<u>Topics in Prime Number Theory</u>

by

Amit Ghosh
B.Sc., Imperial College, London, 1978

Thesis submitted to the University of Nottingham for the degree of Doctor of Philosophy, October, 1981.

TABLE OF CONTENTS

ABSTRACT

The thesis is divided into five sections:

(a) Trigonometric sums involving prime numbers and applications,

(b) Mean-values and Sign-changes of $S(t)$-- related to Riemann's Zeta function,

(c) Mean-values of strongly additive arithemetical functions,

(d) Combinatorial identities and sieves,

(e) A Goldbach-type problem.

Parts (b) and (c) are related by means of the techniques used but otherwise the sections are disjoint.

(a) We consider the question of finding upper bounds for sums like

$$\sum_{p \leq N} e(\alpha p^2),$$

and using a method of Vaughan, we get estimates which are much better than those obtained by Vinogradov. We then consider two applications of these, namely, the distribution of the sequence (αp^2) modulo one. Of course we could have used the improved results to get improvements in estimates in various other problems involving p^2 but we do not do so.

We also obtain an estimate for the sum

$$\sum_{p \leq N} e(\alpha p^3),$$

and get improved estimates by the same method.

(b) Let $N(T)$ denote the number of zeros of $\zeta(s)$ - Riemann's Zeta function. It is well known that

$$N(T) = L(T) + S(T),$$

where

$$L(T) = \frac{1}{2\pi} T \log(T/2\pi) - T/2\pi + 7/8 + O(1/T),$$

but the finer behaviour of $S(T)$ is not known. It is known that

$$S(t) \ll \log t \quad ; \quad \int_0^t S(u)\,du \ll \log t,$$

so that $S(T)$ has many changes of sign. In 1942, A. Selberg showed that the number of sign changes of $S(t)$ for $t \in (0,T)$ exceeds

$$T (\log T)^{1/3} \exp(-A \log\log T), \tag{1}$$

but stated to Professor Halberstam in 1979 that one can improve the constant $1/3$ in (1) to $1 - \varepsilon$. It can be shown easily that the upper bound for the number of changes of sign is $\log T$.

We give a proof of Selberg's statement in (b), but in the process we do much more. Selberg showed that if k is a positive integer then

$$\int_T^{T+H} |S(t)|^{2k}\,dt = C_k H (\log\log T)^k \left\{ 1 + O\left((\log\log T)^{-\frac{1}{2}} \right) \right\}, \tag{2}$$

where $T^{\frac{1}{2}} < H \leq T^2$ and C_k is some explicit constant in k. We have found a simple technique which gives (2) with the constant k replaced by any non-negative real number. Using this type of result, I prove Selberg's statement, with $(\log T)^{-\varepsilon}$ replaced by

$$\exp\left(-A \sqrt{\log\log T}\, (\log\log\log T)^{-\frac{1}{2}} \right).$$

(c) I use the method for finding mean-values above to answer similar questions for a class of strongly additive arithemetical functions.

We say that f is strongly additive if

(1) $f(mn) = f(m) = f(n)$, if m and n are coprime,

(2) $f(p^a) = f(p)$ for all primes p and positive integer a.

Let

$$A_k(x) = \sum_{p<x} \frac{|f(p)|^k}{p} .$$

Halberstam and Delange showed that if then f lies in a certain class, then one can show that for any $k \in \mathbb{N}$

$$\sum_{n \leq x} |f(n) - A_1(x)|^{2k} \sim \mu_{2k} \cdot x \ A_2(x)^k ,$$

where

$$\mu_q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^q e^{-\frac{1}{2}t^2} dt .$$

This is a moment problem and was motivated by a paper due to Erdös and Kac. I use my technique to show that $2k$ can be replaced by any positive real number.

(d) This section contains joint work with Professor Halberstam and is still in its infancy. We have found a general identity and a type of convolution which serves to be the starting point of most investigations in Prime Number Theory involving the local and the global sieves. The term global refers to sieve methods of Brun, Selberg, Rosser and many more. The term local refers to things like Selberg's formula in the elementary proof of the prime number theorem, Vaughan's identity and so on. We have shown that both methods stem from the same source and so leads to a unified approach to such research.

(e) I considered the question of solving the representation of an integer N in the form

$$N = p_1^2 + p_2^2 + K p_3 ,$$

where the p_i's are prime numbers. This problem was motivated by Goldbach's Problem and is exceedingly difficult. So I looked into getting partial answers.

Let E(x) denote the numbers less than x not representable in the required form. Then there is a computable constant $\delta > 0$ such that

$$E(x) \ll x^{1-\delta} .$$

To do this we use a method of Montgomery and Vaughan but the proof is long and technical, and we do not give it here.

We show by sieve methods that the following result holds true:

$$N = p_1^2 + p_2^2 + k\, p_3 p_4 p_5 .$$

We have been unable to replace the product of three primes by two.

Note: k is a constant depending on the residue class of N modulo 12.

ACKNOWLEDGEMENTS

I am grateful to Prof. R. C. Vaughan for suggesting the investigation of Section A, and for his helpful and valuable criticisms; also to Dr. D. R. Heath-Brown for providing me with the inspiration that led to the moment method of Section B. I am very grateful to my supervisor, Prof. H. Halberstam, who has been very encouraging throughout the period of my research and who has kindly allowed me to include the material of Chapter 6.

Finally, I take pleasure in thanking the University of Nottingham for providing me with a Research Studentship which made the work that went towards this thesis possible.

BRIEF INTRODUCTION

Each chapter has a detailed introduction describing the contents and therefore I shall give at this stage only the scope of my thesis.

Chapters 1 and 2

We study exponential sums over prime numbers, using the Vaughan identity. Applications are then made to the distribution of the sequence $\{\alpha p^2 \text{ modulo } 1 \; : \; p \text{ prime numbers}\}$, where α is an irrational number.

Chapters 3 and 4

We introduce a technique that is used to obtain all the moments of $|S(t)|$ and we use these to obtain results on (a) the sign-changes of $S(t)$ and (b) the finer behaviour of the limiting distribution of $S(t)$.

Chapter 5

We apply the method of moments, introduced above, to obtain corresponding results for the limiting distribution for a general class of additive functions.

Chapter 6

This is work done in collaboration with Prof. Halberstam. We introduce a type of arithmetical convolution and a summatory formula which is then used to derive the combinatorial identities underlying all the known sieve methods. This account gives a unified approach to the subject of sieves.

Chapter 7

We apply sieve methods to the problem of representing natural numbers N in the form

$$N = p_1^2 + p_2^2 + K p_3 p_4 p_5 \quad , \quad p_i \text{ primes.}$$

SECTION A. EXPONENTIAL SUMS IN PRIMES

Chapter 1.

The Distribution of αp^2 Modulo 1

§1.1. Introduction

In 1977, Vaughan [5] introduced an elementary method in prime-number theory which enabled him to improve known results on the distribution of the sequence (αp) modulo 1, where α is an irrational number and p runs through the set of prime numbers. We shall consider the corresponding questions for the distribution of (αp^2) modulo 1. The basic result is embodied in

THEOREM 1. *Suppose α is a real number and a and q are positive integers satisfying $(a,q)=1$ and $|\alpha - a/q| < q^{-2}$. Then, for any positive integers H and N, given any real number $\varepsilon>0$, we have*

$$\mathcal{A}(H,N) = \sum_{h=1}^{H}\Big|\sum_{n=1}^{N}\Lambda(n)e(hn^2\alpha)\Big| \ll HN^{1+\varepsilon}\left(q^{-1}+N^{-\frac{1}{2}}+qH^{-1}N^{-2}\right)^{\frac{1}{4}},$$

where the constant implied by the $\ll$ notation depends at most on ε.

Even the case $H=1$ of Theorem 1 appears to be new and we record it as

THEOREM 2. *Suppose that α and N are as specified in Theorem 1. Then, given any real number $\varepsilon>0$, we have*

$$\sum_{n=1}^{N}\Lambda(n)\,e(n^2\alpha) \ll N^{1+\varepsilon}\left(q^{-1}+N^{-\frac{1}{2}}+qN^{-2}\right)^{\frac{1}{4}},$$

where the constant implied depends at most on ε.

Such results, due to Vinogradov [6,7], exist in the literature but are weaker in the sense that they are non-trivial only for much shorter ranges of q. For the ranges of q for which Theorem 2 is non-trivial, Vinogradov appears to have in the exponent nothing better than $\frac{1}{192}$ where we obtain $\frac{1}{4}$, but he has a power of a logarithm in place† of N^{ε}.

In 1958, Chen [1] showed that, for any real $\alpha \leq 3/8$,

$$S_k(N, a/q) = \sum_{p \leq N} e\left(\frac{a}{q} p^k\right) \ll N^{1-\alpha/3},$$

for $q = [N^{1-\alpha}]$, where a, q, N and k are positive integers, and p runs through prime numbers. Also, $(a, q) = 1$. For $k = 2$ and $q = [N^{5/8}]$ this shows that $S_2(N, a/q) \ll N^{7/8}$, whereas Theorem 2 implies that

$$S_2(N, a/q) \ll N^{7/8 + \varepsilon},$$

for all q satisfying $N^{\frac{1}{2}} \leq q \leq N^{3/2}$.

As has been remarked by Vaughan [3], one would like to eliminate the term not inolving q in the estimate in Theorem 2. This would follow if it were possible to replace the term $N^{-\frac{1}{2}}$ by N^{-1}.

We shall give two applications of Theorem 1:

THEOREM 3. Suppose α is a real irrational number and β is an arbitrary real number. Let $\|\theta\|$ denote the least distance of θ from an integer. Then, given any real number $\varepsilon > 0$, there is a positive number $c(\varepsilon)$, depending only on ε, such that

$$\|\alpha p^2 - \beta\| \leq c(\varepsilon) p^{-\frac{1}{8} + \varepsilon}$$

for infinitely many prime numbers p.

---

†The ε derives from repeated use of the classical estimate $d_k(n) \ll n^{\varepsilon}$ for the divisor functions. It follows that, if anything were to be gained by it, then N^{ε} could be replaced everywhere by $\exp(c(\log\log N)^{-1}\log N)$, where c is some suitable positive constant and $N \geq N_0(c)$.

Heilbronn showed in [2] that for any integer $N \geqslant 1$ and every real number α, integers n can be found such that $1 \leqslant n \leqslant N$ and

$$\|n^2\alpha\| \leqslant c(\varepsilon)\, N^{-\frac{1}{2}+\varepsilon}$$

where ε is an arbitrarily small positive number and $c(\varepsilon)$ is a positive constant depending on ε. We call such a result a 'local result'. A 'semi-local result' is a result of the same kind but valid only for $N \geqslant N(\alpha)$. It can be shown that local and even semi-local results are unattainable with n restricted to primes when α is a Louiville number. In general, such results are attainable by our method if the denominators of the convergents of α do not increase too rapidly (in some sense).

THEOREM 4. *Suppose that* $0 \leqslant \gamma < \gamma + \delta \leqslant 1$ *and* α, a, *and* q *are as defined in Theorem* 1. *Let* $\{\theta\}$ *denote the fractional part of* θ *and let* $\pi^*(\gamma, \delta, N)$ *denote the number of prime numbers* $p \leqslant N$ *such that* $\gamma \leqslant \{\alpha p^2\} < \gamma + \delta$. *Then, for any real number* $\varepsilon > 0$,

$$\pi^*(\gamma, \delta, N) - \delta\pi(N) \ll N^{1+\varepsilon}\left(q^{-1} + N^{-\frac{1}{2}} + q\delta N^{-2}\right)^{\frac{1}{4}}(\log q) + N^{1+\varepsilon}(q\delta^{-1})^{\varepsilon}\left(q^{-1} + N^{-1}\right)^{\frac{1}{2}},$$

where $\pi(N)$ *denotes the number of prime numbers not exceeding* N *and the constants implied by the* $\ll$ *notation depend at most on* ε.

In 1946, Vinogradov [8] obtained a result like this but with the weaker error term

$$N\left(q^{-1} + N^{-\frac{1}{2}}\right)^{\theta}, \qquad \theta = \tfrac{1}{32}(\log 24)^{-1},$$

for $q \leqslant N$. Theorem 4 implies a similar error term, valid for

$q \leq N$, with $\theta = \frac{1}{4}$ but at the expense of having the additional term N^{ε} .

All these results will be proved in §§1.4 and 1.5.

§1.2. Notation

Throughout, every opportunity has been taken to make explicit important notation used at the point where it has been introduced.

$\Lambda(n)$ is von Mangoldt's function,

$d_k(n)$ is the number of representations of n as the product of exactly k integers,

$d(n) = d_2(n)$ denotes the divisor function,

$\mu(n)$ is the Mobius function,

$\pi(n)$ denotes number of primes not exceeding n ,

$\vartheta(n) = \sum_{p \leq N} \log p$, where p always denotes a prime,

$[\theta]$ is the integer part of θ ,

$\{\theta\}$ is the fractional part of θ , that is $\theta - [\theta]$,

$\|\theta\|$ is the least distance of θ from an integer, i.e. $\min_{n \in \mathbb{Z}} |n - \theta|$,

$e(\theta) = e^{2\pi i \theta}$

$|\mathcal{G}|$ denotes the cardinality of the set $\mathcal{G}$.

The following shall always denote integers (even if suffixed):

$a, d, h, \ell, m, n, q, r, t, u, y, H, N, V, W.$

The following shall always denote real numbers:

$\varepsilon, \varepsilon_0, \alpha, \beta, \gamma, \delta, \vartheta, \eta, \omega, c, x, z$

We shall use Vinogradov's symbol $\ll$ as an alternative to the O -notation. We say $f = O(g)$ or $f \ll g$ if there is a constant M such that $|f| \leq M g$. In all cases, constants implied by either symbol will be absolute or dependent at most on ε, where the definition of ε is specified in the appropriate context.

§1.3. Auxillary Results

The proof of Theorem 1 will occupy us for most of the chapter. As we shall show, it will be necessary to estimate exponential sums of the type

$$S(H,\mathcal{G}) = \sum_{1\le h\le H} \Big| \sum_{\substack{m,n \\ (m,n)\in\mathcal{G}}} a_m b_n\, e(hm^2n^2\alpha) \Big| ,$$

where H, V, V', W, and W' are positive non-zero integers satisfying $V < V' \le 2V$, $W < W' \le 2W$, and $VW \le N$, and

$$\mathcal{G} = \left\{(m,n) \,:\, V < m \le V',\ W < n \le W',\ mn \le N\right\},$$

with (m,n) running through all the lattice points in the hyperbolic region in $\mathbb{R}\times\mathbb{R}$ as defined by $\mathcal{G}$. Note that $\mathcal{G}$ is empty if $VW \geq N$ and that $|\mathcal{G}| \le N$. The weights $(a_m)_{m\in(V,V']}$ and $(b_n)_{n\in(W,W']}$ will be arbitrary complex numbers satisfying inequalities of the type

$$a_m \ll A \qquad \text{and} \qquad b_n \ll B$$

where A and B will be specified variously later.

In the analysis of $S(H,\mathcal{G})$, two classes of sums need to be distinguished:

Class I: $b_n = 1$ for all n in $(W,W']$;

Class II: $b_n \neq 1$ for some n in $(W,W']$.

As one would expect, Class I sums are easier to estimate and, by the well-known Weyl procedure, Class II sums can be reduced to Class I sums in a small number of steps. Essentially, the idea is to apply Cauchy's inequality to $S(H,\mathcal{G})$ a number of times so as to yield sums of the type

$$\sum_{n_1}\sum_{n_2}\Big|\sum_{n_3} e(f(n_1,n_2)n_3\alpha)\Big| ;$$

here n_3 runs through all the integers in an interval of length x_3 which may depend on n_1 and n_2, and $f(n_1,n_2)$ is a polynomial in n_1 and n_2. Such a sum is then

$$\ll \sum_{n_1}\sum_{n_2} \min.(x_3, \|f(n_1,n_2)\alpha\|^{-1}),$$

and can be estimated by some classical results of Vinogradov (see Lemma 1 below).

We have, from Cauchy's inequality

$$(1.3.1)\quad |S(H,\mathcal{G})| \leq \Big(\sum_{1\leq h\leq H}\sum_{V<m\leq V'} |a_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{1\leq h\leq H}\sum_{m}\Big|\sum_{\substack{n\\(m,n)\in\mathcal{G}}} b_n e(hm^2n^2\alpha)\Big|^2\Big)^{\frac{1}{2}}$$

$$\ll (HVA^2)^{\frac{1}{2}}\Big(\sum_{h}\sum_{m}\sum_{\substack{n\\(m,n)\in\mathcal{G}}} |b_n|^2$$

$$+ \sum_{h}\sum_{m}\sum_{\substack{n_1\neq n_2\\(m,n_i)\in\mathcal{G}\\i=1,2}} b_{n_1}\overline{b_{n_2}}\, e(hm(n_1^2-n_2^2)\alpha)\Big)^{\frac{1}{2}}$$

$$(1.3.2)\qquad \ll (HV)^{\frac{1}{2}} A (H|\mathcal{G}|B^2 + \Sigma_1)^{\frac{1}{2}},$$

say, and Σ_1 will be estimated in one of two ways, according as $S(H,\mathcal{G})$ is in Class I or Class II.

Suppose that $S(H,\mathcal{G})$ is in Class I. Then

$$\Sigma_1 = \sum_{1\leq h\leq H}\sum_{V<m\leq V'}\sum_{\substack{n_1\neq n_2\\(m,n_i)\in\mathcal{G}\\i=1,2.}} e(hm^2(n_1^2-n_2^2)\alpha)$$

(1.3.3) $=$

$$\sum_{1\le h\le H}\ \sum_{V<m\le V'}\ \sum_{1\le|y|<2W}\ \sum_{\substack{n\\(m,n)\in\mathcal{G}\\(m,n+y)\in\mathcal{G}}} e(hm^2y(y+2n)\alpha).$$

The variable n in the multiple sum above actually traverses an interval of length

$$|\{n:(m,n)\in\mathcal{G}\ \text{ and }\ (m,n+y)\in\mathcal{G}\}|\le Nm^{-1},$$

by the definition of $\mathcal{G}$. Hence, from (1.3.3),

$$(1.3.4)\qquad |\textstyle\sum_1| \ll \displaystyle\sum_{1\le h\le H}\ \sum_{V<m\le 2V}\ \sum_{1\le|y|<2W} \min(Nm^{-1},\|2hm^2y\alpha\|^{-1}),$$

and the expression on the right is of a kind to be estimated in Lemma 2 below.

Suppose now that $S(H,\mathcal{G})$ is in Class II. We may write $\sum_1$ in the form

$$(1.3.5)\qquad \textstyle\sum_1 = \displaystyle\sum_{1\le h\le H}\ \sum_{V<m\le V'}\ \sum_{\substack{|t|\ge 1\\(m,t)\in\mathcal{G}_1}} \zeta_1(t)\, e(hm^2t\alpha),$$

where

$$\mathcal{G}_1 = \{(m,t): V<m\le V',\ 1\le|t|\le W'^2,\ |m^2t|\le N^2\},$$

and (m,t) runs through all the lattice points in $\mathcal{G}_1$. Also

$$\zeta_1(t) = \sum_{\substack{n_1 \quad n_2 \\ n_1^2 n_2^2 = t \\ (m,n_i)\in \mathcal{G} \\ i=1,2.}} \sum b_{n_1} \bar{b}_{n_2} \ll B^2 d(|t|) .$$

Changing the order of summation in (1.3.5), we have

$$\Sigma_1 = \sum_{1\le h\le H} \sum_{1\le |t| < W'^2} \zeta_1(t) \sum_{\substack{m \\ (m,t)\in \mathcal{G}_1}} e(htm^2\alpha) ,$$

and this expression will be treated along the same lines as Class I sums. By Cauchy's inequality,

$$(1.3.6) \quad |\Sigma_1| \le \Big(\sum_{1\le h\le H} \sum_{1\le |t| < W'^2} |\zeta_1(t)|^2 \Big)^{\frac{1}{2}} \times$$

$$\Big(\sum_{1\le h\le H} \sum_{1\le |t|\le W'^2} \Big| \sum_{\substack{m \\ (m,t)\in \mathcal{G}_1}} e(htm^2\alpha) \Big|^2 \Big)^{\frac{1}{2}}$$

$$\ll B^2 H^{\frac{1}{2}} \Big(\sum_{1\le t\le 4W^2} d(t)^2 \Big)^{\frac{1}{2}} \Big(\sum_{1\le h\le H} |\mathcal{G}_1| + \Sigma_2 \Big)^{\frac{1}{2}} ,$$

say, and

$$|\mathcal{G}_1| = \sum_{1\le |t| < W'^2} \sum_{\substack{V < m < V' \\ m^2|t| \le N^2}} 1 \ll N \sum_{1\le |t|\le 4W^2} |t|^{-\frac{1}{2}} \ll NW ,$$

and

$$\Sigma_2 = \sum_{1\le h\le H} \sum_{1\le |t| < W'^2} \sum_{\substack{m_1 \ne m_2 \\ (m_i,t)\in \mathcal{G}_1 \\ i=1,2.}} e(ht(m_1^2 - m_2^2)\alpha)$$

$$= \sum_{1\le h\le H} \sum_{1\le |t| < W'^2} \sum_{1\le |y| < V'} \sum_{\substack{m \\ (m,t)\in \mathcal{G}_1 \\ (m,t+y)\in \mathcal{G}_1}} e(hty(y+2m)\alpha) .$$

Then

$$|\Sigma_2| \leqslant \sum_{1\le h\le H}\ \sum_{1\le |t|<4W^2}\ \sum_{1\le |y|<2V} \Big|\sum_{\substack{m\\ (m,t)\in \mathcal{G}_1\\ (m+y,t)\in\mathcal{G}_1}} e(2htym\alpha)\Big| ,$$

and the inner-most sum actually extends over all integers of a certain interval. Thus, summing over m, we have

$$(1.3.7)\qquad |\Sigma_2| \ll \sum_{1\le h\le H}\ \sum_{1\le|t|<4W^2}\ \sum_{1\le|y|<2V} \min\left(L(t,y), \|2hty\alpha\|^{-1}\right),$$

where $L(t,y)$ is the length of the interval traversed by m. Indeed,

$$(1.3.8)\qquad L(t,y) = |\{m : (m,t)\in\mathcal{G}_1 \text{ and } (m+y,t)\in\mathcal{G}_1\}|$$

$$= \sum_{\substack{V<m\le V'\\ m^2|t|\le N^2\\ (m+y)^2|t|\le N^2}} 1 \leqslant (N^2|t|^{-1})^{\frac12} = N|t|^{-\frac12},$$

so that, by (1.3.6), (1.3.7), and (1.3.8),

$$(1.3.9)\qquad \Sigma_1 \ll B^2H^{\frac12}\Big(\sum_{1\le t\le 4W^2} d^2(t)\Big)^{\frac12} \times$$

$$\times\Big(HNW + \sum_h\sum_t\sum_y \min(N|t|^{-\frac12}, \|2hty\alpha\|^{-1}\Big)^{\frac12},$$

and this we shall evaluate after Lemma 1 below, which we shall now state.

LEMMA 1 (Vinogradov). <u>Suppose that</u> X <u>and</u> Y <u>are positive integers</u>. <u>Also suppose that</u> $|\alpha - a/q| < q^{-2}$, <u>where</u> α <u>is a real number with</u> a <u>and</u> q <u>integers satisfying</u> $(a,q)=1$. <u>Then</u>

$$\sum_{x \le X} \min(Y, \|\alpha x\|^{-1}) \ll XYq^{-1} + (X+q)\log 2q \,,$$

$$\sum_{x \le X} \min(XY/x, \|\alpha x\|^{-1}) \ll (XYq^{-1} + X + q)(\log 2XYq) \,.$$

These inequalities are essentially Lemmas 8a and 8b of Chapter 1 of [9]. We are now in a position to prove

LEMMA 2. _Let_ N, H, V, V', W, _and_ W' _be positive integers satisfying_ $V < V' \le 2V$, $W < W' \le 2W$, _and_ $VW \le N$. _Suppose_ α, a, _and_ q _are defined as in Lemma_ 1 _with the additional requirement that_ $\log q \ll \log N$.

Let (a_m) _and_ (b_n) _be two sequences of complex numbers such that there are positive numbers_ A _and_ B, _depending on_ N, _such that_

$$a_m \ll A \qquad \text{and} \qquad b_n \ll B \,.$$

Define the sum

$$S_1 = \sum_{1 \le h \le H} \Big| \sum_{V < m \le V'} a_m \sum_{\substack{W < n \le W' \\ mn \le N}} b_n \, e(\alpha h m^2 n^2) \Big| \,.$$

Then, for any fixed real number $\varepsilon > 0$,

(a) _if_ $b_n = 1$ _for all_ $n \in (W, W']$, _we have_

$$S_1 \ll A (NH)^{\varepsilon} (HN^{\frac{1}{2}} V + HNV^{\frac{1}{2}} q^{-\frac{1}{2}} + (HVq)^{\frac{1}{2}}) \,, \tag{1.3.10}$$

(b) _otherwise, we have_

$$S_1 \ll AB (NH)^{\varepsilon} (HN^{\frac{1}{2}} V^{\frac{1}{2}} + HN^{3/4} W^{1/4} + HNq^{-\frac{1}{4}} + N^{\frac{1}{2}} H^{3/4} q^{\frac{1}{4}}) \,. \tag{1.3.11}$$

Proof. (a) It will suffice to bound the expression in (1.3.4). We put

(1.3.12) $$\ell = 2hm^2y\ , \quad \text{so that} \quad 1 \ll |\ell| \leqslant 8NHV\ ,$$

and ℓ will run through all the integers in the interval shown. Also, the number of representations of ℓ in the form (1.3.12) is not more than $d_3(|\ell|)$. Next, note that $Nm^{-1} = N\,.\,|2hmy|\,.\,|\ell|^{-1} \ll N^2H\,|\ell|^{-1}$. Thus, from (1.3.4),

(1.3.13) $$|\Sigma_1| \ll \sum_{|\ell| \leqslant 8NHV} d_3(|\ell|)\min(N^2H|\ell|^{-1}\ , \|\ell\alpha\|^{-1})$$

$$\ll (NHV)^{\varepsilon} \sum_{|\ell| \leqslant 8NHV} \min(N^2H|\ell|^{-1}, \|\ell\alpha\|^{-1})\ ,$$

and by Lemma 1, this is

$$\ll (N^2H)^{\varepsilon}(HN^2q^{-1} + HNV + q)(\log 2N^2Hq).$$

By the conditions on q, and from (1.3.2) and (1.3.13),

$$S_1 \ll A(HN^{\frac{1}{2}}V^{\frac{1}{2}} + (HV\Sigma_1)^{\frac{1}{2}})$$

$$\ll A(HN^{\frac{1}{2}}V^{\frac{1}{2}} + (HN)^{\varepsilon}\{H^2N^2Vq^{-1} + H^2NV^2 + HVq\}^{\frac{1}{2}})$$

$$\ll A(HN)^{\varepsilon}(HN^{\frac{1}{2}}V + HNV^{\frac{1}{2}}q^{-\frac{1}{2}} + (HVq)^{\frac{1}{2}}).$$

(b) We shall evaluate the multiple sum in (1.3.9). In (1.3.9) put

(1.3.14) $$\ell = 2hty\ , \quad \text{so that} \quad 1 \leqslant |\ell| \leqslant 16HNW.$$

Also,

$$N|t|^{-\frac{1}{2}} = N|\ell|^{-1}.|2ht^{\frac{1}{2}}y| \ll N^2H|\ell|^{-1}.$$

The number of representations of λ in the form (1.3.14) is at most $d_3(|\ell|)$ and so, from (1.3.9),

$$\Sigma_1 \ll B^2 H^{\frac{1}{2}} \Big(\sum_{1 \le t \le 4W^2} d^2(t) \Big)^{\frac{1}{2}} \Big(HNW + \sum_{1 \le |\ell| \le 16HNW} d_3(|\ell|) \min\Big(\frac{N^2 H}{|\ell|}, \|\ell\alpha\|^{-1} \Big)\Big)^{\frac{1}{2}}.$$

For our purposes, it will suffice to have

$$\sum_{1 \le t \le 4W^2} d^2(t) \ll N^{\varepsilon} W^2 .$$

By Lemma 1, we then have that

$$\Sigma_1 \ll N^{\varepsilon} B^2 H^{\frac{1}{2}} W \big(HNW + N^{\varepsilon}(N^2 H q^{-1} + NHW + q)\log(2N^2 Hq)\big)^{\frac{1}{2}},$$

and, since $\log q \ll \log N$,

$$\Sigma_1 \ll B^2 (NH)^{2\varepsilon} \big(HN^{\frac{1}{2}} W^{3/2} + HNW q^{-\frac{1}{2}} + H^{\frac{1}{2}} W q^{\frac{1}{2}} \big)^{\frac{1}{2}},$$

and the result follows from (1.3.1) and (1.3.2).

We now establish a consequence of Lemma 2 which will be a principal tool in the proof of Theorem 1 in §4.

LEMMA 3. _Suppose that_ α, a, q, _and_ N _are as defined in Lemma_ 2. _Let_ M_1, M_2, N_1, N_2 _be positive integers such that_

$$M_1 < M_2, \quad N_1 < N_2, \quad \text{and} \quad M_1 N_1 \le N .$$

Also suppose that (a_m) _and_ (b_n) _are complex sequences as in Lemma_ 2, _with_

$$a_m \ll A \qquad \text{and} \qquad b_n \ll B .$$

Define the sum

$$S_2 = \sum_{1 \le h \le H} \Big| \sum_{M_1 < m \le M_2} a_m \sum_{\substack{N_1 < n \le N_2 \\ mn \le N}} b_n \, e(hm^2 n^2 \alpha) \Big| .$$

Then, given any real number $\varepsilon > 0$,

(a) if $b_n = 1$ for all $n \in (N_1, N_2]$,

(1.3.15) $$S_2 \ll A(NH)^{\varepsilon}(HN^{\frac{1}{2}}M_2 + HNM_2^{\frac{1}{2}}q^{-\frac{1}{2}} + (HM_2q)^{\frac{1}{2}}),$$

(b) otherwise

(1.3.16) $$S_2 \ll AB(NH)^{\varepsilon}(HN^{\frac{1}{2}}M_2^{\frac{1}{2}} + HN^{3/4}N_2^{\frac{1}{4}} + HNq^{-\frac{1}{4}} + H^{3/4}N^{\frac{1}{2}}q^{\frac{1}{4}}).$$

Proof. We shall indicate the proof of (a) only since (b) is analogous. We subdivide the interval $(M_1, M_2]$ (also $(N_1, N_2]$) into $O(\log N)$ subintervals of the type $(V, V']$ (also $(W, W']$) such that $M_1 \leq V < V' \leq 2V \leq M_2$ (also $N_1 \leq W < W' \leq 2W \leq N_2$) respectively. Moreover, we may assume that $VW \leq N$ as otherwise the contribution from integers in these intervals to the sum below is zero. It is clear that

$$S_2 \leq \sum_V \sum_W \Big\{ \sum_{1 \leq h \leq H} \Big| \sum_{V < m \leq V'} a_m \sum_{\substack{W < n \leq W' \\ mn \leq N}} b_n \, e(hm^2n^2\alpha) \Big| \Big\},$$

and the sum in brackets is estimated by Lemma 2 for the cases (a) and (b). For case (a), this gives

$$S_2 \ll A(NH)^{\varepsilon} \sum_V \sum_W \{HN^{\frac{1}{2}}V + HNV^{\frac{1}{2}}q^{-\frac{1}{2}} + (HVq)^{\frac{1}{2}}\}$$

$$\ll A(NH)^{3\varepsilon}(HN^{\frac{1}{2}}M_2 + HNM_2^{\frac{1}{2}}q^{-\frac{1}{2}} + (HM_2q)^{\frac{1}{2}}),$$

which is as required with 3ε replaced by ε .

§1.4. Proof of Theorem 1

We shall show that

$$(1.4.1)\quad \mathcal{A}(H,N) = \sum_{1\le h\le H} \Big| \sum_{1\le n\le N} \Lambda(n)\, e(hn^2\alpha)\Big| \ll HN^{1+\varepsilon}\left(q^{-1}+N^{-\frac{1}{2}}+qH^{-1}N^{-2}\right)^{\frac{1}{4}}.$$

(i) Suppose that $q>N^2H$. Then, the expression on the right of (1.4.1) exceeds $HN^{1+\varepsilon}$. This is clearly a trivial estimate for $\mathcal{A}(H,N)$ and so (1.4.1) holds for $q>N^2H$.

(ii) Next, suppose that $H\geqslant N$ and $q\le N^2H$. By Cauchy's inequality

$$\mathcal{A}(H,N) \leqslant H^{\frac{1}{2}}\Big(\sum_{1\le h\le H}\Big|\sum_{1\le n\le N}\Lambda(n)\, e(hn^2\alpha)\Big|^2\Big)^{\frac{1}{2}}$$

$$\ll H^{\frac{1}{2}}\Big(HN\log N + \sum_{1\le h\le H}\ \sum_{\substack{n_1\ne n_2\\ n_1,n_2\le N}} \Lambda(n_1)\Lambda(n_2)\, e\big(h(n_1^2-n_2^2)\alpha\big)\Big)^{\frac{1}{2}}.$$

Now put

$$S_2(\ell) = \sum_{\substack{n_1\le N\\ n_1^2-n_2^2=\ell\\ n_1\ne n_2}}\ \sum_{n_2\le N} \Lambda(n_1)\Lambda(n_2) \leqslant (\log N)^2\, d(|\ell|).$$

It follows from above that

$$\mathcal{A}(H,N) \ll HN^{\frac{1}{2}}(\log N)^{\frac{1}{2}} + H^{\frac{1}{2}}\Big|\sum_{1\le h\le H}\ \sum_{1\le|\ell|<N^2} S_2(\ell)\, e(h\ell\alpha)\Big|^{\frac{1}{2}}.$$

Changing the order of summation and by Lemma 1, we have

$$\mathcal{A}(H,N) \ll HN^{\frac{1}{2}}(\log N)^{\frac{1}{2}} + H^{\frac{1}{2}}\Big(\sum_{1\le|\ell|<N^2} S_2(\ell)\min(H,\|\ell\alpha\|^{-1})\Big)^{\frac{1}{2}}$$

$$\ll HN^{\frac{1}{2}}(\log N)^{\frac{1}{2}} + H^{\frac{1}{2}}N^{\varepsilon}\left(HN^2q^{-1}+N^2+q\right)^{\frac{1}{2}}$$

$$\ll N^{\varepsilon}\left(HNq^{-\frac{1}{2}} + HN^{\frac{1}{2}} + H^{\frac{1}{2}}N + H^{\frac{1}{2}}q^{\frac{1}{2}}\right)$$

$$\ll N^{\varepsilon}\left(HNq^{-\frac{1}{4}} + HN^{\frac{1}{2}} + H^{3/4}N^{\frac{1}{2}}q^{\frac{1}{4}}\right),$$

since $q \gg 1$ and $H \gg N$, $q \leq N^2 H$.

(iii) We may assume from now on that $H < N$ and $q \leq N^2 H$ so that, in particular, $\log q \ll \log N$.

We shall apply a method of Vaughan which is very efficient in estimating sums of the type $\sum_{n \leq N} \Lambda(n) e(n\alpha)$. For a detailed description of the method, we refer the reader to [4] and [5]. The starting point of the method is the identity

$$\sum_{u < n \leq N} f(1,n) + \sum_{u < n \leq Nu^{-1}} \sum_{u < m \leq Nn^{-1}} \tau_m f(m,n)$$

$$= \sum_{d \leq u} \mu(d) \sum_{u < n \leq Nd^{-1}} \sum_{r \leq Nd^{-1}n^{-1}} f(dr,n),$$

where $\tau_m = \sum_{\substack{d \mid m \\ d \leq u}} \mu(d)$. Our choice of the function $f(m,n)$ is

$$f(m,n) = \begin{cases} \Lambda(n) e(hm^2 n^2 \alpha) & \text{if } u < n \leq Nm^{-1}, \\ 0 & \text{otherwise}, \end{cases}$$

where u is an integer less than $N^{\frac{1}{3}}$ to be defined explicitly later on. Then, we obtain

$$(1.4.2) \qquad \sum_{n \leq N} \Lambda(n) e(hn^2\alpha) = S_1 - S_2 - S_3 + O(N^{1/3})$$

where, using $\sum_{d \mid m} \Lambda(d) = \log m$, we have

$$(1.4.3) \qquad S_1 = \sum_{d \leq u} \mu(d) \sum_{r \leq Nd^{-1}} \sum_{n \leq Nd^{-1}r^{-1}} \Lambda(n) e(hd^2r^2n^2\alpha)$$

$$= \sum_{d \leq u} \mu(d) \sum_{m \leq Nd^{-1}} (\log m)\, e(hd^2m^2\alpha)$$

$$= \int_1^N S_1(\beta) \frac{d\beta}{\beta},$$

with

$$(1.4.4)\qquad \mathcal{S}_1(\beta) = \sum_{d \le \min(u, N\beta^{-1})} \mu(d) \sum_{\beta < k \le Nd^{-1}} e(hd^2k^2\alpha)\ ;$$

next

$$(1.4.5)\qquad \mathcal{S}_2 = \sum_{d \le u} \mu(d) \sum_{n \le u} \Lambda(n) \sum_{r \le Nd^{-1}n^{-1}} e(hd^2n^2r^2\alpha)\ ,$$

$$= \sum_{\ell \le u^2} c_\ell \sum_{r \le N\ell^{-1}} e(h\ell^2r^2\alpha)\ ,$$

where

$$(1.4.6)\qquad c_\ell = \sum_{\substack{d \mid \ell \\ d \le u}} \mu(d)\Lambda(\ell/d)\ ;$$

and finally

$$(1.4.7)\qquad \mathcal{S}_3 = \sum_{u < m \le Nu^{-1}} \tau_m \sum_{u < n \le Nm^{-1}} \Lambda(n)\, e(hm^2n^2\alpha)\ .$$

Observe that $\mathcal{S}_1(\beta)$, $\mathcal{S}_2$, and $\mathcal{S}_3$ are sums of a type discussed in §3. It follows from (1.4.1) and (1.4.7) that

$$(1.4.8)\qquad \mathcal{A}(H,N) \le \sum_{h \le H} (|\mathcal{S}_1| + |\mathcal{S}_2| + |\mathcal{S}_3|) + O(HN^{\frac{1}{3}})\ .$$

We shall denote the first three sums on the right of (1.4.8) by $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$. We proceed to estimate these sums by Lemma 3.

