Classical and quantum modifications of gravity

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Abstract

Einstein's General Relativity has been our best theory of gravity for nearly a century, yet we know it cannot be the final word. In this thesis, we consider modifications to General Relativity, motivated by both high and low energy physics.

In the quantum realm, we focus on Hořava gravity, a theory which breaks Lorentz invariance in order to obtain good ultraviolet physics by adding higher spatial derivatives to the action (improving propagator behaviour in loops) but not temporal (avoiding Ostrogradski ghosts). By using the Stückelberg trick, we demonstrate the necessity of introducing a Lorentz violating scale $M_\star$ into the theory, far below the Planck scale, $M_\star \ll M_{pl}$, to evade strong coupling concerns. Using this formalism we then show explicitly that Hořava gravity breaks the Weak Equivalence Principle, for which there are very strict experimental bounds. Moving on to considering matter in such theories, we construct $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ invariant actions for both scalar and gauge fields at a classical level, before demonstrating that they are only consistent with the Equivalence Principle in the case that they reduce to their covariant form. This motivates us to consider the size of Lorentz violating effects induced by loop corrections of Hořava gravity coupled to a Lorentz invariant matter sector. Our analysis reveals potential light cone fine tuning problems, in addition to evidence that troublesome higher order time derivatives may be generated.

At low energies, we demonstrate a class of theories which modify gravity to solve the cosmological constant problem. The mechanism involves a composite metric with the square root of its determinant a total derivative or topological invariant, thus ensuring pieces of the action proportional to the volume element do not contribute to the dynamics. After demonstrating general properties of the proposal, we work through a specific example, demonstrating freedom from Ostrogradski ghosts at quadratic order (in the action) on maximally symmetric backgrounds. We go on to demonstrate sufficient conditions for a theory in this class to share a solution space equal to that of Einstein's equations plus a cosmological constant, before determining the cosmology these extra solutions may have when present.

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List of papers

This thesis contains material from the following four papers:

[1] *'Lessons from the decoupling limit of Hořava gravity'*
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JHEP **1007** (2010) 014

[2] *'Cleaning up the cosmological constant'*
Ian Kimpton and Antonio Padilla
JHEP **1212** (2012) 031

[3] *'Matter in Hořava-Lifshitz gravity'*
Ian Kimpton and Antonio Padilla
arXiv:1301.6950 [hep-th]

[4] *'Cosmology and new solutions with generalised unimodular gravity'*
Ian Kimpton and Antonio Padilla
(in press)

Notation and conventions

'High-energy physics units' are employed in this thesis, such that $c = 1$ and $\hbar = 1$. However, in Chapters 2 and 3, factors of c are occasionally restored. In these units $G \neq 1$. G is often rewritten as the Planck mass, $M_{pl} = (16\pi G)^{-1} \sim 10^{18}$GeV or the Planck length $l_{pl} = 1/M_{pl} = 16\pi G$. Note that alternative definitions of M_{pl} use prefactors of 8π or unity.

Standard index notation and summation conventions are used throughout this thesis. Greek letters $\mu, \nu, \ldots$ denote spacetime (almost always 4D) indices and Latin letters from the middle of the alphabet $i, j, k, \ldots$ denote purely spatial indices. When summing over spatial indices with a flat metric $g_{ij} = \delta_{ij}$, the distinction between upper/lower indices may be dropped for convenience. Letters from the start of the Latin alphabet, $a, b, c, \ldots$ will be used to denote indices unrelated to spacetime (*e.g.* labelling fields).

The mostly positive signature metric $(- + ++)$ is used. The definition of the Riemann tensor is $R_{\mu\nu\alpha}{}^{\beta} = -2\partial_{[\mu}\Gamma^{\beta}{}_{\nu]\alpha} + 2\Gamma^{\lambda}{}_{\alpha[\mu}\Gamma^{\beta}{}_{\nu]\lambda}$, with the Ricci tensor corresponding to the contraction $R_{\mu\nu} = R_{\mu\lambda\nu}{}^{\lambda}$. Except where explicitly stated, a Levi-Civita connection is assumed.

We will refer to both curvatures of the full spacetime and spacelike hypersurfaces. R is always used to denote the full spacetime curvature, while $R^{(3)}$ is the curvature of a 3D spatial hypersurface.

A distinction is made in Chapters 5 and 6 between a physical (but composite) metric $\tilde{g}_{\mu\nu}$, to which matter couples and on which particles follow geodesics, and a fundamental metric $g_{\mu\nu}$. Quantities such as covariant derivatives ($\tilde{\nabla}$, ∇), Ricci scalars ($\tilde{R}$, R) and scale factors ($\tilde{a}$, a) will be tilded or untilded to denote the metric from which they are built.

Contents

CHAPTER 1

Introduction

Gravity is the fundamental force most familiar directly to our everyday lives. It appears to be a quite different animal to the other forces, many order of magnitude weaker and resisting attempts to be quantised on small scales, yet being the dominating force on large scales in the universe! The theories of Newton and Einstein are the most famous, with both experiencing great success and the latter remaining our best theory of gravity even as we approach its centenary. In this chapter, we will discuss the successes of General Relativity, then motivate the need for its modification, before highlighting the problems which arise when attempting to do so.

1.1 Newtonian gravity

Newton provided the first explanatory theory of gravity. He provided a simple underlying model, with the same force governing falling apples on Earth and the planets in the heavens [5]. He was able to explain previously purely empirical relations such as Kepler's law and the theory experienced particular success with planetary orbits.

In the 18th century, astronomer William Herschel discovered Uranus. However, comparing the orbit with a Newtonian calculation indicated additional perturbations. It appeared that either another planet must exist further out or Newton was wrong. John Couch Adams and Urbain Leverrier independently made predictions for the existence of Neptune and shortly after, the observation by Galle confirmed their predictions [6, 7], a great testament to Newton's theory.

The success of this prediction meant later, when a discrepancy between observed perihelion procession and that calculated from Newton's theory for Mercury was seen, Leverrier again conjectured a planet, ultimately dubbed 'Vulcan', as being responsible [8]. However, this 'dark planet', predicted to be even closer to the sun than Mercury, would turn out not to exist. In fact, it transpired that to match observations of Mercury, we did not need to find a new planet, but change the theory of gravity.

Einstein demonstrated that his General Theory of Relativity solved the problem of Mercury's perihelion precession [9]. Not only this, but the theory also provided a gravitational model solving the inconsistencies between Newtonian gravity and Special Relativity, with gravity only propagating at a finite speed (allowing consistency with locality and causality) and energy, mass and momentum being treated on the same footing. The theory passed the key test of reducing to Newtonian gravity in appropriate limits, ensuring that the successful observations resulting from Newton were explained in the theory, but also made new and differing predictions, such as the extent to which light is bent by the Sun. This prediction of the theory was confirmed by Eddington in 1919 [10].

1.2 General Relativity

Newton famously said of his theory *hypotheses non fingo* ('I frame no hypotheses') [5], reflecting the fact that he understood no deeper meaning for the properties of gravity and the inverse square law of his theory, despite their successes. By contrast, General Relativity (GR) has a very powerful and beautiful interpretation as arising from the dynamics of space itself. In GR, spacetime is described by a pseudo-Riemannian manifold. Freely-falling particles follow geodesics on this manifold, and these geodesics depend on the metric. The dynamics of this metric are the dynamics of gravity, and Einstein's field equations [11],

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

describe how the metric is influenced by the presence of energy and momentum. Note the interplay between matter being influenced by the metric and matter influencing the metric, perhaps most elegantly encapsulated in the quote from John Wheeler [12],

> "Space tells matter how to move...matter tells space how to curve"

1.2.1 Formulations

While the Einstein equations in the form (1.1) are the original and most well-known mathematical description of the theory, there exist other useful and powerful formulations used in modern gravitational work.

Einstein-Hilbert Action

Much of modern theoretical physics is formulated in terms of actions. One great advantage is that one can describe both the classical and quantum dynamics through the action. The classical dynamics are obtained by extremising the functional, while the quantum behaviour derives from the path integral $\int \mathcal{D}\phi e^{\mathrm{i}S[\phi]}$. In addition, the Lagrangian formulation allows theories to be constructed based on their symmetries, and Noether's theorem [13] can also be used to derive the resultant conserved quantities.

The Einstein equations can be derived from the Einstein-Hilbert action[14],

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R, \tag{1.2}$$

where $\sqrt{-g}d^4x$ is the covariant volume element. Minimising $S = S_{EH} + S_m$, where S_m is some matter action, leads to (1.1) with $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}}$. From this action, one can show that energy momentum conservation, $\nabla^\mu T_{\mu\nu} = 0$, is a result of the diffeomorphism invariance.

The Einstein-Hilbert action, rather than the Einstein equations, is often the starting point for modified theories of GR. The action (1.2) also shows that it is consistent with all the symmetries to add a cosmological constant term $-\Lambda\int d^4x\sqrt{-g}$ to the action. This will play an important role in our discussion in Section 1.3.2.

ADM formulation

An alternative formalism exists courtesy of Arnott, Deser and Misner [15]. By making a clever splitting of the metric components, the spacetime can be viewed as a collection of spatial hypersurfaces evolving in time. This turns out to be particularly convenient for canonical approaches to quantum gravity (since it allows one to write a gravitational Hamiltonian), for numerical relativity, and as we shall see, for certain modified theories of gravity.

We split spacetime into a one-parameter family of spacelike hypersurfaces, by defining a timelike vector field $n^\mu = (N, N^i)$. The metric can be written

$$\mathrm{d}s^2 = -N^2c^2\mathrm{d}t^2 + \gamma_{ij}(N^i\mathrm{d}t + \mathrm{d}x^i)(N^j\mathrm{d}t + \mathrm{d}x^j). \tag{1.3}$$

γ_{ij} is the spatial metric on the hypersurfaces and $N(t,\mathbf{x})$, $N^i(t,\mathbf{x})$ are known as the lapse function and shift vector respectively. In General Relativity, these play the role of Lagrange multipliers, leading to the Hamiltonian and momentum constraints respectively. All the dynamics are governed by the spatial metric. Note that despite first appearances, this splitting retains full covariance (see the transformation properties of the metric components below), one can choose some foliation of spacetime but you retain the the full ability to transform the coordinates.

The Einstein-Hilbert action is written in this formalism as

$$S = \frac{1}{16\pi G}\int dt d^3x \sqrt{\gamma} N c \left[K^{ij}K_{ij} - K^2 + R^{(3)}\right], \tag{1.4}$$

where $R^{(3)}$ is the 3D Ricci scalar, K_{ij} is the extrinsic curvature of a slice, $K_{ij} = \frac{1}{2}\pounds_n \gamma_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - 2D_{(i}N_{j)}\right)$, D_i is the covariant derivative on the slice and dots indicate derivatives with respect to time. Note that all the time derivatives are now contained within the extrinsic curvature while the Ricci scalar solely contains spatial derivatives.

The Einstein-Hilbert action is invariant under diffeomorphisms $x^\mu \to x^\mu - \xi^\mu$, which correspond to $g_{\mu\nu} \to g_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}$. The action (1.4) must clearly be invariant under the same transformations, however they take a different form. Considering the infinitesimal transformations $x^i \to x^i - \xi^i(t, \mathbf{x})$, $t \to t - f(t, \mathbf{x})$, the metric fields transform as

$$N \to N + \xi^j \partial_j N + \dot{f} N + f \dot{N} \tag{1.5a}$$

$$N_i \to N_i + \partial_i \xi^j N_j + \xi^j \partial_j N_i + \dot{\xi}^j \gamma_{ij} + \dot{f} N_i + f \dot{N}_i \tag{1.5b}$$

$$\gamma_{ij} \to \gamma_{ij} + 2\partial_{(i}\xi^k \gamma_{j)k} + \xi^k \partial_k \gamma_{ij} + f \dot{\gamma}_{ij}. \tag{1.5c}$$

Note that the last of these can be written in the more familiar form $\gamma_{ij} \to \gamma_{ij} + D_{(i}\xi_{j)} + f\dot{\gamma}_{ij}$.

1.2.2 Properties of GR

GR has a number of interesting properties. We discuss some of these presently, as they will become important for contrast when we discuss modified theories of gravity.

The first property we note is the diffeomorphism invariance of the theory. This is a stronger statement than just the coordinate independence of the theory. In Special Relativity, one has already made an assumption about the geometry of spacetime — it is Riemann flat, $\mathrm{Riem}(g) = 0$. By contrast, the only *a priori* imposition on spacetime in GR is that it can be described by a (smooth) metric. One is then free to choose foliations or coordinates as desired, the physics will remain the same.

Following on from this, GR is a pure tensor theory. This means that it is a theory of a massless spin-2 particle — only the two transverse, traceless, tensor modes propagate. There are no scalar or vector fields involved in the gravitational sector. This follows from the diffeomorphism symmetry, in the same way that the massless and transverse properties of the spin-1 photon follows from the $U(1)$ symmetry of electromagnetism (and again, only two modes propagate). In fact, GR (plus a cosmological constant) corresponds to the *only* four-dimensional, relativistic, divergence-free second-order theory of a spin-2 particle of second order in derivatives (generic theories will be fourth order) [16]. Alternatively, one can start from a particle physics point of view and write down the action of a spin-2 particle on a background order-by-order, starting at quadratic order. It turns out that conservation of energy-momentum implies that the only consistent way to do this is to write down the linearised action for GR [17, 18]! This would seem to provide another piece of evidence for the uniqueness of GR as a theory.

A key stage in the development of GR was Einstein's "happiest thought", the realisation that the effects of gravity and the effects of acceleration were indistinguishable. This idea is a feature of GR and is known as the *Equivalence Principle* (EP). In its weakest form, the *Weak Equivalence Principle* (WEP), this simply states the equality of inertial and gravitational mass for test particles. This occurs in Newton's laws, resulting in the acceleration of freely-falling particles being independent of mass. Einstein strengthened this to the *Einstein Equivalence Principle* (EEP): in addition to the WEP holding, the local outcome of non-gravitational experiments in free-fall is independent of the frame velocity and location in the universe. In GR, the *Strong Equivalence Principle* (SEP) also holds, extending the above ideas to gravitational experiments [19].

Einstein's equations (1.1) are a complicated non-linear system of PDEs. Contrast this with Newton's single second-order ODE, from which the deterministic[1] nature follows from Picard's theorem for the uniqueness of solutions of ODEs. We expect any physical theory of this sort to be expressible as an initial value problem (IVP). It is not immediately obvious this is possible for GR. Indeed, due to the diffeomorphism invariance and the fact one is trying to evolve the metric (including the time components), the nature of the initial conditions are not readily apparent. In fact, GR does admit a well-posed IVP [20], which can be expressed by working in the previously mentioned ADM formalism and picking a gauge.

This is by no means an exhaustive list of GR's properties and theoretical success. For example, black holes are now widely important in astronomy, from stellar evo-

[1]This is the statement that the behaviour, in principle, completely follows from the initial conditions (rather than necessarily being a statement about predictability). In particular, this does not preclude the possibility of chaotic behaviour, a high *sensitivity* to initial conditions, which occurs in the three-body gravitational problem. Obviously this does not apply to measurement in quantum systems other than probabilistically.

lution to galaxies to cosmology; the absence of ghostly or tachyonic instabilities in the theory; the causal and local nature of the theory; and energy-momentum conservation. Many of these theoretical advantages no longer hold when considering modified gravity theories.

1.2.3 Successes of GR

Several of the above properties can be considered theoretical successes of the theory, demonstrating the internal consistency of the theory. GR has also enjoyed remarkable successes in the observational realm, and we now discuss some of these. These tests all indicate a strong agreement between GR and observations, which we expect any modified theory of gravity to match.

Most results concerning the weak-field limit of the theory are described under the heading of the 'solar system tests' of GR. All these weak-field tests derive from considering perturbations about a Minkowksi background. The aforementioned measurements of light bending by the sun and the Mercury's perihelion advance are categorised under this heading, as is Shapiro delay [21]: When light relays travel through the gravitational field of a massive body (such as the sun), they experience time dilation, introducing a time delay relative to the Newtonian prediction. While the Newtonian prediction is not observable, one can measure the difference in time taken for light to travel in different strength gravitiational fields; for example, the difference between superior and inferior conjunction of the interior planets [19]. Recent measurements of all of these have resulted in strong evidence for the validity of Einstein's theory in this régime [22].

Precision testing of GR began with measurements of the gravitational redshift of a photon. By making use of condensed matter effects [23], Pound and Rebka were able to measure a frequency change of $\Delta\nu/\nu \sim 10^{-15}$, in agreement with GR [24], by comparing the frequencies over a height change of just 22m.

Gravitational lensing is a generalisation of the predictions of the sun's light bending to larger scales. Light from a distant object can be distorted due to massive objects in our line of sight, and under the right conditions can result in multiple copies of the object appearing in the sky. This effect was first observed in 1979 [25] and today is used in cosmology to help investigate dark matter [26].

In addition to the weak field limits, GR has also been tested in a stronger field régime. Binary pulsars can be highly relativistic [27] and so determination of their orbits provides tests of a gravity system where relativistic effects are much larger than in the solar system. Strong field systems are also important in terms of gravitational waves, since more strongly gravitating systems produce larger gravitational waves. Binary pulsar systems are again relevant for this and observations of the decay of their orbital periods provide indirect evidence for the existence of gravitational waves [28]. Directly detecting and determining the polarisations of gravitational waves can provide very stringent tests on modifications of gravity, but unfortunately they have not yet been detected (see [29] for a review of gravitational waves and their detection).

There have been tests of all the different forms of the equivalence principle, all returning results consistent with GR [22]. The WEP is measured by comparing the rate at which objects of differing masses are affected by gravity, generally using Earth based torsion balances (but also, famously, by dropping a hammer and feather on the moon). The results are often expressed in terms of the Eötvos parameter, defined as $\eta \equiv 2\left|\frac{a_1-a_2}{a_1+a_2}\right|$ between two objects which experience an acceleration a_1 and a_2. Experiments give very tight bounds on this parameter [30, 31]. Redshift experiments provide an alternative method to test the WEP. The EEP can probed through tests of special relativity. Among the most precise are Hughes-Drever experiments [32, 33], which place constraints on the existence of a preferred frame from very precise measurements of the anisotropy of atoms or nuclei. The EEP is also tested by attempting to measure variation of dimensionless parameters (such as the fine-structure constant) [19]. It is conjectured that GR is the only theory which can satisfy the SEP [19][2], so tests of the SEP are very important in determining the correctness of GR. Violations of the SEP involve searching for evidence of time or space variation of the gravitational constant, or fall at a rate dependent on its self-gravity [35]. A particularly useful system for carrying out these investigations is the Earth-Moon system by using a legacy of the Apollo missions, Lunar Laser Ranging [36].

Most of the tests discussed so far apply on solar system scales or smaller, but GR has also achieved success on the largest scales in the universe. Einstein's equations can be solved for homogeneous and isotropic universes to obtain cosmological solutions. The absence of stable, static cosmological solutions led to Lemaître overthrowing the static model of the universe and predicting it should be expanding [37]. Hubble's confirmation of this prediction validated this GR approach. The modern cosmological model which has emerged from this early work has been very successful, notably in predictions and observations of the cosmic microwave background (CMB) and big bang nucleosynthesis (BBN). Increasingly precise measurements suggest the modern day cosmological model, ΛCDM, in a GR setting, is a very good description of our universe [38].

[2] Although it may turn out to be the case that the SEP is violated even in GR, hinted at by calculations of the self-force of particles in highly curved backgrounds [34].

1.3 The need to modify gravity

Despite the great successes of Einstein's theory, all is not well. This is reminiscent of Newton's theory before it, with evidence that an otherwise very successful theory must be modified. There are two main sources of motivation for these modifications, occurring on the smallest and largest scales in our universe.

1.3.1 Small scales and renormalisability

The standard model (SM) of particle physics is a quantum field theory (QFT) providing an excellent description of microscopic physics [39]. It incorporates the strong, weak and EM forces and all the observed fundamental particles into our best model of particle physics. However, despite modelling all the other particles wonderfully, including predictions of the first 12 digits of the electron's anamolous magnetic moment, it neglects gravity, and so cannot be a full fundamental theory of our universe.

It is natural to try and extend the SM to include gravity in the usual way, by simply minimally coupling matter and adding the Einstein-Hilbert term to the action. In fact, Feynman thought that since gravity was so much weaker[3] than EM, for which perturbative QED is an excellent description since $\alpha_{EM} \sim 1/137$, a QFT describing gravity would be even easier to work with as it was even weaker [40]! However, for reasons I will now explain, the perturbative procedure breaks down and gravity is perturbatively non-renormalisable.

QFTs, in general, contain infinities[4]. These arise from loop integrals, which correspond to corrections to the propagator/vertices arising due to the interactions in the theory. If only a finite number of these vertices contain divergences, then the infinities can be absorbed by redefining a finite number of parameters in the theory, meaning they are physically irrelevant. In this case the theory is said to be *power-counting renormalisable*. The total number of loop integrals is generically infinite, since the expansions of the theory used generically have an infinite number of terms. If loop diagrams of higher order exhibit better convergence behaviour than those of lower order, it is possible that only a finite number of integrals diverge.

We now obtain conditions on the theory for this to be possible. This treatment is

[3]Consider the force between two electrons due to electromagnetism and due to gravity. Then $F_{EM}/F_{grav} \sim 10^{43}$.

[4]Couching the theory in the language of distributions rather than ill-defined integrals can more formally make mathematical sense of the divergences [41].

familiar from standard texts on QFT, *e.g.* [42]. Consider a scalar field theory in d dimensions with an action[5]

$$S = \int d^dx \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{n!}\lambda_n\phi^n \right).$$

The mass dimension of the field ϕ is $[\phi] = d/2 - 1$, and $[\lambda_n] = d + n(1 - d/2)$. Consider corrections to the vertex with V legs, which we denote C_V. Now the dimension of C_V is simply equal to that of the corresponding coupling constant, $[C_V] = [\lambda_V] = d + V(1 - d/2) = V + d(1 - V/2)$. So for example C_2, corrections to the propagator, are of mass dimension 2 and C_4, corrections to the 4-point interaction, are of mass dimension $4 - d$.

But we can also look at the Feynman diagrams for this correction. A diagram at loop order N from the interaction λ_n, with I internal legs and L independent loop momenta, makes a contribution to C_V of the form

$$C_V \supset \lambda_n^N \int \left(\prod_{i=1}^{L} \frac{d^d l_i}{(2\pi)^d} \right) \left(\prod_{j=1}^{I} \frac{1}{p_j^2 + m^2} \right), \tag{1.6}$$

with the p_i some linear combinations of the loop momenta l_i and the external momenta. From this we can also calculate the mass dimension of C_V, $[C_V] = dL - 2I + N[\lambda_n]$. It is also clear that the degree to which the above diagram diverges depends on the powers of momentum appearing in the integral, and to this end, we introduce the *superficial degree of divergence*, $D \equiv dL - 2I$. $D > 0$ corresponds to a power-law divergence, $D = 0$ to a logarithmic divergence, and $D < 0$ to a non-divergence.

By combining our expressions for C_V and D, we obtain

$$D = [C_V] - N[\lambda_n] \qquad \text{with } [C_V] = d - V(d/2 - 1). \tag{1.7}$$

What does this tell us? How many vertices do we expect to diverge? There are potentially an infinite number of vertices, since vertices with arbitrarily many numbers of legs, V, can be generated by quantum corrections. However, the number of legs of a vertex, V, will not trouble us since it is clear that $[C_V]$ will be negative for all $V \geq 2d/(d-2)$, becoming larger in magnitude for larger V. On the other hand, each vertex has corrections going to arbitrarily high loop order, N. These will only be well-behaved if $[\lambda_n] \geq 0$. If $[\lambda_n] < 0$, D becomes positive for high enough N and so one has an infinite number of diverging vertices, indicating that the theory is not renormalisable by power counting.

[5] One can extend the idea to higher spin fields and to multiple interaction terms, but the basic details go through essentially unchanged aside from additional indices and summations.

Note that being renormalisable by power-counting is neither a necessary nor sufficient condition for a theory to be renormalisabile (hence the 'superficial' degree of divergence), but is reliable as a general guide [42].

This discussion of power-counting renormalisability brings us on to the issue of quantum gravity. The coupling constant for gravity is obviously Newton's constant, G. Unfortunately, in d dimensions $[G] = 2 - d$, and this negative mass dimension in $d = 4$ means that the perturbative theory is not viable — higher and higher loop orders exhibit worse divergences.

However, doing the explicit calculation demonstrates that *pure* GR is, in fact, renormalisable *at one loop*. The possible operators generated at one loop are R^2, $R^2_{\mu\nu}$ and $R^2_{\mu\nu\alpha\beta}$. The $R^2_{\mu\nu\alpha\beta}$ terms can be converted into $R^2_{\mu\nu}$ and R^2 terms since the Gauss-Bonnet combination $R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is a topological invariant in four dimensions. The remaining terms are removed by using the equation of motion, $R_{\mu\nu} = 0$ [6]. Thus all the non-renormalisable operators can be removed, rendering the theory one-loop renormalisable [43].

Unfortunately, this pure GR one-loop result is an accident. Continuing the perturbative procedure for pure gravity, one finds that it is non-renormalisable at two loops [7] [44, 45]. When coupled to matter, the quantum behaviour is even worse, being non-renormalisable at one-loop due to the fact that $R_{\mu\nu} \neq 0$ [43].

This result is disastrous! Despite General Relativity's great success at large distances and quantum field theory's great success at small ones, trying to combine the theories appears to lead to inconsistencies. Trying to consistently combine gravity and quantum behaviour has been a big motivator in constructing replacements for GR, including string theory and loop quantum gravity.

This does not mean we can say nothing about GR in the quantum realm. In fact, we can make predictions about quantum gravity using GR, but only in the context of effective field theory (EFT). Working below the Planckian cut-off it is possible, for example, to predict the first-order corrections to Newton's constant [46]. However, an EFT is only valid in a given régime (here energies less than M_{pl}) and we still require a UV completion.

Before moving on, we note an alternative interpretation. It may be possible that these inconsistencies are only an effect of perturbative GR. Considering the full theory non-perturbatively may reveal that the gravitational coupling constant

[6] This is sometimes expressed instead as a field redefinition $g_{\mu\nu} \to g_{\mu\nu}(1 + aR) + bR_{\mu\nu}$. Considering the field equations, however, makes it more explicit why this procedure fails for the case when $G_{\mu\nu} \neq 0$.

[7] One cannot just use $R_{\mu\nu} = 0$ from the equations of motion as before, since we are now working at higher order we must be careful in that $R_{\mu\nu} = 0 +$ one loop corrections.

approaches a non-Gaussian fixed point in the ultraviolet. In this case, the theory would, despite perturbative appearances, be asymptotically safe and valid up to arbitrarily high energies [47, 48]. Since, as we have observed, there are an infinite number of quantum corrections to the Einstein-Hilbert action generated (and thus an infinite number of β functions), one does not know the bare action and so is forced to consider a truncation of the full quantum action. The reliability of this truncation can be checked by investigating the convergence behaviour of the fixed points and critical exponents when higher order terms are added.

1.3.2 Large scales and cosmology

Having mentioned the great success of GR applied to cosmology, it may initially seem odd that cosmology provides a motivation to *alter* GR. However, the success of the standard ΛCDM model may not be all that it seems. Measurements indicate that the universe is of nearly critical density, comprised mainly of pressureless matter and energy in the form of a cosmological constant.

Consider the matter sector — observations such as galaxy rotation curves [49], indicate that the observable matter cannot account for all the material in galaxies. In fact, the most current data suggests that baryonic matter comprises only ∼5% of the overall energy density. This only accounts for 1/6 of the pressureless matter we can infer exists [38]. 'Cold dark matter', a new non-SM particle, has been proposed to account for this discrepancy, with no shortage of candidates (see [50] for a review), but at the time of writing, is still to be directly observed. One may think this story sounds familiar from Newtonian gravity and the unseen 'dark planet' Vulcan. Maybe, instead of new, unseen entities, it is really our theory of gravity that is wrong. Models such as MOND [51] and TeVeS [52] have attempted to explain dark matter (particularly galaxy rotation curves) by modifying gravity rather than introducing dark matter. However, extra particles remain the most popular paradigm for explaining the matter deficit, in part because colliding galaxy clusters have provided evidence favouring dark matter over modifying gravity [26, 53], the evidence that MOND may need (potentially SM) dark matter to explain observations [54], and also the fact that theories such as SUSY give a motivation for the presence of additional particles in our universe.

However, this is not the end of the story for the 'dark sector'. The current expansion of the universe is accelerating [55, 56], a surprising result which indicates that ∼70% of the universe's energy budget is in the form of a negative pressure fluid, dark energy. This mysterious and unexpected fluid component is most commonly explained by adding a constant term $\Lambda_{bare} g_{\mu\nu}$ to Einstein's equations (1.1). At a naïve level this seems perfectly reasonable — this term is consistent with the symmetries of the theory and so there is no reason to exclude it. In fact, if one works with gravity as an effective QFT there are terms generated of the form $\Lambda_{SM} g_{\mu\nu}$. These can be tuned with a bare cosmological constant appearing in the Lagrangian Λ_{bare}, and summing all these contributions leads to the observed value of $\Lambda = \Lambda_{bare} + \Lambda_{SM}$.

So far, so good. What if we now try and calculate the contributions, Λ_{SM}? They arise from zero point energies of the fields, leading to the contribution [57]

$$\frac{1}{2} \sum_{\text{fields}} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sqrt{\mathbf{k}^2 + m^2} \approx \frac{M_{\text{co}}^4}{16\pi} = \frac{\Lambda_{SM}}{8\pi G}, \tag{1.8}$$

where M_{co} is some cut-off of your theory, where the EFT description breaks down. We expect this cut-off to be of order $M_{pl} \sim 10^{18}\text{GeV}$. Unfortunately, the observed value of the cosmological constant takes the approximate value $\Lambda \sim H_0^2 \sim (10^{-33}\text{eV})^2$. In order to agree with the observed value, the bare value must cancel this particle physics contribution to 120 decimal places, a horrendous level of fine tuning! Choosing not to tune is not an option — without fine tuning the particle physics contribution, the contribution from the electron alone would be enough to ensure that the cosmological horizon "would not even reach the moon", as Pauli colourfully observed [58]. Models with supersymmetry reduce the level of fine tuning[8] above the SUSY breaking scale, but even taking TeV scale breaking requires a sixty place cancellation. This problem is compounded by non-perturbative effects, such as phase transitions in the early universe. Undergoing a phase transition introduces another contribution to Λ_{SM}, potentially resulting in large changes in the observed value of Λ (since one can only pick the bare value Λ_{bare} once). The *cosmological constant problem*[9] is sizeable, and many feel that it should not just be swept under the carpet as it is in the ΛCDM model.

There are two tacks taken to deal with this this: ignore it, or set the cosmological constant to zero via some mechanism, and then generate the cosmological constant via some new field or modification of gravity.

Ignoring the large vacuum energy can either be a completely blasé process as in ΛCDM, or one may argue that the cosmological constant takes its value necessarily by the anthropic principle [57], and so if it was not cancelled to a huge extent, there would be no observers to measure it. This idea is sometimes upgraded to a (speculative) model, rather than an accident, by working in a multiverse picture,

[8] Since all particles have a superpartner of opposite spin-statistics, and fermionic and bosonic contributions differ in sign.

[9] We use this here to mean this large degree of fine tuning necessary to obtain viable physics. Some authors also use this terminology to refer to the *coincidence* problem, the question of why we happen to live in an era when $\rho_m \propto 1/a^3$ is of comparable size to $\rho_\Lambda \propto 1$, which is mysterious as it only happens over a very small period in time of the universe's evolution.

based around an eternally inflating universe and the string landscape [59].

One can attempt to set the cosmological constant to zero and then generate the observed value by quantum fluctuations, new dynamical fields[10] or thinking back to the 'dark planet' Vulcan, and trying to modify gravity. This is often achieved by considering higher dimensional physics, such as braneworld models, or by adding additional curvature terms to the Einstein-Hilbert action. (See [60, 61] for a review of relevant models.)

But finding a mechanism to set the cosmological constant to zero is less simple than it might seem. Unbroken SUSY would lead to a vanishing cosmological constant contribution from the SM sector, but this clearly does not describe our universe. There exists a no-go theorem due to Weinberg [57], preventing Λ being dynamically driven to zero. This is a big obstruction to model makers although, as with all no-go theorems, the exceptions can point you towards possible resolutions of the problem. This is expanded upon in Section 5.1.

As a final note, there are also ideas that it is our assumptions about cosmology that are really what's wrong, in particular the Copernican principle. Maybe, in fact we are not in an 'average' place in the universe, instead residing in an underdense void, and the apparent acceleration of the universe results from trying to fit the round peg of a Copernican universe into the square hole of an inhomogenous one [62]. Even in this picture though, one is still required to explain why the vacuum energy does not lead to a large cosmological constant.

1.4 Modifying gravity

How do you go about modifying GR? Well, Lovelock's theorem [16] about the uniqueness of the Einstein equations is a useful starting point for a general consideration, by considering the assumptions required by the theorem. Breaking symmetries (or adding them in the case of SUGRA), working in higher dimensions and adding higher derivatives all result in a theory with different physics to Einstein's. For a general review of modified gravity theories, see [63].

Yet, given GR's great successes, attempting to modify gravity can be fraught with dangers. Even seemingly minor changes to the theory, such as giving a mass to the graviton [64] runs into severe difficulties [65]. In fact, we will regularly return to this example as it highlights many issues in modified gravity theories today.

[10]For example, a scalar field ϕ with a flat potential has $w_\phi = \frac{\dot{\phi}^2-2V(\phi)}{\dot{\phi}^2+2V(\phi)} \approx -1$

1.4.1 Extra modes

General Relativity is a theory of a symmetric tensor with two tensor degrees of freedom. A generic symmetric tensor (in 4D) has ten degrees of freedom, but the gauge symmetries of the theory result in only two modes propagating, a property unique to GR in 4D.

Considering almost any modification of gravity introduces extra degrees of freedom. For example, in massive gravity the linearised action does not retain the symmetry $h_{\mu\nu} \to h_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}$ under the change $x^\mu \to x^\mu - \xi^\mu$, demonstrating the absence of full diffeomorphism symmetry. Even theories such as $f(R)$ gravity, where the action is built from diffeomorphism invariant components, introduce extra degrees of freedom, since the equations of motion do not just correspond to the Einstein tensor. An extra scalar is the new degree of freedom in the case of $f(R)$, and the theory is often rewritten with the presence of this extra mode made manifest.

So, these theories have new degrees of freedom in addition to the two transverse, traceless tensor modes, and in 4D, this means that some of the 4-vector or scalar modes in a general symmetric tensor gain dynamics. It is often very convenient to study such modifications by splitting the theory into the two tensor modes plus the new, additional scalar/vector degrees of freedom. This is the Stückelberg trick [66][11] which we will make use of in Chapter 3. Very often, the behaviour of the tensor modes is satisfactory and in agreement with GR, but it is the new, additional modes responsible for the pitfalls in the theories.

These extra modes often suffer from theoretical disasters: instabilities, tachyons,

[11] A simpler (and the original) example is a massive photon,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A^\mu A_\mu + J^\mu A_\mu.$$

Clearly, this has broken the $U(1)$ gauge symmetry $A_\mu \to A_\mu + \partial_\mu \Lambda$. However, one can artificially restore the gauge invariance by introducing a new field $\hat{A}_\mu = A_\mu + \partial_\mu \phi$, resulting in the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} + \frac{1}{2}m^2(\hat{A}^\mu \hat{A}_\mu + 2\hat{A}^\mu \partial_\mu \phi + \partial_\mu \phi \partial^\mu \phi) + J^\mu(\hat{A}_\mu + \partial_\mu \phi).$$

This now *does* exhibit gauge symmetry, $\tilde{A}_\mu \to \tilde{A}_\mu + \partial_\mu \Lambda, \phi \to \phi - \Lambda$. How does the extra mode behave as we take the limit $m \to 0$? Well, if we canonically normalise, introducing $\hat{\phi} = m\phi$, then

$$\mathcal{L} = -\frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} + \frac{1}{2}m^2 \hat{A}_\mu \hat{A}^\mu + m\hat{A}^\mu \partial_\mu \hat{\phi} + \frac{1}{2}\partial_\mu \hat{\phi}\partial^\mu \hat{\phi} + J_\mu \hat{A}^\mu + \frac{1}{m}J_\mu \partial^\mu \hat{\phi},$$

and the gauge symmetry is $\hat{A}_\mu \to \hat{A}_\mu + \partial_\mu \Lambda, \hat{\phi} \to \hat{\phi} - m\Lambda$. Taking $m \to 0$, all the terms are well behaved, except the final (non-conserved) current term in the action. This diverges, ruining the chances of the usual photon results being recovered. As in gravity, it is the additional modes which have led here to unacceptable behaviour.

ghosts or superluminalities. Superluminal propagation of gravity modes can allow information to be broadcast faster than the speed of light, leading to causal problems in the theory. We will discuss these issues, along with the observational concerns presently.

1.4.2 Ghosts

Generically, modifications to the Einstein-Hilbert action introduce higher order derivatives. These often sound the death-knell for a theory since they almost always introduce ghostly instabilities in the theory. This is due to Ostrogradski's theorem [67, 68], which demonstrates that under very general conditions (non-degeneracy of the symplectic structure[12]), the Hamiltonian of the system contains more than two time derivatives and so is unbounded from below.

Classically this is a headache; quantum mechanically it is a nightmare! Ghosts lead you to either abandon unitary (and thus quantum mechanics) due to the presence of negative norm modes or otherwise accept that there are negative eigenvalues of the Hamiltonian. Thanks to pair creation in quantum mechanics, the latter option leads to a proliferation (in fact, a divergence in Lorentz invariant theories [69]) of particles being pair produced, since lighter particles can decay into heavier ones plus ghosts, or the vacuum into ghosts, to minimise their energy. There are tight constraints on this rapid particle production, meaning only extremely weakly coupled ghosts (which must also be Lorentz violating) can be considered to have any chance of viability [69].

Ghosts arise not only due to higher derivatives. Since theories of canonical spin-0 and spin-2 particles are always attractive [40], attempting to produce repulsive forces (to mimic dark energy), often involves changing the sign of the kinetic piece, leading to a ghostly spin-0 or spin-2 particle.

The presence of these ghosts is not always manifestly obvious. Boulware and Deser [70] discovered that the sixth mode of Pauli-Fierz massive gravity, while hidden on maximally symmetric backgrounds, reappears at higher order or on more general backgrounds as a ghostly instability.

The ghost problem is a highly generic one of modified gravity, though there do exist explicit theories where the ghost can be made safe [71]. See also [68] for a discussion of common misconceptions about ignoring ghosts.

[12]Lagrangians constructed purely from polynomials of the Ricci scalar, $f(R)$ theories, are degenerate so do not necessarily contain ghosts. Of course, in accordance with Lovelock's theorem, they still contain higher derivatives.

1.4.3 Strong coupling and non-linearities

Perturbative GR is well tested, notably binary pulsar system predictions, observations of which won Hulse and Taylor the Nobel Prize [27]. It is, of course, vital that any theory seeking to replace GR should also replicate its successes. In order to match these perturbative results, and indeed to retain predictability, a modified gravity theory should have a well defined perturbative expansion, at least for low energies. Unfortunately, this is not the case in many modified gravity theories. For example, massive gravity exhibits strong coupling at energies of $(m^4 M_{pl})^{1/5}$ [65]. Since $m \ll M_{pl}$, this cutoff is far below the Planck scale.

This is not only a classical issue, but also vital for gravitational theories which claim to be perturbatively renormalisable. If the theory is strongly coupled at low energies, the perturbative expansion has broken down, and there is no reason to expect renormalisability! We will see an example of this in Chapter 3.

However, strong coupling can also bring unexpected benefits to a theory. For example, in massive gravity, features such as light bending and solar system orbits disagree between massless gravity and massive gravity, even in the limit $m \to 0$. This is the vDVZ discontinuity [72, 73] and arises because of the presence of additional graviton modes for all $m \neq 0$. No matter how small the graviton mass is made, you cannot reproduce the successful predictions of GR. But, as is clear from the expression for the strong coupling scale, as $m \to 0$ then the strong coupling scale also $\to 0$, so describing weak-field effects with the perturbative form of the theory is not valid. In fact, Vainshtein showed [74, 75] that around a body of mass M, there is a scale $r_V \sim \left(\frac{M}{M_{pl}^2 m^4}\right)^{1/5}$ within which the GR gravitational results are reobtained. This is because this corresponds to a non-linear régime in the theory, where the strongly coupled tensor and scalar gravitons form a bound state mimicking the GR graviton. This is equivalent to resumming an all-order perturbative expansion [76]. Dvali has conjectured that this mechanism always 'saves' theories of gravity which become strongly coupled at low energies [77].

Non-linear effects can be vital in masking modified gravity theories from violating GR observations, particularly solar system tests. In addition to the Vainshtein effect, Chameleon fields [78] also exhibit such behaviour. These are fields with self-interactions resulting in an environment dependent mass. In more dense environments, such as inside the Earth or Sun, the chameleon has a large mass, while in less dense environments, it has a small mass. This allows chameleons to have order unity matter couplings, and be able to influence cosmology, but not be observable to fifth force experiments, since the range of a force varies with its mass.

1.4.4 Other common pitfalls

The observational successes discussed in 1.2.3 has lead to a standard parametrisation to test post-Newtonian effects and their consistency with observation, the Parametrised Post-Newton (PPN) framework [19]. In this, a theory can be assigned 10 parameters by its behaviour in the weak-field limit, and then the bounds on these standard parameters can be compared to experiment. Since the observed deviations from GR are small (all consistent with zero) [22], one requires the weak-field limit to mimic this behaviour to a very high degree. As previously discussed, one is often reliant on strong coupling effects to ensure that post-Newtonian observations can be matched in solar system tests.

The Equivalence Principle is also generically violated by modified gravity theories. Violations of the WEP and EEP are very stringently constrained, so one must either ensure that extra modes do not couple directly to matter in a non-universal manner, so they are not violated, or else kept very small. As previously mentioned, only GR is thought to satisfy the SEP, so the inevitable deviations must also be kept small. A helpful example is the case of Brans-Dicke gravity, which promotes Newton's constant to a dynamical field [79]. The theory contains a new parameter ω, and in the limit $\omega \to \infty$ the theory is equivalent to GR. Since matter does couples universally to the Jordan frame metric, it does not violate the EEP, but bounds on the SEP give lower bound for ω, which has been increasing as experimental precision improves, resulting in a current 2σ lower bound of 40,000 [80]. While technically in a valid parameter régime in the theory, the fact it is forced to deviate so little from GR has resulted in a loss of interest in the theory as a promising alternative.

Cosmological data is able to place good constraints on the expansion history and present content of the universe. Several approaches to try and parametrise the effect of new theories exist, such as the Parameterised Post-Friedmann model [81]. The new cosmological equations of a theory can generally be written as those of a FLRW universe in GR, plus an additional fluid. Depending on the modification, it may be desired that this fluid mimics dark matter or dark energy, or that it remains sufficiently subdominant to not affect the expansion rate and other observables. Not only can the background cosmological quantities be observed, there is also hope that future measurements will enable the constraint of cosmological perturbations in modified gravity [82, 83]. Another key ingredient in cosmological physics, Big Bang Nucleosynthesis (BBN), places tight constraints on new relativistic degrees of freedom, such as extra gravitational fields, or alterations to the universe's expansion history [39].

Gravitational waves may prove to be a particularly fruitful testing bed. Currently, they have not been directly detected, however the decay of the orbit of binary systems is well tested in GR and if, as in a modified theory, extra modes are present, we expect the orbital decay period to alter, since the additional modes can also carry away energy, generically producing different results to GR. Direct detection is also particularly promising, since the polarisations of gravitational waves (the type and number) can potentially tell us about the nature of the gravitational degrees of freedom. This is analogous to observing EM waves and noticing they are always transverse, backing up the description of a photon as only having the transverse degrees of freedom.

Models attempting to explain the cosmological constant problem via modifying gravity generally involve IR modifications, for example DGP [84]. Despite these theories appearing local at first glance, they may contain 'secret' and non-obvious violations of causality and locality in the models [85].

Alongside these issues there exist many others, including the difficulty of solving the fields equations of many modified gravity theories. The trouble in computing predictions in theories such as string theory is well known, and ideas such as discrete spacetime models require one to do away with infinitesimal calculus.

1.5 Outline

Having demonstrated the successes and drawbacks of GR, and the benefits and pitfalls of modified gravity, I will consider two particular theories for the majority of this thesis.

Chapters 2, 3 and 4 will focus on Hořava gravity, a modification of gravity designed to be renormalisable. Chapter 2 is a review of the ideas and previous work in Hořava gravity. Chapter 3 discusses classical problems and the necessity of introducing a hierarchy of scales into the theory, and is based on [1]. Chapter 4 considers adding classical and quantum matter into the theory, and is based on [3].

Chapters 5 and 6 will focus on a new proposal to solve the cosmological constant problem. Chapter 5 is based on [2] and contains details of the proposal, along with a specific model which may be ghost-free. Chapter 6 is based on [4] and involves understanding the behaviour of non-GR solutions present in the theory, particularly their cosmology.

Discussion and conclusion will take place in Chapter 7. Additional calculational details are contained in Appendices A and B.

Chapter 2

Hořava gravity

Hořava gravity attempts to create a renormalisable theory of gravity by giving up Lorentz invariance. The model, proposed in 2009, rapidly gained a large amount of interest. Here, we review the theory and the associated literature, before moving on to discuss original research in Chapters 3 and 4.

2.1 Motivation

Hořava gravity is a non-relativistic modification to gravity, proposed by Petr Hořava in 2009 [86, 87]. To motivate the theory, consider why gravity is perturbatively non-renormalisable. As discussed in Section 1.3.1, the coupling constant has negative mass dimension $[G_N] = -2$, and so the superficial degree of divergence of each loop order increases. However, one can construct higher derivative theories of gravity which appear to be power-counting renormalisable [88]. These modify the propagator so that rather than falling off like $1/k^2$ it falls off with a greater power of k. This means that loop integrals diverge less quickly[1], and so the superficial degree of divergence reduces — the theory becomes power-counting renormalisable. Unfortunately, this comes at a cost. If one adds, for example, fourth-order derivatives, the resulting propagator $1/(k^2 \mp \lambda k^4)$ can be re-written,

$$\frac{1}{k^2 \mp \lambda k^4} = \frac{1}{k^2} - \frac{1}{k^2 \mp \lambda^{-1}}, \tag{2.1}$$

explicitly revealing the presence of a ghost mode. This is unsurprising given Ostrogradski's theorem, discussed in Section 1.4.2, that theories with more than two time derivatives generically result in such instabilities.

[1]Recall that the degree of divergence depends on the power of k in the numerator vs. that in the denominator. The power of k in the denominator is affected by the propagator and so increasing the powers of momentum in the propagator reduces the divergence of the diagrams.

The question we ask is whether one can obtain the good (renormalisable) behaviour of the theory while avoiding the bad (ghostly pathology)? Hořava's proposal [87] is to break Lorentz invariance, adding higher-order spatial derivatives but remaining second-order in time derivatives. This gives a theory with the good propagator behaviour, while ensuring that there are no higher-order time derivatives to introduce Ostrogradski ghosts.

Such a theory necessarily marks out time as a special 'direction' in spacetime, in contrast with relativity which treats space and time on an equal footing. Hořava drew inspiration from condensed matter models involving systems with preferred directions, first worked on by Lifshitz [89]. Real world examples include ferrormagnets. For this reason, Hořava's proposal is sometimes known as Hořava-Lifshitz (or sometimes just Lifshitz) gravity.

These models make use of *anisotropic scaling.* This means that one of the dimensions scales differently to the others. In Hořava gravity, time and space scale differently,

$$\mathbf{x} \to b\mathbf{x} \qquad t \to b^z t, \tag{2.2}$$

where b is constant. z is known as the 'dynamical critical exponent', and clearly takes the value $z = 1$ in relativistic theories. Let's illustrate how this anisotropic scaling works with a free field theory example. The action

$$S = \int dt d^D\mathbf{x} \left(\dot{\phi}^2 - \frac{\phi \left(-c^2\triangle\right)^z \phi}{M^{2(z-1)}} \right), \tag{2.3}$$

where $\triangle \equiv \partial_i \partial_i$, M is a mass scale and factors of c have been restored, becomes under the anisotropic scaling (2.2),

$$S = \int dt d^D\mathbf{x} b^{-z+D} \left(\dot{\phi}^2 - \frac{\phi \left(-c^2\triangle\right)^z \phi}{M^{2(z-1)}} \right). \tag{2.4}$$

Thus, the theory has anisotropic scaling with critical exponent z. Clearly, this breaks Lorentz invariance; but we can make the theory appear to be Lorentz invariant at low energies by adding to the action

$$c^2 \int dt d^D\mathbf{x} \phi \triangle \phi, \tag{2.5}$$

which is relevant in the IR. At low energies, this will dominate over the $\phi\triangle^{2z}\phi$ term (provided $z > 1$). At low energies, this deformation results in the theory looking like the usual relativistic field theory, but at high energies it exhibits anisotropic scaling. In such a model, its relativistic nature appears as an accidental symmetry at low energies, rather than as a fundamental ingredient of the theory.

Thinking in terms of quantum theory, consider adding a potential term $\lambda_n \phi^n$ to our action (2.3). Recall that in relativistic QFT, the dimension of the coupling constant is $[\lambda_n] = D + 1 - \frac{n}{2}(D-1)$ in $D+1$ dimensions, and so the theory is power counting renormalisable for $n \leq 2\frac{D+1}{D-1}$ since the coupling constant is then non-negative. For example, in $3+1$ dimensions, $\lambda\phi^4$ is the highest power renormalisable term. For a theory exhibiting anisotropic scaling, the appropriate dimensions for time and space are $[t] = -z, [x] = -1$. The scaling dimension[2] of the coupling constant is $[\lambda_n] = D + z - \frac{n}{2}(D-z)$ instead, and the condition to have a non-negative dimension of the coupling constant is

$$[\lambda_n] \geq 0 \Rightarrow \begin{cases} n \leq 2\frac{D+z}{D-z} & \text{for } D > z \\ n \geq -2\frac{z+D}{z-D} & \text{for } z > D \\ \text{all}\, n & \text{for } z = D, \end{cases}$$

or alternatively for a fixed n, the potential term is renormalisable if $D \leq z\frac{\frac{n}{2}+1}{\frac{n}{2}-1}$. For example in $3+1$ dimensions, if $z=2$, potential terms up to ϕ^{10} are renormalisable. If $z=4$, potential terms of higher or equal power to ϕ^{-14} are renormalisable. While for the critical case, $z=3$, all possible polynomial potential terms are renormalisable by power counting. Visser demonstrated some power counting renormalisable anisotropic scalar field theories in more detail in [90, 91]. There is also work pre-dating Hořava's theory, considering the renormalisability of Lorentz violating scalar and fermion fields [92].

2.2 Formulation

We now demonstrate how to incorporate these ideas into a model of quantum gravity. We restrict ourselves to considering Hořava's theory in $3+1$ dimensions, since this corresponds to a theory introducing as little extra baggage relative to GR as possible.

The model is most easily formulated in terms of the ADM formalism, familiar from Section 1.2.1. We write the metric in ADM form,

$$\mathrm{d}s^2 = -N^2c^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{2.6}$$

This corresponds to a spacetime $\mathcal{M}$ and a folation of spacelike hypersurfaces $\mathcal{F}$. In GR, the foliation is purely a choice, which can be made for convenience; but in Hořava gravity the foliation is an additional structure in the theory — different foliations are physically inequivalent. The symmetry group is broken down from full spacetime diffeomorphisms to the subgroup which preserves the foliation structure, $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M}) \subset \mathrm{Diff}(\mathcal{M})$. This group is generated by the transformations

$$t \to \tilde{t}(t), \qquad x^i \to \tilde{x}^i(t, \mathbf{x}). \tag{2.7}$$

[2] Note that scaling, rather than mass, dimension is the relevant dimension here.

These foliations pick out a preferred time direction, and so one can separate into temporal and spatial derivatives in the manner required to create a theory with anisotropic scaling.

Under infinitesimal transformations of the form (2.7), $x^i \to x^i - \xi^i(t, \mathbf{x})$ and $t \to t - f(t)$, the components of the ADM metric transform as

$$N \to N + \dot{f}N + f\dot{N} \tag{2.8a}$$

$$N_i \to N_i + \partial_i \xi^j N_j + \xi^j \partial_j N_i + \dot{\xi}^j \gamma_{ij} + \dot{f} N_i + f \dot{N}_i \tag{2.8b}$$

$$\gamma_{ij} \to \gamma_{ij} + 2\partial_{(i}\xi^k \gamma_{j)k} + \xi^k \partial_k \gamma_{ij} + f\dot{\gamma}_{ij}. \tag{2.8c}$$

At this stage, we note the different variants of Hořava's proposal. One variant, called the *projectable theory* restricts $N = N(t)$, while the *non-projectable theory* keeps $N = N(t, \mathbf{x})$. The motivation behind the projectable theory lies in the fact that N_i and N can be considered as the gauge fields of spatial diffeomorphisms and time reparametrisations. It therefore seems reasonable to restrict the gauge field N to be solely a function of time, to match the property of the time reparametrisations. The two variants have distinct advantages and disadvantages, which will be discussed further when we review the literature in Section 2.3.

Using the transformations (2.8), we can determine the appropriate building blocks for our action. We build our action with terms up to dimension $2z$. For time derivatives, $K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - 2D_{(i}N_{j)}\right)$ is the only combination transforming appropriately under (2.8). This term has dimension z. We thus build the kinetic piece from various contractions of this with the inverse metric γ^{ij}, leading to the kinetic Lagrangian $\mathcal{L}_K = K_{ij}K^{ij} - \lambda K^2$, of dimension $2z$. In GR, λ is unity — the freedom for it to take on other values here is a direct consequence of the reduced diffeomorphisms. This λ will turn out to play a key role in our analysis. Since we are dealing with a quantum theory, we expect all parameters, including λ, to run under the renormalisation group (RG) flow. To mimic GR, we need $\lambda \to 1$ in the IR under the RG flow. We can now write the action for Hořava theory in the ADM form,

$$S_{grav} = \frac{1}{16\pi G} \int dt d^3x \sqrt{\gamma} N c \left(K_{ij}K^{ij} - \lambda K^2\right) + S_V, \tag{2.9}$$

where S_V is an action for the potential, containing all terms consistent with our symmetries up to dimension $2z$. In GR, the symmetries force this to take the unique form $R^{(3)}$ (the 3D Ricci tensor) as a result of the diffeomorphisms mixing the kinetic and potential pieces. The weakened symmetry of foliation-preserving diffeomorphisms means that is not the case here, as our time and spatial derivatives transform completely separately, and so the potential piece transforms independently of the kinetic piece. This extra freedom is precisely what we need to introduce higher spatial derivatives while remaining second-order in time derivatives.

The building blocks for the potential are the 3D Ricci curvature[3] of the spatial slice $R^{(3)}_{ij}$, along with covariant derivatives D_i and contractions obtained with the inverse metric γ^{ij}. In addition, it is consistent with the symmetries of the non-projectable theory to include the acceleration between spatial slices $a_i \equiv \partial_i \log N$. This was not included in the original formulation of the theory, and led to problems we discuss in the literature review in Section 2.3.3. The dimension of these objects is

$$[R^{(3)}_{ij}] = 2 \qquad [a_i] = 1 \qquad [D_i] = 1 \qquad [\gamma^{ij}] = 0.$$

To build a theory with anisotropic scaling z, one should include in the potential Lagrangian all inequivalent combinations of terms built from these objects[4], such that the overall dimension of each is $2z$. For example, with $z = 3$, $R^{(3)}_{ij} a^i D^j R$ is one possible combination of dimension $2z = 6$. There are in fact a very large number of such terms which can be written down (contrasting with the only one possible term in GR). To avoid this plethora of terms, Hořava proposed 'detailed balance' as an *ad hoc* ordering principle allowing one to only consider a few terms, again by analogy with properties of condensed matter models. Detailed balance is expanded upon in Section 2.3.1.

