Interaction of Topology and Algebra in Arithmetic Geometry

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A mis padres y hermana

Ella, divina música!, en el meu cor petit
fa cabre l'infinit, trencades les rescloses,
i se m'emporta lluny dels Nombres i les Coses,
més enllà del desig, quasi fins a l'oblit.

Màrius Torres

Abstract

THIS THESIS studies topological and algebraic aspects of higher dimensional local fields and relations to other neighbouring research areas such as nonarchimedean functional analysis and higher dimensional arithmetic geometry.

We establish how a higher local field can be described as a locally convex space once an embedding of a local field into it has been fixed. We study the resulting spaces from a functional analytic point of view: in particular we introduce and study bounded, c-compact and compactoid submodules of characteristic zero higher local fields. We show how these spaces are isomorphic to their appropriately topologized duals and study the implications of this fact in terms of polarity.

We develop a sequential-topological study of rational points of schemes of finite type over local rings typical in higher dimensional number theory and algebraic geometry. These rings are certain types of multidimensional complete fields and their rings of integers and include higher local fields. Our results extend the constructions of Weil over (one-dimensional) local fields. We establish the existence of an appropriate topology on the set of rational points of schemes of finite type over the rings considered, study the functoriality of this construction and deduce several properties.

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Introduction

A *zero-dimensional* local field is a finite field. An *n-dimensional local field*, for $n \geq 1$, is a complete discrete valuation field such that its residue field is an $(n-1)$-dimensional local field.

Higher local fields and other closely related objects appear as local components in the adeles associated to n-dimensional schemes by Parshin [Par76] and Beilinson [Beĭ80]. As such, they generalise the role that (one-dimensional) local fields play as local components of the adele ring of a projective smooth curve over a finite field.

Establishing results for higher local fields by trying to generalise what is known for one-dimensional local fields is far from an easy job. The algebraic and topological methods which are sufficient and give a rather clear description of the one-dimensional situation turn out to be somewhat limited as soon as we move to dimensions two and higher.

As an example of such behaviour, we could mention higher class field theory: the right object to consider as the domain of the reciprocity map for an n-dimensional local field is not its group of units, but its n-th Milnor K-group, a considerably more complicated object [Par84], [Kat79], [Kat80], [Kat82].

Regarding topological aspects, Parshin initially introduced so-called higher topologies in his work; Fesenko successfully used several topologies on higher local fields, their unit groups and their Milnor K-groups, including sequential saturations, in order to achieve a deeper understanding of local class field theory [Fes01]. This use of several topologies simultaneously is a feature that appears in other parts of higher number theory.

On a different direction, Kato pointed out that the topological language is not well-suited for n-dimensional local fields and advocated in [Kat00] for replacing topology in favour of ind- and pro- categories. This categorical point of view and associated derivations have been further developed in other works such as [Kap01], [Kap], [Pre11].

From this categorical point of view, we deduce many properties from the fact that higher local fields may be described as iterated inductive and projective limits of finite-dimensional objects. For example, in positive characteristic, this point of view leads to the treatment of topologies on higher local fields using linear topological tools and replacing the absence of local compactness by local linear compactness.

One of the achievements of this thesis is the extension of this linear topological point of view to the case of higher local fields of characteristic zero. The absence of a structure of infinite dimensional vector space over a finite field is replaced by the structure of an infinite dimensional vector space over a nonarchimedean local field; we show that such a structure is canonical whenever our characteristic zero higher local field is obtained from a complete flag on an arithmetic scheme (cf. §1.3.3).

The language of locally convex spaces over a local field is suitable for the description of higher topologies, and this triggers the need to develop a systematic study of the properties of characteristic zero higher local fields from the point of view of nonarchimedean functional analysis. This is the object of Chapters §2 and §3.

Further, these are the first steps in the development of a new theory at the intersection of nonarchimedean functional analysis and higher number theory.

In §2 we prove that a higher topology on a two-dimensional local field induces the structure of a locally convex topological vector space, provided an embedding of a local field into the two-dimensional local field has been fixed beforehand. This application of functional theoretic tools to the study of higher local fields opens a new direction of research and several natural questions arise with many associated developments. In the rest of the chapter we study the behaviour of the resulting spaces; it changes dramatically depending on the residue characteristic of the two-dimensional local field. Namely, equal-characteristic two-dimensional fields are bornological, reflexive and nuclear while all such properties may be shown to fail in the mixed characteristic case. Such discrepancy regarding the residue characteristic of the field is a new phenomenon which does not occur in other approaches to the study of higher local fields.

It is also possible to study which properties hold in all cases: two-dimensional local fields are isomorphic to their appropriately topologized dual spaces, cf. Theorem 2.4.2. Moreover, certain subsets of two-dimensional local fields that had not been considered yet arise as the classes of bounded and compactoid submodules. From self-duality we can identify the field and its dual and study polarity, which is a map on the class of submodules of the two-dimensional local field which exchanges open lattices and compactoid submodules. The latter class of submodules is expected to have certain finiteness properties which are suitable generalisations of the role played by compact submodules in the theory of local fields, as they are analogous to linearly compact subspaces of a linear topological space. The results described in this paragraph can be extended to the case of arbitrary dimensional local fields; this is the goal of §3.

One of the general conclusions that we extract from our functional analytic approach is that bornology may be a better language to understand higher local fields in replacement of topology. Just like a topology on a set is the minimum amount of information required to be able to speak about *neighbourhood of a point* and *continuous map*, a bornology on a set is the minimum amount of information required to be able to speak about *bounded set* and *bounded map* (cf. §1.1). We show how either the Von-Neumann bornology or the bornology generated by compactoid submodules on a higher local field define a structure of bornological algebra (cf. Proposition 3.3.4, §5.1). Bornological tools have seldom been used in number theory, although the works of Meyer deserve a mention [Mey04], [Mey05].

Perhaps it is necessary to explain the need to treat the arbitrary dimensional case separately. The two-dimensional case often supplies the first step of induction, or at least the first non-trivial step; therefore it is a good idea to treat it first. At the same time, the cases for dimension greater than two quickly turn into a rather involved exercise in notation and the application of arguments which are familiar from the case $n = 2$. As such, the proof of many results in §3 often refers to §2 for the case $n = 2$ and then indicates how to proceed by induction.

Furthermore, there are many relevant functional analytic properties which may be shown to hold in the two-dimensional case and which fail in greater dimension or which one could only expect to hold in few particular cases; being bornological, reflexive or nuclear is an example of such properties.

It is still possible to give explicit bases for the bornologies of bounded and compactoid $\mathcal{O}$-submodules of F. We also show an explicit self-duality result in Theorem 3.5.3 which generalises Theorem 2.4.2 to higher dimensions.

Out of the work carried in this direction, two research papers [Cáma], [Cámb] have been written and submitted for publication.

Our functional theoretic approach to topologies on higher local fields is of a linear nature. This excludes the study of nonlinear aspects, such as continuity of polynomial maps: it is a well-known fact that a higher local field endowed with a higher topology is not a topological ring.

It is however possible to use sequential topological tools in this setting, since the multiplication map is sequentially continuous. In Chapter §4 we use a sequential approach to the study of rational points over higher local fields and their valuation rings.

Sets of rational points on schemes of finite type over local fields or, more generally, topological rings may be topologized in a satisfactory way [Con]. The locally compact topologies obtained in the particular case of algebraic groups over local fields allow us to introduce and develop the theory of Tamagawa measures and Tamagawa numbers [Wei82].

In §1.2 we introduce the notion of sequential ring: a ring R endowed with a topology for which its underlying additive group is a topological group and such that multiplication $R \times R \to R$ is a sequentially continuous map. The main example we know of sequential ring is a higher local field.

Using sequential topological arguments, we extend the one-dimensional constructions to the arbitrary dimensional case in Chapter §4. The main result in this direction is Theorem 4.2.2: for a sequential local ring R satisfying some conditions and a scheme of finite type $X \to \operatorname{Spec} R$, the set $X(R)$ admits a sequentially saturated topology which is functorial, carries open (resp. closed) immersions of schemes to open (resp. closed) topological embeddings, fibred products to fibred products, gives $\mathbb{A}^1_R(R) = R$

the sequential saturation of the given topology and gives Hausdorff topological spaces when applied to separated schemes. Moreover, these conditions characterize $X(R)$ as a topological space.

Out of this line of work, a research paper [Cámc] has been written and submitted for publication.

Besides the aforementioned chapters, which contain the bulk of our own work, this thesis contains two other chapters. In Chapter §1 we take the time to introduce notions and results in several topics which are relevant to the thesis but which are not original research work: these are a summary of the theory of locally convex vector spaces over a local field, an introduction to several aspects of sequential topologies and a brief introduction to the theory of higher local fields.

We also take the time in §5 to gather some conclusions and outline several new directions of work motivated by our results.

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Chapter 1

Prerequisites

It is our purpose to use this initial chapter of the thesis as a place in which to summarise some preliminary topics. We will also take the chance to fix some notations conveniently. The contents of this chapter are not part of our research, although some definitions have been introduced in our work for the first time and we have supplied proofs for some well-known results for which no proof seemed to be available in the literature. Therefore, we do not claim originality for any result contained in this chapter.

Section §1.1 contains a review of some definitions and results about the theory of locally convex vector spaces over a local field, and will set basic notations necessary for Chapters §2 and §3.

From a different point of view, we discuss a few aspects of sequential topologies on groups and rings in §1.2. These are necessary for the discussions in Chapter §4.

In §1.3 we will introduce higher local fields, describe their structure and discuss higher topologies. This section is key for pretty much any further chapters of the thesis.

Notation. Whenever F is a complete discrete valuation field, we will denote by $\mathcal{O}_F, \mathfrak{p}_F, \pi_F, \overline{F}$ its ring of integers, the unique nonzero prime ideal in the ring of integers, an element of valuation one and the residue field, respectively.

1.1 Locally convex spaces over a local field

In this section we summarise some concepts and fix some notation regarding locally convex vector spaces over a local field.

Let us fix for the rest of this section a characteristic zero local field K, that is, a finite extension of $\mathbb{Q}_p$ for some prime p. The cardinality of the finite field $\overline{K}$ will be denoted by q. The absolute value of K will be denoted by $|\cdot|$, normalised so that $|\pi_K| = q^{-1}$. Due to far too frequent apparitions in the text, we will ease notation by letting $\mathcal{O} := \mathcal{O}_K$ and $\mathfrak{p} := \mathfrak{p}_K$.

The theory of locally convex vector spaces over a nonarchimedean field is well developed in the literature, so we will keep a concise exposition of the facts that we will require later. Both [Sch02] and [PGS10] are very good references on the topic.

Let V be a K-vector space. A *lattice* in V is an $\mathcal{O}$-submodule $\Lambda \subseteq V$ such that for any $v \in V$ there is an element $a \in K^\times$ such that $av \in \Lambda$. This is equivalent to having

$$\Lambda \otimes_{\mathcal{O}} K \cong V$$

as K-vector spaces. A subset of V is said to be *convex* if it is of the form $v + \Lambda$ for $v \in V$ and Λ a lattice in V. A vector space topology on V is said to be *locally convex* if the filter of neighbourhoods of zero admits a collection of lattices as a basis.

A *seminorm* on V is a map $\|\cdot\| : V \to \mathbb{R}$ such that:

(i) $\|\lambda v\| = |\lambda| \cdot \|v\|$ for every $\lambda \in K$, $v \in V$,

(ii) $\|v + w\| \leq \max(\|v\|, \|w\|)$ for all $v, w \in V$.

These conditions imply in particular that a seminorm only takes non-negative values and that $\|0\| = 0$. A seminorm $\|\cdot\|$ is said to be a *norm* if $\|x\| = 0$ implies $x = 0$.

The *gauge seminorm* of a lattice $\Lambda \subseteq V$ is defined by the rule:

$$\|\cdot\|_\Lambda : V \to \mathbb{R}, \quad v \mapsto \inf_{v \in a\Lambda} |a|. \tag{1.1}$$

Given a family of seminorms $\{\|\cdot\|_j\}_{j \in J}$ on V, there is a unique coarsest vector space topology on V making the maps $\|\cdot\|_j : V \to \mathbb{R}$ continuous for every $j \in J$. Such topology is locally convex: since the intersection of a finite number of lattices is a lattice, the *closed balls*

$$B_j(\varepsilon) = \{v \in V;\ \|v\|_j \leq \varepsilon\}, \quad \varepsilon \in \mathbb{R}_{>0}, j \in J$$

supply a subbasis of neighbourhoods of zero consisting of open lattices. Note that the use of the adjective *closed* here is, as usual in this setting, an imitation of the analogous archimedean convention. Topologically, $B_j(\varepsilon)$ and $\{v \in V;\ \|v\|_j < \varepsilon\}$ are both open and closed.

A locally convex topology can be described in terms of lattices or in terms of seminorms; passing from one point of view to the other is a simple matter of language.

A locally convex vector space V is said to be *normable* if its topology may be defined by a single norm. By saying that V is *normed* K-vector space, we will imply that we are considering a norm on it and that we regard the space together with the locally convex topology defined by a norm.

For a locally convex vector space V, a subset $B \subset V$ is *bounded* if for any open lattice $\Lambda \subset V$ there is an $a \in K$ such that $B \subseteq a\Lambda$. Alternatively, B is bounded if for every continuous seminorm $\|\cdot\|$ on V we have

$$\sup_{v \in B} \|v\| < \infty.$$

A locally convex K-vector space V is *bornological* if any seminorm which is bounded on bounded sets is continuous. A linear map between locally convex vector spaces $V \to W$ is said to be *bounded* if the image of any bounded subset of V is a bounded subset of W.

More generally, a *bornology* on a set X is a collection $\mathcal{B}$ of subsets of X which cover X, is hereditary under inclusion and stable under finite union. We say that the elements of $\mathcal{B}$ are *bounded sets* and the pair $(X, \mathcal{B})$ is referred to as a *bornological space* [HN77, Chapter I].

Just like a topology on a set is the minimum amount of information required in order to have a notion of open set and continuous map, a bornology on a set is the minimum amount of information required in order to have a notion of bounded set and *bounded map*, the latter being a map between two bornological spaces which preserves bounded sets. A *basis* for a bornology $\mathcal{B}$ on a set is a subfamily $\mathcal{B}_0 \subset \mathcal{B}$ such that every element of $\mathcal{B}$ is contained in an element of $\mathcal{B}_0$.

The bornology which we have described above for a locally convex vector space V is known as the *Von-Neumann bornology* [HN71, §I.2], and it is compatible with the vector space structure, meaning that the vector space operations are bounded maps. Moreover, the Von-Neumann bornology on a locally convex vector space is *convex*, as it admits a basis given by convex subsets [HN71, §I.6].

Open lattices in a non-archimedean locally convex space are also closed [Sch02, §6]. A locally convex space V is said to be *barrelled* if any closed lattice is open.

Among many general ways to construct locally convex spaces [Sch02, §5], we will require the use of products.

Proposition 1.1.1. *Let $\{V_i\}_{i\in I}$ be a family of locally convex K-vector spaces, and let $V = \prod_{i\in I} V_i$. Then the product topology on V is locally convex.*

Proof. See [Sch02, §5.C]. If $\{\Lambda_{i,j}\}_j$ denotes the set of open lattices of V_i for $i \in I$, then the set of open lattices of V is given by finite intersections of lattices of the form $\pi_i^{-1}\Lambda_{i,j}$.

Equivalently, the product topology on V is the one defined by all seminorms of the form

$$v \mapsto \sup_{i,j} \|\pi_i(v)\|_{i,j},$$

where $\{\|\cdot\|_{i,j}\}_j$ is a defining family of seminorms for V_i for all $i \in I$, $\pi_i : V \to V_i$ is the corresponding projection and the supremum is taken over a finite collection of indices i, j. □

Similarly, if $(X_i, \mathcal{B}_i)_{i\in I}$ is a collection of bornological sets, the *product bornology* on $X = \prod_{i\in I} X_i$ is the one defined by taking as a basis the sets of the form $B = \prod_{i\in I} B_i$ with $B_i \in \mathcal{B}_i$ [HN77, §2.2].

Another construction which we will require is that of inductive limits. Let V be a K-vector space and $\{V_i\}_{i\in I}$ be a collection of locally convex K-vector spaces. Let, for each $i \in I$, $f_i : V_i \to V$ be a K-linear map. The final topology for the collection $\{f_i\}_{i\in I}$ is not locally convex in general. However, there is a finest locally convex topology on V making the map f_i continuous for every $i \in I$ [Sch02, §5.D]. That topology is called the *locally convex final topology* on V. Inductive limits and direct sums of locally convex spaces are particular examples of such construction.

Definition 1.1.2. Suppose that V is a K-vector space and that we have an increasing sequence of vector subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V$ such that $V = \cup_{n \in \mathbb{N}} V_n$. Suppose that for each $n \in \mathbb{N}$, V_n is equipped with a locally convex topology such that $V_n \hookrightarrow V_{n+1}$ is continuous. Then the final locally convex topology on V is called the *strict inductive limit topology.*

Let us fix, from now until the end of the present section, a locally convex vector space V. In order to discuss completeness issues, we require to deal not only with sequences, but arbitrary nets.

Let I be a directed set. A *net* in V is a family of vectors $(v_i)_{i \in I} \subset V$. A *sequence* is a net which is indexed by the set of natural numbers. The net $(v_i)_{i \in I}$ *converges* to a vector v, and we shall write $v_i \to v$, if for any $\varepsilon > 0$ and continuous seminorm $\| \cdot \|$ on V, there is an index $i \in I$ such that for every $j \geq i$ we have $\|v_j - v\| \leq \varepsilon$. Similarly, the net $(v_i)_{i \in I}$ is said to be *Cauchy* if for any $\varepsilon > 0$ and continuous seminorm $\| \cdot \|$ on V there is an index $i \in I$ such that for every pair of indices $j, k \geq i$ we have $\|v_j - v_k\| \leq \varepsilon$.

Definition 1.1.3. A subset $A \subseteq V$ is said to be *complete* if any Cauchy net in A converges to a vector in A.

A *K-Banach space* is a complete normed locally convex vector space. V is said to be a Fréchet space if it is complete and its locally convex topology is metrizable. A locally convex vector space is said to be an *LF-space* if it may be constructed as a countable inductive limit of Fréchet spaces.

Example 1.1.4. K is a Fréchet space. There is a unique locally convex topology on any finite dimensional K-vector space which defines a structure of Fréchet space [Sch02, Proposition 4.13].

In general, the usual topological notion of compactness is not very powerful for the study of infinite dimensional vector spaces over non-archimedean fields. This is why we prefer to use the language of c-compactness, which is an $\mathcal{O}$-linear concept of compactness.

Definition 1.1.5. Let A be an $\mathcal{O}$-submodule of V. A is said to be *c-compact* if, for any decreasing filtered family $\{\Lambda_i\}_{i\in I}$ of open lattices of V, the canonical map

$$A \to \varprojlim_{i\in I} A/\left(\Lambda_i \cap A\right)$$

is surjective.

Example 1.1.6. The base field K is c-compact as a K-vector space [Sch02, §12]. This shows that a c-compact module need not be bounded.

This property may be phrased in a more topological way.

Proposition 1.1.7. *An $\mathcal{O}$-submodule $A \subseteq V$ is c-compact if and only if for any family $\{C_i\}_{i\in I}$ of closed convex subsets $C_i \subseteq A$ such that $\bigcap_{i\in I} C_i = \emptyset$ there are finitely many indices $i_1, \ldots, i_m \in I$ such that $C_{i_1} \cap \ldots \cap C_{i_m} = \emptyset$.*

Proof. See [Sch02, Lemma 12.1.ii and subsequent paragraph]. □

Proposition 1.1.8. *Let $\{V_h\}_{h\in H}$ be a collection of locally convex K-vector spaces, and for each $h \in H$ let $A_h \subseteq V_h$ be a c-compact $\mathcal{O}$-submodule. Then $\prod_{h\in H} A_h$ is c-compact in $\prod_{h\in H} V_h$.*

Proof. [Sch02, Prop. 12.2]. □

Another notion which is used in this setting is that of a compactoid $\mathcal{O}$-module; it is a notion which is analogous to that of relative compactness in the archimedean setting.

Definition 1.1.9. Let $A \subseteq V$ be an $\mathcal{O}$-submodule. A is *compactoid* if for any open lattice Λ of V there are finitely many vectors $v_1, \ldots, v_m \in V$ such that

$$A \subseteq \Lambda + \mathcal{O}v_1 + \cdots + \mathcal{O}v_m.$$

Let $A \subseteq V$ be an $\mathcal{O}$-submodule. If A is c-compact, then it is closed and complete. Similarly, if A is compactoid then it is bounded. [Sch02, §12].

Proposition 1.1.10. *Let $A \subseteq V$ be an $\mathcal{O}$-submodule. The following are equivalent.*

(i) *A is c-compact and bounded.*

(ii) A is compactoid and complete.

Proof. [Sch02, Prop. 12.7]. □

The collection of compactoid $\mathcal{O}$-submodules of V generates a bornology which is a priori weaker than the one given by the locally convex topology.

Remark 1.1.11. It should be pointed out that the locally convex vector spaces that we consider in this work are always defined over a local field, which is discretely valued and, therefore, locally compact and spherically complete. This implies that the general theory of locally convex spaces over a nonarchimedean complete field simplifies in our setting. In particular, for an $\mathcal{O}$-submodule $A \subseteq V$, compactoidicity and completeness imply compactness [PGS10, Theorem 3.8.3]. We choose, however, to use the language of c-compact and compactoid submodules.

If V, W are two locally convex K-vector spaces, a linear map $f : V \to W$ is continuous as soon as the pull-back of a continuous seminorm is a continuous seminorm. We denote the K-vector space of continuous linear maps between V and W by $\mathcal{L}(V, W)$.

The space $\mathcal{L}(V, W)$ may be topologized in the following way. Let $\mathcal{B}$ be a collection of bounded subsets of V. For any continuous seminorm $\|\cdot\|$ on W and $B \in \mathcal{B}$, consider the seminorm

$$\|\cdot\|_B : \mathcal{L}(V, W) \to \mathbb{R}, \quad f \mapsto \sup_{v \in B} \|f(v)\|.$$

Definition 1.1.12. We write $\mathcal{L}_{\mathcal{B}}(V, W)$ for the space of continuous linear maps from V to W endowed with the locally convex topology defined by the seminorms $\|\cdot\|_B$, for every continuous seminorm $\|\cdot\|$ on W and $B \in \mathcal{B}$.

In the particular case in which $\mathcal{B}$ consists of all bounded sets of V, we write $\mathcal{L}_b(V, W)$ for the resulting space, which is then said to have the topology of *uniform convergence*, or *b-topology*. If $\mathcal{B}$ consists only of the singletons $\{v\}$ for $v \in V$, we denote the resulting space by $\mathcal{L}_s(V, W)$ and say that it has the topology of *point-wise convergence*. Finally, if $\mathcal{B}$ is the collection of compactoid $\mathcal{O}$-submodules of V, we denote the resulting space by $\mathcal{L}_c(V, W)$ and say that it has the topology of *uniform convergence on compactoid submodules*, or *c-topology*.

There are two cases of particular interest: the topological dual space $V' = \mathcal{L}(V, K)$, and the endomorphism ring $\mathcal{L}(V) = \mathcal{L}(V, V)$. We denote V'_s, V'_b, V'_c, $\mathcal{L}_s(V), \mathcal{L}_b(V)$ and $\mathcal{L}_c(V)$ for the corresponding topologies of point-wise convergence, uniform convergence and uniform convergence on compactoid submodules, respectively.

The choice of a family of bounded subsets $\mathcal{B}$ of V does not affect $V'_{\mathcal{B}}$ as a set, but it does affect the bidual space. As such, in the category of locally convex vector spaces over K, it is an interesting issue to classify which spaces are isomorphic, algebraically and/or topologically, to certain bidual spaces through the duality maps

$$\delta : V \to (V'_{\mathcal{B}})', \quad v \mapsto \delta_v(l) = l(v). \tag{1.2}$$

The best possible case is when δ induces a topological isomorphism between V and $(V'_b)'_b$; in this case we say that V is *reflexive*.

Proposition 1.1.13. *Every locally convex reflexive K-vector space is barrelled.*

Proof. [Sch02, Lemma 15.4]. □

The notion of polarity plays a role in the study of duality, as it provides us with a way of relating $\mathcal{O}$-submodules of V to $\mathcal{O}$-submodules of V'.

Definition 1.1.14. If $A \subseteq V$ is an $\mathcal{O}$-submodule, we define its *pseudo-polar* by

$$A^p = \left\{ l \in V';\ |l(v)| < 1 \text{ for all } v \in A \right\}.$$

The pseudo-bipolar of A is

$$A^{pp} = \{ v \in V;\ |l(v)| < 1 \text{ for all } l \in A^p \}.$$

Taking the pseudo-polar of an $\mathcal{O}$-submodule of V gives an $\mathcal{O}$-submodule of V'.

We have that $l \in A^p$ if and only if $l(A) \subseteq \mathfrak{p}$. Note that the traditional notion of polar relaxes the condition in the definition of pseudo-polar to $|l(v)| \leq 1$ or, equivalently, $l(A) \subseteq \mathcal{O}$. Introducing the distinction is an important technical detail, as pseudo-polarity is a better-behaved notion in the nonarchimedean setting.

Proposition 1.1.15. *Let $A \subseteq V$ be an $\mathcal{O}$-submodule. We have*

(i) If $A \subseteq B \subseteq V$ is another $\mathcal{O}$-submodule, then $B^p \subseteq A^p$.

(ii) A^p is closed in V'_s.

(iii) Let $\mathcal{B}$ be any collection of bounded subsets of V. If $A \in \mathcal{B}$, then A^p is an open lattice in $V'_{\mathcal{B}}$.

(iv) A^{pp} is equal to the closure of A in V.

Proof. Statements (i), (ii) and (iii) are part of [Sch02, Lemma 13.1]. (iv) is [Sch02, Proposition 13.4]. □

In order to conclude this section we define nuclear spaces. For any submodule $A \subseteq V$, denote $V_A := A \otimes_{\mathcal{O}} K$, endowed with the locally convex topology associated to the gauge seminorm $\|\cdot\|_A$. V_A may not be a Hausdorff space, but its completion

$$\widehat{V_A} := \varprojlim_{n\in\mathbb{Z}} V_A/\pi^n A$$

is a K-Banach space.

Definition 1.1.16. V is said to be *nuclear* if for any open lattice $\Lambda \subseteq V$ there exists another open lattice $M \subseteq \Lambda$ such that the canonical map $\widehat{V_M} \to \widehat{V_\Lambda}$ is compact, that is: there is an open lattice in $\widehat{V_M}$ such that the closure of its image is bounded and c-compact.

Proposition 1.1.17. *We have:*

(i) An $\mathcal{O}$-submodule of a nuclear space is bounded if and only if it is compactoid.

(ii) Arbitrary products of nuclear spaces are nuclear.

(iii) Strict inductive limits of nuclear spaces are nuclear.

Proof. (i) is [Sch02, Proposition 19.2], (ii) is [Sch02, Proposition 19.7] and (iii) is [Sch02, Corollary 19.8]. □

1.2 Sequential topology on groups and rings

We review a few aspects about the category of sequential topological spaces. Details about the topics contained in this section may be found in [Fra65] and [Fra67].