From (1.4.3), it follows that

$$\mathcal{B}_1 = \sum_{h \le H} |\mathcal{S}_1| \le \int_1^N \Big(\sum_{1 \le h \le H} |\mathcal{S}_1(\beta)| \Big) \frac{d\beta}{\beta}\ ,$$

where

$$\sum_{1\le h\le H} |\mathcal{S}_1(\beta)| = \sum_{1\le h\le H} \Big| \sum_{d\le \min(u,N\beta^{-1})} \mu(d) \sum_{\substack{\beta<k\le N\\ dk\le N}} e(hd^2k^2\alpha)\Big| .$$

By Lemma 3(a), with $M_1 = 1$, $M_2 = \min(u, N\beta^{-1}) \le u$, $N_1 = \beta$, and $N_2 = N$, and noting that $|\mu(d)| \le 1$ (so that $A = 1$), we see that it follows from (1.3.15), for any $\varepsilon > 0$, that

$$\sum_{1\le h\le H} |\mathcal{S}_1(\beta)| \ll N^{\varepsilon}\big(HN^{\frac12}u + HNu^{\frac12}q^{-\frac12} + (Huq)^{\frac12}\big).$$

Hence

$$(1.4.9) \qquad \mathcal{B}_1 \ll N^{2\varepsilon}(HN^{\frac12}u + HNu^{\frac12}q^{-\frac12} + (Huq)^{\frac12}).$$

From (1.4.6), we note that $c_\ell \le (\log \ell)\, d(\ell) \ll N^{\varepsilon}$. We write (cf. (1.4.5))

$$(1.4.10) \quad \mathcal{S}_2 = \Big(\sum_{m\le u}\sum_{r\le Nm^{-1}} + \sum_{u<m\le N^{\frac12}}\sum_{u<r\le N^{1/2}} + \sum_{u<m\le N^{\frac12}}\sum_{r\le u} \sum_{N^{\frac12}<m\le u^2}\sum_{r\le Nm^{-1}} + \sum_{u<m\le N^{\frac12}}\sum_{N^{1/2}<r\le Nm^{-1}}\Big) c_m e(hm^2r^2\alpha)$$
$$= \sum_{j=1}^{5} \mathcal{S}_2^{(j)} ,$$

say. Then

$$\mathcal{B}_2 \le \sum_{1\le h\le H} \sum_{j=1}^{5} |\mathcal{S}_2^{(j)}| = \sum_{j=1}^{5} \mathcal{B}_2^{(j)} ,$$

where the meaning of the notation is plain. We now estimate $\mathcal{B}_2^{(j)}$ $(j = 1,2,\ldots,5)$ as follows. For $\mathcal{B}_2^{(1)}$, apply Lemma 3(a) with $M_1 = 1$, $M_2 = u$, $N_1 = 1$, $N_2 = N$, and $A = N^{\varepsilon}$. From (1.3.15), it follows that

$$\mathcal{B}_2^{(1)} \ll N^{2\varepsilon}\left(HN^{\frac{1}{2}}u + HNu^{\frac{1}{2}}q^{-\frac{1}{2}} + (Huq)^{\frac{1}{2}}\right).$$

For $\mathcal{B}_2^{(2)}$, apply Lemma 3(b) with $M_1 = N_1 = u$, $M_2 = N_2 = N^{\frac{1}{2}}$, $B = 1$, and $A = N^{\varepsilon}$. Then

$$\mathcal{B}_2^{(2)} \ll N^{2\varepsilon}\left(HN^{7/8} + HNq^{-\frac{1}{4}} + H^{3/4}N^{\frac{1}{2}}q^{\frac{1}{4}}\right).$$

Trivially,

$$\mathcal{B}_2^{(3)} \ll N^{\varepsilon}\left(N^{\frac{1}{2}}Hu\right).$$

The estimation of $\mathcal{B}_2^{(4)}$ follows from Lemma 3(b) with $M_1 = N_2 = N^{\frac{1}{2}}$, $M_2 = u^2$, $N_1 = 1$, $B = 1$, and $A = N^{\varepsilon}$, so that

$$\mathcal{B}_2^{(4)} \ll N^{2\varepsilon}\left(HN^{\frac{1}{2}}u + HN^{7/8} + HNq^{-\frac{1}{4}} + H^{3/4}N^{\frac{1}{2}}q^{\frac{1}{4}}\right).$$

Finally, by a change in the order of summation in $\mathcal{S}_2^{(5)}$, we have from (1.4.10) that

$$\mathcal{S}_2^{(5)} = \sum_{N^{1/2} < r \le Nu^{-1}} \ \sum_{\substack{u < m \le N^{1/2} \\ rm \le N}} c_m\, e(hr^2m^2\alpha) \ ;$$

and to estimate $\mathcal{B}_2^{(5)}$, we now apply Lemma 3(b) with $M_1 = N_2 = N^{\frac{1}{2}}$, $M_2 = Nu^{-1}$, $N_1 = u$, $A = 1$, and $B = N^{\varepsilon}$. From (1.3.16) it follows that

$$\mathcal{B}_2^{(5)} \ll N^{2\varepsilon}\left(HNu^{-\frac{1}{2}} + HN^{7/8} + HNq^{-\frac{1}{4}} + H^{3/4}N^{\frac{1}{2}}q^{1/4}\right).$$

Combining these five estimates, we arrive at

$$(1.4.11) \qquad \mathcal{B}_2 \ll N^{2\varepsilon}\Big(HNu^{-\frac{1}{2}} + HN^{\frac{1}{2}}u + HN^{7/8} + HNq^{-\frac{1}{4}}$$
$$+ H^{3/4}N^{\frac{1}{2}}q^{\frac{1}{4}} + HNu^{\frac{1}{2}}q^{-\frac{1}{2}} + (Huq)^{\frac{1}{2}}\Big).$$

We have finally to estimate $\mathcal{B}_3$. It is clear that $\tau_m \le d(m) \ll N^{\varepsilon}$. We shall rewrite $\mathcal{S}_3$, from (1.4.7), as

$$(1.4.12)\quad \Big(\sum_{u<m\le N^{1/2}} \sum_{u<n\le N^{1/2}} + \sum_{N^{1/2}<m\le Nu^{-1}} \sum_{u<n\le Nm^{-1}} + \sum_{u<m\le Nu^{-1}} \sum_{N^{1/2}<n\le Nm^{-1}} \Big)$$

$$\times\ \tau_m \Lambda(n)\, e(hm^2n^2\alpha)$$

$$= \mathcal{S}_3^{(1)} + \mathcal{S}_3^{(2)} + \mathcal{S}_3^{(3)},$$

say. Then

$$(1.4.13)\qquad \mathcal{B}_3 \le \sum_{j=1}^{3} \Big(\sum_{1\le h\le H} |\mathcal{S}_3^{(j)}| \Big).$$

We observe that $\mathcal{S}_3^{(3)}$, after a change in the order of summation, is

$$\sum_{N^{1/2}<n\le Nu^{-1}} \Lambda(n) \sum_{u<m\le Nn^{-1}} \tau_m\, e(hm^2n^2\alpha)$$

and this is essentially like $\mathcal{S}_3^{(2)}$ --the only difference being in the position of the weights. Thus, it would suffice to estimate $\sum_{1\le h\le H} |\mathcal{S}_3^{(2)}|$. We put $M_1 = N_2 = N^{\frac{1}{2}}$, $M_2 = Nu^{-1}$, $N_1 = u$, $A = B = N^{\varepsilon}$, in Lemma 3(b), so that

$$\sum_{1\le h\le H} |\mathcal{S}_3^{(j)}| \ll N^{2\varepsilon}(HNu^{-\frac{1}{2}} + HN^{7/8} + HNq^{-\frac{1}{4}} + H^{3/4}N^{1/2}q^{1/4}),$$

for j = 2 and 3.

To estimate $\sum_{1\le h\le H} |\mathcal{S}_3^{(1)}|$, we put $M_1 = N_1 = u$ and $M_2 = N_2 = N^{\frac{1}{2}}$ in Lemma 3(b) with $A = B = N^{\varepsilon}$. Then

$$\sum_{1\le h\le H} |\mathcal{S}_3^{(1)}| \ll N^{3\varepsilon}(HN^{7/8} + HNq^{-\frac{1}{4}} + H^{3/4}N^{\frac{1}{2}}q^{\frac{1}{4}}).$$

Combining these estimates, we see that it follows that

$$(1.4.14) \qquad \mathcal{B}_3 \ll N^{3\varepsilon}\left(HNu^{-1/2} + HN^{7/8} + HNq^{-\frac{1}{4}} + H^{3/4}N^{\frac{1}{2}}q^{\frac{1}{4}}\right).$$

Collecting the estimates for $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ from (1.4.10), (1.4.11), and (1.4.14), we have

$$(1.4.15) \qquad \mathcal{A}(H,N) \ll N^{3\varepsilon}\left(HNq^{-\frac{1}{4}} + HN^{7/8} + H^{3/4}N^{\frac{1}{2}}q^{\frac{1}{4}}\right)$$

$$+ N^{3\varepsilon}\left(HN^{\frac{1}{2}}u + HNu^{-\frac{1}{2}} + HNu^{\frac{1}{2}}q^{-\frac{1}{2}} + (Huq)^{\frac{1}{2}}\right).$$

We minimize the terms in the second expression on the right of (1.4.15) by an appropriate choice of $u < N^{1/3}$, namely by

$$u = \min\left(N^{1/3},\ q^{\frac{1}{2}},\ (N^2Hq^{-1})^{\frac{1}{2}}\right),$$

and note that since $q < N^2H$, this expression is well defined. Such a choice of u gives us (1.4.1) with 3ε replaced by ε.

§1.5. Proof of Theorem 3 and Theorem 4

We shall follow in the spirit of the proof of Theorem 2 in [5]. For each real number η satisfying $0 \le \eta < \frac{1}{2}$, define the function

$$(1.5.1) \qquad f_\eta(x) = \begin{cases} 1, & -\eta \le x < \eta, \\ 0, & -\frac{1}{2} \le x < -\eta \ \text{ or } \ \eta \le x < \frac{1}{2}. \end{cases}$$

We extend the range of definition of $f_\eta(x)$ periodically with period 1 in x. Then, we obtain the Fourier-series expansion of $f_\eta(x)$ as

$$(1.5.2) \qquad f_\eta(x) - 2\eta = \sum_{\substack{-\infty \\ m\neq 0}}^{\infty}{}' \frac{\sin 2\pi m\eta}{\pi m}\, e(mx),$$

where $\sum'$ denotes summation over m with values corresponding to $\pm m$ taken together.

Let α and β be real numbers as defined in the statement of Theorem 3. It follows from (1.5.2) that

$$f_\eta(n^2\alpha-\beta) - 2\eta = \sum_{\substack{-\infty \\ m\neq 0}}^{\infty}{}' \frac{\sin(2\pi m\eta)}{\pi m} e(m(n^2\alpha-\beta)) .$$

We write the sum on the right as

$$\sum_{1\leq |m| \leq H} \frac{\sin 2\pi m\eta}{\pi m} e(m(n^2\alpha-\beta)) + \sum_{|m|>H}{}' \frac{\sin 2\pi m\eta}{\pi m} e(m(n^2\alpha-\beta)) ,$$

for some integer $H>1$. It follows that

$$\sum_{1\leq n\leq N} \Lambda(n) f_\eta(n^2\alpha-\beta) = \mathcal{F}_1 + \mathcal{F}_2 , \tag{1.5.3}$$

where

$$\mathcal{F}_1 = \sum_{1\leq |m|\leq H} \frac{\sin 2\pi m\eta}{\pi m} e(-\beta m) \sum_{1\leq n\leq N} \Lambda(n) e(mn^2\alpha) , \tag{1.5.4}$$

and

$$\mathcal{F}_2 = \sum_{1\leq n\leq N} \Lambda(n) \sum_{|m|>H}{}' \frac{\sin 2\pi m\eta}{\pi m} e(m(n^2\alpha-\beta)) . \tag{1.5.5}$$

Now, let a/q be a convergent to the continued fraction for α, with $(a,q)=1$. Then, $|\alpha - a/q| < q^{-2}$. Next, take

$$H = [Nq\eta^{-1}] + 1 . \tag{1.5.6}$$

We evaluate $\mathcal{F}_2$ as follows. We write $\mathcal{F}_2$ as

$$\frac{1}{2i}\sum_{n\le N}\Lambda(n)\sum_{|m|>H}\frac{1}{\pi m}\left\{e(m(n^2\alpha-\beta+\eta))-e(m(n^2\alpha-\beta-\eta))\right\}. \tag{1.5.7}$$

Note that if either of $n^2\alpha-\beta\pm\eta$ is an integer, for some n, the contribution of this term to the sum over m is zero. We put

$$\theta_n^{\pm} = n^2\alpha-\beta\pm\eta ,$$

and assume that $\theta_n^{\pm}$ is not integral. From (1.5.7),

$$|\mathcal{F}_2| \le \tfrac{1}{2}(\log N)\left\{\sum_{n\le N}\Big|\sideset{}{'}\sum_{|m|>H}\frac{1}{\pi m}e(m\theta_n^{+})\Big| + \sum_{n\le N}\Big|\sideset{}{'}\sum_{|m|>H}\frac{1}{\pi m}e(m\theta_n^{-})\Big|\right\}.$$

Put $\mathcal{F}_n^{\pm}(k) = \sum_{1\le m\le k} e(m\theta_n^{\pm})$. As is well known

$$|\mathcal{F}_n^{\pm}(k)| \le \tfrac{1}{2}\|\theta_n^{\pm}\|^{-1},$$

and therefore, by Abel summation,

$$\Big|\sideset{}{'}\sum_{|m|>H}\frac{1}{m}e(m\theta_n^{\pm})\Big| \ll H^{-1}\|\theta_n^{\pm}\|^{-1}.$$

Moreover, since

$$\Big|\sideset{}{'}\sum_{|m|>H}\frac{1}{\pi m}e(m\theta_n^{\pm})\Big| \ll 1 ,$$

uniformly in H and $\theta_n^{\pm}$, we have

$$\Big|\sideset{}{'}\sum_{|m|>H}\frac{1}{\pi m}e(m\theta_n^{\pm})\Big| \ll \min(1, H^{-1}\|\theta_n^{\pm}\|^{-1}).$$

It follows that

$$\mathcal{F}_2 \ll (\log N)\sum_{n\le N}\min(1, H^{-1}\|\alpha n^2-\beta\pm\eta\|^{-1}). \tag{1.5.8}$$

We define the function

$$\Psi(H, w) = \min(1, H^{-1}\|w\|^{-1}) ,$$

which is periodic, with period 1 in w. This function has a Fourier-series expansion, in the form

$$\Psi(H, w) = \sum_{h=-\infty}^{\infty}{}' c_h \, e(hw) ,$$

where Σ' has the same meaning as in (1.5.2). It is easily verified that

$$c_h \ll \min\left(\frac{1 + \log H}{H} , h^{-1} , Hh^{-2}\right) .$$

We may write

$$\Psi(H, w) = \sum_{1 \le |h| \le H^2} c_h \, e(hw) + \sum_{|h| > H^2}{}' c_h \, e(hw) + O(H^{-1} \log H) .$$

Then, it follows from (1.5.8) that

$$\mathcal{J}_2 \ll (\log NH)^2 \sum_{1 \le |h| \le H^2} \min(H^{-1}, h^{-1}) \Big| \sum_{1 \le n \le N} e(hn^2\alpha) \Big|$$

$$+ (\log N) \sum_{1 \le n \le N} \Big| \sum_{|h| > H^2}{}' c_h \, e(h(\alpha n^2 - \beta \pm \eta)) \Big| + NH^{-1}(\log NH)^2$$

$$= \mathcal{J}_2^{(1)} + \mathcal{J}_2^{(2)} + O(NH^{-1}(\log NH)^2) ,$$

say. We estimate $\mathcal{J}_2^{(2)}$ trivially as

$$\ll (\log N) \sum_{1 \le n \le N} \sum_{|h| > H^2} Hh^{-2} \ll NH^{-1} \log N .$$

To estimate $\mathcal{J}_2^{(1)}$, we put

$$\mathcal{V}(u) = \sum_{1 \le |h| \le u} \Big| \sum_{1 \le n \le N} e(hn^2\alpha) \Big| .$$

Then, the sum of $\tilde{\mathcal{J}}_2^{(1)}$ is

$$\frac{\mathcal{V}(H^2)}{H^2} - \frac{\mathcal{V}(1)}{H} + \int_H^{H^2} z^{-2}\,\mathcal{V}(z)\,dz .$$

To estimate $\mathcal{V}(u)$, we apply Cauchy's inequality to obtain

$$|\mathcal{V}(u)|^2 \leq u \sum_{1\leq|h|\leq u} \Big|\sum_{1\leq n\leq N} e(hn^2\alpha)\Big|^2$$

$$= u\Big(2uN + \sum_{1\leq|h|\leq u} \sum_{\substack{n_1\neq n_2\\ 1\leq n_1,n_2\leq N}} e(h(n_1^2-n_2^2)\alpha)\Big)$$

$$\leq u\Big(2uN + \sum_{1\leq|h|\leq u}\sum_{1\leq|t|<N}\Big|\sum_{1\leq n\leq \min(N,N-t)} e(ht(t+2n)\alpha)\Big|\Big)$$

$$\ll u^2N + u\sum_{1\leq|h|\leq u}\sum_{1\leq|t|<N} \min(N, \|2ht\alpha\|^{-1})$$

$$\ll u^2N + u\sum_{|l|\leq 2Nu} \varsigma_3(l)\min(N, \|l\alpha\|^{-1}),$$

where we have put $l=2ht$ and

$$\varsigma_3(l) = \sum_{1\leq|h|\leq u}\sum_{\substack{1\leq|t|\leq N\\ 2ht=l}} 1 \leq 2d(|l|) \ll (Nu)^{\varepsilon}.$$

Hence, it follows by Lemma 1, that

$$\mathcal{V}(u) \ll (Nuq)^{2\varepsilon}\big(Nuq^{-\frac{1}{2}} + N^{\frac{1}{2}}u + (uq)^{\frac{1}{2}}\big).$$

Next observe that $\mathcal{V}(1)H^{-1} \ll NH^{-1}$. Thus, it follows that

$$\frac{\mathcal{V}(H^2)}{H^2} - \frac{\mathcal{V}(1)}{H} + \int_H^{H^2} z^{-2}\mathcal{V}(z)\,dz \ll (NHq)^{\varepsilon}\big(Nq^{-\frac{1}{2}} + N^{\frac{1}{2}} + q^{\frac{1}{2}}H^{-\frac{1}{2}}\big) + NH^{-1}.$$

Hence, we have shown that

(1.5.9) $$\mathcal{F}_2 \ll (NHq)^{3\varepsilon}(Nq^{-\frac{1}{2}} + N^{\frac{1}{2}} + NH^{-1} + N^{-\frac{1}{2}}q^{\frac{1}{2}})$$
$$\ll (Nq\eta^{-1})^{6\varepsilon}(Nq^{-\frac{1}{2}} + N^{\frac{1}{2}}),$$

by (1.5.6).

We now proceed to evaluate $\mathcal{F}_1$. Define the function

(1.5.10) $$\mathcal{G}(k) = \sum_{1\le m\le k} \Big|\sum_{n\le N} \Lambda(n)\, e(mn^2\alpha)\Big| .$$

In Theorem 1, choose and then fix an $\varepsilon_0 > 0$. Then

$$\mathcal{G}(k) \ll Nk\,\mathcal{D}(k),$$

where

$$\mathcal{D}(k) = N^{\varepsilon_0}(N^{-1/8} + q^{-\frac{1}{4}} + (qN^{-2})^{\frac{1}{4}}k^{-\frac{1}{4}}).$$

From (1.5.4) we have, after an application of Stieltjes integration at the last step,

$$\mathcal{F}_1 \ll \sum_{1\le|m|\le H} \min(\eta, |m|^{-1}) \Big|\sum_{n\le N} \Lambda(n)\, e(mn^2\alpha)\Big|$$
$$\ll \sum_{1\le m\le H} \min(\eta, m^{-1}) \Big|\sum_{n\le N} \Lambda(n)\, e(mn^2\alpha)\Big|$$
$$\ll \frac{\mathcal{G}(H)}{H} - \eta\,\mathcal{G}(1) + \int_{\eta^{-1}}^{H} u^{-2}\,\mathcal{G}(u)\,du .$$

From (1.5.10), we have

(1.5.11) $$\mathcal{F}_1 \ll N\mathcal{D}(H) + N\eta\,\mathcal{D}(1) + N\mathcal{D}(\eta^{-1})(\log H\eta) .$$

Since $\mathcal{D}(k)$ is a decreasing function of k and since $\eta\mathcal{D}(1) \le \mathcal{D}(\eta^{-1})$, we have from (1.5.6) and (1.5.10) that

(1.5.12) $$\mathcal{F}_1 \ll N\,\mathcal{D}(\eta^{-1})\,(\log 2Nq).$$

We now choose $\varepsilon = \frac{1}{6}\varepsilon_0$ in (1.5.9). Then, from (1.5.3), (1.5.12), and (1.5.9),

(1.5.13) $$\sum_{n\le N} \Lambda(n)\left\{ f_\eta(\alpha n^2-\beta) - 2\eta \right\}$$

$$\ll N\,\mathcal{D}(\eta^{-1})(\log 2Nq) + (Nq\eta^{-1})^{\varepsilon_0}(Nq^{-\frac{1}{2}} + N^{\frac{1}{2}}).$$

For the proof of Theorem 3, we put

$$\eta = c(\varepsilon_0)\, N^{-\frac{1}{8}+3\varepsilon_0}$$

where $c(\varepsilon_0)$ is a sufficiently large constant, dependent only on ε_0. Since α is irrational, there are infinitely many values of q satisfying $|\alpha - a/q| < q^{-2}$. Observe that if $N^{\frac{1}{2}} \le q \le N^{3/2}$ then

$$\mathcal{D}(k) \ll N^{-\frac{1}{8}+\varepsilon_0} \qquad \text{for } k \geq 1.$$

For each such q, we choose $N = q$, and so by (1.5.13),

(1.5.14) $$\sum_{n\le N} \Lambda(n)\left\{ f_\eta(\alpha n^2-\beta) - 2\eta \right\} \ll N^{7/8+2\varepsilon_0},$$

for sufficiently large values of N defined as above. Thus, we have shown that there exists an increasing sequence of positive integers $\{N_1, N_2, \dots\}$ such that

$$\sum_{n\le N_i} \Lambda(n)\, f_{\eta_i}(\alpha n^2-\beta) = 2\eta_i \sum_{n\le N_i} \Lambda(n) + O\left(N_i^{7/8+2\varepsilon_0}\right),$$

where $\eta_i = c(\varepsilon_0)\, N_i^{-\frac{1}{8}+3\varepsilon_0}$, for $i = 1,2,\dots$. Hence, for all sufficiently large N_i, from the prime number theorem, (1.5.15) $$\sum_{p < N_i} (\log p)\, f_{\eta_i}(\alpha p^2 - \beta) > c(\varepsilon_0) N_i^{7/8 + 3\varepsilon_0} .$$

The sum on the left is

$$\sum_{\substack{p \le N_i \\ \|p^2\alpha - \beta\| < c(\varepsilon_0) N_i^{-\frac{1}{8} + 3\varepsilon_0}}} (\log p) ,$$

and Theorem 3 now follows.

For the proof of Theorem 4, we may assume $\delta < 1$, as the result is trivial for $\delta = 1$. Note that β in (1.5.13) was an arbitrary real number. We now choose $\beta = \gamma + \frac{1}{2}\delta$ and $\eta = \frac{1}{2}\delta$, in (1.5.13). Then

(1.5.16) $$\sum_{n \le N} \Lambda(n) f_{\delta/2}(\alpha n^2 - \gamma - \tfrac{1}{2}\delta)$$

$$= \delta \sum_{n \le N} \Lambda(n) + O\left(N \varpi(2/\delta)(\log 2Nq) + (Nq\delta^{-1})^{\varepsilon_0}(Nq^{-\frac{1}{2}} + N^{\frac{1}{2}}\right).$$

The error term is

$$\ll N^{2\varepsilon_0}(N^{7/8} + Nq^{-\frac{1}{4}} + N^{\frac{1}{2}}\delta^{\frac{1}{4}}q^{\frac{1}{4}})(\log q) + (Nq\delta^{-1})^{\varepsilon_0}(Nq^{-\frac{1}{2}} + N^{\frac{1}{2}}) .$$

Now, $f_{\delta/2}(\alpha n^2 - \gamma - \frac{1}{2}\delta) = 1$ if and only if there is an integer h such that

$$-\tfrac{1}{2}\delta \le \alpha n^2 - \gamma - \tfrac{1}{2}\delta - h < \tfrac{1}{2}\delta ,$$

that is, such that $\gamma \le \alpha n^2 - h < \gamma + \delta$, and since $\gamma + \delta \le 1$, it follows that $h = [\alpha n^2]$. Hence $f_{\delta/2}(\alpha n^2 - \gamma - \frac{1}{2}\delta) = 1$, if and only if $\gamma \le \{\alpha n^2\} < \gamma + \delta$, which is what we require.

Hence, from (1.5.16), it follows that

$$\sum_{\substack{p \le N \\ \gamma \le \{\alpha p^2\} < \gamma + \delta}} \log p = \delta\,\vartheta(N) + O\Big(N^{2\varepsilon_0}(N^{7/8} + Nq^{-\frac{1}{4}} + N^{\frac{1}{2}}\delta^{\frac{1}{4}}q^{\frac{1}{4}}) \times$$

$$\times (\log q) + (Nq\delta^{-1})^{\varepsilon_0}(Nq^{-\frac{1}{2}} + N^{\frac{1}{2}})\Big),$$

and Theorem 4 follows by partial summation.

§1.6. Appendix

It is crucial, in some applications, to have a power of a logarithm in place of N^{ε} in Theorem 1 (and so Theorem 2). Under the assumptions of Theorem 1, we can prove the following:

$$|\mathcal{A}(H,N)| \ll HN (\log N)^{c_1(\varepsilon)} (q^{-1} + N^{-\frac{1}{2}} + qH^{-1}N^{-2})^{\frac{1}{4}-\varepsilon},$$

where $c_1(\varepsilon)$ is a positive constant depending at most on ε.

The main point is an improvement in the estimate for the sum in (1.3.13). An application of Hölder's inequality (for details of the technique, see the proof of Lemma 4 in §2.2) gives

$$|\Sigma_1| \ll N^2H(\log 2N^2Hq)^{c_2(\varepsilon)}(q^{-1} + VN^{-1} + qH^{-1}N^{-2})^{1-\varepsilon}.$$

A similar improvement is obtained in §1.4(ii). The next improvement is required in the estimate for

$$\sum_{1 \le |t| < W'^2} |\zeta_1(t)|^2,$$

in (1.3.6). We show here that for any $M \ge 1$,

$$A := \sum_{1 \le |\ell| \le M} |\zeta_1(\ell)|^2 \ll M(\log M)^{37/2}.$$

By the definition of $\zeta_1(\ell)$, we note that

$$A \leq \sum_{\substack{1 \leq m_i, n_i \leq M^{\frac{1}{2}} \\ i=1,2, \\ m_1^2+m_2^2 = n_1^2+n_2^2}} \sum d^2(n_1)\, d^2(m_1) \quad . \tag{1.6.1}$$

(a) Suppose that $n_2 = m_2$. Then, we must have $m_1 = n_1$ so that the contribution to A from the sum above is

$$\leq M^{\frac{1}{2}} \sum_{m \leq M^{\frac{1}{2}}} d^4(m) \ll M(\log M)^{15} \tag{1.6.2}$$

(b) Suppose $m_2 = n_2 + t$, and $1 \leq |t| \leq M^{\frac{1}{2}}$. Then, the sum in question is

$$\sum_{\substack{n_1, n_2, |t|, m_1 \\ n_1^2 + n_2^2 = m_1^2 + (n_2+t)^2}} d(n_1)^2\, d(m_1)^2 = \sum_{n_1, m_1 \leq M^{\frac{1}{2}}} d(n_1)^2\, d(m_1)^2 \sum_{\substack{|t|, n_2 \\ t(2n_2+t) = n_1^2 - m_1^2}} 1$$

$$\ll \sum_{n_1, m_1 \leq M^{1/2}} d(n_1)^2\, d(m_1)^2\, d(|n_1^2 - m_1^2|) \quad .$$

Applying Cauchy's inequality twice, to the final sum gives us a contribution of

$$\leq \Big(\sum_{n \leq M^{1/2}} d(n)^4 \Big) \Big(\sum_{n_1, m_1 \leq M^{1/2}} d^2(|n_1^2 - m_1^2|) \Big)^{\frac{1}{2}} , \tag{1.6.3}$$

$$\ll M^{\frac{1}{2}} (\log M)^{15} \Big(\sum_{1 \leq |\ell| \leq M} d^2(|\ell|)\, r(\ell) \Big)^{\frac{1}{2}} ,$$

where

$$r(\ell) = \sum_{\substack{n_1, m_1 \\ \ell = n_1^2 - m_1^2}} 1 \leq d(|\ell|) ,$$

so that (1.6.3) is

$$\ll \quad M^{\frac{1}{2}} (\log M)^{15} \Big(\sum_{1 \le |\ell| \le M} d^3(|\ell|) \Big)^{\frac{1}{2}}$$

$$\ll \quad M (\log M)^{37/2} .$$

Collecting this estimate and that of (1.6.2) gives us, by (1.6.1), the stated result.

There is no other new difficulty elsewhere.

Chapter 2.

Estimate for the Exponential Sum $\sum_{p \le N} e(\alpha p^3)$.

§2.1. Introduction

We use the method given in Chapter 1 to prove the

THEOREM 5. *Suppose α is a real number and a and q are positive integers satisfying $(a,q)=1$ and $|\alpha - a/q| < q^{-2}$. Let N be a positive integer. Then, given any real number $\varepsilon > 0$, there is a positive number $c(\varepsilon)$ such that*

$$\sum_{n \le N} \Lambda(n)\, e(n^3\alpha) \ll \left(N^{\frac{31}{32}+\varepsilon} + N\Delta^{\frac{1}{20}}\right)(\log N)^{c(\varepsilon)}, \tag{2.1.1}$$

where

$$\Delta = \Delta(q) = \left(q^{-1} + N^{-3}q\right), \tag{2.1.2}$$

and the implied constants depend at most on ε.

The result is of a poorer quality than Theorem 2 (one may hope to replace the exponent $\frac{1}{20}$ by $\frac{1}{8}$), but is better than similar results available in the literature. Thus, Vinogradov proves in [8] that the sum is bounded by

$$N^{1-\frac{1}{1536}} + N\Delta^{\frac{1}{1024}}$$

valid for $\log q \gg 7.2^{16}\cdot \log\log N$.

§2.2. Proof of Theorem 5

Throughout this section, $c(\varepsilon)$ will denote a generic constant depending on $\varepsilon > 0$ only. We also put

$$\Delta = (q^{-1} + N^{-3} q).$$

As in the proof of Theorem 1, we shall require some preliminary results corresponding to Lemmas 2 and 3.

LEMMA 4. *Let* N, V, V', W, W' *be positive integers satisfying* $V < V' \leq 2V$, $W < W' \leq 2W$, *and* $VW \leq N$. *Suppose* α, a, *and* q *are defined as in Lemma* 1 *with* $\log q \ll \log N$.

Let (a_m) *and* (b_n) *be two sequences of complex numbers, and define the sum*

$$T_1 = \sum_{V<m\leq V'} a_m \sum_{\substack{W<n\leq W' \\ mn\leq N}} b_n \, e(m^3 n^3 \alpha) \qquad (2.2.1)$$

Then, for any fixed real number $\varepsilon > 0$, *there is a constant* $c(\varepsilon) > 0$ *such that*

(2.2.2) (a) *if* $b_n = 1$ *for all* $n \in (W, W']$, *we have*

$$T_1 \ll_\varepsilon \Big(\sum_{V<m\leq V'} |a_m|^2\Big)^{\frac12} \Big(N^{\frac12} W^{\frac14} + N^{\frac34+\varepsilon} V^{\frac14} + N \Delta^{\frac14-\varepsilon}\Big) (\log N)^{c(\varepsilon)},$$

(2.2.3) (b) *otherwise, we have*

$$T_1 \ll_\varepsilon V^{\frac12} \Big(\sum_{V<m\leq V'} |a_m|^2\Big)^{\frac12} \Big(\sum_{W<n\leq W'} |b_n|^2\Big)^{\frac12} + \Big(\sum_m |a_m|^2\Big)^{\frac12} \Big(\sum_n |b_n|^4\Big)^{\frac14},$$
$$\times \Big(N^{3/8} W^{3/8} + N^{\frac12} W^{\frac{3}{16}+\varepsilon} + N^{\frac12} W^{\frac14} \Delta^{\frac{1}{16}-\varepsilon}\Big) (\log N)^{c(\varepsilon)}.$$

Proof.

(a) By Cauchy's inequality, we have

$$|T_1|^2 \ll \Big(\sum_{V<m\leq V'} |a_m|^2\Big)^{\frac12} \Big(VW + \sum_{V<m\leq V'} \sum \sum_{\substack{n_1 \neq n_2 \\ W<n_i\leq W' \\ mn_i \leq N \\ i=1,2}} e(\alpha m^3 (n_1^3 - n_2^3))\Big) \qquad (2.2.4)$$

We denote the triple sum in (2.2.4) by T_2. Putting

$$n_1 = n_2 + h \quad , \quad \Psi(h,n_2) = h(h^2 + 3hn_2 + 3n_2^2) ,$$

we get

$$(2.2.5) \qquad T_2 = \sum_{V < m \leq V'} \sum_{|h| \geq 1} \sum_{n_2} e(m^3 \Psi(h,n_2)\alpha) ,$$

where the sum is over the integers

$$1 \leq |h| \leq \min(W, \tfrac{2N}{m}) \; ; \quad W < n_2 \leq \min(W', W'-h, \tfrac{N}{m}, \tfrac{N-h}{m}) .$$

Applying Cauchy's inequality to (2.2.5), we get

$$(2.2.6) \qquad |T_2|^2 \ll VW \sum_m \sum_{|h|} \Big| \sum_{n_2} e(m^3 \Psi(h,n_2)\alpha) \Big|^2$$

$$\leq VW \sum_m \sum_{|h|} \Big(W + \sum_{n_3 \neq n_4} e(m^3(\Psi(h,n_3) - \Psi(h,n_4))\alpha) \Big) .$$

Now, note that

$$(2.2.7) \qquad \Psi(h,n_3) - \Psi(h,n_4) = 3h(n_3 - n_4)(h + n_3 + n_4) ,$$

so that putting

$$n_3 = n_4 + t \quad , \quad \Psi_1(h,t,n_4) = 3ht(h + t + 2n_4)$$

we get from (2.2.6) that

$$(2.2.8) \qquad |T_1|^2 \ll VW\Big(VW^2 + \sum_m \sum_{|h|} \sum_{|t| \geq 1} \sum_{n_4} e(m^3 \Psi_1(h,t,n_4)\alpha) \Big)$$

$$\ll VW\Big(VW^2 + \sum_m \sum_{|h|} \sum_{|t| \geq 1} \Big| \sum_{n_4} e(6m^3 htn_4 \alpha) \Big| \Big)$$

$$\ll VW\Big(VW^2 + \sum_m \sum_{|h|} \sum_{|t| \geq 1} \min(W, \|6m^3 ht\alpha\|^{-1}) \Big)$$

Putting

$$(2.2.9) \qquad \nu = 6m^3ht \quad , \quad |\nu| \ll V^3W^2 \ll N^2V \ ,$$

and letting $\rho(\nu)$ denote the number of representations of ν in the form (2.2.9), so that $\rho(\nu)$ is bounded by $d_4(\nu)$, we have from (2.2.8) that

$$(2.2.10) \quad |T_2|^2 \ll VW\Big(VW^2 + \sum_{|\nu| \ll N^2V} d_4(\nu) \min(W, \|\nu\alpha\|^{-1})\Big) .$$

By Hölder's inequality, the sum in (2.2.10) is bounded by

$$(2.2.11) \quad \Big(\sum_{\nu \ll N^2V} d_4(\nu)^{\theta} \cdot W\Big)^{\frac{1}{\theta}} \Big(\sum_{\nu \ll N^2V} \min(W, \|\alpha\nu\|^{-1})\Big)^{1-\frac{1}{\theta}} \quad ,$$

for any $\theta > 1$. Using Lemma 1, and the fact that

$$\sum_{n \le x} d_k(n)^r \ll x (\log x)^{c_{r,k}} \quad ,$$

for some $c_{r,k}$, the expressions in (2.2.11) are bounded by

$$(2.2.12) \quad W^{\frac{1}{\theta}} (AN^2V)^{\frac{1}{\theta}} (\log N)^{c(\theta)} \Big(\frac{N^2VW}{q} + N^2V + q\Big)^{1-\frac{1}{\theta}}$$

$$\ll \big(N^2VW^{1/\theta} + N^3\Delta^{1-\frac{1}{\theta}}\big) (\log N)^{c(\theta)} \ ,$$

and we choose $\theta = \frac{1}{\varepsilon}$. Hence, collecting the estimates from (2.2.12), (2.2.10), and (2.2.4), we have (2.2.2).

(b) By Cauchy's inequality

$$(2.2.13) \quad |T_1|^2 \ll \Big(\sum_{V<m\le V'} |a_m|^2\Big)\Big\{ V \sum_{W<n\le W'} |b_n|^2 +$$

$$+ \sum_{V<m\le V'} \sum_{n_1 \ne n_2} b_{n_1}\overline{b}_{n_2}\, e(m^3(n_1^3 - n_2^3)\alpha)\Big\} .$$

After a change of summation, the inner sum in (2.2.13) is bounded by

$$(2.2.14) \qquad \sum_{n_1 \neq n_2} |b_{n_2}|^2 \left| \sum_m e(\alpha m^3 (n_1^3 - n_2^3)) \right| ,$$

with

$$W < n_1, n_2 \leq W' \quad ; \quad V < m \leq \min(V', Nn_1^{-1}, Nn_2^{-1}) .$$

Put

$$n_1 = n_2 + h \quad ; \quad \psi(h_1, n_2) = h_1(h_1^2 + 3h_1 n_2 + 3n_2^2) ,$$

so that (2.2.14), say T_2, is

$$(2.2.15) \qquad \sum_{|h_1|} \sum_{n_2} |b_{n_2}|^2 \left| \sum_m e(m^3 \psi(h_1, n_2)\alpha) \right| ,$$

with

$$1 \leq |h_1| \leq W \quad ; \quad W < n_2 \leq \min(W', W' - h_1) \quad ;$$

$$V < m \leq \min(V', N(n_2 + h_1)^{-1}, Nn_2^{-1}) .$$

By Cauchy's inequality, we have

$$(2.2.16) \qquad |T_2|^2 \ll W\Big(\sum_{n_2} |b_{n_2}|^4\Big)\Big\{\sum_{|h_1|} \sum_{n_2} \Big| \sum_m e(\psi(h_1,n_2) m^3 \alpha)\Big|^2\Big\}$$

$$= W\Big(\sum_{n_2} |b_{n_2}|^4\Big)\Big\{\sum_{|h_1|} \sum_{n_2} \sum_{m_1 \neq m_2} e(\psi(h_1,n_2)(m_1^3 - m_2^3)\alpha)$$

$$+ O(V^2 W)\Big\} ,$$

the O-term coming from the case $m_1 = m_2$. The multiple sum in (2.2.16) we shall denote by T_3, and we put

$$m_1 = m_2 + t_1 \quad , \quad \psi(t_1, m_2) = t_1(t_1^2 + 3t_1 m_2 + 3m_2^2) ,$$

so that

$$1 \leq |t_1| \leq V \quad ; \quad V < m_2 < \min(V', V' - t_1) ,$$

and we have

$$(2.2.17) \quad T_3 \ll \sum_{|h_1|} \sum_{n_2} \sum_{|t_1|} \left| \sum_{m_2} e(\psi(h_1, n_2)\, \psi(t_1, m_2)\alpha) \right| .$$

By Cauchy's inequality,

$$(2.2.18) \quad |T_3|^2 \ll W^2 V \Big(\sum_{|h_1|} \sum_{n_2} \sum_{|t_1|} \sum_{m_3 \neq m_4} e(\psi(h_1, n_2)(\psi(t_1, m_4) - \psi(t_1, m_3))\alpha) + W^2 V^2 \Big) .$$

Put

$$m_4 = m_3 + t_2 ,$$

$$\psi(t_1, m_4) - \psi(t_1, m_3) = 3 t_1 t_2 (t_1 + t_2 + 2 m_3) = \eta(t_1, t_2, m_3) ,$$

say. Also, note that

$$V < m_3 \leq \min(V', V' - t_2, N(n_2 + h_1)^{-1}, N(n_2 + h_1)^{-1} - t_2, N n_2^{-1}, N n_2^{-1} - t_2) ,$$

$$1 \leq |t_2| \leq \left| \min(V', N(n_2 + h_1)^{-1}, N n_2^{-1}) - V \right| \leq V .$$

The multiple sum in (2.2.18), say T_4, is then

$$(2.2.19) \quad \sum_{|h_1|} \sum_{n_2} \sum_{|t_1|} \sum_{|t_2|} \sum_{m_3} e(\psi(h_1, n_2)\, \eta(t_1, t_2, m_3)\alpha)$$

$$\ll \sum_{|h_1|} \sum_{|t_1|} \sum_{|t_2|} \sum_{m_3} \left| \sum_{n_2} \eta(t_1, t_2, m_3)\, \psi(h_1, n_2)\alpha) \right| .$$

By Cauchy's inequality

$$(2.2.20)\quad |T_4|^2 \ll W V^3 \Big(W^2 V^3 + \sum_{|h_1|} \sum_{|t_1|} \sum_{|t_2|} \sum_{m_3} \sum_{n_3 \neq n_4} e(\eta(t_1, t_2, m_3)\{\Psi(h_1, n_4) - \Psi(h_1, n_3)\}\alpha) \Big).$$

Putting

$$n_4 = h_2 + n_3 \quad ; \quad \eta(h_1, h_2, n_3) = 3h_1h_2(h_1 + h_2 + 2n_3)$$

in the multiple sum in (2.2.20), the said sum is

$$(2.2.21)\quad \ll \sum_{|h_1|} \sum_{|t_1|} \sum_{|t_2|} \sum_{m_3} \sum_{|h_2|} \Big| \sum_{n_3} e(\eta(t_1, t_2, m_3)\eta(h_1, h_2, n_3)\alpha) \Big|$$

$$= \sum_{|h_1|} \sum_{|t_1|} \sum_{|t_2|} \sum_{m_3} \sum_{|h_2|} \Big| \sum_{n_3} e(\eta(t_1, t_2, m_3) . 6h_1h_2n_3\alpha) \Big|$$

$$\ll \sum_{|h_1|} \sum_{|t_1|} \sum_{|t_2|} \sum_{m_3} \sum_{|h_2|} \min(W, \|\eta(t_1, t_2, m_3) . 6h_1h_2\alpha\|^{-1}) .$$

Putting

$$(2.2.22)\quad v = 6h_1h_2\eta(t_1, t_2, m_3) \quad , \quad |v| \ll W^2V^3 \ll N^2V \quad ,$$

and noting that the number of representations of v in the form (2.2.22) does not exceed $d_4(v)$, the sum in (2.2.21) is

$$(2.2.23)\quad \ll \sum_{|v| \ll N^2V} d_4(v) \min(W, \|v\alpha\|^{-1}) ,$$

$$\ll (N^{2+\varepsilon}V + N^3\Delta^{1-\varepsilon}) (\log N)^{c(\varepsilon)} ,$$

by the argument given for the estimate of (2.2.10). Collecting the estimates from (2.2.23), (2.2.20), (2.2.18), (2.2.16) and (2.2.13) gives us the result (2.2.3).

For our purpose, the sequences $\{a_m\}$ and $\{b_n\}$ are going to satisfy the property:

$$\text{(2.2.24)} \qquad \sum_{V<m\leq V'} |a_m|^k \ll V(\log V)^\lambda \ll V(\log N)^\lambda ,$$

$$\sum_{W<n\leq W'} |b_n|^k \ll W(\log W)^\lambda \ll W(\log N)^\lambda ,$$

where λ is dependent on k only, and the value of which need not concern us. Indeed, we shall always absorb λ into $c(\varepsilon)$ in the analysis that follows.