In addition to these terms of dimensionality $2z$ in the potential, one should also add deformations which become important in the IR, of lower dimension. This is vital, since without these terms, there is no way that our action is going to mimic the action of GR. It is only possible to construct terms of even dimension from contracting all the indices in our building blocks, so one must repeat the process above for terms of dimension $2z, 2(z-1), \ldots, 4, 2$, producing a potential action which can be written $S_V = \sum_{p=1}^{z} S_{2p} = S_2 + S_4 + \cdots + S_{2z}$.

Note that the scaling dimension of the coupling constant G in (2.9) can be easily determined to be $[G] = z - 3$. By analogy with the behaviour of the other 3 fundamental forces, we choose the marginal case $z = 3$ so the coupling constant has a (scaling) dimension of zero in the UV. Since the coupling constant no longer has negative mass dimension, it appears that the theory may be power counting renormalisable! The price has been to give up Lorentz invariance as a fundamental symmetry, but one approximately restored when looking in the deep IR. Naïvely this does not appear to be too fanciful — even the LHC only probes energies many orders of magnitude below the Planck scale [39].

[3]Note that since in 3D the Weyl tensor vanishes identically, the Riemann tensor contains no extra information compared to the Ricci tensor.

[4]Since we are dealing with a quantum theory, we expect all terms consistent with the symmetry to be generated.

Since we focus on the $z = 3$ case, we can write the potential action as

$$S_V = \int dt d^3x \sqrt{\gamma} N c \left(M_{UV}^2 V_2 + V_4 + \frac{V_6}{M_{UV}^2} \right), \tag{2.10}$$

where M_{UV} is the energy scale associated with the theory of gravity, and V_2, V_4 and V_6 are the potential pieces of dimension 2, 4 and 6 respectively. M_{UV} is usually taken to be the Planck scale $M_{UV} = M_{pl}$. This is not necessarily the case, however, as we will discuss in Chapter 3. The exact form of these potentials will depend upon the vat of the theory we consider.

We now go on to review Hořava gravity and the associated literature. This will also give us a chance to explore the different variations of the theory and their relative merits.

2.3 Variants and viability

Hořava's proposal of a power-counting renormalisability prompted a flurry of excitement. But is it too good to be true? We now consider the viability of Hořava gravity, and the pros and cons of variants existing in the literature.

The arguments relating to a scalar field differ for gravity in some important ways. The most important is the fact that for the scalar field we broke Lorentz invariance, a symmetry of the background. However, here we are breaking a *dynamical* symmetry of the theory. In addition, it is a spin-2, not a spin-0 theory, and so breaking this dynamical symmetry generically introduces new degrees of freedom, as we discussed in Chapter 1. When we take the GR limit, how does this mode behave? In particular, does it decouple or become strongly coupled? The behaviour of this extra mode will turn out to be key when we discuss the viability of the several variants of Hořava's theory.

Strong coupling is not only a problem in that it prevents us matching to GR observations such as binary pulsars (depending on the scale at which strong coupling occurs), but also it indicates a breakdown of the perturbative expansion in the QFT. Hořava gravity assumes a valid perturbation expansion to argue for renormalisability. Without one, there is no reason to believe the theory is renormalisable.

In order to match GR, we require the parameter $\lambda \to 1$ under the RG flow in the IR. However, no calculation of the RG flow behaviour in Hořava gravity has been performed, and there are other phenomenological results constraining the RG flow [93]. The renormalisablity of the theory beyond power counting has never been demonstrated, as it is generally more difficult to construct renormalisable theories of higher spin fields. This issue has not been discussed much in the literature, though Chapter 4 does touch upon the question of renormalisability once matter is included in the theory.

Lorentz invariance is well tested up to the Planck scale, as the effects of Lorentz violation are expected to propagate down to low energies [94–97]. It also appears that in order for apparent Lorentz invariance to return in the IR, one needs to fine tune various co-efficients in the theory [98]. Not only must Hořava gravity explain why gravity and light appear to have the same propagation speed, but fine tuning seems to be required to explain why all particles observe the same limiting speed c at low energies [99, 100]. This issue will be examined in more detail in Chapter 4.

A more theoretical issue related to Lorentz invariance is that conventionally, particles' spins are determined via the representation of the Lorentz group they transform under. With Lorentz invariance only an accidental symmetry it is not obvious what particles' 'spin' truly means in Hořava gravity.

Before we discuss the variants of the theory, note that while in this thesis we are concerned with treating Hořava gravity as a fundamental theory of gravity, not all of the literature has focused on this possibility. There is a proposal that Hořava gravity is the continuum limit of a discrete model of quantum gravity, causal dynamical triangulations [101]. Evidence has been presented to promote this view, including the similarity of the phase diagram [102] and numerical evidence that both spacetimes exhibit the same spectral dimension behaviour [103]. In addition, an analogue of the AdS-CFT correspondence in Hořava gravity has attracted interest. In this case, Hořava gravity corresponds to the bulk gravity theory, with an anisotropic QFT living on its boundary [104, 105].

2.3.1 Detailed balance

Hořava originally proposed the organising principle of 'detailed balance', to help reduce the number of terms appearing in the theory. Its motivation was the analogy to condensed matter theories, which have detailed balance alongside anisotropic scaling [87]. The potential is restricted to be of the form

$$E^{ij}\mathcal{G}_{ijkl}E^{kl}$$

where $\mathcal{G}_{ijkl}$ is the inverse of the DeWitt metric $G^{ijkl} = \gamma^{i(k}\gamma^{l)j} - \lambda\gamma^{ij}\gamma^{kl}$, and E^{ij} is derivable from an action principle. This means that $\sqrt{\gamma}E^{ij} = \frac{\delta W}{\delta\gamma_{ij}}$ for some action W. Hořava used this to argue that the appropriate form of the dimension six potential was

$$S_{V_6} = \int dt d^3x \sqrt{\gamma} N c \left(\frac{-M_{pl}^2}{w^4} \right) C_{ij} C^{ij}$$

where C_{ij} is the Cotton tensor, $C_{ij} = \epsilon_{ik}{}^{l} D^k \left(R^{(3)}_{jl} - \frac{1}{4} R^{(3)} \delta_{jl} \right)$. The Cotton tensor can be derived from the Chern-Simons action,

$$W = \frac{1}{w^2} \int_\Sigma \mathrm{Tr} \left(\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right).$$

This turns out to be the only suitable candidate satisfying detailed balance and containing three spatial derivatives (and hence $z = 3$ scaling in the UV) [87]. One can add the term $\mu \int d^3x \sqrt{\gamma}(R - 2\Lambda)$ to the action W to introduce relevant deformations, allowing the $z = 3$ theory to mimic GR at low energies.

However, this principle has now been abandoned in most work due to a variety of inconsistencies. Strong coupling at all scales appears to be an inevitable feature of Hořava gravity if we assume detailed balance [106] and take the GR limit, preventing us matching it to observations [27]. This also leads to problems in trying to match the spherical solutions of the theory to general relativity in the infrared limit; namely GR is not recovered at large distances [107, 108]. This is because one cannot just neglect the higher order terms, due to the strong coupling [106]. In addition, one is forced to have a cosmological constant of the wrong sign, unless either detailed balance is abandoned or certain parameters are analytically continued to complex values [107, 109, 110]. Without this, scale invariance is broken in the cosmological perturbations [109]. Cosmology also strongly disfavours the detailed balance scenario, with an extra fine tuning of the cosmological constant necessary to match ΛCDM observations [111].

However, a more recent paper [112] has argued that many of the above issues lie in projectability rather than detailed balance itself, and the non-projectable theory with detailed balance is (mostly) healthy. However, the issue of the large, negative cosmological constant required still results in problematic phenomenology for detailed balance. There is no current candidate for a suitable replacement as an ordering principle, and so abandoning detailed balance results in the theories containing a plethora of potential terms, with arbitrary coefficients.

2.3.2 Projectable theory

As mentioned, there are two distinct variants of Hořava's theory. The projectable branch restricts $N = N(t)$, to match the restriction of reparametrisations of time to $t \to \tilde{t}(t)$. This has the advantage of ensuring that the constraint algebra of the theory closes, which did not occur for the original non-projectable theory [113], as we will discuss below.

Even with detailed balance abandoned, the restriction of $N = N(t)$ is enough to restrict the potential to just eight terms[5],

$$S_V = \int dtd^3x\sqrt{\gamma}Nc\Big[M_{pl}^2 R + A_1(R^{(3)}_{ij})^2 + A_2(R^{(3)})^2 + \frac{B_1}{M_{pl}^2}(R^{(3)})^3 + \frac{B_2}{M_{pl}^2}R^{(3)}(R^{(3)}_{ij})^2 + \frac{B_3}{M_{pl}^2}R^{(3)}{}^i_j R^{(3)}{}^j_k R^{(3)}{}^k_i + \frac{B_4}{M_{pl}^2}R^{(3)}\triangle R^{(3)} + \frac{B_5}{M_{pl}^2}(D_i R^{(3)}_{jk})^2\Big], \quad (2.11)$$

where A_i, B_j are dimensionless coefficients. This action is sufficiently simple that it can be studied in generality [114].

However, the projectable theory faces a serious ghost and instability problem, arising from the extra degree of freedom in the theory [115]. It turns out that one is forced to make a choice between a ghost mode (for $\lambda \in (1/3, 1)$) or a tachyonic instability (for $\lambda < 1/3$ or $\lambda > 1$)[6] [116]. On theoretical and observational grounds, the tachyonic instability is less problematic, provided that the instability is not too strong [117]. However, it turns out that when the $\lambda \to 1$ (GR limit) is taken, the theory becomes strongly coupled [116, 118], meaning that we require a lot of fine tuning to avoid the instability spoiling the theory. However, on de Sitter space the ghost free branch appears to be free of tachyonic instabilities [119]. This is analogous to the situation in massive gravity, where the vDVZ discontinuity disappears in a background with a non-vanishing cosmological constant, and when we take $m \to 0$ faster than the vacuum curvature $\Lambda \to 0$ [120, 121]. However, in Hořava gravity this still requires fine tuning to avoid the instability. The associated constraints can be considered phenomenological constraints on the RG flow [93].

Recall from Section 1.4.3 that strong coupling can sometimes be considered beneficial for theories due to the resulting Vainshtein effect. [122] performed a gradient expansion, indicating the presence of a Vainshtein effect in this case. In addition, [122] also argues that despite the theory becoming strongly coupled in the standard perturbative expansion, the existence of the valid gradient expansion may mean that the arguments for power counting renormalisability still hold (though the resulting action is non-local), though it is not clear that this is the case.

Forcing λ to be very close to one also results in problems with Čerenkov radiation[7]. For $\lambda \sim 1$, the phase velocity of the additional mode is given by $v^2 \approx 1-\lambda$. Clearly

[5]Strictly speaking it depends on when one imposes the projectability condition. If it is only imposed via the equations of motion (*i.e.* on shell), then the potential can contain many more terms, since the quantum mechanical path integral should consider off-shell fluctuations.

[6]One has to treat the cases $\lambda = 0$ and $\lambda = 1/3$ separately, however they are still not well behaved.

[7]Recall that the most familiar case of Čerenkov radiation involves charged particles moving through a medium faster than the phase speed of light in that medium. The particle produces a shock wave analogous to that of a supersonic wave, producing distinctive radiation, losing energy in the process.

as $\lambda \to 1$, $v \to 0$. Therefore any high-speed particles moving through a region where this scalar field is non-zero should Čerenkov radiate, yet we observe high energy rays which travel vast distances without decaying away [123].

The cosmology of the projectable theory presents interesting features. Much of present time cosmology is unchanged, since only the lowest order derivatives matter. However, the theory offers a candidate for dark matter, namely an integration constant [124]. This possibility arises since the Hamiltonian constraint is no longer $\mathcal{H} = 0$, as in GR, but is instead non-local, $\int d^3x\mathcal{H} = 0$, so a homogeneous component can be freely added to the Hamiltonian. This is a direct result of N only being a function of time; the relevant δ-function only kills off the temporal rather than spatial integrals.

Considering early universe cosmology results in significant changes to the standard model. The modifications to gravity can be interpreted as an effective fluid, resulting in dark radiation ($w = 1/3$) and stiff matter ($w = 1$) contributions, important in the early universe [125]. Hořava gravity is able to reproduce a scale invariant spectrum without inflation: one needs only an expanding universe rather than an accelerating one, as a result of the change in the dispersion relation at high energies [126]. It is expected that $\mathcal{O}(1)$ oscillations $\propto k^z$ will appear in the cosmological perturbations, but the frequency of these is so high it is unlikely that they will be observable in future experiments [127]. Bouncing and cyclic cosmology models also arise in Hořava gravity, since the higher curvature terms become important in the early universe [128].

The projectable theory can be considered as a ghost condensate model, demonstrating the existence of issues with caustics — the theory breaks down in finite time [118]. Mukohyama disputes this, arguing that the theory is free of caustics due to higher derivative terms [129], but the authors of [118] argue that the presence of higher derivatives does not affect their conclusions.

It is worth noting that the projectability condition means that while projectable spacetimes cannot be converted into non-projectable ones and vice-versa, GR solutions can always be locally made projectable and for the most interesting physical solutions (Schwarzschild, Reissner-Nordström, Kerr, FLRW — all of which are static or conformally static) this can be done globally [99].

It is, however, not clear how to directly link the IR limit of the projectable theory and GR, due to the restriction of N to be a function solely of time [107]. In addition, the non-locality of the Hamiltonian constraint indicates the absence of local energy conservation in the theory.

2.3.3 Non-projectable theory (original form)

The original formulation[8] of the non-projectable theory neglected the a_i terms, so this section considers the theory in their absence, before discussing their presence in Section 2.3.4.

In this case, the theory is sick. There are issues with the constraint algebra, including the fact that the constraint algebra does not close, is five-dimensional [113] (and so corresponds to two and a half degrees of freedom) and the lapse function is over-constrained [130]. In addition, there are phenomenological concerns. The theory becomes strongly coupled at all scales as one takes the GR limit [106, 118]. This is most simply seen in the relativistic Stückelberg formulation (which will be discussed more fully in Section 3.2). In the IR limit, where the 4 and 6-dimensional terms are not relevant, the theory can be written

$$S = S_{GR} + \frac{(1-\lambda)}{l_{pl}^2}\int d^4x\sqrt{-g}\hat{\mathcal{K}}^2. \tag{2.12}$$

At this point, we can insert the Stückelberg field ϕ using (3.6). We then expand ϕ as $\phi = x^0 + \chi$ on a Minkowski background, since the condition $\phi = const$ for the hypersurfaces replaces $t = const$. To cubic order, one obtains [118]

$$S = S_{GR} + \frac{(1-\lambda)}{l_{pl}^2}\int \mathrm{d}^4x \left[(\triangle\chi)^2 + 2\dot{\chi}\left((\triangle\chi)^2 + 2\partial_i\chi\partial_i\triangle\chi\right)\right]. \tag{2.13}$$

Canonically normalising this results in the strong coupling scale, $\Lambda = \sqrt{|1-\lambda|}/l_{pl}$. Clearly as the GR limit $\lambda \to 1$ is taken, this strong coupling scale goes to zero. However, it might be argued that the extra mode is non-dynamical, since no time derivatives appear at quadratic order, and so it could be integrated out [131, 132]. If this were correct, then strong coupling is not an issue, but this is not the case. [118] showed that if you expand on a nearly flat but non-trivial background (of background curvature L), you obtain the schematic action,

$$S = S_{GR} + \Lambda\int d^4x \left[L^2 v^i\dot{\chi}\partial_i\chi + (\triangle\chi)^2 + \dot{\chi}(\triangle\chi)^2\right], \tag{2.14}$$

where v^i is a unit vector along the extrinsic curvature gradient. This exhibits strong coupling at a scale $\Lambda' = |1-\lambda|^{3/8}/(L^{1/4}l_{pl}^{3/4})$. This action now features time derivatives at the quadratic level, and so is dynamical. This shows that the the strong coupling scale not only tends to zero in the GR limit $\lambda \to 1$, but also in the Minkowski limit $L \to 0$.

[8] In some sense, the 'original' and 'extended' versions of Hořava's theory are the same theory, as a_i terms will be generated by quantum corrections and so the 'extended' theory is just the 'original' theory written in a fully consistent way. However, we make the distinction here as it is important in the context of the history and literature of the theory.

It is also worth noting that the action (2.14) only contains one time derivative in the quadratic piece, which is unusual. It has been conjectured that this is related to the odd dimensionality of the phase space [118]. This is disputed by [113], who question whether it is valid to perform direct comparisons of the perturbative and exact degrees of freedom.

Because of these concerns, this variant of the theory is no longer considered consistent, and so non-projectable Hořava gravity is now studied in the form of the so-called 'extended' non-projectable theory, which we now discuss.

2.3.4 Extended non-projectable theory

As noted by [118], the above analysis of the non-projectable theory does not hold if terms of the form $a_i \equiv \partial_i \log N$ are included in the theory. One can then add a large number of extra terms to the theory, including the dimension two piece $\alpha a^i a_i$ (where α is constant). This provides an additional term in the IR limit compared to (2.12). This ensures that unlike the projectable theory, there is a region of parameter space where the extra mode is ghost-free and avoids tachyonic instabilities, $\lambda > 1$ and $\alpha \in (0, 2)$ [117]. This region is consistent with the hope of the RG flow restoring a GR-like régime via $\lambda \to 1$, $\alpha \to 0$ in the IR. However, it has been claimed that one needs $\lambda < 1/3$ (contradicting GR) to be ghost free and have a stable vacuum from stochastic quantisation arguments [133].

It was further claimed in [117] that this suffices to make the theory free of strong coupling, resulting in it being deemed a 'healthy extension' of Hořava gravity. However, Sotiriou and Papazoglou [134] considered the low energy limit of this theory and showed that it too suffered from problems with strong coupling, by considering only the lowest-order derivatives. In response, the original authors have argued that the strong coupling scale might exceed the cut-off scale for their derivative expansion and thus be rendered meaningless [135]. Obviously, one could force this to be the case by introducing a new scale in the theory. However it is worth asking whether or not this is actually *necessary*, and if so, to what degree — can the strong coupling scale be raised, or removed altogether, simply by including higher-order interactions, but without introducing large dimensionless parameters? One can also question the naturalness of the hierarchy of scales introduced when one attempts this in the extended version of Hořava gravity. A new lower energy scale will also result in Lorentz violating effects kicking in at lower energies, potentially causing contradictions with experimental measurements. In particular, the lowest scale of Lorentz violation in the gravity sector should not induce violations at the same scale in the matter sector, due to constraints from synchrotron radiation from the Crab Nebula [136].

In Chapter 3 we consider these questions in more depth, addressing them in a convenient limit where the extra modes and GR decouple.

The homogeneous and isotropic cosmology of the non-projectable theory coincides with that of the projectable theory — FLRW corresponds to a projectable spacetime and spatial derivatives of N vanish. However, since the Hamiltonian constraint is now local, dark matter can no longer result from an integration constant.

An interesting relation exists between Hořava gravity and Einstein-æther theory [137]. Einstein-æther contains a unit timelike vector field in addition to the metric tensor of GR. Restricting this vector to be hypersurface orthonormal recovers the IR limit of the extended version of Hořava gravity. This allows a number of results to be carried over from Einstein-æther theory, including spherical solutions [137] and slowly rotating black holes [138].

2.3.5 Hořava gravity with an extra $U(1)$ symmetry

A third variant of Hořava gravity was proposed in [139]. The starting point is to note that for the *projectable* theory, the action (2.9) has an Abelian symmetry present at the linearised level about Minkowski. With an expansion $N = 1 + n$, $N_i = n_i$, $\gamma_{ij} = \delta_{ij} + h_{ij}$, the Abelian symmetry acts as

$$\delta_\alpha n_i = \partial_i \alpha, \qquad \delta_\alpha h_{ij} = \delta_\alpha n = 0. \tag{2.15}$$

The symmetry is a global one, so α is constant in time, $\dot{\alpha} = 0$. It will be denoted $U(1)_\Sigma$, since α corresponds to a symmetry for each slice Σ of the foliation.

Acting with (2.15) on (2.9) (with $c = 1$) results in the linearised action

$$\delta_\alpha S_{grav} = -2M_{pl}^2 \int dt d^3x \triangle\alpha \left[(1-3\lambda)\dot{\zeta} - (1-\lambda)\triangle\beta\right], \tag{2.16}$$

where ζ is the trace of the metric tensor γ_{ij} and $\delta_i\beta$ is the scalar perturbation in N_i. If one sets $\lambda = 1$ and uses integration by parts to write the first term as $(1-3\lambda)\zeta\triangle\dot{\alpha}$, then since $\dot{\alpha} = 0$ this means that the linearised action is invariant under this symmetry.

This linearised symmetry only appears when $\lambda = 1$, and so this symmetry may help provide a mechanism to force this to one and hence have a GR-like action. Not only this, but the gauging of this extra symmetry will help us to kill off the non-tensor degrees of freedom, eliminating the pathological modes arising from the loss of Lorentz symmetry. This is not in contradiction with Lovelock's theorem since we have broken Lorentz invariance.

This is only, however, a linearised symmetry on a specific background. Generalising the non-trivial pieces of the symmetry generators to arbitrary background leads to an obstruction to the global symmetry — a piece of the action which does not transform appropriately. This can be 'undone' by introducing a new scalar field ν, which transforms as $\delta_\alpha \nu = \alpha$. Having restored the global symmetry, one is able to introduce a gauge field A and gauge the symmetry via the Noether procedure.

One therefore obtains a theory consistent with Hořava's original idea, but with only two (tensor) propagating degrees of freedom, and $\lambda \equiv 1$. Maximally symmetric and (projectable) Schwarzschild solutions solve the theory, which is favourable for solar system tests.

Unfortunately for the theory, matter coupling is problematic [140]. The A equation of motion forces R to be constant, which is fine in vacuum, but not with matter present in the theory. Therefore, one cannot just minimally couple matter in the usual approach to obtain consistent phenomenology; you are forced to introduce a coupling between the gauge field A and matter. This coupling is particularly problematic for cosmology since $[A] = 2z - 2$, while time derivatives correspond to dimension z. Any coupling of the two will lead to a term of dimension $3z - 2$, which is an irrelevant operator in our case ($z = 3$). While relevant terms can be built with spatial derivatives, in cosmology the isotropy and homogeneity of FLRW geometry forces spatial derivatives to vanish, and the matter fields to depend only on time. However, the matter and gravity sectors cannot be coupled with time derivatives in the theory, ruling out usual FLRW cosmology (unless $H = 0$). Extending the minimal substitution rule proposed in [140] for the theory also indicates that Newtonian gravity is not restored in the weak-field limit [141].

In addition, it turns out that the scalar field ν is too good at its job [140]. The symmetry of the linearised theory on Minkowski required $\lambda = 1$ and we then introduced ν to allow the symmetry to appear on more general backgrounds. But this trick actually results in the global symmetry being present for any λ — ν is too powerful a tool for the job!

Despite this freedom in λ, the scalar modes are still eliminated by the gauge symmetry. It would be an interesting exercise to perform a Stückelberg analysis of the theory to understand what happens to the disappearing Stückelberg mode.

2.4 Outlook

In a relatively short space of time, Hořava gravity has become a popular and well-studied paradigm as a quantum gravity model. In this chapter, we have seen why this theory may stand a hope of being renormalisable, before going on to more precisely formulate the theory. After this, we looked into the viability of the theory, reviewing the associated literature. We demonstrated that there were dangerous shortfalls in several variants of theory. For the most part (but by no means exclusively), these arose from the additional degree of freedom.

From this point onwards we concern ourselves with the 'extended' non-projectable Hořava theory. This appears to have the greatest potential for being 'healthy' — the extra mode can be ghost-free and stable, matter can be consistently coupled, the cosmology is viable and there may be no strong coupling concerns. The following two chapters will further investigate this model. Mostly pure gravitational aspects, including strong coupling and the necessity of introducing a hierarchy of scales will be considered in Chapter 3. Hořava gravity plus matter content will be considered in Chapter 4, including how to build $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ invariant actions and the effect of quantum corrections on the matter sector.

Chapter 3

Extended Hořava gravity and the Decoupling Limit

Having reviewed Hořava gravity generally, we now consider the 'extended' non-projectable theory and discuss the properties of the extra degree of freedom. In particular, we will focus on strong coupling and whether one is forced to introduce a hierarchy of scales into the theory. This chapter is based on the paper [1].

3.1 Introduction

In the previous chapter, we discussed disputes surrounding strong coupling in the 'extended' version of Hořava gravity [117]. Does strong coupling appear in the theory, or is the derivative cutoff in [134] invalid [135]? In particular, can strong coupling be avoided without introducing a new scale into the theory? Is there a Vainshtein mechanism at work in Hořava gravity? Does it help with phenomenology?

These questions revolve around scalar degrees of freedom in the theory. In this chapter we address these and other issues directly by isolating the troublesome extra degree of freedom and studying its properties. This can be done using the Stückelberg trick to fully restore diffeomorphism invariance to all orders, along the lines proposed in [118, 142]. To drastically simplify the analysis we will take a limit in which the Stuckelberg field decouples completely from the spin-2 sector. The decoupled Stückelberg theory ought to capture most of the interesting physics associated with strong coupling, as well as the possible presence of ghosts and other pathologies[1].

[1]Surprisingly, it turns out one cannot hope to capture the proposed resolution of strong coupling presented in [135] unless one retains some coupling between the graviton and the Stuckelberg mode.

Our starting point is the full action for Hořava gravity, which is given by[2] [86, 87, 114, 117]

$$S = S_{grav} + S_m, \tag{3.1a}$$

$$S_{grav} = l_{pl}^2 \int dt d^3x \sqrt{\gamma} N c \left(K_{ij}K^{ij} - \lambda K^2\right) + S_V, \tag{3.1b}$$

$$S_V = \int \mathrm{d}t\mathrm{d}^3x \sqrt{\gamma} N c \left(\frac{1}{l_{UV}^2} V_2 + V_4 + V_6 l_{UV}^2\right), \tag{3.1c}$$

with S_m is the generalised matter action. The potential terms are built from $R^{(3)}_{ij}$ and $a_i \equiv \partial_i \log N$, plus derivatives and contractions. All possible terms of dimension 2, 4 and 6 which are inequivalent (unrelated by integrations by parts, Bianchi identity, *etc.*) and not total derivatives must be included in the action, since this corresponds to the full set of terms satisfying our symmetries. The dimension two and four potential pieces can be written explicitly,

$$V_2 = \alpha(a^i a_i) + \beta R^{(3)}, \tag{3.2a}$$

$$\begin{aligned} V_4 = {} & A_1(a^i a_i)^2 + A_2(a^i a_i)a^j{}_j + A_3(a^i{}_i)^2 + A_4 a^{ij} a_{ij} \\ & + B_1 R^{(3)} a^i a_i + B_2 R^{(3)}_{ij} a^i a^j + B_3 R^{(3)} a^i{}_i + C_1 (R^{(3)})^2 + C_2 (R^{(3)}_{ij})^2, \end{aligned} \tag{3.2b}$$

where we have introduced $a_{ij} \equiv \partial_i \partial_j \log N$. The dimension six piece contains of order sixty terms, so will not be written down explicitly. Note that the term β is not physically relevant — it can be set to any non-zero value by rescaling time. We will use this freedom later to set it to a convenient value. Our full theory is then (3.1) along with (3.2).

3.2 The Stückelberg formulation

The Stückelberg trick can be used to covariantise Hořava's theory, allowing a $4D$ formulation as opposed to the $3+1$ ADM formalism [118, 142]. This covariant Stückelberg formalism allows for very easy separation of the Stückelberg mode and understanding of its dynamics. We start with the action of the 'extended' version of Hořava's theory (3.1).

The first (albeit trivial) step is simply to undo the anisotropic scaling in the coordinates by introducing $x^0 = ct$. We can now introduce spacetime coordinates and four-vectors $x^\mu = (x^0, \mathbf{x})$. Each component now has the same scaling dimension.

The non-relativistic nature of the theory is also encoded in the fact that the spacelike hypersurfaces in the theory are specified by $t = const$. We want to

[2]Note that we work with the Planck length, $l_{pl} = 1/M_{pl}$ and UV length $l_{UV} = 1/M_{UV}$ in this chapter.

remove this explicit breaking of diffeomorphism invariance, and the price to pay will be introducing the extra Stückelberg degree of freedom. The foliation is redefined so that

$$t = const \to \phi(x) = const. \tag{3.3}$$

The foliation-preserving diffeomorphism condition implies that the action is invariant under changes of the form $\phi(x) \to f(\phi(x))$. We can now construct a covariant action, by making use of projection operators and the Gauss-Codazzi relations [20].

To start, we introduce a unit normal to the foliation,

$$u^\mu = \frac{\nabla^\mu \phi}{X}, \qquad X = \sqrt{-\nabla_\mu \phi \nabla^\mu \phi}. \tag{3.4}$$

Our indices are now full spacetime indices, and ∇_μ is the full spacetime covariant derivative.

We promote the induced metric on the spatial slices to a covariant 4D projector onto the slices,

$$\gamma_{ij} \to \gamma_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu. \tag{3.5}$$

It is simple to promote the extrinsic curvature to a four tensor, by using the fact that it is the Lie derivative of the metric along the normal,

$$\frac{1}{c} K_{ij} \to \mathcal{K}_{\mu\nu} = \frac{1}{2} \pounds_u \gamma_{\mu\nu} = \gamma^\alpha_{(\mu} \gamma^\beta_{\nu)} \nabla_\alpha u_\beta. \tag{3.6}$$

In order to promote the Riemann tensor, one uses the Gauss-Codazzi relation and its contractions,

$$R^{(3)}_{ijkl} \to \mathcal{R}_{\mu\nu\rho\sigma} = \gamma^\alpha_\mu \gamma^\beta_\nu \gamma^\gamma_\rho \gamma^\delta_\sigma R_{\alpha\beta\gamma\delta} - \mathcal{K}_{\mu\rho}\mathcal{K}_{\nu\sigma} + \mathcal{K}_{\nu\rho}\mathcal{K}_{\mu\sigma}, \tag{3.7a}$$

$$R^{(3)}_{ij} \to \mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\lambda\nu}{}^{\lambda} = \gamma^\alpha_\mu \gamma^\gamma_\nu \gamma^{\beta\delta} R_{\alpha\beta\gamma\delta} - \mathcal{K}_{\mu\nu}\mathcal{K} + \mathcal{K}_{\mu\lambda}\mathcal{K}^\lambda_\nu, \tag{3.7b}$$

$$R^{(3)} \to \mathcal{R} = \gamma^{\alpha\gamma}\gamma^{\beta\delta} R_{\alpha\beta\gamma\delta} - \mathcal{K}^2 + \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu}, \tag{3.7c}$$

where we have introduced the $4D$ Riemann tensor $R_{\alpha\beta\gamma\delta}$ and a 'covariant $3D$ Riemann tensor' $\mathcal{R}_{\mu\nu\rho}{}^{\sigma}$.

We will also use the standard rules relating derivatives on embedded hypersurfaces to those in the full space [20],

$$D_k T^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \to D_\rho T^{\mu_1 \cdots \mu_m}{}_{\nu_1 \cdots \nu_n} = \gamma^{\mu_1}_{\alpha_1} \cdots \gamma^{\mu_m}_{\alpha_m} \gamma^{\beta_1}_{\nu_1} \cdots \gamma^{\beta_n}_{\nu_n} \gamma^\sigma_\rho \nabla_\sigma T^{\alpha_1 \cdots \alpha_m}{}_{\beta_1 \cdots \beta_n}, \tag{3.8}$$

and these allow us to write

$$a_i \to a_\mu = u^\lambda \nabla_\lambda u_\mu \tag{3.9a}$$

$$a_{ij} \to a_{\mu\nu} = \gamma^\alpha_{(\mu} \gamma^\beta_{\nu)} \nabla_\alpha a_\beta. \tag{3.9b}$$

Using this, one can rewrite the action (3.1) as

$$S = \frac{1}{cl_{pl}^2}\int \mathrm{d}^4x\sqrt{-g}\left[R^{(4)} + (1-\lambda)\mathcal{K}^2\right] + S_V$$
$$= S_{GR} + \Delta S_K + \Delta S_2 + \Delta S_4 + \Delta S_6 + S_m \tag{3.10}$$

where

$$S_{GR} = \frac{1}{l_{pl}^2}\int d^4x\sqrt{-g}R^{(4)} \tag{3.11}$$

is the emergent GR piece, and the beyond GR terms are

$$\Delta S_K = \frac{1-\lambda}{l_{pl}^2}\int d^4x\sqrt{-g}\ \mathcal{K}^2 \tag{3.12a}$$
$$\Delta S_2 = \frac{1}{l_{UV}^2}\int d^4x\sqrt{-g}\ \alpha a^\mu a_\mu + \left(\beta - \frac{l_{UV}^2}{l_{pl}^2}\right)\mathcal{R} \tag{3.12b}$$
$$\Delta S_4 = \int d^4x\sqrt{-g}\ \mathcal{V}_4 \tag{3.12c}$$
$$\Delta S_6 = l_{UV}^2\int d^4x\sqrt{-g}\ \mathcal{V}_6 \tag{3.12d}$$

where

$$\mathcal{V}_4 = A_1(a^\mu a_\mu)^2 + A_2(a^\mu a_\mu)a^\nu{}_\nu + A_3(a^\mu{}_\mu)^2 + A_4 a^{\mu\nu}a_{\mu\nu}$$
$$+ B_1\mathcal{R}a^\mu a_\mu + B_2\mathcal{R}_{\mu\nu}a^\mu a^\nu + B_3\mathcal{R}a^\mu{}_\mu + C_1\mathcal{R}^2 + C_2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}. \tag{3.13}$$

As before, we will not attempt to write down $\mathcal{V}_6$. At this stage we use the fact that the precise value of β is physically irrelevant and set $\beta = \frac{l_{UV}^2}{l_{pl}^2}$, thereby eliminating the last term in ΔS_2. The matter coupling is $S_m = S_m[g_{\mu\nu}, \phi; \Psi]$, whose form we will now discuss.

3.2.1 Matter coupling

An important question for Hořava gravity is how matter couples to the theory. In general relativity, the coupling is determined by demanding that the matter action be invariant under spacetime diffeomorphisms. Obviously this argument does not carry over to Hořava gravity, due to the reduced symmetry in the theory. It should also be noted that the argument here holds in both the projectable and non-projectable theory[3], and was first performed in [1].

The calculation follows the same lines as it does in general relativity. The overall action can be written $S = S_{grav}[g_{\mu\nu}, \phi] + S_m[g_{\mu\nu}, \phi; \Psi]$, where Ψ are the various matter fields appearing in the matter action. The overall action is required to be

[3]However, it does not apply to the variant with the extra $U(1)$, whose matter coupling differs.

invariant under foliation-preserving diffeomorphisms. S_{grav} is the Hořava gravity action, invariant by construction and so $\delta_{\text{Diff}_{\mathcal{F}}} S_{grav} = 0$. So to have $\delta S = 0$ overall[4], we require that $\delta S_m = 0$. Varying S_m results in

$$\begin{aligned}\delta S_m &= \int d^4x \left(\frac{\delta S_m}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\delta S_m}{\delta \phi} \delta \phi + \sum_i \frac{\delta S_m}{\delta \Psi_i} \delta \Psi_i \right) \\ &= \int d^4x \left(\frac{1}{2} \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} + \frac{\delta S_m}{\delta \phi} \delta \phi \right), \end{aligned} \tag{3.14}$$

where the energy-momentum tensor $T^{\mu\nu}$ takes it usual definition, and the final term on the first line vanishes by the Euler-Lagrange equations for the matter fields.

Under a diffeomorphism generated by a vector field ξ^μ, $x^\mu \to x^\mu - \xi^\mu$, the metric and Stückelberg field[5] transforms as

$$\delta g_{\mu\nu} = \pounds_\xi g_{\mu\nu} = 2\nabla_{(\mu} \xi_{\nu)} \qquad \delta \phi = \pounds_\xi \phi = \xi^\mu \partial_\mu \phi. \tag{3.15}$$

Substituting into (3.14) and integrating by parts yields

$$\int d^4x \sqrt{-g} \xi^\mu \left(-\nabla^\nu T_{\mu\nu} + \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta \phi} \partial_\mu \phi \right) = 0, \tag{3.16}$$

and so, since this must hold for arbitrary ξ^μ, we can see how matter couples to the Stückelberg field

$$\frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta \phi} = -\frac{u_\nu}{X} \nabla_\mu T^{\mu\nu} \tag{3.17}$$

where we have used $X^2 = -\nabla_\nu \phi \nabla^\nu \phi$ and $u_\nu = \nabla_\nu \phi / X$. By substituting this expression for $\frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta \phi}$ back into (3.16) and rearranging, one obtains

$$\gamma_{\lambda\nu} \nabla_\mu T^{\mu\nu} = 0. \tag{3.18}$$

This expression explicitly demonstrates that we no longer have full energy-momentum conservation in Hořava gravity. This is in line with the result from the perturbative analysis carried out in [106]. We can explicitly see how it is violated — it is only required that the projection of $\nabla_\mu T^{\mu\nu}$ onto our spacelike hypersurfaces vanishes. In the case where $\phi = x^0$ (our hypersurfaces are of constant time) the relevant constraint on $T^{\mu\nu}$ is $\nabla_\mu T^{\mu i} = 0$. However, it is allowed that $\nabla_\mu T^{\mu 0} \neq 0$, which has particular relevance for cosmology, since this affects the conservation of energy equation for a perfect isotropic fluids, so $\dot{\rho} + 3(\rho + p) \neq 0$.

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[4]We drop the $\text{Diff}_{\mathcal{F}}$ subscript for the rest of this section to avoid clutter.

[5]The change of coordinates $x^\mu \to x^\mu - \xi^\mu$ will produce a shift $\delta\phi \to \delta\phi - \xi^\mu \partial_\mu \phi$ in the Stückelberg field. This in general spoils the foliations, since they are defined by $\phi = const$. Hence we must undo this change. Thus, we require the transformation $\phi(x) \to \tilde{\phi}(\tilde{x}) = \phi(\tilde{x}) + \xi^\mu \partial_\mu \phi$.

3.3 Decoupling limit

Another common tool in analysing modified theories of gravity is taking a decoupling limit. One considers some limit in the theory such that the Stückelberg and general relativistic fields decouple, allowing one to study the dynamics of the new field independently of the known GR dynamics. It is reasonable to consider this limit, since for compatibility with standard GR we expect at large distances that gravity is mediated by a massless graviton propagating at the emergent speed of light c seen by all other particle species.

An example of its use is in DGP gravity [84]. In that case there is a strongly coupled scalar and the key physics can be seen by taking the decoupling limit in which the Planck length is sent to zero, but the scale of strong coupling remains fixed. This isolates the scalar mode from the standard gravitational piece [143, 144]. We will do the same for Hořava gravity shortly.

In Hořava gravity, we hope to recover GR in the limit $\lambda \to 1$, $\alpha \to 0$, however other studies [134] suggest that the Stückelberg field becomes strongly coupled in this limit. However, a question remains whether the truncation of higher derivatives leads one falsely to the conclusion that the theory is strongly coupled [135]. Taking a decoupling limit allows us to investigate, while keeping terms of all relevant orders.

We begin by assuming that the Lorentz symmetry breaking scale is roughly Planckian, which is reasonable on the grounds of naturalness to avoid introducing a new scale into the theory. So, without any further loss of generality, we will take $l_{UV} = l_{pl}$ for the remainder of this chapter. We now introduce two new length scales

$$l_\lambda = \frac{l_{pl}}{\sqrt{|1-\lambda|}}, \qquad l_\alpha = \frac{l_{pl}}{\sqrt{|\alpha|}}. \tag{3.19}$$

In the decoupling limit, $l_{pl} \to 0$, these scales will be held fixed. Of course, this requires that $\lambda \to 1$ and $\alpha \to 0$ which is consistent with their expected running in the infra-red, and will ensure that only S_{GR} contributes to the graviton dynamics. However, by holding the scales (3.19) fixed we enable ΔS_K and ΔS_2 to contribute to the Stückelberg dynamics, along with ΔS_4. The two dynamical sectors completely decouple as $l_{pl} \to 0$, with all the interesting phenomenology appearing in the Stückelberg sector.

We expand about Minkowski, in terms of a canonically normalised graviton, $h_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + l_{pl} h_{\mu\nu}, \tag{3.20}$$

and now raise and lower indices with the Minkowski metric. It then follows that

$$S_{GR} = -\int d^4x \; \frac{1}{2} h^{\mu\nu} \mathcal{E} h_{\mu\nu} + \mathcal{O}(l_{pl}) \tag{3.21a}$$

$$\Delta S_K = \frac{\text{sgn}(1-\lambda)}{l_\lambda^2} \int d^4x \; \tilde{\mathcal{K}}^2 + \mathcal{O}(l_{pl}) \tag{3.21b}$$

$$\Delta S_2 = \frac{\text{sgn}(\alpha)}{l_\alpha^2} \int d^4x \; \tilde{a}^\mu \tilde{a}_\mu + \mathcal{O}(l_{pl}) \tag{3.21c}$$

$$\Delta S_4 = \int d^4x \; \tilde{\mathcal{V}}_4 + \mathcal{O}(l_{pl}) \tag{3.21d}$$

$$\Delta S_6 = \mathcal{O}(l_{pl}^2) \tag{3.21e}$$

$$S_m = S_m[\eta_{\mu\nu}, \phi, \Psi] + l_{pl} \int d^4x \; \frac{1}{2} h_{\mu\nu} T^{\mu\nu} + \mathcal{O}(l_{pl}^2), \tag{3.21f}$$

where $\mathcal{E} h_{\mu\nu}$ is the Einstein tensor linearised about Minkowski, and

$$\tilde{\mathcal{K}}_{\mu\nu} = \left(\delta_\mu^\alpha + \frac{\partial^\alpha\phi\partial_\mu\phi}{\tilde{X}^2}\right)\left(\delta_\nu^\beta + \frac{\partial^\beta\phi\partial_\nu\phi}{\tilde{X}^2}\right)\frac{\partial_\alpha\partial_\beta\phi}{\tilde{X}}, \qquad \tilde{\mathcal{K}} = \eta^{\mu\nu}\tilde{\mathcal{K}}_{\mu\nu} \tag{3.22a}$$

$$\tilde{X} = \sqrt{-\partial_\mu\phi\partial^\mu\phi}, \tag{3.22b}$$

$$\tilde{a}_\mu = \left(\delta_\mu^\alpha + \frac{\partial^\alpha\phi\partial_\mu\phi}{\tilde{X}^2}\right)\frac{\partial^\beta\phi\partial_\alpha\partial_\beta\phi}{\tilde{X}^2} \tag{3.22c}$$

$$\begin{aligned}\tilde{\mathcal{V}}_4 &= A_1(\tilde{a}^\mu\tilde{a}_\mu)^2 + A_2(\tilde{a}^\mu\tilde{a}_\mu)\tilde{a}^\nu{}_\nu + A_3(\tilde{a}^\mu{}_\mu)^2 + A_4\tilde{a}^{\mu\nu}\tilde{a}_{\mu\nu} \\ &+ B_1(\tilde{\mathcal{K}}_{\mu\nu}\tilde{\mathcal{K}}^{\mu\nu} - \tilde{\mathcal{K}}^2)\tilde{a}^\mu\tilde{a}_\mu + B_2(\tilde{\mathcal{K}}_{\mu\alpha}\tilde{\mathcal{K}}_\nu^\alpha - \tilde{\mathcal{K}}\tilde{\mathcal{K}}_{\mu\nu})\tilde{a}^\mu\tilde{a}^\nu \\ &+ B_3(\tilde{\mathcal{K}}_{\mu\nu}\tilde{\mathcal{K}}^{\mu\nu} - \tilde{\mathcal{K}}^2)\tilde{a}^\mu{}_\mu + C_1(\tilde{\mathcal{K}}_{\mu\nu}\tilde{\mathcal{K}}^{\mu\nu} - \tilde{\mathcal{K}}^2)^2 \\ &+ C_2(\tilde{\mathcal{K}}_{\mu\alpha}\tilde{\mathcal{K}}_\nu^\alpha - \tilde{\mathcal{K}}\tilde{\mathcal{K}}_{\mu\nu})(\tilde{\mathcal{K}}_\beta^\mu\tilde{\mathcal{K}}^{\beta\nu} - \tilde{\mathcal{K}}\tilde{\mathcal{K}}^{\mu\nu})\end{aligned} \tag{3.22d}$$

$$\tilde{a}_{\mu\nu} = \left(\delta_\mu^\alpha + \frac{\partial^\alpha\phi\partial_\mu\phi}{\tilde{X}^2}\right)\left(\delta_\nu^\beta + \frac{\partial^\beta\phi\partial_\nu\phi}{\tilde{X}^2}\right)\partial_{(\alpha}\tilde{a}_{\beta)}. \tag{3.22e}$$

Variation of the matter action with respect to the Stückelberg field gives

$$\frac{\delta S_m}{\delta\phi} = -\frac{\tilde{u}_\nu\partial_\mu T^{\mu\nu}}{\tilde{X}} + \mathcal{O}(l_{pl}), \tag{3.23}$$

where $\tilde{u}_\mu = \partial_\mu\phi/\tilde{X}$. The violation of energy-momentum conservation has some characteristic scale Γ, given by

$$\Gamma = \frac{\sqrt{\nabla_\mu T^{\mu\nu}\nabla^\lambda T_{\lambda\nu}}}{T^{\alpha\beta}T_{\alpha\beta}}, \tag{3.24}$$

or schematically $\nabla T^{\mu\nu} \sim \Gamma T^{\mu\nu}$. We will typically take Γ to be much smaller than the overall scale of the energy-momentum tensor. Indeed, by taking $\Gamma \lesssim H_0$, where H_0 is the current Hubble scale it can be argued that violations of energy-momentum conservation would not have been detected during the universe's lifetime.

We are now able to take our decoupling limit by simultaneously taking the following limits,

$$l_{pl} \to 0, \qquad \alpha \to 0, \qquad \lambda \to 1, \qquad T^{\mu\nu} \to \infty, \qquad \Gamma \to 0, \tag{3.25}$$

whilst keeping l_α, l_λ, $l_{pl}T^{\mu\nu}$ and $\Gamma T^{\mu\nu}$ finite. We thus arrive at the decoupled Hořava action

$$S = S_{graviton} + S_{Stuckelberg} \tag{3.26}$$

where

$$S_{graviton} = -\int d^4x\, \frac{1}{2} h^{\mu\nu}\mathcal{E}h_{\mu\nu} + l_{pl}\int d^4x\, \frac{1}{2}h_{\mu\nu}T^{\mu\nu} \tag{3.27}$$

and

$$S_{Stuckelberg} = \int d^4x \left[\frac{\text{sgn}(1-\lambda)}{l_\lambda^2}\tilde{\mathcal{K}}^2 + \frac{\text{sgn}(\alpha)}{l_\alpha^2}\tilde{a}^\mu\tilde{a}_\mu + \tilde{\mathcal{V}}_4\right] + S_m[\eta_{\mu\nu}, \phi, \Psi], \tag{3.28}$$

where we use our $\widetilde{\text{tilde}}$ notation to make explicit we are working with the decoupled quantities. From this point onwards we take no interest in the graviton action (3.27), since the dynamics are clearly identical to GR.

We can derive the equations of motion for the Stückelberg field from the action (3.28),

$$\partial_\nu\left(\tilde{\gamma}^{\mu\nu}\frac{\rho_\mu}{\tilde{X}}\right) = \frac{\tilde{u}_\nu\partial_\mu T^{\mu\nu}}{\tilde{X}}, \tag{3.29}$$

where $\tilde{\gamma}^{\mu\nu} = \eta^{\mu\nu} + \tilde{u}^\mu\tilde{u}^\nu$ and

$$\begin{aligned}\rho^\mu = {} & \lambda^\nu\partial^\mu\tilde{u}_\nu - \partial_\nu\left(\tilde{u}^\nu\lambda^\mu\right) - \partial_\nu\left(\mu^{\mu\nu}\right) + \tilde{a}_\nu\mu^{\mu\nu} \\ & + 2\lambda^{\rho\sigma}u_{(\rho}\tilde{\gamma}_{\sigma)\nu}\partial^\mu\tilde{a}^\nu + 2\lambda^{\mu\sigma}\tilde{u}^{(\alpha}\tilde{\gamma}_\sigma^{\beta)}\partial_\alpha\tilde{a}_\beta,\end{aligned} \tag{3.30}$$

where for ease of reading the equation we leave ρ^μ written implicitly in terms of

$$\begin{aligned}\lambda^\mu = {} & \partial_\nu\left(\frac{\partial\tilde{\mathcal{V}}_4}{\partial\tilde{a}_{\rho\sigma}}\tilde{\gamma}^\nu_{(\rho}\tilde{\gamma}^\mu_{\sigma)}\right) - 2\frac{\text{sgn}(1-\lambda)}{l_\lambda^2}\tilde{\mathcal{K}}\tilde{u}^\mu \\ & - \frac{\partial\tilde{\mathcal{V}}_4}{\partial\tilde{\mathcal{K}}_{\mu\nu}}\tilde{u}_\nu - 2\frac{\text{sgn}(\alpha)}{l_\alpha^2}\tilde{a}^\mu - \frac{\partial\tilde{\mathcal{V}}_4}{\partial\tilde{a}_\mu},\end{aligned} \tag{3.31a}$$

$$\lambda^{\mu\nu} = -\frac{\partial\tilde{\mathcal{V}}_4}{\partial\tilde{a}_{\mu\nu}}, \tag{3.31b}$$

$$\mu^{\mu\nu} = -2\frac{\text{sgn}(1-\lambda)}{l_\lambda^2}\tilde{\mathcal{K}}\eta^{\mu\nu} - \frac{\partial\tilde{\mathcal{V}}_4}{\partial\tilde{\mathcal{K}}_{\mu\nu}}. \tag{3.31c}$$

The derivatives of the potential are given in Appendix A.1. The right-hand side of (3.29) goes like $\Gamma T^{\mu\nu}$ and, as such, remains finite. Note further that the Stückelberg equation of motion (3.29) is invariant under $\phi \to f(\phi)$, as required by foliation preserving diffeomorphisms. In principle one could use this equation to study the response of the Stückelberg field to the presence of a non-trivial source. In particular one can ask whether or not the Stückelberg field gets screened at short distances due to the Vainshtein effect, and we will return to this issue later.

3.4 Perturbations of the Stückelberg field

We now consider fluctuations of the Stückelberg field about the vacuum, $\phi = \bar{\phi} + \chi$, where $\bar{\phi} = x^0$. This choice of vacuum corresponds to choosing a constant time foliation of Minkowski space. To study the dynamics of vacuum fluctuations we simply expand the action (3.28) order by order in χ. For now, we neglect the contribution from the matter sector. The result is

$$S_\chi = \sum_{n=2}^{\infty} S_n[\chi] \tag{3.32}$$

where $S_n[\chi]$ is of order χ^n. On this background, $\bar{a}_\mu = \bar{a}_{\mu\nu} = \bar{\mathcal{K}}_{\mu\nu} = 0$. This drastically reduces the number of terms that appear in the expansion especially at low order. At quadratic and cubic order we find,

$$S_2[\chi] = \int \mathrm{d}^4x \left\{ \frac{\mathrm{sgn}(1-\lambda)}{l_\lambda^2}(\partial^2\chi)^2 + \frac{\mathrm{sgn}(\alpha)}{l_\alpha^2}(\partial_i\dot{\chi})^2 + (A_3 + A_4)(\partial^2\dot{\chi})^2 \right\}, \tag{3.33}$$

where dot is now ∂_0, the spatial Laplacian is $\partial^2 \equiv \partial_i\partial_i$, and

$$\begin{aligned} S_3[\chi] = \int \mathrm{d}^4x \Bigg\{ & 2\frac{\mathrm{sgn}(1-\lambda)}{l_\lambda^2}\left[-2(\partial_i\dot{\chi})(\partial_i\chi)(\partial^2\chi) - \dot{\chi}(\partial^2\chi)^2\right] \\ & + 2\frac{\mathrm{sgn}(\alpha)}{l_\alpha^2}\left[-\ddot{\chi}(\partial_i\chi)(\partial_i\dot{\chi}) - \dot{\chi}(\partial_i\dot{\chi})^2 - (\partial_i\dot{\chi})(\partial_j\chi)(\partial_i\partial_j\chi)\right] \\ & - A_2(\partial_i\dot{\chi})^2\partial^2\dot{\chi} \\ & - 2A_3\left[(\partial_i\ddot{\chi})(\partial_i\chi)(\partial^2\dot{\chi}) + \ddot{\chi}(\partial^2\chi)(\partial^2\dot{\chi}) + (\partial_i\dot{\chi})^2\partial^2\dot{\chi} + \dot{\chi}(\partial^2\dot{\chi})^2\right. \\ & \qquad \left. + (\partial^2\dot{\chi})(\partial_i\partial^2\chi)(\partial_i\chi) + (\partial_i\partial_j\chi)^2\partial^2\ddot{\chi}\right] \\ & - 2A_4\left[(\partial_i\ddot{\chi})(\partial_j\chi)(\partial_i\partial_j\dot{\chi}) + \ddot{\chi}(\partial_i\partial_j\chi)(\partial_i\partial_j\dot{\chi}) + (\partial_i\dot{\chi})(\partial_j\dot{\chi})(\partial_i\partial_j\dot{\chi})\right. \\ & \qquad \left. + \dot{\chi}(\partial_i\partial_j\dot{\chi})^2 + (\partial_i\partial_j\dot{\chi})(\partial_i\partial_j\partial_k\chi)\partial_k\chi + (\partial_i\partial_k\chi)(\partial_j\partial_k\chi)(\partial_i\partial_j\dot{\chi})\right] \\ & - B_3\left[(\partial_i\partial_j\chi)^2\partial^2\dot{\chi} - (\partial^2\chi)^2\partial^2\dot{\chi}\right] \Bigg\}. \end{aligned} \tag{3.34}$$

The quartic term also contains the following contribution

$$S_4[\chi] \supset \int \mathrm{d}^4x \frac{\mathrm{sgn}(\alpha)}{l_\alpha^2}\left[\partial_j\chi\partial_i\partial_j\chi\right]^2, \tag{3.35}$$

which will prove to be of importance later.

We are now in a position to determine the conditions on λ and α to ensure the theory is free from any number of pathologies — in particular ghosts (which violate unitarity), tachyons (which lead to instabilities), and superluminal mode propagation (which violates causality). In addition, we will check the scale of strong-coupling and compare it to the cut-off scale for the low energy effective theory.

3.4.1 Ghosts, tachyons and superluminal propagation

Let us begin by exorcising the ghost and throwing out the tachyon. To this end, it is convenient to rewrite the quadratic action (3.33) as follows

$$S_2[\chi] = \int \mathrm{d}^4x \left\{ \dot{\chi} \left(\frac{\mathrm{sgn}(\alpha)}{l_\alpha^2}\Delta^2 + (A_3 + A_4)\Delta^4 \right) \dot{\chi} + \frac{\mathrm{sgn}(1-\lambda)}{l_\lambda^2}\chi\Delta^4\chi \right\}, \quad (3.36)$$

where we have introduced the operator $\Delta = \sqrt{-\partial^2}$, which measures the magnitude of momentum. To avoid a ghost, we require the kinetic term in the action to be positive, and so

$$\frac{\mathrm{sgn}(\alpha)}{l_\alpha^2} + (A_3 + A_4)\Delta^2 > 0. \quad (3.37)$$

At low energies, this means that we require $\alpha > 0$ to avoid the ghost, whereas at high energies we require $A_3 + A_4 > 0$. It now follows that a tachyonic instability will kick in unless $\mathrm{sgn}(1-\lambda) < 0$, or in other words $\lambda > 1$. So, in summary to avoid both ghosts and tachyons we require

$$\alpha > 0 \qquad \lambda > 1.$$

This can be contrasted with the result obtained in [117], where the conditions were

$$0 < \alpha < 2 \qquad \lambda > 1 \text{ or } \lambda < 1/3.$$

The difference arises due to taking the decoupling limit, in which $\alpha \to 0$, $\lambda \to 1$, and so we only capture the behaviour near these values. Our results are clearly consistent.

