with finite α at a certain critical value of λ (*i.e.* thickness). They calculated, for example, that at a field of ten Tesla the system undergoes a transition to finite α at $\lambda_{critical} = 4.1 \pm 1.5$, which corresponds to a thickness of $t = 1600\mathring{A}$.

5.4 The Density Distribution

5.4.1 The Values for a Uniform System

We would like to be able to study the translationally generated ground states of Musaelian and Joynt in more detail. In order to do this it will be helpful to first build some mathematical tools suited to the study of the 2D system. We will concentrate on the radial density distribution of the system (radial because of circular symmetry) which, in a system with constant kinetic energy, is the most important determinant of the energy of the system.

We begin by examining a classical system of uniform density. Now, we know that (due to circular symmetry) that we can write the density at any point, $\rho_r(\mathbf{r})$, as a function of radius, $r = |\mathbf{r}|$, only. Further, since r can only take positive values we can actually write the density as a function of the variable

$$u = \frac{1}{2}r^2$$

(5.23)

without any loss of generality. We will often use this variable for reasons of convenience, which will become apparent later. We calculate the average value of u^m for the system assuming circular symmetry and that the density is uniform between $r = 0$ and $r = R$, with value ρ, and is zero for all values of r which are greater than R. We then obtain, for a state obeying these conditions, that

$$\left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{2} r_j^2 \right)^m \right\rangle = \frac{1}{\Omega \rho} \int d^2 \mathbf{r} . \rho(\mathbf{r}) \left(\frac{1}{2} r^2 \right)^m . \tag{5.24}$$

which we write in polar coordinates as

$$\left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{2} r_j^2 \right)^m \right\rangle = \frac{1}{\Omega \rho} \int_0^R \int_0^{2\pi} .\rho \left(\frac{1}{2} r^2 \right)^m .r.dr.d\theta \tag{5.25}$$

and transforming by $u = \frac{1}{2} r^2$, $U_R = \frac{1}{2} R^2$ we get

$$\left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{2} r_j^2 \right)^m \right\rangle = \frac{2\pi}{\Omega \rho} \int_0^{U_R} .\rho u^m .du \tag{5.26}$$

and because of

$$\Omega = \pi R^2 = 2\pi U_R \tag{5.27}$$

we find that

$$\left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{2} r_j^2 \right)^m \right\rangle = \frac{1}{U_R} \int_0^{U_R} .u^m .du \tag{5.28}$$

which leaves us with the result that

$$\left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{2} r_j^2 \right)^m \right\rangle = \frac{1}{1+m} U_R^m. \tag{5.29}$$

This is of course an extrinsic quantity, so in order for us to be able to assign meaningful values to our results in the limit as N (*i.e.* R, U_R) becomes infinitely large we use the intrinsic quantities

$$\left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{1}{2} \frac{r_j^2}{N} \right)^m \right\rangle = \frac{1}{1+m} \left(\frac{1}{\nu} \right)^m \tag{5.30}$$

and

$$\left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{u_j}{U_R} \right)^m \right\rangle = \frac{1}{1+m}. \tag{5.31}$$

These quantities give us information about the large scale features of the system *i.e.* the defining shape of the radial density distribution. We see that, for the case of uniform density, we obtain a very simple result. It will be very useful to compare these values to the lowest Landau Level quantum mechanical calculations which we will discuss later.

5.4.2 The Symmetric Gauge and the Area of the System

In the last section we implicitly assumed that the electrons were confined to a bounded area (which may or may not be allowed to become infinitely large) which is circular in shape and centered on the point $z = 0$. Fortunately, it turns out to be the case that any many particle quantum state in the lowest Landau level which has a bounded value of n_{max} will satisfy these conditions (n_{max} is used to denote the highest power of z appearing in the wavefunction when it is represented in the projected Hilbert space of the lowest Landau level (symmetric gauge) as is described in chapter 2).

This can be seen quite easily by considering the single particle basis states of the lowest Landau level (symmetric gauge). These states are given by

$$\phi_n = z^n . e^{-\frac{1}{4}|z|^2} \tag{5.32}$$

which leads to

$$|\phi_n|^2 = |z|^{2n} . e^{-\frac{1}{2}|z|^2} \tag{5.33}$$

and we see that this expression is radially symmetric and can therefore be rewritten in terms of the u variable

$$|\phi_n|^2 = u^n . e^{-u} \tag{5.34}$$

where we have ignored the normalisation constants throughout.

By taking first and second derivatives of this last expression it is readily shown that the probability density takes its maximum value at

$$u_{maximum} = n \tag{5.35}$$

and that its inflexions occur when

$$u_{inflexion} = n \pm \sqrt{n}. \tag{5.36}$$

The points of inflexion are a good guide as to the sharpness of the probability density. We see that the density is centered around $u = n$ within a range of roughly $n - \sqrt{n} \leftrightarrow n + \sqrt{n}$.

As n becomes infinitely large we can say that the density becomes more sharply focussed about the $u = n$ region and that the area encompassed by the particle is $\Omega = 2\pi n$. This is because as n becomes infinitely large the presence of the particle in the area beyond $u = n$ becomes negligible. We see that this extra area, $2\pi\sqrt{n}$, vanishes as a fraction of the area enclosed by the electron, $2\pi n$, as n becomes infinitely large. It is worth noting that the expectation value of the area of the electron agrees with this value. This is easily seen by considering

$$\langle \phi_n | \Omega | \phi_n \rangle = \langle \phi_n | 2\pi u | \phi_n \rangle \tag{5.37}$$
$$= \langle \phi_n | 2\pi \bar{u} | \phi_n \rangle \tag{5.38}$$
$$= \langle \phi_n | 2\pi \partial z | \phi_n \rangle \tag{5.39}$$
$$= 2\pi(n + 1). \tag{5.40}$$

Where, of course, the 1 is negligible as n becomes infinitely large.

This means that whenever a state has a bounded value of n_{max} then we can say that the area of the system is

$$\Omega = 2\pi n_{max} \tag{5.41}$$

in the limit as $n_{max} \to \infty$ and that

$$n_{max} = \frac{1}{\nu} N \tag{5.42}$$

showing that for any state with finite density that n_{max} must be bounded in relation to N, the number of particles.

The Laughlin states (and the translationally generated states) obey the above conditions and hence have an area given by $\Omega = 2\pi n_{max}$ where $n_{max} = N/\nu$ as well as being centered on the point $z = 0$. We can now expect that these quantum states should be comparable to the uniform system discussed in the last section $i.e.$ the Laughlin states have particles at every value of n all the way up to the maximum value n_{max} and we therefore expect the density of the system to be roughly uniform

out to $u = n_{max}$, where it then falls sharply to zero. This leads us conclude that the radial density distributions of these states will be comparable with the density distributions of the uniform systems discussed in the last section.

5.4.3 U and the Angular Momentum

We will now begin to examine the nature of the radial density distribution for the quantum mechanical Laughlin and translationally generated states. In this section we will study the most important of the large scale terms, the first order average of u,

$$\left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{u_j}{U_R} \right) \right\rangle = \left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{r_j^2}{R^2} \right) \right\rangle . \tag{5.43}$$

We are interested in states which are completely contained within the lowest Landau level and so we will represent our wavefunctions and operators in the projected Hilbert space so that

$$\langle \psi | u | \psi \rangle = \langle \psi | \bar{u} | \psi \rangle = \langle \psi | \partial z | \psi \rangle \tag{5.44}$$

where we have used

$$\bar{u} = \overline{\left(\frac{1}{2} z^* z \right)} = \partial z . \tag{5.45}$$

This, of course, trivially extends to the many particle case as

$$\left\langle \psi \left| \frac{1}{N} \sum_{j=1}^{N} \left(\frac{u_j}{U_R} \right) \right| \psi \right\rangle = \left\langle \psi \left| \frac{1}{N} \sum_{j=1}^{N} \left(\frac{\partial_j z_j}{U_R} \right) \right| \psi \right\rangle \tag{5.46}$$

or in simpler operator notation

$$\sum_{j=1}^{N} \bar{u}_j = \sum_{j=1}^{N} \partial_j z_j . \tag{5.47}$$

The main reason for the importance of the first order u operator (equation 5.47) is the fact that it is completely equivalent to the projected total canonical angular momentum operator in the lowest Landau level. We can this see by considering the single particle case as follows:

We have that

$$L = -i\hbar \left[x \partial_y - y \partial_x \right] \tag{5.48}$$

where L is, of course, the z component of the angular momentum but since it is the only component in the two dimensional system we shall continue to refer to it as the angular momentum.

Now, we have a complete basis, $\{\phi_n\}$, for the lowest Landau level, H_0,

$$\phi_n = z^n.e^{-\frac{1}{4}|z|^2} \tag{5.49}$$

$$= (x+iy)^n.e^{-\frac{1}{4}\left(x^2+y^2\right)} \tag{5.50}$$

and it can be easily shown by elementary calculation that

$$L.\phi_n = n\hbar.\phi_n \tag{5.51}$$

which compares with

$$z\partial.\phi_n = n.\phi_n \tag{5.52}$$

and

$$\bar{u}.\phi_n = \partial z.\phi_n = (n+1).\phi_n \tag{5.53}$$

so that we see that

$$L.\psi = \bar{L}.\psi = \hbar z\partial.\psi = \hbar\left(\bar{u}-1\right).\psi \tag{5.54}$$

for any state $\psi \in H_0$.

Extension to the many particle case is straightforward:

$$\bar{L} = \sum_{j=1}^{N} \bar{L}_j \tag{5.55}$$

$$= \hbar \sum_{j=1}^{N} z_j\partial_j \tag{5.56}$$

$$= \hbar \sum_{j=1}^{N} \left(\bar{u}_j - 1\right) \tag{5.57}$$

$$= \hbar \sum_{j=1}^{N} \bar{u}_j - \hbar N. \tag{5.58}$$

We see immediately that, because the projected angular momentum and the first order average u operators differ from one another only by multiplicative constant and

additive constant terms, the projected angular momentum operator will have exactly the same eigenstates as the first order average u operator.

We also see that for any $\psi \in H_0$ that $L\psi = \bar{L}\psi$. This is simply a result of the fact that the action of the angular momentum operator does not alter the kinetic energy in any way *i.e.* the angular momentum operator preserves Landau level. This has an important consequence as can be seen from the following argument:

We know that

$$
\begin{align}
[H, L] &= 0 \tag{5.59} \\
H^\dagger &= H \tag{5.60} \\
L^\dagger &= L \tag{5.61} \\
P_0^\dagger &= P_0 \tag{5.62} \\
P_0^2 &= P_0 \tag{5.63} \\
\bar{H} &= P_0 H P_0 \tag{5.64} \\
\bar{L} &= P_0 L P_0 \tag{5.65} \\
P_0 \psi &= \psi \qquad \forall \psi \in H_0 \tag{5.66}
\end{align}
$$

where H is the unprojected Hamiltonian, L is the unprojected total angular momentum and P_0 is the projection operator onto the lowest Landau level, H_0.

Now, consider

$$
\begin{align}
\langle \psi | \left[\bar{H}, \bar{L} \right] | \psi \rangle &= \langle \psi | \bar{H} \bar{L} | \psi \rangle - \langle \psi | \bar{L} \bar{H} | \psi \rangle && , \ \forall \psi \in H_0 \\
&= \langle \psi | P_0 H P_0^2 L P_0 | \psi \rangle - \langle \psi | P_0 L P_0^2 H P_0 | \psi \rangle \\
&= \langle \psi | H P_0 L | \psi \rangle - \langle \psi | P_0 L P_0 H | \psi \rangle \\
&= \langle \psi | H P_0 L | \psi \rangle - \langle \bar{L}^\dagger \psi | H | \psi \rangle \\
&= \langle \psi | H L | \psi \rangle - \langle L^\dagger \psi | H | \psi \rangle \\
&= \langle \psi | H L | \psi \rangle - \langle \psi | L H | \psi \rangle \\
&= \langle \psi | \left[H, L \right] | \psi \rangle \\
\langle \psi | \left[\bar{H}, \bar{L} \right] | \psi \rangle &= 0 \qquad\qquad \forall \psi \in H_0 \tag{5.67}
\end{align}
$$

and hence

$$\left[\bar{H}, \bar{L}\right] = 0 \tag{5.68}$$

and similarly

$$\left[H, \bar{L}\right] = 0. \tag{5.69}$$

So we see that the projected Hamiltonian and angular momentum operators commute with one another. This is a non trivial result. It is not necessarily the case that when two operators commute that their projections will commute also - for example the z^* and z operators are proof of this. From our earlier discussion it also follows that the projected average u operator commutes with the projected Hamiltonian due to its eigenequivalence with the projected canonical angular momentum, so that we have

$$\left[\bar{H}, \sum_{j=1}^{N} \bar{u}_j\right] = 0 = \left[\bar{H}, \bar{L}\right]. \tag{5.70}$$

This leads us to conclude, by similar arguments as discussed before, that we can expect the ground state of the projected Hamiltonian to also be an eigenstate of the projected average u operator as well as the projected angular momentum operator (and, indeed, we can reach a similar conclusion regarding all of the eigenstates of the projected Hamiltonian).

The Laughlin states (equation 5.2),

$$\psi_L = \prod_{i<j} (z_i - z_j)^{\frac{1}{\nu}} \qquad , \tag{5.71}$$

where $1/\nu = 1, 3, 5, 7, 9 \ldots$

are readily seen to be eigenstates of the projected total u operator,

$$\left(\sum_{l=1}^{N} \bar{u}_l\right).\psi_L = \left(\sum_{l=1}^{N} (p_l + 1)\right).\psi_L \tag{5.72}$$

$$= \left(\frac{1}{\nu}\frac{N(N-1)}{2} + N\right).\psi_L \tag{5.73}$$

where

$$p_l = z_l \partial_l \tag{5.74}$$

is the power operator and has the property

$$p_j.z_j^n = n.z_j^n \tag{5.75}$$

and

$$\bar{L} = \hbar \sum_{j=1}^{N} p_j. \tag{5.76}$$

Calculating the average u value for these Laughlin states, we obtain

$$\left\langle \psi_L \left| \frac{1}{N} \sum_{j=1}^{N} \frac{u_j}{N} \right| \psi_L \right\rangle = \frac{1}{2}\frac{1}{\nu} - \frac{1}{N}\left(\frac{1}{2\nu} - 1\right) \tag{5.77}$$

which becomes

$$\left\langle \psi_L \left| \frac{1}{N} \sum_{j=1}^{N} \frac{u_j}{N} \right| \psi_L \right\rangle = \frac{1}{2}\frac{1}{\nu} \tag{5.78}$$

in the limit of infinitely large N, $i.e.$ $N \to \infty$, which is exactly the same value as we obtained for the system of uniform density in subsection 5.4.1, namely

$$\left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{u_j}{N}\right)^m \right\rangle = \frac{1}{1+m}\left(\frac{1}{\nu}\right)^m \tag{5.79}$$

and

$$\left\langle \frac{1}{N} \sum_{j=1}^{N} \left(\frac{u_j}{U_R}\right)^m \right\rangle = \frac{1}{1+m}. \tag{5.80}$$

The extra $1/N$ terms in equation 5.77 are due to non-large scale properties of the density (such as the tail at the edge of the system where the density falls to zero) which become insignificant in the infinite limit. Indeed since we expect the ground state of the system to be not only an eigenstate of angular momentum but also to have a uniform density (in its large scale features at least) then we can say that we expect the ground state, ψ_0, to be an eigenstate of the projected average u operator with eigenvalue given by

$$\frac{1}{N} \sum_{j=1}^{N} \left(\frac{\bar{u}_j}{N}\right)^m .\psi_0 = \frac{1}{1+m}\left(\frac{1}{\nu}\right)^m .\psi_0 \tag{5.81}$$

in the infinite limit.

We see that the Laughlin states are eigenstates of the projected angular momentum and have density distributions consistent, up to the first power of u, with having a uniform density in the infinite limit .

5.4.4   The U^m Operators

In the last section we examined the average u operator and used it to study the radial density distribution of the system. In this section we will generalise to the average u^m operators and use these to study the density profile of the system in more detail.

The u^m operators suffer from a slight complication caused by the algebra of operators in the lowest Landau level. This is embodied in the relations

$$\overline{u^m} = \overline{\left(\frac{1}{2}z^*z\right)^m} \tag{5.82}$$

$$= \partial^m z^m \tag{5.83}$$

whereas

$$\bar{u}^m = \left(\overline{\frac{1}{2}z^*z}\right)^m \tag{5.84}$$

$$= (\partial z)^m \tag{5.85}$$

$$= (1 + z\partial)^m \tag{5.86}$$

and so we see that

$$\overline{u^m} \neq \bar{u}^m. \tag{5.87}$$

This has the principal effect of making the sums involved when we evaluate expectation values of the average u^m operators extremely tedious to calculate. We can get round this problem by using the following result:

For the many particle system, it is true that

$$\frac{1}{N}\sum_{j=1}^{N}\frac{\overline{u^m}}{N^m} \longleftrightarrow \frac{1}{N}\sum_{j=1}^{N}\frac{p^m}{N^m} \tag{5.88}$$

for any finite m in the limit as $N \to \infty$

or, equivalently,

$$\frac{1}{N}\sum_{j=1}^{N}\frac{\overline{u^m}}{U_R^m} \longleftrightarrow \frac{1}{N}\sum_{j=1}^{N}\frac{p^m}{n_{max}^m} \tag{5.89}$$

for any finite m in the limit as $N \to \infty$

or, again, in another useful form,

$$\frac{1}{N} \sum_{j=1}^{N} \frac{\partial^m z^m}{N^m} \longleftrightarrow \frac{1}{N} \sum_{j=1}^{N} \frac{(z\partial)^m}{N^m} \tag{5.90}$$

for any finite m in the limit as $N \to \infty$.

That is, the above operators are equivalent to one another for any finite value of m in the limit as the number of electrons, N, becomes infinitely large.

This can be seen to be true from the following argument:

We use the commutation relation

$$[\partial, z] = 1 \tag{5.91}$$

to show that (appendix B)

$$\partial^m z^m = \prod_{j=1}^{m} (z\partial + j) \tag{5.92}$$
$$= (1 + z\partial)(2 + z\partial)\ldots\ldots(m + z\partial). \tag{5.93}$$

Now, multiplying out the brackets allows us to rearrange the projected average u^m

operator into the form

$$\frac{1}{N}\sum_{j=1}^{N}\frac{\partial_j^m z_j^m}{N^m} = \frac{1}{N}\sum_{j=1}^{N}\frac{(z_j\partial_j)^m}{N^m}$$

$$+ \quad \frac{1}{N}\sum_{j=1}^{N}\frac{\alpha_{m-1}(m)}{N}\frac{(z_j\partial_j)^{m-1}}{N^{m-1}}$$

$$+ \quad \frac{1}{N}\sum_{j=1}^{N}\frac{\alpha_{m-2}(m)}{N^2}\frac{(z_j\partial_j)^{m-2}}{N^{m-2}}$$

$$+ \quad \cdots \quad \cdots \quad \cdots$$

$$+ \quad \frac{1}{N}\sum_{j=1}^{N}\frac{\alpha_2(m)}{N^{m-2}}\frac{(z_j\partial_j^2)}{N^2}$$

$$+ \quad \frac{1}{N}\sum_{j=1}^{N}\frac{\alpha_1(m)}{N^{m-1}}\frac{(z_j\partial_j)}{N}$$

$$+ \quad \frac{1}{N}\sum_{j=1}^{N}\frac{\alpha_0(m)}{N^m} \tag{5.94}$$

where the $\alpha_k(m)$ are functions of m and are finite valued whenever m is finite valued. Remembering that the average p^m operators are always finite valued for states with finite filling factor (because $n_{max} \sim N$), *i.e.*

$$\frac{1}{N}\sum_{j=1}^{N}\frac{(z_j\partial_j)^m}{N^m} \sim N^0 \tag{5.95}$$

in the limit as $N \to \infty$,

we immediately see that all of the terms on the right hand side of equation 5.94, apart from the first one, vanish for any finite value of m in the limit as N becomes infinitely large (*i.e.* for m independent of N).

It is clear, then, from this expression that

$$\frac{1}{N}\sum_{j=1}^{N}\frac{\partial^m z^m}{N^m} \longleftrightarrow \frac{1}{N}\sum_{j=1}^{N}\frac{(z\partial)^m}{N^m} + O\left(\frac{1}{N}\right) \tag{5.96}$$

for any finite m in the limit as $N \to \infty$

as claimed (where the $O(1/N)$ represents a term which scales with $1/N$ as the number of particles is increased).

We now wish to use this result to study the density distribution of some quantum states in the two dimensional system. Consider the Slater Determinant wavefunctions

$$\psi_{Sl} \quad = \quad \prod_{i<j} \left(z_i^n - z_j^n \right) \tag{5.97}$$

$$= \quad \begin{vmatrix} z_1^0 & \cdots & \cdots & \cdots & \cdots & z_N^0 \\ z_1^n & \cdots & \cdots & \cdots & \cdots & z_N^n \\ z_1^{2n} & \cdots & \cdots & \cdots & \cdots & z_N^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^{n(N-1)} & \cdots & \cdots & \cdots & \cdots & z_N^{n(N-1)} \end{vmatrix} \tag{5.98}$$

where the filling factor is given by $\nu = \frac{1}{n}$ and $n = 1, 3, 5, 7, 9 \ldots$.
(in fact n can also be even but we are not interested in those cases)

For the $\nu = 1$ case then ψ_{Sl} is equal to the corresponding Laughlin wavefunction. These states, like the Laughlin states, have an even spread of single particle $\bar{u}$ eigenfunctions and so we would expect them to be of uniform density distribution in their large scale features in the limit of infinitely large N. That these Slater determinant wavefunctions should have the same large scale density features as the corresponding Laughlin states will be discussed later in section 5.5.3.

Due to the Slater determinant nature of the ψ_{Sl} states the average p^m operator is relatively straightforward to calculate for these wavefunctions and because of the relation

$$\frac{1}{N} \sum_{j=1}^{N} \frac{\partial^m z^m}{N^m} \longleftrightarrow \frac{1}{N} \sum_{j=1}^{N} \frac{(z\partial)^m}{N^m} \tag{5.99}$$

for any finite m in the limit as $N \to \infty$

the average u^m is similarly straightforward to evaluate in the limit as $N \to \infty$ when m is finite. We then have

$$\left\langle \psi_{Sl} \left| \frac{1}{N} \sum_{j=1}^{N} \frac{u^m}{N^m} \right| \psi_{Sl} \right\rangle = \left\langle \psi_{Sl} \left| \frac{1}{N} \sum_{j=1}^{N} \frac{\partial^m z^m}{N^m} \right| \psi_{Sl} \right\rangle \tag{5.100}$$

$$= \left\langle \psi_{Sl} \left| \frac{1}{N} \sum_{j=1}^{N} \frac{(z\partial)^m}{N^m} \right| \psi_{Sl} \right\rangle \tag{5.101}$$

$$= \left(\frac{1}{\nu} \right)^m \frac{1}{N} \sum_{j=0}^{N-1} \left(\frac{j}{N} \right)^m \tag{5.102}$$

$$= \left(\frac{1}{\nu} \right)^m \int_0^1 x^m dx \tag{5.103}$$

$$= \left(\frac{1}{\nu} \right)^m \frac{1}{1+m} \tag{5.104}$$

for m finite and $N \to \infty$

The step where the integral is introduced is explained in appendix C. This is of course exactly the same result which we obtained in equations 5.30 and 5.31. So we see that these Slater determinant wavefunctions are of uniform density in their large scale features (in the limit of an infinitely large number of particles). This agrees with what we would expect to be true of such states and also of the Laughlin wavefunctions and gives us confidence in our results.