When dealing with the elements of a sequence, the expression *almost all* may be safely replaced by *all but finitely many.*

Let (X, τ) be a topological space.

Definition 1.2.1. A sequence $(x_n)_n \subset X$ *converges* to $x \in X$ if for every open neighbourhood U of x in X, almost all of the elements in the sequence $(x_n)_n$ belong to U. That is: there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for $n \geq n_0$.

We will use the notation $x_n \to x$ whenever $(x_n)_n \subset X$ is a sequence which converges to $x \in X$.

In general, specifying the set of convergent sequences of a set X does not determine a unique topology on X, but rather a whole family of topologies. Among them, there is one which is maximal in the sense that it is the finest: the sequential saturation.

Definition 1.2.2. A subset A of X is *sequentially open* if for any sequence $(x_n)_n$ in X convergent to $x \in A$, there is an index $n_0 \in \mathbb{N}$ such that $x_n \in A$ for $n \geq n_0$.

Proposition 1.2.3. *Every open set is sequentially open. The collection of sequentially open sets of X defines a topology τ_s on X, finer than τ.* □

The natural map $(X, \tau_s) \to (X, \tau)$, given by the identity on the set X, is continuous.

Definition 1.2.4. The space (X, τ_s) is called the *sequential saturation* of (X, τ). Whenever $\tau = \tau_s$, we shall simply say that X is *sequential.*

Remark 1.2.5. A subset $C \subseteq X$ is *sequentially closed* if for every $x_n \to x$ in X, $x_n \in C$ for every n implies $x \in C$. This condition is equivalent to $X \setminus C$ being sequentially open. We could have defined τ_s by specifying that its closed sets are all sequentially closed sets in (X, τ).

Proposition 1.2.6. *Let $f : X \to Y$ be a map between topological spaces. The following are equivalent:*

(i) For any convergent sequence $x_n \to x$ in X, the sequence $(f(x_n))_n$ converges to $f(x)$ in Y.

(ii) The preimage under f of any sequentially open set in Y is sequentially open.

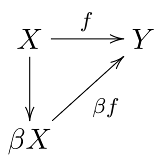
(iii) The map $f : (X, \tau_s) \to Y$ is continuous.

When this situation holds, we say that f is a sequentially continuous map. □

Any continuous map is sequentially continuous. Hence, sequential continuity is a weaker condition than ordinary continuity. In a sequential space, the topology is essentially controlled by sequences.

Example 1.2.7. Any metric space is a sequential space. More generally, any first countable space is sequential [Fra65, §1].

Example 1.2.8. The Stone-Čech compactification of a topological space X is the unique Hausdorff compact space βX provided with a continuous map $X \to \beta X$ such that for any Hausdorff compact space Y and any continuous map $f : X \to Y$ there is a unique map $\beta f : \beta X \to Y$ such that the diagram



commutes. The Stone-Čech compactification of the natural numbers, $\beta\mathbb{N}$, is an example of a space which is not sequential [Gor, Example 1.1]. Despite being compact, $\beta\mathbb{N}$ is not sequentially compact. In this space, the closure of a set does not consist only of the limits of all sequences in that set, and a function from $\beta\mathbb{N}$ to another topological space may be sequentially continuous but not continuous.

Let **Seq** denote the subcategory of sequential topological spaces. Taking the sequential saturation of a topological space defines a functor **Top** $\to$ **Seq**, whose restriction to sequential spaces is the identity functor **Seq** $\to$ **Seq**.

Some of the usual constructions in **Top** are not inherited by **Seq**. The product of topological spaces is a remarkable example of such failure. Other examples of operations which do not behave well with respect to sequential saturation are function spaces and subspaces [Fra65, Example 1.8]. However, open and closed subspaces and open and closed images are closed constructions in **Seq** [Fra65, Prop. 1.9].

These sort of problems may be addressed by performing the usual construction in **Top**, and then taking the sequential saturation of the resulting space. In this fashion, the sequential saturation of the product topology provides a product object in **Seq**. In the words of Steenrod [Ste67], **Seq** is a *convenient* category of topological spaces.

We are interested in the compatibility between sequential topology and algebraic structures.

Definition 1.2.9. A *sequential group* is a group G provided with a topology, such that multiplication $G \times G \to G$ and inversion $G \to G$ are sequentially continuous ($G \times G$ is provided with the product topology). A *homomorphism* of sequential groups is a sequentially continuous group homomorphism. In other words: if (G, τ) is a sequential group, then (G, τ_s) is a group object in **Seq**.

Remark 1.2.10. If (G, τ) is a topological group, then so is (G, τ_s).

When considering rings, we could be interested in topologies for which subtraction and multiplication are sequentially continuous. However, we will deal with rings and topologies on them such that their additive groups are topological groups.

Definition 1.2.11. A *sequential ring* is a commutative ring R provided with a topology and such that:

(i) $(R, +)$ is a topological group.

(ii) Multiplication $R \times R \to R$ is sequentially continuous.

A *homomorphism* of sequential rings is a continuous ring homomorphism.

Remark 1.2.12. We could have required homomorphisms of sequential rings to be sequentially continuous maps. However, there is a good reason to prefer this stronger condition: in this way we preserve the underlying topological structure of the additive groups. Still, a continuous homomorphism of sequential rings $R \to S$ does not furnish

S with the structure of a topological R-module. This is why we consider the notion of sequential module.

Definition 1.2.13. Let M be an R-module. We say that M is a *sequential module* if M is provided with a topology such that $(M, +)$ is a topological group and multiplication

$$R \times M \to M$$

is sequentially continuous. A *homomorphism* of sequential R-modules is a continuous R-module homomorphism.

Note that with this definition R is a sequential R-module. If $R \to S$ is a homomorphism of sequential rings, S is automatically endowed with the structure of a sequential R-module.

When R is a topological ring, it is always possible to provide the units with a group topology. Because we do not demand inversion to be continuous, the correct way to topologize $R^\times$ is by considering the initial topology for the map

$$R^\times \to R \times R, \quad x \mapsto \left(x, x^{-1}\right). \tag{1.3}$$

The situation when R is a sequential ring is not very different, since we do not demand inversion on $R^\times$ to be sequentially continuous. The topology on $R^\times$ given by the sequential saturation of the initial topology for the map (1.3) turns $R^\times$ into a topological group.

Remark 1.2.14. A priori, we do not require sequential groups, rings and modules to be sequential topological spaces.

1.3 A brief introduction to higher local fields

A *zero-dimensional local field* is a finite field. An *n-dimensional local field*, for $n \geq 1$, is a complete discrete valuation field F such that $\overline{F}$ is an $(n-1)$-dimensional local field. Thus, a local field in the usual sense is a one-dimensional local field.

An n-dimensional local field F determines then a collection of fields F_i, $i \in \{0, \dots, n\}$, by letting $F_n = F$, $\overline{F_i} = F_{i-1}$ for $1 \le i \le n$; being n-dimensional is then determined by the finiteness of F_0.

An excellent introduction to this topic may be found in [Mor], we also often refer to results explained in [FK00, §1] and [MZ95].

Example 1.3.1. The fields $\mathbb{F}_q((t))((u))$ and $\mathbb{Q}_p((t))$ are both examples of higher local fields of dimension two. More generally, $\mathbb{F}_q((t_1))\cdots((t_n))$ and $\mathbb{Q}_p((t_1))\dots((t_{n-1}))$ are n-dimensional local fields.

Besides fields of Laurent series, there is another construction which is key in order to work with two-dimensional local fields, and higher local fields in general.

Example 1.3.2. For any complete discrete valuation field L, consider

$$L\{\{t\}\} = \left\{ \sum_{i\in\mathbb{Z}} x_i t^i;\ x_i \in L,\ \inf_{i\in\mathbb{Z}} v_L(x_i) > -\infty,\ x_i \to 0\ (i \to -\infty) \right\},$$

with operations given by the usual addition and multiplication of power series. Note that we need to use convergence of series in L in order to define the product. With the discrete valuation given by

$$v_{L\{\{t\}\}}\left(\sum_{i\in\mathbb{Z}} x_i t^i\right) := \inf_{i\in\mathbb{Z}} v_L(x_i),$$

$L\{\{t\}\}$ turns into a complete discrete valuation field. In the particular case in which L is an $(n-1)$-dimensional local field, the field $L\{\{t\}\}$ is an n-dimensional local field which we call the *standard mixed characteristic field over* L. Its first residue field is $\overline{L}\big((\overline{t})\big)$.

We view elements of L as elements of $L\{\{t\}\}$ in the obvious way. In particular, if π_L is a uniformizer of $\mathcal{O}_L$, it is also a uniformizer of $\mathcal{O}_{L\{\{t\}\}}$; the element $t \in L\{\{t\}\}$ is such that $\overline{t} \in \overline{L}\big((\overline{t})\big)$ is a uniformizer.

Let us fix an n-dimensional local field F. Besides the ring of integers $\mathcal{O}_F$ for the unique normalized discrete valuation of F, we might consider other valuation rings in F which are of arithmetic interest. Fix a *system of parameters* $t_1, \dots, t_n$ of F, that is:

t_n is a uniformizer of F, t_{n-1} is a unit of $\mathcal{O}_F$ such that its image in $\overline{F}$ is a uniformizer, and so on.

We may use our chosen system of parameters to define a valuation of rank n:

$$\mathbf{v} = (v_1, \ldots, v_n) : F^\times \to \mathbb{Z}^n,$$

where $\mathbb{Z}^n$ is ordered with the inverse of the lexicographical order. The procedure we follow is: $v_n = v_F$, $v_{n-1}(\alpha) = v_{\overline{F}}(\alpha t_n^{-v_n(\alpha)})$, and so on.

Although $\mathbf{v}$ does depend on the choice of the system of parameters, the valuation ring

$$O_F = \{\alpha \in F;\ \mathbf{v}(\alpha) \geq 0\}$$

does not [FK00, §1.1]. The ring O_F, which is a local ring with maximal ideal

$$\{\alpha \in F;\ \mathbf{v}(\alpha) > 0\},$$

is called the *rank-n ring of integers of F*. It is a subring of $\mathcal{O}_F$.

Once a system of parameters is chosen, we may consider a valuation of rank r for every $1 \leq r \leq n$ by mimicking the same procedure and stopping after r steps. The valuation rings we obtain are independent of any choice of parameters and are ordered by inclusion:

$$O_F = O_1 \subset O_2 \subset \cdots \subset O_n = \mathcal{O}_F. \tag{1.4}$$

Example 1.3.3. Consider $K = \mathbb{Q}_p \subset \mathbb{Q}_p\{\{t\}\} = F$. For the choice of uniformizer p for v_F, the associated rank-two valuation of F is

$$(v_1, v_2) : F^\times \to \mathbb{Z} \oplus \mathbb{Z}, \quad x = \sum_{i \in \mathbb{Z}} x_i t^i \mapsto \left(\inf_{i \in \mathbb{Z}} v_p(x_i), \inf \left\{ i;\ x_i \notin p^{v_1(x)+1} \mathbb{Z}_p \right\} \right).$$

The restriction of v_1 to K is v_p, while v_2 restricts trivially. The rank-two ring of integers is

$$\mathrm{O}_F = \left\{ \sum_{i \in \mathbb{Z}} x_i t^i \in F;\ x_i \in p\mathbb{Z}_p \text{ for } i < 0 \text{ and } x_i \in \mathbb{Z}_p \text{ for } i \geq 0 \right\}.$$

Example 1.3.4. Consider $K = \mathbb{Q}_p \subset \mathbb{Q}_p(\!(t)\!) = F$. In such case, the rank-two valuation of F associated to the uniformizer t for v_F is

$$(v_1, v_2) : F^\times \to \mathbb{Z} \oplus \mathbb{Z}, \quad \sum_{i \geq i_0} a_i t^i \mapsto (i_0, v_p(a_{i_0})),$$

where we suppose that a_{i_0} is the first nonzero coefficient in the power series. The restriction of v_1 to K is trivial while the restriction of v_2 to K is v_p. In this case we have $\mathrm{O}_F = \mathbb{Z}_p + t\mathbb{Q}_p[\![t]\!]$.

1.3.1 Classification

Higher local fields may be classified up to isomorphism. Since they are particular examples of complete rings, the classification follows from Cohen's work in this more general setting [Coh46].

There are some key facts from Cohen structure theory that we wish to highlight. A local ring A is said to be of *equal characteristic* if A and its residue field k share the same characteristic. Otherwise, A is said to be of *mixed characteristic*.

Whenever A is of equal characteristic, then there do exist *coefficient fields*, that is: ring homomorphisms $k \hookrightarrow A$ [Coh46, Theorem 9]. Moreover, as explained by Cohen immediately after proving this result, when $\operatorname{char} k = 0$ and the field extension $k|\mathbb{Q}$ is not algebraic, a coefficient field $k \hookrightarrow A$ may be chosen in infinitely many ways.

It is immediate to check that the set of ring homomorphisms $k \hookrightarrow A$ is empty whenever $\operatorname{char} A \neq \operatorname{char} k$.

The classification of a higher local field F depends crucially on the relation between the characteristics of F and its residue fields.

Theorem 1.3.5 (Classification of higher local fields)**.** *Let F be an n-dimensional field, and F_i the residue fields associated to F as explained at the beginning of §1.3. We have the following possibilities.*

(i) If $\operatorname{char} F$ is positive, then it is possible to choose $t_1, \dots, t_n \in F$ such that $F \cong F_0(\!(t_1)\!) \cdots (\!(t_n)\!)$.

(ii) If $\operatorname{char} F_1 = 0$, then there are $t_1, \dots, t_{n-1} \in F$ such that $F \cong F_1(\!(t_1)\!) \cdots (\!(t_{n-1})\!)$.

(iii) If none of the above holds, then there is a unique $r \in \{1, \ldots, n-1\}$ such that $\operatorname{char} F_{r+1} \neq \operatorname{char} F_r$. *Then there is a characteristic zero local field L and elements $t_1, \ldots, t_{n-1} \in F$ such that F is a finite extension of*

$$L\{\{t_1\}\} \cdots \{\{t_r\}\}((t_{r+1})) \cdots ((t_{n-1})) .$$

Moreover, if $\operatorname{char} F_0 = p$, *$L$ may be chosen to be the unique unramified extension of $\mathbb{Q}_p$ with residue field F_0.*

Proof. See, for example, [MZ95, §0], [FK00, §1.1] or [Mor, Theorem 2.18]. □

Definition 1.3.6. A *standard* n-dimensional local field of mixed characteristic is one of the form

$$L\{\{t_1\}\} \cdots \{\{t_r\}\}((t_{r+1})) \ldots ((t_{n-1}))$$

with L a characteristic zero local field.

In the case (iii) of the classification theorem it is also possible to show that such an F is contained in a standard field for possibly a different choice of local field L, and the resulting extension of fields is finite [FK00, §1.1].

1.3.2 Higher topology

Fix an n-dimensional local field F.

A higher local field has a topology determined by its unique normalized discrete valuation. This topology turns F into a Hausdorff and complete topological group which is not locally compact if $n > 1$, due to the fact that the residue field is not finite. There are several reasons to consider a different topology, such as the fact that the formal series in fields such as the ones in Example 1.3.1 do not converge in the valuation topology.

The construction of the higher topology on F is explained for example in [Mor, §4] and [MZ95, §1]. It revolves around two basic constructions, which are applied depending on the structure of F. We need to distinguish cases.

First suppose that a L is a field on which a translation invariant and Hausdorff topology has been defined. Let $\{U_i\}_{i \in \mathbb{Z}}$ be a sequence of neighbourhoods of zero of L, with the property that there is an index $i_0 \in \mathbb{Z}$ such that $U_i = L$ for all $i \geq i_0$. The sets of the form

$$\sum_{i \in \mathbb{Z}} U_i t^i := \left\{ \sum_{i \gg -\infty} x_i t^i;\ x_i \in U_i \text{ for all } i \right\} \tag{1.5}$$

describe a basis of neighbourhoods of zero for a translation invariant Hausdorff topology on $L((t))$, for which addition, subtraction and multiplication by a fixed element are continuous [MZ95, §1].

Remark 1.3.7. Algebraically, $L((t))$ can be presented as an inductive limit of projective limits of finite dimensional vector spaces over L in the following way:

$$L((t)) = \varinjlim_{i \geq 0} \varprojlim_{j \geq 0} t^{-i} L[t]/t^j L[t].$$

If a Hausdorff translation invariant topology has already been defined on L, then there is a natural linear topology on $L((t))$ which is defined by taking the product topology on finite-dimensional vector spaces over L and the natural projective limit and linear inductive limit topologies. The resulting topology agrees with the one we have described above, cf. Proposition 2.1.4.

Suppose now that F is an equal characteristic n-dimensional local field. Then the first to last residue field, F_1, is a local field; it is either of the form $\mathbb{F}_q((t_1))$ or a finite extension of $\mathbb{Q}_p$ for some prime p. We consider the usual locally compact topology on F_1. Let us choose an isomorphism

$$F \cong F_1((t_2)) \cdots ((t_{n-1})),$$

and apply inductively the process which has been described for defining a topology on a field of Laurent series on $F_1((t_2)) \cdots ((t_{n-2}))$.

Next, let L be a complete discrete valuation field with $\operatorname{char} L \neq \operatorname{char} \overline{L}$, so that $L\{\{t\}\}$ is a complete discrete valuation field of mixed characteristic. Suppose that a translation invariant and Hausdorff topology has been defined on L. Let $\{V_i\}_{i \in \mathbb{Z}}$ be a sequence of neighbourhoods of zero of L satisfying the following two conditions:

(i) There is $c \in \mathbb{Z}$ such that $\mathfrak{p}_L^c \subset V_i$ for every $i \in \mathbb{Z}$.

(ii) For every $l \in \mathbb{Z}$ there is $i_0 \in \mathbb{Z}$ such that for every $i \geq i_0$ we have $\mathfrak{p}_L^l \subset V_i$. This condition simply means that as $i \to \infty$ the neighbourhoods of zero V_i become bigger and bigger. We will denote this condition by $V_i \to L$ as $i \to \infty$.

The sets of the form

$$\sum V_i t^i := \left\{ \sum_{i \in \mathbb{Z}} x_i t^i \in L\{\{t\}\}\,;\ x_i \in V_i \text{ for all } i \right\} \tag{1.6}$$

constitute the basis of neighbourhoods of zero for a translation invariant and Hausdorff topology on $L\{\{t\}\}$ [MZ95, §1].

Suppose now that F is a standard mixed characteristic n-dimensional local field; in this case there is a local field of characteristic zero K such that we can give an isomorphism

$$F \cong K\{\{t_1\}\}\cdots\{\{t_r\}\}((t_{r+1}))\cdots((t_{n-1}))\,,$$

with $r \in \{1,\dots,n-1\}$ being characterized by the fact that char $F_r \neq$ char F_{r+1}. We topologize F by applying the two constructions specified above. Namely, we endow K with its $\mathfrak{p}$-adic topology, and for every $k \in \{1,\dots,r\}$, we apply on $E\{\{t_k\}\}$ the construction in which open neighbourhoods of zero are given by (1.6), with $E = K\{\{t_1\}\}\cdots\{\{t_{k-1}\}\}$. For $k \in \{r+1,\dots,n-1\}$, we apply on $E((t_k))$ the construction in which neighbourhoods of zero are given by (1.5), with

$$E = K\{\{t_1\}\}\cdots\{\{t_r\}\}((t_{r+1}))\cdots((t_{k-1}))\,.$$

Remark 1.3.8. We have chosen explicit isomorphisms in our construction. If $r < n$ the higher topology on F depends on the choice of isomorphism. More precisely, it depends on a choice of a coefficient field [MZ95, §1.4]. This is due to the fact that, since in this case char $\overline{F} = 0$ and $\overline{F}$ is transcendental over $\mathbb{Q}$, there are infinitely many choices for such an embedding, as explained in §1.3.1.

Similarly, the higher topology may be shown not to depend on any choices as soon as char $F_{n-1} > 0$ [MZ95, §1.3].

We have not described yet how to define the higher topology on a general mixed characteristic local field. Suppose that F is a general n-dimensional local field of mixed characteristic.

The choice of a system of parameters $t_1 = \pi_F, t_2, \ldots, t_{n-1}$ determines a *canonical lifting*, that is: a set-theoretic section of the residue homomorphism

$$h : \overline{F} \to \mathcal{O}_F.$$

with some special properties, see [MZ95, Lemma 1.2] for the actual definition.

Assume that the topology on $\overline{F}$ has already been constructed. Let $\{U_i\}_{i\in\mathbb{Z}}$ be a sequence of open neighbourhoods of zero in $\overline{F}$, and assume that there is an index $i_0 \in \mathbb{Z}$ such that $U_i = \overline{F}$ for all $i \geq i_0$. Define

$$\mathcal{U} = \left\{ \sum_{i \gg -\infty} h(f_i)\pi_F^i \in F;\ f_i \in U_i,\ \text{for all } i \gg -\infty \right\}. \tag{1.7}$$

The sets of the form (1.7) define the basis of open neighbourhoods of zero for a group topology on F [MZ95, §1].

There is an alternative way to define the higher topology on F via choosing a finite standard subfield $M \hookrightarrow F$. After this, we topologize M using the procedure explained above for standard fields and give F the product topology using the linear isomorphism $F \simeq M^{[F:M]}$. We obtain the same topology on F in all cases [MZ95, §1].

Since the topology constructed on F does not depend on any choices when $\mathrm{char}\overline{F} > 0$, we will use the determinate article and speak about *the* higher topology in this case. Whenever $\mathrm{char}\,\overline{F} = 0$, we still use the indeterminate article and speak about *a* higher topology, unless an isomorphism $F \cong \overline{F}((t))$ has been fixed.

We summarize the properties of higher topologies below.

Proposition 1.3.9. *Let F be an n-dimensional local field. Then any higher topology on F satisfies the following properties:*

(i) $(F, +)$ *is a topological group which is complete and Hausdorff.*

(ii) *If $n > 1$, every base of neighbourhoods of zero is uncountable.*

(iii) *If* $n > 1$*, multiplication* $F \times F \to F$ *is not continuous. However, both multiplication* $F \times F \to F$ *and inversion* $F^\times \to F^\times$ *are sequentially continuous.*

(iv) *Multiplication by a fixed nonzero element* $F \to F$ *is a homeomorphism.*

(v) *The residue homomorphism* $\rho : \mathcal{O}_F \to \overline{F}$ *is open.*

Proof. See [FK00, §1.3.2] for parts *(i)* to *(iv)*. Regarding *(v)*, the statement is a well known fact but unfortunately seems to be unavailable in the literature; we shall provide a proof for completeness.

First assume that $F \simeq \overline{F}((t))$ is of equal-characteristic. In such case, let U_0 be an open neighbourhood of zero in $\overline{F}$. Then

$$\rho^{-1}(U_0) = U_0 + \sum_{i \geq 1} \overline{F} t^i = U_0 + t\mathcal{O}_F,$$

which is open in $\mathcal{O}_F$. Moreover, an open neighbourhood of zero $\mathcal{U} \subseteq \mathcal{O}_F$ is of the form $\mathcal{U} = \sum_{i \geq 0} U_i t^i$ where $U_i \subseteq \overline{F}$ are open neighbourhoods of zero, and we have $\rho(\mathcal{U}) = U_0$.

Second, assume that F is of mixed-characteristic. Without loss of generality, we may assume that $F \simeq F'\{\{t\}\}$ is standard, as openness of maps is preserved on finite cartesian products. Let $U_0 \subseteq \overline{F}$ be an open neighbourhood of zero. Then $\rho^{-1}(U_0) = h(U_0) + \pi_F \mathcal{O}_F$, which is open. Finally, if $\mathcal{U}$ is as in (1.7), then $\rho(\mathcal{U}) = U_0$, which is open. □

A higher topology takes into account the topology of the residue field and has some good properties, but in general it is not a ring topology. While the additive group is always a topological group with respect to this topology, multiplication is only sequentially continuous. It is for this reason and the fact that many computations involving higher local fields are of a sequential nature, that sequential topology plays an important role in the theory of higher local fields.

A group topology on an abelian group is said to be *linear* if the filter of neighbourhoods of the identity element admits a collection of subgroups as a basis. A commutative ring R provided with a topology τ which is linear for the additive group and for which multiplication maps

$$R \to R, \quad y \mapsto xy$$

are continuous for all $x \in R$ is said to be a semitopological ring.

In view of (i) and (iv) in Proposition 1.3.9, it is possible to show that a higher local field endowed with a higher topology is a semitopological ring. A theory of semitopological rings has been developed and applied to the study of higher fields by Yekutieli [Yek92] and others.

There seems to be a disagreement between the sequential and linear approaches to topologies on higher local fields.

Theorem 1.3.10. *Let F be a higher local field. Denote τ for the topology on F defined by Proposition 1.3.9, and let τ_s be its sequential saturation. The collections of open subgroups for τ and τ_s agree.*

Proof. We split the proof in several steps. Let $F_i = t^i\mathcal{O}_F$.

Step 1. A subset $A \subseteq F$ is sequentially open if and only if $A \cap F_i$ is sequentially open in F_i for every i.

Suppose that $A \cap F_i$ is sequentially open in F_i for every i. Let $x_n \to x \in A$. Then, there is an index j such that $x_n \in F_j$ for all n, and since F_j is sequentially closed, $x \in F_j$. As $A \cap F_j$ is sequentially open and $x \in A \cap F_j$, almost all of the x_n belong to $A \cap F_j \subset A$, showing that A is sequentially open.

Step 2. Let $Y \subset F_i$ be a subset such that $0 \in Y$. Y is sequentially open if and only if it contains a subgroup $\mathcal{U} = \sum_{j\geq i} U_j t^j$, where the residues of elements in U_j are open subgroups of $\overline{F}$ and $U_j = \overline{F}$ if j is large enough.

Let u be a lift of a uniformizer of $\overline{F}$. If no such subgroup $\mathcal{U}$ is contained in Y, then for every n there is an element $y_n \in t^i\left(u^n O_F + t^n\mathcal{O}_F\right)$. However, $y_n \to 0 \in Y$ and therefore almost all of the y_n must belong to Y, a contradiction.

Step 3. A subgroup of H of F is open for τ if and only if it is open for τ_s.

Suppose that H is open for τ_s. By step 1, $H \cap F_i$ is sequentially open and by step 2 it contains a subset $U_i t^i$, such that the image of U_i in $\overline{F}$ is open and it contains F_n for some n.

Put $\mathcal{U} = \sum U_i t^i \cap F$.

If char $F =$ char $\overline{F}$, U_i may be chosen so that it is an open subgroup of $\overline{F}$ viewed inside F. Otherwise, U_i may be modified to ensure that $\mathcal{U}$ is a subgroup of H such that the image of U_i in $\overline{F}$ is an open subgroup and $\mathcal{U}$ contains some F_n. In both cases, $\mathcal{U}$ is an open subgroup for τ.

Since H is the union of $\mathcal{U}$-cosets, H is also an open subgroup for τ. □

Corollary 1.3.11. *For a general higher local field F and a higher topology τ, we have that τ_s is not a linear topology.*

Proof. A linear topology is completely determined by its collection of open subgroups. After the previous theorem, it suffices to show that $\tau \neq \tau_s$. So we recover a counterexample from [Fes01].