Let us now consider the possibility of superluminal propagation[6]. To get a handle on the speed at which the Stückelberg mode propagates, consider the linearised equation of motion

$$\frac{\mathrm{sgn}(1-\lambda)}{l_\lambda^2}\partial^2(\partial^2\chi) + \frac{\mathrm{sgn}(\alpha)}{l_\alpha^2}(\partial^2\ddot{\chi}) - (A_3 + A_4)\partial^2\partial^2\ddot{\chi} = 0. \quad (3.38)$$

Which leads to the dispersion relation

$$\omega^2 = -\frac{\mathrm{sgn}(1-\lambda)}{\mathrm{sgn}(\alpha)}\left(\frac{l_\alpha}{l_\lambda}\right)^2 \frac{k^2}{1 + (A_3 + A_4)\mathrm{sgn}(\alpha) l_\alpha^2 k^2}. \quad (3.39)$$

At low energies $k < 1/l_\alpha$, the wave propagates with sound speed given by

$$c_s^2 = -\frac{\mathrm{sgn}(1-\lambda)}{\mathrm{sgn}(\alpha)}\left(\frac{l_\alpha}{l_\lambda}\right)^2 = \frac{\lambda - 1}{\alpha}. \quad (3.40)$$

[6]In a Lorentz violating theory, superluminal modes are not necessarily an issue. However, in the far infra-red, our effective theory is designed to be approximately Lorentz invariant, so it is desirable, if not essential, to prohibit superluminal propagation at low energies.

Note that the sound speed is real in the absence of ghosts and tachyons. In addition, to avoid superluminal propagation we also require $c_s \leq 1$, which gives

$$l_\alpha \leq l_\lambda \qquad \Longrightarrow \qquad |\lambda - 1| \leq |\alpha| \,. \tag{3.41}$$

Again, we can contrast (3.40) with the sound speed given in [117, 134],

$$c_s^2|_{\text{exact}} = \frac{2-\alpha}{\alpha}\left(\frac{\lambda-1}{3\lambda-1}\right). \tag{3.42}$$

This expression was derived by working directly with the original action (3.1), without any Stückelberg tricks. Once again we see that it is consistent with the expression (3.40) derived here in the decoupling limit $\lambda \to 1,\ \alpha \to 0$. This demonstrates both the power and limitations of the Stückelberg trick in the decoupling limit. It is clear that working directly with the Stückelberg action (3.28) reveals the key physics more easily than working with the full Hořava action (3.1). This truncation works perfectly well as long as we are happy to stay close to the limiting values of λ and α.

From now on we will assume that $\lambda > 1$, $\alpha > 0$ to avoid ghosts and tachyons, and that the low energy speed of sound is given by $c_s \equiv l_\alpha / l_\lambda \leq 1$.

3.4.2 Strong coupling

It was claimed by Blas *et al* that the so-called 'healthy' theory [117] might be free from strong-coupling [135], contrary to the claims made in [134]. Their argument roughly goes as follows: The analysis of [134] only includes the lowest order terms in the derivative expansion, corresponding to two-dimensional operators. Therefore, the effective theory is only valid at energies below some scale Λ_{hd}. If the strong coupling scale derived by [134] exceeds this cut off, then it cannot be taken seriously since higher-order operators should have been included in the analysis [135]. They give a toy example in which an erroneous strong coupling scale is derived in the effective theory, only to disappear when the higher-order operators are included.

Of course, for the case of Hořava gravity, one can force a suitably low cut-off in the derivative expansion by hand, by introducing a low energy scale, M_*, in the higher derivative terms at quadratic order. This scale corresponds to the new cut-off in the derivative expansion, and should lie below the would-be strong coupling scale. In some sense this is reminiscent of string theory in that the string scale is introduced just below the Planck scale where strong coupling would otherwise occur in gravity. According to [134], the low energy effective theory becomes strongly coupled at a scale $\frac{1}{l_\alpha} \sim \sqrt{\alpha} M_{pl}$, so this suggests we should take $M_* < \sqrt{\alpha} M_{pl}$. As experimental constraints require $\alpha \lesssim 10^{-7}$, this forces M_* to lie three to four orders of magnitude below the Planck scale [135]. If the new scale appears at order $2z$ (in spatial derivatives), we must introduce a dimensionless parameter of order $B \sim \alpha(\frac{M_{pl}}{M_*})^{2z-2} > \alpha^{2-z}$. The proposal to avoid strong coupling requires the scalar mode to enter a phase of anisotropic scaling, with dispersion relation $\omega^2 \propto k^6$, below the strong coupling scale of the low energy effective theory [135]. To get the right anisotropic scaling at the right scale, we therefore need to introduce a large term at $z = 3$, with dimensionless parameter $B > \frac{1}{\alpha} \gtrsim 10^7$.

This proposal corresponds to introducing a large dimensionless coefficient in the potential (specifically, the six derivative piece, $\mathcal{V}_6$). We did not consider this possibility when taking the decoupling limit, preferring instead to keep all dimensionless coefficients of order one, on grounds of naturalness. However, even if we had introduced some new scale $M_* \ll M_{pl}$ and taken the limit $l_{pl} \to 0$ whilst holding M_* fixed, the dispersion relation for the Stückelberg mode would not have coincided with the desired anisotropic scaling for the scalar mode in [135], $\omega^2 \sim k^6/M_*^4$ for large k. To recover this behaviour using the Stückelberg approach, we need to retain some coupling between the graviton and the Stückelberg mode, at least to quadratic order. While this is beyond the scope of this thesis, it does illustrate some of the limitations of the decoupling limit — the decoupling limit could never have captured this.

Nonetheless, it is still worth asking whether or not this brute force approach is absolutely necessary, and to what degree. Can the strong coupling scale be raised, or removed altogether, simply by including higher order interactions, but without introducing large coefficients? Whilst this might be possible in principle[7] let us demonstrate explicitly how it is not the case here, at least at the level of the decoupled Stückelberg theory. To this end, and as we have already emphasised, we will assume that all coefficients A_i, B_i and C_i are $\mathcal{O}(1)$, as are combinations of these coefficients. This enables us to make definite statements in what follows, but is also to be expected on grounds of naturalness. In contrast to [134], however,

[7]For example, consider the following toy model, in which we have a small kinetic term at low energies, as in Hořava gravity, with no additional large parameters introduced,

$$S = \int d^4x \, \left(\frac{1}{2}\epsilon^2 M_{pl}^2 \left[\dot{\psi}^2 - \psi \Delta^2 \psi \right] + \psi \Delta^4 \psi + \frac{1}{M_{pl}^2} (\Delta^2 \psi)^3 + \frac{1}{M_{pl}^2} \psi (\Delta^2 \psi)^3 \right), \qquad \epsilon \ll 1.$$

In the low energy theory, the relativistic kinetic term dominates, and the dominant interaction becomes strongly coupled at the scale $\Lambda_{\text{false}} \sim \epsilon^{2/3} M_{pl}$, which is above the low energy cut-off at ϵM_{pl}. By studying the theory at higher energies $\Delta > \epsilon M_{pl}$ (and canonically normalising appropriately), it can be shown that the dominant interaction actually becomes strongly coupled at the higher scale $\Lambda_{\text{true}} \sim \epsilon^{1/3} M_{pl} > \Lambda_{\text{false}}$.

we will not necessarily restrict attention to the low energy effective field theory.

Now, the first thing to do is to perform a derivative expansion at quadratic order in order to establish both the cut-off and the leading order terms in the effective theory. We find that

$$S_2[\chi] = \int \mathrm{d}^4x \left\{ \frac{1}{l_\alpha^2} \dot{\chi}\Delta^2\dot{\chi} - \frac{1}{l_\lambda^2}\chi\Delta^4\chi \right\} \left(1 + \mathcal{O}(\Delta^2 l_\alpha^2)\right). \tag{3.43}$$

Clearly then the cut off for the effective theory is given by

$$\Lambda_{hd} \sim \frac{1}{l_\alpha}. \tag{3.44}$$

Cubic and quartic order interaction terms are given by (3.34) and (3.35). Generically, there are three types of terms appearing at order n, corresponding to each of the first three terms in the Stückelberg action (3.28). Using dimensional analysis one can easily show that these terms are schematically given by

$$S^\alpha_{(n,a)}[\chi] = \int \mathrm{d}^4x \; \frac{1}{l_\alpha^2} (\partial_0)^a \Delta^{n+2-a} \chi^n \tag{3.45a}$$

$$S^\lambda_{(n,a)}[\chi] = \int \mathrm{d}^4x \; \frac{1}{l_\lambda^2} (\partial_0)^a \Delta^{n+2-a} \chi^n \tag{3.45b}$$

$$S^V_{(n,a)}[\chi] = \int \mathrm{d}^4x \; (\partial_0)^a \Delta^{n+4-a} \chi^n, \tag{3.45c}$$

where a controls the number of time derivatives and n, a are positive integers. To estimate the scale at which these terms become strongly coupled (if at all) we first need to canonically normalise the quadratic part of the action. To this end we set

$$\hat{x}^0 = c_s x^0, \qquad \hat{x}_i = x_i, \qquad \hat{\chi} = \frac{\sqrt{c_s}}{l_\alpha}\chi$$

so that the quadratic part of the action becomes

$$S_2[\hat{\chi}] = \int \mathrm{d}^4\hat{x} \; \left(\partial_{\hat{0}}\hat{\chi}\Delta^2\partial_{\hat{0}}\hat{\chi} - \hat{\chi}\Delta^4\hat{\chi}\right) \tag{3.46}$$

It follows that the interaction terms now go like

$$S^\alpha_{(n,a)}[\hat{\chi}] = \int \mathrm{d}^4x \; \left(\frac{1}{\Lambda^\alpha_{(a,n)}}\right)^{n-2} (\partial_{\hat{0}})^a \Delta^{n+2-a} \hat{\chi}^n, \tag{3.47a}$$

$$S^\lambda_{(n,a)}[\hat{\chi}] = \int \mathrm{d}^4x \; \left(\frac{1}{\Lambda^\lambda_{(a,n)}}\right)^{n-2} (\partial_{\hat{0}})^a \Delta^{n+2-a} \hat{\chi}^n, \tag{3.47b}$$

$$S^V_{(n,a)}[\hat{\chi}] = \int \mathrm{d}^4x \; \left(\frac{1}{\Lambda^V_{(a,n)}}\right)^{n} (\partial_{\hat{0}})^a \Delta^{n+4-a} \hat{\chi}^n, \tag{3.47c}$$

where

$$\Lambda^\alpha_{(a,n)} = \frac{1}{l_\alpha} c_s^{\frac{1}{2} + \frac{2-a}{n-2}}, \qquad \Lambda^\lambda_{(a,n)} = \frac{1}{l_\alpha} c_s^{\frac{1}{2} - \frac{a}{n-2}}, \qquad \Lambda^V_{(a,n)} = \frac{1}{l_\alpha} c_s^{\frac{1}{2} + \frac{1-a}{n}}. \tag{3.48}$$

These scales are the scales at which the corresponding terms become strongly coupled. However, it is important to note that not all of these terms actually appear in the action, as one can immediately see by looking at the cubic term (3.34). Given that $c_s \leq 1$, it turns out (see Appendix A.2) that the lowest of these scales to appear in the action is given by

$$\Lambda^{\alpha}_{(1,3)} = \Lambda^{\alpha}_{(0,4)} = \frac{1}{l_\alpha} c_s^{3/2}. \tag{3.49}$$

This is the scale at which the largest interaction terms become significant and one enters a strongly coupled régime. The strongly coupled terms correspond to the cubic interaction,

$$-2 \int \mathrm{d}^4 x \; \frac{\mathrm{sgn}(\alpha)}{l_\alpha^2} (\partial_i \dot{\chi})(\partial_j \chi)(\partial_i \partial_j \chi) \tag{3.50}$$

and the quartic interaction given by (3.35). We therefore identify the lowest energy strong coupling scale as being

$$\Lambda_{sc} \sim \frac{1}{l_\alpha} c_s^{3/2}. \tag{3.51}$$

Now, since $c_s \leq 1$, it follows that $\Lambda_{sc} \lesssim \Lambda_{hd} \sim 1/l_\alpha$, which means that the derived strong coupling scale does indeed lie below the cut off of the derivative expansion. Of course, one may call into question this conclusion if $c_s \sim 1$, as then we have $\Lambda_{sc} \sim \Lambda_{hd}$.

To allay any possible concerns let us consider what happens at high energies $\Delta \gg 1/l_\alpha$. Then the quadratic part of the action is given by

$$S_2[\chi] = \int \mathrm{d}^4 x \left\{ (A_3 + A_4)\, \dot{\chi} \Delta^4 \dot{\chi} - \frac{1}{l_\lambda^2} \chi \Delta^4 \chi \right\} \left(1 + \mathcal{O}(1/\Delta^2 l_\alpha^2)\right). \tag{3.52}$$

As expected x^0 and x^i scale differently in the UV,

$$x^i \to b^{-1} x^i, \qquad x^0 \to x^0.$$

The fact that x^0 does not scale makes sense given that the dispersion relation (3.39) goes like $\omega \sim$ *constant* for large k. Indeed, to quadratic order the system reduces to a simple harmonic oscillator with fixed frequency of oscillation.

In order to keep $S_2[\chi]$ invariant under the scaling, we must have

$$\chi \to b^{-1/2} \chi.$$

It follows that the interaction terms scale like

$$S^{\alpha}_{(n,a)}[\chi] \to b^{\frac{n}{2} - a - 1} S^{\alpha}_{(n,a)}[\chi] \tag{3.53a}$$
$$S^{\lambda}_{(n,a)}[\chi] \to b^{\frac{n}{2} - a - 1} S^{\lambda}_{(n,a)}[\chi] \tag{3.53b}$$
$$S^{V}_{(n,a)}[\chi] \to b^{\frac{n}{2} - a + 1} S^{V}_{(n,a)}[\chi]. \tag{3.53c}$$

These interactions become relevant in the UV whenever the exponent of b is positive in the above scaling. Interactions with many time derivatives (that is, with large a), are irrelevant in the UV, whereas those with fewer time derivatives become relevant. Indeed, the fourth-order interaction term given by (3.35) is clearly relevant, as it scales like b^3. Therefore there is no reason to expect that the theory is UV finite, even for $c_s \sim 1$. To make sense of the perturbative theory we need to introduce a cut-off given by the strong coupling scale. For $c_s \sim 1$, the only scale we have available is $1/l_\alpha \sim 1/l_\lambda$, so it follows that this corresponds to the scale at which (3.35) becomes large.

In conclusion then, unless we introduce some new scales by brute force, the 'healthy' theory is unlikely to be UV finite since it becomes strongly coupled at a scale

$$\Lambda_{sc} \sim \frac{1}{l_\alpha} c_s^{3/2} = \left(\frac{l_\alpha}{l_\lambda^3}\right)^{1/2} = \frac{1}{l_{pl}} \left[\frac{(\lambda-1)^3}{\alpha}\right]^{1/4}. \tag{3.54}$$

This result agrees with [134], at least when $c_s \sim 1$, but is more robust, having considered the effect of higher-derivative corrections and allowing for $c_s \ll 1$. The correct interpretation of this result is to realise that we must introduce new physics by hand, below the scale Λ_{sc}. We can do this by explicitly introducing a new low scale of Lorentz violation, M_*, in the higher derivative terms, as proposed in [135]. Experimental considerations actually push Λ_{sc}, and by association M_*, to well below the Planck scale.

3.5 Matter sources and the Stückelberg force

It has been suggested that strong coupling problems in some versions of Hořava gravity might be a blessing in disguise, at least from a phenomenological perspective. The claim is that a Vainshtein mechanism might occur, such that non-linear interactions become important, helping to screen any additional force due to the Stückelberg mode. In this section we will derive the size of the Stückelberg force, and compare it to the size of the usual force mediated by the graviton.

Of course, we need a suitable source. To excite the Stückelberg mode, this must violate the usual energy-momentum conservation law, as is clear from (3.29). A simple choice is a time dependent point mass, with energy-momentum tensor

$$T^{\mu\nu} = M(x^0)\delta^{(3)}(\mathbf{x})u^\mu u^\nu. \tag{3.55}$$

Recall that the violation of energy-momentum conservation is characterised by some scale Γ, which we will take to be much less than overall scale of the energy-momentum tensor. In other words

$$\Gamma \sim \frac{|\partial_0 M|}{M} \ll M.$$

We will consider the following simple cases: a slowing decaying point mass,

$$M(x^0) = M_* \exp(-\Gamma x^0), \tag{3.56}$$

or a slowly oscillating point mass,

$$M(x^0) = M_*(1 - \sin(\Gamma x^0)). \tag{3.57}$$

In the decoupling limit given by (3.25), we see that we must take $M_* \to \infty$, $\Gamma \to 0$ and $l_{pl} \to 0$, holding $l_{pl} M_*$ and ΓM_* fixed. At this stage we could, in principle, solve the Stückelberg equation of motion (3.29) to leading order, but we can do better than that. We can take advantage of the strong coupling previously discussed to simplify the full non-linear analysis.

Recall that fluctuations on the trivial vacuum become strongly coupled at a scale Λ_{sc}, given by (3.54). All the features of this strongly coupled theory can be captured by taking the limit $l_\alpha \to 0$, $l_\lambda \to 0$, whilst holding Λ_{sc} fixed. This just means that $l_{pl} \to 0$ faster than $\lambda \to 1$ and $\alpha \to 0$, so in a sense it corresponds to the case where deviations from General Relativity play a maximal role[8]. Note that this implies that the speed of sound $c_s \to 0$. As regards the scaling of matter in this limit, we will assume that both $l_{pl} T^{\mu\nu}/\sqrt{c_s}$ and $\Gamma T^{\mu\nu}$ remain finite so that both the graviton and the Stückelberg sector get non-vanishing source terms, as we will show presently.

In this limit all but the largest interaction terms discussed in the previous section go away, and the full theory is reduced to

$$S_{\hat{\chi}} = \int d^4\hat{x} \left[\partial_{\hat{0}}\hat{\chi}\Delta^2\partial_{\hat{0}}\hat{\chi} - \hat{\chi}\Delta^4\hat{\chi} - \frac{2}{\Lambda_{sc}}\partial_{\hat{0}}\partial_i\hat{\chi}\partial_i\partial_j\hat{\chi}\partial_j\hat{\chi} + \frac{1}{\Lambda_{sc}^2}\left[\partial_j\hat{\chi}\partial_i\partial_j\hat{\chi}\right]^2\right] - \int d^4\hat{x}\, \frac{\hat{\chi}}{\Lambda_{sc}}\partial_\mu T^{\mu 0} \tag{3.58}$$

where we have included the matter coupling, which is indeed finite. This is the exact Stückelberg theory in this limit. Note that this action possesses a symmetry $\hat{\chi} \to \hat{\chi} + f(x^0)$, which is an artifact of foliation preserving diffeomorphisms. The corresponding graviton theory goes like

$$S_{graviton} = -\int d^4\hat{x}\, \frac{1}{2}\hat{h}^{\mu\nu}\mathcal{E}\hat{h}_{\mu\nu} + \frac{l_{pl}}{\sqrt{c_s}}\int d^4\hat{x}\, \frac{1}{2}\hat{h}_{\mu\nu}T^{\mu\nu}, \tag{3.59}$$

[8] In the opposite limit, $l_\lambda \to \infty$, $l_\alpha \to \infty$, there is no deviation from GR whatsoever at low energies, since we might as well just set $\lambda = 1$, $\alpha = 0$ from the outset.

where $\hat{h}_{\mu\nu} = \frac{1}{\sqrt{c_s}} h_{\mu\nu}$. Again, the matter coupling is held finite.

The Stückelberg equations of motion are now given by

$$\frac{1}{\Lambda_{sc}}\partial_\mu T^{\mu 0} = 2\partial^2\partial_{\hat{0}}^2\hat{\chi} - 2\partial^4\hat{\chi} + \frac{1}{\Lambda_{sc}^2}\partial_i\left[(\partial_i\hat{\chi})\partial^2(\partial_j\hat{\chi}\partial_j\hat{\chi})\right] - \frac{2}{\Lambda_{sc}}\partial_i\left[\partial_i\partial_j\partial_{\hat{0}}\hat{\chi}\partial_j\hat{\chi} + \partial_i\partial_j\hat{\chi}\partial_{\hat{0}}\partial_j\hat{\chi} + \partial_{\hat{0}}\partial^2\hat{\chi}\partial_i\hat{\chi}\right] \quad (3.60)$$

whereas the graviton equations of motion are given by

$$\mathcal{E}\hat{h}^{\mu\nu} = \frac{l_{pl}}{2\sqrt{c_s}}T^{\mu\nu}. \quad (3.61)$$

For our slowly varying point sources, we have

$$T^{\mu\nu} \to M_*\delta^{(3)}(\mathbf{x})u^\mu u^\nu, \qquad \partial_\mu T^{\mu 0} \to -\Gamma M_*\delta^{(3)}(\mathbf{x}), \quad (3.62)$$

with ΓM_* and $l_{pl}M_*/\sqrt{c_s}$ held fixed. We shall seek static spherically symmetric solutions to (3.60) of the form $\hat{\chi} = \hat{\chi}(r)$. After being careful to express div, grad and the Laplacian in spherical coordinates, and integrating over a sphere of radius r centred on the origin, we find that

$$\frac{d}{dr}\left(\frac{1}{r^2}\frac{d}{dr}r^2 u\right) - \frac{1}{2\Lambda_{sc}^2}\frac{u}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}u^2\right) = \frac{1}{\Lambda_{sc}}\left(\frac{\Gamma M_*}{8\pi r^2}\right), \quad (3.63)$$

where $u = \hat{\chi}'(r)$. We can solve this equation as a power series in $1/\Lambda_{sc}$. To all orders in the expansion, the unique solution is given by

$$u = -\frac{1}{16\pi}\frac{\Gamma M_*}{\Lambda_{sc}} \qquad \Longrightarrow \qquad \hat{\chi}(r) = c - \frac{1}{16\pi}\frac{\Gamma M_*}{\Lambda_{sc}}r, \quad (3.64)$$

where c is some arbitrary integration constant. Owing to the foliation preserving diffeomorphisms, the Stückelberg mode possesses a shift symmetry $\hat{\chi} \to \hat{\chi} + const.$ We use this symmetry to set $c = 0$, so that the final solution is given by

$$\hat{\chi}(r) = -\frac{1}{16\pi}\frac{\Gamma M_*}{\Lambda_{sc}}r. \quad (3.65)$$

Now consider the graviton (3.61). The solution to this equation is well known, and most conveniently expressed in Newtonian gauge,

$$\hat{h}_{00} = \left(\frac{l_{pl}M_*}{\sqrt{c_s}}\right)\frac{1}{8\pi r}, \qquad \hat{h}_{ij} = \left(\frac{l_{pl}M_*}{\sqrt{c_s}}\right)\frac{1}{8\pi r}\delta_{ij}. \quad (3.66)$$

Now suppose we probe the field generated by the source using a second point mass, with energy-momentum tensor satisfying

$$\tilde{T}^{\mu\nu} \to \tilde{M}_*\delta^{(3)}(\mathbf{x}-\mathbf{y})u^\mu u^\nu, \qquad \partial_\mu\tilde{T}^{\mu 0} \to -\tilde{\Gamma}\tilde{M}_*\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad (3.67)$$

with $\tilde{\Gamma}\tilde{M}_*$ and $l_{pl}\tilde{M}_*/\sqrt{c_s}$ held fixed in the relevant limits. The potential energy of the probe due to the Stückelberg interaction is given by

$$V_{\hat{\chi}} = \int \mathrm{d}^3x \, \frac{\hat{\chi}}{\Lambda_{sc}} \partial_\mu \tilde{T}^{\mu 0} = \frac{1}{16\pi} \frac{1}{\Lambda_{sc}^2} \Gamma M_* \tilde{\Gamma} \tilde{M}_* r. \tag{3.68}$$

It follows that the Stückelberg field mediates a constant attractive force

$$\mathbf{F}_{\hat{\chi}} = -\nabla V_{\hat{\chi}} = -\frac{1}{16\pi} \frac{1}{\Lambda_{sc}^2} (\Gamma M_*)(\tilde{\Gamma}\tilde{M}_*) \underline{\hat{r}}, \tag{3.69}$$

between the two point masses. Thus we have confinement, and in particular, a bound universe. There is no fall off with distance, and no way for the masses to escape the mutual Stückelberg force on each other.

In contrast, the potential energy of the probe due to the graviton interaction is just the Newtonian potential

$$V_{\hat{h}} = -\frac{l_{pl}}{\sqrt{c_s}} \int d^3x \, \frac{1}{2} \hat{h}_{\mu\nu} \tilde{T}^{\mu\nu} = -\frac{l_{pl}^2}{16\pi} \left(\frac{M_* \tilde{M}_*}{c_s} \right) \frac{1}{r}, \tag{3.70}$$

with the usual attractive force satisfying an inverse square law,

$$\mathbf{F}_{\hat{h}} = -\nabla V_{\hat{h}} = -\frac{l_{pl}^2}{16\pi} \left(\frac{M_* \tilde{M}_*}{c_s} \right) \frac{\underline{\hat{r}}}{r^2}. \tag{3.71}$$

Note that there is an additional factor of c_s compared to the standard formula owing to the fact that we are using $\hat{x}^0 = c_s x^0$ as our time coordinate. Now let us compare the two forces, given by equations (3.69) and (3.71). They both mediate an attractive force although they scale differently with distance. At large distances the Stückelberg force dominates, while at smaller distances the graviton force dominates. The two forces are equal at a distance of

$$r_{eq} = \frac{\Lambda_{sc} l_{pl}}{\sqrt{c_s \Gamma \tilde{\Gamma}}} = \frac{\lambda - 1}{\sqrt{\Gamma \tilde{\Gamma}}}. \tag{3.72}$$

For $r \ll r_{eq}$ the Stückelberg force is irrelevant, and one should expect to recover Newtonian gravity for a two particle system. However, we would like to stress that this is not really a Vainshtein effect since it is not the case that non-linear interactions screen the Stückelberg force above a certain scale. In fact, non-linear interactions never become important in this particular example. It is simply the case that the graviton force grows at short distances whereas the Stückelberg force remains constant.

Nonetheless, we expect that provided we have a large enough crossover scale for objects within the solar system, Newton's law always should always hold at this scale. Indeed, if we assume that $\Gamma \sim \tilde{\Gamma} \sim H_0$, then the crossover scale $r_{eq} \sim$

$(\lambda-1)/H_0$. For $\lambda-1\sim 10^{-10}$, we expect to recover Newton's law within the Oort cloud, at distances $r \ll 10^{16}$ m. One might worry that that there are, in principle, many many far away sources for the Stückelberg field that will exert a constant long range force on the objects within the solar system. However, as long as we assume homogeneity and isotropy at large scales, the effect of far away sources should cancel one another out.

In summary then, for point sources, with slowly varying mass, there is a scale at which the graviton force becomes dominant and one is able to recover Newtonian gravity. However, this is *not* a Vainshtein effect as non-linear interactions never play much of a role. Does this mean that strong coupling is not important? Clearly this is unlikely to be the case in less symmetric configurations. It would be interesting to consider alternative sources, in particular a binary system that violates energy conservation. As was pointed out in [106], binary systems are particularly relevant as they represent a direct test of perturbative GR.

Concluding on a much more troubling note, it is clear from (3.72) that the crossover scale generically depends on the variation rate of the probe, $\tilde{\Gamma}$, as well as the rate of the source, Γ. This illustrates the fact that the Stückelberg force violates the (Weak) Equivalence Principle. Indeed, probe masses with different Γ's will experience different accelerations in the presence of a Stückelberg field generated by a point source. Such violations will be of the order $\eta \sim (\Gamma_1-\Gamma_2)/(\Gamma_1+\Gamma_2)$ for different probes with violation rates $\Gamma_1,\ \Gamma_2$. The violation will kick in at large distances, beyond the lesser of the two crossover scales. We expect this to be a generic feature for objects that source the Stückelberg field. Violation of the Equivalence Principle can potentially be used to place phenomenological constraints on Hořava gravity, a fact that had not been noticed previously (and indeed some thought the EP would not be violated [125]).

3.6 Discussion

By taking an appropriate decoupling limit we have obtained new insights into Hořava gravity and its suitability as a quantum gravity candidate. The analysis here focuses on the 'healthy extension' of the theory [117], and specifically the case where Lorentz violation occurs at the Planck scale. We have been particularly interested in the behaviour of the Stückelberg field, the troublesome extra degree of freedom arising from breaking full diffeomorphism invariance. Taking the limit where this decouples from the gravity sector simplifies the calculations significantly. Indeed, both the validity and simplicity of this approach were clearly demonstrated in Section 3.4.1, where we recovered some known results [117, 134] with consummate ease. In particular, we reproduced the strong coupling result of [134] in an elegant manner, and in some respects our analysis is more complete since we do not restrict attention to the low energy effective theory. The correct way to interpret our result is to realise that the only way to avoid strong coupling is to explicitly introduce a new scale in the theory, below the would-be strong coupling scale, giving rise to a much lower scale of Lorentz violation.

Indeed, it has already been proposed that strong coupling can be avoided if one accepts this slightly ad hoc introduction of a new scale in the theory [135]. Since this creates a hierarchy between the Lorentz breaking scale and the Planck scale, we might wish to avoid this on grounds of naturalness[9]. Nevertheless it would be interesting to consider some possible implications of this scenario, as will shortly be discussed. Returning to the case of Planckian Lorentz violation, we find that the strong coupling scale is smaller than the scale of the gradient expansion, confirming the validity of the analysis in [134]. The largest interactions become strongly coupled at the scale, $\Lambda_{sc} \sim \frac{1}{l_\alpha} c_s^{3/2}$, which is less than the scale $1/l_\alpha$ corresponding to the cut-off in the derivative expansion. Of course, this argument relies on the fact that the Stückelberg field does not propagate faster than light ($c_s \leq 1$). However, even if we allow superluminal propagation, which is not unreasonable in a Lorentz violating theory, strong coupling is still a problem. To see this note that the cubic term $\sim \frac{1}{l_\alpha^2} \dot{\chi}(\partial_i \dot{\chi})^2$ becomes strongly coupled at scale $\frac{1}{l_\alpha} c_s^{-1/2}$, which is below the higher derivative cut-off for $c_s > 1$.

Why is strong coupling so bad? In principle, it is not. It depends on the context. QED becomes strongly coupled in the UV due to the presence of a Landau pole, which merely indicates that we should not be considering QED in isolation, while QCD becomes strongly coupled in the IR, but that just tells us we are using the wrong perturbative degrees of freedom. In contrast, we have never known for sure if Hořava gravity was a renormalisable theory. It was only ever suggested by a dubious power counting argument, in which one wrongly infers a schematic form for the action in terms of the perturbative degrees of freedom. The problem is that the Stückelberg mode is essentially ignored. Indeed, if we assume that to leading order one can schematically replace the curvatures with derivatives of the graviton and that there is nothing else to worry about, one might expect the action to resemble those discussed in [90, 91], which *are* power counting renormalisable. But, of course, we should not ignore the Stückelberg mode. Typically, the Stückelberg theory behaves nothing like the renormalisable actions described in [90, 91]. We therefore have little reason to expect renormalisability and little reason to tout Hořava gravity as a UV complete theory of gravity. To get the Stückelberg theory

[9] Note that we are not referring to technical naturalness in the sense of the stability of $M_\star$ against radiative corrections (see footnote 9 of [135] for a fuller discussion of this).

to behave more appropriately in the UV, we need to introduce a new scale of Lorentz violation by hand, and take it to be well below the Planck scale, as suggested by [135].

Of course, even if it is not UV complete (losing much of the original motivation), one could ask if Hořava gravity is a phenomenologically viable modification of GR. What is the significance of strong coupling in this context? The strong coupling scale is the scale at which quantum fluctuations on the vacuum start to interact strongly. In the presence of a perturbative source, this scale can be linked to the scale at which classical linearised perturbation theory breaks down[10]. The original Hořava theory has been argued to not be viable [106] since it was strongly coupled on all scales. This was a problem because it meant that there was no scale at which one could apply the standard linearised theory around a heavy source. Linearised General Relativity around a heavy source has been well tested, at least indirectly, thanks to the Nobel Prize winning binary pulsar observations of Hulse and Taylor [27]. In the extended version of Hořava gravity with Lorentz violation at Planckian scales, strong coupling kicks in at a finite scale $\Lambda_{sc} \sim \frac{1}{l_\alpha} c_s^{3/2} = \frac{1}{l_{pl}} \left(\frac{(\lambda-1)^3}{\alpha}\right)^{1/4}$. This suggests that linearised theory around a heavy source is valid up to some finite scale, although one clearly ought to check that the Stückelberg field does not spoil GR's successful matching to Hulse and Taylor's observations (see [146, 147] for corresponding studies in Brans-Dicke gravity).

However, these arguments are not quite enough to rule out strongly coupled versions of Hořava gravity on phenomenological grounds, since they ignore any possible Vainshtein effect [74]. The Vainshtein effect typically occurs in modifications of GR that exhibit strong coupling. Even if one has too many degrees of freedom at the linear level to mimic General Relativity, non-linear interactions can save the day. Because of strong coupling, bound states form, allowing extra degrees of freedom to be screened and enabling one to recover GR at short enough distances. To study any possible Vainshtein mechanism in Hořava gravity, it is important to understand how the Stückelberg mode couples to matter. We have been able to determine this coupling by making use of the reduced diffeomorphisms, as described in Section 3.2.1. It turns out that one should include some violation of energy-momentum conservation, measured by some scale Γ, to source the Stückelberg field. Note that this is not as crazy as it might sound! — energy-momentum conservation is not required because we do not have full diffeomorphism invariance. From a phenomenological perspective, we can assume that $\Gamma \lesssim H_0$, so that

[10] In DGP gravity, the strong coupling scale is given by $(M_{pl}H_0^2)^{1/3}$ [143, 144]. This scale can be linked to the scale at which classical perturbation theory breaks down around a heavy source. Indeed, for a source of mass M, linearised perturbation theory breaks down at a scale $(MH_0^2)^{(1/3)}$[145].

violations only become apparent on superhorizon scales.

In Section 3.5 we studied the interaction between two point particles, with slowly varying masses. This is probably the simplest way to source the Stückelberg field, in order to see if there is indeed any sort of Vainshtein mechanism at work. Using our results from strong coupling, and taking the limit $c_s \to 0$ while keeping Λ_{sc} finite, we were able to write down the exact action for this system. It turns out that the Stückelberg field gives rise to a constant attractive force between the two particles. This field dominates over the graviton force at large distances and gives rise to confinement. At short distances the graviton force dominates and one can recover Newtonian gravity. However this is not a Vainshtein effect since non-linearities do not play any role in screening the Stückelberg force. In fact, non-linearities play no role at all in this example, although we do not expect this to be true in general. In fact, it is probably just an artefact of our taking the (almost) static limit. It would be interesting to consider alternative sources for the Stückelberg field, most notably a binary system that weakly violates energy-momentum conservation.

In the absence of a Vainshtein mechanism in our example, perhaps the most important result of Section 3.5 is the realisation that Hořava gravity will inevitably lead to violations of the Equivalence Principle. This is because the Stückelberg force depends on the rate of energy-momentum conservation violation Γ of each particle, as well as their masses. Different probes with different Γ's will therefore feel different accelerations. This gives a non-trivial Eötvös parameter, for which we have very tight experimental bounds [30, 31]. Of course, one might hope to evade this issue by imposing, without adequate motivation, that only conserved sources are allowed in this theory. Whilst this can be achieved through a specific choice of matter coupling in the classical Lagrangian, it is clear that loop effects will introduce small corrections, suppressed by the scale of Lorentz symmetry breaking. Even though the resulting "scale of non-conservation", Γ, is small for a generic source, it will crucially be non-zero. This can lead to large effects since the violations of Equivalence Principle will be of the order $\eta \sim (\Gamma_1 - \Gamma_2)/(\Gamma_1 + \Gamma_2)$ for probes that violate energy-momentum conservation at different rates $\Gamma_1 \neq \Gamma_2$. Of course, the effect only kicks in beyond the lesser of the two crossover scales (3.72) for each probe, so this phenomena could be used to place a lower bound on the value of λ. This merits further investigation.

Although some of our results hold only in the "healthy extension" of the Hořava gravity [117] *with Planckian Lorentz violation*, it is clear that all the analysis could be repeated fairly simply for the original non-projectable theory, with roughly similar conclusions. One might even consider extending our method to the projectable theory, by adding a term $S_{gf} = \int dtd^3x\sqrt{\gamma}NQ_ia^i$ into the original action. Q_i is a

Lagrange multiplier, whose equations of motion enforce $a_i = 0 \Rightarrow \partial_i \log N = 0 \Rightarrow N = N(t)$.

Let us now turn to the issue of scales. Note that we have avoided the introduction of additional scales into the theory on ground of naturalness. This manifests itself through the absence of large dimensionless coefficients in the action, and a single Lorentz violating scale l_{UV}, taken to be Planckian. In [135], it is claimed that the strong coupling problems discussed here can be avoided by introducing a dimensionless coefficient $B \gtrsim 10^7$. This means that Lorentz invariance is broken at much lower scale $M_* \ll M_{pl}$ than one might expect. We could modify our decoupling limit by holding M_* fixed as we take $l_{pl} \to 0$, but this would still not give the desired dispersion relation for the Stückelberg mode in the UV ($\omega^2 \propto k^6$) required to 'cure' strong coupling. The conclusion then is that this effect cannot be reproduced in the decoupling limit – one needs to retain some coupling between the graviton and the Stückelberg mode, even if it is just to quadratic order. What we can do is ask what impact this low scale of Lorentz violation has on tests of the Equivalence Principle. As we have discussed, even if one assumes conserved sources classically, quantum mechanically we will get violations of energy-momentum conservation, suppressed by some power of the Lorentz symmetry breaking scale. The lower the scale, the less the suppression, and the larger the generic value of Γ. This would raise the lower bound on λ derived from the crossover scale. It would be interesting to see if this can be made compatible with tests of Lorentz violation which place an upper bound on the value of λ (see for example, [135]).

In conclusion, making use of the powerful tools of the decoupling limit has allowed us to demonstrate that one cannot avoid strong coupling problems in the 'healthy' extension of Hořava gravity *if one assumes Lorentz violation at the Planck scale.* With additional pressure coming from experimental observation, one is forced to introduce a much lower scale of Lorentz violation, along the lines proposed by [135]. Perhaps surprisingly, the details of avoiding strong coupling in that scenario cannot be captured by the decoupling limit. At the level of phenomenology, we have studied the force between two point particles with slowly varying masses. We have found no Vainshtein effect, but we have seen violations of the Equivalence Principle. We believe the latter is a generic feature, but not the former. It is possible that tests of Equivalence Principle will present a challenge to the low scale of Lorentz violation designed to cure strong coupling [135], although a more detailed study is clearly required.

Chapter 4

Matter in Hořava gravity

Any realistic theory of gravity must also include matter fields, and so we now extend our discussion of Hořava gravity to include matter fields, making both classical and quantum considerations. Classically, we construct matter theories consistent with the $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ symmetry. Quantum mechanically, we calculate one-loop corrections to the matter propagators. This chapter is based on the paper [3].

4.1 Introduction

Hořava gravity is conjectured to be a quantum theory of gravity, designed to retain the good features of GR while also being power counting renormalisable. However, much of the discussion has only concerned its properties in vacuum. By contrast, our concern in this chapter will not be the pure gravity sector, but the coupling of Hořava gravity to matter. Gravity theories coupled with matter tend to have worse quantum behaviour than pure gravity theories [43], and so even if pure Hořava gravity is renormalisable, does it remain so when coupled to matter?

Our main focus will lie in one-loop corrections to the matter propagator. Such loops involving non-relativistic gravity fields will generically introduce Lorentz violation in the matter sector. It is sometimes argued that supersymmetry can help suppress radiative corrections that violate Lorentz invariance [148], although there are doubts that it is possible to construct a supersymmetric extension of Hořava gravity [149]. Since Lorentz Invariance is highly constrained by observation, it is important to ask how much Lorentz violation will naturally occur. Furthermore, as shown in Section 3.2.1, Lorentz breaking terms in the matter sector source the Stückelberg mode in Hořava gravity and can then give rise to violations of the Equivalence Principle.

This chapter is made up of two main parts. In the first half we construct the general form of matter Lagrangians consistent with the reduced symmetry group of Hořava gravity. For example, for a scalar field, the breaking of diffeomorphism invariance (Diff) down to foliation-preserving diffeomorphism ($\mathrm{Diff}_{\mathcal{F}}$) allows one to add terms to the Lagrangian such as $\varphi\Delta^2\varphi$, where Δ is the spatial Laplacian. The relevant actions are written in both the ADM and Stückelberg formalisms. Phenomenological difficulties, including Equivalence Principle violations, may arise when matter couples directly to the Stückelberg field. To try and evade these concerns, we establish the conditions for such couplings to be absent. It turns out that they are only absent for the standard Lorentz invariant Lagrangians for both the scalar and the gauge field.

The second half of this chapter focusses on one-loop corrections to the scalar propagator. Quantum scalar fields have been studied in the context of Hořava-Lifshitz gravity at the semi-classical level [150, 151], whereas here we will allow gravitational fields to flow in the loops. We begin, as in [100], by assuming that the tree level theory is Lorentz invariant, and minimally coupled to the full spacetime metric. This is primarily because we do not want to face fine-tuning issues in the limit that gravity decouples (see [100] for discussion on this point). Since the gravity fields couple to the scalar they can flow in loops and this generically introduces Lorentz breaking. Whilst there is some overlap with the analysis of [100], our work differs in some important ways. In particular, [100] only consider constant loop corrections to the light cone, whereas we also consider momentum dependent corrections from the generation of higher-order derivatives. We also use a different method: [100] fix the gauge and then compute one-loop diagrams involving non-diagonal propagators. In contrast, we integrate out the constraints and work with the propagating degrees of freedom directly. While this enables us to avoid non-diagonal propagators, our method is not without some subtleties of its own. Note that we also use dimensional regularisation so we only encounter logarithmic divergences. The quadratic divergences found in [100] manifest themselves as large momentum dependent corrections in our case [152].

Our loop calculations reveal a number of worrying features. The first is the large renormalisation of the light cone ($\sim 1/\alpha \gtrsim 10^7$) at low energies and momentum. This follows from the fact that the scalar graviton is so strongly coupled to matter. However, it can probably be alleviated by modifying the gravitational part of the action to include terms of the form $(D_iK_{jk})^2$. The second issue is the generation of higher derivatives with respect to both space *and time*. The former were expected, and kick in at the Planck scale. It turns out that the UV scaling of the scalar graviton feels Planckian suppression so this is the scale that controls the higher-order corrections. The higher-order time derivatives come as more of a surprise, and not a pleasant one. They suggest the presence of a new heavy ghost degree of freedom, spoiling the unitarity of the theory at high energies. The Lifshitz scaling was designed to avoid introducing precisely these Ostrogradski modes. One finds similar behaviour in perturbative General Relativity coupled to matter, but then the resulting ghost can only propagate beyond the Planck scale, outside of the régime of validity of the effective theory. By contrast, Hořava gravity is intended as a UV complete theory, so if the heavy ghosts are indeed present, there is no safety net offered by an effective field theory cut-off. It may be possible to resolve this issue by extending the tree-level matter action to include non-relativistic terms consistent with the Lifshitz scaling of the gravity sector. We discuss this question in Section 4.6.

The rest of this chapter is arranged as follows: in Section 4.2, we explicitly state the action for the Hořava gravity model we work with. We then embark on the first of our two main topics in Section 4.3, constructing matter Lagrangians that are consistent with the reduced symmetry group of Hořava gravity. In Section 4.4, the second of our topics is considered, focussing on the quantum effects of a relativistic scalar coupled to Hořava gravity and picking out the interesting features. The calculation is repeated for the case of a $U(1)$ gauge field in Section 4.5. Discussion of our results takes place in Section 4.6.

4.2 Non-relativistic gravity

In this section, we make explicit the gravitational action we will be using for our calculations. We will be considering the extended non-projectable Hořava theory, and so our action is given by

$$S_{grav} = M_{pl}^2 \int dt d^3x \sqrt{\gamma} N \left(K_{ij}K^{ij} - \lambda K^2 \right) + S_V, \tag{4.1}$$

where M_{pl} is the Planck mass and S_V the gravitational potential. The gravitational potential is built from objects satisfying the $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ symmetry, up to sixth order in spatial derivatives. The exhaustive list of building blocks is the (inverse) metric γ^{ij}, the Ricci tensor of a slice $R^{(3)}_{ij}$ and $a_i \equiv \partial_i \log N$ [117], the acceleration of spatial slices through the spacetime. We write this piece of the action as

$$S_V = M_{pl}^2 \int dt d^3x \sqrt{\gamma} N \left(R^{(3)} + \alpha a_i a^i + \frac{1}{M_{pl}^2} V_4 + \frac{1}{M_{pl}^4} V_6 \right), \tag{4.2}$$

where the four-derivative V_4 and six-derivative V_6 terms are given by

$$V_4 = A_1 (R^{(3)}_{ij})^2 + A_2 (R^{(3)})^2 + A_3 R^{(3)} D_i a^i + A_4 (D_i a^i)^2 \tag{4.3a}$$

$$V_6 = B_1 (D_i R^{(3)}_{jk})^2 + B_2 (D_i R^{(3)})^2 + B_3 \triangle R^{(3)} D_i a^i + B_4 a^i \triangle^2 a_i \tag{4.3b}$$

where $\triangle \equiv D_i D^i$ and we only include terms which are inequivalent at quadratic order around a Minkowski background[1]. To ensure the absence of strong coupling in the theory (needed to ensure that perturbation theory remains valid and so the power counting argument can hold), one needs to introduce a hierarchy of scales by making the Bs large [1, 117]. For definiteness we assume $A_i \sim \mathcal{O}(1)$ and $B_i \sim 1/\alpha$ [117]. Constraints on λ and α give roughly $|1-\lambda| \sim \alpha \lesssim 10^{-7}$ [135], or $B \gtrsim 10^7$. It turns out that two new scales are introduced, $M_\star \sim \sqrt{\alpha} M_{pl}$ and $M_h \sim \alpha^{1/4} M_{pl}$ [135], the former relating to the scalar graviton, the latter to the tensor. Putting all these pieces together, one obtains our action for Hořava gravity.

As discussed in Section 3.2, it is often more illuminating to write Hořava gravity in a form using covariant 4D spacetime tensors. Our main concern regarding this here is how matter should couple to gravity in this theory. Recall from (3.17) and (3.18)

$$\gamma_{\alpha\nu} \nabla_\mu T^{\mu\nu} = 0 \qquad \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta \phi} = -\frac{1}{X} \nabla_\mu \phi \nabla_\nu T^{\mu\nu}, \tag{4.4}$$

where $T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}$ is the energy-momentum tensor derived from the matter action S_m. Note that matter sources the Stückelberg field directly when there is some violation of energy-momentum conservation. Such violations can, in principle, lead to violations of the Equivalence Principle. This equation will prove useful later when deriving the conditions for our theory to attempt to evade EP constraints.

4.3 Non-relativistic matter

We now consider the first main topic of this chapter: what is the general form of matter Lagrangians consistent with the $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ symmetry of Hořava gravity? Lorentz invariant matter actions, minimally coupled to the spacetime metric are expected to receive quantum corrections via gravity loops that spoil the Lorentz invariance. Indeed, we will later show explicitly that this is the case. For now, however, let us try to formulate the relevant actions for a scalar and a $U(1)$ gauge field, consistent with the foliation of spacetime. Of course, these will differ from the standard Lorentz invariant actions because extra terms are allowed due to the reduced symmetry. If Hořava gravity plus matter is indeed renormalisable, we expect these extra terms to be the only ones generated by quantum corrections. The analysis here is similar to that of [110], but we also consider the possible effect of a_i terms. Note that in keeping with the philosophy of Hořava gravity we only

[1] Some of our expansions will go to higher order, but including just the terms here will capture all the relevant physics.

consider generalisations to the potential and assume the time derivatives enter as in the relativistic theory. This ensures the absence of ghostly instabilities at tree level, but as we will see later on, it is not guaranteed at the one loop level.

4.3.1 Scalar field

For a scalar field φ, the generic action with a $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ invariant potential can be written in the ADM formalism as

$$S_\varphi = \int dt d^3\mathbf{x}\sqrt{\gamma}N\left[\frac{1}{2N^2}(\dot{\varphi} - N^i\partial_i\varphi)^2 - \frac{1}{2}\gamma^{ij}\partial_i\varphi\partial_j\varphi - V(\varphi) - F[\varphi, D_i, R_{ij}, a_i, \gamma^{ij}]\right]. \tag{4.5}$$

The symmetries of the Hořava framework permit additional terms in the theory relative to GR, which F controls ($F = 0$ is the usual minimally-coupled diffeomorphism invariant action). Since we are matching the symmetry of the gravitational sector, we will only consider terms up to scaling dimension 6. These terms can be constructed from φ, D_i, $R^{(3)}_{ij}$, a_i and γ^{ij} and make up a general F. We will enforce P and T symmetries, neglect purely gravitational terms, and only consider terms inequivalent at quadratic order on Minkowski. The result is

$$\begin{aligned} F =&\alpha_1\varphi D^i a_i + \alpha_2\varphi\triangle\varphi + \alpha_3 R^{(3)}\varphi + \frac{\beta_1}{M_{pl}^2}D^i a_i\triangle\varphi + \frac{\beta_2}{M_{pl}^2}\varphi\triangle^2\varphi + \frac{\beta_3}{M_{pl}^2}R^{(3)}\triangle\varphi \\ &+ \frac{\gamma_1}{M_{pl}^4}D^i a_i\triangle^2\varphi + \frac{\gamma_2}{M_{pl}^4}\varphi\triangle^3\varphi + \frac{\gamma_3}{M_{pl}^4}R^{(3)}\triangle^2\varphi, \end{aligned} \tag{4.6}$$

where $\triangle \equiv D_i D^i$. This list is exhaustive given our above restrictions, since terms such $R^{(3)}_{ij}D^iD^j\varphi$ are equivalent to other terms via integration by parts (IBP) and the Bianchi identity, and others such as $R^{(3)}D_i\varphi D^i\varphi$ are ignored since they vanish at quadratic order on Minkowski space.

These expressions can also be re-written using the Stückelberg formulation. Again, as with gravity, the action is simpler in this formalism, and can be written

$$S_\varphi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi) - F\right]. \tag{4.7}$$

where F is now

$$\begin{aligned} F =&\alpha_1\varphi\mathcal{D}_\mu a^\mu + \alpha_2\varphi\mathcal{D}^\mu\mathcal{D}_\mu\varphi + \alpha_3\mathcal{R}\varphi + \frac{\beta_1}{M_{pl}^2}\mathcal{D}_\mu a^\mu\mathcal{D}^\nu\mathcal{D}_\nu\varphi + \frac{\beta_2}{M_{pl}^2}\varphi(\mathcal{D}^\mu\mathcal{D}_\mu)^2\varphi \\ &+ \frac{\beta_3}{M_{pl}^2}\mathcal{R}\mathcal{D}^\mu\mathcal{D}_\mu\varphi + \frac{\gamma_1}{M_{pl}^4}\mathcal{D}_\mu a^\mu(\mathcal{D}^\nu\mathcal{D}_\nu)^2\varphi + \frac{\gamma_2}{M_{pl}^4}\varphi(\mathcal{D}^\nu\mathcal{D}_\nu)^3\varphi + \frac{\gamma_3}{M_{pl}^4}\mathcal{R}(\mathcal{D}^\nu\mathcal{D}_\nu)^2\varphi, \end{aligned} \tag{4.8}$$

where we have introduced the spatially projected covariant derivative,

$$D_i X^{j_1\cdots j_n} \to \mathcal{D}_\mu X^{\alpha_1\cdots\alpha_n} = \gamma^\nu_\mu\gamma^{\alpha_1}_{\beta_1}\cdots\gamma^{\alpha_n}_{\beta_n}\nabla_\nu X^{\beta_1\cdots\beta_n}. \tag{4.9}$$

In this formalism, the recovery of the usual minimally coupled scalar field in the case $F = 0$ is clearer.

It is now possible to ask in what way the action must be constructed to avoid any coupling to the Stückelberg field (which one may want to avoid for reasons of *e.g.* Equivalence Principle [1] or Lorentz invariance violation). In fact, it is easy to show that the only combination of terms which does not couple matter to the Stückelberg field is the usual Lorentz invariant action with $F \equiv 0$.

To see this, we set $\frac{\delta}{\delta\phi}\int d^4x\sqrt{-g}F = 0$. Strictly speaking we only require this to vanish on-shell, but since φ can be coupled to a source independently of its coupling to the Stückelberg field, it is clear that we need to impose $\frac{\delta}{\delta\phi}\int d^4x\sqrt{-g}F = 0$ off-shell in order to guarantee $\frac{\delta S_\varphi}{\delta\phi} = 0$ in all cases. Now, because the necessary cancellation can only occur between terms with the same power of M_{pl} and the same number of φ's, it immediately follows that $\alpha_2 = \beta_2 = \gamma_2 = 0$, and that

$$\frac{\delta}{\delta\phi}\int d^4x\sqrt{-g}\left[\alpha_1\varphi\mathcal{D}_\mu a^\mu + \alpha_3\mathcal{R}\varphi\right] = 0 \tag{4.10a}$$

$$\frac{\delta}{\delta\phi}\int d^4x\sqrt{-g}\left[\beta_1\mathcal{D}_\mu a^\mu\mathcal{D}^\nu\mathcal{D}_\nu\varphi + \beta_3\mathcal{R}\mathcal{D}^\mu\mathcal{D}_\mu\varphi\right] = 0 \tag{4.10b}$$

$$\frac{\delta}{\delta\phi}\int d^4x\sqrt{-g}\left[\gamma_1\mathcal{D}_\mu a^\mu(\mathcal{D}^\nu\mathcal{D}_\nu)^2\varphi + \gamma_3\mathcal{R}(\mathcal{D}^\nu\mathcal{D}_\nu)^2\varphi\right] = 0. \tag{4.10c}$$

Consider equation (4.10a). Introducing $\tilde{\varphi} = \frac{\sqrt{-g}}{\sqrt{\gamma}}\varphi$, this implies that

$$\alpha_3\sqrt{\gamma}\left[\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}\gamma_{\mu\nu}\right]\tilde{\varphi} + \text{terms with derivatives of } \tilde{\varphi} = 0. \tag{4.11}$$

Since this should be true for any φ and $\gamma_{\mu\nu}$, we conclude that $\alpha_3 = 0$. Furthermore, since $\frac{\delta}{\delta\phi}\int d^4x\sqrt{-g}\varphi\mathcal{D}_\mu a^\mu \neq 0$, in general, it also follows that $\alpha_1 = 0$. Similar arguments can be applied to equations (4.10b) and (4.10c) to conclude that $\beta_1 = \beta_3 = 0$, and $\gamma_1 = \gamma_3 = 0$. It now follows that $F \equiv 0$, as previously stated.