5.4.5   The Density Distribution of the Translationally Generated Wavefunctions

We have now established a means of studying the density distributions of quantum states in the lowest Landau level and we have used these methods to verify that the Slater determinant (and hence Laughlin) wavefunctions possess the large scale density features which we would expect *i.e.* those consistent with a system of uniform density. We would now like to apply these methods, in a similar fashion, to study the density distribution of the translationally generated wavefunctions of Musaelian and Joynt (section 5.3).

It can be argued that we should expect the ψ_α states to be uniform, in the limit of infinite number of particles, as we know that the Laughlin states are uniform and the ψ_α states are generated from the Laughlin states by a small translation of some of the nodes in the wavefunction which we would not expect to have any significant effect on the large scale features of the infinitely large system. This is especially clear in the classical plasma picture where we see that the ψ_α states are generated by a small translation of the virtual particles and we would not expect a small, uniform translation to have any effect on the large scale density distribution of the system.

Unfortunately, such arguments ignore the subtleties of working in the infinite limit. The translationally generated states are not, in fact, uniform - not even in the infinite limit. They are, though, uniform in a region about the centre of the system where that region is infinitely far away from the edge of the system. This doesn't help, though, as using the results of the last few sections we can show that the non-uniformity is so significant that it effects even the large scale features of the system. We can see that this is the case from an examination of the translationally generated wavefunctions,

$$\psi_\alpha = \prod_{i<j}(z_i - z_j)(z_i - z_j - \alpha)(z_i - z_j + \alpha), \tag{5.105}$$

which we can expand to give

$$
\begin{aligned}
\psi_\alpha \quad = \quad & \prod_{i<j}(z_i - z_j)^3 \\
+ \quad & \alpha.f_1(\tilde{z}) \\
+ \quad & \alpha^2.f_2(\tilde{z}) \\
+ \quad & \cdots \\
& \vdots \\
+ \quad & \cdots \\
+ \quad & \alpha^{2N(N-1)-1}.f_{2N(N-1)-1}(\tilde{z}) \\
+ \quad & \alpha^{2N(N-1)}.\prod_{i<j}(z_i - z_j).
\end{aligned}
\tag{5.106}
$$

where each $f_l(\tilde{z})$ is an eigenfunction of angular momentum with eigenvalue $3N(N-1) - l$. We note that ψ_α is manifestly not an eigenstate of angular momentum. We

see that the ψ_α wavefunctions are in fact a superposition of a $\psi_L(\nu = 1/3)$ state with $n_{max} = 3(N-1)$, a $\psi_L(\nu = 1)$ state with $n_{max} = N - 1$ and a whole range of states with every possible value of n_{max} between these two extremes. Also these different states all have different expectation values for the average u^m operators which range between the two extremes

$$\prod_{i<j}(z_i - z_j)^3 \Longrightarrow \left\langle \frac{1}{N}\sum_{j=1}^{N}\frac{\overline{u^m}}{N^m}\right\rangle = \frac{3^m}{1+m} \tag{5.107}$$

and

$$\prod_{i<j}(z_i - z_j) \Longrightarrow \left\langle \frac{1}{N}\sum_{j=1}^{N}\frac{\overline{u^m}}{N^m}\right\rangle = \frac{1}{1+m} \tag{5.108}$$

and this, combined with the fact that any antisymmetric polynomial is an eigenstate of the average u^m operator, means that

$$\frac{1}{1+m} < \left\langle \psi_\alpha \left| \frac{1}{N}\sum_{j=1}^{N}\frac{\overline{u^m}}{N^m} \right| \psi_\alpha \right\rangle < \frac{3^m}{1+m} \tag{5.109}$$

for any non-zero, finite value of α. Along with the fact that the value of n_{max} for the ψ_α states is $3N(N-1)$ this shows that the translationally generated states have density distributions whose large scale features are inconsistent with having a uniform density. The actual density distribution of the ψ_α states depends on the value of α. For example, as $\alpha \to 0$ then ψ_α becomes the Laughlin ($\nu = 1/3$) state which is uniform with a radius given by $U_R = 3(N-1)$ (this is the upper limit in equation 5.109). In the other extreme, as $\alpha \to \infty$ then ψ_α becomes the equivalent of a Laughlin ($\nu = 1$) state (effective radius given by $U_R = N - 1$) sitting in the middle of an area (radius given by $U_R = 3(N-1)$) where the density approaches zero as $\alpha \to \infty$. All in all, we see that the translationally generated wavefunctions are a superposition of states with different large scale density distributions and as a result they do not have a uniform density (even in their large scale features) for any non-zero, finite value of α.

5.4.6 The Classical Plasma Picture

We have shown, using quantum mechanical arguments, that the translationally generated wavefunctions do not represent states with a uniform density. The question

then arises as to how this is explained within the classical plasma picture of section 5.3.2. One might argue that since (in the limit of an infinitely large number of particles) the Laughlin plasma is translationally invariant (with uniform density) then a small uniform translation applied to the virtual particles in a uniform manner should not have any effect on the large scale features of this translationally invariant system. This argument, though, falls down on one major obstacle, which is that the Laughlin states are not translationally invariant.

Consider the normal method for constructing translationally invariant systems, which is to construct normalised, translationally invariant single particle states (using periodic boundary conditions) and then to fill these states with a number of particles which is proportional to the area (volume) of the period. This period can then be allowed to become infinitely large, along with the number of particles, so that the density is a constant, independent of the period. This system is truly translationally invariant because the single particle states are translationally invariant.

The Laughlin states (along with any finite density, lowest Landau level states with centre $z = 0$) are somewhat different. Here the single particle states, and hence the multiparticle wavefunction, have an (effectively) bounded area which is given by $\Omega = 2\pi n_{max}$. So we see that such states are never translationally invariant on their largest length scales. As long as n_{max} is proportional to the number of particles (*i.e.* $n_{max} = N/\nu$) we have a state with a finite density and we can consider the limit as $N \to \infty$. But, even in such a limit, the system can, at best, only be translationally invariant over length scales which are small compared to the radius of the system, and only in regions far from the edges of the system. This radius, though, is bounded by a linear relationship with $\sqrt{N}$. We see then, that, no matter how large N is, there is always a length scale (*i.e.* $\sim \sqrt{N}$) over which the system is not translationally invariant. One might think that since this length scale becomes infinitely large as $N \to \infty$ that this difference between the lowest Landau level and the translationally invariant system discussed above is an extremely subtle one of the sort usually ignored by physicists but we shall see (in fact we have already seen) that it does make a serious difference to the physical properties of the system.

The explanation for the non-uniformity which occurs in the translationally gen-

erated wavefunctions is in many ways much clearer in terms of the classical plasma picture than the direct quantum mechanical calculations described in section 5.4.5. Consider figure 5.4.

Imagine that we begin with the Laughlin state, *i.e.* ψ_α with $\alpha = 0$, which we know is uniform in is large scale features. Now, the virtual particles are clumped together in groups of three (corresponding to each electron), all sharing the same position. In the classical plasma the system is uniform and in force equilibrium - *i.e.* the net force repelling any electron away from the centre (due to the virtual particles) is exactly balanced by the attractive force pulling the electron towards the centre (or in other words, the most probable configuration of the classical plasma is the one which has the minimum potential energy and this minimum corresponds to a zero net force on each electron). For the purpose of this discussion, while considering the forces acting on one particular electron, we will divide the virtual particles into two sets in relation to that electron. First, we note that for a particular electron position there will be an area of virtual particles which repel that electron towards the centre. This force towards the centre will be exactly cancelled out by an equal and opposite area of virtual particles just to the other side of the electron. These two virtual particle areas, together, form the first set which, because they have a net zero effect on that electron, we can more or less ignore. The second set of virtual particles is that area left over which is responsible for the total net repulsion from the centre that the electron experiences.

Now, what happens as we increase α ? As α is increased from zero, of every clump of three virtual particles two of them translate away from their original position by α and $-\alpha$ along a direction line defined by the direction of α, as shown in figure 5.4.

How does this movement of virtual particles affect the net force acting on a particular electron ? For the virtual particles in the bulk of the system there is no change because the line density of virtual particles per unit length on the line does not change (there are virtual particles displaced in both directions). For those virtual particles within a distance α from the edge of the system, though, there is a change in the force which they collectively exert on the electron. This is because those virtual particles which are displaced to a position outside the edge of the system can not be replaced
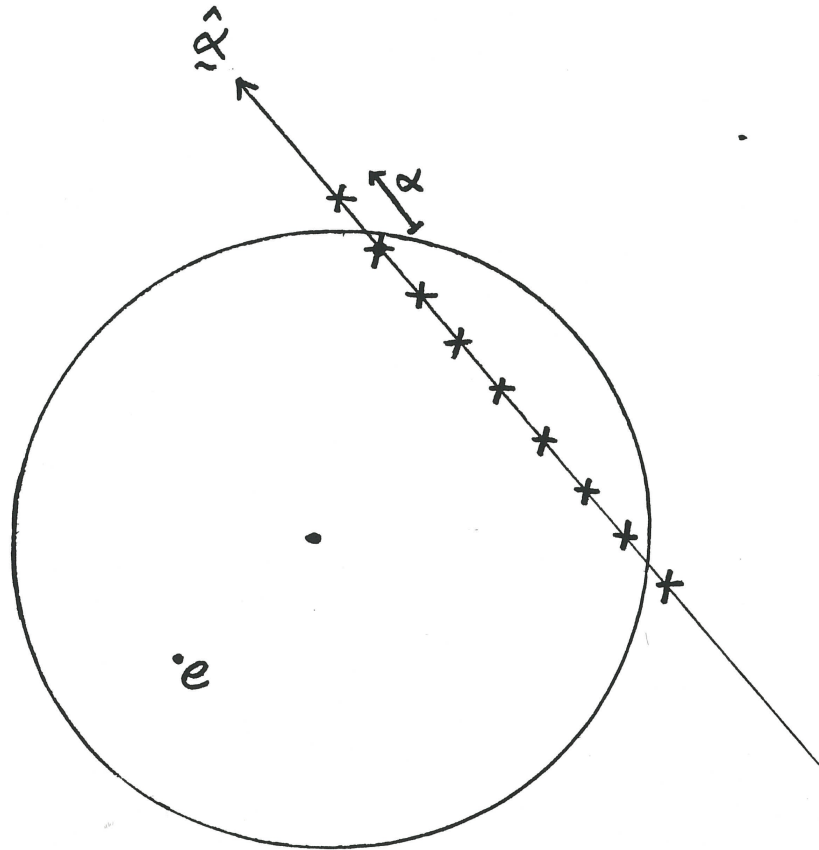
FIGURE 5.4: The Classical Plasma Picture for ψ_α

FIGURE 5.5: Virtual Particles Hopping Positions by $\pm\alpha$. We see that in the middle of the system that the line density of virtual particles does not change - two particles move out of every spot and two move into every spot. On the edges, though, we see that the density does change as particles move into spots which cannot send one back to replace them. This is a result of the bounded size of the system and would not happen in a truly translationally invariant system.

by other virtual particles which are displaced in the opposite direction (this would not happen in a truly translationally invariant system - see figure 5.5). This leads to a net effect where the total distance between the electron and all the virtual particles on the α line is increased which results in a reduction in the net repulsive force pushing the electron away from the centre. So we see that increasing α above zero reduces the repulsive force pushing the electrons away from the centre while the attractive force towards the centre remains the same. This means that the original (uniform) density distribution of $\alpha = 0$ is no longer a configuration of net force equilibrium and hence the original electron configuration is no longer the configuration of lowest potential energy in the classical plasma representation. The only way to restore the net force equilibrium is for the density distribution to change allowing the electrons to move closer in towards the centre of the system, until the increasing repulsion caused by the closer proximity of the virtual particles balances the attractive force towards the centre. So we see that for non-zero α that the configuration of lowest potential energy is one where the electrons are more closely packed in towards the centre of the system and therefore has a non-uniform density distribution (because we know from the quantum mechanical arguments that n_{max} and hence the radius of the system does not change).

In the extreme case where $\alpha \to \infty$ (*i.e.* α becomes infinitely larger than the size of the system) then we see that all of the displaced virtual particles are so far away that they have no effect on any of the electrons and are therefore negligible. This leaves only those virtual particles which are tied to the electron positions - *i.e.* one virtual particle per electron. We know that the net force equilibrium configuration of this system is the same as the $\nu = 1$ Laughlin state (where we have one virtual particle per electron). So we see that as $\alpha \to \infty$ that the system collapses to a $\nu = 1$ Laughlin state with radius $U_R = N - 1$ surrounded by an area of radius $U_R = 3(N - 1)$ (the original radius of the system) where the density tends to zero as α tends to infinity. This is the same result as we obtained quantum mechanically in section 5.4.5 for $\alpha \to \infty$.

We see that the classical plasma picture supports the direct quantum mechanical analysis of section 5.4.5 which concluded that the translationally generated wavefunc-

tions possessed a non-uniform density distribution. We have also seen how this is a direct result of the fact that the Laughlin wavefunctions are not truly translationally invariant states.

5.4.7 The Angular Momentum Eigenoperator

The fact that the translationally generated wavefunctions of Musaelian and Joynt have a non-uniform density distribution casts some doubt on their results suggesting that the Laughlin state could become energetically unstable to these states for softened interaction potentials. It seems very unlikely that any such interaction could favour a state which is non-uniform in its large scale features over one which is uniform over such scales. Musaelin and Joynt, themselves, expressed the belief that the ψ_α states were of uniform density. While it is extremely unlikely that such non-uniform states could have lower energy than the Laughlin wavefunction, non-uniformity is not, on its own, a sufficient enough condition to guarantee that it is impossible.

What is a sufficient condition, though, is the fact that the ψ_α state is a superposition of different eigenstates of $\bar{L}$, the angular momentum, each of different radius (*i.e.* n_{max}) and hence of different density. We can write

$$\psi_\alpha = \sum_M a_M \phi_M \qquad (5.110)$$

where the ϕ_M are eigenstates of angular momentum with eigenvalue M and the $a'_M s$ are chosen so that the state is normalised. We then have that

$$\langle \psi_\alpha | H | \psi_\alpha \rangle = \sum_M |a_M|^2 \langle \phi_M | H | \phi_M \rangle \qquad (5.111)$$

where we have used the fact that the Hamiltonian, H, conserves angular momentum, L, *i.e.*

$$[H, L] = 0. \qquad (5.112)$$

¿From this we see that the energy of the ψ_α wavefunction is simply the weighted average of the energies of all the ϕ_M angular momentum eigenstates. Therefore we can obtain a state with lower energy than ψ_α simply by choosing whichever of the

ϕ_M states has the lowest energy. We can be sure that this will be the one with the lowest density and this state turns out to be

$$\prod_{i<j}(z_i - z_j)^3 \tag{5.113}$$

which is, of course, just the $\nu = 1/3$ Laughlin wavefunction. So we conclude that it is impossible for the translationally generated wavefunctions to ever have a lower energy than the Laughlin state.

5.4.8 The Calculations for the Translationally Generated States

We have concluded that it is impossible for the translationally generated wavefunctions to ever have a lower energy than the Laughlin state but this is contradicted by the results of Musaelian and Joynt [57]. They used numerical methods to calculate that their ψ_α states did, indeed, have lower energy than the Laughlin state for sufficiently softened interaction potentials. Where have they gone wrong and what can be learned from their mistake ?

Essentially they have assumed that the translationally generated states have a uniform density and this false assumption has crept into every aspect of their calculation. They have assumed that the radial distribution function does not depend on the position of the electron chosen which, for a non-uniform system, it does. In formula 5.22 they have used the average density of the system when, because their calculation concentrated on those electrons near the centre of the system where the density of the ψ_α states is higher than the average, they should have used a higher value. A more fundamental problem is that they tried to calculate a figure for the energy per particle by considering only a finite number of electrons near the centre of the system. This is only valid if the system does have a uniform density.

Their numerical studies did not pick up on the non-uniformity of the states. This is because the ψ_α states are uniform over a large area at the centre of the system. In fact, in the infinite limit the translationally generated states are uniform over an infinite area. Unfortunately they have a non-uniform density over an area twice as large as the uniform one (figure 5.6). This means that if one studies only a finite
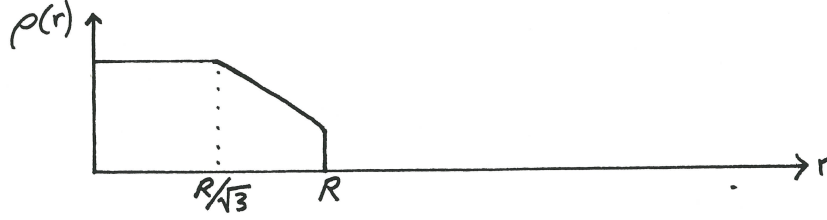
FIGURE 5.6: Radial Density Distribution for ψ_α States. We see that although in the limit $N \to \infty$ there is a region, of area $\to \infty$, with uniform density that there is an area twice as large as this with a non-uniform density.

number of particles, near the centre of the system, then it is possible to believe that the system has a uniform density and although the density of this central area is higher than the average value, for small α the difference is very slight.

We see that physically relevant, as well as subtle and interesting, effects can occur in the infinite limit of the two dimensional electron system in the lowest Landau level. We see that we must be very careful about studying effects in this infinite limit.

5.5 Rotationally Generated Variational Ground States

5.5.1 Conservation of Density Distribution and Other Conditions to be Satisfied by Candidate Ground States

We have seen that the translationally generated wavefunctions of Musaelian and Joynt are not good candidate ground states for the two dimensional electron system. We believe, though, that the essential idea of moving the zeros of the wavefunction (*i.e.* the virtual particles in the plasma picture) out away from the electrons is a good one. This is because to do so will create the wider, shallower correlation hole around each electron which we know will give a lower energy wavefunction for softer potentials. We saw, though, that their attempt to put this idea into practice suffered from serious problems. We would like to try to implement this essential idea while avoiding the obstacles which tripped up the ψ_α wavefunctions. In order to do this we will first look at the conditions which we would expect a good candidate ground state to satisfy.

The first condition is that the wavefunction should be an eigenstate of angular momentum, which is equivalent to being an eigenstate of the operators

$$\frac{1}{N} \sum_{l=1}^{N} \frac{\bar{u}_l}{N} \tag{5.114}$$

and

$$\frac{1}{N} \sum_{l=1}^{N} \frac{p_l}{N}. \tag{5.115}$$

Because we also expect that the ground state, ψ_{gs}, should be uniform in its large scale density features, we know that these eigenoperators should lead to eigenvalues of

$$\left(\frac{1}{N} \sum_{l=1}^{N} \frac{\bar{u}_l}{N} \right) . \psi_{gs} = \frac{1}{2} \left(\frac{1}{\nu} \right) . \psi_{gs} \tag{5.116}$$

and

$$\left(\frac{1}{N} \sum_{l=1}^{N} \frac{p_l}{N} \right) . \psi_{gs} = \frac{1}{2} \left(\frac{1}{\nu} \right) . \psi_{gs} \tag{5.117}$$

in the limit as $N \to \infty$

where we note that the eigenvalues are both the same in the infinite limit. We note that both the Laughlin states, ψ_L 's, and the Slater determinant states, ψ_{Sl} 's, obey these conditions.

Because we expect any good candidate ground state to have uniform large scale density features, we also have the conditions that

$$\left\langle \psi_{gs} \left| \frac{1}{N} \sum_{l=1}^{N} \left(\frac{u_l}{N} \right)^m \right| \psi_{gs} \right\rangle = \left\langle \psi_{gs} \left| \frac{1}{N} \sum_{l=1}^{N} \left(\frac{p_l}{N} \right)^m \right| \psi_{gs} \right\rangle = \frac{1}{1+m} \left(\frac{1}{\nu} \right)^m \tag{5.118}$$

for all finite values of m in the limit as $N \to \infty$.

Again we note that both the Laughlin, ψ_L 's, states and the Slater determinant, $\psi'_{Sl}s$, states obey these conditions.

We have seen already (for example, in section 5.4.6) that the system described by the Laughlin wavefunction is not translationally invariant. The Laughlin states (and any state in the lowest Landau level with centre $z = 0$ and finite density) correspond

to a system with a circularly symmetric boundary - though that boundary may well be infinitely far away from the centre of the system. That the Laughlin wavefunctions are not translationally invariant is readily checked by direct inspection. We have that

$$\left|\psi_{L,\nu=\frac{1}{m}}(\tilde{z})\right|^2 = \prod_{i<j} |z_i - z_j|^{2m} . e^{-\frac{1}{2}\sum_{l=1}^N |z_l|^2} \tag{5.119}$$

whereas we have that

$$\left|\psi_{L,\nu=\frac{1}{m}}(\tilde{z} - \tilde{\alpha})\right|^2 = \prod_{i<j} |z_i - z_j|^{2m} . e^{-\frac{1}{2}\sum_{l=1}^N |z_l - \alpha|^2} \tag{5.120}$$

showing, quite clearly, that the Laughlin states are not translationally invariant. This is a direct consequence of the fact that the underlying single particle basis states (in which ψ_L can be represented as a bounded superposition) are themselves not translationally invariant.

The Laughlin wavefunctions are, though, rotationally invariant as should not be surprising considering the geometry of the system that they represent. That they are rotationally invariant can be seen by direct inspection. We have (using the fact that $f(e^{i\theta}z)$ is the same shape as the function $f(z)$ but rotated by $-\theta$) that

$$\begin{aligned}\left|\psi_{L,\nu=\frac{1}{m}}\left(\widetilde{e^{i\theta}z}\right)\right|^2 &= \prod_{i<j} \left|e^{i\theta}(z_i - z_j)\right|^{2m} . e^{-\frac{1}{2}\sum_{l=1}^N \left|e^{i\theta}z_l\right|^2} \\ &= \prod_{i<j} |z_i - z_j|^{2m} . e^{-\frac{1}{2}\sum_{l=1}^N |z_l|^2} \\ &= \left|\psi_{L,\left(\nu=\frac{1}{m}\right)}(\tilde{z})\right|^2 \end{aligned} \tag{5.121}$$

Showing, quite definitely, that the Laughlin states are rotationally invariant. This, in turn, is a direct consequence of the fact that the underlying single particle basis states used are rotationally invariant. In the same way, it is easily verified that the

Slater determinant states,

$$\psi_{Sl} = \prod_{i<j} \left(z_i^n - z_j^n \right) \tag{5.122}$$

$$= \begin{vmatrix} z_1^0 & \cdots & \cdots & \cdots & \cdots & z_N^0 \\ z_1^n & \cdots & \cdots & \cdots & \cdots & z_N^n \\ z_1^{2n} & \cdots & \cdots & \cdots & \cdots & z_N^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^{n(N-1)} & \cdots & \cdots & \cdots & \cdots & z_N^{n(N-1)} \end{vmatrix} \tag{5.123}$$

with filling factor given by $\nu = \frac{1}{n}$ and $n = 1, 3, 5, 7, 9 \ldots$,

are also rotationally invariant but not translationally invariant. Indeed, if we look at the Hamiltonian of the system, remembering that there is a circularly symmetric boundary (even though it may be infinitely far away from the centre), we have

$$H = \sum_{i<j} v \left(|z_i - z_j| \right) + \sum_{l=1}^{N} U \left(|z_l| \right). \tag{5.124}$$

We see that the Hamiltonian is rotationally invariant but not translationally invariant. As a result of the rotational invariance of the system we would certainly expect that any good candidate ground state would also be rotationally invariant. In fact the condition that a wavefunction be rotationally invariant is entirely equivalent to the condition that the wavefunction is an eigenstate of angular momentum.