LEMMA 5. Subject to the conditions of Theorem 5 and the conditions given above, in (2.2.24), let M_1, M_2, N_1, N_2 be positive integers so that

$$M_1 < M_2 \quad , \quad N_1 < N_2 \quad , \quad M_1 N_1 \leq N .$$

Define the sum

$$\text{(2.2.25)} \qquad T = \sum_{M_1<m\leq M_2} a_m \sum_{\substack{N_1<n\leq N_2 \\ mn\leq N}} b_n \, e(m^3 n^3 \alpha) .$$

Then, for any $\varepsilon>0$, there is a constant $c(\varepsilon)>0$ such that

(2.2.26) (a) if $b_n = 1$ for all $n \in (N_1, N_2]$, then

$$T \ll \left(M_2^{3/4} N^{3/4+\varepsilon} + M_2^{\frac{1}{2}} N \Delta^{\frac{1}{4}-\varepsilon}\right)(\log N)^{c(\varepsilon)} ;$$

(2.2.27) (b) otherwise,

$$T \ll \left(M_2^{\frac{1}{2}} N^{\frac{1}{2}} + N_2^{\frac{1}{8}} N^{7/8} + M_2^{\frac{1}{16}} N^{\frac{15}{16}+\varepsilon} + N\Delta^{\frac{1}{16}-\varepsilon}\right)(\log N)^{c(\varepsilon)} .$$

Proof. This follows on using (2.2.24), with Lemma 4 in a manner analogous to the proof of Lemma 3.

For the proof of Theorem 5, we may assume that $q < N^3$ since the result follows trivially otherwise. So, we have $\log q < 3 \log N$. Using Vaughan's identity (cf. §1.4), with

$$f(m,n) = \begin{cases} \Lambda(n)\, e(m^3 n^3 \alpha) & , \quad u < n \le N^3 m^{-1} \\ 0 & , \quad \text{otherwise} \end{cases}$$

where u is an integer not exceeding $N^{1/3}$, and is made explicit in (2.2.42), we have

$$\text{(2.2.28)} \qquad \sum_{n \le N} \Lambda(n)\, e(n^3 \alpha) = \int_1^N S_1(\beta) \frac{d\beta}{\beta} - S_2 - S_3 + O(N^{1/3}) ,$$

with

$$\text{(2.2.29)} \qquad S_1(\beta) = \sum_{d \le \min(u, N\beta^{-1})} \mu(d) \sum_{\beta < k \le N d^{-1}} e(d^3 k^3 \alpha) ,$$

$$\text{(2.2.30)} \qquad S_2 = \sum_{d \le u^2} c_d \sum_{r \le N d^{-1}} e(d^3 r^3 \alpha) ,$$

with c_d as in (1.4.6), and finally

$$\text{(2.2.31)} \qquad S_3 = \sum_{u < m \le N u^{-1}} \tau_m \sum_{u < n \le N m^{-1}} \Lambda(n)\, e(m^3 n^3 \alpha) ,$$

where

$$\tau_m = \sum_{d | m,\, d \le u} \mu(d) .$$

Note that the sequences $\{\mu_d\}$, $\{c_d\}$, $\{\tau_m\}$, and $\{\Lambda(n)\}$ satisfy the condition stated in (2.2.24).

By Lemma 5(a), with $M_1 = 1$, $M_2 = \min(u, N\beta^{-1})$, $N_1 = \beta$, $N_2 = N$, it follows that

$$\mathcal{S}_1(\beta) \ll \left(u^{3/4} N^{3/4+\varepsilon} + u^{\frac{1}{2}} N \Delta^{\frac{1}{4}-\varepsilon} \right) (\log N)^{c(\varepsilon)} ,$$

so that

$$(2.2.32) \quad \int_1^N \frac{\mathcal{S}_1(\beta)}{\beta}\, d\beta \ll \left(u^{3/4} N^{3/4+\varepsilon} + u^{\frac{1}{2}} N \Delta^{\frac{1}{4}-\varepsilon} \right) (\log N)^{c(\varepsilon)} .$$

To estimate $\mathcal{S}_2$, we write it as

$$(2.2.33) \quad \Bigg(\sum_{m \le u} \sum_{r \le N m^{-1}} + \sum_{u < m \le N^{1/2}} \sum_{u < r \le N^{1/2}} + \sum_{u < m \le N^{1/2}} \sum_{r \le u} +$$

$$+ \sum_{N^{\frac{1}{2}} < m \le u^2} \sum_{r \le N m^{-1}} + \sum_{u < m \le N^{1/2}} \sum_{N^{1/2} < r \le N m^{-1}} \Bigg) c_m\, e(m^3 r^3 \alpha) ,$$

$$= \sum_{j=1}^{5} \mathcal{S}_2^{(j)} ,$$

say. We then have by Lemma 5(a), with $M_1 = N_1 = 1$, $M_2 = u$, $N_2 = N$,

$$(2.2.34) \quad \mathcal{S}_2^{(1)} \ll \left(u^{3/4} N^{3/4+\varepsilon} + u^{\frac{1}{2}} N \Delta^{\frac{1}{4}-\varepsilon} \right) (\log N)^{c(\varepsilon)} ;$$

by Lemma 5(b), with $M_1 = N_1 = u$, $M_2 = N_2 = N^{\frac{1}{2}}$,

$$(2.2.35) \quad \mathcal{S}_2^{(2)} \ll \left(N^{\frac{31}{32}+\varepsilon} + N \Delta^{\frac{1}{16}-\varepsilon} \right) (\log N)^{c(\varepsilon)} ;$$

trivially

$$(2.2.36) \quad \mathcal{S}_2^{(3)} \ll u N^{\frac{1}{2}} (\log N)^{c(\varepsilon)} ;$$

from Lemma 3(b) with $M_1 = N_2 = N^{\frac{1}{2}}$, $M_2 = u^2$, $N_1 = 1$,

$$(2.2.37)\qquad \mathcal{S}_2^{(4)} \ll \left(N^{\frac{1}{2}}u + u^{\frac{1}{8}}N^{\frac{15}{16}+\varepsilon} + N\Delta^{\frac{1}{16}-\varepsilon}\right)(\log N)^{c(\varepsilon)};$$

and from Lemma 3(b) with $M_1 = u$, $M_2 = N_1 = N^{\frac{1}{2}}$, $N_2 = Nu^{-1}$,

$$(2.2.38)\qquad \mathcal{S}_2^{(5)} \ll \left(N^{\frac{31}{32}+\varepsilon} + Nu^{-1/8} + N\Delta^{\frac{1}{16}-\varepsilon}\right)(\log N)^{c(\varepsilon)}.$$

Collecting the estimates together from (2.2.34), (2.2.35), (2.2.36), (2.2.37) and (2.2.38), and using (2.2.33), we have

$$(2.2.39)\qquad \mathcal{S}_2 \ll \Big(u^{3/4}N^{3/4+\varepsilon} + N^{\frac{31}{32}+\varepsilon} + Nu^{-1/8} + u^{1/2}N\Delta^{\frac{1}{4}-\varepsilon} + N\Delta^{\frac{1}{16}-\varepsilon}\Big)(\log N)^{c(\varepsilon)}.$$

To estimate $\mathcal{S}_3$, we write it as (after a change in the order of summation)

$$(2.2.40)\qquad \Bigg(\sum_{u<m\le N^{1/2}}\ \sum_{\substack{u<n\le Nu^{-1}\\ mn\le N}} \tau_m\Lambda(n) + \sum_{u<n\le N^{1/2}}\ \sum_{\substack{u<m\le Nu^{-1}\\ mn\le N}} \Lambda(n)\tau_m\Bigg)e(m^3n^3\alpha).$$

Applying Lemma 5(b) to both sums in (2.2.40) with $M_1 = N_1 = u$, $M_2 = N^{1/2}$ and $N_2 = Nu^{-1}$, we get

$$(2.2.41)\qquad \mathcal{S}_3 \ll \left(N^{\frac{31}{32}+\varepsilon} + Nu^{-1/8} + N\Delta^{\frac{1}{16}-\varepsilon}\right)(\log N)^{c(\varepsilon)}.$$

Collecting the estimates from (2.2.41), (2.2.39), (2.2.32) and (2.2.28), we have

$$\sum_{n\le N}\Lambda(n)\,e(n^3\alpha) \ll \Big(u^{3/4}N^{3/4+\varepsilon} + Nu^{-1/8} + u^{\frac{1}{2}}N\Delta^{\frac{1}{4}-\varepsilon} + N^{\frac{31}{32}+\varepsilon} + N\Delta^{\frac{1}{16}-\varepsilon}\Big)(\log N)^{c(\varepsilon)}.$$

Choose

$$u = \min\left(N^{2/7} , \Delta^{-2/5} \right) , \tag{2.2.42}$$

to get the result.

SECTION B. THE METHOD OF MOMENTS AND ITS APPLICATIONS

Chapter 3.

On Riemann's Zeta-function--Sign Changes of $S(t)$

§3.1. Introduction

Let

$$S(t) = \pi^{-1} \operatorname{am.} \zeta(\tfrac{1}{2}+it),$$

where the amplitude is obtained by continuous variation along the straight lines joining 2, $2+it$ and $\frac{1}{2}+it$, starting with the value zero. When t is equal to the imaginary part of a zero of $\zeta(s)$, we put

$$S(t) = \lim_{\varepsilon \to 0} \tfrac{1}{2}\left\{S(t+\varepsilon) + S(t-\varepsilon)\right\}.$$

The variation of $S(t)$ is thus closely related to the distribution of the imaginary parts of the non-trivial zeros of $\zeta(s)$. It is also well-known that

$$N(t) = S(t) + \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + O\left(\frac{1}{t}\right)$$

where the term $O(t^{-1})$ is continuous in t.

In a comment on Littlewood's pioneering work on $S(t)$, Selberg is reported as having said that "it is possible to show that the number of changes of sign of $S(t)$, for $t \in (0,T)$, exceeds $T(\log T)^{1-\varepsilon}$"; and indeed, he added the observation[†] that with a bit more effort, one could even replace $(\log T)^{-\varepsilon}$ by an explicit slowly decreasing function.

[†]In some notes given to H. Halberstam, one of the editors of Littlewood's collected papers, to be published by Clarendon, Oxford. The above comment is in connection with Littlewood's paper "On the zeros of Riemann's Zeta-Function," Proc. Camb. Phil. Soc. 22 (1924), 295-318.

The best, unconditional, result available in the literature is due to Selberg [1] himself, and has

$$(\log T)^{\frac{1}{3}} \exp\left(-A\sqrt{\log\log T}\right),$$

in place of $(\log T)^{1-\varepsilon}$ above.

We provide a proof of Selberg's statements.

THEOREM 1. *If* $(T+1)^{\alpha} \leq H \leq T$, *where* $\frac{1}{2} < \alpha \leq 1$, *and is an arbitrarily small real number, there is an* $A = A(\alpha, \delta) > 0$ *and a* $T_0 = T_0(\alpha, \delta) > 0$ *such that when* $T > T_0$, $S(t)$ *changes its sign at least*

$$H(\log T)\ \exp\left(-A(\log\log T)(\log\log\log T)^{-\frac{1}{2}+\delta}\right),$$

times in the interval $(T, T+H)$.

In particular, there exist positive constants $A(\delta)$ *and* $T_0(\delta)$ *such that* $S(t)$ *changes sign at least*

$$T(\log T)\ \exp\left(-A(\log\log T)(\log\log\log T)^{-\frac{1}{2}+\delta}\right),$$

times in the interval $(0, T)$, *for* $T > T_0(\delta)$.

Note. If we assume the Riemann Hypothesis, the theorem is true with $0 < \alpha \leq 1$.

For the proof of this theorem, we shall require asymptotic formulae for integrals of the type

$$\int_T^{T+H} |S(t)|^{2k}\, dt,$$

and

$$\int_T^{T+H} |S_1(t+h) - S_1(t)|^{2k}\, dt,$$

where

$$S_1(t) = \int_0^t S(u)\, du,$$

with the error terms uniform in integers $k \geqslant 1$ and $h > 0$, with a suitable value of h. (The results given in this direction in [1] are not uniform in k.) These formulae are established in Sections 3.2-3.4. We shall prove

THEOREM 2. *If* $T^{\alpha} \leqslant H \leqslant T$, *where* $\frac{1}{2} < \alpha \leqslant 1$, *there is a constant* $A_1 = A_1(\alpha) > 0$ *such that for any integer* k *satisfying*

$$1 \leqslant k \ll (\log\log T)^{1/6},$$

$$\int_T^{T+H} |S(t)|^{2k}\,dt = \frac{(2k)!}{k!}\left(\frac{1}{2\pi}\right)^{2k} H(\log\log T)^k + O\left((A_1 k)^k H(\log\log T)^{k-\frac{1}{2}}\right),$$

where the implied constants depend at most on α, *and*

THEOREM 3. *If* $(T+h)^{\alpha} \leqslant H \leqslant T$, *for* $\frac{1}{2} < \alpha \leqslant 1$, *there is an absolute constant* $A_2 = A_2(\alpha) > 0$ *such that for any integer* k, *with*

$$1 \leqslant k \ll (\log\log T)^{1/6},$$

and any h *satisfying*†

$$(\log T)^{\frac{1}{2}} < h^{-1} < \frac{1}{10k}\log T,$$

$$\int_T^{T+H} |S_1(t+h) - S_1(t)|^{2k}\,dt = \frac{(2k)!}{k!}\left(\frac{h}{2\pi}\right)^{2k} H\left(\log h^{-1}\right)^k +$$

$$+ O\left((A_2 k)^k H h^{2k} (\log\log T)^{k-\frac{1}{2}}\right).$$

---

†The lower bound given here can be relaxed, as is clear from the proof, but is itself more than we will need.

As a consequence of these theorems, we shall prove in Sections 3.5 and 3.6 the following two results.

THEOREM 4. *If* $T^{\alpha} \leq H \leq T$ *and* $\frac{1}{2} < \alpha \leq 1$, *given any* $\delta > 0$, *we have*

$$\int_T^{T+H} |S(t)|\, dt = \frac{2}{\sqrt{\pi}} \frac{H}{2\pi} \sqrt{\log\log T} + O_\delta\left(H \sqrt{\log\log T} \, (\log\log\log T)^{-\frac{1}{2}+\delta} \right),$$

the implied constants depending at most on α *and* δ .

THEOREM 5. *If* $(T+h)^{\alpha} \leq H \leq T$ *and* $\frac{1}{2} < \alpha \leq 1$, *given any* $\delta > 0$ *and any* h *satisfying*

$$(\log T)^{\frac{1}{2}} < h^{-1} < \varepsilon_1 \frac{\log T}{\log\log\log T} ,$$

for some suitably chosen constant $\varepsilon_1 = \varepsilon_1(\alpha) > 0$, *then*

$$\int_T^{T+H} | S_1(t+h) - S_1(t) |\, dt = \frac{2}{\sqrt{\pi}} \frac{Hh}{2\pi} \sqrt{\log h^{-1}} + O\left(Hh \sqrt{\log\log T} \, (\log\log\log T)^{-\frac{1}{2}+\delta} \right),$$

where the implied constant depend at most on α *and* δ .

Notation. All implied constants are absolute and depend at most on α . Moreover, we shall use A to denote the generic constant of this kind (thus, for example, we might write $A^2 = A$ etc.).

The rest of the notation is made clear in the context, with p and q (with or without suffixes) denoting prime numbers.

§3.2. Preliminary Lemmas

We first state a lemma which will be needed later, on several occasions.

LEMMA 1. *Let τ be a real positive number and suppose that $\delta(n)$ are complex numbers, satisfying*

$$|\delta(n)| \leq C$$

for some fixed constant $C > 0$. Then for any integer $k \geq 1$, we have

$$(3.2.1) \quad \sum_{\substack{p_1,\dots,p_k<y \\ q_1,\dots,q_k<y \\ p_1\cdots p_k = q_1\cdots q_k}} \frac{\delta(p_1)\dots\delta(p_k)\,\delta(q_1)\dots\delta(q_k)}{(p_1\cdots p_k q_1\cdots q_k)^{\tau}} = k!\left(\sum_{p<y}\frac{\delta^2(p)}{p^{2\tau}}\right)^k$$

$$+ O\left(C^{2k} k! \left\{\sum_{p<y} p^{-2\tau}\right\}^{k-2}\left\{\sum_{p<y} p^{-4\tau}\right\}\right).$$

Proof. Suppose first that all the p_i's in the sum on the left of (3.2.1) are different. Then we shall have a contribution of

$$(3.2.2) \qquad k! \sum_{p_1,\dots,p_k<y}{}' \frac{\delta^2(p_1)\cdots\cdots\delta^2(p_k)}{(p_1\cdots\cdots p_k)^{2\tau}}$$

where Σ' denotes the sum over those primes $p_1,\dots\dots,p_k$ such that

$$p_i \neq p_j \qquad , \quad i \neq j \; ,$$

since to each choice of distinct $p_1,\dots,p_k$, there are exactly $k!$ ways of choosing $q_1,\dots,q_k$ so that $q_1\cdots q_k = p_1\cdots p_k$. We can write the sum in (3.2.2) as

$$(3.2.3)\quad \Big(\sum_{p<y} \frac{\delta^2(p)}{p^{2\tau}}\Big)^k - \sum_{\substack{p_1,\dots,p_k<y \\ p_i \neq p_j \text{ some } i\neq j}} \frac{\delta^2(p_1)\cdots\delta^2(p_k)}{(p_1\cdots p_k)^{2\tau}} .$$

The second expression in (3.2.3) is

$$\ll c^{2k} \sum_{p_1,\dots,p_{2k-2}<y} (p_1\cdots p_{2k-2})^{-2\tau} \sum_{p_{k-1}<y} p_{k-1}^{-4\tau},$$

$$= c^{2k}\Big(\sum_{p<y} p^{-2\tau}\Big)^{k-2}\Big(\sum_{p<y} p^{-4\tau}\Big).$$

Next, the contribution from the remaining terms in the sum on the left of (3.2.1) is

$$(3.2.4)\quad \ll c^{2k} \sum_{\substack{p_1,\dots,p_k<y \\ q_1,\dots,q_k<y \\ p_1\cdots p_k = q_1\cdots q_k \\ p_i=p_j \text{ some } i\neq j}} (p_1\cdots p_k)^{-2\tau},$$

$$= c^{2k} \sum_{\substack{p_1,\dots,p_{k-2}<y \\ q_1,\dots,q_{k-2}<y}} (p_1\cdots p_{k-2})^{-2\tau} \sum_{\substack{p_{k-1},q_{k-1}<y \\ p_1\cdots p_{k-2}p_{k-1}^2 = q_1\cdots q_{k-2}q_{k-1}^2}} p_{k-1}^{-4\tau},$$

Now, there are at most $k-1$ ways of choosing q_{k-1} such that q_{k-1} and p_{k-1} are the same. Hence, the above estimate for (3.2.4) is

$$(3.2.5)\quad \ll k(k-1)c^{2k} \sum_{\substack{p_1,\dots,p_{k-2}<y \\ q_1,\dots,q_{k-2}<y \\ p_1\cdots p_{k-2}=q_1\cdots q_{k-2}}} (p_1\cdots p_{k-2})^{-2\tau} \sum_{p<y} p^{-4\tau},$$

since there are at most k choices for p_{k-1}. We may now show, by an inductive procedure, that (3.2.5) is

$$\ll k(k-1)(k-2)!\, C^{2k} \Big(\sum_{p<y} p^{-2\tau}\Big)^{k-2} \Big(\sum_{p<y} p^{-4\tau}\Big)$$

$$+ \quad C^{2k} k(k-1)(k-2)! \Big(\sum_{p<y} p^{-2\tau}\Big)^{k-4} \Big(\sum_{p<y} p^{-4\tau}\Big)^{2},$$

and the result follows.

Define the number $\sigma_{x,t}$ for $x \geqslant 2$, $t > 0$ in the following manner:

(3.2.6) $$\sigma_{x,t} = \frac{1}{2} + 2 \max_{\rho} \left(\beta - \frac{1}{2} , \frac{2}{\log x} \right),$$

where ρ runs through all those zeros $\beta + i\gamma$ of $\zeta(s)$ for which

(3.2.7) $$|t - \gamma| \leqslant x^{3(\beta - \frac{1}{2})} (\log x)^{-1} .$$

LEMMA 2. <u>Suppose that</u> $T^{\alpha} \leqslant H \leqslant T$, <u>where</u> $\frac{1}{2} < \alpha \leqslant 1$, <u>and</u> $x \geqslant 2$, $1 \leqslant \xi \leqslant x^{8k}$, $x^3 \xi^2 \leqslant (HT^{-\frac{1}{2}})^{1/4}$. Then, we have for $0 \leqslant \nu \leqslant 8k$

(3.2.8) $$\int_T^{T+H} \left(\sigma_{x,t} - \tfrac{1}{2}\right)^{\nu} \xi^{\sigma_{x,t} - \frac{1}{2}}\, dt \ll A^k \nu!\, H (\log x)^{-\nu}.$$

This is Lemma 12 of [1], but with the error term made uniform in k and ν. The result is implicit in the proof of Lemma 12.

LEMMA 3. *Let* $H > 1$, $k \geq 1$ *and* $1 < y \leq H^{1/k}$. *Suppose that* α_p *are complex numbers satisfying*

(3.2.9) $$|\alpha_p| < B \frac{\log p}{\log y} , \quad \text{for } p < y .$$

Then

(3.2.10) $$\int_0^H \Big| \sum_{p<y} \alpha_p p^{-\frac{1}{2}-it} \Big|^{2k} dt \ll (AB^2 k)^k H ,$$

and if

(3.2.11) $$|\alpha_p'| < B ,$$

then

(3.2.12) $$\int_0^H \Big| \sum_{p<y} \alpha_p' p^{-1-2it} \Big|^{2k} dt \ll (AB^2 k)^k H .$$

Proof. Write

$$\Big(\sum_{p<y} \alpha_p p^{-s} \Big)^k = \sum_{n<y^k} \beta_n n^{-s} .$$

It follows that

$$|\beta_n| < B^k \sum_{\substack{p_1,\dots,p_k<y \\ n=p_1\cdots p_k}} \frac{\log p_1}{\log y} \cdots \frac{\log p_k}{\log y} ,$$

$$< B^k \Big(\sum_{\substack{p|n \\ p<y}} \frac{\log p}{\log y} \Big)^k < B^k \Big(\frac{\log n}{\log y} \Big)^k ,$$

$$< (Bk)^k .$$

Hence, we have

$$(3.2.13)\qquad \int_0^H \Big|\sum_{p<y} \alpha_p\, p^{-\frac{1}{2}-it}\Big|^{2k} dt = \int_0^H \Big|\sum_{n<y^k} \beta_n\, n^{-\frac{1}{2}-it}\Big|^2 dt ,$$

$$(3.2.14)\qquad \ll (Bk)^k \Big\{ H \sum_{n<y^k} |\beta_n| n^{-1} + \sum_{\substack{m,n \\ m<n<y^k}} \frac{|\beta_n|}{\sqrt{mn}} \Big|\log \frac{m}{n}\Big|^{-1} \Big\}.$$

Now

$$\sum_{n<y^k} |\beta_n| n^{-1} \leq \Big(\sum_{p<y} |\alpha_p| p^{-1}\Big)^k < B^k (\log y)^{-k} \Big(\sum_{p<y} \frac{\log p}{p}\Big)^k \ll (AB)^k .$$

Also,

$$(3.2.15)\qquad \sum_{m<n<y^k} \frac{|\beta_n|}{\sqrt{mn}} \Big|\log \frac{m}{n}\Big|^{-1}$$

$$= \sum_{n<y^k} \frac{|\beta_n|}{\sqrt{n}} \Big\{ \sum_{m\leq n/2} \frac{1}{\sqrt{m}} \Big(\log \frac{n}{m}\Big)^{-1} + \sum_{\frac{n}{2}<m<n} \frac{1}{\sqrt{m}} \Big(\log \frac{n}{m}\Big)^{-1} \Big\} ,$$

$$\ll \sum_{n<y^k} \frac{|\beta_n|}{\sqrt{n}} \Big\{ \sqrt{n} + \sqrt{n} \sum_{m<n} \frac{1}{m} \Big\} ,$$

$$\ll k \log y \sum_{n<y^k} |\beta_n| \ll k \log y \Big(\sum_{p<y} |\alpha_p|^k\Big) ,$$

$$(3.2.16)\qquad \ll (AB)^k y^k (\log y)^{1-k} \leq (AB)^k H .$$

Substituting these estimates in (3.2.14), we obtain (3.2.10).

To estimate (3.2.12), write

$$\Big(\sum_{p<y} \alpha_p'\, p^{-1-2it}\Big)^k = \sum_{n<y^k} \beta_n'\, n^{-1-2it} ,$$

where

$$|\beta_n'| < B^k \sum_{\substack{p_1,\ldots,p_k < y \\ n = p_1 \cdots p_k}} 1 = B^k c_n ,$$

say. Hence, the integral in (3.2.12) is

(3.2.17) $$\ll H \sum_{m<y^k} |\beta_m'| m^{-2} + \sum_{m<n<y^k} |\beta_m' \beta_n'| m^{-1} n^{-1} \left|\log \frac{m}{n}\right|^{-1} ,$$

$$\ll B^{2k} H \sum_{m<y^k} c_m^2 m^{-2} + B^{2k} \sum_{m<n<y^k} c_n^2 m^{-1} n^{-1} \left|\log \frac{m}{n}\right|^{-1} .$$

Now

(3.2.18) $$\sum_{m<y^k} c_m^2 m^{-2} = \sum_{\substack{p_1,\ldots,p_k < y \\ q_1,\ldots,q_k < y \\ p_1 \cdots p_k = q_1 \cdots q_k}} (p_1 \cdots p_k)^{-2}$$

By Lemma 1, with $\tau = 1$ and $C = 1$, the right hand side of (3.2.18) is

(3.2.19) $$\ll k! \left(\sum_{p<y} p^{-2} \right)^k \ll A^k k!$$

Next,

$$\sum_{m<n<y^k} c_n^2 (mn)^{-1} \left(\log \frac{n}{m}\right)^{-1} \ll \sum_{n<y^k} c_n^2 n^{-1} \log n ,$$

by the argument leading to (3.2.16). This is now

$$(3.2.20) \quad \ll k(\log y) \sum_{n<y^k} c_n^2 n^{-1} = k \log y \sum_{\substack{p_1,\dots,p_k<y \\ q_1,\dots,q_k<y \\ p_1\cdots p_k = q_1\cdots q_k}} \frac{1}{p_1\cdots p_k} ,$$

$$\ll k \cdot k!(\log y)\Big(\sum_{p<y} \frac{1}{p}\Big)^k \ll (k+1)!\,(\log H)(\log\log H)^k ,$$

by Lemma 1, with $C=1$ and $\tau=\frac{1}{2}$. The result now follows by substituting (3.2.19) and (3.2.20) into (3.2.17).

Let $x>1$ and write

$$\Lambda_x(n) = \begin{cases} \Lambda(n) , & 1 \le n \le x \\ \Lambda(n) \dfrac{\log^2(x^3/n) - 2\log^2(x^2/n)}{2\log^2 x} , & x \le n \le x^2 \\ \Lambda(n) \dfrac{\log^2(x^3/n)}{2\log^2 n} , & x^2 \le n \le x^3 . \end{cases}$$

We quote the following result from [1].

LEMMA 4. <u>Suppose that</u> $t \in [T, T+H]$, <u>where</u> $T^\alpha \le H \le T$ <u>and</u> $\frac{1}{2} < \alpha \le 1$. <u>Put</u>

$$x = T^{\frac{\alpha - \frac{1}{2}}{60k}} .$$

<u>Then</u>

$$(3.2.21) \quad S(t) + \pi^{-1} \sum_{p<x^3} \frac{\sin(t\log p)}{\sqrt{p}} \ll (|E_1| + \dots + |E_4|)(t) ,$$

<u>where</u>

$$(3.2.22) \quad E_1(t) = \sum_{p<x^3} \frac{\Lambda(p) - \Lambda_x(p)}{\sqrt{p}\,\log p}\, p^{-it} ,$$

$$(3.2.23)\qquad E_2(t) = \sum_{p<x^{3/2}} \frac{\Lambda_x(p^2)}{p\log p}\, p^{-2it},$$

$$(3.2.24)\qquad E_3(t) = (\sigma_{x,t} - \tfrac{1}{2}) \log T ,$$

and

$$(3.2.25)\qquad E_4(t) = (\sigma_{x,t} - \tfrac{1}{2})\, x^{\sigma_{x,t} - \frac{1}{2}} \int_{\frac{1}{2}}^{\infty} x^{\frac{1}{2}-\sigma} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log xp}{p^{\sigma+it}} \right| d\sigma .$$

LEMMA 5. Suppose that $T^{\alpha} \le H \le T$ and $\frac{1}{2} < \alpha \le 1$. Then

$$(3.2.26)\qquad \int_T^{T+H} \left| S(t) + \pi^{-1} \sum_{p<T^{(\alpha-\frac{1}{2})/20k}} \frac{\sin(t\log p)}{\sqrt{p}} \right|^{2k} dt \ll (Ak)^{4k} H .$$

Proof. By Lemma 4

$$(3.2.27)\qquad \left| S(t) + \pi^{-1} \textstyle\sum_1(t) \right|^{2k} \ll A^k \left(|E_1(t)|^{2k} + \cdots + |E_4(t)|^{2k} \right),$$

where the notation is plain. Now, by (3.2.10), with $B = O(1)$,

$$(3.2.28)\qquad \int_T^{T+H} |E_1(t)|^{2k}\, dt \ll (Ak)^k H ;$$

and by (3.2.12), with $B = O(1)$,

$$(3.2.29)\qquad \int_T^{T+H} |E_2(t)|^{2k}\, dt \ll (Ak)^k H ;$$

By Lemma 2, with $\xi = 1$ and $\nu = 2k$, we obtain

$$(3.2.30)\qquad \int_T^{T+H} |E_3(t)|^{2k}\,dt \ll (Ak)^{2k} H (\log x)^{-2k} (\log T)^{2k},$$

$$\ll (Ak)^{4k} H.$$

Next

$$(3.2.31)\qquad \int_T^{T+H} |E_4(t)|^{2k}\,dt = \int_T^{T+H} (\sigma_{x,t}-\tfrac{1}{2})^{2k} x^{2k(\sigma_{x,t}-\frac{1}{2})}$$

$$\times \left\{ \int_{\frac{1}{2}}^{\infty} x^{\frac{1}{2}-\sigma} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log xp}{p^{\sigma+it}} \right| d\sigma \right\}^{2k} dt,$$

$$\leqslant \left(\int_T^{T+H} (\sigma_{x,t}-\tfrac{1}{2})^{4k} x^{4k(\sigma_{x,t}-\frac{1}{2})}\,dt \right)^{\frac{1}{2}}$$

$$\times \left(\int_T^{T+H} \left\{ \int_{1/2}^{\infty} x^{\frac{1}{2}-\sigma} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log xp}{p^{\sigma+it}} \right| d\sigma \right\}^{4k} \right)^{\frac{1}{2}},$$

and by Lemma 2, with

$$\nu = 4k, \quad \xi = x^{4k},$$

the first factor in the product on the right of (3.2.31) is

$$(3.2.32)\qquad \ll \left((Ak)^{4k} H (\log x)^{-4k} \right)^{\frac{1}{2}} \ll (Ak)^{2k} H^{\frac{1}{2}} (\log x)^{-2k}.$$

The second factor is, by Hölder's inequality

$$(3.2.33)\qquad \leqslant \left(\int_T^{T+H} \left\{ \int_{\frac{1}{2}}^{\infty} x^{\frac{1}{2}-\sigma}\,d\sigma \right\}^{4k-1} \left\{ \int_{\frac{1}{2}}^{\infty} x^{\frac{1}{2}-\sigma} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log xp}{p^{\sigma+it}} \right|^{4k} d\sigma \right\} dt \right)^{\frac{1}{2}},$$

$$= (\log x)^{2k+\frac{1}{2}} \left(\int_{\frac{1}{2}}^{\infty} x^{\frac{1}{2}-\sigma} \int_T^{T+H} \left| \sum_{p<x^3} \frac{\Lambda_x(p)\log xp}{p^{\sigma+it}} \right|^{4k} dt\,d\sigma \right)^{\frac{1}{2}},$$

and by (3.2.10) with $B=O(1)$, the integral over t is

$$(3.2.34) \qquad \ll (Ak)^{2k} H .$$

Hence, (3.2.33) is

$$\ll (Ak)^{k} \cdot H^{\frac{1}{2}} \cdot (\log x)^{2k} .$$

Thus, from (3.2.32) and (3.2.34)

$$(3.2.35) \qquad \int_{T}^{T+H} |E_4(t)|^{2k} \, dt \ll (Ak)^{3k} H .$$

Hence, combining (3.2.27) with the estimates (3.2.28), (3.2.29), (3.2.30) and (3.2.35) the result follows.

As a consequence of Lemma 5, we have

THEOREM 6. <u>Suppose that</u> $T^{\alpha} \leq H \leq T$ <u>with</u> $\frac{1}{2} < \alpha \leq 1$. <u>Then</u> <u>if</u> $k \geq 1$ <u>is an integer and</u>

$$(3.2.36) \qquad x^{3} = T^{\frac{\alpha - \frac{1}{2}}{20k}} \leq z \leq H^{1/k} ,$$

$$(3.2.37) \qquad \int_{T}^{T+H} \Big| S(t) + \pi^{-1} \sum_{p<z} \frac{\sin(t \log p)}{\sqrt{p}} \Big|^{2k} dt \ll (Ak)^{4k} H .$$

<u>Proof</u>. We may write the expression between modulus signs in the integrand of (3.2.37) as

$$S(t) + \pi^{-1} \sum\nolimits_1(t) + \pi^{-1} \sum_{x^3 < p < z} \frac{\sin(t \log p)}{\sqrt{p}} .$$

Taking the $2k$-th power and then integrating over $[T, T+H]$, the integral on the left of (3.2.37) is

$$(3.2.38) \quad \ll 4^k \left\{ \int_T^{T+H} \left| S(t) + \pi^{-1} \textstyle\sum_1(t) \right|^{2k} dt \; + \int_T^{T+H} \left| \sum_{x^3 < p < z} p^{-\frac{1}{2} - it} \right|^{2k} dt \right\}.$$

By Lemma 5, the first integral in (3.2.38) is

$$\ll (Ak)^{4k} H \;,$$

while by (3.2.10, with $B = O(1)$ the second integral is

$$\ll (Ak)^{4k} H \;.$$

(In the notation of Lemma 3,

$$\alpha_p = 1 = \frac{\log p}{\log z} \frac{\log z}{\log p} \ll \frac{\log p}{\log z} \;,$$

so that (3.2.9) is satisfied with z in place of y.)

§3.3. Proof of Theorem 2

Put

$$(3.3.1) \qquad z = T^{\frac{\alpha}{5k}} \;,$$

$$(3.3.2) \qquad \Delta_z(t) = \Delta(t) = S(t) + \pi^{-1} \sum_{p<z} \frac{\sin(t \log p)}{\sqrt{p}} \;.$$

Then

$$(3.3.3) \quad S(t)^{2k} = \left(\pi^{-1} \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \right)^{2k} + \sum_{\ell=1}^{2k} \binom{2k}{\ell} \Delta^{\ell}(t) \left(-\pi^{-1} \sum_{p<z} \frac{\sin(t \log p)}{\sqrt{p}} \right)^{2k-\ell} .$$

The last sum on the right of (3.3.3) is at most

$$|\Delta(t)| \sum_{\ell=1}^{2k} \binom{2k}{\ell} |\Delta(t)|^{\ell-1} \Big| \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \Big|^{2k-\ell},$$

$$= |\Delta(t)| \sum_{\ell=0}^{2k-1} \binom{2k}{\ell+1} |\Delta(t)|^{\ell} \Big| \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \Big|^{2k-1-\ell},$$

$$\ll k|\Delta(t)| \sum_{\ell=0}^{2k-1} \binom{2k-1}{\ell} |\Delta(t)|^{\ell} \Big| \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \Big|^{2k-1-\ell},$$

$$= k\,|\Delta(t)| \Big(|\Delta(t)| + \Big| \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \Big| \Big)^{2k-1},$$

$$\ll 4^k k\,|\Delta(t)| \Big(|\Delta(t)|^{2k-1} + \Big| \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \Big|^{2k-1} \Big).$$

Thus, we have

$$\int_T^{T+H} |S(t)|^{2k}\,dt - \frac{1}{\pi^{2k}} \int_T^{T+H} \Big(\sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \Big)^{2k} dt \tag{3.3.4}$$

$$\ll A^k \int_T^{T+H} |\Delta(t)|^{2k}\,dt + A^k \int_T^{T+H} |\Delta(t)| \Big| \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \Big|^{2k-1} dt, \tag{3.3.5}$$

$$\ll A^k \int_T^{T+H} |\Delta(t)|^{2k}\,dt$$
$$+ A^k \Big(\int_T^{T+H} |\Delta(t)|^{2k}\,dt \Big)^{1/2k} \Big(\int_T^{T+H} \Big| \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \Big|^{2k} dt \Big)^{1-\frac{1}{2k}}, \tag{3.3.6}$$

by Hölder's inequality. Hence, by Theorem 6,(3.3.5) is

$$\ll (Ak)^{4k} H + (Ak)^2 H^{1/2k} \Big(\int_T^{T+H} \Big| \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \Big|^{2k} dt \Big)^{1-\frac{1}{2k}}, \tag{3.3.7}$$

subject to the conditions of Theorem 6, which are easily verified.

To evaluate

$$(3.3.8) \qquad \int_T^{T+H} \left| \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} \right|^{2k} dt ,$$

we write

$$(3.3.9) \qquad \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} = \frac{i}{2}(\eta - \bar{\eta}) ,$$

where

$$(3.3.10) \qquad \eta = \eta(t) = \sum_{p<z} p^{-\frac{1}{2}-it} ,$$

and the integral (3.3.8) then has the binomial expansion

$$(3.3.11) \qquad \left(\frac{i}{2}\right)^{2k} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} \int_T^{T+H} \eta^j \bar{\eta}^{2k-j} \, dt$$

$$= 4^{-k} \frac{(2k)!}{(k!)^2} \int_T^{T+H} |\eta(t)|^{2k} dt + O\left(4^{-k} \sum_{\substack{j=0 \\ j\neq k}}^{2k} \binom{2k}{j} \left| \int_T^{T+H} \eta^j \bar{\eta}^{2k-j} dt \right| \right) .$$

The integral inside the error term in (3.3.11) is

$$(3.3.12) \qquad \ll \sum_{p_1,\ldots,p_j<z} \; \sum_{q_1,\ldots,q_{2k-j}<z} (p_1\cdots p_j q_1 \cdots q_{2k-j})^{-\frac{1}{2}} \left| \log \frac{p_1\cdots p_j}{q_1\cdots q_{2k-j}} \right|^{-1} .$$

Using the fact that

$$\left|\log\left(\frac{a}{b}\right)\right| \geq \min\left(\frac{1}{a}, \frac{1}{b}\right) ,$$

for any two distinct positive integers a and b , the logarithmic term in (3.3.12) is greater than

$$z^{-2k} ,$$

so that the contribution from (3.3.12) is

$$\ll z^{2k}\Big(\sum_{p<z} p^{-\frac{1}{2}}\Big)^{2k} \ll A^k z^{3k} \ll A^k H,$$

and the error term in (3.3.11) is then easily seen to be

$$\ll A^k H. \tag{3.3.13}$$

Next

$$\int_T^{T+H} |\eta(t)|^{2k}\,dt = H \sum_{\substack{p_1,\dots,p_k<z \\ q_1,\dots,q_k<z \\ p_1\cdots p_k = q_1\cdots q_k}} (p_1\cdots p_k)^{-1} \tag{3.3.14}$$

$$+ O\Big(\sum_{\substack{p_1,\dots,p_k,q_1,\dots,q_k<z \\ p_1\cdots p_k \neq q_1\cdots q_k}} (p_1\cdots p_k q_1\cdots q_k)^{-\frac{1}{2}} \Big|\log \frac{p_1\cdots p_k}{q_1\cdots q_k}\Big|^{-1}\Big).$$

The error term in (3.3.14) is, as in (3.3.12)

$$\ll A^k H. \tag{3.3.15}$$

By Lemma 1, with $C=1$ and $\tau=\frac{1}{2}$, the first term in (3.3.14) is

$$k!\,H\Big(\sum_{p<z}\frac{1}{p}\Big)^k + O\Big(k!H\Big(\sum_{p<z}p^{-1}\Big)^{k-2}\Big).$$

Now

$$\sum_{p<z} p^{-1} = \log\log z + C + O(|z|^{-1}), \tag{3.3.16}$$

so that from (3.3.14), (3.3.15) and (3.3.16)

$$\int_T^{T+H} |\eta(t)|^{2k}\,dt = k!\,H(\log\log T)^k + O\big((Ak)^k H(\log\log T)^{k-1}\big). \tag{3.3.17}$$

Hence, from (3.3.8), (3.3.11), (3.3.13) and (3.3.17), we obtain

$$(3.3.18)\quad \int_T^{T+H} \left| \frac{\sin(t\log p)}{\sqrt{p}} \right|^{2k} dt = \frac{(2k)!}{k!} 4^{-k} H (\log\log T)^k$$

$$+ O\left((Ak)^k H (\log\log T)^{k-1} \right).$$

This, when substituted into (3.3.7) gives us the estimate for the error term as

$$(3.3.19)\quad \ll (Ak)^{4k} H + (Ak)^k H (\log\log T)^{k-\frac{1}{2}} .$$

Hence, from (3.3.4), (3.3.5), (3.3.18) and (3.3.19), our theorem follows.

