A FRAMEWORK FOR UNDERSTANDING WHAT ALGEBRAIC THINKING IS

by

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ABSTRACT

In relation to the learning of mathematics, algebra occupies a very special place, both because it is in itself a powerful tool for solving problems and modelling situations, and also because it is essential to the learning of so many other parts of mathematics.

On the other hand, the teaching of algebra has proven to be a difficult task to accomplish, to the extent of algebra itself being sometimes considered the border line which separates those who can from those who cannot learn mathematics.

A review of the research literature shows that no clear characterisation of the algebraic activity has been available, and that for this reason research has produced only a local understanding of aspects of the learning of algebra.

The research problem investigated in this dissertation is precisely to provide a clear characterisation of the algebraic activity.

Our research has three parts:

(i) a theoretical characterisation of *algebraic thinking*, which is shown to be distinct from *algebra*; in our framework we propose that *algebraic thinking* is

- thinking *arithmetically*,
- thinking *internally*, and
- thinking *analytically*.

and each of those characteristics are explained and analysed;

(ii) a study of the historical development of *algebra* and of *algebraic thinking*; in this study it is shown that our characterisation of *algebraic thinking* provides an adequate framework for understanding the tensions involved in the production of an algebraic knowledge in different historically situated mathematical cultures, and also that the characteristics of the algebraic knowledge of each of those mathematical cultures can only be understood in the context of their broader assumptions, particularly in relation to the concept of *number*.

(iii) an experimental study, in which we examine the models used by secondary school students, both from Brazil and from England, to solve "algebraic verbal problems" and "secret number problems"; it is shown that our characterisation of *algebraic thinking* provides an adequate framework for distinguishing different types of solutions, as well as for identifying the sources of errors and difficulties in those students' solutions.

The key notions elicited by our research are those of:

(a) *intrasystemic* and *extrasystemic meaning*;

(b) different *modes of thinking* as operating within different *Semantical Fields*;

(c) the development of an algebraic mode of thinking as a process of cultural immersion—both in history and for individual learners;

(d) *ontological* and *symbolical* conceptions of number, and their relationship to *algebraic thinking* and other modes of manipulating arithmetical relationships;

(e) the *arithmetical ariculation* as a central aspect of *algebraic thinking*; and,

(f) the place and role of algebraic notation in relation to *algebraic thinking*.

The findings of our research show that although it can facilitate the learning of certain early aspects of algebra, the use of non-algebraic models—such as the scale-balance or areas—to "explain" paricular algebraic facts, contribute, in fact, to the constitution of obstacles to the development of an algebraic mode of thinking; not only because the sources of *meaning* in those models are completely distinct from those in *algebraic thinking*, but also because the *direct* manipulation of numbers as measures, by manipulating the objects measured by the numbers, is deeply conflicting with a *symbolic* understanding of number, which is a necessary aspect of *algebraic thinking*.

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Chapter 1
Introduction

1.0 THE RESEARCH PROBLEM

Algebra has always been a problematic area in school mathematics, while at the same time being one of the essential parts of mathematics to grasp if one wishes to learn and understand science and mathematics beyond the most elementary level.

A good deal of effort has been put both into developing new teaching programmes and into developing theoretical frameworks which support the development of such teaching programmes.

At the beginning of our research, our main interest was in how people give meaning to the symbolism of mathematics; for this reason, we have always been aware that dealing with the same expression can be done on the basis of different understandings, and that attributing the possession of a certain form knowledge by simply verifying the ability to deal with certain types of expressions is an approach bound to produce incorrect readings of the learners' knowledge. At this point Dr Alan Bell suggested that we concentrate our study in the field of algebra, both because of the need to restrict and delimit the mathematical topic of our research—for the obvious reason of the time available—and because of the explicit emphasis of symbolism in algebra.

Gradually, we had become more and more aware of the fact that there was a clear difference between students explanations of their solutions of "algebraic verbal problems" and that which would correspond to a verbal description of an "algebraic" solution.

In reviewing the research carried out until now on the difficulties faced by students in the learning and use of algebra, we were led to two conclusions:

(i) apart from the general theories of intellectual development, which are too general and provide little insight into the nature of the mathematical activity, no clear characterisation of *algebraic thinking* was available;

(ii) as a consequence, research into the learning and use of algebra was ill-informed, and unable to produce deep and unifying results or insights; as we will show on chapter 2, most results from research on the learning and use of algebra are local and descriptive of failure, rather than offering a positive characterisation of students knowledge.

The research problem we decided to investigate, then, was twofold. First, and crucial, we had to develop a characterisation for *algebraic thinking*, in order to be able to compare students' solutions with that which we would expect to be present in an algebraic solution. We also decided that such a characterisation would have to be useful not only to produce understanding of what happens with students, but also to produce understanding of the historical development of algebra, and to offer a framework applicable from elementary school algebra to abstract algebra. Second, we decided to investigate students' solutions to "algebraic verbal problems" in order to understand what mode of thinking—if not an algebraic one—those students were using to approach the problems. This was also essential, both because we would be able to test our characterisation's ability to distinguish different types of solution and identify sources of errors, and because by understanding the models used by the students we would be in the position of better understanding the possible obstacles they would face in the learning of algebra.

By providing such a characterisation of *algebraic thinking*, we also produced a much better understanding of what it is that we want our students to learn when we teach them algebra.

In the process of our investigation, both in the theoretical and experimental parts, many new aspects of the research problem were revealed, and they are discussed in different parts of this dissertation. To try an exhaustive presentation of those many aspects at this early point is, we think, inadequate, mainly because only in the the light of specific parts of the argument their relevance is understood. We prefer, thus, to describe our research problem, at this point, in its simplest form: to provide a characterisation of *algebraic thinking*, to test the adequacy of this characterisation in the examination of students' mathematical activity, and to investigate a specific part of that activity, namely, the solution of "algebraic verbal problems."

In Chapter 5, *General Discussion*, we will further examine general issues related to our characterisation, but from the point of view of the detailed insights accumulated along our investigation.

1.1 THE NATURE OF MATHEMATICS

The first task we must face in order to provide a clear picture of our research approach, is to clarify what we understand to be the nature of mathematics.

Our main concern will not be with the *internal* nature of mathematics, eg, how it is organised, or how its statements are shown to be correct (and what this means); we will rather examine the place mathematics occupies within the frame of human thinking.

A dichotomy which has been discussed in various forms and which provides a useful starting point, is that composed, on the one hand, by mathematics as something that exists "in the world," and as such is independent of the existence of human beings, and on the other hand, by mathematics as a creation of human mind, and only existing within each human being.

The central problems with such radical formulations are these. If one follows the former position, ie, the Platonic idealism, it is difficult to explain why it took so long for many aspects of mathematics which are conceptually simple to define to appear, as, for example, the notion of group structure, which can be immediately grasped from number systems. The second formulation brings with it a different problem; if "it all happens within our minds," how is it possible at all that mathematical knowledge accumulates, once everything would have to be re-created from the beginning by each individual.

These are, of course, simplifications of the problems faced by each of the two positions, but they provide enough ground for one to appreciate the value of the contribution offered by Leslie White in relation to the subject of the place—or places—occupied by mathematics in the framework of human existence.

In an extremely interesting paper, White (1956) discusses precisely why it is not correct to *oppose* those two views, and offers a third way, which not only solves the difficulties we have mentioned, but also opens a new perspective on the learning and understanding of mathematics.

Briefly stated, White's thesis is that mathematics is part of cultures. From this point of view, it *is* independent of individual human minds, which have to "discover" it in the process of learning the existing mathematical knowledge, but at the same time, mathematics *is* a human invention, and as part of culture totally dependent on the existence of human beings.

According to White,

> "*Culture* is the anthropologist's technical term for the mode of life of any people, no matter how primitive or advanced. It is the generic term of which *civilization* is a specific term. The mode of life, or culture, of the human species is distinguished from that of all other species by the use of symbols...Every people lives not merely in a habitat of mountains or plains, of lakes, woods, and starry heavens, but in a setting of beliefs, customs, dwellings, tools, and rituals as well." (op. cit., pp2351-2352)

It is crucial to understand that, according to this view, mathematics is part of a culture's way of "seeing" the world, and consequently of its way of organising it; in this context, our conception of mathematics molds and is molded by our conception of the world as much as it happens with religious affairs.

On the one hand, it seems undoubted that the whole content of mathematics *could* be reconstructed in a historical development beginning with, say, a group of Amazonian native indians, but to say it *could* happen, is only to affirm our belief that all human beings share the same type of "hardware," the same physiological conditions to do it—neurologically or otherwise. But as White (op. cit., p2352) says, "every individual is born into a man-made world of *culture*, as well as the world of nature." It is that culture that provides the "template," not "raw nature" or some "primitive nature": "Had Newton been reared in Hotentot culture, he would have calculated as a Hotentot." (White, op. cit., p2353)

This is our point of departure: *Algebraic Thinking* as a particular way of organising the world, as a way of modelling it and of manipulating those models. The central aim of this dissertation is, thus, to establish what this form of modelling the world—*algebraic thinking*—is, the tensions involved in different manifestations of it, and how this mode of thinking might develop within or be barred from the conceptual framework of different mathematical cultures.

From this point of view, our study of the history of algebra and of algebraic thinking will be conducted as much as possible within the framework of each culture examined, and not in a search of chained results across time and not in search of "origins" as such; our historical study will concentrate, however, on the mathematical cultures, rather than exploring all other cultural factors, like economy, social and political organisation, religions, and educational systems. It is not the case that this "epistemological closure," as Rashed (1984) calls it, goes without we paying the price of missing important information regarding which kinds of cultural contexts make a suitable ground for given types of mathematical cultures. Nevertheless, we think that ours is a *necessary first step*, that it is necessary to study the articulation of *algebraic thinking* within different mathematical cultures; in many instances, however, we will be able to establish some links between mathematical cultures and the broader context of the cultures where they belong.

When we say that mathematics is part of culture, we are not referring only to the contents of mathematics[1], but also—and from the point of view of our research, much

[1] Eg, theorems, algorithms, etc.

more important—we are referring to those forms of mathematical activity, those modes of mathematical thinking, which are seen as relevant, or even *legitimate*, within a culture. Within a given culture, number and geometric magnitudes may be understood exclusively as distinct and irreconcileable things; in another culture, to associate numbers and things may be understood as a magic act—with its possible consequences—and specific diagrams may represent deities or magic beings. In yet another, there may be an explicit antagonism to too much explanation, as one would find in Euclid's proofs, for example.

1.2 TWO CASES

FROM CULTURAL STUDIES IN PSYCHOLOGY

Our first example of how mathematical activity presents itself as a cultural trait, is taken from the work of the Soviet psychologist A.R. Luria, who was a distinguished member of the group of psychologists who studied, under the direction and inspiration of L.Vygotsky, the impact of the new Revolutionary order—in post-1917 Soviet Union—on people's consciousness and knowledge.

Luria (1976, p3) presents the research problem that is examined, by saying that,

> "It seems surprising that the science of psychology has avoided the idea that many mental processes are social and historical in origin, or that important manifestations of human consciousness have been directly shaped by the basic practices of human activity and the actual forms of culture."

In this book (Luria, 1976), one finds a number of interview transcriptions, in which the subjects are either illiterate peasants or peasants who had been to school or engaged in activities of political organisation. Luria and his assistants asked them simple questions involving, for example, the classification of objects—Chapter 3, "Generalization and Abstraction," the chapter from which we will extract our examples.

The crucial point in the theoretical framework used by Luria to analyse the responses is that,

> "...higher cognitive activities remain sociohistorical in nature, and that the structure of mental activity—**not just the specific content but also the general forms basic to all cognitive processes**—change in the course of historical development." (op. cit., p8) (our emphasis)

and children's intellectual development is also understood from this perspective (op. cit., p9).

The typical experiment in Chapter 3, is to present the subject with drawings of objects and ask for the one that "doesn't belong" to the group.

We quote a somewhat long protocol, from pages 59-60:

> "Subject: Abdy-Gap., age sixty-two, illiterate peasant from remote village. After the task is explained, he is given the series: *knife-saw-wheel-hammer*.
>
> 'They're all needed here. Every one of those things. The saw to chop firewood, the others for other jobs.'
>
> *Evaluates objects in terms of 'necessity' instead of classifying them.*
>
> **No, three of those things belong in one group. You can use one word for them that you can't for the other one.**
>
> 'Maybe it's hammer? But it's also needed. You can drive nails with it.'
>
> The principle of classification is explained: three of the objects are 'tools.'
>
> 'But you can sharpen things with a wheel. If it's a wheel from an *araba* [kind of bullock cart], why'd they put it here?'
>
> Subject's ability to learn the principle of classification is tested through another series: *bayonet-rifle-sword-knife*.
>
> 'There's nothing you can leave out here! The bayonet is part of the gun. A man's got to wear the dagger on his left side and the rifle on the other.'
>
> *Again employs the idea of necessity to group objects.*
>
> The principle of classification is explained: three of the objects can be used to cut, but the rifle cannot.
>
> 'It'll shoot from a distance, but up close it can also cut.'
>
> He is then given the series *finger-mouth-ear-eye* and told that three objects are found on the head, the fourth on the body.
>
> 'You say the finger isn't needed here. But if a fellow is missing an ear, he can't hear. All these are needed, they all fit in. If a man's missing a finger, he can't do a thing, not even move a bed.'
>
> *Applies same principle as in preceding response.*
>
> Principle is explained once again.
>
> 'No that's not true, you can't do it that way. You have to keep all these things together."

Luria himself expresses the central character of this passage very clearly:

> "One could scarcely find a more clear-cut example to prove that for some people abstract classification is **a wholly alien procedure**." (ibid) (our emphasis)

The distinction Luria uses throughout the chapter is that between "situational" or "concrete" thinking, and "abstract" or logical" thinking. The former two refer to classifications based on everyday practical usage, while the other two refer to classification based on properties of those objects such as to produce classifications like tools, animals, etc..

Luria's comment on the procedure being alien to that subject is extremely significant, specially because in many of the other protocols one finds the subjects admitting that an "abstract" classification could indeed apply to those objects, but still then refuse to use it unless prompted to (eg, op. cit., p61).

The important suggestion contained here, which Luria naturally elaborates further, is that it is the culture in which those subjects live, their cultural practice—and not their intellectual development in the sense of stages of development somewhat "natural" to the human race—which predominantly molds their responses.

A similar situation was observed in other studies, for example, in Gay and Cole (1967), where the sorting abilities of people of the Kpelle of Liberia was tested.

Another instance, which is somewhat distinct, but has strong implications to the issues in question, is to be found in Rik Pinxten's study of the North American Navajo Indian's conception of space (Pinxten, 1988); Pinxten found that for the Navajos, the world is in perpetual motion, and can only be understood so, and it is, thus, described in terms of movement, not in terms of static objects. In this context, the Navajos had no word for *angle*, as in each case the movement producing it was described instead; the process of introducing a new word to denominate angle in a static manner, a requirement for the people of a Navajo Reservation to approach White geometry, was important enough to require a discussion in the Council of the tribe, and the condition that the word would not be known to any white person, as it meant a weakening of their cultural position[2]. The

[2]This episode was described in detail in a presentation given by Pinxten at Cambridge University, in 1988.

reader is enthusiastically referred to Pinxten's book, as it provides vivid and illuminating insights for anyone interested in the process of cultural interaction, in particular those involving concepts we—White Men—would classify as mathematical matters.

FROM THE CLASSROOM

Our second instance, is presented in Freudenthal (1978, p242ff). It is essentialy a teaching experiment, probably aimed at evaluating the efficiency of a certain teaching method[3]; we are not told explicitly of the original aims.

Two groups, A and B, each of which composed by 25 students, were taught the same subject, elements of statistics; in group A, the teaching used 70 minutes, in group B, 130 minutes. This difference in time spent was allowed so to guarantee that each group had worked through the material at a convenient pace.

Group A belonged to a school leading to University and higher vocational studies; group B belonged to a school leading to lower vocational instruction. In both cases they were 7th graders (13-14 years-old).

The teaching method employed in both groups was based on investigation and discussion of topics related to everyday life, such as going to the cinema. A test was applied, at the end of the teaching period, aimed "at ascertaining whether the [students] had understood the importance of size and representativity of samples in a qualitative sense." (op. cit., p243)

One of the questions in the test was,

> "In order to investigate how many people watch a certain television programme, the N.O.S. arbitrarily chose 1500 people to fill out each day on a form which programme they had watched that day.
> Right/Wrong Explanation." (ibid)

Freudenthal says that in group A the students' answers were "predominantly satisfactory," but that 22 out of the 25 students in group B "did not grasp what was at stake," and quotes the answers of five girls:

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[3]One can guess, given the approach of the teaching as described in the book, that it was part of developing the "Realistic" approach to teaching statistics.

"[1] Wrong, because the people can know themselves which programme they like to watch.

[2] Wrong, I find it ridiculous to do this.

[3] Wrong, it is not normal, it only costs the people postage.

[4] Wrong, I think it is not their business, the people must know themselves which TV programme they want to watch.

[5] Wrong, because it is none of their business which programmes they watched that day." (op. cit., p244)

Freudenthal's comments go directly to the heart of the matter:

> "It is a paragon of—catastrophic—failure to grasp the context—I mean the context which was of course intended, the mathematical context. The 22 pupils who failed did see a context—the social one. They could not free themselves from it, they could not achieve the required change of perspective. Was this so silly? The longer I think about it, the more I become prone to answer the query in the negative and to ask a counter query: Which screw was loose with the pupils of group A (and the three girls of group B who did it well) that they obeyed the crooked wishes of the mathematician, obediently disregarded the social context, and had no problems in accepting the mathematical context?" (op. cit., p245)

There are some very important points here.

First, the distinction between the *social context* and the *mathematical context*; the former could be substituted by *situational context*, providing an adequate link with Luria's subjects.

The second point is that we are led to the need to investigate and characterise the *contexts of mathematics*, ie, the modes of thinking which make the intended mathematical activities meaningful, but also the ways in which students in group B made sense of the material presented to them so as to convince the teacher that they were progressing through the material. To understand the *contexts of mathematics* is, we think, a necessary condition to be fulfilled if we—researchers and teachers—are to understand what it is that we want our children to learn.

Finally, when Freudenthal speaks of a "change of perspective," and of a "loose screw," we think that a correct interpretation has to lead to the fact that an "immersion" into the *mathematical context* is a necessary condition for the learning of the various aspects and parts of the mathematical knowledge, and we are again led to the need of closely investigating which are the *mathematical contexts* we are presenting to the students, and which kind of thinking is necessary to operate successfully within those contexts.

It is the central aim of the research work presented in this dissertation to provide such a characterisation in the case of *algebraic thinking*, and to show that there is an *intention* that drives the development and acquisition of algebraic knowledge.

From a much broader point of view, Bishop (1988) discusses the process of learning and developing mathematical knowledge as a cultural process, ie, one which requires the immersion into and acceptation of another culture—or *ethos*, as it is sometimes more adequate to say, notably in relation to children— or a complex, and many times slow, transformation of a mathematical culture (eg, the acceptance of negative numbers as "equals" to positive numbers).

Among many interesting and well supported points, Bishop contrasts knowledge as "a way of doing" and knowledge as "a way of knowing." (op. cit., p3) The importance of this distinction is to provide a way of characterising mathematics (Which ways of knowing does it comprise?) which makes of mathematics a driving force in producing knowledge of certain kinds and in certain ways. The emphasis on the plural is important: it accounts for different modes of thinking *within* mathematics, and also for individual differences within and across those modes. *Algebraic thinking* is one of the modes of thinking *within* mathematics.

1.3 WHAT *ALGEBRAIC THINKING* IS

We now proceed to present our characterisation of *algebraic thinking*.

The first point on which we will insist, is that there is a clear distinction between *algebra* and *algebraic thinking*. This distinction is not related to a separation between *process* and *product*, nor it is intended to distinguish "what goes inside our minds" from "what is outside our minds."

"Thinking" in *algebraic thinking*, has to be understood as an indication of *algebraic thinking* referring to a way of producing meaning, while *algebra* can be understood as a content to be made sense of; it is possible, of course, to make sense of *algebra* in many different ways, and *algebraic thinking* is only one of them.

"Thinking" in *algebraic thinking* can also be understood as in expressions like *religious thinking* or *political thinking*. In both cases we have forms of organising the world: in the former, through dealing with transcendental aspects of existence, in the latter through dealing with the structures of power and representation of individual and collective rights[4]. *Algebraic thinking* is a way of organising the world by modelling situations and manipulating those models in a certain way, which we will describe a few paragraphs ahead. All three modes of thinking mentioned here can be valued differently by different societies, and they can, indeed, be altogether ignored by some of them, or be a dominant form of organising the world, as it is the case of *algebraic thinking* in many contemporary societies, specially through science and technology (see, for example, Davis and Hersh, 1988).

In our characterisation, *algebraic thinking* is better understood as an *intention*, ie, "a way in which I want to do things," even in the cases in which the concepts or methods necessary to carry through that intention are not available or developed. It is only by adopting this approach that we can understand the mechanisms involved in the algebraic development of an algebraic knowledge, be it in historically situated cultures or in children's learning; the *intention* of manipulating an equation algebraically must necessarily precede the technical ability to do it, unless we postulate that people learning it find out purely by chance a method that "works" and only then reflects upon it and transforms it into a piece of knowledge that can be deliberately used. It is true, however, that the development of such an *intention* is many times produced through the exposition to other people doing it, for example the teacher solving equations on the blackboard, a picture which remains for many students as inexplicable as it was when they first saw it, while for others it may provide the paradigm that molds the intention and gives meaning to the whole activity, possibly in a way very similar to that by which some people become political thinkers by being immersed into an—at first inexplicable—environment in which questions relative to power and the representation of individual and collective rights are in evidence.

This is not to say, of course, that "teaching by example" is in itself the best, or even a good, teaching approach, but only to show that the way in which our characterisation is developed can account for the well known fact that even the most rigid and thoughtless presentation of algebra will almost certainly produce a couple of pupils who "understand

[4] It is important to observe that, no matter how tempting these propositions might be, "God" is not a necessary content of *religious thinking*, and that "State" is not a necessary content of *political thinking*. Modes of thinking, as we understand them, have no "necessary content," as there are other factors which strongly influence the production of "content," such as material needs—the problems to be solved, for example—and the overall possibilities of the culture in which the process is developing.

it." In fact, were it not for this effect, mathematicians would almost certainly be an extinct species...

But our objective as educators must not be only the perpetuation of the *homo mathematicus*, but rather to offer to the largest number possible of people the greatest variety possible of ways of organising the world, and given the conceptual framework in which we understand *algebraic thinking*, this must mean that teaching has to address *directly* the fact that thinking algebraically requires a shift of perspective, a "loose screw," a specific *intention*, and this can only be achieved by consciously comparing different ways of modelling the same situation, and openly discussing the characteristics, virtues, and difficulties of each method used.

We finally arrive at the direct characterisation: *To think algebraically is,*

(i) *To think ARITHMETICALLY,* ***and***
(ii) *To think INTERNALLY,* ***and***
(iii) *To think ANALITICALLY.*

First we will explain what we mean by each of those characteristics, and then we will discuss their relevance for characterising *algebraic thinking*.

Characteristic (i), the *arithmeticity* of *algebraic thinking*, might initially sound almost paradoxical, particularly because mathematical educators have for a long time adopted the position of opposing arithmetical and algebraic solutions to verbal problems. It is true, however, that the basic material of both arithmetic and of elementary algebra is the same: numbers and arithmetical operations.

In the sense used in our characterisation, *arithmeticity* means precisely "modelling in numbers," which naturally implies the use of the arithmetical operations in order to produce the relationships which constitute the model. Descartes' Analytical Geometry is "modelling in numbers," as is al-Khwarizmi's algebraic method for solving problems; but "problems in numbers" can be as well modelled by using geometry or whole-part relations, which are non-arithmetical models.

Aruthmeticity means, for example, that a problem involving the determination of a speed, a distance, a weight, or a the size of the sun is seen as the problem of determining *a number* which satisfies some given *arithmetical* relationships. Any other considerations, such as the maker of the car, the unit of measurement—miles or kilometres—the shape of the object, or the colour of the sunlight, are irrelevant as soon as the necessary arithmetical relationships are established.

As we have said before, a "problem in numbers" can be solved by modelling it back into, for example, a geometrical configuration or a whole-part relationship. Let us examine an example.

Suppose that a given problem leads to the determination of a number which satisfies the equation

$$3x + 150 = 450 \tag{I}$$

An algebraic solution is immediately visible, and we will make no comments on it. It is possible, however, that the solver produces the following solution:

> "The 450 is composed of two parts, one of which is 150, the problem tells me. So, if from the whole, ie, 450, I remove one of the parts, in this case, 150, I will obtain the other part. So, the other part is 300. But this other part is composed of three smaller parts. In order to determine each of them, I would have to share the 300 into 3 parts, ie, each of the small parts is 100."

Of course, this solution produces a correct result, and in fact this kind of solution is many times taught to students as a way of "explaining" equations.

The true character of this type of solution—the use of a whole-part model—only becomes apparent when we try to apply it to other "formally" identical equations, for example,

$$3x + 150 = 60 \tag{II}$$

or

$$3.7x + 10 = 94 \tag{III}$$

In equation (II), the first half of the previous whole-part model does not apply, as the whole is smaller than one of the parts; in equation (III), the second half of the model is difficult to apply, because the "sharing" into a non-integer "number" of parts is, to say the least, a not very "natural" way of putting it.

There are other difficulties, such as dealing with equations like,

$$150 - 3x = 94 \qquad (IV)$$

but those difficulties will be dealt with on the chapter on the Experimental Study.

What we wanted to make clear, is the essential difference between dealing with those equations *internally*, ie, by reference only to the properties of the operations and the equality relation, and dealing with them by modelling them back into a non-arithmetical context; the *internalism* in our characterisation of *algebraic thinking* is precisely intended at enabling us to distinguish *internal* solutions, ie, those which proceed within the boundaries of the *Semantical Field* of numbers and arithmetical operations, and **not** by the manipulation of non-arithmetical (in our sense) models.

The notion of *Semantical Field* appears first in linguistics (see, for example, Miller and Johnson-Laird, 1976; Miller, 1978; Grandy, 1987), where it is used as a tool for explaining how words—as opposed to sentential expressions—acquire meaning. A technical discussion of *Semantical Fields* can be found in Grandy (op. cit), and it is completely beyond the scope of this dissertation. Our own version of a *Semantical Field*, which in fact had been elaborated before we learned of its existence in linguistics, is much simpler than its linguistic counterpart; in our sense, a *Semantical Field* denotes a set of meanings generated by a given "way of knowing." Mathematical expressions, an equation, for example, have different meanings within the *Semantical Field* of numbers and arithmetical operations and within the *Semantical Field* of whole-part relationships[5], as also have the arithmetical operations.

Within the *Semantical Field* of numbers and arithmetical operations, arithmetical operations are *objects*, ie, they have properties and provide information on what can and must be done to manipulate a relationship to a required effect; within other *Semantical Fields*, as for example in the non-algebraic solution of equation (I) presented above, the arithmetical operations are used only as *tools* which allow us to evaluate parts as necessary.

It is characteristic of algebraic thinking that arithmetical operations become objects, while also being used as tools and this is only a consequence of

[5]Winston et al. (1987), describes "A taxonomy of part-whole or meronymic relations...to explain the ordinary English-speaker's use of the term 'part of' and its cognates." In a sense, Vergnaud's analysis of additive problems produces on a taxonomy of whole-part relations as applied to modelling those problems.

the combined requirements of the *arithmeticism* and of the *internalism* of *algebraic thinking*.

Third, and finally, the *analiticity* of *algebraic thinking*.
Pappus says:

> "Now, *analysis* is a method of taking what is sought as though it were admitted and passing from it through its consequences in order to something which is admitted as a result of synthesis; for in analysis we suppose that which is sought to be already done, and we inquire what it is from which this comes about, and again what is the antecedent cause of the latter, and so on until, by retracing our steps, we light upon something already known or ranking as a first principle; and such a method we call analysis, as being a reverse solution. (...) But in *synthesis*, proceeding in the opposite way, we suppose to be already done that which was last reached in the analysis, and arranging in their natural order as consequents what were formerly antecedents and linking them one with another, we finally arrive at the construction of what was sought; and this we call synthesis." (Fauvell and Gray, 1990, p209)

In *synthesis*, one deals only with "what is known and true," and through a chain of logical deductions, other true statements are obtained; it is the method exclusively used in the whole of Euclid's *Elements*[6]. In *analysis*, on the other hand, what is "unknown" has to be taken as "known," with the "unknown" elements being used "as if they were know," as part of the relationships which are to be manipulated until one arrives at "something already known," ie, the "unknown" elements have to be manipulated on the basis of properties general to the class of objects to which they belong, and not as an actual manipulation of a given, specific, object. This seemingly innocuous situation in *analysis*, is strongly relevant in relation to Greek mathematics, as we will see in Chapter 3, precisely because

> "...analysis is immediately concerned with the generality of the procedure, [while] synthesis is, in accordance with the fundamental Greek conception of the objects of mathematics, obliged to 'realize' this general procedure in an *unequivocally determinate object*." (Klein, 1968, p163)

[6]In the chapter on the history of algebra and of algebraic thinking, we will examine in detail why the generality of Euclid's results could only be achieved through synthetic proofs, and what forms analysis takes in Greek mathematics.

Pappus distinguishes, moreover, two types of *analysis*:

> "...one, whose object is to seek the truth, being called *theoretical* [*zetetic*, from 'to search'], and the other, whose object is to find something set for finding, being called *problematical* [*poristic*, from 'to supply']." (Fauvell and Gray, 1990)

Seen from this point of view, the *analiticity* of *algebraic thinking* serves to characterise it as a "method for searching the truth"[7]—as one sees in Diophantus, in the Islamic algebraists, in Vieta and in Descartes—but also to characterise the fact that in *algebraic thinking* the "unknown" is treated as "known."

The explicit association of *algebra* and *analysis* is found in many authors throughout history, but the forms and the reach of the analytic processes in algebra vary tremendously in different mathematical cultures, a theme that we will examine closely. Nevertheless, *analiticity* is clearly not sufficient to characterise *algebraic thinking*; as Barrow said,

> "...to be sure analysis...seems to belong to mathematics no more than to physics, ethics or any other science. For this is merely a part or species of logic, or a manner of using reason in the solution of questions and in the finding or proof of conclusions, and of a kind not rarely made use of in all other sciences. Therefore it is not a part or species but rather the servant of mathematics; and no more is synthesis, which is a manner of demonstrating theorems opposite and converse to analysis." (quoted in Whiteside, 1962, p198)[8]

* * *

[7]As Klein (1968, p279) says, "Algebra for Vieta meant a special procedure for discovery. It was analysis in the sense of Plato, who opposed it to synthesis. Theon of Alexandria, who introduced the term 'analysis,' defined it as the process that begins with the assumption of what is sought and by deduction arrives at a known truth. This is why Vieta called his algebra the analytic art. It performed the process of analysis, particularly for geometric problems."
[8]Euler (1840, p2) identifies algebra and analysis, but in a footnote we read about the dissenting voices: "Several mathematical writers make a distinction between *Analysis* and *Algebra*. By the term *Analysis*, they understand the method of determining those general rules which assist the understanding in all mathematical investigations; and by *Algebra*, the instrument which this method employs for accomplishing that end. This is the definition given by M. Bezout in the preface to his Algebra."

So, these are the three characteristics of *algebraic thinking*: *arithmeticity*, *internalism*, and *analiticity*. We will now discuss some implications of this characterisation.

The first important point to be highlighted, is that our characterisation of *algebraic thinking* does not imply in any form or to any extent, that *algebraic thinking* can only happen in the context of symbolic—literal or other—notation. However, and this is certainly a very attractive consequence of our characterisation, the compact algebraic notation as it has developed—borrowing from the notation of arithmetic—is not only ***possible*** in the context of *algebraic thinking*, but also ***adequate***.

The reason for both its possibility and its adequacy is in the fact that the operations used for manipulating algebraic expressions are exactly the same used to constitute them in the first place: the arithmetical operations. When operating in *Semantical Fields* other that that of numbers and arithmetical operations, the manipulation of the model is done, for example, through composition and decomposition of wholes and parts, operations which are simply and adequately described verbally or with the help of diagrams, while the actual evaluation of the parts is done by using the arithmetical operations. There is nothing in the algebraic manipulation of an algebraic expression that is not related to the elements (of the base set of the operations), the operations and the equality: the "basic objects" of *algebraic thinking* form a domain tight enough to permit the compact notation, as geometric configurations in problems, for example, become irrelevant.

The second aspect which is highlighted by our characterisation, is the fact that in the context of *algebraic thinking*, numbers can only be understood *symbolically*. By this we do **not** mean the use of "symbolic notation," but that numbers are meaningful only in relation to the properties of the operations that operate on them, and not in relation to any possible interpretation of them in other mathematical or non-mathematical contexts. The notion of "symbolic number" is discussed in much greater detail on Chapter 3.

Third, our model shows that by equating the learning of algebra with developing the ability of "doing algebra," be it solving equations or squaring binomials, the mathematical educator is naturally led to incorrect readings of the didactic situation, as, for example, legitimate models for solving one type of equation might well be meaningless in relation to other types; unless we understand the *models* guiding the use of any piece of knowledge, we are bound to *impose* our understanding on other people's actions—a behaviour which leads, more often than not, to some form of misguided and authoritarian cultural action[9].

In the more specific case of algebra, the explanations for, for instance, students being able to solve some linear equations, but not others, have ranged from "stages of intellectual development" to "misconceptions derived from arithmetic," but little has been done in the direction of providing a framework in relation to which those difficulties can be understood without recourse to *ad hoc* hypothesis[10]. We think that our characterisation of *algebraic thinking* provides precisely a framework in which pupils' solutions can be examined and understood, and which can guide the teaching of algebra in a much more coherent and fruitful way than the previous models.

Fourth, and finally, we must stress that according to our characterisation, *algebraic thinking* is not *a priori* a more powerful or more adequate mode of thinking than others, not even within mathematics: it is simply different from other modes of thinking. From this point of view, learning to think algebraically is as important as learning to think geometrically or combinatorially; from a broader perspective, it is as important as learning to think politically or religiously. It is the possibility of examining the world from different, complementary and possibly conflicting, perspectives, that makes learning each of those modes of thinking important[11].

* * *

Our characterisation of *algebraic thinking* puts much emphasis on the *numerical* character of algebraic modelling.

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[9] It may be useful to think of a related behaviour in a different context. The teacher complains to the school's psychologist: "The drawings Little Rom brings from home are all in purple and black. I am a bit worried." The psychologist examines the drawings and agrees that they depict a "heavy" atmosphere: "Maybe the family is going through some crisis!" etc.. In the worst case, the parents will be called and some form of counselling suggested; it may well be that the parents do not really understand what is going on and are frightened and agree. But, I say, it may well be the case that all the other colour pencils were lost by Little Rom, or even that his family's cultural background is one in which black and purple do not have the same connotations as in the teacher and psychologist's æsthetics... The case of "black and purple drawings" is a real one, told to me by a teacher who was alert enough to investigate the matter properly.

[10] In the case of the stage theories, the *ad hoc* element is provided by a characterisation of the algebraic thinking that necessarily forces the conclusions arrived at by the theories, and this results in a crystallisation of prejudices, rather than in understanding. In the next chapter we return to this point.

[11] I think it was Proust who pointed out that the true journey is not seeing a thousand places with a pair of eyes, but to see one place with a thousand pairs of eyes.

If we consider abstract algebra, but also groups of permutations, groups of symmetries, and polynomial rings, for example, it seems that such emphasis is not only restrictive, but also incorrect, even if we limit ourselves to discussing "elementary algebra.". We will argue, however, that this is not the case.

The central notion in the arithmetical operations, is that of "combining" two elements of the base set to "produce" another element of the base set. Put in more formal terms, the two original elements are not literally "combined," as this would imply the need of an explicit law of "combination." We use, instead, the term "law of *composition*," and say that this law of composition *associates* to each ordered pair of elements of the base set, another element of the base set. It is perfectly clear not only that the "law of composition" formulation is "inspired" by the arithmetical operations, but also that even when dealing with an algebraic system in which the laws of composition are as abstract as one can imagine, we are still psychologically satisfied that $a \otimes a^{-1}=1$ is like "calculating." And it is, in fact, technically irrelevant whether we think or not of "calculations," as long as we do not require that the actual "law" be exhibited.

The other important aspect here, is that of *number*. For the ancient Greek, irrationals were not *numbers*, and negative numbers were simply *unthinkable*. The Chinese accepted negative numbers in specific mathematical contexts, but the notion was not generalised. In Islamic mathematics, both zero and negative numbers were largely disregarded, but surds were treated in some great detail. Even in the 19th century, there were critics of negative numbers, and it was a long time before mathematicians fully accepted imaginary numbers, while, in fact, they were "calculating" with them much before a foundation was provided. It is clear that in Cardano, for example, $\sqrt{-4}$ does not "measure" anything, nor has any similitude with any of the previously accepted *numbers*, and, rightfully, they were called "quantities," not "numbers."

Today we call negatives, surds, fractions, complex, e, and π, *numbers*. We do not call quaternions *numbers*, but we naturally should, as there is nothing to distinguish their general "outlook" from that of complex *numbers*; in the same way, we may ask ourselves Why not to call polynomials, matrices, permutations, etc., *numbers* ? Certainly there is no technical damage done.

Strictly speaking, the following "definition" is technically acceptable:

> "Number is any element of the base set of an algebraic system."

As we said before, once one is thinking algebraically, *numbers* are understood *symbolically*, ie, they are dealt with only by reference to the properties of the arithmetical operations. But this is exactly the case with polynomials, matrices and permutations when they are *collapsed* [12] into elements of the base set of an algebraic system; the notion of isomorphism between algebraic structures highlights precisely this aspect.

Of course, the definition we provided does not correspond to the way in which we use the word *number.*

Nevertheless, we think that by using *arithmeticity,* instead of a more sophisticated form of characterisation for this aspect of *algebraic thinking*, at least two important functions are fulfilled: (i) the *intuition* generated by the arithmetical operations is clearly preserved in our characterisation, in a way which is useful in extending *algebraic thinking* for situations in which the base set is not a "numerical" set; and, (ii) the notion of *symbolic number* is highlighted, as our characterisation emphasises the distinction between the *symbolic* treatment of *number*, ie, in the context of *algebraic thinking*, and other models for representing and manipulating relationships involving numbers—as measure, for example.

Throughout the rest of this dissertation, we will keep in line with the conventional usage, and reserve the word *number* only for those mathematical objects which are so normally called. Nevertheless, the reader should bear in mind, at all times, that the essential notion behind our choice of *arithmeticism* as a name for a characteristic of *algebraic thinking* reflects the precise and crucial fact that *algebraic thinking* is a mode of thinking for which the *external interpretations*—on the basis of which much of the usage of the word *number* has been built—are irrelevant references.

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[12]We will return to this very essential and illuminating notion on Chapter 3. For the moment, the following example should be sufficient: a polynomial $f(x)$ in the formal variable x is formally defined as an expression of the form $a_0+a_1x+...+a_nx^n$, and with this "internal" structure in view, we can speak, for example, of complete and incomplete polynomials, etc. When we speak of an algebraic system in which the base set is a set of polynomials, however, this "internal" structure is—at least temporarily—*collapsed,* and the elements become f, g, etc.; it is only the properties of the operations which operate on them that are relevant, here, not how thy eventually "deal" with the "internal" structure of the polynomials.

* * *

We think that one last word of explanation is necessary.

The result of our research effort is, without doubt, to "isolate" a mode of thinking—*algebraic thinking*—and characterise it. This is not, however, part of a "dissectionist" approach, a fact which will become even more evident in the subsequent chapters, as we make clear that it is only possible to understand what *algebraic thinking* is by articulating it and contrasting it to other modes of thinking within mathematics, and all this in the broader context of the mathematical culture in question.

The importance of placing this observation here, is to discourage *immediately* anyone from taking the work contained in this dissertation as an invitation to develop "a new course in algebra," or even from reading it as a preliminary effort in that direction.

The main educational objective of our research work is precisely to enable teaching to proceed in an as open framework as possible, by providing the tools for the teacher to distinguish and understand, *on the fly*, the thinking and learning processes which are developing on the part of the learner. As we have mentioned before, the specificities of *algebraic thinking* are best grasped only by contrasting it with other modes of thinking, and this is a possibility which only an open, investigative teaching setting can provide.

Chapter 2
A Study of Previous Research

2.1 INTRODUCTION

Algebra has been seen, for a long time, as a difficult, although important, area of school mathematics, and as a consequence a huge number of studies have been carried out on the subject.