With the above conditions in mind, we can now look for candidate variational ground states for the two dimensional system.

5.5.2 The Rotationally Generated Wavefunctions

Bearing in mind the discussion from the last section, we suggest a rotationally generated variational wavefunction of the (unnormalised) form, (for $\nu = 1/3$),

$$\psi_\theta = \prod_{i<j} \left(z_i - z_j \right) \left(z_i - e^{i\theta} z_j \right) \left(z_i - e^{-i\theta} z_j \right) . e^{-\frac{1}{4} \sum_{l=1}^{N} |z_l|^2} \tag{5.125}$$

or, in the lowest Landau level notation,

$$\psi_\theta = \prod_{i<j} \left(z_i - z_j \right) \left(z_i - e^{i\theta} z_j \right) \left(z_i - e^{-i\theta} z_j \right). \tag{5.126}$$

We see that the ψ_θ wavefunctions have their zeros shifted away from the electron positions - creating the wider, shallower correlation hole around each electron which we know leads to lower energies for sufficiently soft interaction potentials. With the ψ_θ states, though, the zeros are rotated away from the electron positions (by an angle θ) rather than being linearly translated away from the electrons. This use of rotations, rather than translations, respects the symmetry of the system and so avoids the problems encountered by the translationally generated states (discussed in the last few sections).

The ψ_θ states are antisymmetric, which is, of course, necessary if they are to represent a fermionic system. The rotationally generated wavefunctions are, quite clearly, eigenstates of angular momentum and have the same eigenvalue as the Laughlin state and the Slater determinant state. In the infinite limit we therefore have that

$$\left(\frac{1}{N} \sum_{l=1}^{N} \frac{p_l}{N} \right) . \psi_\theta = \frac{1}{2} \left(\frac{1}{\nu} \right) . \psi_\theta \tag{5.127}$$

in the limit as $N \to \infty$

which is the only possible result consistent with a system possessing uniform large scale density features.

We have seen then that the rotationally generated wavefunctions are antisymmetric, rotationally invariant, eigenstates of angular momentum and they have the angular momentum eigenvalue which corresponds to a state of large scale uniform density features in the limit of infinite number of particles. In order to study the large scale density features of the ψ_θ states we shall next look at the classical plasma picture.

5.5.3 The Classical Plasma Picture

In the classical plasma picture the rotationally generated state, ψ_θ, leads to a potential energy

$$E_\theta(\tilde{z}) = \frac{1}{\beta}\left(-2\sum_{i<j}\left\{\ln\left|z_i - z_j\right| + \ln\left|z_i - e^{i\theta}z_j\right| + \ln\left|z_i - e^{-i\theta}z_j\right|\right\} + \frac{1}{2}\sum_{k=1}^{N}|z_k|^2\right).$$
(5.128)

So we see that, this time, the virtual particles are rotated away from the electron positions while the attraction to the centre of the system is again unaffected. See figure 5.7.

In order to study the large scale density distribution of the rotationally generated wavefunctions we imagine that we begin with the state $\theta = 2\pi/3$. This is the Slater determinant state, ψ_{Sl}, which we know (from direct quantum mechanical calculation as well as simple inspection) has a density distribution which is uniform in its large scale features (in the infinite limit). We then allow the value of the angle parameter, θ, to change. See figure 5.8.

We will consider the forces acting on a particular electron. For the purposes of our discussion we will divide the virtual particles into thin annuli, all centered on the centre of the system. At our original value of θ the attractive force towards the centre of the system, which acts on the electron, is balanced by the repulsive force from the annuli of virtual particles. As we change θ these virtual particles are rotated within their respective annuli. We see that, because the system is rotationally invariant, the number and density of virtual particles within any annulus remains constant as θ is varied. This in turn means that the repulsive force acting on the electron remains constant and is therefore independent of the value of θ. This means that the force equilibrium configuration is one of large scale uniform density distribution (i.e. the same as the original density distribution with which we started). This, in turn, means that the configuration of lowest potential energy is one of large scale uniform density distribution. Therefore we see, using the classical plasma picture, that all of the rotationally generated, ψ_θ, states represent a system with large scale uniform density features. Indeed, since the Laughlin state, ψ_L, is nothing other than
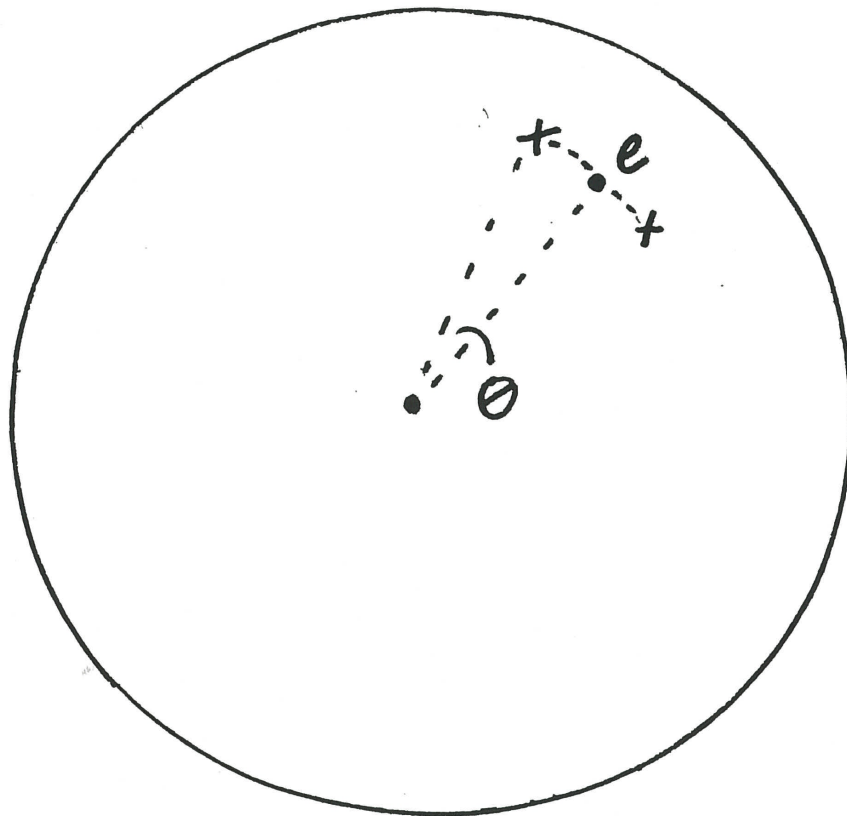
FIGURE 5.7: Virtual Particle Plasma for ψ_θ
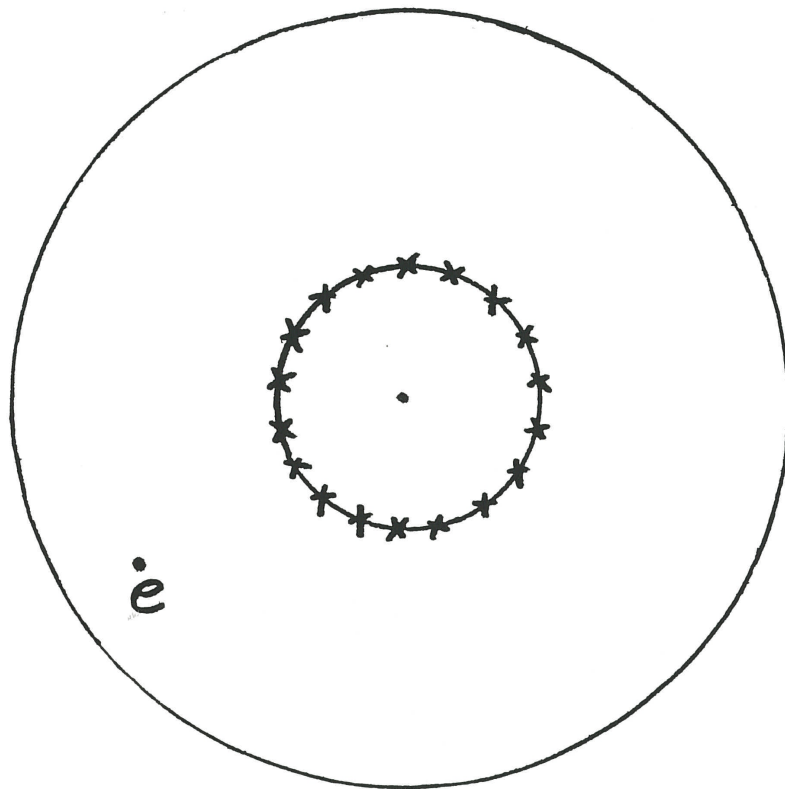
FIGURE 5.8: The Classical Plasma Picture for ψ_θ

the special case $\psi_{\theta=0}$ we see that the above argument constitutes a proof that the Laughlin state is one with large scale uniform density features in the infinite limit. We note the importance of the fact that we were rotating the virtual particles in order to fit in with the rotationally invariant nature of the system and how this avoided the problems encountered when the virtual particles were translated in a non-translationally invariant system.

5.5.4   More General Quantum Mechanical Results

We have looked at some (hopefully) good candidate ground states in the form of the rotationally generated wavefunctions. These states are very unlikely to be the exact ground states of the system (except for, perhaps, special interaction potentials) and so must involve some approximation. We would like to understand, in some detail, whatever this approximation might be.

We begin by studying what it is possible to say about the wavefunction for the system in an exact manner. We know that any wavefunction in the lowest Landau level with finite density and centre $z = 0$ must have a value of n_{max} which is bound by a linear relationship with N ($n_{max} = N/\nu$ as $N \to \infty$ and for the ψ_θ states, $n_{max} = (N-1)/\nu$). This means that any lowest Landau level wavefunction with finite density and centre $z = 0$ can be written as a multivariable polynomial in the z_i's with each z_i being taken to the power of n_{max} somewhere in the polynomial (due to the antisymmetric nature of the wavefunction). From the fundamental theorem of algebra, this means that we can write

$$\psi = P_i(z_i) = A_0^i . \left(z_i - A_1^i\right) . \left(z_i - A_2^i\right) \dots \dots \left(z_i - A_{n_{max}}^i\right) \tag{5.129}$$

$$\text{where each } A_j^i = A_j^i(z_1, z_2, \dots, z_{i-1}, z_{i+1}, \dots, z_N)$$

for any particle index, i.

We now make the assumption that the wavefunction contains only pairwise interaction terms (there are reasons to expect that this is a very good approximation for the Laughlin fractions - see section 5.6 - in fact an exact result for the idealised system). We can then use the fundamental theorem of algebra and the fact that the

state is an eigenstate of angular momentum to show that the A_j^i functions are each linear functions of a single particle complex coordinate (see appendix D). This is as expected as the state must be rotationally invariant.

This means that, taking into account the antisymmetric nature of the state (which includes the fact that no two electrons can ever occupy the same position), we now have (for $\nu = 1/3$)

$$\psi = \prod_{i<j} \left(z_i - z_j\right) \left(z_i - \alpha.z_j\right) \left(z_i - \frac{1}{\alpha}z_j\right) \tag{5.130}$$

where α is a complex constant.

By considering the classical plasma arguments of section 5.5.3 we see that the only values of α which will give ψ a density distribution which is uniform in its large scale features are $\alpha = e^{i\theta}$ where θ is a real number.

We now see that, within the constraints of the ground state being of finite density, centre $z = 0$, an eigenstate of angular momentum and of uniform density distribution in its large scale features, the rotationally generated states,

$$\psi_\theta = \prod_{i<j} \left(z_i - z_j\right) \left(z_i - e^{i\theta}z_j\right) \left(z_i - e^{-i\theta}z_j\right) . e^{-\frac{1}{4}\sum_{l=1}^{N}|z_l|^2}, \tag{5.131}$$

are in fact the most general possible wavefunctions within the approximation of only considering pairwise interaction terms. Therefore, we see that the only way in which it would be possible to obtain a better approximation to the real ground state wavefunction than the $\psi_\theta's$ would be to use a wavefunction which included three particle interaction terms (and/or higher order terms).

5.5.5 From Very Hard to Very Soft Interaction Potentials

We have now established solid reasons why we believe that the rotationally generated wavefunctions, ψ_θ, are good variational ground states for the two dimensional electron system. The question arises, though, as to what is the exact nature of the difference between states with different values of the angle parameter, θ, and what is the relevance of this difference with respect to the energy of the system. The answer

is that the larger the value of θ the wider and shallower the correlation hole around each electron. We therefore expect that the softer the interaction potential of the system the larger the lowest energy value of θ will be. We know that the Laughlin wavefunction, $\psi_L = \psi_{\theta=0}$, is the exact ground state in the limit of the interaction potential becoming infinitely hard [56]. We also know that the Slater determinant state, $\psi_{Sl} = \psi_{\theta=2\pi/3}$, is the exact ground state in the limit as the interaction potential becomes infinitely soft (*i.e.* the Slater determinant state is the natural state of non-interacting electrons and it is the state with the widest, shallowest correlation hole). This suggests the possibility that we have a natural progression starting from $\theta = 0$, for the infinitely hard potential, with the lowest energy value of θ increasing as the potential softens reaching the limiting value of $2\pi/3$ for the infinitely soft potential. It seems unlikely that the Laughlin wavefunction will be the best approximation for the unsoftened Coulomb potential if it is also the best ground state for the infinitely hard interaction. This suggests that the rotationally generated states may not only provide better approximate ground states for thick systems but may well provide a better approximation to the ground state for the unsoftened Coulomb potential (presumably with a very small value of θ).

5.5.6 Rotationally Generated Ground States for Different Filling Factors

So far we have concentrated only on wavefunctions with filling factor $\nu = 1/3$. Generalisation to any of the Laughlin fractions, $\nu = 1/m$, is quite straightforward. For example, the class of rotationally generated wavefunctions, which represent a $\nu = 1/5$ state, is given by

$$\psi_{\theta,\phi} = \prod_{i<j} \left(z_i - z_j\right) \left(z_i - e^{i\theta}z_j\right) \left(z_i - e^{-i\theta}z_j\right) \left(z_i - e^{i\phi}z_j\right) \left(z_i - e^{-i\phi}z_j\right) e^{-\frac{1}{4}\sum_{l=1}^{N}|z_l|^2}.$$

$$(5.132)$$

These states satisfy the same conditions and arguments as the $\nu = 1/3$ states of the last few sections. Clearly, this can be extended to any filling factor of the form $\nu = 1/m$.

For the more general fractions

$$\nu = \frac{n}{2mn+1},$$ (5.133)

corresponding to the integer quantum Hall effect of composite fermions, Jain suggests [58] a wavefunction of the form

$$\nu = P_0. \prod_{i<j} (z_i - z_j)^{2m} . \phi_n$$ (5.134)

where ϕ_n is the wavefunction corresponding to the ground state of the system with $\nu = n$ (*i.e.* in the absence of Landau level mixing this would be the state with n filled Landau levels). If we ignore Landau level mixing and put $n = 1$ then we have

$$\psi = \prod_{i<j} (z_i - z_j)^{2m} . \prod_{i<j} (z_i - z_j)$$ (5.135)

$$= \prod_{i<j} (z_i - z_j)^{2m+1}$$ (5.136)

which is simply the Laughlin state for $\nu = 1/M = 1/(2m+1)$. So we see that these composite fermion ground states are simply the extension of the Laughlin wavefunctions to other filling factors, obtained by replacing the filled Landau level state in the last equation with the state ϕ_n.

We see that, like the Laughlin wavefunctions, these composite fermion states contain no variable parameters which might enable the variational wavefunctions to vary in order to fit different interaction potentials. We may suppose that, like the Laughlin states, the composite fermion wavefunctions of equation 5.134 are best suited to very hard potentials. We therefore suggest that for softer potentials rotationally generated composite fermion wavefunctions might produce a better variational bound to the energy of the system. For example, when $m = 1$ we suggest

$$\psi_\theta = P_0. \prod_{i<j} \left(z_i - e^{i\theta} z_j \right) \left(z_i - e^{-i\theta} z_j \right) . \phi_n$$ (5.137)

and when $m = 2$

$$\psi_{\theta,\phi} = P_0. \prod_{i<j} \left(z_i - e^{i\theta} z_j \right) \left(z_i - e^{-i\theta} z_j \right) \left(z_i - e^{i\phi} z_j \right) \left(z_i - e^{-i\phi} z_j \right) . \phi_n.$$ (5.138)

For the general case we suggest a variational wavefunction of the form

$$\psi_{\{\theta_l\}} = P_0 . \prod_{i<j} \prod_{l=1}^{m} \left(z_i - e^{i\theta_l} z_j\right) \left(z_i - e^{-i\theta_l} z_j\right) . \phi_n \tag{5.139}$$

i.e.

$$\psi_{\{\theta_l\}} = P_0 . \prod_{i<j} \left(z_i - e^{i\theta_1} z_j\right) \left(z_i - e^{-i\theta_1} z_j\right) \left(z_i - e^{i\theta_2} z_j\right) \left(z_i - e^{-i\theta_2} z_j\right) \ldots \ldots$$
$$\ldots \ldots \left(z_i - e^{i\theta_m} z_j\right) \left(z_i - e^{-i\theta_m} z_j\right) . \phi_n . \tag{5.140}$$

These wavefunctions represent a system where the electrons capture vortices as in equation 5.134 but now the vortices are bound to positions slightly displaced from the electron positions. This, we expect, would lead to the wider, shallower correlation hole around each electron that is more suitable for softer potentials.

5.5.7   Symmetry, Rotational and Translational Invariance and the Infinite Limit

We have seen that the system described by the Laughlin states (and, indeed, any lowest Landau level states with centre $z = 0$ and finite density) are not translationally invariant.

Why is it that the single particle basis states,

$$\phi_n = z^n e^{-\frac{1}{4}|z|^2}, \tag{5.141}$$

in the lowest Landau level do have a centre ($z = 0$) and are not translationally invariant? This is because, out of the infinite choice of possible basis sets, we chose that one in which the states were also eigenstates of angular momentum. Now, if we look at (the interaction part of) the Hamiltonian

$$H = \sum_{i<j} v(z_i - z_j) \tag{5.142}$$

we see that this Hamiltonian is not only translationally invariant but that it is also rotationally invariant about any centre. When an angular momentum operator, L_{z_0}, is defined it must be with respect to some centre, z_0. This means that this Hamiltonian commutes with every one of the angular momentum operators, L_{z_0}. When we chose

our basis we assumed a centre of $z_0 = 0$. This assumption of centre $z = 0$ along with our requirement that the system has a finite density is what gives us our bounded value of n_{max} and a system which is rotationally invariant but not translationally invariant.

This is no bad thing. A more realistic Hamiltonian for the system would also include an external potential term (*i.e.* a confining barrier and/or the potential of the neutralising positive background) so that

$$H = \sum_{i<j} v(z_i - z_j) + \sum_{l=1}^{N} U(z_l). \tag{5.143}$$

The confining barrier will not be translationally invariant but it is possible that it has a centre and is rotationally invariant about that centre. In such a circumstance the eigenstates of the angular momentum about that same centre are a very natural basis for the system, as they will match the symmetry of the system. Indeed, such a confining barrier (even if the barrier is infinitely far away) is the only physically plausible reason for identifying a special central point in the system.

So we see that the Laughlin states (and the other states we have discussed) are only potential ground states in the presence of a rotationally invariant confining barrier (though that barrier may be infinitely far removed form the centre, in the infinite limit, where we hope that it won't effect any of the bulk properties of the system). In fact without such a barrier the Laughlin states would be energetically degenerate, as we could create different states with the same energy by simply using the translation operator (*i.e.* we could create Laughlin states with different centres). It is also true, for any system with a bounded number of electrons, that a confining barrier is a necessary requirement if we are to have a ground state with a finite density.

That whether or not an infinitely far away barrier is rotationally invariant may seem irrelevant. We have seen earlier, though, that the fundamental symmetries of the system can have important effects and that the infinite limit can be quite subtle.

One example of the importance of the rotational symmetry is that because the translationally generated states are not rotationally invariant they are necessarily degenerate. This can be seen by considering a particular ψ_α and another $\psi_{\alpha'}$ where

α' is of the same magnitude as α but in a different direction. It is obvious that both of these states have the same energy (because of the rotational invariance of the Hamiltonian) but they are different states, showing that they must be at least two fold degenerate. On the other hand, the rotationally generated states have no such problem due to the fact that angle,*i.e.* θ, is only a one dimensional parameter.

5.5.8 Numerical Calculations

There is a lot of potentially interesting information about the rotationally generated wavefunctions which we have not been able to glean from our purely theoretical approach. Things like the energy of the ψ_θ states and the dependency of θ on the form of the interaction potential are probably best studied numerically. It has not been within the scope of this work to carry out any numerical calculations based on the theory that has been developed. The rotationally generated wavefunctions can be represented in the classical plasma picture in a similar fashion as the translationally generated states were. Since this plasma picture allowed numerical work to be carried out on those states it might be hoped that it might allow similar numerical studies to be carried out with the ψ_θ wavefunctions.

A potential problem with this, though, might be the necessity of restricting any numerical study to a relatively small number of particles (we saw earlier that this caused problems for the study of the translationally generated wavefunctions). This might be important as regards the distance by which the virtual particles are displaced away from the electron positions. This distance, because θ is a constant parameter, is a linearly increasing function of radius. This could easily be problematic in a numerical study that only considered the two hundred or so electrons nearest the centre of the system.

If $\theta \sim 1/\sqrt{N}$ (as seems plausible) so that the maximum vortex (arclength) displacement, $l_R = \theta R$ is of (small) finite size (where R is the radius of the system) then one suggestion for coping with this potential problem is as follows:

If we consider a region situated far away from the centre we see that the arclength vortex displacement will be very close to being a uniform translation over that local

area (*i.e.* the curvature is very small). This suggests that the change in energy per particle (*i.e.* per unit area) could be calculated, to a good approximation, by simply considering the displacement to be a uniform linear translation over that region. If we then sum over all such regions we should obtain a value for the average energy per particle for the whole system. So that if we can find the functional dependency that the energy per particle, $E(l)$, has on the uniformly applied linear vortex displacement, l, (where $l = \theta r \leftrightarrow r \sin \theta$ for a region at radius r from the centre - the limit holding as $\theta \to 0$ and θR remaining finite) in the form of a Taylor series

$$E(l) = \sum_{m=0}^{\infty} a_m . l^m \tag{5.144}$$

then we can calculate the average energy per particle for the whole system using

$$\bar{E}(\theta) = \frac{1}{N} \int_0^{2\pi} \int_0^R \rho E(\theta r) r \, dr \, d\phi \tag{5.145}$$

$$= \frac{2}{(\theta R)^2} \int_0^{\theta R} E(l) . l . dl \tag{5.146}$$

$$= \sum_{m=0}^{\infty} \frac{2}{2+m} a_m (\theta R)^m . \tag{5.147}$$

This value has resulted from a few assumptions. First we have neglected the curvature of the vortex displacement by assuming that it is possible to treat the displacement as a translation applied uniformly over a small (infinitesimal) local area. This should be okay whenever the small region is infinitely far away from the centre, *i.e.* for the area given by

$$r_a > \frac{R}{a} \tag{5.148}$$

where a is a number which can be made very large but must still be such that $r_a \to \infty$ as $R \to \infty$. The ratio of the area inside this radius to the area of the whole system is

$$\frac{\pi r_a^2}{\pi R^2} = \frac{1}{a^2} \tag{5.149}$$

and we see that this central area becomes of negligible importance as a becomes very large. This suggests that the approximation of the vortex displacements as uniformly applied translations over the small local areas should be acceptable in the limit as the area of the system is made to be infinitely large.