§3.4. Proof of Theorem 3

With the notation of Theorem 6, put

$$\Delta_z(t) = \Delta(t) = S(t) + \pi^{-1} \sum_{p<z} \frac{\sin(t\log p)}{\sqrt{p}} = S(t) + \pi^{-1} \textstyle\sum_2(t),$$

say. Then

$$S_1(t+h) - S_1(t) = \int_t^{t+h} S(u)\,du = -\pi^{-1} \int_t^{t+h} \textstyle\sum_2(u)\,du + \int_t^{t+h} \Delta(u)\,du ,$$

and

$$(3.4.1)\quad \left| \int_t^{t+h} S(u)\,du \right|^{2k} = \frac{1}{\pi^{2k}} \left| \int_t^{t+h} \textstyle\sum_2(u)\,du \right|^{2k}$$

$$+ O\left(A^k \left| \int_t^{t+h} \Delta(u)\,du \right|^{2k} \right) + O\left(A^k \left| \int_t^{t+h} \Delta(u)\,du \right| \left| \int_t^{t+h} \textstyle\sum_2(u)\,du \right|^{2k-1} \right),$$

exactly as in the analysis of (3.3.4). Now

$$\left|\int_t^{t+h} \Delta(u)\,du\right|^{2k} \leqslant h^{2k-1}\int_t^{t+h} |\Delta(u)|^{2k}\,du \quad ,$$

and therefore, with the help of Hölder's inequality,

$$(3.4.2)\quad \int_T^{T+H}\left|\int_t^{t+h} S(u)\,du\right|^{2k} dt = \frac{1}{\pi^{2k}}\int_T^{T+H}\left|\int_t^{t+h}\Delta(u)\,du\right|^{2k} dt$$

$$+\, O\left(A^k h^{2k-1}\int_T^{T+H}\int_t^{t+h}|\Delta(u)|^{2k}\,du\,dt\right)$$

$$+\, O\left(A^k\left\{h^{2k-1}\int_T^{T+H}\int_t^{t+h}|\Delta(u)|^{2k}\,du\,dt\right\}^{1/2k}\right.$$

$$\left.\times\left\{\int_T^{T+H}\left|\int_t^{t+h}\textstyle\sum_2(u)\,du\right|^{2k} dt\right\}^{1-\frac{1}{2k}}\right).$$

Next, note that

$$(3.4.3)\quad \int_T^{T+H}\int_t^{t+h}|\Delta(u)|^{2k}\,du\,dt = \int_0^h du\int_{T+u}^{T+u+H}|\Delta(t)|^{2k}\,dt$$

so that by Theorem 6 with

$$(T+h)^{\alpha} \leqslant H \leqslant T \quad ; \qquad \tfrac{1}{2}<\alpha\leqslant 1 \quad ,$$

and

$$(T+h)^{(\alpha-\frac{1}{2})/20k} \leqslant z \leqslant H^{1/k}$$

$$(3.4.4)\quad \int_T^{T+H}|\Delta(t)|^{2k}\,dt \ll (Ak)^{4k}H \, .$$

Thus, subject to these restrictions, we have

$$(3.4.5)\quad \int_T^{T+H} \left| \int_t^{t+h} S(u)\,du \right|^{2k} dt = \pi^{-2k} \int_T^{T+H} \left| \int_t^{t+h} \Sigma_2(u)\,du \right|^{2k} dt$$

$$+ O\Bigg((Ak)^{4k} h^{2k} H + (Ak)^2 H^{1/2k} \times h \left(\int_T^{T+H} \left| \int_t^{t+h} \Sigma_2(u)\,du \right|^{2k} dt \right)^{1-\frac{1}{2k}} \Bigg).$$

We now proceed to evaluate the main term in (3.4.5). The integral is

$$(3.4.6)\quad \int_T^{T+H} \left| \sum_{p<z} \frac{\cos((t+h)\log p) - \cos(t\log p)}{\sqrt{p}\,\log p} \right|^{2k} dt .$$

Now

$$(3.4.7)\quad \cos((t+h)\log p) - \cos(t\log p) = \tfrac{1}{2} p^{it}(p^{ih}-1) + \tfrac{1}{2} p^{-it}(p^{-ih}-1),$$

so that

$$(3.4.8)\quad \sum_{p<z} \frac{\cos((t+h)\log p) - \cos(t\log p)}{\sqrt{p}\,\log p} = \tfrac{1}{2}(\eta + \bar{\eta}),$$

where

$$(3.4.9)\quad \eta = \eta(t) = \sum_{p<z} p^{-\frac{1}{2}-it} (\log p)^{-1} (p^{-ih}-1) .$$

The integral in (3.4.6) is therefore equal to

$$(3.4.10)\qquad 2^{-2k}\frac{(2k)!}{(k!)^2}\int_T^{T+H}|\eta(t)|^{2k}\,dt \;+\; O\Big(4^{-k}\sum_{\substack{j=0\\ j\neq k}}^{2k}\binom{2k}{j}\Big|\int_T^{T+H}\eta^{j}\,\bar\eta^{2k-j}\,dt\Big|\Big).$$

Now, if $j\neq k$, the integral within the sum in the error term in (3.4.10) is

$$(3.4.11)\qquad \ll \sum_{\substack{p_1,\dots,p_j<z\\ q_1,\dots,q_{2k-j}<z}} (p_1\cdots p_j\, q_1\cdots q_{2k-j})^{-\frac12}$$

$$\times \prod_{m=1}^{j}\frac{|p_m^{ih}-1|}{\log p_m}\prod_{n=1}^{2k-j}\frac{|q_n^{ih}-1|}{\log q_n}\,\Big|\log\frac{p_1\cdots p_j}{q_1\cdots q_{2k-j}}\Big|^{-1},$$

$$\ll z^{2k}h^{2k}\Big(\sum_{p<z}p^{-\frac12}\Big)^{2k}$$

since

$$|p^{ih}-1| = 2\,|\sin(\tfrac12 h\log p)| \le h\log p,$$

and by the analysis used in dealing with (3.3.12). Put

$$z = T^{\frac{\alpha}{5k}}.$$

Then, from (3.4.11), the error term in (3.4.10) is

$$(3.4.12)\qquad \ll h^{2k}\cdot A^{k}H.$$

Now

$$(3.4.13)\qquad \int_T^{T+H}|\eta(t)|^{2k}\,dt = H\sum_{\substack{p_1,\dots,p_k<z\\ q_1,\dots,q_k<z\\ p_1\cdots p_k=q_1\cdots q_k}}(p_1\cdots p_k\,q_1\cdots q_k)^{-\frac12}$$

$$\times\prod_{j=1}^{k}\frac{(p_j^{ih}-1)(q_j^{-ih}-1)}{(\log p_j)(\log q_j)} + O(h^{2k}A^{k}H),$$

in exactly the same manner as we evaluated (3.3.14) and (3.4.10). By Lemma 1, with $\tau = \frac{1}{2}$,

$$\delta(p_j) = \begin{cases} \left(p_j^{ih} - 1\right)\left(\log p_j\right)^{-1} , & 1 \le j \le k \\ \left(p_j^{-ih} - 1\right)\left(\log p_j\right)^{-1} , & k+1 \le j \le 2k , \end{cases}$$

and $C = h$, the sum in (3.4.13) is

$$(3.4.14) \quad k! \Big(\sum_{p<z} \frac{|p^{ih}-1|^2}{p \log^2 p} \Big)^k H + O\Big(h^{2k} k! \Big(\sum_{p<z} \frac{1}{p} \Big)^{k-2} H \Big) ,$$

$$= k!\, H \Big(4 \sum_{p<z} \frac{1}{p} \Big\{ \frac{\sin(\frac{1}{2} h \log p)}{\log p} \Big\}^2 \Big)^k + O\Big((Ak)^k h^{2k} (\log\log T)^{k-2} H \Big) .$$

Now, assuming that

$$1 < h^{-1} < \frac{1}{10k} \log T < \log z ,$$

we write the sum in (3.4.14) as

$$(3.4.15) \quad \Big(\sum_{p<e^{1/h}} + \sum_{e^{1/h} \le p < z} \Big) \frac{1}{p} \Big\{ \frac{\sin(\frac{1}{2} h \log p)}{\log p} \Big\}^2 .$$

The first sum in (3.4.15) is then

$$(3.4.16) \quad \frac{h^2}{4} \sum_{p<e^{1/h}} p^{-1} + O\Big(h^2 \sum_{p<e^{1/h}} \frac{h^2 \log^2 p}{p} \Big) = \frac{h^2}{4} \log h^{-1} + O(h^2) ,$$

by (3.3.16). The second sum in (3.4.15) is

$$(3.4.17) \quad \ll \sum_{e^{1/h} < n < z} \frac{1}{n \log^3 n} \ll \log^{-2}(e^{1/h}) \ll h^2$$

whence we have

$$(3.4.18) \qquad \sum_{p<z} \frac{1}{p} \left\{ \frac{\sin(\frac{1}{2}h \log p)}{\log p} \right\}^2 = \frac{h^2}{4} \log h^{-1} + O(h^2).$$

Collecting these estimates in (3.4.13) gives us

$$\int_T^{T+H} |\eta(t)|^{2k} dt = k!\, H h^{2k} \left(\log h^{-1} + O(1) \right)^k + O\left((Ak)^k h^{2k} H (\log\log T)^{k-2} \right).$$

Substituting this result and (3.4.12) into (3.4.5) gives us Theorem 3.

§3.5. Proof of Theorem 4

The main idea in the proof is to relate the integral to integrals involving the even powers of $S(t)$. This is done by noting that for any real number F,

$$(3.5.1) \qquad |F| = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin(|F|u)}{u} \right)^2 du .$$

Put

$$(3.5.2) \qquad \mathcal{L} = \mathcal{L}(T) = \frac{1}{4\pi^2} \log\log T ,$$

and

$$(3.5.3) \qquad W(t) = \mathcal{L}^{-\frac{1}{2}} S(t) .$$

Then, we may write

$$(3.5.4) \qquad |W(t)| = \frac{2}{\pi} \int_0^\lambda \left(\frac{\sin(W(t)u)}{u} \right)^2 du + \frac{2}{\pi} \int_\lambda^\infty \left(\frac{\sin(W(t)u)}{u} \right)^2 du ,$$

for any real number $\lambda \geqslant 0$ (but λ will be chosen sufficiently large later on). The second integral in (3.5.4) is, trivially, at most $2/\pi \ \lambda^{-1}$. Therefore

$$(3.5.5) \quad \int_T^{T+H} |W(t)|\, dt = \frac{2}{\pi} \int_0^\lambda \int_T^{T+H} \left\{ \frac{\sin(W(t)u)}{u} \right\}^2 du + O(H\lambda^{-1}) .$$

Next, note that

$$(3.5.6) \qquad \sin^2 x = \frac{1}{2} \sum_{j=1}^{N} \frac{(-1)^{j+1} (2x)^{2j}}{(2j)!} + O\left(\frac{(2x)^{2N+2}}{(2N+2)!} \right) ,$$

so that the main term in (3.5.5) is

$$(3.5.7) \quad \frac{1}{\pi} \int_0^\lambda \frac{1}{u^2} \sum_{j=1}^{N} \frac{(-1)^{j+1}(2u)^{2j}}{(2j)!} \int_T^{T+H} |W(t)|^{2j}\, dt$$

$$+ O\left(\frac{4^N}{(2N+2)!} \left(\int_0^\lambda u^{2N}\, du \right) \left(\int_T^{T+H} |W(t)|^{2N+2}\, dt \right) \right),$$

where N is chosen to be of the form

$$(3.5.8) \qquad \varepsilon \log \mathcal{L} ,$$

with ε a sufficiently small positive absolute constant. Thus, by Theorem 2, the error term in (3.5.7) is

$$(3.5.9) \qquad \ll 4^N \frac{H}{(2N+2)!} \frac{(2N+2)!}{(N+1)!} \frac{1}{2N+1} \lambda^{2N+1} ,$$

$$\ll H \frac{(2\lambda)^{2N}}{N!} \cdot \frac{\lambda}{N^2} .$$

Now, again by Theorem 2, noting the condition on $k=j$, which is satisfied, the main term in (3.5.7) is

$$(3.5.10)\quad \frac{H}{\pi}\int_0^{\lambda}\frac{1}{u^2}\sum_{j=1}^{N}\frac{(-1)^{j+1}}{(j)!}(2u)^{2j}\,du \;+\; O\Big(H\mathcal{L}^{-\frac{1}{2}}\sum_{j=1}^{N}\frac{(Aj)^j}{(2j)!}\int_0^{\lambda}u^{2j-2}du\Big).$$

The error term in (3.5.10) is

$$(3.5.11)\qquad \ll\; H\mathcal{L}^{-\frac{1}{2}}A^N\sum_{j=1}^{N}\frac{\lambda^{2j-1}}{j!}\;,$$

$$\ll\; H\mathcal{L}^{-\frac{1}{2}}A^N e^{\lambda^2}.$$

In the leading term in (3.5.10), the sum is

$$(3.5.12)\qquad 1-e^{-4u^2}\;+\;O\Big(\frac{1}{N!}(2u)^{2N+2}\Big)\;.$$

Hence, the error term in (3.5.12) contributes to the leading term in (3.5.10)

$$(3.5.13)\qquad \ll\; 4^N\frac{H}{N!}\int_0^{\lambda}u^{2N}du \;\ll\; \frac{H}{N!}(2\lambda)^{2N}\cdot\frac{\lambda}{2N+1}\;.$$

The first term in (3.5.12) contributes, when inserted in (3.5.10),

$$(3.5.14)\quad \frac{H}{\pi}\int_0^{\infty}\frac{1-e^{-4u^2}}{u^2}du \;-\;\frac{H}{\pi}\int_{\lambda}^{\infty}\frac{1-e^{-4u^2}}{u^2}du \;=\;\frac{2}{\sqrt{\pi}}H\;+O(H\lambda^{-1}).$$

Collecting all the estimates together, we have

$$(3.5.15)\quad \int_T^{T+H}|W(t)|\,dt \;=\;\frac{2}{\sqrt{\pi}}H\;+\;O\Big(H\lambda^{-1}+H\frac{(2\lambda)^{2N}}{N!}\frac{\lambda}{N}\;+\;H\mathcal{L}^{-\frac{1}{2}}A^N e^{\lambda^2}\Big),$$

from (3.5.9), (3.5.11), (3.5.13) and (3.5.14).

Choose

$$\lambda = N^{\frac{1}{2}-2\delta}$$

where $\delta > 0$ is an arbitrarily small absolute constant. This gives us

$$(3.5.16) \quad \int_T^{T+H} |W(t)|\, dt = \frac{2}{\sqrt{\pi}} H + O_\delta\left(H \left(\log \mathcal{L}\right)^{-\frac{1}{2}+\delta}\right),$$

and hence Theorem 4, by the definition of $W(t)$.

§3.6. Proof of Theorem 5

The proof of this theorem can be carried out in exactly the same way as the proof of Theorem 4. We put

$$W_1(t) = W_1(t,h) = h^{-1} \mathcal{L}_1^{-\frac{1}{2}} \int_t^{t+h} S(u)\, du ,$$

where

$$\mathcal{L}_1 = \frac{1}{4\pi^2}\left(\log h^{-1}\right) .$$

To use Theorem 3 in the same way as we used Theorem 2 in the previous section, we shall assume that

$$\left(\log T\right)^{\frac{1}{2}} < h^{-1} < \frac{1}{10N} \left(\log T\right)$$

with N defined in (3.5.8). So, we may use Theorem 3 for each $k \leq N$. The analysis then carries through as in Section 5, with $W_1(t)$ in place of $W(t)$ and $\mathcal{L}_1$ in place of $\mathcal{L}$ giving us

$$\int_T^{T+H} |W_1(t)|\, dt = \frac{2}{\sqrt{\pi}} H + O_\delta\left(H \left(\log \mathcal{L}\right)^{-\frac{1}{2}+\delta}\right),$$

with the previous value of λ. Theorem 5 now follows from the definition of $W_1(t)$.

§3.7. Proof of Theorem 1

The method of proof is standard and is given here only for completeness. Put

$$h = \Phi(T)(\log T)^{-1},$$

where (cf. (3.5.3))

$$\varepsilon \log \mathcal{L} < \Phi(T) < (\log T)^{\frac{1}{2}}. \tag{3.7.1}$$

Define

$$I(h,t) = I = \int_t^{t+h} S(u)\, du,$$

and

$$J(h,t) = J = \int_t^{t+h} |S(u)|\, du.$$

Let E denote the subset of $(T, T+H)$ such that for each $t \in E$,

$$J(h,t) > |I(h,t)|.$$

Then

$$\int_E J(t)\, dt = \int_E (J(t) - |I(t)|)\, dt + \int_E |I(t)|\, dt$$

$$\geq \int_T^{T+H} J(t)\, dt - \int_T^{T+H} |I(t)|\, dt.$$

Also

$$\int_E J(t)\, dt \leq m(E)^{\frac{1}{2}} \left(\int_T^{T+H} J^2(t)\, dt \right)^{\frac{1}{2}},$$

where $m(E)$ denotes the measure of E. So, we have

$$(3.7.2) \qquad m(E) \geq \frac{\left(\int_T^{T+H} J(t)\, dt - \int_T^{T+H} |I(t)|\, dt \right)^2}{\int_T^{T+H} |J(t)|^2\, dt} .$$

The numerator in the expression given in (3.7.2) is, by Theorem 4 and Theorem 5

$$(3.7.3) \qquad \frac{2}{\sqrt{\pi}} \frac{hH}{2\pi} \left\{ \sqrt{\log\log T} - \sqrt{\log h^{-1}} \right\}$$

$$+ O_\delta\left(Hh\sqrt{\log\log T} \, (\log\log\log T)^{-\frac{1}{2}+\delta} \right),$$

subject to (3.7.1) and the conditions of this theorem (this is because

$$\int_T^{T+H} J(t)\, dt = \int_0^h \int_{T+u}^{T+u+H} |S(t)|\, dt\, du \quad).$$

Now

$$\sqrt{\log\log T} - \sqrt{\log h^{-1}} = \frac{\log \phi(T)}{\left(\sqrt{\log\log T} + \sqrt{\log h^{-1}}\right)} > \frac{1}{2} \frac{\log \phi(T)}{\sqrt{\log\log T}} .$$

So, if T is sufficiently large (i.e. $T \geq T_0(\delta, \alpha)$), and

$$(3.7.4) \qquad \phi(T) > \exp\left(C_1 \log\log T \, (\log\log\log T)^{-\frac{1}{2}+\delta} \right),$$

for some suitable $C_1 = C_1(\delta, \alpha) > 0$, we have the expression (3.7.3) as

$$> C_2(\delta)\, hH \frac{\log \phi(T)}{\sqrt{\log\log T}} ,$$

for some suitable $C_2(\delta) > 0$. The denominator in (3.7.2) is simply

$$(3.7.5) \qquad \leq h \int_T^{T+H} \int_t^{t+h} |S(u)|^2 \, du \, dt \,,$$

$$= h \int_0^h \int_{T+u}^{T+u+H} |S(t)|^2 \, dt \, du \,,$$

$$\ll h^2 H \log\log T \,,$$

by Theorem 2. Thus, we have seen, subject to (3.7.4), and substituting (3.7.5) into (3.7.2) that

$$m(E) > C_3(\delta) \, H \left(\frac{\log \phi(T)}{\log\log T} \right)^2 ,$$

for some suitable $C_3(\delta) > 0$.

Now, divide the interval $(T, T+H)$ into $[Hh^{-1}]$ subintervals $G_1, \ldots, G_{[Hh^{-1}]}$, where each of the intervals is of length h , except possibly for the last. If G_i contains a point of E , then $S(t)$ must change sign either in G_i or G_{i+1} . Since at least $(h^{-1} m(E) - 2)$ of these intervals contain a point of E , then $S(t)$ must change sign at least

$$\frac{1}{2} \left(h^{-1} m(E) - 2 \right) > C_4(\delta) \, H \, \frac{\log T}{\phi(T)} \left(\frac{\log \phi(T)}{\log\log T} \right)^2 ,$$

$$> H \log T \, \exp\left(-C_5(\delta) \log\log T \left(\log\log\log T \right)^{-\frac{1}{2}+\delta} \right),$$

times in the interval $(T, T+H)$, where $C_4(\delta)$ and $C_5(\delta)$ are suitable positive constants.

Chapter 4.

Mean-Values and the Distribution of

§4.1. Introduction

In Chapter 3, we showed how one can derive the asymptotic formula for the first mean of $|S(t)|$. In this chapter, we shall prove the following extensions of Theorem 2 and Theorem 4 of Chapter 3.

THEOREM 1. *Suppose* $T^{\alpha} \leq H \leq T$, *where* $\frac{1}{2} < \alpha \leq 1$. *Let* λ *be a positive real number. Then, for any* $\delta > 0$ *and for* $T \geq T_0 = T_0(\alpha, \delta)$, *some suitably large* $T_0 > 0$, *we have*

$$(4.1.1) \qquad \frac{1}{H}\int_T^{T+H} |S(t)|^{\lambda}\,dt - \nu_{\lambda} \ll \begin{cases} \lambda^{-1} Z_T^{-(\frac{1}{2}-\delta)\lambda}, & 0 < \lambda \leq 1, \\ \mu_{\lambda} Z_T^{-(\frac{1}{2}-\delta)}, & \lambda \geq 1, \end{cases}$$

subject to the condition that

$$(4.1.2) \qquad \lambda = o\left(Z_T^{\frac{1}{2}-\delta} \right),$$

where

$$(4.1.3) \qquad Z_T = \frac{\log\log\log T}{\log\log\log\log T},$$

and

$$(4.1.4) \qquad \nu_{\lambda} = \frac{1}{\pi^{\lambda+\frac{1}{2}}}\,\Gamma\left(\frac{\lambda+1}{2}\right).$$

Corollary. If $T^{\alpha} \leq H \leq T$, $\frac{1}{2} < \alpha \leq 1$, *then for any* $\lambda \geq 0$ *and* $T \geq T_0(\lambda)$, *for some sufficiently large number* $T_0(\lambda) > 0$, *we have*

$$(4.1.5) \qquad \int_T^{T+H} |S(t)|^\lambda \, dt = \frac{\Gamma(\frac{\lambda+1}{2})}{\pi^{\lambda+\frac{1}{2}}} H(\log\log T)^{\lambda/2}(1 + o(1)) .$$

We then apply Theorem 1 with $\lambda \in \mathbb{Z}$ to obtain some information on the limiting distribution of $|S(t)|$. Put

$$(4.1.6) \qquad W_T(t) = W(t) = \frac{|S(t)|}{\sqrt{\log\log T}} ,$$

and denote by $M(\sigma, T)$ the measure of the subset in $(T, T+H)$ such that

$$|W(t)| < \sigma ,$$

where σ is a non-negative number. Put

$$P(\sigma, T) = H^{-1} M(\sigma, T) .$$

Then, $P(\sigma, T)$ is a distribution function with characteristic function, say, $\phi_T(\zeta)$, defined by

$$\phi_T(\zeta) = H^{-1} \int_T^{T+H} e^{i\zeta W(t)} \, dt .$$

Now, $P(\sigma, T)$ tends, weakly, to a limiting distribution, namely

$$P(\sigma, T) \longrightarrow P(\sigma) = \frac{2}{\sqrt{\pi}} \int_0^\sigma e^{-z^2} dz .$$

The corresponding characteristic function is

$$\phi(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{i\zeta t - t^2} dt .$$

Our purpose is to evaluate the discrepancy between $P(\sigma, T)$ and $P(\sigma)$ for T sufficiently large. We do this by means of the Berry-Esseen Theorem (see §5.3). We find a relationship between $\phi_T(\zeta)$ and $\phi(\zeta)$ by means of moments, using Theorem 1 and prove

THEOREM 2. <u>For any</u> $\sigma \geqslant 0$, <u>we have</u>

$$P(\sigma, T) = P(\sigma) + O_\sigma\left((\log Z_T)^{-\frac{1}{2}} \right) ,$$

<u>where</u> Z_T <u>was defined in</u> (4.1.3).

We then have the following corollary, which we label as

THEOREM 3. <u>For</u> $T^\alpha \leqslant H \leqslant T$, $\frac{1}{2} < \alpha \leqslant 1$, <u>the measure of the set contained in</u> $(T, T+H)$ <u>for which</u>

$$|S(t)| < \sigma \sqrt{\log\log T} ,$$

<u>for any</u> $\sigma \geqslant 0$ <u>is</u>

$$H\, P(\sigma) + O\left(A(\sigma)\, H\, (\log Z_T)^{-\frac{1}{2}} \right) ,$$

<u>where</u> $A(\sigma)$ <u>depends at most on</u> σ <u>and</u> α .

In particular, we have the

<u>Corollary</u>. <u>Subject to the conditions on</u> H , <u>as above, the measure of the set contained in</u> $(T, T+H)$ <u>for which</u>

$$|S(t)| < \delta \sqrt{\log\log T}$$

<u>for any</u> $\delta > 0$, <u>is</u> $O(\delta H)$ <u>for</u> $T \geqslant T_0(\delta)$, <u>for some</u> $T_0(\delta) > 0$.

It is clear from the mean-value theorems that the measure of the set contained in $(T, T+H)$ for which

$$|S(t)| > \delta^{-1} \sqrt{\log\log T} ,$$

for any $\delta > 0$, is also $O(\delta H)$ for $T \geqslant T_0(\delta)$. The result of our corollary appears to be new and says that $|S(t)|$ has normal order.

We defer the proof of Theorems 2 and 3 to the next chapter, where corresponding proofs are given for additive functions (the method being identical).

§4.2. Proof of Theorem 1

Define $W(t)$ as in (4.1.6) so that by Theorem 2 of Chapter 3

$$\frac{1}{H}\int_T^{T+H} W(t)^{2k}\,dt = \nu_{2k} + O\left((Ak)^k \mathcal{L}^{-\frac{1}{2}}\right), \tag{4.2.1}$$

for

$$1 \leq k \ll (\log\log T)^{\frac{1}{6}}. \tag{4.2.2}$$

We consider two cases, depending on the size of λ.

(a) $0 < \lambda \leq 1$.

We start with the formula

$$|F|^{\lambda} = \frac{1}{c_\lambda}\int_0^{\infty} \frac{(\sin |F| u)^2}{u^{1+\lambda}}\,du \tag{4.2.3}$$

$$= \frac{1}{c_\lambda}\int_0^{X} \frac{(\sin |F| u)^2}{u^{1+\lambda}}\,du + O\left(\frac{1}{\lambda c_\lambda} X^{-\lambda}\right),$$

for any complex number F, and real number $X>0$ and with

$$c_\lambda = \int_0^{\infty} \frac{\sin^2(u)}{u^{1+\lambda}}\,du. \tag{4.2.4}$$

Note that in the proof of Theorem 4 of Chapter 3, we had used $\lambda = 1$.

(b) Define the non-negative integer m and the real number θ uniquely by

$$\lambda = 2m+1+\theta, \quad 0 < \theta < 2, \quad \lambda > 1. \tag{4.2.5}$$

We use the formula

$$(4.2.6)\qquad |F|^{\lambda} = \frac{|F|^{2m}}{D_\theta}\int_0^\infty \frac{(\sin |F|u)^4}{u^{2+\theta}}\,du\ ,$$

$$= \frac{|F|^{2m}}{D_\theta}\int_0^X \frac{(\sin |F|u)^4}{u^{2+\theta}}\,du + O\left(\frac{|F|^{2m}}{D_\theta}X^{-1-\theta}\right),$$

with

$$(4.2.7)\qquad D_\theta = \int_0^\infty \frac{(\sin u)^4}{u^{2+\theta}}\,du\ .$$

The proof of Theorem 1 for $0<\lambda\le 1$ is similar to the proof of the special case, namely Theorem 4 of Chapter 3, and so, we shall give the details only for Theorem 1, for $\lambda>1$.

Put

$$F = W(t)\ ,$$

and integrating over t, we have from (4.2.6)

$$(4.2.8)\qquad \frac{1}{H}\int_T^{T+H} W^{\lambda}(t)\,dt = \frac{1}{D_\theta}\int_0^X u^{-2-\theta}\left\{\frac{1}{H}\int_T^{T+H} W(t)^{2m}\sin^4(W(t)u)\,dt\right\}du$$

$$+\ O\left(X^{-1-\theta}\cdot\frac{1}{H}\int_T^{T+H} W(t)^{2m}\,dt\right).$$

We assume that

$$(4.2.9)\qquad X \ll (\log\log T)^{\frac{1}{6}}$$

so that by (4.2.1), we have the error term in (4.2.8) bounded by

$$(4.2.10)\qquad \nu_{2m}\,X^{-1-\theta} + (A_m)^m\,\mathcal{L}^{-\frac{1}{2}}\,X^{-1-\theta}\ .$$

Next, by Taylor's Theorem, with the remainder term, we have

$$(4.2.11) \qquad (\sin u)^4 = \frac{1}{8} \sum_{j=2}^{N} b_j u^{2j} + O\left(\frac{(4u)^{2N+2}}{(2N+2)!} \right),$$

where

$$(4.2.12) \qquad b_j = \frac{(-1)^{j+1}}{(2j)!} 4^{j+1} (4^{j-1} - 1) ,$$

and N is an integer, exceeding 2, which will be made explicit later on. The error term in (4.2.11) contributes, in (4.2.8), an amount

$$(4.2.13) \qquad \ll \frac{1}{H} \int_T^{T+H} W(t)^{2m} \frac{(4W(t))^{2N+2}}{(2N+2)!} \int_0^X u^{2N-\theta} \, du \, dt ,$$

$$\ll \frac{4^{2N+2}}{(2N+2)!} \left(\frac{1}{H} \int_T^{T+H} W(t)^{2(m+N+1)} dt \right) \frac{X^{2N+1-\theta}}{2N+1-\theta} ,$$

$$\ll \frac{4^{2N+2}}{(2N+2)!} \frac{X^{2N+1-\theta}}{2N+1-\theta} \left\{ \nu_{2(M+1)} + (AM)^M \mathcal{L}^{-\frac{1}{2}} \right\} ,$$

where we have used (4.2.1), with the notation

$$M = m + N \ll (\log\log T)^{1/6} .$$

The main term in (4.2.11) contributes, in (4.2.8)

$$(4.2.14) \qquad \frac{1}{D_\theta} \cdot \frac{1}{H} \int_T^{T+H} W(t)^{2m} \int_0^X u^{-2-\theta} \cdot \frac{1}{8} \sum_{j=2}^{N} b_j (W(t)\, u)^{2j} \, du \, dt ,$$

$$= \frac{1}{8D_\theta} \int_0^X u^{-2-\theta} \sum_{j=2}^{N} b_j u^{2j} \left\{ \frac{1}{H} \int_T^{T+H} W(t)^{2(m+j)} dt \right\} du ,$$

$$= \frac{1}{8D_\theta} \int_0^X u^{-2-\theta} \sum_{j=2}^{N} b_j u^{2j} \nu_{2(m+j)} \, du$$

$$+ O\left(\frac{\mathcal{L}^{-\frac{1}{2}}}{D_\theta} \int_0^X u^{-2-\theta} \sum_{j=2}^{N} |b_j| u^{2j} (AM)^{m+j} du \right) .$$

The error term here is simply

$$(4.2.15) \qquad \ll \mathcal{L}^{-\frac{1}{2}} \sum_{j=2}^{N} |b_j| \, (AM)^{(m+j)} \, \frac{X^{2j-1-\theta}}{2j-1-\theta} \, ,$$

$$\ll \mathcal{L}^{-\frac{1}{2}} (AM)^{M} \, X^{-1-\theta} \, e^{X^2} \, .$$

To estimate the main term in (4.2.14), we write

$$\nu_{2m+2j} = \frac{2}{\pi^{2m+2j+\frac{1}{2}}} \int_0^\infty \alpha^{2(m+j)} \, e^{-\alpha^2} \, d\alpha \, ,$$

so that we get

$$(4.2.16) \qquad \frac{1}{8D_\theta} \int_0^X u^{-2-\theta} \, \frac{2}{\pi^{2m+\frac{1}{2}}} \int_0^\infty e^{-\alpha^2} \left\{ \sum_{j=2}^{N} b_j \left(\frac{u\alpha}{\pi}\right)^{2j} \right\} \alpha^{2m} \, d\alpha \, du \, ,$$

$$= \frac{2}{\pi^{2m+\frac{1}{2}}} \cdot \frac{1}{D_\theta} \int_0^\infty \alpha^{2m} e^{-\alpha^2} \int_0^X u^{-2-\theta} \left\{ \sin^4\left(\frac{u\alpha}{\pi}\right) + O\left(\frac{1}{(2N+2)!} \left(\frac{4u\alpha}{\pi}\right)^{2N+2}\right) \right\} du \, d\alpha$$

The interchanging of integrands justified by absolute convergence.

The error term in (4.2.16) is

$$(4.2.17) \qquad \ll \frac{1}{D_\theta} \left(\int_0^\infty \alpha^{2N+2} e^{-\alpha^2} \, d\alpha \right) \left(\int_0^X u^{2N-\theta} \, du \right) \frac{16^N}{(2N+2)! \, \pi^{2M+2+1/2}} \, ,$$

$$\ll \frac{\nu_{2M+2}}{D_\theta} \, \frac{X^{2N+1-\theta}}{2N+1-\theta} \, \frac{16^N}{(2N+2)!} \, ,$$

$$\ll \frac{\nu_{2M+2}}{D_\theta} \, \frac{(4X)^{2N+1-\theta}}{N \, (2N+2)!} \, .$$

The contribution from the main term in (4.2.16) is

$$(4.2.18)\quad \frac{2}{\pi^{2m+\frac{1}{2}}}\int_0^\infty \alpha^{2m} e^{-\alpha^2}\left\{\frac{1}{D_\theta}\int_0^\infty \frac{\sin^4(\frac{1}{\pi}u\alpha)}{u^{2+\theta}}\,du + O\left(X^{-1-\theta}\right)\right\} d\alpha ,$$

$$= \frac{2}{\pi^{2m+\frac{1}{2}}}\int_0^\infty \alpha^{2m} e^{-\alpha^2}\left(\frac{\alpha}{\pi}\right)^{1+\theta} d\alpha + O\left(\nu_{2m} X^{-1-\theta}\right),$$

$$= \frac{2}{\pi^{2m+1-\theta}}\,\frac{1}{\sqrt{\pi}}\int_0^\infty \alpha^{2m+1+\theta} e^{-\alpha^2} d\alpha + O\left(\nu_{2m} X^{-1-\theta}\right),$$

$$= \nu_\lambda + O\left(\nu_{2m} X^{-1-\theta}\right).$$

Collecting the error terms from (4.2.18), (4.2.17), (4.2.15), (4.2.13) and (4.2.10) we have shown that

$$(4.2.19)\quad \frac{1}{H}\int_T^{T+H} W(t)^\lambda\,dt - \nu_\lambda \ll \nu_{2m} X^{-1-\theta} + \mathcal{L}^{-\frac{1}{2}}(AM)^M X^{-1-\theta} e^{X^2}$$

$$+ \frac{16^N X^{2N+1-\theta}}{N\,(2N+2)!}\left\{\nu_{2M+2} + (AM)^M \mathcal{L}^{-\frac{1}{2}}\right\}.$$

We use the bound

$$\nu_{2m+2N+2} \le (2m)^{N+2}\,\nu_{2m}\,\nu_{2N+2} \ll (2m)^{N+2}\,\nu_{2m}\,(AN)^N .$$

We make the following choices:

$$X = \frac{1}{16m} N^{\frac{1}{2}-\varepsilon},$$

$$N = \varepsilon\,\frac{\log \mathcal{L}}{\log\log \mathcal{L}},$$

and assume that

$$m = o\left(N^{\frac{1}{2}-\varepsilon}\right),$$

where $\varepsilon > 0$ is a small positive number. The errors induced by these choices in (4.2.19) are

$$\ll \quad \nu_\lambda \left(\frac{\log \mathcal{L}}{\log \log \mathcal{L}} \right)^{-\frac{1}{2} - \delta} ,$$

for some positive number δ . This concludes the proof.

Chapter 5.

On the Distribution of a Class of Additive Functions

§5.1. Introduction

Let $f(n)$ be a real function, for each positive integer n, satisfying the following conditions (of strong additivity):

(i) $f(mn) = f(m) + f(n)$ for all integers $(m,n) = 1$,

(ii) $f(p^a) = f(p)$ for all primes p and all integers $a \geqslant 1$.

For any $k \geqslant 1$, put

$$A_k(x) = \sum_{p<x} \frac{|f(p)|^k}{p},$$

where p runs through the primes.

Now suppose that $f(n)$ belongs to the class of additive arithmetical functions satisfying:

(5.1.1) (a) $A_2(x)$ tends to infinity as x tends to infinity,

(5.1.2) (b) there exists a number M such that

$$|f(p)| \leqslant M$$

for all primes p.

We shall denote this class by $\mathcal{C}$.

It was shown by Halberstam [6] and Delange [3] that the following theorem then holds:

THEOREM. <u>If</u> $f \in \mathcal{C}$, <u>then for any integer</u> $k \geqslant 1$, <u>and for</u> x <u>sufficiently large, we have</u>

$$(5.1.3) \qquad \sum_{n \leqslant x} \left(f(n) - A_1(x)\right)^{2k} = \mu_{2k}\, x\, A_2(x)^k + o\left(x\, A_2(x)^k\right),$$

where

$$\mu_q = \sqrt{\frac{2}{\pi}} \int_0^\infty t^q e^{-\frac{1}{2}t^2} dt .$$

This complemented the famous result due to Erdős and Kac [4] that, if $N_0(x,\omega)$ is the number of integers n not exceeding x such that

(5.1.4) $$f(n) \leq A_1(x) + \omega A_2^{\frac{1}{2}}(x) ,$$

then for any real number ω, as x tends to infinity,

(5.1.5) $$\frac{1}{x} N(x,\omega) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\omega} e^{-\frac{1}{2}t^2} dt .$$

The main purpose here is to show that asymptotic formulae of the type (5.1.3) can be obtained for any real number $\lambda \geq 0$ in place of $2k$. We shall prove

THEOREM 1. *Suppose* $f \in \mathcal{E}$. *Let* λ *be any positive real number. Then, for any* $\varepsilon > 0$ *and for* x *sufficiently large, we have*

(5.1.6) $$\frac{1}{x} \sum_{n \leq x} \left| \frac{f(n) - A_1(x)}{\sqrt{A_2(x)}} \right|^{\lambda} - \mu_\lambda \ll \begin{cases} \lambda^{-1} Z_x^{-(\frac{1}{2}-\varepsilon)\lambda} & , \ 0 < \lambda \leq 1 \\ \mu_\lambda Z_x^{-(\frac{1}{2}-\varepsilon)} & , \ \lambda \geq 1 \end{cases}$$

subject to the condition that

$$\lambda = o\left(Z_x^{\frac{1}{2}-\varepsilon} \right) ,$$

where

(5.1.7) $$Z_x = \frac{\log A_2(x)}{\log\log A_2(x)} ,$$

with the implied constants depending at most on M *and* ε .

In particular, we have

Corollary. Suppose $f \in \mathcal{C}$. For any $\lambda \geq 0$ and $x \geq x_0(\lambda)$, for some sufficiently large number $x_0(\lambda)$, we have

$$(5.1.8) \qquad \sum_{n \leq x} |f(n) - A_1(x)|^{\lambda} = \frac{2^{\lambda/2}}{\sqrt{\pi}} \Gamma\left(\frac{\lambda+1}{2}\right) \times A_2(x)^{\lambda/2}\left(1 + o(1)\right).$$

This follows clearly from Theorem 1 on noting that

$$(5.1.9) \qquad \mu_{\lambda} = \frac{2^{\lambda/2}}{\sqrt{\pi}} \Gamma\left(\frac{\lambda+1}{2}\right).$$

Such results were previously not within the scope of the methods of Halberstam and Delange. More specifically, their methods made use of the fact that since there are no modulus signs in the sum in (5.1.3), one can interchange orders of summation in the subsequent analysis. This, of course, would not be possible in (5.1.6).

Our proof uses the method introduced in Chapter 3 and Chapter 4. We shall require the result (5.1.3) with an explicit error term uniform in k. We have

THEOREM 2. If $f \in \mathcal{C}$, for any integer $k \geq 1$, and x sufficiently large, there is a constant $A = A(M)$ such that

$$(5.1.10) \qquad \sum_{n \leq x} |f(n) - A_1(x)|^{2k} - \mu_{2k} \times A_2(x)^{k} \ll (Ak)^{4k} \times A_2(x)^{k - \frac{1}{2}},$$

where the implied constants are absolute.