Our research has a "foundational" character, rather then a "didactic" one[1], and for this reason we will not include in our examination the many teaching approaches and experiments in algebra produced in the past years, as, for example, Alan Bell's richly suggestive teaching experiment (Bell, 1989b); there are two exceptions, namely Lesley Booth's further investigation into the difficulties identified by the CSMS algebra survey, which throws light into the survey itself, and Davydov's approach to the teaching of algebra in elementary school, which embodies a theoretical approach to the problem which is radically different from the approaches we find in "Western" literature.

The review of the relevant literature which follows, is primarily aimed at three aspects of the research on the learning of algebra: (i) the topics examined by research; (ii) the underlying epistemological and methodological assumptions of those researches; and, (iii) the issues raised by them.

We will not, however, present a thorough account of the available literature; we choose, instead, to examine here only a selection of material which seemed sufficient to allow a reflection on the research on the learning of algebra *as a whole*.

2.2 CRITICAL REVIEW OF THE PREVIOUS RESEARCH

THE SOLO TAXONOMY

The SOLO Taxonomy was developed by Biggs and Collis in order to provide educators with a general framework for assessing the *quality* of learning. In Biggs and Collis (1982), *quality* is characterised as the answer to the question "how well," and opposed to the *quantitative* aspect of learning, which is characterised as the answer to "how much." At the very beginning of the preface, they say:

> "In this book, we suggest that the evaluation of thought, from childhood to adulthood, gives an important clue as to quality. That clue is *structural*

[1] It is never too much to emphasise that although *at this point* we are concerned primarily with the "foundational" side of our research, it naturally aims at providing a solid foundation for the development of an approach and programme for the teaching of algebra, as well as at providing a better understanding of the issues involved in research on the learning of algebra.

organisation, which discriminates well learned from poorly learned material in a way not unlike that in which mature thought is distinguishable from immature thought." (op. cit., pxi)

The key characteristic of the SOLO Taxonomy, is that it examines the outcome of learning focusing on how the response is structured, rather than on whether a given content was or was not learned. Although postulating that the structure of the responses can be characterised by levels—from "concrete" to "abstract"—of progressing complexity, they examined the characteristics of the traditional models of Stage Theories of development, and concluded that they are inadequate to deal with the assessment of the *quality* of learning, as: (i) they postulate a stability for the stages that is not confirmed by research, ie, the same student answers at different levels at different times and in relation to different situations; the concept of *décalage*, used by Piagetians to account for this phenomenon, is too common to be only an exception, Biggs and Collis say; (ii) they are intended to predict, on the basis of logically related tests, how a person will respond to a given test; this possibility is based both on the stability of the stages and on the measurability of the *hypothetical cognitive structure* (op. cit., p22).

The crucial difference between the approach in the SOLO Taxonomy and the Stage Theories, is that in the latter it is the *learner* that is categorised, whereas in the former it is the *outcome*. This shift removes the need to appeal to the concept of décalage as a corrective device, and at the same time makes for a better educational instrument[2]; *hypothetical cognitive structure* is replaced by the SOLO Taxonomy in a way similar to replacing *ability* by *attainment*. Biggs and Collis say that *hypothetical cognitive structure* is not, in most cases, an issue to the teachers (see note 1).

The SOLO Taxonomy distinguishes 5 levels of outcome, Prestuctural, Unistructural, Multistructural, Relational, and Extended Abstract, which are characterised in relation to three "dimensions": (i) working memory capacity; (ii) relating operation—the way in which cue and response relate; and (iii) closure and consistency. A detailed explanation of those three aspects is provided in Biggs and Collis (op. cit.).

In Chapter 4, the SOLO Taxonomy is used to analyse the responses to some test-items given to students. We will briefly examine aspects of their analysis of one of the items[3].

[2]The stage theorist, on the basis of standard tests, suppose the adequacy of predicting the possibility of a learner learning a given material, to the extent of considering that "...reading, as a symbolic and verbal activity, should not be taught until the high-school years." (Furth's position, in Biggs and Collis, 1982, p21; see also p23)

[3]The same criticism presented here, applies to the other sections of the chapter on mathematics, and for this reason we will not examine them directly.

The following problem was proposed[4]:

You are to decide whether the following statements are true always, sometimes, or never. Put a circle around the right answer. If you put a circle around "sometimes" explain when the statement is true. All letters stand for whole numbers or zero (eg, 0, 1, 2, 3, etc.)

1.	$a + b = b + a$	Always Never Sometimes, that is, when ____
2.	$m + n + q = m + p + q$	Always Never Sometimes, that is, when ____
3.	$a + 2b + 2c = a + 2b + 4c$	Always Never Sometimes, that is, when ____

According to the SOLO Taxonomy, the different levels would be indicated by the following behaviours:

> "*Unistructural responses.* At this level of response the students saw each letter as representing one and only one number...If...one trial did not give a satisfactory result, they gave up working on that item.
>
> *Multistructural responses.* Students giving responses at this level tried a couple of numbers and if they satisfied the relationship they drew their conclusions on this basis...
>
> *Relational responses.* At this level the students seemed to have extracted a concept of 'generalised' number by which a symbol b, say, could be regarded as an entity in its own right but having the same properties **as any number with which they had previous experience...** [our emphasis]...Even though the responses showed that they possessed the concept of generalised number, students responding at this level were unable

[4]Although in the book we find a "minimal age" associated to the levels, the ages of the students answering the tests are irrelevant *for the purpose of examining the difference in outcomes from the point of view of the theory.*

to cope adequately with the problem of making the necessary deduction in the final step of the second and third items...Again with the third item...it is the next level of abstraction, that of a pronumeral as a variable, where thinking of zero occurs so that the number system is consistent with itself. *Extended abstract responses*. Responses in this category demonstrate an ability to view a pronumeral as a variable and thus enable the final deduction necessary in the second and third items to be made..." (op. cit., p69ff)

At this point, a strong objection to Biggs and Collis' analysis must be raised. In all cases they are assuming that the students are working with numbers *as such*, ie, that there are no *nonnumerical* models guiding their judgement. If we accept, instead, the possibility that the students could be thinking of the letters as naming segments of lines, much of the analysis could be different: (i) in the third item, the crucial question would be the possibility of using a numerical model, as "zero" cannot be represented as a segment; (i) in the second item, the difficulty could be related to the practice—common and, in fact, necessary in life—of not giving the same object (a line segment, in this case) two names; (iii) in the first item, an "always" without calculations could well mean the obvious fact that if you conjoin two segments of line, the total will always be the same. In relation to (ii), even in the case of a numerical model being used, the *mathematical usefulness **and acceptability*** of the possibility of two letters representing the same number might play a crucial role, ie, the case is not considered because the student does not *know* that it can be so. The possibility of this gap highlights the fact that there is never any attempt—in this section or elsewhere in the chapter on mathematics—to relate the types of responses to schooling conditions, such as the sequence of the topics taught and the characteristics of the teaching material used[5].

It could be argued that the students had been *told* that the letters stood for numbers, but this is not sufficient to determine which model is used to guide the manipulation of relations involving those numbers. From the text of the book, it is not possible to know which—if any—indications the students tested gave of using nonnumerical models, but the simple fact that this possibility is not mentioned or discussed is indicative that the authors were probably unaware of the distinct possibilities it would bring.

[5] It is true that the book deals with "implications of SOLO for the teaching of mathematics." Nevertheless, the definition of school mathematics adopted ("...a logical system or structure of relationships that has as its base a set of elements and a clearly defined method of operating on them...") naturally excludes the two considerations we have mentioned.

In the same way in which Biggs and Collis pointed out that several variables may interfere with the production of a response, and, thus, the stages theories are not a good model for assessing the quality of the response, we must point out that the quality of a response, in the sense of the SOLO Taxonomy, can only be evaluated from the point of view of the mathematical framework within which the learner is operating, ie, his or her mathematical conceptualisations. Strictly speaking, the failure to answer correctly a test-item, *analysed in the absence of a knowledge of the model in relation to which the learner tried to solve it*, can only mean that "the learner was not able to deal with that test-item." We must make clear, nevertheless, that our criticism is only directed at the impossibility for the SOLO Taxonomy to elucidate, by itself, the characteristics of the learner's *mathematical ethos*, and in particular, the model used as a support in any specific problem solving situation.

At the same time, it is clear to us that our characterisation of *algebraic thinking* is not capable of, nor aimed at, distinguishing responses in a manner similar to the SOLO Taxonomy. Instead, it is aimed exactly at distinguishing between different models used to deal with and produce algebraic knowledge. The first phrase of Biggs and Collis (1982) is, "In this book, we are concentrating on a common learning situation: one that involves *the meaningful learning of existing knowledge*, or reception learning." It is precisely because one speaks of *meaning*, that it is necessary to determine which is the conceptual framework in which this knowledge is supposed to be inserted, and the central aim of this dissertation is to provide the means to to this determination in the case of algebraic knowledge.

The interpretation given by Biggs and Collis to the responses, depends on a second assumption, namely that the mathematical context of the response rests defined by a *content*, in this case, that composed by the algebraic expressions proposed—this meaning precisely a combination of letters and arithmetical symbols—together with the knowledge that is required to answer correctly the questions *if they are treated numerically*. As this knowledge cannot be communicated to the solver, or the questions would not be questions, we are left with the algebraic expressions as supposedly defining the mathematical context of the questions in the view of the researchers. On page 87 we read:

> "The necessity to communicate parts of the structure [mathematics] to others gives rise to a formal symbolism that takes in both the elements and the operations. The mathematical statement 4(a+b)=4a+4b may be used to demonstrate the point. The elements involved in the statement are numbers and variables; **the operations to be carried out on the elements,**

> **multiplication and addition are clearly defined**...and the statement itself indicates a link between two sections of the mathematical structure, that concerned with addition and that concerned with multiplication." (our emphasis)

The possibility of the mathematical expression representing a statement about areas is simply not considered.

THE CSMS ALGEBRA SURVEY

The objective of the CSMS project was to produce a survey of secondary school mathematics, in a number of areas. The main results of the survey are reported in Hart (1984).

One of the areas of interest, in the CSMS survey is the understanding children have of letters in mathematics. In Hart (op. cit.) the results are presented under the title of "Algebra," but in Küchemann (1978) they are described as an "investigation of children's understanding of **generalised arithmetic**."

In order to analyse the results of the testing, six categories were created, describing different ways in which letters could be used in the context of the test-items; those categories were based on earlier work by Collis. The six categories are (Hart, op. cit., p104):

> (i) letter **evaluated**: "This category applies where the letter is assigned a numerical value from the outset."
>
> (ii) letter **ignored**: "Here the children ignore the letter, or at best acknowledge its existence but without giving it a meaning."
>
> (iii) letter **as object**: "The letter is regarded as a shorthand for an object or as an object in its own right."
>
> (iv) letter **as specific unknown**: Children regard a letter as a specific but unknown number, and can operate upon it directly."
>
> (v) letter **as generalised number**: "The letter is seen as representing, or at least being able to take, several values rather than just one."
>
> (vi) letter **as variable**: "The letter is seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values."

The actual results have no direct relevance for our research, so we will not examine them in any detail. We will focus instead on the aims of the CSMS research on algebra, and some aspects of its methodology.

First and of foremost importance, the study reports a link between the different uses of a letter and Piaget's levels of intellectual development, but does not take into consideration, at any time, the instruction received by those students on the topics tested; Booth's follow-up study of the survey, which we will analyse a few paragraphs ahead, shows that this is an aspect of crucial importance in relation to the results collected by the survey. It also shows, however, a conception of knowledge and of knowing well in line with the Piagetian tradition of the "little-lone-scientist."

Second, the survey does not examine whether there was consistency within the answers of single students, and thus, the validity of the association with developmental levels is seriously jeopardised.

From a more general point of view, Bell (1987, 1989b) showed that the six categories are not adequate to describe all the different situations that may arise in the algebraic activity; also, in focusing the investigation on simple and immediate uses of letters, the survey does not provide any insights into the processes by which the different uses proposed are developed or interrelated.

As we have already said, Lesley Booth produced a follow-up study of the algebra part of the CSMS survey; the results are reported in Booth (1984). As with the CSMS survey, her study deliberately concentrated on the use of letters in "generalised arithmetic." The aim of Booth's study was,

> ...to investigate the reasons underlying particular errors in generalised arithmetic which the earlier CSMS (mathematics) project had shown to be widely prevalent among 2nd to 4th year children in English secondary schools, and to explore the effectiveness of short teaching modules designed to help children to correct or avoid these errors." (op. cit., p1)

Two hypothesis are investigated: the dependence of errors on the interpretations given to the letters, and on the use of procedures that are imported by the children from the solution of arithmetical problems.

The main conclusions of the study can be thus summarised:

(i) there seems to be support to the view that the possibility of using letters in different ways is related to a cognitive awareness;

(ii) part of the difficulties faced by the children result from the use of "informal" methods, which are methods which are elicited by specific aspects of a problem, rather than general solution or manipulation procedures; Booth points out that it is highly relevant that even after being taught formal methods, many children continue to use the informal ones, and considers the possibility of interpreting this on the basis of Collis interpretation of "concrete thinking," according to which the "child's thinking is restricted to concrete-empirical experience so that the child tends to operate in terms of the particular situation presented" (cf. Booth, op. cit., p89). She also points out that children "do not look beyond the particular solution of immediate, concrete, problems," (ibid) but indicates that children benefited from teaching in overcoming this situation, in that it assisted them "to move towards operating in the more formal systems"; this last result seems to disagree with the idea that only when reaching the level of formal operational thinking they would be able to think within formal systems.

(iii) the notational conventions of arithmetic might influence children's construction of meaning for algebraic expressions. An important result, is that the "acceptance of lack closure" (see, for example, Biggs and Collis, 1982) was shown to be much less resistant to teaching than expected, leading Both to consider that "the acceptance of lack of closure, and the view of letters as generalized rather than particular number, may relate to different levels of conceptual difficulty, rather than be manifestations of a single cognitive structure as suggested by the Collis-Piaget formulation." (op. cit., p91)

It seems, from the written report of the research, that by "informal methods" Booth always means "informal numerical methods," as in for example, dividing 525 by 5 by separating 525 in 500 and 25, dividing each part by 5, and adding the partial results, rather than considering which is the model for quantities guiding this process (it could be, for example, a whole-part model, or it could be a model based on properties of the notational system). She suggests that further research is needed on the informal methods used by children in generalised arithmetic, and of the five points she highlights, two are more directly related to our research: "How do those informal methods develop?," and "Why do many children fail to assimilate the formal taught procedures."

Z.P. DIENES ON THE TEACHING OF ALGEBRA

In this section, we want to examine briefly Dienes' conception of what should be aimed at by the teaching of algebra, by summarising Chapter 4 of his *Building up mathematics* (Dienes, 1960).

First, Dienes points out that the learning of arithmetic requires in fact the learning of some "algebraic facts," and also that symbolization should follow the development of algebraic concepts, not precede it. He argues that "It is no earthly use to put a variable [in the form of a letter] before a child until he has seen it vary." (op. cit., p76)

As it is well-known, Dienes conceives the construction of mathematical knowledge by children as abstracting the mathematical structure from experience with a number of mathematically similar situations (the Principle of Perceptual Variability), so he proposes that from activities with tiles and scale-balances, the laws of algebra (eg: $A \times B = B \times A$) be *derived* as abstractions.

Dienes also suggests that equations be set and solved with concrete material, and then symbolised, and that in a similar way, formulas such as for the square of a binomial, and procedures for factoring quadratic polynomials, be derived. The use of concrete models, however, precludes the same approach with expressions involving negative quantities, but Dienes justifies the correctness of the approach saying that,

> "We are quite happy to tell children that $X^2+1=0$ has no solutions, and yet proceed happily to contradict ourselves a few years later. The same should apply to any stage of learning in which only a restricted field of numbers is considered." (op. cit., p100)

The use of concrete models in this manner, to justify and illustrate the rules and procedures of algebra, has certainly become influential (see section *Research...reported at PME*, below), but some authors have considered that features of the concrete models can stay too firmly tied to the mathematical construction (eg, Booth, 1987), and also that children do not see the relationship between the concrete model used and the mathematical concepts which they are supposed to illustrate, although the concrete model was seen as "useful" by children (eg, Hart, 1988, 1989).

Summarising, we can say that Dienes view of the algebraic knowledge that is to be achieved by the children, corresponds more to the *content* of algebra, ie, its laws and rules of manipulation, and less to a *mode of thinking* according to which those aspects are more meaningful; according to Dienes' approach, the means of providing meaning to algebra is to relate its laws and procedures to a model that can be directly and concretely manipulated, and not by appealing to properties of the algebraic expressions as expressions of numerical relations; in many ways, Dienes' approach amounts to providing an *ontology* for the objects being manipulated, ie, to say "what they are," and from then derive the properties of operations on them[6].

[6]The notion of an *ontology* is discussed in detail in Chapter 3, on the historical development of algebra.

RESEARCH ON THE LEARNING OF ALGEBRA REPORTED AT PME[7]

Since its first conference in 1977, the PME group has been recognised as the most important international research forum in the Psychology of Mathematical Education. The interests of people belonging to PME range over a variety of themes, from more theoretical issues (eg, Wachsmuth, 1981), to concept formation (eg, Meira, 1990), to the use of non-specific computer software to promote the learning of specific aspects of mathematics (eg, Sutherland, 1989).

The interest in algebra and algebraic learning has varied over the years: 7 papers in 1981, 1 paper in 1982, 16 in 1987, 6 in 1988, 5 in 1989, 14 in 1990, 13 in 1991.[8]

Those papers can be roughly divided into three main areas[9]: (A) difficulties in algebra caused by the use of literal notation; (B) difficulties in algebra caused by an insufficient understanding of arithmetic; (C) models for characterising the algebraic activity. We will briefly examine those areas in turn.

(A) Difficulties in algebra caused by the use of literal notation

A common approach here is to propose test-items in algebra and to analyse the distribution and types of errors. Pereira-Mendoza (1987) examines the way in which students make incorrect generalisations of algorithms to deal with expressions in arithmetic, and apply them to algebraic (literal) expressions; he distinguishes the "arithmetic space" from the "algebraic space." Becker (1988) does a similar investigation, but focusing on the role of the literal symbolism on the formation of errors.

A second approach is to investigate directly the characteristics of the algebraic symbolism. Kirshner (1987 and 1990); in the first paper he examines the syntax of algebraic symbolism from the point of view of the *parsing* of expressions, and in the second paper he examines issues on the acquisition of algebraic language from the point of view of a model for its syntax. Filloy (1987) also examines algebra from a linguistic point of view, but in a broader perspective, relating the linguistic issues with the tension between semantic and syntax, arguing, with Thorndike, that emphasis must be put on practice with the syntax in order to free the individuals attention from the syntax

[7]The annual conferences of the International Group on the Psychology of Mathematical Education. PME is a group within ICME, the International Conference on Mathematical Education, which meets every four years.
[8]Papers on functions were not included, unless they focused on algebraic aspects of functions.
[9]Many papers, of course, examine more than one of those aspects.

and allow him to concentrate on other—less "automatic"—aspects of the problem he is handling.

In Gallardo and Rojano (1987), a number of specific aspects of the use of literal symbolism are examined, with almost total reference to "the unknown" in the context of solving equations; the paper refers to the "didactic cut" that happens when the students are requested to deal with equations in which the unknown appears on both sides, and account for this difficulty on the basis of a refusal to "operate on the unknown."

(B) Difficulties in algebra caused by an insufficient understanding of arithmetic

In the past five years, very few PME papers deal directly with this aspect of the learning of algebra. Booker (1987) provides a brief review of the main issues examined until then. Booth (1989) also provides a brief review, and examines the results of an experimental study designed to investigate students' understanding of inverse operations, association and commutativity, and relates those results with possible consequences to the learning of algebra. Booth's study is based on students' ability to manipulate arithmetical expressions with varying degrees of complexity, and the use of non-numerical models by the students is not examined; she argues for the teaching of arithmetic to put more emphasis on the structural properties of numbers, which, in fact, would amount to a greater degree of *algebraisation* of the teaching of arithmetic.

In the beginning of the 1980's, the interest in the transition between arithmetic and algebra was more intense than today, with papers such as Kieran's (1981), examining both the difficulties introduced in algebra by the undue transfer of concepts and procedures from arithmetic, and the ability of some pre-algebra students to understand intuitively some aspects of algebra, as the solution of simple linear equations.

To some great extent, the main issues related to this theme were "exhausted," but failed to produce a deeper understanding of the learning of algebra, as many of the students who had a good understanding of arithmetic also faced sharp difficulties with algebra. Nevertheless, those studies informed teaching in a very useful way, pointing out that merely "generalising" arithmetic was not sufficient to lead to the learning of algebra, and let the field open to other investigations.

(C) Characterisations of the algebraic activity

The papers under this heading are of three kinds.

First, there is a small group of papers where the algebraic activity is organised around the uses of algebra. Bell (1987), discussing the basis for designing an algebra curriculum, argues that such curriculum should be organised around different modes of algebraic activity, of which he distinguishes four: generalising; forming, solving and interpreting equations; functions and formulæ; and, general number properties. He opposes his proposal to the traditional organisation around different types of algebraic manipulation, and to the organisation around the different uses of letters. Bell's model is flexible and designed to provide students with a sense of *purpose* for algebra, but a discussion of the mathematical nature of the algebraic knowledge is not provided. Lee (1987), [Ursini] Legovich (1990) and Ursini (1991), examine algebra in the context of generalisation. In all three cases, the usefulness—as perceived by the students—of algebra, in expressing generality, is examined, and also how the use of algebra is often replaced, by students, with other models, in dealing with the generality of, for example, patterns. In her paper, Lee points out to four major conclusions:

> "1. A majority of students do not appreciate the implicit generality of algebraic statements involving variables.
> 2. For most students, numerical instances of generalisation carry more conviction than an algebraic demonstration of the generalisation.
> 3. Many students do no appreciate that a single numerical counterexample is sufficient to disprove a hypothesised generalisation.
> 4. **Students who can competently handle the forms and procedures of algebra rarely turn spontaneously to algebra to solve a problem even when other methods are more lengthy and less sure.**" (our emphasis)

There seem to be two possibilities, here. First, that the students did not consider the possibility of modelling those patterns *in numbers*, and for this reason refused to use algebra to manipulate the—non-numerical—generality they perceived. Second, that precisely because the generality perceived by the students was not an *arithmo-algebraic* one, it was not *visible* in the algebraic statements, as there is an implicit shift in the *objects* in the process of modelling a situation algebraically. In Lee, we find some of the attempts to manipulate the generality of a pattern directly, through the manipulation of the geometric configurations that generated it.

In Friedlander et al. (1989), "visual" and "numerical" forms of justifying the solution of "algebraic" problems are examined.

In second group of papers, the algebraic activity is examined by organising it around the *content* of algebra: equations, equation solving, variables, expressions (Kieran, 1988 and 1991; Linchevsky and Vinner, 1990; Rubio, 1990), and specific difficulties examined.

In a number of papers in this group, the use of non-algebraic models to provide "meaning" for algebra is advocated (Cortez and Vergnaud, 1990, scale-balance; Garançon et al., 1990, computer-aided arithmetic model; Filloy, 1991, scale-balance and areas) or the procedures that can be generated through such support models examined (Carraher and Schliemann, 1987, scale-balance used in a professional context; Sutherland, 1989, Logo; Rojano and Sutherland, 1991, spreadsheet). Only Booth (1987), however, examines the effect of using such models in the conceptualisation of the algebraic activity that is produced by the students; she points out that "...careful thought needs to be given to the kind of [concrete] model used, to the ways in which the model is related to the formal procedure, and to the limitations and misleading notions that might be inherent in the particular models adopted." She does not consider, however, the possibility of mistakes observed in students of algebra being due to the "background," ie, not explicit, use of such models[10].

The third, and last, group of papers, is quite limited in size, and varied in approaches. It is composed by attempts, more or less comprehensive, to characterise the algebraic activity in itself, ie, to characterise the mode of thinking that is peculiar to it, and not through its content[11].

Sfard (1987, 1989), develops the distinction between the *operational* and *structural* aspects of mathematical—and in particular, algebraic—notions; to the former, she associates *processes*, and to the latter, *static* "entities." Sfard's model is intended to characterise the passage from simple to complex levels of the algebraic activity, based on the mechanism of "reïfication" of processes into "compact static wholes." In both papers she analyses the learning of the concept of function from the point of view of her framework, concludes that "the fully fledged structural conception of function is rather rare in high-school students," and draws possible implications for

[10]Rosamund Sutherland, of the Institute of Education, University of London, is at present carrying out an investigation aimed at eliciting the models used by pupils who solve "algebraic verbal problems" using a spreadsheet.
[11]The model proposed in Harper (1981), and its developments, will be analysed separately, in a later section.

the teaching of mathematics. Sfard's model will be more closely examined a few paragraphs ahead.

Arzarello (1991), prefers the distinction between *procedural* and *relational*, but uses it as a "double-polarity" which "lives in every solution of an algebraic problem." Arzarello briefly points out to the use of "[the] subject's actions, the very process of their constructions and generations, every other extramathematical information about them," to "express the meaning of mathematical objects." Arzarello says:

> "a. The discovery-construction of an algebraic rule is not a trivial process of generalization from particular to general, but it is stirred by the strained connections between the two polarities. Typically, the dialectic between the two polarities marks the birth of algebraic work."

indicating that his model intends to characterise a mode of thinking *first*, and then examine the nature of the objects generated, from the point of view of the requirements of this mode of thinking.

Sfard's model is strictly within the structuralist tradition, and inherits its difficulties; for example, it fails to provide a *reason* for the passage from procedure to structure—even in the case of functions, which she examines in some detail. Arzarello's model, on the other hand, correctly points out to the fact that the objects of algebra are generated in the process of dealing with situations or problems with different *intentions*, ie, aiming at different aspects of the model.

None of the two models, however, provide any indication of which is the *intention* that drives the production of an algebraic knowledge or of an algebraic mode of thinking.

The characterisation of *algebraic thinking* that is the object of this dissertation, was first presented—in provisional form—in Lins (1990), a PME paper which belong to the small group of papers we have just examined.

LEARNING AND THE HISTORICAL DEVELOPMENT OF ALGEBRA

In this section we will examine three approaches to this question, all of which have in common the fact that they accept, as a principle or as a hypothesis to be investigated, the notion that the learning of algebra by individuals, closely recapitulates the historical development of the subject; it is usual to refer to this hypothesis by saying that "ontogenesis—the development of the individual—parallels philogenesis—the development within the history of the human race." Garcia and Piaget prefer "psychogenesis" to "ontogenesis," and this choice, far from casual, indicates an emphasis on the "psychological," in the "internal," side of the individual, which is in agreement with Piaget's understanding of knowledge, while with "ontogenesis" the many possibilities of the "being" remain open. Similarly, "philogenesis" may be replaced by "sociogenesis," but it also implies a sort of judgement of the crucial aspect in "philo." We will adopt "ontogenesis" and "sociogenesis," in agreement with our position, made clear in the previous chapter, that the social factor is a necessary and determining feature of the human endeavour.

Eon Harper and three uses of letters in algebra

The essence of Harper's approach to this question is the following:

> "It is generally accepted by historians of mathematics that algebra has passed through three important stages: rethorical, syncopated, and symbolic." (Harper, 1987, p77)

and from that point of departure, ie, the classification of the uses of letters in algebra in those three categories—which we will subsequently examine—to analyse the responses of children of various ages to test-items especially devised.

We will first examine the historical aspect[12].

The three stages to which Harper refers, were in fact proposed by Nesselmann, in his *Die Algebra der Griechen*, published in 1842. Heath (1964, p49) points out that Nesselmann speaks of the three stages "In order to show in what place, **in respect of systems of algebraic notation**, Diophantus stands..." (our emphasis)

The three stages are thus characterised:

> "(1) The first stage Nesselmann represents by the name *Rethorical Algebra* or 'reckoning by complete words." The characteristic of this stage is the absolute want of all symbols, the whole of the calculation being carried on by means of complete words, and forming in fact continuous prose...(2) The second stage Nesselmann proposes to call the *Syncopated Algebra*. This stage is essentially *rethorical*, and therein like the first in its treatment of the questions; but we now find for often-recurring operations and quantities,

[12] We prefer to do it here rather than to refer the reader to the chapter on the historical development of algebra, both because there are specific issues which would be lost in a more general discussion, and because this discussion is not of special interest in the context of our historical investigation.

certain abbreviational symbols...(3) To the third stage Nesselmann gives the name *Symbolic Algebra*, which uses a complete system of notation by signs having no visible connexion with the words or things which they represent..."

Most of the agreement to which Harper refers, stops here. Heath, while using Nesselmann's classification, gives his own interpretation, saying that Vieta belongs to the third stage, while Klein (1968, p146) informs us that "according to Nesselmann **even Vieta belongs to the stage of syncopated algebra**," and points out that Rodet, in 1881, "opposed this tripartite division with the thesis that only two types of algebra should be recognized, namely 'l'algèbre des abbréviations et des données numériques' and 'l'algèbre symbolique.'" M. Kline (1990) remarks, almost casually, that, "Because he does use some symbolism, Diophantus' algebra has been called syncopated...", and this is the only mention to the three stages, and van der Waerden (1985) ignores altogether Nesselmann's classification. Moreover, Whiteside (1962, p197) says that,

> "The development of the concept of variable is very closely tied up with the notation used to express it...But the variable is something more than its mere symbolic denotation and Nesselmann's classification is perhaps a little too narrow and rigid, and certainly arbitrary."

Harper makes a claim which is historically inaccurate. He claims that,

> "The use of the letter as a representation of a 'given' quantity (Vieta called his letters 'species') **introduces** a new numerical concept into mathematics—the 'algebraic number concept' (Harper, 1979) or 'symbolic number concept' (Klein, 1968)" (our emphasis)[13]

It is true that Klein uses the term "symbolic number" to denote the conception that underlies Vieta's *species*, but he also says that,

> "The new [symbolic] 'number' concept...already controlled, although not explicitly, the algebraic expositions and investigations of Stifel, Cardano, Tartaglia..." (Klein, 1968, p178)

[13]There is in fact an improper use of the term "algebraic number," a notion which only appears when Legendre conjectures that π is not a root of a polynomial with rational coefficients, and a term very clearly understood in mathematics.

an aspect that we will examine in more detail on Chapter X. Vieta's *species* are a remarkably useful condensation of the "symbolic number," and not that which *introduces* it.

This distinction is important because it is precisely on the basis of its *lacking* that Harper uses Nesselmann's classification to analyse pupils' work, as he characterises the solutions according to how they are *presented*, rather than how they are *produced.*

One of the problems proposed by Harper, and in which responses he bases most of his argument, is the following:

> "If you are given the sum and the difference of any two numbers show that you can always find out what the numbers are. Make your answer as general as possible."

and in Harper (1981) we find what each of the three types of solution would be:

> "(i) Rethorical: The pupil typically writes down little except perhaps two numbers to represent a sum and a difference, and the 'solution': 'You add the sum and the difference together and divide by two. That gives you one number. Take the difference from the sum and divide by two and that gives you the other number.'
> (ii) Diophantine: The pupil chooses a particular sum and difference, writes down two equations containing two unknowns, and solves them. He (she) often suggests, verbally or in writing, that the same method can be used whatever the numbers chosen for the sum and the difference.
> (iii) Vietan: The pupil writes down two simultaneous equations involving two unknowns and a letter for each of the sum and the difference. These are solved to produce, for example: $x = \frac{a + b}{2}$, $y = \frac{a - b}{2}$

The data obtained indicates a clear swing from "Rethorical" to "Diophantine" and then to "Vietan" responses, from Year 1 (11y9m, average) to A-level (17y3m), which Harper (1981) sees as, "an age-related transition Rethorical → Diophantine → Vietan." He considers the possibility of an influence of teaching, but counters that possibility by arguing that,

> "(i) pupils in the school were not encouraged to provide rethorical type responses in any of their work

(ii) pupils were introduced to 'letters for unknowns' and were expected to use these in problem solving activities during Year 1 and onwards
(iii) pupils were using letters as 'givens' in the context of functions, and to make generalisations as early as Year 2
(iv) simultaneous equations were introduced in Year 2." (Harper, 1987)

We think that the reason why the students did not use the techniques they had been taught, may be related to the fact that the problem itself is, probably, unusual for them, as it is not asking them to solve a problem, but rather to show that it can always be done. The subtle, but crucial, difference, is similar to that which exists between the problems in Diophantus' *Arithmetica*, and in Euclid's *Data*, in which only the possibility of a construction is required to be shown[14].

As we said before, Harper's categorisation of the answers focus strongly on the way in which the solutions are *presented*, and does not examine in detail how they are *produced*. In relation to this, we think that a few observations are relevant.

First, if a mathematician gives the "rethorical" response in reply to the question, classifying it as "rethorical," could not imply a cognitive impossibility on the part of the solver. But if this is the case, it implies that categorising children's responses, and considering a possible correlation between the types of responses and levels of cognitive development, depends precisely on the special assumption that the choice of a specific approach means something different in children and in adults, and, as a consequence, history could not inform Harper's model, unless he is prepared to assume that Diophantus' was at a lower intellectual level—in a developmental sense—than Vieta.

Second, as we have pointed out, in Bombelli one finds a symbolical conception of number, but not the adoption of generic coefficients; as a consequence, *historically informed only*, there is no way to characterise the "Diophantine" solutions as indicating a lack of such symbolic understanding of number. What characterises the "symbolic number" of Klein, is not a notational form *per se*, but the way in which number is understood, as intending the "things" which are measured by it, or, instead, *symbolically*, as meaningful only in relation to the—algebraic—system in which it is defined, ie, in relation to the properties of the operations of that system.

The difficulties in Harper's model suggest two areas in which extreme care has to be taken, if we are to elicit the informative value—if there is any—of history to research into cognition in mathematics: (i) the problems used have to aim deeper in the

[14]We will examine this difference in Chapter 3.

students' knowledge than the presentation of the solutions; and, (ii) history cannot be arbitrarily dissected and reassembled into a lifeless, linear, progression from the particular to the general, from the simple to the complex, from the primitive to the sophisticated.

Anna Sfard and the process of *reification*

Anna Sfard proposes a model of concept formation in mathematics, a model which is based on the distinction between two ways in which mathematical objects can be perceived: as a process—the *operational* aspect—or as product—the *structural* aspect. She examines the concept of function from this point of view: *operationally*, functions are "certain computational procedures"; *structurally*, functions are "aggregates of ordered pairs." (Sfard, 1989)

Central in Sfard's model is the thesis that the *operational* aspect precedes the *structural* aspect; on one hand, she argues that the latter is much more "abstract" than the former, and that,

> "...in order to speak about mathematical objects one must to be able to focus on input-output relations ignoring the intervening transformation. Thus, to expect that the student would understand a structural definition without some previous experience with the underlying processes seems as unreasonable as hoping that he or she would comprehend the two-dimensional scheme of a cube without being acquainted with its "real-life" three-dimensional model. In the classroom, therefore, the operational approach should precede the structural." (Sfard, 1989)

while at the same time she says that,

> "...a close look at the history of such notions as number or function will show that they had been conceived operationally long before their structural definitions and representations were invented." (Sfard, 1991)

There is a difficulty with Sfard's model. One can reasonably say, that it is not a good idea to introduce the notion of functions as elements of an algebraic system—for example, the additive group of polynomials in a formal variable, and with coefficients in **Q**—before the learner has had plenty of opportunities to add polynomials, to deal with their additive inverses, and to examine the properties of those polynomials in relation to that operation. But it is a totally different matter to say that one has to "substitute a lot of values for x and calculate the result," in order to be able to understand the algebraic system described above. In the former, one has to see polynomials as *formal expressions*, in the latter as *formulæ*, and, in fact, given the similarity of the notation—a situation which has its advantages—the two notions are *conflicting*. The difficulty, then, consists in defining exactly what *operational* means, if it means "using to calculate," or if it means "doing calculations on," or something else.

Similarly, Sfard never defines "structural," let alone "structure," directly. As a consequence, *structural*, which is a word with a rich—to say the least—net of meanings around it, has to be re-understood on the basis of her use of it.

Sfard says that,

> "Of the two kinds of mathematical definitions, the structural descriptions seem to be more abstract. Indeed, in order to speak about mathematical *objects*, we must be able to deal with the *products* of some process without bothering about the processes themselves. In the case of functions and sets (in their modem sense) we are even compelled to ignore the very question of their constructivity. It seems, therefore, that the structural approach should be regarded as the more advanced stage of concept development." (Sfard, 1991, p10)

The word "structural" appears twice: in "structural description" and in "structural approach...to concept development."

In the former, we can take it as meaning, for example, "functions can be described in different ways, one of them is as a set of ordered pairs, which we will call structural." But *why* should we call that form of description "structural," instead of "static"? Does it *reveal* the *structure* of a function? Sfard also offers (1991) a *structural* definition of "circle": "The locus of all points equidistant from a given point," while an *operational* definition would be "[a curve obtained by] rotating a compass around a fixed point." But if I define "circle" as "$x^2+y^2=r^2$," without adding "the set of points such that..." or "plotting the set of points such that...", it seems that the distinction does not work.

In the latter of the two uses of "structural," the more likely meaning is that "concept development will be seen as the progressive unveiling of *the* structure of the concepts in question." From this point of view, history and learning should necessarily follow a similar path, precisely because in both cases human beings are unveiling the same structure, ie, along history Man *learns* this structure. But this can only be true *if the structure is a "property" of the concepts*, and moreover, if this structure is "deposited" somewhere. The second of those conditions we have addressed in Chapter

1[15]. The first condition is a key one in Sfard's model, as she postulates that without *operational* understanding—as indicated by a given *definition—structural* understanding is not *possible*. But this means only that *given that a structural understanding is a form of abstraction from an operational understanding*—a form of abstraction that Sfard calls *reification*, the transformation into "object"—*it is not possible to have structural understanding before operational understanding*. In other words, the vicious circle is forced by the attempt to prove the precedence of *operational* over *structural*, when *structural* is defined precisely as a transformation of *operational*.

If instead, we consider that there are plenty of situations from which to construct a notion of function that does not depend *at all* on the notion of "calculation"—for example, water from a tap filling a tank, pupils being paired in preparation to a game, the length of the shade of a stick vertically set during various hours of the day, *or even using throws of dice*—it becomes clear that the precedence of *operational* over *structural* cannot be established in general; a table is no less a way of "calculating" the value of a function for a given "input" than formulæ. Sfard herself accepts that "Geometric ideas...can probably be conceived structurally even before full awareness of the alternative procedural descriptions has been achieved." (Sfard, 1991, p10)

Sfard's approach to historical research is at least incomplete; saying that the "transition from computational operations to abstract objects is a long and inherently difficult process," (Sfard, 1991) does not help, unless this difficulty is justified. The **historical** example of the distinct speed of developments in algebra and in geometry seems to suggest that such explanation is still some way from being reached, and a number of historians do not hesitate in calling it "a paradox."

The question that has to be asked in relation to history, is about which were the conditions in which a given conception was "natural," and also which aspects of those conditions could make the development of another, given, conception—the modern one, *for example*—impossible. It is precisely from this point of view that history can inform education, by revealing ways in which mathematical knowledge is biased and "organic" within a culture. As we had pointed out in relation to Harper's attempt at linking history and learning, Sfard's model is based in a "progressivist" reading of history, which means that she looks at history as some sort of struggle to unearth true knowledge from the depths of...some sort of "structure" living in a Platonic world of ideas. Jacob Klein's (Klein, 1968) analysis of the conditions in which Vieta's symbolic invention was produced, clearly indicates that there is a strong shift in the *intention* that animated Diophantus' and Vieta's concept of number, and that the mathematics in the

[15]On the section "On the nature of mathematics."

former **cannot** be seen as a "primitive stage" of the mathematics in the latter. We are again led to stress that the "progressivist" reading of history is, in fact, a projection of the modern understanding, conceptualisation, into the historical texts, and its "result" is not an understanding of history, but the reconstruction of history according to a pre-fixed hierarchy of contents and concepts. In Chapter X we provide some of the elements necessary to redress the relations between history and learning.

The difficulties in Sfard's model are due to two factors.

First, it fails to appreciate that the obstacles identified in the transition from an *operational* to a *structural* conception, implicitly assume the previous existence of the former; as we saw in the case of function, it is possible—*precisely because we, educators, already **know** the "ordered pairs" definition*—to consider situations where the "ordered pairs" conception is achieved without going through the *operational* one as Sfard defined it, ie, it is possible to present the "much more abstract" form *directly*.