Another assumption is that the quantity $E(l)$ is meaningful, *i.e.* that the energy of a small area is determined by the positions of the electrons and vortices in a surrounding area which is sufficiently finite that we can approximate the vortex displacements as uniform displacements over that area. This standard of locality does not seem implausible even though the system is a very highly correlated one (indeed such a locality assumption is often implicit in finite systems studies).

We would hope that, the above outlined approach being valid, that the quantity $E(l)$ could be calculated using translationally invariant classical plasmas with uniformly applied vortex translations. This could then be used to calculate a value for $\bar{E}$ as suggested above. We have included these ideas as suggestions which might encourage numerical work in others and we hope that those of our colleagues more knowledgeable and experienced in such computational methods might take up this work.

5.6 An Explanation of the Incompressible Nature of the Laughlin 1/m Fractions in Terms of the Plasma of Virtual Particles

The incompressibility of the Laughlin wavefunctions has been discussed in some detail elsewhere [55,59]. In this section we discuss an explanation of the incompressibility of the Loughlin wavefunction using the virtual particle plasma picture. This explanation is also valid for the rotationally generated wavefunctions and hence shows that they too are incompressible (in the sense that there is a discontinuity in the energy per particle as one particle is added to or removed from these states).

Consider the classical plasma picture described earlier. The position of the virtual particles (zeros in the wavefunction with respect to the z_i coordinate) determine the regions from which the ith electron is excluded. We know that there must be one zero at the position of every other electron because of the indistinguishable and antisymmetric nature of the electrons. We now wish to consider in what manner the arrangement of the zeros will affect the energy per particle of the wavefunction.

We will assume that the inter-electron potential is a strongly repulsive one (we will discuss the quantification of 'strongly' later). Now, for a sufficiently strong repulsion, the lowest energy state (for a given density) will be one in which the ith electron is excluded from the region around every other electron. This means that the zeros of the z_i wavefunction must be closely gathered around the positions of the other electrons.

In such a state, we have from symmetry that there must be the same number of zeros (in the z_i wavefunction) associated with each of the other electrons. This means that we need to have a certain number of zeros, m, associated with each of the other electron positions. This is only possible if the state is represented by a two particle interaction wavefunction as it is only such a state which allows the zeros of the ith electron to be exclusively associated with individual coordinates of the other electrons.

This can be seen by noting that for each zero of the ith electron to be a function of one electron coordinate only implies that the linear factorisation of the total wavefunction into zeros of z_i (which is always possible by the fundamental theorem of algebra) must contain only factors involving z_i and one other electron coordinate. From symmetry this must be true for all i. Therefore we have a two particle interaction wavefunction.

To see this in another way, consider a q particle interaction wavefunction. This means that the linear factorisation of the total wavefunction in powers of z_i will contain factors which involve q particle coordinates. It is these factors which determine the position of the zeros of the ith electron. If q is larger than two then, by definition, there will be some factors which involve z_i and a function of $q - 1$ other electron coordinates. Since these zeros depend on the coordinates of more than one other electron they will not be able to hug an individual electron position very closely but will have to be shared between the particles involved. We therefore suggest that the ground state of the idealised system will always be a two particle interaction wavefunction whenever it is possible for it to be so. We mean this exactly as opposed to a variational approximation as used in [45, 55, 59].

We therefore see that the two particle interaction wavefunctions represent a unique

state of the system which leads to the lowest possible energy per particle (for a sufficiently strong repulsive potential). We know, though, from appendix D that the indistinguishable and antisymmetric nature of the electrons coupled with the quantisation of area in the single particle lowest Landau level states means that a two particle interaction wavefunction is only possible for the filling factors

$$\nu = \frac{1}{m}\frac{N}{N-1} \to \frac{1}{m} \quad \text{as} \quad N \to \infty. \tag{5.150}$$

Where the m even system is forced to include a term $(z_i + z_j)$ and hence forces some of the zeros to be distanced from the other electrons. Therefore it is only for m odd that all the zeros can be attached to individual electrons.

We see that if we are at one of these filling factors and we try to add or remove an electron (some electrons) this will force the whole system to undergo a transition from this low energy phase, two particle interaction wavefunction to some higher energy, many particle interaction wavefunction. We see, then, that there is an energy gap for adding or removing an electron whenever we are in the ground state of one of the Laughlin fractions. Indeed, we see that these two particle interaction wavefunctions resist any change in ν from the Laughlin values - they are incompressible.

We see that this first principles argument leads directly to the prediction that the $\nu = 1/m$ filling factor ground states are a low energy phase of the system and that adding or removing an electron from one of these states causes a phase transition to a higher energy (per electron) state. The argument is very simple and we see that the effect is caused solely by the non-local effect of indistinguishable antisymmetry, the magnetic quantisation of area in the lowest Landau level and the presence of a sufficiently strongly repulsive inter-electron potential. We see that the incompressability of the 1/m fractions is not caused by anything special about the 1/m wavefunctions (which simply occupy the lowest energy state) but is caused by the fact that the antisymmetric and indistinguishable nature of the electrons (Pauli exclusion principle) prevents the wavefunctions reaching a similar low energy state at densities in the region of the 1/m fractions.

*Note: This argument suggests that for any sufficiently strong repulsive potential

(*i.e.* any potential strong enough to favour the association of zeros with electrons) that the actual ground state of the ideal system must be a two particle interaction wavefunction for the Laughlin fractions. This means that the ideal ground state must be one of the rotationally generated wavefunctions, ψ_θ, where the value of θ will depend on the detailed nature of the potential.

5.7 Summary

Based on the idea of vortex displacement (virtual particle displacement in the classical plasma picture) we have suggested a set of (rotationally generated) variational wavefunctions, ψ_θ, which can have lower energy than the Laughlin wavefunction ($\psi_L = \psi_{\theta=0}$) for sufficiently soft interaction potentials. We discussed the relationship of these states to the symmetries and constraints of the system. We have shown that these states were the most general wavefunctions possible within the two particle interaction approximation. We suggested an extension of these vortex displaced wavefunctions to all of the composite fermion filling factors.

We put forward an argument which suggested that the exact ground state of the (idealised) system is always one of the rotationally generated wavefunctions (for a sufficiently strong repulsive potential). We also showed that the rotationally generated wavefunctions are, like the Loughlin wavefunctions, incompressible in the presence of a strongly repulsive interaction potential.

CHAPTER 6

Rotationally Invariant Excited States

6.1 Overview

We analyse the collective excitations of the system, paying particular attention to its symmetries. We find that the inherently non-uniform nature of the lowest Landau level wavefunctions causes complications in the analysis. We overcome this by studying the system in angular momentum space. We suggest the Fourier density operator in angular momentum space as one which can generate good variational excited states from ground states which are uniform in their large scale features. We note that this operator is the single particle rotation operator (for the lowest Landau level) in normal coordinate space. We discuss the application of these operators to studying excitations of finite systems (such as quantum dots). We suggest a generalised version of the rotation operator which may be useful for studying excitations of non-uniform ground states in finite and infinite systems.

6.2 Rotations and Translations

6.2.1 Symmetries of the System

We have seen in the last chapter that the natural symmetry of the system under study is one of rotational invariance. We saw that trying to use methods and techniques suited to translationally invariant systems led to serious problems - even in the infinite limit. We further saw that, by maintaining the rotational invariance of the

system, we could identify the class of rotationally generated wavefunctions as good variational ground states for different electron interaction potentials. In this chapter we wish to study the low energy collective excited states of these rotationally generated wavefunctions. We will begin by looking at what sort of conditions we might expect the excited states of the system to satisfy.

The first condition is rotational invariance. The system is represented by a rotationally invariant Hamiltonian and so it is reasonable to expect the excited states of the system to be rotationally invariant as well as the ground state.

A second condition, equivalent to the first one, is that the excited state should be an eigenstate of angular momentum (projected angular momentum). This is because the (projected) Hamiltonian commutes with the (projected) angular momentum. Indeed, we can go further, because the value of the angular momentum is related to the radial density distribution of the system and because we expect that any low energy excitation will have the same large scale density features as the ground state we can conclude that any low energy excitation will not only be an eigenstate of angular momentum but that it will have the same eigenvalue as the ground state.

We can, in fact, be more general than simply requiring that any low lying excited state should have the same value of angular momentum as the ground state. Because we expect such low energy excitations to have the same large scale density features as the ground state we know that such excited states must have the same values as the ground state for all of the average u^m and p^m operators, $i.e.$

$$\left\langle \psi_{ground} \left| \frac{1}{N} \sum_{j=1}^{N} \frac{\partial^m z^m}{N^m} \right| \psi_{ground} \right\rangle = \left\langle \psi_{excited} \left| \frac{1}{N} \sum_{j=1}^{N} \frac{\partial^m z^m}{N^m} \right| \psi_{excited} \right\rangle. \qquad (6.1)$$

An equivalent condition is that the excited state has the same energy with respect to the confining boundary potential as the ground state does. This means that all of the energy of excitation should come from the bulk electron-electron interaction and none of it from relative movement with respect to the boundary.

With these conditions in mind we will examine some possibilities for the low energy collective excitations of the system.

6.2.2 Rotational Operators

The projected density operators, $\bar{\rho}_k$, generate magnetoroton wavefunctions [50] in the lowest Landau level by translating individual electrons by a distance $-ik$ (perpendicular to the direction of k). We see that such an operator does not preserve the values of the u^m operators (including angular momentum) and that it would interfere with any circularly symmetric boundary. These effects might become negligible in the infinite limit but either way they highlight the fact that the projected density operators are translation operators. It seems possible, given our discoveries in the last chapter, that some method of generating excited states using rotations may lead to a more natural representation than the translations used in the magnetoroton theory.

The projected density operators,

$$\bar{\rho}_k = \sum_{j=1}^{N} e^{-ik\partial_j} e^{-ik^* \frac{1}{2} z_j}, \tag{6.2}$$

are in effect single electron linear position shifting operators, which translate an individual particle (or at least its average position) by $(k_y, -k_x)$. It is an interesting possibility, then, that an excited state generated by a single particle rotation operator may fit in better with the rotational symmetry of the system. Examining the relationship,

$$e^{-i\theta z \partial} . z^n = e^{-i\theta n} z^n \tag{6.3}$$

$$= \left(e^{-i\theta} z \right)^n, \tag{6.4}$$

(where the z^n states form a single particle basis in the lowest Landau level)

it is simple to see that

$$\bar{R}_\theta = \sum_{j=1}^{N} e^{-i\theta z_j \partial_j} \tag{6.5}$$

$$= \sum_{m=0}^{\infty} \frac{(-i)^m \theta^m}{m!} \sum_{j=1}^{N} (z_j \partial_j)^m \tag{6.6}$$

is the single particle rotation operator in the lowest Landau level. The rotation operator $\bar{R}_\theta$ rotates each individual particle by an angle θ about the centre of the

system (*i.e.* $z = 0$). The rotation operators are clearly lowest Landau level operators and, like the translation operators, they obey the condition

$$\bar{R}_\theta^\dagger = \bar{R}_{-\theta}. \tag{6.7}$$

We suggest that the wavefunction

$$\phi_\theta = \bar{R}_\theta . \psi_{gs}, \tag{6.8}$$

where ψ_{gs} is the ground state of the system,

is worth studying as a variational excited state of the rotationally invariant system.

We notice immediately that the $\bar{R}_\theta$ operators commute with all of the $\overline{u^m}$ operators,

$$\left[\bar{R}_\theta, \sum_{j=1}^N \partial^m z^m \right] = 0. \tag{6.9}$$

This is as we would expect since the state $\phi_\theta = \bar{R}_\theta . \psi_{gs}$ is generated from the ground state by rotations only and so has the same radial density distribution as the ground state, ψ_{gs}.

The above commutation relations also mean that the $\bar{R}_\theta \psi_{gs}$ state has the same energy as the ground state with respect to any circularly symmetric external potential (*i.e.* the confining barrier). This is simply because any circularly symmetric potential can be written as

$$\bar{U} = \sum_{j=0}^N \sum_{m=0}^\infty a_m \partial_j^m z_j^m \tag{6.10}$$

so that

$$\left[\bar{U}, \bar{R}_\theta \right] = 0 \tag{6.11}$$

and hence

$$\left\langle \psi \left| \bar{R}_\theta^\dagger \left[\bar{H} + \bar{U}, \bar{R}_\theta \right] \right| \psi \right\rangle = \left\langle \psi \left| \bar{R}_\theta^\dagger \left[\bar{H}, \bar{R}_\theta \right] \right| \psi \right\rangle \tag{6.12}$$

(where H is the interparticle energy and U is the circularly symmetric potential)

and so we see that the energy change caused by the rotation operator comes entirely from the interaction part of the Hamiltonian and is completely independent from the

boundary potential (except insofar as the boundary may affect the form of the ground state).

A special case of the above commutation relations is that

$$\left[\bar{R}_\theta, \bar{L}\right] = 0 \tag{6.13}$$

and the fact that $\bar{R}_\theta$ commutes with the angular momentum ensures that $\bar{R}_\theta \psi_{gs}$ is an eigenstate of angular momentum with the same eigenvalue as the ground state.

Physically, these operators are nothing else other than single particle rotation operators. The projected density operators created a higher energy wavefunction by forming a superposition of states which were each the same as the ground state but with one particle slightly translated away from its lowest energy position. The rotation operators are entirely analogous. They form a higher energy wavefunction by forming a superposition of states which are each the same as the ground state except for having one particle rotated away from its lowest energy position.

Indeed, from arguments similar to those used at the end of the last chapter, we can see that at large distances from the centre a small rotation is roughly equivalent to a uniform translation in a localised region. This suggests that the rotation operators should have a very similar physical effect as the projected density operators while also maintaining the advantages of being fully compatible with the rotational symmetry of the system.

6.2.3 Fourier Operators

While drawing analogies with the magnetoroton wavefunctions we should notice that the projected density operators may well be closely related to the single particle translation operators but that they are principally the Fourier density operators projected onto the lowest Landau level. It is this relationship to the density distribution of the system which gives us the property that

$$\begin{aligned}
\langle \psi \left| \bar{\rho}_k \right| \psi \rangle &= \langle \psi \left| \rho_k \right| \psi \rangle \tag{6.14} \\
&= \int d^2 r.e^{-ik.r} \langle \psi \left| \rho(r) \right| \psi \rangle . \tag{6.15} \\
&= 0 \tag{6.16}
\end{aligned}$$

for any state ψ which is of uniform density and completely contained within the lowest Landau level. This property is quite vital as it is responsible for the fact that the state $\bar{\rho}_k\psi$ is orthogonal to the state ψ. Without this property the magnetoroton wavefunction could not be considered to be an variational excited state and all the variational calculations based on the state would be useless. (This relationship is not strictly true for any state whose density distribution has a Fourier transform which behaves like a delta function under integral signs but it is true for translationally invariant states. This is because translationally invariant states are eigenstates of momentum and the Fourier density operator is a momentum shifting operator. For non-translationally invariant states the '= 0' will, at best, only hold for certain values of k, for example if

$$f(t) = \begin{cases} 1 & |t| \leq a \\ 0 & \text{otherwise} \end{cases} \tag{6.17}$$

then

$$\mathcal{F}[f(t)](\omega) = F(\omega) = 2\frac{\sin(\omega a)}{\omega} \tag{6.18}$$

and so such a distribution will have a Fourier transform that equals zero whenever $\omega = m\pi/a$ where m is a non-zero integer. In fact it is this discreteness of the zeros in the Fourier transform of the density which gives the density of states of the $\bar{\rho}_k\psi$ states.)

This fact suggests that the analogy which we should draw on is not necessarily the one between translation and rotation operators but one between translational and rotational/radial Fourier operators. Consider the operator defined by

$$S_\theta = \sum_{j=1}^{N} e^{-i\theta u_j} \tag{6.19}$$

where $u_j = \frac{1}{2}|r_j|^2 = \frac{1}{2}|z_j|^2$.

This operator is the natural Fourier density operator in u space for a rotationally

invariant system. To see this consider

$$S_\theta = \sum_{j=1}^{N} e^{-i\theta u_j} \tag{6.20}$$

$$= \int e^{-i\theta u} \sum_{j=1}^{N} \delta^2(\underline{\mathbf{r}} - \underline{\mathbf{r}}_j) d^2\mathbf{r} \tag{6.21}$$

$$= \int e^{-i\theta u} \rho(\underline{\mathbf{r}}) d^2\mathbf{r} \tag{6.22}$$

where

$$\rho(\underline{\mathbf{r}}) = \sum_{j=1}^{N} \delta^2(\underline{\mathbf{r}} - \underline{\mathbf{r}}_j) \tag{6.23}$$

is the density operator. Now, using polar coordinates we have

$$S_\theta = \int_0^{2\pi} d\phi \int_0^\infty e^{-i\theta u} \rho_r(\underline{\mathbf{r}}) r.dr \tag{6.24}$$

and we have define

$$\rho_r(r) = \frac{1}{2\pi} \int_0^{2\pi} \rho(\underline{\mathbf{r}}) d\theta \tag{6.25}$$

so that the number of particles between r and $r + dr$ is $2\pi\rho_r(r)r.dr$. Now, using $du = r.dr$ we obtain

$$S_\theta = 2\pi \int_0^\infty e^{-i\theta u} \rho_u(u) du \tag{6.26}$$

where we have defined

$$\rho_u(u) = \rho_r(\sqrt{2u}) \tag{6.27}$$

so that $\rho_u(u)$ is the density of the system in u space, *i.e.* the number of particles between u and $u + du$ is given by $2\pi\rho_u du$ and for a rotationally invariant system $\rho_u(u)$ is the number of particles per unit area at the radius $r = \sqrt{2u}$. So we see that

$$S_\theta = \sum_{j=1}^{N} e^{-i\theta u_j} \tag{6.28}$$

$$= 2\pi \int_0^\infty e^{-i\theta u} \rho_u(u) du \tag{6.29}$$

is nothing more than the half range Fourier transform of the u space density of the system. This means that we can interpret the θ parameter as a frequency in u space.

This operator is the natural Fourier density operator in a rotationally invariant system and from our discussion on the merits of Fourier operators we suggest that the state

$$\phi_\theta = \bar{S}_\theta . \psi_{gs} \tag{6.30}$$

is a wavefunction worth studying as a variational excited state, where

$$\bar{S}_\theta = \sum_{j=1}^{N} \overline{e^{-i\theta u_j}} \tag{6.31}$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{N} \frac{(-i)^m \theta^m}{m!} \overline{u_j^m} \tag{6.32}$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{N} \frac{(-i)^m \theta^m}{m!} \partial_j^m z_j^m \tag{6.33}$$

is the projection of the u space Fourier density operator onto the lowest Landau level.

The $\bar{S}_\theta$ operator commutes with the angular momentum, the boundary potential and the u^m operators in exactly the same way as the $\bar{R}_\theta$ operator and so it has all the advantages of fitting in with the rotational invariance of the system which were discussed earlier. It also has the advantage of being directly related to the density of the system via a Fourier relationship which is likely to give a density of $\bar{S}_\theta \psi_{gs}$ states which are orthogonal to the ground state, ψ_{gs}.

6.2.4   Relationship between the Fourier and Rotational Operators

The Fourier, $\bar{S}_\theta$, and the rotational, $\bar{R}_\theta$, operators are not independent of one another. This can be seen quite readily from their power series expansions,

$$\bar{S}_\theta = \sum_{j=1}^{N} \overline{e^{-i\theta u_j}} \tag{6.34}$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{N} \frac{(-i)^m \theta^m}{m!} \overline{u_j^m} \tag{6.35}$$

$$= \sum_{m=0}^{\infty} \sum_{j=0}^{N} \frac{(-i)^m \theta^m}{m!} \partial_j^m z_j^m \tag{6.36}$$

and

$$\bar{R}_\theta \;=\; \sum_{j=1}^{N} e^{-i\theta \bar{u}_j} \tag{6.37}$$

$$=\; \sum_{m=0}^{\infty} \sum_{j=0}^{N} \frac{(-i)^m \theta^m}{m!} \bar{u}_j^m \tag{6.38}$$

$$=\; \sum_{m=0}^{\infty} \sum_{j=0}^{N} \frac{(-i)^m \theta^m}{m!} \left(z_j \partial_j\right)^m. \tag{6.39}$$

Indeed, we see that if it were not for the fact that

$$\overline{u^m} \neq \bar{u}^m \tag{6.40}$$

the $\bar{R}_\theta$ and $\bar{S}_\theta$ operators would be exactly equal to one another.

We can show, however, that, in the infinite limit, these two operators are equal to one another for a certain range of θ. Consider

$$\frac{1}{N}\bar{S}_\theta \;=\; \frac{1}{N}\sum_{j=1}^{N} \overline{e^{-i\theta u_j}} \tag{6.41}$$

$$=\; \frac{1}{N}\sum_{j=1}^{N} \overline{e^{-i\alpha \frac{u_j}{U_R}}} \tag{6.42}$$

$$=\; \sum_{m=0}^{\infty} \frac{(-i)^m \theta^m}{m!} \frac{1}{N}\sum_{j=1}^{N} \partial_j^m z_j^m \tag{6.43}$$

$$=\; \sum_{m=0}^{\infty} \frac{(-i)^m \alpha^m}{m!} \frac{1}{N}\sum_{j=1}^{N} \frac{\partial_j^m z_j^m}{U_R^m} \tag{6.44}$$

and

$$\frac{1}{N}\bar{R}_\theta \;=\; \frac{1}{N}\sum_{j=1}^{N} e^{-i\theta \bar{u}_j} \tag{6.45}$$

$$=\; \frac{1}{N}\sum_{j=1}^{N} e^{-i\alpha \frac{\bar{u}_j}{U_R}} \tag{6.46}$$

$$=\; \sum_{m=0}^{\infty} \frac{(-i)^m \theta^m}{m!} \frac{1}{N}\sum_{j=1}^{N} \left(z_j \partial_j\right)^m \tag{6.47}$$

$$=\; \sum_{m=0}^{\infty} \frac{(-i)^m \alpha^m}{m!} \frac{1}{N}\sum_{j=1}^{N} \frac{\left(z_j \partial_j\right)^m}{U_R^m} \tag{6.48}$$

where we have written

$$\theta = \frac{\alpha}{U_R}. \tag{6.49}$$

Now, equation 5.96 tells us that

$$\frac{1}{N} \sum_{j=1}^{N} \frac{\partial_j^m z_j^m}{U_R^m} \longleftrightarrow \frac{1}{N} \sum_{j=1}^{N} \frac{(z_j \partial_j)^m}{U_R^m} \tag{6.50}$$

in the limit as $N \to \infty$ for any finite value of m. In fact, this relationship holds for any m such that

$$\frac{m^2}{U_R} = \nu \frac{m^2}{N} \to 0, \tag{6.51}$$

as is shown in appendix E.