Delange's method will give this result without difficulty. In fact, what we need is to prove Theorem B of [3] with an explicit error term. Let

$$f_y(n) = \sum_{p|n,\, p<y} f(p) ,$$

$$M(y) = \sum_{p<y} |f(p)| .$$

THEOREM 3. <u>Let</u> $f \in \mathcal{C}$, <u>and</u> k <u>be any positive integer.</u> <u>Suppose that</u> y <u>is a real number satisfying</u>

$$y \leq x \quad ; \quad y \longrightarrow \infty \quad \text{as} \quad x \longrightarrow \infty$$

<u>Then, for</u> x <u>sufficiently large, there is a constant</u> $B = B(M)$ <u>such that</u>

$$(5.1.11) \quad \sum_{n \leq x} |f_y(n) - A_1(y)|^{2k} - \mu_{2k}\, x\, A_2(y)^k$$

$$\ll (Bk)^{2k}\, x\, A_2(y)^{k-\frac{1}{2}} + B^k M(y)^{2k} ,$$

<u>where the implied constants are absolute.</u>

Our next object is to apply Theorem 1 with $\lambda \in \mathbb{Z}$ to the question of the limiting distribution of $f(n)$, to obtain a result of the type (5.1.5). Put

$$g_k(n) = |f(n) - A_1(k)|\, A_2(k)^{-\frac{1}{2}} ,$$

and let $N(\sigma, k)$ denote the number of integers $\ell \leq k$ such that

$$g_k(\ell) < \sigma ,$$

where σ is a non-negative number.

Put

$$P(\sigma, k) = \frac{1}{k} N(\sigma, k) .$$

Then, $P(\sigma, k)$ is a distribution function with characteristic function, say, $\phi_k(\zeta)$, defined by

$$(5.1.12) \qquad \phi_k(\zeta) = \frac{1}{k} \sum_{\ell \leq k} e^{i\zeta g_k(\ell)} .$$

Now, $P(\sigma,k)$ tends, weakly, to a limiting distribution, namely

$$P(\sigma) = \sqrt{\frac{2}{\pi}} \int_0^{\sigma} e^{-\frac{1}{2}x^2} dx ,$$

for $\sigma \geq 0$. The corresponding characteristic function is

$$\phi(\zeta) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{i\zeta t - t^2/2} dt .$$

Our purpose is to evaluate the discrepancy between $P(\sigma,k)$ and $P(\sigma)$ for k sufficiently large. We do this in §5.3 by means of the Berry-Esseen Theorem (see [2] and [5]). We find a relationship between $\phi_k(\zeta)$ and $\phi(\zeta)$ by means of moments (using Theorem 1) and prove

THEOREM 4. _For any_ $f \in \mathcal{C}$ _and_ $\sigma \geq 0$

$$P(\sigma, k) = P(\sigma) + O\left((\log Z_k)^{-\frac{1}{2}} \right) ,$$

where Z_k _is defined as before_.

We then have the following

Corollary. _The number of integers_ $\ell \leq k$ _such that_

$$|f(\ell) - A_1(k)| < \sigma A_2^{\frac{1}{2}}(k) ,$$

for any $\sigma \geq 0$ _is_

$$k P(\sigma) + O\left(k (\log Z_k)^{-\frac{1}{2}} \right) .$$

Results of this type were obtained before by LeVeque [8], Kubilius [7], Turan and Renyi [9] and others, with estimates of the discrepancy substantially better than that above, but for very special classes of functions or for a general class (cf. [7]) but making use of the probabilistic model for additive functions. Since our method depends on the moments of $f(n)$, one would need to improve the error-terms of Theorem 1 to get the corresponding improvement in Theorem 4.

Notation. Throughout, A will denote a generic constant depending at most on M (thus, for example, we will write $A^2 = A$ etc.).

The letter p (with or without suffixes) is reserved for prime numbers.

Other notations are made clear in the context.

§5.2. Proof of Theorem 1

Define

(5.2.1) $$W(n) = \left(f(n) - A_1(x)\right) A_2(x)^{-\frac{1}{2}} .$$

Then, Theorem 2 implies that

(5.2.2) $$\frac{1}{x} \sum_{n \le x} |W(n)|^{2k} = \mu_{2k} + O\left((Ak)^{4k} \mathcal{L}^{-1} \right) ,$$

where we denote $A_2^{\frac{1}{2}}(x)$ by $\mathcal{L}$, for simplicity.

We shall consider two cases, depending on the size of λ.

(a) $0 < \lambda \le 1$

We start from the formula

$$|\alpha|^{\lambda} = \frac{1}{C_\lambda} \int_0^{\infty} \frac{(\sin |\alpha| u)^2}{u^{1+\lambda}} du ,$$

$$= \frac{1}{C_\lambda} \int_0^{T} \frac{(\sin |\alpha| u)^2}{u^{1+\lambda}} du + O\left(\frac{1}{\lambda C_\lambda} T^{-\lambda} \right),$$

for any complex number α , real number $T>0$ and with

$$c_\lambda = \int_0^\infty \frac{(\sin u)^2}{u^{1+\lambda}}\,du .$$

(b) $\lambda > 1$.

Define the non-negative integer m and the real number θ uniquely by

$$\lambda = 2m+1+\theta \quad , \quad 0<\theta<2 .$$

Here, we shall use

$$(5.2.3) \qquad |\alpha|^\lambda = \frac{|\alpha|^{2m}}{D_\theta} \int_0^\infty \frac{(\sin|\alpha|u)^4}{u^{2+\theta}}\,du ,$$

$$= \frac{|\alpha|^{2m}}{D_\theta} \int_0^T \frac{(\sin|\alpha|u)^4}{u^{2+\theta}}\,du + O\left(\frac{|\alpha|^{2m}}{D_\theta} T^{-1-\theta}\right),$$

with

$$D_\theta = \int_0^\infty \frac{(\sin u)^4}{u^{2+\theta}}\,du .$$

We give the details of the analysis only for case (b) as case ( ) is similar. So, putting

$$\alpha = W(n) ,$$

in (5.2.3), and summing over all n not exceeding x , we get

$$(5.2.\ ) \qquad \frac{1}{x}\sum_{n\le x} |W(n)|^\lambda$$

$$= \frac{1}{D_\theta}\int_0^T u^{-2-\theta}\left\{\frac{1}{x}\sum_{n\le x}|W(n)|^{2m}\,(\sin W(n)u)^4\right\} du$$

$$+ O\left(\frac{1}{x}\sum_{n\le x}|W(n)|^{2m}\;T^{-1-\theta}\right).$$

By Theorem B, in the form (5.2.2), the error term in (5.2.4) is bounded by

$$(5.2.5) \qquad \mu_{2m} T^{-1-\theta} + (A_m)^{4m} \mathcal{L}^{-1} T^{-1-\theta} .$$

Now, by Taylor's theorem with the remainder term, we have

$$(5.2.6) \qquad (\sin x)^4 = \frac{1}{8} \sum_{j=2}^{N} b_j x^{2j} + O\left(\frac{(4x)^{2N+2}}{(2N+2)!} \right) ,$$

where

$$b_j = \frac{(-1)^{j+1}}{(2j)!} 4^{j+1} (4^{j-1} - 1) ,$$

and N is an integer (exceeding 2) which will be made explicit later on.

The error term in (5.2.6) contributes, in (5.2.4), an amount

$$(5.2.7) \quad \ll \frac{1}{x} \sum_{n \leq x} |W(n)|^{2m} \frac{(4 W(n))^{2N+2}}{(2N+2)!} \int_0^T u^{2N-\theta} \, du ,$$

$$\ll \frac{4^{2N+2}}{(2N+2)!} \left(\frac{1}{x} \sum_{n \leq x} |W(n)|^{2(m+N+1)} \right) \cdot \frac{T^{2N+1-\theta}}{2N+1-\theta} ,$$

$$\ll \frac{4^{2N+2}}{(2N+2)!} \frac{T^{2N+1-\theta}}{2N+1-\theta} \left\{ \mu_{2(M+1)} + (AM)^{4M} \mathcal{L}^{-1} \right\} ,$$

where we have used (5.2.2), with the notation

$$M = m + N .$$

The main term in (5.2.6) contributes, in (5.2.4)

$$(5.2.8) \qquad \frac{1}{D_\theta} \cdot \frac{1}{x} \sum_{n \le x} W(n)^{2m} \int_0^T u^{-2-\theta} \frac{1}{8} \sum_{j=2}^{N} b_j \left(W(n)u\right)^{2j} \, du,$$

$$= \frac{1}{8D_\theta} \int_0^T u^{-2-\theta} \sum_{j=2}^{N} b_j u^{2j} \left\{ \frac{1}{x} \sum_{n \le x} \left(W(n)\right)^{2(m+j)} \right\} du,$$

$$= \frac{1}{8D_\theta} \int_0^T u^{-2-\theta} \sum_{j=2}^{N} b_j u^{2j} \mu_{2(m+j)} \, du$$

$$+ O\left(\mathcal{L}^{-1} D_\theta^{-1} \int_0^T u^{-2-\theta} \sum_{j=2}^{N} |b_j| u^{2j} (AM)^{4(m+j)} du \right).$$

The error term here is simply

$$(5.2.9) \qquad \ll \mathcal{L}^{-1} \sum_{j=2}^{N} |b_j| (AM)^{4(m+j)} \frac{T^{2j-1-\theta}}{2j-1-\theta},$$

$$\ll \mathcal{L}^{-1} (AM)^{4M} T^{-1-\theta} \sum_{j=2}^{N} \frac{T^{2j}}{(2j)!},$$

$$\ll \mathcal{L}^{-1} (AM)^{4M} T^{-1-\theta} e^{T^2}.$$

To estimate the main term in (5.2.8), we write

$$\mu_{2m+2j} = \sqrt{\frac{2}{\pi}} \int_0^\infty t^{2(m+j)} e^{-\frac{1}{2}t^2} \, dt,$$

so that we get

$$(5.2.10) \quad \frac{1}{8D_\theta}\int_0^T u^{-2-\theta} \cdot \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{1}{2}t^2}\left\{\sum_{j=2}^{N} b_j (ut)^{2j}\right\} t^{2m}\, dt\, du ,$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{1}{D_\theta}\int_0^\infty t^{2m} e^{-\frac{1}{2}t^2}\int_0^T u^{-2-\theta}\left\{\sin^4(ut) + O\left(\frac{(4ut)^{2N+2}}{(2N+2)!}\right)\right\} du dt ,$$

the interchanging of integrands justified by absolute convergence. The error term in (5.2.10) is

$$(5.2.11) \quad \ll \frac{1}{D_\theta}\left(\int_0^\infty t^{2M+2} e^{-\frac{1}{2}t^2} dt\right)\left(\int_0^T u^{2N-\theta} du\right) \frac{16^N}{(2N+2)!} ,$$

$$\ll \frac{1}{D_\theta}\ \mu_{2M+2}\ \frac{T^{2N+1-\theta}}{2N+1-\theta} \cdot \frac{16^N}{(2N+2)!} ,$$

$$\ll \frac{1}{D_\theta}\ \mu_{2M+2}\ \frac{(4T)^{2N+1-\theta}}{N\,(2N+2)!} .$$

The contribution from the main term in (5.2.10) is simply

$$(5.2.12) \quad \sqrt{\frac{2}{\pi}}\int_0^\infty t^{2m} e^{-\frac{1}{2}t^2}\left\{\frac{1}{D_\theta}\int_0^\infty \frac{\sin^4(ut)}{u^{2+\theta}} du + O(T^{-1-\theta})\right\} dt ,$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty t^{2m+1-\theta} e^{-\frac{1}{2}t^2} dt + O(\mu_{2m}\, T^{-1-\theta}) ,$$

$$(5.2.13) \quad = \frac{2^{\lambda/2}}{\sqrt{\pi}}\,\Gamma\left(\frac{\lambda+1}{2}\right) + O(\mu_{2m}\, T^{-1-\theta}) .$$

Collecting all the error terms from (5.2.13), (5.2.11), (5.2.9), (5.2.7) and (5.2.5), we have proved that

$$\frac{1}{x}\sum_{n\le x}|W(n)|^{\lambda} - \frac{2^{\lambda/2}}{\sqrt{\pi}}\Gamma\left(\frac{\lambda+1}{2}\right) \ll \mu_{2m}T^{-1-\theta} \tag{5.2.14}$$

$$+ \frac{16^N\, T^{2N+1-\theta}}{N\,(2N+2)!}\left\{\mu_{2M+2} + (AM)^{4M}\mathcal{L}^{-1}\right\}$$

$$+ \mathcal{L}^{-1}(AM)^{4M}\,T^{-1-\theta}\,e^{T^2}.$$

We shall use the bound

$$\mu_{2m+2N+2} \le (2m)^{N+2}\mu_{2m}\,\mu_{2N+2}\,, \tag{5.2.15}$$

$$\ll (2m)^{N+2}\mu_{2m}\,(AN)^N,$$

which is easily verified. We now make the following choices:

$$T = \frac{1}{16m}\,N^{\frac{1}{2}-\varepsilon}, \tag{5.2.16}$$

$$N = \varepsilon\,\frac{\log\mathcal{L}}{\log\log\mathcal{L}}, \tag{5.2.17}$$

and assume

$$m = o\left(N^{\frac{1}{2}-\varepsilon}\right),$$

where ε is a small positive number. So, from (5.2.15), (5.2.16) and (5.2.17), the errors in (5.2.14) are bounded by

$$\mu_{\lambda}\left(\frac{\log\mathcal{L}}{\log\log\mathcal{L}}\right)^{-\frac{1}{2}+\delta}$$

for some positive number δ. This concludes the proof of Theorem 1.

§5.3. Proof of Theorem 4

We first state the Berry-Esseen Theorem:

If $F(x)$ and $G(x)$ are two distribution functions, $G'(x)$ exists for all x and $|G'(x)| \leq \alpha$, $f(u)$ and $g(u)$ the characteristic functions of $F(x)$ and $G(x)$ respectively, and the following condition is satisfied:

$$(5.3.1) \qquad \int_{-T}^{T} \frac{|f(u) - g(u)|}{u}\, du < \varepsilon_1 ,$$

then for $-\infty < x < \infty$,

$$(5.3.2) \qquad |F(x) - G(x)| < K\left(\varepsilon_1 + \frac{\alpha}{T}\right),$$

where K is an absolute constant.

We shall apply this result with $G(x) = P(x)$ (implying $\alpha = \sqrt{\frac{2}{\pi}}$), and restrict ourselves to the domain $[0,\infty)$. We put

$$f(\zeta) = \phi_k(\zeta) .$$

Now, by Taylor's theorem, we have

$$e^x = \sum_{j=0}^{N-1} \frac{x^j}{j!} + O\left(\frac{|x|^N}{N!}\right) .$$

Substituting this into (5.1.12), we get

$$(5.3.3) \qquad \phi_k(\zeta) = \sum_{j=0}^{N-1} \frac{(i\zeta)^j}{j!} \left\{\frac{1}{k} \sum_{\ell \leq k} g_k^j(\ell)\right\} + O\left(\frac{|\zeta|^N}{N!} k^{-1} \sum_{\ell \leq k} g_k^n(\ell)\right).$$

We apply Theorem 1 to the above, noting that for the case $j = 0$, the error term does not exist, to get

$$(5.3.4)\qquad \phi_k(\zeta) = \sum_{j=0}^{N-1} \frac{(i\zeta)^j}{j!}\mu_j + O\Big(\sum_{j=1}^{N-1} \frac{|\zeta|^j}{j!}\mu_j Z_k^{-(\frac{1}{2}-\varepsilon)}\Big)$$

$$+ O\Big(\frac{|\zeta|^N}{N!}\mu_N + \frac{|\zeta|^N}{N!}\mu_N Z_k^{-(\frac{1}{2}-\varepsilon)}\Big),$$

where Z_k is as in Theorem 1 (with x replaced by k), and subject to the restriction

$$N = o\big(Z_k^{\frac{1}{2}-\varepsilon}\big).$$

The main term on the right hand side of (5.3.4) is

$$(5.3.5)\qquad \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{1}{2}t^2} \Big(\sum_{j=0}^{N-1} \frac{(i\zeta t)^j}{j!}\Big)\, dt,$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{1}{2}t^2} \Big\{ e^{i\zeta t} + O\Big(\frac{|\zeta t|^N}{N!}\Big)\Big\}\, dt,$$

$$= \phi(\zeta) + O\Big(\frac{|\zeta|^N}{N!}\mu_N\Big).$$

Collecting the error terms from (5.3.4) and (5.3.5), we get

$$\phi_k(\zeta) = \phi(\zeta) + O\Big(\frac{|\zeta|^N}{N!}\mu_N + Z_k^{-(\frac{1}{2}-\varepsilon)} \sum_{1\le j\le N-1} \frac{|\zeta|^j}{j!}\mu_j\Big).$$

We apply this inside the formula (5.3.1) to get the bound

$$(5.3.6)\qquad \int_{-T}^{T} \frac{|\phi_k(\zeta) - \phi(\zeta)|}{|\zeta|}\, d\zeta \ll \int_0^T \frac{\zeta^{N-1}}{N!}\mu_N\, d\zeta$$

$$+ Z_k^{-(\frac{1}{2}-\varepsilon)} \sum_{1\le j\le N-1} \frac{\mu_j}{j!} \int_0^T \zeta^{j-1}\, d\zeta,$$

$$\ll \frac{\mu_N}{N.N!} T^N + Z_k^{-(\frac{1}{2}-\varepsilon)} e^{T^2}.$$

We choose

$$N = Z_k^{\frac{1}{2} - 2\varepsilon} \quad , \qquad T = \delta \sqrt{\log Z_k}$$

and

$$\varepsilon_1 = T^{-1} ,$$

where $\delta > 0$ is a suitably small absolute constant. Then, the left of (5.3.6) is

$$\ll \left(\log Z_k\right)^{-\frac{1}{2}} .$$

and Theorem 4 follows from (5.3.2).

§5.4. Proof of Theorem 3

For each positive integer q , put

$$F_q(x, y) = \sum_{n \leq x} f_y(n)^q \quad ,$$

and

$$\nu_y(n) = \sum_{p|n,\, p<y} 1 .$$

For any positive real number y , and any integer $n \geq 1$

$$f_y(n)^q = \sum_{\substack{\alpha_1 + \cdots + \alpha_r = q \\ r \leq \nu_y(n)}} \sum_{\substack{p_1 < \cdots < p_r < y \\ p_1|n, \ldots, p_r|n}} \frac{q!}{\alpha_1! \cdots \alpha_r!} f(p_1)^{\alpha_1} \cdots f(p_r)^{\alpha_r} ,$$

where the notation is clear. Summing over n , we get (cf. §2.1 of [2])

$$F_q(x,y) = \sum_{\substack{\alpha_1 + \cdots + \alpha_r = q \\ r \leq \pi(y)}} \sum_{p_1, \ldots, p_r < y} \frac{q!}{\alpha_1! \cdots \alpha_r!} f(p_1)^{\alpha_1} \cdots f(p_r)^{\alpha_r} \left[\frac{x}{p_1 \cdots p_r}\right] .$$

Omission of the integer-part brackets introduces an error of

$$\ll \sum_{\substack{\alpha_1+\cdots+\alpha_r=q \\ r\le\pi(y)}} \frac{q!}{\alpha_1!\cdots\alpha_r!} \sum_{p_1<\cdots<p_r<y} |f(p_1)|^{\alpha_1}\cdots|f(p_r)|^{\alpha_r} ,$$

so that

$$(5.4.1) \qquad F_q(x,y) = x \sum_{\underline{\alpha}} \sum_{\underline{p}} \frac{q!}{\alpha_1!\cdots\alpha_r!} \frac{f(p_1)^{\alpha_1}\cdots f(p_r)^{\alpha_r}}{p_1\cdots p_r} +$$

$$+ \theta_q(x,y)\, M(y)^q ,$$

where

$$|\theta_q(x,y)| \le 1 .$$

Define the entire function

$$(5.4.2) \qquad G_y(z) = \prod_{p<y} \left(1 + \frac{e^{z f(p)} - 1}{p} \right) .$$

On expanding $G_y(z)$ in a Taylor expansion

$$(5.4.3) \qquad G_y(z) = \sum_{j=0}^{\infty} a_j(y)\, z^j ,$$

it is easily checked, from (5.4.1), that

$$(5.4.4) \qquad F_q(x,y) = x\, q!\, a_q(y) + \theta_q(x,y)\, M(y)^q .$$

Now,

$$\left(f_y(n) - A_1(y)\right)^q = \sum_{h=0}^{q} (-1)^h \binom{q}{h} A_1(y)^h f_y(n) ,$$

implies that

$$(5.4.5)\quad \sum_{n \leq x} \left(f_y(n) - A_1(y)\right)^q = \sum_{h=0}^{q} (-1)^h \binom{q}{h} A_1(y)^h F_{q-h}(x,y) ,$$

$$= x \sum_{h=0}^{q} (-1)^h \binom{q}{h} (q-h)!\, A_1(y)^h\, a_{q-h}(y) \quad +$$

$$+ \sum_{h=0}^{q} (-1)^h \binom{q}{h} A_1(y)^h\, \theta_q$$

The second sum in (5.4.5) is, in absolute value, at most

$$(5.4.6)\quad \sum_{h=0}^{q} \binom{q}{h} |A_1(y)|^h\, |M(y)|^{q-h} = \left(|A_1(y)| + |M(y)|\right)^q \leq 2^q M(y)^q .$$

On the other hand, the first sum on the right of (5.4.5), namely

$$q! \sum_{h=0}^{q} \frac{(-1)^h}{h!} A_1(y)^h\, a_{q-h}(y) ,$$

is the coefficient of z^q in the Taylor expansion of

$$G_y(z) \exp\left(-A_1(y) z\right) = \sum_{j=0}^{\infty} b_j(z)\, z^j ,$$

so that

$$(5.4.7)\quad \sum_{n \leq x} \left(f_y(n) - A_1(y)\right)^q = x\, q!\, b_q(y) + 2^q \theta_q^*(x,y) M(y)^q ,$$

where

$$|\theta_q^*(x,y)| \leq 1 .$$

Define the entire function

$$(5.4.8)\quad H_y(z) = \prod_{p<y} \left(1 + \frac{e^{zf(p)} - 1}{p}\right) \exp\left(-\frac{e^{zf(p)} - 1}{p}\right) ,$$

$$= \sum_{j=0}^{\infty} c_j(y)\, z^j ,$$

where, obviously,

$$c_0(y) = H_y(0) = 1 .$$

It is easily shown (cf. [3], §2.51) that $H_y(z)$ converges uniformly in any compact set as $y \to \infty$, to $H(z)$, say. Now

$$G_y(z) \exp(-A_1(y) z) = H_y(z) \exp\left\{ \sum_{p<y} \frac{e^{zf(p)} - zf(p) - 1}{p} \right\} ,$$

so that

$$(5.4.9) \quad \sum_{j=0}^{\infty} b_j(y) z^j = \left(\sum_{j=0}^{\infty} c_j(y) z^j \right) \exp\left\{ \sum_{k=2}^{\infty} \frac{A_k(y)}{k!} z^k \right\} .$$

We are interested in $b_q(y)$. For this, it is necessary to find a bound for

$$|c_j(y)| , \qquad 0 \leq j \leq q$$

uniformly in y and q. We have

$$c_j(y) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{H_y(z)}{z^{j+1}} \, dz ,$$

where $\mathcal{C}$ denotes the unit circle with centre at the origin. So, we have

$$|c_j(y)| \leq \frac{1}{2\pi} \int_0^{2\pi} |H_y(e^{i\theta})| \, d\theta .$$

Putting

$$x_1 = \frac{e^{zf(p)} - 1}{p} ,$$

$$|x_1| \leq \frac{e^{M|z|} + 1}{p} = \frac{e^M + 1}{p} \qquad \text{on} \quad |z| = 1 .$$

Hence

$$|H_y(e^{i\theta})| \leqslant \prod_{p<y} \left(1 + \sum_{j=1}^{\infty} \frac{j}{(j+1)!} (e^M+1)^{j+1} p^{-(j+1)} \right),$$

$$\leqslant \prod_{p<y} \left(1 + (e^M+1)^2 p^{-2} \sum_{j=0}^{\infty} \frac{(e^M+1)^j}{(j+1)!} p^{-j} \right),$$

$$\leqslant \prod_{p<y} \left(1 + \left(\frac{e^M+1}{p}\right)^2 \exp\left(\frac{e^M+1}{p} \right) \right),$$

$$\leqslant \prod_{p<y} (1 + A p^{-2}) \quad \leqslant A .$$

Hence, we have

$$|c_j(y)| \leqslant A .$$

Now, from (5.4.9), it is clear that $b_q(y)$ can be written as a polynomial involving $c_0(y), \ldots, c_q(y)$ and $A_2(y), A_3(y), \ldots$, which is linear in terms of the $c_j(y)$'s . Put

(5.4.10) $$b_q(y) = \mathcal{P}_q [c_j(y) ; A_k(y)] ,$$

where $\mathcal{P}_q [c_j(y) ; A_k(y)]$ is the polynomial expression. Replacing z by λz , we get the transformations

$$b_j(y) \longmapsto \lambda^j b_j(y) ;$$

$$c_j(y) \longmapsto \lambda^j c_j(y) ;$$

$$A_j(y) \longmapsto \lambda^j A_j(y) ,$$

so that

$$\lambda^q b_q(y) = \lambda^q \mathcal{P}_q[c_j(y); A_k(y)] ,$$
$$= \mathcal{P}_q[\lambda^j c_j(y); \lambda^k A_k(y)] .$$

From this, we deduce that the terms in the polynomial are of the form

$$e(j, \alpha_2, \alpha_3, \ldots) \; c_j(y) \; A_2^{\alpha_2}(y) \ldots A_r^{\alpha_r}(y) \ldots$$

where

$$j + 2\alpha_2 + 3\alpha_3 + \ldots r\alpha_r = q .$$

The constant depends only on $j, \alpha_2, \ldots$ and not on y . Since

$$|A_k(y)| \leq M^{k-2} A_2(y) ,$$

we have

$$(5.4.11) \quad |c_j(y) A_2(y)^{\alpha_2} \ldots A_r(y)^{\alpha_r}| \leq A \; |A_2(y)|^{\alpha_2 + \cdots + \alpha_r} \; M^{\alpha_3 + \cdots + (r-1)\alpha_r} .$$

Now, we have

$$(5.4.12) \qquad \alpha_2 + \alpha_3 + \cdots + \alpha_r = \tfrac{1}{2}(q - j - \alpha_3 - \cdots - (r-2)\alpha_r) ,$$
$$\leq \tfrac{1}{2}(q-1) ,$$

unless all the $j, \alpha_3, \ldots, \alpha_r$ are zero, in which case we get

$$\alpha_2 + \alpha_3 + \cdots + \alpha_r = \tfrac{1}{2}q .$$

Put

$$q = 2k \quad , \quad k \geq 1 \quad \text{an integer.}$$

Then, from (5.4.12), the right hand side of (5.4.11) is bounded by

$$A \; |A_2(y)|^{k-\frac{1}{2}} \; M^{2k} ,$$

if at least one of $j, \alpha_3, \alpha_4, \ldots$ is non-zero. Consequently, the contribution of these terms of the polynomial (5.4.10) is

$$(5.4.13) \qquad \leqslant A\, |A_2(y)|^{k-\frac{1}{2}} M^{2k} \sum_{\substack{j,\alpha_2,\ldots,\alpha_r \\ j+2\alpha_2+\cdots+r\alpha_r = q}} |\, e(j,\alpha_2,\ldots,\alpha_r)\,| \ .$$

The main term comes from the polynomial when

$$j = \alpha_3 = \cdots = \alpha_r = 0 \ ,$$

to give the contribution

$$\mu_{2k} \ A_2(y)^k$$

It remains to find an upper bound for the sum in (5.4.13). First note that each of the summands is non-negative, and the sum is really the coefficient of z^q in

$$\Big(\sum_{j=0}^{\infty} z^j \Big) \exp\Big(\sum_{k=2}^{\infty} \tfrac{1}{k!} z^k \Big) \ ,$$

$$= \quad \frac{1}{1-z} \exp\big(e^z - z - 1 \big) \ ,$$

where we assume that $|z| < 1$. Hence, the sum in (5.4.13) is

$$(5.4.14) \qquad \frac{1}{2\pi i} \int_{\mathcal{C}_1} \frac{1}{1-z} \exp(e^z - z - 1)\ z^{-q-1}\, dz \ ,$$

where $\mathcal{C}_1$ is the circle of radius $\frac{1}{2}$ with centre at the origin. The expression in (5.4.14) is easily

$$\ll 2^q = 4^k \ .$$

We have shown that

$$b_q(y) = b_{2k}(y) = \mu_{2k}\, A_2(y)^k + O\big(4^k\, |A_2(y)|^{k-\frac{1}{2}} M^{2k} \big) \ .$$

Substituting into (5.4.7) gives us the theorem.

§5.5. Proof of Theorem 2

Put

(5.5.1) $$\delta(x,y) = \{f(n) - f_y(n)\} - \{A_1(x) - A_1(y)\}.$$

Then, we have

(5.5.2) $$f(n) - A_1(x) = f_y(n) - A_1(y) + \delta(x,y),$$

so that

(5.5.3) $$(f(n) - A_1(x))^{2k} = (f_y(n) - A_1(y))^{2k} + \sum_{\ell=1}^{2k} \binom{2k}{\ell} (f_y(n) - A_1(y))^{2k-\ell} \delta(x,y)^{\ell}.$$

Summing over n in the expression in (5.5.3), we have

(5.5.4) $$\sum_{n\le x} (f(n) - A_1(x))^{2k} = \sum_{n\le x} (f_y(n) - A_1(y))^{2k} + \sum_{\ell=1}^{2k} \binom{2k}{\ell} \delta(x,y)^{\ell} \sum_{n\le x} \{f_y(n) - A_1(y)\}^{2k-\ell}.$$

The last sum above is bounded by

$$\sum_{\ell=1}^{2k} \binom{2k}{\ell} \delta(x,y)^{\ell}\, x^{\frac{1}{2}} \Big\{ \sum_{n\le x} \{f_y(n) - A_1(y)\}^{2(2k-\ell)} \Big\}^{\frac{1}{2}}.$$

Next, we have that

(5.5.5) $$f(n) - f_y(n) = \sum_{p|n,\, p\ge y} f(p) \ll M \sum_{p|n,\, p\ge y} 1,$$

$$\ll M \sum_{p|n,\, p\ge y} \frac{\log p}{\log y} \ll M \frac{\log x}{\log y}$$

Also,

$$(5.5.6) \qquad A_1(x) - A_1(y) = \sum_{y \le p < x} \frac{f(p)}{p} \ll M \sum_{y \le p < x} \frac{1}{p} ,$$

$$\ll M \left\{ \log \frac{\log x}{\log y} + O\left(\frac{1}{y} \right) \right\} ,$$

$$\ll M \frac{\log x}{\log y} ,$$

using the well-known fact that

$$(5.5.7) \qquad \sum_{p \le x} \frac{1}{p} = \log\log x + c + O\left(\frac{1}{x} \right) ,$$

for some constant c. So, from (5.5.6), (5.5.5) and (5.5.1), we have

$$(5.5.8) \qquad \delta(x,y) \ll \frac{\log x}{\log y} .$$

Applying Theorem 3 to the sums in (5.5.4), and applying the bound (5.5.8), we obtain the estimate of

$$(5.5.9) \ll A^k x^{\frac{1}{2}} \sum_{\ell=1}^{2k} \binom{2k}{\ell} \left(\frac{\log x}{\log y} \right)^{\ell} \left\{ \mu_{2(2k-\ell)} \, x \, A_2(y)^{2k-\ell} + \right.$$

$$\left. + \, x \, (Ak)^{4k} |A_2(y)|^{2k-\ell-\frac{1}{2}} + A^{2k} M(y)^{2(2k-\ell)} \right\}^{\frac{1}{2}} ,$$

$$\ll (Ak)^{2k} x^{\frac{1}{2}} \sum_{\ell=1}^{2k} \binom{2k}{\ell} \left(\frac{\log x}{\log y} \right)^{\ell} \left\{ x^{\frac{1}{2}} A_2(y)^{k-\ell/2} \right.$$

$$\left. + \, x^{\frac{1}{2}} A_2(y)^{k-\frac{1}{2}\ell-\frac{1}{4}} + M(y)^{2k-\ell} \right\} ,$$

$$\ll (Ak)^{2k} x^{\frac{1}{2}} \sum_{\ell=0}^{2k-1} \binom{2k-1}{\ell} \left(\frac{2k}{\ell+1} \right) \left(\frac{\log x}{\log y} \right)^{\ell+1} \left\{ x^{\frac{1}{2}} |A_2^{\frac{1}{2}}(y)|^{2k-\ell-1} + M(y)^{2k-\ell-1} \right\} ,$$

$$\ll (Ak)^{2k} \left(\frac{\log x}{\log y} \right) \left\{ x \left(\frac{\log x}{\log y} + A_2^{\frac{1}{2}}(y) \right)^{2k-1} + x^{\frac{1}{2}} \left(\frac{\log x}{\log y} + M(y) \right)^{2k-1} \right\} .$$

We choose

$$(5.5.10) \qquad y = x^{\frac{1}{4k}} ,$$

and with the condition

$$(5.5.11) \qquad k = o(\log x) ,$$

we see that y has the necessary property. Moreover,

$$M(y) = \sum_{p<y} |f(p)| \leq M\pi(y) \ll Mk \frac{x^{\frac{1}{4k}}}{\log x} .$$

Since

$$\frac{\log x}{\log y} = 4k ,$$

the error from (5.5.9) is clearly

$$(5.5.12) \quad \ll (Ak)^{4k} \left\{ x A_2(y)^{k-\frac{1}{2}} + x^{\frac{1}{2}} \left(\frac{x^{\frac{1}{4k}}}{\log x} \right)^{2k-1} \right\} ,$$

$$\ll (Ak)^{4k} x A_2(y)^{k-\frac{1}{2}} ,$$

as desired in Theorem 2. To evaluate the second sum in (5.5.4), we apply Theorem 3, to get an error term bounded by

$$(5.5.13) \quad (Ak)^{2k} x A_2(x^{\frac{1}{4k}})^{k-\frac{1}{2}} + A^k x^{\frac{1}{2}} \left(\frac{x^{\frac{1}{4k}}}{\log x} \right)^{2k} ,$$

$$\ll (Ak)^{4k} x |A_2(y)|^{k-\frac{1}{2}} .$$

Finally, the main term in Theorem 3 contributes

$$(5.5.14) \qquad \mu_{2k}\, x\, A_2\left(x^{\frac{1}{4k}}\right)^k$$

$$= \mu_{2k}\, x\, A_2(x)^k + O\left(\mu_{2k}\, x \sum_{l=1}^{k} \binom{k}{l} \left|A_2(x) - A_2\left(x^{\frac{1}{4k}}\right)\right|^l A_2(x)^{k-l}\right),$$

and the error term is easily contained in (5.5.13), by noting that

$$\left|A_2(x) - A_2\left(x^{\frac{1}{4k}}\right)\right| \ll M^2 \sum_{x^{\frac{1}{4k}} \le p < x} \frac{1}{p},$$

$$\ll \log\left(\frac{\log x}{\log x^{1/4k}}\right) + x^{-\frac{1}{4k}},$$

$$\ll \log 4k .$$

SECTION C. SIEVES AND AN APPLICATION

Chapter 6. Combinatorial Identities and Sieves

Chapter 7. An Analogue of Goldbach's Problem

Chapter 6.

Combinatorial Identities and Sieves
(with H. Halberstam)

§6.1. Introduction

The theory of sieves has played a very important role in the general study of the distribution of prime numbers and various representation problems involving primes. Sieves fall into two general categories: Local Sieves and Global Sieves.

Local sieves derive from functions that are, essentially, weighted characteristic functions of primes or of numbers with at most a fixed number of prime factors. Such are the Generalised von Mangoldt functions $\Lambda_k(n)$, defined by

$$\Lambda_k(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right)(\log d)^k , \qquad k = 1, 2, \ldots$$

which have the important property

$$\Lambda_k(n) = 0 \qquad \text{if} \qquad \nu(n) > k \geq 1 ,$$

(that is, if n has more than k distinct prime factors).

Perhaps the first, and most celebrated, instance of a local sieve is the famous Selberg formula, used in the elementary proof of the prime number theorem (see [14]):

$$(6.1.1) \qquad \sum_{n \leq x} \Lambda_2(n) = x \log x + O(x) ,$$

or what is the same thing,

$$(6.1.2) \qquad \psi(x) \log x + \sum_{n \leq x} \Lambda(n)\, \psi\left(\frac{x}{n}\right) = 2x \log x + O(x) .$$

The Global sieves rest on the observation that if one sifts out, from a finite set of integers, all multiples of small primes, then, the remaining--unsifted--integers have only large prime factors, and consequently only a few of them. In particular, one has the inclusion-exclusion principle

$$a(1) + \sum_{N^{\frac{1}{2}} < p \leq N} a(p) = \sum_{d \mid P_0(N^{1/2})} \mu(d) \sum_{1 \leq m \leq Nd^{-1}} a(dm) ,$$

for any function $a(\cdot)$, where $\mu(\cdot)$ is the Mobius function, and for $z \gg 2$

$$P_0(z) = \prod_{p<z} p .$$

A particular choice of $a(\cdot)$ (letting $a(\cdot)$ be the characteristic function of the odd natural numbers) yields the Erathosthenes-Legendre formula

$$\pi(N) = \pi(N^{\frac{1}{2}}) - 1 + \sum_{2 < d \mid P_0(N^{1/2})} \mu(d) \left[\frac{N}{2d} \right] ,$$

where $[x]$ denotes the integer-part of x .

Developments of these ideas have led to many varieties of Global sieves, e.g. the Brun sieves, the Selberg sieve, the Rosser-Iwaniec sieve and more recently, Greaves' sieve.

As far as the mechanism of the two categories of sieves is concerned, one can say that

(i) local sieves isolate numbers with few prime factors, and

(ii) global sieves isolate numbers without small prime factors.

There have been instances recently where a combination of both kinds of sieves have led to significant progress: for example, the work of Iwaniec, with Jutila [8] and then with Heath-Brown [9], showing that the interval $(x - x^{\frac{11}{20}}, x)$ contains primes if x is large enough.

My purpose here is to report on work done, with Prof. Halberstam, which unifies in one way the two categories of sieves. This leads to a general and elegant approach to sieve theory, from which all known sieves can be deduced as special cases.

In §6.2, we introduce a new type of inversion principle which will then lead us to the Fundamental Sieve Identity. In §6.3, we discuss Local Sieves and in §6.4, Global Sieves.

§6.2. The Fundamental Sieve Identity

Let $\mathcal{H}(N)$ denote the hyperbolic region in $\mathbb{Z} \times \mathbb{Z}$ defined by

$$(6.2.1) \qquad \mathcal{H}(N) = \{ (m,n) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq mn \leq N \} , \qquad N \geq 1 .$$

Let $f(m,n)$ be a function defined over $\mathcal{H}(N)$, and with it associate the summatory function $F(m,n)$ by the relationship

$$(6.2.2) \qquad F(m,n) = \sum_{d \mid n} f(md, \tfrac{n}{d}) , \qquad (m,n) \in \mathcal{H} .$$

We shall introduce a convention whereby the original functions will be in lower case while the associated functions will be in upper case e.g. $f \longleftrightarrow F$, $\theta \longleftrightarrow \Theta$, etc.

If $f(m,n) = f(n)$ for all m , and if

$$F(n) = \sum_{d \mid n} f(n/d) = \sum_{d \mid n} f(d) ,$$

then, by the Mobius inversion theorem, we have

$$(6.2.3) \qquad f(n) = \sum_{d \mid n} \mu(d) F(n/d) .$$

These may be considered to be "projections" of our definitions above to the one-dimensional case (i.e. $\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$). We have the following

LEMMA 2.1 (Inversion principle). *If* f *and* F *are defined and associated as above, we have*

$$(6.2.4) \qquad f(m,n) = \sum_{d\mid n} \mu(d)\, F(md, \tfrac{n}{d}) .$$

Proof. By (6.2.2), the expression on the right hand side of (6.2.4) is

$$\sum_{d\mid n} \mu(d) \sum_{k\mid n/d} f(mdk, \tfrac{n}{dk}) ,$$

$$= \sum_{kd\mid n} \mu(d)\, f(mdk, \tfrac{n}{dk}) = \sum_{l\mid n} f(ml, \tfrac{n}{l}) \sum_{d\mid l} \mu(d) ,$$

$$= f(m,n) .$$

Let θ and Θ be another pair of functions related in the same way. Then, we have

LEMMA 2.2 (Hyperbolic Inversion Principle)

$$(6.2.5) \qquad \sum_{(m,n)\in\mathcal{H}(N)} \Theta(m,n)\, f(n,m) = \sum_{(l,k)\in\mathcal{H}(N)} \theta(l,k)\, F(k,l)$$

Proof. By (6.2.2), we may write the sum on the left hand side in (6.2.5) as

$$\sum_{1\le mn\le N} f(n,m) \Big(\sum_{d\mid n} \theta(md, \tfrac{n}{d}) \Big) = \sum_{1\le mkd\le N} f(kd,m)\,\theta(md,k) ,$$

$$= \sum_{1\le lk\le N} \theta(l,k) \Big(\sum_{d\mid l} f(kd, \tfrac{l}{d}) \Big) = \sum_{1\le lk\le N} \theta(l,k)\, F(k,l) .$$

Corollary 2.1 (Fundamental Sieve Identity)

$$(6.2.6) \qquad \sum_{1\le m\le N} f(1,m)\,\Theta(m,1) = \sum_{1\le mn\le N} \theta(m,n)\,F(n,m)$$

$$- \sum_{1<n\le N}\ \sum_{1\le m\le Nn^{-1}} \Theta(m,n)\,f(n,m)\,.$$

This simple identity forms the basis from which everything else will follow by suitable choices of the functions $f(m,n)$ and $\theta(m,n)$.