Second, it fails to consider that what we find in history are *mathematical conceptual systems* which belong "organically" the whole of each culture; as one changes, so does the other. To say that an "object" is more abstract than another one is, *a priori*, a statement that depends on a given formalisation; unless Sfard—or, for that matter, anyone—is able to prove that for a given mathematical concept, or "object" A, there can be no interpretation in which A does not depend on the "reification" of another "object" B, any attempt at postulating the precedence of B over A, purely on the basis of **one** possible interpretation, is bound to meet the vicious circle we have indicated to exist in Sfard's model.

It is a good point in Sfard's work, that she prefers *dualities* to *dichotomies*, but the route she actually takes in the three papers we have examined, leads in fact to *hierarchies*. It is very good that she says,

> "When analyzing the process of learning mathematics, one should be aware of the crucial role played by such epistemological issues as students' implicit beliefs about the nature of mathematics on the whole, and of mathematical entities in particular" (Sfard, 1989)

but similar observations apply to the researchers' beliefs about the nature of history and about the nature of learning.

Rolando Garcia and Jean Piaget

In a book published for the first time in 1982, *Psicogénesis e Historia de la Ciencia* (Garcia and Piaget, 1984)[16], Rolando Garcia, a physicist and epistemologist, and Jean Piaget, a psychologist and epistemologist, approach the question of which are the basic *mechanisms* involved in the production of knowledge in mathematics and in physics. They look into two directions, into history and into stages of cognitive development. They say about the objective of their investigation, that,

> "...it is not, in any way, to put into correspondence the succession in history with those revealed by the psychogenetic analysis, by highlighting contents. It is, on the contrary, an entirely different objective: to show that the mechanism of transition between historical periods are analogous to the mechanisms of transition between psychogenetic stages." (op. cit., p33)

They claim that two of those mechanisms can be identified both in history and in psychogenesis. The first is

> "...a general process that characterises any cognitive progress: every time there is a breakthrough, that which is surpassed is in some way integrated into that which surpasses it..." (ibid)

The "nature" of what is surpassed or surpasses is not clarified, and the word used in Spanish for "breakthrough," and "surpass," come from the same root, "rebasar," which means "to go beyond"; this means that, from the point of view of this mechanism, no hierarchies are established, but it is stated that the "initial configuration" plays a key role in the process of producing knowledge, and also that it is, in fact, an essential element in this process.

The second mechanism is described by them as the process which produces a succession of three stages: *intra-objectal*, the analysis of the objects, *inter-objectal*, the study of relations and transformations involving those objects, and *trans-objectal*, the construction of structures. According to Garcia and Piaget, reaching stage j is a *necessary* condition for reaching stage $j+1$, but, we must add, it is not a *sufficient* condition; we will return to those two points later.

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[16]As far as we could find, there is no English translation of the book, and we will quote our own translations of the original Spanish.

In relation to algebra, which study is on Chapter V of their book, Garcia and Piaget make a clear-cut choice: it is only with Vieta's symbolic invention that one can speak of the beginning of algebra. They claim to have found the historical support in Jacob Klein's *Greek Mathematical Thought and the Origins of Algebra* (Klein, 1968), a work to which we will many times return on Chapter X of this dissertation. The key notion that they borrow from Klein, is the distinction between the conceptualisations of number in Diophantus and in Vieta, that being a *symbolic number* in the latter.

The "object" that replaces the general place-holder in the three stages described two paragraphs above, is "operation"; so, in algebra, they study the succession from *intra-operational*, through *inter-operational*, to *trans-operational*. The text where those three stages are characterised, is quite obscure, so we present it in full:

> "The intra-operational stage is characterised by intra-operational relations that present themselves as detachable forms, without transformations from one to another which imply the existence of invariants, and without composition among them that conduce to the definition of structures...The inter-operational stage is characterised by correspondence and transformations between the detachable forms of the previous stage, with the invariants which such transformations require...The trans-operational stage is characterised by the construction of structures which internal relations correspond to the inter-operational transformations." (op. cit., p134)

Some of the examples they provide to characterise each of the stages are: (i) Cardano and the algebraists of the Renaissance are in the *intra-operational* stage, as they work with solutions for various and isolated problems; (ii) Lagrange is at the *inter-operational* stage, as he examines the nature of the methods employed successfully to solve cubic and quartic polynomial equations; and, (iii) Galois "opens" the *trans-operational* stage. Other examples are analysed, such as Gauss's work with quadractic forms.

On the side of psychogenesis, Garcia and Piaget briefly examine the development of the notion of conservation of equality in relation to the action of adding to both sides of the equality, and conclude that the mechanisms observed there are the same they explored in relation to history.

It is not our intention to go beyond this short account, which, nevertheless, provides elements for a reflection on their model, and the reader is referred to the book for a much fuller account of the authors' points of view.

It is clear that the model is strongly characterised by the *assumption* of the necessity of the succession *intra, inter, trans*; Garcia and Piaget attempt to solve the difficulty of accounting for the necessary order of succession by saying that,

> "We could also come to sustain that the [*intra, inter, trans*] successions plunge their roots in biology: they [the successions] are that which justify the dream of an integral constructionism, that will link, through all the necessary intermediate steps, the biological structures which are at the point of departure and the logico-mathematical creations which are in the point of arrival." (op. cit., p172)

The unavailability of such link with biology, which would establish the necessity of the succession, leaves open other possibilities to investigate. One of them is to consider that in history, for example in the 18th century, the notion of structure as we have now had not been established, and that it may be possible to construct new mathematical objects from the initial construction of a general structure within which those new objects can be given meaning[17].

A difficulty in examining those successions in history, is that one has a double possibility: (i) to examine history "searching" for such successions, ie, choosing an initial object and attempting to trace the corresponding succession; or, (ii) to examine each mathematical culture in order to understand the developments within that culture in terms of its own possibilities, ie, from the point of view of its own conception. If approach (ii) is adopted, as it is by Garcia and Piaget, than one is left with the task of explaining why the succession did not take less time to be completed, and also why it happens for some initial objects but not for others; but this can only be understood by using approach (i). As we saw with Sfard and with Harper, the "progressivist" reading of history presents other difficulties.

To give more flexibility to the model, Garcia and Piaget propose that within each stage, there are sub-stages, which follow the same sequence: *intra, inter, trans*. From this perspective, they identify in the development of the Theory of Categories, three sub-stages, *trans-intra*, *trans-inter*, and *trans-trans*. Because the *trans* stage is "stronger" than the other two, there seems to be no difficulty here, but can we think of *intra-intra*, *intra-inter*, and *intra-trans* sub-stages? Would it not be true that in this case the characterisation of the stages cannot be directly applied, or we would meet a contradiction, namely that we reach the last stage in the course of completing the first?

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[17]For example, to define negative numbers directly as additive algebraic inverses of positive numbers, and not as "debts," or as directed numbers in the sense of using the number line to define them. On the conclusions to Chapter 3 we examine this possibility in some more detail.

The authors emphasise the "dialectic" character of their model, but we think the inflexibility of the model creates, at this particular point, for example, an unnecessary conflict.

Another difficulty is this. Although Garcia and Piaget aim at a general succession, one that is not content dependent[18], one would have to explain why the constitution of the notion of "structure" in one branch of mathematics does not immediately sets the paradigm which is followed by other branches; it is true that one hundred years after Galois, the notion of structure was firmly in place within mathematics, but mathematics itself was not reduced to the study of abstract structures, although it may be seen as the abstract study of structures; the subtle distinction indicates that the tension between the "initial objects" and the "final structure" has not been resolved, and we think that, in fact, it cannot be totally resolved if mathematics is to remain meaningful within a culture[19]. In relation to psychogenesis, the phenomenon is more complex to study, and Piaget had to take refuge in the notion of *décalage*, in order to explain the failure of the model to account for differences in cognitive developments where they should not exist according to it (see, in this chapter, the sub-section on the SOLO Taxonomy).

Underlying Garcia and Piaget's model, we have the notions of *assimilation* and of *accommodation* (op. cit., p246ff), which give to the model its constructivist character, and leave open the possibility of explaining the interaction between the individual and the social context. They also say that,

> "...we must differentiate, on the one hand, the mechanisms of acquisition of knowledge that an individual has at his disposition, and on the other, the form under which it is presented the object which will be assimilated by that individual. Society modifies the latter, but not the former." (op. cit., p245)

Garcia and Piaget's position amounts to say that the internal character of the cognitive apparatus of individuals is that which keeps knowledge on the tracks, so to speak, of the successions; another possibility to consider, would be that culture is precisely that which focus the enormous power of our cognitive apparatus in one direction or the other, but they reject this possibility:

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[18] Cf Garcia and Piaget (op. cit., p33), quoted at the beginning of this subsection.

[19] Not only because it is through this tension that mathematical modelling becomes possible, but also because it allows mathematics to retain an unified character that does not depend on strong reductions such as a set-theoretical foundational program.

"That the attention of the subject be directed to certain objects (or situations) and not to others; that the objects be situated in certain contexts and not in others; that the actions on the objects be directed in a certain way and not in others: all this is strongly influenced by the social and cultural environment (or by that which we call the social paradigm). But all those conditions do not modify *the mechanisms* that such a particular biological species—human beings—needs to *acquire* a knowledge of those objects, in *given* contexts, with all the particular significations, socially determined, that have been assigned to them." (op. cit., p245)

As pointed out by Collis, it remains to be proved that those "ultimate" mechanisms can be directly examined, a possibility on which the correctness of Garcia and Piaget's model depends. It is important to emphasise that, as we saw in the first paragraphs of this this sub-section, the succession which they present is introduced as the result of a process which is never discussed directly: we know about it only through its result, the succession.

SOVIET RESEARCH ON THE TEACHING OF ALGEBRA

If not for anything else, Soviet research in the field can be immediately distinguished from its "Western" counterpart by its explicit interest in the teaching of algebra at the lower grades of elementary school. There is at once a conflict between such approach and the canons of Piagetian and other stage-theories of intellectual development, in particular in relation to the belief that "algebra" requires "formal operational thinking," and, thus, it cannot—or it should not—be taught to children younger than 13 or 14 years-old. It is very likely, that Soviet research could proceed with its investigations precisely for its isolation from Western research, although it is true that Professor Davydov himself faced opposition, from teachers, to the implementation of his teaching programme[20]. There are in general very few sources available on Soviet research in education, and in particular on the teaching of algebra[21]. We will rely on a paper by Freudenthal, and on an English translation of a paper by Davydov; that the paper by Freudenthal was published in 1974, but almost no reference to Soviet research is made by Western researchers on the subject, is at the same time sad and remarkable, and it is a strong indication of how difficult it can be to absorb that which contradicts our deep beliefs, even if scientifically supported.

[20] Personal communication from Dr M. Wolters, from the Dept. for Developmental Psychology, University of Utrecht, The Netherlands.
[21] A number of papers have been translated into Dutch and German, but very few into other languages.

A paper by V.V. Davydov

We will examine now, the paper *An experiment in introducing elements of algebra in elementary school*, by V.V. Davydov. It was first published in the Sovetskaia pedagogika, in 1962, and later translated into English (Davydov, 1962).

The paper is divided in two parts. In the first, Davydov presents the rationale for the pedagogical approach adopted, and in the second he describes briefly the resulting teaching programme.

As Davydov sees it, the most important reason for introducing elements of algebra in the first grades of elementary school, is the need to provide a *scientific*, as opposed to a *practical*, mathematical education. But this has to be understood correctly, as in fact he does not mean, by *scientific*, an education that is "theoretical" in the sense of its links with "reality" being severed. On the contrary, he believes that a teaching programme to achieve such *scientific* education, must meet three requirements:

> "1) To overcome the existing gap between the content of mathematics in elementary and secondary schools[22]; 2) to provide a system of knowledge of the chief laws of quantitative relationships in the objective world; **the properties of numbers as a special form of expressing quantity must become a special but not the main section of the program**; 3) to cultivate in the pupils mathematical thinking methods, and not calculating habits; this involves building a system of problems which is based on a deeper study of the sphere of dependencies of real magnitudes (the connections of mathematics with physics, chemistry, biology, and other sciences dealing with specific magnitudes)..." (op. cit., p30)[23] (our emphasis)

The *scientific* education proposed by Davydov, is one in which the systematic examination of the mathematical material support the development of the mathematical technique and its applications. In relation to algebra, the basis of this *scientific* mathematical education is to be found in quantitative relationships[24], which, Davydov says, "[as] numerous observations made by psychologists and educators...[indicate,]

[22]Such gap exists in the Soviet school system and it certainly still exists in most Western school systems.
[23]There is a fourth point, related to the simplification of calculation, but in view of the availability of electronic calculators and computers, it tends to become completely irrelevant.
[24]Quantitative relationships, as used by Davydov, are those implied in a whole-part model.

arise in children long before they acquire a knowledge of numbers and methods of operating with them." (ibid.). Here lies the strength of Davydov's approach: on the one hand, the introduction of algebra is not seen as a "generalisation" of the arithmetico-numerical knowledge, and, thus, it does not face the problems identified by so many researchers in the transition between arithmetic and algebra; on the other hand, on the basis of those first algebraic elements, the construction of a number system is much more solid, *as it is not done on the basis of a collection of procedures and ad hoc justifications, but on the basis of a mode of thinking*. Moreover, Davydov observes that the tendency to call those quantitative conceptions "pre-mathematical," is derived from an undue—according to Davydov—association between "an object's quantitative characterization with a number":

> "And it sometimes happens that the depth of these allegedly 'pre-mathematical formations' is more important for the development of the child's own mathematical thinking than knowledge of the fine points of calculating techniques and the ability to find purely numerical dependencies." (ibid)

We will now present a summary of Davydov's programme for the first half of the first year of elementary school; in Soviet Union, at that time, pupils entered elementary school at the age of seven.

<u>Theme I. Comparison of magnitudes</u>:

1. Operations involving practical equaling-out and matching of things by length, volume, weight, composition, etc;
   - a) selecting the 'same article' (a sample is given) according to a given parameter from the set;
   - b) making the 'same article' (a sample is given) according to a given parameter.

...

2. Comparing things according to given parameters and recording the result of comparison in letter symbols:
   - a) actual comparison of things...
   - b) recording the results
     - ° first only by the symbols >, =, <, without designating the things
     - ° then, recording the things compared by symbols and drawings

° finally, by symbols and letters: $A=B$, $A>B$, $A<B$, solving problems of the type 'compare those things by...and write down the result as a formula.'

c) deriving by a formula the relationships of equality and inequality: 'If $A=B$ then $B=A$; if $A>B$ the $B<A$," etc..

<u>Theme II. Disturbance of Equality and its preservation. Introducing addition and subtraction</u>

1. Disturbance of equality if one of its elements increases or diminishes.

$A=B \rightarrow A + e > B$ [etc.]

2. Preserving of equality by a corresponding 'balancing-out.'

$A=B$ ($\rightarrow A+e>B$) $\rightarrow A+e=B+e$ [etc.]

3. Solving problems in which these relationships appear

<u>Theme III. Reduction to Equality</u>

1. $A<B \rightarrow A+e=B$ [or] $A=B-e$ ...('e' is equal to the difference between A and B) [etc.]

2. $A+e=B \rightarrow A<B$ [etc.]

3. Solving relevant problems [two baskets of apples, with A and B apples, and A is more than B, etc.]

<u>Theme IV. Dependencies Between Elements of Structural Equality</u>

$A+e=B$	$A-e=B$
$A<B$ (by e)	$A>B$ (by e)
$A=B-e$	$A=B+e$
$e=B-A$	$e=A-B$

On the paper a more detailed description of the teaching process is provided.

On the second half of the first year, numbers are introduced as measure—"the relationship of the magnitude under examination to that accepted as the unit of measure"—and the arithmetical operations treated on the basis of the preceding development:

<u>Theme VI. Addiction and subtraction of numbers (by reducing inequality to equality) introducing the 'x'</u>

$3<7$

$3(+x)=7$ [sic]

$x=(7-3)$ [etc.]

Multiplication and division are also understood in relation to the "formulas."

The paper's content does not allow us to have any detailed insight into the exact results of the experiment, but, overall, the indication is that they were positive[25]. A few comments, however, are possible.

First, there is the distinctive intention of founding the learning of arithmetic on a more general framework, in particular the characterisation of pairs of inverse operations in relation to the equality relationship, which is mathematically sound, as the "undo" character is more closely related to the idea of inverse elements, and not to inverse operations. Second, by presenting the notation *before* the formal introduction of numbers, the problem of "if it is *any* number, why not choose one and use it?", but also, and of immense significance, the idea of "different uses of letters" simply does not arise: there are, instead, different uses of that algebraic knowledge, an idea which is in agreement with Bell's conception of a curriculum for algebra (Bell, 1988). Third, the concept of equality is presented from the beginning as a symmetric relationship, and as an *object*, with its properties highlighted.

It is clear that much refinement of the approach is possible, and the task has been taken on by a group of Soviet educationalists, to which we will refer in the next paragraphs, and also by Dutch educationalists, who developed a programme for the first two grades of elementary school based on the results of Soviet research, but have also extended those results considerably (see, for example, Wolters, 1983 and 1991)

Freudenthal on Soviet research on the teaching of algebra

Freudenthal (1974) published a paper centrally concerned with reporting and analysing the contents of three chapters of a book edited by Davydov, which was then—and still is today, as far as we know—only available in Russian (Davydov, 1969).

The paper concentrates on chapters IV, V, and VI, respectively, *Psychological foundations of solving problems with literal data*, by G.G. Mikulina, *Developing general solving methods*, by G.I. Minskaja, and *Developing a general method of solving problems with young children*, by F.G. Bodanskij.

We will concentrate in collecting Freudenthal's comments, rather than the actual content of the chapters, which are conveniently summarised in the paper, where we also find diagrams produced by pupils and extracts of transcriptions from actual lessons.

[25]A test was applied, and the results are presented. Through the test, however, we can only assess the direct retention of formulæ manipulation rules, but not the overall impact in the children's thought.

The first task the paper undertakes, is to understand the principles on which the traditional teaching of mathematics in the elementary grades—numbers and arithmetic first—is based, and what kind of support is offered to the alternative proposal.

In the Soviet Union, the traditional teaching of mathematics is justified by the existence of four "levels of abstraction": in arithmetic, the first is the level of whole-numbers (7 to 10 years-old), the second is the level of fractions—or *quantity relations*—(11 to 12 years-old). In those two levels the numbers are "empirical." The third level is that of "arbitrary non-empirical numbers, indicated by letters," (13 years-old), and the fourth level is that of "ratios and equations, the laws of numerical relations." (op. cit., p392ff). Expressing a very strong judgement, which is in agreement with the results of the research carried out by Davydov and others following his ideas, Freudenthal says that,

> "I think that this order of succession is based upon tradition rather than upon independent research; just as elsewhere theories are more often created in order to justify old habits than to create new ones." (op. cit., p393, footnote 3)

Davydov's approach has already been characterised a few paragraphs above.

According to the tradition in Soviet schools, where teaching the solution of word problems takes a good part of the programme, the introduction of elements of algebra has to be analysed from that perspective. Freudenthal comments on the traditional use of "arithmetical methods," and concludes that,

> "The fallacy of traditional didactics is the diversity of methods according to the—direct or indirect—wording of the problem. There should be a unique method, which, however, cannot be realized unless letters are used to indicate unknown magnitudes. But even this is not enough; the technique of solving equations can be better acquired within the explicit context of literal calculus." (op. cit., p395)

The central notion that is to be used in the new programme, is that of *whole and parts*, which can be perceived—although not directly mentioned—in Davydov's paper.

To those general considerations, there follows a summary of the teaching activity using the notion of whole and part, diagrams of various kinds, and literal notation.

The real merit of this approach emerges in full when problems are solved using the knowledge about quantitative relations gathered in the first parts of the teaching, and the examples provided on pages 399-400. We present an extract of the teaching activity, involving, as far as one can gather from the paper, children 8 to 9 years-old.

> "An example from the 37th lesson.
>
> The text was: 'One day a boy read a pages of a book, the next day k and both days together c.' It was noted down in three formulas ($c=a+k$, $k=c-a$, $a=c-k$). The teacher asked the class to substitute numbers for a and c.
>
> Gena F.: a is equal to 5, and c is equal to 2.
>
> Misa Z.: Wrong, c cannot be 2. This is very small.
>
> Teacher: Why not?
>
> Ljuda B.: It was 5 pages the first day, and c is the whole. The whole cannot be smaller than a part, thus c cannot be 2, for example, it can be 10 or 8.
>
> Teacher: Well let us write c is equal to 8. We still have the magnitude k left. I propose to write k=4. Or is there another proposal?
>
> Andrej K.: It is equal to 3.
>
> Teacher: Who proposes another number?
>
> Sasa Z.: k is equal to 8.
>
> Teacher: Still another proposal?
>
> Misa P.: We cannot think up the magnitude k. It is precisely fixed. This number must be computed, but not thought up. k equals 3.
>
> Teacher: According to which formula must we compute k ?
>
> Anderj S.: k equals c minus a.
>
> Andrej M.: k equals 8 minus 5, that is, 3." (op. cit., p400)

The considerations of Mikulina, the author of the chapter from which this passage was extracted, concludes that it is perfectly possible to teach young children to deal with literal representation of whole-part relations even before they learn numbers, and that this knowledge can be purposefully used in the solution of literal problems. Moreover, and crucially important, we think, the use of such approach avoids the distinction between "direct and indirect problems," as both types are treated in the same way; more than an unity in relation to solving problems, it is the unity of a mathematical model that is being developed, and this unity may well serve as paradigm for examining other problems.

The treatment of more advanced topics, in grades 2nd to 4th, is described in the following section. The conclusions of the author of the chapter, Minskaja, point out that the continuation of the teaching approach in those grades proved possible; she also highlights the fact that

> "Compared with traditional views, the algebrisation of initial mathematics is closely connected to a qualitatively different interpretation of generalisation and abstraction." (op. cit., p406)

After examining the solution of problems with equations, in the four initial grades, Freudenthal comes to his final conclusions. First, he indicates his disagreement with using the approach only in relation to a small range of types of problems, suggesting that the approach could be used in the context of more meaningful problems, but then he says,

> "I started my appreciation with pointed criticism in order to finish with well deserved praise of what is valuable. In vivid contrast with the stress on subject matter and the complete disregard for all details of teaching method and style which prevails in Western literature, one is struck by the manifestation of scrupulous care for details and the clear image of the didactic process...
> What is more important is what I called in the introduction a sound pedagogical idea behind the experiments. I mean the idea that abstraction and generality are—in many cases—not reached by abstracting and generalising from a large number of concrete and special cases. They are rather reached by one—paradigmatic—example, or if this is not available—as in algebra—by a straightforward abstract and general approach. Algebra as it is traditionally taught, by making algebraic ideas and laws plausible through ridiculous examples, is a fake, which does not serve any reasonable aim. The experiments convincingly show that algebra can be taught more adequately, and at an even earlier age than it is now." (op. cit., p412)

2.3 CONCLUSIONS TO THE CHAPTER

Although not covering in detail all the research into the learning of algebra, this survey clearly shows that no generally accepted characterisation of *algebraic thinking* is available.

Most researchers approach the learning of algebra as the process of abstracting and generalising from the arithmetic knowledge learned at the initial series of elementary school; the Soviet research provides the only exception to this approach that we could find.

Underlying this *evolutionist* approach, there are three main beliefs. First, that thinking algebraically is *doing* or *using* algebra, usually including the notion of "calculating with letters." Second, that algebra is, in some sense which is not always made very clear, a *generalisation of arithmetic*; this position has been criticised, but it still is quite common. Third, that there exist age-related levels of intellectual development, and that *algebraic thinking* can only be achieved by people at the level of formal operations; difficulties with stage theories have been pointed out, particularly the lack of stability, within the stages, of the answers given by a same person.

Soviet research has challenged all three beliefs, and as far as we can know, successfully; the key notion of their approach, is that achieving generality and abstraction can be done directly, rather than through processes of generalisation and abstraction from, respectively, particular cases and "concrete" cases. Its theoretical foundations indicate that it might be the case that building "structures" by first dealing with the "elements," presents an obstacle that is not totally inherent to "structure," but to this specific process for constituting them.

The contrast between the SOLO Taxonomy and the Stage Theories, highlights the difference that there is between categorising responses and categorising individuals' thinking as a whole; although the latter is an obvious aim of epistemology, and a most valued would-be tool for educators, it is not clear at all that it is possible to achieve it.

The approach of categorising responses, however, is not enough to reveal how that knowledge is situated within the learner's mathematical *ethos*, and for this reason the technique seems to be better used in the context of broader examination of students' mathematical performance.

In most of the approaches we have examined, "learning algebra" is strongly—if not totally—identified with "learning the contents and techniques of algebra." What remains hidden in such approaches, is the fact that the content of algebra can be produced, in many cases, by *non-algebraic* means, as for example, using areas to prove that $(a+b)^2=a^2+2ab+b^2$. In fact, the use of non-numerical models to teach the contents and techniques of algebra, for example scale-balances and areas, is seen as a correct way of smoothing the transition from "arithmetic" to its generalised counterpart, "algebra." The Soviet teaching approach for elementary school does use whole-part models to generate the relationships which are to be later manipulated in "literal" form, but this "handicap" is to some extent compensated by the firm commitment to progressing from there to a clearly algebraic approach, as it is seen, for example, in the treatment of inverse operations[26].

Very few researchers actually examined the implications of using geometric and other analogies in doing algebra, Lesley Booth being a remarkable exception; it seems, indeed, that this is an area, within the broader subject of learning and using algebra, that badly needs more investigation.

The distinction between *algebraic* and *non-algebraic thinking* in algebra has to be clearly understood, and the interplay between them examined. The primary aim of this dissertation, is to establish a characterisation for *algebraic thinking* that enables us to approach those questions on a sound basis; moreover, in the course of making clear the adequacy and usefulness of our characterisation of *algebraic thinking*, we examine some aspects of *non-algebraic thinking* in algebra.

From the analysis of the research previously carried out, four points emerged, in relation to which our research exercises as much care as possible:

(i) to avoid focusing the analysis on the use of a given notational form, in particular the use of letters, *unless there is other evidence to support that its use or lack of use corresponds to, or tells us about, the underlying mode of thinking*;

(ii) to examine pupils' solutions always aiming at the underlying model that guided the solution process—be the solution correct or incorrect; the "outcome" is to be understood as the "visible" solution together with underlying model. Whenever it is possible, we will examine the possibilities and impossibilities of the model used by the students in relation to the problem proposed;

(iii) in the analysis of the history of algebra, to avoid a "progressivist" reading; each mathematical culture will be examined "internally," ie, in relation to its own conceptualisations, possibilities and impossibilities. Only from this perspective, the relation between different mathematical cultures is to be analysed: the assimilation, rejection or re-interpretation of "imported" knowledge into the conceptual framework of a given culture;

(iv) overall, to examine the relationship between *algebraic thinking*—as we define it—and the algebraic activity, in order to understand in which ways

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[26] It would be unwise to believe that there can be an approach which completely avoids the problem of generating the first relationships to be examined; it seems, though, that if the step towards examining those relationships algebraically is taken soon enough, subsequent difficulties are minimised. Also, there are clear advantages in not associating numbers, as measures, to the parts and wholes, because we can than focus on a general reasoning procedure which is not dependent of or based on calculating particularities.

the former may drive the latter, but also the ways in which the latter highlights the former.

Chapter 3
Historical Study

3.1 General Introduction to the Historical Research

THE NEED FOR AND THE ADEQUACY OF THE HISTORICAL RESEARCH

Our discussion of mathematics as part of a culture, and our understanding of learning as a culturally bound process, naturally lead to the need of investigating the learning of algebra from that perspective. It is not reasonable, however, to expect that by directly questioning our students on what they think about numbers, algebra, solving problems, or mathematics in general, we can get consistent, precise information, exactly because such "metamathematical" considerations are usually not a part of their lives; in many cases those questions are simply considered absurd by them. If we ask the mathematician, we will, of course, get answers that reflect a modern conceptualisation of mathematics; the discussion itself may serve to raise a number of interesting points about this conceptualisation, but usually it sheds little light on other forms of conceptualising mathematics.

The study of the historical development of algebra, on the other hand, is the perfect source for such an inquiry. Our informants are mature thinkers, well used to thinking about their activity, and, more often than not, they do not represent only themselves, but a trend, as people who achieved some degree of public recognition. By studying their mathematical production—which many times include "nonmathematical," ie, non-technical, considerations—we can learn about the *intention* of their work, about the ways in which mathematical objects and concepts are treated, and we can determine, at least in most cases, around which of those the algebraic activity is organised. The study of history, then, can provide us with *clusters* of mathematical concepts and objects and of conceptualisations of the mathematical activity, and those *clusters*, in turn, provide patterns against which students' mathematical activity can be examined.

What this historical inquiry *cannot* provide, however, is a way of arranging the different aspects and modes of the algebraic activity in a "linear progression" which could be used to justify, in some sense, the adequacy of this or that order of presentation of the content in a programme for teaching algebra[1], and if such "linear progression" is seen in history by some authors, it is precisely because they are not following history, but their own conceptual frameworks. Our investigation of the historical development of algebra will establish the truth of this claim.

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[1]To exemplify it briefly: the concept of number in Babylonia is much richer than its counterpart in Classic Greece; the same Vieta that introduces literal notation for the coefficients of an equation rejects negative numbers; and in the 17th century Pascal and Barrow—in his time considered a mathematician second only to Newton—objected algebra because it lacked justification (Cf. M. Kline, 1990, p279).

In the context of our research, there are three objectives to be achieved with a study of the historical development of algebra.

First, we want to determine to what extent it is possible to identify, in the mathematical cultures examined, a knowledge that can be said to correspond to what we call today "algebra." The central criteria used to identify "algebraic knowledge," will be that of a piece of knowledge that *explicitly* deals with manipulating relationships involving number-expressions[2] and arithmetical operations. It is in this tradition that algebra develops historically, and until quite recently in history it was in fact the *only* tradition in algebra. That in many cases numbers are *explicitly* associated with geometric magnitudes, does not affect our criteria, but if it is the case that a mathematical object which is clearly recognisable as *number* is dissociated from another mathematical object, we will not recognise knowledge related to the latter as having to do with algebra. In the case of Greek mathematics, this aspect will be examined in some detail, and our distinction shown to be adequate.

Second, this algebraic knowledge that we identify, has to be understood in the context of the cultures where it was created, ie, we must determine which is the *meaning* of that knowledge *within those cultures*. This is not only a requirement for the correct understanding of the knowledge achieved in a historical perspective—as we will show in the following paragraphs—but it is essential if we want to get, from history, insights into the process of learning algebra and developing an algebraic mode of thinking, by individuals. A piece of algebraic knowledge has to be characterised in relation to: (i) the possibilities of the mathematical culture where it is produced, ie, the ways in which the mathematical activity, mathematical concepts, and mathematical objects are conceived; and (ii) the *intention* of the knowledge produced, ie, the scope and character of that knowledge *as perceived within the mathematical culture which produces it*.

Third, it is precisely from that perspective that the methodology employed to study the historical development of algebra can be seen as paradigmatic for the study of the development of an algebraic mode of thinking by individuals, as long as this development is understood—as we do—as the insertion into an aspect of a mathematical culture, and the mastery of its technical means. It is important, then, that along our historical study, the reader's mind is focused on the relation between the way in which mathematical objects are conceived, the ways in which mathematical methods—in particular algebraic methods—*intend* their objects in different

[2]We could take, for example, Martin Ohm's definition of "expression": "...an arbitrary numerical symbol or as an arbitrary symbol which has the nature of a numerical symbol." (see, Novy, 1973, p86)

mathematical cultures, and the limits to the production of algebraic knowledge intrinsically expressed in those conditions.

In our historical investigation, we will be concerned with the broader cultural context to which each of the mathematical cultures we will examine belongbut only to some extent, as we explained on Chapter 1. We will be concerned with vertical developments—ie, along time, within a same culture—but only to the extent to which such development can elucidate changes in conceptualisation, and not in relation to technical developments *per se*. The question of "sources," for example whether al-Khwarizmi's *Algebra* is or not a compilation of Babylonian and Hindu mathematics, is not a central concern, unless it can help us to understand the conceptual framework of a period, or to highlight the fact that a given mathematical culture deliberately disregarded technical achievements it could have borrowed from another culture.

In their technical aspect, the mathematical results of none of the cultures examined will be described in detail, apart from the few cases where we judged them to be worth as illustrations of the points we wanted to make, or to make possible the comparison with other results.

RATIONALE FOR THE METHODOLOGY USED

Broadly speaking, the historians' approach to the history of mathematics can be divided into two groups.

The first group, to which Bourbaki and van der Waerden, belong, see the history of mathematics as the history of the production of mathematical results. Matzloff (1988, p5) points out that one of the central characteristics of this approach is that "there is only one universal science, teleologically structured from its origins according to categories of thought comparable to those of present day science." (our translation)

To a second group, to which we can associate the names of Rashed, Martzloff, Unguru, and Jacob Klein, the history of mathematics has to be studied as a history of mathematical cultures. Klein will adopt the view that it is necessary to understand the philosophical context underlying a culture, if we are to understand the mathematics it produces, and obtains very elegant and deep results with this approach; Rashed will prefer what he calls an "epistemological closure," ie, to examine the development of algebra "internally,"

> "Par 'clôture épistemologique', je voudrais dire simplement qu'à partir d'un certain seuil, à partir d'un certain stade de développement de la science, un théorème de l'algèbre est

> produit, et seulement produit, par une séries d'autres théorèmes qui existaient auparavant; il n'y a pas des raisons extérieures." (Rashed, 1984, p67)

Such "epistemological closure" has the merit of forcing us to look much more closely to the presentation of the mathematical treatises, and produces important insights, but it also points out to the possibility of examining the mathematical production of an individual—a student solving problems, or a carpenter using mathematics in his job—and to investigate not only the technical content of the mathematics being used, but also the way in which this knowledge is organised and treated.

The differences between the two approaches—the result-wise and the culture-wise investigations—have some far reaching consequences.

More frequently than not, isolating the technical result from the cultural context produces strong distortions of the historical reading. Martzloff (1988, p57), for example, argues that the technique of "translating" ancient terms by means of modern terminology involves that assumption that, as one obtains the same results, ancient and modern procedures are but superficially different, two forms of expression of a same "deep reality." On the other hand, he says, there is a great risk involved, as modern concepts are more general, and one can easily attribute to the ancient terms more than they actually meant or intended. He also quotes, on a footnote, Marrou, who says that "Sous prétexte d'atteindre à la réalité profonde, on substitue en toute ingénuité au réel authentique un jeu d'abstractions réifiées..." We will show, in the course of our investigation of the historical development of algebra, that Diophantus' algebra does not admit the substitution of letters as generic coefficients for the specific coefficients he uses, and that negative numbers in the Chinese *fang cheng* cannot be understood as a general mathematical object transferable to other methods within Chinese mathematics.

A result-wise reading of history is also bound to produce the impression that mathematics proceeds linearly, from counting stones to the sophisticate theories of the 20th century, which is, of course, untrue. Novy (1973, p1) reminds us that

> "The nature of mathematics, more than of any other discipline, tempts one to interpret the history of mathematics only as a sequence of logically linked discoveries which culminates in the present state of science..."

but such an approach does not tell us anything about the factors that precluded, in a given mathematical culture, the development of "stronger" results or methods—as, for example, in the case of Diophantus not dealing directly with "generic" coefficients—nor it tells us of why an axiomatic treatment of algebra was not developed in Chinese mathematics. Summing up beautifully, Rashed (1984, p259) says,

> "Comment, en effet, déterminer les véritables changements de style qui purent survenir alors, et localiser avec rigueur leurs manifestations, si Bachet et Fermat succèdent tout simplement à Euclide et Diophante? Comment, dans ces conditions, se garder d'un jugement qui n'exprime le plus souvent que l'incapacité de discerner les différences?"

In relation to the overall objective of our research, we are exactly interested in learning about the different ways in which algebraic knowledge can be conceived and produced, interested in understanding what precludes or bolsters the development of an algebraic mode of thinking, and the only useful reading of the history of algebra is one that explores how those aspects are manifest in different mathematical cultures.

THE RELEVANCE OF THE HISTORICAL RESEARCH IN THE OVERALL RESEARCH

The findings of our historical research will help us to establish at once the cultural character of the development of an algebraic knowledge and of the development of an algebraic mode of thinking. In different mathematical cultures, we will find a variety of approaches to *number*, providing a number of insights into how individuals' conceptions of *number* may affect their understanding of algebra; we will also find different ways of characterising and organising the mathematical activity, and, again, a number of insights important to mathematical education are produced.

As we have said before, it must be clearly understood that in no instance it is our objective to produce any sort of "hierarchy" of levels of development of *algebraic thinking*, as it is exactly our thesis that *algebraic thinking* must be understood as an *intention*, and the development of an algebraic knowledge seen both as a result of employing *algebraic thinking* and as the development of tools that give greater power and reach to *algebraic thinking*. As we learn from history, *algebraic thinking drives the development of algebra*, but not exclusively, *although it is only the realisation that extrasystemic interpretations have no relevance to the algebraic activity that makes possible the establishment of algebra as a theoretical discipline*, with the subsequent changes in the character of the algebraic activity.

The historical development of algebra shows that the algebraic activity involves a tension between the inner structure of the elements in an algebraic system—for example, what complex numbers or negative numbers "are," or the fact that permutations do not "look like" numbers—and thinking algebraically. We think that there is an extremely important insight for the teaching of algebra, here, namely, that the teaching of algebra has to address this tension directly, and this implies that the development of an algebraic mode of thinking should become an explicit objective of teaching, rather than wishing that pupils would simply "absorb it" through the learning of algebraic techniques.

It will also be seen that there is a tension—of a different sort, though—between "solving problems" and making *algebraic thinking* explicit, and Vieta's *Analytical Art* has the double merit of highlighting this tension and of providing a notational form which will allow algebra to develop in the direction of "method," rather than that of "solving problems." Traditionally, algebra is introduced in school through "solving problems with equations." Our findings suggest that this might not be the best approach, but this suggestion only implies that "solving problems algebraically" be taken as *distinct from*, not *secondary to*, activities which aim is deliberately the development of an algebraic mode of thinking; moreover, we think that the activity of "solving problems algebraically" is better understood as *modelling*, in which case the nature of an algebraic model can be distinguished from that of a geometric, combinatorial, or functional model, and the nature of *algebraic thinking* can undergo further clarification.

3.2 Aspects of Greek Mathematical Culture

Greek Doctrines of Number

The three doctrines which we will examined, are associated to the names of Pythagoras, Plato, and Aristotle. These three philosophers are of particular interest to us not only because their work had an immense impact in the formation of our modern western civilisation, but also because there we find a discussion of the Greek conceptions about mathematics, and in particular Greek conceptions about numbers. There should be no doubt that Greek mathematics—or Greek philosophy, for that matter—was not as homogeneous and linearly developing as our exposition might make it seem, and also that what we present here is a compact version of a complex subject. In respect to the relation between Greek philosophy and mathematics, we think that Jacob Klein's *Greek Mathematical Thought and the Origin of Algebra* is unsurpassed, and should be a central reference in any study concerned with the subject.

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Pythagoras, the first philosopher we will consider, lived in southern Italy about 582-500 BC. He—or more precisely, his school, the Pythagoreans—is credited with the notion that everything in the Universe *is number*. An example is that of the relation between the lengths of strings and the tones they produce, so an octave in relation to the original tone is produced by a string which length is in the ratio 1:2 to another string (the other characteristics of the strings being the same), and a fifth is produced when the ratio of the lengths is 2:3.

The distinctive aspect of this Pythagorean notion, is that what it is saying is not that "the Universe *can be expressed through quantitative relationships*," as a modern physicist might say, but that "the being of the Universe ***is** numbers*." The two pillars supporting this conception are exactly those which define the Pythagorean concept of *number*. First, *number* is only a whole number, and even more, *a definite number of **things***. Second, *number* for them, could not be understood "outside" the world of things[3,4]. In other words, *number* is only manifested in the *manyness* of a collection of things[5], at the same time it was that which allowed us to know the Universe[6]. It is as an immediate consequence of the nature of *number* being assimilated to that of counted collections, that there has always to be a *unit*, representing "what is being counted," and only whole numbers can be conceived. Ratios of whole numbers are never taken as "fractional numbers" in our sense (as we will see in many passages ahead).