If we write

$$A_m = \frac{1}{N} \sum_{j=1}^{N} \frac{\partial_j^m z_j^m}{U_R^m} \tag{6.52}$$

and

$$B_m = \frac{1}{N} \sum_{j=1}^{N} \frac{(z_j \partial_j)^m}{U_R^m} \tag{6.53}$$

then we have

$$\bar{S}_\theta = \sum_{m=0}^{\infty} \frac{(-i)^m \alpha^m}{m!} A_m \tag{6.54}$$

$$= \sum_{m=0}^{\infty} x_m \tag{6.55}$$

and

$$\bar{R}_\theta = \sum_{m=0}^{\infty} \frac{(-i)^m \alpha^m}{m!} B_m \tag{6.56}$$

$$= \sum_{m=0}^{\infty} y_m. \tag{6.57}$$

Applying the ratio test to these series gives

$$\left| \frac{x_{m+1}}{x_m} \right| = \left| \frac{\alpha}{m+1} \right| \left| \frac{A_{m+1}}{A_m} \right| \tag{6.58}$$

and

$$\left| \frac{y_{m+1}}{y_m} \right| = \left| \frac{\alpha}{m+1} \right| \left| \frac{B_{m+1}}{B_m} \right|. \tag{6.59}$$

We know that

$$\left|\frac{A_{m+1}}{A_m}\right| \quad, \quad \left|\frac{B_{m+1}}{B_m}\right| \leq 1 \tag{6.60}$$

and that

$$\left|\frac{A_{m+1}}{A_m}\right| \quad, \quad \left|\frac{B_{m+1}}{B_m}\right| \sim N^0 \tag{6.61}$$

i.e. these quantities do not scale with N.

This means that the series both begin to converge at values of m where

$$\left|\frac{\alpha}{m+1}\right| < 1 \tag{6.62}$$

and an arbitrary accuracy $(\pm 1/\epsilon)$ can be achieved by terminating the series at a value of m such that (see appendix F)

$$\left|\frac{\alpha}{m+1}\right| < \frac{1}{1+\epsilon} \tag{6.63}$$

which is to say when $m > (1+\epsilon)\alpha$.

This means that for values of $\alpha \sim N^a$ for $a < 1/2$ the series can be truncated within arbitrary (finite) accuracy at values of $m \sim N^a$, *i.e.* values of m such that

$$\frac{m^2}{N} \to 0, \tag{6.64}$$

and so, for such values of α, it is possible to truncate the series at values of m for which the relation 6.50 holds. Therefore, for all such values of α it is true that

$$\frac{1}{N}S_\theta \longleftrightarrow \frac{1}{N}R_\theta + O(\frac{1}{N}) \tag{6.65}$$

in the limit as $N \to \infty$ for $\alpha \sim N^a$ where $a < 1/2$, *i.e.* $\theta \sim 1/N^{1-a}$, that is
$$\theta \sim 1/N^c \text{ where } c > 1/2.$$

At values of θ which scale with N above this limit it is no longer possible to truncate the series at a point where equation 6.50 is valid and hence the $\bar{S}_\theta$ and $\bar{R}_\theta$ operators no longer approach one another in the infinite limit.

Consider a state which satisfies

$$\left\langle \psi \left| \frac{1}{N}\sum_{j=1}^N \frac{\partial_j^m z_j^m}{U_R^m} \right| \psi \right\rangle = \frac{1}{1+m} \tag{6.66}$$

for all m such that $\frac{m^2}{N} \to 0$. This is the condition for a state to have a uniform density in its large scale features (from last chapter) and we note that the ψ_{Sl} states (and hence the rotationally generated states - including the Laughlin state) satisfy this condition up to a term of order $1/N$. For values of $\theta \sim < 1/\sqrt{N}$ we then have

$$\langle \psi | \bar{S}_\theta | \psi \rangle = \langle \psi | \bar{R}_\theta | \psi \rangle \tag{6.67}$$

$$= N \sum_{m=0}^{\infty} \frac{(-i)^m \alpha^m}{m!} \frac{1}{1+m} \tag{6.68}$$

$$= \frac{N}{-i\alpha} \sum_{n=1}^{\infty} \frac{(-i)^n \alpha^n}{n!} \tag{6.69}$$

$$= \frac{N}{i\alpha} \left[1 - e^{-i\alpha} \right] \tag{6.70}$$

$$= \frac{N}{i\theta b} \left[1 - e^{-i\theta b} \right] \tag{6.71}$$

where $b = U_R$.

This is the exact half range Fourier transform of a system with uniform density out to a value of $u = b = U_R$ and zero density for all values of u beyond this value. We see that, for this range of θ,

$$\langle \psi | \bar{S}_\theta | \psi \rangle = \langle \psi | \bar{R}_\theta | \psi \rangle + O(\frac{1}{N}) = 0 + O(\frac{1}{N}) \tag{6.72}$$

whenever

$$\theta = \frac{2\pi}{b} j \tag{6.73}$$

which gives a density of states of $b/2\pi = N/2\pi\nu$.

We see that the projected Fourier u space density operator, $\bar{S}_\theta$, behaves like the rotation operator, $\bar{R}_\theta$, for values of $\theta < 1/\sqrt{N}$. We should note that the limit $\theta < 1/\sqrt{N}$ is an important one. This is because the radius of the system scales with $\sqrt{N}$ and hence an angle of rotation $\theta \sim 1/\sqrt{N}$ leads to a finite displacement of each particle, $l = R\theta \sim N^0$. It is interesting that it is at this scale that the relationship between $\bar{S}_\theta$ and $\bar{R}_\theta$ breaks down.

6.3 The Fourier Operators and Uniform Density Wavefunctions

6.3.1 Fourier Operators for Perfectly Uniform States

We have seen that the $\bar{S}_\theta$ operators can be used to generate states,

$$\phi_\theta = \bar{S}_\theta . \psi_{gs}, \tag{6.74}$$

which are consistent with the necessary symmetries of the system. We saw, also, that for values of $\theta \sim< 1/\sqrt{N}$ this operator had the physical properties of a single particle rotation operator but that it diverged from this behaviour for larger values of θ. If the state $\bar{S}_\theta . \psi_{gs}$ is to be a good variational excited state of the system then one vital property that it must have is that it must be orthogonal to the ground state, ψ_{gs}.

To study the property of orthogonality to the ground state we begin by recalling the Fourier relationship between the S_θ operator and the density distribution of the system, *i.e.*

$$S_\theta = \sum_{j=1}^{N} e^{-i\theta u_j} \tag{6.75}$$

$$= 2\pi \int_0^\infty e^{-i\theta u} \rho_u(u) du \tag{6.76}$$

so that

$$\langle \psi | S_\theta | \psi \rangle = 2\pi \int_0^\infty e^{-i\theta u} \langle \psi | \rho_u(u) | \psi \rangle du. \tag{6.77}$$

Now, if we have a system with a perfectly uniform density,

$$\rho_u(u) = \begin{cases} \frac{N}{2\pi b} & 0 \le u \le b \\ 0 & \text{otherwise} \end{cases}, \tag{6.78}$$

where $b = U_R$ is the extent of the system and can be allowed to go to infinity as

$$N \to \infty,$$

and if the state is completely contained within the lowest Landau level, then we have that

$$\langle \psi | \bar{S}_\theta | \psi \rangle = \langle \psi | S_\theta | \psi \rangle \tag{6.79}$$

$$= \frac{N}{b} \int_0^b e^{-i\theta u} \tag{6.80}$$

which gives

$$\langle \psi \left| S_\theta \right| \psi \rangle \quad = \quad \frac{N}{i\theta b} \left[1 - e^{-i\theta b} \right] \tag{6.81}$$

$$= \quad \frac{2\pi \rho_0}{i\theta} \left[1 - e^{-i\theta b} \right] \tag{6.82}$$

where $\rho_0 = \frac{N}{2\pi b} = \frac{N}{\pi R^2}$ is the density of the system.

We see that this agrees with the result from equation 6.67. This means that

$$\langle \psi \left| \bar{S}_\theta \right| \psi \rangle = 0 \tag{6.83}$$

whenever

$$\theta = \frac{2\pi}{b} j \tag{6.84}$$

for any integer, j. This leads to a density of states of

$$\rho_s = \frac{b}{2\pi} = \frac{\Omega}{4\pi^2}. \tag{6.85}$$

So we see that the state $\bar{S}_\theta.\psi_{gs}$ is orthogonal to the ground state, ψ_{gs}, for these values of θ whenever ψ_{gs} represents a system with a perfectly uniform density.

*Note: actually the condition for orthogonality should strictly be that

$$\frac{\langle \psi \left| \bar{S}_\theta \right| \psi \rangle}{\sqrt{\langle \psi \left| \bar{S}_\theta^\dagger \bar{S}_\theta \right| \psi \rangle}} = 0 \tag{6.86}$$

but the condition that

$$\langle \psi \left| \bar{S}_\theta \right| \psi \rangle = 0 \tag{6.87}$$

is obviously equivalent as long as the magnitude of the state is non-zero. Since this can only occur, for the values of θ which we are interested in, when $\bar{S}_\theta$ is an eigenoperator of the ground state, ψ_{gs}, it is not something that we really need to worry about (apart from the Slater determinant states which are eigenstates of the $\bar{S}_\theta$ operator).

6.3.2   Fourier Operators for Lowest Landau Level States

We saw in the last section that for a perfectly uniform system, completely contained in the lowest Landau level, that $\bar{S}_\theta.\psi$ had a zero overlap with ψ for a density of orthogonal states $\rho_s = b/2\pi$. In this section we will study some actual lowest Landau level wavefunctions and compare them with this ideal uniform result.

We will begin by considering the (normalised) single particle basis states

$$\phi_n = \frac{1}{\sqrt{2\pi 2^n n!}} z^n e^{-\frac{1}{4}|z|^2}. \tag{6.88}$$

Their square modulus is more conveniently written in u space as

$$|\phi_n|^2 \;=\; \frac{1}{2\pi n!}\frac{1}{2^n}|z|^{2n}e^{-\frac{1}{2}|z|^2} \tag{6.89}$$

$$=\; \frac{1}{2\pi n!}u^n e^{-u} \tag{6.90}$$

which is, of course, the density distribution of the single particle state. This has the Fourier transform

$$\left\langle \phi_n \left| e^{-i\theta u} \right| \phi_n \right\rangle \;=\; \int_0^{2\pi}\int_0^\infty e^{-i\theta u}\left|\phi_n\right|^2 r.dr.d\theta \tag{6.91}$$

$$=\; 2\pi \int_0^\infty e^{-i\theta u}\left|\phi_n\right|^2.du \tag{6.92}$$

$$=\; \frac{1}{n!}\int_0^\infty e^{-i\theta u}u^n e^{-u}.du \tag{6.93}$$

$$=\; \frac{1}{n!}\int_0^\infty e^{-(1+i\theta)u}u^n.du \tag{6.94}$$

$$=\; \left(\frac{1}{1+i\theta}\right)^{n+1} \tag{6.95}$$

$$=\; \left(\frac{e^{-i\phi}}{\sqrt{1+\theta^2}}\right)^{n+1} \tag{6.96}$$

where $\phi = \tan^{-1}\theta$.

In general the multiparticle wavefunctions in which we are interested in (*i.e.* good candidate ground states) are quite complicated, interacting systems. To calculate the Fourier density distribution of these wavefunctions would involve extensive numerics. There is one class of wavefunctions, though, which are readily amenable to direct

analytical examination. This is the class of Slater determinant states, the ψ_{Sl} 's. These will give us, at least, an indication of how wavefunctions behave in the lowest Landau level. We are particularly interested in the filled Landau level state, $\psi_{Sl}(\nu = 1)$, which is, of course, equal to the Laughlin wavefunction at $\nu = 1$. This is the most interesting wavefunction in the lowest Landau level, for our purposes in this section, because it is the most uniform possible wavefunction in the lowest Landau level. This means that an examination of the filled Landau level state should help us understand some of the limits under which more general lowest Landau level wavefunctions exist.

The Slater determinant wavefunctions are given by

$$\psi_{Sl} = \prod_{i<j} \left(z_i^n - z_j^n \right) \tag{6.97}$$

$$= \begin{vmatrix} z_1^0 & \cdots & \cdots & \cdots & \cdots & z_N^0 \\ z_1^n & \cdots & \cdots & \cdots & \cdots & z_N^n \\ z_1^{2n} & \cdots & \cdots & \cdots & \cdots & z_N^{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_1^{n(N-1)} & \cdots & \cdots & \cdots & \cdots & z_N^{n(N-1)} \end{vmatrix} \tag{6.98}$$

with filling factor given by $\nu = \frac{1}{n}$ and $n = 1, 3, 5, 7, 9 \ldots$,

and we will be principally interested in the case $\nu = 1, n = 1$. The Fourier density of the state ψ_{Sl} is quite easy to calculate:

$$\langle \psi_{Sl} | \bar{S}_\theta | \psi_{Sl} \rangle = \langle \psi_{Sl} | S_\theta | \psi_{Sl} \rangle \tag{6.99}$$

$$= \left\langle \psi_{Sl} \left| \sum_{j=1}^{N} e^{-i\theta u_j} \right| \psi_{Sl} \right\rangle \tag{6.100}$$

$$= \sum_{m=0}^{N-1} x^{nm+1} \tag{6.101}$$

$$= x. \sum_{m=0}^{N-1} (x^n)^m \tag{6.102}$$

where

$$x = \frac{1}{1 + i\theta} = \frac{e^{-i\phi}}{\sqrt{1 + \theta^2}} \quad , \quad \phi = \tan^{-1}\theta \tag{6.103}$$

This series is easily evaluated to give

$$\langle \psi_{Sl} | S_\theta | \psi_{Sl} \rangle = x\frac{1 - x^{nN}}{1 - x^n} \tag{6.104}$$

$$= \left(\frac{1}{1 + i\theta}\right)^{N+1} \cdot \frac{(1 + i\theta)^{nN} - 1}{(1 + i\theta)^n - 1} \tag{6.105}$$

$$= \frac{e^{-i\phi}}{\sqrt{1 + \theta^2}} \left(\frac{1 - e^{-inN\phi}(1 + \theta^2)^{-\frac{nN}{2}}}{1 - e^{-in\phi}(1 + \theta^2)^{-\frac{n}{2}}}\right) \tag{6.106}$$

and, putting $n = 1$ to obtain the filled Landau level ($\nu = 1$) values, we obtain

$$\langle \psi_{Sl} | S_\theta | \psi_{Sl} \rangle = \frac{e^{-i\phi}}{\sqrt{1 + \theta^2}} \left(\frac{1 - e^{-iN\phi}(1 + \theta^2)^{-\frac{N}{2}}}{1 - e^{-i\phi}(1 + \theta^2)^{-\frac{1}{2}}}\right). \tag{6.107}$$

We see that this is not obviously comparable to the expression obtained for the perfectly uniform state in section 6.3.1. This is interesting since we know that the filled Landau level state is uniform in its large scale features (see last chapter). If we look at expression 6.107, though, we see that it is impossible for $\langle \psi_{Sl} | S_\theta | \psi_{Sl} \rangle$ to equal zero except in the infinite limit, $N \to \infty$, and only then if

$$\theta^2 N \to 0 \implies \theta \sim< \frac{1}{\sqrt{N}} \tag{6.108}$$

which is, of course, exactly the condition used in section 6.2.4 and which is identical to the condition that the state is uniform up to mth order in u where

$$\frac{m^2}{N} \to 0 \tag{6.109}$$

which is the quantified version of our statement that the filled Landau level is uniform in its large scale features.

Indeed, if we have that

$$\theta \sim \frac{1}{N^a} \quad \text{where} \quad a > \frac{1}{2} \tag{6.110}$$

then we can take the limit $N \to \infty$ in equation 6.104 to obtain

$$\langle \psi_{Sl} | S_\theta | \psi_{Sl} \rangle = \frac{1}{in\theta} \left[1 - e^{-inN\theta}\right] \tag{6.111}$$

and using the fact that $b = n(N - 1) \sim nN$ gives

$$\langle \psi_{Sl} | S_\theta | \psi_{Sl} \rangle = \frac{N}{ib\theta} \left[1 - e^{-ib\theta} \right] \tag{6.112}$$

which is exactly the same result as we obtained in section 6.2.4. This shows that the filled Landau level is uniform for u space frequencies $\sim < 1/\sqrt{N}$ but we also see that the expression in equation 6.107 cannot possibly equal zero for any value of

$$\theta \sim \geq \frac{1}{\sqrt{N}}. \tag{6.113}$$

For example, consider the case of finite θ (or any value of θ such that $(1+\theta^2)^{N/2} \to \infty$) in the limit as $N \to \infty$. Then we have

$$\langle \psi_{Sl} | S_\theta | \psi_{Sl} \rangle = \frac{e^{-i\phi}}{(1 + \theta^2)^{\frac{1}{2}} - e^{-i\phi}} \tag{6.114}$$

which is never zero.

This means that the filled Landau level is non-uniform for all periods (in u space)

$$T_{cut} \sim \leq \frac{2\pi}{\theta_{cut}} = \sqrt{N}. \tag{6.115}$$

This means that the filled Landau level system possesses an area which scales as $\sqrt{N}$ over which the state is non-uniform. This non-uniform area, in turn, destroys the density of orthogonal states for the $\bar{S}_\theta.\psi$ wavefunctions for all values of θ which scale with N as $1/\sqrt{N}$ and at all values of θ larger than this. The filled Landau level state is the most uniform possible wavefunction in the lowest Landau level so it is safe to conclude that this cut off (due to non-uniformity) occurs for all states in the lowest Landau level.

The non-uniform area is, in fact, the area around the edge of the system (as is shown in appendix G). The non-uniform area is an annulus around the circumference of the system with a width which scales independently of N (area $= width \times 2\pi\sqrt{N}$).

This cut off in the $\bar{S}_\theta$ states at θ_{cut} is most unsatisfactory. It is possible that we could consider the $\bar{S}_\theta.\psi$ states to be good variational excited states for a perfectly uniform system and then we might expect these states to be perturbed by the addition

of a boundary (edge effects). It is also not obvious just exactly how much difference it would make if the quantity

$$\frac{\langle\psi|\bar{S}_k|\psi\rangle}{\sqrt{\langle\psi|\bar{S}_k^\dagger\bar{S}_k|\psi\rangle}} \tag{6.116}$$

tended to zero in the limit as $N \to \infty$ rather than actually equalled zero. This would be possible if the denominator scaled as $\sqrt{N}$ which would not be surprising, although we have not studied this in detail. All in all, these problems discourage us from pursuing this approach any further. We will adopt a different approach in the next section.

*Note: This non-uniformity of the lowest Landau level states means that we also expect the quantity $\langle\psi|\rho_k|\psi\rangle$ to be non-zero for values of k above some cut off value, k_{cut}. The quantity

$$\frac{\langle\psi|\rho_k|\psi\rangle}{\sqrt{\langle\psi|\bar{\rho}_k^\dagger\bar{\rho}_k|\psi\rangle}} \tag{6.117}$$

should still tend to zero in the limit as $N \to \infty$ as the denominator scales with $\sqrt{N}$. In what way the fact that the overlap with the ground state only tends to zero rather than equals zero might affect any of the magnetoroton calculations which were based on the Laughlin states is not considered here. It seems likely, though, that the magnetoroton states would no longer be good (in the quantum sense) excited states but would be affected by a perturbation representing the edge effects. That the non-zero overlap of $\langle\psi|S_\theta|\psi\rangle$ is related to the non-zero overlap of $\langle\psi|\rho_k|\psi\rangle$ can be also be seen by considering

$$S_\theta = \sum_{j=1}^N e^{-i\theta\frac{1}{2}r_j^2} \tag{6.118}$$

$$= \int_\Omega e^{-i\theta\frac{r^2}{2}} \sum_{j=1}^N \delta^2(r-r_j).d^2r \tag{6.119}$$

$$= \int_\Omega e^{-i\theta\frac{r^2}{2}} \rho(r).d^2r \tag{6.120}$$

$$= \frac{1}{\Omega} \sum_k f_\theta(k)\rho_k \tag{6.121}$$

where we have used

$$\rho(r) = \frac{1}{\Omega} \sum_k e^{ik.r} \rho_k \tag{6.122}$$

and

$$f_\theta(k) = \int_\Omega e^{-i\theta \frac{r^2}{2}} e^{ik.r} d^2 r. \tag{6.123}$$

¿From this we see that

$$f_\theta(0) = \frac{2\pi}{i\theta} \left[1 - e^{-i\theta \frac{\Omega}{2\pi}} \right] \tag{6.124}$$

so that $f_\theta(0) = 0$ for the values of θ which we are interested in. So we see that for S_θ to give a non-zero value it is necessary for some of the ρ_k 's to also give a non-zero value.

6.4 The Rotational Operators and Angular Momentum Space

6.4.1 Rotations and the Rotational Invariance of the System

At the beginning of this chapter we suggested two different operators for generating rotationally invariant excited states, the $\bar{R}_\theta$ and the $\bar{S}_\theta$ operators. We have seen that the S_θ operators have problems with orthogonality which are caused by the non-uniformity of the lowest Landau level wavefunctions. We will, therefore, return to studying the single particle rotation operators, $\bar{R}_\theta$.

In the limit as $N \to \infty$ we know that the $\bar{R}_\theta.\psi$ wavefunctions have a density of orthogonal states for $\theta \sim< 1/\sqrt{N}$ (the region in which the $\bar{R}_\theta$ and $\bar{S}_\theta$ operators are equivalent to one another). It seemed likely at the time that the S_θ operators would be more likely to generate orthogonal states above this range because of their Fourier relationship to the density of the system. We saw, though, that it was exactly this relationship which caused the problems. Perhaps, then, the rotation operators, which lose their Fourier relationship to the density for values of θ outside this range, will maintain their orthogonality properties for larger values of θ.

6.4.2 Angular Momentum Space

In order to study the lowest Landau level rotation operator, $\bar{R}_\theta$, it would be helpful to know what operator it is the projection of. Obviously, this is not unique but we shall see that there is one very natural and useful operator which projects onto the lowest Landau level to give the rotation operator.

We define

$$
\begin{aligned}
L_j &= -i\left[x\partial_y - y\partial_x\right] && (6.125) \\
&= \frac{1}{\hbar}\left(\text{angular momentum of } jth \text{ particle}\right) && (6.126)
\end{aligned}
$$

which is the angular momentum of the jth particle measured in units of $\hbar$ and is therefore a unitless quantity. We know that

$$
L.\phi_n = n.\phi_n \tag{6.127}
$$

and (for any natural number m)

$$
L^m.\phi_n = n^m.\phi_n \tag{6.128}
$$

where

$$
\phi_n = \frac{1}{\sqrt{2\pi n! 2^n}} z^n e^{-\frac{1}{4}|z|^2} \tag{6.129}
$$

and so the set $\{\phi_n\}$ forms a basis for the lowest Landau level. We therefore have that

$$
\bar{L} = z\partial \tag{6.130}
$$

and hence

$$
\overline{L^m} = (z\partial)^m = \bar{L}^m \tag{6.131}
$$

which is a non-trivial result in the lowest Landau level (for example, remember that $\overline{u^m} \neq \bar{u}^m$). This means that if we define

$$
R_\theta = \sum_{j=1}^{N} e^{-i\theta L_j} \tag{6.132}
$$

we have that

$$\overline{R_\theta} \;=\; \overline{\sum_{j=1}^{N} e^{-i\theta L_j}} \tag{6.133}$$

$$=\; \sum_{j=1}^{N} e^{-i\theta \bar{L}_j} \qquad \text{because } \overline{L^m} = \bar{L}^m \tag{6.134}$$

$$=\; \sum_{j=1}^{N} e^{-i\theta z_j \partial_j} \tag{6.135}$$

and so we see that the lowest Landau level rotation operator is the projection of the operator

$$R_\theta = \sum_{j=1}^{N} e^{-i\theta L_j} \tag{6.136}$$

onto $\mathcal{H}_0$. We also see that

$$\overline{R_\theta R_\phi} = \bar{R}_\theta \bar{R}_\phi \tag{6.137}$$

and similarly for any number of rotation operators. This is a simple consequence of the fact that L and $\bar{L}$ are equal to each other in actional form, differing from one another only in their domains.