§6.3. The Local Sieves

Choose

$$(6.3.1) \qquad f(n,m) = a(nm)\,\Lambda_k(m) \quad , \quad k\geq 1,$$

where $a(\cdot)$ is an arithmetical function. Then, it is clear that

$$F(n,m) = \sum_{d\mid m} f(nd, m/d) = \sum_{d\mid m} a(nm)\,\Lambda_k(m/d)\,,$$

$$= a(nm)\,(\log m)^k\,.$$

Substituting this into (6.2.6), we arrive at

LEMMA 3.1 (Local Sieves Identity)

$$(6.3.2) \qquad \sum_{1\le m\le N} a(m)\,\Lambda_k(m)\,\Theta(m,1) = \sum_{1\le mn\le N} a(nm)(\log m)^k\,\theta(m,n)$$

$$- \sum_{1<n\le N} \Lambda_k(n) \sum_{1\le m\le Nn^{-1}} a(nm)\,\Theta(m,n)\,.$$

We shall derive from this various well-known formulae and identities in the next sections.

§6.3.1. The Selberg Formula

We take $k=1$ and choose θ in the following way:

$$\theta(m,n) = \mu(n)\log m, \tag{6.3.3}$$

so that

$$\Theta(m,n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\log(md),$$

$$= \sum_{d|n} \mu(n/d)\log d + \log m \sum_{d|n} \mu(d).$$

Thus, if $n \neq 1$, it is clear that

$$\Theta(m,n) = \Lambda(n),$$

and we get

$$\Theta(m,n) = \begin{cases} \log m, & n=1 \\ \Lambda(n), & \text{otherwise}. \end{cases} \tag{6.3.4}$$

Substituting into (6.3.2) gives us

$$\sum_{1\leq m\leq N} a(m)\Lambda(m)\log m = \sum_{1\leq mn\leq N} a(mn)\mu(n)\log^2 m - \sum_{1\leq mn\leq N} a(mn)\Lambda(m)\Lambda(n). \tag{6.3.5}$$

Of course, this formula is no other than

$$\sum_{1\leq m\leq N} a(m)\Lambda_2(m) = \sum_{1\leq mn\leq N} a(mn)\mu(n)\log^2 m, \tag{6.3.6}$$

$$= \sum_{m\leq N} a(m)\left\{\sum_{d|m} \mu(d)\log^2\frac{m}{d}\right\},$$

but it is interesting that it should come from specializing (6.3.2), and, more important, it suggests other variations.

In [14], Selberg made the choice $a(n) = 1$, and after some calculation, derived the formula (6.1.2). It is also noted here that Bombieri [2] shows the existence of formula of the type (6.1.1) with weights a_n which are constrained to satisfy certain conditions (see §6.3.3 in these notes).

§6.3.2. A Variation on the Selberg Formula

The formula (6.1.2) allows one to deduce information about $\psi(x)$ from properties of the average function

$$(6.3.7) \qquad \psi_1(x) = \sum_{n \le x} \psi(n) .$$

We show here that one can get a similar formula involving $\psi_1(x)$.

LEMMA 3.2.

$$(6.3.8) \qquad \psi_1(x) \log x + \sum_{n \le x} n \Lambda(n) \psi_1\left(\frac{x}{n}\right) = x^2 \log x + O(x^2) .$$

Proof. First of all, write

$$(6.3.9) \qquad \psi_1(x) = \sum_{n \le x} \psi(n) = \sum_{n \le x} \sum_{m \le n} \Lambda(m) ,$$

$$= \sum_{m \le x} \Lambda(m) \sum_{m \le n \le x} 1 = \sum_{m \le x} \Lambda(m)([x] - m + 1) ,$$

$$= x \sum_{m \le x} \Lambda(m)\left(1 - \frac{m}{x}\right) + O(x) ,$$

$$= x S(x) + O(x) ,$$

say. Take $k=1$ and let

$$a(n) = \left(1 - \frac{n}{x}\right) \quad , \quad n \leq x ,$$

and

(6.3.10) $$\theta(m,n) = \mu(n)\left(1 - \frac{\log n}{\log x}\right) \quad , \quad 1 \leq mn \leq x ,$$

so that

(6.3.11) $$\Theta(m,n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right)\left(1 - \frac{\log n/d}{\log x}\right) ,$$

$$= \sum_{d \mid n} \mu(d) - \frac{1}{\log x} \sum_{d \mid n} \mu(d) \log d ,$$

$$= \begin{cases} \Lambda(n)/\log x & , \quad n > 1 , \\ 1 & , \quad n = 1 . \end{cases}$$

Thus, we have from (6.3.2) that

(6.3.12) $$\log x \sum_{1 \leq m \leq x} \Lambda(m)\left(1 - \frac{m}{x}\right) + \sum_{1 \leq n \leq x} \Lambda(n) \sum_{1 \leq m \leq x/n} \Lambda(m)\left(1 - \frac{mn}{x}\right)$$

$$= \sum_{1 \leq n \leq x} \mu(n) \log \frac{x}{n} \sum_{1 \leq m \leq x/n} \log m \left(1 - \frac{mn}{x}\right) .$$

Observe that we may write the expression on the left hand side of the equality in the form

$$S(x) \log x + \sum_{n \leq x} \Lambda(n) S(x/n) ,$$

which is, by (6.3.9)

$$(6.3.13)\qquad \frac{\psi_1(x)}{x}\log x \;+\; \frac{1}{x}\sum_{n\le x} n\Lambda(n)\,\psi_1\!\left(\frac{x}{n}\right) \;+\; O(x).$$

So, to prove the lemma, we need only find an asymptotic expression for the right hand side of the equality in (6.3.12). To do this, we use the following, easily verified, formulae:

$$(6.3.14)\qquad \sum_{n\le x}(\log n)\left(1-\frac{n}{x}\right) = \frac{1}{2}x\log x + c_1 x + O(\log x),$$

$$\sum_{n\le x}\frac{1}{n}\log n = \frac{1}{2}(\log x)^2 + c_2\log x + c_3 + O\left(\frac{\log x}{x}\right),$$

$$\sum_{n\le x}\frac{1}{n} = \log x + c_4 + O\left(\frac{1}{x}\right),$$

$$x\sum_{d\le x}\frac{\mu(d)}{d} = O(x),$$

and

$$\sum_{d\le x}\log^2(x/d) = O(x),$$

where $c_1,\dots,c_4$ are absolute constants whose exact values need not concern us. Hence, we have

$$(6.3.15)\qquad \sum_{n\le x}\mu(n)\log\frac{x}{n}\sum_{m\le x/n}\log m\left(1-\frac{mn}{x}\right)$$

$$= \sum_{n\le x}\mu(n)\log\frac{x}{n}\left\{\frac{1}{2}\frac{x}{n}\log\frac{x}{n} + c_1\frac{x}{n} + O\left(\log\frac{x}{n}\right)\right\},$$

$$= \frac{1}{2}x\sum_{n\le x}\frac{\mu(n)}{n}\log^2\frac{x}{n} + c_1 x\sum_{n\le x}\frac{\mu(n)}{n}\log\frac{x}{n} + O(x).$$

To estimate the first sum here, we put (using (6.3.14)),

$$(6.3.16)\quad \tfrac{1}{2}x \sum_{n\le x} \frac{\mu(n)}{n}\log^2\frac{x}{n} = x\sum_{n\le x}\frac{\mu(n)}{n}\Big\{\sum_{m\le x/n}\frac{\log m}{m}$$

$$- c_2\log\frac{x}{n} - c_3 + O\Big(\frac{\log x/n}{x/n}\Big)\Big\},$$

$$= x\sum_{n\le x}\frac{1}{n}\Lambda(n) - c_2 x\sum_{n\le x}\frac{\mu(n)}{n}\log\frac{x}{n} + O(x)\ .$$

The first sum in (6.3.16) is simply, by Mertens' Theorem

$$\log x + O(1)\ ,$$

so that we need only estimate

$$\sum_{n\le x}\frac{\mu(n)}{n}\log\frac{x}{n} = \sum_{n\le x}\frac{\mu(n)}{n}\Big\{\sum_{m\le x/n}\frac{1}{m} - c_4 + O\Big(\frac{n}{x}\Big)\Big\}\ ,$$

$$= \sum_{mn\le x}\frac{\mu(n)}{mn} + O(1)$$

$$= O(1)\ .$$

Collecting these estimates together give us the lemma.

At this stage, it is worthwhile pointing out that formulae of the type (6.1.2) and (6.3.8) can give upper (or lower) bounds provided we have the corresponding lower (or upper) bounds by some other means. In particular, it was shown by Erdős in [5] that

$$(6.3.17)\quad (\log 2 - \varepsilon)x \leq \psi(x) \leq (2\log 2 + \varepsilon)x\ ,$$

for any $\varepsilon > 0$ and $x \geq x_0(\varepsilon)$, for some suitably large $x_0(\varepsilon)$. By formula (6.1.2), it is clear that the lower bound in (6.3.17) gives the better upper bound

$$\psi(x) \leq (2 - \log 2 + \varepsilon)\, x \,,$$

without any further work. (This has not appeared in print before, although the result (6.3.17) has been frequently quoted.)

§6.3.3. The Twin-prime Problem

In [1], Bombieri showed that

$$(6.3.18) \qquad \sum_{n \leq x} \Lambda_2(n)\, \Lambda(n+2) \sim 4 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) . N \log N \,, \quad (N \to \infty),$$

on the assumption of the Halberstam-Richert conjecture:

$$(6.3.19) \qquad \sum_{q < N^{1-\varepsilon}} \max_{(a,q)=1} \max_{M \leq N} \left| \psi(M; q, a) - \frac{M}{\phi(q)} \right| \ll N (\log N)^{-A} \,,$$

for any fixed $\varepsilon > 0$ and every large A. The method of Bombieri is an instance of the local sieve (with $a(m) = \Lambda(m+2)$ in formula (6.3.2)) but an additional idea (using the Global sieves) was required to carry the analysis through. It has not been noticed before that the result (6.3.18) also follows, more simply, from the following

Conjecture. Let $Q = N^{1-\delta(N)}$ where $\delta(N)$ is a positive function such that

$$(6.3.20) \qquad \delta(N) (\log N)^{2/3} \longrightarrow 0 \qquad \text{as} \quad N \to \infty .$$

Then

$$(6.3.21) \qquad \sum_{q \approx Q} \mu(q) \log^2 \frac{N}{q} \left\{ \psi(N; q, 2) - \frac{N}{\phi(q)} \right\} = o(N \log N) .$$

This conjecture contains the stronger form of the Halberstam-Rickert conjecture, where $N^{1-\varepsilon}$ is replaced by $\exp\{-(\log N)^{\alpha}\}$,

$0 < \alpha < \frac{1}{3}$. The novelty of (6.3.21) is that the changes of sign of $\mu(n)$ is left intact and so, it may be that (6.3.21) is easier to show than (6.3.19).

To prove (6.3.18) subject to the conjecture, we write

$$\log^2 \frac{m}{d} = \log^2 \frac{N}{d} - 2\left(\log\frac{N}{m}\right)\left(\log\frac{m}{d}\right) - \log^2\frac{N}{m} ,$$

in (6.3.6), to get

$$(6.3.22) \quad \sum_{m \le N} a(m)\Lambda_2(m) = \sum_{m\le N} a(m) \sum_{d|m} \mu(d)\log^2\frac{N}{d}$$

$$- 2\sum_{m\le N} a(m)\log\frac{N}{m} \sum_{d|m}\mu(d)\log\frac{m}{d} - \sum_{m\le N} a(m)\log^2\frac{N}{m}\sum_{d|m}\mu(d) ,$$

$$= \sum_{m\le N} a(m) \sum_{d|m}\mu(d)\log^2\frac{N}{d} - 2\sum_{m\le N} a(m)\log\frac{N}{m}\Lambda(m)$$

$$- a(1)\log^2 N .$$

We choose

$$(6.3.23) \qquad a(m) = \Lambda(m+2) ,$$

so that the second sum in (6.3.22) can be written as

$$(6.3.24) \qquad \int_1^N \left(\sum_{m\le t}\Lambda(m)\Lambda(m+2)\right)\frac{dt}{t} ,$$

and by Selberg's sieve

$$\sum_{m\le t}\Lambda(m)\Lambda(m+2) \ll t ,$$

so that (6.3.24) contributes an error of $O(N)$ in (6.3.22). Rewrite the main term in (6.3.22) as

$$(6.3.25)\quad \sum_{m\le N} \Lambda(m+2)\left\{ \sum_{\substack{d|m\\ d\le Q}} \mu(d)\log^2\left(\frac{N}{d}\right) + \sum_{\substack{d|m\\ d>Q}} \mu(d)\log^2\left(\frac{N}{d}\right)\right\},$$

$$= \sum_{d\le Q} \mu(d)\log^2\left(\frac{N}{d}\right) \sum_{\substack{m\le N\\ m\equiv 0\ (\mathrm{mod}\ d)}} \Lambda(m+2) \quad +$$

$$+ O\left(\log^2\frac{N}{Q} \sum_{Q<m\le N} \Lambda(m+2) \sum_{\substack{d|m\\ d>Q}} |\mu(d)| \right).$$

The error term in (6.3.25) can be rewritten as

$$\log^2\frac{N}{Q} \sum_{Q<m\le N} \Lambda(m+2) \sum_{\substack{k|m\\ k<m/Q}} \mu^2\left(\frac{m}{k}\right),$$

$$\ll \log^2\frac{N}{Q} \sum_{k<NQ^{-1}} \sum_{\substack{Q<m\le N\\ m\equiv 0 \bmod k}} \Lambda(m+2),$$

$$\ll \log^2\frac{N}{Q}\left\{ \sum_{\substack{k<NQ^{-1}\\ (k,2)>1}} \log k + \sum_{\substack{k<NQ^{-1}\\ 2\nmid k}} \log N\left(\pi(N+2;k,2)-\pi(Q+2;k,2)\right)\right\},$$

so that by the Brun-Titchmarsh inequality, this is

$$(6.3.26)\quad \ll \log^2\frac{N}{Q}\cdot\log N \sum_{k<NQ^{-1}} \frac{N-Q}{\phi(k)\log(N/2k)},$$

$$\ll \log^2\frac{N}{Q}\cdot\log N\cdot N\,\frac{\log(NQ^{-1})}{\log Q},$$

$$= o(N\log N),$$

subject to the conditions on Q. The main term in (6.3.25) is simply

(6.3.27)† $$\sum_{d \le Q}{}' \mu(d) \log^2 \frac{N}{d} \Psi(N+2\,;\,d,2) + O(Q \log Q)\,.$$

Put

$$\Psi(N\,;\,d,2) = \frac{N}{\phi(d)} + E(N\,;\,d,2)\,.$$

Then, we have the main term in (6.3.27) as

$$\sum_{d \le Q}{}' \mu(d) \log^2 \frac{N}{d} \cdot \frac{N}{\phi(d)} + o(N \log N)\,,$$

subject to the conjecture. It is then easily seen that

(6.3.28) $$N \sum_{d \le Q}{}' \frac{\mu(d)}{\phi(d)} \log^2 \frac{N}{d} = N \sum_{d \le Q}{}' \frac{\mu(d)}{\phi(d)} \log^2 \frac{Q}{d} +$$

$$+ 2N \log \frac{N}{Q} \sum_{d \le Q}{}' \frac{\mu(d)}{\phi(d)} \log Q/d + N \log^2 \frac{N}{Q} \sum_{d \le Q}{}' \frac{\mu(d)}{\phi(d)}\,,$$

$$= 4N \log N \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) + o(N \log N)\,,$$

after a bit of calculation. Collecting together the estimates in (6.3.22), (6.3.24), (6.3.26), (6.3.27), and (6.3.28) gives us Bombieri's result, subject to the conjecture.

---

† $\sum'$ means sum over all odd integers.

§6.3.4. A Variation on the Local Sieve Identity

We now choose, in (6.3.2), with $k = 1$,

$$\theta(m,n) = \begin{cases} \mu(n)\log m & , \; n \le V < N , \\ 0 & , \; n > V . \end{cases} \tag{6.3.29}$$

Then

$$\Theta(m,n) = \sum_{\substack{d \mid n \\ \frac{n}{d} \le V}} (\log md)\,\mu\left(\frac{n}{d}\right) = \sum_{\substack{d \mid n \\ d \le V}} \left(\log \frac{mn}{d}\right)\mu(d) , \tag{6.3.30}$$

$$= \log m \sum_{\substack{d \mid n \\ d \le V}} \mu(d) + \sum_{\substack{d \mid n \\ d \le V}} \mu(d)\left(\log \frac{n}{d}\right) ,$$

$$= \beta_n \log m + \gamma_n ,$$

say, where the meaning is clear. So, we have

$$\Theta(m,n) = \begin{cases} \log m & , \; n = 1 , \\ \Lambda(n) & , \; 1 < n \le V , \\ \beta_n \log m + \gamma_n & , \; V < n \le N m^{-1} , \end{cases} \tag{6.3.31}$$

and on substituting into (6.3.2) gives us

$$\sum_{m \le N} f(1,m) \log m + \sum_{1 \le n \le V} \Lambda(n) \sum_{m \le N n^{-1}} f(n,m) \tag{6.3.32}$$

$$+ \sum_{V < n \le N} \sum_{m \le N n^{-1}} (\beta_n \log m + \gamma_n) f(n,m)$$

$$= \sum_{n \le V} \mu(n) \sum_{m \le N n^{-1}} F(n,m) \log m .$$

We now have the following identities:

Put

$$f(n,m) = a(mn)\Lambda(m) \quad ; \quad F(n,m) = a(nm)\log m \ ,$$

to give

$$(6.3.33) \quad \sum_{m\le N} a(m)\left\{ \Lambda(m)\log m + \sum_{\substack{n|m \\ n\le V}} \Lambda(n)\Lambda\left(\frac{m}{n}\right)\right\}$$

$$+ \sum_{m\le N} a(m)\left\{ \sum_{\substack{n|m \\ n>V}} \beta_n \log\frac{m}{n} + \gamma_n \right\}$$

$$= \sum_{n\le V} \mu(n) \sum_{m\le Nn^{-1}} a(mn)\log^2 m \ ,$$

which is a generalisation of Selberg's formula. Of course, we can truncate this even further, namely by putting

$$f(n,m) = \begin{cases} a(mn)\Lambda(n) & , \quad U < m \le Nn^{-1} \\ 0 & , \quad m \le U \ , \end{cases}$$

with $UV=N$. Then, a simple calculation gives us

$$F(n,m) = a(nm)\left(\log m - \sum_{\substack{k|m \\ k\le U}} \Lambda(k)\right),$$

and we arrive at the identity

$$(6.3.34) \quad \sum_{U<m\le N} \left\{ \Lambda(m)\log m + \sum_{\substack{n|m \\ n\le V \\ \frac{m}{n}>U}} \Lambda(n)\Lambda\left(\frac{m}{n}\right)\right\} a(m)$$

$$= \sum_{n\le V} \mu(n) \sum_{U<m\le Nn^{-1}} a(nm)\log^2 m$$

$$- \sum_{n\le V} \mu(n) \sum_{U<m\le Nn^{-1}} a(nm)\,\delta_m \log m \ ,$$

where we have put

$$(6.3.35) \qquad \delta_m = \sum_{\substack{d|m \\ d \le U}} \Lambda(d) .$$

§6.3.5. The Vaughan Identity

The identity of Lemma 3.1 may be considered to be a general form of the famous Vaughan identity, which we derive here. We put

$$\theta(m,n) = \theta(n) \quad ; \quad \Theta(m,n) = \Theta(n) ,$$

and suppose that $1 \le U \le N$. Then we may write (6.3.2) as

$$(6.3.36) \qquad \sum_{1 \le m \le N} \Lambda(m)\, a(m) = \sum_{1 \le m \le U} \Lambda(m)\, a(m)$$

$$- \sum_{1 \le n \le N} \Big(\sum_{\substack{m|n \\ m \le U}} \Lambda(m)\, \theta\Big(\frac{n}{m}\Big)\Big) \sum_{\ell \le Nn^{-1}} a(n\ell)$$

$$+ \int_1^N \Big(\sum_{1 \le n \le Nt^{-1}} \theta(n) \sum_{t < m \le Nn^{-1}} a(mn) \Big) \frac{dt}{t}$$

$$- \sum_{1 < n \le NU^{-1}} \Theta(n) \sum_{U < m \le Nn^{-1}} \Lambda(m)\, a(nm) ,$$

(we are using $k = 1$, but a similar formula will also hold for each $k = 1, 2, \ldots$).

The particular form of the identity (6.3.36) known as the Vaughan Identity is derived by putting

$$\theta(n) = \begin{cases} \mu(n) , & n \le V < N , \\ 0 , & n > V , \end{cases}$$

so that

$$\Theta(n) = \sum_{\substack{k \mid n \\ k \le V}} \mu(k) = \beta_n ,$$

where $V \le N$ is otherwise arbitrary.

An exposition of this identity can be found in Vaughan [15]. We have already used Vaughan's identity in Chapter 1 but it is not clear what advantage we would get by choosing the function $\theta(\cdot)$ differently.

To close this section, let us remark that there is no reason why we should not use other arithmetical functions in place of $\Lambda(n)$ to get an identity like (6.3.2).

For example, if we took

$$f(n,m) = \mu(m) \qquad \text{for all} \quad m, n ,$$

then

$$F(n,m) = \sum_{d \mid m} \mu(d) = \begin{cases} 1 , & m = 1 , \\ 0 , & m > 1 . \end{cases}$$

Then, formula (6.2.6) would give us

$$\Theta(1) \sum_{1 \le m \le N} \mu(m) + \sum_{1 < n \le N} \Theta(n) \sum_{m \le N n^{-1}} \mu(m) = \sum_{n \le N} \theta(n) ,$$

so that the choice

$$\theta(n) = \mu(n) \left(1 - \frac{\log n}{\log N} \right) , \qquad n \le N ,$$

gives us the well-known formula

$$M(N) \log N + \sum_{n \le N} \Lambda(n) M\left(\frac{N}{n}\right) = O(N) ,$$

where

$$M(N) = \sum_{n \le N} \mu(n) .$$

§6.4. The Global Sieve

Let $\mathcal{P}_0$ denote the set of prime numbers and $\mathcal{P}$ a subset of $\mathcal{P}_0$. Write

$$(6.4.1) \qquad P(z) = \prod_{\substack{p\in\mathcal{P}\\ p<z}} p \, , \qquad (z \geqslant 2) \, ,$$

and

$$(6.4.2) \qquad P(z_1, z) = \frac{P(z)}{P(z_1)} = \prod_{\substack{p\in\mathcal{P}\\ z_1 \leq p<z}} p \, , \qquad (2 \leq z_1 \leq z) \, ;$$

define

$$(6.4.3) \qquad b_{z_1, z}(m) = \begin{cases} 1 \, , & (m, P(z_1, z)) = 1 \, , \\ 0 \, , & \text{otherwise} \, , \end{cases}$$

and put

$$(6.4.4) \qquad b_z(m) = b_{2, z}(m) \, .$$

Also, define $p(1) = \infty$, and for $m > 1$, let $p(m)$ denote the least prime factor of m.

Now, let $\mathcal{B}(N)$ denote the region contained in $\mathcal{H}(N)$, and defined by:

$$(6.4.5) \qquad (m, n) \in \mathcal{H}(N) \, , \quad n \mid P(z_2, z) \, , \quad (m, P(z_1, z)) = 1 \, ,$$

where $2 \leqslant z_1 \leqslant z_2 \leqslant z$. Then, define $f(n, m)$, for $(n, m) \in \mathcal{B}(N)$, by

$$(6.4.6) \qquad f(n,m) = \begin{cases} a(nm) & , \quad \left(m, \frac{P(z_2,z)}{n}\right) = 1 , \\ 0 & , \quad \text{otherwise} . \end{cases}$$

So, we have

$$(6.4.7) \qquad F(n,m) = a(nm) \sum_{\substack{d \mid m , \\ nd \mid P(z_2,z) , \\ \left(\frac{m}{d}, \frac{P(z_2,z)}{nd}\right) = 1 , \\ \left(\frac{m}{d}, P(z_1,z_2)\right) = 1 .}} 1 \quad , \quad (m,n) \in \mathcal{B}(N) .$$

The sum in (6.4.7) is simply

$$\sum_{\substack{d \mid m , \\ \left(m, \frac{P(z_2,z)}{n}\right) = d , \\ \left(\frac{m}{d}, P(z_1,z_2)\right) = 1}} 1 \quad = \quad 1 \quad , \quad (m,n) \in \mathcal{B}(N) .$$

So, we have

$$(6.4.8) \qquad F(n,m) = a(nm) \quad , \quad (m,n) \in \mathcal{B}(N) ,$$

and zero otherwise. Substituting these into the fundamental identity in (6.2.6) gives us

LEMMA (The Global Sieve Identity).

$$(6.4.9)\qquad \sum_{m \le N} b_{z_1,z}(m)\, a(m)\, \Theta(m,1)$$

$$+ \sum_{\substack{1<n\le N \\ n \mid P(z_2,z)}} \ \sum_{\substack{1\le m\le Nn^{-1} \\ \left(m,\, \frac{P(z_2,z)}{n}\right)=1}} b_{z_1,z_2}(m)\, a(nm)\, \Theta(m,n)$$

$$= \sum_{\substack{1\le n\le N \\ n \mid P(z_2,z)}} \ \sum_{m\le Nn^{-1}} b_{z_1,z_2}(m)\, a(nm)\, \theta(m,n) .$$

Corollary 1. Suppose $\theta(1)=1$. Then

$$(6.4.10)\qquad \sum_{m\le N} b_{z_1,z}(m)\, a(m) = \sum_{\substack{1\le n\le N \\ n\mid P(z_2,z)}} \theta(n) \sum_{m\le Nn^{-1}} b_{z_1,z_2}(m)\, a(nm)$$

$$- \sum_{\substack{1<n\le N \\ n\mid P(z_2,z)}} \left\{ \theta\!\left(\frac{n}{p(n)}\right) + \theta(n) \right\} \sum_{1\le m\le Nn^{-1}} b_{z_1,p(n)}(m)\, a(nm) .$$

Proof. We take $\theta(m,n)=\theta(n)$ for all (m,n), so that it suffices to show that the second term in (6.4.9) is the same as the last term in (6.4.10). For this, we use the method given for the proof of equation (1.8) in [7]--which is a special case of our (6.4.10). It is clear that

$$\Theta(n) = \sum_{d \mid \frac{n}{p(n)}\cdot p(n)} \theta(d) = \sum_{d \mid \frac{n}{p(n)}} \left\{ \theta(d) + \theta(d\, p(n)) \right\} .$$

So, the second term in (6.4.9) is

(6.4.11)
$$\sum_{\substack{1<n \mid P(z_2,z) \\ n \le N}} \sum_{d \mid \frac{n}{p(n)}} \{\theta(d)+\theta(d\,p(n))\} \sum_{\substack{m \le Nn^{-1} \\ (m,\, \frac{P(z_2,z)}{n})=1}} b_{z_1,z_2}(m)\, a(nm) \quad ,$$

$$= \sum_{\substack{1<k \mid P(z_2,z) \\ k \le N}} \{\theta(\tfrac{k}{p(k)})+\theta(k)\} \sum_{\substack{\ell \mid \frac{P(z_2,z)}{k} \\ p(k)<p(\ell)}} \sum_{\substack{m \le Nk^{-1}\ell^{-1} \\ (m,\, \frac{P(z_2,z)}{k\ell})=1}} b_{z_1,z_2}(m)\, a(k\ell m) \quad ,$$

$$= \sum_{\substack{1<k \mid P(z_2,z) \\ k \le N}} \{\theta(\tfrac{k}{p(k)})+\theta(k)\} \sum_{t \le Nk^{-1}} a(tk) \left\{ \sum_{\substack{\ell \mid t, \\ \ell \mid \frac{P(z_2,z)}{k}, \\ p(\ell)>p(k), \\ (\frac{t}{\ell},\, \frac{P(z_2,z)}{k\ell})=1, \\ (\frac{t}{\ell},\, P(z_1,z_2))=1}} 1 \right\} .$$

The last sum in $\{\cdots\}$ can be written as

(6.4.12)
$$\sum_{\substack{\ell \mid (t,\, \frac{P(z_2,z)}{k}), \\ p(\ell)>p(k), \\ \ell = (t,\, \frac{P(z_2,z)}{k}), \\ (\frac{t}{\ell},\, P(z_1,z_2))=1 .}} 1 \quad .$$

The conditions

$$\ell = \left(t, \frac{P(z_2, z)}{k}\right) \qquad \text{and} \qquad p(\ell) > p(k),$$

are satisfied if and only if

(6.4.13) $$(t, P(z_2, p(k))) = 1 .$$

This means that ℓ can only have prime factors exceeding $p(k)$. The condition

$$\left(\frac{t}{\ell}, P(z_1, z_2)\right) = 1 ,$$

is, therefore, the same as the condition

(6.4.14) $$(t, P(z_1, z_2)) = 1 .$$

Combining (6.4.13) and (6.4.14) shows us that the sum (6.4.12) is 1 if and only if

$$(t, P(z_1, p(k))) = 1 ,$$

as required.

Following the example of an argument given in [7], we now put

$$\theta(n) = \mu(n)\, \chi(n) ,$$

where $\chi(1) = 1$, but is otherwise arbitrary. (Actually we need only have $\chi(1) \neq 0$). The function $\chi(\cdot)$ acts as a sort of characteristic function for some subset of the divisors of $P(z_2, z)$. We then have a generalised form of the identity (1.8) given in [7].

Corollary. If $\chi(1)=1$, then

$$(6.4.15) \qquad \sum_{m\le N} b_{z_1,z}(m)\, a(m) = \sum_{\substack{1\le n\le N \\ n\mid P(z_2,z)}} \mu(n)\chi(n) \sum_{m\le Nn^{-1}} b_{z_1,z_2}(m)\, a(mn)$$

$$+ \sum_{\substack{1<n\mid P(z_2,z) \\ n\le N}} \mu(n)\left\{\chi\left(\frac{n}{p(n)}\right) - \chi(n)\right\} \sum_{m\le Nn^{-1}} b_{z_1,p(n)}(m)\, a(mn) .$$

The sieve methods of Brun, Selberg, Jurkat-Richert and Rosser-Iwaniec each correspond to a specific choice of $\chi(\,)$ in (6.4.15). Later on , we shall also show how the sieve of Greaves' fits into our scheme of things. It can also be shown that the sharp form of the linear sieve as formulated recently by Iwaniec[†] [10] follows from choosing as a certain interval function (letter from Motohashi to Halberstam).

§6.4.1. The Inclusion-Exclusion Principle

Take

$$\chi(n)=1 \qquad \text{for all} \quad n .$$

Then, (6.4.15) becomes

$$\sum_{m\le N} b_{z_1,z}(m)\, a(m) = \sum_{\substack{n\le N \\ n\mid P(z_2,z)}} \mu(n) \sum_{m\le Nn^{-1}} b_{z_1,z_2}(m)\, a(mn) ,$$

so that on choosing $z_2=z_1$, we have the familiar formula

$$\sum_{m\le N} b_{z_1,z}(m)\, a(m) = \sum_{\substack{n\le N \\ n\mid P(z_1,z)}} \mu(n) \sum_{m\le Nn^{-1}} a(mn) .$$

[†]This paper is not without errors but Motohashi and Vaughan have correct versions of the proofs of the results.

with the choice $\mathcal{P} = \mathcal{P}_0$, $z_1 = 2$ and $z = N^{\frac{1}{2}}$, we arrive at

$$a(1) + \sum_{N^{\frac{1}{2}} < p \leq N} a(p) = \sum_{\substack{d \mid P(N^{1/2}) \\ d \leq N}} \mu(d) \sum_{m \leq N d^{-1}} a(md) .$$

§6.4.2. Generalised Form of the Meissel-Busthab Identity

Take

$$\chi(n) = \begin{cases} 1 & , \ n = 1 \\ 0 & , \ n > 1 . \end{cases}$$

Then, it is clear

$$\chi\left(\frac{n}{p(n)}\right) - \chi(n) = \begin{cases} 1 & , \ n \text{ is prime} , \\ 0 & , \ \text{otherwise} . \end{cases}$$

Thus, we arrive at

$$(6.4.16) \qquad \sum_{m \leq N} b_{z_1, z}(m) \ a(m) = \sum_{m \leq N} b_{z_1, z_2}(m) \ a(m)$$

$$- \sum_{z_2 \leq p < z} \ \sum_{1 \leq m \leq N p^{-1}} b_{z_1, p}(m) \ a(pm) .$$

The special case of this identity, obtained when $z_1 = z_2 = 2$ and $a(n) = 1$ for all n , is the famous Busthab identity. We shall digress here for a moment and give an application of this version of Busthab's identity.

§6.4.3. A Weighted Version of a Chen Identity

Let $R(n)$ denote the number of representations of an even integer n as the sum of two odd prime numbers. By the sieve methods, it is possible to show that

$$(6.4.17) \qquad R(n) \leq 8\,\mathfrak{S}(n)\,\frac{n}{(\log n)^2},$$

where

$$\mathfrak{S}(n) = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{\substack{p\mid n\\ p>2}}\frac{p-1}{p-2}.$$

Improvement of the constant 8 in (6.4.17) is very difficult, and recently, Chen (see Pan [13]) showed that 8 can be replaced by 7.988. To do this, a crucial role was played by a formula, which is a special case of the following identity:

$$(6.4.18) \qquad \sum_{a\in\mathcal{A}} b_z(a) = \sum_{a\in\mathcal{A}} b_{z_2}(a) - \sum_{z_1\leq p<z_2} f(p)\sum_{\substack{a\in\mathcal{A}\\ a\equiv 0 \pmod p}} b_{z_2}\left(\frac{a}{p}\right)$$

$$+ \sum_{z_2\leq p_1<p_2<p_3<z} f(p_3)\sum_{\substack{a\in\mathcal{A}\\ p_1p_2p_3\mid a}} b_{p_2}\left(\frac{a}{p_1}\right)$$

$$+ \sum_{z_2\leq p_1<p_2<z}\{f(p_1)+f(p_2)-1\}\sum_{\substack{a\in\mathcal{A}\\ p_1p_2\mid a}} b_{p_2}\left(\frac{a}{p_1}\right)$$

$$- \sum_{z_2\leq p<z}(1-f(p))\sum_{\substack{a\in\mathcal{A}\\ p\mid a}} b_z\left(\frac{a}{p}\right),$$

valid for any sequence $\mathcal{A}$ of square-free numbers, with $2 \le z_2 \le z$ and any function $f(\cdot)$. Chen's formula ([13]) follows with the choice

$$f(p) = \tfrac{1}{2} \qquad \text{for all } p .$$

The question which arises automatically is whether one can do better than Chen's result by a different choice of $f(\cdot)$. For the problem for $R(n)$, one is led to solving an extremal problem but which I have been unable to solve, yet. At any rate, it appears that if $f(p) = \frac{1}{2}$ is not the optimal function, it is not very far from it and the ensuring improvement would be very small indeed.

Remark. If the sequence $\mathcal{A}$ is not square-free, if $|f(p)| \le M$ for all p, and $\sup\{|a| : a \in \mathcal{A}\} \le N$ with $z \le N$, the identity (6.4.18) is still valid but one has an error term $\ll MN z_2^{-1}$.

Proof of (6.4.18). Define the function

$$w(a) = \sum_{\substack{p \mid a \\ z_2 \le p < z}} f(p) ,$$

and zero whenever the sum is empty. Then, it is obvious that

$$(6.4.19) \qquad \sum_{a \in \mathcal{A}} b_z(a) = \sum_{a \in \mathcal{A}} b_z(a) \left(1 - w(a)\right) ,$$

since $w(a)$ is zero under the conditions. By the Generalised Busthab identity (6.4.16), with $a(m)$ the characteristic function of the sequence $\mathcal{A}$, with weight $1 - w(m)$, with $z_1 = z$ and $N = \infty$, the sum on the right of (6.4.19) is

$$(6.4.20) \qquad \sum_{a \in \mathcal{A}} b_{z_2}(a)\{1 - w(a)\} \quad - \sum_{z_2 \le p < z} \sum_{\substack{a \in \mathcal{A} \\ p | a}} b_p\left(\frac{a}{p}\right)\{1 - w(a)\} .$$

The first sum in (6.4.20) is

$$(6.4.21) \qquad \sum_{a \in \mathcal{A}} b_{z_2}(a) \quad - \sum_{z_2 \le p < z} f(p) \sum_{\substack{a \in \mathcal{A} \\ p | a}} b_{z_2}(a)$$

The second sum in (6.4.20) is, after another application of (6.4.16), $J_1 + J_2$, where

$$(6.4.22) \qquad J_1 \quad = \sum_{z_2 \le p < z} \sum_{\substack{a \in \mathcal{A} \\ p | a}} b_z\left(\frac{a}{p}\right)\{1 - w(a)\} ,$$

where in this expression, there is exactly one prime factor of a in the given interval, namely p . So, we have $w(a) = f(p)$. This gives us

$$(6.4.23) \qquad J_1 \quad = \sum_{z_2 \le p < z} \{1 - f(p)\} \sum_{\substack{a \in \mathcal{A} \\ p | a}} b_z\left(\frac{a}{p}\right) .$$

Next, we have

$$(6.4.24) \qquad J_2 \quad = \sum_{z_2 \le p_1 < p_2 < z} \sum_{\substack{a \in \mathcal{A} \\ p_1 p_2 | a}} \{1 - \omega(a)\}\, b_{p_2}\left(\frac{a}{p_1}\right) .$$

Now, because the sequence $\mathcal{A}$ is square-free, $1-w(a)$ is really

$$f(p_1) + f(p_2) + \sum_{\substack{p_2 < p < z \\ p \mid a}} f(p) ,$$

so that (6.4.24) is

$$(6.4.25) \qquad \sum_{z_2 \le p_1 < p_2 < z} \{1 - f(p_1) - f(p_2)\} \sum_{\substack{a \in \mathcal{A} \\ p_1 p_2 \mid a}} b_{p_2}\left(\frac{a}{p_1}\right)$$

$$- \sum_{z_2 \le p_1 < p_2 < p_3 < z} f(p_3) \sum_{\substack{a \in \mathcal{A} \\ p_1 p_2 p_3 \mid a}} b_{p_2}\left(\frac{a}{p_1}\right) .$$

Collecting the expressions from (6.4.20), (6.4.21), (6.4.23) and (6.4.25) gives us (6.4.18).

<u>Remark</u>. In [4], Chen improves 7.988 to 7.8342 by using many very complicated formulae. Prof. H-E Richert, in a letter to Prof. Halberstam has shown that those formulae can be deduced from (6.4.15).

§6.4.4. <u>The Brun Sieves</u>

For k a positive integer, take

$$(6.4.26) \qquad \chi(n) = \begin{cases} 1 , & \nu(n) \le k , \ n \mid P(z_2, z) , \\ 0 , & \text{otherwise} , \end{cases}$$

in (6.4.15), where $\nu(n)$ denotes the number of distinct prime factors of n. This gives us

$$(6.4.27)\qquad \sum_{m\le N} b_{z_1,z}(m)\,a(m) = \sum_{\substack{n|P(z_2,z)\\ n\le N\\ \nu(n)\le k}} \mu(n) \sum_{m\le Nn^{-1}} b_{z_1,z_2}(m)\,a(mn)$$

$$+\ (-1)^{k+1} \sum_{\substack{n|P(z_2,z)\\ 1<n\le N\\ \nu(n)=k+1}} \ \sum_{1\le m\le Nn^{-1}} b_{z_1,p(n)}(m)\ a(mn)\ ,$$

since

$$\chi\left(\frac{n}{p(n)}\right) - \chi(n) = \begin{cases} 1\ , & \nu(n)=k+1\ ,\\ 0\ , & \text{otherwise}. \end{cases}$$

If the $a(\cdot)$ is non-negative, (6.4.27) tells us that, for any pair of non-negative integers r and s,

$$(6.4.28)\qquad \sum_{\substack{n|P(z_2,z)\\ n\le N\\ \nu(n)\le 2s+1}} \mu(n) \sum_{m\le Nn^{-1}} b_{z_1,z_2}(m)\,a(mn) \le \sum_{m\le N} b_{z_1,z}(m)\,a(m)$$

$$\le \sum_{\substack{n|P(z_2,z)\\ n\le N,\ \nu(n)\le 2r}} \mu(n) \sum_{m\le Nn^{-1}} b_{z_1,z_2}(m)\ a(mn)\ ,$$

which represents the content of Brun's "Pure" Sieve. This simple beginning suggests the following more general assertion, whenever $a(n)$ is non-negative.