Let $F = \mathbb{F}_p((u))((t))$. Let $C = \{t^a u^{-c} + t^{-a} u^c,\ a, c \geq 1\}$. Then $W = F \setminus C$ is open for τ_s.

Suppose that $U_i \subset \mathbb{F}_p((u))$ are open subgroups such that $U_i = \mathbb{F}_p((u))$ if i is large enough and such that $\mathcal{U} = \sum U_i t^i \cap F$ is contained in W. Then, for any positive c such that $u^c \in U_{-a}$ we would have $t^a u^{-c} + t^a u^c \in W$, a contradiction. Hence, W is not open for τ. □

Although the approach to higher topologies by using linear topologies and semitopological rings is useful for the study of the construction of higher adeles by means of ind-pro functors, the sequential approach is very important from the point of view of higher class field theory. When dealing with rational points over higher local fields, the sequential approach will allow us to say something about the continuity of polynomial maps $R^n \to R$, whereas this is not possible with a semitopological ring.

1.3.3 Higher local fields arising from arithmetic geometric contexts

Chapters §2 and §3 require us to acknowledge that whenever n-dimensional fields arise from an arithmetic geometric setting they always present themselves in the company of an embedding of a (one-dimensional) local field into them.

This idea was first introduced in [Mor10a] for the two-dimensional case; allow us to explain why this is the case in general.

Let S be the spectrum of the ring of integers of a number field and $f : X \to S$ be an arithmetic scheme of dimension n (for our purposes, it is enough to suppose that X is an

n-dimensional regular scheme and that f is projective and flat). Given a complete flag of irreducible subschemes $\eta_n \in \overline{\{\eta_{n-1}\}} \subset \cdots \subset \overline{\{\eta_0\}} = X$, and assuming for simplicity that η_n is regular in $\overline{\{\eta_i\}}$ for each $0 \leq i \leq n-1$, define $A^n = \widehat{\mathcal{O}_{X,\eta_n}}$ and

$$A^i = \widehat{A^{i+1}_{\eta_i}}, \quad i \in \{0, \ldots, n-1\}.$$

It can be shown [Mor, Remark 6.12] that $F = A^0$ is an n-dimensional local field. The ring homomorphism $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$ induces a field embedding $K \hookrightarrow F$, where $K = \operatorname{Frac}\left(\widehat{\mathcal{O}_{S,f(x)}}\right)$.

This is the reason why in Chapters §2 and §3 we study n-dimensional local fields not as fields F, but as pairs of a field F and a field embedding $K \hookrightarrow F$. We shall refer to such a pair as an *n-dimensional local field over K*. A morphism of higher local fields over K is therefore a commutative diagram of field embeddings

$$\begin{array}{ccc} F_1 & \longrightarrow & F_2 \\ \uparrow & \nearrow & \\ K & & \end{array}$$

where F_1 and F_2 are higher local fields and $F_1 \to F_2$ is an extension of complete discrete valuation rings.

1.3.4 Other types of higher fields

There are other fields which can be defined in a similar way as a higher local field, and for which the construction of a higher topology is still valid. We give here two important examples of such. Let k be any perfect field.

Definition 1.3.12. A *zero-dimensional complete field* is a perfect field. For $n \geq 1$, an *n-dimensional complete field* is a complete discrete valuation field such that its residue field is an $(n-1)$-dimensional complete field. If F is an n-dimensional complete field with last residue field k, we say that F is an n-dimensional complete field over k.

Example 1.3.13. Higher local fields coincide with higher dimensional complete fields over a finite field.

Higher dimensional complete fields over arbitrary perfect fields may be classified using the same techniques which are used for the classification of higher local fields [MZ95, Theorem in §0]. For example, if char $F > 0$ and the last residue field is k, we may choose a system of parameters $t_1, \ldots, t_n$ such that

$$F \simeq k((t_1)) \cdots ((t_n)) .$$

The reason why we can consider topological issues in this more general setting is because the construction of a higher topology does not use at any point the fact that the last residue field is finite. The constructions which have been outlined in §1.3.2 in the particular case of higher local fields apply also in this case, see [Fes95, §2] for details. In this work, Fesenko considered the class field theory of n-dimensional complete fields over a perfect field of positive characteristic. Regarding higher topology, the main result of interest, which completely resembles Proposition 1.3.9, is the following.

Proposition 1.3.14. *An n-dimensional complete field F over a perfect field of positive characteristic may be topologized in such a way that:*

(i) $(F, +)$ *is a topological group which is complete and Hausdorff.*

(ii) *If $n > 1$, every base of open neighbourhoods of zero is uncountable.*

(iii) *Multiplication $F \times F \to F$ and inversion $F^\times \to F^\times$ are sequentially continuous.*

(iv) *Multiplication by a fixed non-zero element $F \to F$ is a homeomorphism.*

(v) *The residue homomorphism $\rho : \mathcal{O}_F \to \overline{F}$ is open.*

Proof. Consider the last residue field of F as a topological field with respect to the discrete topology, and apply the inductive process described in §1.3.2. □

Hence, these are more general examples of sequential rings.

A notion of higher archimedean local field exists, and these may be topologized applying the same tools.

Definition 1.3.15. The only *one-dimensional archimedean local fields* are $\mathbb{R}$ and $\mathbb{C}$. Let $n > 1$. An *n-dimensional archimedean local field* is a complete discrete valuation field such that its residue field is a $(n-1)$-dimensional archimedean local field.

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

Remark 1.3.16. The dimension of an archimedean higher local field does not agree with its dimension as a higher complete field over $\mathbb{K}$: an n-dimensional archimedean local field is an $(n-1)$-dimensional complete field over $\mathbb{K}$. The reason for this shift in the dimension is because, since $\mathbb{K}$ is itself already a local field, we want to put fields such as $\mathbb{R}((t))$ and $\mathbb{Q}_p((t))$ in the same box; both fields are two-dimensional with the definitions we have taken.

As the field $\mathbb{K}$ is of characteristic zero, higher archimedean local fields are easily classified: they are isomorphic to Laurent power series fields over $\mathbb{K}$.

Proposition 1.3.17. *Let F be an n-dimensional archimedean local field. Then, there are parameters $t_2, \ldots, t_n \in F$ such that*

$$F \simeq \mathbb{K}((t_2)) \cdots ((t_n)) .$$

Proof. Since $\operatorname{char} \mathbb{K} = 0$, all the residue fields of F have characteristic zero; the result follows by induction in the dimension and by Cohen structure theory for equal dimensional complete fields. □

Regarding the way to topologize such fields; we will always consider the euclidean topology on $\mathbb{K}$. This topology satisfies the conditions by which we have constructed a topology on fields of Laurent series, in which neighbourhoods of zero are of the form (1.5). We apply the construction specified in the aforementioned Proposition inductively, in order to obtain the following result.

Proposition 1.3.18. *An n-dimensional archimedean local field F may be topologized in such a way that:*

(i) $(F, +)$ *is a topological group which is complete and Hausdorff.*

(ii) *If $n > 1$, every base of open neighbourhoods of zero is uncountable.*

(iii) *Multiplication $F \times F \to F$ and inversion $F^\times \to F^\times$ are sequentially continuous.*

(iv) *Multiplication by a fixed non-zero element $F \to F$ is a homeomorphism.*

(v) *The residue homomorphism $\rho : \mathcal{O}_F \to \overline{F}$ is open.*

Remark 1.3.19. In the case of archimedean higher local fields, a higher topology does depend on the choice of a system of parameters, as we have that both the field and its first residue field are of characteristic zero.

Remark 1.3.20. Higher rank valuation rings (1.4) may also be defined in the case of archimedean higher local fields and higher complete fields. In the first case, the system of parameters has $n-1$ elements and we can only construct a chain of $n-1$ subrings by this procedure.

1.3.5 The unit group of a higher local field

Let F be an n-dimensional local field. There are several approaches to constructing topologies on $F^\times$, which we will describe here.

As described in §1.2, the way in which we shall endow the unit group of F with a topology is by embedding it into two copies of F using the map given by (1.3).

Another topology on $F^\times$ is often used. After choosing parameters for F, we obtain a decomposition

$$F^\times \simeq \mathbb{Z}t_n \oplus \cdots \oplus \mathbb{Z}t_1 \oplus F_0^\times \oplus V_F, \tag{1.8}$$

where F_0 is the last residue field and V_F is the group of principal units. $F^\times$ is provided with a topology by (1.8) once $\mathbb{Z}t_n \oplus \cdots \oplus \mathbb{Z}t_1 \oplus F_0^\times$ is given the discrete topology and $V_F \subset \mathcal{O}_F$ the subspace topology for the higher topology.

It is important to remark that both topologies on $F^\times$ agree after taking sequential saturations.

Proposition 1.3.21. *The two topologies described above for $F^\times$ have the same sequential saturation.*

Proof. Denote the topology defined by (1.8) by τ, and the initial topology defined by (1.3) by λ. Although this is implicitly included in [Fes01], we describe the explicit argument here.

A sequence $a_m = t_n^{i_{n,m}} \cdots t_1^{i_{1,m}} u_m$ of elements of $F^\times$, with u_m in the unit group of the ring of integers of F with respect to any of its discrete valuations of rank n tends to 1 in λ if and only if the sequence $u_m - 1$ tends to 0 with respect to the higher dimensional topology on F (described by Proposition 1.3.9) and for every j such that $1 \leq j \leq m$, we have $i_{j,m} = 0$ for sufficiently large m. But this last condition is equivalent to the sequence a_m converging to 1 with respect to τ. □

We summarize the properties of this topology on units below.

Proposition 1.3.22. *The topology we have defined on $F^\times$ satisfies:*

(i) The group $F^\times$ is a topological group with respect to this topology only when F is of dimension at most 2.

(ii) The topology does not depend on the choice of a system of parameters.

(iii) $F^\times$ is complete.

(iv) Multiplication on $F^\times$ is sequentially continuous.

Proof. See [MZ95, §3] for *(i)* to *(iii)*, [FK00, §1.4] for the rest. □

Example 1.3.23. The n-th Milnor K-group of any field F is presented as the term in the right in the following exact sequence of abelian groups:

$$0 \to I_n \to F^\times \otimes_{\mathbb{Z}} \overset{(n)}{\cdots} \otimes_{\mathbb{Z}} F^\times \to K_n(F) \to 0,$$

where $I_n = \langle a_1 \otimes \cdots \otimes a_n;\ a_i + a_j = 1 \text{ for some } i \neq j \rangle_{\mathbb{Z}}$. By definition, $K_1(F) = F^\times$ and by convention $K_0(F) = \mathbb{Z}$.

When F is an n-dimensional local field, K_n generalises the role of $K_1 = \mathbb{G}_m$ in describing abelian extensions of the field [FK00]. From this point of view, it provides a correct higher dimensional generalization of the group of units. The functor K_n is not representable for $n \geq 2$, meaning that in general $K_n(F)$ is not the set of F-rational points on any scheme.

Let F be an n-dimensional local field, and consider the topology on F (resp. $F^\times$) given by Proposition 1.3.9 (resp. 1.3.22).

Consider the finest topology on $K_m(F)$ for which:

(i) The symbol map $F^\times \times \cdots \times F^\times \to K_m(F)$ is sequentially continuous.

(ii) Subtraction $K_m(F) \times K_m(F) \to K_m(F)$ is sequentially continuous.

This topology is sequentially saturated [Fes01, §4, Remark 1].

The topological m-th Milnor K-group is

$$K_m^{\mathrm{t}}(F) = K_m(F)/\Lambda_m(F), \tag{1.9}$$

where $\Lambda_m(F)$ is the intersection of all open neighbourhoods of zero.

A sequentially continuous Steinberg map $F^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} F^\times \to G$ where G is a Hausdorff topological group induces a continuous homomorphism $K_m^{\mathrm{t}}(F) \to G$. The Artin-Schreier-Parshin pairing, the Vostokov pairing and the tame symbol are examples of such continuous homomorphisms defined on $K_m^{\mathrm{t}}(F)$ [FK00, I.6.4].

Chapter 2

Functional analysis on two-dimensional local fields

In this chapter we will concern ourselves with the study of two-dimensional local fields over a fixed (one-dimensional) local field of characteristic zero.

Notation. Throughout this chapter, K will denote a characteristic zero local field, that is, a finite extension of $\mathbb{Q}_p$ for some prime p. The cardinality of the finite field $\overline{K}$ will be denoted by q. The absolute value of K will be denoted by $|\cdot|$, normalised so that $|\pi_K| = q^{-1}$. Due to far too many appearances in the text, we will ease notation by letting $\mathcal{O} := \mathcal{O}_K$, $\mathfrak{p} := \mathfrak{p}_K$ and $\pi := \pi_K$.

The conventions $\mathfrak{p}^{-\infty} = K$, $\mathfrak{p}^{\infty} = \{0\}$ and $q^{-\infty} = 0$ will be used.

There are two particular local fields which play a very distinguished role when these objects are to be studied from a functional analytic point of view. Those are the fields $K(\!(t)\!)$ and $K\{\{t\}\}$. As we will see, most topological properties which hold in these particular cases will hold in general after taking restrictions of scalars or a base change over a finite extension which topologically is equivalent to taking a finite cartesian product. It is for this reason that we will work from now on with these two particular examples of two-dimensional local fields. We will explain how our results extend to the general case in §2.5.

Notation. When working with the two-dimensional local fields $F = K\{\{t\}\}$ or $F =$

$K((t))$, for any collection $\{A_i\}_{i\in\mathbb{Z}}$ of subsets of K, we will denote

$$\sum_{i\in\mathbb{Z}} A_i t^i = \left\{ \sum_i x_i t^i \in F;\ x_i \in A_i \text{ for all } i \in \mathbb{Z} \right\}.$$

We will also denote $\mathcal{O}_{K\{\{t\}\}} = \mathcal{O}\{\{t\}\}$. After all, this ring consists of all power series in $K\{\{t\}\}$ all of whose coefficients lie in $\mathcal{O}$.

2.1 Higher topologies are locally convex

In this section we will explain how the higher topology on $K((t))$ and $K\{\{t\}\}$ is a locally convex topology.

We are forced to study both cases separately.

2.1.1 Equal characteristic

We recall from §1.3.2 that the higher topology on $K((t))$ is defined as follows. Let $\{U_i\}_{i\in\mathbb{Z}}$ be a collection of open neighbourhoods of zero in K such that, if i is large enough, $U_i = K$. Then define

$$\mathcal{U} = \sum_{i\in\mathbb{Z}} U_i t^i. \tag{2.1}$$

The collection of sets of the form $\mathcal{U}$ defines the set of neighbourhoods of zero of the higher topology.

Proposition 2.1.1. *The higher topology on $K((t))$ defines the structure of a locally convex K-vector space.*

Proof. As K is a local field, the collection of open neighbourhoods of zero admits a collection of open subgroups as a filter, that is: the basis of neighbourhoods of zero for the topology is generated by the sets of the form

$$\mathfrak{p}^n = \{a \in K;\ v_K(a) \geq n\},$$

where $n \in \mathbb{Z}\cup\{-\infty\}$. These closed balls are not only subgroups, but $\mathcal{O}$-fractional ideals.

This in particular means that the sets of the form

$$\Lambda = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{n_i} t^i \subseteq K((t)), \tag{2.2}$$

where $n_i = -\infty$ for large enough i, generate the higher topology. Moreover, they are not only additive subgroups, but also $\mathcal{O}$-modules.

If $x = \sum_{i \geq i_0} x_i t^i \in K((t))$ is an arbitrary element, and i_1 is such that $n_i = -\infty$ for all $i > i_1$ then we have the possibilities:

(i) $i_0 > i_1$, in which case $x \in \Lambda$.

(ii) $i_0 \leq i_1$. In such case, let

$$n = \max\left(\max_{i_0 \leq i \leq i_1} n_i, 0 \right).$$

Then $\pi^n \in \mathcal{O}$ satisfies $\pi^n x \in \Lambda$.

Thus, Λ is a lattice and the higher topology is locally convex. □

As a consequence of the previous proposition, it is possible to describe the higher topology in terms of seminorms.

Corollary 2.1.2. *For any sequence $(n_i)_{i \in \mathbb{Z}} \subset \mathbb{Z} \cup \{-\infty\}$ such that there is an integer k satisfying $n_i = -\infty$ for all $i > k$, define*

$$\| \cdot \| : K((t)) \to \mathbb{R}, \qquad \sum_{i \gg -\infty} x_i t^i \mapsto \max_{i \leq k} |x_i| q^{n_i}. \tag{2.3}$$

Then $\| \cdot \|$ is a seminorm on $K((t))$ and the higher topology on $K((t))$ is the locally convex topology defined by the family of seminorms given by (2.3) as $(n_i)_{i \in \mathbb{Z}}$ varies over all sequences specified above.

Proof. This result is a consequence of Proposition 2.1.1 and of the fact that the gauge seminorm attached to a lattice of the form

$$\Lambda = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{n_i} t^i$$

with $n_i = \infty$ for all $i > k$ is precisely the one given by (2.3). In order to see that, let $x = \sum_{i \geq i_0} x_i t^i \in K((t))$ and $a \in K$. We have that $x \in a\Lambda$ if and only if $x_i \in a\mathfrak{p}^{n_i}$ for every $i \geq i_0$. This is the case if and only if we have

$$|x_i| q^{n_i} \leq |a|$$

for all $i \geq i_0$. The infimum value of $|a|$ for which the above inequality holds is precisely the supremum of the values of $|x| q^{n_i}$ for $i \geq i_0$. $\square$

The seminorm $\|\cdot\|$ from the previous corollary is associated to and does depend on the choice of the sequence $(n_i)_{i \in \mathbb{Z}}$. If we have chosen notation not to reflect this fact, it is in hope that a lighter notation will simplify reading and that the sequence of integers defining $\|\cdot\|$, when needed, will be clear from the context.

Remark 2.1.3. As F is a field, it is worth asking ourselves whether the seminorm (2.3) is multiplicative. It is very easy to check that for $i, j \in \mathbb{Z}$,

$$\|t^i\| \cdot \|t^j\| = q^{n_i + n_j},$$

while

$$\|t^{i+j}\| = q^{n_{i+j}}.$$

These two values need not coincide in general.

The field of Laurent series $K((t))$ has been considered previously from the point of view of the theory of locally convex spaces in the following manner. The ring of Taylor series $K[[t]]$ is isomorphic to $K^{\mathbb{N}}$ as a K-vector space, and thus might be equipped with the product topology of countably many copies of K. Moreover, we have

$$K((t)) = \cup_{i \in \mathbb{Z}} t^i K[[t]], \tag{2.4}$$

with $t^i K[[t]] \cong K^{\mathbb{N}}$. Therefore, we may topologize $K((t))$ as a strict inductive limit.

In the result below, we explain how the higher topology on $K((t))$ agrees with this description. We will immediately deduce most of the analytic properties of $K((t))$ from this result.

Proposition 2.1.4. *The higher topology on* $K((t))$ *agrees with the strict inductive limit topology given by (2.4).*

Proof. The open lattices for the product topology on $K^{\mathbb{N}}$ are exactly the ones obtained as the intersection of finitely many lattices of the form

$$\prod_{i\in I}\Lambda_i \times \prod_{i\notin I} K,$$

where I is a finite subset of $\mathbb{N}$ and Λ_i are open lattices in K, that is, integer powers of $\mathfrak{p}$. This description agrees with the description of the open lattices in $K[\![t]\!]$ for the subspace topology induced by the higher topology.

Further, if $\Lambda = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{n_i} t^i$ is an open lattice for the higher topology on $K((t))$, for any $j \in \mathbb{Z}$, we have that $\Lambda \cap t^j K[\![t]\!] = \sum_{i\geq j} \mathfrak{p}^{n_i} t^i$ is an open lattice for the product topology on $t^j K[\![t]\!] \cong K^{\mathbb{N}}$.

Finally, a basis of open lattices for the strict inductive limit $\bigcup_{j\in\mathbb{Z}} t^j K[\![t]\!]$ is determined by a collection of open lattices $\Lambda_j \subseteq t^j K[\![t]\!]$ for each $j \in \mathbb{Z}$ [Sch02, Lemma 5.1.iii], which we may assume to be of the form $\Lambda_j = \sum_{i\geq j} \mathfrak{p}^{n_{i,j}} t^i$ for some sequence $(n_{i,j})_{i\geq j} \subset \mathbb{Z} \cup \{-\infty\}$ for which there is an index $k_i \geq i$ such that $n_{i,j} = -\infty$ for all $j \geq k_i$. The fact that the inductive limit is strict amounts to the following: if $i_1 < i_2$ then we have $n_{i_1,j} = n_{i_2,j}$ for every $j \geq i_2$ and, in particular, $k_{i_1} = k_{i_2}$. Altogether, this determines a sequence $(n_i)_{i\in\mathbb{Z}} \subset \mathbb{Z} \cup \{-\infty\}$ and an index $k \in \mathbb{Z}$ such that $n_i = -\infty$ for every $i \geq k$. Under the identification $K((t)) = \bigcup_{j\in\mathbb{Z}} t^j K[\![t]\!]$, the lattice associated to $(\Lambda_j)_{j\in\mathbb{Z}}$ is $\Lambda = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{n_i} t^i$, which is open for the higher topology. □

Remark 2.1.5. The higher topology on $K((t))$ also admits the following description as an inductive limit. For each $i \in \mathbb{Z}$ and $j \geq i$, $t^i K[t]/t^j K[t]$ is a finite dimensional K-vector space and we endow it with its unique Hausdorff locally convex topology. The field of Laurent series might be constructed as

$$K((t)) = \varinjlim_{i\in\mathbb{Z}} \varprojlim_{j\geq i} t^i K[t]/t^j K[t];$$

the higher topology on it agrees with the one obtained by endowing the direct and inverse limits in the above expression with the corresponding direct and inverse limit locally convex topologies. The proof of this statement is a restatement of Proposition 2.1.4.

Corollary 2.1.6. *$K((t))$ is an LF-space. As a locally convex space, it is complete, bornological, barrelled, reflexive and nuclear.*

Proof. After Proposition 2.1.4, in order to see that $K((t))$ is an LF-space it suffices to check that the locally convex space $K^{\mathbb{N}}$ endowed with the product topology is a Fréchet space. This follows from the fact that K itself is Fréchet and that a countable product of Fréchet spaces is a Fréchet space [PGS10, Corollary 3.5.7].

Completeness follows from being a strict inductive limit of complete spaces [Sch02, Lemma 7.9]. Being bornological follows from [Sch02, Proposition 8.2] and [Sch02, Examples after Prop. 6.13].

Reflexivity follows from [PGS10, Corollary 7.4.23], if one notes that $K((t)) \cong \bigoplus_{\mathbb{N}} K \oplus \prod_{\mathbb{N}} K$ in the category of locally convex K-spaces; barrelledness follows from reflexivity (cf. Proposition 1.1.13). Finally, nuclearity follows from Proposition 1.1.17 □

2.1.2 Mixed characteristic

The higher topology on $K\{\{t\}\}$ may be described as follows.

Let $\{V_i\}_{i\in\mathbb{Z}}$ be a sequence of open neighbourhoods of zero in K such that

(i) There is $c \in \mathbb{Z}$ such that $\mathfrak{p}^c \subseteq V_i$ for every $i \in \mathbb{Z}$.

(ii) For every $l \in \mathbb{Z}$ there is an index $i_0 \in \mathbb{Z}$ such that $\mathfrak{p}^l \subseteq V_i$ for every $i \geq i_0$.

Then define

$$\mathcal{V} = \sum_{i\in\mathbb{Z}} V_i t^i \subset K\{\{t\}\}\,. \tag{2.5}$$

The higher topology on $K\{\{t\}\}$ is the group topology defined by taking the sets of the form $\mathcal{V}$ as the collection of open neighbourhoods of zero [MZ95, §1].

Again, as K is a local field, the collection of neighbourhoods of zero admits the collection of open subgroups as a filter. These are not only subgroups but $\mathcal{O}$-fractional ideals, namely the integer powers of the prime ideal $\mathfrak{p}$.

Proposition 2.1.7. *Let $(n_i)_{i\in\mathbb{Z}} \subset \mathbb{Z}\cup\{-\infty\}$ be a sequence restricted to the conditions:*

(i) There is $c \in \mathbb{Z}$ such that $n_i \leq c$ for every i.

(ii) For every $l \in \mathbb{Z}$ there is an index $i_0 \in \mathbb{Z}$ such that $n_i \leq l$ for every $i \geq i_0$.

The set

$$\Lambda = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{n_i} t^i \tag{2.6}$$

is an $\mathcal{O}$-lattice. The sets of the form (2.6) generate the higher topology on $K\{\{t\}\}$, which is locally convex.

Condition (ii) is equivalent, by definition of limit of a sequence, to having $n_i \to -\infty$ as $i \to \infty$; we will phrase it this way in the future.

Proof. It is clear that Λ is an $\mathcal{O}$-module, and that the conditions imposed on the indices n_i imply that it is a basic neighbourhood of zero for the higher topology.

Given an arbitrary element $x = \sum_{i=-\infty}^{\infty} x_i t^i \in F$, we must show the existence of an element $a \in K^\times$ such that $ax \in \Lambda$. Indeed, a power of the uniformizer does the trick: we have that $\pi^n x \in \Lambda$ if and only if $\pi^n x_i \in \mathfrak{p}^{n_i}$ for every $i \in \mathbb{Z}$, and this is true if and only if

$$n + v_K(x_i) \geq n_i$$

for all $i \in \mathbb{Z}$. In other words, such an n exists if and only if the difference

$$n_i - v_K(x_i)$$

cannot be arbitrarily large. But on one hand there is an integer c that bounds the n_i from above, and on the other hand the values $v_K(x_i)$ are bounded below by $v_F(x)$. We may take $n = c - v_F(x)$.

Because the integer powers of $\mathfrak{p}$ generate the basis of neighbourhoods of zero of the topology on K, the lattices of the form (2.6) generate the higher topology. In particular, the higher topology on $K\{\{t\}\}$ is locally convex. □

We wish to point out that condition (ii) for the sequence $(n_i)_{i\in\mathbb{Z}}$ has not been used in the proof. Indeed, such a condition may be suppressed and we would still obtain a locally convex topology on $K\{\{t\}\}$, if only finer: see Remark 2.2.8 for a description of the topology obtained in such case.

Once we know that the higher topology is locally convex, we can describe it in terms of seminorms.

Corollary 2.1.8. *For any sequence $(n_i)_{i\in\mathbb{Z}} \subset \mathbb{Z} \cup \{-\infty\}$ satisfying the conditions:*

(i) there is $c \in \mathbb{Z}$ such that $n_i \leq c$ for all $i \in \mathbb{Z}$,

(ii) $n_i \to -\infty$ as $i \to \infty$,

consider the seminorm

$$\| \cdot \| : K\{\{t\}\} \to \mathbb{R}, \quad \sum_{i\in\mathbb{Z}} x_i t^i \mapsto \sup_{i\in\mathbb{Z}} |x_i| q^{n_i}. \tag{2.7}$$

The higher topology on $K\{\{t\}\}$ is the locally convex topology generated by the family of seminorms defined by (2.7), as $(n_i)_{i\in\mathbb{Z}}$ varies over the sequences specified above.