We will refer to the operator R_θ as the L space Fourier density. This is because if we consider the angular momentum, L, to be a coordinate for the system then

$$R_\theta \;=\; \int_0^\infty e^{-i\theta L} \rho_L(L).dL \tag{6.138}$$

$$=\; \int_0^\infty e^{-i\theta L} \sum_{j=1}^{N} \delta(L - L_j).dL \tag{6.139}$$

$$=\; \sum_{j=1}^{N} e^{-i\theta L_j} \tag{6.140}$$

so we see that R_θ is the Fourier transform of the density of the system in angular momentum space (half range Fourier transforms and full range Fourier transforms are equivalent in L space because $L \geq 0$).

We construct angular momentum space wavefunctions according to the rule

$$|\phi_n|^2 \longleftrightarrow \delta(L - n) \tag{6.141}$$

i.e.

$$\phi_n \longleftrightarrow \delta^!(L - n) \tag{6.142}$$

where the wavefunction $\delta^!(L - n)$ is, like ϕ_n, the single particle eigenstate of angular momentum, L, which is simply a multiplication operator in angular momentum space (for a discussion of the canonically conjugate operator to L, ∂_L, and its relationship to angle see appendix H).

The angular momentum and angular momentum eigenstates in this L space representation are analogous to the normal one dimensional coordinate space representation where position, x, is a multiplication operator and the eigenstates of x form delta functions where all of the particle probability is focussed at one position. In x space the operator

$$r_k = \sum_{j=1}^{N} e^{-ikx_j} \tag{6.143}$$

is the fourier density operator and therefore we can interpret

$$R_\theta = \sum_{j=1}^{N} e^{-i\theta L_j} \tag{6.144}$$

as fulfilling the same role in L space.

This representation is useful to us because we now see that the many particle wavefunctions in the lowest Landau level are isomorphic to a one dimensional system in x space where the particles are only allowed to exist in evenly separated discrete lattice sites (remember that L_j can only take the values $n = 0, 1, 2, 3, 4, \ldots$). So the lowest Landau level system is analogous to a one dimensional crystal lattice of interacting electrons. That the system should be in one to one correspondence with a one dimensional system is only natural. The basis $\{\phi_n\}$ is only one dimensional, after all. This is due to the quantisation of the angular momentum. We see that the spatial quantisation of the single particle states, caused by the magnetic field, is a most important property of the system with very fundamental effects.

*Note: This does not mean that we are considering a system where there is one electron at each lattice site. This would be a non-interacting system and would therefore be represented by a single Slater determinant, *i.e.* ψ_{Sl}. The system we have

in mind will be, in general, a superposition of many Slater determinants representing the interacting nature of the electrons as they mutually repel one another. Indeed, the crystal system will be represented by exactly the same superposition of Slater determinant states as the actual lowest Landau level wavefunction which we wish to represent but with the single particle ϕ_n states replaced with $\delta'(x - n)$ states.

6.4.3 Density of States

In normal coordinate space we found that the lowest Landau level wavefunctions always suffered from a certain amount of non-uniformity. It was this non-uniformity that caused the orthogonality problems of the $\bar{S}_\theta.\psi$ states. In angular momentum space, however, it is easy to construct wavefunctions which have a perfectly uniform L space density, $\rho_L(L)$, and are completely contained within the lowest Landau level (perfect, that is, within the discrete nature of possible density distributions due to the restriction of L to integer values only). For a state to have a uniform (discretised) density in L space is merely equivalent to it having an even spread of single particle angular momentum eigenstates. Further, such states are the ones which represent those wavefunctions with the most uniform possible densities in coordinate space (as discussed in appendix G). The Slater determinant states, ψ_{Sl}, the Laughlin wavefunctions, ψ_L, and the rotationally generated states, ψ_θ, all have such a uniform L space density (the ψ_θ states may contain an added modulation of period $T = 1/\nu$ in L space).

We see then that a wavefunction with a uniform spread of angular momentum will have an L space density

$$\rho_L(L) \sim \sum_{m=0}^{b} \delta(L - m) \tag{6.145}$$

so that

$$\langle R_\theta \rangle \sim \sum_{m=0}^{b} e^{-im\theta} \tag{6.146}$$

which equals zero whenever

$$\theta = \frac{2\pi}{b+1}j \approx \frac{2\pi}{b}j \tag{6.147}$$

giving a density of states

$$\rho_s = \frac{b+1}{2\pi} = \frac{\Omega}{4\pi^2} + \frac{1}{2\pi}. \tag{6.148}$$

These values of θ will correspond to excited states (orthogonal to the ground state) as long as the quantity $\langle\psi|R_\theta^\dagger R_\theta|\psi\rangle$ does not equal zero (we will refer to this quantity as the rotational structure factor or as the L space structure factor). Since, for these values of angle, this can only happen if R_θ is an eigenoperator of the ground state we shouldn't have to worry about in the presence of large numbers of particles (with the exception of the Slater determinant states) - see appendix I.

We expect that, in general, the rotationally generated states may have a modulation of their L space density of period $1/\nu$. This does not (nor, indeed, will any finite modulation) affect the density of orthogonal states for the $R_\theta.\psi\,'s$ but it does affect the cut off in the density of states. For a modulation of period $\alpha = 1/\nu$ in the L space density we will find a cut off at $\theta_{cut} = 2\pi/\alpha$, *i.e.* that $\langle\psi|R_{\theta_{cut}}|\psi\rangle$ is non-zero. This is a result of elementary Fourier transform theory (see appendices J and K). We notice that a cut off at $\theta_c = 2\pi/\alpha$ and a density of states of $\rho_s = (b+1)/2\pi$ leads to

$$\text{INT}\,(\theta_c\rho_s) + 1 = N \tag{6.149}$$

different orthogonal states (the $+1$ represents the ground state, $R_0.\psi = \psi$). That we have N modes in an N particle system (which, as we have seen, is fundamentally one dimensional) is by no means a surprising result. We also expect that

$$\langle\psi|R_\phi R_\theta|\psi\rangle = 0 \tag{6.150}$$

as long as $\phi+\theta \neq \theta_{cut}$ in analogy with normal coordinate space density operators [60, pages 71-77]. This in conjunction with the fact that

$$[R_\theta, R_\phi] = \left[\bar{R}_\theta, \bar{R}_\phi\right] = 0 \tag{6.151}$$

leads us to believe that we have N independent modes of excitation, each labeled by an angle, θ.

*Note: In the same way that k is conserved by the interparticle Hamiltonian because it represents the momentum of the system (or in the lowest Landau level it

represents the position of the centre of mass of the particles) we expect the angle quantity, θ, to be a good label for wavefunctions as the combined interparticle and boundary Hamiltonian will conserve total angle of the system (*i.e.* the Hamiltonian of the system will not cause spontaneous rotations any more than it will cause spontaneous translations - see appendix H).

*Note: The rotation operators studied here are ideally suited to generating excitations of ground states which are uniform in their large scale features. These operators can be generalised, though, in order to create excitations of blatantly non-uniform ground states. The generalised versions no longer have the simple physical interpretation of single particle rotation operators. Indeed their physical action is much more of a squishy nature and we refer to them as squishy operators - see appendix L.

*Note: It is interesting to notice that the area swept out by one of these Discrete Jump ANGular excitatiONS (djangons) is equal to

$$\frac{b}{b+1}\frac{\Omega}{F} \tag{6.152}$$

where $F = B\Omega/(h/e)$ is the total number of magnetic flux quanta over the area of the system. Hence, in the limit as $b \to \infty$ we have that the quantised unit of area associated with these excitations (the area swept out by one djangon) is equal to the area of one magnetic flux quantum.

*Note: The rotation operators all commute with one another and they are equivalent to one dimensional Fourier density operators in angular momentum space. This suggests that the bosonisation scheme

$$R_\theta = A(\theta)\left[a_\theta^\dagger + a_\theta\right] \tag{6.153}$$

may well be a viable one. The details of this, though, depend on the form of the Hamiltonian and we do not consider them here.

6.4.4 Rotationally Generated Excited States

The magnetoroton states are suited to a perfectly translationally invariant system. They are created via linear translations of individual particles. We conjecture that an

explanation of the physical relationship between the rotational excitations, $R_\theta.\psi$, and the magnetoroton (translational) excitations is given by the following description.

If we imagine a perfectly uniform and translationally invariant system with the translational excitations (magnetorotons) as the excited states and if we then add a boundary potential to the Hamiltonian, we would expect the magnetoroton wavefunctions to be perturbed by this extra term in the Hamiltonian. We could then imagine that the boundary has the effect of turning the translations of the magnetoroton aside in order to match the direction of the boundary and the symmetry of the whole system - see figure 6.4.4.

We therefore conjecture that the rotational excitations are related to the translational (magnetoroton) excitations of a translationally invariant system via the perturbative effect of the boundary potential in the Hamiltonian. We see that this effect is very much dependent on the presence of a high magnetic field as it is the angular momentum space representation of the lowest Landau level wavefunctions which led to the existence of the rotationally excited states. It is also worth noting that the Laughlin states (on which many magnetoroton calculations are based) are not translationally invariant but that their existence depends upon the implicit assumption that a confining boundary potential is present.

In spherical systems (which are topologically similar to translationally invariant systems) the magnetoroton wavevector (particle displacement) has been identified with an angular (angular momentum) quantity [45, page 325]. It is not clear what would correspond to the rotational excitations in such a system - perhaps this same quantity? or perhaps there is no correspondence as the rotational excitations are very much a product of the disc topology which is quite different to that of the sphere. We hope that future numerical work will study the rotational excitations in more detail. With the finite limited nature of much numerical study we next, therefore, look at the rotational excitations in small systems.
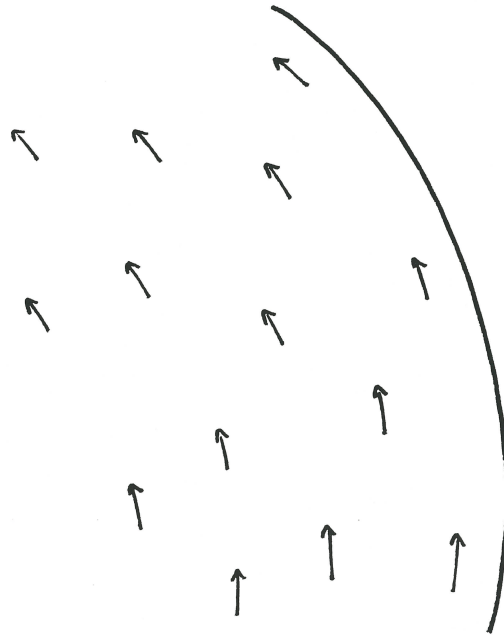
FIGURE 6.1: Translational Excitations Strongly Perturbed by a Boundary. In the translationally invariant system translational excitations have no problems with the symmetry of the system. When a boundary is added, however, the translations at the edge are turned to match the direction of the boundary. This, in turn, affects those translations further inside the sample, forcing them to turn in the same direction. This carries throughout the whole system until we have a rotationally generated excited state (for a circular boundary). That the presence of a boundary can have an influence throughout the area of the system should not be surprising in a highly interacting system.

6.4.5 Finite Systems

The rotationally invariant nature of the $R_\theta.\psi$ states is encouraging as regards generating excited states for finite system wavefunctions as well as infinite systems. In a finite system the rotation operator still has all the advantages of the rotational symmetry and the state $R_\theta.\psi$ will be orthogonal to ψ for any value of θ for which the Fourier transform of the L space density of ψ equals zero. In systems with a small number of particles, obviously, we have to be more careful with some of our results which depend on large scale systems like those regarding the density of states in appendices J and K which apply when density modulations are of a period small compared to the size of the system (*i.e.* uniformity in large scale features).

As the most simple possible example, we will study the two particle wavefunctions which are eigenstates of angular momentum with eigenvalue 4. This value corresponds to a filling factor of 1/3, although, because of the small number of particles it does not actually have this density. We can write the most general possible (lowest Landau level) wavefunction as

$$
\psi \;=\; (z_1 - z_2)(z_1 - \alpha z_2)\left(z_1 - \frac{1}{\alpha}z_2\right) \tag{6.154}
$$

$$
\;=\; (z_1^3 - z_2^3) + \left(1 + \alpha + \frac{1}{\alpha}\right)(z_1 z_2^2 - z_1^2 z_2) \tag{6.155}
$$

$$
\;=\; \begin{vmatrix} z_1^0 & z_2^0 \\ z_1^3 & z_2^3 \end{vmatrix} + \left(1 + \alpha + \frac{1}{\alpha}\right)\begin{vmatrix} z_1^1 & z_2^1 \\ z_1^2 & z_2^2 \end{vmatrix} \tag{6.156}
$$

where the wavefunction is necessarily of the two particle interaction form as there are only two particles. We can readily calculate the overlap between this state and the rotationally excited state $R_\theta.\psi$,

$$
\frac{\langle \psi | R_\theta | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{3a + b\left|1 + \alpha + \frac{1}{\alpha}\right|^2}{3 + \left|1 + \alpha + \frac{1}{\alpha}\right|^2} \tag{6.157}
$$

where

$$
a \;=\; e^{-i\theta 0} + e^{-i\theta 3} = 1 + x^3 \tag{6.158}
$$

$$
b \;=\; e^{-i\theta 1} + e^{-i\theta 2} = x + x^2 \tag{6.159}
$$

where

$$x = e^{-i\theta}. \tag{6.160}$$

So we see that the overlap is zero if and only if

$$3 + Ax + Ax^2 + 3x^3 \quad = \quad 0 \tag{6.161}$$

$$\Downarrow \tag{6.162}$$

$$(x + 1)(3x^2 + (A - 3)x + 3) \quad = \quad 0 \tag{6.163}$$

$$\Downarrow \tag{6.164}$$

$$(x - e^{-i\pi})(3x^2 + (A - 3)x + 3) \quad = \quad 0 \tag{6.165}$$

where $A = \left|1 + \alpha + \frac{1}{\alpha}\right|^2$.

The other two solutions for x are given by

$$x \quad = \quad \frac{3 - A}{6} \pm \sqrt{\frac{A^2 - 6A - 27}{36}} \tag{6.166}$$

$$= \quad \frac{3 - A}{6} \pm i\sqrt{\frac{-A^2 + 6A + 27}{36}}. \tag{6.167}$$

For values of $A > 9$ the solutions for x fail to have the property $|x| = 1$ and hence are inconsistent with the R_θ operator. These values of A can be easily dealt with using the generalised operators of appendix L but we shall not consider them here. Restricting $A \le 9$ is equivalent to putting $\alpha = e^{i\phi}$ (*i.e.* making ψ a rotationally generated state) which straightforward calculation readily shows guarantees the condition $|x| = 1$. Therefore we see that the rotation operators can create orthogonal states for any rotationally generated ground state.

For the wavefunction of uniform density,

$$\psi = (z_1 - z_2) \left(z_1 - e^{i\phi}z_2\right) \left(z_1 - e^{-i\phi}z_2\right) \tag{6.168}$$

with $\cos \phi = \frac{\sqrt{3}-1}{2}$ so that $\phi = 1.196$ radians,

the R_θ operator produces a zero overlap for $\theta = \pi/2$, π and $3\pi/2$. The π state is actually equal to zero as R_π is an eigenoperator of the above ground state, as is consistent with appendix I. This means that the L space structure factor is zero

for this state. This makes sense physically as the wavefunction is made up of two Slater determinants, one where the particles are separated by 1 in L space and one where they are separated by 3. They are never separated by 2 which is the period corresponding to a frequency of π. The other two states produced are are on either side of the eigenoperator and are equal to one another apart from a phase factor. Therefore there are two distinct rotational modes for this ground state, $R_0.\psi = \psi$ and $R_{\pi/2}.\psi$ as we would expect.

We see, then, that the rotationally excited states (and, most certainly, there generalised versions) are potentially good variational excited wavefunctions for finite systems as well as infinite ones. We believe that, therefore, these rotation operators may well prove fruitful in the study of quantum dots which can be modeled as a finite number of electrons in a rotationally invariant system.

6.5   Summary

We studied the possibility of generating excitations of the system (mainly considering the ground state to be one of the rotationally generated wavefunctions of last chapter - but we did briefly discuss the generalisation to arbitrary wavefunctions) which were compatible with the rotational invariance of the system. We discovered a set of rotationally generated variational excited states which were shown to be potentially interesting for the study of finite systems (such as quantum dots) as well as the more 'normal' infinite systems. We also discussed a generalisation of the rotational operators which were applicable to any arbitrary wavefunction whether it was uniform or non-uniform in its large scale features.

CHAPTER 7

Summary

We have proposed a set of sum rules which the projected correlation functions must obey. These may be useful as checks on numerical calculations of these functions.

We studied the possibility of constructing a bosonisation scheme for the projected density operators in the lowest Landau level. We suggested a bosonisation which was an exact solution of the commutation relations and discussed an approximation which allowed calculations to be made within the scheme.

We examined the possible ground states of the system and suggested a set of variational wavefunctions which, we believe, will have lower energy than the Loughlin wavefunction for sufficiently soft potentials. We also showed that the exact ground state of the idealised system will be one of the rotationally generated wavefunctions for sufficiently repulsive electron-electron interactions.

We proposed a set of variational excited states which we believe are particularly well suited to the symmetries of the system. We discussed their applicability to finite as well as infinite systems and we suggest that they will be particularly well suited to the study of quantum dots.

Appendix A

Sums and Integrals

We often use summations over discrete sets of vector indices for all sorts of symbolic manipulation and we also prefer such a discrete picture when thinking about the physical processes involved - unfortunately in order that any calculation can be performed these summations have to be replaced by integrals. It is not only for technical convenience that integrals are to be used over summations - it is that the difference between summation over a discrete set and integration over a continuous one represents the difference between a finite system and an infinite one. This appendix shows how the integrals are constructed and why it is still sensible to carry out manipulations using summations as long as they are replaced by integrals at the end of the calculation.

We start with the potential Hamiltonian for N electrons on the x, y plane,

$$V = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^{N} v(r_i - r_j), \tag{A.1}$$

where $v(r)$ is the interaction potential between two electrons separated by r. We rewrite this as

$$V = \frac{1}{2} \sum_{i,j}^{N} v(r_i - r_j) - \frac{1}{2} N v(0). \tag{A.2}$$

The last term is a constant and we ignore it (even if it is infinite). Now, we consider the system of electrons to be contained within an area of size Ω and therefore there is some maximum distance which any two electrons can have from one another (*i.e.* for a circular area no two electrons can have a separation greater than the diameter of

the circle). We are therefore able to replace $v(r)$ with any other function which has the same value as $v(r)$ over all the region of r corresponding to the electrons both being confined to the area. We therefore extend v periodically and expand it as a Fourier series in a wavevector q.

$$v(r_i - r_j) = \frac{1}{\Omega} \sum_q \tilde{v}_q e^{iq.r_i} e^{-iq.r_j}$$

$$\text{where} \quad \tilde{v}_q = \int_\Omega d^2r.v(r)e^{-iq.r}.$$

$$(A.3)$$

Now, we can rewrite the Hamiltonian as

$$V = \frac{1}{2\Omega} \sum_{i,j}^N \sum_q \tilde{v}_q e^{iq.r_i} e^{-iq.r_j}$$

$$= \frac{1}{2\Omega} \sum_q \tilde{v}_q \rho_q^\dagger \rho_q.$$

$$(A.4)$$

Where the set of points at which $\tilde{v}_q$ is defined forms a grid in q space with a density of states

$$\rho = \frac{\Omega}{4\pi^2} \tag{A.5}$$

In order to consider the infinite system we now allow Ω to tend to infinity (the whole x, y plane). In this limit $\rho.d^2q$ is the number of grid points contained in the element of q space, of area d^2q, and we obtain for the Hamiltonian:

$$V = \frac{1}{8\pi^2} \int d^2q.\tilde{v}(q)\rho_q^\dagger \rho_q. \tag{A.6}$$

This procedure is quite similar to that which quantum field theorists use to quantise free particles (which have a continuous range of allowed momentum). They put the particle in a box of finite volume (area in 2D) so that the particle has only a discrete set of allowed momentums which forms a grid in q space, which gives a density of momentum states, which in turn allows integrals to be defined in the limit that the volume becomes infinite and the momentum range becomes continuous. The density

operators $\rho_q^\dagger$ are shift operators for the total momentum of the many body system and what we have done is equivalent to putting the whole system in a box of finite area so that its total momentum is quantised with a density of states in q space and then finding the integrals for the true continuous range of momentum in the limit that the box is infinite.

Sometimes we wish to define a set of raising and lowering operators $a_k, a_k^\dagger$ for a set of independent modes labeled by k. Consider, first, the case of a system of finite area Ω where, from above, the Hamiltonian is

$$
\begin{aligned}
V &= \frac{1}{2\Omega} \sum_q \tilde{v}_q \rho_q^\dagger \rho_q \\
\bar{V} &= \frac{1}{2\Omega} \sum_q \tilde{v}(q) \bar{\rho}_q^\dagger \bar{\rho}_q
\end{aligned}
$$

$$(A.7)$$

where the allowed values of q form a grid in q space. Therefore it is only for those values of q corresponding to grid points that the operator $\bar{\rho}_q^\dagger$ has any relevance to the energy of the system. Therefore when we seek to write $\bar{\rho}_q^\dagger$ as some series of raising and lowering operators we expect that it is only possible (or at least sensible) to do this for values of q which lie on the grid. We expect that the modes labeled by this q must then also correspond to a discrete set of values lying on this same grid. Therefore we can consider these modes, when we take the limit of infinite area, in exactly the same way as before, with the same density of states as calculated above. This allows us to express summations of these raising and lowering operators as integrals in the limit of infinite area. Quantising the modes in this way has the same effect as it would have done had we considered a single quasiparticle and quantised its possible momenta in the standard way, imagining it to be a single particle.

Therefore we can always construct our algebra in terms of summations over the grid points in q space (whether over the Fourier components in the Hamiltonian or over the quasiparticle modes) as long as at the end of any calculation we take the limit of the area of the system going to infinity, which has the result that we can

simply replace all the summations with integrals according to the rule:

$$\frac{1}{\Omega} \sum_q \longmapsto \frac{1}{4\pi^2} \int d^2q.$$

(A.8)

and in order to remain consistent we must replace Kronecker deltas in the summation notation with Dirac deltas in the integrals according to:

$$\delta_{k,q} \longmapsto \frac{4\pi^2}{\Omega} \delta^2(k-q)$$

(A.9)

Appendix B

Relationship Between Powers of u and p

Using the fact that

$$[\partial^k, z] = k\partial^{k-1} \tag{B.1}$$

so that

$$\partial^k z = z\partial^k + k\partial^{k-1} \tag{B.2}$$

and writing

$$
\begin{aligned}
\partial^m z^m &= \partial^m z z^{m-1} & \text{(B.3)} \\
&= \left(z\partial^m + m\partial^{m-1}\right) z^{m-1} & \text{(B.4)} \\
&= (z\partial + m)\partial^{m-1} z^{m-1} & \text{(B.5)} \\
&= (z\partial + m)(z\partial + m - 1)\partial^{m-2} z^{m-2} & \text{(B.6)} \\
&\vdots \quad \vdots & \text{(B.7)} \\
&= (1 + z\partial)(2 + z\partial)\ldots\ldots(m + z\partial) & \text{(B.8)} \\
&= \prod_{j=1}^{m}(j + z\partial) & \text{(B.9)}
\end{aligned}
$$

we prove the required result.