Given any pair of functions χ^+ and χ^- (with $\chi^{\pm}(1)=1$), and satisfying

$$(6.4.29)\qquad \mu(n)\left\{\chi^+\left(\frac{n}{p(n)}\right) - \chi^+(n)\right\} \le 0\ ,\qquad 1<n\mid P(z_2,z)\ ,$$

and

(6.4.30) $$\mu(n)\left\{\chi^-\left(\frac{n}{p(n)}\right) - \chi^-(n)\right\} \geqslant 0 \quad , \quad 1<n \mid P(z_2,z) ,$$

we obtain from (6.4.27)

(6.4.31) $$\sum_{n \mid P(z_2,z)} \mu(n)\chi^-(n) \sum_{mn \le N} b_{z_1,z_2}(m)\, a(nm)$$

$$\leqslant \sum_{m \le N} b_{z_1,z}(m)\, a(m) \leqslant \sum_{n \mid P(z_2,z)} \mu(n)\chi^+(n) \sum_{mn \le N} b_{z_1,z_2}(m)\, a(nm) ,$$

and now, we need to make an optimal choice of χ^+ and χ subject to (6.4.29) and (6.4.30). Indeed, Brun gave an example which is superior to the choice (6.4.26), as follows:

Put $z_1 = z_2 = 2$ in (6.4.31). Relative to a suitable partition

$$2 = x_r < x_{r-1} < \cdots\cdots < x_1 < x_0 = z ,$$

of $[2,z)$, take

(6.4.32) $$\chi^+(n) = \begin{cases} 1 \quad , & \nu\big((n, P(x_i,z))\big) \leqslant 2(i+b-1) \quad , \quad i = 1,\ldots,r , \\ 0 \quad , & \text{otherwise} , \end{cases}$$

where b is an additional parameter (as are $x_1, \ldots\ldots, x_r$). Note that $\chi^+(n) = 1$ if n has not more than $2b$ prime factors of $\mathcal{P}$ from $[x_1,z)$, not more than $2(b+1)$ from $[x_2,z)$ and so on. We check that the definition (6.4.32) indeed satisfies (6.4.29).

Suppose $\mu(n) = 1$ and suppose that

$$\chi^+\left(\frac{n}{p(n)}\right) - \chi^+(n) > 0 .$$

This is only possible if

$$\nu\left(\left(\frac{n}{p(n)}, P(x_i, z)\right)\right) \leq 2(i+b-1) \quad , \quad i = 1, \ldots, r \ ,$$

and

$$\nu\left((n, P(x_{i_0}, z)\right) \geq 2(i_0+b-1)+1 \quad , \quad \text{some } i_0 \leq r \, .$$

This means that

$$p(n) \mid P(x_{i_0}, z)$$

and in fact, there are exactly $2(i_0+b-1)+1$ prime factors of n in $[x_{i_0}, z)$. But these account for all of them and leads to a contradiction because it implies $\mu(n)=-1$. If we assume $\mu(n)=-1$, and

$$\chi^+\left(\frac{n}{p(n)}\right) - \chi^+(n) < 0 \ ,$$

this implies that

$$\nu\left((n, P(x_i, z))\right) \leq 2(i+b-1) \quad , \quad i = 1, \ldots, r \ ,$$

but

$$\nu\left(\left(\frac{n}{p(n)}, P(x_{i_0}, z)\right)\right) > 2(i_0+b-1)+1 \quad , \quad \text{some } i_0 \leq r \, .$$

This is clearly impossible. Thus, the choice of Brun (6.4.32) is indeed a legitimate χ^+ and leads to an upper bound sieve. A similar construction is available for a χ^- .

§6.4.5. <u>Selberg's (Upper) Sieve</u>

Let λ_n ($n \mid P(z_2, z)$) be any real numbers subject only to the condition $\lambda_1 = 1$. Now, in (6.4.15), take

(6.4.33) $$\chi(n) = \mu(n) \sum_{\{n_1, n_2\} = n} \lambda_{n_1} \lambda_{n_2} \ ,$$

where $\{n_1, n_2\}$ denotes the least common multiple of n_1 and n_2. Then, we have

$$(6.4.34)\quad \chi\left(\frac{n}{p(n)}\right) - \chi(n) = -\mu(n)\left\{\sum_{\{n_1,n_2\}=\frac{n}{p(n)}} \lambda_{n_1}\lambda_{n_2} + \sum_{\{d_1,d_2\}=n} \lambda_{d_1}\lambda_{d_2}\right\},$$

$$= -\mu(n)\left\{\sum_{\{n_1,n_2\}=\frac{n}{p(n)}} \lambda_{n_1}\lambda_{n_2} + \lambda_{n_1p(n)}\lambda_{n_2} + \lambda_{n_1}\lambda_{n_2p(n)} + \lambda_{n_1p(n)}\lambda_{n_2p(n)}\right\},$$

$$= -\mu(n)\sum_{\{n_1,n_2\}=\frac{n}{p(n)}} \left(\lambda_{n_1} + \lambda_{n_1p(n)}\right)\left(\lambda_{n_2} + \lambda_{n_2p(n)}\right)$$

Substituting this into (6.4.15), we get

$$(6.4.35)\quad \sum_{m\le N} b_{z_1,z}(m)\, a(m)$$

$$= \sum_{\substack{n \mid P(z_1,z)\\ n\le N}} \left(\sum_{\{n_1,n_2\}=n} \lambda_{n_1}\lambda_{n_2}\right) \sum_{\substack{m\le N\\ n\mid m}} b_{z_1,z}\left(\frac{m}{n}\right) a(m)$$

$$- \sum_{\substack{1<n\le N\\ n\mid P(z_1,z)}} \left\{\sum_{\{n_1,n_2\}=\frac{n}{p(n)}} \left(\lambda_{n_1}+\lambda_{n_1p(n)}\right)\left(\lambda_{n_2}+\lambda_{n_2p(n)}\right)\right\} \sum_{\substack{m\le N\\ n\mid m}} b_{z_1,p(n)}(m)\, a(m).$$

The first sum on the right of the equality, in (6.4.35), is

(6.4.36)
$$\sum_{m\le N} b_{z_1,z_2}(m)\, a(m) \sum_{\substack{n\mid P(z_2,z)\\ n\mid m}} \sum_{\{n_1,n_2\}=n} \lambda_{n_1}\lambda_{n_2} ,$$

$$= \sum_{m\le N} b_{z_1,z_2}(m)\, a(m) \Big(\sum_{n\mid (m,P(z_2,z))} \lambda_n \Big)^2 .$$

The last expression in (6.4.35) can be rewritten as

(6.4.37)
$$\sum_{p\mid P(z_2,z)} \sum_{\substack{k\mid P(z_2,z)\\ p(k)>p}} \Big(\sum_{\{n_1,n_2\}=k} (\lambda_{n_1}+\lambda_{n_1 p})(\lambda_{n_2}+\lambda_{n_2 p}) \Big) \times$$

$$\times \sum_{\substack{m\le N\\ kp\mid m}} b_{z_1,p}\Big(\frac{m}{kp}\Big)\, a(m) ,$$

$$= \sum_{p\mid P(z_2,z)} \sum_{\substack{m\le N\\ p\mid m}} b_{z_1,p}(m)\, a(m) \sum_{\substack{k\mid m\\ k\mid P(z_2,z)\\ p(k)>p}} \Big(\sum_{\{n_1,n_2\}=k} (\lambda_{n_1}+\lambda_{n_1 p})(\lambda_{n_2}+\lambda_{n_2 p}) \Big) ,$$

$$= \sum_{p\mid P(z_2,z)} \sum_{\substack{m\le N\\ p\mid m}} b_{z_1,p}(m)\, a(m) \Big(\sum_{\substack{k\mid m\\ k\mid P(p^+,z)}} (\lambda_p + \lambda_{kp}) \Big)^2 ,$$

where p^+ denotes the prime number which is the successor of p, in $\mathcal{P}$. Collecting (6.4.35), (6.4.36) and (6.4.37), we have

$$(6.4.38)\qquad \sum_{m\le N} b_{z_1,z}(m)\,a(m) = \sum_{m\le N} b_{z_1,z_2}(m)\,a(m)\Big(\sum_{\substack{n|m\\ n|P(z_2,z)}} \lambda_n\Big)^2$$

$$-\sum_{p|P(z_2,z)}\ \sum_{\substack{m\le N\\ p|m}} b_{z_1,p}(m)\,a(m)\Big(\sum_{\substack{k|m\\ k|P(p^+,z)}} (\lambda_k+\lambda_{kp})\Big)^2 .$$

For non-negative $a(\cdot)$, with $z_1 = z_2 = 2$, this gives at once the well-known form of Selberg's upper sieve; but the point of identity (6.4.38) is that it suggests that more might be true than is usually deduced by Selberg's procedure. For example, while the classical Selberg result reads

$$(6.4.39)\qquad \sum_{m\le N} b_z(m)\,a(m) \leqslant \sum_{m\le N} a(m)\Big(\sum_{\substack{d|m\\ d|P(z)}} \lambda_d\Big)^2 ,$$

(6.4.38) implies that

$$(6.4.40)\qquad \sum_{m\le N} b_z(m)\,a(m) \leqslant \sum_{m\le N} a(m)\Big(\sum_{\substack{d|m\\ d|P(z)}} \lambda_d\Big)^2$$

$$-\sum_{p|P(z)} (1+\lambda_p)^2 \sum_{m\le Np^{-1}} b_z(m)\,a(pm) .$$

It is not known if there is a better choice of λ's than the Selberg λ's, which optimize (6.4.39). Heath-Brown has raised the problem of optimising the λ's in (6.4.40) subject to

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_p + 1 = \left(\frac{\log p}{\log z}\right)^{\frac{1}{2}} .$$

For the next two sections, we introduce the following: we write a typical factor n of $P(z_1, z)$, $n>1$ in the form

(6.4.41) $$n = p_1 p_2 \ldots p_s \quad , \quad (p_1 > p_2 > \cdots > p_s) ,$$

and give $\chi(n)$ the structure

(6.4.42) $$\chi(n) = \chi(p_1 \ldots p_s) = \eta(p_1)\eta(p_1 p_2) \ldots \eta(p_1 \cdots p_s) ,$$

where $\eta(\cdot)$ is an arithmetic function to be chosen as follows:

§6.4.6. The Sieve of Jurkat-Richert

With parameters $\beta > 0$ and $y \geq z_2$, let

(6.4.43) $$\eta(p_1 \cdots p_r) = \begin{cases} 1 & , \quad p_r^{\beta+1} p_{r-1} \cdots p_1 < y , \\ 0 & , \quad \text{otherwise} , \end{cases}$$

so that

(6.4.44) $$\chi\left(\frac{n}{p(n)}\right) - \chi(n) = \chi\left(\frac{n}{p(n)}\right)(1 - \eta(n)) ,$$

$$= \begin{cases} 1 & , \quad p_s^{\beta+1} p_{s-1} \cdots p_1 > y \quad \text{and} \\ & \quad\; p_i^{\beta+1} p_{i-1} \cdots p_1 \leq y \quad \text{for all } i = 1, \ldots, s-1 , \\ 0 & , \quad \text{otherwise} , \end{cases}$$

whence, in particular, we get a non-zero contribution when

(6.4.45) $$\frac{y}{p(n)^{\beta}} \leq n < y .$$

This choice of χ then leads to the sieve of Jurkat and Richert [11] and may be considered as the sieve that results when one iterates the Busthab identity infinite times.

§6.4.7. The Sieve of Rosser-Iwaniec

Again, let $\beta > 0$ and $y \geq 2$ be parameters, and suppose that χ has the structure (6.4.42). Now, the expression

$$(6.4.46) \qquad \mu(n)\left\{\chi\left(\frac{n}{p(n)}\right) - \chi(n)\right\} = \mu(n)\,\chi\left(\frac{n}{p(n)}\right)\left(1-\eta(n)\right),$$

is non-positive (so that χ is a χ^+) provided that

$$(6.4.47) \qquad \eta(n) = \eta^+(n) = \begin{cases} 1, & \mu(n) = 1, \\ \in [0,1], & \mu(n) = -1, \end{cases}$$

and (6.4.46) is non-negative (so that χ is a χ^-) provided that

$$(6.4.48) \qquad \eta(n) = \eta^-(n) = \begin{cases} 1, & \mu(n) = -1 \\ \in [0,1], & \mu(n) = 1 \end{cases}.$$

Accordingly

$$\chi^+(n) = \eta^+(p_1)\,\eta^+(p_1p_2p_3)\cdots\eta^+(p_1p_2\cdots p_{2\ell+1}),$$

where $\ell = [\frac{1}{2}(s-1)]$, with η^+ satisfying (6.4.47), leads to an upper bound sieve, while

$$\chi^-(n) = \eta^-(p_1p_2)\,\eta^-(p_1p_2p_3p_4)\cdots\eta^-(p_1\cdots p_{2\ell}),$$

where $\ell = [\frac{1}{2}s]$, and η^- satisfying (6.4.48) leads to a lower bound sieve; both in the case of non-negative $a(\cdot)$. On choosing

$$\eta^+(p_1\cdots p_s) = \begin{cases} 1, & p_s^{\beta+1}p_{s-1}\cdots p_1 < y, \\ 0, & \text{otherwise}, \end{cases}$$

for s odd, and the same choice for $\eta^-(p_1\cdots p_s)$ with s even gives us the Rosser-Iwaniec sieve.

§6.4.8. The Weighted Sieve of Greaves

It has been shown by Kuhn [12], Chen [3] and others that if certain weight functions are attached to the well-known sieves, and the resulting expression rewritten in such a way that each component is really an unweighted sieve, then the results known for such sieves tend to give improvements over those sieves without any weights. In particular, these weights may not be non-negative but the expressions can be reduced to several others where the weights are non-negative, and then, say, the Rosser-Iwaniec sieve could be used to treat the individual terms.

Recently, Greaves [6] introduced a weighted sieve where the sieve problem is attacked directly (that is, not by reducing it to expressions as described before). But, he gives the impression that the Rosser-Iwaniec sieve is necessarily bound up with his weighted one. We give here the general setting from which Greaves' sieve follows.

Put

$$W(\ell) = 1 - \sum_{p \mid \ell} w(p) \qquad ; \qquad \ell > 1, \tag{6.4.49}$$

and

$$W(1) = 1,$$

where $w(\cdot)$ is as yet an arbitrary function. Then,

$$\sum_{\ell \mid k} W(\ell)\mu(\ell) = \begin{cases} 1 & , \quad k = 1, \\ w(p) & , \quad k = p \ - \text{prime}, \\ 0 & , \quad \text{otherwise}. \end{cases} \tag{6.4.50}$$

Now, put

$$(6.4.51) \qquad \theta(m,n) = \theta_1(m,n)\, W(n) ,$$

and let $\Theta(m,n)$ and $\Theta_1(m,n)$ be the associated functions as defined in (6.2.2). Then, if $n \mid P(z_2,z)$, we have

$$(6.4.52) \qquad \Theta(m,n) = \sum_{d \mid n} \theta(md, \tfrac{n}{d}) = \sum_{d \mid n} \theta_1(md, \tfrac{n}{d})\, W(\tfrac{n}{d}) ,$$

and since $(\frac{n}{d}, d) = 1$, we have

$$W(n) = W(\tfrac{n}{d}) - \sum_{p \mid d} w(p) .$$

So, putting this in (6.4.52) gives us

$$(6.4.53) \qquad \Theta(m,n) = W(n) \sum_{d \mid n} \theta_1(md, \tfrac{n}{d}) + \sum_{d \mid n} \theta_1(md, \tfrac{n}{d}) \sum_{p \mid d} w(p) ,$$

$$= W(n)\, \Theta_1(m,n) + \sum_{p \mid n} w(p)\, \Theta_1(mp, \tfrac{n}{p}) .$$

We now choose

$$\theta(n) = \theta_1(n) W(n) \quad ; \quad \theta(1) = 1 ,$$

in the identity (6.4.16), so that from (6.4.51), we get

$$(6.4.54) \qquad \sum_{m \le N} b_{z_1,z}(m)\, a(m) = \sum_{\substack{n \mid P(z_2,z) \\ n \le N}} \theta_1(n) W(n) \sum_{m \le Nn^{-1}} b_{z_1,z_2}(m)\, a(mn)$$

$$- \sum_{\substack{1 < n \mid P(z_2,z) \\ n \le N}} \Theta(n) \sum_{\substack{m \le Nn^{-1} \\ (m, \frac{P(z_2,z)}{n}) = 1}} b_{z_1,z_2}(m)\, a(nm) .$$

By (6.4.53), the last expression (6.4.54) is $S_1 + S_2$, where

$$S_1 = \sum_{\substack{1<n|P(z_2,z)\\ n\le N}} \oplus_1(n)W(n) \sum_{\substack{m\le Nn^{-1}\\ (m,\frac{P(z_2,z)}{n})=1}} b_{z_1,z_2}(m)\, a(mn)\,,$$

and

$$(6.4.55)\qquad S_2 = \sum_{\substack{1<n|P(z_2,z)\\ n\le N}} \sum_{p|n} w(p)\oplus_1\left(\frac{n}{p}\right) \sum_{\substack{m\le Nn^{-1}\\ (m,\frac{P(z_2,z)}{n})=1}} b_{z_1,z_2}(m)\, a(nm)\,,$$

$$= \sum_{\substack{k|P(z_2,z)\\ k\le N}} \oplus_1(k) \sum_{p|\frac{P(z_2,z)}{k}} w(p) \sum_{\substack{m\le Nk^{-1}p^{-1}\\ (m,\frac{P(z_2,z)}{kp})=1}} b_{z_1,z_2}(m)\, a(mkp),$$

$$= \sum_{p|P(z_2,z)} w(p) \sum_{\substack{m\le Np^{-1}\\ (m,\frac{P(z_2,z)}{p})=1}} b_{z_1,z_2}(m)\, a(mp)$$

$$+ \sum_{1<k|P(z_2,z)} \oplus_1(k) \sum_{p|\frac{P(z_2,z)}{k}} w(p) \sum_{\substack{m\le Nk^{-1}p^{-1}\\ (m,\frac{P(z_2,z)}{kp})=1}} b_{z_1,z_2}(m)\, a(mkp)\,,$$

$$= S_3 + S_4\,,$$

say. Observe that S_3 is actually equal to

$$(6.4.56)\qquad \sum_{\substack{m\le N\\ (m,P(z_2,z))=p}} w(p)\, b_{z_1,z_2}(m)\, a(m)\,.$$

We shall now simplify S_1 and S_4 further. We have by the argument given in Corollay 1, that

$$(6.4.57) \qquad S_1 = \sum_{\substack{1<l \mid P(z_2,z) \\ l \leq N}} \left\{ \theta_1\left(\frac{l}{p(l)}\right) + \theta_1(l) \right\} \times$$

$$\times \sum_{\substack{t \mid \frac{P(z_2,z)}{l}, \\ p(t) > p(l)}} W(lt) \sum_{\substack{m \leq N l^{-1} t^{-1} \\ \left(m, \frac{P(z_2,z)}{lt}\right) = 1}} b_{z_1,z_2}(m)\, a(mlt) ,$$

and since $(l, t) = 1$, we have

$$W(lt) = W(t) - \sum_{p \mid l} w(p) ,$$

so that from (6.4.57),

$$(6.4.58) \qquad S_1 = \sum_{\substack{1<l \mid P(z_2,z) \\ l \leq N}} \left\{ \theta_1\left(\frac{l}{p(l)}\right) + \theta_1(l) \right\} \left\{ \left(\sum_{\substack{t \mid P(z_2,z) \\ p(t) > p(l)}} W(t) \times \right.\right.$$

$$\times \sum_{\substack{m \leq N l^{-1} t^{-1} \\ \left(m, \frac{P(z_2,z)}{lt}\right) = 1}} b_{z_1,z_2}(m)\, a(mlt) \Bigg) - \Bigg(\sum_{p \mid l} w(p) \Bigg) \times$$

$$\left.\times \sum_{\substack{t \mid \frac{P(z_2,z)}{l} \\ p(t) > p(l)}} \sum_{\substack{m \leq N l^{-1} t^{-1} \\ \left(m, \frac{P(z_2,z)}{lt}\right) = 1}} b_{z_1,z_2}(m)\, a(mlt) \Bigg) \right\} ,$$

$$= S_5 - S_6 ,$$

say. By a similar reasoning, we also have

(6.4.59) $$S_4 = \sum_{\substack{1<l \mid P(z_2,z) \\ l \leq N}} \left\{ \theta_1\left(\frac{l}{p(l)}\right) + \theta_1(l) \right\} \times$$

$$\times \sum_{\substack{t \mid \frac{P(z_2,z)}{l} \\ p(t) > p(l)}} \sum_{p \mid \frac{P(z_2,z)}{lt}} w(p) \sum_{\substack{m \leq N l^{-1} t^{-1} p^{-1} \\ \left(m, \frac{P(z_2,z)}{ltp}\right) = 1}} b_{z_1,z_2}(m)\, a(ltpm).$$

Now, in the expression in (6.4.59), we may write

$$\frac{P(z_2,z)}{lt} = \frac{P(p(l),z)}{lt} \cdot P(z_2, p(l)) ,$$

so that

$$\sum_{p \mid \frac{P(z_2,z)}{lt}} w(p) = \sum_{p \mid \frac{P(p(l),z)}{lt}} w(p) + \sum_{p \mid P(z_2, p(l))} w(p) .$$

Putting this in (6.4.59), and writing $k = pt$, we get

(6.4.60) $$S_4 = \sum_{1<l \mid P(z_2,z)} \left\{ \theta_1\left(\frac{l}{p(l)}\right) + \theta_1(l) \right\} \sum_{k \mid \frac{P(p(l),z)}{l}} \left(\sum_{p \mid k} w(p) \right) \times$$

$$\times \left\{ \sum_{\substack{m \leq N l^{-1} k^{-1} \\ \left(m, \frac{P(z_2,z)}{lk}\right) = 1}} b_{z_1,z_2}(m)\, a(mlk) \right\}$$

$$+ \sum_{1<l \mid P(z_2,z)} \left\{ \theta_1\left(\frac{l}{p(l)}\right) + \theta_1(l) \right\} \sum_{p \mid P(z_2, p(l))} w(p) \sum_{\substack{t \mid \frac{P(z_2,z)}{l} \\ p(t) > p(l)}} \sum_{\substack{m \leq N (lpt)^{-1} \\ \left(m, \frac{P(z_2,z)}{ltp}\right) = 1}} b_{z_1,z_2}(m)\, a(mltp).$$

The first expression in (6.4.60) is actually equal to

$$(6.4.61)\quad \sum_{1<l \mid P(z_2,z)} \left\{ \theta_1\left(\frac{l}{p(l)}\right) + \theta_1(l) \right\} \sum_{k \mid \frac{P(p(l),z)}{l}} \sum_{\substack{m \le N l^{-1} k^{-1} \\ \left(m, \frac{P(z_2,z)}{lk}\right)=1}} b_{z_1,z_2}(m)\, a(mlk)$$

$$- S_5 ,$$

$$= S_7 - S_5 ,$$

say. Denote the second expression in (6.4.61) by S_8. So, we have

$$\begin{aligned}(6.4.62)\quad S_1 + S_2 &= (S_5 - S_6) + (S_3 + S_4) ,\\ &= (S_5 - S_6) + (S_3 + S_7 - S_5 + S_8) ,\\ &= S_3 + S_7 + S_8 - S_6 .\end{aligned}$$

Now, the method given in Corollary 1 allows one to treat the inner sums in S_7, S_8 and S_6. We get

$$(6.4.63)\quad S_7 - S_6 = \sum_{1<l \mid P(z_2,z)} \left\{ \theta_1\left(\frac{l}{p(l)}\right) + \theta_1(l) \right\} W(l) \sum_{m \le N l^{-1}} b_{z_1,p(l)}(m)\, a(ml) ,$$

and

$$(6.4.64)\quad S_8 = \sum_{1<l \mid P(z_2,z)} \left\{ \theta_1\left(\frac{l}{p(l)}\right) + \theta_1(l) \right\} \sum_{\substack{m \le N l^{-1} \\ (m, P(z_2, p(l)))=p}} w(p)\, b_{z_1,z_2}(m)\, a(ml) ,$$

and combining $S_7 - S_6 + S_8$, we can write it as

$$(6.4.65)\quad \sum_{1<l \mid P(z_2,z)} \left\{ \theta_1\left(\frac{l}{p(l)}\right) + \theta_1(l) \right\} \sum_{\substack{m \le N l^{-1} \\ (m, P(z_1,z_2))=1}} a(ml) \left\{ \sum_{d \mid (m, P(z_2, p(l)))} \mu(d) \times \right.$$

$$\left. \times \left(W(d) - \sum_{p \mid t} w(p) \right) \right\} .$$

Combining all these estimates together, taking S_3 to the left hand side of (6.4.54), and simplifying, we get

LEMMA. For any function $\theta(.)$, with $\theta(1) = 1$, we have

(6.4.66)
$$\sum_{m \leq N} a(m)\, b_{z_1,z_2}(m) \Big(\sum_{d \mid (m, P(z_2,z))} \mu(d) W(d) \Big)$$

$$= \sum_{n \mid P(z_2,z)} \theta(n) W(n) \sum_{m \leq Nn^{-1}} b_{z_1,z_2}(m)\, a(nm)$$

$$- \sum_{1<n \mid P(z_2,z)} \Big\{ \theta\Big(\frac{n}{p(n)}\Big) - \theta(n) \Big\} \sum_{m \leq Nn^{-1}} b_{z_1,z_2}(m)\, a(nm) \Big(\sum_{d \mid (m, P(z_2,p(n)))} \mu(d) W(dn) \Big) .$$

Observe that this is a generalisation of the identity (6.4.16), on choosing

$$w(p) = 0 \qquad , \qquad p \mid P(z_2, z)$$

so that

$$W(n) = 1 \qquad , \qquad n \mid P(z_2, z) .$$

The combinatorial argument behind Greaves' sieve is obtained on choosing $z_1 = z_2 = 2$, and with the $\theta(n) = \mu(n)\chi(n)$ such that the $\chi(n)$ are those defined for the Rosser-Iswaniec sieve.

Chapter 7.

An Analogue of Goldbach's Problem†

§7.1. Introduction

Goldbach's Problem says that if N is an even integer, then it has a representation in the form

$$N = p_1 + p_2$$

where the p_i's shall denote prime numbers. On squaring the above equality, we arrive at

$$(*) \qquad N^2 = p_1^2 + p_2^2 + 2p_1p_2 .$$

This is our motivation in asking for the solability of the equation

$$(**) \qquad N = p_1^2 + p_2^2 + K p_3 ,$$

where K depends on the residue class of N modulo 12. The problem (**) is very difficult but is true for almost all integers (as can be proved by the Circle method.). In particular, I have shown that the measure of the set containing even (odd) integers $n \leq N$ not satisfying the representation

$$n^2 = p_1^2 + p_2^2 + 2p_3 \qquad , \quad n \text{ even} ,$$

$$n^2 = p_1^2 + p_2^2 + p_3 \qquad , \quad n \text{ odd} ,$$

is $O(N^{1-\delta})$ for some $\delta > 0$. This was motivated by (*) and the proof is long and complicated--using the Circle method as developed for such a purpose in Goldbach's Problem by Montgomery and Vaughan [4].

In 1973, J-R Chen showed that every sufficiently large even integer can be represented in the form

†Notation in this chapter follows that of Reference [2].

$$N = p_1 + Q_2 ,$$

where Q_r is a number with at more r prime factors. We ask the corresponding question for our equation (**), and we show the following

THEOREM 1. Let

$$\mathcal{A}^K = \left\{ \frac{N - p_1^2 - p_2^2}{K} : 3 \leq p_1, p_2 \leq M = [N^{\frac{1}{2}}] , \ p_i \text{ primes}, i = 1,2, \ N \equiv p_1^2 + p_2^2 \pmod K \right\}.$$

Then, for sufficiently large

$$|\{a \in \mathcal{A}^K : a = Q_3\}| \geq \frac{26}{3} \mathfrak{S}_K(N) \frac{N}{(\log N)^3} ,$$

where $\mathfrak{S}_K(N) > 0$, is defined later (see §7.5), while K takes the values

$$\begin{cases} 1 , & N \equiv 1, 3, 7, 9, \\ 3 , & N \equiv 5, 11 , \\ 2 , & N \equiv 0, 4 , \\ 6 , & N \equiv 8 , \\ 8 , & N \equiv 6, 10 , \\ 24 , & N \equiv 2 \end{cases} \pmod{12} .$$

In particular, every sufficiently large integer N can be represented in the form

$$N = p_1^2 + p_2^2 + KQ_3 .$$

We have been unsuccessful in proving the theorem with Q_3 replaced by Q_2 --the method of Chen does not apply here because we don't know how to sieve sums of two squares of primes effectively.

The method of proof of Theorem 1 is by means of the linear sieve as given in [2]. The main tool is Theorem 9.1 of [2] but, unfortunately, the condition $\Omega_2(\kappa, L)$ is not satisfied by this problem. So, in the Appendix at the end of this section, we verify Theorem 9.1 of [2] under the weaker condition $\Omega_3(\kappa, L)$, defined as

$$(\Omega_3(\kappa,L)) : \quad \left| \sum_{w \le p < z} \frac{\omega(p)}{p} \log p \quad - \kappa \log \frac{z}{w} \right| \le L \ , \quad (2 \le w \le z),$$

where L may depend on X (to be defined later). For the sake of simplicity, we also introduce the condition

$$(\Omega_0^*) : \qquad \frac{\omega(p)}{p} \log p \le A \qquad , \text{ for some absolute constant } A .$$

(Note that Ω_0^* is a consequence of $\Omega_2(\kappa, L)$ and is satisfied in most applications.) We shall also need the following conditions:

$$(\Omega_1) : \qquad 0 \le \frac{\omega(p)}{p} \le 1 - \frac{1}{A_1} \qquad \text{for some } A_1 \geqslant 1 ,$$

$$(R(1,\alpha)) : \qquad \sum_{d \le X^{\alpha}(\log X)^{-A_2}} \mu^2(d) \, 3^{\nu(d)} |R_d| \le A_3 \, X (\log X)^{-2} .$$

Then, we can prove the following:

THEOREM 2.[†] (Ω_1) ; $(\Omega_3(1, L))$; $(R(1,\alpha))$.

<u>If</u> u <u>and</u> v <u>are two numbers</u> (<u>independent of</u> X) <u>satisfying</u>

$$\alpha^{-1} < u < v \ ,$$

<u>and</u>

$$0 < \lambda < A_4 \qquad , \text{ \underline{for some absolute} } A_4 ,$$

<u>then</u>

$$W(\mathcal{A}; \mathfrak{P}, u, v, \lambda) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(X^{1/v}))=1}} \left\{ 1 - \lambda \sum_{\substack{X^{1/v} < p < X^{1/u} \\ p | a, \, p \in \mathfrak{P}}} \left(1 - u \frac{\log p}{\log X} \right) \right\} ,$$

[†]This is Theorem 9.1 of [2] with different conditions but same notation.

satisfies

$$W(\mathcal{A};\mathcal{P},v,u,\lambda) \geqslant X\, W(x^{\frac{1}{v}}) \cdot$$

$$\times \left\{ f(\alpha v) - \lambda \int_u^v F(v(\alpha - \tfrac{1}{t}))(1-\tfrac{u}{t}) \frac{dt}{t} - \frac{C_2 L}{(\log x)^{1/14}} \right\},$$

for X sufficiently large, with $C_2 = C_2(u,v;A_1,\dots,A_4,\alpha)$, and

$$\log x \gg L^{15}.$$

§7.2. Proof of the Theorem 1.

We now proceed with the task of proving that the conditions $\Omega_3(\kappa, L)$, Ω_0, Ω_1 and $R(1,\alpha)$ are satisfied. Throughout, $\mathcal{P}_0 = \{p$: set of primes$\}$. Put

$$P(z) = \prod_{p<z} p \,.$$

We need to estimate $|\mathcal{A}_d^K|$ for $d \mid P(z)$ where

$$(7.2.1)\qquad |\mathcal{A}_d^K| = \sum_{\substack{3\le p,q\le M \\ N\equiv p^2+q^2 \pmod K \\ \frac{N-p^2-q^2}{K}\equiv 0 \pmod d}} 1 ,$$

$$= \sum_{\substack{3\le p,q\le M \\ N\equiv p^2+q^2 \pmod{Kd} \\ (pq,Kd)=1}} 1 + \sum_{\substack{3\le p,q\le M \\ N\equiv p^2+q^2 \pmod{Kd} \\ (pq,Kd)>1}} 1 .$$

The second sum is

$$\ll \sum_{\substack{3 \le p \le M \\ p \mid Kd}} \sum_{\substack{3 \le q \le M \\ q \nmid Kd \\ N \equiv p^2+q^2 \pmod{Kd}}} 1 \quad + \sum_{\substack{3 \le p, q \le M \\ p \mid Kd,\ q \mid Kd \\ N \equiv p^2+q^2 \pmod{Kd}}} 1 ,$$

$$\ll \left(\frac{M}{Kd}+1\right) \sum_{p \mid Kd} 1 \quad + \left(\sum_{p \mid Kd} 1\right)^2 ,$$

$$\ll \frac{M}{d} \log Kd \quad + \log^2 Kd \quad = R_1(d) ,$$

say. Let $\rho(d)$ denote the number of solutions to:

$$1 \le u, v \le d \quad ; \quad (uv, d) = 1 \quad , \quad N \equiv u^2 + v^2 \pmod d .$$

The first sum in (7.2.1) is

$$\sum_{\substack{1 \le u, v \le Kd \\ N \equiv u^2+v^2 \pmod{Kd} \\ (uv, Kd) = 1}} \Bigg(\sum_{\substack{3 \le p \le M \\ p \equiv u \pmod{Kd}}} 1 \Bigg) \Bigg(\sum_{\substack{3 \le q \le M \\ q \equiv v \pmod{Kd}}} 1 \Bigg) ,$$

$$= \sum_{\substack{1 \le u, v \le Kd \\ N \equiv u^2+v^2 \pmod{Kd} \\ (uv, Kd) = 1}} \pi(M; Kd, u)\, \pi(M; Kd, v) .$$

We put $\pi(M; Kd, u) = \dfrac{L_i M}{\phi(Kd)} + E(u, Kd)$. Then, the above sum is

$$\frac{(L_i M)^2}{\phi^2(Kd)} \rho(Kd) \quad + \quad R_2(d) ,$$

where $R_2(d) = \eta_1(d) + \eta_2(d)$, defined as

$$\eta_1(d) = \frac{(Li\, M)}{\phi(Kd)} \sum_{\substack{u,v \ (\mathrm{mod}\ Kd) \\ N \equiv u^2+v^2 \ (\mathrm{mod}\ Kd) \\ (uv,\, Kd)=1}} \{E(u,Kd) + E(v,Kd)\} ,$$

and

$$\eta_2(d) = \sum_{\substack{u,v \ (\mathrm{mod}\ Kd) \\ N \equiv u^2+v^2 \ (\mathrm{mod}\ Kd) \\ (uv,\, Kd)=1}} E(u,Kd)\, E(v,Kd) .$$

We now define

$$X = (Li\, M)^2 ,$$

$$\omega_K(d) = \frac{d\ \rho(Kd)}{\phi^2(Kd)} ,$$

$$|R_d| = |R_1(d)| + |R_2(d)| .$$

We first verify $R(1,\alpha)$ for some α. Clearly

(7.2.2) $$\sum_{d \le X^{\alpha} (\log x)^{-A_2}} \mu^2(d)\, 3^{\nu(d)} |R_1(d)| \ll M \log x \sum_{d \le X^{\alpha} (\log x)^{-A_2}} \frac{\mu^2(d)}{d} 3^{\nu(d)}$$

$$+ \log^2 x \sum_{d \le X^{\alpha} (\log x)^{-A_2}} \mu^2(d)\, 3^{\nu(d)} .$$

Since d is square-free,

$$3^{\nu(d)} = \tau(d)^{\log_2 3} \leqslant \tau(d)^2 ,$$

and for sufficiently large A_2, the expression on the right of (7.2.2) is

$$\ll M(\log x)^5 + X^{\kappa}(\log x)^{-2} .$$

To estimate the contribution from $R_2(d)$ we follow the method of Greaves [1]. We give the proof, with $\alpha = \frac{1}{2}$, for completeness. First

$$\eta_1(d) = \frac{(\mathrm{Li}\, M)}{\phi(Kd)} \sum_{\substack{u,v \bmod Kd \\ N \equiv u^2+v^2 \pmod{Kd} \\ (uv, Kd)=1}} \{E(u,Kd) + E(v,Kd)\} ,$$

$$\ll \frac{(\mathrm{Li}\, M)}{\phi(d)} \sum_{\substack{c \pmod d \\ (c,d)=1}} |E(c,d)|$$

since K is a bounded absolute constant. For any $B>0$, this is

$$\ll \frac{(\mathrm{Li}\, M)}{\phi(d)} \sum_{\substack{c \pmod d \\ (c,d)=1}} \left\{ \frac{\mathrm{Li}\, M}{\phi(d)}(\log x)^{-B} + \frac{(\log x)^B \phi(d) E^2(c,d)}{\mathrm{Li}\, M} \right\} ,$$

$$\ll \frac{(\mathrm{Li}\, M)^2}{\phi(d)}(\log x)^{-B} + (\log x)^B \sum_{\substack{c \pmod d \\ (c,d)=1}} E^2(c,d) ,$$

$$\ll \frac{X}{\phi(d)}(\log x)^{-B} + (\log x)^B \sum_{\substack{c \pmod d \\ (c,d)=1}} E^2(c,d).$$

Also

$$\eta_2(d) \leq \frac{1}{2} \sum_{\substack{u,v \bmod Kd \\ N \equiv u^2+v^2 \pmod{Kd} \\ (uv,Kd)=1}} \{E^2(u,Kd) + E^2(v,Kd)\} ,$$

$$\ll \sum_{\substack{u \bmod Kd \\ (u,Kd)=1}} E^2(u,Kd)$$

$$\ll \sum_{\substack{u \bmod d \\ (u,d)=1}} E^2(u,d) .$$

Hence, we have

$$R_2(d) \ll \frac{x}{\phi(d)} (\log x)^{-B} + (\log x)^B \sum_{\substack{u \bmod d \\ (u,d)=1}} E^2(u,d) .$$

Consequently

$$\sum_{d \leq x^\alpha (\log x)^{-A_2}} \mu^2(d)\, 3^{\nu(d)} |R_2(d)| \ll x (\log x)^{-B} \sum_{d \leq x^\alpha (\log x)^{-A_2}} \frac{\tau^2(d)}{\phi(d)}$$

$$+ (\log x)^B \sum_{d \leq x^\alpha (\log x)^{-A_2}} \mu^2(d)\, \tau^2(d) \sum_{\substack{c \bmod d \\ (c,d)=1}} E^2(c,d) .$$

Using the Brun-Titchmarsh theorem in the form

$$\pi(x; k, l) \ll \frac{x}{\phi(k) \log \frac{x}{k}} ,$$

valid for $1 \leq k < x$ and $(k,l)=1$, we have

$$\sum_{d \le X^{\alpha}(\log X)^{-A_2}} \tau^2(d) \sum_{\substack{c \,(\mathrm{mod}\, d)\\ (c,d)=1}} E^2(c,d) \ll \Big(\sum_{d \le X^{\alpha}(\log X)^{-A_2}} \phi(d)\, \tau^4(d) \max_{(c,d)=1} |E(c,d)|^2 \Big)^{\frac{1}{2}}$$

$$\times \Big(\sum_{d \le X^{\alpha}(\log X)^{-A_2}} \sum_{\substack{c \,(\mathrm{mod}\, d)\\ (c,d)=1}} E^2(c,d) \Big)^{\frac{1}{2}},$$

and so is

$$\ll \Big(M^2 \sum_{d \le X^{\alpha}(\log X)^{-A_2}} \tau^4(d)/\phi(d) \Big)^{\frac{1}{2}} \Big(\sum_{d \le X^{\alpha}(\log X)^{-A_2}} \sum_{\substack{c \,(\mathrm{mod}\, d)\\ (c,d)=1}} E^2(c,d) \Big)^{\frac{1}{2}}.$$

The inequality of Barban-Davenport-Halberstam in the form

$$\sum_{l \le x(\log x)^{-A_5}} \sum_{\substack{c \,(\mathrm{mod}\, l)\\ (c,l)=1}} E^2(c,l,x) \ll x^2 (\log x)^{5-A_5},$$

gives us

$$\sum_{d \le X^{\frac{1}{2}}(\log X)^{-A_2}} \mu^2(d)\, 3^{\nu(d)} |R_2(d)| \ll X (\log X)^{-B+5}$$

$$+ X (\log X)^{B+14-\frac{1}{2}A_2},$$

and a suitable choice of A_2 and B gives us $R(1,\alpha)$ with $\alpha = \frac{1}{2}$.

§7.3. <u>Evaluation of $\rho(d)$</u>.