The well known proof that the diagonal of a square is not commensurable with its side, deeply shook the Pythagoreans' beliefs, and one has to have in mind that Pythagoras was not "simply a mathematician"; mathematics occupied a very central role in his philosophy, which embraced mystical, cosmological and moral considerations (Abbott, 1985). Nevertheless, nor the Pythagoreans neither the other Greek philosophers opted for "extending" the notion of *number* to accommodate those new "ir-rational" quantities; instead, their mathematics, following the philosophical demands, adheres to a strict separation between numbers and geometrical magnitudes,

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[3]Klein (1968, p67): "Aristotle stresses again and again that it is characteristic of the Pythagorean view that 'they do not make number *separable* [from the things]; this means that they do not go so far as to suppose the existence of 'pure' numbers of 'pure' units, although *they were the very men* who concerned themselves with numbers not for a practical but for a theoretical purpose, who conceived of the *arithmos* as *arithmos mathematikos*, as scientific number."

[4]Morris Kline (1990, p29): "When the early Pythagoreans said that all objects were composed of (whole) numbers or that numbers were the essence of the universe, they meant it literally, because numbers were to them like atoms are to us."

[5]Klein (1968, p65): "The Pythagorean mode of definition is, then, characterized by the attempt to define the *being* of things by reducing and *assimilating* them to conditions 'primarily' exhibited in the realm of counted collections as such..."

[6]Klein (1968, p63)" Their [the Pythagoreans] chief object was to understand the 'order within heavens'"; also, page 67, "Thus the science of the Pythagoreans is an *ontology of the cosmos*, a doctrine concerning the mode of being of the world and of the things comprised in it."

and geometry, developing free of such "limitation", is definitely brought to the forefront of Greek mathematics.

In respect to our overall argument, the important point here is that the Pythagoreans did not deny the study of "irrationals," but the only *model* that allowed them to continue their study was that of geometry, ie, they made sense of *irrationality* in the context of geometric *figures*; not only they did not, they could not conceive the study of geometry as relating to that of numbers[7]. Nevertheless, the Pythagorean study of *number* will continue to make use of *forms* (eg, the *gnomon*, as well as the notion of *figurate number*[8]), which we may well read as *geometric*, but which have in fact a deeper significance to the Greek study of *number*, as we will see in relation to the notion of *eidos*.

In Plato, who lived in 427-347 BC, in Athens, we find a reformation of the Pythagorean conception of number, mainly in that for the Pythagoreans *number* was the being of things themselves, whereas in Plato, the possibility of *counting*, which was on the basis of any knowledge of *number* (Cf. Klein, 1968, p46)[9], is derived from the existence of a realm of *pure monads*[10], or units, distinct from that of the counted things[11,12]. According to Morris Kline (1990, p43), the distinction between objects of sense and objects of thought—which will remain in Aristotle—is probably of Socratic origin[13]. In the Pythagorean conception, the fact that *number* was always "a number of something," and that *number* always intended the counted things themselves, in their multitude, accounted both for the determinedness of each number and for the fact that *number* is always a *definite number*. In the Platonic view, however, there are no "specific" collections in the realm of pure monads, and the latter can only be accounted by introducing the notion of *eidos* ("literally: 'looks'; kind, form, species, 'idea';

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[7]As we will see, this alone is sufficient to seriously undermine the claim of a *geometrical algebra* to be found in Euclid's *Elements*.

[8]Square, triangular, pentagonal, etc.

[9]And, thus, any possibility of a negative or irrational *number* is completely precluded.

[10]Which are not in any way presupposed by the Pythagorean conception, as Klein notes on page 69.

[11]Klein (1968, p70): "Especially in discussing numbers, Aristotle never tires of stressing that Plato, in opposition to the Pythagoreans, made them 'separable' from objects of sense, so that they appear '*alongside* perceptible things' as a separate realm of being."

[12]Klein (1968, p50): "...now our concern is rather with understanding the very possibility of this activity [counting], with understanding the meaning of the fact that *knowing* is involved and that there must therefore be a corresponding *being* which possesses that *permanence* of condition which first makes it capable of being known'...What is required [in the Platonic doctrine] is an object which has a purely noetic [*noetón*, object of thought] character and which exhibits at the same time all the characteristics of the *countable* as such. This requirement is exactly fulfilled by the 'pure' units, which are 'nonsensual,' accessible only to the understanding, indistinguishable from one another, and resistant to all partition (Cf. Pp. 23ff and 39ff, also p53 [of the *Republic*])."

[13]Morris Kline (1990, p43) refers to this distinction as that "between abstraction and material objects"; although tempting, given the modern conceptualisation, Kline's formulation does not apply correctly. In Plato, the pure monads are *not* abstractions.

sometimes: 'figure'"; Translator's note to Klein, 1968)[14,15,16]. Here, it is the *eidos*, and not *number*, that is to be the object of *arithmetic* ("Only the *arithmoi eidetikoi* make something of the nature of number possible in this our world." Klein, op. cit., p92).

Before we progress any further, it is necessary to clarify a distinction essential in Greek mathematics, that between *arithmetic* and *logistic*. In Heath (1981, vol 1, p13ff) we find that,

> "Arithmetic, says Geminus [Rhodes, 1st century BC], is divided into the theory of linear numbers, the theory of plane numbers, and the theory of solid numbers. It investigates in and by themselves, the species of number as they are successively evolved from the unit ...As for the [logistic], it is not in and by themselves that he considers the properties of numbers but with reference to sensible objects; ... The scholiast to [Plato's] *Charmides* is fuller still: 'Logistic is the science that deals with numbered things, not numbers ... Its subject-matter is everything that is numbered. Its branches include the so-called Greek and Egyptian methods in multiplication and division..."

Arithmetic is a *science*, an *episteme*, and *logistic* an *art*. The crucial reason for this distinction lies in the indivisibility of the unit. The logistician can speak of and operate with fractions by virtue of the bodily nature of the objects being counted, which may be divided at will, while the arbitrarily assumed unit of calculation, an apple, for example, still remains intact. This is not that case in the realm of pure monads, as the division of the unit can only produce—paradoxically—an increase in the number of units, as they are all the same.

In Plato, then, the *eidos* provide a *delimited* object, and a notion that solves the difficulty of *number* being one and many at the same time: *number* is always "many,"[17] and the *eidos* to which it belongs is "one"[18].

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[14] The notion of *eidos* in Greek mathematics is a complex one, but instead of trying to offer a downright "definition," we prefer to let it gain substance as we repeatedly use it in our argument.

[15] Klein (1968, p56): "Precisely because *the arithmos as such is not one but many, its delimitation in particular cases can be understood only by finding the eidos which delimits its multiplicity, in other words, by means of* arithmetike *as a theoretical discipline.*"

[16] Typical examples of *eide* are the odd, the even, the odd times even, for example. Also, the triangular, the square, etc, as in figurate numbers.

[17] Klein (1968, p46): "...the *arithmos [number]* indicates in each case a *definite number of definite things*...it intends the *things* insofar as they are present in this number, and cannot, at least at first, be separated from things at all."

[18] Also, Klein (1968, p59): "...the absence of any mention of either *arithmos* or *arithmoi* in the definitions of arithmetic and logistic in the *Gorgias* and in the *Charmides* ... expresses the fact that the multitude of arbitrarily chosen assemblages of monads is accessible to *episteme* only through the determinate *eide* which can always be found for these assemblages..."

The introduction of the pure monads, which may seem a simple step in view of our "modern" conceptualisation of number, is crucial enough to produce Klein's observation that ,

> "The thought of 'pure' numbers separated from all body is originally so remote that it becomes the philosopher's task precisely to point out emphatically the fact that they are *independent* and *detached*, and to secure this fact against all doubt." (op. cit., p71ff)

Now, this "somatic" nature of numbers which is to be substituted by Plato's construction, seems to be the source of many obstacles students face in dealing with the *internalism* of *algebraic thinking*, for example in relation to negative numbers; also, many of the students we worked with in the experimental part of our research, failed to produce "purely numerical" models to solve the problems we proposed, suggesting that the "unknown" or the "indeterminate" number could only be dealt with by recourse to a "somatic" interpretation of some kind. It is true that Plato's model certainly does not allow for negative numbers, as the pure monads are conceived in a way to allow the "replication" of collections of counted things, but at the same time, it is this construction that gives *arithmetic* the status of *episteme*, and allows Aristotle to elaborate further to achieve a conceptualisation flexible enough to provide grounds for Diophantus' work[19]. We think it is adequate, thus, to point out at this early stage, the roots of such a deep reaching process, so we can be alert to other aspects in it that may provide us with insights into the obstacles faced by our students.

Plato's construction involves a much less evident difficulty: since the *eide* are the *objects* of *arithmetic*, the general notion of *number* is not possible, once—as Aristotle noticed and criticised—each *eidos* has its own nature[20]. Plato's project of a *theoretical logistic* is prevented by this difficulty[21], and only with Aristotle it becomes possible.

Aristotle (384-322 BC, born in Macedonia) was for 20 years pupil and colleague of Plato. In 355 BC he founds the Lyceum, in Athens (which comes to be

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[19]Which, in turn, becomes the object of a much later reinterpretation that is to a good extent responsible for our modern view of number and of algebra.

[20]In Aristotle, this difficulty is solved by attributing to the *eide* a classificatory role, but not a constitutory one.

[21]The objective of a *theoretical logistic* would be to offer a "scientific" treatment of *number as counted*, ie, in its manyness, as opposed to the treatment offered, by *arithmetic*, to *number as one*, ie, the *eide*. As Klein (page 23) puts it, "...theoretical logistic arises from practical logistic when its practical applications are neglected and its presuppositions are pursued for its own sake." With *number as counted*, ie, with the logisticians, fractions were allowed by virtue of the bodily nature of the objects being counted, but this fractioning of the unit is exactly what is not possible in the realm of pure monads and that which led Plato to turn to the *eidos* as the *object* of *arithmetic*.

known as the Peripatetic school). In his doctrine, Aristotle operates a radical transformation of the Pythagorean and Platonic conceptions of *number*. Instead of positing, with Plato, that there is a separate, independent, realm of pure monads, Aristotle argues that the pure "numbers" are obtained *by abstraction* from (definite) collections of things. It is necessary, we think, to emphasise that this position is also substantially different from that of the Pythagoreans, as for them, *number* is *identical* with the being of things, whereas in Aristotle they are distinct, although inevitably dependent of, the being of things. This situation is arrived at by postulating that the "pure" numbers arise by *disregarding* the sense-related qualities of the counted collections[22], and at the same time asserting that *number* exists only as long as things are being counted[23]. In Aristotle's framework, three types of *numbers* are distinguished: (i) the *arithmos eidetikos*, the idea-number; (ii) the *arithmos aisthetos*, which corresponds to the things themselves, which are present for perception in this number (amount); and, (iii) the *arithmos mathematikos* or *monadikos*, which "shares with the first the 'purity' and 'changelessness' and with the second its manyness and reproducibility." (Klein, 1968, p91). The *numbers* with which the arithmetician deals are *objects of thought*, although abstracted from collections of sensible objects, and the *noetic*—as opposed to "somatic"—character introduced by Plato is preserved.

In Aristotle, "A number is [only] that which has been counted or can be counted." (quoted by Klein, op. cit., p107) *Number* is revealed only in the process of counting, and not by virtue of each number by which we count being available through a "pure" *number* that exists independently and before any counting. It is in this sense that *number* is "...*derived* from the experience of counting multitudes and of culling from them those different formations 'by abstraction'." (Klein, p107) A most important consequence of all this, is that it is impossible, in the context of the Aristotelian conception, to conceive a *number* that is neither known nor *intended* to be known immediately.

One crucial aspect in Aristotle's conception of number, in fact that which makes Plato's project of a *theoretical logistic* possible, appears in his solution to the problem of the dual "one-many" nature of numbers. In the Aristotelian framework, this question is solved by observing that counting is possible only insofar as the things being

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[22]Klein (1968, p104ff)"[Aristotle:] 'The mathematician makes those things which arise from abstraction his study, for he views the after having *drawn off* all that is sensible...and he *leaves* only the [object of the question] 'how many' and continuous magnitudes.(Metaphysics, K3, 1061 a 28ff)'...Not *original* 'detachment' but *subsequent* 'indifference' characterizes the mode of being of pure numbers..."

[23]Klein (1968, p101): "...the assertion 'three trees' presuppose[s] the assertion 'three,' but what the assertion 'three' intends has no existence 'outside of the trees of which there are said to be three...At the root of this Aristotelian conception lies the 'natural' meaning of *arithmos*; the assertion that certain things are present 'in a certain number' means only that *such* a thing is present in just this definite multitude: 'To be present in number is to be some number of a [given] object.'"

counted—after the "disregarding" of its visible qualities—become homogeneous, ie, they are all the same. But this sameness is expressed exactly in the existence of a *common measure*, a *unit* :

> "[Aristotle:] 'For each number is 'many' because each is [made up of] 'ones' and because each is measured by [its own] 'one'.' (Metaphysics, I 6, 1056 b 23 f....) In *this* sense the 'one' (or the *one* thing subjected to counting) makes counting and thus the 'counting-number' possible...The priority of the one over number does not follow from a relationship of genus over species, but rather from the character of the one as 'measure'...We comprehend a number as *one* because we do our counting over one and the same thing, because our eyes remain fixed on *one and the same thing*." (Klein, 1968, p108)

This approach enables one to deal with fractional parts, not by "fractioning" the unit—which is, of course, indivisible—but by using different units: to speak of $\frac{5}{2}$ is simply to speak of five $\frac{1}{2}$'s, where $\frac{1}{2}$ is a unit, and not a *number* in its own. With Aristotle, "number is a multitude measured by a unit" (Klein, p109; our emphasis). A crucial shift from Plato and Pythagoras, is that here the pure unit is *the property of being a measure*, rather than being *a thing itself*. It is precisely this characteristic that produces the flexibility necessary to Diophantus' work, and explains why his main work can be called *Arithimetica*, a *science*, at the same time it deals with fractional parts, an activity previously restricted to *logistic*. A second shift is seen in the role played by the *eide*, which are now much less significant[24]; we'll see, in fact, that in Diophantus they have only an *instrumental* function, whereas before they were part of the core of the possibility of understanding *number*.

Summarising, we saw in the course of this brief examination of the three most influential doctrines concerning *number* in Greek mathematics, that the conceptions contained in each of those doctrines, far from simply being a matter "for philosophy," played a major role in determining what could and could not be done in the Greek study of numbers. Plato's framework allowed for a somewhat "general" treatment of *number* through the study of the *eide*, but it made any attempt to include fractions in this study, impossible. Aristotle's framework, on the other hand, allowed for the treatment of fractions in the form of "numbers of fractional parts," but limited the study of numbers

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[24]Klein (1968, p110): "The 'even,' the 'odd,' the 'even-times-odd,' etc.,...are now no more than the 'peculiar characteristics' of numbers...They represent merely a *quality* of numbers...The 'what' of each number insofar as it is a number is precisely that quantity which it indicates; thus 'six' units are not in themselves 'two times three' units or 'three times two' units, for this indicates only their 'composite quality,' but 'once six'..."

to that of numbers that are either *known* or only *as yet unknown*, ie, ***intended to be known***. Moreover, by determining what is to be called *number*, those frameworks suggest—if not determine—what *can* be done with those *numbers*: From Plato to Aristotle, we move towards a more "natural" conception of *number*, but as a result we are held back to a context in which numbers are very much like things, and neither the assimilation of *counting* to *measuring* nor the assertion that mathematics deals with objects of thought will take us far away from the context of "the natural world." This is hardly surprising, as the objective of Greek *episteme* grew more and more to be the understanding of the natural world; the association between mathematics and "the world" as we find in Pythagoras ("everything *is number*"), in Plato's postulating of the existence of a world of ideas independent of us, or in Aristotle's "natural" conception of *number*, all point in that direction.

Common to the three doctrines we examined, are the *indivisibility of the unit*, and the conception that *one is the principle of number but not [a] number itself*. Also, in all three cases, *number* means "whole number" and "a number of..."; *number is a definite number of things, be they pure monads or objects of the sense.*

Another feature common to them is that *number* has a *discrete* nature, as a consequence of them always arising in relation to *counting*. Geometric magnitudes, on the other hand, are always continuous, and on this basis alone a first distinction could be established between the two realms, as the Pythagoreans in fact did. On the arithmetical Books—VII, VIII, and IX—Euclid represents *numbers* by lines, but this is to be seen in the framework of the Aristotelian conception of *number*, as the possibility of representing number, understood as a *measured multitude*, in a convenient way[25,26]. It is not the case that in Euclid *number* become *continuous*; the true conception has to be permanently kept in mind, or we are bound to misunderstand the texts.

Before moving to the Greek mathematical production "proper"—Euclid and Diophantus, in this case—we have to deal a little more with the problem of incommensurability. It is frequently asserted that the discovery made by a Pythagorean was that "the ratio of the hypothenuse to either side [of an isosceles rectangle triangle]

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[25]Klein (1968, p11): "The 'arithmetical' books of Euclid (VII, VII, IX) directly mirror this ontological transformation...The 'pure' units of which the numbers to be studied are compounded are here understood precisely only as *'units of measurement'* such as can be represented most simply by straight lines which are *directly measurable* (rather than by points...), quite independently of whether they form a 'linear' (prime), 'plane,' or 'solid' number. The same approach is indicated by Definitions 8, 9, 11, 12, 14 of the seventh book (namely that of even-times-even, even-times-odd, odd-times-odd, prime and composite number...which define the nature of each number with respect to the *measuring character* of its factors..."

[26]Morris Kline (1990, p136), refers to the use of lines to represent *numbers* as a way of *visualising* them.

is the irrational number $\sqrt{2}$ " (eg, Abbott, 1985, p110). Although, of course, correct from the point of view of our understanding of number, this formulation hides many of the problems faced by the Greeks. What in fact they concluded was that the ratio between the side and the diagonal of a square is *not* the ratio between two *numbers*, ie, whole numbers. The Pythagorean "numerical" theory of proportions could not deal with incommensurability, so the finding did hurt not only the non-mathematical, so to speak, aspects of their philosophy, but also the certainty of proofs that depended on such a theory of proportions (Heath, 1981, vol 1, p326).

The theory of proportions developed by Eudoxus (Asia Minor, c408-c355 BC) solves the problem of incommensurability, but not, as Dedekind for example did, by legitimating the existence of ***irrational numbers***. Instead, Eudoxus' theory is exclusively concerned with *geometric magnitudes*, and not *intended* to be applied to *numbers*[27]. In a very reassuring passage, Morris Kline (1990, p48ff) says that,

> "Eudoxus introduced the notion of a magnitude...**It was not a number**, but stood for entities such as line segments, angles, areas, volumes, and times which could vary, as we would say, continuously. Magnitudes were opposed to numbers, which jumped from one value to another, as from 4 to 5. **No quantitative values were assigned to magnitudes**. Eudoxus then defined a ratio of magnitudes and a proportion, that is, an equality of two ratios, to cover commensurable and incommensurable ratios. **However, again, no numbers were used to express such ratios. The concepts of ratio and proportion were tied to geometry**...What Eudoxus accomplished was to avoid irrational as numbers." (our emphasis)

Now, neither *arithmetic* nor a *theoretical logistic*—at the time of Eudoxus still not possible— could deal with incommensurability, precisely because *number* was always a whole number, and only Eudoxus' theory provided a way of dealing with it. As a result, geometry and number are forced apart, and geometry assumes the leading role by virtue of offering a way out of the central ontological problem of Greek mathematics of that time. Morris Kline (1990, p49) points out that "The Eudoxian solution to the problem of treating incommensurable lengths...actually reversed the emphasis of previous Greek mathematics. The early Pythagorean had certainly emphasised number as the fundamental concept..."

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[27]Heath (1990, vol 1, p90): "This subject [the irrationals] was regarded by the Greeks as belonging to geometry rather than arithmetic. The irrationals in Euclid, Book X, are straight lines or areas, and Proclus mentions as special topics in geometry matters relating (1) to *positions* (for numbers have no positions), (2) to *contacts* (for tangency is between *continuous* things), and (3) to *irrational straight lines* (for where there is division *ad infinitum*, there is also the irrational)."

It is then, the solution to the problem, and not "the problem" itself, that turns Greek mathematics towards geometry; as we will see, the situation in Chinese, Hindu, and Arab mathematics was quite distinct, and the irrationals are absorbed as numbers. The effect of such a solution, however, can only be understood, as we indicated before, in the context of the Greek conception of number.

What we have said so far, immediately enables us to make one very important point. It is certainly beyond all doubt that all the Greek thinkers mentioned here were mature thinkers, and indeed sophisticated thinkers. One naturally asks, then, "How could they had held such 'simplistic' and limiting conceptions about numbers?" This question is the more relevant to our research as we remember that children, too, have difficulties in grasping the notion of a fractional *number*, of a negative *number*, and even more that of an irrational *number*. And we do not mean providing sound logical foundations for them, but only accepting their being. What the example of Greek mathematics shows us, is that underlying conceptions, and not intellectual power, are responsible for the situation that resulted. This is not to say, of course, that a seven years-old child is as able as an Aristotle to deal with such matters, but simply to point out that such conceptions, which are unequivocally *cultural*, part of their *culture*, of their whole system of ideas, can and do prevent powerful minds from accepting or producing some forms of knowledge, and thus, they can and do prevent the production of whole systems of knowledge—**which, in fact, would have no place in that culture**. The parallel with children's learning should not be made on the basis of the *empirical finding* that "these and those conceptions imply this and that that difficulty,"[28] but rather in terms of the *overall conclusion* that "my understanding and learning depends on the knowledge being offered having a place in my conceptual world.

Two other schools should be mentioned in the context of Greek philosophy. The Ionian school, founded and led by Thales (Mileto, c640-c546 BC), is credited with starting the drive towards a rational knowledge of nature and with providing the first definition of *number*, "...defined as a collection of units, 'following the Egyptian view'," according to Iamblichus quoted in Heath (1981, vol 1, p69ff); the Eleatic school, to which Zeno and Parmenides (5th century BC) belonged, is better known by the studies carried there about continuity and the infinitely small (as seen, for example, in Zeno's paradox about the impossibility of Achilles beating a tortoise in a race), but

[28]An approach that would certainly produce the most paradoxical didactic situations, as even if a seven years-old child in today's world thinks that *number* can only be a whole number, her or his experiences with numbers—telephone numbers, house numeration, car plates, prices, and so on—are infinitely distant from that of Pythagoras.

they also produced results in geometry proper, for example Democritus' discovery that the volume of the cone is one-third of the volume of the cylinder with the same base and height (M. Kline, 1990, p37). The contributions of both those school to the understanding of *number*, however, are far less important than the ones we have examined in some detail.

We will now turn our attention to the work of two Greek mathematicians: Euclid and Diophantus, who belong to the Alexandrian—or Hellenistic—period of Greek culture, which succeeds the Classical period.

Alexandria, the geographical centre of this new phase of Greek culture, was founded in northern Egypt, in 322 BC, by Alexander of Macedonia, son of Phillip of Macedonia, the conqueror of Athens, and himself a conqueror of Greece and Egypt. In the context of this new culture, the old belief that educated people should not be concerned with an *art* such as *logistic*, was slowly discredited. It is also probable that the much more intense and deliberate exchange with other cultures—by Alexander's designation—brought into Greek mathematics many new elements, for example a concern with producing the means for dealing with more "practical" problems. As Morris Kline observes, "It might be logically satisfactory to think of $\sqrt{2} \cdot \sqrt{3}$ as an area of a rectangle, but if one needed to know the product in order to buy floor covering, he would not have it."; Kline also says that, "...the mathematicians of the Alexandrian period severed their relation with philosophy and allied themselves with engineering." Archimedes, we are reminded, was Alexandrian. Alexandrian mathematics, however, does maintain the Classical approach of considering the objects of mathematics as objects of thought[29].

This is the context in which the shift towards arithmetic produced in the Alexandrian mathematics has to be understood: not only a *theoretical logistic* is made possible by the Aristotelian framework and by the imports from other cultures, but is in fact *required* by the enterprises and scientific context of the time. Rubens Lintz in his *História da Matemática* (of which I only had access to the manuscript version), supported by a substantial historical research and a convincing argument, suggests that in fact one should consider Diophantus not as part of a then declining Greek tradition, but rather as part of a new, emerging, tradition[30].

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[29]A fine example of this was Archimedes acceptance of mechanical analogies as means to suggest the truth of theorems, but not as means to prove them, for which task geometry was essential (Cf. Heath, 1981, vol 2, p21).

[30]Lintz's argument in this respect is mainly based on the fact that Diophantus work is—in relation to Lintz's framework—more akin to the *magic culture* of the Arabs, than to the *plastic culture* of the Greeks. In the context of the *magic culture*, the solution of an equation corresponds to the—almost liturgical—process of *revealing what is hidden in the equation*, ie, the unknown number.

The work of the Alexandrians Euclid and Apollonius (in relation to his geometry) are exceptional in this context, but one has to remember that in both cases, although more particularly in the case of Euclid, what we have is a reorganisation of the Greek Classical mathematics; Euclid is at Alexandria only 30 years after its foundation, and Apollonius' work both in astronomy and on irrational numbers are influenced by the Alexandrian culture (M. Kline, 1990, 104).

EUCLID

The first Greek mathematician whose work we will examine is Euclid. We know of his life that he has probably studied in Plato's school in Athens, and after that moved to Alexandria, where he founded his own school. (Heath, 1981, vol 1, p356).

We will restrict our examination of Euclid's work to his *Elements*, more specifically some parts of the *Elements* which are relevant to our research, ie, those explicitly concerned with *number* (yet in the Greek sense), ie, the arithmetical Books VII, VIII, and IX, and those parts which could be interpreted—from the point of view of our modern mathematical notions—as referring to numbers, ie, the "geometric algebra," in particular Book II.

Analysis and Synthesis in Euclid

An important aspect of the *Elements* we would like to emphasise, is made clear in the words of Heath (1981, vol 1, p371):

> "The *Elements* is a synthetic treatise in that it goes directly forward the whole way, always proceeding from the known to the unknown, from the simple and particular to the more complex and general; hence *analysis*, which reduces the unknown or the more complex to the known, has no place in the exposition, though it would play an important part in the discovery of the proofs."

In the case of geometric propositions, the proofs always contain the construction of the elements sought, so in II,11, for example,

> "To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment." (Fauvel and Gray, 1987, p119)

the solution consists in the construction of the sought cut, followed by the proof that such cut is actually the required one.

There are two points to consider here. First, not only the *Elements*, but the general lack of Classical Greek mathematical texts dealing with the process by which theorems and proofs are suggested, indicate the extent to which the ultimate aim of mathematical activity was to provide proofs for mathematical facts: that is what remained in the final form of the texts. Second, as Heath points out, one should be aware that some form of *analysis* must have been used in order to find the constructions that are part of the proof, and we shall investigate to some extent, what form this *analysis* took in Greek mathematics.

We will examine the second point. An indispensable source on the Greek use of *analysis* is Pappus' *On the Treasury of Analysis*, to which we have already referred as containing a most clear definition of *analysis* and *synthesis*. In Pappus' words,

> "The so-called *Αναλυομενος* [*τοπος*, *The Treasury of Analysis*] is...a special body of doctrine provided for the use of those who, after finishing the ordinary Elements, are desirous of acquiring the power of solving problems which may be set them involving (the construction of) lines, and it is useful for this alone. It is the work of three men, Euclid,...Apollonius of Perga and Aristæus the elder, and proceeds by way of analysis and synthesis." (quoted in Heath, 1981, vol 2, p400)

The first book listed by Pappus as belonging to *The Treasury*, is Euclid's *Data*, in which the propositions are intended to prove that, "...if in a given figure certain parts or relations are given, other parts or relations are also given, in one or another of these senses [to be found in the Definitions]...It is clear that a systematic collection of *Data* such as Euclid's would very much facilitate and shorten the procedure in *analysis*." (Heath, 1981, vol 1, p422) The example provided by Heath of Prop. 59 of the *Data* (op. cit., p423), is illustrative. *Analysis*, then, is to be understood as the process that goes like, "I want to solve this problem. If X and Y were given, I could solve the problem; but to construct X and Y, I would need to know Z and W, etc.." At some point I either arrive at the need of magnitudes that can be constructed only by using the ones given in the problem—and the problem can be *synthetically* solved—or I conclude that some required construction is contradictory with the problems data—in which case the problem is impossible. It is in this latter sense that the *reductio ad absurdum* is a form of *analysis*[31].

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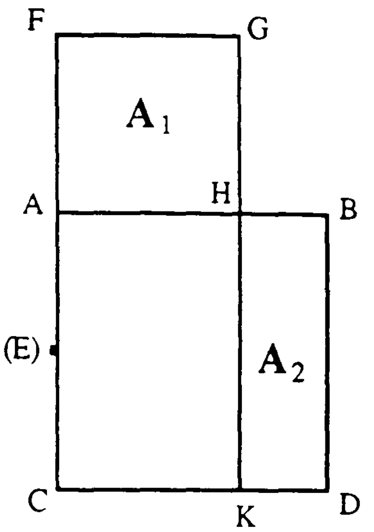
[31]Cf. Heath, 1981, vol 1, p372. We must add that when Euclid uses this type of proof, as, for example in IX.20 ("Prime numbers are more than any assigned multitude of prime numbers.") he is always dealing with determinate numbers, and what is supposed is a property of that number, only.

We now offer a possible way in which Euclid's *analysis* leading to the solution of II,11 might have taken place. The figure bellow is used, which depicts the problem as if it had been solved. AB is the given line, E is the middle point of AC, and the letters are used in exactly the same way as in Fauvel and Gray (1990, p119ff), where Euclid's demonstration is given, so the reader can easily follow the "way back," ie, the synthesis. **T**(x) denotes the square with side x, and **O**(x,y) denotes the rectangle with sides x and y, following Mueller's notation (Mueller, 1981, pp42 and 45)



Euclid might have thought: "If the problem had been solved, then $A_1=A_2$, ie, O(CF,FA)=T(AB). Now, using II,6 I could relate O(CF,FA) to T(AE) and T(FE), because II,6 says that O(CF,FA)+T(AE)=T(FE). Good. But, wait...all this means that T(AB)+T(AE)=T(FE). Hmmm...it smells Pythagoras, this one. Let me look at the drawing again...Of course!! FE has to be **made** the same as EB!!"[32]

Characteristic of *analysis* used in this way, one is always looking for ways of producing other magnitudes from the already known, as the objective of the *analysis* is exactly to provide the construction of the required magnitudes. *Analysis* does not prove, it only shows how the proof can be effected. In *algebraic thinking*, however, the central aspect of the process is exactly the *analysis*, to the extent that establishing rules by which one can move from *the supposition of the unknown being known* to *the actual production of the unknown* become a central part of the method, in the same way that a book like Euclid's *Data* —by providing a knowledge of what can be

[32]In his 1976 article *Defence of a "Shocking" Point of View*, quoted in Flauvel and Gray (1987), van der Waerden states that "al-Khwarizmi's solution of quadratic equations is equivalent to Euclid's procedure," and in van der Waerden (1983, p83ff) he offers his reasons for stating it. Having read both al-Khwarizmi's and Euclid's books, I was not satisfied with the first assertion, as the only way in which it could make sense of it was to take it as meaning that by both procedures one would arrive at the same final solution, what is hardly surprising, once they are both correct, and his later "explanation" is artificial—although, of course, possible. It was, thus, to my great pleasure, that I worked out the solution here presented, totally geometrical and **leading directly to Euclid's construction and *synthesis*.**

obtained from a given geometric configuration—would greatly help the *geometric analysis*. The crucial difference is that *algebraic thinking **intends** analysis*, whereas the *Data **intends** the possibility of constructions*.

In the solutions given by the students in our Experimental Study, we frequently observed *analysis* used in the Euclidean sense, and in many cases only the steps that actually produce the answer are exhibited.

The claim of a Geometric Algebra in Greek Mathematics

In recent years a debate involving historians of mathematics and mathematicians concerned with the history of mathematics, has developed around the interpretation of what came to be known as the Greek "geometric algebra." According to Klein (p122), "[Hieronymus Georg] Zeuthen was not the first to understand the ancient mode of presenting mathematical facts as a 'geometrical algebra,' although he was the first to use the term consistently." We will examine the merits of arguments for and against the "geometric algebra" interpretation, not with the objective of producing an answer to the question of whether this interpretation is accurate—although in the course of our examination a negative answer is produced, at least in the case of the *Elements*—but rather aiming at the arguments themselves and to the conceptual frameworks which support them. As a result, we will learn about the impact of conceptual frameworks in the interpretation of mathematical knowledge, which is closely related, we think, to their impact on the acquisition and understanding of such knowledge, but we will also learn about some specific aspects of the context of this debate—namely, about geometric models and about the use of algebraic notation in the context of those models.

Behind the idea of a "geometric algebra" to be found in the Greeks, is the understanding that a substantial part of the "geometrical" theorems are, in fact, "algebraic" theorems "dressed up" in a geometrical form[33]. In his *Science Awakening*, van der Waerden goes as far as to say that,

> "Presently we shall make clear that this geometric algebra is the continuation of Babylonian algebra. The Babylonians also used the terms 'rectangle' for xy and 'square' for x^2, but beside these and alternating with them, such arithmetic expressions as

[33]It is clear, from for example the quotation that immediately follows this note in the main body of the text, that the "geometric algebra" refers to numbers, and not, as one might conceive, to a geometric "calculus" where propositions are proved to be used later on. Would this be the case, there would be no case at all, apart from dismissing the terminology as inappropriate. A strong motivation for the "geometric algebra" hypothesis, seems to be the desire to account for the lack of an "arithmetical" treatment of irrational numbers. The lack exists, it is true, but it is just a consequence of the Greek conceptual framework for mathematics.

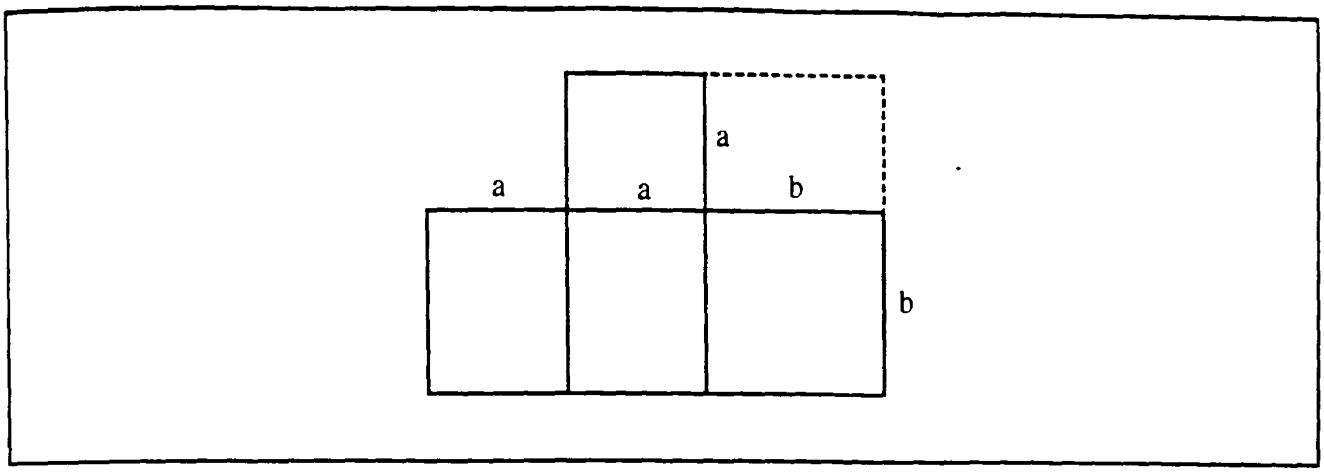
multiplication, root extraction, etc. occur as well. The Greeks, on the other hand, consistently avoid such expressions...everything is translated into geometric terminology. But since it is indeed a translation which occurs here and the line of thought is algebraic, there is no danger of misrepresentation, if we reconvert the derivations into algebraic language and use modern notation." (quoted in Fauvel, 1990, p142)

In Euclid, the most relevant Book in relation to the debate about "geometric algebra" is Book II. The "translation" according the "geometric algebra" interpretation, gives for the first few propositions:

Prop. I:	$a\,(b+c+d+\ldots) = ab+ac+ad+\ldots$
Prop. II:	$(a+b)a+(a+b)b = (a+b)^2$
Prop. III:	$(a+b)a = ab+a^2$
Prop. IV:	$(a+b)^2 = a^2+2ab+b^2$
Prop. V:	$ab+\left\{\frac{1}{2}(a+b) - b\right\}^2 = \left\{\frac{1}{2}(a+b)\right\}^2$
Prop. VI:	$(2a+b)b + a^2 = (a+b)^2$

If we understand those propositions as meaning what the use of the algebraic notation suggest—ie, *numerical equalities*—we have to assume their *symmetry*. But in this case, propositions I and IV put together make proposition VI in the most direct way[34]. In Euclid, however, the construction *has* to be effected, because the geometrical configuration that results from Prop. IV (the well known square divided into two squares and two rectangles) cannot but by means of a geometrical construction be associated to the geometrical configuration resulting from Prop. VI, no matter how evident the equality of areas is from its diagram (see figure bellow).

[34]And, "There are, for example, simple algebraic derivations of [Prop. II and III] from [Prop. I]...Similarly, II,3 is a consequence of II,1 because $(x+y)y=y(x+y)=yx+yy=yx+y^2$. Since Euclid normally takes for granted such geometrically obvious assertions as $T(x)\equiv O(x,x)$ and $O(x,y)\equiv O(y,x)$ [where $T(x)$ is a square with side x, and $O(x,y)$ a rectangle with sides x and y], he could have carried out geometrized versions of these arguments." (Mueller, 1981, p46) Heath also points out that "It appears to be Heron [of Alexandria, c. 250 AD] who first introduced the easy but uninstructive semi-algebraical method of proving the propositions II.2-10 [in the *Elements*] which is now so popular. On this method the propositions are proved 'without figures' as consequences of II.1 corresponding to the algebraical formula $a\,(b + c + d + \ldots) = ab + ac + ad + \ldots$. Heron explains that it is not possible to prove II.1 without drawing a number of lines (ie, without drawing the actual rectangles), but the following propositions can be proved by merely drawing one line." (1981, vol 2, p311)

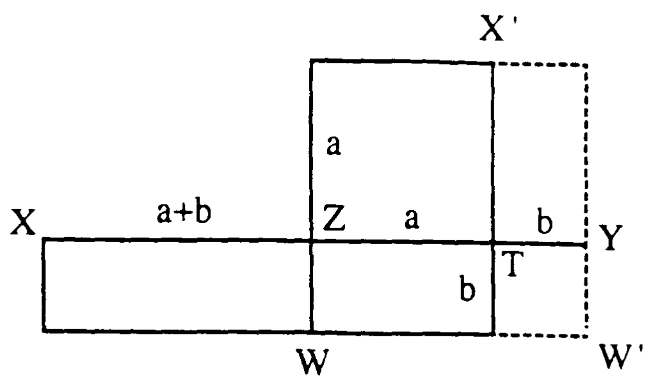


Prop. VI, Book II (the *Elements*)

The "algebraic translation" of Prop. V certainly is not immediately identifiable with that of Prop. VI. The "translation" presented above is to be found in, for example, Morris Kline (1990, p65), and corresponds literally to the text in Euclid, which is,

> "II,5. If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half." (van der Waerden, 1983, p78)

Van der Waerden himself, when trying to fit the algebraic notation to the propositions, notices that Prop. V and Prop. VI correspond exactly to the same algebraic identity (v.d. Waerden, 1983, p78ff), and says that "This shows that there is something wrong [in the way he is trying to make sense of the propositions].", but does not consider the possibility of the "geometric algebra" not being a sensible interpretation. An adequate way of understanding the essential difference between the two propositions is this: in the figure bellow (Prop. V), we represent proposition V, with XY corresponding to the "whole", Z the middle point of XY, and T the point of the "unequal section." It is immediately clear that Prop. V can be "translated" into $(2a+b)b+a^2=(a+b)^2$, which is exactly the translation of Prop. VI. However, and this is *the* crucial difference between the two propositions, in Prop. VI only a rectangle (corresponding, for example, to rectangle X'Y in the diagram for Prop. V) is required to be "moved," while in Prop. VI, the rectangle *and* one of the squares are required to be "moved" (altogether, rectangle XW, which has to be proven identical with rectangle X'W'). The two propositions are *geometrically* different, and the inclusion of both offers strong support to the view that the *intended objects* are in fact geometric ones, making the hypothesis of a geometric algebra untenable.