4.3.2 Gauge field

We also consider a vector field A^μ invariant under a $U(1)$ gauge symmetry[2] (see also [153]). Our analysis will run along much the same lines as for the scalar field. The general action, invariant under $\mathrm{Diff}_\mathcal{F}(\mathcal{M})$ can be written in terms of ADM

[2] Considering a non-Abelian gauge field would introduce a internal index on the F, $F_{\mu\nu} \to F^a_{\mu\nu}$ and thus also $B_i \to B^a_i$. In the action one takes a trace over these indices. We choose to focus on $U(1)$ since (i) our results would not be substantially altered here aside from the clutter of additional notation and (ii) we will consider quantum corrections only for the simplest case of $U(1)$ symmetries.

variables as,

$$S_A = \frac{1}{4}\int dt d^3\mathbf{x}\sqrt{\gamma}N\left[\frac{2}{N^2}\gamma^{ij}(F_{0i}-F_{ki}N^k)(F_{0j}-F_{lj}N^l)-F_{ij}F_{kl}\gamma^{ik}\gamma^{jl}-G\right], \tag{4.12}$$

where $G=0$ in the familiar relativistic case and $F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu$. There is an additional constraint on the possible terms in G since we are demanding that the theory remain gauge invariant with respect to the $U(1)$. In order to add higher spatial derivatives, it is convenient to write the higher-order terms containing the vector field in terms of the magnetic field,

$$B^i = \frac{1}{2}\frac{\varepsilon^{ijk}}{\sqrt{\gamma}}F_{jk}, \tag{4.13}$$

where ε^{ijk} is the Levi-Civita symbol. The magnetic field corresponds to the only gauge invariant way that higher-order spatial derivatives of A can enter[3]. G can be built therefore from $B^i, D_i, R^{(3)}_{ij}, a_i, \gamma^{ij}$. Assuming P and T symmetry again, the terms inequivalent at quadratic level on Minkowski and up to scaling dimension 6 are

$$\begin{aligned}G &= \alpha_1 a_i B^i+\alpha_2 B_i B^i+\frac{\beta_1}{M_{pl}^2}a_i\triangle B^i+\frac{\beta_2}{M_{pl}^2}B_i\triangle B^i+\frac{\beta_3}{M_{pl}^2}(D_iB^i)^2+\frac{\beta_4}{M_{pl}^2}R^{(3)}D_iB^i\\ &+\frac{\gamma_1}{M_{pl}^4}B_i\triangle^2 a^i+\frac{\gamma_2}{M_{pl}^4}B_i\triangle^2 B^i+\frac{\gamma_3}{M_{pl}^4}(D_iD_jB^j)^2+\frac{\gamma_4}{M_{pl}^4}R^{(3)}\triangle D_iB^i.\end{aligned} \tag{4.14}$$

In order to also write the vector field action in the Stückelberg approach, we need a 4-vector expression for B_i. The appropriate expression is

$$\mathcal{B}^\mu = \frac{1}{2}\frac{\epsilon^{\nu\mu\rho\sigma}}{\sqrt{-g}}F_{\rho\sigma}u_\nu. \tag{4.15}$$

The Stückelberg field couples here via the normal 4-vector u_ν. Proceeding as before, our action is now the familiar

$$S_A = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-G\right]. \tag{4.16}$$

By making the substitutions $B_i\to\mathcal{B}_\mu$, $D_i\to\mathcal{D}_\mu$, $a_i\to a_\mu$, $R^{(3)}_{ij}\to\mathcal{R}_{\mu\nu}$ and $\gamma^{ij}\to\gamma^{\mu\nu}$ into (4.14), the general expression for G can be written in this formalism. In this case, the Stückelberg field couples through the projection operator to the matter field, as well as to the magnetic field through the normal. As with the scalar field, repeating the same procedure informs us that the only way to prevent a coupling between the matter field and the Stückelberg field is to set $G\equiv 0$ in our action.

[3]The electric field corresponds to time derivatives, so additional electric field terms result in higher-order time derivatives.

4.4 Quantum corrected scalar fields

We now turn to our second major topic in this chapter: quantum corrections to relativistic matter Lagrangians. In particular, in this section we consider one loop corrections to the relativistic scalar field action with mass m and a φ^4 interaction,

$$S_{\varphi}^{tree} = \int dtd^3\mathbf{x}\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\mu}{4!}\varphi^4\right]. \tag{4.17}$$

Gauge fields are considered in Section 4.5.

Since we are interested in the role played by the Lorentz violating gravity sector, we will include loops of gravity fields, and expect these to induce some Lorentz violation in the scalar field theory. We will concentrate on corrections to the scalar field propagator, including the contribution from higher-order spatial derivatives, in contrast to [100] who only considered constant corrections to the light cone. Our method also differs to that in [100]: they fix the gauge and work with non-diagonal propagators, whereas we integrate out the constraints and work directly with the dynamical degrees of freedom. This method has the advantage of allowing us to work with diagonal propagators, but is not without its subtleties as we will illustrate by means of a toy model in the next section. Note that the effective action we obtain is consistent in that the resulting classical dynamics is independent of the order in which we impose the constraints *i.e.* before we compute the equations of motion, or after.

4.4.1 Toy model

As a warm up to the main event we consider the following toy model of a dynamical scalar, ϕ, coupled to a non-dynamical scalar, A.

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}A\triangle A - \frac{1}{2}m^2A^2 + \lambda\phi^2 A, \tag{4.18}$$

where $\triangle \equiv \partial_i\partial^i$. Our interest lies in the one-loop corrections to the propagator for ϕ. We can compute this in two ways: directly from the Lagrangian (4.18) by defining a propagator for both ϕ and A, or by integrating out the non-dynamical field, A, and only working with the dynamical degree of freedom, ϕ. We will compare the two methods, beginning with the former.

The Lagrangian (4.18) gives rise to the following field equations

$$\frac{\delta}{\delta\phi}\int d^4x\mathcal{L} = \Box\phi + 2\lambda\phi A = 0 \tag{4.19a}$$

$$\frac{\delta}{\delta A}\int d^4x\mathcal{L} = -(\Delta + m^2)A + \lambda\phi^2 = 0, \tag{4.19b}$$

and the set of Feynman rules shown in Figure 4.1. At one loop the correction to the ϕ propagator comes from the Feynman diagrams shown in Figure 4.2. The

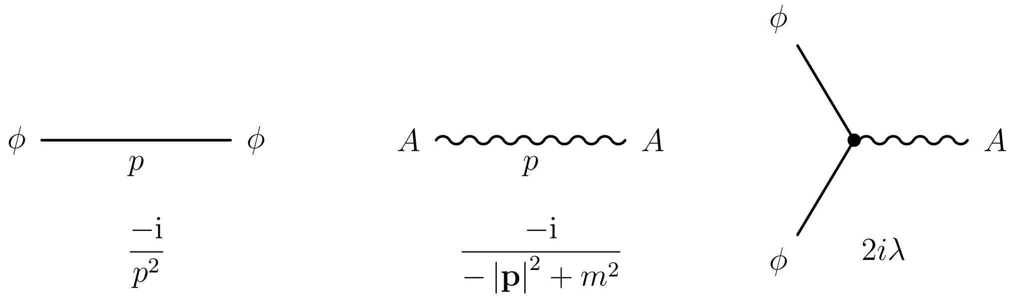


Figure 4.1: The Feynman rules for the Lagrangian (4.18)

(a) 1PI contribution (b) Tadpole contribution

Figure 4.2: One-loop diagrams for the ϕ propagator

one-loop correction contains a one-particle irreducible (1PI) contribution and a tadpole contribution. In contrast to QED, here the tadpole contribution need not vanish. Indeed, from Figure 4.2a, we find the 1PI contribution to be

$$\begin{aligned}
1\mathrm{PI} &= \left(\frac{-\mathrm{i}}{k^2}\right)^2 (2\mathrm{i}\lambda)^2 \int \frac{d^4p}{(2\pi)^4}\frac{-\mathrm{i}}{p^2}\frac{-\mathrm{i}}{-|\mathbf{k}-\mathbf{p}|^2+m^2} \\
&= -\frac{4\lambda^2}{k^4}\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2\left(-|\mathbf{k}-\mathbf{p}|^2+m^2\right)}.
\end{aligned} \tag{4.20}$$

whereas from Figure 4.2b, we find the tadpole contribution to be

$$\text{tadpole} = \frac{1}{2}\left(\frac{-\mathrm{i}}{k^2}\right)^2 (2\mathrm{i}\lambda)^2 \int \frac{d^4p}{(2\pi)^4}\frac{-\mathrm{i}}{p^2}\frac{-\mathrm{i}}{m^2} = -\frac{2\lambda^2}{k^4}\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2m^2}. \tag{4.21}$$

Now a non-vanishing tadpole is the same as saying that the vacuum expectation value (vev) of the field A is non-vanishing. One could add a counterterm to the Lagrangian of the form $\Delta\mathcal{L} = (\text{constant})A$ in order to eliminate this, and therefore eliminate the tadpole. The spirit of this discussion is particularly relevant for matter loops in Hořava gravity to be studied in subsequent sections. The point is that in Hořava gravity matter loops also endow the gravitational fields with a non-trivial vev because the theory offers no solution to the cosmological constant problem. By inserting a bare cosmological constant into the action as a counterterm one can eliminate the vevs of those fields. In the subsequent section we will assume that this has been done by neglecting the tadpole contribution from the relevant diagrams.

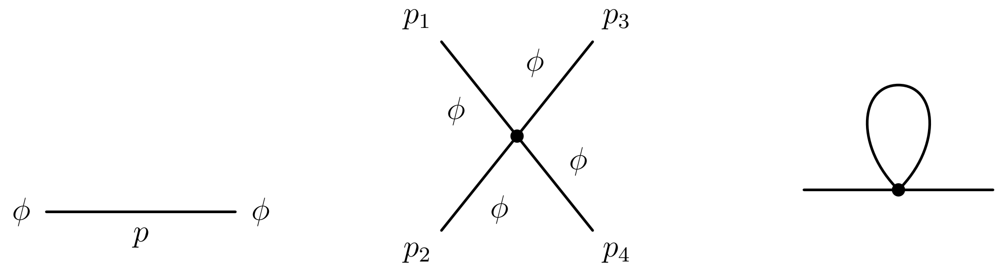
To be able to neglect the tadpoles, we need to understand how they manifest themselves when we integrate out the offending fields. To this end we integrate out the field A in the Lagrangian (4.18) using the constraint (4.19b). Substituting the constraint back in we obtain

$$\mathcal{L}_{reduced} = -\frac{1}{2}(\partial_\mu\phi)^2 + \frac{\lambda^2}{2}\phi^2\frac{1}{\triangle + m^2}\phi^2, \tag{4.22}$$

where the term containing the inverse of $\triangle$ is formally defined using a Fourier transformation. The resulting equation of motion is given by

$$\frac{\delta}{\delta\phi}\int d^4x\mathcal{L}_{reduced} = \Box\phi + 2\lambda^2\phi\frac{1}{\triangle + m^2}\phi^2 = 0 \tag{4.23}$$

Note that one obtains exactly the same equation from substituting the constraint (4.19b) into the ϕ equation of motion (4.19a), thereby illustrating the consistency of our method. The Feynman rules for the reduced Lagrangian (4.22) are now shown[4] in Figure 4.3, along with the only one-loop contribution to the propagator correction. Computing our solitary Feynman diagram, we obtain



(a) $\frac{-\mathrm{i}}{p^2}$ (b) $\frac{\mathrm{i}\lambda^2}{2}\sum_{\sigma_i\in\{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4\}}\frac{1}{-|\sigma_1+\sigma_2|+m^2}$ (c)

Figure 4.3: (a) and (b) The Feynman rules for the reduced Lagrangian (4.22); (c) One-loop diagrams for the ϕ propagator.

$$\frac{1}{2}\left(\frac{-\mathrm{i}}{k^2}\right)^2\int\frac{d^4p}{(2\pi)^4}\left(\frac{-\mathrm{i}}{p^2}\right)\frac{\mathrm{i}\lambda^2}{2}\sum_{\text{perms}}\frac{1}{-|\mathbf{p}_3+\mathbf{p}_4|^2+m^2}, \tag{4.24}$$

where $\mathbf{p}_3$ and $\mathbf{p}_4$ take values in $\{\mathbf{k}, -\mathbf{k}, \mathbf{p}, -\mathbf{p}\}$. Before proceeding further, we need to consider all the permutations. Essentially, we need to find all the permutations

[4]Note the permutations of $\sigma_1 \neq \sigma_2$ across the set of $\mathbf{p}_i$s.

pairing elements of the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ with $\{\mathbf{k}, -\mathbf{k}, \mathbf{p}, -\mathbf{p}\}$, the momenta of each leg of the vertex in the relevant diagram. Eight permutations result in $\mathbf{p}_3 + \mathbf{p}_4 = \mathbf{p}+\mathbf{k}$, four in $\mathbf{p}_3+\mathbf{p}_4 = \mathbf{p}-\mathbf{k}$, four in $\mathbf{p}_3+\mathbf{p}_4 = -\mathbf{p}+\mathbf{k}$ and eight in $\mathbf{p}_3+\mathbf{p}_4 = 0$. Using the fact we are integrating over p and only care about the modulus squared, we can rewrite these as sixteen giving $\mathbf{k} - \mathbf{p}$ and eight permutations giving 0. So, the one-loop correction to the propagator gives

$$\frac{-\lambda^2}{k^4} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} \left[\frac{4}{-|\mathbf{k}-\mathbf{p}|^2 + m^2} + \frac{2}{m^2} \right]. \tag{4.25}$$

Clearly, the first term in (4.25) is equal to the 1PI contribution (4.20) derived earlier, while the second is equal to the tadpole contribution (4.21). Therefore, if we want to neglect the tadpole contributions for the reasons described above we need to take care with "bubblegum diagrams" with the generic shape shown in Figure 4.3c. In particular we should not include permutations that lead to vanishing combinations of momenta in the relevant 4-vertex. Upon integrating the non-dynamical field *back in* we now understand this as vanishing momenta being transferred to a loop by the propagator for the non-dynamical field in the tadpole diagram. We keep this in mind when computing bubblegum diagrams in Hořava gravity.

4.4.2 Reduced action for a scalar field

Our goal is to identify one-loop corrections to the relativistic propagator for a scalar field coupled to Hořava gravity. At tree level, this theory is described by the following action

$$S = S_{grav} + S_{\varphi}^{tree}, \tag{4.26}$$

where S_{grav} is given by the action for Hořava gravity (4.1) and S_{φ}^{tree} is given by the relativistic action (4.17) for a scalar field of mass m and potential φ^4, coupled to the spacetime metric. The Hamiltonian and momentum constraints for this theory are

$$\begin{aligned}
\mathcal{C}_N = \frac{\delta S}{\delta N} &= M_{pl}^2\sqrt{\gamma}\left[-K_{ij}K^{ij} + \lambda K^2 + R^{(3)} - \alpha a^i a_i - 2\alpha D_i a^i\right] \\
&+ \sqrt{\gamma}\left[A_1(R^{(3)}_{ij})^2 + A_2(R^{(3)})^2 + A_3\left(R^{(3)}D_i a^i + \frac{1}{N}\triangle(NR^{(3)})\right)\right. \\
&\qquad \left. + A_4\left((D_i a^i)^2 + \frac{2}{N}\triangle(ND_i a^i)\right)\right] \\
&+ \frac{\sqrt{\gamma}}{M_{pl}^2}\left[B_1(D_i R^{(3)}_{jk})^2 + B_2(D_i R^{(3)})^2 + B_3\left(\triangle R^{(3)} D_i a^i + \frac{1}{N}\triangle(N\triangle R^{(3)})\right)\right. \\
&\qquad \left. + B_4\left(D_i a^i \triangle D_j a^j + \frac{1}{N}\triangle(N\triangle D_i a^i) + \frac{1}{N}\triangle^2(ND_i a^i)\right)\right] \\
&- \sqrt{\gamma}\left[\frac{1}{2N^2}\left(\dot{\varphi} - N^i\partial_i\varphi\right)^2 + \frac{1}{2}D^i\varphi D_i\varphi + \frac{1}{2}m^2\varphi^2 + \frac{\mu}{4!}\varphi^4\right] \qquad (4.27a) \\
\mathcal{C}_i = \frac{\delta S}{\delta N_i} &= 2M_{pl}^2\sqrt{\gamma}\left(D_j K^{ij} - \lambda D^i K\right) + \frac{\sqrt{\gamma}}{N}\left[-\dot{\varphi}D^i\varphi + D^i\varphi N_j D^j\varphi\right]. \qquad (4.27b)
\end{aligned}$$

We need to establish the form of the reduced action for the dynamical fields, having integrated out the constraints up to the appropriate order. To this end, we begin by perturbing our ADM fields about Minkowski,

$$N = 1 + \epsilon n \tag{4.28a}$$

$$N_i = \epsilon\left(\partial_i\beta + S_i\right) \qquad \text{where } \partial^i S_i = 0 \tag{4.28b}$$

$$\gamma_{ij} = \delta_{ij}\left(1 + 2\frac{\epsilon}{M_{pl}}\zeta\right) + 2\epsilon\partial_i\partial_j E + 2\epsilon\partial_{(i}V_{j)} + \frac{\epsilon}{M_{pl}}h_{ij} \tag{4.28c}$$

$$\text{where } \partial_i V^i = \partial_i h^{ij} = h^i_i = 0,$$

where we have introduced the expansion parameter ϵ, and we have assumed units in which the emergent speed of light $c = 1$. Note that once this expansion has been made, we will not be concerned with distinguishing between upper or lower indices, since they will all correspond to coordinates on flat space. For the matter sector we have to ensure we replace $\varphi \to \epsilon\varphi$, since we are also considering these as leading order perturbations to a vacuum Minkowski background.

Having performed a helicity decomposition on the metric components it is convenient to introduce projection operators,

$$\pi_{ij} \equiv \delta_{ij} - \frac{\partial_i\partial_j}{\triangle} \qquad \frac{1}{2}\Pi_{ij|kl} \equiv \frac{1}{2}\left(2\pi_{i(k}\pi_{l)j} - \pi_{ij}\pi_{kl}\right), \tag{4.29}$$

which project out the transverse and transverse-traceless components respectively[5]. When we switch to Fourier space, these will have a vector as a superscript, *e.g.* $\pi^{\mathbf{k}}_{ij}$ in which case one replaces $\partial_i \to \mathbf{k}_i$ in the above expressions. We will also find it

[5] Note that the factor of $\frac{1}{2}$ in the definition of $\Pi_{ij|kl}$ has no significance beyond notation.

useful to define

$$f(\mathbf{k}) := \frac{1 - \frac{A_3}{M_{pl}^2}\mathbf{k}^2 + \frac{B_3}{M_{pl}^4}\mathbf{k}^4}{\alpha + \frac{A_4}{M_{pl}^2}\mathbf{k}^2 + \frac{B_4}{M_{pl}^4}\mathbf{k}^4}. \tag{4.30}$$

Some of the unphysical metric degrees of freedom can be removed by gauge fixing, others we will have to integrate out[6]. It is clear from the foliation preserving transformations (2.8) that we may choose the gauge conditions

$$V_i = 0 \qquad E = 0, \tag{4.31}$$

without losing knowledge of our constraints. We have now reduced our expansion of γ_{ij} to the physical scalar and tensor. The pieces arising from N and N_i will be removed only when we integrate out the corresponding constraints. Expanding the action order by order in ϵ, we find that

$$S = \epsilon^2 S^{(2)} + \frac{\epsilon^3}{M_{pl}} S^{(3)} + \frac{\epsilon^4}{M_{pl}^2} S^{(4)} + \mathcal{O}\left(\epsilon^5\right), \tag{4.32}$$

where

$$\begin{aligned} S^{(2)} = \int dtd^3\mathbf{x} \Bigg[& \frac{1}{2}\varphi\left(-\partial_t^2 + \triangle - m^2\right)\varphi + \frac{1}{4}h_{ij}\left(-\partial_t^2 + \triangle + \frac{A_1}{M_{pl}^2}\triangle^2 - \frac{B_1}{M_{pl}^4}\triangle^3\right)h_{ij} \\ & + M_{pl}^2 n\left(\alpha\triangle - \frac{A_4}{M_{pl}^2}\triangle^2 + \frac{B_4}{M_{pl}^4}\triangle^3\right)n - M_{pl}^2(1-\lambda)\beta\triangle^2\beta + \frac{1}{2}M_{pl}^2 S_i\triangle S_i \\ & + 3(1-3\lambda)\dot{\zeta}^2 - 2\zeta\triangle\zeta + \frac{(6A_1 + 16A_2)}{M_{pl}^2}\zeta\triangle^2\zeta - \frac{(6B_1+16B_2)}{M_{pl}^2}\zeta\triangle^3\zeta \\ & + nC_N^{(1)} + n_i C_i^{(1)}\Bigg], \end{aligned} \tag{4.33}$$

$$\begin{aligned} S^{(3)} = \int dtd^3\mathbf{x} \Bigg\{ & \frac{3}{2}\zeta\dot{\varphi}^2 - \frac{1}{2}\zeta\partial_i\varphi\partial^i\varphi - \frac{3}{2}m^2\zeta\varphi^2 + \frac{1}{2}h^{ij}\partial_i\varphi\partial_j\varphi \\ & + \alpha M_{pl}^2\left[-\zeta\partial_i n\partial_i n + 2M_{pl} n\partial_i n\partial_i n\right] - 4A_3\triangle\zeta\partial_i n\partial_i n - 4\frac{B_3}{M_{pl}^2}\triangle^2\zeta\partial_i n\partial_i \\ & + A_4\left[\zeta(\triangle n)^2 + 2M_{pl}n(\triangle n)^2 - 2\partial_i\zeta\partial_i n\triangle n + 4M_{pl}\partial_i n\partial_i n\triangle n\right] \\ & + \frac{B_4}{M_{pl}^2}\Big[2M_{pl}\triangle n\triangle(n\triangle n) + 2M_{pl}\triangle n\triangle(n\triangle n) + 2M_{pl}\triangle n\triangle(\partial_i n\partial_i n) \\ & \quad + \zeta\triangle n\triangle^2 n - \partial_i\zeta\partial_i n\triangle^2 n - \partial_i\zeta\partial_i\triangle n\triangle n + 2\triangle n\triangle(\zeta\triangle n) - \triangle n\triangle(\partial_i\zeta\partial_i n)\Big] \\ & + 2M_{pl}^3 n\left(\partial_i\partial_j\beta\right)^2 - 2M_{pl}^3\lambda n(\triangle\beta)^2 - 2M_{pl}^2(1-3\lambda)\dot{\zeta}n\triangle\beta + 2M_{pl}^3 n\partial_{(i}S_{j)}\partial_i S_j \\ & + 4M_{pl}^3 n\partial_i\partial_j\beta\partial_i S_j + M_{pl}^2\zeta(\partial_i\partial_j\beta + \partial_{(i}S_{j)})(\partial_i\partial_j\beta + \partial_{(i}S_{j)}) - M_{pl}^2\lambda\zeta(\triangle\beta)^2 \\ & + 4M_{pl}^2\partial_i\zeta(\partial_j\beta + S_j)(\partial_i\partial_j\beta + \partial_{(i}S_{j)}) - 2M_{pl}^2(1-\lambda)\partial_i\zeta(\partial_i\beta + S_i)\triangle\beta \\ & + M_{pl}\left(nC_N^{(2)} + n_i C_i^{(2)}\right)\Bigg\} + \ldots, \end{aligned} \tag{4.34}$$

[6]Recall in EM, one can obtain an action in solely the two degrees of freedom by removing the longitudinal part with the transverse gauge fixing $\partial^i A_i = 0$ and integrating out the non-dynamical field A_0, at the expense of losing *manifest* locality and Lorentz invariance.

$$\begin{aligned}
S^{(4)} = \int \mathrm{d}t d^3\mathbf{x} \Bigg\{ & \frac{3}{4}\zeta^2\dot{\varphi}^2 + \frac{1}{4}\zeta^2\partial_i\varphi\partial^i\varphi - \frac{3}{4}m^2\zeta^2\varphi^2 + \frac{3}{2}\zeta h^{ij}\partial_i\varphi\partial_j\varphi \\
& - \frac{1}{8}h_{ij}h^{ij}\dot{\varphi}^2 + \frac{1}{8}h_{ij}h^{ij}\partial_k\varphi\partial^k\varphi - \frac{1}{2}h^{ik}h^j_k\partial_i\varphi\partial_j\varphi + \frac{1}{8}m^2h^{ij}h_{ij}\varphi^2 \\
& - \frac{1}{2}M_{pl}^2 n^2\dot{\varphi}^2 - M_{pl}^2 nn^i\dot{\varphi}\partial_i\varphi - \frac{1}{2}M_{pl}^2 n^i n^j\partial_i\varphi\partial_j\varphi - \frac{1}{4!}M_{pl}^2\mu\varphi^4 \\
& + M_{pl}^2\left(nC_N^{(3)} + n_iC_i^{(3)}\right)\Bigg\} + \ldots,
\end{aligned} \tag{4.35}$$

and "…" denote terms which are irrelevant to our subsequent calculations, and the constraints are expanded as $\mathcal{C}_N = \epsilon C_N^{(1)} + \epsilon^2 C_N^{(2)} + \epsilon^3 C_N^{(3)} + \mathcal{O}(\epsilon^4)$ and $\mathcal{C}_i = \epsilon C_i^{(1)} + \epsilon^2 C_i^{(2)} + \epsilon^3 C_i^{(3)} + \mathcal{O}(\epsilon^4)$. Note that we do not need to consider interactions beyond fourth order since for 1-loop corrections to the propagator we will only encounter up to four point vertices.

We now integrate out the constraints by setting $C_N^{(1)} = -\epsilon C_N^{(2)} - \epsilon^2 C_N^{(3)} - \epsilon^3 C_N^{(4)} + \mathcal{O}(\epsilon^4)$ and $C_i^{(1)} = -\epsilon C_i^{(2)} - \epsilon^2 C_i^{(3)} - \epsilon^3 C_i^{(4)} + \mathcal{O}(\epsilon^4)$, or more specifically,

$$\begin{aligned}
2\epsilon M_{pl}^2\left(-\alpha + \frac{A_4\triangle}{M_{pl}^2} - \frac{B_4\triangle^2}{M_{pl}^4}\right)\triangle n = 4\epsilon M_{pl}&\left(1 + \frac{A_3\triangle}{M_{pl}^2} + \frac{B_3\triangle^2}{M_{pl}^4}\right)\triangle\zeta \\
&+ \frac{\epsilon^2}{2}\left(\dot{\varphi}^2 + \partial_i\varphi\partial^i\varphi + m^2\varphi^2\right) + \epsilon^2 H_2 \\
&- \epsilon^3\Big[n\dot{\varphi}^2 + n^i\partial_i\varphi\dot{\varphi} + \frac{1}{2}h^{ij}\partial_i\varphi\partial_j\varphi. \\
&\qquad + \zeta\partial_i\varphi\partial^i\varphi\Big] + \epsilon^3 H_3 + \mathcal{O}(\epsilon^4)
\end{aligned} \tag{4.36a}$$

$$\begin{aligned}
2\epsilon M_{pl}^2(1-\lambda)\triangle^2\beta &= 2\epsilon M_{pl}(1-3\lambda)\triangle\dot{\zeta} - \epsilon^2\partial^i\left(\dot{\varphi}\partial_i\varphi\right) + \epsilon^2 P_2 \\
&\quad + \epsilon^3\partial^i\left(\partial_i\varphi\partial_j\varphi(\partial^j\beta + S^j)\right) + \epsilon^3 P_3 + \mathcal{O}(\epsilon^4)
\end{aligned} \tag{4.36b}$$

$$\begin{aligned}
\epsilon M_{pl}^2\triangle S_i &= -\epsilon^2\pi_{ij}\dot{\varphi}\partial^j\varphi + \epsilon^2 Q_{2i} + \epsilon^3 Q_{3i} \\
&\quad + \epsilon^3\pi_{ij}\left(\partial^j\varphi\partial^k\varphi(\partial_k\beta + S_k)\right) + \mathcal{O}(\epsilon^4)
\end{aligned} \tag{4.36c}$$

where

$$H_q = H_q(h_{ij}, \zeta, n, \beta, S_i) = -\frac{d^q}{d\epsilon^q}\left(\frac{1}{q!}\frac{\delta S_{grav}}{\delta N}\right)\Bigg|_{\epsilon=0} \tag{4.37a}$$

$$P_q = P_q(h_{ij}, \zeta, n, \beta, S_i) = \partial^i\frac{d^q}{d\epsilon^q}\left(\frac{N\gamma_{ij}}{q!}\frac{\delta S_{grav}}{\delta N_j}\right)\Bigg|_{\epsilon=0} \tag{4.37b}$$

$$Q_q^i = Q_q^i(h_{ij}, \zeta, n, \beta, S_i) = \pi^{ij}\frac{d^q}{d\epsilon^q}\left(\frac{N\gamma_{jk}}{q!}\frac{\delta S_{grav}}{\delta N_k}\right)\Bigg|_{\epsilon=0}. \tag{4.37c}$$

Of course, we obtain three equations from the two constraints as the momentum constraint can be split into its transverse and longitudinal parts, yielding two equations. Note that H_q, P_q and Q_q contain n, β and S_i, which can be removed iteratively to any desired order by re-substituting (4.36) into the resulting expression.

We now use the equations (4.36) to eliminate the non-dynamical field n, β and S_i from the action, thereby arriving at the reduced action for the dynamical fields,

h_{ij}, ζ and φ.

$$S_{reduced} = \int dt d^3\mathbf{x}\ \epsilon^2 \left[\frac{1}{2} h_{ij} O^{ij|kl} h_{kl} + \frac{1}{2}\zeta O^{\zeta}\zeta + \frac{1}{2}\varphi(-\partial_t^2 + \triangle - m^2)\varphi\right]$$
$$+ \epsilon^3 \left[V_{h\varphi^2} + V_{\zeta\varphi^2} + \ldots\right] + \epsilon^4 \left[V_{\varphi^4} + V_{h^2\varphi^2} + V_{\zeta^2\varphi^2} + \ldots\right] + \mathcal{O}\left(\epsilon^5\right) \quad (4.38)$$

where $O^{ij|kl}$ and O^{ζ} denote complicated operators for the leading order kinetic terms for h_{ij} and ζ. There are two important three point vertices and three important four point vertices: the $h_{ij}\varphi^2$ vertex denoted by $V_{h\varphi^2}$, the $\zeta\varphi^2$ vertex denoted by $V_{\zeta\varphi^2}$, the φ^4 vertex denoted by V_{φ^4}, the $h_{ij}h_{kl}\varphi^2$ vertex denoted by $V_{h^2\varphi^2}$, and the $\zeta^2\varphi^2$ vertex denoted by $V_{\zeta^2\varphi^2}$. Again, the "…" correspond to terms that will play no role in the 1-loop correction to the scalar propagator, namely pure gravity vertices[7].

Feynman Rules

The precise form of these operators is best expressed in terms of the corresponding Feynman rules. Working in Fourier space with four-momentum k^μ split into energy $\omega_{\mathbf{k}}$ and three-momentum $\mathbf{k}$, we have the following tree-level propagators, as shown in Figure 4.4:

$$i\tilde{\triangle}^{\varphi}(k) = \frac{1}{-\omega_{\mathbf{k}}^2 + |\mathbf{k}|^2 + m^2} \quad (4.39a)$$

$$i\tilde{\triangle}^{h}_{ij|kl}(k) = \frac{\frac{1}{2}\Pi^{\mathbf{k}}_{ij|kl}}{-\omega_{\mathbf{k}}^2 + |\mathbf{k}|^2 + \frac{A_1}{M_{pl}^2}|\mathbf{k}|^4 + \frac{B_1}{M_{pl}^4}|\mathbf{k}|^6} =: \frac{1}{2}\Pi^{\mathbf{k}}_{ij|kl} i\tilde{\triangle}^h(k) \quad (4.39b)$$

$$-i\left(\tilde{\triangle}^{\zeta}(k)\right)^{-1} = \frac{2}{\alpha + \frac{A_4}{M_{pl}^2}|\mathbf{k}|^2 + \frac{B_4}{M_{pl}^4}|\mathbf{k}|^4}\left[(2-\alpha)|\mathbf{k}|^2 - (A_4 + 4A_3)\frac{|\mathbf{k}|^4}{M_{pl}^2}\right.$$
$$\left. + \left(4B_3 - B_4 + 2A_3^2\right)\frac{|\mathbf{k}|^6}{M_{pl}^4} - 4A_3B_3\frac{|\mathbf{k}|^8}{M_{pl}^6} + 2B_3^2\frac{|\mathbf{k}|^{10}}{M_{pl}^8}\right] \quad (4.39c)$$
$$-\frac{3\lambda - 1}{\lambda - 1}\omega_{\mathbf{k}}^2,$$

where the indexless $\tilde{\triangle}^h$ has been introduced to allow us to separate the projection operator and the Green's function. The relevant vertices are shown in Figure 4.5. Our convention is that all momenta point *into* the vertex. The detailed form for each vertex is presented in Appendix A.3.

4.4.3 One-loop corrections to a scalar field propagator

We are now ready to compute the one-loop correction to the scalar propagator. To this end, the relevant 1PI graphs are shown in Figure 4.6. As usual, the

[7] In addition, the $\varphi^2\zeta h_{ij}$ vertex is not pure gravity but cannot contribute at one loop, as the

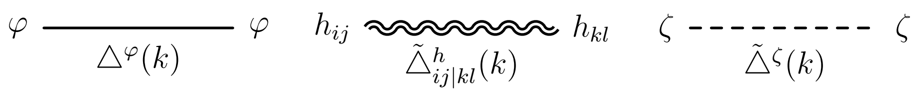


Figure 4.4: Propagators for the dynamical fields

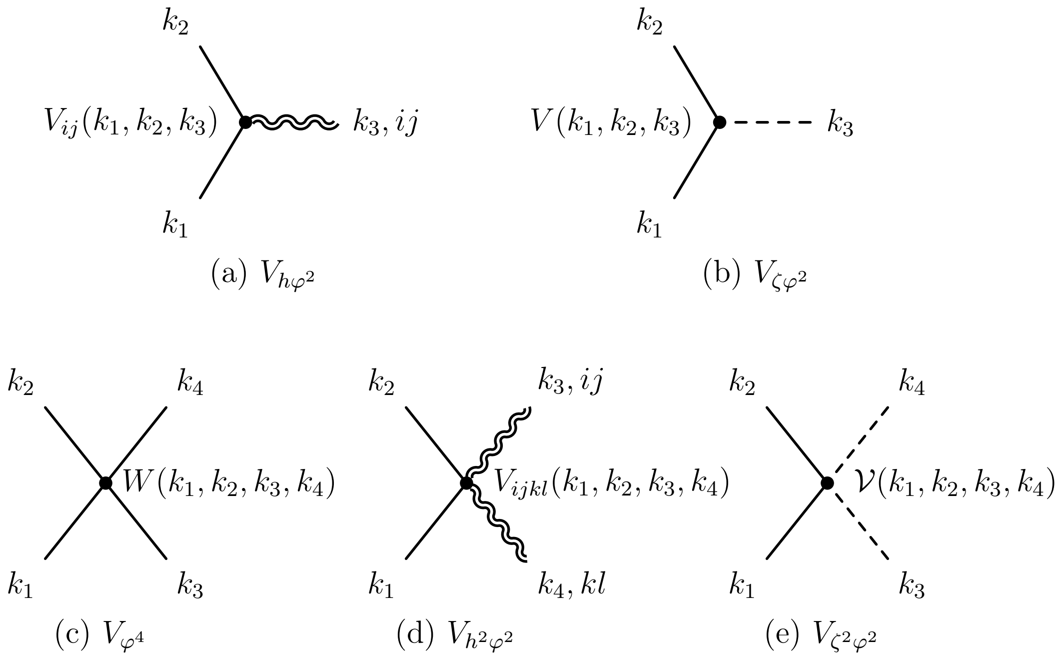


Figure 4.5: Three and four-point vertices for the dynamical fields. The precise form of these is presented in Appendix A.3.

renormalised two-point vertex for the scalar $\Gamma^{ren}_{\varphi\varphi} = \Gamma^{tree}_{\varphi\varphi} - \Sigma$, where $\Gamma^{tree}_{\varphi\varphi} = (\tilde{\Delta}^{\varphi})^{-1}$ is the tree-level vertex and Σ is the self energy (at one loop).

Let us now compute the contributions to the self energy for each diagram. Our expressions will be given in terms of the integrations over internal momenta, although we will explicitly drop terms that will obviously vanish when this integration is performed (such as terms linear in $\omega_{\mathbf{p}}$).

We begin with the pure scalar bubblegum diagram shown in Figure 4.7. The appropriate contraction of the legs introduces a symmetry factor of two, so we

internal legs cannot contract all their indices appropriately.

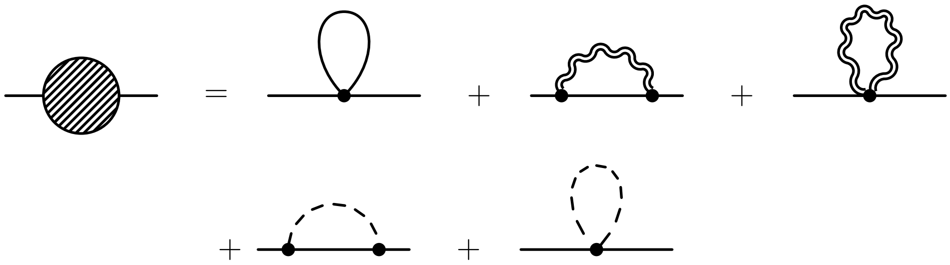


Figure 4.6: 1-Loop corrections to the scalar propagator

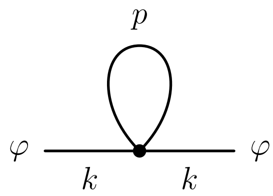


Figure 4.7: The pure scalar bubblegum diagram with a φ^4 vertex

find that the contribution to the self-energy is given by Σ_{φ^4}, where

$$\begin{aligned}
M_{pl}^2\Sigma_{\varphi^4} &= M_{pl}^2\int d\omega_{\mathbf{p}} d^3\mathbf{p}\tilde{\Delta}^{\varphi}(p)\frac{W(k,-k,p,-p)}{2}\\
&= \int d\omega_{\mathbf{p}} d^3\mathbf{p}\tilde{\Delta}^{\varphi}(p)\Bigg[-\frac{1}{2}\frac{\omega_{\mathbf{k}}^2\omega_{\mathbf{p}}^2+\left(\mathbf{k}\cdot\mathbf{p}+m^2\right)^2}{\alpha\left|\mathbf{p}+\mathbf{k}\right|^2+\frac{A_4}{M_{pl}^2}\left|\mathbf{p}+\mathbf{k}\right|^4+\frac{B_4}{M_{pl}^4}\left|\mathbf{p}+\mathbf{k}\right|^6}\\
&-\frac{\left(\omega_{\mathbf{p}}^2\mathbf{k}^2+\omega_{\mathbf{k}}^2\mathbf{p}^2\right)}{\left|\mathbf{k}+\mathbf{p}\right|^2}+\left(1-\frac{1}{2(1-\lambda)}\right)\frac{\left[(\mathbf{k}+\mathbf{p})\cdot\mathbf{k}\right]^2\omega_{\mathbf{p}}^2+\left[(\mathbf{k}+\mathbf{p})\cdot\mathbf{p}\right]^2\omega_{\mathbf{k}}^2}{\left|\mathbf{k}+\mathbf{p}\right|^4}-\frac{\mu}{2}M_{pl}^2\Bigg].
\end{aligned} \tag{4.40}$$

Note that since we are computing a bubblegum diagram we have taken care to neglect the 'tadpole-like' contributions as discussed in Section 4.4.1.

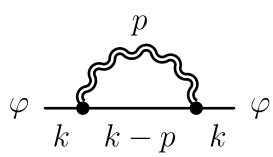


Figure 4.8: Diagram containing two $h_{ij}\varphi^2$ vertices.

Next we consider the diagram containing $h_{ij}\varphi^2$ vertices, shown in Figure 4.8. As this is not a bubblegum diagram we don't need to worry about tadpole effects. Taking into account the symmetries we find that the contribution to the self energy is $\Sigma_{h\varphi^2}$, where

$$M_{pl}^2\Sigma_{h\varphi^2} = M_{pl}^2 \int d\omega_{\mathbf{p}} d^3\mathbf{p} V_{ij}(k, p-k, -p) V_{kl}(-k, k-p, p) \tilde{\Delta}^{\varphi}(k-p) \tilde{\Delta}^{h}_{ijkl}(p)$$
$$= \int d\omega_{\mathbf{p}} d^3\mathbf{p} \left[-\frac{1}{2} \Pi^{\mathbf{p}}_{ij|kl} \mathbf{k}_i \mathbf{k}_j \mathbf{k}_k \mathbf{k}_l \tilde{\Delta}^{\varphi}(k-p) \tilde{\Delta}^{h}(p) \right]. \quad (4.41)$$

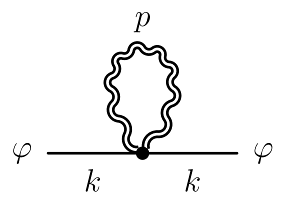


Figure 4.9: Bubblegum diagram with a tensor graviton in the loop and a $h_{ij}h_{kl}\varphi^2$ vertex.

Now we consider another bubblegum diagram, this time with the tensor graviton propagating around the loop, as shown in Figure 4.9. The diagram contains a $h_{ij}h_{kl}\varphi^2$ vertex and, given the symmetry factor of two, contributes a self-energy $\Sigma_{h^2\varphi^2}$ where

$$\begin{aligned} M_{pl}^2\Sigma_{h^2\varphi^2} =& \frac{1}{2} M_{pl}^2 \int d\omega_{\mathbf{p}} d^3\mathbf{p} V_{ijkl}(k, -k; p, -p) \cdot \tilde{\Delta}^{h}_{ijkl}(p) \\ =& \int d\omega_{\mathbf{p}} d^3\mathbf{p} \tilde{\Delta}^{h}(p) \left[\frac{1}{2}\left(-\omega_{\mathbf{k}}^2 + |\mathbf{k}|^2 + m^2\right) - \pi^{\mathbf{p}}_{ij} \mathbf{k}_i \mathbf{k}_j \right]. \end{aligned} \quad (4.42)$$

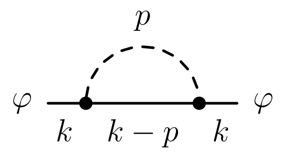


Figure 4.10: Diagram containing two $\zeta\varphi^2$ vertices

The diagram with the $\zeta\varphi^2$ vertices is shown in Figure 4.10. With the appropriate symmetry factors this gives a self-energy contribution $\Sigma_{\zeta\varphi^2}$ where

$$\begin{aligned} M_{pl}^2\Sigma_{\zeta\varphi^2} =& M_{pl}^2 \int d\omega_{\mathbf{p}} d^3\mathbf{p} V(k, p-k, -p) V(-k, k-p, p) \tilde{\Delta}^{\varphi}(k-p) \tilde{\Delta}^{\zeta}(p) \\ =& \int d\omega_{\mathbf{p}} d^3\mathbf{p} \tilde{\Delta}^{\varphi}(k-p) \tilde{\Delta}^{\zeta}(p) \Bigg[(3 + 2f(\mathbf{p}))\, \omega_{\mathbf{k}} (\omega_{\mathbf{k}} - \omega_{\mathbf{p}}) - (1 - 2f(\mathbf{p}))\, \mathbf{k} \cdot (\mathbf{k} - \mathbf{p}) \\ & - (3 - 2f(\mathbf{p}))\, m^2 + \frac{1-3\lambda}{1-\lambda} \left[\frac{\omega_{\mathbf{p}}^2}{|\mathbf{p}|^2} \mathbf{p} \cdot \mathbf{k} + \frac{\omega_{\mathbf{p}}\omega_{\mathbf{k}}}{|\mathbf{p}|^2} \left(|\mathbf{p}|^2 - 2\mathbf{p} \cdot \mathbf{k} \right) \right] \Bigg]^2 . \end{aligned}$$
$$(4.43)$$

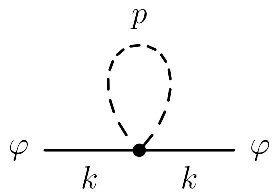


Figure 4.11: Bubblegum diagram with a scalar graviton in the loop and a $\zeta^2\varphi^2$ vertex

Finally, we consider a third bubblegum diagram, shown in Figure 4.11. This has the scalar graviton running through the loop with a $\zeta^2\varphi^2$ vertex. Taking care to neglect 'tadpole-like' contributions, we find that the contribution to the self-energy is given by $\Sigma_{\zeta^2\varphi^2}$ where

$$\begin{aligned} M_{pl}^2\Sigma_{\zeta^2\varphi^2} =& M_{pl}^2 \int d\omega_{\mathbf{p}} d^3\mathbf{p} \Delta^\zeta(p) \mathcal{V}(k,-k;p,-p) \\ =& \int d\omega_{\mathbf{p}} d^3\mathbf{p} \Delta^\zeta(p) \left[\frac{1}{2}\left(3+8f(\mathbf{p})^2\right)\omega_{\mathbf{k}}^2 + \frac{1}{2}\left[1-8f(\mathbf{p})\right]|\mathbf{k}|^2 \right. \\ & \left. - 2\left(\frac{1-3\lambda}{1-\lambda}\right)^2 \omega_{\mathbf{p}}^2 \frac{(\mathbf{k}\cdot\mathbf{p})^2}{|\mathbf{p}|^4} + \frac{3}{2}m^2 \right]. \end{aligned} \tag{4.44}$$

We cannot hope to solve these integrals exactly, but we can get a handle on their schematic properties by making some approximations. We will examine the leading order behaviour at low spatial momentum $k \lesssim M_*$ and assume for simplicity that the scalar potential vanishes ($m = \mu = 0$) and that $|\alpha| \sim |1-\lambda| \ll 1$. In each case we Wick rotate to Euclidean signature, and perform the integration over $\omega_{\mathbf{p}}$ followed by the integration over $\mathbf{p}$. For the latter, we approximate $|\mathbf{k} \pm \mathbf{p}|^2 \approx |\mathbf{k}|^2 + |\mathbf{p}|^2$, so that we can integrate out the angular components. We also split the integration over $|\mathbf{p}|$ into different régimes, approximating the integrand accordingly. This will hopefully be evident from the example we will work through shortly. Before doing so, however, let us quote some useful integral formulae, in particular [154, 155]

$$I_n = \int_0^\infty dz \frac{z^n}{z^2+A^2} = \frac{A^{n-1}}{2}\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right). \tag{4.45}$$

For even integer values of $n = 2N$, this integral gives

$$I_{2N} = \frac{(-1)^N A^{2N-1}\pi}{2}, \tag{4.46}$$

whereas for odd integer values $n = 2N+1$ it is divergent. We can regulate the divergence using dimensional regularisation, such that

$$I_{2N+1} = \lim_{\epsilon\to 0}\int_0^\infty \frac{d^{1+\epsilon}z}{\mu^\epsilon}\frac{z^{2N+1}}{z^2+A^2} = (-1)^N A^{2N}\left[-\frac{2}{\epsilon} + \ln\left(\frac{\mu^2}{4\pi A^2}\right) - \gamma\right], \tag{4.47}$$

where γ is the Euler-Mascheroni constant and μ is the renormalisation scale. We will also make use of the following integral which is finite for integer values of N

$$\int_0^\infty dz \frac{z^{2N}}{(z^2+A^2)(z^2+B^2)} = \frac{(-1)^N \pi}{2}\left[\frac{A^{2N-1}-B^{2N-1}}{A^2-B^2}\right]. \tag{4.48}$$

Let us now work through the simplest example to illustrate our methods. Consider $\Sigma_{h^2\varphi^2}$ as given by the integral expression (4.42). Schematically, we write this as

$$\Sigma_{h^2\varphi^2} \approx \frac{1}{M_{pl}^2}\left[\#\omega_{\mathbf{k}}^2 \int d\bar{\omega}_{\mathbf{p}} d^3\mathbf{p} \frac{1}{\bar{\omega}_{\mathbf{p}}^2+|\mathbf{p}|^2 X(|\mathbf{p}|)} + \#|\mathbf{k}|^2 \int d\bar{\omega}_{\mathbf{p}} d^3\mathbf{p} \frac{1}{\bar{\omega}_{\mathbf{p}}^2+|\mathbf{p}|^2 X(|\mathbf{p}|)}\right] \tag{4.49}$$

where $\#$ denotes (not necessarily equal) numbers of order one, and $X(z) = \# + \#\frac{z^2}{M_{pl}^2} + \#\frac{z^4}{M_h^4}$ with $M_h \sim M_{pl}\alpha^{1/4}$ being the scale of Lorentz violation in the tensor sector [117]. Here we are obviously being sloppy with tensor structure and have used the fact that, upon Wick rotating the energy, $\omega_{\mathbf{p}} \to -i\bar{\omega}_{\mathbf{p}}$, we have $\tilde{\Delta}^h(p) = \frac{\#}{\bar{\omega}_{\mathbf{p}}^2+|\mathbf{p}|^2 X(|\mathbf{p}|)}$. We begin by using equation (4.46) to do the integration over $\omega_{\mathbf{p}}$, and then do the angular integration yielding

$$\Sigma_{h^2\varphi^2} \approx \frac{1}{M_{pl}^2}\left(\#\omega_{\mathbf{k}}^2 \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|}{\sqrt{X(|\mathbf{p}|)}} + \#|\mathbf{k}|^2 \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|}{\sqrt{X(|\mathbf{p}|)}}\right) \tag{4.50}$$

Now for $|\mathbf{p}| \ll M_h$, we have $X \sim \#$, whereas for $|\mathbf{p}| \gg M_h$ we have $X \sim \#|\mathbf{p}|^4/M_h^4$. Thus we split this integral up into two domains and approximate it as follows

$$\int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|}{\sqrt{X(|\mathbf{p}|)}} \approx \# \int_0^{M_h} d|\mathbf{p}||\mathbf{p}| + \# \int_{M_h}^\infty d|\mathbf{p}| \frac{M_h^2}{|\mathbf{p}|} \tag{4.51}$$

Note that $\int_{M_h}^\infty d|\mathbf{p}| \frac{M_h^2}{|\mathbf{p}|} \approx M_h^2 \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|}{|\mathbf{p}|^2+M_h^2} - \int_0^{M_h} d|\mathbf{p}||\mathbf{p}|$, and so using the formula (4.47) for $N=0$, we obtain

$$\Sigma_{h^2\varphi^2} \approx \frac{M_h^2}{M_{pl}^2}\left(\frac{\#}{\epsilon} + \#\ln\frac{\mu^2}{M_h^2} + \#\right)\left(\#\omega_{\mathbf{k}}^2 + \#|\mathbf{k}|^2\right). \tag{4.52}$$

This reveals a logarithmic divergence and finite pieces that simply renormalise the constant part of the light cone, but by an amount that is suppressed by a factor of $\sqrt{\alpha} = \frac{M_h^2}{M_{pl}^2}$.

Using similar techniques, we arrive at the following approximations for the other contributions to the self-energy

$$\Sigma_{\varphi^4} \approx \left(\frac{\#}{\epsilon} + \#\ln\frac{\mu^2}{M_*^2} + \# + \#\frac{|\mathbf{k}|^2}{M_*^2}\right)\left(\#\omega_{\mathbf{k}}^2 + \#|\mathbf{k}|^2\right) \tag{4.53a}$$

$$\Sigma_{h\varphi^2} \approx \left[\# + \#\ln\frac{|\mathbf{k}|^2}{M_h^2}\right]\frac{|\mathbf{k}|^4}{M_{pl}^2} \tag{4.53b}$$

$$\begin{aligned}\Sigma_{\zeta\varphi^2} \approx& \frac{1}{M_*^2}\left(\# + \#\ln\frac{|\mathbf{k}|^2}{M_*^2}\right)\left(\#\omega_{\mathbf{k}}^4 + \#\omega_{\mathbf{k}}^2|\mathbf{k}|^2 + \#|\mathbf{k}|^4\right)\\ &+ \frac{1}{\alpha}\left(\frac{\#}{\epsilon} + \#\ln\frac{\mu^2}{M_{pl}^2} + \#\right)\left(\omega_{\mathbf{k}}^2 + \#|\mathbf{k}|^2\right)\end{aligned} \tag{4.53c}$$

$$\begin{aligned}\Sigma_{\zeta^2\varphi^2} \approx& \alpha\left(\frac{\#}{\epsilon} + \#\ln\frac{\mu^2}{M_h^2} + \frac{\#}{\alpha}\right)\omega_{\mathbf{k}}^2\\ &+ (1+\#\alpha)\left(\frac{\#}{\epsilon} + \#\ln\frac{\mu^2}{M_h^2} + \#\right)|\mathbf{k}|^2\end{aligned} \tag{4.53d}$$

where we have set $\omega_{\mathbf{k}} = 0$ in the denominator of the integrands for $\Sigma_{h\varphi^2}$ and $\Sigma_{\zeta\varphi^2}$, corresponding to equations (4.41) and (4.43) respectively.

The first thing to note is that we have at most logarithmic divergences on account of the fact that we have used dimensional regularisation. Focussing on the finite terms it is clear that we generate terms of the form

$$\frac{1}{\alpha}\varphi\left[a_0 + a_1\frac{\Delta}{M_{pl}^2} + a_2\frac{\partial_t^2}{M_{pl}^2} + + \ldots\right]\ddot{\varphi}, \qquad \frac{1}{\alpha}\varphi\left[b_0 + b_1\frac{\Delta}{M_{pl}^2} + \ldots\right]\Delta\varphi, \tag{4.54}$$

where we have neglected the contribution from the $\ln\frac{|\mathbf{k}|^2}{M_*^2}$ terms as they are not expected to be important when we properly take into account infra-red corrections arising from a non-trivial potential (*i.e.* $m \neq 0, \mu \neq 0$). There are a number of important features to dwell upon. The first is the potentially large leading order correction to the light cone, of order $\delta c^2 \sim (a_0 - b_0)/\alpha \gtrsim 10^7$. This large factor is a direct result of the strong coupling between matter and the scalar graviton and suggests an unpalatable amount of fine tuning of the light cone for different particle species. Of course, the effect may be reproduced in exactly equal measure for all particles in which case there is nothing to worry about. It is beyond the scope of this work to establish whether or not such an optimistic scenario occurs. This point is discussed further in Section 4.6.

Beyond the leading order terms, we have higher derivatives with an additional Planckian suppression. This is the relevant scale because the scalar graviton propagator, $\tilde{\Delta}^\zeta$ only feels the $z = 3$ scaling at beyond the Planck scale[8]. Higher spatial derivatives were anticipated in Section 4.3, and may have been expected from the quadratic divergences that appeared in [100]. Because they were seen to remove

[8]From equation (4.39c) we see that the scalar graviton propagator behaves roughly as $\tilde{\Delta}^\zeta(k) \sim$

these divergences, it has been suggested [152] that the inclusion of terms such as $(D_i K_{jk})^2$ will help suppress these operators in the UV, beyond the scales M_* and M_h. However, our integrals are evaluated for low momenta $k < M_*$ so we do not probe the very high energy corrections in this work.

In contrast, we did not anticipate the terms $\frac{1}{M_{pl}^2\alpha}\varphi\Delta\ddot{\varphi}$ and $\frac{1}{M_{pl}^2\alpha}\varphi\partial_t^4\varphi$ in Section 4.3, even though they are compatible with the $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ symmetry. This is because we did not endeavour to generalise to terms involving temporal derivatives, consistent with the original formulation of the gravitational action. However, we now see that such terms are generated by loop corrections, and that they alter the temporal part of the propagator in the UV. This is dangerous and will generically lead to ghosts. Indeed, the fourth-order time derivative can be identified with a new degree of freedom corresponding to an Ostrogradski ghost [67].