Proof. The gauge seminorm associated to the lattice Λ is (2.7). The argument is the same as the proof of Corollary 2.1.2 and we omit it. □

The seminorms in Corollary 2.1.8 are well defined because they arise as gauge seminorms attached to lattices. If we forget this fact for a moment, let us examine the values $|x_i|q^{n_i}$.

On one hand, when i tends to $-\infty$, the values $|x_i|$ tend to zero while the values q^{n_i} stay bounded. On the other hand, when i tends to $+\infty$ the values $|x_i|$ stay bounded and q^{n_i} tends to zero. In conclusion, the values $|x_i|q^{n_i}$ are all positive and tend to zero when $|i| \to +\infty$; this implies the existence of their supremum.

Just like in the equal characteristic case, a defining seminorm $\|\cdot\|$ is not multiplicative, for the same reason.

A mixed characteristic two-dimensional local field cannot be viewed as a direct limit in a category of locally convex K-vector spaces in the fashion of Remark 2.1.5. However, such an approach is valid from an algebraic point of view in a category of $\mathcal{O}$-modules.

2.1.3 First properties

For starters, let us recall some properties from §1.3.2. A two-dimensional local field $K \hookrightarrow F$ endowed with a higher topology is a Hausdorff topological group. Moreover, multiplication by a fixed nonzero element defines a homeomorphism $F \to F$ and the residue map $\mathcal{O}_F \to \overline{F}$ is open when $\mathcal{O}_F$ is given the subspace topology and the local field $\overline{F}$ is endowed with its usual complete discrete valuation topology.

Remark 2.1.9. In order to show that $K((t))$ or $K\{\{t\}\}$ is Hausdorff, it suffices to show that given a nonzero element x, there is a continuous seminorm $\|\cdot\|$ for which $\|x\| \neq 0$. This is obvious.

Multiplication $\mu : F \times F \to F$ fails to be continuous when the product topology is considered on the left hand side. However, μ is separately continuous as explained above.

No basis of open neighbourhoods of zero for a higher topology is countable. In other words, these topologies do not satisfy the first countability axiom. This implies that the set of seminorms defining the higher topology is uncountable. From the point of view of functional analysis, this shows that two-dimensional local fields are not Fréchet spaces, as such topologies are not metrizable.

Definition 2.1.10. We will call seminorms of the form (2.3) in the equal characteristic case and (2.7) in the mixed characteristic case *admissible.*

In both cases, admissible seminorms are attached to a sequence $(n_i)_{i\in\mathbb{Z}} \subset \mathbb{Z}\cup\{-\infty\}$, subject to different conditions, but satisfying the formula

$$\left\|\sum_i x_i t^i\right\| = \sup_i |x_i| q^{n_i};$$

the reasons why this formula is valid differ in each case.

Remark 2.1.11. Power series expressions of the form $x = \sum_i x_i t^i$ define convergent series with respect to the higher topology, in the sense that the net of partial sums $\left(\sum_{i\leq n} x_i t^i\right)_{n\in\mathbb{Z}}$ converges to x. If we let $S_n = \sum_{i\leq n} x_i t^i$ and $\|\cdot\|$ be any admissible seminorm, then

$$\|x - S_n\| = \left\| \sum_{i>n} x_i t^i \right\|$$

may be shown to be arbitrarily small if n is large enough.

Another well-known fact is that rings of integers $K[\![t]\!]$ and $\mathcal{O}\{\{t\}\}$ are closed but not open. In the first case, consider the set of open (and closed) lattices

$$\Lambda_n = \sum_{i \leq 0} \mathfrak{p}^n t^i + K[\![t]\!], \quad n \geq 0$$

to find that $K[\![t]\!] = \bigcap_{n \geq 0} \Lambda_n$ is closed. In the second case, consider the open (and closed) lattices:

$$\Lambda_n = \sum_{i<n} K t^i + \mathcal{O} t^n + \sum_{i>n} K t^i, \quad n \in \mathbb{Z}$$

and obtain that $\mathcal{O}\{\{t\}\} = \bigcap_{n \in \mathbb{Z}} \Lambda_n$ is closed. In order to see that these rings are not open, it is enough to say that they do not contain any open lattice.

A very similar argument shows that the rank-two rings of integers $\mathcal{O} + tK[\![t]\!]$ and $\sum_{i<0} \mathfrak{p} t^i + \sum_{i \geq 0} \mathcal{O} t^i$ are closed but not open.

After the previous remark, we get the following result.

Proposition 2.1.12. *The field $K\{\{t\}\}$ is not barrelled.*

Proof. The ring of integers $\mathcal{O}\{\{t\}\}$ is a lattice which is closed but not open. □

2.2 Bounded sets and bornology

Let us describe the nature of bounded subsets of $K((t))$ and $K\{\{t\}\}$. We will supply a description of a basis for the Von-Neumann bornology of these fields.

Example 2.2.1. Let $\|\cdot\|$ be an admissible seminorm, attached to the sequence $(n_i)_{i \in \mathbb{Z}}$. The values of $\|\cdot\|$ on $\mathcal{O}$ only depend on n_0. If $n_0 = -\infty$ then the restriction of $\|\cdot\|$ to $\mathcal{O}$ is identically zero. Otherwise, for any $x \in \mathcal{O}$ we have $\|x\| \leq q^{n_0}$ and therefore $\mathcal{O}$ is bounded.

Similarly, if $n_0 > -\infty$, we may find elements $x \in K$ making the value $|x|q^{n_0}$ arbitrarily large. Hence, K is unbounded.

Proposition 2.2.2. *For any sequence* $(k_i)_{i\in\mathbb{Z}} \subset \mathbb{Z}\cup\{\infty\}$ *such that there is an index* $i_0 \in \mathbb{Z}$ *for which* $k_i = \infty$ *for every* $i < i_0$*, consider the* $\mathcal{O}$*-submodule of* $K((t))$ *given by*

$$B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{k_i} t^i. \tag{2.8}$$

The bornology of $K((t))$ *admits as a basis the collection of* $\mathcal{O}$*-submodules given by (2.8) as* $(k_i)_{i\in\mathbb{Z}}$ *varies over the sequences specified above.*

Proof. First, the $\mathcal{O}$-submodule B given by (2.8) is bounded: suppose that $\|\cdot\|$ is an admissible seminorm on $K((t))$ given by the sequence $(n_i)_{i\in\mathbb{Z}}$ and that k is the index for which $n_i = -\infty$ for every $i > k$.

If $k < i_0$, then the restriction of $\|\cdot\|$ to B is identically zero. Otherwise, for $x = \sum_{i\geq i_0} x_i t^i \in B$,

$$\|x\| = \max_{i_0\leq i\leq k} |x_i| q^{n_i} \leq \max_{i_0\leq i\leq k} q^{n_i - k_i},$$

and the bound is uniform for $x \in B$ once $\|\cdot\|$ has been fixed.

Next, we study general bounded sets. From Example 2.2.1 we deduce that if a subset of $K((t))$ contains elements for which one coefficient can be arbitrarily large, then the subset is unbounded in F. Therefore, any bounded subset of $K((t))$ is included in a subset of the form

$$\sum_{i\in\mathbb{Z}} \mathfrak{p}^{k_i} t^i, \quad k_i \in \mathbb{Z}\cup\{\infty\}.$$

In order to prove our claim, it is enough to show that the indices $k_i \in \mathbb{Z}\cup\{\infty\}$ may be taken to be equal to ∞ for all small enough i.

We will show the contrapositive: a subset $D \subset K((t))$ cannot be bounded as soon as there is a decreasing sequence of indices $(i_j)_{j\geq 0} \in \mathbb{Z}_{<0}$ satisfying that for every $j \geq 0$ there is an element $\xi_{i_j} \in D$ with a nonzero coefficient in degree i_j, which we denote $x_{i_j} \in K$.

For, if such is the case, let

$$n_i = \begin{cases} -\infty, & i \neq i_j \text{ for any } j \geq 0 \text{ or } i > 0, \\ -i_j + v_K(x_{i_j}), & i = i_j \text{ for some } j \geq 0; \end{cases}$$

and consider the associated admissible seminorm $\|\cdot\|$ on $K((t))$. We have

$$\|\xi_{i_j}\| \geq |x_{i_j}| q^{n_{i_j}} = q^{-i_j}$$

for every $j \geq 0$, and this shows that D is not bounded. □

Corollary 2.2.3. *If $\|\cdot\| : K((t)) \to \mathbb{R}$ is a seminorm which is bounded on bounded sets, then there is an index $i_0 \in \mathbb{Z}$ such that $\|t^i\| = 0$ for all $i \geq i_0$.*

Proof. Suppose that for every $i_0 \in \mathbb{Z}$ there is an $i \geq i_0$ such that $\|t^i\| \neq 0$. If i is such that $\|t^i\| > 0$, take $k_i \in \mathbb{Z}$ such that

$$q^{-k_i}\|t^i\| \geq q^i.$$

If i is such that $\|t^i\| = 0$, take $k_i = 0$. By Proposition 2.2.2, the set

$$B = \sum_{i \geq 0} \mathfrak{p}^{k_i} t^i$$

is bounded. Let $x_j = \pi_K^{k_j} t^j$ for every $j \geq 0$. We have that $\|x_j\| = q^{-k_j}\|t^j\|$. Our hypothesis implies that the sequence of real numbers $(\|x_j\|)_{j\geq 0}$ is unbounded, and therefore $\|\cdot\|$ is not bounded on B. □

Proposition 2.2.4. *Consider a sequence $(k_i)_{i\in\mathbb{Z}} \subset \mathbb{Z} \cup \{\infty\}$ which is bounded below. The bornology of $K\{\{t\}\}$ admits the $\mathcal{O}$-submodules of the form*

$$B = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{k_i} t^i \tag{2.9}$$

as a basis.

Proof. First, let us show that B is bounded. Assume all the k_i in (2.9) are bounded below by some integer d. Let $\|\cdot\|$ be an admissible seminorm on $K\{\{t\}\}$ defined by a sequence $(n_i)_{i\in\mathbb{Z}} \subset \mathbb{Z} \cup \{-\infty\}$. In particular, there is an integer c such that $n_i \leq c$ for every $i \in \mathbb{Z}$.

Then, if $\sum x_i t^i \in B$, we have that

$$\left\|\sum x_i t^i\right\| = \sup_i |x_i| q^{n_i} \leq q^{c-d},$$

and the bound is uniform on B once $\|\cdot\|$ has been fixed.

Next, we study general bounded sets. Again, from Example 2.2.1 we may deduce that a subset of $K\{\{t\}\}$ which contains elements with arbitrarily large coefficients cannot be bounded. Therefore, any bounded subset of $K\{\{t\}\}$ is contained in a set of the form

$$\sum_{i\in\mathbb{Z}} \mathfrak{p}^{k_i} t^i, k_i \in \mathbb{Z} \cup \{\infty\}.$$

In order to prove our claim, it is enough to show that the indices k_i may be taken to be bounded below.

Suppose that $D \subset K\{\{t\}\}$ is not contained in a set of the form (2.9). Then it must contain elements with arbitrarily large coefficients. More precisely, at least one of the following must happen:

1. There is a decreasing sequence $(i_j)_{j\geq 0} \subset \mathbb{Z}_{<0}$ and a sequence $(\xi_{i_j})_{j\geq 0} \subset D$ such that, if $x_{i_j} \in K$ denotes the coefficient in degree i_j of ξ_{i_j}, we have $|x_{i_j}| \to \infty$ as $j \to \infty$.

2. There is an increasing sequence $(i_j)_{j\geq 0} \subset \mathbb{Z}_{\geq 0}$ and a sequence $(\xi_{i_j})_{j\geq 0} \subset D$ such that, if $x_{i_j} \in K$ denotes the coefficient in degree i_j of ξ_{i_j}, we have $|x_{i_j}| \to \infty$ as $j \to \infty$.

If condition 1 holds, consider the admissible seminorm $\|\cdot\|$ associated to the sequence

$$n_i = \begin{cases} 0, & \text{if } i \leq 0, \\ -\infty, & \text{if } i > 0. \end{cases}$$

We have that $\|\xi_{i_j}\| \geq |x_{i_j}|$ for all $j \geq 0$ and this implies that D cannot be bounded.

If condition 1 does not hold, then condition 2 must hold. In such case, define

$$n_{i_j} = \begin{cases} \frac{v_K(x_{i_j})-1}{2} & \text{if } v_K(x_{i_j}) \text{ is odd}, \\ \frac{v_K(x_{i_j})}{2}, & \text{if } v_K(x_{i_j}) \text{ is even}. \end{cases}$$

Furthermore, let $n_i = -\infty$ for any $i < 0$ and $n_l = n_{i_j}$ for any index l such that

$i_j \leq l < i_{j+1}$. With such choices, the following three facts hold:

(i) The sequence $(n_{i_j} - v_K(x_{i_j}))_{j\geq 0}$ tends to infinity.

(ii) The sequence $(n_i)_{i\in\mathbb{Z}}$ is bounded above.

(iii) For any $l \in \mathbb{Z}$, there is an index i_0 such that $n_i \leq l$ for all $i \geq i_0$.

After (ii) and (iii), let $\|\cdot\|$ be the admissible seminorm associated to $(n_i)_{i\in\mathbb{Z}}$. We have, for every $j \geq 0$, $\|\xi_{i_j}\| \geq |x_{i_j}|q^{n_{i_j}}$, and thus D cannot be bounded. □

Definition 2.2.5. Given that they constitute a basis for the Von-Neumann topology, we will refer any $\mathcal{O}$-submodule of the form (2.8) (resp. (2.9)) as a *basic bounded $\mathcal{O}$-submodule* of $K((t))$ (resp. $K\{\{t\}\}$).

Corollary 2.2.6. *If $\|\cdot\| : K\{\{t\}\} \to \mathbb{R}$ is a seminorm which is bounded on bounded sets, then there is a real number $C > 0$ such that $\|t^i\| < C$ for every $i \in \mathbb{Z}$.*

Proof. Suppose that $\|\cdot\|$ is a seminorm such that the sequence of real numbers $(\|t^i\|)_{i\in\mathbb{Z}}$ is not bounded. Consider the bounded set

$$\mathcal{O}\{\{t\}\} = \sum_{i\in\mathbb{Z}} \mathcal{O}t^i,$$

and the sequence $(t^i)_{i\in\mathbb{Z}} \subset \mathcal{O}\{\{t\}\}$. The seminorm $\|\cdot\|$ is not bounded on $\mathcal{O}_F$. □

Contrary to the situation in the equal characteristic case, in the mixed characteristic setting we get the following.

Corollary 2.2.7. *The space $K\{\{t\}\}$ is not bornological.*

Proof. It is enough to supply a seminorm which is bounded on bounded sets but not continuous.

Consider the norm on $K\{\{t\}\}$ given by

$$\left\| \sum_{i\in\mathbb{Z}} x_i t^i \right\| = \sup_{i\in\mathbb{Z}} |x_i|, \tag{2.10}$$

which is the absolute value on $K\{\{t\}\}$ related to the valuation v_F. If B is a basic bounded set as in (2.9), then

$$\sup_{x\in B}\|x\| = \sup_{i\in\mathbb{Z}} q^{-k_i},$$

and hence $\|\cdot\|$ is bounded on bounded sets. However, the norm $\|\cdot\|$ is not continuous on $K\{\{t\}\}$ because

$$\mathcal{O}\{\{t\}\} = \{x \in K\{\{t\}\}\,;\ \|x\| \leq 1\}$$

is closed but not open in $K\{\{t\}\}$. □

Remark 2.2.8. When defining the higher topology on $K\{\{t\}\}$, an admissible seminorm was attached to a sequence $(n_i)_{i\in\mathbb{Z}} \subset \mathbb{Z}\cup\{-\infty\}$ subject to two conditions:

(i) The n_i are bounded above.

(ii) We have $n_i \to -\infty$ as $i \to \infty$.

However, in the proof of Proposition 2.1.7 we did not require to make use of condition (ii).

Indeed, if we remove condition (ii) and allow all sequences $(n_i)_{i\in\mathbb{Z}}$ satisfying only condition (i), we obtain a locally convex topology. Let us describe it: on one hand, the norm on $K\{\{t\}\}$ given by

$$\sum_{i\in\mathbb{Z}} x_i t^i \mapsto \sup_{i\in\mathbb{Z}} |x_i|,$$

becomes continuous, as it corresponds to taking $n_i = 0$ for all $i \in \mathbb{Z}$. Hence, the resulting locally convex topology is both finer than the higher topology and finer than the complete discrete valuation topology. It is an immediate exercise to see that under such a topology the ring of integers $\mathcal{O}\{\{t\}\}$ is a bounded open lattice and this is equivalent to the locally convex topology being defined by a single seminorm [Sch02, Proposition 4.11]. We conclude that the resulting topology is the complete discrete valuation topology.

It is immediate to check that the complete discrete valuation topology on $K\{\{t\}\}$ defines a Banach K-algebra structure with very nice analytic properties. However, it is unclear whether this structure is of any arithmetic interest.

Proposition 2.2.9. *Let $F = K((t))$ or $K\{\{t\}\}$. The multiplication map $\mu : F\times F \to F$ is bounded with respect to the product bornology on the domain.*

Proof. Let $B_1 = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{m_i}t^i$ and $B_2 = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{n_j}t^j$ be two bounded $\mathcal{O}$-submodules of F. The product bornology on $F\times F$ is generated by sets of the form $B_1 \times B_2$. We have that $\mu(B_1, B_2) = \sum_{k\in\mathbb{Z}} V_k t^k$ with $V_k = \sum_{k=i+j} \mathfrak{p}^{m_i}\mathfrak{p}^{n_j} = \sum_{k=i+j} \mathfrak{p}^{m_i+n_j}$. We distinguish cases.

If $F = K((t))$, $m_i = \infty$ and $n_j = \infty$ if i and j are small enough. In this case, the sum defining V_k is actually finite and there is $l_k \in \mathbb{Z}\cup\{\infty\}$ such that $V_k \subset \mathfrak{p}^{l_k}$. Moreover, we actually have $V_k = \{0\}$ if k is small enough and therefore $\mu(B_1, B_2) \subset F$ is bounded.

If $F = K\{\{t\}\}$, then there are integers c and d such that $m_i \geq c$ for all $i \in \mathbb{Z}$ and $n_j \geq d$ for all $j \in \mathbb{Z}$. This implies that $V_k \subset \mathfrak{p}^{c+d}$ for every k and that it is bounded. □

2.3 Complete, c-compact and compactoid $\mathcal{O}$-submodules

In this section we will study relevant $\mathcal{O}$-submodules of $K((t))$ and $K\{\{t\}\}$, including rings of integers and rank-2 rings of integers.

We start dealing with completeness of rings of integers.

Proposition 2.3.1. *The rings of integers $K[[t]]$ and $\mathcal{O}\{\{t\}\}$ are complete $\mathcal{O}$-submodules of $K((t))$ and $K\{\{t\}\}$, respectively.*

In the case of $K[[t]] \subset K((t))$, the result follows because $K((t))$ is complete and $K[[t]]$ is a closed subset. However, it is also immediate to give an argument by hand.

Proof. Let I be a directed set and $(x_i)_{i\in I}$ a Cauchy net in the ring of integers. We distinguish cases below.

In the case of $K[[t]]$, we write $x_i = \sum_{k\geq 0} x_{k,i}t^k$ with $x_{k,i} \in K$. Since $(x_i)_{i\in I}$ is a Cauchy net in $\mathcal{O}_F$, we have that $(x_{k,i})_{i\in I}$ is a Cauchy net in K and hence converges to an element $x_k \in K$ for every $k \geq 0$. The element $x = \sum_{k\geq 0} x_k t^k$ is the limit of the Cauchy net.

In the case of $\mathcal{O}\{\{t\}\}$, the procedure is very similar. We write $x_i = \sum_{k\in\mathbb{Z}} x_{i,k}t^k$ with $x_{i,k} \in \mathcal{O}$. Since $\mathcal{O}$ is complete and $(x_{i,k})_{i\in I}$ is a Cauchy net, it converges to an element $x_k \in \mathcal{O}$ for every $k \in \mathbb{Z}$. It is elementary to check that as $k \to -\infty$, we have $x_k \to 0$ and therefore $x = \sum_{k\in\mathbb{Z}} x_k t^k$ is a well-defined element in $\mathcal{O}\{\{t\}\}$ which is the limit of the Cauchy net. □

Corollary 2.3.2. *The rank-2 rings of integers of $K((t))$ and $K\{\{t\}\}$ are complete.*

Proof. It follows from the previous proposition due to the fact that they are closed subsets of complete $\mathcal{O}$-submodules. □

Next we will study rings of integers from the point of view of c-compactness and compactoidicity.

Proposition 2.3.3. *$K[\![t]\!]$ is c-compact.*

Proof. As a locally convex K-vector space, $K[\![t]\!]$ is isomorphic to $K^{\mathbb{N}}$ (Proposition 2.1.4). The field K is c-compact (Example 1.1.6). Finally, a product of c-compact spaces is c-compact (Proposition 1.1.8). □

Corollary 2.3.4. *The rank-2 ring of integers of $K((t))$, $\mathcal{O} + tK[\![t]\!]$, is c-compact.*

Proof. After the previous proposition, the result follows from the fact that $\mathcal{O} + tK[\![t]\!] \subset K[\![t]\!]$ is closed, as c-compactness is hereditary for closed subsets [Sch02, Lemma 12.1.iii]. □

Corollary 2.3.5. *The rings $K[\![t]\!]$ and $\mathcal{O} + tK[\![t]\!]$ are not compactoid.*

Proof. This follows from the fact that they are both c-compact, unbounded, complete and Proposition 1.1.10. □

The compactoid submodules of a locally convex vector space define a bornology. Since every compactoid submodule is bounded, in our case it is important to decide which basic bounded submodules of $K((t))$ and $K\{\{t\}\}$ are compactoid.

Since $K((t))$ is a nuclear space, the class of bounded $\mathcal{O}$-submodules and compactoid $\mathcal{O}$-submodules coincide [Sch02, Proposition 19.2].

It is in any case easy to see that any basic bounded subset

$$B = \sum_{i \geq i_0} \mathfrak{p}^{k_i} t^i, \quad k_i \in \mathbb{Z} \cup \{\infty\}$$

is compactoid: suppose that $\Lambda = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{n_i} t^i$ with $n_i \in \mathbb{Z} \cup \{-\infty\}$ and such that for every $i > i_1$ we have $n_i = -\infty$. If $i_1 < i_0$ then $B \subset \Lambda$ and there is nothing to show.

Otherwise, let $l_i = \min(n_i, k_i)$ for $i_0 \leq i \leq i_1$. Then

$$B \subseteq \Lambda + \sum_{i=i_0}^{i_1} \mathcal{O} \cdot \pi^{l_i} t^i,$$

which shows that it is compactoid.

Corollary 2.3.6. *The basic bounded $\mathcal{O}$-submodules of $K((t))$ are c-compact.*

Proof. In view of Proposition 1.1.10, it is enough to show that a submodule B as in the proof of the previous proposition is complete for nets. But the argument for showing completeness of such $\mathcal{O}$-submodules is the same as in the proof of Proposition 2.3.1 and we shall omit it. □

In the case of $K\{\{t\}\}$ there is a difference between bounded and compactoid $\mathcal{O}$-submodules. For the proof of the following proposition we will consider the projection maps

$$\pi_j : K\{\{t\}\} \to K, \quad \sum_{i \in \mathbb{Z}} x_i t^i \mapsto x_j, \quad j \in \mathbb{Z}.$$

These are examples of continuous nonzero linear forms on $K\{\{t\}\}$.

Proposition 2.3.7. *The only compactoid submodules amongst the basic bounded submodules of $K\{\{t\}\}$ are the ones of the form*

$$B = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{k_i} t^i, \tag{2.11}$$

with $k_i \in \mathbb{Z}$ bounded below and such that $k_i \to \infty$ as $i \to -\infty$. In particular, the submodules of the form (2.11) describe a basis for the bornology on $K\{\{t\}\}$ generated by compactoid submodules.

Proof. Let B be a basic bounded submodule as in (2.11), with the $k_i \in \mathbb{Z} \cup \{\infty\}$ bounded below.

On one hand, assume that $k_i \to \infty$ as $i \to -\infty$. Let $\Lambda = \sum_{i \in \mathbb{Z}} \mathfrak{p}^{n_i} t^i$ be an open lattice and assume that B is not contained in Λ, as otherwise there is nothing to prove. When $i \to \infty$, the k_i are bounded below and $n_i \to -\infty$. Similarly, when $i \to -\infty$, $k_i \to \infty$ and the n_i are bounded above. Hence, the following two statements are true:

(i) There is an index i_0 such that for every $i < i_0$, $k_i \geq n_i$.

(ii) There is an index i_1 such that for every $i > i_1$, $k_i \geq n_i$.

We have $i_0 \leq i_1$, as otherwise B is contained in Λ. Let $l_i = \min(k_i, n_i)$ for $i_0 \leq i \leq i_1$. Then we have

$$B \subseteq \Lambda + \sum_{i=i_0}^{i_1} \mathcal{O} \cdot \pi^{l_i} t^i,$$

which shows that B is compactoid.

On the other hand, suppose that the k_i do not tend to infinity as $i \to -\infty$. In such case, there is a decreasing sequence $(i_j)_{j\geq 0} \subset \mathbb{Z}_{<0}$ such that $(k_{i_j})_{j\geq 0}$ is bounded above. Let $M \in \mathbb{Z}$ be such that $k_{i_j} < M$ for every $j \geq 0$.

Let

$$\Lambda = \sum_{i<0} \mathfrak{p}^M t^i + \sum_{i\geq 0} K t^i \subset K\{\{t\}\},$$

which is an open lattice. Suppose that $x_1, \ldots, x_m \in K\{\{t\}\}$ satisfy that $B \subseteq \Lambda + \mathcal{O}x_1 + \cdots + \mathcal{O}x_m$. We denote $x_l = \sum_{i\in\mathbb{Z}} x_{l,i} t^i$, with $x_{l,i} \in K$, for $1 \leq l \leq m$.

We know that for $1 \leq l \leq m$, we have $x_{l,i} \to 0$ as $i \to -\infty$. Therefore, there is an index $j_0 \geq 0$ such that for every $j \geq j_0$ we have $v_K(x_{l,i_j}) > M$. Then we have

$$\pi_{i_{j_0}}(B) \subseteq \pi_{i_{j_0}}(\Lambda + \mathcal{O}x_1 + \cdots + \mathcal{O}x_m),$$

from where we deduce

$$\mathfrak{p}^{k_{i_{j_0}}} \subseteq \mathfrak{p}^M + \mathfrak{p}^{v_K(x_{1,i_{j_0}})} + \cdots + \mathfrak{p}^{v_K(x_{m,i_{j_0}})} = \mathfrak{p}^M.$$

However, this inclusion contradicts the fact that $k_{i_{j_0}} < M$. □

Definition 2.3.8. We will refer in the sequel to the $\mathcal{O}$-submodules of the form (2.8) (resp. (2.11)) as *basic compactoid submodules* of $K((t))$ (resp. $K\{\{t\}\}$).