Prop. V, Book II (the *Elements*)

Van der Waerden believes that the "geometric algebra" of the Greeks actually intended numbers—rational and irrational—but represented them with lines and areas. As we have shown above, this cannot be the case. On the one hand, the Greek distinction between *number* and geometric magnitudes is sharp; on the other hand, *number* is always a whole number, never an irrational magnitude. Had Euclid simply used the geometric representation to avoid the problem of incommensurability, he would have certainly considered that whole numbers and fractions were particular cases which were "included" in the general treatment using geometric magnitudes, and a substantially self-contained treatment of *number*, as we have in the arithmetical Books of the *Elements*, would not have been necessary[35]. Szabó (quoted in Berggren, 1984, p397) says that the term *Geometric algebra* should be replaced by *Geometry of Areas*, "...in order to emphasize that the theorems are geometric theorems, used to prove other theorems in geometry, and that there is no concrete evidence that pre-Euclidean Greeks took over Babylonian algebra and recast it in geometric form." Mueller (1981, p44) considers a geometric interpretation "...sufficiently plausible to render the importation of algebraic ideas unnecessary."

Of great importance to our understanding of *algebraic thinking*, the "translation" into algebraic notation that van der Waerden considers harmless (and that many times is assumed as the only difference between "the problem" and "the algebraic expression") creates a situation where the intended objects are replaced without this fact becoming apparent: the *arithmetisation* of Greek geometry it produces could never be accepted—and, thus, understood—by the very men who produced it, as much as Euclid would certainly dismiss—probably as ignorant, possibly as mad—anyone that proposed him

[35]Mueller (1981, p107): "It is striking...that although Euclid's arithmetic thought is often governed by geometric analogies, nothing in books VII-IX which has been discussed involves an actual transference of a geometric truth into arithmetic. In particular, although such notions as those of plane and square numbers seem to invite the use of geometric algebra, we have seen no cases in which it has been used." M. Kline (1990, p77) also observes that many of the propositions of the *arithmetical* Books are "proved again," when they could be referred to propositions already proven in Book V.

to consider non-Euclidean geometries. The "translation" of the propositions in Book II of the *Elements* hides the true geometric nature of the objects intended.[36] By studying the debate about the notion of a Greek "geometric algebra," we have become more able to understand the process by which a conceptualisation—and, thus, an *intention*—is imposed on the reading of mathematical production or knowledge, in this case, the imposition of a much more general framework, leading to the introduction—in the conclusions by the one imposing his or her views (the *impose-tor*), **but hidden from his or her eyes**—of improper elements through an improper interpretation. In the study of history, this leads at least to superficiality, and at the worst to paradoxes, but in the case of mathematical education it easily leads to misguided didactic efforts.

A second point that emerges from investigating the adequacy or not of the "geometric algebra" interpretation, is that, as we saw with propositions V and VI, if the *intended objects* are geometric ones—even when they are being used to represent numbers—the geometric configuration in which they are displayed, and the manipulation of that configuration which takes us to a solution of the problem, play a central role in the solution process; properties of the geometric *objects* will be guiding the solution process. We exemplify. If a square is drawn and lines used to cut the square in four parts as to illustrate the equality $(a+b)^2 = a^2+2ab+b^2$, the insight is easily achieved, and the proposition means simply that the square "on the left" can be decomposed into the pieces represented "on the right". If, however, the proposition is looked at "backwards," ie, as representing $a^2+2ab+b^2 = (a+b)^2$, a number of difficulties arise; with a substantial amount of goodwill (or mathematical enculturation), one will agree that the proposition is saying that the pieces "on the left" can be assembled to produce the square "on the right". But the pieces "on the left" could be in any of many different configurations—they could even be scattered; from a geometric point of view, the problem is ill-formulated. Only when a precise configuration is required to be shown transformable into the square "on the right" is the problem clear, and that is exactly why Euclid proves "twice" the "algebraic" proposition, in Prop. II and Prop. VI.

The third point we want to make here, is in connection with Klein's strong argument against the "geometric algebra":

36When we look at students in our experimental study that can do "pure calculations" with negative numbers, and also to solve successfully the equation $100\text{-}3x=10$ by doing: "100-10=90; 90/3=30," but fail to solve the equation $100\text{-}3x=190$, we are led to think that the *intended objects* of the first process were not ***numbers***, but possibly the elements in a whole-part relationship.

"This interpretation can arise only on the basis of an insufficient distinction between the *generality of the method* and the *generality of the object of investigation*. Thus Zeuthen...immediately relates his concept of 'geometric algebra' to that of 'general magnitude.'...[A]ncient mathematics is characterised precisely by a tension between method and object. The objects in question (geometric figures and curves, their relations, proportions of commensurable and incommensurable geometric magnitudes, numbers, ratios) give the inquiry its direction, for they are both its point of departure and its end...The problem of the 'general' applicability of a method is therefore for the ancients the problem of the 'generality'...of the mathematical objects themselves, and this problem *they* can solve only on the basis of an ontology of mathematical objects." (1968, p122)

He directly points out to a necessary distinction between *method* and *object* in Greek mathematics. *Objects* in Greek mathematics are, as Klein lists, lines, figures, numbers, ratios. One is always speaking of the *objects that are "in fact" manipulated*.[37] To say, as Klein does, that, "The problem of the 'general' applicability of a method is...for the ancients the problem of the 'generality'...of the mathematical objects themselves," is to say that the nature, the constitution, of the objects determine in which ways they can be manipulated, and, thus, what can be done to solve problems or prove theorems *about* them—never *using* them. As we will see, a central aspect of the "symbolic invention" of Vieta, is that the focus of attention is explicitly directed to the method. The predominance of object over method in Greek mathematics, precludes operations from becoming *objects*; once they are understood only as natural possibilities derived from the ontological nature of the *objects proper*, studying them is equivalent to studying the *objects proper*. Allowing the operations to have an independent existence is not possible in Greek mathematics, precisely because there would be no insight into the *objects* on which they operate, and, again, were this insight produced, it could only come from examining the *objects proper* directly, and the independence of the operations would rest annihilated.

In many cases, the students in our experimental study behaved very much in that way: the "operations" which they use to manipulate the *objects* present in the model, ie, to solve the problems, are directly dependent of or derived from properties that those *objects* are perceived as having. For instance, if two parts make up a whole and one of them is removed, we are left with the other part; the "removal" is possible

[37]The inclusion, in Klein's list, of *relations*, must be understood as meaning a specific geometric configuration, or the relation between two consecutive triangular numbers and a square number, and *not* as it might mean, for example, the equality relation, which is only a tool (as in the Common Notions in the *Elements*) and never the object of study.

precisely because of the whole being conceived as composed by its parts[38], and the mentioned property of "removal" is a consequence of that and of the non-overlapping of the parts, ie, it is reduced to properties of a whole and its parts, rather than irreducibly belonging to the "removal" itself.

The Arithmetical Books

Books VII, VIII, and IX of the *Elements* are known as the *arithmetical* books, in which we find 102 propositions about whole numbers and ratios of whole numbers, most of them dealing with properties of divisibility and proportion.

As we have said before, the arithmetical books are mostly self-contained (Mueller, 1981, p58). As Mueller also observes, given the independence of those books, one would expect to find in them specific postulates for *arithmetic*, but what we find, instead, are 23 definitions, in which *number*, prime and composite *numbers*, etc. are defined. Definitions 3 and 4 deal with the notions of *part* and *parts*:

> "(3) A number is part of a number, the less of the greater, when it measures the greater.
> (4) But parts when it does not measure." (Mueller, 1981, p337)

The interesting point about those definitions, is that they reproduce in a very natural way—on the basis of the notion of a *number* "measuring" another—the notion of part; if the whole number b is a divisor of the whole number a, then there is a whole number c such that $a=bc$. If b is taken as the divisor (as we would tend to do when we say that "*b divides a*"), it means that c goes into a, b times; but we can also understand, in a more direct way, that it is b that goes into a an exact number of times, in which case a can be decomposed into exactly c parts, each of "size" b. Euclid's definition is elegant, in that it does not deal with "how many parts," but only with the fact that b is naturally a part of a. Definition 4 says *parts*, on the other hand, because: (i) the greater and the lesser number being whole numbers, there is always a common measure (in the worst case the unit); (ii) this common measure is a *part* of the greater number; (iii) it is also a *part* of the lesser number, which can be said to be composed by a *number* of them. So, in the lesser number, we have (a *number* of) *part*'s of the greater number.

Those definitions establish the character of the use of lines as a notational form in the context of the *arithmetical* Books: not as continuous lines, but as objects

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[38]This remark may seem somewhat circular, but it is not. The notion of a whole and its parts is independent of whatever one wishes to do or does with them. "The whole is greater than the part" is in fact the only Common Notion stated by Euclid in relation to the whole and its parts.

measurable by a unit (Klein, 1968, p11). Moreover, considering the question of "how many times one into another," ie, considering the c in $a=bc$, is not possible in Euclid, as it would impply the acceptance of fractional *numbers*.

Only one operation is defined, multiplication (Definition 16), in which definition addition is taken for granted[39]. We suggest an interpretation for the adoption of that definition which is compatible with the Greek commitment to an ontology of the mathematical objects. In Euclid, adding is seen as concatenation (Mueller, 1981, p70), and the nature of the object produced by addition is obviously the same as that of the numbers being added, as in fact the parts added are both contained in the result; with the multiplication of numbers, however, a definition is required exactly to guarantee that the result is still a *number*[40]; a further requirement is that the commutativity of multiplication be proved, as it is not "obvious" as in the case of the "geometric multiplication" of lines—and which Euclid takes for granted—and this is done in Prop. 16 of Book VII. It is now possible to represent *numbers* always by lines, and the expressions plane, square, solid, and cube *numbers* refer only to their composition in terms of factors, and not to a geometric nature: literally read, Euclid adds "plane" (*number*) with a *number* that is one of its "sides," a procedure unthinkable in relation to true geometric *objects*.

Of interest to us, is the way in which the representation by lines is used in the *arithmetical* Books. In those Books, the lines representing *numbers* are never used geometrically in the sense of, for example, Book II, although both multiplication and proportion could be dealt with by using "true" geometric constructions—as Thales' theorem, or some of Euclid's own constructions—and thus avoiding the problem of representing multiplication by the construction of a rectangle, which would limit the number of factors to three (unacceptable, for example, in Euclid's demonstration of the infinitude of the set of prime numbers). Instead, the lines in the *arithmetical* Books are used either as a mnemonic device (to indicate, for example, the order of the sizes of different numbers involved, as in Prop. VII,14, or to indicate that the sum of certain numbers produce another one, as in Prop. VII, 22), or to support a combinatorial argument.

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[39]Mueller (p59ff) observes that in Euclid's definition of multiplication, *number* is used both as a *number proper* and as a "metalanguage variable or subscript," and that "Such usage is impossible within first order logic but not in an extension to higher order logic...[which] incorporates within itself all of elementary arithmetic." We think that this double usage is a natural consequence of the nature of the Greek *number*, which is inextricably associating with counting. As we have mentioned before, the operations are subordinated to the *objects proper*, and addition here is no exception; in the framework of Greek mathematics, the formal distinction between the two usages (*number proper* and "subscript") is not required.

[40]The importance of this step can be better appreciated if we consider that the "multiplication" of lines produce a rectangle, and not another line.

We would like to remark, at this point, that in the use of geometric objects to deal with problems requiring the determination of a number, our students' methods resemble much more the "geometric analogies" used in the *arithmetical* Books, than a fully fledged "geometric algebra"—the latter taken in the sense intended by van der Waerden. In fact, whenever multiplication is represented in that way, the multiplier is only understood as Euclid's multiplier, ie, as the number of times a line is being added[41]. If in Euclid the definition of multiplication is natural in the context of the Greek *number*, as he is dealing only with whole numbers, in our students, who are sufficiently acquainted with the multiplication of decimal numbers, this behaviour must represent a restriction imposed by the model being used, an aspect that is examined in detail in the chapter in the Experimental Study.

Proportion appears in Definition 21, and it does not involve multiplication:

> "(21) Numbers are proportional when the first is an equal multiple of the second and the third of the fourth, or they are the same part or parts." (Mueller, 1981, p338)

In view of our interpretation of *part* and *parts*, this definition should be understood as follows. In the case of equal multiples and equal *part*, it simply states that the lesser *numbers* determine the same number of parts in the corresponding greater *numbers*. In the case of equal *parts*, it would say that the number of *parts* of the greater *numbers* to be found in the corresponding lesser *numbers* are equal; but if we remember that in the notion of *parts* any measure common to both numbers will do, a serious problem arise, because unless the common measure is in each case the greatest possible, the number of them in each of the lesser numbers would always have to be compared with reference to the total number of common measures in the corresponding greater numbers, and we would return to the original problem[42]. It is probably for this reason that Prop. 2 of Book VII is,

> "2. Given two numbers not prime to one another, to find their greatest common measure." (Mueller, 1981, p339)

and Prop. 1 is precisely a preliminary step for the Euclidean algorithm for determining the GCD of tho whole numbers:

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[41]In the Ticket & Driving group of problems, for example, it will be seen that this restriction is responsible for difficulties when the multiplier is not a whole number.
[42]Using the greatest common measure corresponds to taking both ratios in its least terms, in which case the proportionality is reduced to an *identity* of ratios.

> "1. If two unequal numbers are set out and the lesser is always subtracted in turn from the greater...then, if the remainder never measures the number before it until a unit is left, the original numbers will be prime to one another." (ibid.)

The importance of understanding in some detail the *arithmetic* definition of proportion (VII,21) is to enable us to compare it to the *geometric* definition of proportion—or rather, of equality of ratios—that is given in V,5. The *geometric* definition of proportion is:

> "Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples of the second and fourth, the former equimultiples alike exceed, are alike equal, or alike fall short of, the latter equimultiples taken in corresponding order." (Heath, 1981, vol 1, p385)[43]

Euclid's Book V contains in fact Eudoxus' theory of proportion, and as we have seen before, no magnitudes or ratios were expressed by *numbers*. Bearing this in mind, we might represent Definition V,5 as:

$$a:b :: c:d \iff \forall\, m,n \in \mathbb{N} \text{ then, } \quad \begin{array}{l} m \cdot a > n \cdot b \iff m \cdot c > n \cdot d \\ m \cdot a = n \cdot b \iff m \cdot c = n \cdot d \\ m \cdot a < n \cdot b \iff m \cdot c < n \cdot d \end{array}$$

An essential difference between VII,21 and V,5 is this: because in the case of incommensurable magnitudes the notions of *part* and *parts* do not apply, Eudoxus is forced to define his "general" proportion in terms of a criteria that cannot be finitely verified, as opposed to the *arithmetic* one, which immediately allows the development of an algorithm by which equality of ratios of whole numbers can be verified. The difficulty here involves the essential difference between the continuity of geometric magnitudes and the discreteness of *number*[44]. In the context of Greek mathematics a (general) theory of proportions cannot be developed on the basis of the equivalence,

$$a:b :: c:d \iff ad = bc$$

precisely because with the difficulties with the definition of multiplication[45].As we have pointed out, the general applicability of the method depends, in the conceptual

[43]We preferred to use Heath's version of the text, which in this case is clearer than Mueller's whose text we have followed until here.

[44]Mentioning Proclus, Heath (1981, vol 1, p90) observes that "...*irrational straight lines* [is a topic in geometry matters] (for where there is division *ad infinitum*, there is also the irrational)."

[45]In our "translation" of V,5, it must be clear that the "multiplication" means only that the geometric magnitude is to be taken that number of times.

framework of Greek mathematics, on the generality of the object, and the development of a theory based on multiplication could not be generally applicable, as we would need distinct definitions of multiplication for different objects[46]. The theory of proportion in Book V of the *Elements*, achieves its generality—in the sense of a theory generally applicable to all geometric magnitudes—by dealing only with ratios as *objects proper*[47].

In relation to our research problem, that of characterising *algebraic thinking* and understanding how different conceptualisations of number and of mathematics can promote or hinder its development, the comparison of the two definitions of proportion throws light into important aspects.

First, the non-homogeneity of the realm of geometric magnitudes presents a problem for the development of an algebraic mode of thinking; the use of a geometric model to produce algebra, be it in the form of a "geometric algebra" supported by Book II of the *Elements*, or in analogies like the use of a diagram to "prove" the "formula" for the square of the sum of two terms, will only introduce or reinforce the non-homogeneity. Euclid's solution in the *arithmetic* Books, ie, to force a definition of multiplication that directly produces the sought homogeneity is adequate in this aspect. The modern notion of operation addresses the difficulty correctly.

Second, a model for numbers based on properties of whole numbers present difficulties beyond the obvious inadequacy of Euclid's definition of multiplication for a multiplier that is not a whole number. In themselves, notions as those of *part* and *parts* suggest the "counting" role of a multiplier; moreover, the notions of addition and subtraction naturally remain too tightly linked to that of *counting*, posing an obstacle, for example, to the acceptance of negative numbers or to the acceptance of 6+7 as an expression in its own right.

Third, any ontology of irrational numbers derived from or based in rational numbers will inevitably have to involve either a potential—as in the Intuitionistic version—or an actual—as in the Formalist version—notion of infinity. Nevertheless, and this is a key distinction, *algebraic thinking* is only concerned with the way in which the operations defined on those elements work, their properties, and not an ontology of the elements on which it operates.

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[46]We remind the reader, if at all necessary, that before the theory itself is established it is not possible to define a general multiplication in terms of, for example, Thales' theorem. The product of two lines can be defined as a rectangle, but the problem with the multiplication of two rectangles, for example, is unsolvable in Euclid's geometry.
[47]It must not be understood, however, that those ratios are "abstract" and generally applicable: they are always ratios of geometric magnitudes, and never of *numbers*. See Unguru (1979, p559ff)

Diophantus lived in Alexandria, and his main work, the *Arithmetica*, is dated by historians as being produced about 250 AD. Diophantus' other works include *On Polygonal Numbers*, of which only fragments survived, and the *Porisms*, a collection of propositions from which existence we know only through its mention in three propositions of the *Arithmetica*. The *Arithmetica* was originally composed, according to a remark by Diophantus in its text, by thirteen volumes, but until recently only six of those had been recovered[48]. In 1976 Jacques Sesiano completed the translation of another four Books, which were translated from Arabic manuscripts; his translation was published in the book form, which is Sesiano (1982).

It is almost unnecessary to point out the importance of Diophantus' in the history of mathematics. That his name is attached to Diophantine Analysis, and that Vieta' *Analytical Art* was inspired by the *Arithmetica* seem to be sufficient indication.

From the point of view of our research, however, there are specific reasons for investigating in some details aspects of the *Arithmetica*. First, Diophantus is a Greek, but his work departs in many aspects from the previous Greek mathematics; as Morris Kline (1990, p143) observes, "...we cannot find traces of Diophantus' work in his predecessors." We will examine his work in order to identify the conceptual framework that makes it "possible" in comparison with the previously existing Greek mathematics. Second, the *Arithmetica* undoubtedly involves algebra, and we shall investigate what form algebra and *algebraic thinking* took in Diophantus, *particularly against the background of Greek mathematics*.

We begin by briefly comparing the *arithmetical* Books of the *Elements* with the *Arithmetica* of Diophantus. In the *arithmetic* Books we have a study of the properties of whole numbers and of proportions involving whole numbers, whereas in the *Arithmetica* we have a collection of problems solved with the aid of equations. The former is systematic, the latter only insofar as to "...arranging the mass of material at his disposal...[in order to] make the beginner's course easier and to fix what he learns in his memory." (Heath, 1964, p131). Euclid, as we saw, represents numbers by lines, Diophantus uses an "arithmetical" notation, which we will examine further ahead. Finally, the numbers in the *arithmetic* Books are never specific, while in the *Arithmetica* they are all—including the "unknown" ones—specific.

In view of all that, both works would seem to have no connection possible, but this is not the case. In both of them, *number* is the Aristotelian *number*, ie, "a

[48]For a thorough examination of the history of the manuscripts and translations of the *Arithmetica* up to 1910, the reader is referred to pages 14-31 of T.H. Heath's edition of the *Arithmetica* (1964).

multitude measured by a unit."[49] As we saw, this is what allows Euclid to represent *numbers* by lines—which are not made into continuous magnitudes because of it—and it is also what allows Diophantus to speak of "fractions," as a *number* of "fractional parts."

The question of why Diophantus' does not solve his problems for "generic" numbers, *although he always proposes them in "generic" terms*, is a most important, and at the same time, a difficult one to answer. One possibility is that the notation available to Diophantus prevented him of doing so, but in view of Euclid's use—five and a half centuries before Diophantus—of lines to represent *numbers*, and of associating letters to the lines so he could easily refer to them in the text, it would be puzzling that Diophantus, whom almost certainly knew the *Elements*, had not borrowed the notation for the *Arithmetica* had he intended the "generality" of the numbers involved in the problem in the sense of our "general" coefficients of equations. Only to put the problem in a more complex, but certainly more interesting perspective, Diopahntus' *did* use, in his *On Polygonal Numbers* the same type of line-and-letter notation employed in the *arithmetic* Books (see, Heath, 1964, p247ff). The subject of *On Polygonal Numbers* being obvious, we are left to say that it is collection of propositions, all proved in all the possible generality, ie, no particular cases are taken to be solved as paradigmatic, and it proceeds *synthetically*[50].

We must emphasise that the question of "generality" in Diophantus is not one of historical interest only; a number of issues in the learning of algebra have been related to it, as we saw in the review of previous research on the subject, and as we will show, precious insights can be gained in the process of clarifying and finally answering the question.

We shall now examine Diophantus' notational system.

Specific numbers in Diophantus are written using the Greek alphabetic notation for numbers, which is described in detail by Heath (1981, vol 1, p36ff). In this system we would have, for example, $\bar{\rho}$ representing 100, $\bar{\kappa}$ representing 20, and $\overline{\sigma\eta}$ representing 208; the stroke on the top of the letters was one of the forms used to distinguished them from verbal text. For the unknown, Diophantus used the final ς[51], and for the "powers of the unknown" he used: Δ^Y for the square, K^Y for the cube, $\Delta^Y\Delta$ for the fourth power (square-square), ΔK^Y for the fifth power, and K^YK for

[49]On the Peripatetic character of Diophantus' work, see also pages 112, 113, 133 and 143 of Klein (1968).
[50]The first proposition of the *On Polygonal Numbers* in Heath's version is, "If there are three numbers with a common difference, then 8 times the product of the greatest and the middle + the square of the least = a square, the side of which is the sum of the greatest and twice the middle number."
[51]See Heath (1964, p32-38) for a thorough discussion on the origins of the symbol.

the sixth power. The word used for square, in Greek, was δυναμις, which means "power," and whose first two letters capitalised become ΔΥ; the same happens with κυβος, "cube." We see that Diophantus in fact created, from the limited stock at his disposition, new symbols; the argument raised that he did not solve the problems in their "generic" form because no letters were available is, thus, awkward, once he could have easily made clear the fact that he would use small letters for numbers, with the stroke, and capital letters for "generic" coefficients.

Other difficulty with his notational system would be the lack of a symbol for a second, third, etc., unknown. This could be solved, for example, by adding dots on the top of the ς, with $\ddot{ς}$, for example, being used for a second "unknown," etc. The problem would arise with the representations of the powers, but a solution is not difficult to be worked out. Curiously, we find in Heath himself, one of the proposers of the "lack of letters" theory, that,

> "Again we find two cases, II,28 and 29, where for the proper working out of the problem two unknowns are imperatively necessary. We should of course use x and y; but Diophantus calls the first ς as usual; the second, for want of a term, he agrees to call '*one unit*,' ie, 1. Then later having completed the part of the solution necessary to find ς, he substitute its value, and uses ς over again to denote what he had originally called '1'—the second variable—and so finds it. This is the most curious case of all, and the way in which Diophantus, after having worked with this '1' along with other numerals, is yet able to put his finger upon the particular places where it has passed to, so as to substitute ς for it, is very remarkable. This could only be possible in particular cases such as those which I have mentioned; but even here, it seems scarcely possible now to work out the problem using x and 1 for the variables as originally taken by Diophantus without falling into confusion. Perhaps, however, in working out the problems before writing them down as we have them Diophantus may have given the '1' which stood for the [second] variable some mark by which he could recognise it and distinguish it from other numbers." (Heath, 1964, p52)

The idea of using numeral-letters plus a special sign to distinguish them as a symbol for an "unknown," which would not be operated with the normal numbers could also have been considered. So, we have to look bellow the surface of the problem.

A few paragraphs above, we enclosed *powers of the unknown* in quotes for a very specific reason. Given our modern conceptualisation of algebra, it is only natural to expect the "unknown" to be defined first, and only then "the powers of the unknown," but this is not the case in Diophantus. *First*, he defines *number* (which are all "...made up of some multitude of units...") and the five *eide* which we have termed "powers of the unknown," *and only then he introduces the notion of the "unknown" and a symbol for it.* It is truly amazing that of all the books we have consulted on Diophantus (Heath, 1964 and 1981; M. Kline, 1990; Klein, 1968; Lintz, undated manuscript; van der Waerden, 1983), only Klein's book takes notice of this fact. This "inversion" is crucial in determining the character of Diophantus' algebra, and we must examine it[52].

First, it is necessary to remember that in the Aristotelian framework for *number* to which the *Arithmetica* belongs, a *number* is always determinate or intended to be determined. With this in mind, we understand that the "unknown" *number* in Diophantus can only be *as yet* indeterminate, or, as Klein puts it, "provisionally indeterminate," and not "potentially determinate only." (p140) After defining the *eide*, Diophantus' says that, "It is from the addition, subtraction or multiplication of these numbers or from the ratios which they bear to one another or to their sides respectively that most arithmetical problems are formed...[and] each of these numbers...is recognised as an element in arithmetical inquiry." (Heath, 1964, p130)[53] This is the firm foundation which allows the notion of arithmetic problem to be formed, and it is this, the problem, that constitutes the "eidos"—to use a very stretched, but illuminating, metaphor—of the "unknown": "...as the concept of [indeterminate number] becomes fully understandable only on the basis of *figures* 'similar' to one another (ie, given only in shape and not determinate in size), so also is the unknown to be understood...from the point of view of the completed solution...and as a number *which is about to be exactly determined in its multitude...*" (Klein, p140), and, we should emphasise, a number that rests characterised by the conditions of the problem.

We are now in a position from which we can elucidate why Diophantus does not *solve* the problems in their "generic" form, *although he proposes them so.* In the Diophantine framework, *to solve a problem can only mean to exhibit in full the number or numbers that satisfy a given, definite problem.* Unless the problem is given in definite terms, the "eidos" of the "unknown"—ie, the equation—is not established, and the "unknown" itself cannot make sense. To do as we would today, ie, to exhibit the *potential only* solubility of a problem by using an algebraic expression such as

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[52]If only for it pointing out the inversion, we would already be greatly indebted for Klein's work. However, he also sets with his overall analysis, the only context in which the problem could be solved. I cannot think of a finer piece of historical analysis in all the very many texts I have consulted during the research for preparing this text, and I am only obliged, and delighted, to follow closely his line of reasoning in this part of my exposition.

[53]It is worth noticing that, naturally, each of those *eide* have its *side*, which is not, however, its reason of being nor its "origin," as we would understand nowadays.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

is precisely a *non-solution* in the framework imposed by the ontological presuppositions of the *Arithmetica*. As we have conclusively shown, a notation for generic coefficients was certainly *possible* from the notational point of view only, *but we now see that it was also meaningless in the context of* ***solving arithmetical problems***. Neither our "extension" of Diophantus' notation nor Euclid's lines and letters notation had a reason to be in the *Arithmetica*. Euclid can use it in the *arithmetic* Books because he is not solving problems, he is proving theorems; his procedure is totally *synthetic*, which means that all numbers are definite numbers. Diophantus' procedure, however, is *analytic*, and as each element in the *presuppositions* that form the equations has to be determined either in its manyness or in its form, the requirement of a determinate *eidos* is imperative for a *number* that is not known in its multitude.

The other difficulty to be explained, that of using only one symbol for "unknown," can be elucidated in similar lines.

We chose the detour of first trying to offer a "surface" solution for the questions on Diophantus' notational system in order to create a true question about the generality or not of his solutions, one that was to be answered by our analysis. We can now safely say that his solutions were truly general, but not in the sense conveyed by expressing a general solution in algebraic notation. The detour, moreover, highlights the key role of conceptualisation in the understanding of mathematical knowledge, a crucial point in our overall argument.

After introducing the definition of number, the *eide*, and the "unknown," Diophantus introduces a sign, $\overset{o}{\mu}$, "...denoting that which is invariable in determinate numbers, namely the unit..." (Diophantus, in Heath 1964, p130), and the notation for the reciprocals of the *eide*, which uses a sign that we will, for the lack of a better typographical sign, represent by ^{x}. For example, $\Delta^{Y}\Delta^{x}$ meant the reciprocal of $\Delta^{Y}\Delta$, and Δ^{Yx} the reciprocal of $\Delta^{Y}\Delta$, etc..

Diophantus uses no special sign for addition; the "forthcoming" terms—the terms being added—are simply juxtaposed. For the "wanting" terms—the terms being subtracted—he uses a specially created sign, a monogram: ⋔[54]. Expressions in

[54]In explaining the process—as he sees it—by which Diophantus generated his sign for *wanting*, Heath says that the use of the initial Λ in λειψις (or the inflected form λιπ) would not be

Diophantus are typically composed of two blocks, the "forthcoming" and the "wanting", which are characteristics of the expression and not of the numbers involved, and any association of the "wantness" with negative numbers can make no sense in that context. That Diophantus had a rule for multiplying expressions involving "wanting," is well known; the rules are justified in a combinatorial way, very similar to the inclusion-exclusion principle (see, for example, Anderson, 1989, p67). A sign for multiplication is not used, because, as Heath (1964, p39) indicates, "...it is rendered unnecessary by the fact that his coefficients are all definite numbers or fractions, and the results are simply put down without any preliminary step which would call for the use of a symbol." For our "=" Diophantus had the sign ι^{σ}, an abbreviation of ισος, equal.

Further discussion of Diophantus' notational system is irrelevant to our purposes, but we think it is worth "tasting" Diophantus' notation "in action," so we examine a sample solution using it. In the original form, the equations were written into the course of the speech, ie, they were not displayed each step on a separate line. The example bellow is extracted from Heath (1964, p48), and the arrangement in lines is credited to Maximus Planudes (about 1260-1310 AD); we added the algebraic form, in brackets, to make the comparison of the two systems easier[55]. The problem is Diophantus' I,28, "To find two numbers such that their sum and the sum of their squares are given numbers." Notice how Diophantus actually solves the problem of finding half the difference between the two numbers. (figure follows on next page)

acceptable, as it already denoted a number, and "Therefore an addition is necessary," the adopted one being a monogram for ΛΙ.

[55]We believe that this illustration should be enough to convince the reader that getting used to Diophantus' notation is not a difficult task.

[given numbers:]	$\overset{o}{\mu}\,\bar{\kappa}$		$\overset{o}{\mu}\,\overline{\sigma\eta}$
	(20, the sum)		(208, the sum of sq's)
setting out:	$\varsigma\bar{\alpha}\ \overset{o}{\mu}\,\bar{\iota}$		$\overset{o}{\mu}\,\bar{\iota}\ \pitchfork\ \varsigma\bar{\alpha}$
	(x + 10)		(10 — x)
squaring:	$\Delta^{Y}\bar{\alpha}\,\varsigma\varsigma\bar{\kappa}\,\overset{o}{\mu}\,\bar{\rho}$		$\Delta^{Y}\bar{\alpha}\,\overset{o}{\mu}\,\bar{\rho}\ \pitchfork\ \varsigma\varsigma\bar{\kappa}$
	(x^2 + 20x + 100)		(x^2 +100 — 20x)
adding:	$\Delta^{Y}\bar{\beta}\ \overset{o}{\mu}\,\bar{\sigma}$	ι^{σ}	$\overset{o}{\mu}\,\overline{\sigma\eta}$
	(2x^2 + 200)	(=)	(208)
subtracting:	$\Delta^{Y}\bar{\beta}$	ι^{σ}	$\overset{o}{\mu}\,\bar{\eta}$
	(2x^2)	(=)	(8)
dividing:	$\Delta^{Y}\bar{\alpha}$	ι^{σ}	$\overset{o}{\mu}\,\bar{\delta}$
	([1]x^2)	(=)	(4)
	$\varsigma\,\bar{\alpha}$	ι^{σ}	$\overset{o}{\mu}\,\bar{\beta}$
	([1]x)	(=)	(2)
result:	$\overset{o}{\mu}\,\overline{\iota\beta}$		$\overset{o}{\mu}\,\bar{\eta}$
	(12)		(8)

Solution of a problem using Diophantus' notation.

The *eide* are never used on their own, not even when there is only "one square," as in the line *squaring*, or "one unknown," as in the line *setting out*[56], indicating that the *eide* are *denominations* rather then *numbers proper*. It is also

[56]Actually, in Heath's book one does find in the line *squaring*, on the right-hand side, $\Delta^{Y}\overset{o}{\mu}\,\bar{\rho}\ \pitchfork\ \varsigma\varsigma\bar{\kappa}$, which can only be a misprint, as we were not able to find such usage in any other book where Diophantus' notation is discussed.

interesting that the ς is declined—for example in the line *squaring*, where we find ςςκ̄ , a practice that would produce, in our modern notation, something like $20x$'s !

We shall now summarise and add some conclusions to our analysis of Diophantus and the *Arithmetica.*

Undoubtedly, the *Arithmetica* of Diophantus has many points of contact with that which we came to call algebra. In this sub-section, we set out to investigate what form algebra took in the *Arithmetica*, in particular, in what sense and to what extent it could be said to deal with each problem proposed "in generality." The removed paradigm of algebra, against which Diophantus' is to be examined is our literal or symbolic calculus, and the question naturally arises, in view of the lack of such calculus in Diophantus, "...whether Diophantine logistic may not contain within itself the possibility of a symbolic calculating technique." (Klein, 1968, p139). Klein says, moreover, that,

> "Since Vieta this question has been...answered positively...by those who see the Diophantine science merely as the primitive 'preliminary stage' of modern algebra. From the point of view of modern algebra only a single additional step seems necessary to perfect Diophantine logistic: the thoroughgoing substitution of 'general' numerical expressions for the 'determinate numbers,' of symbolic for numerical values." (ibid)

Through our study of Diophantus' work, we were led to conclude that such a substitution is simply not possible in the *Arithmetica*, not for circumstantial reasons such as a "lack of letters," nor, it goes without saying, for a supposition of Diophantus' intellectual limitations[57]. Instead, it is the very possibility *and intention* of his *episteme*, to show how, in each specifically given case, the problem can be solved. In the *Arithmetica*, to solve a problem is to show actual numbers that satisfy the given conditions, not just to assert the possibility of determining them, and this as a consequence of Diophantus' conceptualisation of *number* and of his *theoretical logistic*, which by virtue of the Aristotelian conception of *number*, can now be named also as *arithmetic*. A deep aspect of this knowledge is that the *eidos* to which the "unknown" belongs, its *species*, that without which the "unknown in multitude" is even unthinkable, is exactly the problem, or, more exactly, the relationships given in the problem, which when presupposed in the process of *analysis* blur the distinction

[57]What I have in mind here, is the surrealistic phrase "Diophantus had not reached the intellectual stage of formal operations," which although never uttered in my presence, I sometimes believe to have seen its ghost.

between known and unknown, and through which the problem is finally solved: the equations.

But we can now ask about someone involved in learning algebra—"our" algebra—the same question Klein asks about history, thus construed: "Does the learning of techniques to solve equations in x and possibly y, with specific numbers as coefficients, contain *in itself* the possibility of a symbolic calculating technique?" The case of Diophantus has certainly provided us with richly suggestive insights as to how approach this question.

CONCLUSIONS

The richness of the insights both into *algebraic thinking* and into a methodology for the research in the history of mathematics produced in this section, fully vindicates, we think, our choice of Greek mathematics as the first historical period to be presented.

From the methodological point of view, Klein's approach to the history of mathematics must have been felt throughout this section, by anyone who read his book on the origins of algebra. The benefits of studying the history of mathematics from the point of view of the conceptual framework of those who produced it are immense, and they range from the possibility of understanding ways of doing mathematics that otherwise remain obscure or paradoxical—as the lack of "generic" coefficients in Diophantus' solutions—to understanding how a conceptualisation of mathematics and mathematical objects interacts with the production of mathematical knowledge. More important, however, in relation to our research, this approach actually provides us with specific instances of this interaction, and those specific instances form, in turn, a rich model for understanding processes involved in the acquisition of *algebraic thinking* by individuals.

From the point of view of *algebraic thinking*, then, our study of aspects of Greek mathematics showed that:

(i) The knowledge of a *calculating practice* with numbers, in which different types of numbers are dealt with, does not imply *per se* the possibility of establishing a theoretical study of it, and it is only through the transformation of *tool-operations* into *object-operations* that *algebraic thinking* becomes possible.

(ii) There is a tension—potentially difficult to overcome—between an ontological understanding of number and the transformation of arithmetical operations into *objects*; one way of overcoming this tension is by *collapsing*[58] ontologically defined numbers into "dimensionless" elements, which become simply "the elements on which the operations operate." In order to do this and still retain the possibility of investigating propositions involving those elements, meaning is shifted to the operations, ie, they become *objects*, although having been conceived as more or less natural consequences of an ontology. The problem with this approach is that the stricter the ontological commitment is, the greater the difficulty of introducing new elements—numbers—that are consistent with the operations but not with the ontology of the "primitive" elements.

(iii) Arithmetic operations are *homogeneous*, ie, if a and b are numbers, and $\oplus$ is an arithmetical operation, then $a \oplus b$ is, whenever defined[59], also a number. This clearly distinguishes the arithmetical treatment of numbers from, for example, a geometric treatment in which the multiplication of two lines is a rectangle, which cannot be directly added to another line. If the elements of an operation are *collapsed*, "dimensionless" elements, as in (ii), it means that they are not distinguished from one another by a possible ontology, and the operation is *homogeneous*. The *arithmeticity* of *algebraic thinking*, in our theoretical model, asserts the *homogeneity* of the operations which become *objects* of in *algebraic thinking*.

(iv) *Internalism*, in our theoretical model, means disregarding any ontology of the elements of the operation. As we saw in (ii), this abandonment may be provisional only, as the degree of autonomy given to the operations depends on the strength of a possible commitment to an ontology of its elements.

(v) In Diophantus' *Arithmetica*, *analysis* is central and directly dealt with; in the *arithmetic* Books of Euclid's *Elements*, and in Diophantus' *On Polygonal Numbers*, it is auxiliary and kept hidden. In those works, the possibility of manipulating given but non-specific numbers, as in the latter, or the requirement of specific numbers, as in the former, are determined by the ontology of number to which those mathematicians are committed.

58As, for example, in collapsing a "window" in the graphical interface of a computer's operating system, into an "icon," which may then be manipulated in its character of being an "icon" only, irrespective of being "the icon of a window" and not "the icon of a text document," "the icon of a graphics document," or "the icon of a programme." Later in this dissertation we will examine this metaphor again. For the moment it suffices to say that this notion of *collapsed elements* is similar to what Klein (1968, p109) calls *"reduced" structures*.

59We are using, of course, the word "operation" in the sense in which subtraction is called an "operation," which it is not, for example, if we consider only positive numbers.

Greek ontological commitments are strong enough to keep *numbers* and geometric magnitudes apart, even if, from the point of view of the modern conceptualisation of mathematics, *numbers* can be taken as particular cases of "magnitudes." The separate treatment of proportions involving each of the two types of mathematical objects, suggest that we should be aware of the possibility of finding such strong ontological commitments in learners, with the difficulties that would follow.

3.3 ISLAMIC ALGEBRA

INTRODUCTION

The culture of Islam has its historical beginning at a very precise date, the year 622 AD, when Muhammad, the Prophet, travels from Mecca to Medina. Before that time, Arabic peoples lived within a tribalistic social structure; the emergence of Islam answers to the challenge of reforming the old tribal order, and the teachings of the Coran, the Sacred Book of Revelations, will produce a unity unprecedented in the Arab world (Pryce-Jones, 1989). In less than a century from Muhammad's journey, Islamism will have extended over the Middle East, North Africa and Spain.

In one essential point the Islamic culture differs from Greek culture. In Islam the religious aspect takes over all other aspects of life[60]; *faith* and *revelation* are central notions, and, in fact, "The very word *islam* means both 'submission' and 'peace'—or 'being at one with the Divine Will'." (Nasr, 1968, p22). But, Nasr (op. cit., p23) points out, Islam has three levels of meaning: (i) all men are Muslims, by the mere fact that they were created by God in that way, and have no alternative to it, as much as a flower cannot escape being a flower; (ii) there are those who surrender their will to the law of Islam, as the warrior who, leaving for battle, says, "And now, God, take my soul."; and (iii) there is the gnostic, who surrenders his whole being to God, in his way to achieve pure knowledge and understanding. Islam, then, did not imply a religious dogmatism that prevented the search for knowledge, and, as Nasr (ibid) puts it,

> "...'knowledge' and 'science' are defined as basically different from mere curiosity and even from analytical speculation. The gnostic is from this point of view 'one with nature'; he understands it 'from the inside,' he has become in fact the channel of grace for the universe. His *islam* and the *islam* of Nature are now counterparts."