We shall consider the properties of the function $\rho(d)$ which will be required to verify Ω_1, Ω_0^* and $\Omega_3(1,L)$. The function $\rho(d)$ is a multiplicative function of d : for if $d = d_1 d_2$ such that $(d_1, d_2) = 1$, we have

$$\rho(d_1 d_2) = \sum_{\substack{u,v \pmod{d_1 d_2} \\ (uv, d_1 d_2) = 1 \\ N \equiv u^2 + v^2 \pmod{d_1 d_2}}} 1 = \sum_{\substack{u_1, v_1 \pmod{d_1} \\ (u_1 v_1, d_1) = 1}} \sum_{\substack{u_2, v_2 \pmod{d_2} \\ (u_2 v_2, d_2) = 1 \\ N \equiv (u_1^2 + v_1^2) d_2^2 + (u_2^2 + v_2^2) d_1^2 \\ \pmod{d_1 d_2},}} 1 ,$$

where we have put $u = u_1 d_2 + u_2 d_1$; $v = v_1 d_2 + v_2 d_1$. Since $(d_1, d_2) = 1$,

$$\rho(d_1 d_2) = \sum_{\substack{u_1, v_1 \pmod{d_1} \\ (u_1 v_1, d_1) = 1}} \sum_{\substack{u_2, v_2 \pmod{d_2} \\ (u_2 v_2, d_2) = 1}} 1 ,$$
$$N \equiv u_1^2 + v_1^2 \pmod{d_1}$$
$$N \equiv u_2^2 + v_2^2 \pmod{d_2}$$

$$= \rho(d_1)\, \rho(d_2) .$$

We next consider the properties of $\rho(p)$ where p is a prime:

$$\rho(p) = \frac{1}{p} \sum_{a=1}^{p} B^2(a,p)\, e\left(-\frac{a}{p} n\right) ,$$

where

$$B(a,n) = \sum_{\substack{r=1 \\ (r,n)=1}}^{n} e\left(\frac{a}{n} r^2\right) .$$

So

(7.3.1) $$\rho(p) = \frac{\phi^2(p)}{p} + \frac{1}{p} \sum_{a=1}^{p-1} B^2(a,p)\, e\left(-\frac{a}{p} N\right) .$$

(i) Suppose $p \mid N$. Then the second sum in (7.3.1) is

$$\sum_{a=1}^{p-1} B^2(a,p) = \sum_{a=1}^{p-1} \left(S(a,p) - 1\right)^2 , \tag{7.3.2}$$

where

$$S(a,p) = \sum_{r=1}^{p} e\left(\frac{a}{p} r^2\right) .$$

We use the well-known properties of $S(a,p)$ namely:

(a) $$S(a,p) = \left(\frac{a}{p}\right)_L S(1,p) , \quad \text{if} \quad p \nmid a ,$$

(b) $$S(1,p) = \begin{cases} \sqrt{p} & , \; p \equiv 1 \pmod 4 , \\ 0 & , \; p = 2 , \\ i\sqrt{p} & , \; p \equiv 3 \pmod 4 . \end{cases}$$

Hence (7.3.2) is

$$\sum_{a=1}^{p-1} \left\{ S^2(1,p) + 1 - 2\left(\frac{a}{p}\right)_L S(1,p) \right\} = \phi(p) \left\{ S^2(1,p) + 1 \right\}$$

since

$$\sum_{a=1}^{p-1} \left(\frac{a}{p}\right)_L = 0 .$$

(ii) Suppose $p \nmid N$. Then the sum in (7.3.1) is

$$\left\{ S^2(1,p) + 1 \right\} \sum_{a=1}^{p-1} e\left(-\frac{aN}{p}\right) - 2\, S(1,p) \sum_{a=1}^{p-1} \left(\frac{a}{p}\right)_L e\left(-\frac{aN}{p}\right) .$$

Since

$$\sum_{a=1}^{p-1} e\left(-\frac{aN}{p}\right) = -1 ,$$

and

$$\sum_{a=1}^{p-1} \left(\frac{a}{p}\right)_L e\left(-\frac{aN}{p}\right) = \left(\frac{N}{p}\right)_L \overline{S(1,p)} ,$$

we have

$$\rho(p) = \begin{cases} 2(p-1) , & p \mid N , \quad p \equiv 1 \pmod 4 , \\ 0 , & p \mid N , \quad p \equiv 3 \pmod 4 , \\ 1 , & p = 2 , \quad 2 \mid N , \\ \dfrac{(p-1)^2 - 1 - p\delta_N(p)}{p} , & p \nmid N , \quad p \equiv 1 \pmod 4 , \\ \dfrac{(p-1)^2 + 2p - 1 - p\delta_N(p)}{p} , & p \nmid N , \quad p \equiv 3 \pmod 4 , \\ 0 , & p = 2 , \quad 2 \nmid N \end{cases}$$

where

$$\delta_N(p) = 1 + 2\left(\frac{N}{p}\right)_L = \begin{cases} 3 , & N \text{ is Quadratic residue mod } p , \\ -1 , & \text{otherwise} \end{cases}$$

We next note that

$$(7.3.3) \qquad w_K(p) = \frac{\rho(K)}{\phi^2(K)} w_1(p) , \qquad p \nmid K ,$$

$$= \frac{\omega_1(p^{a+1})\, \omega_1(Kp^{-a})}{K} , \qquad p^a \| K .$$

(we have used the multiplicative structure of $\rho(\cdot)$ here). We shall need explicit estimates for $\rho(2)$, $\rho(3)$, $\rho(4)$, $\rho(8)$, $\rho(9)$ and $\rho(24)$ later on. The first two are trivial. We give the last four:

$$\rho(4) = \begin{cases} 4 & , \quad N \equiv 2 \pmod 4, \\ 0 & , \quad \text{otherwise}, \end{cases}$$

$$\rho(8) = \begin{cases} 20 & , \quad N \equiv 2 \pmod 3, \\ 0 & , \quad \text{otherwise}, \end{cases}$$

$$\rho(9) = \begin{cases} 14 & , \quad N \equiv 1, 2 \text{ or } 4 \pmod 9, \\ 0 & , \quad \text{otherwise}, \end{cases}$$

$$\rho(24) = \begin{cases} 36 & , \quad N \equiv 2 \text{ or } 10 \pmod{16}, \\ 0 & , \quad \text{otherwise} \end{cases}$$

§7.4. Verification of Ω_1 and $\Omega_3(1,L)$.

We now verify conditions Ω_1 and $\Omega_3(1,L)$. We consider the six sets $\mathcal{A}^1$, $\mathcal{A}^3$, $\mathcal{A}^2$, $\mathcal{A}^6$, $\mathcal{A}^8$ and $\mathcal{A}^{24}$.

Case (i).

N is not a quadratic residue modulo 3. Hence

$$\omega(p) = \begin{cases} 2 + \frac{2}{p-1} & , \quad p \mid n, \quad p \equiv 1 \pmod 4, \\ 0 & , \quad p \mid N, \quad p \equiv 3 \pmod 4, \\ 1 - \frac{\delta_N(p)}{p-1} - \frac{\delta_N(p)+1}{(p-1)^2} & , \quad p \nmid N, \quad p \equiv 1 \pmod 4, \\ 1 + \frac{2-\delta_N(p)}{p-1} - \frac{\delta_N(p)-1}{(p-1)^2} & , \quad p \nmid N, \quad p \equiv 3 \pmod 4, \\ 0 & , \quad 2 \nmid N, \quad p = 2. \end{cases}$$

Ω_1 is satisfied by all $p \neq 3$. When $p = 3$, if $3 \mid N$, then $\omega(3) = 0$. If $3 \nmid N$ and N is a quadratic residue modulo 3, then $\omega_1(3) < 1$. Hence, Ω_1 is satisfied for all p. Also

$$\frac{\omega_1(p)}{p} = \begin{cases} \frac{1}{p} + O\left(\frac{1}{p^2}\right) & , \text{ if } p \nmid N , \\ \frac{2}{p} + O\left(\frac{1}{p^2}\right) & , p \mid N , \ p \equiv 1 \ (\mathrm{mod}\ 4) , \\ 0 & , p \nmid N , \ p \equiv 3 \ (\mathrm{mod}\ 4) . \end{cases}$$

Hence

$$\sum_{w \le p < z} \frac{\omega_1(p)}{p} \log p = \sum_{w \le p < z} \frac{\log p}{p} + \sum_{\substack{w \le p < z \\ p \mid N \\ p \equiv 1 \ (\mathrm{mod}\ 4)}} \frac{\log p}{p} - \sum_{\substack{w \le p < z \\ p \mid N \\ p \equiv 3 \ (\mathrm{mod}\ 4)}} \frac{\log p}{p}$$

$$+ O(1) .$$

Hence

$$\left| \sum_{w \le p < z} \frac{\omega_1(p) \log p}{p} - \log \frac{z}{w} \right| \le \sum_{\substack{w \le p < z \\ p \mid N}} \frac{\log p}{p} + O(1) ,$$

$$\le C \log\log 3N ,$$

say, where C is an absolute constant. We put $L = C \log\log N$, and note that the above argument is valid for all the six sets $\mathcal{A}^i$ to verify $\Omega_3(1, L)$. We need only verify Ω_1 for the rest of the cases.

Case (ii).

Clearly, we need only verify that when $p=3$ and $3 \nmid N$, then $\frac{1}{3}\omega_3(3) \le 1 + A_1^{-1}$ for some $A_1 \gg 1$. We have, from (7.3.3)

$$\frac{1}{3} \omega_3(3) = \frac{\omega_1(9)}{9} = \frac{\rho(9)}{\phi^2(9)} \le \frac{14}{36} .$$

Case iii).

We now have to consider the case $p = 2 \mid N$, while case (i) provides the rest of the verification for Ω_1. Now, from (7.3.3)

$$\frac{1}{2}\varpi_2(2) = \frac{\omega_1(4)}{4} = \frac{\rho(4)}{\phi^2(4)} = 0 ,$$

since $N \equiv 0 \pmod 4$.

Case (iv).

Here we consider the case $p=3$ and $p=2$ when $3 \nmid N$. We have

$$\frac{\omega_6(3)}{3} = \frac{\omega_1(9)\,\omega_1(2)}{18} = \frac{\rho_1(9)\,\rho_1(2)}{\phi^2(9)\,\phi^2(2)} < \frac{14}{36} ,$$

and

$$\frac{\omega_6(2)}{2} = \frac{\omega_1(4)\omega_1(3)}{12} = \frac{\rho(4)\,\rho(3)}{\phi^2(4)\,\phi^2(3)} = 0$$

since $N \equiv 0 \pmod 4$,

Case (v).

Here we consider the case $p=2$. Hence, by (7.3.3)

$$\frac{\omega_8(2)}{2} = \frac{\omega_1(16)}{16} = \frac{\rho(16)}{\phi^2(16)} \leq \frac{36}{64} .$$

Case (vi).

It remains to consider both the cases 3 and 2 when $3 \nmid N$. Then

$$\frac{\omega_{24}(3)}{3} = \frac{\omega_1(9)\,\omega_1(8)}{72} = \frac{\rho(9)}{\phi^2(9)}\,\frac{\rho(8)}{\phi^2(8)} \leq \frac{35}{72} ,$$

and

$$\frac{\omega_{24}(2)}{2} = \frac{\omega_1(16)\omega_1(3)}{48} = \frac{\rho(16)\,\rho(3)}{\phi^2(16)\,\phi^2(3)} \leq \frac{36}{64} .$$

Hence Ω_1 is satisfied for all p, for each of the individual sets.

§7.5. Completion of Proof of Theorem 1

We now complete the proof of Theorem 1, with the help of Theorem 2. We define

$$W_K(z) = \prod_{p<z}\left(1 - \frac{\omega_K(p)}{p}\right),$$

and

$$\mathfrak{S}_K(N) = \prod_p \left(1 - \frac{\omega_K(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-1}.$$

Now, by Lemma 5.3 in [2] (see Appendix)

$$W_K(z) = \mathfrak{S}_K(N)\frac{e^{-\gamma}}{\log z}\left(1 + O\left(\frac{1}{\log z}\right)\right)\left(\exp\left(O\left(\frac{L}{\log z}\right)\right)\right),$$

and using $z = X^{1/8}$, it follows that

$$(7.5.1) \qquad W_K(X^{1/8}) = 8\,\mathfrak{S}_K(N)\frac{e^{-\gamma}}{\log X}\left(1 + O\left(\frac{\log\log N}{\log X}\right)\right).$$

Let us choose

$$u = 8/3 \quad , \quad v = 8.$$

Then, by Lemma 9.1 of [2]

$$f(4) - \lambda\int_{8/3}^{8} F\left(4 - \frac{8}{t}\right)\left(1 - \frac{8}{3t}\right)\frac{dt}{t} = \frac{e^{\gamma}}{2}\left(1 - \frac{2}{3}\lambda\right)\log 3.$$

Hence, by Theorem 2, we have

$$(7.5.2) \qquad W(\mathcal{A}^K; \mathcal{P}_0, 8, \tfrac{8}{3}, \lambda)$$

$$\geqslant \frac{4\,\mathfrak{S}_K(N)\,X}{\log X}\left\{\left(1 - \frac{2}{3}\lambda\right)\log 3 - \frac{c_5 \log\log X}{(\log X)^{1/14}}\right\}.$$

We just note that $\mathfrak{S}_K(N) > 0$ but there is no real need to work them out in full (though, we have all the necessary information to do so). As an example, we give

$$\mathfrak{S}_1(N) = 2 \prod_{\substack{2<p \mid N \\ p \equiv 1 \pmod 4}} \left(1 - \frac{3}{p}\right) \prod_{\substack{p \nmid N \\ N\, QR \bmod p \\ p \equiv 1 \pmod 4}} \left(1 + \frac{3p+1}{(p-1)^3}\right) \prod_{\substack{p \nmid N \\ N\, QR \pmod p \\ p \equiv 3 \pmod 4}} \left(1 + \frac{p+5}{(p-1)^3}\right) \times$$

$$\times \prod_{\substack{p \nmid N,\, N\, QNR \bmod p, \\ p \equiv 1 \bmod 4}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \nmid N,\ p \equiv 3 \pmod 4 \\ N\, QNR \pmod p}} \left(1 - \frac{3p-1}{(p-1)^3}\right) .$$

We now interpret the result given by Theorem 2. We shall work with the general set $\mathcal{A}^K$ as the argument is valid for all the values of K we are considering.

We first discard all those numbers from $\mathcal{A}^K$ which are not square-free with respect to the primes in the interval $[x^{\frac{1}{8}}, x^{3/8}]$. The number of such elements is clearly at most

$$\sum_{x^{1/8} < p_1 < x^{3/8}} \sum_{\substack{p, q \le M \\ N \equiv p^2 + q^2 \pmod{p_1^2}}} 1 \ll \sum_{x^{1/8} < p_1 < x^{3/8}} \left(M^2 p_1^{-2} + M\right) \ll x^{7/8 + \varepsilon} ,$$

say. We may absorb this estimate in the error term in (7.5.2). The remaining elements of $\mathcal{A}^K$ are square-free with respect to the primes

$$x^{1/8} < p_1 < x^{3/8} .$$

Let b denote a non-excluded member of $\mathcal{A}^K$ which gives a positive contribution to the weight

$$1 - \lambda \sum_{\substack{x^{1/8} \le p_1 \le x^{3/8}, \\ p_1 \mid b,\ p_1 \in \mathcal{P}_0}} \left(1 - \frac{8}{3} \frac{\log p_1}{\log x}\right) .$$

Clearly, b does not have any prime divisors less than $x^{1/8}$. Hence, the weight of b is at most

$$1 - \lambda\left(\Omega(b) - \frac{8}{3} \frac{\log |b|}{\log x}\right) .$$

Suppose $\lambda > 3/4$. Now, if $\Omega(b) \geq 4$, the weight of b would be at most

$$1 - \lambda\left(4 - \frac{8}{3}\ \frac{\log 3N}{\log x}\right),$$

(where we have used $|b| \leq 3N$). If N is sufficiently large, this would not be positive, which contradicts the definition of b. Hence, it follows that

$$|\{a \in \mathcal{A}^K : a = Q_3\}| \geq W(\mathcal{A}^K; \mathcal{P}_0, 8, 8/3, \lambda) + O(X^{7/8+\varepsilon}),$$

provided one can find a $\lambda > 3/4$. But since

$$\left(1 - \frac{2}{3}\lambda\right) \log 3 > \frac{13}{24},$$

when $\lambda = \frac{3}{4}$, we have shown that such a λ exists and moreover,

$$W(\mathcal{A}^K; \mathcal{P}_0, 8, 8/3, \lambda) \geq \frac{13}{6}\, \mathfrak{S}_K(N) \frac{X}{\log X},$$

for sufficiently large X. Theorem 1 now follows.

Remark. Halberstam and Richert remarks on pp. 252 of [2] that an improvement of α in Bombieri's Theorem would lead to the replacement of the Q_3's by Q_2's in the quasi-Goldbach problem. For our purposes, we are not so fortunate as Montgomery in [3] has shown that

$$\sum_{q \leq Q} \sum_{\substack{c \bmod q \\ (c,q)=1}} \left(\psi(x;q,c) - \frac{\operatorname{Li} x}{\phi(q)}\right)^2 = Qx\log x + \frac{\zeta(2)\zeta(3)}{\zeta(6)} x^2 \log Qx^{-1}$$

$$- Qx + A_7 x^2 + O(Qx(\log x)^{-A_8}),$$

for $Q \geq x$.

Hence, for our purposes, $Q = x(\log x)^{-B}$ seems best possible and this is simply $\alpha = \frac{1}{2}$. Consequentially, a new argument is required (or perhaps the use of a different system of weights specially designed for the problem), for a further qualitative requirement.

§7.6. Appendix

We shall state modified Lemmas and Theorems which appear in [2], and these shall be used to prove Theorem 2. Throughout, we shall assume conditions Ω^*, Ω_1, $\Omega_3(1, L)$, $R(1, \alpha)$.

LEMMA* 5.2. If $2 \leq w \leq z$,

$$\left| \sum_{w \leq p < z} \frac{\omega(p)}{p} - \kappa \log \frac{\log z}{\log w} \right| \leq \frac{L}{\log w},$$

and

$$\left| \sum_{w \leq p < z} \frac{g(p)}{p^{s+1}} - \kappa \sum_{w \leq p < z} \frac{1}{p^{s+1}} \right| \ll \frac{L + \kappa}{\log w},$$

uniformly in s.

Proof. The first assertion follows directly as in [2]. For the second assertion, we put $s = 0$ and use

$$g(p) = \frac{\omega(p)}{p} + \frac{\omega(p)}{p} g(p).$$

Then

$$\sum_{w \leq p < z} \frac{\omega(p)}{p} g(p) \leq A_1 \sum_{w \leq p < z} \frac{\omega^2(p)}{p^2} = A_1 \sum_{w \leq p < z} \frac{\omega(p)}{p \log p} \cdot \frac{\omega(p)}{p} \log p,$$

$$\leq A_1 A \sum_{w \leq p < z} \frac{\omega(p)}{p \log p}.$$

Partial summation gives

$$\sum_{w \leq p < z} \frac{\omega(p)}{p \log p} = \kappa \frac{\log(z/w)}{(\log z)(\log w)} + O\left(\frac{L}{(\log w)(\log z)}\right).$$

The result then follows for $s = 0$. The general case then follows by partial summation.

LEMMA* 5.3. If $2 \leq w \leq z$

$$\prod_{p} \left(1 + \frac{g(p)}{p^{s}}\right)\left(1 - \frac{1}{p^{s+1}}\right)^{\kappa} = \exp\left(O\left(\frac{L}{\log w}\right)\right),$$

$$\frac{W(w)}{W(z)} = \left(\frac{\log z}{\log w}\right)^{\kappa} \exp\left(O\left(\frac{L}{\log w}\right)\right),$$

$$W(z) = \prod_{p}\left(1 - \frac{\omega(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-\kappa} \frac{e^{-\gamma\kappa}}{(\log z)^{\kappa}} \exp\left(O\left(\frac{L}{\log w}\right)\right).$$

Proof. This is exactly as in [2] with Lemma* 5.2 replacing Lemma 5.2.

LEMMA* 5.4. Suppose that $\log z > 3L$. Then

$$\frac{1}{G(z)} = W(z)\, e^{\kappa\gamma}\, \Gamma(\kappa+1)\left\{1 + O\left(\frac{L}{\log z}\right)\right\}.$$

Proof. The argument used in the proof of Lemma 5.4 is valid until one reaches line 3 of pp. 149. The sum concerned is

$$O\left(\sum_{\sqrt{x/d} \leq p < x/d} g(p)\right).$$

Using

$$g(p) \leq A_1 \frac{\omega(p)}{p},$$

and $\Omega_3(\kappa, L)$, the sum is

$$O\left(\sum_{\sqrt{x/d} \leq p < x/d} \frac{\omega(p)}{p}\right) = O\left(\frac{L}{\log x/d}\right) = O(L).$$

Hence

$$\sum_{\substack{d<x \\ d|P(z)}} g(d) \log d = \sum_{\substack{\frac{x}{z} \leq d < x \\ d|P(z)}} g(d) \left(\kappa \log \frac{x}{d} + O(L) \right)$$

$$+ \sum_{\substack{d<x/z \\ d|P(z)}} g(d) \left(\kappa \log z + O(L) \right) + O(L\, G(x,z)) .$$

But this is precisely the line after it. The only other point in the proof we need to justify is (3.11), which states

$$G(y) \ll (\log y)^{\kappa} .$$

This follows from Lemma 4.1, with $x = z$, $A_2 = L$ and our assumption that $L < \frac{1}{8} \log z$. Application of Lemma* 5.3 gives the result. There is no other change needed in the proof.

LEMMA* 6.1. Suppose $\log z \gg \max(L^2, L\tau^{2\kappa+1})$. Then

$$\frac{1}{G(\xi,z)} = W(z) \left\{ \frac{1}{\sigma_{\kappa}(2\tau)} + O\left(\frac{L}{\log z} \left(\tau^{2\kappa+1} + \tau^{-\kappa+1} \right) \right) \right\},$$

if $z \leq \xi$ and

$$\tau = \frac{\log \xi}{\log z} .$$

<u>Proof</u>. There is essentially nothing new to add as Lemma 6.1 is proved under the condition $\log z \gg L\tau^{2\kappa+1}$. Moreover, the variable w appearing in (4.7) is perfectly safe since it is always chosen to satisfy

$$w \gg z .$$

THEOREM* 6.3. Suppose $\log z \gg \max(L^2, L\tau^{2\kappa+1})$. Then

$$S(\mathcal{A}_q, \mathcal{P}, z) \leq \frac{\omega(q)}{q} X W(z) \left\{ \sigma_{\kappa}^{-1}(2\tau) + O\left(\frac{L}{\log z}(\tau^{-\kappa-1} + \tau^{2\kappa+1})\right) + \sum_{\substack{d \mid P(z) \\ d < \xi^2}} 3^{\nu(d)} |R_{dq}|,\right.$$

for $\tau > 0$ and $z < \xi$.

<u>Proof</u>. This follows from Theorem 6.1 and Lemma* 6.1. Note that Theorem 6.1 needs no verification.

LEMMA* 7.2. Suppose $\log z_1 \gg L^2$, and

$$z_1 \leq z_2 \leq \xi \quad , \quad z \geq z_2$$

Let $\Psi(t)$ be a non-negative, monotonic and continuous function for $t \geq 1$ and define

$$M = \max_{z_1 \leq w \leq z_2} \Psi\left(\frac{\xi^2/w}{\log w}\right).$$

Then

$$\sum_{z_1 \leq p \leq z_2} \frac{\omega(p)}{p} W(p) \Psi\left(\frac{\log \xi^2/p}{\log p}\right) = \kappa W(z)\left(\frac{\log z}{\log \xi^2}\right)^{\kappa} \int_{\frac{\log \xi^2}{\log z_2}}^{\frac{\log \xi^2}{\log z_1}} t^{\kappa-1} \Psi(t-1)\, dt$$

$$+ O\left(\frac{L M W(z) (\log z)^{\kappa}}{(\log z_1)^{\kappa+1}}\right).$$

THEOREM* 6.1. We have for any $\xi > 1$, that

$$S(\mathcal{A}_q, \mathcal{P}, z) \leq \frac{\omega(q)}{q} \frac{X}{G(\xi, z)} + \sum_{\substack{d \mid P(z) \\ d < \xi^2}} 3^{\nu(d)} |R_{dq}|$$

LEMMA* 4.1. Suppose that $\log z > L^2$. Then

$$\frac{1}{G(x, z)} \leq W(z)\left\{1 + O\left(\exp\left\{-\lambda \frac{\log x}{\log z} + \frac{2\kappa}{\lambda} + \frac{L}{\log z}\right\}\right)\right\}.$$

Proof. The proof of this follows Lemma 4.1 using

$$c_2 = 2\mathcal{K}e + \frac{e}{3} + \log 2 .$$

THEOREM* 6.2. Suppose $\log z > L^2$. If $\xi \geq z$, we have

$$S(\mathcal{A}_q; \mathcal{P}, z) \leq \frac{\omega(q)}{q} X W(z) \Big\{ 1 + O\Big(\exp\{-\tau(\log \tau + 2)\}\Big)\Big\}$$

$$+ \sum_{\substack{d|P(z) \\ d<\xi^2}} 3^{\nu(d)} |R_{qd}|$$

where

$$\tau = \frac{\log \xi}{\log z} .$$

Proof. For $\tau > z$, we use the proof given on pp. 192, Theorem 6.2. So, assume that $\tau \leq z$. Then Theorem* 6.1 and Lemma* 4.1 give us

$$S(\mathcal{A}_q, \mathcal{P}, z) \leq \frac{\omega(q)}{q} X W(z) \Big\{ 1 + O\Big(\exp\big(-\lambda\tau + \big(\frac{2\mathcal{K}}{\lambda} + \frac{L}{\log z}\big) e^{\lambda}\big)\Big)\Big\} .$$

Assume $\log \tau > e^{10}$. Otherwise, we get a result of the quality of Theorem 4.1 (which one obtains on applying Lemma* 4.1). Now put

$$\lambda = \log \tau + \tfrac{1}{2} \log\log \tau .$$

Then

$$\Big(\frac{2}{\lambda} + \frac{L}{\log z}\Big) e^{\lambda} \leq 3\tau .$$

The result then follows.

THEOREM* 7.1. Suppose $\log z > L^3$ and $\xi \geq z$. Then we have

$$S(\mathcal{A}_q, \mathcal{P}, z) = \frac{\omega(q)}{q} X W(z) \Big\{ 1 + O\big(\exp(-2\tau)\big)\Big\}$$

$$+ \theta \sum_{\substack{d<\xi^2 \\ d|P(z)}} 3^{\nu(d)} |R_{dq}| ,$$

where

$$\tau = \frac{\log \xi}{\log z} ,$$

and $|\theta| \leq 2$.

Proof. As in the proof of Theorem 7.1, we need only prove the lower bound. By Lemma 7.1, with $\log z_1 = 2L^2$, we have

$$S(\mathcal{A}_q ; \mathcal{R}, z) - \frac{\omega(q)}{q} X W(z) = S(\mathcal{A}_q ; \mathcal{R}, z_1) - \frac{\omega(q)}{q} X W(z_1)$$

$$- \sum_{\substack{z_1 \leq p < z \\ p \in \mathcal{R}}} \left\{ S(\mathcal{A}_{pq}, \mathcal{R}, p) - \frac{\omega(pq)}{pq} X W(p) \right\} .$$

For the sum, we note that we may apply Theorem* 6.2 with p replacing z since

$$\log p \geq \log z_1 > L^2 .$$

The argument given in the proof of Theorem 7.1 for the estimation of the sum is now valid because we can now use our previous results. It remains to estimate

$$S(\mathcal{A}_q ; \mathcal{R}, z_1) - \frac{\omega(q)}{q} X W(z_1) .$$

By Theorem 6.2, this is

$$\ll \frac{\omega(q)}{q} X W(z_1) \exp(-2\tau_1) + \sum_{\substack{d < \xi^2 \\ d | P(z)}} 3^{\nu(d)} |R_{dq}| ,$$

where

$$\tau_1 = \frac{\log \xi}{\log z_1} = \frac{\log \xi}{2L^2} .$$

Also, by Lemma* 5.3

$$W(z_1) = W(z) \left(\frac{\log \xi}{\log z_1} \right)^{\kappa} \exp\left(O\left(\frac{L}{\log z_1} \right) \right) .$$

It therefore remains to show that

$$\left(\frac{\log z}{\log z_1} \right)^{\kappa} \exp\left(-2\tau_1 + O\left(\frac{L}{\log z_1} \right) \right) \ll \exp(-2\tau) .$$

Since $L(\log z_1)^{-1} \ll L^{-1}$, and τ_1 is large, we may discard the error term. Hence, the above is

$$\ll \quad \exp\left(-2\tau_1 + \kappa \log \frac{\log z}{\log z_1} \right),$$

$$\ll \quad \exp\left(-2 \frac{\log \xi}{\log z} \cdot \frac{\log z}{\log z_1} + \kappa \log \frac{\log z}{\log z_1} \right),$$

and since $\frac{\log z}{\log z_1} > 1$, the above expression is

$$\ll \quad \exp(-2\tau)$$

The result then follows with the same proof as in Theorem 7.1.

<u>The Linear Sieve: $\kappa = 1$</u>

LEMMA* 8.1. Suppose $\log z_1 \gg L^2$ and $z_1 \le z \le \xi$. Then, for any v,

$$W(z)\ \phi_v\left(\frac{\log \xi^2}{\log z}\right) = W(z_1)\ \phi_v\left(\frac{\log \xi^2}{\log z}\right)$$

$$- \sum_{z_1 < p \le z} \frac{\omega(p)}{p} W(p)\ \phi_{v+1}\left(\frac{\log \xi^2/p}{\log p}\right) + O\left(\frac{W(z) L \log z}{(\log z_1)^2}\right).$$

<u>Proof</u>. One applies Lemma* 7.2 in place of Lemma 7.2 and the result follows.

LEMMA* 8.2. Let $\log z_1 \gg L^2$. Also suppose that

$$z \le \xi^2, \qquad 2 \le z_1 \le \mu,$$

where

$$\mu = \min(z, \xi^{2/3})$$

Then

$$\sum_{z_1 \le p < \mu} \frac{\omega(p)}{p} W(p) \exp\left(-\frac{\log \xi^2/p}{\log p}\right) \le \frac{W(z)\, e}{3} \exp\left\{-\frac{\log \xi^2}{\log z}\right\}\left(1 + O\left(\frac{L \log \xi}{\log^2 z_1}\right)\right).$$

Proof. Again, replace Lemma 7.2 by Lemma* 7.2.

THEOREM* 8.2. Suppose that $\log z_1 \gg L^2$ and $z_1 \leq z \leq \xi$. Set

$$\xi_j^2 = \frac{\xi^2}{p_1 \cdots p_j} \quad , \quad j = 1, 2, \cdots .$$

Then, the expression holds.

Proof. The first point to note is that the various substitutions, namely p taking the place of z , is perfectly valid since $p > z_1$ and so

$$\log p > \log z_1 \gg L^2 ,$$

a condition needed to be satisfied by $\log z_1$ for us to use the previous results above.

THEOREM* 8.3. Suppose that

$$\xi \gg z \quad , \quad \log \xi \gg L^{15} .$$

Then

$$S(\mathcal{A}_q, \mathcal{R}, z) \leq \frac{\omega(q)}{q} \times W(z) \left\{ F\left(\frac{\log \xi^2}{\log z} \right) + O\left(\frac{L}{(\log \xi)^{1/14}} \right) \right\}$$

$$+ \sum_{\substack{n \leq \xi^2 \\ n | P(z)}} 3^{\nu(n)} |R_{qn}| ,$$

and

$$S(\mathcal{A}_q, \mathcal{R}, z) \geq \frac{\omega(q)}{q} \times W(z) \left\{ f\left(\frac{\log \xi^2}{\log z} \right) + O\left(\frac{L}{(\log \xi)^{1/14}} \right) \right\}$$

$$- \sum_{\substack{n < \xi^2 \\ n | P(z)}} 3^{\nu(n)} |R_{nq}| .$$

Proof. Our assumptions above allows us to choose z_1 as in Theorem 8.3. Moreover the use of Theorem* 7.1 allows us the choice

$$z_1 \leq z \leq \xi .$$

The proof now follows exactly as that of Theorem 8.3. One should note that the various interchanges occurring at the bottom of page 232 are valid since

$$\log p_i \geq \log z_1 = (\log \xi)^{7/10} > L^3 .$$

Consequently, we may use our previous results.

THEOREM* 8.4. For $z \leq X$ and $\log z \gg L^{15}$, we have

$$S(\mathcal{A};\mathcal{P},z) \leq X W(z) \left\{ F\left(\alpha \frac{\log X}{\log z}\right) + B_{21} \frac{L}{(\log X)^{1/14}} \right\} ,$$

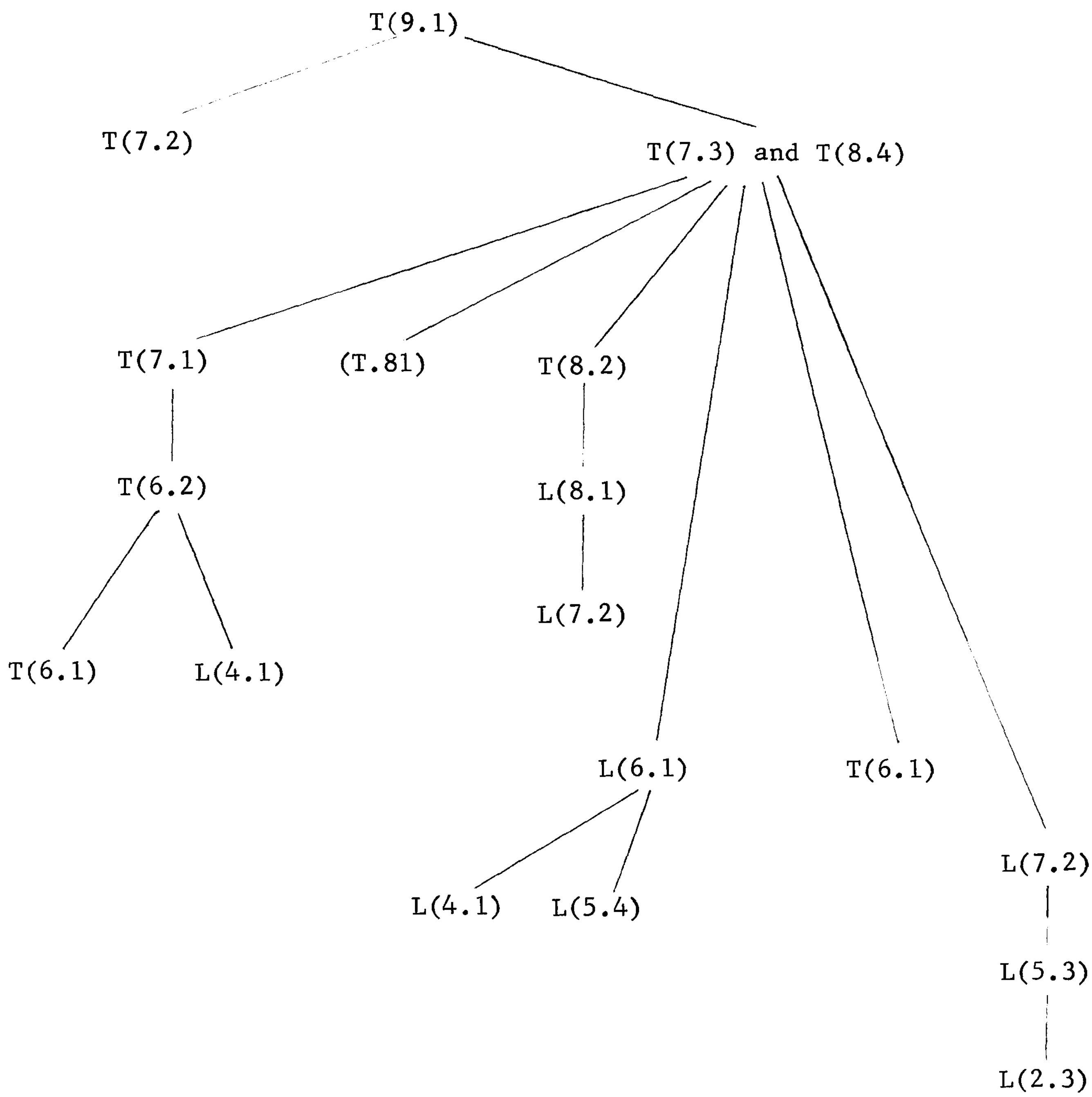
and

$$S(\mathcal{A};\mathcal{P},z) \geq X W(z) \left\{ f\left(\alpha \frac{\log X}{\log z}\right) - B_{22} \frac{L}{(\log X)^{1/14}} \right\} ,$$

with the functions F and f as defined there.

Theorem 9.1 (our Theorem 2) now follows with the aid of Theorem* 8.4, Theorem* 8.2 and Lemma* 7.2. There is no change in the argument.

Note. It has been assumed throughout that L is not bounded as otherwise we may replace it with $\Omega_2(\mathcal{K})$ or even $\Omega_2(\mathcal{K},L)$.



This is a chart interconnecting the relevant Lemmas (L) and Theorems (T) required in the analysis for Theorem 9.1.

REFERENCES

Section A

[1] Chen, J. R., "Estimates for trigonometric sums," Acta. Math. Sinica 14 (1964), 765-768.

[2] Heilbronn, H., "On the distribution of the sequence $\{n^2\theta\}$ mod 1," Quart J. Math. Oxford 19 (1948), 249-256.

[3] Vaughan, R. C., "Mean value theorems in prime number theory," JLMS (2) 10 (1975), 153-162.

[4] Vaughan, R. C., "Sommes trigonometriques sur les nombres premiers," C. R. Acad. Sci. Paris, Ser. A 285 (1977), 981-983.

[5] Vaughan, R. C., "On the distribution of αp modulo 1," Mathematika 24 (1977), 136-141.

[6] Vinogradov, I. M., "A new estimate of a trigonometric sum containing primes," Bul. Acad. Sci. USSR Ser. Math. 2 (1938), 1-13.

[7] Vinogradov, I. M., "A general distribution law for the fractional parts of values of a polynomial with variable running over primes," Ibid., 51 (1946), 491-2.

[8] Vinogradov, I. M., The method of trigonometric sums in the theory of numbers. Translated from the Russian, revised and annotated by K. F. Roth and A. Davenport (Interscience, New York, 1954).

Section B

[1] Selberg, A., "Contributions to the theory of the Riemann Zeta-Function," Arch. Math. Naturvid 48 (1946), 89-155.

[2] Berry, A. C., "The accuracy of the Gaussian approximation to the sum of independent variates," Trans. Amer. Math. Soc. 49 (1941), 122-136.

[3] Delange, H., "Sur un theorem d'Erdös et Kac," Bull. Belgian Royal Acad., Sr. 5 (1956), 130-144.

[4] Erdös, P. and Kac, M., "The Gaussian law of errors in the theory of additive number theoretic functions," Amer. J. Math. 62 (1940), 738-742.

[5] Esseen, C. G., "Fourier analysis of distribution functions, A mathematical study of the Laplace-Gaussian law," Acta. Math. 77 (1945), 1-125.

[6] Halberstam, H., "On the distribution of additive number-theoretic functions," Journ. Lond. Math. Soc. 30 (1955), 43-53.

[7] Kubilius, J. P., Probablistic Methods in the Theory of Numbers, Vol. 11, AMS Transl. 1964.

[8] LeVeque, W. J., "On the size of certain number theoretic functions," Trans. Amer. Math. Soc. 66 (1949), 440-463.

[9] Renyi, A. and Turan, P., "On a theorem of Erdös-Kac," Acta Arith. 4 (1958), 71-84.

Section C: Chapter 6

[1] Bombieri, E., "On twin almost primes," Acta Arith. 28 (1975), 177-193 and Corrigendum ibidem, pp. 457-461.

[2] Bombieri, E., "The Asymptotic Sieve," Rend. Accad. Naz. XL (5) 1/2 (1975/76), 243-269.

[3] Chen, J. R., "On the representation of a large even integer as the sum of a prime and the product of at most two primes," Sci. Sin. 16 (1973), 157-176.

[4] Chen, J. R., "On the Goldbach Problem and the Sieve method," Sci. Sin. 21 (1978), 701-739.

[5] Erdös, P., Acta Univ. Szegedrensis Szeged, Hungary 5 (1932), 194-198.

[6] Greaves, G., "A weighted sieve of Brun's Type," to appear Acta Arith.

[7] Halberstam, H. and Ric ert, H. E., Sieve Methods. Acad. Press (Lond.), 1974.

[8] Iwaniec, H. and Jutila, M., "Primes in short-intervals," in press, 1979.

[9] Iwaniec, H. and Heath Brown, D. R., "On the difference between consecutive primes," Inv. Math. 55 (1979), 49-70.

[10] Iwaniec, H., "A new form of the error-term in the linear sieve," Acta Arith., in press, 1979.

[11] Jurkat, W. B. and Richert, H. E., "An improvement of Selberg's sieve method I," Acta. Arith. 11 (1965), 217-240.

[12] Kuhn, P., "Zur Viggo Burn'schen Siebmethode. I," Norske Vid. Selsk. Forh., Trondhjem 14 (1941), no. 39, 145-148.

[13] Pan, C., "A new Mean-Value Theorem and its applications," Recent Progress in Analytic Number Theory, Vol. 1, Acad. Press (to appear).

[14] Selberg, A., "An elementary proof of the prime-number theorem," Ann. of Math. (2) 50 (1949), 305-313.

[15] Vaughan, R. C., An elementary method in prime number theory. Lecture Notes, Institute Mittag-Leffler.

Section C: Chapter 7

[1] Greaves, G., "An application of a theorem of Barban-Davenport-Halberstam," BLMS 6 (1974), 1-9.

[2] Halberstam, H. and Richert, H. E., Sieve Methods. Academic Press (London), 1974).

[3] Montgomery, H. L., Topics in Multiplicative Number Theory. Springer Verlag Notes No. 227.

[4] Montgomery, H. L. and Vaughan, R. C., "On the exceptional set in Goldbach's Problem," Acta. Arithm. 27 (1975), 353-370.