We deduce several consequences of this result.

Corollary 2.3.9. *The field $K\{\{t\}\}$ is not a nuclear space.*

Proof. After the previous Proposition and (i) in Proposition 1.1.17, the result follows by observing that $K\{\{t\}\}$ contains $\mathcal{O}$-submodules, such as $\mathcal{O}\{\{t\}\}$, which are bounded but not compactoid. □

Corollary 2.3.10. *The basic compactoid submodules of $K\{\{t\}\}$ are c-compact.*

Proof. Again, in view of Proposition 1.1.10, it is enough to show that the $\mathcal{O}$-submodule B as in (2.11) is complete. The argument is the same as in the proof of Proposition 2.3.1 and we omit it. □

The proof of the following corollary is immediate after Proposition 2.3.7.

Corollary 2.3.11. *$\mathcal{O}\{\{t\}\}$ and the rank-two ring of integers of $K\{\{t\}\}$ are not compactoid nor c-compact.*

Proof. The fact that these rings are not compactoid follows from Proposition 2.3.7. The fact that they are not c-compact follows from the fact that, on top of not being compactoid, they both are bounded and complete. □

Corollary 2.3.12. *Multiplication $\mu : K\{\{t\}\} \times K\{\{t\}\} \to K\{\{t\}\}$ is also bounded when $K\{\{t\}\}$ is endowed with the bornology generated by compactoid $\mathcal{O}$-submodules in the codomain, and the product of two copies of such bornology in the domain.*

Proof. Let $B_1 = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{m_i} t^i$ and $B_2 = \sum_{j\in\mathbb{Z}} \mathfrak{p}^{n_j} t^j$ be two basic compactoid submodules of $K\{\{t\}\}$; let V_k as in the proof of Proposition 2.2.9. Just like in the aforementioned proof, V_k is a contained in a fractional ideal of K and is therefore bounded. Moreover, it is possible to choose $l_k \in \mathbb{Z} \cup \{\infty\}$ such that $l_k \to \infty$ as $k \to -\infty$ and $V_k \subset \mathfrak{p}^{l_k}$; this proves that $\mu(B_1, B_2)$ is contained in a compactoid $\mathcal{O}$-submodule of $K\{\{t\}\}$. □

2.4 Duality

Let us describe some duality issues of two-dimensional local fields when regarded as locally convex vector spaces over a local field.

Much is known about the self-duality of the additive group of a two-dimensional local field. From [Fes03, §3], if F is a two-dimensional local field, once a nontrivial continuous character

$$\psi : F \to S^1 \subset \mathbb{C}^\times$$

has been fixed, the group of continuous characters of the additive group of F consists entirely of characters of the form $\alpha \to \psi(a\alpha)$, where a runs through all elements of F. This result is entirely analogous to the one-dimensional theory [Tat67, Lemma 2.2.1].

In the case of $K((t))$ and $K\{\{t\}\}$, self-duality of the additive group follows in an explicit way from two self-dualities: that of the two-dimensional local field as a locally convex K-vector space, and that of the additive group of K as a locally compact abelian group. Since the second is sufficiently documented [Tat67, §2.2], let us focus on the first one.

We have already exhibited nontrivial continuous linear forms on a two-dimensional local field. Let $F = K((t))$ or $K\{\{t\}\}$; the map

$$\pi_i : F \to K, \quad \sum x_j t^j \mapsto x_i \tag{2.12}$$

is a continuous nonzero linear form for all $i \in \mathbb{Z}$.

Consider now the following map:

$$\gamma : F \to F', \quad x \mapsto \pi_x,$$

with

$$\pi_x : F \to K, \quad y \mapsto \pi_0(xy).$$

More explicitly, if $x = \sum x_i t^i$ and $y = \sum y_i t^i$, then

$$\pi_x(y) = \sum x_i y_{-i}.$$

The map γ is well-defined because π_x, being the composition of multiplication by a fixed element $F \to F$ and the projection $\pi_0 : F \to K$, is a continuous linear form. Besides that, γ is K-linear and injective.

Remark 2.4.1. Regarding topologies on dual spaces, we have that $K((t))'_c = K((t))'_b$

after Proposition 1.1.17.(i). However, the topology of $K\{\{t\}\}'_c$ is strictly weaker than the one of $K\{\{t\}\}'_b$: consider the seminorm

$$|\cdot|_{\mathcal{O}\{\{t\}\}} : K\{\{t\}\}' \to \mathbb{R}, \quad l \mapsto \sup_{x\in\mathcal{O}\{\{t\}\}} |l(x)|,$$

which is continuous with respect to the b-topology. If $|\cdot|_{\mathcal{O}\{\{t\}\}}$ was continuous with respect to the c-topology, there would be a basic compactoid submodule $B \subset K\{\{t\}\}$ and a constant $C > 0$ such that $|l|_{\mathcal{O}\{\{t\}\}} \leq C|l|_B$ for all $l \in K\{\{t\}\}'$.

However, suppose that $B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{k_i} t^i$ with $k_i \to \infty$ as $i \to -\infty$. For any real number $C > 0$ there is an index $j \in \mathbb{Z}$ such that $Cq^{-k_j} < 1$. This implies the inequality

$$C|\pi_j|_B < |\pi_j|_{\mathcal{O}\{\{t\}\}}.$$

This shows that $|\cdot|_{\mathcal{O}\{\{t\}\}}$ is not continuous in the c-topology.

Theorem 2.4.2. *The map $\gamma : F \to F'_c$ is an isomorphism of locally convex K-vector spaces.*

Before we prove this result, we need an auxiliary result.

Lemma 2.4.3. *Let $w \in F'$ and define, for every $i \in \mathbb{Z}$, $a_i = w(t^{-i})$. Then the formal sum $\sum a_i t^i$ defines an element of F.*

Proof. We distinguish cases. If $F = K((t))$, it is necessary to show that $a_i = 0$ for all small enough indices i. In other words, that there is an index $i_0 \in \mathbb{Z}$ such that for every $i \geq i_0$ we have $w(t^i) = 0$. Without loss of generality, we may restrict ourselves to a continuous linear form $w : K[\![t]\!] \to K$. In this case we get our result from the following isomorphisms: first $K[\![t]\!] \cong K^{\mathbb{N}}$, second $(K^{\mathbb{N}})' \cong \bigoplus_{\mathbb{N}} K'$ [PGS10, Theorem 7.4.22], and third $K' \cong K$.

In the case in which $F = K\{\{t\}\}$, we need to show that the values $|a_i|$ for $i \in \mathbb{Z}$ are bounded and that $|a_i| \to 0$ as $i \to -\infty$. On one hand, the subset $\mathcal{O}\{\{t\}\} \subset F$ is bounded after Proposition 2.2.4 and $t^i \in \mathcal{O}\{\{t\}\}$ for every $i \in \mathbb{Z}$. As w is continuous, the set $w(\mathcal{O}\{\{t\}\}) \subset K$ is bounded and therefore the values $w(t^i)$ are bounded. On the other hand, the net $\left(t^i\right)_{i\in\mathbb{Z}}$ tends to zero in $K\{\{t\}\}$ as $i \to \infty$. As w is continuous, $a_i = w(t^{-i}) \to 0$ as $i \to -\infty$. ∎

Proof of Theorem 2.4.2. As explained above, the map γ is well-defined, K-linear and injective.

Let $w \in F'$. Define $x = \sum_i a_i t^i \in F$ with a_i as in Lemma 2.4.3. Then, for $y = \sum y_i t^i \in F$, we have

$$w\left(\sum y_i t^i\right) = \sum y_i w(t^i) = \sum y_i a_{-i} = \pi_0(xy)$$

(the first equality follows from Remark 2.1.11). Therefore, $w = \pi_x$ and the map δ is surjective.

In order to show bicontinuity, let us first work out what continuity means in this setting. For any $\varepsilon > 0$ and B a set in the bornology generated by compactoid submodules, we must show that there are $\delta > 0$ and an admissible seminorm $\|\cdot\| : F \to K$ such that $\|x\| \leq \delta$ implies $|\pi_x|_B \leq \varepsilon$.

Without loss of generality, we may replace ε and δ by integer powers of q, and the generic bounded set B by a basic compactoid submodule of F, which is of the form (2.8) in the equal characteristic case or (2.11) in the mixed characteristic case. For convenience, let us write

$$B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{k_i} t^i, \quad k_i \in \mathbb{Z} \cup \{\infty\}$$

by allowing, in the equal characteristic case, $k_i = \infty$ for every small enough i.

Now, let $n \in \mathbb{Z}$. We take $n_i = -k_{-i}$ for every $i \in \mathbb{Z}$. Because of the conditions defining B, the sequence $(n_i)_{i\in\mathbb{Z}}$ defines an admissible seminorm $\|\cdot\|$ in both cases. Now, for $x = \sum x_i t^i$, we have that $\|x\| \leq q^n$ if and only if for every index $i \in \mathbb{Z}$ we have

$$n_i - n \leq v_K(x_i). \tag{2.13}$$

Similarly, $|\pi_x|_B \leq q^n$ if and only if for every index $i \in \mathbb{Z}$ we have

$$-k_{-i} - n \leq v_K(x_i). \tag{2.14}$$

By direct comparison and substitution between (2.13) and (2.14), we have that with our choice of admissible seminorm $\|\cdot\|$,

$$\|x\| \leq q^n \text{ if and only if } |\pi_x|_B \leq q^n,$$

which shows bicontinuity. □

Remark 2.4.4. In the mixed characteristic case we may ask ourselves if it is possible to exhibit any self-duality result involving F'_b, that is, topologizing the dual space according to uniform convergence over all bounded sets.

It can be seen from the proof of Theorem 2.4.2 that this is not the case. Any bornology $\mathcal{B}$ stronger than the one generated by compactoid submodules will stop the map $\gamma : F \to F'_{\mathcal{B}}$ from being continuous.

We remark that if there were no other bounded sets in $K\{\{t\}\}$ besides the ones generated by compactoid submodules, it would be possible to show that such a locally convex vector space is bornological.

From the failure of $K\{\{t\}\}$ at being bornological we may deduce that Theorem 2.4.2 is the best possible result.

By applying Theorem 2.4.2 twice on $K((t))$, we recover the fact that this locally convex space is reflexive. Indeed, we have made this fact explicit via the choice of duality pairing:

$$K((t)) \times K((t)) \to K, \quad (x, y) \mapsto \pi_0(xy).$$

Corollary 2.4.5. *The field $K\{\{t\}\}$ is not reflexive.*

Proof. Since any reflexive space is barrelled (Proposition 1.1.13), the result follows from Proposition 2.1.12. □

In order to conclude this section let us describe polars and pseudo-polars of the $\mathcal{O}$-submodules which we have studied in §2.3.

After Theorem 2.4.2, the topological isomorphism given by γ allows us to identify F with F'_c, and in particular lets us relate their $\mathcal{O}$-submodules.

Definition 2.4.6. Let $F = K((t))$ or $K\{\{t\}\}$. Let $A \subset F$ be an $\mathcal{O}$-submodule. We let

$$A^\gamma = \gamma^{-1}(A^p) \subset F$$

and refer to it, by abuse of language, as the *pseudo-polar* of A.

Proposition 2.4.7. *Consider the $\mathcal{O}$-submodule*

$$A = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{k_i} t^i, \quad k_i \in \mathbb{Z} \cup \{\pm\infty\}$$

of $F = K((t))$ or $K\{\{t\}\}$. Then, we have

$$A^\gamma = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{1-k_{-i}} t^i.$$

Proof. Let $B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{1-k_{-i}} t^i$.

On one hand, suppose $x = \sum x_i t^i \in B$. We have, for every $y = \sum y_i t^i \in A$,

$$|\pi_x(y)| = \left|\sum x_{-i} y_i\right| \leq \sup |x_{-i}||y_i| = \sup q^{-1+k_i-k_i} < 1$$

and, therefore, $B \subseteq A^\gamma$.

On the other hand, suppose that $x = \sum x_i t^i \in A^\gamma$. Then, by definition, we have

$$|\pi_x(y)| < 1, \quad \text{for any } y \in A.$$

In particular, let $y = \pi^{k_i} t^i$. Then the inequality

$$|\pi_x(\pi^{k_i} t^i)| = |x_{-i} \pi^{k_i}| < 1$$

implies that $v_K(x_{-i}) \geq 1 - k_i$. Therefore $x_{-i} \in \mathfrak{p}^{1-k_i}$. Since our conclusion holds for any $i \in \mathbb{Z}$, we have that $x \in A^\gamma$ and, therefore, $B \subset A^\gamma$. □

After (iv) in Proposition 1.1.15, we may think of the following corollary as a proof that the submodule A in the statement of the previous Proposition is closed, as it is equal to its pseudo-bipolar; proofs to this result and the following two corollaries are immediate.

Corollary 2.4.8. *For an $\mathcal{O}$-submodule A as in the previous Proposition, we have $A^{pp} = A$.* □

Corollary 2.4.9. *We have $K[\![t]\!]^\gamma = K[\![t]\!]$. For the rank-2 ring of integers, we have $(\mathcal{O} + tK[\![t]\!])^\gamma = \mathfrak{p} + tK[\![t]\!]$.* □

Corollary 2.4.10. *We have $(\mathcal{O}\{\{t\}\})^\gamma = \mathfrak{p}\{\{t\}\}$. For the rank-2 ring of integers, we have $(\sum_{i<0} \mathfrak{p}t^i + \sum_{i\geq 0} \mathcal{O}t^i)^\gamma = \sum_{i\leq 0} \mathcal{O}t^i + \sum_{i>0} \mathfrak{p}t^i$.* □

By Proposition 1.1.15 and Theorem 2.4.2, pseudo-polarity exchanges open lattices and basic compactoid submodules. Under the characterization given by Proposition 2.4.7, the relation is evident.

The same arguments exposed apply to compute that the polar of the $\mathcal{O}$-submodule $\sum_{i\in\mathbb{Z}} \mathfrak{p}^{k_i}t^i$, $k_i \in \mathbb{Z} \cup \{\pm\infty\}$ is $\sum_{i\in\mathbb{Z}} \mathfrak{p}^{-k_{-i}}t^i$. As such, the polar of an open lattice is a compactoid lattice and vice versa.

Let us write down a table with pseudo-polars and polars of relevant $\mathcal{O}$-submodules:

A	$\mathbf{A}^\gamma$	**polar of A**
$K[\![t]\!]$	$K[\![t]\!]$	$K[\![t]\!]$
$\mathcal{O} + tK[\![t]\!]$	$\mathfrak{p} + tK[\![t]\!]$	$\mathcal{O} + tK[\![t]\!]$
$\mathcal{O}\{\{t\}\}$	$\mathfrak{p}\{\{t\}\}$	$\mathcal{O}\{\{t\}\}$
$\sum_{i<0} \mathfrak{p}t^i + \sum_{i\geq 0} \mathcal{O}t^i$	$\sum_{i\leq 0} \mathcal{O}t^i + \sum_{i>0} \mathfrak{p}t^i$	$\sum_{i\leq 0} \mathcal{O}t^i + \sum_{i>0} \mathfrak{p}^{-1}t^i$
$\Lambda = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{n_i}t^i$ (open lattice)	$B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{1-n_{-i}}t^i$ (compactoid)	$B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{-n_{-i}}t^i$ (compactoid)
$B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{k_i}t^i$ (basic compactoid)	$\Lambda = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{1-k_{-i}}t^i$ (open lattice)	$B = \sum_{i\in\mathbb{Z}} \mathfrak{p}^{-k_{-i}}t^i$ (open lattice)

The isomorphism $\gamma : F \to F'_c$ is not unique, as it depends on choosing a nonzero linear form on F, which in our case is π_0. For example, replacing π_0 by π_1 in the definition of γ would lead to an identical result. The actual shape of A^γ, for a given $\mathcal{O}$-submodule $A \subset F$, depends heavily on γ. However, the fact that polarity exchanges open lattices with compactoid submodules does not depend on γ.

In conclusion, taking the pseudo-polar or polar is a self-map on the set of $\mathcal{O}$-submodules of $K((t))$ or $K\{\{t\}\}$ which reverses inclusions, gives basic compactoid submodules when applied to open lattices and vice versa, and whose square equals the identity map when restricted to closed $\mathcal{O}$-submodules.

2.5 General two-dimensional local fields

In the previous sections of this chapter we have developed a systematic study of $K((t))$ and $K\{\{t\}\}$ from the point of view of the theory of locally convex spaces over K. Let us explain how the previous results extend to a general characteristic zero two-dimensional local field $K \hookrightarrow F$. The moral of the story is that we can link the higher topology on F to the constructions on $K((t))$ and $K\{\{t\}\}$ that we have performed in the preceding sections of this work by performing operations such as restriction of scalars along a finite extension and taking finite products, and due to their finite nature, none of these operations modifies the properties of the resulting locally convex spaces.

Due to the difference in their structures, we consider the equal characteristic and mixed characteristic cases separately.

2.5.1 Equal characteristic

Assume that $K \hookrightarrow F$ is a two-dimensional local field and that $\operatorname{char} F = \operatorname{char} \overline{F}$. In this case, as explained in §1.3.1, the choice of a field embedding $\overline{F} \hookrightarrow F$ determines an isomorphism $F \cong \overline{F}((t))$.

Denote the algebraic closure of K in F by $\widetilde{K}$. The extension $\widetilde{K}|K$ is finite and $\widetilde{K} \hookrightarrow F$ is the only coefficient field of F which factors the field inclusion $K \hookrightarrow F$ [Mor10a, Lemma 2.7], and this is the only coefficient field of F that we will take into account in our constructions. Whenever the dimension of F is greater than two such a choice of coefficient field is not available: it is true that the algebraic closure $\widetilde{K}$ of K inside F factors the field embedding $K \hookrightarrow F$, but we only obtain a coefficient field of F in the two-dimensional case.

Remark 2.5.1. It is a well-known fact that in this case the higher topology of F depends on the choice of a coefficient field [Yek92, Example 2.1.22]. This is why we stress that in this work the only coefficient field we consider is $\widetilde{K} \hookrightarrow F$ because the field embedding $K \hookrightarrow F$ is given a priori.

The $\widetilde{K}$-vector space $F \cong \widetilde{K}((t))$ is a complete, bornological, barrelled, reflexive and nuclear locally convex space by direct application of Corollary 2.1.6. The higher topology on F only depends on the choice of the embedding $\widetilde{K} \hookrightarrow F$ and, therefore, does not change by restriction of scalars along $K \hookrightarrow \widetilde{K}$.

Let us explain this fact with more detail. On one hand, all open lattices Λ are $\mathcal{O}$-modules by restriction of scalars. On the other hand, if $x \in F$, there is a positive power of $\pi_{\widetilde{K}}$ that maps x to Λ by multiplication. These facts are enough to deduce local convexity over K.

The absolute value on $\widetilde{K}$ restricts to the absolute value of K and therefore Corollary 2.1.2 describes the admissible seminorms of $K \hookrightarrow F$ without any changes.

Moreover, after Proposition 2.1.4 we have that the higher topology on $\widetilde{K}((t))$ agrees with the strict inductive limit topology given by

$$\widetilde{K}((t)) = \bigcup_{i \in \mathbb{Z}} t^i \widetilde{K}[\![t]\!] \,,$$

which is also a union of K-vector spaces. Since the extension $\widetilde{K}|K$ is finite, we also have that $\widetilde{K}[\![t]\!]$ is isomorphic to a product of countably many copies of K and is therefore a Fréchet K-vector space. Hence, we get that F is an LF-space over K and in particular we may deduce from Proposition 2.1.6 that F is a complete, bornological, barrelled, reflexive and nuclear K-vector space.

Because admissible seminorms do not change after restricting scalars to K, Proposition 2.2.2 describes a basis of bounded $\mathcal{O}$-submodules of F. These are complete, and from nuclearity we deduce that the classes of bounded $\mathcal{O}$-submodules and compactoid $\mathcal{O}$-submodules of F agree.

Since $\widetilde{K}$ is a finite dimensional K-vector space, it is c-compact. The ring of integers $\mathcal{O}_F = \widetilde{K}[\![t]\!]$ is therefore c-compact, being isomorphic to a product of copies of $\widetilde{K}$. It is unbounded, complete and not compactoid after Proposition 1.1.10. Similarly, the rank-2 ring of integers of F shares all these properties with $\mathcal{O}_F$.

Regarding duality, the fact that the map $\gamma : F \to F'_c$ is an isomorphism of locally convex spaces does not change when we restrict scalars to K. Explicit nonzero linear forms $F \to K$ may be constructed by composing the maps $\pi_i : F \to \widetilde{K}$ as in (2.12) with $\mathrm{Tr}_{\widetilde{K}|K}$.

Problem. It is relevant to decide whether the class of bounded sets of F changes along with the change of vector space and locally convex structures associated to the choice of a different coefficient field.

2.5.2 Mixed characteristic

If $\mathrm{char}\, F \neq \mathrm{char}\, \overline{F}$, then, as explained in §1.3.1, there is a unique field embedding $\mathbb{Q}_p \hookrightarrow F$. If we denote the algebraic closure of $\mathbb{Q}_p$ in F by $\widetilde{K}$, the field inclusion $K \hookrightarrow F$ may be factored into the following diagram of field embeddings

$$\begin{array}{ccccc} & & K\{\{t\}\} & \longrightarrow & \widetilde{K}\{\{t\}\} & \longrightarrow & F\ , \\ & & \uparrow & & \uparrow & & \\ \mathbb{Q}_p & \longrightarrow & K & \longrightarrow & \widetilde{K} & & \end{array}$$

in which all horizontal arrows are finite extensions.

The inclusions $K \hookrightarrow K\{\{t\}\}$ and $\widetilde{K} \hookrightarrow \widetilde{K}\{\{t\}\}$ correspond to the situation we have been dealing with in the preceding sections of this work. Let $n = [F : K\{\{t\}\}]$.

As K-vector spaces, we have

$$F \cong K\{\{t\}\}^n .$$

The higher topology on F may be defined as the product topology on n copies of the higher topology on $K\{\{t\}\}$ [MZ95, 1.3.2]. Furthermore, it does not depend on any choices of subfields $K\{\{t\}\} \subset F$ [Kat00, §1]. Hence, since the product topology on a product of locally convex vector spaces is locally convex, the inclusion $K \hookrightarrow F$ gives a locally convex K-vector space.

We may describe the family of open lattices or, equivalently, continuous seminorms, from the corresponding lattices or seminorms for $K\{\{t\}\}$ and Proposition 1.1.1.

The situation for the ring of integers $\mathcal{O}_F$ is as follows. The inclusion $\mathcal{O}\{\{t\}\} \hookrightarrow \mathcal{O}_F$ turns $\mathcal{O}_F$ into a rank-n free $\mathcal{O}\{\{t\}\}$-module. Therefore the subspace topology on $\mathcal{O}_F \subset F$ coincides with the product topology on $\mathcal{O}_F \cong \mathcal{O}\{\{t\}\}^n$. From here, it is possible to show that $\mathcal{O}_F$ is a bounded and complete $\mathcal{O}$-submodule of F which is neither c-compact nor compactoid. It is however closed, but not open, and this proves that F is not barrelled. The norm attached to the valuation v_F provides a example of a seminorm which is bounded on bounded sets but not continuous, as its unit ball, $\mathcal{O}_F$, is not open. Hence $K \hookrightarrow F$ is not bornological.

From the self-duality of $K\{\{t\}\}$, we obtain a chain of isomorphisms of locally convex K-vector spaces

$$F'_c \cong (K\{\{t\}\}^n)'_c \cong \left(K\{\{t\}\}'_c\right)^n \cong K\{\{t\}\}^n \cong F,$$

which shows that F is also self-dual. Explicit nonzero linear forms may be constructed in this case composing the trace map $\mathrm{Tr}_{F|K\{\{t\}\}}$ with the maps $\pi_i : K\{\{t\}\} \to K$ as in (2.12). Finally, as $\mathcal{O}_F$ is a bounded $\mathcal{O}$-submodule which is not compactoid, we deduce that $K \hookrightarrow F$ is not nuclear.

2.6 A note on the archimedean case

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We will denote by $|\cdot|$ either the usual absolute value on $\mathbb{R}$, or the module on $\mathbb{C}$.

In this section we will consider the study of archimedean two-dimensional local fields. An archimedean two-dimensional local field is a complete discrete valuation field F whose residue field is an archimedean (one-dimensional) local field. Hence, we have a non-canonical isomorphism $F \cong \mathbb{K}((t))$ for one of our two choices of $\mathbb{K}$. Once an inclusion of fields $\mathbb{K} \subset F$ has been fixed and t has been chosen, a unique such isomorphism is determined.

The theory of locally convex vector spaces over $\mathbb{K}$ was developed much earlier than the analogous non-archimedean theory and is well explained in, for example, [Köt69]. Let V be a $\mathbb{K}$-vector space and $C \subseteq V$. The subset C is said to be convex if for any

$v, w \in C$, the segment

$$\{\lambda v + \mu w; \lambda, \mu \in \mathbb{R}_{\geq 0}, \lambda + \mu = 1\}$$

is contained in C. The subset C is said to be absolutely convex if, moreover, we have $\lambda C \subseteq C$ for every $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$.

We may associate a seminorm p_C to any convex subset $C \subseteq V$ by the rule

$$p_C : V \to \mathbb{R}, \quad x \mapsto \inf_{\rho > 0,\, x \in \rho C} \rho.$$

This seminorm satisfies the usual triangle inequality, but not the ultrametric inequality.

Definition 2.6.1. The $\mathbb{K}$-vector space V is said to be locally convex if it is a topological vector space such that its topology admits a basis of neighbourhoods of zero given by convex sets.

It may be shown that if V is locally convex its filter of neighbourhoods of zero also admits a basis formed by absolutely convex subsets [Köt69, §18.1].

The higher topology on $\mathbb{K}((t))$ is defined following the procedure outlined in §1.3.2. In this case, we consider the disks of $\mathbb{K}$ centered at zero and of rational radius; this defines a countable basis of convex neighbourhoods of zero for the euclidean topology on $\mathbb{K}$. Denote

$$D_\rho = \{a \in \mathbb{K};\ |a| < \rho\}, \quad \rho \in \mathbb{Q}_{>0} \cup \{\infty\}.$$

Given a sequence $(\rho_i)_{i \in \mathbb{Z}} \subset \mathbb{Q}_{>0} \cup \{\infty\}$ such that there is an index i_0 satisfying that $\rho_i = \infty$ for all $i \geq i_0$, consider the set

$$\mathcal{U} = \sum_{i \in \mathbb{Z}} D_{\rho_i} t^i \subset F. \tag{2.15}$$

The sets of the form (2.15) form a basis of neighbourhoods of zero for the higher topology on F.

Proposition 2.6.2. *The higher topology on F is locally convex, in the sense of Definition 2.6.1.*

Proof. As the discs D_{ρ_i} are convex, given two elements $x, y \in \mathcal{U}$, it is easy to check that the segment

$$\{\lambda x + \mu y;\ \lambda, \mu \in \mathbb{R}_{\geq 0};\ \lambda + \mu = 1\}$$

is contained in $\mathcal{U}$ by checking on each coefficient separately.