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[60]For example, "...Muslim philosophers were Muslim first and philosophers second." (Qadr, 1990, p9)

The Coran itself is unmistakably clear:

> "Whoever wishes to have the benefits of the immediate world let him acquire knowledge; whoever wishes to have the benefits of the Hereafter, let him acquire knowledge and whoever wishes to have both together, let him acquire knowledge."
> (quoted in Qadr, 1990, p16)

to what Qadr immediately adds, "Further it may be noted that Islam favours both rational and empirical knowledge. No dogma, however sacred and ancient it might be, is acceptable in Islam and to Muslims unless it stands the test of reason."

The central notion of Islam is unity, not a unity produced by intellect alone, by a systematisation of our understanding of Nature, but an *original* unity, one emanating from God. The prohibition of portraying individual objects in Islamic art has to be understood in this context, as the avoidance of the particular[61]; it is also in this context that the importance of mathematics in the Islamic culture has to be understood, as a way to overcome the distance between the multiplicity *exhibited* in Nature and the unity *underlying* Nature.

It would be impossible for us—in the context of this dissertation—to go any deeper into the study of the influence of the Coran in Islamic science, but the important point to be made is this: the Coran provided not only a code for the restructuring of the tribal social structure of the Arab world of the time, but also, and for us of more interest, it provided a drive towards the search for knowledge. This is a key aspect of the Islamic culture, as it prepares the ground for the study, by Muslim scientists and philosophers, of the work of the Greeks.

From the Greek philosophers, Pythagoras, Plato, and Aristotle were more deeply studied by the Arabs. Nasr (1968, p70) argues that the interest in the Greek philosophers probably arose from the position of inferiority in which early Islamic theologians and philosophers found themselves, unable to defend the precepts of Islam against Christians and Jew thinkers, who were—specially the former—an important source for Greek knowledge in the Islamic culture. From the Pythagorean tradition, its interest in the mystic aspects of numbers, in its aspect of making possible an

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[61] For a good sample of Islamic art, see Prisse d'Avenne (1989), where on page 10 we read that, "...freezes bearing great foliated scrolls intermixed with human and animal figures, must have appeared to the Arabs as monstrous manifestations of the warped imagination of pantheism."

understanding of the world, was taken by Arab *falsafah*—philosophy. The influence of Plato, and in particular of Aristotle was much greater[62].

In view of the importance of mathematics as a "ladder" to higher levels of understanding (Nasr, 1968, p147), together with the importance given to the reading and interpretation of Greek philosophy, it is almost paradoxical that one will not find in Islamic philosophers the same kind of discussion of number, for example, that is found in Plato and Aristotle. In itself, this is a strong indication that the ontological commitment of the Greek had to a great extent been abandoned, and this for the reasons that follow. Although it can be said that the Arabs shared with the Greek the urge to know Nature, within the Islamic culture the Greek dismissal of empirical knowledge as lesser and even misleading was rejected. Number as *used* in all sorts of situations seems to be the number dealt with by Islamic mathematicians, and not the ontologically determined number of Plato and Aristotle. There should be no doubt that the Arabs knew the incommensurability problem, as Euclid's *Elements* were know to Islam by al-Khwarizmi's time[63], and it would be unreasonable to think that not being able to understand it properly, they dismissed it. It seems, instead, that in Islamic mathematics the factor determining *number* was the possibility of calculating with them, and as a consequence the philosophical discussion about number was substituted by a technical one, as one finds, for example, in their Numerical Analysis (see Rashed, 1978).

About the history of Islamic mathematics in the period before al-Khwarizmi, Rashed says that little is known—apart from studies in Combinatorial Analysis, which is, however, always presented in the form of dogmatic rules and in the context of linguistic and lexicography—and that a patient effort is required to try and reconstitute it (Rashed, 1984, p18, footnote 6).

In order to understand the concept of number adopted by Islamic mathematicians, we will, then, examine directly the mathematical text, and where possible, complement this study with references to other Islamic authors.

AL-KHWARIZMI

Know almost universally as the author of the treatise from which the word "algebra" is derived, al-Khwarizmi (c. 780-c. 863, born in eastern Persia) was more than that. Nasr considers him one of the "universal figures" of Islam, and tells us that "He wrote the first extensive Muslim work on geography, revising much of Ptolemy

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[62]For a more comprehensive study of the influence of Plato and Aristotle in Islamic philosophy, the reader is referred to Walzer (1963), O'Leary (1948), and Peters (1968).

[63]Translated by Al-Hajag, a contemporary of al-Khwarizmi in the "House of Wisdom." (Cf. Rashed, 1984, p21)

and drawing new geographical and celestial maps. His astronomical tables are among the best in Islamic astronomy." (Nasr, 1968, p45).

Al-Khwarizmi lived in Baghdad, where he wrote his famous treatise, "Kitab al-mukhtasar fi hisab al-jabr wa'l-muqabalah," or "The Brief Book on the Calculus (hisab) of algebra and muqabala."[64] The importance of this book in the history of mathematics can be measured by the fact that it became a standard textbook on algebra in medieval Europe, but also for other reasons. First, al-Khwarizmi's treatment of algebra is not to be found in his Arab predecessors nor in Diophantus, and as we will show, it has an immediate influence in its contemporaries. Second, because its approach to algebra represents an important step in the constitution of algebra as an independent discipline in mathematics. Before examining al-Khwarizmi's *Algebra*, we will try and establish a few aspects of the broader context in which the work was produced.

Baghdad was founded in 762 AD, at at time when Basra was the principal city in the region, and the *Algebra* was written there between 813 and 833, the period when al-Mamun reigned, and established the "House of Wisdom," with a library, an observatory, and a department for translation (Qadr, 1990, p36); al-Khwarizmi was a member of the "House of Wisdom."

Almost a contemporary of al-Khwarizmi in Baghdad, was al-Kindi (801-873 AD), the founder of the Islamic Peripatetic school, and the first in a long line of great Islamic philosophers[65]. The importance of mentioning al-Kindi here, is to establish that, at least at a more formal level, we should not expect to find in al-Khwarizmi a strong influence of the Aristotelian doctrine of number, and in fact this is the case.

Another reference which we think to be necessary, is to the "...Brethren of Purity... a group of scholars, probably from Basra, who in the fourth [Hegira]/tenth [AD] century produced a compendium of the arts and sciences in fifty-two epistles." (Nasr, 1968, p152) Their approach to numbers is Pythagorean in the sense that numbers are studied with a mystical interest (op. cit., pp153, 155 and 157). On their distinction between "number" and "numbered," however, they are closer to the Platonic doctrine of number (op. cit., p154).

The important aspect of the text, however, is the importance given by them to the written form of numbers. On pages 154-155, properties of the first twelve numbers are given, and we read,

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[64]"Le livre concis du calcul d'algèbre et d'al-muqabala," which is the translation into French by Rashed (1984, p17), and which seemed to convey best the purpose and content of the book.
[65]See, for example, Peters (1968, p158ff).

"The property of one is that it is the principle and origin of numbers...And the property of two is that, speaking absolutely, it is the first number...The property of three is that it is the first of the odd numbers; by it one can measure one-third of all numbers...The property of four is that it is the first square number...The property of five is that it is the first circular number [?]...The property of six is that it is the first complete number [equal to the sum of its divisors...The property of eight is that it is the first cubic number. The property of nine is that it is the first odd square and the last of the single digits. The property of ten is that it is the first of the two-digit numbers..."[66]

Of interest to us, "nine" and "ten" are assigned properties that only make sense in relation to the notational system; in view of the mixture of notational, mathematical and mystical properties, one could suppose that we are in the presence of an ill-informed text, but as we said before, the Brethren of Purity was composed by a group of scholars, and one has to suppose that care was taken as to present only that which the authors considered as well supported knowledge, a Coranic requirement. But we are to find the clearest justification for the acceptability of the notational criteria, at the very beginning of the section to which the description of properties belong:

"**Unity and Multiplicity**: Expressions indicate meaning; meaning is that which is named, and expressions are names." (op. cit., p154)

An immediate consequence of this, is that as long as a number can be said or written, it must be meaningful; it is quite revealing that in the quotation on the properties of numbers, it says that "two" is the first number *absolutely speaking*, as one senses in the whole passage a tension between an attempt to provide a Greek-like ontology, and a much more flexible—although mathematically unsound, of course—understanding of number. It is in this sense, as an expression, that "square root of five," for example, acquires meaning, the meaning of a number one can calculate with, and can thus be uttered and written[67].

Rosen (al-Khwarizmi, 1831, p ix) indicates that there is evidence that al-Khwarizmi work was informed by the work of the Hindus[68], but that it is highly

[66]It is inevitable to notice the similarity with "an even number is a number that ends in 0, 2, 4..."
[67]Sabra (in the entry *'Ilm al-Hisab* (arithmetic), in Lewis et al., 1971, vol III, p1138) says that "Like their Greek predecessors, Arabic authors on the whole considered irrational magnitudes, the subject of Bk. X of the *Elements*, as belonging to geometry rather than arithmetic." The examination of the work of leading Arabic algebraists—al-Khwarizmi, Abu Kamil, al-Karaji and al-Khayyam—shows that Sabra's statement lacks precision.
[68]Van der Waerden (1985, p10ff) and Nasr (op. cit., p168ff) also indicate that the *Siddhanta* of Brahmagupta served as a basis for the production of al-Khwarizmi's astronomical tables.

improbable that he knew of the work of Diopahntus, as his *Arithmetica* was translated into Arabic only in the 10th century.

We shall now turn our attention to al-Khwarizmi's *Algebra* itself. We will always refer to Rosen's translation of the *Algebra* (al-Khwarizmi, 1831), unless otherwise stated[69].

The characteristic aspects of the *Algebra* are three. First, the *Algebra* is not a collection of solved problems, as in Diophantus and in the Chinese and Hindu algebras. It *begins* with a theoretical part, in two sections, where the fundamental concepts are introduced, and the necessary algebraic techniques presented. Second, in the *Algebra* not only an irrational number is accepted as the solution of an equation, but we also find the beginnings of an arithmetical treatment of *surds*[70]. Third, and most important, the *Algebra* is conceived as a method which can be applied equally to geometric and arithmetic problems. (Rashed, 1984, p20)

The *Algebra* is completely *in words*; even numbers, in the body of the text, are written in full, and as algebraic symbolism is so commonly associated to *algebraic thinking*, a closer examination of this aspect of the text is necessary. The use, by the Arabs, of the Hindu notational system for numbers, was certainly a way of acknowledging its usefulness, and so the question arises, as to why not even in this case—writing down specific numbers—we will find a symbolic notation[71]. Anbouba gives an explanation which seems—specially in al-Khwarizmi's case, at an early stage of the Islamic culture—the most likely:

> "Elle [the *Algebra*] est entièrement parlée et les nombres mêmes y sont écrits en toutes lettres ce qui en assure une énonciation déclinée conforme aux règles de la grammaire, **question d'une importance presque religiseuse pour un Arabe**." (Anbouba, 1978, p68ff) (our emphasis)

The importance of *words* in the Islamic world cannot be over-emphasized. The Coran, probably following the same traditions to be found in the Old Testament, identifies "knowledge" and the "names of things," (Qadr, 1990, p5) and as the Coran spread, carrying with it Arabic, with the status of "sacred language" (Nasr, 1968, p30),

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69 Rosen's translation has been criticised for its inaccuracy, but we preferred to use it than to rely only on fragments in secondary texts. In some cases we could use available passages of Karpinsky's English rendition of Robert de Chester's translation, quoted in Nasr (1968, p158ff).

70 Al-Khwarizmi, however, does not consider negative numbers.

71 Cf. Rosen's Introduction to the *Algebra*: "Numerals in the text of the work are always expressed by words: figures are only used in some diagrams, and in a few marginal notes."

Greek, Syriaic, Pahlavi and Sanskrit texts began to be translated into Arabic (op. cit., p69ff). We have also already mentioned the Brethren of Purity in relation to names and meaning.

What form, then, does al-Khwarizmi adopts in order to overcome the obvious difficulty of expressing his *Algebra* in words? Rashed points out the importance of the *canon*, the solving procedure for each of the prototypical equations, and which assume in each case a "standard" verbal form. When dealing with the manipulation of expressions, however, this strategy is not available. Al-Khwarizmi's approach to this question is truly remarkable. In the first section on the manipulation of expressions, *On Multiplication* (p21ff), he uses, **in all the examples**, the number "ten": "ten and one to be multiplied by ten and two," "ten less one to be multiplied by ten less one," "ten and two to be multiplied by ten less one," and then, "ten less thing to be multiplied by ten," until "ten and thing to be multiplied by thing less ten." The use of "ten" has, we think, a very special importance here: it is a *unit*, as the scholars in the Brethren of Purity called it (Nasr, 1968, p154), but also it is technically more useful than "one" in that when multiplied—by itself or by another number—it would "leave its mark." In two times one the "one" is "invisible" in the result, but not the "ten" in two times ten. This procedure is not followed in the subsequent sections (*On Addition and Subtraction* and *On Division*), but then the need to identify terms in the resulting expression and the procedure by which it is obtained is less pressing. We think that this usage is consciously directed at fixing the reader's attention in the process, at the same time it lends generality to the "specific" examples.

The somewhat lengthy description of the to initial parts of the *Algebra*—which we will now present—is necessary to allow a correct understanding of the treatise. The usual concise presentations, as one finds for example in van der Waerden (1985), or the even more concise one in Taton (1948), only produce the characterisation of the *Algebra* as a primitive textbook in algebra, which lacks any form of "algebraic" symbolism (letters) and presents no result of interest, a book which only merit seems, at times, to be its age[72].

The *Algebra* begins by clarifying its point of departure:

> "...reflecting that all things which men need require computation, I discovered that all things involve number..." (Karpinsky's English edition of Robert de Chester's Latin translation, quoted in Nasr, 1968, p158)

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[72]On the "Index Historique," Dieudonné (1987) refers to al-Khwarizmi as "...the author of a treatise on algebra that lacks originality..."

and then it briefly explains the nature of the decimal notation for whole numbers:

> "Moreover, I found that any number, which may be expressed from one to ten, surpasses the preceding by one unit: afterwards the ten is doubled or tripled, just as before the units were: thus arise twenty, thirty. &c., until a hundred; then the hundred is doubled and tripled in the same manner as the units and the tens, up to a thousand; then the thousand can be thus repeated at any complex number; and so forth to the utmost limit of numeration." (al-K., p5)

There are two points of interest, here: (i) the direct association of number and its notation—in fact the generation of numbers is explained by the possibility of expressing them—which supports our interpretation that the meaning of number in al-Khwarizmi—possibly in all of Islamic mathematics—was associated with the possibility of expressing it and calculating with it; and (ii) there is no mention of fractions or surds, the former being introduced in relation to the *root* , and the surds appearing almost casually later in the book.

As opposed to Diophantus' *Arithmetica*, the *Algebra* first "defines" the *root* [73] and then the *square*; *simple numbers* are said to be "...any number which may be pronounced without reference to root or square." At one time, the definition of square is *arithmetical*, and the "independent terms," so to speak, are not characterized as a "number of monads," but in themselves.

From there, the *Algebra* sets out to announce the six prototypical equations to which all problems will be reduced[74,75].

First case: "squares are equal to roots" (p6)	$cx^2 = bx$
Second case: "squares are equal to numbers" (p7)	$ax^2 = b$
Third case: "roots are equal to numbers" (p7)	$ax = b$
Fourth case: "roots and squares are equal to numbers" (p8)	$ax^2 + bx = c$
Fifth case: "squares and numbers are equal to roots" (p11)	$ax^2 + b = cx$
Sixth case: "roots and numbers are equal to squares" (p12)	$ax^2 = bx + c$

[73] p6, "...any quantity which is to be multiplied by itself, consisting of units, or numbers ascending, or fractions descending."

[74] The existence of six types is mostly due to al-Khwarizmi rejection of negative numbers.

[75] It is clear that the six types are determined by a combinatorial consideration.

In each case a numerical example is given and solved by a standard solving procedure, or *canon*. Rashed considers the notion of *canon* to be a key one in the *Algebra*,

> Devant la diversité des 'êtres mathématiques' — geométriques, arithmétiques — l'unité de l'objet algébrique est fondée seulement par la généralité des opérations nécessaires pour ramener un problème quelconque à une forme d'équation ou encore de préférence à l'un des six types canoniques énoncés par al-Khwarizmi ... d'une part, et par la généralité des opérations pour déduire des solutions particulières, c'est-à-dire un canon, d'autre part." (Rashed, 1984, p249)

The merit of al-Khwarizmi's work is precisely this: the elaboration of a theory is possible because al-Khwarizmi intends the method by which the solutions are found, and the examples he uses in this first part are at one time illustrations and conveyors of the general solutions[76]. On the other hand, this method is still aimed at solving practical, "concrete," problems, and an "actual" solution is required[77]. The true significance of the disposition of the contents in the *Algebra*, and also of the statement which opens the book, is that all the problems in the later parts of the book, be they geometric or concerning inheritance, whether they require the determination of a measure or an amount, or an answer to "how much," "how many," or "how long," they will be always solved "by numbers" **and** using the same methods in each case, without reference to the problems' contexts. Moreover, it is clear that *equation* is an *object* in the *Algebra*, as not only they are used to provide the prototypical problems, but also, the *normal form* of equations is distinguished[78].

Immediately after the six prototypes, al-Khwarizmi gives demonstrations for the solution procedure in the case of three specific equations. Bellow is the demonstration of the case $x^2+10=39$, to be found on page 13ff.

> "*Demonstration of the Case: 'a Square and ten Roots are equal to thirty-nine Dirhems'* :
> The figure to explain this a quadrate, **the sides of which are unknown**. It

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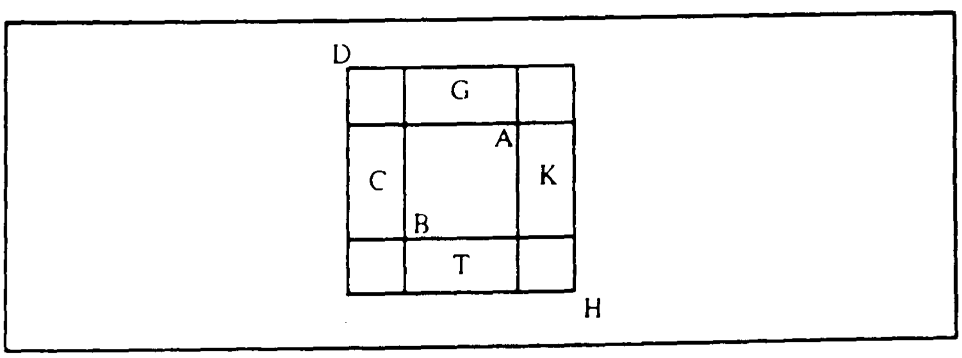
[76]On page 3, still in the author's introduction, we find that this is a "short work," and that it is "[confined] to what is easiest and most useful in arithmetic..." We are led to believe that al-Khwarizmi is stating that a more complete treatise could be composed, but it is not possible to know if the treatise would be extended in its mathematical part or by presenting a more complete selection of application problems.

[77]Externally, then, al-Khwarizmi is close to Diophantus in this respect; it is necessary, however, to have clearly in mind that in each case the requirement of "actual" solutions is justified by totally different conditions. In Diophantus, as we saw, a "generic" solution is impossible, whereas in al-Khwarizmi it only lacks further motivation.

[78]Whenever necessary, al-Khwarizmi remarks that if there is more **or less** than one square, it must be reduced to one square, and the other terms in the equation accordingly adjusted.

represents the square, the which, or the root of which, you wish to know. This is the figure AB, each side of which may be considered as one of the roots; and if you multiply one of these sides by any number, then the amount of that number may be looked upon as the number of the roots which are added to the square. Each side of the quadrate represents the root of the square; and, as in the instance, the roots were connected with the square, we may take one-fourth of ten, that is to say, two and a half, and **combine** it with each of the four sides of the figure. Thus with the original quadrate AB, four new parallelograms are combined, each having a side of the quadrate as its length, and the number of two and a half as its breadth [not constructible]; they are the parallelograms C, G, T, and K. We have now a quadrate of equal, though unknown sides; but in each of the four corners of which a square piece of two and a half multiplied by two and a half is **wanting**. In order to compensate for this want and to complete the quadrate, we must add (to that which we have already) four times the square of two and a half, that is, twenty-five. We know (by the statement) that the first figure, namely, the quadrate representing the square, together with the four parallelograms around it, which represent the ten roots, is equal to thirty-nine of numbers. If to this we add twenty-five, which is the equivalent of the four quadrates at the corners of the figure AB, by which the great figure DH is completed, then we know that this together makes sixty-four. One side of this great quadrate is its root, that is, eight. If we subtract twice a fourth of ten, that is five, from eight, **as from the two extremities of the side of the great quadrate DH**, then the remainder of such a side will be three, and that is **the root of the square, or the side of the original figure AB**. It must be observed, that we have halved the number of the roots, and added the product of the moiety multiplied by itself to the number thirty-nine, in order to complete the great figure in its four corners; because the fourth of any number multiplied by itself, and then by four, is equal to the product of the moiety of that number multiplied by itself [$4\left(\frac{b}{4}\right)^2 = \left(\frac{b}{2}\right)^2$]. Accordingly, we multiplied only the moiety by itself, instead of multiplying its fourth by itself, and then by four. This is the figure: [bellow]" (our emphasis)



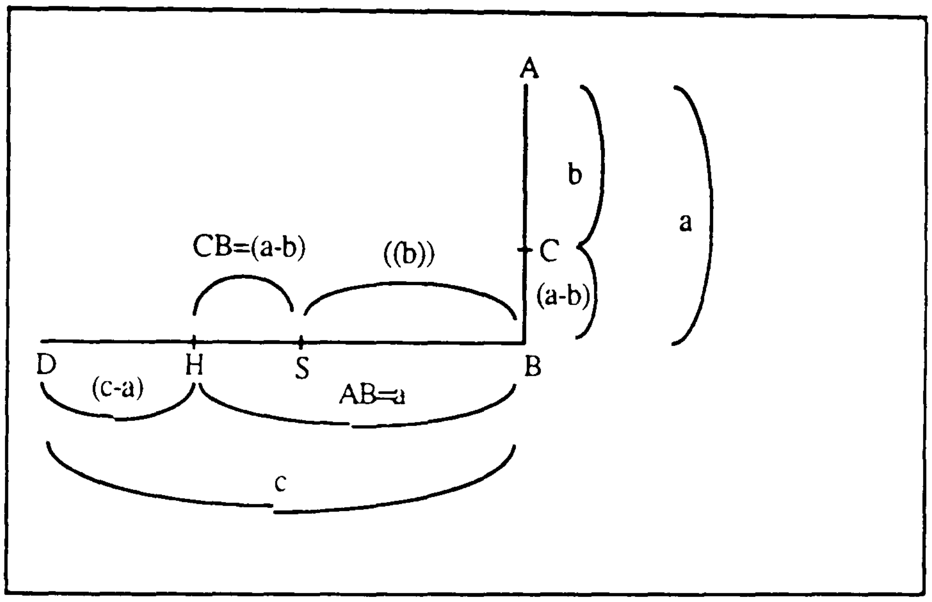
Although using geometric figures, al-Khwarizmi's demonstrations should not be called "geometric." Anbouba (1978) prefers "proof by figures," to use, he says, al-Khwarizmi's own words (Al-K., p27), and Rosen himself calls them "geometric illustrations" (eg, p13); M. Kline (1990, p193) suggests an influence of the Greek "geometric algebra," but van der Waerden (1985) correctly observes that for an insufficiency in his proof of Pythagoras' theorem (p74ff of the *Algebra*), which is proved only for the case of an isosceles rectangle triangle, one should be quite sure that al-Khwarizmi's source is not Euclid. In all his demonstrations we find lines—and squares—of unknown measure, ie, they are *analytical*; if it were indeed the case that he used the "geometric algebra" from some Greek source, a strong reinterpretation must have taken place, as a geometric construction involving a line of unknown length is not possible. It seems, instead, that his solutions followed a model like Heron's *dissolutio* and *compositio* method, ie the "splitting-up" and "composition" of rectangles and squares (Heath, 1981, vol 2, p311).

Following the first part treating the solution of equations, al-Khwarizmi proceeds to explain the rudiments of an algebraic calculus, and few aspects of his exposition are worth examining.

In relation to multiplication, he begins by giving a definition that is nowhere used in the rest of the book: "Whenever a number is to be multiplied by another, the one must be repeated as many times as the other contains units." (p21) This definition applies only to products where the multiplier is an integer number, but al-Khwarizmi explicitly deals with the product of fractions and *surds*. The rules he gives for the multiplication of binomials are perfectly general, in the sense that they not only cover all possible cases, but also in that they are given first in a general form, and only then particular examples are examined. The first section of the multiplication rules assumes that each term is a known number, but they are immediately and with no further justification extended to those cases where one of the terms of each monomial is the unknown: "I have explained this, that it might serve as an introduction to the multiplication of unknown sums..." (p23). On pages 27-31, the basis for an arithmetic treatment of *surds* is given, in the form of rules such as $n\sqrt{x^2} = \sqrt{n^2x^2}$ and $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$. The rules are given through specific examples, but their general validity is always stated following the examples; also, specific examples with irrational numbers are provided, for example,

$$\sqrt{\tfrac{1}{2}} \cdot \sqrt{\tfrac{1}{3}} = \sqrt{\tfrac{1}{2} \cdot \tfrac{1}{3}} = \sqrt{\tfrac{1}{6}}.$$

In the case of the addition and subtraction of expressions, he again treats first expressions involving only known numbers, but this time beginning with specific examples. On the first two examples he uses irrational numbers as terms in the expressions to be added and subtracted ($\sqrt{200}$), and then gives examples involving quadratic trinomials in an unknown number. For adding and subtracting expressions, al-Khwarizmi provides "...the reason...by a figure." (p27) The argument (p31ff) that provides the "reason" is, in a general form, the following (figure bellow): "To show that $(c - a)+(a - b)=c - b$". Make AB=a, and AC=b. Then we immediately have, CB=$a - b$. Now, make BD=c, and HB=AB(=a). We have, again immediately, DH=$c - a$. Now make HS=CB(=$a - b$), and as HB=AB, we have SB=AC, and as a result, DH(=$c - a$) + HS(=$a - b$) = DB-SB(=$c - b$). The core of the argument is simply a whole-part relationship applied to lines, as in AC+BC=AB $\Rightarrow$ AB-AC=BC.



Geometrically the argument is not easy applied in the case of the third example he considers, namely, $50 + 10x - 2x^2 + (100 + x^2 - 20x)$. Al-Khwarizmi is well aware of this difficulty and says that,

> "...this does not admit of any figure, because there are three different species, *viz.* squares, and roots, and numbers, and nothing corresponds to them by which they might be represented [at the same time]. We had, indeed, contrived to construct a figure also for this case, but it was not sufficiently clear...The elucidation by words is very easy. You know that you have a hundred and a square, minus twenty roots. When you add to this fifty and ten roots, it becomes a hundred and fifty and a square, minus ten roots. The reason for these ten negative roots is, that from the twenty negative roots ten positive roots were subtracted by reduction. This being done, there remains a hundred and fifty and a square, minus ten roots. With the hundred a square is connected. If you subtract from this hundred and square the two squares negative connected with fifty,

then one square disappears by reason of the other, and the remainder is a hundred and fifty, minus a square, and minus ten roots...This is what we wished to explain."

Al-Khwarizmi is careful to manipulate the expressions as to avoid a "negative" term—"wanting" in the Greek—to be dealt with without a "positive" term to which it is attached. At the same time, the step in which the two squares "negative" are finally added, al-Khwarizmi shows he is aware of the property that a "wanting" added to a "positive" will make the latter "disappear," but this could not be translated into a true arithmetical property, simply because in al-Khwarizmi—and in Islamic mathematics for many years after him—zero is not considered as a number. Finally, it is clear that the objective of the "elucidation" is to provide the reader with some knowledge of the mechanism of manipulating the proposed expressions, not to provide a "logical foundation"; the difficulties with the geometrical representation are put aside simply by not using it. The expressions are treated as wholes and parts, with the added feature of "negative parts", that although not explicitly stated, are skilfully used.

We can now have a better evaluation of the character and importance of al-Khwarizmi's *Algebra*.

As we have already said, its two distinctive characteristics are the form of the presentation and its acceptance of *surds* as numbers within the calculus.

The central *object* of the book is clearly the *equation*, which appears from the beginning by itself, heading, not following, a problem: each prototypical equation represents a whole class of problems. The tension between method and object, which we had discussed in relation to Greek mathematics, is much weaker here; it is around the equations that solution methods are organised, but it is precisely the generality of the method of solution for each of the six types of equation that gives them their character as *objects*.

It is significant that the instruments by which the expressions in the equations are manipulated, the calculus with algebraic expressions, is treated separately. In Diophantus, the manipulation of the expressions themselves was only instrumental, and almost casually mentioned in the introductory part of the *Arithmetica*, but in al-Khwarizmi the subject is given much more autonomy. As Rashed (1984, p25) says,

"Ces chapitres sont bien plus important par l'intention qui les anime que par les résultats qu'ils renferment. Si l'on considère en effet les déclarations d'al-Khwarizmi, la place qu'il attribue à ces chapitres...et enfin l'autonomie qu'il restitue à chacun d'eux, il apparaît que l'auter a voulu entreprendre pour elle-même l'étude du calcul algébrique,

> c'est-à-dire des propriétés des binômes et trinômes associés aux équations considérées dans la précédente partie de son livre."

The absence of a "geometric illustration" of the rules for multiplying binomials, which are substituted by examples with specific numbers *worked*, however, *exactly as the expressions should be*; the rules for multiplication of radicals that include examples involving *surds*, the reduction of all sorts of problems to problems *in number*: in all this an *algebraic algebra* is anticipated in al-Khwarizmi, but also a different understanding of number is produced, allowing more freedom to the arithmetical operations, and, consequently, the extension of the possibilities of an algebraic calculus. Rashed (1984, p250) sees in the development of an algebraic calculus more than a technical achievement:

> "Les successeurs d'al-Khwarizmi, tout en poursuivant ses recherches, ont réagi ... contre l'insuffissance de la démonstration geométrique en algèbre. Cependant, la nécessité pressantie d'une démonstration numérique n'a été elle-même possible qu'au terme d'une extension du calcul algébrique et de son domain, puis de sa systématisation. Les successeurs immédiats d'al-Khwarizmi se mirent à cette tâche sans tarder...L'extension et la systématisation du calcul algébrique ont permis de formuler l'idée de demonstration algébrique dans la mesure où elles ont fourni les éléments d'une réalisation possible. Au début du XI^{e} siècle ... al-Karaji (fin du X^{e} siècle), s'engage à donner, outre la démonstration géometrique, une autre démonstration, celle-là algébrique, des problèmes qu'il considère."

The requirement of a "numerical"—in Rashed's words—demonstration, as opposed to a "geometric" one, which in al-Khwarizmi is essentially a combinatorial proof using lines and areas, *precedes and motivates* the development of an algebraic knowledge. In the process initiated by al-Khwarizmi's *Algebra*, *algebraic thinking* means an intention that drives the development of the means necessary to fulfil it.

The first consequences of al-Khwarizmi's *Algebra* were soon felt, with mathematicians engaging in the task of developing both the theory of equations and the algebraic calculus. Rashed (1984) indicates that al-Mahani "translates into algebra some biquadratic problems of Book X of the *Elements*, and cubic problems from Archimedes," (p27) and that Abu Kamil and Sinan ibn al-Fath extend the notion of algebraic powers[79] (p21); ibn al-Fath, for example, solves equations involving the terms ax^{2p+n}, bx^{p+n}, and cx^n (Anbouba, 1978, p79).

Abu Kamil, an Egyptian naval engineer (fl. Cairo, about 850 AD), produced an algebra that is more accomplished than al-Khwarizmi's, both by systematically providing proofs for the rules in the *Algebra*, and by treating a far greater variety of problems. For example, al-Khwarizmi had solved the equation

$$\frac{x}{10-x} + \frac{10-x}{x} = 2\frac{1}{6} \qquad \text{(I)}$$

but he passes from (I) to

$$x^2 + (10-x)^2 = 2\frac{1}{6}(10-x) \qquad \text{(II)}$$

without providing any justification. Abu Kamil, however, inserts between (I) and (II) a demonstration, "by segments," (Anbouba, 1978, p81) of

$$\frac{a}{b} = \frac{a \cdot a}{b \cdot a} \quad \text{and} \quad \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab}$$

In Anbouba (op. cit., p83), we also find a demonstration—which we reproduce bellow, in our translation into English—of the transformation of the equation

$$\frac{a}{x} - \frac{a}{x + c} = d$$

into a "recognisable" quadratic:

[79]In a footnote to this observation, we find the words of Sinan ibn al-Fath, in which he explains the series of ascending powers. It is interesting that he gives a nomenclature for them, and then says, "You are allowed to change those names after you have understood the intention," which again shows the attention paid to words in Islamic culture.

Make the rectangles ABCD and AEFH, with area equal to a, and such that AB=x and

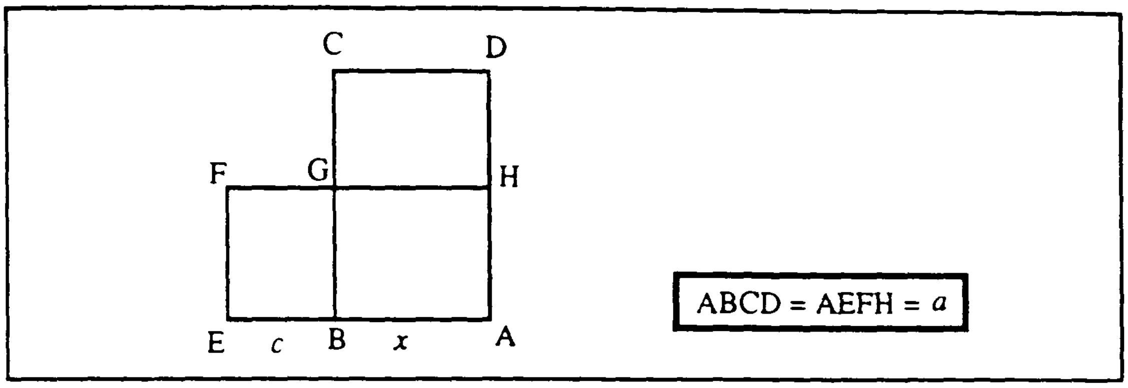
AE=$x+c$. (1)

Thus, AD=$\frac{a}{x}$, AH=$\frac{a}{x+c}$. (2)

But then, DH=d, and DHGC=dx. (3)

As EBFG = DHGC = dx, then EF = $\frac{dx}{c}$ (4)

and one finally has, AEFH = $\frac{dx}{c}(x+c) = a$ (5)



The demonstration is, as in al-Khwarizmi, a combinatorial one, but with a higher degree of sophistication that allows Abu Kamil to manipulate the model in a much more powerful way. In the theoretical part of al-Khwarizmi's *Algebra*, the area of a rectangle is never explicitly associated with a numerical value, and it is mainly its support for the *dissolutio* and *compositio* that is sought; it is only in the introduction of the section *On Mensuration* that the numerical link is directly established[80]. In Abu Kamil, this link is much more skilfully explored: in (2) the length of a side is derived, as a division, from the area and the other side. This use of the "geometrical illustration" provides the necessary support to deal directly with complex expressions, but as we saw, Abu Kamil also treated a transformation such as that between (I) and (II) as an *arithmetical* transformation—demonstrated, it is true, in a proof with lines.

The firm link of geometric figures and numbers, and the submission of the geometric model to the operative aspects of the *arithmetical* treatment, are clear. As Gandz (1947, p114) says,

> "...In EUCLID, geometry is mistress and algebra is hidden and ancillary. With AL-KHUWARIZMI, algebra predominates and the geometric demonstration is

[80]"Know that the meaning of the expression 'one by one' is mensuration...Every quadrangle of equal sides, which has one yard for every side, has also *one* for its area. Has such a quadrangle two yards for its side, then the area of the quadrangle is four times the area of a quadrangle, the side of which is one yard. The same takes place with three by three, and so on, ascending or descending: for instance, a half by a half, which gives a quarter...In the same manner...two-thirds by a half, or more or less than this, always according to the same rule." (al-K., p70ff)

auxiliary...The most important contribution of ABU KAMIL is that he combines the algebra of AL-KHUWARIZMI with the demonstrations of EUCLID..."

First with al-Khwarizmi's simpler use, and then with Abu Kamil's refinement, it is the notion of *being measurable with numbers* that provides the possibility of such constructions, which are, of course, symbolic.

A key point in the demonstration presented above is very illuminating: viewed simply as a geometric demonstration, it is not saying absolutely anything; viewed as a statement about the numbers involved, it is not saying anything either. In fact, it is purposeful only as producing a new *representation* of the original relationship, and the notion of *representation* must be understood not in respect to any sort of symbolism, but in relation to the *arithmetical articulation of the terms involved in the equations*. It is in this sense that *arithmeticism* —in Islamic algebra, as well as in our theoretical model—is characterised by an acceptance of the arithmetical operations as *objects*, while the process Rashed calls "arithmetisation of algebra" must be understood, in this context, as the drive towards an *arithmetical internalism*, ie, accepting the justification of the procedures of algebra only by its internal coherence in reference to the arithmetical operations and to an numerical notion of equality[81], and not by reference to any sort of geometric intuition.

Abu Kamil's methods of demonstrations are dominated by an *arithmetical* intention, but they are still, however, dependent on geometrical *objects*, in a very specific sense: it is the whole-part properties of the figures that support the manipulation of the equality relationships in the process, and the specificity of the geometric configuration used is crucial[82]. In the course of the development of Islamic algebra, this tension will be resolved in two ways. In one line of development, "pure" algebraic proofs-solutions,ie, *internal* and *arithmetical* will be required; in the other, investigation in algebra will continue to make use of geometry, with the particular addition of the solution of equations by the intersection of curves. The culmination of the first tradition in Islamic mathematic is to be seen in al-Karaji, and that of the second, in al-Khayyam, whose work we will examine later.

To resume the chronology, we saw that around 820 the *Algebra* of al-Khwarizmi is produced, and that still in the 9th century, the study and development of the theory of equations and of the algebraic calculus are lively pursued. The *Elements* of Euclid had been translated into Arabic by al-Hajag, a contemporary of

[81] A geometric notion of equality would be the notion of a figure being tansformable into another by means of a geometric construction.
[82] In the case of the demonstration we have presented, it is crucial that the two rectangles are produced in exactly that configuration, directly producing the equality between CDGH and FGEB.

al-Khwarizmi, but it seems that it had little or no direct influence on the *Algebra*. The algebra of Abu Kamil can be considered the culmination of the efforts of the 9th century.

In the beginning of the 10th century, Qusta Ibn Luqa translates Diophantus' *Arithmetica* into Arabic, and Thabit Ibn Qurra translates the *Conics* of Apollonius, works of Archimedes and the *Introduction to Arithmetic* by Nicomachus (Nasr, 1968, p149). In the second half of the 10th century, Abu al-Wafa al-Buzjani wrote two commentaries, one on the *Algebra* of al-Khwarizmi, another on the *Arithmetica* of Diophantus; he also wrote a "Book of the proofs of the propositions used by Diophantus in his work..." (Cf. Heath, 1964, p6).

Al-Haytham (c.965-1039) works with prime numbers and the Chinese Remainder problem, and although Banu Musa[83] had already refused, in the 9th century, the geometric interpretation of arithmetic operations (Cf. Rashed, op. cit., p192), during the 10th century the work of the Islamic algebraists was still largely connected to geometry, as in the work of ibn Qurra (see van der Waerden, 1985, p18ff)[84], and of al-Buzjani (see Nasr, op. cit., p149). It is only in the work of al-Karaji (about the end of 10th-beginning of 11th century), that the project of an *arithmetically internal* algebra begins to materialise.

Al-Karaji[85] wrote a treatise on algebra, the *Fakhri*, of which an abridged edition—in French—was given by Woepcke in 1853, reprinted in 1982 (in the bibliography, Woepcke, 1982). An aspect of his exposition that distinguishes it from the algebra treatises of his predecessors, is the fact that it begins with a theory of an algebraic calculus (Rashed, op. cit., p32). According to Rashed,

> "Cet exposé a pour but plus ou moins explicite la recherche des moyens de réaliser l'autonomie et la spécificité de l'algèbre afin d'être en mesure de refuser, en particulier, la représentation géométrique des opérations algébriques." (ibid.)