Let us consider this fourth order time derivative more closely. It stems from the $\Sigma_{\zeta\varphi^2}$ contribution to the self-energy, and in particular the piece proportional to $\omega_{\mathbf{k}}^4$,

$$\Sigma_{\zeta\varphi^2} \supset \frac{i}{2M_{pl}^2}\bar{\omega}_{\mathbf{k}}^4 \int d\bar{\omega}_{\mathbf{p}} d^3\mathbf{p} \tilde{\Delta}^{\varphi}(k-p)\tilde{\Delta}^{\zeta}(p)\left(3+2f(\mathbf{p})\right)^2 \tag{4.55}$$

where the $\bar{\omega}$ indicates explicitly that we have performed a Wick rotation $\omega \to -i\bar{\omega}$ on all internal and external energies. In our rough evaluation of this integral, we set $\bar{\omega}_{\mathbf{k}} = 0$ inside the scalar part of the loop. One might worry that this eliminates an important correction, so let us see what happens when we leave it in. The *Wick rotated* propagators have the approximate form

$$\tilde{\Delta}^{\varphi}(k-p) \sim \frac{1}{(\bar{\omega}_{\mathbf{k}}-\bar{\omega}_{\mathbf{p}})^2 + |\mathbf{k}-\mathbf{p}|^2 + m^2}, \qquad \tilde{\Delta}^{\zeta}(p) \sim \frac{\alpha}{\bar{\omega}_{\mathbf{p}}^2 + c^2(|\mathbf{p}|)|\mathbf{p}|^2} \tag{4.56}$$

where $c(|\mathbf{p}|)$ is given in footnote 8. Using the Feynman trick, then integrating over $\bar{\omega}_{\mathbf{p}}$ we obtain,

$$\Sigma_{\zeta\varphi^2} \supset \frac{\pi i}{4M_{pl}^2}\alpha\bar{\omega}_{\mathbf{k}}^4 \int_0^1 dx \int d^3\mathbf{p} \frac{\left(3+2f(\mathbf{p})\right)^2}{[x(|\mathbf{k}-\mathbf{p}|^2+m^2) + (1-x)c^2(|\mathbf{p}|)|\mathbf{p}|^2 + x(1-x)\bar{\omega}_{\mathbf{k}}^2]^{3/2}}. \tag{4.57}$$

Since we are interested in the role of higher-order time derivatives, we may as well set the external 3 momentum to vanish, $\mathbf{k} = 0$. Now performing the integration

---

$\frac{\alpha}{\omega_{\mathbf{k}}^2 - c^2(|\mathbf{k}|)|\mathbf{k}|^2}$ where

$$c(|\mathbf{k}|) \sim \begin{cases} 1 & |\mathbf{k}| < M_* \\ M_*^2/|\mathbf{k}|^2 & M_* < |\mathbf{k}| < M_h \\ |\mathbf{k}|^2/M_{pl}^2 & |\mathbf{k}| > M_h \end{cases}$$

.

over x and then the angles, we obtain,

$$\Sigma_{\zeta\varphi^2} \supset \frac{2\pi^2 i}{M_{pl}^2}\alpha\bar{\omega}_{\mathbf{k}}^4 \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|\,(3+2f(|\mathbf{p}|))^2\,(\sqrt{|\mathbf{p}|^2+m^2}+|c(|\mathbf{p}|)||\mathbf{p}|)}{|c(|\mathbf{p}|)|\sqrt{|\mathbf{p}|^2+m^2}[(\sqrt{|\mathbf{p}|^2+m^2}+|c(|\mathbf{p}|)||\mathbf{p}|)^2+\bar{\omega}_{\mathbf{k}}^2]}$$
$$= \frac{2\pi i}{M_{pl}^2}\alpha\bar{\omega}_{\mathbf{k}}^4 \sum_{n=0}^{\infty} \frac{\bar{\omega}_{\mathbf{k}}^{2n}}{n!}\mathcal{I}_n, \tag{4.58}$$

where in the last line we have performed a Taylor expansion about $\bar{\omega}_{\mathbf{k}}^2 = 0$, with

$$\mathcal{I}_n = (-1)^n \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|\,(3+2f(|\mathbf{p}|))^2}{|c(|\mathbf{p}|)|\sqrt{|\mathbf{p}|^2+m^2}(\sqrt{|\mathbf{p}|^2+m^2}+|c(|\mathbf{p}|)||\mathbf{p}|)^{2n+1}}. \tag{4.59}$$

Now the crucial point is that, generically, each of the $\mathcal{I}_n$ is finite so the Taylor expansion is valid in some neighbourhood of $\bar{\omega}_{\mathbf{k}}^2 = 0$. This suggests that the higher-order time derivatives are a real phenomena and not some artifact of our rough approximations[9]. We will discuss the pathological implications of these higher-order time derivatives and how they may be avoided in more detail in Section 4.6.

4.5 Quantum corrected gauge fields

The Standard Model of particle physics contains three gauge fields, twelve fermions and one scalar field. The existence of the vectors and fermions has long been established, but many years passed with no fundamental scalar fields being discovered. In July 2012 a new particle, thought to be the Higgs particle, was discovered at the LHC [156, 157]. This particle is a boson and most likely spin-0 [158], but further measurements are required to confirm its spin. It remains possible that there exist no fundamental scalars in nature, in which case: what can be made of the analysis in Section 4.4? Even if the particle is confirmed as a scalar particle, the data on its properties are nowhere near as precise as the well known vector and fermion fields. This motivates us to consider a gauge field[10], specifically the $U(1)$ of electromagnetism, coupled to Hořava gravity, and investigate the resultant effects of the one-loop corrections to the propagator, with greater certainty of the physical relevance.

[9] This is basically saying that the expansion of the integral about $\bar{\omega}_{\mathbf{k}}^2 = 0$ does not contain negative powers of $\bar{\omega}_{\mathbf{k}}^2$ that cancel off the overall factor of $\bar{\omega}_{\mathbf{k}}^4$.

[10] We do not consider fermions due to the extra complications in the analysis — one cannot simply minimally couple the flat space action, but instead is required to introduce vierbeins.

4.5.1 Reduced action for a gauge field

Our tree level matter action is now

$$S = S_{grav} + S_A^{tree}, \tag{4.60}$$

where

$$S_A^{tree} = -\frac{1}{4}\int dt d^3\mathbf{x}\sqrt{-g}F_{\mu\nu}F^{\mu\nu}. \tag{4.61}$$

Again, we will integrate out the constraints to work directly with the dynamical degrees of freedom. This will also be necessary for the matter field in this case since it carries a gauge symmetry.

The Hamiltonian and momentum constraints along with that from the vector field component A_0 are

$$\begin{aligned}
\mathcal{C}_N = \frac{\delta S}{\delta N} &= M_{pl}^2\sqrt{\gamma}\left[-K_{ij}K^{ij} + \lambda K^2 + R^{(3)} - \alpha a^i a_i - 2\alpha D_i a^i\right] \\
&+ \sqrt{\gamma}\left[A_1(R_{ij}^{(3)})^2 + A_2(R^{(3)})^2 + A_3\left(R^{(3)}D_i a^i + \frac{1}{N}\triangle(NR^{(3)})\right)\right. \\
&\qquad\left. + A_4\left((D_i a^i)^2 + \frac{2}{N}\triangle(ND_i a^i)\right)\right] \\
&+ \frac{\sqrt{\gamma}}{M_{pl}^2}\left[B_1(D_i R_{jk}^{(3)})^2 + B_2(D_i R^{(3)})^2 + B_3\left(\triangle R^{(3)}D_i a^i + \frac{1}{N}\triangle(N\triangle R^{(3)})\right)\right. \\
&\qquad\left. + B_4\left(D_i a^i\triangle D_j a^j + \frac{1}{N}\triangle(N\triangle D_i a^i) + \frac{1}{N}\triangle^2(ND_i a^i)\right)\right] \\
&- \frac{\sqrt{\gamma}}{4}\left[\frac{2}{N^2}\gamma^{ij}(F_{0i} - F_{ki}N^k)(F_{0j} - F_{lj}N^l) + F_{ij}F^{ij}\right] \qquad (4.62a) \\
\mathcal{C}_i = \frac{\delta S}{\delta N_i} &= 2M_{pl}^2\sqrt{\gamma}\left(D_j K^{ij} - \lambda D^i K\right) - \frac{\sqrt{\gamma}}{N}F^{ij}\left(F_{0j} - F_{kj}N^k\right) \qquad (4.62b) \\
\mathcal{C}_A = \frac{\delta S}{\delta A_0} &= \partial_j\left(\frac{\sqrt{\gamma}}{N}\gamma^{ij}\left(F_{0i} - F_{ki}N^k\right)\right) \qquad (4.62c)
\end{aligned}$$

We perform an expansion about Minkowski, given by (4.28). For the vector field, we work in the gauge $A_\mu \to \epsilon A_\mu$. We work in Coulomb gauge, so our full set of gauge conditions is

$$V_i = 0 \qquad E = 0 \qquad \partial_i A^i = 0, \tag{4.63}$$

meaning A_i is purely transverse. This will reduce our expansions of γ_{ij} to the physical scalar and tensor, and of A_i to the physical vector. The pieces corresponding to N, N_i and A_0 will be eliminated when we integrate out the constraints (4.62). Expanding the action (4.60) order by order in ϵ, we find that

$$S = \epsilon^2 S^{(2)} + \frac{\epsilon^3}{M_{pl}}S^{(3)} + \frac{\epsilon^4}{M_{pl}^2}S^{(4)} + \mathcal{O}\left(\epsilon^5\right), \tag{4.64}$$

where

$$S^{(2)} = \int dtd^3\mathbf{x}\Bigg[\frac{1}{2}A_i\left(-\partial_t^2+\triangle\right)A_i + \frac{1}{4}h_{ij}\left(-\partial_t^2+\triangle+\frac{A_1}{M_{pl}^2}\triangle^2-\frac{B_1}{M_{pl}^4}\triangle^3\right)h_{ij}$$
$$+M_{pl}^2 n\left(\alpha\triangle-\frac{A_4}{M_{pl}^2}\triangle^2+\frac{B_4}{M_{pl}^4}\triangle^3\right)n - M_{pl}^2(1-\lambda)\beta\triangle^2\beta+\frac{1}{2}M_{pl}^2 S_i\triangle S_i$$
$$+3(1-3\lambda)\dot{\zeta}^2-2\zeta\triangle\zeta+\frac{(6A_1+16A_2)}{M_{pl}^2}\zeta\triangle^2\zeta-\frac{(6B_1+16B_2)}{M_{pl}^2}\zeta\triangle^3\zeta$$
$$-\frac{1}{2}(\partial_i A_0)^2+nC_N^{(1)}+n_iC_i^{(1)}+A_0C_A^{(1)}\Bigg], \tag{4.65}$$

$$S^{(3)} = \int dtd^3\mathbf{x}\Bigg\{\frac{1}{2}\zeta F_{0i}^2-\frac{1}{2}h^{ij}F_{0i}F_{0j}+\frac{1}{4}\zeta F_{ij}^2+\frac{1}{2}h^{jk}F_{ij}F_{ik}$$
$$-\alpha M_{pl}^2\left[\zeta\partial_i n\partial_i n-2M_{pl}n\partial_i n\partial_i n\right]-4A_3\triangle\zeta\partial_i n\partial_i n-4\frac{B_3}{M_{pl}^2}\triangle^2\zeta\partial_i n\partial_i n$$
$$-A_4\left[2\partial_i\zeta\partial_i\zeta\triangle n-\zeta(\triangle n)^2-4M_{pl}\triangle n\partial_i n\partial_i n-2M_{pl}n(\triangle n)^2\right]$$
$$-\frac{B_4}{M_{pl}^2}\big[-3\zeta\triangle n\triangle^2 n+\partial_i\zeta\partial_i n\triangle^2 n-2\partial_i\zeta\triangle n\partial_i\triangle n-2\triangle\zeta(\triangle n)^2$$
$$+2\partial_i\partial_j\zeta\partial_i\partial_j n\triangle n-2M_{pl}n\triangle n\triangle^2 n-2M_{pl}\partial_i n\partial_i n\triangle^2 n$$
$$-8M_{pl}\partial_i n\triangle n\partial_i\triangle n-2M_{pl}(\triangle n)^3-4M_{pl}(\partial_i\partial_j n)^2\triangle n\big]$$
$$+2M_{pl}^3 n\left(\partial_i\partial_j\beta\right)^2-M_{pl}^3 2\lambda n(\triangle\beta)^2+2M_{pl}^3 n\partial_{(i}S_{j)}\partial_i S_j$$
$$-2M_{pl}^2(1-3\lambda)\dot{\zeta}n\triangle\beta+4M_{pl}^3 n\partial_i\partial_j\beta\partial_i S_j+M_{pl}^2\zeta(\partial_i\partial_j\beta)^2$$
$$-M_{pl}^2\lambda\zeta(\triangle\beta)^2+M_{pl}^2\zeta\partial_{(i}S_{j)}\partial_i S_j+2M_{pl}^2\zeta\partial_i\partial_j\beta\partial_i S_j$$
$$+2M_{pl}^2\partial_i\zeta\partial_j\beta\partial_i\partial_j\beta+2M_{pl}^2\partial_i\zeta S_j\partial_i\partial_j\beta-M_{pl}^2(1-\lambda)\partial_i\zeta\partial_i\beta\triangle\beta$$
$$-M_{pl}^2(1-\lambda)\partial_i\zeta S_i\triangle\beta+2M_{pl}^2\partial_i\zeta\partial_j\beta\partial_i S_j+2M_{pl}^2\partial_i\zeta S_j\partial_i S_j$$
$$+nC_N^{(2)}+n_iC_i^{(2)}+A_0C_A^{(2)}\Bigg\}+\ldots, \tag{4.66}$$

$$S^{(4)} = \int dtd^3\mathbf{x}\Bigg\{-\frac{1}{2}M_{pl}^2n^2F_{0i}^2-\frac{1}{4}\zeta^2F_{0i}^2+\frac{1}{2}\zeta h^{ij}F_{0i}F_{0j}-\frac{1}{8}h_{jk}^2F_{0i}^2$$
$$+\frac{1}{2}h^{ik}h_k^jF_{0i}F_{0j}-3M_{pl}^2n(\partial_j\beta+S_j)F_{ji}F_{0i}-\frac{1}{2}M_{pl}^2n_jn_kF_{ji}F_{ki}$$
$$-\frac{3}{8}\zeta^2F_{ij}^2-\zeta h^{jk}F_{ij}F_{ik}+\frac{1}{16}h_{kl}^2F_{ij}^2-\frac{1}{2}h^{jl}h_l^kF_{ij}F_{ik}$$
$$-\frac{1}{4}h^{ik}h^{jl}F_{ij}F_{kl}+nC_N^{(3)}+n_iC_i^{(3)}+A_0C_A^{(3)}\Bigg\}+\ldots, \tag{4.67}$$

and as before, "…" denote terms which are irrelevant to our subsequent calculations, and the gravitational constraints are expanded as $\mathcal{C}_N = \epsilon C_N^{(1)} + \epsilon^2 C_N^{(2)} + \epsilon^3 C_N^{(3)} + \mathcal{O}(\epsilon^4)$ and $\mathcal{C}_i = \epsilon C_i^{(1)} + \epsilon^2 C_i^{(2)} + \epsilon^3 C_i^{(3)} + \mathcal{O}(\epsilon^4)$. We expand the gauge field constraint as $\mathcal{C}_A = \epsilon C_A^{(1)} + \epsilon^2 C_A^{(2)} + \mathcal{O}(\epsilon^3)$, since we will not be sensitive to $\mathcal{O}(\epsilon^3)$ terms. Again, we do not need to consider interactions beyond fourth order since for 1-loop corrections to the propagator we will only encounter up to four point vertices.

We now integrate out the constraints by setting $C_N^{(1)} = -\epsilon C_N^{(2)} - \epsilon^2 C_N^{(3)} - \epsilon^3 C_N^{(4)} + \mathcal{O}(\epsilon^4)$, $C_i^{(1)} = -\epsilon C_i^{(2)} - \epsilon^2 C_i^{(3)} - \epsilon^3 C_i^{(4)} + \mathcal{O}(\epsilon^4)$ and $C_A^{(1)} = -\epsilon C_A^{(2)} - \epsilon^2 C_A^{(3)} + \mathcal{O}(\epsilon^3)$, or more specifically,

$$\begin{aligned}2\epsilon M_{pl}^2\left(-\alpha+\frac{A_4\triangle}{M_{pl}^2}-\frac{B_4\triangle^2}{M_{pl}^4}\right)\triangle n = 4\epsilon M_{pl}\left(1+\frac{A_3\triangle}{M_{pl}^2}+\frac{B_3\triangle^2}{M_{pl}^4}\right)\triangle\zeta \\ +\frac{\epsilon^2}{2}\left(F_{0i}F_0{}^i+\frac{1}{2}F_{ij}F^{ij}\right)+\epsilon^2 H_2+\epsilon^3 H_3 \\ -\epsilon^3\Big[(\zeta+n)F_{0i}F_0{}^i+F_{0i}F^{ji}n_j+\frac{1}{2}h^{ij}F_{0i}F_{0j} \\ +F^{ij}F_{ij}\zeta+\frac{1}{2}F_{ij}F^{kj}h^i_k\Big]+\mathcal{O}(\epsilon^4)\end{aligned} \tag{4.68a}$$

$$\begin{aligned}2\epsilon M_{pl}^2(1-\lambda)\triangle^2\beta =& 2\epsilon M_{pl}(1-3\lambda)\triangle\dot\zeta-\epsilon^2\partial^i\left(F_i{}^jF_{0j}\right)+\epsilon^2P_2 \\ &+\epsilon^3\partial^i\left(2\zeta F_i{}^jF_{0j}+h^{jk}F_{ik}F_{0j}+F_i{}^jF_{kj}n^k\right) \\ &+\epsilon^3P_3+\mathcal{O}(\epsilon^4)\end{aligned} \tag{4.68b}$$

$$\begin{aligned}\epsilon M_{pl}^2\triangle S_i =& -\epsilon^2\pi^{ik}\left(F_k{}^jF_{0j}\right)+\epsilon^2 Q_{2i} \\ &+\epsilon^3\pi_{ij}\left(2\zeta F_k{}^jF_{0j}+h^{jl}F_{kl}F_{0j}+F_k{}^jF_{lj}n^l\right) \\ &+\epsilon^3 Q_{3i}+\mathcal{O}(\epsilon^4)\end{aligned} \tag{4.68c}$$

$$\begin{aligned}\epsilon\triangle A_0 =& \epsilon^2\Big[\partial_i(\zeta-n)\dot A_i-h_{ij}\partial_j\dot A_i-\partial_i(n_jF_{ji}) \\ &-\partial_i(\zeta-n)\partial_iA_0+h_{ij}\partial_i\partial_jA_0\Big]+\mathcal{O}(\epsilon^3)\end{aligned} \tag{4.68d}$$

where H_q, P_q and Q_q are defined as in (4.37).

We now use the equations (4.68) to eliminate the non-dynamical field n, β, S_i, A_0 from the action, thereby arriving at the reduced action for the dynamical fields, h_{ij}, ζ and A_i.

$$\begin{aligned}S_{reduced} = \int dt d^3\mathbf{x}\,\epsilon^2\left[\frac{1}{2}h_{ij}O^{ij|kl}h_{kl}+\frac{1}{2}\zeta O^\zeta\zeta+\frac{1}{2}A_i(-\partial_t^2+\triangle)A_i\right] \\ +\epsilon^3\left[V_{hA^2}+V_{\zeta A^2}+\ldots\right]+\epsilon^4\left[V_{A^4}+V_{h^2A^2}+V_{\zeta^2A^2}+\ldots\right]+\mathcal{O}\left(\epsilon^5\right)\end{aligned} \tag{4.69}$$

where $O^{ij|kl}$ and O^ζ denote complicated operators for the leading order kinetic terms for h_{ij} and ζ. As before, there are two important three-point vertices and three important four-point vertices: the $h_{ij}A_kA_l$ vertex denoted by V_{hA^2}, the ζA_iA_j vertex denoted by $V_{\zeta A^2}$, the $A_iA_jA_kA_l$ vertex denoted by V_{A^4}, the $h_{ij}h_{kl}A_mA_n$ vertex denoted by $V_{h^2A^2}$, and the $\zeta^2A_iA_j$ vertex denoted by $V_{\zeta^2A^2}$. Note that integrating out introduces an A^4 vertex, which appears absent prior to integrating out. This also occurs in QED coupled to matter, if one works with just the dynamical degrees of freedom and is not problematic for the theory.

Feynman Rules

Working in Fourier space with four-momentum k^μ split into energy $\omega_{\mathbf{k}}$ and three-momentum $\mathbf{k}$, we have the following tree-level propagators, as shown in Figure 4.12:

$$i\tilde{\triangle}^A_{ij}(k) = \frac{\pi^{\mathbf{k}}_{ij}}{-\omega^2_{\mathbf{k}} + |\mathbf{k}|^2 + m^2} =: \pi^{\mathbf{k}}_{ij} i\tilde{\triangle}^A(k) \tag{4.70a}$$

$$i\tilde{\triangle}^h_{ij|kl}(k) = \frac{\frac{1}{2}\Pi^{\mathbf{k}}_{ij|kl}}{-\omega^2_{\mathbf{k}} + |\mathbf{k}|^2 + \frac{A_1}{M^2_{pl}}|\mathbf{k}|^4 + \frac{B_1}{M^4_{pl}}|\mathbf{k}|^6} =: \frac{1}{2}\Pi^{\mathbf{k}}_{ij|kl} i\tilde{\triangle}^h(k) \tag{4.70b}$$

$$\begin{aligned} -i\left(\tilde{\triangle}^\zeta(k)\right)^{-1} = \frac{2}{\alpha + \frac{A_4}{M^2_{pl}}|\mathbf{k}|^2 + \frac{B_4}{M^4_{pl}}|\mathbf{k}|^4} &\left[(2-\alpha)|\mathbf{k}|^2 - (A_4 + 4A_3)\frac{|\mathbf{k}|^4}{M^2_{pl}}\right. \\ &\left. + \left(4B_3 - B_4 + 2A_3^2\right)\frac{|\mathbf{k}|^6}{M^4_{pl}} - 4A_3B_3\frac{|\mathbf{k}|^8}{M^6_{pl}} + 2B_3^2\frac{|\mathbf{k}|^{10}}{M^8_{pl}}\right] \\ &- \frac{3\lambda - 1}{\lambda - 1}\omega^2_{\mathbf{k}}, \end{aligned} \tag{4.70c}$$

where the indexless $\tilde{\triangle}^A$ and $\tilde{\triangle}^h$ have been introduced to allow us to separate the projection operators and the Green's function. The relevant vertices are shown in

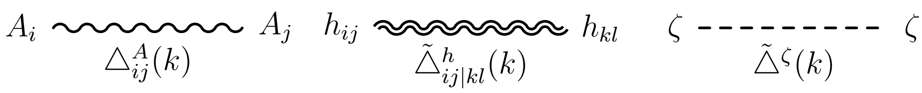


Figure 4.12: Propagators for the dynamical fields in the vector case

Figure 4.13. Recall our convention that all momenta point *into* the vertex.

4.5.2 One-loop corrections to a gauge field propagator

We are now ready to compute the one-loop correction to the vector propagator. To this end, the relevant 1PI graphs are shown in Figure 4.14. As usual, the renormalised two-point vertex for the vector $\Gamma^{ren}_{AA} = \Gamma^{tree}_{AA} - \Sigma$, where $\Gamma^{tree}_{AA} = (\tilde{\triangle}^A)^{-1}$ is the tree-level vertex and Σ is the self energy (at one loop). We have used the fact that the propagator will remain transverse (since we have maintained rotational invariance) to write $\Sigma_{ij} = \pi_{ij}\Sigma$, and reduce the number of indices to wade through.

Let us now compute the contributions to the self energy for each diagram. Our expressions will again be given in terms of the integrations over internal momenta, explicitly dropping terms that will obviously vanish when this integration is performed.

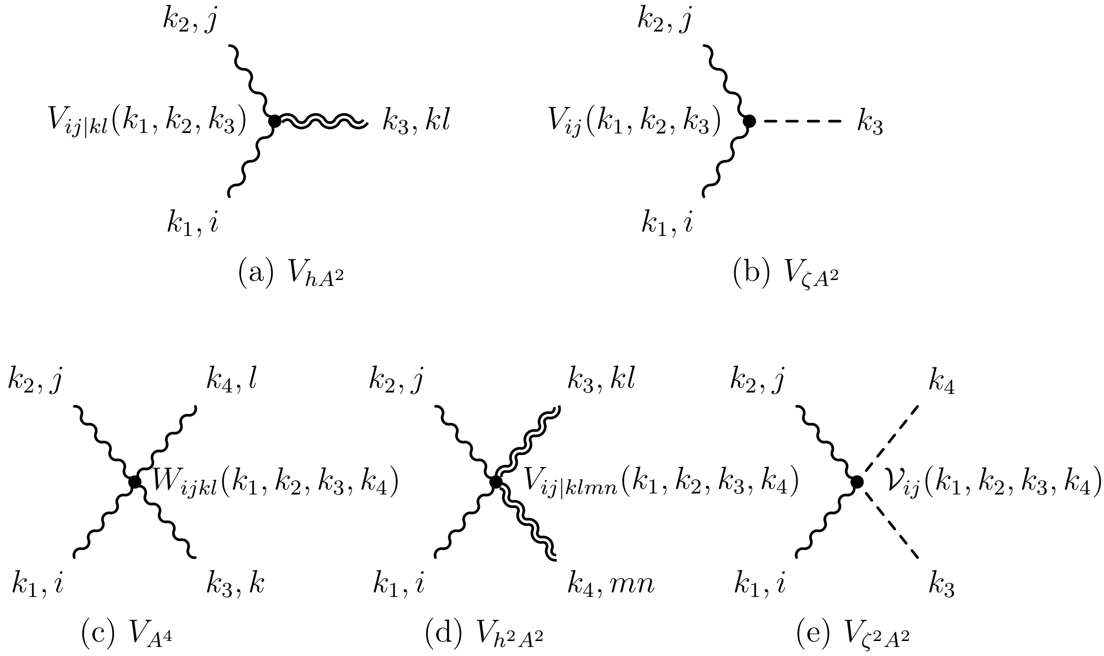


Figure 4.13: Three and four point vertices for the dynamical fields in the vector case.

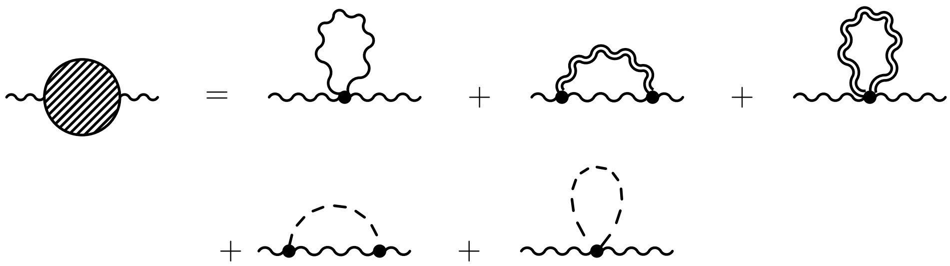


Figure 4.14: 1-Loop corrections to the vector propagator

Begin with the pure vector bubblegum diagram shown in Figure 4.15. The appropriate contraction of the legs introduces a symmetry factor of two. We find that

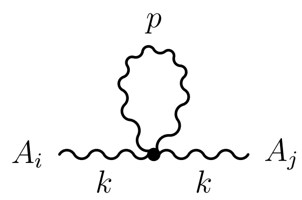


Figure 4.15: The pure vector bubblegum diagram with a A_i^4 vertex

the contribution to the self-energy is given by Σ_{A^4}, where

$$
\begin{aligned}
M_{pl}^2 \Sigma_{A^4} \pi_{ij}^{\mathbf{k}} &= M_{pl}^2 \int d\omega_{\mathbf{p}} d^3\mathbf{p} \tilde{\Delta}_{kl}^A(p) \frac{W_{ijkl}(k,-k,p,-p)}{2} \\
&= \int d\omega_{\mathbf{p}} d^3\mathbf{p} \pi_{im}^{\mathbf{k}} \pi_{jn}^{\mathbf{k}} \pi_{kl}^{\mathbf{p}} \tilde{\Delta}^A(k) \\
&\quad \left\{ -\frac{1}{2} \left[\frac{\omega_{\mathbf{k}}^2 \omega_{\mathbf{p}}^2 \delta_{mn}\delta_{kl} + (\mathbf{k}\cdot\mathbf{p}\delta_{mk} - \mathbf{p}_m\mathbf{k}_k)(\mathbf{k}\cdot\mathbf{p}\delta_{nl} - \mathbf{p}_n\mathbf{k}_l)}{\alpha|\mathbf{k}+\mathbf{p}|^2 + A_4\frac{|\mathbf{k}+\mathbf{p}|^4}{M_{pl}^2} + B_4\frac{|\mathbf{k}+\mathbf{p}|^6}{M_{pl}^4}} \right] \right. \\
&\qquad - \frac{\omega_{\mathbf{p}}^2 \left(\delta_{ml}\delta_{kn}|\mathbf{k}|^2 + \delta_{mn}\mathbf{k}_k\mathbf{k}_l\right) + \omega_{\mathbf{k}}^2 \left(\delta_{ml}\delta_{nl}|\mathbf{p}|^2 + \mathbf{p}_m\mathbf{p}_n\delta_{kl}\right)}{|\mathbf{k}+\mathbf{p}|^2} \\
&\qquad + \left(1 - \frac{1}{2(1-\lambda)}\right) \frac{\omega_{\mathbf{p}}^2 \left[\left(|\mathbf{k}|^2 + \mathbf{k}\cdot\mathbf{p}\right)\delta_{mk} - \mathbf{p}_m\mathbf{k}_k\right]\left[\left(|\mathbf{k}|^2 + \mathbf{k}\cdot\mathbf{p}\right)\delta_{nl} - \mathbf{p}_n\mathbf{k}_l\right]}{|\mathbf{k}+\mathbf{p}|^4} \\
&\qquad \left. + \left(1 - \frac{1}{2(1-\lambda)}\right) \frac{\omega_{\mathbf{k}}^2 \left[\left(|\mathbf{p}|^2 + \mathbf{p}\cdot\mathbf{k}\right)\delta_{mk} - \mathbf{p}_m\mathbf{k}_k\right]\left[\left(|\mathbf{p}|^2 + \mathbf{p}\cdot\mathbf{k}\right)\delta_{nl} - \mathbf{p}_n\mathbf{k}_l\right]}{|\mathbf{k}+\mathbf{p}|^4} \right\},
\end{aligned}
\tag{4.71}
$$

where we again take care to neglect the 'tadpole-like' contributions.

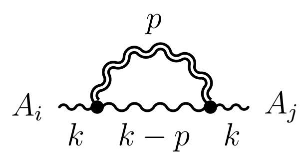


Figure 4.16: Diagram containing two $h_{ij}A_i^2$ vertices.

Next we consider the diagram containing $h_{ij}A_i^2$ vertices, shown in Figure 4.16. As this is not a bubblegum diagram we don't need to worry about tadpole effects. Taking into account the symmetries we find that the contribution to the self energy is $\Sigma_{h\varphi^2}$, where

$$\begin{aligned}
M_{pl}^2\Sigma_{hA^2}\pi^{\mathbf{k}}_{ij} =& M_{pl}^2\int d\omega_{\mathbf{p}}d^3\mathbf{p}V_{ik|mn}(k,p-k,-p)V_{jl|pq}(-k,k-p,p)\tilde{\triangle}^A_{kl}(k-p)\tilde{\triangle}^h_{mnpq}(p)\\
=& \int d\omega_{\mathbf{p}}d^3\mathbf{p}\pi^{\mathbf{k}}_{ir}\pi^{\mathbf{k}}_{js}\pi^{\mathbf{p}+\mathbf{k}}_{kl}\cdot\frac{1}{2}\Pi^{\mathbf{p}}_{mn|pq}\tilde{\triangle}^A(k-p)\tilde{\triangle}^h(p)\\
&\left\{\left[\delta_{rm}\delta_{nk}\left[\omega_{\mathbf{k}}(\omega_{\mathbf{k}}+\omega_{\mathbf{p}})-\mathbf{k}\cdot(\mathbf{k}+\mathbf{p})\right]-\delta_{rk}\mathbf{k}_m\mathbf{k}_n+2\mathbf{k}_m\delta_{n(r}\mathbf{p}_{k)}\right]\right.\\
&\left.\left[\delta_{sp}\delta_{ql}\left[\omega_{\mathbf{k}}(\omega_{\mathbf{k}}+\omega_{\mathbf{p}})-\mathbf{k}\cdot(\mathbf{k}+\mathbf{p})\right]-\delta_{sl}\mathbf{k}_p\mathbf{k}_q+2\mathbf{k}_p\delta_{q(s}\mathbf{p}_{l)}\right]\right\}.
\end{aligned} \tag{4.72}$$

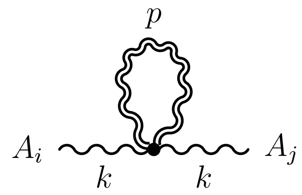


Figure 4.17: Bubblegum diagram with a tensor graviton in the loop and a $h^2_{ij}A^2_i$ vertex.

Now we consider another bubblegum diagram, this time with the tensor graviton propagating around the loop, as shown in Figure 4.17. The diagram contains a $h^2_{ij}A^2_i$ vertex and, given the symmetry factor of two, contributes a self-energy $\Sigma_{h^2A^2}$ where

$$\begin{aligned}
M_{pl}^2\Sigma_{h^2A^2}\pi^{\mathbf{k}}_{ij} =& \frac{1}{2}M_{pl}^2\int d\omega_{\mathbf{p}}d^3\mathbf{p}V_{ij|klmn}(k,-k;p,-p)\cdot\tilde{\triangle}^h_{klmn}(p)\\
=& \int d\omega_{\mathbf{p}}d^3\mathbf{p}\pi^{\mathbf{k}}_{ip}\pi^{\mathbf{k}}_{jq}\frac{1}{2}\Pi^{\mathbf{p}}_{kl|mn}\tilde{\triangle}^h(p)\\
&\left\{\omega_{\mathbf{k}}^2\left(\delta_{pk}\delta_{qm}\delta_{ln}-\frac{1}{4}\delta_{pq}\delta_{km}\delta_{ln}\right)+\frac{1}{4}\delta_{pq}\delta_{km}\delta_{ln}\left|\mathbf{k}\right|^2\right.\\
&-\delta_{ln}\left(\delta_{pk}\delta_{qm}\left|\mathbf{k}\right|^2+\delta_{pq}\mathbf{k}_k\mathbf{k}_m\right)-\delta_{pk}\left(\delta_{ql}\mathbf{k}_k\mathbf{k}_l-\delta_{qm}\mathbf{k}_l\mathbf{k}_n\right)\\
&\left.-\frac{\omega_{\mathbf{k}^2}}{\left|\mathbf{k}+\mathbf{p}\right|^2}\mathbf{k}_k\mathbf{k}_m\delta_{ln}\right\}.
\end{aligned} \tag{4.73}$$

Note that the final term in the above equation comes from integrating out the A_0 constraint.

The diagram with the ζA^2_i vertices is shown in Figure 4.18. With the appropriate

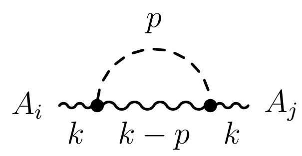


Figure 4.18: Diagram containing two ζA_i^2 vertices

symmetry factors this gives a self-energy contribution $\Sigma_{\zeta A^2}$ where

$$
\begin{aligned}
M_{pl}^2 \Sigma_{\zeta A^2} \pi_{ij}^{\mathbf{k}} = {} & M_{pl}^2 \int d\omega_{\mathbf{p}} d^3\mathbf{p} V_{ik}(k, p-k, -p) V_{jl}(-k, k-p, p) \tilde{\triangle}_{kl}^{A}(k-p) \tilde{\triangle}^{\zeta}(p) \\
= {} & \int d\omega_{\mathbf{p}} d^3\mathbf{p} \pi_{ai}^{\mathbf{k}} \pi_{bj}^{\mathbf{k}} \pi_{kl}^{\mathbf{p}+\mathbf{k}} \tilde{\triangle}^{A}(k-p) \tilde{\triangle}^{\zeta}(p) \\
& \left\{ (1+2f(\mathbf{p})) \left[\delta_{ij} \omega_{\mathbf{k}} (\omega_{\mathbf{k}} + \omega_{\mathbf{p}}) + \mathbf{k} \cdot (\mathbf{k} + \mathbf{p}) \delta_{ij} - \mathbf{k}_i \mathbf{k}_j \right] \right. \\
& \left. - \frac{1-3\lambda}{1-\lambda} \frac{\omega_{\mathbf{p}}}{|\mathbf{p}|^2} \left[\omega_{\mathbf{p}} (\mathbf{k} \cdot \mathbf{p} \delta_{ij} - \mathbf{p}_i \mathbf{k}_j) + \omega_{\mathbf{k}} (2\mathbf{k} \cdot \mathbf{p} \delta_{ij} + |\mathbf{p}|^2 \delta_{ij} - \mathbf{p}_i \mathbf{k}_j - \mathbf{p}_i \mathbf{p}_j) \right] \right\} \\
& \left\{ (1+2f(\mathbf{p})) \left[\delta_{kl} \omega_{\mathbf{k}} (\omega_{\mathbf{k}} + \omega_{\mathbf{p}}) + \mathbf{k} \cdot (\mathbf{k} + \mathbf{p}) \delta_{kl} - \mathbf{k}_k \mathbf{k}_l \right] \right. \\
& \left. - \frac{1-3\lambda}{1-\lambda} \frac{\omega_{\mathbf{p}}}{|\mathbf{p}|^2} \left[\omega_{\mathbf{p}} (\mathbf{k} \cdot \mathbf{p} \delta_{kl} - \mathbf{p}_k \mathbf{k}_l) + \omega_{\mathbf{k}} (2\mathbf{k} \cdot \mathbf{p} \delta_{kl} + |\mathbf{p}|^2 \delta_{kl} - \mathbf{p}_k \mathbf{k}_l - \mathbf{p}_k \mathbf{p}_l) \right] \right\}.
\end{aligned}
\tag{4.74}
$$

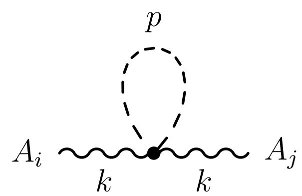


Figure 4.19: Bubblegum diagram with a scalar graviton in the loop and a $\zeta^2 A_i^2$ vertex

Finally, we consider a third bubblegum diagram, shown in Figure 4.19. This has the scalar graviton running through the loop with a $\zeta^2 A_i^2$ vertex. Taking care to neglect 'tadpole-like' contributions, we find that the contribution to the self-energy is given by $\Sigma_{\zeta^2 A^2}$ where

$$\begin{aligned}M_{pl}^2\Sigma_{\zeta^2A^2}\pi^{\mathbf{k}}_{ij} =& M_{pl}^2\int d\omega_{\mathbf{p}} d^3\mathbf{p}\tilde{\Delta}^{\zeta}(p)\mathcal{V}_{ij}(k,-k;p,-p)\\ =& \int d\omega_{\mathbf{p}} d^3\mathbf{p}\pi^{\mathbf{k}}_{ik}\pi^{\mathbf{k}}_{jl}\tilde{\Delta}^{\zeta}(p)\\ & \Bigg\{-\frac{1}{2}\left(1+8f(\mathbf{p})-8f(\mathbf{p})^2\right)\omega_{\mathbf{k}}^2\delta_{kl}-\frac{1}{2}(3+8f(\mathbf{p}))\left|\mathbf{k}\right|^2\delta_{kl}\\ & -2\left(\frac{1-3\lambda}{1-\lambda}\right)^2\frac{\omega_{\mathbf{p}}^2}{|\mathbf{p}|^4}\left[(\mathbf{k}\cdot\mathbf{p})^2\delta_{kl}+|\mathbf{k}|^2\mathbf{p}_k\mathbf{p}_l\right]-\frac{|\mathbf{p}|^2\omega_{\mathbf{k}}^2}{|\mathbf{k}+\mathbf{p}|^2}\delta_{kl}\Bigg\}.\end{aligned} \tag{4.75}$$

The above expressions can all be marginally simplified by multiplying through by $\pi^{\mathbf{k}}_{ij}$ and using $\pi^{\mathbf{k}}_{ij}\pi^{\mathbf{k}}_{ij}=2$, if one desires.

We are again interested in calculating these integrals, and do so by approximation methods similar to those adopted for the scalar in Section 4.4.3, obtaining the following expressions,

$$\Sigma_{A^4} \approx \left(\frac{\#}{\epsilon}+\#\ln\frac{\mu^2}{M_*^2}+\#+\#\frac{|\mathbf{k}|^2}{M_*^2}\right)\left(\#\omega_{\mathbf{k}}^2+\#|\mathbf{k}|^2\right) \tag{4.76a}$$

$$\begin{aligned}\Sigma_{hA^2} \approx& \left(\#+\#\ln\frac{|\mathbf{k}|^2}{M_h^2}\right)\frac{|\mathbf{k}|^4}{M_{pl}^2}+\left(\#+\#\ln\frac{|\mathbf{k}|^2}{M_h^2}++\#\frac{|\mathbf{k}|^2}{M_{pl}^2}\right)\frac{\omega_{\mathbf{k}}^4}{M_{pl}^2}\\ &+\left(\frac{\#}{\epsilon}+\#\ln\frac{\mu^2}{M_{pl}^2}+\#\right)\omega_{\mathbf{k}}^2+\#\frac{M_h^2}{M_{pl}^2}|\mathbf{k}|^2\end{aligned} \tag{4.76b}$$

$$\begin{aligned}\Sigma_{h^2A^2} \approx& \frac{M_h^2}{M_{pl}^2}\left(\frac{\#}{\epsilon}+\#\ln\frac{\mu^2}{M_h^2}+\#\right)\left(\#\omega_{\mathbf{k}}^2+\#|\mathbf{k}|^2\right)\\ &+\frac{|\mathbf{k}|^2}{M_{pl}^2}\left(\#\omega_{\mathbf{k}}^2+\#\,|\mathbf{k}|^2\right)\end{aligned} \tag{4.76c}$$

$$\begin{aligned}\Sigma_{\zeta A^2} \approx& \frac{1}{M_*^2}\left(\#+\#\ln\frac{|\mathbf{k}|^2}{M_*^2}\right)\left(\#\omega_{\mathbf{k}}^4+\#\omega_{\mathbf{k}}^2|\mathbf{k}|^2+\#|\mathbf{k}|^4\right)\\ &+\frac{1}{\alpha}\left(\frac{\#}{\epsilon}+\#\ln\frac{\mu^2}{M_{pl}^2}+\#\right)\left(\#\omega_{\mathbf{k}}^2+\#|\mathbf{k}|^2\right)\end{aligned} \tag{4.76d}$$

$$\begin{aligned}\Sigma_{\zeta^2A^2} \approx& \alpha\left(\frac{\#}{\epsilon}+\#\ln\frac{\mu^2}{M_h^2}+\frac{\#}{\alpha}\right)\omega_{\mathbf{k}}^2\\ &+(1+\#\alpha)\left(\frac{\#}{\epsilon}+\#\ln\frac{\mu^2}{M_h^2}+\#\right)|\mathbf{k}|^2.\end{aligned} \tag{4.76e}$$

Comparing with the scalar field case in (4.52), (4.53), we see that the broad pattern of these is unchanged in the vector case. The only additional terms are those in the tensor graviton h vertices, due to the fact that our matter field carries a spacetime index. These terms are subdominant to those in (4.76d) at any rate, so are not expected to lead to new effects.

The matter field still couples strongly to the scalar graviton ζ, as can be seen in (4.76d). This suggests that the vector field will also experience large corrections to the light cone. These are not necessarily observable, if all light cones are renormalised in the same manner. Indeed, [100] argue that most large corrections cancel out if one compares $\delta c^2_{vector} - \delta c^2_{scalar}$, provided some additional terms are added into the potential (these will be discussed further in Section 4.6). Unfortunately, our approximation scheme loses specific numerical coefficients so we cannot independently verify this, but a naïve hard cut-off analysis of our results suggests no disagreement with [100]. It would be interesting to repeat the analysis for fermionic fields, to investigate whether the light cone fine tuning is resolved in that case.

An unexpected feature of the scalar field result was the presence of higher time derivatives. Are these dangerous terms still present in the vector field case? Yes, and in fact in our case there are contributions from both the scalar graviton (4.76d) as before and also the tensor graviton (4.76b). For photons, not only are the Ostrogradski ghosts dangerous, but experimentally the other higher derivative terms may be too, as future experiments may be able to push the limits of the suppression scale of dimension 6 operators[11] on the photon above $M_\star$. Similar restrictions do exist for the spin-$\frac{1}{2}$ sector, which we discuss in the next section.

It appears that regardless of rank of the field, coupling Hořava gravity to matter runs into the same, serious problems.

4.6 Discussion

Hořava gravity has attracted much interest in its gravitational sector. However, the knottier issue of matter in the theory is still relatively new. In this chapter we have looked at both classical and quantum effects of Hořava gravity coupled to matter.

Having reviewed pure Hořava gravity in Section 4.2, we investigated Hořava-like matter theories in Section 4.3. We constructed the most general (at quadratic order around a Minkowski background) $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ invariant action of matter coupled to gravity, obeying the usual power-counting renormalisability conditions used in Hořava gravity and assuming the temporal derivatives are as in the relativistic theory. This corresponds to all the terms which can be generated if the theory is renormalisable. We constructed these fields both in the usual ADM composition

[11] Note that the standard terminology for the dimension in this case includes the dimension of the *field*, so since $[\varphi] = 1$ when canonically normalised and $[\triangle] = 2$, the term $\varphi\triangle^2\varphi$ is of dimension *six* in this language.

and the Stückelberg formalism. Using this, it was easy to demonstrate that the only way of coupling matter to gravity but not the new mode (in order to evade Lorentz invariance or Equivalence Principle violations) is the standard Lorentz invariant matter action.

Up to this point, we worked classically. However, in Sections 4.4 and 4.5, we considered the quantum corrections. In particular we studied one loop corrections to the propagator for scalar and vector matter fields. Our approach differed somewhat from that of [100] in that we integrated out the constraints and worked directly with the propagating degrees of freedom. We also used dimensional regularisation to (roughly) evaluate our loop integrals thereby eliminating the quadratic and quartic divergences that appeared in [100]. These divergences now manifest themselves as large momentum dependent corrections.

This analysis has revealed some potentially worrying features. The first is the large renormalisation of the light cone ($\sim 1/\alpha \gtrsim 10^7$) at low energies and momentum. This arises because the scalar graviton couples so strongly to the matter sector and was not noticed in [100] since they only focussed on divergences. Whether or not this means light cones for different particle species must be fine tuned to one part in 10^7 remains to be seen. What we can say is that the situation can probably be improved by modification of the Hořava action to include terms such as $(D_i K_{jk})^2$, provided they are introduced sufficiently far below the Planck scale. Such terms were originally proposed by [100] to alleviate quadratic divergences in the the relative light cones of different species. Here they will act to modify the propagator for the scalar graviton such that it becomes more weakly coupled to matter with increasing momentum.

The consistency of including these terms can be questioned, however, since they correspond to scaling dimension-8, and so the gravitational potential should be modified to include dimension-8 terms. It is also not clear that these are the only terms to be added — it may be that two-loop calculations imply a need to introduce dimension-10 terms, and so on for all loop orders. In addition, [100] only consider terms of the form $(D_i K^{ij})^2$ to avoid modifying the tensor graviton kinetic piece. However, this is an artificial restriction, and since terms of the form $(D_i K_{jk})^2$ are permitted by the symmetries, one expects them to be generated quantum mechanically. A full analysis of the theory with these additional terms included would be very interesting, particularly as the matter actions would also gain new pieces.

Another important issue may be the effect of having interacting matter fields, since then the RG flow can lead to corrections pushing both fields towards the same light cone [159]. The strength of these corrections depends on the strength of the force, meaning there is no reason in Hořava's theory to expect gravity to experience the same light cone as matter, despite the $\sim 1\%$ experimental constraints from binary pulsars [100].

The second significant feature revealed by our analysis is the generation of higher-order temporal derivatives. These are perfectly compatible with the $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ symmetry, but are generically associated with Ostrogradski ghosts [67]. Higher-order time derivatives are also generated in perturbative General Relativity although the corresponding ghosts have Planckian mass and so do not propagate when the effective theory is valid. In contrast, Hořava gravity is touted as a UV complete theory, rather than an effective theory only valid up to some cut-off, so we can always get a ghost to propagate because we can go to arbitrarily high energies.

Can we avoid this problem by modifying the gravitational part of the action? This seems unlikely since the origin of the higher-order time derivatives term can be traced back to the relativistic matter Lagrangian with minimal coupling to gravity. Indeed, consider the standard action

$$S \sim \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi, \tag{4.77}$$

If one expands $g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{pl}} h_{\mu\nu}$, then one obtains for the $h_{\mu\nu}\varphi^2$ vertex $V^{\mu\nu} = \frac{1}{M_{pl}} \left[k \cdot q \eta^{\mu\nu} - 2k^{(\mu} q^{\nu)}\right]$, where k, q are the energy-momenta of the scalars and p is the energy-momentum of the graviton. Working through, one arrives at the contribution to the scalar propagator of

$$\sim \frac{1}{M_{pl}^2} \int d^4p \mathcal{V}^{\mu\nu\rho\sigma}(k,p) \tilde{\triangle}^{\varphi}(k-p) \tilde{\triangle}^{grav}_{\mu\nu\rho\sigma}(p), \tag{4.78}$$

where $\mathcal{V}^{\mu\nu\rho\sigma}(k,p) = \left[k \cdot q \eta^{\mu\nu} - 2k^{(\mu} q^{\nu)}\right] \left[k \cdot q \eta^{\rho\sigma} - 2k^{(\rho} q^{\sigma)}\right]$ and $\tilde{\triangle}^{grav}_{\mu\nu\rho\sigma}(p)$ is some generalised graviton propagator (which may be a sum of different helicity propagators, *e.g.* spin-2 tensor and spin-0 scalar gravitons), and $q = k - p$.

If one splits the spacetime indices (μ, ν, ρ, σ) into temporal (0) and spatial (i, j, k, l) indices, then $\mathcal{V}^{ijkl}$, V^{00ij} and $\mathcal{V}^{0000}$ contain $\omega_{\mathbf{k}}^4$. This suggests that fourth-order time derivatives will generically be generated, though we cannot rule out the possibility that the details of the graviton propagator may be such that the ω^4 dependence disappears from the integral. Given the discussion at the end of the previous section, it seems a little optimistic to expect that this could be achieved by a small modification of the gravitational action in Hořava gravity.

Can we avoid the higher time derivatives by modifying the matter action? Naively one might be a little more optimistic for the following reason. Consider the offending contribution to the self-energy given by equation (4.55) but with the Wick rotated scalar propagator given by

$$\tilde{\triangle}^{\varphi}(p) \sim \frac{1}{\bar{\omega}_{\mathbf{p}}^2 + \mathcal{Q}^2(|\mathbf{p}|)}, \tag{4.79}$$

Working through the analysis as at the end of the previous section we find that

$$\Sigma_{\zeta\varphi^2} \supset \frac{2\pi^2 i}{M_{pl}^2}\alpha\bar{\omega}_{\mathbf{k}}^4 \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|\,(3+2f(|\mathbf{p}|))^2\,(|\mathcal{Q}(|\mathbf{p}|)| + |c(|\mathbf{p}|)||\mathbf{p}|)}{|c(|\mathbf{p}|)||\mathcal{Q}(|\mathbf{p}|)|[(|\mathcal{Q}(|\mathbf{p}|)| + |c(|\mathbf{p}|)||\mathbf{p}|)^2 + \bar{\omega}_{\mathbf{k}}^2]} \tag{4.80}$$

If we imagine that both propagators have a pure $z = 3$ scaling *i.e.* $\mathcal{Q}(|\mathbf{p}|) \sim \mathcal{Q}_0|\mathbf{p}|^3$, $c(|\mathbf{p}|)|\mathbf{p}| \sim c_0|\mathbf{p}|^3$ and take $f(|\mathbf{p}|) \sim f_0$, constant then the integral evaluates as $\propto 1/\bar{\omega}_{\mathbf{k}}^2$, so that the higher-order time derivatives are eliminated[12]. Of course, given that such terms are generated anyway by quantum corrections perhaps it is natural to consider matter Lagrangians that include an explicit $z = 3$ scaling in addition to the leading order relativistic piece. However, the leading order relativistic piece will almost certainly spoil the neat cancellation we have just described which relied on exclusively $z = 3$ scalings. This question deserves further investigation, not forgetting the phenomenological implications of introducing Lorentz violating contributions to the classical matter action. Along similar lines, it would be interesting to calculate the one-loop corrections to the *gravity* propagators, to investigate whether the generation of Ostrogradski ghosts occurs in the gravitational sector[13].

It would also be interesting to investigate whether fermions also have light cones which can avoid fine tuning in the same manner as scalars and vectors, or not. In addition, [136] has argued that Crab Nebula synchrotron radiation forces the scale of Lorentz violation in the fermionic sector to be Planckian, a result which they argue also applies to higher-order operators. For both our scalar and gauge fields, it appears from (4.53c) and (4.76d) that the relevant scale of suppression is $M_\star$ not M_{pl}. If this also holds in the fermionic case, it suggests we also need an additional mechanism to suppress these corrections. These issues indicate that extending our analysis for the case of fermions, while technically more involved, would be fruitful.

Our analysis has revealed dangerous issues for matter in Hořava gravity, and points towards a need for more detailed studies which may help rule out or save the theory.

[12] This pure $z = 3$ theory is equivalent to calculating corrections for $1 + 1$ gravity, which is renormalisable, as can be seen by making the substitution $|\mathbf{p}|^3 = |\mathbf{q}|$ inside the integral.

[13] Naïvely this seems reasonable — if terms of $2z$ are present in the matter sector, they should also appear in the gravitational sector.

Chapter 5

Cleaning up the cosmological constant problem

We now move on to consider a theory of gravity motivated by the need to alter cosmology and low-energy physics. We introduce a class of models which resolve the cosmological constant problem, before demonstrating some properties of an explicit example. Note that in this chapter, the physical spacetime metric is denoted $\tilde{g}_{\mu\nu}$, with $g_{\mu\nu}$ reserved for a fundamental spin-2 field. This chapter is based on the paper [2].

5.1 Introduction

Small scale physics is not the only indication that we need to alter our theory of gravitation. On the largest scales, the expansion of our universe indicates a need for new physics. Observations ranging from supernovae [55, 56] to the cosmic microwave background [160] indicate that standard model particles make up only 5% of the universe. The rest is in the form of (roughly 25%) dark matter and (roughly 70%) dark energy [38]. This can be explained within general relativity, but at a cost — one must introduce some new particles to explain dark matter, and add a cosmological constant to account for dark energy.

While there is significant motivation (supersymmetry *etc.*) for introducing new, stable, heavy particles into our fundamental theories, which may then also be able to account for dark matter, accounting for dark energy is more troublesome. The cosmological constant, and the associated accelerated expansion of the universe, can be explained simply by adding a cosmological constant into the Einstein-Hilbert action. Recall from our discussion in Section 1.3.2 that such a cosmological constant results in a horrendous degree of fine tuning. The observed cosmological constant is given by $\Lambda = \Lambda_{bare} + \Lambda_{SM}$. Now, $\Lambda_{SM} \sim M_{pl}^2$, but $\Lambda \sim H_0^2 \sim$

$(10^{-33}\mathrm{eV})^2$. This means that Λ_{bare} must cancel Λ_{SM} in its first 120 digits!

If we choose to worry about this 'cosmological constant problem', then we want some mechanism to explain the discrepancy between the vacuum energy from particle physics and the vacuum curvature of the universe. A nice approach would appear to be finding some mechanism to set $\Lambda = 0$ exactly, and then generate the acceleration via another mechanism; a scalar field or modified gravity for example. Unfortunately, methods of forcing Λ to zero tend to fall foul of Weinberg's 'no-go' theorem [57], ruling them out. However, as with all no-go theorems, there is a silver lining. It doesn't only prohibit theories, but points you towards how to create a theory where the issues may be evaded, giving the potential for viable self-tuning theories.