Thus, the basis of open neighbourhoods of zero described by the sets of the form (2.15) consists of convex sets, and hence the higher topology on F is locally convex. □

As we have done in the rest of cases, we may now describe the higher topology in terms of seminorms.

Proposition 2.6.3. *Let $k \in \mathbb{Z}$. Given a sequence $\rho_i \in \mathbb{Q}_{>0} \cup \{\infty\}$ for every $i \leq k$, such that $\rho_k < \infty$, consider the seminorm*

$$\| \cdot \| : \mathbb{K}(\!(t)\!) \to \mathbb{R}, \quad \sum_{i \geq i_0} x_i t^i \mapsto \max_{i \leq k} \frac{|x_i|}{\rho_i}, \tag{2.16}$$

having in mind the convention that $a/\infty = 0$ for every $a \in \mathbb{R}_{\geq 0}$. The higher topology on F is defined by the set of seminorms specified by (2.16).

Proof. We will show that the seminorm $\| \cdot \|$ defined by (2.16) is the gauge seminorm attached to the basic open neighbourhood of zero $\mathcal{U}$ given by (2.15).

Let $x = \sum_{i \geq i_0} x_i t^i \in F$ and $\rho > 0$. If $k < i_0$, we may take $\rho = 0$ and deduce that $q(x) = 0$.

Otherwise, $x \in \rho\mathcal{U}$ if and only if $x_i \in \rho D_{\rho_i}$ for every $i_0 \leq i \leq k$.

From this, we may deduce that $x \in \rho\mathcal{U}$ if and only if

$$\frac{|x_i|}{\rho_i} < \rho \quad \text{for every } i_0 \leq i \leq k. \tag{2.17}$$

Finally, the infimum value of ρ satisfying (2.17) is precisely the maximum of the values $|x_i|/\rho_i$ for $i_0 \leq i \leq k$. □

We have described the higher topology on $\mathbb{K}(\!(t)\!)$ in a fashion that matches what has been done in the previous sections. However, this locally convex space often arises in functional analysis in the following way. We write

$$\mathbb{K}(\!(t)\!) = \cup_{i \in \mathbb{N}} t^{-i}.\mathbb{K}[\![t]\!]\,,$$

Each component in the union is isomorphic to $\mathbb{K}^{\mathbb{N}}$, topologized using the product topology, and the limit acquires the strict inductive limit locally convex topology.

It is known that $\mathbb{K}[\![t]\!]$ is a Fréchet space, that is, complete and metrizable. As such, the two-dimensional local field $\mathbb{K}(\!(t)\!)$ is an LF-space and many of its properties may be deduced from the general theory of LF-spaces, see for example [Köt69, §19]. In particular, $\mathbb{K}(\!(t)\!)$ is complete, bornological and nuclear.

2.7 A note on the characteristic p case

Let $k = \mathbb{F}_q$ be a finite field of characteristic p. In this section we will consider the two-dimensional local field $F = k(\!(u)\!)(\!(t)\!)$. It is a vector space both over the finite field k and over the local field $k(\!(u)\!)$.

The higher topology on F may be dealt with in two ways from a linear point of view. The first approach was started by Parshin [Par84], and it regards F as a k-vector space. In this approach, k is regarded as a discrete topological field and the tools used are those of linear topology, see [Kap, §1] for an account. Linear topology was first introduced by Lefschetz [Lef42].

The work developed in the previous sections of this work may be applied and we may regard F a locally convex $k(\!(u)\!)$-vector space. In this section we will explain that in this case we have obtained nothing new.

A topology on a k-vector space is said to be linear if the filter of neighbourhoods of zero admits a collection of linear subspaces as a basis. A linearly topological vector space V is said to be linearly compact if any family $A_i \subset V$, $i \in I$ of closed affine subspaces such that $\bigcap_{i\in J} A_i \neq \emptyset$ for any finite set $J \subset I$, then $\bigcap_{i\in I} A_i \neq \emptyset$. Finally, a linearly topological vector space is locally linearly compact if it has a basis of neighbourhoods of zero formed by linearly compact open subspaces.

Let $\mathbf{Vect}_k$ be the category of linearly topological k-vector spaces. Similarly, let $\mathbf{Vect}_{k(\!(u)\!)}$ be the category of locally convex $k(\!(u)\!)$-vector spaces.

Proposition 2.7.1. *The rule*

$$\mathbf{Vect}_{k(\!(u)\!)} \to \mathbf{Vect}_k, \tag{2.18}$$

which restricts scalars on $k((u))$-vector spaces along the inclusion $k \hookrightarrow k((u))$ and preserves topologies and linear maps, is a functor. Furthermore, if V is a locally convex $k((t))$-space which is linearly compact as a k-space, then it is a c-compact $k((u))$-space.

Proof. Let V be a locally convex $k((u))$-vector space, and let Λ denote an open lattice. As the lattice Λ is an $\mathcal{O}_{k((u))}$-module and we have the inclusion $k \hookrightarrow \mathcal{O}_{k((u))} = k[[u]]$, it is also a k-vector space by restriction of scalars.

As the collection of open lattices Λ is a basis for the filter of neighbourhoods of zero, V is a linearly topological k-vector space and the first part of the proposition follows.

The fact that a $k((u))$-space V which is a linearly compact k-space must be a c-compact $k((u))$ space in the first place comes from the fact that any closed convex subset of V as a $k((u))$ space is a closed affine k-subspace by restriction of scalars. □

It is not true in general that restriction of scalars on a c-compact $k((u))$-vector space yields a linearly compact k-vector space: $k((u))$, being spherically complete, is a c-compact $k((u))$-vector space [Sch02, §12] which is not a linearly compact k-vector space.

The lack of an embedding of a finite field into a characteristic zero two-dimensional local field makes the linear topological approach unavailable in that setting; the locally convex approach to these fields is therefore to be regarded as analogous to the linear approach in positive characteristic. Similarly, the language of locally convex spaces is to be regarded as one which unifies the approach to the zero characteristic and positive characteristic cases.

Chapter 3

Locally convex structures on higher local fields

Let us explain how to extend some of the results of Chapter §2 to the general case of arbitrary dimension.

In this chapter, K still denotes a fixed characteristic zero local field, and the notations that we have used regarding K in the previous chapter also carry on to the present chapter.

Notation. We fix from now on, and until the beginning of §3.6,

$$F = K\{\{t_1\}\}\cdots\{\{t_r\}\}((t_{r+1}))\cdots((t_{n-1}))$$

with $0 \leq r \leq n-1$. The extremal case $r=0$ (resp. $r=n-1$) stands for $F = K((t_1))\cdots((t_{n-1}))$ (resp. $F = K\{\{t_1\}\}\cdots\{\{t_{n-1}\}\}$). We also let

$$L = K\{\{t_1\}\}\cdots\{\{t_r\}\}((t_{r+1}))\cdots((t_{n-2})),$$

by which we simply mean that L is the subfield of F consisting of power series in $t_1, \dots, t_{n-2}$.

It will be extremely convenient to use multi-index notation. For this purpose, let $I = \mathbb{Z}^{n-1}$ and $J = \mathbb{Z}^{n-2}$. For $l \in \{1, \dots, n-1\}$, if we fix indices $(i_l, \dots, i_{n-1}) \in \mathbb{Z}^{n-l-1}$, we will denote

$$I(i_l,\dots,i_{n-1}) = \left\{\alpha \in I;\ \alpha = (j_1,\dots,j_{l-1},i_l,\dots,i_{n-1}) \text{ for some } (j_1,\dots,j_{l-1}) \in \mathbb{Z}^{l-1}\right\}.$$

Any element $x \in F$ can be written uniquely as a power series

$$x = \sum_{i_{n-1} \gg -\infty} \cdots \sum_{i_{r+1} \gg -\infty} \sum_{i_r \in \mathbb{Z}} \cdots \sum_{i_1 \in \mathbb{Z}} x_{i_1,\dots,i_{n-1}} t_1^{i_1} \cdots t_{n-1}^{i_{n-1}},$$

with $x_{i_1,\dots,i_{n-1}} \in K$. We will abbreviate such an expression to

$$x = \sum_\alpha x(\alpha) t^\alpha,$$

for $\alpha \in I$ and $x(\alpha) \in K$. Finally, denote $-\alpha = (-i_1,\dots,-i_{n-1})$ for any $\alpha = (i_1,\dots,i_{n-1}) \in I$.

Several proofs will use induction arguments. For such, it will be convenient to denote elements of L as $\sum_\beta x(\beta)t^\beta$ for $\beta \in J$ and $x(\beta) \in K$, with this notation being analogous to that adopted for elements in F. The statement $\alpha = (\beta, i)$ for $\alpha \in I$, $\beta \in J$ and $i \in \mathbb{Z}$ means that if $\beta = (i_1,\dots,i_{n-2})$, then $\alpha = (i_1,\dots,i_{n-2},i)$.

When necessary, the set I will be ordered with the inverse lexicographical order, that is: $(i_1,\dots,i_{n-1}) < (j_1,\dots,j_{n-1})$ if and only if for an index $l \in \{1,\dots,n-1\}$ we have $i_l < j_l$ and $i_m = j_m$ for $l < m \leq n-1$.

We will construct objects such as $\mathcal{O}$-submodules of and seminorms on F attached to a given net in $\mathbb{Z} \cup \{\pm\infty\}$. Instead of using notation $(n_\alpha)_{\alpha \in I}$, which is standard for sequences and was used thoroughly in Chapter §2, we will denote the elements of a net by $(n(\alpha))_{\alpha \in I}$, the net of coefficients $(x(\alpha))_{\alpha \in I} \subset K$ attached to an element $x = \sum_\alpha x(\alpha)t^\alpha \in F$ being a first example. We will ease notation by letting, when given a net $(n(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{\pm\infty\}$, $n(\beta, i) := n((\beta, i))$ for $\beta \in J$, $i \in \mathbb{Z}$.

3.1 Local convexity of higher topologies

Proposition 3.1.1. *The higher topology on F is locally convex. It may be described as follows. For any net $(n(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{-\infty\}$ subjected to the conditions:*

(i) *For any $l \in \{r+1, \ldots, n-1\}$ and fixed indices $i_{l+1}, \ldots, i_{n-1} \in \mathbb{Z}$, there is a $k_0 \in \mathbb{Z}$ such that for every $k \geq k_0$ we have*

$$n(\alpha) = -\infty \quad \textit{for all } \alpha \in I(k, i_{l+1}, \ldots, i_{n-1}).$$

(ii) *For any $l \in \{1, \ldots, r\}$ and fixed indices $i_{l+1}, \ldots, i_{n-1}$, there is an integer $c \in \mathbb{Z}$ such that*

$$n(\alpha) \leq c \quad \textit{for every } \alpha \in I(i_{l+1}, \ldots, i_{n-1}),$$

and we have that

$$n(\alpha) \to -\infty, \quad \alpha \in I(k, i_{l+1}, \ldots, i_{n-1}), \ \textit{as } k \to \infty.$$

Then, the open lattices of F are those of the form

$$\Lambda = \sum_{\alpha} \mathfrak{p}^{n(\alpha)} t^{\alpha}. \tag{3.1}$$

Remark 3.1.2. Let us clarify what the second part in condition (ii) above stands for. The condition is that for any $l \in \{1, \ldots, r\}$ and fixed indices $i_{l+1}, \ldots, i_{n-1}$, given $d \in \mathbb{Z}$ there is an integer k_0 such that for every $k \geq k_0$ and $\alpha \in I(k, i_{l+1}, \ldots, i_{n-1})$ we have $n(\alpha) \leq d$.

Proof. We will prove the result by induction on n. For $n = 2$, see Propositions 2.1.1 and 2.1.7. Suppose $n > 2$. Then write $L = K\{\{t_1\}\} \cdots \{\{t_r\}\}((t_{r+1})) \cdots ((t_{n-2}))$, with $r \in \{0, n-2\}$. By induction hypothesis, the higher topology on L is locally convex and its open lattices are of the form

$$M = \sum_{\beta \in J} \mathfrak{p}^{n(\beta)} t^{\beta}, \tag{3.2}$$

with $(n(\beta))_{\beta \in J} \subset \mathbb{Z} \cup \{-\infty\}$ a net satisfying the conditions in the statement of the proposition.

Now we need to distinguish two cases. First, if $r \leq n-2$, we must apply the construction in which neighbourhoods of zero are of the form (1.5), as $F = L((t_{n-1}))$.

So we let

$$M_i = \sum_{\beta \in J} \mathfrak{p}^{n(\beta,i)} t^{\beta}, \quad i \in \mathbb{Z},$$

with the property that there is an $i_0 \in \mathbb{Z}$ such that for all $i \geq i_0$, $M_i = L$. This last condition is equivalent to setting $n(\beta, i) = -\infty$ for all $\beta \in J$ and $i \geq i_0$. As the M_i describe a basis of neighbourhoods of zero for the higher topology on L, the higher topology on F admits a basis of neighbourhoods of zero formed by sets of the form

$$\Lambda = \sum_{i \in \mathbb{Z}} M_i t_{n-1}^i.$$

By induction hypothesis, the M_i are all $\mathcal{O}$-lattices, which is enough to show that Λ is an $\mathcal{O}$-lattice. So, in this case, we let $\alpha = (\beta, i)$ so that a basis of neighbourhoods of zero for the higher topology is described by sets $\Lambda = \sum_{\alpha \in I} \mathfrak{p}^{n(\alpha)} t^{\alpha}$. On top of the conditions which the indices $n(\beta, i)$ satisfy by the induction hypothesis for $\beta \in J$, we must add the further condition that there is an integer i_0 such that for all $i \geq i_0$, $n(\alpha) = -\infty$ for all $\alpha \in I(i)$. This shows that our claim holds in this case.

The second case is the one in which $r > n - 2$ and we must apply the construction in which neighbourhoods of zero are given by sets of the form (1.6), as $F = L\{\{t\}\}$. So we set

$$M_i = \sum_{\beta \in J} \mathfrak{p}^{n(\beta,i)} t^{\beta}, \quad i \in \mathbb{Z},$$

subject to the properties:

(i) There is an integer c such that for every $i \in \mathbb{Z}$, $\mathfrak{p}_L^c \subset M_i$. By induction hypothesis, this means that $n(\beta, i) \leq c$ for every $\beta \in J$ and $i \in \mathbb{Z}$.

(ii) $M_i \to L$ as $i \to \infty$. This is equivalent to $n(\beta, i) \to -\infty$ for $\beta \in J$ as $i \to \infty$.

As M_i describe a basis of neighbourhoods of zero of the topology of L, the sets of the form

$$\Lambda = \sum_{i \in \mathbb{Z}} M_i t_{n-1}^i$$

describe a basis of neighbourhoods of zero for the higher topology on F. Since the M_i are $\mathcal{O}$-lattices, we get that Λ is an $\mathcal{O}$-lattice. Again, we let $\alpha = (\beta, i)$, so that the

$\mathcal{O}$-lattice Λ may be described as $\Lambda = \sum_{\alpha \in I} \mathfrak{p}^{n(\alpha)} t^\alpha$. On top of the conditions satisfied by the $n(\alpha)$ which are inherited by induction, there are the two new conditions:

(i) There is an integer c such that $n(\alpha) \geq c$ for all $\alpha \in I$.

(ii) $n(\alpha) \to -\infty$ for $\alpha \in I(i)$, as $i \to \infty$.

This shows that the result also holds in this case. □

After showing that the higher topology on $K \hookrightarrow F$ is locally convex, it is natural to describe it in terms of seminorms.

Proposition 3.1.3. *The higher topology on F is the locally convex K-vector space topology defined by the seminorms of the form*

$$\| \cdot \| : F \to \mathbb{R}, \quad \sum_\alpha x(\alpha) t^\alpha \mapsto \sup_\alpha |x(\alpha)| q^{n(\alpha)} \tag{3.3}$$

as $(n(\alpha))_{\alpha \in I} \subset \mathbb{Z} \cup \{-\infty\}$ varies over the nets described in the statement of Proposition 3.1.1.

Proof. It is necessary to show that the gauge seminorm associated to the open lattice Λ as in (3.1) is the one given by (3.3).

The gauge seminorm defined by Λ is by definition

$$\|x\| = \inf_{x \in a\Lambda} |a|, \quad \text{for } x \in F.$$

Let $x = \sum_\alpha x(\alpha) t^\alpha$. We have that $x \in a\Lambda$ if and only if $x(\alpha) \in a\mathfrak{p}^{n(\alpha)}$ for every $\alpha \in I$. That is, if and only if

$$|x(\alpha)| q^{n(\alpha)} \leq |a| \quad \text{for all } \alpha \in I. \tag{3.4}$$

The infimum of the values $|a|$ for which (3.4) holds is the supremum of the values $|x(\alpha)| q^{n(\alpha)}$ as $\alpha \in I$. □

Definition 3.1.4. The seminorms on F defined in the previous proposition will be referred to as *admissible seminorms.*

An admissible seminorm $\|\cdot\|$ is attached to a net $(n(\alpha))_{\alpha\in I} \subset \mathbb{Z}\cup\{-\infty\}$. If we have chosen notation not to reflect this fact it is in pursue of a lighter reading and understanding that the net $(n(\alpha))_{\alpha\in I}$, when needed, will be clear from the context.

3.2 First properties

We recall some properties from §1.3.2 and make some immediate deductions from the fact that these topologies are locally convex. We also state some properties which do not hold in general because they are known not to hold already for $n = 2$.

The field F, equipped with a higher topology, is a locally convex K-vector space, as shown in Proposition 3.1.1. As such, it is a topological vector space. It is a previously known fact that higher topologies are Hausdorff. In order to show that this property holds in our setting it is enough to show that, given $x \in F^\times$, there is an admissible seminorm $\|\cdot\|$ for which $\|x\| \neq 0$. If the α-coefficient of x is nonzero, any admissible seminorm for which $n(\alpha) > -\infty$ suffices.

Moreover, the reduction map $\mathcal{O}_F \to \overline{F}$ is open when $\mathcal{O}_F$ is given the subspace topology and $\overline{F}$ a higher topology compatible with the choice of coefficient field if $\operatorname{char} F \neq \operatorname{char} \overline{F}$.

Multiplication $F \to F$ by a fixed nonzero element induces a homeomorphism of F, but multiplication $\mu : F \times F \to F$ is not continuous; the immediate reason why this is the case being that for any open lattice Λ we have $\mu(\Lambda, \Lambda) = F$.

Higher topologies are not first-countable and therefore not metrizable. Moreover, in general, F is not bornological, barrelled, reflexive nor nuclear and its rings of integers are not c-compact nor compactoid, the first counterexample being the field $K\{\{t\}\}$, as exposed throughout Chapter §2.

Remark 3.2.1. Power series in the system of parameters $t_1, \ldots, t_{n-1}$ are convergent in the higher topology. If $x = \sum_{\alpha\in I} x(\alpha)t^\alpha \in F$, we define a net $(s(\alpha))_{\alpha\in I} \subset F$ by taking $s(\alpha) = \sum_{\alpha'\leq\alpha} x(\alpha')t^{\alpha'}$. If $\|\cdot\|$ is an admissible seminorm on F, then as $\alpha \in I$ grows, the value $\|x - s(\alpha)\|$ becomes arbitrarily small.

3.3 Bounded $\mathcal{O}$-submodules

Let us study the bounded $\mathcal{O}$-submodules of F. We start by describing a basis for the Von-Neumann bornology on $K \hookrightarrow F$.

Proposition 3.3.1. *Let $(k(\alpha))_{\alpha\in I} \subset \mathbb{Z} \cup \{\infty\}$ be a net subjected to the conditions:*

(i) For every $l \in \{r+1, \dots, n-1\}$ and indices $i_{l+1}, \dots, i_{n-1} \in \mathbb{Z}$ there is an index $j_0 \in \mathbb{Z}$ such that for every $j < j_0$ we have $k(\alpha) = \infty$ for all $\alpha \in I(j, i_{l+1}, \dots, i_{n-1})$.

(ii) For every $l \in \{1, \dots, r\}$ and $i_{l+1}, \dots, i_{n-1} \in \mathbb{Z}$, there is an integer d such that $k(\alpha) \geq d$ for all $\alpha \in I(i_{l+1}, \dots, i_{n-1})$.

The Von-Neumann bornology of F admits as a basis the collection of $\mathcal{O}$-submodules

$$B = \sum_{\alpha\in I} \mathfrak{p}^{k(\alpha)} t^{\alpha} \tag{3.5}$$

as $(k(\alpha))_{\alpha\in I}$ varies over the nets specified above.

Proof. First, let us show that the sets B are bounded. As we will use induction on n, the case $n = 2$ is thoroughly explained in Propositions 2.2.2 and 2.2.4.

Let $\|\cdot\|$ be an admissible seminorm attached to the net $(n(\alpha))_{\alpha\in I} \subset \mathbb{Z} \cup \{-\infty\}$.

Let $x = \sum_{\alpha} x(\alpha) t^{\alpha}$. We have

$$\|x\| = \sup_{\alpha} |x(\alpha)| q^{n(\alpha)} \leq \sup_{\alpha} q^{n(\alpha)-k(\alpha)}$$

and therefore it is enough to prove that the set $\{n(\alpha) - k(\alpha)\}_{\alpha\in I} \subset \mathbb{Z} \cup \{-\infty\}$ is bounded above.

We distinguish two cases. Suppose first that $r \leq n-2$. Then $F = L(\!(t_{n-1})\!)$. On one hand, there is an index $j_0 \in \mathbb{Z}$ such that $k(\alpha) = \infty$ for every $\alpha \in I(j)$, $j < j_0$. On the other hand, there is an index $j_1 \in \mathbb{Z}$ such that $n(\alpha) = -\infty$ for every $\alpha \in I(j)$, $j > j_1$. It is therefore enough to show that each of the finitely many sets

$$N(j) = \{n(\alpha) - k(\alpha);\ \alpha \in I(j)\}, \quad j_0 \leq j \leq j_1$$

are bounded above. But for each $j \in \mathbb{Z}$, the net $(n(\alpha))_{\alpha \in I(j)}$ defines an admissible seminorm and $\sum_{\beta \in J} \mathfrak{p}^{k(\beta,j)} t^{\beta} \subset L$ is a bounded $\mathcal{O}$-submodule; this implies the boundedness of $N(j)$.

The case in which $r = n-1$, and therefore $F = L\{\{t_{n-1}\}\}$, is simpler: we have that all the $n(\alpha)$ are bounded above and all the $k(\alpha)$ are bounded below; therefore the differences $n(\alpha) - k(\alpha)$ are bounded above.

Second, we have to show that any bounded subset of F is contained in an $\mathcal{O}$-submodule of F of the form (3.5).

The elements of any bounded subset $D \subset F$ cannot have t_{n-1}-expansions with arbitrarily large coefficients in L in a fixed degree: otherwise, suppose that this is not the case and that for $j \in \mathbb{Z}$ the j-th coefficients in the t_{n-1}-expansions of the elements in D may be arbitrarily large. By choosing any admissible seminorm with $n(\alpha) > -\infty$ for some $\alpha \in I(j)$ we easily obtain that D is not bounded.

Hence, D is contained in an $\mathcal{O}$-submodule of the form $C = \sum_{i \in \mathbb{Z}} C_i t_{n-1}^i$ with $C_i \subset L$ bounded $\mathcal{O}$-submodules. By induction hypothesis, let us write

$$C_i = \sum_{\beta \in J} \mathfrak{p}^{k(\beta,i)} t^{\beta}, \quad i \in \mathbb{Z},$$

with $k(\beta, i) \in \mathbb{Z} \cup \{\infty\}$ satisfying the conditions exposed in the statement of the proposition.

By letting $\alpha = (\beta, i) \in I$, we may write $C = \sum_{\alpha \in I} \mathfrak{p}^{k(\alpha)} t^{\alpha}$ and we only have to show that the indices $k(\alpha)$ might be taken to satisfy the conditions exposed in the proposition. Suppose that this is not the case, and let us consider separate cases again.

First, if $r \leq n-2$ and $F = L((t_{n-1}))$, the indices $k(\alpha)$ may be taken to satisfy condition (ii) by induction hypothesis. Condition (i) is also satisfied by induction hypothesis for every $l \in \{r+1, \ldots, n-2\}$. So we only have to show that if the $k(\alpha)$ may not be taken to satisfy condition (i) in the case $l = n-1$, then D cannot be bounded.

If the condition does not hold, then there is a decreasing sequence $(j_h)_{h \geq 0} \subset \mathbb{Z}_{<0}$, an index $\alpha_h \in I(j_h)$ and an element $\xi_h \in D$ such that its α_h-coefficient, which we label

$x(\alpha_h)$, is nonzero. Let

$$n(\alpha) = \begin{cases} -j_h + v(x(\alpha_h)), & \text{if } \alpha = \alpha_h \in I(j_h), \\ -\infty & \text{otherwise.} \end{cases}$$

The net $(n(\alpha))_{\alpha \in I}$ defines an admissible seminorm $\| \cdot \|$. Since $\|\xi_h\| \geq |x(\alpha_h)| q^{n(\alpha_h)} = q^{-j_h}$, D cannot be bounded.

Second, suppose that $r = n-1$ and $F = L\{\{t_{n-1}\}\}$. By induction hypothesis, we know that condition (ii) holds for every $l \in \{1, \ldots, n-2\}$, so we suppose that it does not hold in the case $l = n-1$. In such case, at least one of the following must happen:

1. There is a decreasing sequence $(j_h)_{h \geq 0} \subset \mathbb{Z}_{<0}$, an index $\alpha_h \in I(j_h)$ and $\xi_h \in D$ such that, if $x(\alpha_h)$ denotes the α_h-coefficient of ξ_h, we have $|x(\alpha_h)| \to \infty$ as $h \to \infty$.

2. There is an increasing sequence $(j_h)_{h \geq 0} \subset \mathbb{Z}_{\geq 0}$, an index $\alpha_h \in I(j_h)$ and $\xi_h \in D$ such that, if $x(\alpha_h)$ denotes the α_h-coefficient of ξ_h, we have $|x(\alpha_h)| \to \infty$ as $h \to \infty$.

Suppose that condition 1 holds. In this case, let

$$n(\alpha) = \begin{cases} 0, & \text{if } \alpha = \alpha_h \text{ for some } h \geq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

The net $(n(\alpha))_{\alpha \in I}$ defines an admissible seminorm $\| \cdot \|$. Now, for $h \geq 0$, we have $\|\xi_h\| \geq q^{-v(x(\alpha_h))}$ and hence D cannot be bounded.

Finally, if condition 1 does not hold, then condition 2 must happen. In such case, let

$$n(\alpha) = \begin{cases} (v(x(\alpha_h)) - 1)/2, & \text{if } \alpha = \alpha_h \text{ for some } h \geq 0 \text{ and } v(x(\alpha_h)) \text{ odd.} \\ v(x(\alpha_h))/2, & \text{if } \alpha = \alpha_h \text{ for some } h \geq 0 \text{ and } v(x(\alpha_h)) \text{ even.} \\ -\infty & \text{otherwise.} \end{cases}$$

The net $(n(\alpha))_{\alpha \in I}$ defines an admissible seminorm. Moreover, we have

$$n(\alpha_h) - v(x(\alpha_h)) \to \infty \quad \text{as } h \to \infty.$$

We have that $\|\xi_h\| \geq |x(\alpha_h)|q^{n(\alpha_h)} = q^{n(\alpha_h)-v(x(\alpha_h))}$ and therefore D cannot be bounded.