In this alone, a conceptual change can be seen, but the mathematical quality of the book is also outstanding. Without going into a detailed analysis of the *Fakhri*[86], we

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[83]As were collectively called the three sons of Shakir ibn Musa, Muhammad, Ahmad, and Hasan (Cf. Nasr, 1968, p149).

[84]Van der Waerden points out that ibn Qurra uses a *unit* (e) in order to be able to link geometric magnitudes and numbers, and move from $x^2+mx=n$ to $x^2+mex=ne^2$.

[85]This is the spelling used by Rashed, which we adopt. In Woepke's edition of the *Fakhri* (Woepke, 1982), Alkarkhi is used instead.

[86]Beside Woepcke's edition of the *Fakhri*, the reader is referred, for a more complete assessment of the content of the book, to R. Rashed's article on Al-Faraji, originally in the Dictionary of Scientific Biography (1973, v VII, pp240-246), but also a chapter of Rashed (1984).

may remark that in the book al-Karaji uses a notation (name) for a second unknown (Woepcke, p11), and that he defines division and taking the square root as inverse operations to multiplication and squaring (p53ff). The key technical achievement of the *Fakhri*, however, is that it represents the first systematic treatment of an algebra of polynomials (Rashed, op. cit., p33). The extraction of the root of a polynomial expression is restricted to the square root, and division is restricted to division by a monomial, but the fact that the algebraic calculus is treated *in itself and without any recourse to "geometrical illustrations,"* is remarkable. Al-Karaji's "arithmetical intention" is made explicit, as he

> "...fait souvent observer qu'on doit être préparé l'intelligence des règles du calcul algébrique...par les règles de l'arithmétique vulgaire..." (Woepcke, op. cit., p7)

instead of simply letting specific examples to slide casually into the exposition; Rashed (op. cit., p35) remarks that the interest of the "arithmetic algebraists" was to know better the operative structure of the realm of numbers, and not to construct it rigourously[87]. We have seen that in Abul Kamil numbers are intrinsically associated to geometric magnitudes, as if "natural" and not requiring any further explanation. Al-Karaji, however, adopts the definition of incommensurability and irrationality from Book X of the *Elements*, and says,

> "Je montre comment ces quantités [incommensurables, irrationelles] sont transposées en nombres." (quoted in Rashed, op. cit., p36)

an approach that highlights the fact that geometric magnitudes are *modelled* with numbers in the process of using algebra to deal with them, ie, an *algebraic algebra*—as opposed to a *geometric* one, where numbers are modelled with geometric magnitudes—emerges.

The subsequent developments in the process of "arithmetisation" of algebra are described in detail and depth in Rashed (1984)—it is in fact the central objective of the book to study this process from al-Khwarizmi onwards.

We think, however, that it is worth mentioning the 12th century Islamic mathematician as-Samaw'al, author of the *al-Bahir*, where we find a full statement of

[87]"Dans la tradition de cette algèbre, al-Kharaji et As-Samaw'al purent étendre leurs opérations algébriques aux quantités irrationelles, sans s'interroguer sur les raisons de leur succès, ni justifier cette extension." (Rashed, 1984)

the "rule of signs" for the multiplication, in which zero is accepted as a number (Rashed, op. cit., p46). It is as-Samaw'al who says that

> "[algebra is concerned] d'opérer sur les inconnues au moyen de tous les instruments arithmétiques, comme l'arithméticien opère sur les connues." (op. cit., p27)

the motto of the "arithmetic algebraists" finally uttered in full. Together with the mathematical activity proper (new methods and new results), as-Samaw'al develops a reflection on the subject: he identifies algebra and analysis[88] and proposes the classification of propositions in algebra into 3 sub-classes: necessary, possible and impossible (A more complete analysis of this classification is to be found in Rashed, op. cit., pp51-52). The classification is remarkable in that: (i) it distinguishes—within the subclass of the necessary propositions—between propositions that hold for all numbers (the identities), and three classes of propositions where only a restricted set of numbers—finite or infinite—satisfy the relationships given (the problems); (ii) it explicit includes the *conjectures*, "propositions" to which one could not find yet neither a demonstration of its truth nor of its falsehood; and (iii) it introduces the notion of "proof by absurd" into the field of algebra, to characterise the *impossible* propositions.

In as-Samaw'al we have an indication of the level of maturity reached, by the 12th century, in the development of an algebraic knowledge that is driven by and obtained through an *algebraic mode of thinking*.

A NOTE ON AL-KHAYYAM AND THE GEOMETRIC TREND IN ISLAMIC ALGEBRA

Beside the "arithmetic algebra" trend, there was also, as we have said, the development of algebra in another direction, namely the incorporation of geometric methods to it. One of the most important names in the group of Islamic mathematicians working on those lines—if not *the* most important—is that of Omar al-Khayyam. We will examine a few aspects of his work, but only to the extent to which those aspects help us to establish a distinction between the arithmetic and the geometric approaches to algebra.

The Persian al-Khayyam lived in the 11th century, and is known in the West almost exclusively for his *Rubaiyat*, a collection of around 600 short poems; beside being a poet, al-Khayyam was a fine mathematician and astronomer, and an important philosopher (see Nasr, 1968, pp33ff, 52ff, and 160).

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[88] According to Rashed, as-Samaw'al wrote a book entirely dedicated to the theme of analysis and synthesis, which is lost.

Rather than attempting any comprehensive account or analysis of his work, we will instead produce a very short collection of quotes—both from al-Khayyam and from the analysis of his work—which provide material for us to characterise the distinction mentioned above:

* "No attention should be paid to the fact that algebra and geometry are different in appearance. Algebras (*jabre* and *maghabeleh*) are geometric facts which are proved by propositions 5 and 6 of Book II of the *Elements*." (al-Khayyam, in Fauvel and Gray, 1990, p226)
* "*Square square* (x^2) which is known to the algebraists as the product of *square* (x^2) by *square* (x^2) has no meaning in continuous values. How is it possible to multiply a square which is an area by itself? A surface is of two dimensions and the product of two dimensions by two dimensions would be four dimensions, and an object of more than three dimensions is impossible." (al-Khayyam, in Fauvel and Gray, 1990, p226)
* "Omar knows very well that earlier authors sometimes equated geometrical magnitudes with numbers. He avoids this logical inconsistency by a trick, introducing a unit of length. He writes: 'Every time we shall say in this book 'a number is equal to a rectangle', we shall understand by the 'number' a rectangle of which one side is unity, and the other a line equal in measure to the given number, in such a way that each of the parts by which it is measured is equal to the side we have taken as unity'." (van der Waerden, 1985, p24)
* "...observe that the proof of these methods by geometry is not a substitute for a proof by numbers (al-jabr) if the subject is a number and not a mensurable quantity. Do you not see that Euclid proved [the theorems about] proportional quantitative unknowns when their subject is a number, in Book VII?" (Al-Khayyam, in Nasr, 1968, p164)
* "The algebraists have said that...a cube plus a root equal to a square is equivalent to a square plus a number equal to a root...but they gave no proof in the case where the subject of the problems was mensurable. However, when the subject of the problems is a number, that is evident from the Book of Elements, and I shall prove those of them which are geometrical." (Al-Khayyam, in Nasr, 1968, p164)

The distinction between geometric magnitudes and numbers is strict, even implying that an algebraic "reduction" (division of all terms of an equation by the same number) is not seen as proving a similar equivalence in terms of geometric configurations; this indicates that for al-Khayyam, solving a geometric problem *in numbers only* is not *per se* an accepted procedure, and a geometric demonstration has always to be provided. On the other hand, numerical problems require a numerical-algebraic treatment. To the predominance of a geometric perception (the use of a unit to provide homogeneity), one should add al-Khayyam's truly geometric solutions of cubic equations[89], and the geometric character of his algebra is then well established.

CONCLUSIONS

We shall now examine the overall dominant characteristics of the development of Islamic algebra and relate them more closely to our research question.

We begin with two key aspects of that process. First, *algebraic thinking* was, as an *intention*, the driving force behind the development of the algebraic knowledge. Second, the way in which *algebraic thinking* provides a paradigm for this development, is by turning the arithmetical operations into *objects* through the requirements of an *arithmetical internalism.*

The former aspect provides, we think, an important insight into the epistemology of algebra, by making clear that: (i) *algebraic thinking* must be distinguished from algebraic knowledge, if we are to understand the dynamics of the development of the latter; and, (ii) as a consequence, the research on this dynamics must necessarily include a study of the mode of thinking supporting the production of that algebraic knowledge. Seen through the filter of mathematical education, this insight points out to the fact that the ability to cope with literal notation, for example, cannot be taken as a safe indication that *algebraic thinking* is involved, and, thus, it does not serve as an useful indicator of the possibilities of further development or use of that particular knowledge, precisely because the applicability of an specific piece of algebraic knowledge might be tightly bound to the conditions set by the underlying model, as in the case of al-Khayyam, where the reduction of an equation—legitimate in the context of a problem involving numbers—does not imply the correctness of a totally corresponding reduction when the objects are geometric magnitudes. In a more specifically didactic context, we may think of students easily solving the equation $100\text{-}3x=16$, but having great difficulty with $100\text{-}3x=190$, even if they are proficient in dealing with negative numbers, a phenomenon which is investigated in our Experimental Study (Chapter 4).

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[89]"The method employed is not very helpful in numerical calculations. The numerical solution was obtained by approximation and trial." (in the entry *al-Djabr wa 'l-Mukabala*, in Lewis et al., 1965, vol II, p362)

We learned from our study of the development of Islamic algebra, how it begins—in al-Khwarizmi—with the *equation* being transformed into an *object*, through which whole classes of problems are represented and around which the solution methods are organised; then—in Abu Kamil—the algebraic calculus gains in importance, and finally—in al-Karaji and as-Samaw'al—the equation is to a great extent absorbed into a much more general framework, in which the central notion is what we call *arithmetical articulation.* It is in the *arithmetical articulation* that the role of the arithmetical operations as *objects* become clear; when Rashed says that the intention of the algebraic calculus in al-Khwarizmi is more important then the actual results he presents, he is highlighting the fact that the "algebraic" approach and the development of an algebraic calculus were a consequence of the *arithmetical internalism*, but at the same time they make possible the achievement of a higher degree of *arithmetical internalism* in al-Karaji and as-Samaw'al, suggesting that *arithmetical internalism*, algebraic calculus, and the "transformation" approach, all belong naturally to a same *Semantical Field*; it also suggests an understanding of "solving equations *algebraically*" as a particular instance of a knowledge developed within this *Semantical Field*, and meaningful only within it.

There is, then, an important consequence for the teaching of algebra, as the main objective shifts into establishing an *algebraic mode of thinking* which drives the development of the instruments to operate algebraically—instruments which will support and clarify that mode of thinking; the natural context for this process seems to be not that of solving numerical equations—by itself or as tools to solve problems—but that of transforming, *arithmetically, internally* and in purposeful ways, algebraic expressions.

In almost all Islamic algebraists, we find the use of "geometric illustrations" at one point or another. As a rule, those diagrams incorporate lines and areas of unknown length, which are essential part of the proofs; this is an instance of an *analytical* but *non-algebraic* model, one which can offer us insights into how learners can deal with the unknown and at the same time avoid its *arithmetical* manipulation. The use of those models, however, restricts to positive only the numbers that algebra can deal with; also, the acceptance of zero as a number is problematic, as it has no sensible geometric representation.

The concept of number that seems to underlie the drive towards an *arithmetic internalism* in Islamic algebra, is one derived from the possibility of calculating with them, whole-numbers, fractions, or *surds*. The existence of an arithmetical treatment of *surds* from al-Khwarizmi on, indicates, however, that irrational numbers were accepted in themselves, and not legitimated only by their rational approximations.

The school students which took part in our Experimental Study, had a considerable experience in calculating, and for them, whole-numbers, fractions, decimals, *surds*, and negative numbers, are all numbers one can calculate with. Nevertheless, in the process of solving problems, different types of numbers are many times dealt with differently, indicating that: (i) the models underlying the solutions were not *arithmetically internal*; and, (ii) as a consequence, one should expect to find out that for those students the development of an algebraic knowledge is not perceived, *in that context*, as a suitable pursuit: if able to do it for the sake of school, the "rules" are usually forgotten as soon as the context ceases to exist (leaving school or changing the subject of the maths classes); otherwise, they sadly fail to grasp the most basic principles of the subject. That the introduction of algebra represents one of the critical points in school mathematics, is well known, and we think our interpretation provides an explanation for a substantial part of the obstacle.

We finish this section, by again stressing that the development of an algebraic knowledge in Islamic mathematics is strongly related to the requirement of producing an arithmo-algebraic solution of problems; transposed to the didactic context, this points out to the need to shift the focus of the teaching of algebra—and for that matter, of the research on the teaching and learning of algebra—from the contents of algebra into the ways of producing it, and from *solving problems* to *exploring representations of a situation*, ie, *exploring arithmetico-algebraic models*.

3.4 Hindu Arithmetic and Algebra of the Period AD 200-1200

In this section we will present some facts about Hindu arithmetic and algebra, but with the only and express purpose of highlighting some characteristics of the development of Islamic algebra that cannot be understood without this added information.

Because we will concentrate on factual information only, we decided to use a single source, a reliable one in this aspect, namely, Morris Kline's *Mathematical Thought From Ancient to Modern Times*. His remarks on Hindu arithmetic and algebra are contained on pages 184 to 188 of volume 1, and the title of the section in which it is presented is exactly the title we decided to use in this section, if not for its precise descriptive value, as a sincere acknowledgement of the source.

Kline says that up to 200 AD, Hindu mathematics is limited to a few geometric and arithmetic formulas. He also says that during the first part of the period in question, Hindu mathematics was influenced by Greek mathematics, but he does not say in what precise way. We leave the matter of the possible sources of Hindu mathematics here, as it is not relevant to our purposes.

The names of Aryabhata (b. 476), Brahmagupta (b.598), Mahavira (9th century), and Bhaskara (b. 1114), are given as the important ones in Hindu mathematics of the period.

Since at least Mahavira, zero was accepted as a "full" number, as he "...says that multiplication of a number by 0 gives 0, and that subtracting 0 does not diminish a number." (p185). Also, at least since Brahmagupta, negative numbers were used to represent debts.

In the Hindus, Kline says—without, however, specifying a date—we not only find the reduction of quadratic equations to only one type, a reduction made possible by the fact that they accepted zero as a number, which could stand for itself at one side of an equation[90], but we also find the acceptance of negative roots of an equation, and the acknowledgement of two roots for quadratics. Those characteristics are certainly present in the algebra of Bhaskara, to whom it is said we owe the general formula for the solution of quadratics—with the exception, of course, of complex roots[91].

Hindu mathematics also acknowledge *surds* as numbers, and had an arithmetical treatment of them—again, the earliest date we find in Kline is that of Bhaskara (12th century), who says, "Term the sum of two irrationals the greater surd; and twice their product the lesser one. The sum and difference of them reckoned like integers are so," together with the numerical example,

$$\sqrt{3} + \sqrt{12} = \sqrt{(3 + 12) + 2\sqrt{3 \cdot 12}} = \sqrt{27} = 3\sqrt{3}$$

which, Kline observes, is the application of

$$a + b = \sqrt{(a+b)^2} = \sqrt{a^2+b^2+2ab}$$

with $a=\sqrt{3}$ and $b=\sqrt{12}$.

Their arithmetic was completely independent of geometry.

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[90]Kline says that the reduction to one type of quadratic was due to the acceptance of some of the coefficients being negative, but this is clearly insufficient.

[91]The idea of the square root of a negative number is considered by Bhaskara, to be discarded as impossible: no square gives a negative number.

In relation to the use of symbolism, we think it is better to quote a full paragraph in Kline:

> "They used abbreviations of words and a few symbols to describe operations. As in Diophantus, there was no symbol for addition; a dot over the subtrahend indicated subtraction; other operations were called for by key words or abbreviations; thus *ka* from the word *karana* called for the square root of what followed. For the unknowns, when more then one was involved, they had words that denoted colors. The first one was called the unknown and the remaining ones black, blue, yellow, and so forth. The initial letter of each word was also used as a symbol. This symbolism, though not extensive, was enough to classify Hindu algebra as almost symbolic and certainly more so than Diopahntus' syncopated algebra." (p186)

As we have already said, Islamic mathematicians, from al-Khwarizmi on, were informed on Hindu mathematics. The question arises, then, as why in Islamic mathematics, the acceptance of zero as a number has to wait until as-Samaw'al (12th century), while in the Hindus it appears as early as the 9th century. A provisional answer might be provided, by referring to the tension between arithmetical treatment and geometrical demonstrations, as well as by a reference to an influence of the Greek conception of number as "number of something"; we saw, however, that the Islamic commitment to an ontologically defined *number* is much weaker than in the Greeks, which leaves the former as the most likely answer, specially when we consider that the Hindus never provided any sort of proof, and the obstacle of a "geometric illustration" would not arise. Nevertheless, more important, to us, than to answer such historical question, is to point out to the clear fact that the technical aspect of a knowledge, "adaptable" as it might seem from the point of view of our conceptualisation, to another culture's body of knowledge, will only be accepted if it has a place in the conceptual framework of the "adopting" culture, and while the dispute between arithmetic and geometry as the foundation for algebra is not resolved, zero as a number cannot belong to Islamic algebra.

On the other hand, the clear fact that the Hindu notion of number is based on a calculating practice, ie, in numbers as they are used, provides us with another indication of the Islamic conception of number, as they certainly borrowed in notation and calculating techniques from the Hindus.

In relation to the use of symbolism and syncopated forms of notation, one has again to raise the question as to why the Hindu custom failed to motivate Islamic authors, and we are again led to the importance of the written word in Islamic culture, as well as to the question of the conditions under which that which might appear to us as a mere "technical" aspect, can be subscribed by another culture.

3.5 ASPECTS OF CHINESE MATHEMATICS

INTRODUCTION

The Chinese civilisation is a phenomenon unequalled in the history of mankind. Its beginning is dated at about the 21st century AD, with the first Xia Dynasty, and until today Chinese culture is seen as retaining a strong personality, despite the transformation of China into a Republic (1912) and despite the substantial changes in cultural policy undergone after the Communist Revolution led by Mao Tse Tung of 1949.

The *I Ching*, or *Book of Changes*, used as an oracle, but also gradually transformed into a manual on how to conduct wars and public affairs, pre-dates the 11th century BC, and it is seen by many as the first book ever produced[92]. Between 770 and 480 BC we find a decimal system for numeration, and it is plausible that by 220 BC counting rods—which we will later describe—are already commonly used for calculations (Yan and Shiran, 1987, p7). During the first Han Dynasty, in the Western Han period (206 BC-24 AD), the *Zhoubi suanjing*, the oldest mathematical classic of China, appears, and in it we find the *Gougu*, "Pythagoras' theorem." During the Eastern period of the Han Dynasty (25-220 AD), the *Nine Chapters on the Mathematical Art* appears, the book which is considered to be "the most important of all ancient Chinese mathematical books." (Yan and Shiran, 1987, p270) The whole chronology of Chinese mathematics, can be found in Yan and Shiran (op. cit., Appendix 3) and also an excellent description of the Chinese mathematical treatises, with comments, in Martzloff (1987).

It is only in the 17th century that the *Elements* of Euclid are translated into Chinese (by Xu Guangqi and Matteo Ricci, a Jesuit missionary; by 1607 the first six Books had been translated, while the other Books had to wait until 1856), and Yan and Shiran (op. cit., p190) call this period "the first entry of Western mathematics into China." Chinese mathematics certainly interacted with Hindu and Islamic mathematics, at different periods and to different extents. According to Mikami (1913, p56), the contact with the Hindus became "official" in 65 AD, when "...the Han Emperor Ming-Ti dreaming of a 'golden man' had sent a messenger to India"; Mikami says that

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[92]C.G. Jung, the psychoanalyst, took a great interest in the *I Ching* as a powerful symbolism, which can be used in our search for an understanding of the mind; in 1949, he wrote a preface to the English edition of Wilhelm's translation of the *I Ching*. Jung was particularly interested in the concept of *synchronicity* as an "acausal connecting principle.".

the Hindus exerted considerable influence in art, literature and, to a lesser extent, astronomy and "calendrical arts," but Chinese arithmetic remained unaffected, although there is clear evidence that Hindu mathematics had been studied by the Chinese[93]. The contact with the Arabs can be traced back to the 7th century, through the Tazy Sarracens (Mikami, op. cit., p98), but it seems that this exchange will intensify only after the Mongol invasion (1271, the beginning of the Yuan Dynasty of Kublai Khan), which will affect both Muslims and Chinese (see Martzloff, 1988, p94ff, and Mikami, 1913, p98ff) The possible influence that Mikami, for example, finds from the Arabs, is always connected with Chinese astronomy. But, Mikami says,

> "...we are utterly at a loss when we try to illustrate concretely the influences exercised from without upon the mathematics of the Chinese." (op. cit., p108)

The important aspect here, is that although a contact is certain to have occurred, the actual influence, in the form of directly absorbed methods, translations, or foreign mathematicians being quoted, is visibly small. As a result, one can safely look at Chinese mathematics, in particular Chinese arithmetic and "algebra," as a self-contained body of knowledge, and reasonably expect characteristics of Chinese conceptualisation to apply with some uniformity to different historic periods.

As we will see, the whole of Chinese mathematics leans towards the "concrete"; in relation to number, and several Chinese authors express an understanding of number that could be easily identified with the Pythagorean view, both in its mystical and in its ontological aspects. This view however, is, in Chinese mathematics, earlier than Pythagoras.

SOME ASPECTS OF CHINESE "KNOWING"

Master Chen talks to Rong Fan, a student who could not grasp the principal idea in his explanation, even after a few days:

> "This is because you are not familiar with your own thought. ... you still have not got things clear. that is to say, you still cannot generalise what you have learnt. The method of calculation is very simple to explain, but it is of wide application. This is because 'man has a wisdom of analogy', that is to say, after

[93] It is far beyond the possibilities of this dissertation to undertake the study of this historical question, but the right line of inquiry should be, we think, to examine Hindu and Chinese conceptualisations of mathematics and try and determine to what extent the Chinese framework could not absorb specific parts Hindu mathematics.

understanding a particular line of argument one can infer various kinds of similar reasoning. ... So by having people learn similar things and observe similar situations one can find who is intelligent and who is not. To be able to deduce and then to generalize, that is the mark of an intelligent man. ... If you cannot generalize you have not learnt well enough." (Yan and Shiran, 1987, p28)

This passage is exemplary of the Chinese way of saying things: don't say too much, don't explain too much. In the preface to Martzloff (1988), Jacques Gernet says that this inclination towards allusion and conciseness is well in accord "au génie de leur langue," and that the Chinese, on the same basis, "desecrated the stiffness of formal proofs." Moreover,

"...cette horreur du discursif va de pair avec une predilection pour le concret. Leur pédagogie mathématique le montre bien, où le cas particulier suffit à illustrer le général, où les comparaisons, les rapprochements, les manipulations de chiffres, les decoupages, recompositions et retournements de figures permettent de constater sur le champ et *de visu* l'exactitud des solutions." (Gernet, in Martzloff, op. cit., pVII)

Argumentation in Chinese mathematics—in particular in commentaries—is based on *methods*, for example (i) going from the particular to the general, using a well chosen example, (ii) reasoning by comparison, transposing the debate to a situation better known but semantically distinct, (iii) analogies, as in explaining the extraction of a cube root by evoking the process for extracting the square root, (iv) recourse to heuristic procedures, as recommending the use of dissection in dealing with geometrical figures. The closest to the notion of proof that traditional Chinese mathematics gets is the notion of *making visible the mathematical phenomenon* (Martzloff, op. cit., p70), as they are manifest within the tangible things and not within abstract essences.

From this point of view, we should not try to apply the distinction between "theory" and "practice" to Chinese mathematical texts. Martzloff (op. cit., p40) suggests that instead we use the distinction between "pedagogical tools" for the former—used to teach the calculation techniques—and "application manuals" for the latter—or, how to employ the calculating techniques[94].

According to Martzloff (op. cit., p48ff), it is possible to group the problems used in Chinese books into four broad types: (i) Real problems, (ii) Pseudo-real problems, (iii) Recreational problems, and (iv) Speculative problems. Real problems are so faithful to the actual situations of the time, that their texts can be used to support

[94]In all cases, the form of presentation is "statement of problem + numerical solution + statement of solving procedure."

research on the social-economical life of then. Pseudo-real problems are provided to overcome a situation—well-known to us, mathematical educators—where the real problems offered only too simple or too complex problems, and which can be better seen as *exercises*. With a similar objective were produced the Recreational problems. Speculative problems are not very abundant, and Martzloff actually says that it is surprising they existed at all, given the contextualised nature of Chinese mathematics[95].

The evidence on the two last paragraphs must not be taken as meaning that Chinese mathematics can be reduced to an empirical and utilitarian body of knowledge; the extent of its achievement goes against such interpretation, providing far more than the "necessary" for practical uses[96]. We must understand, instead, that Chinese mathematics is contextualised in a slightly different sense, namely, that mathematical concepts and objects are not elaborated independently of the problems they are intended to solve:

> "Les termes chinois ne sont pas définis **in abstracto** à l'issue de procédures platoniciennes, mais se trouvent plutôt engagés dans une dynamique incessante qui les rend objet de continuelles négotiations de sens." (Martzloff, op. cit., p59)

In view of such a conceptualisation, the damage caused by "translating" Chinese mathematics into our modern algebraic notation is immense, not only because we will be attaching to the objects it denotes a generality they do not necessarily have, but also, and more misleading, we will introduce a permanence of that object *across* the whole of mathematics, which many times, as will see, does not exist.

SOME ASPECTS OF CHINESE MATHEMATICS

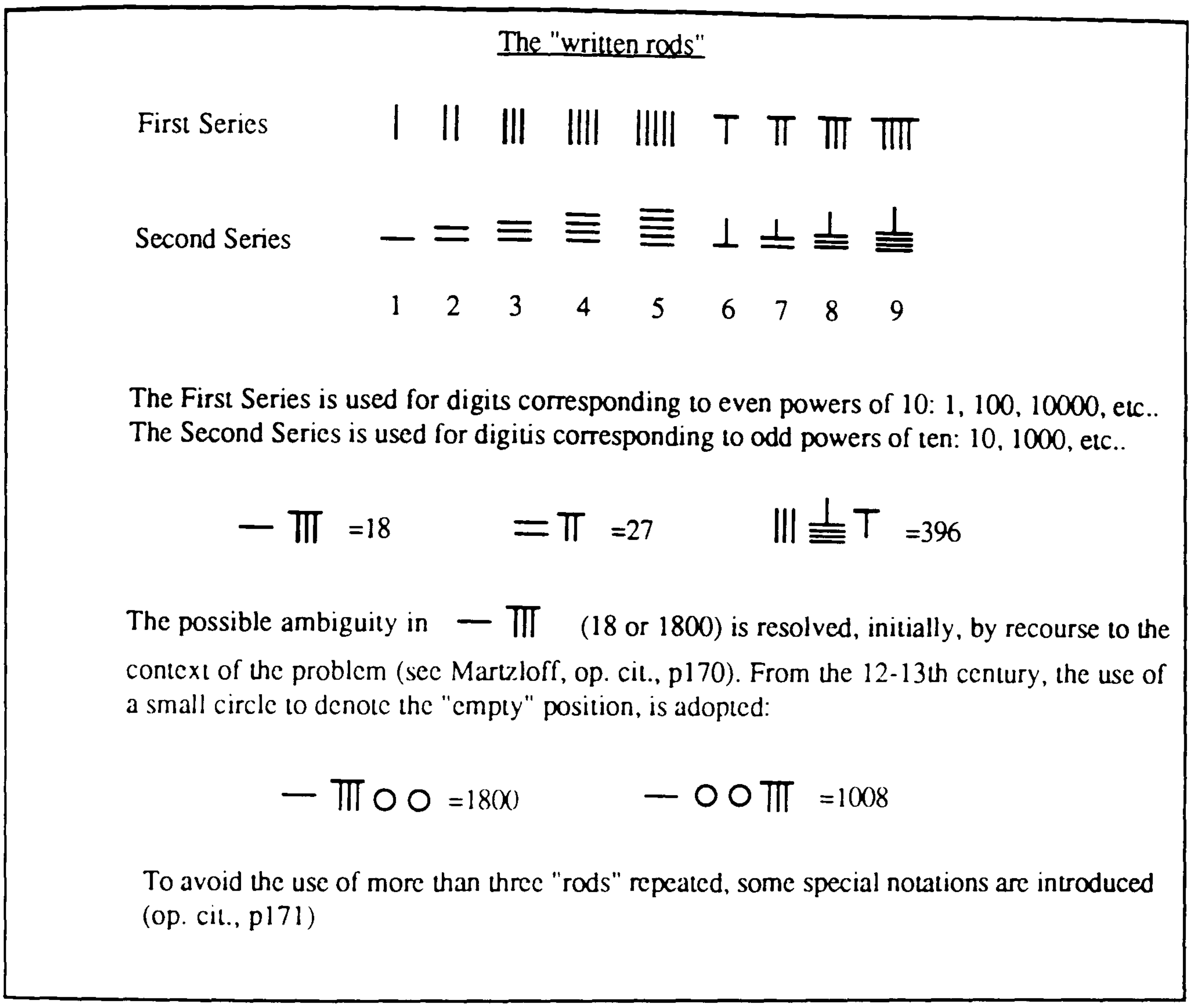
The unique feature of Chinese mathematics, from a very early time, is the pervasive use of counting rods, which cannot be traced back to any other mathematical culture[97]. Those rods were "...small bamboo rods. Ancient Chinese mathematicians operated with these short bamboo rods by arranging them into different configurations to represent numbers and then performed calculations using these rods." (Yan and Shiran, 1987, p6) The extent to which the calculating rods are characteristically linked

[95] "Aucun auteur chinois ne se préoccupe de 'théorie des nombres' (sauf tardivement, au 19e siècle). Certes, il est bien exact que dans maintes oeuvres mathématiques chinoses on trouve des triangles rectangles dont les longueurs des côtés sont des nombres entiers ... Mais aucun auteur chinois n'écrit jamais *explicitement* que la determination de triangles rectangles en nombres entiers constitue le but de ses recherches." (Martzloff, op.cit., p280)

[96] Neugebauer, quoted by Martzloff (op. cit., p42) says that, "The mathematical requirements for even the most developed economic structures of antiquity can be satisfied with elementary household arithmetic which no mathematician would call mathematics."

[97] Cf. Mikami (op. cit., p99)

to Chinese mathematics, is indicated by the fact that the specific names used to designate "mathematics" are many times a composition involving the unit *suan*, which originally designates "a set of concrete objects used to calculate—the rods." (Martzloff, op. cit., p36)[98]



The use of counting rods had at least two consequences of immediate importance: (i) the natural development of the practice of recording numbers and calculating processes using faithful copies of the arrangements with the rods, a step which allows direct calculation even in the absence of the rods; and (ii) as a consequence, the introduction of a matrix-like notation as a standard form of representation, which assumes different roles in different contexts. The "written counting boards" are used to represent fractions (Yan and Shiran, op. cit., p17), the elements in the process of extracting the square root (Martzloff, op. cit., p213), or, in a form very similar to our "matrix of coefficients," the basic setting in the process of solving sets of simultaneous linear equations (Yan and Shiran, op. cit., p47).

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[98] Yan and Shiran (op. cit., p11) add that, "Calculating by means of counting rods is the key to understanding the mathematics of ancient China."

Martzloff (op. cit., p181ff) shows that a number of Chinese texts indicate that the idea of a decimal notation probably arose as a generalisation of the practice of calculating with counting rods[99]; decimal numbers, however, are almost always linked to units of measurement, and a number as 9.62 would never appear in itself, as a "pure" number, but "comme: 9◊ 6◊ 2◊, où les ◊ représentent des caratères chinois désignant des unités concrètes," ie, the ◊ can be, for example, the equivalent of "metre, decimetre, centimetre." The actual practice in using the "rods notation," was to indicate only the principal unit, in a way that resembles our use of the decimal point.

The Chinese developed a method to solve sets of simultaneous linear equations, called the *Fang Cheng—Method of Rectangular Arrays*, which is, in form, Gauss's elimination method. The *fang cheng* is first introduced in the *Nine Chapters on the Mathematical Arts* (around 200 AD), and in the same treatise are introduced the notions of positive and negative numbers, and the methods for adding and subtracting them (Yan and Shiran, op. cit., p46). Negative numbers are used to represent, for example, "paying out" and positive numbers to represent "receiving money." In representing the coefficients, either rods with different cross sections or colours—when using the actual rods—or different colours, special marks, or writing the rods obliquely—in the case of the written form— are used to distinguish positive and negative numbers.

Negative numbers, however, are never to be found, in Chinese mathematics, in the statement of a problem, or as a solution to a problem: *they belong entirely to the context of the method*, without which it would be impracticable[100]. The same is true of the zero, which use seems to be initially linked to representing the absence of a digit, or in this case, of a coefficient. Martzloff (op. cit., p186) notes that the use of negative numbers was not transferred to other mathematical contexts in which their use would allow a much greater simplicity of treatment. This aspect of Chinese mathematics is of extreme interest to us, because the same kind of strong link between the acceptance of specific mathematical objects and specific mathematical contexts can be observed in the mathematical behaviour of learners.

The *fang cheng* did not directly use multiplication and division, only addition and subtraction, no matter the size of the numbers involved; it is natural, thus—in the perspective of the "method-restricted" approach of Chinese mathematics—that only rules involving addition and subtraction of positive and negative numbers are given,

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[99]"Le **Xiahou Yang suanjing** ... explique que, pour multiplier ou diviser un nombre par 10, 100, 1000, 10000, il suffit de faire avancer ou bien de faire reculer les baguettes qui le représentent de 1, 2, 3, 4 rangs decimaux sur la table à compter."

[100]Martzloff offers a way of emphasising this aspect, by noticing that negative numbers appear in Chinese mathematics before than in any other mathematical culture: "Cela peut sembler paradoxal dans la mesure où en Chine plus que partout ailleurs la notion de nombre s'est toujours inscrite dans le concret." (1988, p185)

including those involving zero. In the 3rd century AD, Liu Hui, a commentator on the *Nine Chapters...* proposed a solution to avoid the possibly huge number of subtractions to be performed in the *fang cheng*, by using a "cross multiplication" similar to that used in school algebra solutions; despite its practical advantage, Liu Hui's method faced the problem of overcoming the "debts-credit" interpretation of negative and positive numbers, which seriously hindered the acceptance of their multiplication; for many centuries, Liu Hui's version of the *fang cheng* is not widely accepted. It is only in 1299, and also in the context of the "refined" *fang cheng*, that the rules for the multiplication of positive and negative numbers are stated, without—as one would expect—any justification.

There is another development we want to examine. In the *Nine Chapters...*, methods are given for the extraction of the square and cube roots of a number. Both methods depend on the positional decimal notation, and also on the equalities

$$(a+b)^2 = a^2+2ab+b^2 = a^2+ (2a+b)b \quad \text{and}$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3 = a^3+[3a^2+3(a+b)b]b$$

and are essentially the same methods used today. It is important to notice that those methods were based on a geometric perception of the equalities above (Martzloff, op. cit., p210ff; Yan and Shiran, op. cit., p53)[101].

At that point, the important technical achievement was to extend the method and apply it to the numerical solution of quadratic and cubic equations (see Yan and Shiran, op. cit., p52ff). In the middle of the 11th century, Jia Xian introduces another method for extracting square and cube roots, which is easily generalised for extracting roots of any degree (op. cit., p120). The extension of the method of Jia Xian to be used to the numerical solution of higher degree polynomial equations, required the acceptance **within this method**, of negative coefficients in all positions, where before, in the methods for the extraction of square and cube roots, and in their extension to solve quadratic and cubic equations, the coefficients were required to be positive. The extension is achieved between the second half of the 12th century and the first half of the 13th century (op. cit., p128).

The extension of the methods of Jia Xian, then, involves two aspects worth highlighting: (i) the acceptance of negative coefficients is local and belongs to the

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[101] The generic term to indicate the extraction of roots, **kaifang**, literally means "to open", "to dissect", "to dissociate", "to decompose", the square or the cube, and the expressions used to designate some of the coefficients used in the calculations use the terms **yu** and **lian**, that mean "corner" and "border", respectively. Martzloff says that "Rien de cela n'est gratuite. C'est la marque indélibile du rapport étroit qui existe entre la geométrie et la logique des opérations." (1988, p211, our emphasis)

method—although by that time negative coefficients had been used in the *fang cheng* for more than a thousand years; and (ii) the development of a general method for solving polynomial equations numerically implies the abandonment of the geometrical intuition, certainly in favour of a numerical one—based on the possible generality of the calculating board.

Point (i) simply reinforces an aspect of Chinese mathematics we had already examined.

Point (ii), however, brings a new insight. We can look at the development of Chinese mathematics in two directions. It certainly lacks a "horizontal" generalisation, ie, the concepts and objects of one method do not naturally "spill" into other methods. The extension of Jia Xian's method however, takes a "vertical" line, that of a development *within the method.* If we examine the conceptual changes involved, it is clear that there is an "arithmetisation" of the problem—in quotes to avoid confusing it with the arithmetisation of Islamic algebra, at the same time the *form* of the method is to be preserved, ie, the type of manipulation involving the coefficients as "digits." It is because this is firmly established that the acceptance of negative coefficients can occur: because the method will still be recognisable[102].

In relation to the notation used in Chinese "algebra", it is clear that the the counting board and its written counterpart provide a notational form that is very strong in supporting the development of the methods of Chinese arithmetic-algebra. In relation to the *fang cheng*, for example, the number of "unknowns" is not limited by any notational restriction, as the board can be extended at will; through the use of the board, any numerical example is made perfectly general, specially given the freedom in the use of negative coefficients. In other cases it is not so, until related conceptual problems are resolved, but the board-form remains in use. Martzloff (op. cit., p249ff) also points out to the conciseness of the Chinese language, as offering a compact description of mathematical statements. At the beginning of the 18th century, there is a first attempt at introducing Western algebra in China, through the efforts of Jesuit missionaries, but it fails. On a second attempt, the missionaries develop a new notational system, which, however, is not superior to those employed by 13th century Chinese mathematicians, and is thus refused. As a result, one finds that around 1850 Chinese mathematics were practically ignoring all forms of modern algebraic symbolism (op. cit., p196).

In Chinese mathematics, notation and nomenclature, as much as mathematical concepts and objects, belong, to a great extent, to each method, and the resistance to

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[102]Martzloff (op. cit., p218), says that the historical evidence available is not enough to allow us to understand which type of generalisation the passage from 2 or 3 dimensions to higher dimensions was involved in Chinese mathematics. We think that although not offering a full answer, our approach offers a fruitful line of investigation.

accepting Western algebraic notation probably reflects the rejection of a uniformity and "decontextualisation"—mathematical contexts, that is—which does not fit into the framework of Chinese mathematics.

CONCLUSIONS

In Chinese mathematics, there is a lack of *intention* to investigate the mathematical instruments that make possible, extend, and justify the methods—eg, the algebraic calculus. On the other hand, although numerical problems originate in real or pseudo-real statements, their treatment is numerical—as opposed to geometric, for example. The tension between the "concreteness" of the problems and the mathematical freedom of the methods, begins to be resolved as soon as negative number are admitted into the *fang cheng*, implying a degree of *internalism*; the *analiticity* of the method is to be seen, for example, in the fact that after the "elimination" is completed, the values of the unknown that have been determined are "substituted back," ie, the "unknown" is indeed represented in the configuration using the "written counting board." Although algebra is not constituted into a theory, *algebraic thinking* is behind the development of many of the methods.

From the point of view of the learning of algebra—and, most probably, from the point of view of the learning of other pieces of mathematical knowledge—the most relevant aspect of Chinese mathematics is the independence between what we termed *horizontal* and *vertical* development. It is a crucial point, in relation to our research question, that this *vertical* development is by no means sufficient to guarantee that a *horizontal* development will also occur; the question naturally arises as to which are the conditions under which the *vertical* development holds at least a good chance of resulting in *horizontal* development. Given the example of Chinese mathematics, it seems that the condition which was lacking there, and which establishment should be a target of the teaching of algebra, is the notion of a *theory*, ie, a body of knowledge that aims at itself, no matter what the original motivations might be, and that is intended to amplify the possibilities of accomplishing what begins as an *intention* scarcely realised. In the case of *algebraic thinking*, this *intention* may begin as that of modelling different problems with numbers, and from there evolving to an *internal* way of treating this *numerical* model; or it may begin as the *intention* of examining what can be found in common in locally distinct methods.