What principles does Weinberg's no-go theorem rely on? It assumes that your theory that avoids the cosmological constant problem is based in four spacetime dimensions and uses scale invariance to set Λ to zero. In such a theory there is no scale, so a cosmological constant (which is dimensionful and hence corresponds to a scale) cannot be generated. However, there must exist a goldstone boson from breaking this scale invariance (which clearly is broken at some level, since we do observe particle masses). In terms of quantum corrections, there is nothing to stop an exponential potential term being generated for the goldstone boson — the presence of such a term would drive the boson to an infinitely large value, corresponding to unbroken scale invariance. While the cosmological constant is zero in this case, we also have unbroken scale invariance, ruled out by experiment. Hence one concludes scale invariance cannot be used to fix the cosmological constant problem. If scale invariance is not responsible for setting $\Lambda = 0$, but a different mechanism, the theorem can be evaded. An example is the supersymmetric large extra dimensions (SLED) proposal [161], where SUSY is responsible for the cancellation of the cosmological constant. Other assumptions in the no-go theorem include Poincaré symmetry in the scalar sector, which is broken at the level of the solution in the *Fab Four* [162] proposal. There are also several other approaches in the literature [163–167].

However, it is notable that most attempted resolutions of the cosmological constant problem seek to explain why the cosmological constant is *smaller than expected.* We intend to ask instead why it *gravitates less than expected.* This shift in viewpoint forms the basis for our proposal.

5.2 A novel way to screen the vacuum energy

The idea for avoiding the cosmological constant problem can actually be framed quite simply. All matter is coupled minimally to a physical metric, which we denote $\tilde{g}_{\mu\nu}$. This is actually a composite of fundamental fields, which we denote ϕ_a,[1] so that $\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu}(\phi_a, \partial\phi_a, \partial\partial\phi_a, \ldots)$. Note that here we suppress the tensor rank for brevity, and so the ϕ_a are not all spacetime scalars. In particular, $\{\phi_a\}$ contains a fundamental metric $g_{\mu\nu}$. Unless otherwise stated, we raise and lower spacetime indices with the *physical metric* in this section.

We start with the observation that all the contributions to the effective cosmological constant in the action, both from the bare value and the various particle physics contributions take the form

$$\int d^4x \sqrt{-\tilde{g}} \times const. \tag{5.1}$$

This can be prevented from having any influence on the dynamics if

$$\frac{\delta}{\delta\phi_a} \int d^4x \sqrt{-\tilde{g}} = 0 \qquad \forall \,\text{fields}\, \phi_a. \tag{5.2}$$

It is only possible for us to satisfy this equation if $\tilde{g}_{\mu\nu}$ is not a fundamental field, but is instead a composite, with $\sqrt{-\tilde{g}}$ a topological invariant and/or total derivative of the fundamental fields ϕ_a. In fact, this demonstrates that this proposal is not for a new model, but rather a new *class* of models, corresponding to different choices of $\{\phi_a\}$ and the various topological invariants one can build from them.

Since the vacuum energy completely drops out of the dynamic equations, the vacuum curvature (the curvature of physical spacetime) is now completely distinct to the vacuum energy and can be chosen arbitrarily. We then no longer need to concern ourselves with worries about stability of the vacuum curvature against radiative corrections in the Standard Model sector! It is worth stressing that the exact value of Λ is not explained by this proposal, however within it, one does have the freedom to arbitrarily choose the cosmological constant to agree with observation with a clean conscience.

To illustrate the idea more concretely, consider the case where $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ are conformally related, so that

$$\tilde{g}_{\mu\nu} = \Omega(\phi_a, \partial\phi_a, \partial\partial\phi_a) g_{\mu\nu}, \tag{5.3}$$

with $\Omega > 0$, which transforms as a scalar. Since $\sqrt{-\tilde{g}} = \Omega^2\sqrt{-g}$, one could achieve the required property of a total derivative by choosing $\Omega^2 = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}A^\mu)$. Not

[1] As in GR, we assume that at least some of the fields do not admit a *locally* conserved energy-momentum tensor. This ensures that we evade the no-go theorems presented in [168], *i.e.* the fact that we already have a tensor means we are not building tensors out of lower spin objects.

that one does not necessarily have to choose metrics of the form (5.3), disformal metrics [169] could alternatively be considered.

Choosing the appropriate constituents and form of $\sqrt{-\tilde{g}}$ is only actually half the story. We have proposed how the action should be built for the matter sector, by simply minimally coupling it to the physical metric $\tilde{g}_{\mu\nu}$, but have not commented on how to build the action for the gravity sector. One could choose standard kinetic terms for the ϕ_a, but the resulting theory would modify gravity in ways that are highly constrained by a variety of observational and theoretical considerations, particularly solar system tests. To evade these tight constraints, screening effects such as the Vainshtein effect [74] and chameleon mechanism [78] are often vital to suppress the modifications to gravity.

One obvious way to ensure compatibility with solar system tests is to take the purely gravitational piece of the action to be the Eintein-Hilbert action, built out of the physical metric, $\int d^4x\sqrt{-\tilde{g}}R(\tilde{g})$. Thus, if $\tilde{g}_{\mu\nu}$ is any solution of GR with an arbitrary cosmological constant, it will also solve this theory, since it will also extremise this action. Since (de Sitter) Schwarschild is an excellent approximation to our solar system, this suggests that we can easily meet the observational constraints. Unfortunately, this convenient choice comes with a price to pay — in order to satisfy (5.2), $\tilde{g}_{\mu\nu}$ is required to depend on derivatives of the ϕ_a (or be unimodular gravity, see below). It is thus likely that $\sqrt{-\tilde{g}}R(\tilde{g})$ contains higher derivatives, and is afflicted by Ostrogradski ghosts [67]. However, we will show later that (at second order in perturbations on maximally symmetric backgrounds), ghosts *can be avoided* by choosing the conformal factor appropriately.

The proposal therefore contains a wide class of theories, dependent on your choice of $\sqrt{-\tilde{g}}$ and kinetic terms for ϕ_a. From this point onwards, we will consider more specific examples of theories in this class, but we first discuss unimodular gravity and its relation to this proposal.

5.2.1 Unimodular gravity

We now compare and contrast this model with unimodular gravity [170]. Unimodular gravity relies on a similar idea, but approached from a different angle. The symmetry group of the theory is restricted to those diffeomorphisms that preserve the volume element[2] $\sqrt{-g}d^4x$, whereas we remain fully diffeomorphism invariant. This results in a theory where the curvature of the vacuum is arbitrary.

[2]Often referred to as restricting the volume element to unity, although in fact the real restriction is to coordinate transformations where the Jacobian is one.

Unimodular gravity can be derived by considering the action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}\left[\tilde{R} - 2\tilde{\Lambda} + \mu\left(\sqrt{-\tilde{g}} - \sqrt{-f}\right)\right] + S_m[\tilde{g};\Psi], \tag{5.4}$$

where μ is a Lagrange multiplier, $f_{\mu\nu}$ is a fixed metric and $S_m[\tilde{g},\Psi]$ is a matter action. The Euler-Lagrange equations deriving from this action are

$$\frac{\delta S}{\delta g_{\mu\nu}} = -\frac{\sqrt{-\tilde{g}}}{16\pi G}\left[\tilde{G}^{\mu\nu} - 8\pi G\tilde{T}^{\mu\nu} + \left(\tilde{\Lambda} - \frac{1}{2}\mu\right)\tilde{g}^{\mu\nu}\right] = 0, \tag{5.5}$$

along with the constraint $\sqrt{-\tilde{g}} = \sqrt{-f}$. By taking the trace of (5.5), one can eliminate μ, resulting in

$$\frac{\sqrt{-\tilde{g}}}{16\pi G}\left(\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\right)\left[\tilde{G}^{\mu\nu} - 8\pi G\tilde{T}^{\mu\nu}\right] = 0. \tag{5.6}$$

This just tells us that the traceless part of Einstein's equations must vanish. Note the disappearance of Λ from the equations of motion (5.6). Now, one obtains an equation for the trace by acting on (5.6) with the derivative operator $\tilde{\nabla}$, the covariant derivative associated with $\tilde{g}$. From the Bianchi identity and energy conservation[3], this implies that

$$\tilde{\nabla}_\mu(\tilde{R} + 8\pi G\tilde{T}) = 0. \tag{5.7}$$

Then the combination $\tilde{R}+8\pi G\tilde{T}$ is constant. In other words, the value of the trace of Einstein's equations just corresponds to a boundary condition, and picking this boundary condition specifies the cosmological constant.

Our proposal can actually be considered as a *generalisation* of unimodular gravity, as it includes unimodular gravity as a special case. To see this, one can set Ω in (5.3) to $\Omega^2 = \frac{\sqrt{-f}}{\sqrt{-g}}$, where $f_{\mu\nu}$ is a fixed metric. In this case, one obtains a theory with a fixed volume element since $\sqrt{-\tilde{g}} = \sqrt{-f}$, which is precisely unimodular gravity, and $\sqrt{-\tilde{g}}$ is clearly a topological invariant (as it is a constant). This is, however, just one possibility and so the proposal describes a wide class of models which are mostly new but also reduces to unimodular gravity in this specific case.

This relationship between the theories is useful as there are interesting results concerning unimodular gravity, which gives one hope these will hold in our more general theory. For example, it is argued that the Hamiltonian is non-zero in unimodular gravity, resolving the problem of time [171]. There is also evidence that not only does unimodular gravity solve the cosmological constant problem at the quantum level, but it may also have an UV completion via a full RG analysis [172]. The relevance of these results to our proposal merits further research.

[3]Energy conservation cannot be derived from the action since we do not have full diffeomorphism invariance — it must instead be imposed by hand.

5.2.2 Conformally related theories

Until now, we have considered the proposal in full generality. We now wish to specialise to the case where the physical and fundamental metrics are conformally related, as in (5.3). Explicitly, consider an action of the form

$$S[\phi_a;\Psi_b] = \frac{1}{16\pi G}\int d^4x\sqrt{-\tilde{g}}R(\tilde{g}) + S_m[\tilde{g}_{\mu\nu};\Psi_a], \tag{5.8}$$

where S_m is the matter action describing matter fields Ψ_a minimally coupled to the physical metric $\tilde{g}_{\mu\nu}$. From this, we can calculate the equations of motion. To do this, we make use of the chain rule for functional derivatives $\frac{\delta S}{\delta\phi_a(x)} = \int d^4y \frac{\delta S}{\delta\tilde{g}_{\mu\nu}(y)}\frac{\delta\tilde{g}_{\mu\nu}(y)}{\delta\phi_a(x)}$. One can then calculate[4] that $\frac{\delta\tilde{g}_{\mu\nu}(y)}{\delta\phi_a(x)} = \Omega(\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{4}\tilde{g}^{\alpha\beta}\tilde{g}_{\mu\nu})\frac{\delta g_{\alpha\beta}(y)}{\delta\phi_a(x)} + \frac{1}{2\sqrt{-\tilde{g}}}\tilde{g}_{\mu\nu}\frac{\delta}{\delta\phi_a(x)}\sqrt{-\tilde{g}}(y)$. Turning the handle yields

$$\frac{\delta S}{\delta\phi_a} = \sqrt{-\tilde{g}}\Omega\left(\tilde{E}^{\mu\nu} - \frac{1}{4}\tilde{E}\tilde{g}^{\mu\nu}\right)\frac{\partial g_{\mu\nu}}{\partial\phi_a} + \frac{1}{2}\mathcal{O}_a(\tilde{E}) = 0, \tag{5.9}$$

where

$$\tilde{E}^{\mu\nu} = \frac{1}{\sqrt{-\tilde{g}}}\frac{\delta S}{\delta\tilde{g}_{\mu\nu}} = -\frac{1}{16\pi G}\left[\tilde{G}^{\mu\nu} - 8\pi G\tilde{T}^{\mu\nu}\right], \tag{5.10}$$

and $\tilde{E} = \tilde{E}^\mu_\mu = \tilde{g}^{\mu\nu}\tilde{E}_{\mu\nu}$. Again, I emphasise that we are raising and lowering with the *tilded metric*, $\tilde{g}^{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. Note that the first term in (5.9) only appears in the $g_{\mu\nu}$ equation of motion, and no others, since

$$\frac{\partial g_{\mu\nu}}{\partial\phi_a} = \begin{cases} 1 & \phi_a = g_{\mu\nu} \\ 0 & \phi_a = \text{all other fields}\end{cases}. \tag{5.11}$$

$\tilde{T}^{\mu\nu} = \frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S_m}{\delta\tilde{g}_{\mu\nu}}$ is the physical energy-momentum tensor. The final piece of (5.9) to mention is the linear operator $\mathcal{O}_a$. It acts on scalars, and is defined as

$$\begin{aligned}\mathcal{O}_a(Q) &:= \int d^4y Q(y)\frac{\delta}{\delta\phi_a(x)}\sqrt{-\tilde{g}}(y) \qquad (5.12)\\ &= Q(x)\frac{\partial\sqrt{-\tilde{g}}(x)}{\partial\phi_a(x)} - \frac{\partial}{\partial x^\mu}\left(Q(x)\frac{\partial\sqrt{-\tilde{g}}(x)}{\partial\partial_\mu\phi_a(x)}\right) + \frac{\partial^2}{\partial x^\mu x^\nu}\left(Q(x)\frac{\partial\sqrt{-\tilde{g}}(x)}{\partial\partial_\mu\partial_\nu\phi_a(x)}\right) - \cdots\end{aligned}$$

It is then clear that for any constant c,

$$\mathcal{O}_a(c) = c\frac{\delta}{\delta\phi_a(x)}\int d^4y\sqrt{-\tilde{g}}(y) = 0, \tag{5.13}$$

by use of the condition (5.2). This is vital in eliminating the vacuum energy from our equations. Note that (5.9) clearly reduces to the equations of motion of unimodular gravity (5.6) in the case $\Omega^2 = \frac{\sqrt{-f}}{\sqrt{-g}}$.

[4] We use the product and chain rule for functional derivatives, and our other intermediate steps include $\frac{\delta\tilde{g}_{\mu\nu}}{\delta\phi_a} = \int d^4y\left[\Omega\frac{\delta g_{\mu\nu}}{\delta\phi_a} + \frac{g_{\mu\nu}}{2\Omega^2}\frac{\delta\Omega^2}{\delta\phi_a}\right] = \int d^4y\left[\Omega\frac{\partial g_{\mu\nu}}{\partial\phi_a} + \frac{1}{2\sqrt{-\tilde{g}}}\left(\frac{\delta\sqrt{-\tilde{g}}}{\delta\phi_a} - \frac{\delta\sqrt{-g}}{\delta\phi_a}\right)\tilde{g}_{\mu\nu}\right]$, noted here to aid following the calculation.

We now explicitly demonstrate how the vacuum energy drops out of the dynamics, showing along the way that any solution of Einstein's equations (with any value of the cosmological constant) satisfies (5.9). Now, to begin, we split up the energy momentum tensor into the vacuum energy contribution $\sigma\tilde{g}_{\mu\nu}$ and the fluctuations above the vacuum $\tilde{\tau}_{\mu\nu}$. We use

$$\tilde{G}_{\mu\nu} = \tilde{\tau}_{\mu\nu} - \tilde{\Lambda}\tilde{g}_{\mu\nu}, \qquad 8\pi G\tilde{T}_{\mu\nu} = \tilde{\tau}_{\mu\nu} + \sigma\tilde{g}_{\mu\nu}, \tag{5.14}$$

where $\tilde{\Lambda}$ and σ are constant, to satisfy the field equations. The first of these is simply Einstein's equation sourced solely by the fluctuations above the vacuum with an arbitrary cosmological constant $\tilde{\Lambda}$, note that this is not related to the vacuum energy σ. The second is our splitting of the energy-momentum tensor into its vacuum energy and fluctuations above the vacuum. Clearly with this choice $\tilde{E}_{\mu\nu} = \frac{1}{16\pi G}(\tilde{\Lambda} + \sigma)\tilde{g}_{\mu\nu}$ and $\tilde{E} = \frac{4}{16\pi G}(\tilde{\Lambda} + \sigma)$, which is just a constant. Thus $\mathcal{O}_a(\tilde{E}) = 0$, and also clearly $\tilde{E}_{\mu\nu} - \frac{1}{4}\tilde{E}\tilde{g}_{\mu\nu} = 0$, which clearly satisfies (5.9). Therefore, any solution of Einstein's equations with an arbitrary cosmological constant is a solution of this theory, and the vacuum curvature is *independent of the vacuum energy.* To simplify the equations, we consider only $8\pi G\tilde{T}_{\mu\nu} = \tilde{\tau}_{\mu\nu}$, neglecting the vacuum energy σ since we have shown it does not contribute to the dynamics.

We can therefore choose ΛCDM solutions with an arbitrary cosmological constant with a clean conscience — there is no fine-tuning involved in the value of the cosmological constant. I reiterate that since we can regain all GR solutions for $\tilde{g}_{\mu\nu}$, vacuum solutions, such as (de Sitter) Schwarschild also solve this theory. This ensures that we see no discrepancy from GR tests in the solar system.

Indeed, at first glance, we do not expect to see any deviations from GR, since the physical metric can always just match the GR solution. However, there are a number of points to note about this.

Firstly, GR solutions are not necessarily the only solutions of (5.9). Whilst in unimodular gravity, the equations of motion turn out to be equivalent to Einstein's equations (modulo a cosmological constant), that will not in general be true for this proposal[5]. We can consider both the classical and quantum stability of these branches of solutions to investigate the observation consequences of the extra solutions. This may have implications for Birkhoff's theorem [173]. This is important because the theorem provides evidence that it is reasonable to extend our Newtonian notion that the gravitational effect of a sufficiently distant masses is negligible, and therefore we can treat Minkowski space as the weak-field limit of GR [63]. In Chapter 6 we will go to consider these extra solutions from a classical

[5]Sufficient conditions for the only solutions for $\tilde{g}_{\mu\nu}$ to be those of Einstein's equation are proved in Chapter 6 and Appendix B.3.

perspective.

Secondly, since $\tilde{g}_{\mu\nu}$ is not a fundamental field, the perturbative structure of the theory may differ from GR. Of particular concern is the potential presence of Ostrogradski ghost modes, arising due to the potential existence of higher-derivative terms in the theory. Not only that but perturbative GR is very well tested, for example in calculating energy loss from binary stars [27]. Altering the perturbative structure by having extra modes (even if they are not ghostly) raises doubts as to whether these successes of GR can be matched. In the next section, we consider the perturbative structure of a particular realisation of our proposal, and demonstrate that it is not pathological (at quadratic order on maximally symmetry spaces).

5.3 Avoiding Ostrogradski ghosts: an example

Now, we investigate the perturbative structure of the theory, in particular whether the theory is ghost-free. $\overline{\text{Overbars}}$ will be used to denote background quantities in this section. In this section, we will raise and lower indices with the *un*tilded metric $g_{\mu\nu}$. Our interest is restricted to the case of conformally related metrics (5.3).

Consider perturbations about a maximally symmetric vacuum, with physical Riemann curvature $R_{\mu\nu\rho}{}^{\sigma}(\tilde{g}) = \frac{\tilde{\Lambda}}{3}\left(\tilde{g}_{\mu\rho}\delta^{\sigma}_{\nu} - \tilde{g}_{\nu\rho}\delta^{\sigma}_{\mu}\right)$. For simplicity, assume that the conformal factor has a constant background value, $\Omega = \overline{\Omega} =$ constant. We can then write the Riemann tensor of the fundamental metric as $R_{\mu\nu\rho}{}^{\sigma}(g) = \frac{\Lambda}{3}\left(g_{\mu\rho}\delta^{\sigma}_{\nu} - g_{\nu\rho}\delta^{\sigma}_{\mu}\right)$, where $\Lambda = \overline{\Omega}\tilde{\Lambda}$.

5.3.1 General perturbative result

Following a lengthy calculation, full details of which can be found in Appendix B.1, we obtain an expression for the perturbative action to quadratic order

$$\delta_2 S = \overline{\Omega}\left[\delta_2 S_{GR}[g] + \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Delta\mathcal{L}\right], \tag{5.15}$$

where $\delta_2 S_{GR}$ is the expansion to quadratic order (in $\delta g_{\mu\nu}$) of the usual Einstein-Hilbert action with a cosmological constant, $S_{GR}[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R(g) - 2\Lambda\right)$, with $R(g)$ the Ricci scalar built from $g_{\mu\nu}$. We will denote the fluctuations in the metric as $\delta g_{\mu\nu} = h_{\mu\nu}$. The second term in (5.15) is a perturbative correction arising due to our modification of gravity,

$$\Delta\mathcal{L} = \frac{1}{4}\frac{\delta\Omega^2}{\overline{\Omega}^2}\left(2\delta R(g) - \frac{3}{2}\frac{\Box\delta\Omega^2}{\overline{\Omega}^2} - 2\Lambda\frac{\delta\Omega^2}{\overline{\Omega}^2}\right), \tag{5.16}$$

where $\delta R(g) = \nabla_\mu \nabla_\nu \left(h^{\mu\nu} - h g^{\mu\nu}\right) - \Lambda h$ is the linearised Ricci scalar, $h = h^\mu_\mu$ and ∇ denotes the covariant derivative using the metric connection for $g_{\mu\nu}$.

5.3.2 A ghost-free example

The above equation (5.15) holds for any choice of Ω^2 satisfying (5.2) and (5.3). Let us now consider the following specific case,

$$\Omega^2 = \frac{R_{GB}(\Xi, g)}{\mu^4}, \tag{5.17}$$

where μ is some mass scale and

$$R_{GB}(\Xi, g) = \frac{1}{4} \delta^{\mu_1 \cdots \mu_4}_{\nu_1 \cdots \nu_4} R_{\mu_1 \mu_2}{}^{\nu_1 \nu_2}(\Xi, g) R_{\mu_3 \mu_4}{}^{\nu_3 \nu_4}(\Xi, g) \tag{5.18}$$

is the Gauss-Bonnet combination. The generalised Kronecker delta is given by $\delta^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_n} = n! \delta^{\mu_1}_{[\nu_1} \cdots \delta^{\mu_n}_{\nu_n]}$, and $\Xi^\lambda{}_{\mu\nu}$ is a torsion free-connection *independent of the metric*, from which we construct the corresponding Riemann tensor,

$$R_{\mu\nu}{}^{\alpha\beta} = g^{\lambda\alpha} R_{\mu\nu\lambda}{}^\beta(\Xi) = g^{\lambda\alpha}\left(-2\partial_{[\mu} \Xi^\beta{}_{\nu]\lambda} + 2\Xi^\kappa{}_{\lambda[\mu} \Xi^\beta{}_{\nu]\kappa}\right). \tag{5.19}$$

$\sqrt{-g} R_{GB}(\Xi, g)$ is the well-known Gauss-Bonnet invariant. In 4D, this is a topological invariant and so (5.2) is satisfied. Note that this remains true in spite of the fact we are considering a Palatini variation where the Riemann tensor is not built from the metric connection, but from an independent one.

The full equations of motion are

$$\frac{1}{\sqrt{-\tilde{g}}\Omega} \frac{\delta S}{\delta g_{\mu\nu}} = \left(\delta^\mu_\alpha \delta^\nu_\beta - \frac{1}{4} \tilde{g}_{\alpha\beta} \tilde{g}^{\mu\nu}\right) \tilde{E}^{\alpha\beta} \tag{5.20a}$$

$$\frac{\delta S}{\delta \Xi^\lambda{}_{\mu\nu}} = \frac{1}{2} \mathcal{O}_{\Xi^\lambda{}_{\mu\nu}}(\tilde{g}_{\alpha\beta} \tilde{E}^{\alpha\beta}) = \frac{1}{2} \int (\tilde{g}_{\alpha\beta} \tilde{E}^{\alpha\beta}) \frac{\delta \sqrt{-\tilde{g}}}{\delta \Xi^\lambda{}_{\mu\nu}}. \tag{5.20b}$$

Before moving on to perturbations, we make some comments about the background solutions. It is clear from (5.20a) that the traceless part of Einstein's equations holds (for the physical metric $\tilde{g}_{\mu\nu}$). Note that cosmological constant-like terms will never enter into this traceless piece. One can apply $\tilde{\nabla}_\mu$ (the covariant derivative with respect to the metric connection of $\tilde{g}_{\mu\nu}$) to (5.20a). Since $\tilde{\nabla}_\mu \tilde{E}^{\mu\nu} = 0$ by the Bianchi identity and energy-momentum conservation, one is left with $\partial_\mu(\tilde{g}_{\alpha\beta} \tilde{E}^{\alpha\beta}) = 0$, or the trace[6] of Einstein's equations is constant. In fact, the arbitrary integration constant arising from the previous equation gives precisely the arbitrary cosmological constant present in the theory, which is also exactly what happens in unimodular gravity. Thus, the $g_{\mu\nu}$ equation of motion

[6] In this case, with respect to the *tilded* metric.

(5.20a) leads necessarily to Einstein's equations with an arbitrary cosmological constant for $\tilde{g}_{\mu\nu}$. The $\Xi^\lambda{}_{\mu\nu}$ equation of motion (5.20b) is automatically satisfied from this requirement for any $\Xi^\lambda{}_{\mu\nu}$, since $\mathcal{O}_a(c)$ vanishes for constant c. Let us then assume that the connection $\Xi^\lambda{}_{\mu\nu}$ coincides with the metric connection on the background, $\overline{\Xi}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} \left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}\right)$.

Moving on to perturbations, the gauge invariant fluctuation of Ξ can be written,

$$\delta(\Xi^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\mu\nu}) = \mathcal{B}^\lambda{}_{\mu\nu}. \tag{5.21}$$

It then follows that $\overline{\Omega}^2 = \frac{8\Lambda^2}{3\mu^4}$, and

$$\frac{\delta\Omega^2}{\overline{\Omega}^2} = \frac{1}{2\Lambda}\left(\delta R(g) + \nabla_\mu \chi^\mu\right), \tag{5.22}$$

where $\chi^\mu = \mathcal{B}^{\mu\nu}{}_\nu - \mathcal{B}_\nu{}^{\nu\mu}$. So, the additional contribution to our perturbative action (5.15) that arises from modifying gravity can be written

$$\Delta\mathcal{L} = \frac{1}{4}\psi\left(2\delta R(g) - \frac{3}{2}\nabla^\mu\nabla_\mu\psi - 2\Lambda\psi\right) + \lambda(\delta R(g) + \nabla_\mu\chi^\mu - 2\Lambda\psi), \tag{5.23}$$

where λ is a Lagrange multiplier fixing the newly-introduced ψ to coincide with $\frac{\delta\Omega^2}{\overline{\Omega}^2}$.

Now, the perturbative structure of the theory can only differ from GR in the scalar sector[7]. So, we have two healthy spin-2 modes and none of spin-1, but the behaviour of the spin-0 modes may cause us some concern. In fact, for the specific case we have chosen in (5.17), the scalar modes are not pathological and ,in fact, are absent! This statement is true to at least quadratic order on maximally symmetric spaces. To see this, note that we can rewrite our perturbative action as

$$\delta_2 S = \frac{\overline{\Omega}}{16\pi G}\delta_2 S_{GR}[e^{\psi/2}g] + \int d^4x\sqrt{-g}\lambda(\delta R(g) + \nabla_\mu\chi^\mu - 2\Lambda\psi), \tag{5.24}$$

understanding ψ to be small. That the effective action could ultimately be written like this is obvious given the form of the full non-linear theory (5.8). Taking the χ^μ equation of motion immediately yields $\partial_\mu\lambda = 0$, and so (assuming asymptotically vanishing boundary conditions), it follows that $\lambda = 0$. Hence the above action (5.24) reduces to

$$\delta_2 S = \frac{\overline{\Omega}}{16\pi G}\delta_2 S_{GR}[e^{\psi/2}g]. \tag{5.25}$$

This is now just the effective action for metric fluctuations of the Einstein-Hilbert action, with a cosmological constant, on a maximally symmetric spacetime. The

[7] While the penultimate term in (5.23) looks like it may be a vector, note that the derivative operator ensures only the transverse scalar component is picked out.

scalar ψ serves to renormalise the scalar modes, but does not alter the fact that none of them propagate, as is shown explicitly in Appendix B.2. This means that, to this order at least, the theory has the same perturbative structure as GR, with just two propagating tensor degrees of freedom and no ghost. Thus, we have potentially a ghost-free theory of gravity which can evade the cosmological constant problem and pass all solar system tests.

Unfortunately, it remains possible that for the specific model in (5.17), if one considers non-maximally symmetric spacetimes, or goes higher than quadratic order in perturbation theory, the presence of ghosts could be revealed. Such a scenario is precedented, for example in massive gravity when the ghostly sixth mode makes its presence known on non-trivial backgrounds or at higher order in perturbations [70]. In fact, the idea of being able to remove an Ostrogradski ghost using constraints to all perturbative orders has been questioned recently [174]. It is a matter for future work whether the constraint procedure which exorcises the ghost here is able to remove them to all orders or on all backgrounds.

5.4 Discussion

A novel way to clean up the cosmological constant problem has been proposed. By coupling matter to a composite metric, $\tilde{g}_{ab}(\phi, \partial\phi, \ldots)$, satisfying the property (5.2), we have been able to eliminate the troublesome vacuum energy from contributing to the dynamics of the system. Thus one ought to be able to choose the vacuum curvature to take on an empirical value, as dictated by observation, with a clean conscience. The challenge is now to build a model incorporating this idea into a viable model of gravity.

To this end we have proposed a model that exploits the neat idea, and at the same time ought to be ghost-free and compatible with solar system physics and cosmological tests. This example contains a fundamental metric $g_{\mu\nu}$, and an independent torsion-free connection, $\Xi^{\lambda}{}_{\mu\nu}$. It is described by the action (5.8) with

$$\tilde{g}_{\mu\nu} = \Omega g_{\mu\nu}, \qquad \Omega^2 = \frac{R_{GB}(\Xi, g)}{\mu^4} \tag{5.26}$$

and the matter fields Ψ_a are minimally coupled to the composite metric $\tilde{g}_{\mu\nu}$. Note that this choice of Ω^2 is far from unique. One is free to add a whole host of terms to its definition without introducing any unwanted pathologies. This includes terms proportional to $R_{GB}(\Gamma, g)$, and the Pontryagin term $R \wedge R$. Indeed, all we need is for the conformal factor to contain some sort of 'auxiliary' field whose equation of motion constrains the Lagrange multiplier to vanish. (In the example presented here, the role of the 'auxiliary' field is played by the independent connection, $\Xi^{\lambda}_{\mu\nu}$.)

If that is the case, the perturbative analysis goes through untroubled and there are no ghosts, at least not on maximally symmetric spaces. On more general spaces, one may expect ghosts to (re)appear, however this requires further investigation. One is able to consider far more exotic examples of theories satisfying (5.2) than the conformal theories mentioned, which may remain ghost-free via alternative mechanisms.

It was demonstrated that this theory can be considered a generalisation of unimodular gravity. While we considered the theory at the purely classical level, recent (and not-so recent) results in the literature about the quantum behaviour of unimodular gravity [171, 172], including the possible existence of a UV completion via the renormalisation group, gave us hope that our proposal may be valid when thinking fully quantum mechanically (modulo questions of ghostly pathologies). While the inclusion of a spin-3 field might seem at odds with the results of [168] about the consistency of higher-spin theories, note that approaches such as Loop Quantum Gravity are able to treat the connection as an independent field. Questions of quantum behaviour of the theory deserve further investigation.

As we have seen, any solution to GR, with an arbitrary cosmological constant, is a solution to this theory. However, it is possible that the reverse may not be true and a generic realisation of the theory is expected to permit solutions that are not present in GR. For the model given by (5.17), we noted from (5.20) that no further solutions exist, and the extension to GR is fully encoded by the arbitrariness in Λ. This suggests that the number of propagating degrees of freedom should be equivalent to GR. More general models may yield more exotic solutions containing interesting and potentially testable new features. Chapter 6 considers the effect these new features may have in cosmological scenarios.

Chapter 6

Cosmology and new solutions with generalised unimodular gravity

In Chapter 5, a mechanism to evade the cosmological constant problem was proposed. This was demonstrated to be in some sense a generalisation of unimodular gravity, opening up a new class of potential models. In this chapter, a specific realisation of the idea will be discussed, and the cosmological behaviour of the theory considered. The background cosmology of the theory will be determined, considering both the GR solutions and the new, non-GR solutions available.

Even the GR solutions require work beyond the usual GR result. While the physical metric solves the usual Friedman equations (ensuring that observations agree with GR), we will also be concerned with the dynamics of the fundamental metric. Unlike GR, the physical and fundamental metric are different objects, and so can exhibit different dynamics.

This chapter is based on [4]. Note that we raise and lower indices throughout with the physical metric $\tilde{g}_{\mu\nu}$.

6.1 Determining the model and solution possibilities

For simplicity, we follow our work in Chapter 5 and work only in the case where the physical and fundamental metrics are conformally related, $\tilde{g}_{\mu\nu} = \Omega g_{\mu\nu}$. We now wish to consider theories which possess additional physical metric solutions to those in GR and get a feel for their behaviour. To understand how to choose such a model, it is instructive to prove the following statement:

Statement. *If $\sqrt{-\tilde{g}}$ contains no derivatives of $g_{\mu\nu}$ and $\nabla^{\mu\nu}_{\mu}$, then the only so-*

lutions for $\tilde{g}_{\mu\nu}$ correspond to Einstein's equation with an arbitrary cosmological constant.

Proof. Begin by writing out the operator $\mathcal{O}_{g_{\mu\nu}}(Q)$ (for sufficiently differentiable Q) explicitly,

$$\begin{aligned}\mathcal{O}_{g_{\mu\nu}}(Q) &= \int d^4x Q \frac{\delta\sqrt{-\tilde{g}}}{\delta g_{\mu\nu}} \\ &= Q\frac{\partial\sqrt{-\tilde{g}}}{\partial g_{\mu\nu}} - \partial_\alpha\left(Q\frac{\partial\sqrt{-\tilde{g}}}{\partial(\partial_\alpha g_{\mu\nu})}\right) + \partial_\alpha\partial_\beta\left(Q\frac{\partial\sqrt{-\tilde{g}}}{\partial(\partial_\alpha\partial_\beta g_{\mu\nu})}\right) + \cdots \\ &= -\partial_\alpha Q\frac{\partial\sqrt{-\tilde{g}}}{\partial(\partial_\alpha g_{\mu\nu})} + \partial_\alpha\partial_\beta Q\frac{\partial\sqrt{-\tilde{g}}}{\partial(\partial_\alpha\partial_\beta g_{\mu\nu})} + 2\partial_\alpha Q\partial_\beta\left(\frac{\partial\sqrt{-\tilde{g}}}{\partial(\partial_\alpha\partial_\beta g_{\mu\nu})}\right) + \cdots,\end{aligned}$$

where the dots indicate higher derivatives of $\sqrt{-\tilde{g}}$, and going from the second to the final line, we have used that fact that $\sqrt{-\tilde{g}}$ is a total derivative. Clearly, the last line tells us that if there are no derivatives of $g_{\mu\nu}$ in $\sqrt{-\tilde{g}}$, $\mathcal{O}_{g_{\mu\nu}}(Q) = 0$.

From (5.9), if $\mathcal{O}_{g_{\mu\nu}}(Q) = 0\ \forall Q$, then the $g_{\mu\nu}$ equation of motion can be written

$$\frac{\delta S}{\delta g_{\mu\nu}} = \sqrt{-\tilde{g}}\Omega\left(\tilde{E}^{\mu\nu} - \frac{1}{4}\tilde{E}\tilde{g}^{\mu\nu}\right) = 0. \tag{6.1}$$

Since the metric is non-degenerate and the conformal factor non-zero, we can divide through by $\sqrt{-\tilde{g}}\Omega$. Therefore the traceless part (nine of the ten components) of Einstein's equations must necessarily be zero. Recall from Chapter 5 that we can apply the operator $\tilde{\nabla}$ to the traceless Einstein equations, to obtain the result that the trace is a constant, provided that we also enforce energy-momentum conservation, $\nabla_\mu T^{\mu\nu}$.

This is then equivalent to all ten components of Einstein's equation being satisfied, up to the arbitrariness of the cosmological constant. We have had no freedom in our choice of this, and so this is clearly the unique equation which $\tilde{g}_{\mu\nu}$ must solve. □

Remark. One could alternatively, as shown in Appendix B.3, weaken the condition (still only a sufficient condition) to $\left(\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\right)\tilde{\nabla}_\alpha\mathcal{O}_{g_{\mu\nu}}(Q) = 0$. However, it is harder to get a handle on the meaning of this, and it is not at all obvious that it is possible to construct a theory such that the weaker condition holds but there are derivatives of g present.

Remark. Note that this does not imply a uniqueness of the solutions for the *fundamental* fields. In fact, the opposite is true, since for $\tilde{E} = \mathit{constant}$, the equations of motion (5.9) will clearly be satisfied identically for arbitrary choices of all fields in $\{\phi_a\} \setminus \{g_{\mu\nu}\}$.

What is the importance of this statement? Well, it highlights the fact that a sufficient[1] condition for there to be no solutions in addition to those in GR is if, as in the theory (5.17), no derivatives of $g_{\mu\nu}$ appear in the conformal factor. So, since we are interested in a an understanding of the solution space beyond the GR solutions, we must consider a theory where derivatives of $g_{\mu\nu}$ appear in the conformal factor Ω.

In order to construct a theory with $\mathcal{O}_{g_{\mu\nu}}(Q) \neq 0$ we, analogously with (5.17), consider adding a Gauss-Bonnet term to the conformal factor Ω. In this case, we add $R_{GB}(\Gamma, g)$, the Gauss-Bonnet term built from the metric connection $\Gamma^{\lambda}_{\mu\nu}$ associated with $g_{\mu\nu}$. By choosing a conformal factor

$$\Omega^2 = \frac{1}{\mu^4}\left[R_{GB}(\Xi, g) + \epsilon R_{GB}(\Gamma, g)\right] \tag{6.2}$$

where ϵ is some dimensionless number, we obtain a theory exhibiting additional solutions for $\tilde{g}_{\mu\nu}$ to GR's provided $\epsilon \neq 0$. The work in Chapter 5 suggests that the constraint arising from $R_{GB}(\Xi, g)$ may also be able to ensure this theory is ghost-free, assuming that Ξ is equal to the metric connection of g on the background.

The full Ξ equation of motion resulting from (6.2) can be written

$$\begin{aligned} &\frac{\delta S}{\delta \Xi^{\kappa}{}_{\mu\nu}} = \frac{1}{2}\mathcal{O}_{\kappa}{}^{\mu\nu}(\tilde{E}) = 0 \\ \Rightarrow \quad &0 = \partial_\lambda \tilde{E}\left(-2R^{\lambda(\mu\nu)}{}_{\kappa} - 4R^{\mu\nu}\delta^{\lambda}_{\kappa} + 4R^{\lambda(\mu}\delta^{\nu)}_{\kappa} + g^{\mu\nu}\delta^{\lambda}_{\kappa}R - g^{\lambda(\mu}\delta^{\nu)}_{\kappa}R\right), \end{aligned} \tag{6.3}$$

where the Riemann and Ricci tensors above are built from the connection Ξ. This demonstrates that for $\tilde{E} = const.$, the non-metric fundamental fields are arbitrary.

Meanwhile, the full $g_{\mu\nu}$ equation of motion is[2]

$$\frac{\delta S}{\delta g_{\mu\nu}} = \sqrt{-\tilde{g}}\Omega\left(\tilde{E}^{\mu\nu} - \frac{1}{4}\tilde{E}\tilde{g}^{\mu\nu}\right) + \frac{2\epsilon}{\mu^4}\sqrt{-g}P^{\mu\alpha\nu\beta}\nabla_\alpha\nabla_\beta\tilde{E} = 0, \tag{6.4}$$

where $P^{\mu\nu\alpha\beta}$ is the double-dual[3] of the Riemann tensor,

$$P^{\mu\nu\alpha\beta} = -R^{\mu\nu\alpha\beta} + 2R^{\mu[\alpha}g^{\beta]\nu} - 2R^{\nu[\alpha}g^{\beta]\mu} - Rg^{\mu[\alpha}g^{\beta]\nu},$$

[1] In fact, we conjecture that it is a *necessary and sufficient* condition that $\sqrt{-\tilde{g}}$ contains no derivatives of $g_{\mu\nu}$ in order for $\tilde{g}$ to only possess solutions which also solve the Einstein equations. We know of no counterexample, *i.e.* a theory with derivatives of g which does not permit additional solutions. Intuitively, this appears reasonable as it seems odd that a quantity built from g should be covariantly constant with respect to the connection compatible with a different metric $\tilde{g}$. For the purposes of this chapter, we will not be concerned with the necessary condition at any rate.

[2] A convenient shortcut is to use the equations of motion for V_{ringo} in [175], provided one is careful with signs.

[3] That is, one takes the Hodge dual of each of the pairs of indices in the Riemann tensor, $P^{\mu\nu\alpha\beta} \propto \epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\kappa\lambda}R_{\rho\sigma\kappa\lambda}$. Its trace gives the Einstein tensor [12].

with the curvature tensors here built from the metric connection Γ associated with $g_{\mu\nu}$. It is clear to see that this equation is consistent with Einstein's equations $\tilde{E}^{\mu\nu}$ for $\tilde{g}_{\mu\nu}$, being satisfied up to an arbitrary trace (our arbitrary cosmological constant). Note that acting on the right-hand most term of (6.4) with $\tilde{\nabla}_\mu$ results in a non-zero answer, so there are additional solutions to those in GR present.

We now go on to consider solutions to the theory (6.2) and their cosmology. In Section 6.2, we investigate the cosmology of the GR-like solutions of the theory, in particular the behaviour of the fundamental modes. Having gained this insight in the simplest case, we move on in Section 6.3 to consider the non-GR solutions of the theory and the effect on physical cosmology as well as the fundamental modes. We conclude in Section 6.4.

6.2 GR-like cosmology

Now that we have the full equations of motion for our model (6.2), given by (6.3) and (6.4), we can begin investigating different branches of solutions in the theory. We start with an analysis of the branch familiar to us from GR, which is always also a solution here.

Cosmology is our concern here, so we focus on homogeneous and isotropic solutions. The physical metric can then be written

$$\mathrm{d}s^2 = \tilde{a}^2 \left(-\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1 - \tilde{\kappa} r^2} + r^2 \mathrm{d}S_2 \right), \tag{6.5}$$

where $\tilde{a}$ is the physical scale factor and we work, for now, in conformal time. This makes life easier at this stage since we are dealing with the two conformally related metrics.

A quick aside on notation: Derivatives with respect to conformal time will be denoted by $'$ and $\tilde{\mathcal{H}} \equiv \frac{\tilde{a}'}{\tilde{a}}$. For simplicity, we will make use of the symbol $\gamma_{\mu\nu}$, defined by $\tilde{g}_{\mu\nu} = \tilde{a}^2 \gamma_{\mu\nu}$. Cosmic time will also be used, with derivatives denoted by $\dot{}$ and $\tilde{H} \equiv \frac{\dot{\tilde{a}}}{\tilde{a}}$. This convention carries over to the fundamental metric, just without the tildes: a is the scale factor, $\mathcal{H} \equiv \frac{a'}{a}$ and $H \equiv \frac{\dot{a}}{a}$. Our universe will be filled with a perfect fluid, with stress-energy tensor given by $\tilde{T}_{\mu\nu} = \tilde{a}^2 \mathrm{diag}\,(-\rho, p, p, p)$ (with[4] $\rho \neq -p$). Since the two metrics are related by $\tilde{g}_{\mu\nu} = \Omega g_{\mu\nu}$, the scale factor a for the fundamental metric will be related to that of the physical metric by $\tilde{a} = \sqrt{\Omega} a$.

To gain a first understanding of solutions of the theory, start with the GR solution (for the physical metric). The traceless part of the Einstein equations are satisfied,

[4]This requirement is made simply because $\rho = -p$ corresponds to vacuum energy-like contributions, which we know will not affect our dynamics.

and the trace $\tilde{E}$ is a constant which, for easy comparison with GR, can be set to $\tilde{E} = \frac{4\tilde{\Lambda}}{16\pi G}$. Combining these traceless and trace equations (and taking some linear combinations), we obtain the usual conformal time Friedmann equations,

$$\tilde{\mathcal{H}}^2 + \tilde{\kappa}^2 = \frac{8\pi G}{3}\tilde{a}^2\tilde{\rho} \tag{6.6a}$$

$$\tilde{\mathcal{H}}' = -\frac{4\pi G}{3}\tilde{a}^2\left(\tilde{\rho} + 3\tilde{p}\right), \tag{6.6b}$$

where $\tilde{\rho} = \rho + \frac{\tilde{\Lambda}}{8\pi G} \equiv \rho + \rho_\Lambda$ and $\tilde{p} = p - \frac{\tilde{\Lambda}}{8\pi G}$. In addition, since $\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0$,

$$\rho' + 3\tilde{\mathcal{H}}(\rho + p) = 0, \tag{6.7}$$

for our fluid. If it is made from several (non-interacting) components, $\rho = \sum_a \rho_a$, then the above energy conservation equation holds for each component.

Combining (6.6) and (6.7), one obtains the usual background results in cosmology and in particular, with judicious choices of fluid and $\tilde{\Lambda}$, can match observations in our universe to very good accuracy [38]. However, this is not the full story since we have only determined the solution for the *physical metric*. We really want to understand the dynamics of the fundamental degrees of freedom, Ξ and g.

In terms of solutions for Ξ, it is clear from (6.3) that since $\tilde{E} = const$, the Ξ equation of motion will be satisfied for any connection! Motivated by our work in the previous chapter, we choose the solution where Ξ coincides with Γ, the metric connection for $g_{\mu\nu}$. Ξ is then fully determined by the solution for $g_{\mu\nu}$. Even though the physical metric is clearly well behaved, we want to ensure that there are no hidden singularities in the cosmic evolution of the fundamental fields.

Determining the solutions for g is equivalent to understanding the dynamics of the fundamental scale factor a. Since a is conformally related to $\tilde{a}$, we use our knowledge of $\tilde{a}$ and the behaviour of the conformal factor to determine a. Since we have made a FRW ansatz for g, it follows that $R_{GB}(\Gamma, g) = \frac{24}{a^4}\mathcal{H}'\left(\mathcal{H}^2 + \kappa\right)$. Since we are considering the solution where Ξ coincides with the metric connection for g, the conformal factor (6.2) becomes

$$\tilde{a}^4 = \Omega^2 a^4 = \frac{(1+\epsilon)}{\mu^4} R_{GB}(\Gamma, g) a^4 = \frac{24(1+\epsilon)}{\mu^4}\mathcal{H}'\left(\mathcal{H}^2 + \kappa\right). \tag{6.8}$$

Since $\tilde{a}$ is known, we can (with appropriate boundary conditions) solve this ODE to obtain the behaviour of a. This equation also demonstrates that the sign of $1+\epsilon$ is physically irrelevant, since the change $a \to 1/a$ will compensate for such a sign change. We restrict to $1 + \epsilon > 0$ from now on. Note that rearranging (6.8) reveals that for $\kappa \geq 0$, $H' > 0$, so H is monotonically increasing in conformal time.

Having derived our equations in conformal time, it proves easier to actually solve them in cosmic time. In particular, the physical interpretation of the time coordinate will be simpler in this case. Our system of equations to solve can be rewritten in terms of cosmic time derivatives as,

$$\tilde{H}^2 = \frac{8\pi G}{3}\left(\rho_\gamma + \frac{\tilde{E}}{2}\right) \tag{6.9a}$$

$$\dot{\tilde{H}} = -\frac{8\pi G}{2}(\rho + p) \tag{6.9b}$$

$$\dot{\rho}_\gamma = -3\tilde{H}\gamma\rho_\gamma \tag{6.9c}$$

$$\dot{H} = \frac{\mu^4}{12(1+\epsilon)H^2}\left(\frac{1}{2} - \frac{12(1+\epsilon)H^3}{\mu^4}\tilde{H}\right), \tag{6.9d}$$

where $\tilde{E}$ is a constant, which can be written in terms of the usual cosmological constant as $\tilde{E} = \frac{\tilde{\Lambda}}{8\pi G}$ and ρ_γ is a fluid with equation of state $\gamma \equiv \frac{\rho_\gamma + p_\gamma}{\rho_\gamma}$.

6.2.1 Analytic cases

We cannot solve for general cosmologies in closed form, but we can consider some simple cases of single fluids[5] with $\gamma \neq 0$, or alternatively consider universes containing a pure cosmological constant, and calculate the consequences. This will allow us to gain an understanding of the general behaviour and appropriate initial conditions before we solve more complicated (realistic) theories numerically. For simplicity, we restrict to spatially flat universes, $\kappa = \tilde{\kappa} = 0$, which are in good agreement with observational data [38].

By re-arranging (6.9d), we obtain,

$$\frac{\mu^4}{24(1+\epsilon)} = (\dot{H} + H\tilde{H})H^2. \tag{6.10}$$

Since the left-hand side is positive,the right hand side must always be positive. In particular, we note that if H changes sign, $H^2 \to 0$ at some point in our time evolution. This means that $\dot{H} \to \infty$ at some finite time to maintain the relation (6.10). Since this means that $a \notin C^2(\mathbb{R})$, we choose to exclude this possibility by demanding $H > 0$. Of course, the same conclusions would be reached by working in conformal time from (6.8).

We use the initial conditions $\tilde{a}(t=0) = 1$, $H(t=0) = H(0)$, $\tilde{H}(t=0) = \tilde{H}_0$ to obtain the behaviour for the fundamental Hubble factor H in the case of a single

[5]It is also possible to determine some multiple fluid behaviour, but the analytic forms are less instructive, especially since the late-time behaviour will always take on single field form.

fluid with equation of state $\gamma \neq 0, -2$,[6]

$$H^3 = \left(H(0)^3 - \frac{1}{12\tilde{H}_0(2+\gamma)}\frac{\mu^4}{1+\epsilon}\right)\left(1+\frac{3}{2}\gamma\tilde{H}_0 t\right)^{-2/\gamma} + \frac{1}{12\tilde{H}_0(2+\gamma)}\frac{\mu^4}{1+\epsilon}\left(1+\frac{3}{2}\gamma\tilde{H}_0 t\right). \quad (6.11)$$

While for a pure cosmological constant,

$$H^3 = \left(H(0)^3 - \frac{1}{24\tilde{H}_0}\frac{\mu^4}{1+\epsilon}\right)e^{-3\tilde{H}_0 t} + \frac{1}{24\tilde{H}_0}\frac{\mu^4}{1+\epsilon}. \quad (6.12)$$

The initial value of H always decays away, and the late time dependence is either just proportional to $t^{1/3}$ or constant, so the late time behaviour is of exponential form in both cases,

$$a \sim \begin{cases} \exp\left[\frac{3}{8}\left(\frac{\gamma}{3\gamma-2}\right)^{1/3}\left(\frac{\mu^4}{1+\epsilon}\right)^{1/3} t^{4/3}\right] & \gamma \neq 0 \\ \exp\left[\frac{1}{2(1+\epsilon)^{1/3}}\left(\frac{\mu^4}{\tilde{H}_0}\right)^{1/3} t\right], & \gamma = 0 \end{cases} \quad (6.13)$$

or in terms of the conformal factor,

$$\Omega^2 \propto \begin{cases} \left(\tilde{H}_0 t\right)^{8/3\gamma}\exp\left[-\frac{3}{2}\left(\frac{\gamma}{3\gamma-2}\right)^{1/3}\left(\frac{\mu^4}{1+\epsilon}\right)^{1/3} t^{4/3}\right] & \gamma \neq 0 \\ \exp\left[\left(\tilde{H}_0 - \frac{1}{(24)^{1/3}(1+\epsilon)^{1/3}}\left(\frac{\mu^4}{\tilde{H}_0}\right)^{1/3}\right) t\right]. & \gamma = 0 \end{cases} \quad (6.14)$$

Alternatively, one can solve for a in terms of $\tilde{a}$ in order to determine that, for $\gamma \geq 0$ and at late times

$$a \sim \begin{cases} \exp\left[\left(\frac{1}{12(\gamma+2)}\frac{\mu^4}{(1+\epsilon)\tilde{H}_0^4}\right)^{1/3}\frac{1}{2\gamma}\tilde{a}^{2\gamma}\right] & \gamma > 0 \\ \exp\left[\left(\frac{1}{24}\frac{\mu^4}{(1+\epsilon)\tilde{H}_0^4}\right)^{1/3}\ln\tilde{a}\right], & \gamma = 0 \end{cases} \quad (6.15)$$

with the conformal factor looking like

$$\Omega^2 \sim \begin{cases} \tilde{a}^4\exp\left[-\left(\frac{1}{12(\gamma+2)}\frac{\mu^4}{(1+\epsilon)\tilde{H}_0^4}\right)^{1/3}\frac{2}{\gamma}\tilde{a}^{2\gamma}\right] & \gamma > 0 \\ \exp\left[4\left(1-\left(\frac{1}{24}\frac{\mu^4}{(1+\epsilon)\tilde{H}_0^4}\right)^{1/3}\right)\ln\tilde{a}\right]. & \gamma = 0 \end{cases} \quad (6.16)$$

[6]Considering more general fluids than $\gamma \in [0,2]$ will be helpful when we come to consider the non-GR solutions. For $\gamma = -2$,

$$H^3 = H(0)^3(1-3\tilde{H}_0 t) - \frac{\mu^4}{8(1+\epsilon)}\frac{1}{\tilde{H}_0}\frac{\log(1-3\tilde{H}_0 t)}{1-3\tilde{H}_0 t}$$

These explicitly demonstrate that H is, as we expected, a growing function with time, as is a. The fundamental scale factor and the conformal factor are both clearly well behaved for the lifetime of the universe in the theory — blow-ups only occur when $\tilde{H}$ diverges or $\tilde{a}$ goes to zero, which can occur if *e.g.* $\gamma < 0$, or if there is negative energy density.

The above equations only hold for a single contribution to the energy-momentum of the universe. However, when the universe has multiple non-interacting fluid components, the universe's late time behaviour will be governed by a single component (since $\rho \propto a^{-3\gamma}$), and so the behaviour of H at late times can then be determined by (6.11) or (6.12) for the appropriate component. For example, if there is any contribution to the energy-momentum tensor with $\gamma = 0$, this will dominate the energy budget of the universe at late times, so one expects H to obey (6.12) at late times. If a universe just contained radiation ($\gamma = 4/3$) and pressureless dust ($\gamma = 1$), then (6.11) with $\gamma = 4/3$ would describe the early-time behaviour, and with $\gamma = 1$ would describe late-time behaviour. Our calculations in this section then demonstrate that H will be well behaved throughout the history of the universe. Numerical support for this will be demonstrated in Section 6.2.3.

6.2.2 Dynamical systems analysis

Before moving on to consider more general cosmological systems, we re-write our system in an alternative form, using the tools of dynamical systems to determine critical points in phase space and their stability. The advantages of this formalism are not readily apparent in this case, but it will prove very useful when we consider non-GR solutions to the theory.

Start from our expression for the Hubble parameter (in cosmic time), (6.9a). Introduce the following dimensionless quantities,

$$x \equiv \frac{8\pi G\tilde{E}}{6\tilde{H}^2} \qquad z \equiv \frac{4\epsilon H^3}{\mu^4}\tilde{H}, \tag{6.17}$$

and divide (6.9a) by $\tilde{H}^2$ to obtain the constraint equation

$$\frac{8\pi G}{3\tilde{H}^2}\rho_\gamma = 1 - x. \tag{6.18}$$

This can be re-written $\Omega_\gamma + \Omega_\star = 1$, where the density parameters Ω_γ and $\Omega_\star$ are given by

$$\Omega_\gamma \equiv \frac{8\pi G}{3\tilde{H}^2}\rho_\gamma \qquad \Omega_\star \equiv x. \tag{6.19}$$

Point	$\bar{x}$	$\bar{z}$	Eigenvalues	$\Omega_\star$	Stability
A	0	$\frac{1}{3(\gamma+2)}$	$\left(3\gamma, -\frac{3}{2}(\gamma+2)\right)$	0	Saddle
B	1	$\frac{1}{6}$	$(-3\gamma, -3)$	1	Stable

Table 6.1: Critical points for the set of equations (6.20).

It is now possible to use our equations to determine that this system evolves as

$$\frac{dx}{dN} = 3\gamma x(1-x) \tag{6.20a}$$

$$\frac{dz}{dN} = \frac{1}{2} - 3z - \frac{3}{2}\gamma z(1-x), \tag{6.20b}$$

where $\frac{d}{dN} = \frac{d}{d\ln a} = \frac{1}{H}\frac{d}{d\ln a}$.