□

Definition 3.3.2. We will say that an $\mathcal{O}$-submodule of the form (3.5) is a *basic bounded submodule* of F.

The following result is necessary in order to compare compactoids and c-compacts in the sequel.

Proposition 3.3.3. *The submodules B in Proposition 3.3.1 are complete.*

Proof. Let $B = \sum_{\alpha\in I} \mathfrak{p}^{k(\alpha)} t^\alpha$ with $(k(\alpha))_{\alpha\in I} \subset \mathbb{Z}\cup\{\infty\}$ satisfying conditions (i) and (ii) in the statement of Proposition 3.3.1.

Let H be a directed set and $(x(h))_{h\in H} \subset B$ a Cauchy net. Let us denote, for each $h \in H$, $x(h) = \sum_\alpha x(h)(\alpha)t^\alpha$ with $x(h)(\alpha) \in \mathfrak{p}^{k(\alpha)}$ for $\alpha \in I$.

We have that, for a fixed $\alpha \in I$, $(x(h)(\alpha))_{h\in H} \subset \mathfrak{p}^{k(\alpha)}$ is a Cauchy net. As $\mathfrak{p}^{k(\alpha)}$ is complete, the net converges to an element $x(\alpha) \in \mathfrak{p}^{k(\alpha)}$.

If the power series $\sum_\alpha x(\alpha)t^\alpha$ defines an element x in F, then $x \in B$ and $x(h) \to x$. This is easy to check by induction on n (the case $n = 2$ may be found in §2.3). □

As we have explained, multiplication $\mu : F \times F \to F$ is not a continuous map. However, we may shown that it is bounded.

Proposition 3.3.4. *Multiplication $\mu : F \times F \to F$ is a bounded map.*

Proof. The argument for the proof is by induction on n. The case $n = 2$ is dealt with in Proposition 2.2.9, and the same argument applies when looking at $F = L\{\{t_{n-1}\}\}$ or $L((t_{n-1}))$ □

3.4 Compactoid $\mathcal{O}$-submodules

The result below outlines which basic bounded submodules of F are compactoid, and thus describes a basis for the bornology on F generated by compactoid $\mathcal{O}$-submodules.

Proposition 3.4.1. *Let $(k(\alpha))_{\alpha\in I} \subset \mathbb{Z}\cup\{\infty\}$ be a net satisfying the conditions:*

(i) For every $l \in \{r+1, \ldots, n-1\}$ and indices $i_{l+1}, \ldots, i_{n-1} \in \mathbb{Z}$ there is an index $j_0 \in \mathbb{Z}$ such that for every $j < j_0$ we have $k(\alpha) = \infty$ for all $\alpha \in I(j, i_{l+1}, \ldots, i_{n-1})$.

(ii) For every $l \in \{1, \ldots, r\}$ and $i_{l+1}, \ldots, i_{n-1} \in \mathbb{Z}$, there is an integer $d \in \mathbb{Z}$ such that $k(\alpha) \geq d$ for all $\alpha \in I(i_{l+1}, \ldots, i_{n-1})$ and we have that $k(\alpha) \to \infty$ for $\alpha \in I(j, i_{l+1}, \ldots, i_{n-1})$, as $j \to -\infty$.

Then the $\mathcal{O}$-submodule of F given by

$$B = \sum_{\alpha} \mathfrak{p}^{k(\alpha)} t^{\alpha} \tag{3.6}$$

is compactoid.

The $\mathcal{O}$-submodules of the form (3.6) are the only compactoid submodules amongst basic bounded submodules, and define a basis for the bornology on F defined by compactoid submodules.

In the proof to this proposition we shall need to consider the projection maps to the coefficients of the α-expansions of elements of F. For every $\alpha_0 \in I$, consider the continuous linear form:

$$\pi_{\alpha_0} : F \to K, \quad \sum_{\alpha} x(\alpha) t^{\alpha} \mapsto x(\alpha_0).$$

Proof. The result holds for $n = 2$ as shown in §2.3.

First, let us show that the submodule B as in (3.6) is compactoid. Let $\Lambda = \sum_{\alpha} \mathfrak{p}^{n(\alpha)} t^{\alpha}$ be an open lattice. We will show that there exist elements $x_1, \ldots, x_m \in F$ such that $B \subset \Lambda + \mathcal{O}x_1 + \cdots + \mathcal{O}x_m$.

Regardless of the value of $r \in \{0, \ldots, n-1\}$, there are two indices $j_0, j_1 \in \mathbb{Z}$ such that

$$k(\alpha) \geq n(\alpha), \quad \text{for all } \alpha \in I(j) \text{ with } j < j_0 \text{ or } j > j_1. \tag{3.7}$$

if $j_0 > j_1$ then $B \subset \Lambda$ and we are done. Henceforth, we assume $j_0 \leq j_1$.

Let us examine the situation for $j_0 \leq j \leq j_1$. For a fixed such j, let $\alpha = (\beta, j)$ with

$\beta \in J$. By induction hypothesis, the $\mathcal{O}$-submodule

$$\sum_{\beta \in J} \mathfrak{p}^{k(\beta,j)} t^{\beta} \subset L$$

is compactoid. Similarly, for a fixed such j, $\sum_{\beta} \mathfrak{p}^{n(\beta,j)} t^{\beta}$ is an open lattice in L. Therefore, there exist a finite number of elements $y_{j,1}, \ldots, y_{j,m_j} \in L$ for which we have, for $j_0 \leq j \leq j_1$,

$$\sum_{\beta} \mathfrak{p}^{k(\beta,j)} t^{\beta} \subset \sum_{\beta} \mathfrak{p}^{n(\beta,j)} t^{\beta} + \mathcal{O} y_{j,1} + \cdots + \mathcal{O} y_{j,m_j}.$$

Now, this implies that

$$\sum_{j=j_0}^{j_1} \left(\sum_{\beta \in J} \mathfrak{p}^{k(\beta,j)} t^{\beta} \right) t_{n-1}^{j} \subset \sum_{j=j_0}^{j_1} \left(\sum_{\beta \in J} \mathfrak{p}^{n(\beta,j)} t^{\beta} + \sum_{s=1}^{m_j} \mathcal{O} y_{j,s} \right) t_{n-1}^{j}.$$

We rewrite this last equation as:

$$\sum_{\substack{\alpha \in I(j) \\ j_0 \leq j \leq j_1}} \mathfrak{p}^{k(\alpha)} t^{\alpha} \subset \sum_{\substack{\alpha \in I(j) \\ j_0 \leq j \leq j_1}} \mathfrak{p}^{n(\alpha)} t^{\alpha} + \left(\sum_{j=j_0}^{j_1} \sum_{s=1}^{m_j} \mathcal{O} y_{j,s} t_{n-1}^{j} \right). \tag{3.8}$$

The fact that

$$B \subset \Lambda + \left(\sum_{j=j_0}^{j_1} \sum_{s=1}^{m_j} \mathcal{O} y_{j,s} t_{n-1}^{j} \right)$$

follows from (3.7) and (3.8).

Second, let us show how any compactoid $\mathcal{O}$-submodule of F is contained in one of the form (3.6). Since compactoid $\mathcal{O}$-submodules are bounded, it is enough to show that any basic bounded submodule $C = \sum_{\alpha} \mathfrak{p}^{k(\alpha)} t^{\alpha}$ is compactoid if and only if the indices $k(\alpha)$ satisfy conditions (i) and (ii) in the statement of the Proposition. We proceed by induction, the result holds for $n = 2$ as mentioned above.

Now, suppose C is compactoid. Then, for every $j \in \mathbb{Z}$, the $\mathcal{O}$-submodule of L given by

$$C_j = \sum_{\beta} \mathfrak{p}^{k(\beta,j)} t^{\beta}$$

is compactoid.

Next, we distinguish cases. Suppose that $r \leq n-2$, so that $F = L(\!(t_{n-1})\!)$. In such case, by induction hypothesis, we only need to check that the indices $k(\alpha)$ satisfy condition (i) for $l = n-1$. But if this condition does not hold, then from the proof of Proposition 3.3.1 we deduce that C cannot be compactoid, as it is not bounded. So there is nothing more to say in this case.

Finally, suppose that $r = n-1$, so that $F = L\{\{t_{n-1}\}\}$. By hypothesis induction, the indices $k(\alpha)$ satisfy condition (ii) in the statement of the Proposition for $1 \leq l \leq n-2$. If condition (ii) for $l = n-1$ does not hold, then there is a decreasing sequence $(j_h)_{h\geq 0} \subset \mathbb{Z}_{<0}$ and an index $\alpha_h \in I(j_h)$ for each $h \geq 0$ such that the sequence $(k(\alpha_h))_{h\geq 0}$ is bounded above. Let $M \in \mathbb{Z}$ be such that $k(\alpha_h) < M$ for every $h \geq 0$. Let

$$n(\alpha) = \begin{cases} M, & \text{if } \alpha = \alpha_h \text{ for some } h \geq 0, \\ -\infty & \text{otherwise .} \end{cases}$$

The net $(n(\alpha))_{\alpha\in I}$ defines an open lattice $\Lambda = \sum_\alpha \mathfrak{p}^{n(\alpha)} t^\alpha$. Now, suppose that there are $x_1, \ldots, x_m \in F$ for which $C \subset \Lambda + \mathcal{O}x_1 + \cdots + \mathcal{O}x_m$. Let us write, for $1 \leq l \leq m$, $x_l = \sum_\alpha x_l(\alpha) t^\alpha$.

Since $x_l(\alpha) \to 0$ for $\alpha \in I(j)$ as $j \to -\infty$, we have that there is an index $k \in \mathbb{Z}$ such that for every $j \leq k$, we have $v(x_l(\alpha)) > M$ for every $\alpha \in I(j)$ and $1 \leq l \leq m$. Fix an $h \geq 0$ such that $j_h \leq k$. Now, for such an h, we have

$$\pi_{\alpha_h}(C) \subset \pi_{\alpha_h}(\Lambda + \mathcal{O}x_1 + \cdots + \mathcal{O}x_m),$$

which implies

$$\mathfrak{p}^{k(\alpha_h)} \subset \mathfrak{p}^M + \mathfrak{p}^{v(x_1(\alpha_h))} + \cdots + \mathfrak{p}^{v(x_m(\alpha_h))} = \mathfrak{p}^M.$$

This last inclusion implies that $M \leq k(\alpha_h)$, a contradiction. Hence, we must have $k(\alpha) \to \infty$ for $\alpha \in I(j)$ as $j \to \infty$. □

Definition 3.4.2. We will refer to the $\mathcal{O}$-submodules of F of the form (3.6) as *basic compactoid submodules of* F.

Corollary 3.4.3. *The basic compactoid* $\mathcal{O}$*-submodules of* F *are c-compact.*

Proof. An $\mathcal{O}$-submodule of a locally convex K-vector space is c-compact and bounded if and only if it is compactoid and complete [Sch02, Prop. 12.7]. So the result follows from the fact that these $\mathcal{O}$-submodules are bounded, compactoid and complete. □

3.5 Duality

Let us discuss some issues regarding the dual space of F. We showed in Theorem 2.4.2 how in the two-dimensional case F is isomorphic in the category of locally convex K-vector spaces to F'_c, its continuous dual space topologized using the c-topology, that is: the topology of uniform convergence on compactoid submodules.

The c-topology is defined on the continuous dual of any n-dimensional F by the collection of seminorms

$$|\cdot|_B : F' \to \mathbb{R}, \quad l \mapsto \sup_{x\in B} |l(x)|,$$

for any basic compactoid submodule $B \subset F$.

Our first goal in this section is to construct an isomorphism of locally convex vector spaces $F \cong F'_c$.

We have already come across some continuous nonzero linear forms on F in the previous section, we recall that these were the projections, for every $\alpha_0 \in I$:

$$\pi_{\alpha_0} : F \to K, \quad \sum_{\alpha\in I} x(\alpha)t^{\alpha} \mapsto x(\alpha_0).$$

In particular, denote by π_0 the continuous linear form on F constructed as in the previous example for $\alpha_0 = (0, \ldots, 0) \in I$.

We relate F and its continuous dual space. Define

$$\gamma : F \to F', \quad x \mapsto \pi_x, \tag{3.9}$$

with

$$\pi_x : F \to K, \quad y \mapsto \pi_0(xy).$$

Lemma 3.5.1. *If* $x = \sum_{\alpha\in I} x(\alpha)t^{\alpha}$ *and* $y = \sum_{\alpha\in I} y(\alpha)t^{\alpha}$ *are elements in* F*, we have* *that*

$$\pi_x(y) = \sum_{\alpha \in I} x(-\alpha) y(\alpha) = \sum_{i_{n-1} \in \mathbb{Z}} \cdots \sum_{i_1 \in \mathbb{Z}} x_{(-i_1, \ldots, -i_{n-1})} y_{(i_1, \ldots, i_{n-1})}.$$

Proof. The result becomes clear once notation is unwinded and the 0-th coefficient of xy is taken for each parameter t_l separately, for l descending from $n-1$ to 1. □

Lemma 3.5.2. *Let $w \in F'$. Define, for each $\alpha \in I$, $x(\alpha) = w(t^{-\alpha}) \in K$. Then, the expression $\sum_{\alpha \in I} x(\alpha) t^\alpha$ defines an element in F.*

Proof. The result may be shown by induction; the argument is the same as in Lemma 3.5.2. □

Theorem 3.5.3. *The map $\gamma : F \to F'_c$ is an isomorphism of locally convex vector spaces.*

Proof. The map γ is linear and injective. Surjectivity follows from Lemma 3.5.2; as if $w \in F'$, we apply the lemma to obtain $x = \sum_{\alpha \in I} x(\alpha) t^\alpha \in F$. Then, for any $y = \sum_{\alpha \in I} y(\alpha) t^\alpha \in F$, we have

$$w(y) = w\left(\sum_\alpha y(\alpha) t^\alpha\right) = \sum_\alpha y(\alpha) w(t^\alpha) = \sum_\alpha y(\alpha) x(-\alpha) = \pi_x(y).$$

The second equality follows from Remark 3.2.1.

In order to show bicontinuity of γ, the argument is very similar to the one given in the proof of Theorem 2.4.2; given a basic compactoid $\mathcal{O}$-submodule $B = \sum_{\alpha \in I} \mathfrak{p}^{k(\alpha)} t^\alpha$ of F as in Proposition 3.4.1, the net $(-k(-\alpha))_{\alpha \in I}$ defines an admissible seminorm $\| \cdot \|$ on F. We have that, for $x \in F$,

$$\|x\| \leq q^n \text{ if and only if } |\pi_x|_B \leq q^n,$$

which concludes the proof. □

If $A \subset F$ is an $\mathcal{O}$-submodule, we denote $A^\gamma = \gamma^{-1}(A^p) \subset F$, with $A^p \subset F'$ being the pseudo-polar of A.

Proposition 3.5.4. *Let $A = \sum_{\alpha \in I} \mathfrak{p}^{k(\alpha)} t^{\alpha} \subset F$ be an $\mathcal{O}$-submodule, with $k(\alpha) \in \mathbb{Z} \cup \{\pm\infty\}$. We have that*

$$A^{\gamma} = \sum_{\alpha \in I} \mathfrak{p}^{1-k(-\alpha)} t^{\alpha}.$$

Proof. The argument is the same as the one exposed in the proof of Proposition 2.4.7. □

The isomorphism γ is not unique as, for example, choosing π_α for any $\alpha \in I$ instead of π_0 in the definition of γ would have given a different isomorphism. Thus, the shape of A^γ depends ultimately on our choice of γ.

However, there are certain facts which are general for pseudo-polars of $\mathcal{O}$-submodules in any locally convex K-vector space. As such, we recall that taking the pseudo-polar exchanges open lattices and compactoid $\mathcal{O}$-submodules, and that the pseudo-bipolar of an $\mathcal{O}$-submodule is equal to its closure.

These facts are highlighted in the previous Proposition for the $\mathcal{O}$-submodules of the form $\sum_\alpha \mathfrak{p}^{k(\alpha)} t^\alpha$, which are closed. The facts that for an open lattice Λ we have that Λ^γ is compactoid and that for a basic compactoid set B we have that B^γ is an open lattice are evident by checking that for the nets $(n(\alpha))_{\alpha \in I}$ and $(1 - n(-\alpha))_{\alpha \in I}$, one of them satisfies conditions (i) and (ii) in Proposition 3.1.1 if and only if the other one satisfies conditions (i) and (ii) in Proposition 3.4.1.

3.6 The general case

In this section, let $K \hookrightarrow F$ be a general n-dimensional local field over K. By structure theory, there is an $r \in \{0, \ldots, n\}$ such that F is a finite extension of $F_0 := K\{\{t_1\}\} \cdots \{\{t_r\}\}((t_{r+1})) \cdots ((t_{n-1}))$ as explained in §1.3.1. Denote the degree of such extension by e. The higher topology on F may be defined as the product topology on $F \cong (F_0)^e$.

Since the product topology on a product of locally convex vector spaces is again locally convex, a higher topology on F is locally convex. Open lattices (resp. continuous seminorms) on F may be described using Proposition 3.1.1 (resp. Proposition 3.1.3) and Proposition 1.1.1.

Finally, from Theorem 3.5.3 we recover the existence of an isomorphism $F \cong F_c'$ via the chain

$$F' \cong (F_0^e)' \cong (F_0')^e \cong ((F_0)_c')^e \cong F_0^e \cong F.$$

Explicit nonzero continuous linear forms on F may be obtained by composing the forms $\pi_\alpha : F_0 \to K$ for $\alpha \in I$ with $\mathrm{Tr}_{F|F_0}$.

3.7 Other types of higher local fields

Let us center our attention, for the sake of completeness, on the higher local fields which we have not treated in the previous.

First, suppose that char $F = p$. In such case, as explained in §1.3.1, there is a finite field $\mathbb{F}_q$ and elements $t_1, \ldots, t_n \in F$ such that

$$F \cong \mathbb{F}_q((t_1)) \cdots ((t_n)) .$$

The field $\mathbb{F}_q((t_1))$ is a local field and the results in this work may be applied to F if we let $K = \mathbb{F}_q((t_1))$. However, after §2.7, we are only stating that the higher topology on F is a linear topology when we regard F as a vector space over $\mathbb{F}_q$. The choice between linear-topological structures over $\mathbb{F}_q$ or locally convex structures over $\mathbb{F}_q((t_1))$ is merely a matter of language in our case.

Now let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and denote the usual absolute value by $|\cdot|$. The theory of higher local fields is also developed by looking at complete discrete valuation fields F that have an n-dimensional structure on them and such that $F_1 = \mathbb{K}$. For these, there are $t_1, \ldots, t_{n-1} \in F$ for which

$$F \cong \mathbb{K}((t_1)) \cdots ((t_{n-1})) .$$

As hinted at in §2.6, we can apply the archimedean theory of locally convex spaces to study these fields.

The open disks

$$D_\rho = \{a \in \mathbb{K};\ |a| < \rho\}, \quad \rho \in \mathbb{Q}_{>0} \cup \{\infty\}$$

supply a basis of convex sets for the euclidean topology on $\mathbb{K}$.

The higher topology on F is constructed by iterating the construction in §2.6.

Proposition 3.7.1. *Let $I = \mathbb{Z}^{n-1}$ and $(\rho(\alpha))_{\alpha \in I} \subset \mathbb{Q}_{>0} \cup \{\infty\}$ be a net restricted to the condition:*

For any $l \in \{1, \dots, n-1\}$ and fixed indices $i_{l+1}, \dots, i_{n-1} \in \mathbb{Z}$ there is a $k_0 \in \mathbb{Z}$ such that for every $k \geq k_0$ we have $\rho(\alpha) = \infty$ for all $\alpha \in I(k, i_{l+1}, \dots, i_{n-1})$.

The higher topology on F is locally convex and it is defined by the seminorms of the form

$$\| \cdot \| : F \to \mathbb{R}, \sum_{\alpha \in I} x(\alpha) t^{\alpha} \mapsto \sup_{\alpha \in I} \frac{|x(\alpha)|}{\rho(\alpha)}, \tag{3.10}$$

with the convention that $a/\infty = 0$ for any $a \in \mathbb{R}_{\geq 0}$.

Proof. The result follows from Propositions 2.6.2 and 2.6.3 and straightforward adaptation of the arguments used in Proposition 3.1.1 and Corollary 3.1.3. □

Chapter 4

Topology on rational points over higher local fields

In this chapter we will explain how to use sequential topologies in order to study sets of rational points on schemes of finite type over higher local fields.

More generally, we will work with schemes of finite type defined over a sequential ring: a ring R endowed with a topology for which its additive group is a topological group and such that multiplication $R \times R \to R$ is sequentially continuous. Our main example of sequential ring is a higher local field and its rings of integers; our results will apply immediately to this case.

Our starting point towards endowing sets of rational points over sequential rings with topologies is similar to what is a standard procedure in the case of topological rings, considered for the first time in [Wei82]. As such, we closely follow arguments and notations in [Con], introducing appropriate changes when convenient in order to deal with sequential topologies.

4.1 Affine case

Definition 4.1.1. Let R be a sequential ring, and $X \to \operatorname{Spec} R$ an affine scheme of finite type. Let $X = \operatorname{Spec} A$. By using the topology on R, it is possible to topologize $X(R)$ as follows. There is a natural inclusion of sets

$$X(R) = \operatorname{Hom}_{\mathbf{Sch}_R}(\operatorname{Spec} R, X) = \operatorname{Hom}_{\mathbf{Alg}_R}(A, R) \subset \operatorname{Hom}_{\mathbf{Sets}}(A, R) = R^A.$$

We endow R^A with the product topology and provide $X(R) \subset R^A$ with the sequential saturation of the subspace topology.

Remark 4.1.2. Before taking the sequential saturation, the inclusion map

$$X(R) \hookrightarrow R^A$$

is an embedding. After taking the sequential saturation, it is continuous but not an embedding in general. As we will see in §4.2, taking a sequential saturation in Definition 4.1.1 is important.

Remark 4.1.3. Every element $a \in A$ induces a map $\varphi_a : X(R) \to R$ by evaluating R-algebra homomorphisms $A \to R$ at a. Such a map agrees with the composition

$$X(R) \hookrightarrow R^A \to R$$

where the second map is given by projection to the a-th coordinate. By the previous remark, φ_a is continuous. We could have defined the topology on $X(R)$ as the initial topology for the maps $\{\varphi_a\}_{a\in A}$ in **Seq**.

There is another way to construct a topology on $X(R)$ which is more explicit. The choice of a closed embedding into an affine space identifies $X(R)$ with a subset of R^n for some n, and we may endow R^n with the product topology and $X(R) \subset R^n$ with the saturation of the subspace topology. If this procedure is taken, it is necessary to show that this topology on $X(R)$ does not depend on the choice of embedding into affine space. We explain how this works, and show that it is essentially equivalent to Definition 4.1.1.

Let us choose an R-algebra isomorphism

$$A \simeq R[t_1, \ldots, t_n]/I \tag{4.1}$$

and identify $X(R)$ with the set $V(I)$ of elements in R^n on which all polynomial functions belonging to the ideal I vanish. Consider the product topology on R^n and endow $X(R)$ with the sequential saturation of the subspace topology.

The choice of isomorphism corresponds to the choice of elements $a_i \in A$ that map to $t_i \pmod I$ under (4.1), for $0 \leq i \leq n$. These induce a continuous map $R^A \to R^n$ by projecting to the coordinates indexed by $(a_1, \ldots, a_n)$. The inclusion $X(R) \subset R^n$ factors then into

$$X(R) \hookrightarrow R^A \to R^n.$$

On one hand, the topology on $X(R)$ given by Definition 4.1.1 makes all inclusions $X(R) \hookrightarrow R^n$ given by choosing an R-algebra isomorphism continuous and hence provides a stronger topology.

On the other hand, assume that $X(R)$ is topologized according to an embedding into R^n determined by (4.1). Every element of A is an R-polynomial in $a_1, \ldots, a_n$, and R is a sequential ring. Hence, all polynomial maps $R^n \to R$ are sequentially continuous. It follows that the inclusion map $X(R) \hookrightarrow R^A$ is sequentially continuous. But on a sequential space a sequentially continuous map is necessarily continuous.

Remark 4.1.4. Over the affine line $\mathbb{A}^1_R = \mathrm{Spec} R[t]$, the topology we have just described on $\mathbb{A}^1_R(R) = R$ is the sequential saturation of the topology on R.

We summarize the construction discussed above in the statement below, along with some properties.

Theorem 4.1.5. *Let R be a sequential ring. There is a unique covariant functor*

$$\mathbf{AffSch}_R \to \mathbf{Seq}, \quad X \mapsto X(R)$$

which carries fibred products to fibred products, closed immersions to topological embeddings and gives $\mathbb{A}^1_R(R) = R$ the sequential saturation of its topology.

If R is Hausdorff and $X \to \mathrm{Spec} R$ is affine and of finite type, then $X(R)$ is Hausdorff and closed immersions $X \hookrightarrow X'$ induce closed embeddings $X(R) \hookrightarrow X'(R)$.

Proof. Regarding uniqueness, although it follows from [Con, proof to Proposition 2.1] and the universal property of sequential saturation, it is perhaps a good idea to explain the argument. Let $X \to \mathrm{Spec}\, R$ be a finite type affine scheme. Choose a closed embedding $X \hookrightarrow \mathbb{A}^n_R$ for some $n \geq 0$. We look at the induced map on R-points: since we have compatibility with fibred products and we may view $\mathbb{A}^n$ as a product of n copies of $\mathbb{A}^1$, and since closed immersions carry on to topological embeddings, $X(R) \subset R^n$ is an embedding into R^n viewed as an object in **Seq**. Hence, the topology is unique and $X(R)$ is Hausdorff whenever R is Hausdorff, for a product of Hausdorff spaces in **Seq** is Hausdorff.

Regarding existence, take the topology on $X(R)$ given by Definition 4.1.1; we check the rest of the claimed properties. All unjustified claims are to be found in [Con, Proposition 2.1].

If $X \to Y$ is a morphism of affine R-schemes, the natural map $X(R) \to Y(R)$ is continuous before taking saturations and hence stays continuous after saturating.

If $X \hookrightarrow Y$ is a closed immersion, then the map $X(R) \hookrightarrow Y(R)$ is a closed embedding before taking saturations, and this property stays true after saturating.

Finally, regarding compatibility with fibred products, the argument from [Con, loc. cit.] applies with the only change that products of topological spaces are taken in **Seq**. □

Now we wish to study the behaviour of the topology defined in 4.1.1 with respect to base change. Let $R \to S$ be a morphism of sequential rings, as in Definition 1.2.11, and let $X \to \operatorname{Spec} R$ be an affine scheme of finite type. We will identify

$$X(S) = X_S(S),$$

where $X_S = X \times_R S$. As $X_S \to \operatorname{Spec} S$ is an affine scheme of finite type, the topology we will consider on $X_S(S)$ is that given by applying Definition 4.1.1 to $X_S \to \operatorname{Spec} S$.