This can also be seen, more directly, by studying the single particle basis states

of the lowest Landau level,

$$
\begin{aligned}
(\partial^m z^m)z^n &= \frac{(m+n)!}{n!}z^n & \text{(B.10)}\\
&= (n+m)(n+m-1)\ldots\ldots(n+2)(n+1)z^n & \text{(B.11)}\\
&= (m+z\partial)(m-1+z\partial)\ldots\ldots(2+z\partial)(1+z\partial)z^n & \text{(B.12)}\\
&= \prod_{j=1}^{m}(j+z\partial), & \text{(B.13)}
\end{aligned}
$$

which is also a proof, as the z^n states form a single particle basis for the lowest Landau level.

We can also use the single particle basis states to demonstrate the convergence $\partial^m z^m \longleftrightarrow (z\partial)^m$ as $N \to \infty$. Consider

$$
\begin{aligned}
\frac{\partial^m z^m}{N^m}z^n &= \frac{(m+n)!}{n!N^m}z^n & \text{(B.14)}\\
&= \left(\frac{1}{N}+\frac{n}{N}\right)\left(\frac{2}{N}+\frac{n}{N}\right)\ldots\ldots\left(\frac{m}{N}+\frac{n}{N}\right)z^n & \text{(B.15)}\\
&= \left(\frac{n}{N}\right)^m z^n & \text{(B.16)}\\
&= \left(\frac{z\partial}{N}\right)^m z^n & \text{(B.17)}
\end{aligned}
$$

where we have assumed that m is finite and $N \to \infty$. The n terms do not disappear because $n_{max} \sim N$.

Appendix C

Sums to Integrals

Consider the Riemann integral

$$\int_0^a f(x)dx \longleftrightarrow \sum_{i=0}^{n-1} f(x_i)\Delta \qquad \text{(C.1)}$$

$$\text{as } n \to \infty$$

where $x_i = 0 + i\Delta$, $\Delta = \frac{x_n}{n}$ and $x_n = n\Delta$.

We therefore have that

$$\int_0^1 x^m dx = \sum_{j=0}^{n-1} (j\Delta)^m \, \Delta \qquad \text{(C.2)}$$

which, putting $\Delta = 1/N$ and $n = N$, becomes

$$\int_0^1 x^m dx = \sum_{j=0}^{N-1} \left(\frac{j}{N}\right)^m \frac{1}{N} \qquad \text{(C.3)}$$

and therefore

$$\sum_{j=0}^{N-1} \left(\frac{j}{N}\right)^m \frac{1}{N} = \left[\frac{x^{m+1}}{1+m}\right]_0^1 = \frac{1}{1+m} \qquad \text{(C.4)}$$

for any finite m as $N \to \infty$.

We can examine the condition under which the convergence of the summation and integral occurs in more detail. In order for the summation to be an accurate representation of the integral we need to know that the value of $f(x)$ does not change significantly over the distance Δ, *i.e.*

$$f'(x)\Delta \to 0, \qquad \text{(C.5)}$$

which for $f(x) = x^m$ means

$$mx^{m-1}\Delta \to 0. \tag{C.6}$$

Since we know that $0 \leq x \leq 1$ and $\Delta = 1/N$ we have the condition

$$\frac{m}{N} \to 0. \tag{C.7}$$

Therefore, as long as this condition is satisfied we can replace the summation with the integral as above.

Appendix D

Linear Decomposition of Angular Momentum Eigenstates for Two-Particle Interaction Wavefunctions

We know that any wavefunction with a bounded n_{max} (*i.e.* centre $z = 0$, finite density) can be written in the form of equation 5.129,

$$\psi = P_i(z_i) = A_0^i . \left(z_i - A_1^i\right) . \left(z_i - A_2^i\right) \ldots \ldots \left(z_i - A_{n_{max}}^i\right) \tag{D.1}$$

$$\text{where each } A_j^i = A_j^i(z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_N) \text{ ,}$$

A symmetric/antisymmetric two particle interaction state (one which in the classical plasma picture represents a system where the electrons interact via a two body force) can always be written in the form

$$\psi = \prod_{i<j} \phi(z_i, z_j). \tag{D.2}$$

For the antisymmetric case we can write this as

$$\psi = \prod_{i<j}(z_i - z_j) \prod_{i<j} X(z_i, z_j) \tag{D.3}$$

where, because of the fundamental theorem of algebra (above) and the symmetry of the wavefunction, the $X(z_i, z_j)$ must be a polynomial in z_i (and z_j) of degree $m-1$

(where m is an integer). Because there are $N - 1$ pairs of particles which include z_i this means that $n_{max} = m(N - 1)$ and because $\Omega = 2\pi n_{max}$ we have that

$$\nu = \frac{1}{m}\frac{N}{N - 1} \tag{D.4}$$

which are, of course, the Loughlin fractions (the Loughlin states only have a filling factor of $1/m$ in the limit as $N \to \infty$). This means that it is only possible for the wavefunction to be a two particle interaction state when the filling factor has one of the Loughlin values. We have good reason for believing that this particularly simple form of wavefunction only being possible on the Loughlin fractions is not a coincidence - see section 5.6.

For the whole wavefunction to be an eigenstate of angular momentum $X(z_i, z_j)$ must be an eigenstate of angular momentum of eigenvalue $m - 1$ (this fact has the added effect of making sure that ψ has an angular momentum eigenvalue which is consistent with a state uniform in its large scale features).

¿From symmetry we must also have that

$$X(z_i, z_j) = X(z_j, z_i). \tag{D.5}$$

We can, therefore, write

$$
\begin{aligned}
X(z_i, z_j) &= z_i^{m-1} + q_1 z_j z_i^{m-2} + \ldots + z_j^{m-1} & \text{(D.6)} \\
&= (z_i - f_1(z_j))(z_i - f_2(z_j))\ldots(z_i - f_{m-1}(z_j)) & \text{(D.7)} \\
&= (z_j - f_1(z_i))(z_j - f_2(z_i))\ldots(z_j - f_{m-1}(z_i)) & \text{(D.8)} \\
&= (z_j - f_1^{-1}(z_i))(z_j - f_2^{-1}(z_i))\ldots(z_j - f_{m-1}^{-1}(z_i)) & \text{(D.9)}
\end{aligned}
$$

where we have made frequent use of the fundamental theorem of algebra and the symmetry properties of the many particle wavefunction.

We now want to solve for the functions f_k. We have that (where the $'$ denotes a summation where the different indices never take the same value and different

orderings are not repeated)

$$\sum_{k}{}' f_k(z_j) = -q_1 z_j \tag{D.10}$$

$$\sum_{k,l}{}' f_k(z_j) f_l(z_j) = q_2 z_j^2 \tag{D.11}$$

$$\vdots \quad \vdots \quad \vdots \tag{D.12}$$

$$\sum_{k_1,k_2,\ldots,k_r}{}' f_{k_1}(z_j) f_{k_2}(z_j) \ldots f_{k_r}(z_j) = (-1)^r q_r z_j^r. \tag{D.13}$$

The fundamental theorem of algebra tells us that these equations have a unique solution for the values of each $f_k(z_j)$ for every value of z_j. Hence we know that a unique solution always exists for the functions f_k (unique up to reordering, of course). We now only have to find one solution to know that we have the unique solution. We put the substitution $f_k(z_j) = a_k z_j$ into equation D.10 to obtain

$$\sum_{k}{}' a_k = -q_1 \tag{D.14}$$

$$\sum_{k,l}{}' a_k a_l = q_2 \tag{D.15}$$

$$\vdots \quad \vdots \quad \vdots \tag{D.16}$$

$$\sum_{k_1,k_2,\ldots,k_r}{}' a_{k_1} a_{k_2} \ldots a_{k_r} = (-1)^r q_r. \tag{D.17}$$

The fundamental theorem of algebra tells us that a unique solution always exists for the a_k's in these equations (see, for example, [61, pages 73-74]). This means that we can always write (*i.e.* rewriting equation D.6)

$$X(z_i, z_j) = z_i^{m-1} + q_1 z_j z_i^{m-1} + \ldots + z_j^{m-1} \tag{D.18}$$

$$= (z_i - a_1 z_j)(z_i - a_2 z_j) \ldots (z_i - a_{m-1} z_j) \tag{D.19}$$

$$= (z_j - a_1 z_i)(z_j - a_2 z_i) \ldots (z_j - a_{m-1} z_i) \tag{D.20}$$

$$= (z_j - \frac{1}{a_1} z_i)(z_j - \frac{1}{a_2} z_i) \ldots (z_j - \frac{1}{a_{m-1}} z_i). \tag{D.21}$$

We see that the f_k functions are simple linear multiplicative operators. We also see that when m is odd we have

$$X(z_i, z_j) = \prod_{h=1}^{(m-1)/2} (z_i - \alpha_h z_j)(z_i - \frac{1}{\alpha_h} z_j) \tag{D.22}$$

and when m is even we have

$$X(z_i, z_j) = (z_i + z_j) \prod_{h=1}^{(m-2)/2} (z_i - \alpha_h z_j)(z_i - \frac{1}{\alpha_h} z_j) \tag{D.23}$$

because 'multiply by one' is the only linear multiplicative operator which equals its own inverse.

By putting our expression for X back into the equation for ψ we see that the claim made in the text is justified.

Appendix E

Convergence of powers of u and p

Consider (from chapter 5)

$$
\frac{1}{N}\frac{1}{N^m}\sum_{j=0}^{N}\partial_j^m z_j^m \;=\; \frac{1}{N}\frac{1}{N^m}\sum_{j=0}^{N}\prod_{k=1}^{m}(z_j\partial_j + k) \tag{E.1}
$$

$$
= \; \frac{1}{N}\frac{1}{N^m}\sum_{j=0}^{N}(1 + z_j\partial_j)(2 + z_j\partial_j)\ldots\ldots(m + z_j\partial_j). \tag{E.2}
$$

Now, multiplying out the brackets allows us to rearrange the projected average u^m operator into the form

$$
\frac{1}{N}\sum_{j=1}^{N}\frac{\partial_j^m z_j^m}{N^m} = \frac{1}{N}\sum_{j=1}^{N}\frac{(z_j\partial_j)^m}{N^m}
$$

$$
+ \frac{1}{N}\sum_{j=1}^{N}\frac{\alpha_{m-1}(m)}{N}\frac{(z_j\partial_j)^{m-1}}{N^{m-1}}
$$

$$
+ \frac{1}{N}\sum_{j=1}^{N}\frac{\alpha_{m-2}(m)}{N^2}\frac{(z_j\partial_j)^{m-2}}{N^{m-2}}
$$

$$
+ \quad \cdots \quad \cdots \quad \cdots
$$

$$
+ \frac{1}{N}\sum_{j=1}^{N}\frac{\alpha_2(m)}{N^{m-2}}\frac{(z_j\partial_j^2)}{N^2}
$$

$$
+ \frac{1}{N}\sum_{j=1}^{N}\frac{\alpha_1(m)}{N^{m-1}}\frac{(z_j\partial_j)}{N}
$$

$$
+ \frac{1}{N}\sum_{j=1}^{N}\frac{\alpha_0(m)}{N^m}. \tag{E.3}
$$

We can put limits on the values of the α_r 's. Consider the single particle expression

$$
\partial^m z^m = \prod_{j=1}^{m}(z\partial + j) \tag{E.4}
$$

$$
= (1 + z\partial)(2 + z\partial)\ldots\ldots(m + z\partial). \tag{E.5}
$$

Because $z\partial$ is always positive we then have

$$
(z\partial)^m < \partial^m z^m \leq (m + z\partial)^m \tag{E.6}
$$

(where the equality only holds for $m = 1$). Using the binomial theorem,

$$
(m + n)^m = \sum_{r=0}^{m}\left(\frac{m!}{(m-r)!r!}\right)m^r n^{m-r} \tag{E.7}
$$

$$
= n^m + m^2 n^{m-1} + \frac{1}{2}m^2(m-1)n^{m-2} + \ldots\ldots, \tag{E.8}
$$

we see that

$$\frac{1}{N}\frac{1}{N^m}\sum_{j=0}^{N}\partial_j^m z_j^m \longleftrightarrow \frac{1}{N}\frac{1}{N^m}\sum_{j=0}^{N}(\partial_j z_j)^m \qquad (\text{E.9})$$

whenever $\frac{m^2}{N} \to 0$.

We can see that $\frac{m^2}{N} \to 0$ is a necessary as well as a sufficient condition for the convergence of these two operators by noticing the fact that

$$\alpha_{m-1} \;=\; 1+2+3+\ldots\ldots+m \qquad (\text{E.10})$$

$$\;=\; \frac{1}{2}m(m+1) \qquad (\text{E.11})$$

so that convergence does not occur if $\frac{m^2}{N} \nrightarrow 0$.

We therefore see that

$$\frac{1}{N}\frac{1}{N^m}\sum_{j=0}^{N}\partial_j^m z_j^m \longleftrightarrow \frac{1}{N}\frac{1}{N^m}\sum_{j=0}^{N}(\partial_j z_j)^m \qquad (\text{E.12})$$

if and only if $\frac{m^2}{N} \to 0$.

Appendix F

Accuracy of Truncated Series

Put

$$S = \sum_{m=0}^{n} x_m + \sum_{m=n+1}^{\infty} x_m \tag{F.1}$$

where we have chosen

$$n = \epsilon\alpha \tag{F.2}$$

where ϵ is taken to be a large number. We have

$$\left|\frac{x_{m+1}}{x_m}\right| \leq \left|\frac{\alpha}{m+1}\right| \tag{F.3}$$

so that

$$\left|\frac{x_{m+1}}{x_m}\right| \leq \left|\frac{\alpha}{n+1}\right| \tag{F.4}$$

$$\leq \left|\frac{\alpha}{\epsilon\alpha+1}\right| \tag{F.5}$$

$$\leq \frac{1}{\epsilon} \tag{F.6}$$

$$\implies |x_{m+1}| \leq \frac{1}{\epsilon}|x_m| \tag{F.7}$$

$$\implies |x_{m+k}| \leq \left(\frac{1}{\epsilon}\right)^k |x_n| \tag{F.8}$$

for $m \geq n$.

Therefore, we know that

$$\left| \sum_{m=n+1}^{\infty} x_m \right| \leq \sum_{m=n+1}^{\infty} |x_m| \tag{F.9}$$

$$\leq |x_n| \sum_{k=1}^{\infty} \left(\frac{1}{\epsilon} \right)^k \tag{F.10}$$

$$\leq |x_n| \frac{1}{\epsilon}. \tag{F.11}$$

where we used

$$\sum_{k=1}^{\infty} \left(\frac{1}{\epsilon} \right)^k = \frac{1}{\epsilon - 1} \approx \frac{1}{\epsilon}. \tag{F.12}$$

We also have, using Stirling's formula for the factorial, that

$$|x_n| \leq \left| \frac{\alpha^n}{n!} \right| \tag{F.13}$$

$$\leq \frac{1}{\sqrt{2\pi n} \left(\frac{\epsilon}{e} \right)^n} \tag{F.14}$$

$$\ll 1. \tag{F.15}$$

We therefore have that

$$\left| \sum_{m=n+1}^{\infty} x_m \right| < \frac{1}{\epsilon} \tag{F.16}$$

which proves the desired result, *i.e.* that

$$S = \sum_{m=0}^{\infty} x_m = \sum_{m=0}^{n} x_m \pm \frac{1}{\epsilon} \tag{F.17}$$

where $n = \epsilon \alpha$.

Appendix G

Uniformity of Lowest Landau Level States

We will concentrate on the filled Landau level state since we expect it to be the most uniform possible state. The other possible lowest Landau level wavefunctions will not be more uniform than this one. The density of the filled Landau level state is (in particles per unit area)

$$\rho(u) = \frac{1}{2\pi} \left(\sum_{m=0}^{b} \frac{u^m}{m!} \right) e^{-u} \tag{G.1}$$

where $b = n_{max}$ is the measure we have used to represent the extent of the system ($b = N - 1$ for the filled Landau level). We will justify the use of this measure in this appendix.

We define the quantities

$$\Delta(u) = 2\pi \left[\rho(0) - \rho(u) \right] \tag{G.2}$$

and

$$\delta(u) = 1 - \Delta(u). \tag{G.3}$$

These quantities measure the deviation of the density from the expected uniform value of $1/2\pi$.

Inside b

We begin by considering values of $u < b$. We write

$$\Delta(u) = \left(\sum_{m=b+1}^{\infty} \frac{u^m}{m!} \right) e^{-u}. \tag{G.4}$$

Now, we know that the series ratio is given by

$$\left|\frac{x_{m+1}}{x_m}\right| = \left|\frac{u}{m+1}\right| \leq \left|\frac{u}{b+1}\right| < 1 \tag{G.5}$$

where the inequality holds for all $u \leq b$. We then have from appendix F that

$$\frac{u}{b+1}\frac{u^b}{b!}e^{-u} < \Delta(u) < 2\frac{u}{b+1}\frac{u^b}{b!}e^{-u}. \tag{G.6}$$

We wish to study the limit as b is very large, so we use Stirling's formula

$$b! \approx \sqrt{2\pi b}\left(\frac{b^b}{e^b}\right) \tag{G.7}$$

to rewrite the above as

$$X = \frac{u^b e^{-u}}{b!} = \frac{u^b e^{-u} e^b}{\sqrt{2\pi b}b^b}. \tag{G.8}$$

We now put

$$u = (1-\beta)b \tag{G.9}$$

to obtain

$$X = \frac{e^{[\ln(1-\beta)+\beta]b}}{\sqrt{2\pi b}}. \tag{G.10}$$

If we expand the power in the exponential, using

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots \tag{G.11}$$

we obtain

$$[\ln(1-\beta)+\beta]b = -\frac{1}{2}\beta^2 b - \ldots. \tag{G.12}$$

We see then that the value

$$\beta = \frac{\gamma}{\sqrt{b}} \tag{G.13}$$

marks the boundary between X exponentially tending to zero as b tends to infinity and X tending to a constant over $\sqrt{b}$. We therefore have, from equation G.6, that

$$u^n.\Delta(u) \to 0 \quad \text{for} \quad u < (1-\beta)b \tag{G.14}$$

and

$$u^n.\Delta(u) \nrightarrow 0 \quad \text{for} \quad u \geq (1-\beta)b \tag{G.15}$$

where $\beta = \frac{\gamma}{\sqrt{b}}$ and γ is any constant parameter.

We, therefore, see that the system behaves as if it is uniform up to a value of $u = (1-\beta)b = b - \gamma\sqrt{b}$ and is non-uniform between this limit and $u = b$ (in the limit as $b \to \infty$).

Outside b

We now consider the case when $u > b$. We write

$$\delta(u) = 1 - \Delta(u) = 2\pi\rho(u) = \left(\sum_{m=0}^{b} \frac{u^m}{m!}\right)e^{-u}. \tag{G.16}$$

If we consider the series ratio

$$\left|\frac{x_{m-1}}{x_m}\right| = \left|\frac{m}{u}\right| \leq \left|\frac{b}{u}\right| < 1 \tag{G.17}$$

where the inequality holds for all $u > b$. We have then, from appendix F, that

$$\frac{u^b}{b!}e^{-u} < \delta(u) < 2\frac{u^b}{b!}e^{-u}. \tag{G.18}$$

Again, using Stirling's formula (for large b), we have

$$X = \frac{u^b}{b!}e^{-u} = \frac{u^b e^{-u} e^b}{\sqrt{2\pi b b^b}}. \tag{G.19}$$

We now put

$$u = (1 + \beta)b \tag{G.20}$$

and obtain

$$X = \frac{e^{[\ln(1+\beta)-\beta]b}}{\sqrt{2\pi b}}. \tag{G.21}$$

Now, expanding the power in the exponential gives

$$[\ln(1+\beta) - \beta]b = -\frac{1}{2}\beta^2 b + \ldots. \tag{G.22}$$

and we again see that the value

$$\beta = \frac{\gamma}{\sqrt{b}} \tag{G.23}$$

marks the same boundary as it did before. We therefore have, from equation G.18, that

$$u^n.\delta(u) \to 0 \quad \text{for} \quad u > (1+\beta)b \tag{G.24}$$

and

$$u^n.\delta(u) \nrightarrow 0 \quad \text{for} \quad u \leq (1+\beta)b \tag{G.25}$$

where $\beta = \frac{\gamma}{\sqrt{b}}$ and γ is any constant parameter.

We, therefore, see that the system behaves as if it is uniform for values of $u > (1+\beta)b = b + \gamma\sqrt{b}$ and is non-uniform between this limit and $u = b$ (in the limit as $b \to \infty$).

Uniform and Non-Uniform Regions

We see that (in the limit as $b \to \infty$) the system is uniform from $u = 0$ up to $(1 - \beta)b$, non-uniform between $(1-\beta)b$ and $(1+\beta')b$ and uniform from $(1+\beta')b$ upwards (where $\beta = \gamma/\sqrt{b}$ and $\beta' = \gamma'/\sqrt{b}$). This means that the system has a non-uniform region of width $\sqrt{b}(\gamma + \gamma')$ in u space. So we see that the non-uniformity of the system starts at periods,

$$T \sim \frac{1}{\sqrt{N}}, \tag{G.26}$$

and is present at all periods smaller than this. This corresponds to the non-uniformity being present for all u space Fourier frequencies

$$\theta = \frac{2\pi}{T} \sim \sqrt{N} \tag{G.27}$$

and above. This is exactly what we found in our analysis of S_θ.

A width in u space of $\sqrt{b}$ corresponds to a region of the system of area $\sqrt{b}$. In normal coordinate space this region is an annulus of finite width around the perimeter of the system.

Appendix H

Angle in the Lowest Landau Level

We made the correspondence

$$\phi_n = \frac{1}{\sqrt{2\pi 2^n n!}} z^n \longleftrightarrow \delta(L - n) \tag{H.1}$$

between wavefunctions in the lowest Landau level complex coordinate space and those in angular momentum space. These states are the eigenstates of angular momentum which, in L space, is simply a multiplication operator:

$$L.\delta(L - n) = n.\delta(L - n). \tag{H.2}$$

The canonically conjugate operator to L, the L space momentum, is $-i\partial_L$ and has the commutation relation

$$[-i\partial_L, L] = -i. \tag{H.3}$$

We treat these operators in exactly the same manner as we would treat the operators x and $p = -i\partial_x$. The eigenstates of the L space momentum are $e^{i\theta L}$ so that

$$-i\partial_L.e^{i\theta L} = \theta e^{i\theta L} \tag{H.4}$$

and

$$e^{i\theta L} = {}_{n_x}\overset{Lim}{\to} \infty \int_0^{n_x} e^{i\theta n}\delta(L - n)dn. \tag{H.5}$$

These eigenstates do not belong in the lowest Landau level as they require the inclusion of $\delta(L - n)$ states with non-integer values of n. Another way of seeing this is to note that the eigenstates of $-i\partial_L$ are continuous functions which are non zero over the whole range of L and hence do not have a polynomial factor.

We project these states onto the lowest Landau level by noting that the state $\delta(L - n)$ belongs in the lowest Landau level if n is a natural number and it is not in the lowest Landau level if n is anything else. We therefore obtain a set of states

$$\psi_\theta = \frac{1}{\sqrt{1 + n_x}} \sum_{n=0}^{n_x} e^{i\theta n} \delta(L - n) \tag{H.6}$$

which are contained within the lowest Landau level.

We can write these states in the complex coordinate space representation using equation H.1 as

$$\psi_\theta = \frac{1}{\sqrt{1 + n_x}} \sum_{n=0}^{n_x} e^{-i\theta n} \phi_n \tag{H.7}$$

where we have introduced the minus sign for future convenience.