We are interested in the critical points of the theory, *i.e.* $\frac{dx}{dN} = \frac{dz}{dN} = 0$. These can be easily calculated to be $(\bar{x}, \bar{z}) = \left(0, \frac{1}{3(\gamma+2)}\right)$, which we denote Point A, and $(\bar{x}, \bar{z}) = \left(1, \frac{1}{6}\right)$, which we denote Point B. The stability of these points can be determined by calculating the eigenvalues of the stability matrix[7]. Negative eigenvalues indicate stable directions, positive eigenvalues indicate unstable ones.

As is clear from Table 6.1, Point A is a saddle point and Point B is a stable point. Physically this is easy to understand. Point A corresponds to a universe containing solely the matter density ρ_γ and no cosmological constant, while Point B corresponds to a cosmological constant dominated universe.

If there is a non-zero cosmological constant ($x \neq 0$), this will dominate at sufficiently late times over the matter ρ_γ. This means that Point A is unstable in this direction, but Point B is stable. For $\gamma > 0$, both points are stable in the z direction. This is reasonable, as can be seen from our result for H^3 in (6.11). The late-time behaviour clearly tends to constant z, and moving in the z direction does not introduce any cosmological constant. Thus Point A has stable and unstable directions, and is hence a saddle point, while Point B is stable in all directions, and we thus expect it to be an attractor.

Again, note that this has not necessarily given us any new information to that in Section 6.2.1, but it has demonstrated an alternative approach to deriving the behaviour we obtained in that section. In Section 6.3.3, when we consider non-GR solutions, it will turn out that we are not be able perform the equivalent analytical calculations to those in Section 6.2.1, but we will be able to gain insight into the behaviour of the theory using the dynamical systems approach.

[7] One performs the expansion $x = \bar{x} + \delta x$, $z = \bar{z} + \delta z$ to first order on the right-hand side of (6.20), and writes the resultant equations as a matrix equation $\begin{pmatrix} \frac{dx}{dN} \\ \frac{dz}{dN} \end{pmatrix} = M \begin{pmatrix} \delta x \\ \delta z \end{pmatrix}$. One then determines the eigenvalues of M.

6.2.3 Numerics

It is also possible to consider more general cosmologies via a numerical approach. In Figure 6.1, we plot the behaviour of various objects in the theory in a realistic

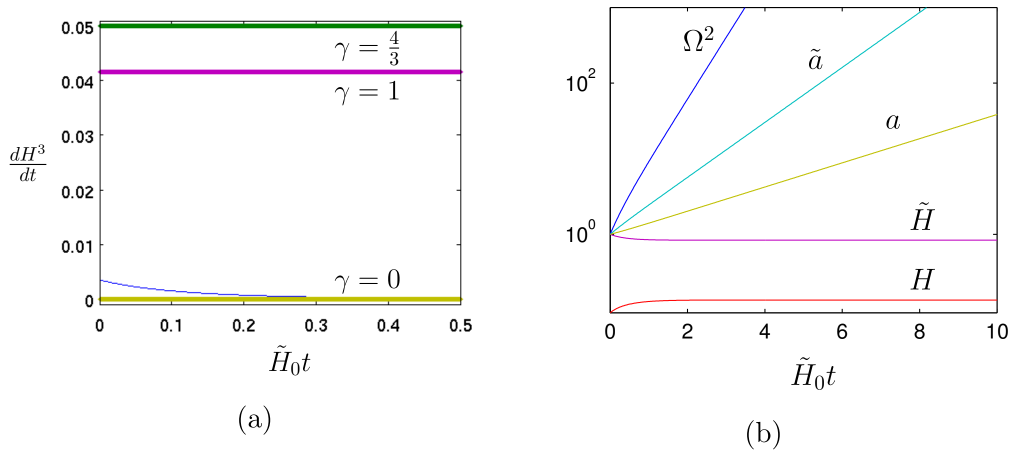


Figure 6.1: Numerical results for a universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$.
(a) The time derivative of H^3 (blue line). The other lines show the analytic results for specific values of γ.
(b) The evolution of various quantities with time in this universe.

cosmology made up of 70% cosmological constant and 30% pressureless matter. At late times, this universe is fully dominated by a cosmological constant. Figure 6.1a, shows the derivative of H^3 against time, and shows that at late times, the behaviour mimics that of a cosmological constant, as expected from our earlier discussion. Figure 6.1b shows the evolution of various quantities with time, demonstrating that the physical cosmology is as expected ($\tilde{H} \to const.$ at late times) and that the quantities introduced by the presence of the fundamental field, a, H and Ω remain well behaved, not only having singularities, but also being smooth.

A wider range of numerical results shows agreement with our expectations from the analytic case, a and H are smooth, increasing functions of time. The physical cosmology is, of course, in line with what is expected from ordinary cosmologies (with a cosmological constant) in all cases, since this is insensitive to the fundamental field dynamics.

6.3 Behaviour of new solutions in the theory

Now that we understand the behaviour of the GR-like solution of the theory, we investigate how the cosmology differs for the additional, non-GR solutions present in our theory (6.2). We begin by going back to (6.2), and no longer work under the assumption that $\tilde{E}$ must be constant.

6.3.1 Behaviour of $\tilde{a}$

Since we are still considering cosmology, we again substitute in our FRW ansätze for $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$. The only non-zero components of the P tensor are

$$P^{0i0j} = \frac{1}{a^6}\gamma^{ij}\left(\mathcal{H}^2 + \kappa\right) \qquad P^{ikjl} = -\frac{2}{a^6}\gamma^{i[j}\gamma^{l]k}\mathcal{H}', \tag{6.21}$$

and components related by symmetry. Using these, we can write

$$\frac{1}{2}\mathcal{O}_{g_{00}}(\tilde{E}) = -\frac{6}{a^6}\frac{\epsilon}{\mu^4}\sqrt{-g}(\mathcal{H}^2+\kappa)\mathcal{H}\tilde{E}' \tag{6.22a}$$

$$\frac{1}{2}\mathcal{O}_{g_{ij}}(\tilde{E}) = \frac{2}{a^6}\frac{\epsilon}{\mu^4}\gamma^{ij}\sqrt{-g}\left[\left(2\mathcal{H}' - \mathcal{H}^2 - \kappa\right)\mathcal{H}\tilde{E}' + \left(\mathcal{H}^2+\kappa\right)\tilde{E}''\right] \tag{6.22b}$$

Since $\mathcal{O}_{g_{\mu\nu}}(\tilde{E})$ is clearly not traceless, we can obtain an evolution equation by taking the trace of (6.4). The resulting equation is

$$\tilde{E}'' = -2\mathcal{H}\tilde{E}'\frac{\mathcal{H}'}{\mathcal{H}^2+\kappa}. \tag{6.23}$$

Note that if $\tilde{E}' = 0$ at some time, then using the above equation (and differentiating), all higher derivatives of $\tilde{E}$ are also zero at that time. Thus if $\tilde{E}' = 0$ at any point, $\tilde{E} = \textit{constant}$ from that time onwards. So if at *any point* in its evolution $\tilde{E}'$ is zero, it will remain fixed, and so the theory will behave like Einstein gravity with a cosmological constant. In fact, as we shall see once we have determined the conformal factor in Section 6.3.3, a non-zero $\tilde{E}'$ will be driven towards zero in general.

This conclusion should be tempered somewhat, since it only holds in cosmic time if $\tilde{H} > 0$, since the left-hand side of (6.23) will pick up a $\tilde{H}\dot{\tilde{E}}$ piece. Thus $\ddot{\tilde{E}}$ will only be opposite in sign to $\dot{\tilde{E}}$ if $\tilde{H} > 0$. This does not, however, seem too restrictive — every case we considered in Section 6.2 had this property, and the observed universe today certainly obeys it, but it is a caveat we will need to bear in mind and return to later.

The traceless part of (6.4) becomes, with the appropriate ansätze substituted in,

$$4\tilde{\mathcal{H}} - 2\frac{\tilde{a}''}{\tilde{a}} + 2\tilde{\kappa} - 8\pi G(\rho+p)\tilde{a}^2 - \frac{8\epsilon}{\mu^4}\frac{1}{\tilde{a}^2}\mathcal{H}(\mathcal{H}^2+\kappa)(-16\pi G)\tilde{E}' = 0, \tag{6.24}$$

which we rewrite as

$$4\tilde{\mathcal{H}} - 2\frac{\tilde{a}''}{a} + 2\kappa - 8\pi G(\tilde{\rho}+\tilde{p})\tilde{a}^2 = 0, \tag{6.25}$$

where $\tilde{\rho} = \rho + \rho_\star$, $\tilde{p} = p + p_\star$ and

$$\rho_\star + p_\star = -\frac{16}{\tilde{a}^4}\frac{\epsilon}{\mu^4}\mathcal{H}(\mathcal{H}^2 + \kappa)\tilde{E}'. \tag{6.26}$$

The 'starred' contribution to the energy-momentum tensor corresponds to our modification of gravity. Note for consistency that if $\tilde{E} = constant$, $\rho_\star = -p_\star$ and the additional contribution is just a pure cosmological constant.

Next, use our expression for $\tilde{E}$,

$$\tilde{E} = \left(-\frac{1}{16\pi G}\right)\frac{1}{\tilde{a}^2}\left(-6\frac{\tilde{a}''}{\tilde{a}} + 8\pi G\tilde{a}^2(\rho - 3p)\right). \tag{6.27}$$

Rearranging, and introducing $\rho_\star$ and $p_\star$ again, we determine using (6.26) that

$$\rho_\star = \frac{\tilde{E}}{2} - \frac{12}{\tilde{a}^4}\frac{\epsilon}{\mu^4}\mathcal{H}\tilde{E}'\left(\mathcal{H}^2 + \kappa\right) \tag{6.28a}$$

$$p_\star = -\frac{\tilde{E}}{2} - \frac{4}{\tilde{a}^4}\frac{\epsilon}{\mu^4}\mathcal{H}\tilde{E}'\left(\mathcal{H}^2 + \kappa\right), \tag{6.28b}$$

where $\tilde{E}$ evolves according to (6.23). Rearranging (6.28) shows that $\gamma_\star \equiv \frac{\rho_\star + p_\star}{\rho_\star}$ can take any real value, including being infinite at $\tilde{E} = \frac{\mathcal{H}}{\mathcal{H}'}\tilde{E}'$ (with the sign determined by the direction of approach). Of course, this $\gamma_\star$ is a function of time in general, but we will be most interested in situations where it mimics a fluid with constant equation of state, which will happen to be at critical points of our dynamical systems analysis. It is also clear that our work is consistent with our previous work in Section 6.2, since when $\tilde{E}' = 0$, $\gamma_\star = 0$ and this is just GR with a cosmological constant.

We now write the Friedmann equation

$$\tilde{\mathcal{H}}^2 + \tilde{\kappa} = \frac{8\pi G}{3}\tilde{a}^2\tilde{\rho} \tag{6.29a}$$

$$\tilde{\mathcal{H}}' = -\frac{4\pi G}{3}\tilde{a}^2(\tilde{\rho} + 3\tilde{p}). \tag{6.29b}$$

If we make the (usual) assumption that the fluid is made from several non-interacting components,

$$\rho_a' = -3\tilde{\mathcal{H}}(\rho_a + p_a) \tag{6.30a}$$

$$\rho_\star' = -3\tilde{\mathcal{H}}(\rho_\star + p_\star), \tag{6.30b}$$

where $p_a + \rho_a$ can be determined from the equation of state γ_a, or from (6.26) for the starred component. These equations also need to be supplemented by an equation for the evolution of $\tilde{E}$, which is given by (6.23).

As before, these equations are sufficient for us to understand the evolution of $\tilde{a}$. It obeys the usual Friedmann equations from GR, but also contains an extra fluid component due to the modification of gravity. We also need to understand how the fundamental fields behave. Again, we restrict ourselves to spatially flat spacetimes, $\kappa = \tilde{\kappa} = 0$ from now on. The important equations for us to understand our fundamental fields are (6.2) and our Ξ equation of motion, (6.3).

6.3.2 Behaviour of Ξ

The key equation for Ξ to solve is, of course, its equation of motion (6.3). It will be easiest to separate Ξ into the metric connection of g and the difference between the two connections, a tensor ξ,

$$\Xi^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \xi^{\lambda}{}_{\mu\nu}. \tag{6.31}$$

We are only concerned with background quantities, and so by the symmetries of our theory, we immediately deduce that $\xi^{i}{}_{00} = \xi^{0}{}_{0i} = 0$. Since we assume $\tilde{\kappa} = 0$, it follows that $\xi^{i}{}_{jk} = 0$. We introduce the following scalars to help us parametrise ξ,

$$\xi^{0}{}_{00} = A \qquad \xi^{i}{}_{0j} = \delta^{i}_{j} B \qquad \xi^{0}{}_{ij} = \delta_{ij} C \tag{6.32}$$

where A, B, C are (yet to be determined) functions of η.

Since we are looking at non-GR solutions, we must assume that $\partial_\mu \tilde{E} \neq 0$, and so the remaining terms in (6.3) must sum to zero. It only makes sense to consider the λ index $= 0$ pieces of these, since the $\lambda = i$ pieces are multiplied by $\partial_i \tilde{E}$ which is identically zero since we are working with a homogeneous and isotropic system.

The two equations we obtain are

$$\mathcal{H}^2 - 2\mathcal{H}' - 2B' + 2AB - 2B^2 + BC + 2\mathcal{H}A - \mathcal{H}B + \mathcal{H}C = 0 \tag{6.33a}$$

$$3\mathcal{H}^2 - 2\mathcal{H}' - 2B' + 2AB - 2B^2 + 3BC + 2\mathcal{H}A + \mathcal{H}B + 3\mathcal{H}C = 0. \tag{6.33b}$$

Note that there is still a degree of arbitrariness here — we have two equations for three unknowns. At first it is unclear how to tackle this. If we try and replicate the symmetry of the Levi-Civita connection case across A, B, C then $A = B = C = 0$, and so necessarily $\mathcal{H} = 0$, too. Unfortunately, this means that the conformal factor will be zero (since $R_{GB}(\Xi, g) = R_{GB}(\Gamma, g) = 0$) and the theory is singular, so we avoid this possibility.

To make progress, we rewrite A, B, C as

$$A = \beta B \qquad B = B \qquad C = \gamma B, \tag{6.34}$$

where β and γ are functions of conformal time. Taking the difference of (6.33), the resultant quadratic yields solutions for B of $B = -\mathcal{H}$ and $B = -\mathcal{H}/\gamma$.

Both of these cases lead to the Riemann tensor (of Ξ) vanishing, and so $R_{GB}(\Xi, g) = 0$. This is a perfectly valid background solution, since the conformal factor Ω has a contribution from $R_{GB}(\Gamma, g)$ too, and so will not necessarily vanish. However, $R_{GB}(\Xi, g) = 0$ may be problematic if trying to construct a ghost-free theory, since our demonstration of the absence of ghosts relied on Ξ's conformal factor being non-zero (and, specifically, a Levi-Civita connection).

This demonstrates that the dynamics of Ξ decouple from the rest of our problem, so we are unaffected by the remaining arbitrariness in the solution. We can now discuss $\tilde{a}$ and a independently of Ξ's behaviour. We will however, return to discussing Ξ in our conclusions in Section 6.4.

6.3.3 Behaviour of a

We now want to understand the dynamics of a. Start by considering the equation $\tilde{a} = \sqrt{\Omega} a$. Our expression for Ω, (6.2), can now be simplified to $\Omega^2 = \frac{\epsilon}{\mu^4} R_{GB}(\Gamma, g)$ since $R_{GB}(\Xi, g) = 0$. Since with our ansatz for g, $R_{GB}(\Gamma, g) = \frac{24}{a^4}\mathcal{H}'(\mathcal{H}^2 + \kappa)$, we can determine a by solving (remembering that we set $\kappa = 0$),

$$\tilde{a}^4 = \frac{24\epsilon}{\mu^4}\mathcal{H}'\mathcal{H}^2, \tag{6.35}$$

with appropriate boundary conditions, assuming $\tilde{a}$ is known. As in Section 6.2, we can use the fact that $\tilde{a}^4$ and $\mathcal{H}^2$ are greater than zero to determine that[8] $\mathcal{H}' > 0$, or $\dot{H} + \tilde{H}H > 0$. As before, taking $\mathcal{H} > 0$, (6.23) tells us that $\tilde{E}'$ will tend to zero regardless of the original sign of $\tilde{E}'$. It is easy to see that this also holds for the cosmic time derivative $\dot{\tilde{E}}$ *under the assumption that* $\tilde{H} > 0$. This is helpful, since it tells us that given enough time, the additional fluid should approximate a cosmological constant.

Now we have all the equations, we attempt to proceed as before and solve the full system of equations. Previously, we could solve for the physical scale factor $\tilde{a}$ before worrying about the fundamental field behaviour, but the way the equations are coupled prevents that approach working in this case (except for $\tilde{E}' = 0$). We can gain some intuition for the behaviour of H from our equations (6.11) and (6.12), since these cover all $\gamma \in \mathbb{R}$, even though $\gamma_\star$ is non-constant. We expect the fundamental field's Hubble parameter cubed H^3 to have (once any transient has decayed away) an linear relationship with time, $\sim At + B$, unless an effective cosmological constant dominates, it which case, H will just exhibit constant behaviour.

[8] We consider $\epsilon > 0$, which is sufficient to capture all the dynamics since a change in sign in ϵ is equivalent to changing $a \to 1/a$.

The relevant equations for us to solve are (6.23), (6.29), (6.30) and (6.35), as well as needing to use (6.28) for our 'modified gravity fluid'. For clarity, these can all be viewed together as a group in Appendix B.4. As before, we will find it most useful to work with the cosmic time equations,

$$\tilde{H}^2 = \frac{8\pi G}{3}\left(\rho_\gamma + \frac{\tilde{E}}{2} - 3J\dot{\tilde{E}}\right) \tag{6.36a}$$
$$\dot{\tilde{H}} = -\frac{8\pi G}{2}\left(\gamma\rho_\gamma - 4J\dot{\tilde{E}}\right) \tag{6.36b}$$
$$\dot{\rho}_\gamma = -3\tilde{H}\gamma\rho_\gamma \tag{6.36c}$$
$$\ddot{\tilde{E}} = -\dot{\tilde{E}}\left(\frac{1}{3J} + \tilde{H}\right) \tag{6.36d}$$
$$\dot{J} = \frac{1}{2} - 3J\tilde{H}, \tag{6.36e}$$

where $J \equiv \frac{4\epsilon}{\mu^4}H^3$, a combination which has been defined for convenience.

Dynamical systems analysis

We cannot perform a straightforward analytical calculation of the solutions of the theory as we did before, though we can formally write the solutions to the above equations. Note in particular

$$J(t) = J(0)e^{-3\int_0^t \tilde{H}(t')dt'} + \frac{1}{2}\int_0^t e^{-3\int_{t'}^t \tilde{H}(t'')dt''}dt', \tag{6.37}$$

and

$$\dot{\tilde{E}}(t) = \dot{\tilde{E}}(0)e^{-\int_0^t\left(\frac{1}{3J(t')}+\tilde{H}(t')\right)dt'}. \tag{6.38}$$

These confirm that $H > 0$ and $\dot{\tilde{E}} \to 0$ as $t \to \infty$, as well as the rapid decay of any initial transient in H, *under the assumption* $\tilde{H} > 0$. Unfortunately, solving this system in full generality will not be possible, so to gain further insight, we use a dynamical systems approach.

Introduce the dimensionless variables

$$x \equiv \frac{8\pi G\tilde{E}}{6\tilde{H}^2} \qquad y \equiv -\frac{8\pi GJ\dot{\tilde{E}}}{\tilde{H}^2} \qquad z \equiv J\tilde{H} \equiv \frac{4\epsilon}{\mu^4}H^3\tilde{H}, \tag{6.39}$$

and, as in Section 6.2.2, obtain a constraint equation by dividing (6.36a) through by $\tilde{H}^2$, resulting in

$$x + y + \frac{8\pi G\rho_\gamma}{3\tilde{H}^2} = 1. \tag{6.40}$$

If we make the (reasonable) assumption that $\rho_\gamma \geq 0$, then (6.40) implies,

$$x + y \leq 1. \tag{6.41}$$

Point	$\bar{x}$	$\bar{y}$	$\bar{z}$	$\Omega_\star$	$\gamma_\star$
A	0	0	$\frac{1}{3(\gamma+2)}$	0	undefined
B	1	0	$\frac{1}{6}$	1	0
C	$\frac{5}{14}$	$\frac{9}{14}$	$\frac{7}{60}$	1	$\frac{6}{7}$

Table 6.2: Critical points for the set of equations (6.45). Their stability is given in Table 6.3.

Our density parameters for the components in our theory are given by

$$\Omega_\gamma \equiv \frac{8\pi G\rho_\gamma}{3\tilde{H}^2} \qquad \Omega_\star \equiv x + y, \tag{6.42}$$

and, clearly from (6.40), $\Omega_\gamma + \Omega_\star = 1$. This also makes it clear that if we demand $\Omega_\star \geq 0$, then we obtain the constraint,

$$x + y \geq 0. \tag{6.43}$$

Since we are treating the modifications to gravity as an effective fluid component, we can define its equation of state,

$$\gamma_\star \equiv \frac{4y}{3(x+y)}. \tag{6.44}$$

So if $y = 0$, then $\gamma_\star = 0$ and it behaves like a cosmological constant. If $y = 3x$, it behaves like a matter component since $\gamma_\star = 1$, and so on. In general, since x and y will change with time, $\gamma_\star$ will change with time, though it will take on fixed values at the critical points.

By taking the time derivative and again introducing a dimensionless time N via $\frac{d}{dN} = \frac{d}{d\ln a} = \frac{1}{\tilde{H}}\frac{d}{dt}$ (and assuming $\tilde{H} > 0$),

$$\frac{dx}{dN} = -\frac{y}{6z} + 4xy + 3\gamma x(1 - x - y) \tag{6.45a}$$

$$\frac{dy}{dN} = \frac{y}{6z} - 4y(1 - y) + 3\gamma y(1 - x - y) \tag{6.45b}$$

$$\frac{dz}{dN} = \frac{1}{2} - 3z - 2yz - \frac{3}{2}\gamma z(1 - x - y). \tag{6.45c}$$

Note that if we restrict ourselves to the surface in $\{x, y, z\}$ space defined by $y = 0$, these equations coincide with those we obtained for the GR branch in (6.20).

We are again able to find the critical points of this system, which we display in Table 6.2, along with their stability in Table 6.3. In addition to the two points on the $y = 0$ plane which we observed previously, A and B, whose stability properties remain unchanged, there is also a new saddle point introduced, Point C. This corresponds to a tuning of $\tilde{E}$ and $\dot{\tilde{E}}$ such that their ratio to each other and to the

Point	Eigenvalues	Stability
A	$\left(3\gamma, -6+7\gamma, -\frac{3}{2}(\gamma+2)\right)$	Saddle
B	$(-3\gamma, -3, -3)$	Stable
C	$\left(-\frac{3}{7}(2+\sqrt{74}), \frac{3}{7}(-2+\sqrt{74}), -\frac{3}{7}(-6+7\gamma)\right)$	Saddle

Table 6.3: Stability of the critical points listed in Table 6.2, for the set of equations (6.45).

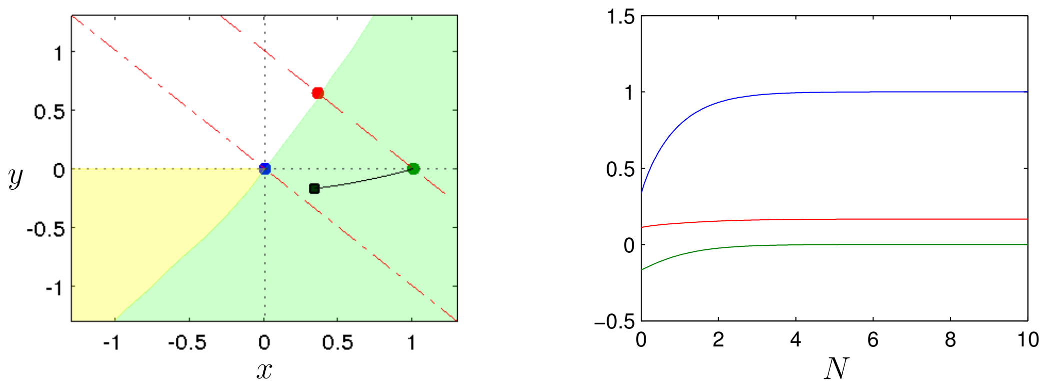
Hubble parameter $\tilde{H}$ remains fixed. This is, unsurprisingly, a saddle point since moving away from the point breaks the tuning.

Since we have a point (B) whose stability matrix eigenvalues are all negative, it is tempting to conclude that we have an attractor in the theory corresponding to the modification to gravity behaving like a cosmological constant at late times, regardless of the initial conditions. Unfortunately, this is not necessarily true. The stumbling block is our assumption that $\tilde{H} > 0$ throughout the universe's evolution. $\tilde{H} < 0$ causes us three problems. In order to change sign and go from positive to negative, $\tilde{H} = 0$ at some point, and it is easy to see that our variables x, y blow up in this limit, meaning our approach breaks down. In addition, since $\tilde{H} < 0$ changes our direction of time ($N \to -N$), the time variable we chose for the system is non-monotonic. This reversal of the time direction also means that we should reverse the eigenvalues of our stability matrix. Stable points thus become unstable ones, and so point B is no longer an attractor in this case. We will alter our formalism to try and get around these problems. First, however, we illustrate a specific example of initial conditions which lead to $\tilde{H} > 0$ and the behaviour we expect, as well as highlighting the regions in initial condition space which correspond to such an evolution.

Consider Figure 6.2, where we numerically evolve a universe with initial conditions and $\tilde{H} > 0$ throughout. Figure 6.2a shows a projection onto the x–y plane and the behaviour of the values of x, y, z with time is shown in Figure 6.2b. Figure 6.2b makes it clear that the system reaches the stable point B, and does not evolve from there.

Figure 6.2a is more complicated and contains a number of lines, points and shadings which need to be explained. The black square indicates the projection of the initial conditions chosen onto the x–y plane, and the system's evolution follows the black line. The red, green and blue circles denote Points A, B and C respectively. Given this, it can then be seen that the system's evolution is from the initial conditions at $(\frac{1}{3}, -\frac{1}{6})$ to the attractor at Point B.

The red lines mark our assumptions on the non-negativity of Ω_γ and $\Omega_\star$. The



(a) A projection onto the x–y plane of phase space.

(b) The evolution of x (blue), y (green) and z (red) with time.

Figure 6.2: The evolution resulting from the initial conditions $(x, y, z) = (\frac{1}{3}, -\frac{1}{6}, \frac{1}{9})$.

region to the top and right of the dashed red line corresponds to $x + y > 1$, or $\Omega_\gamma < 0$, which is considered unphysical (although good late-time behaviour can still be obtained in some of this region!). The region to the bottom and left of the dot-dashed red line corresponds to $x + y < 0$, or $\Omega_\star < 0$. We do not necessarily have as good a motivation to demand that this region is avoided, since $\Omega_\star$ is just some effective fluid rather than a true one[9].

The remaining shadings represent different late time behaviours obtained by starting the system in different regions of the x–y plane[10]. Evolution starting in the light green region will end at the attractor, Point B, and have $\tilde{H} > 0$ throughout. Outside this region, $\tilde{H}$ becomes negative at some time. In the yellow region, the sign of $\tilde{H}$ oscillates, and it ultimately ends up at Point B. In the unshaded region, $\tilde{H} \to -\infty$ in finite time, and so a big crunch occurs, and Point B is not approached. The behaviour of these two regions will be demonstrated in the next section.

A more traditional phase portrait is shown in Figure 6.3. To produce this, we have restricted our dynamical system to the $z = \frac{1}{9}$ plane. This kind of representation of the phase plane can be misleading because of our restriction to the $z = \frac{1}{9}$ plane. Whilst we could calculate the directional trajectory for any point in the $\{x, y, z\}$ space, displaying this information graphically in a clear manner is non-trivial. To avoid potential confusion, we instead use our projections onto the $\{x, y\}$ plane, as

[9] For example, it is not inconsistent for the density parameter for the curvature, $\Omega_{\tilde{\kappa}} \equiv -\frac{\tilde{\kappa}}{\tilde{a}^2 \tilde{H}^2}$ to be negative.

[10] To obtain these colours, one also has to make an assumption about the initial condition of z. Here $z = 1/9$ initially. The choice of z will slightly move the borders of the regions, but make no qualitative change to the discussion.

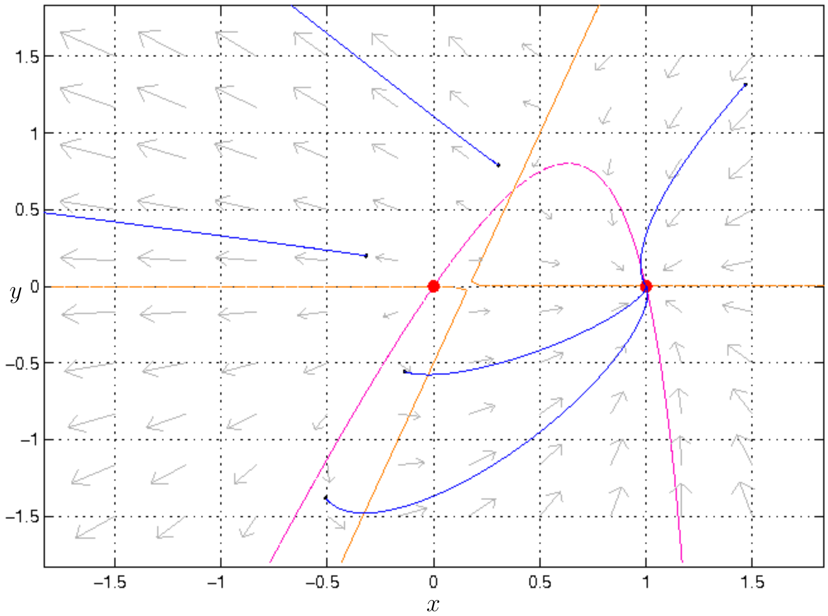


Figure 6.3: A phase portrait of the $z = \frac{1}{9}$ plane in $\{x, y, z\}$ space. The red spots indicate fixed points, the blue lines are the evolution of the system from a selection of initial conditions. The orange and pink lines are nullclines. The arrows indicate the projection into the $z = \frac{1}{9}$ plane of the gradient vector $\frac{d\mathbf{x}}{dN}$, evaluated for various points in the $z = \frac{1}{9}$ plane.

in Figure 6.2a. However, this diagram is still informative since it still accurately displays the flow directions over much of the phase space, in particular the stability properties of Points A and B are clear from this diagram.

Dynamical systems analysis in new variables

The evolution for all starting points in the green shaded region of Figure 6.2a can be well-described using the formalism we have detailed, but for the remainder of phase space, it will prove useful to define new variables which remain finite as $\tilde{H} \to 0$.

We introduce the dimensionless variables,

$$X \equiv \frac{\kappa^2 \tilde{E}}{6\left(\tilde{H}^2 + \tilde{H}_0^2\right)} \qquad Y \equiv -\frac{\kappa^2 J \dot{\tilde{E}}}{\left(\tilde{H}^2 + \tilde{H}_0^2\right)} \qquad Z \equiv J\tilde{H}_0 \qquad W \equiv \frac{\tilde{H}}{\tilde{H}_0}, \tag{6.46}$$

where $\tilde{H}_0 \equiv \tilde{H}(t = 0)$. These are related to our original variables by

$$x = \frac{1 + W^2}{W^2} X \qquad y = \frac{1 + W^2}{W^2} Y \qquad z = WZ. \tag{6.47}$$

W clearly has no analogue in our original variables. Clearly, $\tilde{H} \to 0 \Rightarrow W \to 0$, and in this limit X, Y, Z and W are all well-behaved, whereas x and y blow up.

We also encountered a problem with $\frac{d}{dN} = \frac{1}{\tilde{H}}\frac{d}{dt}$ changing sign, resulting in blow-ups and our time coordinate being non-monotonic. To combat this we introduce a new dimensionless time

$$T = \tilde{H}_0 t, \tag{6.48}$$

which is clearly monotone for monotonically increasing t.

From this, we write down the evolution equations for the system $\{X, Y, Z, W\}$,

$$\frac{dX}{dT} = -\frac{Y}{6Z} + 4WXY + 3\gamma WX\left(\frac{W^2}{1+W^2} - X - Y\right) \tag{6.49a}$$
$$\frac{dY}{dT} = \frac{Y}{6Z} - 4WY(1-Y) + 3\gamma WY\left(\frac{W^2}{1+W^2} - X - Y\right) \tag{6.49b}$$
$$\frac{dZ}{dT} = \frac{1}{2} - 3WZ \tag{6.49c}$$
$$\frac{dW}{dT} = -\left(1+W^2\right)\left[2Y + \frac{3}{2}\gamma\left(\frac{W^2}{1+W^2} - X - Y\right)\right]. \tag{6.49d}$$

Our stable point B[11] corresponds to a critical line along the W axis in these coordinates as $(X, Y, Z) = \left(\frac{W^2}{1+W^2}, 0, \frac{1}{6W}\right)$. Working in the subspace orthogonal to W, this looks like a point, and so we carry out the standard linear stability analysis. For fixed W, the eigenvalues of the stability matrix are

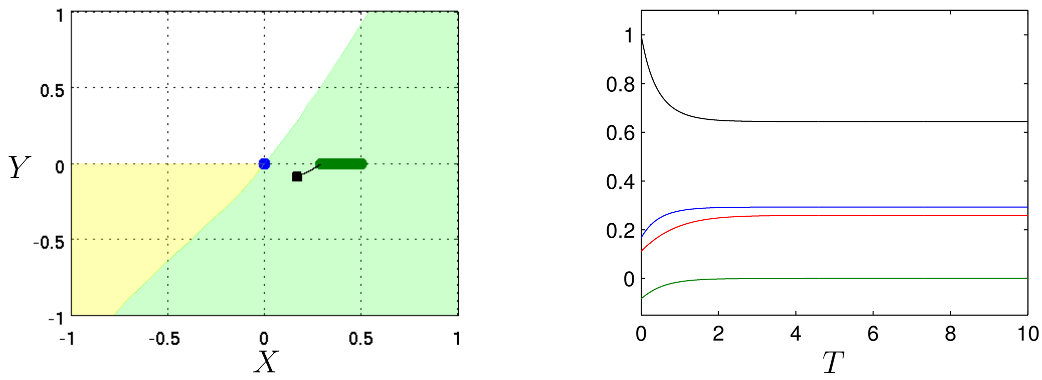
$$\left(-3W, -3W, \frac{-3W^3}{1+W^2}\right).$$

This point is stable if $W > 0$, but unstable in all directions if $W < 0$. This agrees with our earlier observation that Point B was only an attractor if $\tilde{H} > 0$.

As a first stage in our analysis, we now repeat the calculation leading to Figure 6.2 in these new variables. These are shown in Figure 6.4. Note that, as shown in (6.47), the critical points and the red lines drawn in 6.2a will move as the system evolves and W changes, so we do not plot the red lines or the Point C to avoid cluttering the diagram. The X–Y phase plane plot in Figure 6.4a demonstrates the same features as before. The system evolves from its starting black square to the (moving) green circle representing Point B. This interpretation can also be obtained by considering Figure 6.4b, where we see $X \to \frac{W^2}{1+W^2}$, $Y \to 0$, $Z \to \frac{1}{6W}$ and $W \to const$, at which point the system stops evolving and becomes stable.

We now move on to consider the unshaded region in our x–y or X–Y plots, and consider its intersection with the $0 \leq x + y \leq 1$ 'tramlines' of Figure 6.2a, where $\Omega_\gamma \geq 0$ and $\Omega_\star \geq 0$. The evolution of a universe with initial conditions in this region is shown in Figure 6.5, where we show the behaviour of $\gamma_\star$ alongside the diagrams we have previously shown. As is clear from Figure 6.5b, $\tilde{H}$ becomes

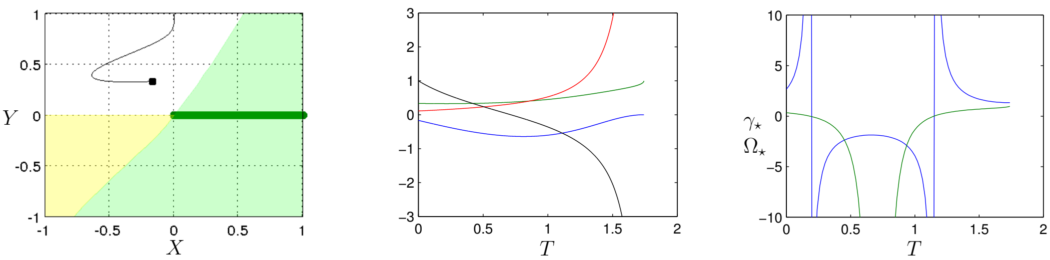
[11] Points A and C are not stable points in this system.



(a) A projection onto the X–Y plane of phase space.

(b) Evolution of X (blue), Y (green), Z (red) and W (black).

Figure 6.4: The evolution of the same system as Figure 6.2, but in our new variables. The initial conditions are now $(X, Y, Z, W) = (\frac{1}{6}, -\frac{1}{12}, \frac{1}{9}, 1)$.



(a)

(b)

(c)

Figure 6.5: The evolution of a system with initial conditions $(X, Y, Z, W) = (-\frac{1}{6}, \frac{1}{3}, \frac{1}{9}, 1)$.
(a) Shows a projection onto the X–Y plane of phase space.
(b) Shows X (blue), Y (green), Z (red) and W (black) evolving.
(c) The evolution of the effective equation of state $\gamma_\star$ (blue) and density parameter $\Omega_\star$ (green).

negative in the evolution, before diverging to $-\infty$ in finite time. Figure 6.5a shows that the system never reaches Point B. Point B ‘moves’ along the X-axis to the origin as $W \to 0$, before moving away from the origin as W^2 grows with W having passed through zero. Figure 6.5c shows that the effective fluid never has a constant equation of state.

At this stage, we have demonstrated that there are regions of phase space where either Point B is reached with $\tilde{H} > 0$, or else $\tilde{H}$ becomes negative and diverges to $-\infty$ in finite time, producing a big crunch with $\dot{\tilde{H}} < 0$ throughout. The former provides a good fit to our Λ dominated universe, while the latter does not describe our universe well, especially since we know $\dot{\tilde{H}} > 0$ presently. As previously alluded to, there is a third region, coloured yellow on the x–y and X–Y planes such as

Figure 6.2a, where $\tilde{H}$ becomes negative during its evolution, but still ultimately reaches Point B. We now consider this possibility.

As we approach the yellow region in the x–y plane, the trajectories is phase space start to loop before reaching the attractor at Point B, as can be seen in Figure 6.6. $\tilde{H} > 0$ throughout so we can use our original variables x, y, z to express this.

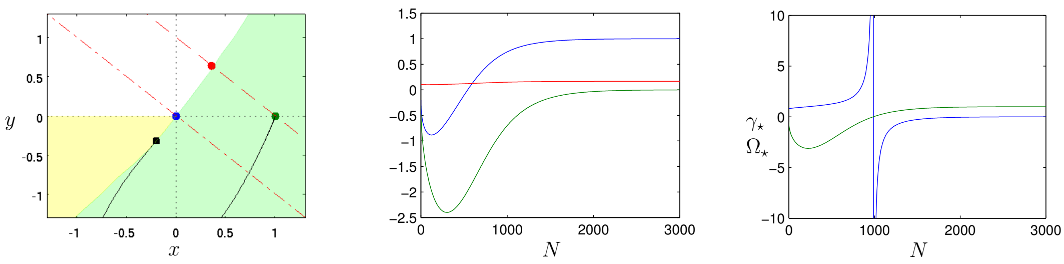
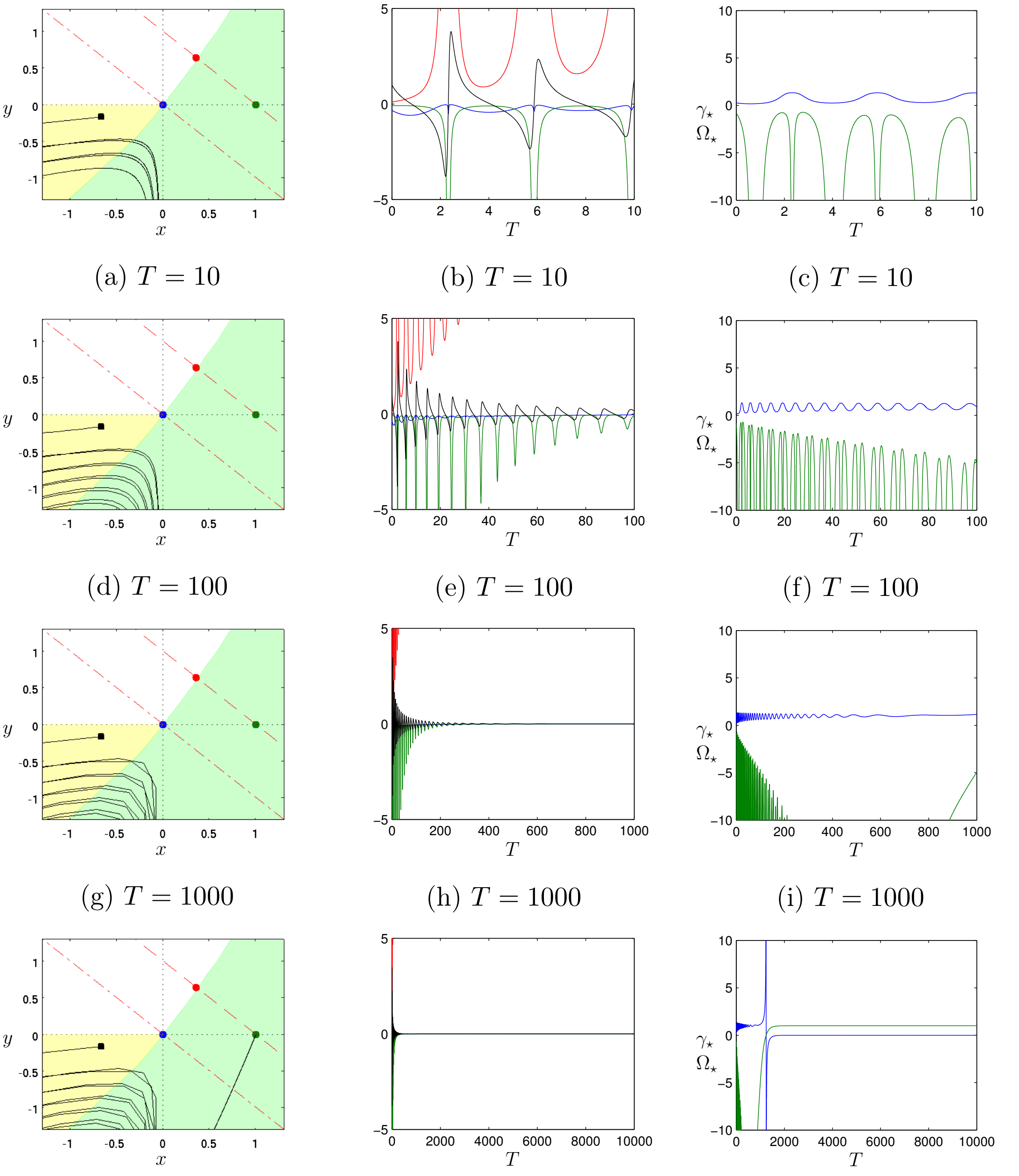


Figure 6.6: The evolution of a system with initial conditions $(x, y, z) = (-\frac{1}{5}, -\frac{8}{25}, \frac{1}{9})$.
(a) Shows a projection onto the x–y plane of phase space.
(b) Shows x (blue), y (green), z (red) evolving.
(c) The effective equation of state $\gamma_\star$ (blue) and density parameter $\Omega_\star$ (green).

While $\dot{\tilde{H}} < 0$ initially, $\dot{\tilde{H}} \to 0$ and the system does not pass through $\tilde{H} = 0$. However, if instead we move our initial conditions away slightly, into the yellow region, $\tilde{H}$ will become negative at some point in the evolution. We can no longer describe this using our (x, y, z) variables.

Such a scenario is demonstrated in Figure 6.7, which shows snapshots at various times in the system's evolution. In this case, we performed our numerical calculations using the $\{X, Y, Z, W\}$ system. However, since the resultant pictures are clearer, the phase plane plotted is the x–y plane, *not* the X–Y plane. These variables have been reconstructed using the relations (6.47). The time evolution in this plane is still the monotonically increasing T. Interpretation is simpler in this plane since the critical points do not move. The contents of Figure 6.7 clearly show that the system undergoes decaying oscillations in $\tilde{H}$, as well as $\dot{\tilde{E}}$ and J. Over time, the magnitude of these oscillations weakens, and over very long timescales, the system eventually reaches the attractor at Point B. The presence of these oscillating solutions the phase space is interesting and they do not offer a physically accurate match to the observed universe.

We have thus seen that there exist three régimes of solutions in this theory, based on initial conditions in the x–y plane. Regardless of whether one imposes the constraints of one or both of the red lines in our x–y diagrams, around half the phase space results in a viable, cosmological constant dominated, cosmology. This



(a) $T = 10$ (b) $T = 10$ (c) $T = 10$

(d) $T = 100$ (e) $T = 100$ (f) $T = 100$

(g) $T = 1000$ (h) $T = 1000$ (i) $T = 1000$

(j) $T = 10000$ (k) $T = 10000$ (l) $T = 10000$

Figure 6.7: The evolution of a system with initial conditions $(x, y, z) = (-\frac{2}{3}, -\frac{1}{6}, \frac{1}{9})$, shown at several snapshots of time T.
(a),(d),(g),(j) Show a projection onto the x–y plane of phase space.
(b),(e),(h),(k) Show X (blue), Y (green), Z (red), W (black) evolving.
(c),(f),(i),(l) Effective equation of state $\gamma_\star$ (blue) and density parameter $\Omega_\star$ (green).

at least suggests that one does not require fine tuning to exhibit cosmological constant-like behaviour in these non-GR solutions of the theory, although, of course, one must pick a more restrictive region to obtain a value for this cosmological constant consistent with observations.

6.4 Discussion

We have investigated cosmology in a generalisation of unimodular gravity, considering non-GR cosmologies and their qualitative behaviour. Having derived the equations of motion of a model we know possesses non-GR solutions, we first determined the cosmological behaviour of the system (fundamental and physical metrics) for the GR-like solutions in the model. We demonstrated that in this case, the system behaved like GR plus a cosmological constant, and the fundamental metric was well-behaved.

Having gained this insight, we moved on to the non-GR solutions. It transpired that there were three classes of solutions. One régime led to late time cosmological constant-like behaviour, with $\gamma \to 0$ as $t \to \infty$, mimicking the GR solution of the theory. We understood this as corresponding to an attractor in the theory. However, this critical point was only an attractor in the case that $\tilde{H} > 0$. The two other régimes involved the physical Hubble parameter $\tilde{H}$ becoming negative, and resulted in wildly different cosmologies. One resulted in an oscillating cosmology, and the other in a 'big crunch' cosmology. Both of these are considerably different cosmologies to what we observe. However, around half the parameter space appeared to exhibit late-time cosmological constant-type behaviour, meaning that despite the presence of extra solutions, it does not appear that unreasonable fine tuning is necessary to obtain Λ-like behaviour, at least at the level of background cosmology.

These extra solutions arise by not imposing the condition $\tilde{E} = const$. Unfortunately, this turns out to also affect the connection Ξ we introduced. With $\tilde{E} \neq const$, Ξ can no longer be chosen arbitrarily, but instead is chosen dynamically. This results in a solution with $R_{GB}(\Xi, g) = 0$. This is problematic, since our demonstration of the ghost-free property depended on the form of Ξ, and so we have no reason to expect that this theory is ghost-free. However, this may not be the case with all ghost-free variants of the theory, which may be able to remain viable despite the fact that $\dot{\tilde{E}} \neq 0$. This issue deserves further consideration.

Ultimately, it appears that while there is naïve similarity between the GR and certain non-GR solutions, at a deeper level the additional solutions allow pathologies such as ghosts to enter the theory. One might want to restrict the theory to the GR solution with a Levi-Civita connection for Ξ by setting $\dot{\tilde{E}} = 0$ by fiat, but there is no good reason for forcing this condition to hold. It appears that the most promising approach may well be to construct theories which do not have any additional solutions to those in GR (such as those without derivatives of g in Ω), which are guaranteed to match Einstein's equations with an arbitrary cosmological constant, and the freedom that exists to choose the fundamental fields other than the metric may help permit a viable theory to exist. An alternative approach may be to attempt to identify some symmetry or other mechanism for ensuring that only the GR solutions of the theory are ever realised.

Chapter 7

Discussion

7.1 Summary

This thesis has discussed models of modified gravity, in the quantum and classical régimes. Even as GR approaches its centenary, there is no obvious candidate for a usurper. The two specific examples of modified gravity considered here both indicate in their own ways how hard finding viable alternatives to GR is.

In Chapter 1, we highlighted the successes of General Relativity, at both a theoretical and observational level. We then moved on to discussing its flaws, in particular evidence from both the high and low ends of the energy spectrum that GR must ultimately be modified. The perils of such approaches were then discussed, in particular noting that the additional modes present in alternative theories often prove dangerous for the viability of the theory.

In Chapter 2, we introduced a modified theory of gravity for the *quantum* realm, Hořava gravity. The set-up of the model was given, showing how Lorentz invariance can be broken in the gravitational sector in order to improve quantum behaviour. Loop corrections diverge less, since there are higher order spatial derivatives in the propagator, but Ostrogradski ghosts are avoided by remaining second order in time. While the potential for good quantum behaviour is exciting, we also highlighted some of the downsides of such a model, particularly focusing on the behaviour of the additional mode which arises due to the breaking of diffeomorphism invariance. By reviewing the literature, we suggested that the 'healthy extension' was the variant of the theory with the most promise, avoiding strong coupling, tachyonic and ghostly instabilities, and issues with matter coupling and cosmology. But there remained disputes in this theory about the presence of strong coupling and the necessity of introducing a hierarchy of scales.

In Chapter 3, we set out to resolve these disputes. By using the Stückelberg trick and employing a decoupling limit we were able to isolate the behaviour of the extra mode from the general relativistic behaviour. After showing that we could recover previously obtained results in our limit, we then demonstrated that not only was it *necessary* to introduce a hierarchy of scales to evade strong coupling, but that avoiding strong coupling was not possible if the theory possessed our decoupling limit. Having obtained a formalism where the Stückelberg dynamics were easier to work with, we used this to our advantage, explicitly demonstrating that Hořava gravity violated the Weak Equivalence Principle. This potentially pathological issue has not been noted previously, and raised the prospect of being able to use Equivalence Principle tests to constrain the theory.

In Chapter 4, we considered the effect of adding matter into Hořava's theory. Working at a classical level, we constructed actions for scalar and gauge fields consistent with the $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ symmetry of Hořava gravity. We then proved that these actions would lead to violations of the Equivalence Principle unless the new terms permitted by the reduced symmetry vanished. While at a classical level it is permissible to set coefficients in the action to zero by fiat to remove unwanted terms, this isn't true quantum mechanically — unless protected by a symmetry, quantum effects can (and generically will) re-introduce such terms.

Motivated by this, we moved on to calculate quantum corrections to a tree level Lorentz invariant matter (scalar or gauge) field coupled to Lorentz violating gravity, focusing on one-loop corrections to the propagator. We elected to work with the propagating degrees of freedom: the matter scalar, tensor graviton and one scalar graviton. To achieve this, we expanded around Minkowski space, performed a partial gauge fixing, and integrated out the remaining non-physical modes using the constraints. Despite the fact that we were forced to approximate the answers to the resultant loop integrals, we were able to find interesting features. The large coupling between the scalar graviton and matter sector resulted in a $1/\alpha \sim 10^7$ correction to the light cone. Unless replicated for all particles, this would result in a fine tuning to ensure all particles see the same speed of light. We saw that generically, higher order spatial derivatives would be generated, with a suppression scale $M_\star$. This was expected since these were consistent with the symmetries of the theory and by the old adage of quantum theory: "if it's not forbidden, it's compulsory" they would appear. Unfortunately, they were joined by higher order time derivatives. The $\mathrm{Diff}_{\mathcal{F}}(\mathcal{M})$ symmetry was designed precisely to protect against these sort of terms and the associated ghosts. We argued that their generation might be related to the fact we did not have a purely $z = 3$ theory — if the only terms across the matter and gravity sector were scaling dimension 6, we could rewrite the action as a $1 + 1$D model, which is conformally invariant, but this was spoilt by the presence of scaling dimension 2 and dimension 4 operators.

In Chapter 5, we moved on to discuss a *classical* modification of gravity. In particular, a new model to tackle the cosmological constant problem was proposal. By coupling matter to a *composite* metric, we were able to remove any contribution to the gravitational field equations from the vacuum energy by making the volume element a total derivative. The value of the cosmological constant just arose as an integration constant, and could be freely chosen without any worries about fine tuning it against particle physics contributions. By constructing the gravitational action as the Ricci scalar of the composite metric, we also saw we could obtain a theory whose solutions for the physical metric were equal to those of Einstein's equations, ensuring it passed solar system tests. After demonstrating that this idea could be considered a generalisation of unimodular gravity, we derived the equations of motion in the case that the composite metric was conformally related to a fundamental one. Since it appeared that our proposal had a generic risk of encountering Ostrogradski ghosts, we calculated a specific example in which (to the second-order action on a maximally symmetric background) this was not the case. By considering a theory with an additional, independent connection Ξ, a constraint arose which was able to exorcise the ghost, and the remaining degrees of freedom were just the same as GR.

In Chapter 6, we moved on to discuss cosmology and non-GR solutions in this proposal. While a solution of GR is always a solution in this theory (for the composite metric), the converse does not necessarily hold. We proved a sufficient condition for a theory's solution space to match that of GR, using this to help motivate a choice of model with non-GR solutions. Wanting to understand the effect of these new solutions on the physical cosmology and the fundamental fields, we considered the GR-like solutions to gain some relevant intuition. Having understood their properties, and demonstrated that the fundamental field was well behaved, we moved on to the non-GR solutions. Over roughly half the phase space, the fundamental field was well behaved and the effect on physical cosmology was consistent with adding a cosmological constant. Unfortunately, choosing to consider non-GR solutions came at a cost, since it forced us to break assumptions on the connection Ξ that we used to derive the ghost-free property of the theory. We concluded by suggesting that restricting to a theory with only GR solutions may be the safest approach to accurately describe our univese.

7.2 Future directions

While, as previously mentioned, modifying gravity (successfully) is very hard, the models here are able to tell us something about the viability of different approaches. In fact, since the same features are generic, occurring across multiple different theories, further investigations into these are likely to be able to tell us about more general modified gravity models.

Research in Hořava gravity would benefit greatly from additional phenomenological work. Recall that the avoidance of strong coupling and solar system tests place *upper* limits on $M_\star$, while Equivalence Principle and Lorentz Invariance violations can place *lower* limits. It is possible that tightening these limits will lead to an inconsistency of the two bounds, ruling out the theory. An example of this would be to use our work on loop corrections to ordinary matter in Hořava gravity, and calculate the resultant size of Equivalence Principle violations.

Lorentz violation is also a looming spectre for Hořava gravity, and the results may turn out to be applicable in other Lorentz violating theories of gravity. Many of our best tests of Lorentz invariance involve fermionic systems and so it would be a very worthwhile exercise to repeat our analysis of Chapter 4 for fermions. This procedure will be more involved than our calculation, due to the necessity of introducing vielbeins, but the results (and their comparison with those for scalar and vector fields) may be directly able to rule out the theory.