Proposition 4.1.6. *Let $R \to S$ be a morphism of sequential rings and $X \to \operatorname{Spec} R$ an affine scheme of finite type. Then the natural map $X(R) \to X(S)$ is continuous. Moreover, if $R \to S$ is an open (resp. closed) embedding, then $X(R) \to X(S)$ is an open (resp. closed) embedding.*

Proof. The natural map $X(R) \to X(S)$ is continuous when $X(R)$ and $X(S)$ are viewed as subspaces of R^n and S^n. Therefore, we may take sequential saturations. Suppose now that $R \to S$ is a closed immersion. Then, $R^n \to S^n$ is also a closed immersion and, by restricting and taking sequential saturations, so is the map $X(R) \to X(S)$. This is also the case when we deal with an open immersion. □

A construction which may also be considered in this setting is the Weil restriction. Let $R \to S$ be an extension of rings, and $Y \to \operatorname{Spec} S$ be an affine scheme of finite type. Consider the functor $X : \mathbf{Sch}_R \to \mathbf{Sets}$ defined by

$$X(T) = \operatorname{Hom}_{\mathbf{Sch}_S}(T \times_R S, Y), \quad T \in \operatorname{Ob}(\mathbf{Sch}_R).$$

Whenever X is representable, the associated scheme is called the Weil restriction of Y along $R \to S$.

Assume that $R \hookrightarrow S$ is an injective morphism of sequential rings such that S is a finite type, locally free R-module and the topology on S is the quotient topology from a presentation (equivalently, any presentation) as a quotient of a finite type free R-module. In particular, as S is a projective R-module, the inclusion map $R \to S$ admits an R-linear splitting and R may be viewed as a subspace of S.

Example 4.1.7. A finite extension of higher local fields $F \hookrightarrow L$ satisfies the above conditions, and so does the extension of valuation rings $\mathcal{O}_F \hookrightarrow \mathcal{O}_L$.

The conditions stated above are enough to guarantee the existence of the Weil restriction of $Y \to \operatorname{Spec} S$ along $R \hookrightarrow S$ [BLR90, §7.6].

Proposition 4.1.8. *Let $R \hookrightarrow S$ be an injective morphism of sequential rings, such that S is a locally free, finite type R-module. Let $Y \to \operatorname{Spec} S$ be an affine scheme of finite type . The Weil restriction $\mathcal{R}_{S|R}(Y) = X \to \operatorname{Spec} R$ is an affine scheme of finite type. By definition,*

$$X(R) = \operatorname{Hom}_{\mathbf{Sch}_S}(\operatorname{Spec} R \times_R S, Y) = \operatorname{Hom}_{\mathbf{Sch}_S}(\operatorname{Spec} S, Y) = Y(S).$$

The two topologies we have defined on this set agree.

Proof. The reason why this is true is because the topologies on $X(R)$ and $Y(S)$ already coincide before taking sequential saturations [Con, Example 2.4].

□

4.2 The general case

In order to adapt the construction explained in §4.1 to general schemes of finite type $X \to \operatorname{Spec} R$, with R a sequential ring, our argument has two steps. First, we take an affine open cover of X and use Theorem 4.1.5 to topologize the affine open sets in the cover. Second, we need a compatibility condition in order to be able to define a topology on X.

There are a few obstacles before we can apply this argument, which will be sorted out by assuming further properties on the sequential ring R. Namely, we need to show that for any open immersion of affine schemes $U \to X$, we have an open embedding of topological spaces $U(R) \hookrightarrow X(R)$. For a general ring R (even for a topological ring) this property will not hold in general and $U(R) \to X(R)$ may even fail to be a topological embedding.

There is one elementary example that illustrates this situation. Suppose that R is a topological ring and consider $U = \mathbb{G}_m$ as the complement of the origin in the affine line $\mathbb{A}^1_R$. The map $U(R) \to \mathbb{A}^1_R(R)$ is the inclusion $R^\times \to R$, which is not an embedding unless inversion on $R^\times$ is continuous, since the topology on $\mathbb{G}_m$ is given by Theorem 4.1.5.

From this, we gather two necessary conditions that need to be imposed on a topological ring R, if we want to be able to use our argument. The first of these conditions is continuity of the inversion map, so that $R^\times \to R$ is an embedding. The second condition is that $R^\times$ is an open subset of R, so that the embedding is open.

For a sequential ring, the above example gives the inclusion $R^\times \subset R$, where both sets have the topologies described in §4.1, which are sequentially saturated. In order for this inclusion to be an open embedding, it is enough to require inversion on R to be sequentially continuous and the subset $R^\times$ to be open in R.

There is another aspect we need to worry about. Whenever $X = \cup_i U_i$ is an affine cover, we need $X(R)$ to be covered by the subsets $U_i(R)$. This will be the case whenever R is local.

Remark 4.2.1. The three conditions imposed on R (sequential continuity of inversion, openness of the unit group and being local) are analogous to the restrictions for the similar argument to work when R is a topological ring [Con, Proposition 3.1].

The above conditions are sufficient to extend Theorem 4.1.5 to general schemes of finite type.

Theorem 4.2.2. *Let R be a local sequential ring such that $R^\times \subset R$ is open and such that inversion $R^\times \to R^\times$ is sequentially continuous. There is a unique covariant functor*

$$\mathbf{Sch}_R \to \mathbf{Seq}$$

which carries open (resp. closed) immersions of schemes to open (resp. closed) topological embeddings, fibred products to fibred products, and giving $\mathbb{A}^1_R(R) = R$ the sequential saturation of its topology.

This agrees with the construction in Theorem 4.1.5 for affine schemes. If R is Hausdorff and $X \to \operatorname{Spec} R$ is separated and of finite type, then $X(R)$ is Hausdorff.

Proof. We remark that under the hypotheses on R, the inclusion $R^\times \hookrightarrow R$ is an open embedding in **Seq**.

The key to the argument is showing that whenever $U \to X$ is an open immersion of affine schemes of finite type over R, then $U(R) \hookrightarrow X(R)$ is an open embedding. However, as explained in [Con, Proposition 3.1], via reducing ourselves to principal open sets and using compatibility for fibred products in the affine case, it is enough to consider the case $U = \mathbb{G}_m$, $X = \mathbb{A}^1_R$. But, after saturating, $R^\times \to R$ is an open embedding by hypothesis.

Finally, if $X \to \operatorname{Spec} R$ is separated, then the diagonal morphism $\Delta : X \to X \times_R X$ is a closed embedding. This implies that the induced diagonal map $X(R) \to X(R) \times X(R)$ is a closed embedding of topological spaces (we take the product topology on the right hand side; the claim is true already before taking sequential saturation of the product topology and stays true after taking it). This suffices to show that $X(R)$ is Hausdorff. $\square$

Remark 4.2.3. Let R be a sequential ring and $G \to \operatorname{Spec} R$ a finite type group scheme.

The structure morphisms

$$\mu : G \times_R G \longrightarrow G,$$
$$\varepsilon : \operatorname{Spec} R \longrightarrow G,$$
$$\iota : G \longrightarrow G,$$

commonly referred to as multiplication, unit and inversion, together with their properties, turn the set $G(R)$ into a group after taking R-points. Since the topology we have constructed is functorial, $G(R)$ becomes a topological group.

Remark 4.2.4. This result cannot be applied in full generality to a higher local field F equipped with a higher topology. F is a sequential local ring in which $F^\times \subset F$ is open, as it is the complement of a singleton in a Hausdorff space. However, the inversion map $x \mapsto x^{-1}$ fails to be sequentially continuous.

Remark 4.2.5. Since the residue map $\rho : \mathcal{O}_F \to \overline{F}$ is continuous and the topology on $\overline{F}$ is Hausdorff, the maximal ideal $\mathfrak{p}_F = \rho^{-1}(\{0\})$ is a closed set. Since $\mathcal{O}_F$ is local, $\mathcal{O}_F^\times = \mathcal{O}_F \setminus \mathfrak{p}_F$ is open in $\mathcal{O}_F$. Inversion on $\mathcal{O}_F^\times$ is sequentially continuous, and this means that Theorem 4.2.2 may also be applied to the ring $\mathcal{O}_F$.

Remark 4.2.6. By iteration of the argument in the previous remark, Theorem 4.2.2 may also be applied to O_F and to any of the higher rank valuation rings of F, displayed in (1.4).

Regarding the behaviour of this topology under base change, Proposition 4.1.6 holds if we remove the affineness condition on X. The reason is that in order to check the conditions stated, we may restrict ourselves to an affine open subscheme, for which Proposition 4.1.6 is valid. For the sake of completeness, we state this result as a proposition.

Proposition 4.2.7. *Let $R \to S$ be a morphism of sequential rings with R and S satisfying the hypotheses of Theorem 4.2.2. Let $X \to \operatorname{Spec} R$ be a scheme of finite type. Then, the map $X(R) \to X(S)$ is continuous. Moreover, if $R \to S$ is an open (resp. closed) immersion, then $X(R) \to X(S)$ is also an open (resp. closed) immersion.*

Chapter 5

Conclusions and future work

This final chapter will serve as a place where we can collect some final conclusions and outline several open directions of work. We have separated these into two categories, according to whether they arise from Chapters §2 and §3 on one hand, or Chapter §4 on the other.

5.1 On the interaction of functional analysis with higher local fields

We hope that our description of higher topologies using locally convex structures makes the topic more accessible to wider audiences as, particularly, our approach to the topic using seminorms gives a flavour of these spaces being *metric-like*, in the sense that the topology can be described using open balls.

Among the new objects established after proving local convexity of higher topologies, the associated Von-Neumann bornology and the bornology generated by compactoid submodules are worth remarking. In particular, compactoidicity seems to be a concept that plays well with open lattices through duality and polarity.

Our results open an interesting new connection between functional analysis over nonarchimedean fields and higher number theory, and the results in this thesis are merely the first steps in the exploration of this connection.

Nonarchidemean functional analysis is a theory which, although having developed initially by mirroring the analogous archimedean theory, finds applications to several areas of number theory, such as the theory of p-adic representations and p-adic differential equations. From this point of view, it does not come as a surprise to realise that it provides a language specially suited for dealing with spaces which are *large*, as is the case of a higher local field.

We outline some directions which we consider interesting to explore in order to apply and extend the results in this work.

$\mathcal{O}$-linear locally convex approach to higher topology. In Corollary 2.1.6 we have shown how $K((t))$ is an LF-space, and we have been able to deduce many properties from this fact. This is not the case in mixed characteristics: the field $K\{\{t\}\}$ is not a direct limit of K-vector spaces. It is, however, a direct limit of $\mathcal{O}$-modules by construction.

The development of a theory of locally convex $\mathcal{O}$-modules, with topologies defined by seminorms, and the constructions arising within that theory, particularly those of initial and final locally convex topologies, would allow us to recover on one hand the results we have established for $K((t))$, and on the other hand they would let us describe $K\{\{t\}\}$ as a direct limit of perhaps *nice* $\mathcal{O}$-modules; this could be an extremely helpful contribution to the study of mixed characteristic two-dimensional local fields.

More generally, structure and topology on higher local fields may be studied successfully as an iteration of applications of inverse limits in the form of completions and direct limits in the form of localizations. In order to describe functional analytic structures that hold in any dimension and regardless of the characteristic type of F, it seems that two initial ingredients are necessary: a theory of locally convex $\mathcal{O}$-modules and a study of which functional analytic properties of these modules are preserved after taking direct and inverse limits.

Study of $\mathcal{L}(F)$. As we have explained in Chapter §2, the ring of continuous K-linear endomorphisms of a two-dimensional local field can be topologized and studied from a functional analytic point of view. It contains several relevant two-sided ideals defined by imposing certain finiteness conditions to endomorphisms. The most important of such ideals is the subspace of nuclear maps. Nuclear endomorphisms of a locally convex space play a distinguished role in the study of the properties of such space. In particular, the usual trace map on finite-rank operators extends by topological arguments to the subspace of nuclear endomorphisms. Establishing a characterization of nuclear endomorphisms of two-dimensional local fields is an affordable goal.

Multiplicative theory of higher local fields. Multiplication $\mu : F \times F \to F$ on a higher local field F is not continuous as explained in §3.2. It is a well-known fact that the map μ is sequentially continuous, and the sequential topological properties of higher topologies have been studied and applied successfully to higher class field theory [Fes01]. We have also applied them successfully to the study of rational points of schemes over higher local fields in Chapter §4.

Besides this, for any $x \in F$, the linear maps

$$\mu(x, \cdot) : F \to F,$$
$$\mu(\cdot, x) : F \to F$$

are continuous. This means that, in the terms of [Sch02, §17], μ is a separately continuous bilinear map and therefore induces a continuous linear map of locally convex spaces

$$\mu : F \otimes_{K,\iota} F \to F, \tag{5.1}$$

where $F \otimes_{K,\iota} F$ stands for the tensor product $F \otimes_K F$ topologized using the inductive tensor product topology.

This suggests that besides the applications of the theory of semitopological rings to the study of arithmetic properties of higher local fields [Yek92], we have the following new approach to the topic: a higher local field F is a locally convex K-vector space endowed with a continuous linear map $\mu : F \otimes_{K,\iota} F \to F$ satisfying the usual axioms of multiplication.

After Proposition 3.3.4, another possible way to look at a higher local field and deal with its multiplicative structure is as a bornological K-algebra, that is: F is a K-algebra endowed with a bornology (that generated by bounded submodules or compactoid submodules) such that all K-algebra operations

$$\begin{aligned} \sigma &: F \times F \to F \quad \text{(addition)}, \\ \varepsilon &: K \times F \to F \quad \text{(scalar multiplication)}, \\ \mu &: F \times F \to F \end{aligned}$$

are bounded.

It is interesting to decide if the arithmetic properties of F can be recovered from these contexts, and it would even more interesting to establish new connections between this functional analytic approach to higher topology and the arithmetic of F.

Functional analysis on adelic rings and modules over them. There are several two-dimensional adelic objects which admit a formulation as a restricted product of two-dimensional local fields and their rings of integers, which in our characteristic zero context were introduced by Beilinson [Beĭ80] and Fesenko [Fes10] (see [Mor, §8] for a discussion of the topic). From what we have exhibited in Chapter §2, these adelic objects may be studied using the theory of locally convex spaces, archimedean or nonarchimedean.

Topological approach to higher measure and integration. The study of measure theory, integration and harmonic analysis on two-dimensional local fields is an interesting problem. A theory of measure and integration has been developed on two-dimensional local fields F by lifting the Haar measure of the local field $\overline{F}$ [Fes03], [Mor10b]. This theory relies heavily on the relation between F and $\overline{F}$. The approach to measure and integration on F using the functional theoretic tools arising from the relation between F and K could yield an alternative integration theory.

Establishing a relevant class of subsets of a higher local field which is independent of choice of coefficient fields. The dependence of higher topologies on choices of coefficient fields as soon as char $\overline{F} = 0$ is a well-known handicap of the theory. For this reason, showing a class of subsets of F with an interesting topological or functional analytic property and which would remain stable under change of coefficient field would be extremely important.

Functional analytic characterization of rings of integers. We have not dealt with the different rings of integers of a higher local field in Chapter §3 on purpose: although they are $\mathcal{O}$-submodules which are very relevant for arithmetic purposes, as highlighted already by comparing $K((t))$ and $K\{\{t\}\}$ in Chapter §2, the functional analytic properties of such rings change drastically according to the characteristic of the residue field. It would also be interesting to establish whether there is a relevant topological or analytic property which highlights the relevance of these arithmetically interesting $\mathcal{O}$-submodules. In particular, can we characterize any of the rings of integers of a higher local field as the maximal subring of the field having a particular analytic property?

5.2 Rational points on schemes over higher local fields

After introducing the topologies on rational points over higher local rings of integers, it is natural to ask ourselves what are the properties of the resulting spaces, and the work presented here only encompasses the very first steps in this direction.

On one hand, using sequential topologies is a good approach to the topic as, after all, the problem that we tackle requires us to be able to say something about continuity of polynomial maps $R^n \to R$; if R is a topological ring this is the case and if R is a sequential ring then such maps are sequentially continuous. If we use any other of the existing approaches to higher topologies, their linear nature makes it not possible to deduce anything about such polynomial maps.

On the other hand, determining a topology up to its sequential saturation in practice means that the only thing we know about this topology is the behaviour of its sequences; this does not determine the topology unless the space is, for example, first-countable. Despite many of the computations involving continuity on a higher local field are of a sequential nature, it is unclear whether sequential information will suffice for the study of spaces of rational points over higher local fields.

Allow us to suggest a list of problems and directions of work regarding rational points of schemes over higher local fields.

Reduction maps. These are a particular case of base change. For a higher local field F of dimension $n \geq 1$ endowed with a higher topology, $\overline{F}$ also is endowed with a higher topology (if $\overline{F}$ is finite, we consider the discrete topology), and the topologies on F and $\overline{F}$ are compatible.

If $X \to \operatorname{Spec} \mathcal{O}_F$ is of finite type, flat and irreducible, then the reduction map

$$\rho : X(\mathcal{O}_F) \to X_{\overline{F}}(\overline{F})$$

is surjective [Liu02, Prop. 10.1.36]. Besides the map ρ being continuous by Proposition 4.2.7, one can say more in certain cases. For example, if $X = \mathbb{A}^1_{\mathcal{O}_F}$, then ρ is open: we are dealing with the map $\rho : \mathcal{O}_F \to \overline{F}$ with respect to the sequential topologies and, since these are group topologies, it is enough to check that the image of a sequentially open neighbourhood of zero in $\mathcal{O}_F$ is sequentially open in $\overline{F}$. The obvious argument works because we are able to choose very particular lifts of elements of a sequence converging to zero in $\overline{F}$. The same argument could be adapted to the case $X = \mathbb{A}^n_{\mathcal{O}_F}$.

A general argument along these lines would not be possible: affine schemes may be viewed as closed embeddings into affine spaces and the restriction of an open map to a closed subspace is not necessarily open.

It would be very interesting to know if the two topologies on either side of the map ρ are related. More precisely: is ρ open, perhaps under certain conditions on X? If not, is it at least a quotient mapping?

Integration on rational points over two-dimensional local fields. The extension of two-dimensional integration to sets of rational points over higher local fields would be a very important achievement in the direction of a better understanding of the arithmetic of higher dimensional schemes.

More precisely, harmonic analysis on rational points over higher local fields would be very helpful for the study of sets of rational points over higher adelic rings and establishment of a theory of higher dimensional Tamagawa numbers.

It seems that a good understanding of the topology of these spaces, as well as the relation between topology and two-dimensional measure, could be an important contribution in this direction.

Points over higher adelic rings. There are several higher adelic objects which may be defined as a restricted product of higher local fields.

In the work of Fesenko in dimension two [Fes10] there are several rings that may be realised as a restricted product of higher local fields and rings.

In Beilinson's general simplicial approach to higher adeles on noetherian schemes [Hub91], some of the adelic objects described are also definable in terms of restricted products of higher local fields.

It seems that an approach to endowing sets of rational points over such higher adelic rings with topologies would require the use of the results in this work as a starting point.

Representation theory of algebraic groups over two-dimensional local fields. Such representation theory has been developed by Gaitsgory and Kazhdan [GK04] and Braverman and Kazhdan [BK06] among others.

Algebraic groups over two dimensional local fields and their central extensions are a generalization of formal loop groups and are related to a generalization of a class of affine Kac-Moody groups. Hence the results in this work can find applications in the corresponding representation theory.

Properness and a sequential analogue of compactness. Suppose that K is a one-dimensional local field, and that X is an algebraic variety over K. Then we have the following result.

Theorem 5.2.1. *X is complete if and only if for every finite field extension $L|K$, the space $X(L)$ is compact.*

See [Lor07] for a beautiful proof to this theorem, the main ingredients of which are the valuative criterion for properness and Hensel's Lemma. A possible interpretation of this result is that compactness of the spaces of rational points of an algebraic variety over a local field characterizes completeness.

If we replace K by a higher local field F, the spaces $X(F')$ for proper finite type schemes $X \to \operatorname{Spec} F$ and finite extensions $F'|F$ are the relevant higher dimensional analogues. Is there any notion of compactness for these spaces which would enable us to prove a generalization of this one-dimensional theorem?

Bibliography

[Beĭ80] A. A. Beĭlinson. Residues and adèles. *Funktsional. Anal. i Prilozhen.*, 14(1):44–45, 1980.

[BK06] Alexander Braverman and David Kazhdan. Some examples of Hecke algebras for two-dimensional local fields. *Nagoya Math. J.*, 184:57–84, 2006.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.

[Cáma] Alberto Cámara. Functional analysis on two-dimensional local fields. 34p. To appear in *Kodai Mathematical Journal*. `arXiv:1210.2995`.

[Cámb] Alberto Cámara. Locally convex structures on higher local fields. 15p. Submitted. `arXiv:1210.8068`.

[Cámc] Alberto Cámara. Topology on rational points over higher local fields. 23p. Awaiting for publication at *Journal of Algebra and Number Theory*. `arXiv:1106.0191v1`.

[Coh46] I. S. Cohen. On the structure and ideal theory of complete local rings. *Trans. Amer. Math. Soc.*, 59:54–106, 1946.

[Con] Brian Conrad. Weil and Grothendieck approaches to adelic points. To appear in *L'Enseignement Mathématique*. Available at `http://math.stanford.edu/~conrad/papers/adelictop.pdf`.

[Fes95] Ivan B. Fesenko. Abelian local p-class field theory. *Math. Ann.*, 301(3):561–586, 1995.

[Fes01] I. B. Fesenko. Sequential topologies and quotients of Milnor K-groups of higher local fields. *Algebra i Analiz*, 13(3):198–221, 2001.

[Fes03] Ivan Fesenko. Analysis on arithmetic schemes. I. *Doc. Math.*, (Extra Vol.):261–284 (electronic), 2003. Kazuya Kato's fiftieth birthday.

[Fes10] Ivan Fesenko. Analysis on arithmetic schemes. II. *J. K-Theory*, 5(3):437–557, 2010.

[FK00] Ivan Fesenko and Masato Kurihara, editors. *Invitation to higher local fields*, volume 3 of *Geometry & Topology Monographs*. Geometry & Topology Publications, Coventry, 2000. Papers from the conference held in Münster, August 29–September 5, 1999.

[Fra65] S. P. Franklin. Spaces in which sequences suffice. *Fund. Math.*, 57:107–115, 1965.

[Fra67] S. P. Franklin. Spaces in which sequences suffice. II. *Fund. Math.*, 61:51–56, 1967.

[GK04] D. Gaitsgory and D. Kazhdan. Representations of algebraic groups over a 2-dimensional local field. *Geom. Funct. Anal.*, 14(3):535–574, 2004.

[Gor] Anthony Goreham. Sequential convergence in topological spaces. `arXiv:math/0412558v1`.

[HN71] H. Hogbe-Nlend. *Théorie des bornologies et applications.* Lecture Notes in Mathematics, Vol. 213. Springer-Verlag, Berlin, 1971.

[HN77] Henri Hogbe-Nlend. *Bornologies and functional analysis.* North-Holland Publishing Co., Amsterdam, 1977. Introductory course on the theory of duality topology-bornology and its use in functional analysis, Translated from the French by V. B. Moscatelli, North-Holland Mathematics Studies, Vol. 26, Notas de Matemática, No. 62. [Notes on Mathematics, No. 62].

[Hub91] A. Huber. On the Parshin-Beĭlinson adèles for schemes. *Abh. Math. Sem. Univ. Hamburg*, 61:249–273, 1991.

[Kap] Mikhail Kapranov. Semiinfinite symmetric powers. `arXiv:math/0107089v1`.

[Kap01] M. Kapranov. Double affine Hecke algebras and 2-dimensional local fields. *J. Amer. Math. Soc.*, 14(1):239–262 (electronic), 2001.

[Kat79] Kazuya Kato. A generalization of local class field theory by using K-groups. I. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 26(2):303–376, 1979.

[Kat80] Kazuya Kato. A generalization of local class field theory by using K-groups. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(3):603–683, 1980.

[Kat82] Kazuya Kato. A generalization of local class field theory by using K-groups. III. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 29(1):31–43, 1982.

[Kat00] Kazuya Kato. Existence theorem for higher local fields. In *Invitation to higher local fields (Münster, 1999)*, volume 3 of *Geom. Topol. Monogr.*, pages 165–195. Geom. Topol. Publ., Coventry, 2000.

[Köt69] Gottfried Köthe. *Topological vector spaces. I.* Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159. Springer-Verlag New York Inc., New York, 1969.

[Lef42] Solomon Lefschetz. *Algebraic Topology.* American Mathematical Society Colloquium Publications, v. 27. American Mathematical Society, New York, 1942.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Lor07] Oliver Lorscheid. Completeness and compactness for varieties over a local field. *Arch. Math. (Basel)*, 88(4):344–348, 2007.

[Mey04] Ralf Meyer. Bornological versus topological analysis in metrizable spaces. In *Banach algebras and their applications*, volume 363 of *Contemp. Math.*, pages 249–278. Amer. Math. Soc., Providence, RI, 2004.

[Mey05] Ralf Meyer. On a representation of the idele class group related to primes and zeros of L-functions. *Duke Math. J.*, 127(3):519–595, 2005.

[Mor] Matthew Morrow. An introduction to higher local fields. Available at `http://math.uchicago.edu/~mmorrow/Morrow%20Intro%20to%20HLF.pdf`.

[Mor10a] Matthew Morrow. An explicit approach to residues on and dualizing sheaves of arithmetic surfaces. *New York J. Math.*, 16:575–627, 2010.

[Mor10b] Matthew Morrow. Integration on valuation fields over local fields. *Tokyo J. Math.*, 33(1):235–281, 2010.

[MZ95] A. I. Madunts and I. B. Zhukov. Multidimensional complete fields: topology and other basic constructions. In *Proceedings of the St. Petersburg Mathematical Society, Vol. III*, volume 166 of *Amer. Math. Soc. Transl. Ser. 2*, pages 1–34, Providence, RI, 1995. Amer. Math. Soc.

[Par76] A. N. Paršin. On the arithmetic of two-dimensional schemes. I. Distributions and residues. *Izv. Akad. Nauk SSSR Ser. Mat.*, 40(4):736–773, 949, 1976.

[Par84] A. N. Parshin. Local class field theory. *Trudy Mat. Inst. Steklov.*, 165:143–170, 1984. Algebraic geometry and its applications.

[PGS10] C. Perez-Garcia and W. H. Schikhof. *Locally convex spaces over non-Archimedean valued fields*, volume 119 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[Pre11] Luigi Previdi. Locally compact objects in exact categories. *Internat. J. Math.*, 22(12):1787–1821, 2011.

[Sch02] Peter Schneider. *Nonarchimedean functional analysis*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2002.

[Ste67] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.

[Tat67] J. T. Tate. Fourier analysis in number fields, and Hecke's zeta-functions. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 305–347. Thompson, Washington, D.C., 1967.

[Wei82] André Weil. *Adeles and algebraic groups*, volume 23 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1982. With appendices by M. Demazure and Takashi Ono.

[Yek92] Amnon Yekutieli. An explicit construction of the Grothendieck residue complex. *Astérisque*, (208):127, 1992. With an appendix by Pramathanath Sastry.