We see that the parameter defining these states is periodic,

$$\psi_{\theta + 2\pi k} = \psi_\theta, \tag{H.8}$$

because all the n in equation H.7 are integers.

We see that these states form an orthonormal set,

$$\langle \psi_\theta | \psi_\phi \rangle = \frac{1}{1 + n_x} \sum_{n=0}^{n_x} e^{i(\theta - \phi)n} = \delta_\theta^\phi, \tag{H.9}$$

as long as

$$\theta = \theta_j = \frac{2\pi}{n_x + 1} j \tag{H.10}$$

and similarly for ϕ.

We see that they form a complete orthonormal basis by noting that each of the ϕ_n basis states can be written as a superposition of the ψ_θ states,

$$\phi_m = \frac{1}{\sqrt{1 + n_x}} \sum_{\theta=0}^{2\pi} e^{i\theta m} \psi_\theta, \tag{H.11}$$

where the summation over θ is over the values

$$\theta = \theta_j = \frac{2\pi}{n_x + 1} j \tag{H.12}$$

for $0 \leq j \leq n_x$,

so we have that

$$\phi_m \ = \ \frac{1}{\sqrt{1+n_x}} \sum_{\theta=0}^{2\pi} e^{i\theta m} \psi_\theta \tag{H.13}$$

$$= \ \frac{1}{\sqrt{1+n_x}} \sum_{j=0}^{n_x} e^{i\theta_j m} \psi_{\theta_j} \tag{H.14}$$

$$= \ \sum_{n=0}^{n_x} \frac{1}{1+n_x} \sum_{j=0}^{n_x} e^{i\theta_j(m-n)} \phi_n \tag{H.15}$$

$$= \ \sum_{n=0}^{n_x} \delta_n^m \phi_n \tag{H.16}$$

$$= \ \phi_m. \tag{H.17}$$

We therefore have that the ψ_θ states are a complete orthonormal basis for the lowest Landau level (which can be obtained from the ϕ_n states by unitary transformation - as above). This means that we can define an Hermitian eigenoperator for these states by

$$\hat{\theta} = \sum_\theta |\psi_\theta\rangle \theta \langle\psi_\theta| \tag{H.18}$$

so that

$$\hat{\theta}.\psi_\theta = \theta.\psi_\theta. \tag{H.19}$$

We have that

$$\psi_0(z) \ = \ \frac{1}{\sqrt{1+n_x}} \sum_{n=0}^{n_x} \phi_n(z) \tag{H.20}$$

$$= \ \frac{1}{\sqrt{1+n_x}} \sum_{n=0}^{n_x} \frac{1}{\sqrt{2\pi 2^n n!}} z^n e^{-\frac{1}{4}|z|^2} \tag{H.21}$$

and

$$\psi_\theta(z) \ = \ \frac{1}{\sqrt{1+n_x}} \sum_{n=0}^{n_x} e^{-i\theta n} \phi_n \tag{H.22}$$

$$= \ \frac{1}{\sqrt{1+n_x}} \sum_{n=0}^{n_x} \frac{1}{\sqrt{2\pi 2^n n!}} e^{-i\theta n} z^n e^{-\frac{1}{4}|z|^2} \tag{H.23}$$

$$= \ \psi_0(e^{-i\theta} z). \tag{H.24}$$

This shows us that the state ψ_θ is simply the state ψ_0 rotated by an angle θ. Clearly, we see that, in general, rotation of the state ψ_θ by an angle ϕ produces the orthogonal state $\psi_{\theta+\phi}$. We therefore see that the $\hat{\theta}$ operator defined in equation H.18 is, in fact, an angle operator - where the angle is measured relative to the orientation of the ψ_0 state which, of course, depends on the choice of coordinate system. Once a coordinate system has been chosen, though, the $\hat{\theta}$ operator is an absolute measure of angle in the lowest Landau level. We will refer to $\hat{\theta}$ as the angle operator.

We see from the relation

$$e^{-i\theta z \partial}.\psi_\phi = \psi_{\phi+\theta} \tag{H.25}$$

that the rotation operator,

$$R_\theta = \sum_{j=1}^{N} e^{-i\theta L_j} = \sum_{j=1}^{N} e^{-i\theta z_j \partial_j}, \tag{H.26}$$

is a shift operator for the total angle,

$$\hat{\theta} = \sum_{j=1}^{N} \hat{\theta}_j. \tag{H.27}$$

We expect, due to the central nature of the interparticle forces, that the Hamiltonian of the system will conserve the relative total angle and hence the angle label on the rotation operators is a good quantum number and hence the θ label on the states $R_\theta \psi$ is a good one for labeling eigenstates of the Hamiltonian.

We also note that the ψ_θ states are well suited to extension in the infinite limit, *i.e.* we can write

$$\psi_\theta = n_x \overset{Lim}{\to} \infty \frac{1}{\sqrt{1+n_x}} \sum_{n=0}^{n_x} e^{-i\theta n} \phi_n \tag{H.28}$$

and so produce a complete basis for the lowest Landau level in the limit of infinite area. Following the discussion of sections 5.5.7 and 5.4.6 we suggest that the ψ_θ basis would be suitable for studying systems which have no boundary (certainly more suitable than the ϕ_n basis).

It would be interesting to write the angle operator in terms of z and ∂ and then to see if we can write the Hamiltonian in terms of the angle and angular momentum (radius) of the particles. Such an analysis has not been considered here.

Appendix I

Zeros in the L Space Structure Factor

For the Slater determinant state with $\nu = 1/\alpha$ the Fourier transform of the L space density is zero whenever

$$\theta = \frac{2\pi}{b + \alpha} j. \tag{I.1}$$

For a completely uniform state in L space the Fourier transform is zero whenever

$$\theta = \frac{2\pi}{b + 1} j. \tag{I.2}$$

For states with L space densities between these two extremes, the zeros will start at equation I.1 and they will be shifted slightly as the state is changed to have a more uniform spread of angular momentum (by the process of conservation of zeros - appendix K) until it is completely uniform and we have the distribution of zeros in equation I.2.

For the L space structure factor to be zero R_θ must have an eigenvalue of zero for every Slater determinant state in the expansion of ψ, including the ψ_{Sl} states. We see that this will only happen when $\psi = \psi_{Sl}$ as we would expect (*i.e.* rotating a single particle is not going to make a difference to the ψ_{Sl} states as they are each symmetric under rotation of individual particles and represent non-interacting systems. Therefore we do not have to worry about the possibility of the L space structure factor equaling zero and spoiling the orthogonality of our excited states with the ground state.

Appendix J

The Cut Off in the Rotational Excitations

If the L space density has a modulation of (maximum) period T then the Fourier transform of the density will have a peak at $\theta = 2\pi/T$. As long as T is finite then the density of zeros in the Fourier transform will remain (nearly) the same in the region between 0 and $2\pi/T$ by the arguments in appendix K though the positions of the zeros will be slightly shifted (depending on the detailed shape of ρ_L) by the interference of the convoluted terms fixed on every delta function in the Fourier transform.

Appendix K

Conservation of Zeros

Consider two continuous functions, $f(t)$ and $g(t)$, each of which have the have the same period of zeros and oscillate between positive and negative values. The sum of these two functions, $f(t) + g(t)$, will have at least as many zeros as the original functions.
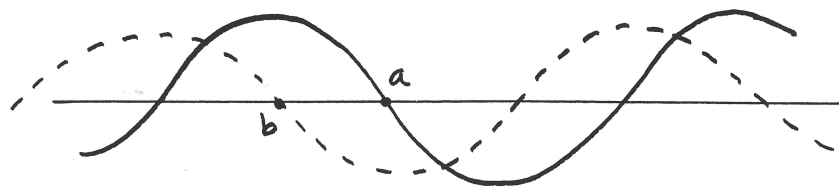

Figure K.1: Addition of Periodic Functions

Consider figure K.1. At point a the sum of the functions must be negative whilst at point b it must be positive. Therefore the sum of the functions must pass through zero somewhere between these two points. We see that every time there is exactly one zero of the first function in-between two zeros of the second function that the sum of functions must have a zero in the near vicinity (*i.e.* within a period).

Indeed we see that this argument works even when the two function are of slightly different periods as a zero is still created every time exactly one zero of the first function lies between two zeros of the second function. In this case the density of

zeros of the summed function will be at least as much as the original function with the lowest density out of the two.

Appendix L

Squishy Operators: Generalisation of Rotational Excitations to Arbitrary Wavefunctions

We have mainly considered the rotational operators

$$R_\theta = \sum_{j=1}^{N} e^{-i\theta L_j} \tag{L.1}$$

and discussed them in the context of creating excitations of ground state wavefunctions which are uniform in their large scale features (*i.e.* any departure from uniformity in the L space density is periodic and of a period small in comparison to the extent of the system, b).

In this section we will discuss a generalisation of the rotation operators which will enable us to generate excitations of any arbitrary wavefunction - whether it is uniform in its large scale features or not. We define

$$R_\alpha = \sum_{j=1}^{N} e^{-\alpha L_j} \tag{L.2}$$

$$= \sum_{j=1}^{N} e^{-(\sigma+i\theta)L_j} \tag{L.3}$$

where $\alpha = \sigma + i\theta$ is a complex number. For $\sigma = 0$ this operator is simply equivalent to the rotation operator. For non-zero σ, though, it is no longer simply a single particle

rotation operator. We see from

$$e^{-\alpha L} z^n e^{-\frac{1}{4}|z|^2} = e^{-n\sigma} \left(e^{-i\theta} z \right)^n e^{-\frac{1}{4}|z|^2} \tag{L.4}$$

that if it wasn't for the Guassian term on the end this would be a rotation plus expansion/contraction operator. Because of the form of the lowest Landau level states, though, the action of the operator can be divided into two parts. The first part is just the rotation caused by θ. The second is the selective weighting of different single particle basis states in favour of large (small) values of L (*i.e.* radius and u) when σ is negative (positive). The R_α operator commutes with all the same operators as the rotation operators and hence it cannot eliminate L values in a state nor can it create new L values. But, by reweighting the Slater determinant basis states in a wavefunction it squishes the system up against its outer perimeter for negative σ and squishes the particles in towards the centre for positive σ. For this reason we refer to these operators as squishy operators.

The rotation operators were seen to be the Fourier density operators in L space. The squishy operators are, in fact, the simple generalisation of this, they are the Laplace transformed density operators in L space. Consider

$$\begin{aligned}
R_\alpha &= \sum_{j=1}^{N} e^{-\alpha L_j} &\text{(L.5)} \\
&= \int_0^\infty e^{-\alpha L} \sum_{j=1}^{N} \delta(L - L_j) dL &\text{(L.6)} \\
&= \int_0^\infty e^{-\alpha L} \rho_L(L) dL &\text{(L.7)} \\
&= \mathcal{L}_\alpha \left[\rho_L(L) \right] &\text{(L.8)}
\end{aligned}$$

which shows that R_α is the half range Laplace transform of the L space density - the natural generalisation of the Fourier transform which represents the rotation operators.

The R_α operators are not going to generate uniform excitations from uniform ground states except when $\sigma = 0$. They are capable, though, of generating orthogonal excited states from any arbitrary ground state (finite or infinite). To see this, consider

a ground state with any arbitrary density distribution in L space,

$$\langle\psi|\rho_L(L)|\psi\rangle = \sum_{m=0}^{b} a_m \delta(L-m), \tag{L.9}$$

so that

$$
\begin{aligned}
\langle\psi|R_\alpha|\psi\rangle &= \mathcal{L}_\alpha\left[\rho_L(L)\right] \tag{L.10} \\
&= \sum_{m=0}^{b} a_m e^{-m\alpha} \tag{L.11} \\
&= \sum_{m=0}^{b} a_m z^m \tag{L.12}
\end{aligned}
$$

where $z = e^{-\alpha}$.

This is simply a polynomial in z of degree b. Therefore by the fundamental theorem of algebra we know that $\langle R_\alpha\rangle$ will have b zeros - *i.e.* b states (*i.e.* b values of α) which have zero overlap with the ground state. Of course, it is possible that many of these states will be equivalent to one another and that some may have a zero modulus but we would still expect that this method would generate genuine excited states for many values of α.

*Note: This method works for finite as well as infinite systems.

*Note: Our arguments in appendix K can be seen to be equivalent to saying that for a system which is uniform in its large scale features (as discussed above) that the a_m coefficients, which describe the angular momentum distribution, will be such that the polynomial

$$\sum_{m=0}^{b} a_m z^m \tag{L.13}$$

will have (nearly) b zeros which lie on the unit circle, $|z| = 1$.

Effect of a Finite Period Modulation in Density on the Average Powers of u

Consider the case where

$$\rho(u) = \rho_0(u) + \rho_1(u) \tag{M.1}$$

where ρ_0 is the density of a system with uniform density and ρ_1 is a periodic modulation in the density of period T. We know that

$$\int_v^{v+T} \rho_1(u) du = 0 \tag{M.2}$$

and hence over one period the area of ρ_1 above the zero axis must equal the area below the axis. We call this area w.

Now, consider the integral

$$\frac{1}{N^{m+1}} \int_0^b \rho(u) u^m . du = \frac{1}{N^{m+1}} \int_0^b \left[\rho_0(u) + \rho_1(u) \right] u^m . du \tag{M.3}$$

which will equal the value for that of a uniform system if

$$\frac{1}{N^{m+1}} \int_0^b \rho_1(u) u^m . du \to 0. \tag{M.4}$$

We know that

$$\left| \int_v^{v+T} \rho_1(u) u^m . du \right| \le (v+T)^m w - v^m w. \tag{M.5}$$

Since u^m has an always increasing gradient the largest slope in the range of integration occurs at the end of the range. Therefore

$$\left| \int_v^{v+T} \rho_1(u) u^m . du \right| \le m(v+T)^{m-1} T w. \tag{M.6}$$

We now have

$$\left| \int_0^b \rho_1(u) u^m . du \right| \le mw \sum_{j=1}^n (jT)^{m-1} T \tag{M.7}$$

where $n = b/T$. But by considering the graphs we see that

$$\int_0^b u^{m-1} du \le \sum_{j=1}^n (jT)^{m-1} T \le \int_0^b (u+T)^{m-1} du. \tag{M.8}$$

We now have that

$$\left| \int_0^b \rho_1(u) u^m . du \right| \le mw \int_0^b (u+T)^{m-1} du \tag{M.9}$$

$$\le w(b+T)^m - wT^m \tag{M.10}$$

$$\le wb^m \left(1 + \frac{T}{b} \right)^m. \tag{M.11}$$

Hence we see that

$$\left| \frac{1}{N^{m+1}} \int_0^b \rho_1(u) u^m . du \right| \le \frac{b^m}{N^{m+1}} w \left(1 + \frac{T}{b} \right)^m \to 0. \tag{M.12}$$

as $b \to \infty$ whenever w, T and m are finite. We see then that any finite modulation of the density of finite period still produces a system which is uniform in its large scale features.

Appendix N

Solving the B Equations

The B equations are

$$B(k, h) = B^*(-k, h + k) \qquad (N.1)$$

and

$$B(q, h).B(k, q + h) - B(k, h).B(q, h + k) = \phi(k, q).B(k + q, h) \qquad (N.2)$$

where

$$\phi(k, q) = e^{\frac{k^* q}{2}} - e^{\frac{q^* k}{2}} \qquad (N.3)$$

$$= 2i e^{\frac{1}{2} k.q} \sin\left(\frac{k \wedge q}{2}\right). \qquad (N.4)$$

We make the substitution (without loss of generality)

$$B(k, q) = 2i e^{-\frac{1}{4}|k|^2} b(k, q) \qquad (N.5)$$

so that the B equations now become

$$b(k, q) = -b^*(-k, q + k) \qquad (N.6)$$

and

$$b(q, h).b(k, q + h) - b(k, h).b(q, h + k) = \sin\left(\frac{k \wedge q}{2}\right).b(k + q, h). \qquad (N.7)$$

We now proceed in several stages.

Step 1

We put $q = -k$ into equation N.7 and use equation N.6 to obtain

$$b(-k, h).b(k, h - k) - b(k, h).b(-k, h + k) \;\; = \;\; 0 \tag{N.8}$$

$$-b(-k, h).b^*(-k, h) + b(k, h).b^*(k, h) \;\; = \;\; 0 \tag{N.9}$$

$$\Rightarrow |b(k, h)| \;\; = \;\; |b(-k, h)| \tag{N.10}$$

$$\forall k, h.$$

Step 2

We now use the last result along with equation N.6 to obtain

$$|b(k, q)| \;\; = \;\; |b(-k, q + k)| \tag{N.11}$$

$$= \;\; |b(k, q + k)| \tag{N.12}$$

$$= \;\; |b(-k, q + 2k)| \tag{N.13}$$

$$= \;\; |b(k, q + 2k)| \tag{N.14}$$

$$\vdots \quad \vdots \tag{N.15}$$

$$|b(k, q)| \;\; = \;\; |b(k, q + mk)| \tag{N.16}$$

$$\forall k, q \quad \text{and} \quad \forall m \in \mathcal{Z} \, .$$

Step 3

We put $k = nq$ into equation N.7 to obtain (where n is an integer)

$$b(q, h).b(nq, q + h) - b(nq, h).b(q, h + nq) = 0. \tag{N.17}$$

We then use the last result to write

$$b(q, h + nq) = e^{ix}.b(q, h) \tag{N.18}$$

and hence obtain

$$|b(nq, q + h)| = |b(nq, h)|. \tag{N.19}$$

We now substitute $k = nq$ to get

$$|b(k, h)| = |b(k, h + \frac{1}{n}k)|. \tag{N.20}$$

We then use this result repeatedly to obtain

$$|b(k, q)| = |b(k, q + \frac{1}{n}k)| \tag{N.21}$$

$$= |b(k, \left[q + \frac{1}{n}k\right] + \frac{1}{n}k) \tag{N.22}$$

$$= |b(k, q + \frac{2}{n}k)| \tag{N.23}$$

$$= |b(k, q + \frac{3}{n}k)| \tag{N.24}$$

$$\vdots \quad \vdots \tag{N.25}$$

$$|b(k, q)| = |b(k, q + \frac{m}{n}k)| \tag{N.26}$$

$$|b(k, q)| = |b(k, q + \alpha k)| \tag{N.27}$$

$$\forall k, q \quad \text{and} \quad \forall m, n \in \mathcal{Z} \quad \text{and} \quad \forall \alpha \in \mathcal{Q}.$$

The rationals are an everywhere dense set in the reals [62, 63] and hence in order for the b functions to be continuous or in any way physically sensible it must be true that

$$|b(k, q)| = |b(k, q + \alpha k)| \tag{N.28}$$

$$\forall k, q \quad \text{and} \quad \forall \alpha \in \mathcal{R}.$$

This means that the b function depends only on k and that part of q which is perpendicular to k. We can therefore write, without loss of generality,

$$|b(k, q)| = |b(k, k \wedge q)|. \tag{N.29}$$

Step 4

We now make some physical assumptions based on the expected symmetry of the B coefficients. The $B(k, q)$ coefficients depend on four variables: $|k|, |q|, \theta_k$ and θ_q. We make the isotropic assumption that the coefficients do not depend on the orientation of the k and q vectors separately but only depend on the angle between them, $\theta_k - \theta_q$.

This means that the coefficients, and hence the $b(k, q)$ functions, can be written as functions of the variables: $|k|, |q|, k.q$ and $k \wedge q$. Now, we see from the last result that the only q dependence in $b(k, q)$ comes from $k \wedge q$ and so this means that $|b(k, q)|$ is a function of $|k|$ and $k \wedge q$ only. Further, we see by considering equation N.7 that if $b(k, q)$ depends on $|k|$ that this introduces a $k.q$ dependency on the right hand side which cannot be matched on the left hand side. Therefore there can be no $|k|$ dependency. This means that

$$|b(k, q)| = |b(k \wedge q)|. \tag{N.30}$$

Step 5

Strictly speaking most of our results only refer to the modulus of the b function. We will assume that they refer to the whole function and worry about the phase later. Equation N.7 can now be written as

$$b\left(\frac{q \wedge h}{2}\right).b\left(\frac{k \wedge (q + h)}{2}\right) - b\left(\frac{k \wedge h}{2}\right).b\left(\frac{q \wedge (k + h)}{2}\right) = \sin\left(\frac{k \wedge q}{2}\right).b\left(\frac{(k + q) \wedge h}{2}\right)$$
$$\tag{N.31}$$

where the factor of 2 is arbitrary but we have included it because of the sine term.

We expand $b(u)$ in a Fourier series/transform (we shall use summations but integrals will give the same answer),

$$b(u) = \sum_w a_w e^{iwu}. \tag{N.32}$$

We then substitute this expansion into equation N.31. We see, though, that on the right hand side the $k \wedge q$ terms can only have the frequencies ± 1 (the sine term). This then gives us that

$$b(u) = a.e^{iu} + c.e^{-iu} \tag{N.33}$$

where $a = -a^*$ and $c = -c^*$ from equation N.6 implying that both coefficients are purely imaginary.

If we then substitute this expression into the equation we find that it is a solution if and only if the coefficients obey the relations

$$a^2 = \frac{a}{2i} \tag{N.34}$$

and

$$c^2 = -\frac{c}{2i}. \tag{N.35}$$

These have the four possible solutions

$$
\begin{aligned}
(a, c) &= (0, 0) &\text{(N.36)} \\
&= (0, -\frac{1}{2i}) &\text{(N.37)} \\
&= (\frac{1}{2i}, 0) &\text{(N.38)} \\
&= (\frac{1}{2i}, -\frac{1}{2i}). &\text{(N.39)}
\end{aligned}
$$

These correspond to

$$
\begin{aligned}
b(k, q) &= 0 &\text{(N.40)} \\
&= -\frac{1}{2i} e^{-i\frac{k \wedge q}{2}} &\text{(N.41)} \\
&= \frac{1}{2i} e^{i\frac{k \wedge q}{2}} &\text{(N.42)} \\
&= \sin\left(\frac{k \wedge q}{2}\right) &\text{(N.43)}
\end{aligned}
$$

and accordingly

$$
\begin{aligned}
B(k, q) &= 0 &\text{(N.44)} \\
&= -e^{-\frac{1}{4}|k|^2} e^{-i\frac{k \wedge q}{2}} &\text{(N.45)} \\
&= e^{-\frac{1}{4}|k|^2} e^{i\frac{k \wedge q}{2}} &\text{(N.46)} \\
&= 2i.e^{-\frac{1}{4}|k|^2} \sin\left(\frac{k \wedge q}{2}\right). &\text{(N.47)}
\end{aligned}
$$

The first solution is trivial and of no use to us (it would mean that there is no non-zero solution to the commutation equations - *i.e.* that all the nth order coefficients are zero). The next two do not satisfy the finite reliability criterion as they do not go to zero as k goes to zero. The last one is, therefore, the only solution which satisfies our requirements.

  *Note: If we now try for a solution of the form

$$b'(k, q) = e^{ix(k,q)}.b(k, q) \tag{N.48}$$

we find that we need

$$x(q, h) + x(k, q + h) = x(k, h) + x(q, k + h) = x(k + q, h). \qquad \text{(N.49)}$$

Solutions of this are of the form

$$x(k, q) = e^{a.k} \qquad \text{(N.50)}$$

where a is a constant vector. We can set a to zero on symmetry grounds (or notice that it can be swallowed by the raising and lowering operators using the simple unitary transform $a_q = e^{-ia.q}c_q$).

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