It has been argued schematically that adding terms like $(D_iK)^2$ to the theory would help avoid the fine tuning of light cones and it would be interesting to repeat our analysis and that of [100] to determine whether this effect really does resolve this issue. We would also like to determine whether it has any effect on the higher dimension (particularly time) operators. Altering the Lifshitz scaling in this manner also permits additional terms to be added to the action. We could investigate which new terms are consistent with the symmetry classically, and which terms generated by quantum corrections. For example $\partial_t^2\partial_i^2$ terms in the tensor gravity or matter sectors may be generated, and it would be interesting to see if the presence of these terms can be made consistent with observation.

The parameter λ and its renormalisation group flow is very important in Hořava gravity. It is desired that it flows under the RG to 1 in the IR (though it should not take the value $\lambda = 1$ precisely), but this has never been demonstrated. This may be possible in the context of holography, by considering a 4+1D Lorentz violating version of Randall-Sundrum II, where varying the size of the extra dimension corresponds to varying the energy.

The proposal for cleaning up the cosmological constant also has a number of interesting directions. It would be interesting to try and work with the full (non-linear) degrees of freedom by using the Hamiltonian. By trying to keep as much generality as is practicable, it may be possible to understand what conditions on the proposal allow one to obtain a ghost-free theory. This may be a particularly useful approach, since far more exotic models than the conformal ones we considered are possible. Disformally related metrics may prove a fruitful avenue, or considering alternative kinetic terms to avoid the ghost (and rely on non-linear effects to pass solar-system tests). All of these merit further investigation.

The theory has only been considered classically thus far, but quantum effects must ultimately be investigated to know whether the proposal is a viable model of the universe. The analogy with unimodular gravity may be highly useful in this context. It would be particularly interesting to understand whether, as in unimodular gravity, a renormalisation group analysis reveals the potential for a UV fixed point.

Understanding the extra solutions of the theory, by either proving or finding a counterexample to our conjecture about GR solutions would be a helpful step. It would be very useful to know whether allowing non-GR solutions always ruins your ghost-freeness property and whether the existence of a large region of parameter space where the theory mimics a cosmological constant, even in a supposedly non-GR solution, is a generic feature. In addition, we have considered only background cosmology thus far. Theories which mimic ΛCDM at the background level can have differing predictions at the level of perturbations. While ΛCDM can be matched when the degrees of freedom are the same as those of GR and we consider GR solutions, working with the non-GR solutions and extra or differing degrees of freedom may result in this not being the case.

Investigating these non-GR solutions outside the realm of cosmology would be very interesting. For example, is there any additional effect which occurs when considering the analogue of the Schwarschild solution for the non-GR branch? Do Birkhoff's theorem or the non-hair theorems still hold? This could have direct phenomenological consequences for observations of stars or black holes. The arbitrariness of Ξ in the case where we had a GR solution, $\dot{\hat{E}} = 0$, suggests the presence of some sort of symmetry. Understanding the precise nature of this symmetry might be a interesting avenue, in particular whether it is able to help provide a mechanism to restrict us to the GR solutions when they are not necessarily the only possible ones.

Appendices

Appendix A

Hořava gravity calculation details

A.1 Derivatives of potential terms

Below are the first derivatives of the potential $\mathcal{V}_4$ used in chapter 3.

$$\begin{aligned}\frac{\partial \mathcal{V}_4}{\partial \tilde{a}_\mu} =& 4A_1(\tilde{a}^\nu \tilde{a}_\nu)\tilde{a}^\mu + 2A_2 \tilde{a}^\mu \tilde{a}^\nu{}_\nu \\ &+ 2B_1(\tilde{\mathcal{K}}_{\rho\sigma}\tilde{\mathcal{K}}^{\rho\sigma} - \tilde{\mathcal{K}}^2)\tilde{a}^\mu + 2B_2(\tilde{\mathcal{K}}_{\mu\alpha}\tilde{\mathcal{K}}^\alpha_\nu - \tilde{\mathcal{K}}\tilde{\mathcal{K}}_{\mu\nu})\tilde{a}^\nu\end{aligned} \tag{A.1}$$

$$\begin{aligned}\frac{\partial \mathcal{V}_4}{\partial \tilde{a}_{\mu\nu}} =& A_2(\tilde{a}^\rho \tilde{a}_\rho)\eta^{\mu\nu} + 2A_3(\tilde{a}^\rho{}_\rho)\eta^{\mu\nu} + 2A_4 \tilde{a}^{\mu\nu} \\ &+ B_3(\tilde{\mathcal{K}}_{\sigma\rho}\tilde{\mathcal{K}}^{\sigma\rho} - \tilde{\mathcal{K}}^2)\eta^{\mu\nu}\end{aligned} \tag{A.2}$$

$$\begin{aligned}\frac{\partial \mathcal{V}_4}{\partial \tilde{\mathcal{K}}_{\mu\nu}} =& 2B_1(\tilde{\mathcal{K}}^{\mu\nu} - \tilde{\mathcal{K}}\eta^{\mu\nu})\tilde{a}^\rho \tilde{a}_\rho + B_2(2\tilde{\mathcal{K}}^{(\mu}_\rho \delta^{\nu)}_\sigma - \eta^{\mu\nu}\tilde{\mathcal{K}}_{\rho\sigma} - \tilde{\mathcal{K}}\delta^{(\mu}_\rho \delta^{\nu)}_\sigma)\tilde{a}^\rho \tilde{a}^\sigma \\ &+ 2B_3(\tilde{\mathcal{K}}^{\mu\nu} - \tilde{\mathcal{K}}\eta^{\mu\nu})\tilde{a}^\rho{}_\rho + 4C_1(\tilde{\mathcal{K}}_{\rho\sigma}\tilde{\mathcal{K}}^{\rho\sigma} - \tilde{\mathcal{K}}^2)(\tilde{\mathcal{K}}^{\mu\nu} - \tilde{\mathcal{K}}\eta^{\mu\nu}) \\ &+ 2C_2(2\tilde{\mathcal{K}}^{\alpha(\mu}\delta^{\nu)}_\rho - \eta^{\mu\nu}\tilde{\mathcal{K}}^\alpha_\rho - \tilde{\mathcal{K}}\delta^{(\mu}_\rho \eta^{\nu)\alpha})(\tilde{\mathcal{K}}^\rho_\beta \tilde{\mathcal{K}}^\beta_\alpha - \tilde{\mathcal{K}}\tilde{\mathcal{K}}^\rho_\alpha)\end{aligned} \tag{A.3}$$

A.2 Determining the strong coupling scale

In section 3.4, we stated that the appropriate strong coupling scale was given by the interaction term in equation (3.35), leading to a strong coupling scale of $\Lambda_{sc} \sim \frac{1}{l_\alpha} c_s^{3/2}$. Here we will demonstrate that this is the appropriate scale, by virtue of having the highest power of c_s. Recall that the scales at which the various terms become strongly coupled are given by equation (3.48), repeated here,

$$\Lambda^\alpha_{(a,n)} = \frac{1}{l_\alpha} c_s^{\frac{1}{2} + \frac{2-a}{n-2}}, \qquad \Lambda^{\mathcal{K}}_{(a,n)} = \frac{1}{l_\alpha} c_s^{\frac{1}{2} - \frac{a}{n-2}}, \qquad \Lambda^V_{(a,n)} = \frac{1}{l_\alpha} c_s^{\frac{1}{2} + \frac{1-a}{n}}. \tag{3.48}$$

Let us begin by considering the $\Lambda^\alpha_{(a,n)}$ terms, We want to identify the terms with the highest powers of c_s, which will result in the lowest strong coupling scale, since

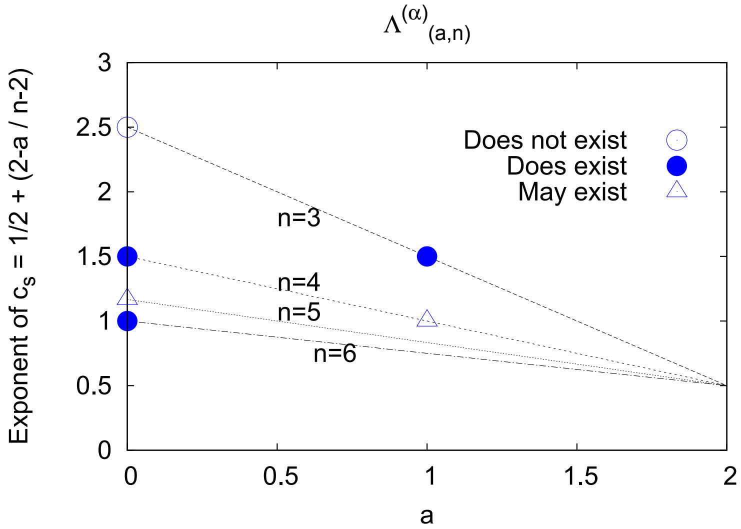


Figure A.1: The exponent of c_s to which various potential strong coupling terms in $\Lambda^{\alpha}_{(a,n)}$ are raised.

$c_s \leq 1$. It is also necessary to bear in mind that n, a are restricted to be integers and that not all the 'possible' terms appear in the perturbative expansion. The exponent of c_s in the $\Lambda^{\alpha}_{(a,n)}$ terms is plotted in Figure A.1. It is clear that the strongest coupling would be given by the term $(a = 0, n = 3)$, but this is not present in the expansion. Instead, the two terms corresponding to $(a = 1, n = 3)$ and $(a = 0, n = 4)$ have the greatest exponent of c_s of any terms present and so result in the lowest energy scale $\Lambda^{\alpha}_{(1,3)} = \Lambda^{\alpha}_{(0,4)} = \Lambda_{sc} \sim \frac{1}{l_\alpha} c_s^{3/2}$. Furthermore, it is clear from equation (3.48) that that is the smallest scale, since $\min \Lambda^{\mathcal{K}}_{(a,n)} = \frac{1}{l_\alpha} c_s^{1/2} = \Lambda_{sc}/c_s$ and $\min \Lambda^{V}_{(a,n)} = \frac{1}{l_\alpha} c_s^{5/6} = \Lambda_{sc}/c_s^{4/6}$ will always be larger than Λ_{sc} for $c_s \leq 1$.

Thus the terms which lead to strong coupling, and set the strong coupling scale, are the $(a = 1, n = 3)$ and $(a = 0, n = 4)$ terms from $S^{\alpha}_{(a,n)}$.

A.3 Scalar Field Vertices

This section contains the explicit form of the vertices shown in Figure 4.5, repeated below as Figure A.2 for convenience. In some cases, the permutations will be done explicitly. In other cases, this will not be done (for clarity). Where it is not performed, a summation $\sum_\pi$ is written explicitly, with the explicit permutations π written under the vertex. Recall that all momentum k_i are ingoing. In this section only, M_{pl} has been set equal to one, it is trivial to restore these factors by dimensional analysis.

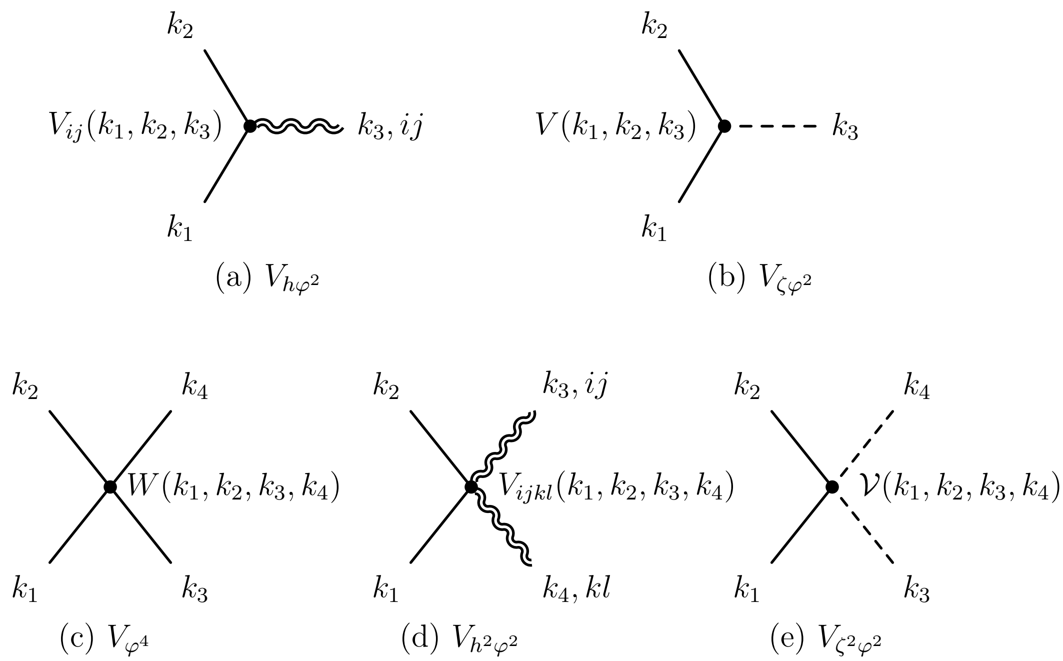


Figure A.2: A repeat of Figure 4.5. Three and four point vertices for the dynamical fields.

Three point φh vertex — $V_{h\varphi^2}$

The diagram for the $h_{ij}\varphi^2$ vertex is shown in Figure 4.5a where

$$V_{ij}(k_1, k_2; k_3) = -\mathbf{k}_{1(i}\mathbf{k}_{2j)} \tag{A.4}$$

Three point $\varphi\zeta$ vertex — $V_{\zeta\varphi^2}$

The diagram for the $\zeta\varphi^2$ vertex is shown in Figure 4.5b, where

$$\begin{aligned} V(k_1, k_2; k_3) = &-3\omega_1\omega_2 + \mathbf{k}_1 \cdot \mathbf{k}_2 - 3m^2 - 2f(\mathbf{k}_3)(\omega_1\omega_2 + \mathbf{k}_1 \cdot \mathbf{k}_2 - m^2) \\ &- \frac{1-3\lambda}{1-\lambda}\frac{\omega_3}{|\mathbf{k}_3|^2}(\mathbf{k}_1 + \mathbf{k}_2) \cdot (\omega_1\mathbf{k}_2 + \omega_2\mathbf{k}_1) \end{aligned} \tag{A.5}$$

Four point φ vertex — V_{φ^4}

The diagram for the φ^4 vertex is shown in Figure 4.5c, where

$$\begin{aligned}W(k_1,k_2,k_3,k_4) =& -\frac{1}{16}\sum_{\pi}\frac{1}{|\mathbf{k}_3+\mathbf{k}_4|^2}\frac{\left(\omega_1\omega_2+\mathbf{k}_1\cdot\mathbf{k}_2-m^2\right)\left(\omega_3\omega_4+\mathbf{k}_3\cdot\mathbf{k}_4-m^2\right)}{\alpha+\frac{A_4}{M_{pl}{}^2}|k_3+\mathbf{k}_4|^2+\frac{B_4}{M_{pl}{}^4}|k_3+\mathbf{k}_4|^4}\\ &+\frac{1}{4}\frac{1}{1-\lambda}\sum_{\pi}\frac{1}{|\mathbf{k}_3+\mathbf{k}_4|^4}\omega_1\omega_3(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{k}_2(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{k}_4\\ &-\frac{1}{2}\sum_{\pi}\frac{1}{|\mathbf{k}_3+\mathbf{k}_4|^2}\omega_1\omega_3\pi_{ij}^{\mathbf{k}_3+\mathbf{k}_4}\mathbf{k}_{2i}\mathbf{k}_{4j}-\mu\end{aligned} \tag{A.6}$$

where $\sum_\pi$ means you sum over all permutations of $\{1,2,3,4\}$.

Four point φh vertex — $V_{h^2\varphi^2}$

The diagram for the $h_{ij}h_{kl}\varphi^2$ vertex is shown in Figure 4.5d, where

$$\begin{aligned}V_{ijkl}(k_1,k_2;k_3,k_4) =& \frac{1}{2}\delta_{i(k}\delta_{l)j}\left(\omega_1\omega_2-\mathbf{k}_1\cdot\mathbf{k}_2+m^2\right)+\delta_{jl}\left(\mathbf{k}_{1i}\mathbf{k}_{2k}+\mathbf{k}_{1k}\mathbf{k}_{2i}\right)\\ &+\frac{1}{|\mathbf{k}_3+\mathbf{k}_4|^2}\left[\frac{1}{\alpha}\langle\mathcal{H}_2\rangle\left(\omega_1\omega_2+\mathbf{k}_1\cdot\mathbf{k}_2-m^2\right)\right.\\ &\qquad-\frac{i}{1-\lambda}(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{k}_2\omega_1+\mathbf{k}_1\omega_2)\langle\mathcal{P}_2\rangle\\ &\qquad\left.+2(\omega_1\mathbf{k}_{2m}+\omega_2\mathbf{k}_{1m})\langle\mathcal{Q}_2\rangle_m\right]\end{aligned} \tag{A.7}$$

and

$$\begin{aligned}\langle\mathcal{H}_2\rangle =& \frac{1}{4}\delta_{i(k}\delta_{l)j}\left[-\omega_3\omega_4-A_1|\mathbf{k}_3|^2|\mathbf{k}_4|^2+B_1\mathbf{k}_3\cdot\mathbf{k}_4|\mathbf{k}_3|^2|\mathbf{k}_4|^2\right]\\ &+\frac{1}{2}\sum_{3\leftrightarrow4}\delta_{i(k}\delta_{l)j}\left[|\mathbf{k}_3|^2+\frac{3}{4}\mathbf{k}_3\cdot\mathbf{k}_4\right]\left[1-A_3|\mathbf{k}_3+\mathbf{k}_4|^2+B_3|\mathbf{k}_3+\mathbf{k}_4|^4)\right]\\ &-\frac{1}{4}\sum_{3\leftrightarrow4}\left[1-A_3|\mathbf{k}_3+\mathbf{k}_4|^2+B_3|\mathbf{k}_3+\mathbf{k}_4|^4)\right]\delta_{jl}\mathbf{k}_{3k}\mathbf{k}_{4i}\end{aligned} \tag{A.8a}$$

$$\langle\mathcal{P}_2\rangle = \frac{1}{2}\sum_{3\leftrightarrow4}(\pm i)\omega_4\left[\delta_{jl}\mathbf{k}_{4i}\mathbf{k}_{3k}-\frac{1}{2}(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{k}_3\delta_{i(k}\delta_{l)j}-\lambda|\mathbf{k}_3+\mathbf{k}_4|^2\delta_{i(k}\delta_{l)j}\right] \tag{A.8b}$$

$$\langle\mathcal{Q}_2\rangle_m = \frac{1}{2}\sum_{3\leftrightarrow4}\omega_4\pi_{mn}^{\mathbf{k}_3+\mathbf{k}_4}\left[-\mathbf{k}_{4i}\delta_{jl}\delta_{nk}+\frac{1}{2}\mathbf{k}_{3j}\delta_{i(k}\delta_{l)j}\right] \tag{A.8c}$$

Four point $\varphi\zeta$ vertex — $V_{\zeta^2\varphi^2}$

The diagram for the $\zeta^2\varphi^2$ vertex is shown in Figure 4.5e, where

$$
\begin{aligned}
&\mathcal{V}(k_1,k_2;k_3,k_4) = \\
&-3\omega_1\omega_2 - \mathbf{k}_1\cdot\mathbf{k}_2 - 3m^2 + 8f(\mathbf{k}_3)f(\mathbf{k}_4)\omega_1\omega_2 - \frac{1}{2}\left(\frac{1-3\lambda}{1-\lambda}\right)^2 \frac{\omega_3\omega_4}{|\mathbf{k}_3|^2|\mathbf{k}_4|^2}\sum_\pi(\mathbf{k}_1\cdot\mathbf{k}_3)(\mathbf{k}_2\cdot\mathbf{k}_4) \\
&-2\frac{1-3\lambda}{1-\lambda}\left(f(\mathbf{k}_3)\frac{\omega_4}{|\mathbf{k}_4|^2}\mathbf{k}_4 + f(\mathbf{k}_4)\frac{\omega_3}{|\mathbf{k}_3|^2}\mathbf{k}_3\right)\cdot(\omega_1\mathbf{k}_2+\omega_2\mathbf{k}_1) - 2\frac{1-3\lambda}{1-\lambda}\sum_\pi \frac{\omega_2\omega_4\mathbf{k}_3\cdot\mathbf{k}_1}{|\mathbf{k}_3|^2}f(\mathbf{k}_4) \\
&+\frac{1}{4\alpha}\frac{\omega_1\omega_2+\mathbf{k}_1\cdot\mathbf{k}_2-m^2}{|\mathbf{k}_1+\mathbf{k}_2|^2}\langle H_2\rangle + 4\mathbf{k}_1\cdot\mathbf{k}_2\left(f(\mathbf{k}_3)+f(\mathbf{k}_4)\right) - 16f(\mathbf{k}_3)f(\mathbf{k}_4)\omega_1\omega_2 \\
&-\frac{i}{1-\lambda}\frac{(\mathbf{k}_1+\mathbf{k}_2)}{|\mathbf{k}_1+\mathbf{k}_2|^2}\cdot(\mathbf{k}_2\omega_1+\mathbf{k}_1\omega_2)\langle P_2\rangle + 2\frac{\pi_{ij}^{\mathbf{k}_1+\mathbf{k}_2}}{|\mathbf{k}_1+\mathbf{k}_2|^2}\left(\omega_1\mathbf{k}_{2j}+\omega_2\mathbf{k}_{1j}\right)\langle Q_2\rangle_i \\
&+\frac{1}{|\mathbf{k}_1+\mathbf{k}_2|^2}(\omega_1\omega_2+\mathbf{k}_1\cdot\mathbf{k}_2-m^2)\left\{(\mathbf{k}_1+\mathbf{k}_2)\cdot[\mathbf{k}_3f(\mathbf{k}_3)+\mathbf{k}_4f(\mathbf{k}_4)] + f(\mathbf{k}_3)f(\mathbf{k}_4)\left[\mathbf{k}_3\cdot\mathbf{k}_4+(\mathbf{k}_1+\mathbf{k}_2)\cdot(\mathbf{k}_3+\mathbf{k}_4)\right]\right\} \\
&-\frac{4}{\alpha}\sum_\pi\left\{\left[A_3|\mathbf{k}_4|^2 - B_3|\mathbf{k}_4|^2\right]f(\mathbf{k}_3)\frac{\mathbf{k}_3\cdot(\mathbf{k}_1+\mathbf{k}_2)}{|\mathbf{k}_1+\mathbf{k}_2|^2}(\omega_1\omega_2+\mathbf{k}_1\cdot\mathbf{k}_2-m^2)\right\} \\
&+\frac{A_4}{\alpha}\sum_\pi(\omega_1\omega_2+\mathbf{k}_1\cdot\mathbf{k}_2-m^2)\Bigg\{f(\mathbf{k}_3)\left[|\mathbf{k}_3|^2 - \mathbf{k}_4\cdot(\mathbf{k}_1+\mathbf{k}_2)\frac{|\mathbf{k}_3|^2}{|\mathbf{k}_1+\mathbf{k}_2|^2} - \mathbf{k}_3\cdot\mathbf{k}_4\right] \\
&\qquad\qquad + f(\mathbf{k}_3)f(\mathbf{k}_4)\left[2\frac{|\mathbf{k}_3|^2|\mathbf{k}_4|^2}{|\mathbf{k}_1+\mathbf{k}_2|^2} + 4|\mathbf{k}_3|^2 + 4\mathbf{k}_4\cdot(\mathbf{k}_1+\mathbf{k}_2)\frac{|\mathbf{k}_3|^2}{|\mathbf{k}_1+\mathbf{k}_2|^2} + 2\mathbf{k}_3\cdot\mathbf{k}_4\right]\Bigg\} \\
&+\frac{B_4}{2\alpha}\sum_\pi(\omega_1\omega_2+\mathbf{k}_1\cdot\mathbf{k}_2-m^2)\Bigg\{f(\mathbf{k}_3)\Bigg[-|\mathbf{k}_3|^4 - |\mathbf{k}_3|^2|\mathbf{k}_1+\mathbf{k}_2|^2 + \mathbf{k}_3\cdot\mathbf{k}_4 + \mathbf{k}_4\cdot(\mathbf{k}_1+\mathbf{k}_2)\frac{|\mathbf{k}_3|^2}{|\mathbf{k}_1+\mathbf{k}_2|^2} \\
&\qquad\qquad + \mathbf{k}_4\cdot(\mathbf{k}_1+\mathbf{k}_2)|\mathbf{k}_3|^2 + \mathbf{k}_3\cdot\mathbf{k}_4|\mathbf{k}_3|^2 - 2|\mathbf{k}_3|^2|\mathbf{k}_4+\mathbf{k}_1+\mathbf{k}_2|^2 \\
&\qquad\qquad - 2|\mathbf{k}_3+\mathbf{k}_4|^2\mathbf{k_3}^2 + |\mathbf{k}_3+\mathbf{k}_4|^2\mathbf{k}_3\cdot\mathbf{k}_4 + |\mathbf{k}_3|^2\frac{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_4|^2}{|\mathbf{k}_1+\mathbf{k}_2|^2}\mathbf{k}_4\cdot(\mathbf{k}_1+\mathbf{k}_2)\Bigg] \\
&\qquad\qquad + 2f(\mathbf{k}_3)f(\mathbf{k}_4)\Bigg[4\mathbf{k}_3\cdot(\mathbf{k}_1+\mathbf{k}_2)\frac{|\mathbf{k}_4|^2}{|\mathbf{k}_1+\mathbf{k}_2|^2} + 2\mathbf{k}_3\cdot\mathbf{k}_4|\mathbf{k}_1+\mathbf{k}_2|^2 \\
&\qquad\qquad\qquad + 2|\mathbf{k}_3+\mathbf{k}_4|^2|\mathbf{k}_4|^2 + 2|\mathbf{k}_3|^2|\mathbf{k}_4|^2\frac{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|^2}{|\mathbf{k}_1+\mathbf{k}_2|^2} \\
&\qquad\qquad\qquad + 2|\mathbf{k}_4|^2|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_4|^2 + 2|\mathbf{k}_3+\mathbf{k}_4|^2\mathbf{k}_3\cdot\mathbf{k}_4 \\
&\qquad\qquad\qquad + 4|\mathbf{k}_4|^2\mathbf{k}_3\cdot(\mathbf{k}_1+\mathbf{k}_2)\frac{|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|^2}{|\mathbf{k}_1+\mathbf{k}_2|^2}\Bigg]\Bigg\} \\
&-\frac{2}{\alpha}\frac{(1-3\lambda)^2}{1-\lambda}(\omega_1\omega_2+\mathbf{k}_1\cdot\mathbf{k}_2-m^2)\frac{\omega_3\omega_4}{|\mathbf{k}_1+\mathbf{k}_2|^2}\left\{1-\frac{1}{1-\lambda}\left[\frac{(\mathbf{k}_3\cdot\mathbf{k}_4)^2}{|\mathbf{k}_3|^2|\mathbf{k}_4|^2}-\lambda\right]\right\} \\
&-8\frac{1-3\lambda}{(1-\lambda)^2}\sum_\pi f(\mathbf{k}_4)\omega_1\omega_3\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2)\left[\frac{((\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{k}_3)^2}{|\mathbf{k}|_3{}^2|\mathbf{k}_1+\mathbf{k}_2|^4} - \frac{\lambda}{|\mathbf{k}_1+\mathbf{k}_2|^2}\right] \\
&+4\frac{1-3\lambda}{1-\lambda}\sum_\pi f(\mathbf{k}_3)\frac{\omega_1\omega_4}{|\mathbf{k}_1+\mathbf{k}_2|^2}\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2) - 8\frac{1-3\lambda}{1-\lambda}\sum_\pi f(\mathbf{k}_4)\mathbf{k}_3\cdot(\mathbf{k}_1+\mathbf{k}_2)\frac{\pi_{ij}^{\mathbf{k}_1+\mathbf{k}_2}\mathbf{k}_{2i}\mathbf{k}_{3j}\omega_1\omega_3}{|\mathbf{k}_3|^2|\mathbf{k}_1+\mathbf{k}^2|^2} \\
&+2\frac{1-3\lambda}{1-\lambda}\frac{\omega_1\omega_3}{|\mathbf{k}_3|^2}\frac{\pi_{ij}^{\mathbf{k}_1+\mathbf{k}_2}}{|\mathbf{k}_1+\mathbf{k}_2|^2}\left\{\mathbf{k}_{3i}\left[(\mathbf{k}_3+\mathbf{k}_4)\cdot(\mathbf{k}_1+\mathbf{k}_2)+2\mathbf{k}_3\cdot\mathbf{k}_4\right] + \mathbf{k}_{4i}\left[\mathbf{k}_3\cdot(\mathbf{k}_1+\mathbf{k}_2)-(1-\lambda)|\mathbf{k}_3|^2\right]\right\} \\
&+\frac{1-3\lambda}{(1-\lambda)^2}\sum_\pi\frac{\omega_1\omega_3}{|\mathbf{k}_1+\mathbf{k}_2|^2}\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2)\Bigg\{2\frac{[(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{k}_3)]^2}{|\mathbf{k}_1+\mathbf{k}_2|^2|\mathbf{k}_3|^2} - 2\lambda + 4\frac{\mathbf{k}_3\cdot(\mathbf{k}_1+\mathbf{k}_2)\mathbf{k}_4\cdot(\mathbf{k}_1+\mathbf{k}_2)}{|\mathbf{k}_1+\mathbf{k}_2|^2|\mathbf{k}_3|^2} \\
&\qquad\qquad + \frac{4(\mathbf{k}_3\cdot\mathbf{k}_4)\mathbf{k}_3\cdot(\mathbf{k}_1+\mathbf{k}_2)}{|\mathbf{k}_1+\mathbf{k}_2|^2|\mathbf{k}_3|^2} - 2(1-\lambda)\frac{\mathbf{k}_3\cdot\mathbf{k}_4}{|\mathbf{k}_3|^2} - 2(1-\lambda)\frac{\mathbf{k}_4\cdot(\mathbf{k}_1+\mathbf{k}_2)}{|\mathbf{k}_1+\mathbf{k}_2|^2}\Bigg\}
\end{aligned}
\tag{A.9}
$$

where $\sum_\pi$ means you permute over $\{1,2\}\{3,4\}$ and

$$\begin{aligned}
\langle H_2\rangle = {}& -\frac{1-3\lambda}{1-\lambda}\omega_3\omega_4\left[1+3\lambda+\frac{1-3\lambda}{1-\lambda}\left(\frac{(\mathbf{k}_3\cdot\mathbf{k}_4)^2}{|\mathbf{k}_3|^2|\mathbf{k}_4|^2}-\lambda\right)\right]+\frac{1}{2}\sum_{3\leftrightarrow 4}(4|\mathbf{k}_3|^2+6\mathbf{k}_3\cdot\mathbf{k}_4)\\
&+4\alpha\Big[f(\mathbf{k}_3)\Big(|\mathbf{k}_3|^2+\mathbf{k}_3\cdot\mathbf{k}_4\Big)+f(\mathbf{k}_3)f(\mathbf{k}_4)\Big(2|\mathbf{k}_3|^2-\mathbf{k}_3\cdot\mathbf{k}_4\Big)\Big]\\
&+\frac{1}{2}\sum_{3\leftrightarrow 4}\Bigg\{-|\mathbf{k}_3|^2|\mathbf{k}_4|^2(5A_1+16A_2)-A_1(\mathbf{k}_3\cdot\mathbf{k}_4)^2\\
&\qquad+A_3\Big[4|\mathbf{k}_3|^2\Big(|\mathbf{k}_4|^2+\mathbf{k}_3\cdot\mathbf{k}_4\Big)-8f(\mathbf{k}_3)\Big(|\mathbf{k}_4|^4+|\mathbf{k}_3|^2|\mathbf{k}_4|^2\Big)\\
&\qquad\quad+|\mathbf{k}_3+\mathbf{k}_4|^2\Big(-6\mathbf{k}_3\cdot\mathbf{k}_4-16|\mathbf{k}_3|^2-8f(\mathbf{k}_3)|\mathbf{k}_4|^2\Big)\Big]\\
&\qquad+A_4\Big[4f(\mathbf{k}_3)|\mathbf{k}_3|^2\Big(\mathbf{k}_3\cdot\mathbf{k}_4-|\mathbf{k}_3|^2\Big)+4f(\mathbf{k}_3)f(\mathbf{k}_4)\Big(2|\mathbf{k}_3|^4-|\mathbf{k}_3|^2|\mathbf{k}_4|^2\Big)\\
&\qquad\quad+4f(\mathbf{k}_3)|\mathbf{k}_3+\mathbf{k}_4|^2\Big(\mathbf{k}_3\cdot\mathbf{k}_4-2|\mathbf{k}_3|^2\Big)+8f(\mathbf{k}_3)f(\mathbf{k}_4)|\mathbf{k}_3+\mathbf{k}_4|^2\mathbf{k}_3\cdot\mathbf{k}_4\Big]\Bigg\}\\
&+\frac{1}{2}\sum_{3\leftrightarrow 4}\Bigg\{\mathbf{k}_3\cdot\mathbf{k}_4\Big[|\mathbf{k}_3|^2|\mathbf{k}_4|^2(5B_1+16B_2)+B_1(\mathbf{k}_3\cdot\mathbf{k}4)^2\Big]\\
&\qquad+B_3\Big[-12|\mathbf{k}_4|^6-4\mathbf{k}_3\cdot\mathbf{k}_4|\mathbf{k}_4|^4+8f(\mathbf{k}_3)(|\mathbf{k}_4|^4|\mathbf{k}_3|^2-|\mathbf{k}_4|^6)\\
&\qquad\quad+|\mathbf{k}_3+\mathbf{k}_4|^2\Big(-8f(\mathbf{k}_3)|\mathbf{k}_4|^4+8|\mathbf{k}_4|^4-4\mathbf{k}_3\cdot\mathbf{k}_4|\mathbf{k}_4|^2\Big)\\
&\qquad\quad+|\mathbf{k}_3+\mathbf{k}_4|^4\Big(6\mathbf{k}_3\cdot\mathbf{k}_4+16|\mathbf{k}_4|^2\Big)\Big]\\
&\qquad+B_4\Big[4f(\mathbf{k}_3)(3|\mathbf{k}_3|^6-\mathbf{k}_3\cdot\mathbf{k}_4|\mathbf{k}_3|^4)+4f(\mathbf{k}_3)f(\mathbf{k}_4)(|\mathbf{k}_3|^2|\mathbf{k}_4|^4-2|\mathbf{k}_4|^6)\\
&\qquad\quad+2f(\mathbf{k}_3)|\mathbf{k}_3+\mathbf{k}_4|^2(8|\mathbf{k}_3|^4-2\mathbf{k}_3\cdot\mathbf{k}_4|\mathbf{k}_3|^2)\\
&\qquad\quad+2f(\mathbf{k}_3)|\mathbf{k}_3+\mathbf{k}_4|^4(4|\mathbf{k}_3|^2-2\mathbf{k}_3\cdot\mathbf{k}_4)\\
&\qquad\quad+8f(\mathbf{k}_3)f(\mathbf{k}_4)|\mathbf{k}_3+\mathbf{k}_4|^4(-|\mathbf{k}_4|^2-\mathbf{k}_3\cdot\mathbf{k}_4)\Big]\Bigg\}
\end{aligned}\tag{A.10a}$$

$$\begin{aligned}
\langle P_2\rangle = {}& \frac{1}{2}\sum_{3\leftrightarrow 4}i\omega_4\Bigg\{\frac{1-3\lambda}{1-\lambda}f(\mathbf{k}_3)\Big[-2|\mathbf{k}_3+\mathbf{k}_4|^2+2((\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{k}_4)^2\Big]\\
&+\frac{1-3\lambda}{1-\lambda}\frac{1}{|\mathbf{k}_4|^2}\Big[-2(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{k}_3(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{k}_4+(1-\lambda)|\mathbf{k}_3+\mathbf{k}_4|^2\mathbf{k}_3\cdot\mathbf{k}_4\Big]\\
&+(2\mathbf{k}_3-(1-9\lambda)\mathbf{k}_4)\cdot(\mathbf{k}_3+\mathbf{k}_4)+\frac{1-3\lambda}{1-\lambda}(\mathbf{k}_3+\mathbf{k}_4)\cdot(\mathbf{k}_4-3\mathbf{k}_3)\\
&\qquad-2\lambda\left(3-\frac{1-3\lambda}{1-\lambda}\right)|\mathbf{k}_3+\mathbf{k}_4|^2\Bigg\}
\end{aligned}\tag{A.10b}$$

$$\begin{aligned}
\langle Q_2\rangle_i = {}& \frac{1}{2}\sum_{3\leftrightarrow 4}\omega_4\pi_{ij}^{\mathbf{k}_3+\mathbf{k}_4}\Bigg\{\mathbf{k}_{3j}\left[\frac{1-3\lambda}{1-\lambda}\frac{\mathbf{k}_4\cdot(\mathbf{k}_3+\mathbf{k}_4)}{|\mathbf{k}_4|^2}-2+3\frac{1-3\lambda}{1-\lambda}\right]\\
&\qquad+\mathbf{k}_{4j}\Bigg[2f(\mathbf{k}_3)\mathbf{k}_4\cdot(\mathbf{k}_3+\mathbf{k}_4)+\frac{1-3\lambda}{1-\lambda}\mathbf{k}_3\cdot(\mathbf{k}_3+\mathbf{k}_4)\\
&\qquad\quad+(1-9\lambda)-\frac{1-3\lambda}{1-\lambda}\frac{\mathbf{k}_3\cdot\mathbf{k}_4}{|\mathbf{k}_4|^2}\Bigg]\Bigg\}
\end{aligned}\tag{A.10c}$$

Appendix B

Cleaning up the cosmological constant problem calculational details

B.1 General perturbative result derivation

We consider the vacuum case of the theory and so $\tilde{E}^{\mu\nu} = -\frac{1}{16\pi G}(\tilde{G}^{\mu\nu} + \tilde{\Lambda}\tilde{g}^{\mu\nu})$ and $\tilde{E} = \frac{1}{16\pi G}(\tilde{R} - 4\tilde{\Lambda})$. Recall that for the relevant work we raise and lower with the *untilded metric*. Using the equations of motion (5.9), we obtain[1]

$$\delta_2 S = \frac{1}{2}\frac{1}{16\pi G}\int \left[-\overline{\Omega}\sqrt{-\tilde{g}}\left(\tilde{g}^{\mu\alpha}\tilde{g}^{\nu\beta} - \frac{1}{4}\tilde{g}^{\alpha\beta}\tilde{g}^{\mu\nu}\right)\delta\tilde{E}_{\alpha\beta}\delta g_{\mu\nu} + \frac{1}{2}\sum_a \delta\phi_a \mathcal{O}_a(\delta\tilde{R})\right], \tag{B.1}$$

where $\delta\tilde{E}_{\alpha\beta} = \left(\delta\tilde{G}_{\alpha\beta} + \Lambda\delta g_{\alpha\beta}\right)$, using the relation between $\tilde{\Lambda}$ and Λ to remove the tilde on $g_{\mu\nu}$ in the last term. We have also used the linearity of $\mathcal{O}_a$ and the fact $\mathcal{O}_a(const) = 0$. Next, we use the relation

$$\sum_a \frac{1}{2}\int \delta\phi_a \mathcal{O}_a(Q) = \sum_a \frac{1}{2}\int Q\delta\phi_a \frac{\delta\sqrt{-\tilde{g}}}{\delta\phi_a} = \frac{1}{2}\int Q\delta\left(\sqrt{-\tilde{g}}\right), \tag{B.2}$$

which follows from the definition of the functional derivative, to rewrite the quadratic action (introducing $\delta g_{\mu\nu} = h_{\mu\nu}$) as

$$\delta_2 S = \frac{\overline{\Omega}}{32\pi G}\int \left[-\sqrt{-g}H^{\mu\nu}\delta\left(\tilde{G}_{\mu\nu} + \Lambda g_{\mu\nu}\right) + \frac{1}{2\overline{\Omega}}\delta\tilde{R}\delta\left(\sqrt{-\tilde{g}}\right)\right], \tag{B.3}$$

[1] Making use of a convenient shorthand way for calculating a quadratic action,

$$\delta_2 S = \frac{1}{2}\sum_a \int \delta\phi_a \delta\left(\frac{\delta S}{\delta\phi_a}\right).$$

where $H_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{4} g_{\mu\nu} h$. Expanding the Einstein tensor, using the fact that $H^{\mu\nu}$ is traceless and that $H^{\mu\nu} \delta \tilde{g}_{\mu\nu} = \Omega H^{\mu\nu} h_{\mu\nu}$, we rewrite this as

$$\delta_2 S = \frac{\overline{\Omega}}{32\pi G} \int \left[-\sqrt{-g} H^{\mu\nu} \delta \left(\tilde{R}_{\mu\nu} - \Lambda g_{\mu\nu} \right) + \frac{1}{2\overline{\Omega}} \delta \tilde{R} \delta \left(\sqrt{-\tilde{g}} \right) \right]. \tag{B.4}$$

It will be useful to consider how this compares with the perturbative action arising from the Einstein-Hilbert action $S_{GR} = \frac{1}{16\pi G} \int \sqrt{-g}(R - 2\Lambda)$, which by the standard calculational techniques can be shown to be

$$\begin{aligned} \delta_2 S_{GR} &= -\frac{1}{32\pi G} \int \sqrt{-g} h^{\mu\nu} \delta \left(G_{\mu\nu} + \Lambda g_{\mu\nu} \right) \\ &= -\frac{1}{32\pi G} \int \sqrt{-g} \bar{h}^{\mu\nu} \delta \left(R_{\mu\nu} - \Lambda g_{\mu\nu} \right), \end{aligned} \tag{B.5}$$

where $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h$.

Next, we introduce $\Delta_2 = 16\pi G (\frac{\delta_2 S}{\overline{\Omega}} - \delta_2 S_{GR})$. This will then tell us the way in which this action deviates from GR (and remove some messy factors). From (B.4) and (B.5),

$$\Delta_2 = \frac{1}{2} \int \left[-\sqrt{-g} \left(H^{\mu\nu} \delta \tilde{R}_{\mu\nu} - \bar{h}^{\mu\nu} \delta R_{\mu\nu} \right) + \sqrt{-g} \Lambda h^2 + \frac{1}{2\overline{\Omega}} \delta \tilde{R} \delta \left(\sqrt{-\tilde{g}} \right) \right]. \tag{B.6}$$

Now, using the standard expressions for the Ricci tensor and scalar of conformally related metrics (see *e.g.* [20]), we can write

$$\delta \tilde{R}_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} \frac{\nabla_\mu \nabla_\nu \delta \Omega^2}{\overline{\Omega}^2} - \frac{1}{4} g_{\mu\nu} \frac{\Box \delta \Omega^2}{\overline{\Omega}^2} \tag{B.7a}$$

$$\delta \tilde{R} = \frac{1}{\overline{\Omega}} \left(\delta R - \frac{3}{2} \frac{\Box \delta \Omega^2}{\overline{\Omega}^2} - 2\Lambda \frac{\Box \delta \Omega^2}{\overline{\Omega}^2} \right), \tag{B.7b}$$

using the fact that $\overline{\Omega} = const$ and being careful to vary the factor of $1/\Omega$ appearing in the general expression for $\tilde{R}$. We will also make use of

$$\delta \left(\sqrt{-\tilde{g}} \right) = \overline{\Omega}^2 \sqrt{-g} \left(\frac{\delta \Omega^2}{\overline{\Omega}^2} + \frac{1}{2} h \right) \tag{B.8a}$$

$$\delta \tilde{g}_{\mu\nu} = \overline{\Omega} \left(\frac{1}{2} g_{\mu\nu} \frac{\delta \Omega^2}{\overline{\Omega}^2} + h_{\mu\nu} \right). \tag{B.8b}$$

Manipulating the first term in the square brackets in (B.6), including an integration by parts, we obtain

$$-\sqrt{-g} \left(H^{\mu\nu} \delta \tilde{R}_{\mu\nu} - \bar{h}^{\mu\nu} \delta R_{\mu\nu} \right) = -\sqrt{-g} \left(\frac{1}{4} h \delta R - \frac{1}{2} \frac{\delta \Omega^2}{\overline{\Omega}^2} \nabla_\mu \nabla_\nu h^{\mu\nu} + \frac{1}{8} \frac{\delta \Omega^2}{\overline{\Omega}^2} \Box h + \Lambda h^2 \right). \tag{B.9}$$

Clearly, this results in the Λh^2 term in (B.6) being cancelled off. Moving on to the remaining part of the square brackets in (B.6), we get

$$\frac{1}{2\overline{\Omega}}\delta\tilde{R}\delta\left(\sqrt{-\tilde{g}}\right) = \sqrt{-g}\Bigg(\frac{1}{4}h\delta R - \frac{3}{8}\frac{\delta\Omega^2}{\overline{\Omega}^2}\Box h + \frac{1}{2}\frac{\delta\Omega^2}{\overline{\Omega}^2}\delta R - \frac{3}{4}\frac{\delta\Omega^2}{\overline{\Omega}^2}\Box\frac{\delta\Omega^2}{\overline{\Omega}^2} - \Lambda\left(\frac{\delta\Omega^2}{\overline{\Omega}^2}\right)^2 - \frac{1}{2}\Lambda h\frac{\delta\Omega^2}{\overline{\Omega}^2}\Bigg). \quad \text{(B.10)}$$

Putting this all together, we obtain

$$\Delta_2 = \frac{1}{2}\int\sqrt{-g}\left[\frac{1}{2}\left(\delta R + \nabla_\mu\nabla_\nu h^{\mu\nu} - \Box h - \Lambda h\right)\frac{\delta\Omega^2}{\overline{\Omega}^2} - \frac{3}{4}\frac{\delta\Omega^2}{\overline{\Omega}^2}\Box\frac{\delta\Omega^2}{\overline{\Omega}^2} - \Lambda\left(\frac{\delta\Omega^2}{\overline{\Omega}^2}\right)^2\right]. \quad \text{(B.11)}$$

Noting that $\delta R = \nabla_\mu\nabla_\nu h^{\mu\nu} - \Box h - \Lambda h$ and recalling our definition of Δ_2, trivial algebraic manipulations allow us to rewrite this as

$$\delta_2 S = \overline{\Omega}\left[\delta_2 S_{GR}[g] + \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Delta\mathcal{L}\right], \quad \text{(B.12)}$$

with

$$\Delta\mathcal{L} = \frac{1}{4}\frac{\delta\Omega^2}{\overline{\Omega}^2}\left(2\delta R(g) - \frac{3}{2}\frac{\Box\delta\Omega^2}{\overline{\Omega}^2} - 2\Lambda\frac{\delta\Omega^2}{\overline{\Omega}^2}\right), \quad \text{(B.13)}$$

as stated in Section 5.3.1.

B.2 No extra scalars propagating in de Sitter

After (5.25), it was stated that the scalar modes of the perturbations are simply renormalised relative to GR, but there are no propagating spin-0 modes, this will be demonstrated here.

Scalar perturbations of de Sitter metric (in flat FRW form) can be written as

$$g_{\mu\nu}dx^\mu dx^\nu = -(1+\alpha)^2 dt^2 + e^{2Ht}e^{2\xi}\delta_{ij}(dx^i + \nabla^i\beta dt)(dx^j + \nabla^j\beta dt), \quad \text{(B.14)}$$

where $H^2 = \frac{\Lambda}{3}$, ∇_i is the covariant derivative associated with the spatial metric and ∇^2 is the spatial Laplacian. The gauge has been chosen (without loss of generality) such that δg_{ij} is a pure trace, thereby dropping terms of the form $(\nabla_i\nabla_j - \frac{1}{3}\delta_{ij}\nabla^2)\nu$, where ν is the non-trace scalar perturbation in g_{ij}.

For the conformally related metric

$$e^{\psi/2}g_{\mu\nu}dx^\mu dx^\nu = -(1+\tilde{\alpha})^2 dt^2 + e^{2Ht}e^{2\tilde{\xi}}\delta_{ij}(dx^i + \nabla^i\tilde{\beta}dt)(dx^j + \nabla^j\tilde{\beta}dt), \quad \text{(B.15)}$$

where

$$\tilde{\alpha} = \alpha + \frac{\psi}{4}, \qquad \tilde{\xi} = \xi + \frac{\psi}{4}, \qquad \tilde{\beta} = \beta, \quad \text{(B.16)}$$

are the renormalised scalar modes. We can then write the perturbative GR action of the conformally related metric as

$$\delta_2 S_{GR}[e^{\psi/2}g] = \int dt d^3x e^{3Ht}\Big[-6\dot{\tilde{\xi}}^2 - 2e^{-2Ht}\tilde{\xi}\nabla^2\tilde{\xi} - 6H^2\tilde{\alpha}^2 + 12H\tilde{\alpha}\dot{\tilde{\xi}} \\ -4e^{-2Ht}\tilde{\alpha}\nabla^2\tilde{\xi} + 4e^{-2Ht}\dot{\tilde{\xi}}\nabla^2\tilde{\beta} - 4He^{-2Ht}\tilde{\alpha}\nabla^2\tilde{\beta}\Big]. \quad \text{(B.17)}$$

As usual, we can use the Hamiltonian and momentum constraints, $\tilde{\alpha} = \frac{\dot{\tilde{\xi}}}{H}$ and $\tilde{\beta} = -\frac{\tilde{\xi}}{H}$ to integrate out $\tilde{\alpha}$ and $\tilde{\beta}$ (the lapse and shift). In doing this, we note that the action (B.17) reduces to

$$\delta_2 S_{GR}[e^{\psi/2}g] = -2\int dt d^3x e^{Ht}\nabla^2\tilde{\xi}\left[\tilde{\xi} + \frac{1}{H}\dot{\tilde{\xi}} + \frac{1}{H}\dot{\tilde{\xi}}\right]. \quad \text{(B.18)}$$

Performing some integrations by parts on the final term, one can manipulate it into the form $e^{Ht}\frac{\dot{\tilde{\xi}}}{H}\nabla^2\tilde{\xi} \to e^{Ht}\left(-\frac{\dot{\tilde{\xi}}}{H} - \xi\right)\nabla^2\tilde{\xi}$, and so the action completely vanishes. Thus there are no propagating spin-0 degrees of freedom.

B.3 Proof of more general condition on the operator $\mathcal{O}$

Here, we prove a more general condition to ensure that only solutions for $\tilde{g}_{\mu\nu}$ that satisfy Einstein's equation are permitted.

Take the trace of (5.9). Since the traceless part of $\tilde{E}^{\mu\nu}$ necessarily vanishes, this imposes the on-shell condition

$$\tilde{g}_{\mu\nu}\mathcal{O}_{g_{\mu\nu}}(\tilde{E}) = 0. \quad \text{(B.19)}$$

This is an on-shell condition, *i.e.* imposed by the equations of motion, not a condition which we must impose in the the theory ourselves. With this on-shell condition, and after dividing through by $\sqrt{-\tilde{g}}\Omega$, we rewrite (5.9) as

$$\left(\delta_\alpha^{(\mu}\delta_\beta^{\nu)} - \frac{1}{4}\tilde{g}^{\mu\nu}\tilde{g}_{\alpha\beta}\right)\left(\tilde{E}^{\alpha\beta} + \frac{1}{2\sqrt{-\tilde{g}}\Omega}\mathcal{O}_{g_{\alpha\beta}}(\tilde{E})\right) = 0. \quad \text{(B.20)}$$

Applying $\tilde{\nabla}_\mu$, and using the Bianchi identities and enforcing conservation of energy to determine $\tilde{\nabla}_\mu\tilde{E}^{\mu\nu} = 0$,

$$-\frac{1}{4}\tilde{\nabla}_\mu\tilde{E} + \frac{1}{2\sqrt{-\tilde{g}}}\left(\delta^\nu_{(\alpha}\tilde{g}_{\beta)\mu} - \frac{1}{4}\delta^\nu_\mu\tilde{g}_{\alpha\beta}\right)\tilde{\nabla}_\nu\left(\frac{\mathcal{O}_{g_{\alpha\beta}}(\tilde{E})}{\Omega}\right) = 0 \quad \text{(B.21)}$$

Now, in order for the unique solution to (B.21) to be $\tilde{E} = \mathit{const.}$, a sufficent condition is clearly

$$\left(\delta^\nu_{(\alpha}\tilde{g}_{\beta)\mu} - \frac{1}{4}\delta^\nu_\mu\tilde{g}_{\alpha\beta}\right)\tilde{\nabla}_\nu\left(\frac{\mathcal{O}_{g_{\alpha\beta}}(Q)}{\Omega}\right) = 0 \quad \text{(B.22)}$$

for all sufficiently differentiable Q, implying that (B.21) simply reduces to $\partial_\mu \tilde{E} = 0$. Substituting $\tilde{E} = const.$ into (B.20), one obtains that the traceless Einstein equations must also vanish, which is the result that we require.

This is more general than the condition derived in Chapter 6, but it is less clear how to gain a feel for the equations using this; and so it is less useful than the other condition in gauging what sort of model we need to get a feel of the solution space.

B.4 System of differential equations

In this section, we explicitly state the ODEs which we solve in Chapter 6, in order that all the relevant equations can be clearly seen together. We use the notation $\tilde{\rho} = \sum_a \rho_a + \rho_\star$, $\gamma_a \equiv \frac{p_a+\rho_a}{\rho_a}$.

B.4.1 Conformal time

The ODEs are

$$\tilde{\mathcal{H}}' = -\frac{4\pi G}{3}\tilde{a}^2\left(\tilde{\rho}+3\tilde{p}\right), \tag{B.23a}$$

$$\rho_a' = -3\tilde{\mathcal{H}}\gamma_a\rho_a, \tag{B.23b}$$

$$\rho_\star' = \frac{48\epsilon}{\mu^4}\frac{\tilde{\mathcal{H}}\mathcal{H}^3}{\tilde{a}^4}\tilde{E}', \tag{B.23c}$$

$$\tilde{E}'' = -2\tilde{E}'\frac{\mathcal{H}'}{\mathcal{H}}, \tag{B.23d}$$

$$\mathcal{H}' = \frac{\mu^4}{24\epsilon}\frac{\tilde{a}^4}{\mathcal{H}^2}, \tag{B.23e}$$

with the auxiliary equations

$$\mathcal{H}^2 = \frac{8\pi G}{3}\tilde{a}^2\tilde{\rho}, \tag{B.24a}$$

$$\rho_\star = \frac{\tilde{E}}{2} - \frac{12\epsilon}{\mu^4}\frac{\mathcal{H}^3}{\tilde{a}^4}\tilde{E}', \tag{B.24b}$$

$$p_\star = -\frac{\tilde{E}}{2} - \frac{4\epsilon}{\mu^4}\frac{\mathcal{H}^3}{\tilde{a}^4}\tilde{E}'. \tag{B.24c}$$

Clearly, $\rho_\star$ and $p_\star$ can be eliminated from this system of equations by using (B.24b) and (B.24c).

B.4.2 Cosmic time

From the conformal time equations, we can derive the following cosmic time equations,

$$\dot{\tilde{H}} = -4\pi G\left(\tilde{\rho} + \tilde{p}\right), \tag{B.25a}$$

$$\dot{\rho}_a = -3\tilde{H}(1 + w_a)\rho_a, \tag{B.25b}$$

$$\dot{\rho}_\star = \frac{48\epsilon}{\mu^4}\frac{\tilde{H}H^3}{\tilde{a}^4}\dot{\tilde{E}}, \tag{B.25c}$$

$$\ddot{\tilde{E}} = -\dot{\tilde{E}}\frac{2\dot{H} + 3\tilde{H}H}{H}, \tag{B.25d}$$

$$\dot{H} = \frac{\mu^4}{24\epsilon}\frac{1}{H^2} - \tilde{H}H, \tag{B.25e}$$

with the auxiliary equations

$$\tilde{H}^2 = \frac{8\pi G}{3}\tilde{\rho}, \tag{B.26a}$$

$$\rho_\star = \frac{\tilde{E}}{2} - \frac{12\epsilon}{\mu^4}H^3\dot{\tilde{E}}, \tag{B.26b}$$

$$p_\star = -\frac{\tilde{E}}{2} - \frac{4\epsilon}{\mu^4}H^3\dot{\tilde{E}}. \tag{B.26c}$$

Again, we eliminate $\rho_\star$ and $p_\star$ using (B.26b) and (B.26c).

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