The existence of a "standard" notational form—such as the "written calculating board"—that is unable to provide, by its form, the link between the different methods in Chinese mathematics, points out to the fact, already examined in previous sections of this chapter—but here strongly highlighted—that the use of any notation, in any context, can only be understood in view of an understanding of the *objects* intended by that notation; in other words, but with a slight twist, the uniformity of notation does not guarantee the generality of the object if intends, nor the general applicability of the method it describes. We have pointed out, earlier in this section, that the "translation" of Chinese mathematics into our algebraic notation would introduce a *horizontal* reach that the concepts and objects do not necessarily have. From the point of view of mathematical education, the forced and undue *horizontalisation* of mathematical concepts and objects—be it through a notational uniformity or through an idealistic, intention-imposing, reading of the learner's knowledge—can lead to tensions which remain hidden and are difficult to locate and resolve.

3.6 ASPECTS OF THE DEVELOPMENT OF ALGEBRA IN EUROPE

INTRODUCTION

It would be totally beyond the possibilities of this dissertation, to attempt even a modestly thorough examination of the historical development of algebra in Western culture, from the Middle Ages onwards. A number of books have covered the subject, from different perspectives and in varying depth: in Kline (1990), we find what is probably the most complete survey of the historical development of the whole of mathematics, and it provides substantial material for one to investigate the development of algebra within the various branches of mathematics; Bottazzini (1986) and Crowe (1967) deal extensively with the history of branches of mathematics that are closely connected with the development of algebra, the former with Calculus, the latter with Vectorial Analysis; Novy (1973), van der Waerden (1985), and Klein (1968), all examine the historical development of algebra, but from points of view which are quite distinct, and to a great extent complementary.

Our approach will consist in examining, with the support of a few selected examples, two aspects of the development of algebra: (i) its gradual *internalisation*, ie, the abandonment of *extrasystemic* interpretations of algebra as a way of justifying its procedures; and, (ii) the development of new forms of notation.

As we have seen in the previous sections, in each of the mathematical cultures examined the tension between *method* and *object* was dealt with differently. In Greek mathematics, the *object* is always established by its ontology, and the methods develop around those ontologically determined objects; in Islamic mathematics, the ontological commitment is much weaker, but the tension still exists, and can be seen in the dependence of its procedures on geometrical models; in Chinese mathematics, the methods are local, and mathematical objects and concepts "belong," to a great extent, to each method; finally, Hindu mathematics presents itself with a much greater technical freedom than the other three, but fail to examine and organise the body of knowledge they had produced or absorbed, and the tension stays completely hidden, as the focus remains on simply providing "solving formulas" for specific problems[103].

We shall now investigate how this tension presents itself and is eventually resolved in Western mathematics.

THE NOTION OF NUMBER AND THE SOLUTION OF EQUATIONS

Historians of mathematics generally agree that the first name worth mentioning in European Middle Ages, is that of Fibonacci (b. 1170)[104]. In a number of studies, the similarity between Fibonacci's work and that of Islamic mathematicians has been pointed out[105]. Fibonacci is also mentioned by Cardano as "a trustworthy source" on the "art of Mahomet the son of Moses the Arab [al-Khwarizmi]," (Cardano, 1968, p7) and it is known that he travelled extensively and studied Islamic mathematics in North Africa and other places (Cf. Fibonacci, 1987, pxvi).

His two main works are the *Liber Abacci* and the *Book of Squares*. The *Liber Abacci* is a book on arithmetic and algebra, which was for a long time a standard textbook and introduced the Hindu-Arabic notation for numbers in Europe[106]; the *Book of Squares* is a collection of propositions on square numbers and indeterminate analysis. As opposed to Diophantus' *Arithmetica*, Fibonacci's book always provides solutions that are general not only in content, but also in form (letters are used, sometimes as in denoting a segment of line, eg, *.ab.*), but they are, however, *synthetical*; the problems are solved with little recourse to geometric arguments[107]. He was not troubled by *surds*.

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[103] See van der Waerden (1985, p190).

[104] Cf. van der Waerden (1985), M. Kline (1990).

[105] For a brief survey of those studies, see Sigler's preface to the *Book of Squares* (Fibonacci, 1987).

[106] A few excerpts of the *Liber Abacci*, translated into English, can be found in Fauvel and Gray (1987).

[107] In the preface to his edition of the *Book of Squares* (Fibonacci, 1987), Sigler says that, "The geometrical algebra used in Leonardo [Pisano, Fibonacci] is that presented by Euclid in the Elements..." This "geometrical algebra" must be understood in the sense of the numerical reinterpretation—already undertaken in Islamic algebra—of Euclid's *Elements*; the geometric diagrams used in the *Book of Squares* are rarely more than a support for the letters used in the text, or support for a combinatorial argument. In relation to Fibonacci's use of lines to represent numbers, Woepcke (1982, p27, footnote) says that, "Fibonacci se sert de ces lignes uniquement pour désigner, d'une manière plus concise, les quantités qui sont l'objet ou les résultat des opérations algébriques."

By the time of the European Renaissance, the efforts in algebra are directed towards the algebraic solution of cubic and quartic equations, and it is in this context that Cardano publishes, in 1545, his *Ars Magna* (Cardano, 1968).

Cardano's book is throughout concerned with solving quadratic and cubic equations, and although also presenting and solving geometric and "real-life" problems, it is clear that the book primarily intends the mathematical procedures. Irrationals are treated effortless, and both algebraic treatment of equations and geometrical demonstrations of solving rules are found in the book. We think, however, that it is in the contents of Chapter XXXVII ("On the Rule for Postulating a Negative"), which represents a remarkable intellectual achievement for the time, that we will find the theme through which we can follow the development of algebra in the Western culture, at the same time it contains the seed of the approach by which this difficulty is finally solved.

Under the heading of *Rule II* (op. cit., p219), Cardano solves the problem of dividing 10 into two parts such that their product is 40. He says that "it is clear that this case is impossible,"[108] but, nevertheless, he takes on the problem, applies the procedure for solving the quadratic equation to which the problem is reduced, and as a result reaches the expressions $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ [(109)]. To check that those two expressions indeed verify the problems conditions, Cardano simply multiply them *arithmetically*: $(5 + \sqrt{-15})(5 - \sqrt{-15}) = 5{\cdot}5 - 5{\cdot}\sqrt{-15} + 5{\cdot}\sqrt{-15} - (\sqrt{-15})^2$, which, of course produces $25-(-15) = 40$. It is not shown that the sum of the two expressions is 10. Witmer, the translator of (Cardano, op. cit.), observes that the original expression accompanying the multiplication, *dimmissis incruciationibus*, can mean both "putting aside the mental tortures," which is used in the main text, but also "the cross-multiples having cancelled out," used in Smith (1959, p202), a play on words.

Nowhere else in the *Ars Magna* square roots of negative numbers are mentioned, and the subject is left to one of the last chapters of the book (which is in forty chapters). Van der Waerden (1985, p56) points out that in Chapter I, where the number of positive and negative roots of cubic equations are discussed, Cardano carefully avoids the imaginaries—which appear in the *casus irreducibilis* when the solution is done by radicals—by an adequate choice of the coefficients, and Sanford (in Smith, 1959, p201) points out that "Cardano...spoke of the complex roots of a certain equation as 'impossible'." Negative roots are normally accepted, but negative numbers are called "fictitious" numbers, as opposed to "true" (positive) ones (Cardano, op. cit., p15); a problem which cannot be solved "with" a positive nor "with" a negative number is a "false problem." (op. cit., p217).

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[108]Possibly because the maximum real value is obtained by squaring half of the ten.

[109]It is clear that the geometric demonstration of the solving formulas does not apply any longer.

Cardano's "play on words" would be, then, an indication of the tension between the "ontological" impossibility and the "operational" reality of imaginary quantities in the mind of the mathematician; an *internal* meaning is given to the otherwise unintelligible $\sqrt{-1}$, and it is acceptable enough to deserve mention in the *Ars Magna*, but not to be normally used in the rest of the book, where it would provide Cardano with the means to achieve a much greater conciseness and unity for the theory of quadratic and cubic equations. At the same time he acknowledges some form of "legitimacy" for the imaginary quantities, and together with the "sophistication" of the subject, Cardano asserts its uselessness (op. cit., p220)[110].

No ontology of imaginary quantities was available, and the manipulation of those "things" did, in fact, simply follow the rules of the arithmetic of real numbers, ie, they had a purely symbolic character. It is interesting to observe that their "right of existence", in Bombelli and Cardano, for example, is tied to the algebraic method, as negatives were tied to the *fang cheng* method in Chinese mathematics. But with the *fang cheng*, they are introduced only as a *necessary element of the method*, whereas in the case of complex numbers, they are at the same time a necessary element of the method and the result of exploring the possibilities of the method, ie, a *theoretical result*.

There is another aspect of the *Ars Magna* which is of interest. Cardano certainly had some insight into the relationship between the degree of a polynomial equation and the number of roots it has; this insight, however, was not entirely explored by him. First, because of the need to fully acknowledge complex roots, but also because of the need to acknowledge zero as a possible root. Second, and more important from the point of view of our research, there was the obstacle of the multiple roots.

The notion of "root" in Cardano—and in all algebraists before him, and also, for some time, after him—is thoroughly associated to that of a number which satisfies the arithmetical relationship proposed in the equation; the notion of root that allows for the understanding of multiple roots, is that of the decomposition of a polynomial into linear factors: $x^2+6x+9=(x+3)(x+3)$, and -3 is a double root of $x^2+6x+9=0$. The former notion is concerned only with an equation as a predicate, while in the latter it is

[110]Compare Cardano's opinion with Girard, in his *L'Invention nouvelle en l'algèbre* (1629) where he says that, "One could say: Of what use are these impossible solutions [complex roots]? I answer: For three things — for the certitude of the general rules, for their utility, and because there are no other solutions." (quoted in M. Kline, 1990, p253)

the root as part of an *arithmetical articulation*, that is central. The full recognition of multiple roots seems to be associated with the appearance of Coordinate Geometry[111].

This brief examination of the work of Cardano indicates the two trails to be pursued in our investigation: the transformation of the notion of number, and the changes in the understanding of the algebraic activity.

In 1545, Cardano accepted negative numbers, to the extent of having them as solutions of equations, and he also found a place for imaginary quantities in his work.

In relation to negative numbers, the development is far from "linear": still around the time of Cardano, Vieta (1540-1603) completely rejected negative numbers, but Harriot (1560-1621) would accept a negative number "by itself on one side of an equation," (Kline, 1990, p252); Stifel called them "absurd," but Bombelli (born c. 1530) decided "to consider the majority of the authors who up to now have written about [algebra], so I can fill in what they have missed out" (in Fauvel and Gray, 1987, p263) and produced not only an understanding of negative numbers, and rules to operate with them *by themselves*—and not only as terms in expressions—but also explicit rules to operate with "píu di meno," the square root of minus one (Fauvel and Gray, op. cit., p.265).

Still many centuries ahead, in the first half of the 19th century, the debate about whether negative numbers were "acceptable" was not yet settled when Peacock's Symbolical Algebra appears. Pycior (1982, p397) says that,

> "...even after exposure to De Morgan's defense of the negative and imaginaries, Frend...clung to his 'contentual' view of the mathematical sciences, according to which symbols stood only for clear and distinct ideas."

The objections to negative number were altogether simple: "How can a quantity be less than zero?" M. Kline mentions a more sophisticate objection and a consequence:

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[111] Montucla (quoted in a footnote by the translator, in Cardano, op. cit., p13) says that "Simple arithmetic would have thrown no light on the subject and it is only the application of algebra to curves which can make one understand the distinction of which we speak." As a matter of fact, it is only after Descartes and Fermat, that the Fundamental Theorem of Algebra is stated in full, although in 1629 Girard had asserted, without proving, that any complete algebraic equation—ie, one where none of the coefficients is zero—has as many solutions as the exponent of the highest term; the distinctive aspect of Girard's assertion, is that he "pointed out that if an equation admits fewer roots than its degree indicates, it is useful to introduce as many *impossible* [ie, complex] *solutions* as will make the total number of roots and impossible solutions equal the degree of the equation" (C.R. Adams, in Smith, 1959, p292).

"An interesting argument against negative numbers was given by Antoine Arnauld (1612-94), a theologian and mathematician who was a close friend of Pascal. Arnauld questioned that -1:1=1:-1 because, he said, -1 is less than +1; hence, How could a smaller be to a greater as a greater to a smaller? The problem was discussed by many men. In 1712 Leibniz agreed that there was a valid objection but argued that one can calculate with such proportions because their form is correct, just as one calculates with imaginary quantities." (1990, p252)

There are three points of interest here. First, the whole objection arises because the proportion in question is examined from the point of view of a concept of *order* which takes "being a part" for "being smaller," indicating an incorrect understanding of the structure of the real numbers[112]. Second, because the intelligibility of negative numbers had, since ancient cultures, been resolved by an appeal to *order*, the possibility of understanding their algebraic properties was serious hindered; we think that this is a most valuable insight for mathematical education, as it suggests that careful attention should be paid to distinguishing those two mathematical aspects of number. Third, although apparently unaware of the difficulties above, Leibniz "settles" the question by appealing to the *internal* consistency of the calculations done with such "unreal" numbers, an approach which, in fact, corresponds to assuming that the algebraic structure alone should provide intelligibility, ie, *meaning*, for those numbers; that he does not distinguish, from this point of view, negative and complex numbers, serves to clarify his approach .

At this point, it should be perfectly clear that the problem with negative numbers was that they were ontologically unsupported. But if in Greek mathematics, irrational numbers were not numbers, precisely because an extension of the accepted ontology for whole numbers could not be provided that accounted for the "incommensurability"— they resisted being counted—how come such objections were not raised against them in Europe?

The answer seems to be, that there had been, as we saw in relation to the Hindus and Chinese and in relation to Islamic mathematics, a substitution of a "calculating" understanding for the Greek-style ontology, but also, in the process, an association between positive numbers and geometric magnitudes was established, the "concreteness" of numbers tightly linked to the geometric figures themselves, as a representation of the *continuum*[113,114]. Much of Stevin's (c.1548-c1620) criticism of

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[112]See Novy (1973, p16ff), where it is pointed out that based on axioms derived from a geometric intuition, one arrives at statements such as $a>b \Rightarrow a+i>b+i$; see also Smith (1959, p59), where it can be seen that Wessel was aware of this difficulty and that he had correctly overcome it.

[113]M. Kline (1990, p251ff) says that around 1650, "... Pascal and Barrow said that a number such as $\sqrt{3}$ can be understood only as a geometric magnitude; irrational numbers are mere symbols that have

the Greek concept of *number*, is based on the properties of the Hindu-Arabic notational system, and he finds it worth to put forward and justifying the thesis that "one" is indeed a number, and also that "zero" is the "true and natural beginning," a "zero" that is totally identified with its notation, and only within the notational system acquires its meaning (Klein, 1968, p191ff). Wallis (1616-1703) had no restrictions against irrationals, and regarded Book V of the *Elements* as arithmetical in nature, while Descartes (1596-1650) accepted them as independent, but pointed out their adequacy to represent continuous magnitudes (Cf. M. Kline, 1990, p252).

The problem with complex numbers presented a much stronger challenge.

In Cardano, complex numbers are dealt with *internally*, but in Bombelli we find a much more dramatic situation.

He solves the equation

$$x^3=15x+4 \qquad \text{(I)}$$

following Cardano's rule, and arrives at

$$x=\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}} \qquad \text{(II)}$$

It is obvious, however, that $x=4$ is a solution of (I). Van der Waerden (1985) says that Bombelli,

> "...now investigates whether he can attach a meaning to the cubic root of complex number. More precisely, he tries to equate the first cube root [in (II)] with a complex number $p+\sqrt{-q}$"

and he finally arrives at,

$$\sqrt[3]{2+\sqrt{-121}}=2+\sqrt{-1} \qquad \text{(III)}$$

Bombelli's result raises an interesting question. In the case of linear equation, if the solving procedure results in, for example, $x+10=5$, a person that does not *conceive*

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no existence independent of continuous geometrical magnitude, and the **logic of operations with irrationals must be justified by the Eudoxian theory of magnitudes**." (our emphasis)

[114]Euler (1840, p2) says that, "...a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit...From this it appears, that all magnitudes may be expressed in numbers..."

negative numbers may rightly say, "the original equation has no solution"; in the case of quadratic equations, the situation is completely similar, because a negative discriminant, for example, immediately means that the original equation has no solution. In both cases the process is simply reduced to that case in analysis in which one arrives, from the initial suppositions, at a false statement, and the problem is found to be impossible.

With cubic equations a much different situation arises. Suppose, again, a person that does not *conceive* of imaginary quantities, and that person solves, as Bombelli did, equation (I) and arrives at the expression (II). Following the same reasoning as the one used with linear and quadratic equations, (I) has no real solution. But it has, and we are now faced with the *fact* that the method used—which is thoroughly based on clear assumptions, such as the possibility of substitutions and the solution of auxiliary quadratics—is not good enough to give the—already found by inspection—solution.

Bombelli's solution of the dilemma is paradigmatic of the way by which algebra will develop in Europe, and it involves two important steps: (i) to assume that the method of solution as an *invariant*, ie, to postulate that it indeed produces a solution if one exists; and, (ii) as a solution exists, and the method is correct, the *expression* reached must be transformable into a "recognisable" form. In both aspects, it is necessary that the reasoning be conducted *internally*—as it is the application of the method that produces the "discrepancy"—and *arithmetically*—as one is attempting to preserve the consistency of a method based on properties of the arithmetical operations. Finally, the process by which the expressions in (II) are given meaning, is *analytical*, as one starts with the presupposition of two *arithmetical articulations* being *equal*, and from there deriving the conditions that make the equality true. Above all, it is the preservation of *meaning* that is aimed at.

It is clear that the concept of equality has to undergo a substantial change if this process is to be possible. The notion of *calculation* cannot be any longer that of the possibility of applying algorithms that produce an *answer*; it is, instead, that of producing another expression which has a different *arithmetical articulation* but which can be *substituted* for the original one in all cases where it would belong[115]. In Bombelli and Cardano, this understanding is only anticipated, and it is not introduced as a paradigm for algebra. In the 20th century, however, we read in A. Robinson (1951, p4):

[115]This notion of equality applies, of course, to the solution of equations as we find in Diophantus and al-Khwarizmi, but it is important to emphasise that this was not the *intended* understanding in those mathematicians.

"1.5. *Definition of Equality.* We shall say that a relation of equality is defined in a given system of axioms if it includes a relation $E(x,y)$ which is symmetrical, reflexive and transitive, and such that every relation $F(x_1,\ldots,x_n)$ included in the system, it can be proved that equal objects can be substituted for one another as arguments,

$$(x_1)\ldots(x_n)(y_1)\ldots(y_n)\ [[E(x_1,y_1) \wedge E(x_2,y_2) \wedge \ldots \wedge E(x_n,y_n)] \supset [F(x_1,\ldots,x_n) \supset F(y_1,\ldots,y_n)]$$

where $\wedge$ and $\supset$ denote conjunction and implication respectively."

Although the technical development was available, the situation was not satisfactory, because of the lack of "logical explanation" for the imaginary quantities.

Descartes rejected complex roots of equations because although negative roots could be made positive by a suitable transformation of the equations, that is not the case with complex roots, and Newton (1642-1727) identified the existence of physical or geometrical solutions with the existence of non-complex roots for the corresponding equation.

The usefulness of complex numbers in algebra was gradually established, and to refuse them simply because they did not correspond to anything "in the real world" was to become a lost cause. However, mathematicians were still searching for a model that would render them more "acceptable." Argand (1768-1822) points out that his geometric representation of complex numbers[116]

"...tend, premièrement, à donner une signification intelligible à des expressions qu'on était forcé d'admettre dans l'analyse, mais qu'on n'avait pas cru jusqu'ici pouvoir rapporter à aucune quantitité conue et évaluable." (Novy, 1973, p120)

and Gauss gives a geometric *interpretation* of complex numbers, but never speaks of calculating with line segments or vectors. (op. cit., p123)[117,118]

Warren (1829) carefully examined and discussed the objections held against imaginary numbers, starting with the observation that "imaginary 'quantities' were capable of undergoing operations analogous to those upon ordinary quantities." (Nagel, 1935, p444) His conclusion was that "the operations of algebra were more comprehensive than the definitions and fundamental principles [of the ordinary quantities]" (ibid.) Warren further explains that imaginary numbers are a sign of

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[116]See, for example, Abbot (1985, p11ff)

[117]Gauss prefers, instead, the form $a+bi$ for complex numbers.

[118]Crowe (1967, p26) says that Gauss rejected the geometric interpretation of complex numbers probably because he had already discovered non-Euclidean geometry.

impossibility only in the same sense that in a problem which does not admit a fractional answer, to arrive at an equation which admits only fractional roots is a sign of impossibility. According to Nagel, this step shows that "impossible" has to be taken as a relative term, and that the question of "impossibility" is not to be settled before any interpretation is given to the algebraic result.

Another approach worth considering is that of Hamilton (1805-1865). Hamilton's project was to provide the science of algebra with firm foundations[119], or, in his own words, to establish it as "...independent...[and] deduced by valid reasonings from its own intuitive principles..." as Euclid had done for geometry. (Crowe, 1967, p24). Following, it appears, Kant's assertion of time and space as the two *a priori* given categories of knowledge, and as space provides the intuition for geometry, Hamilton states that time provides the intuition for algebra, and attempts to develop the number system on that basis (ibid.). Technically, however, he defines complex numbers as being ordered pairs of real numbers, a treatment which, we think, requires no further explanation. It is clear that this treatment reduces the intelligibility of complex numbers to that of real numbers.

Although clearly different, the three approaches[120] converge in a very important aspect. In none of the cases the right to use the imaginary in calculations, ie, their *legitimacy*, is questioned; what is really being attempted is to provide a model for the imaginary quantities from which the calculations with them can be safely justified, as the application of the definition of square root results paradoxical in their case[121], indicating that the traditional intuition about numbers is not enough[122]. There is a difference between providing ***the ontology***—as the Greeks understood their ontology of *number*—and providing ***a foundation***. In the former case the very nature of the object is determined, and from there, what can be done with it; it is not the case of *reducing*, for example, *number*, to other intelligible things, *but of determining its very essence, of reaching its being*. To provide a *foundation*, on the other hand, aims

[119]He distinguished three understandings of algebra: as a practical Art, the Language of Algebra, and algebra as a Science (Cf. Crowe, 1967, p23)

[120](i) providing a visual image for them, and at the same time reducing their arithmetic to "calculable," numbers (Argand and Gauss); (ii) to give "autonomy" to the arithmetical operations, and admit that they produce more then what they were originally intended to, and it is precisely for this reason that the "monsters" they generate behave, under them, exactly as the typical numbers it originally intended; and, (iii) showing that their structure is perfectly acceptable by finding another—numerical— interpretation of the complex numbers which does not hurt the prevailing numerical intuition.

[121]As the square of a number cannot result in a negative number.

[122]Warren's quoted statement implies that the original intuition has to be abandoned, and, as a consequence, the operations must be studied in themselves, instead of trying to make sense of the different types of number separately.

precisely at making the object intelligible by showing how it can be construed from other intelligible concepts. An ontology intends *"what it is"*, while a foundation intends *"how it works"*. Hamilton's approach is exemplary of a foundational effort, as his construction does not directly link the square of $\sqrt{-1}$ and -1 as they appear in arithmetic and algebra, but it rather shows that there exists an intelligible system in which there is an element which "works" as -1 and another which "works" as $\sqrt{-1}$, and that the latter does not depend on the notion of an area with negative value.

In the case of the Greek ontology for *number*, we saw that it precluded any scientific treatment of fractions *as such*, and even in Diophantus they must be understood as "a *number* of fractional parts," and those fractional parts understood as units, not as true parts of a unit, and there was no way in which *numbers* and incommensurability could be articulated together. In the case of the models for providing intelligibility for complex numbers, the articulation between them and real numbers is established by showing that a in restricted part of the model the "behaviour," or to put it in modern terms, the structure, of real numbers, was present.

Seen in the context of mathematical education, this distinction suggests that we examine the difficulties faced by the learners from this point of view, ie, mathematical objects and concepts ontologically determined, and given a name, may constitute an obstacle for the learning of objects and concepts that "go under the same name" but do not fit into the ontology.

THE DEVELOPMENT OF THE ALGEBRAIC NOTATION AND VIETA'S *ANALYTICAL ART*

Our concern with algebraic notation, here, will be particularly focused on the use of letters to designate both known and unknown numbers in *arithmetico-algebraic* expressions. Apart from the importance of this notation as a powerful tool which "reduces the cognitive strain of keeping the whole relevant information accessible," (Skemp, 1987, p79), its development also reflects changes in conceptual understanding.

Of the four non-European mathematical cultures we have examined in the previous sections, only in Islamic mathematics we find a "theoretical" treatment of algebra, both in the sense that equations are studied in themselves and apart from problems to solve, but also in the sense that the means to deal algebraically with those equations are investigated. In all four cases, however, equations always intend the determination of a number or numbers, that are somewhat "hidden" in them.

We can quote Euler saying, in 1770, that

> "The principal object of Algebra...is to determine the value of quantities that were before unknown; and this is obtained by considering attentively the conditions given, which are always expressed in known numbers. For this reason, Algebra has been defined, *The science which teaches how to determine unknown quantities by means of those that are known.*" (Euler, 1840, p186)

and Gauss, in 1801, in the preface to his *Disquisitiones Arithmeticæ*, saying that algebra is "the art of reducing and solving equations." (Gauss, 1986, pxvii)

The whole of Jacob Klein's book *Greek Mathematical Thought and the Origin of Algebra* (Klein, 1968) is dedicated to showing that Vieta's invention, the use of letters for both known and unknown values in an equation—fully stated in 1591, in his *Introduction to the Analytical Art*—is the crystallisation of a new concept of number, namely, that of *symbolic number*. In Klein, a symbolic number is a number without an ontology, ie, a number that acquires meaning only in relation to the properties of the operations to which it is subjected, a conception that is clearly present in Bombelli and in Cardano, although in restricted terms. In terms of our framework, the concept of symbolic number is produced through an *arithmetical internalism*.

There are, however, other points of view from which Vieta's invention has to be examined.

First, and most important, we have to examine the use Vieta himself made of his notation in his mathematical work. Cajori (1928, p185), tells us that

> "Vieta distinguished between number and magnitude even in his notation. In numerical equations the unknown number is no longer represented by a vowel; the unknown number and its powers are represented, respectively, by N (*numerus*), Q (*quadratus*), C (*cubus*), and combinations of them."

We also know that he completely rejected negative numbers (Cf. M. Kline, 1990, p252), a fact which is reflected in his adoption of two separate symbols for subtraction: "—," to be used when we were sure of the first number being greater than the second, and "=," to be used when we were not[123]. (Vieta, 1968, p331ff). M. Kline also points out that,

[123]For the equality, Vieta used the word æquetur or a contraction of it.

"The motivation for much of the algebra that appears in Vieta's *In Artem Analytiam Isagoge*, is solving geometric problems and systematizing geometrical constructions. Typical of the application of algebra to geometry by Vieta is the following problem from his *Zeteticorum Libri Quinque*: Given the area of a rectangle and the ratio of its sides, to find the sides of the rectangle. ... Vieta then shows how from th[e final] equation *A* [the length of the larger side] can be constructed by ruler and compass starting from the known quantities..." (op. cit., p279)

If not the unrestricted acceptance of whatever could come from an *arithmetical internalism*, be it negative or imaginary quantities, nor from a total abstract approach in which the distinction between numbers and magnitudes would be irrelevant, what could be the motive driving Vieta to substitute letters for numbers altogether in the *Analytical Art* ? To put it briefly, Vieta's *intention* is to present a method and to affirm its transparency against the illusion of virtuosity:

"Diophantus in those books which concern arithmetic employed zetetics most subtly of all. But he presented it as if established by means of numbers and not also by species (which, nevertheless, he used), in order that his subtlety and skill might be more admired; inasmuch as those things that seem more subtle and more hidden to him who uses the reckoning by numbers (logistice numerosa) are quite common and immediately obvious to him who uses the reckoning by species (logistice speciosa)." (Vieta, 1968)

or, as van der Waerden puts it (1985, p62), "His aim was to revive the method of analysis explained by Pappos in his great 'Collection' and to combine it with the methods of Diophantos."

Given the emphasis put on Vieta as "the founder of algebra," we think that two remarks are necessary. First, none of the transformations of equations proposed by Vieta are new. They are clearly stated by Islamic algebraists from al-Khwarizmi onwards, as is a general algebraic calculus[124]. Second, the "letters" of the *Analytical Art* are not as general as a less observant eye might believe: the "Law of Homogeneity" addresses exactly the problem caused by the geometric character of the *species*:

"...for Vieta the ultimate aim of this procedure is indeed to find geometric constructions and numbers; in the latter case, this means finding 'possible' numbers, that is, according to the passage from the *Apollonius Gallus*, such numbers as have a direct geometric interpretation." (Klein, 1968, p158)

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[124]We remind the reader that Islamic algebraists had also envisaged algebra as a method for solving problems in geometry and arithmetic, as we saw on Section 2 of this chapter.

The reason why the calculus in the *Analytical Art* is so cluttered with rules about "homogeneity" is precisely because the operations he envisaged were not *homogeneous*, were not "Laws of Composition," but geometric constructions. And he went to such lengths in explaining it—although when dealing with numbers he would not care about it—because he wanted to expose the method in its generality and still avoid a careless dimensional treatment. More than 600 years before Vieta, al-Khayyam had solved this difficulty by implicitly introducing a unit of length (Cf. van der Waerden, 1985, p24)

The *Analytical Art* produces, as it had intended to, and to its great credit, a shift from "solving problems" to "a method for solving problems."[125] It is extremely telling that not a single problem is solved or even mentioned in the *Analytical Art*. It is also telling that the what we see today as one of the greatest technical improvements of all times in mathematics, did not cause the same impression in Vieta's time. M. Kline (1990, p262) says that,

> "...as far as one can judge, the introduction of letters for classes of numbers was accepted as a minor move in the development of symbolism. The idea of literal coefficients slipped almost casually into mathematics. ... Improvements in Vieta's use of letters are due to Descartes ... However, like Vieta, Descartes used letters for positive numbers only ... Not until John Hudde (1633-1704) did so in 1657, was a letter used for positive and negative numbers."

It maybe that in itself, specially if we consider the clumsiness produced by the "Law of Homogeneity," Vieta's invention did not have much to offer for those concerned only with "solving problems," and for this reason the use of letters in Vieta's manner took some time to be absorbed by mathematicians.

By using letters for the coefficients, however, not only *method* is highlighted, but a change in the nature of the expressions of algebra occurs: the *arithmetical articulation* is in evidence, and the manipulation of equations gradually assumes the character of manipulation of (algebraic) *forms*. Algebra had, of course, always proceeded, from Diophantus onwards, by manipulating the forms in the equations, but Vieta's notation brings the *arithmetical articulation* to the forefront, by avoiding it to be "absorbed," at each step of the solution process, because of the actual calculations.

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[125]The *Analytical Art* ends with the phrase "Finally, the analytical art, having at last been put into the threefold form of zetetic, poristic, and exegetic, appropriates to itself by right the proud problem of problems, which is: TO LEAVE NO PROBLEM UNSOLVED."

Already in 1631, however, Harriot explores the "...the true construction of *Compound Equations* and how they be raised by a multiplication of *Simple Equations*, and may therefore be resolved into such," (Wallis, in Fauvel and Gray, 1987, p294), and in 1637, Descartes' *Geometry* makes a totally new use of algebra:

> "If then, we wish to solve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction — to those that are unknown as well to those that are known. Then, making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows most naturally the relations between those lines, until we find it possible to express a single quantity in two ways. This will constitute an equation, since the terms of of one of these two expressions are together equal to the terms of the other." (Descartes, in Flauvel and Gray, op. cit., p399) (our emphasis)

By 1795, Lagrange makes full use of this aspect of algebra—the *arithmetical articulation*—to show that the "the general expression of the roots of an equation of the third degree in the irreducible case cannot be rendered independent of imaginary quantities," beginning by stating that

> "Let us take...the equation $x^3+px+q=0$, and let us suppose that its three roots are a, b, c. By the theory of equations, the left-hand side of the preceeding expression is the product of three quantities $x-a$, $x-b$, $x-c$ ..."(Lagrange, 1901, p83ff)

But another important aspect of algebra is highlighted by the use of the literal notation. Because one is not concerned with actual calculation, the question of whether the letters are standing for whole numbers, irrationals, negative or imaginary quantities becomes very much secondary, and it is the properties of the arithmetical operations that play the main role in the algebraic manipulation proper. In other words, different types of numbers, each one with its own ontology or foundational model, are *collapsed* into a single object, NUMBER, which meaning is given *internally*, or, as Nagel (1935, p458) puts it, "the intrasystemic meanings of the signs (their syntax, or modes of combination) [are kept distinct] from the extrasystemic interpretation which may be given them."

The notion of a *collapsed* object also plays a decisive role in the development of a broader understanding of algebra, because as *collapsed* objects, polynomials, matrices, permutations, etc. can become *objects* of an algebraic system. If we consider, for example, the field of the invertible 2x2 real matrices , we see that the *object* "matrix" is defined, then an addition and multiplication for them, and it is shown that those operations have such and such properties, and from then on, we can deal with the 2x2 matrices as *collapsed* objects—if this is what we wish, of course—as if we had never known that they are "tables" of real numbers and that the operations have this or that effect on those "tables." The case of *abstract* algebraic systems is different only insofar as in them we give up altogether any ontology, foundation or extrasystemic interpretation *for good*, and we intend only the properties of a given algebraic system, ie, *the only meaning available to the elements of an abstract algebraic system is the intrasystemic meaning, the meaning provided by the properties of the operations operating on them.*

The symbolic character of the elements in an algebraic system—numbers being a particular case—then, depends on the mathematician's willingness to *collapse* those objects, to disregard their inner structures, to disregard *extrasystemic interpretations.* This was true for operating with irrational numbers and with complex numbers, as it was, in fact, the conceptualisation that made possible for permutations to be operated "as if they were numbers," (see, Vuillemin, 1960, p16) or for Gauss's treatment of quadratic forms (see Gauss, 1986, and also Bourbaki, 1976, p79ff). Moreover, as the possibility of disregarding extrasystemic interpretations is taken aboard, the only obstacle for the development of abstract algebra is the resistance, *from inside of the mathematical community*, to a "useless" mathematical theory: Hamilton spent the rest of his life after inventing the quaternions, searching for physical applications for his theory (Crowe, 1967, p30), and Peacock refused to give up the "Principle of Permanence of Equivalent Forms"[126]:

> "But could not a symbolic algebra be constructed independently of any of the suggesting science, it may be asked. Is not the function of the suggesting science merely psychological, and does not the equivalence of forms in the algebra depend upon its *own* assumed general rules of operation? Peacock considered the idea, only to reject it, because in that case 'we should be altogether without any means of interpreting either our operations or their results, and the science thus formed would be one of symbols only, admitting of no application whatever." (Nagel, 1935, p455)

We think that one last remark must be made in relation to the use of algebraic symbolism. It is clear that *historically*, it provided a solid base from which concepts and conceptualisations could develop; it also provided a strongly suggestive notational form. Nevertheless, we must keep in mind that from a mathematical point of view, it is

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126"'*Whatever algebraical forms are equivalent, when the symbols are general in form but specific in value, will be equivalent likewise when the symbols are general in value as well as in form.*'...It will follow from this principle, that all the results of Arithmetical Algebra [where only positive numbers are allowed] will be results likewise of Symbolical Algebra..." (Peacock, 1845, p59) not *essential* to those developments. Indeed, it was none other than van der Waerden who said that,

> "...many non-mathematicians...grossly overestimates the importance of symbolism in mathematics. These people see our papers full of formulæ, and they think that these formulæ are an essential part of mathematical thinking. We, working mathematicians, know that in many cases the formulæ are not essential, only convenient." (in Fauvel and Gray, 1987, p143)

and Gauss candidly said, in relation to Wilson's Theorem,

> "It was first published by Waring and attributed to Wilson...But neither of them was able to prove the theorem, and Waring confessed that the demonstration seemed more difficult because no *notation* can be devised to express a prime number. **But in our opinion truths of this kind should be drawn from notions rather than notations.**" (Gauss, 1986, p50) (our emphasis in bold)

CONCLUSIONS

At the beginning of this section, we have said that two aspects of the development of algebra in Europe would constitute our main concern: (i) the process of *internalisation*; and, (ii) the development of new forms of notation.

We have said, moreover, that those two developments should be examined in two directions: changes in the notion of number, and changes in the character of algebraic activity.

About the development of new forms of notation, and about the extension of the notion of number, we think that what we have said so far is sufficient to clarify the matter.

The process of *internalisation* has been made thoroughly clear; not only by the acceptance of complex numbers long before an acceptable foundational model had been provided for them, but also for the gradual realisation that extrasystemic interpretations did not affect algebraic activity itself. The radicalisation of such understanding led to the development of abstract algebra.

In relation to the algebraic structure of number systems, it is only natural to call it *arithmetic internalism*. We must now consider if algebraic operations can be, in some sense, be said to be "arithmetic," and if this usage can produce useful insights.

Algebraic operations are *finite*, ie, they take a finite number of operands. Algebraic operation are closed, and as it has to be defined in all cases, it means that in a system with more than one operation defined, they operate *homogeneously*. In those important aspects, algebraic operations "behave" like arithmetic operations. It is true, of course, that "arithmetical operations are particular cases of algebraic operations, and, thus, they should behave accordingly." But this is not the point in question here: the *intuition* on which algebraic operations are based *is* that of arithmetic operations, and *arithmetic* in Arithmetic has to do with the compositional aspect of the operations, ie, with their *algebraic* aspect as "Composition Laws," and not with *numbers as special objects*. As long as we can speak *properly* of adding and multiplying quaternions, and we speak of the arithmetic of the quaternions, there seems to be no reason why specific, "intuitive," properties should be attached to the notion of "arithmetic" in the sense we are proposing to use it[127].

The term *arithmetic*, used together with *internalism*, does indeed suggest that within the *Semantic Field* produced by *algebraic thinking*, *meaning* results from, and only from, the properties of the operations. By analogy with the traditional use of *arithmetic*, we build a notion, we make *algebraic thinking* intelligible, in the same way that by analogy with real numbers Cardano made complex numbers intelligible—long before any foundational model was available, and the use of *arithmetical internalism* as part of a characterisation of *algebraic thinking* is, thus, justified.

While we are dealing with the solution of equations, the *analyticity* of *algebraic thinking* is clear, and directly relates to "assuming the unknown as known, and from the relationships established determining its value." As the expressions of algebra become less "parts of equations" and more *algebraic expressions in their own right*, as the *arithmetical articulation* of those expressions become the focus of attention, the notion of *analyticity* has also to be seen in a different guise. The distinction between "known" and "unknown" is not only blurred because—as in as-Samaw'al—one operates on both following exactly the same rules, but also, and more important, because the central notion becomes that of *expressing those algebraic expressions under different forms*, and the equivalence of the three equations

$$a+b=c \quad \text{(I)}$$
$$a=c-b \quad \text{(II)}$$
$$b=c-a \quad \text{(III)}$$

has to be understood as the fact that from (I) an expression for a can be deduced, as in (II) and (III), etc.. In this context, *analysis* has to be understood as going from the

[127]Commutativity, for example. And associativity in the case of the octonions.

more complex and general to the more simple and particular, ie, deriving a particular expression for a, from a *supposition*—equation (I), for example—which "indifferently" involves a, b, and c. The element a may be "known" in the sense that it represents "any number," for example, but its *representation in terms of c and b* is, in (I), "unknown."

3.7 Conclusions to the Chapter

We think that the most important general result of our historical investigation, is that it has unfolded several ways in which the algebraic activity can present itself, and also that those modes of presentation formed, in each case, an "organic" whole, within which contradiction, difference, and agreement are dealt with, rather than a somewhat blind struggle to produce more and more mathematical results—be they theorems or methods.

As a first consequence, it becomes clear that to search for a "line of progress" that we can follow through history, is the surest recipe to meet the "void" mentioned by Rashed[128]. Moreover, that general result suggests and informs a rich possibility for the study of the learning of algebra and the development of an algebraic mode of thinking by individuals, namely, to study the mathematical culture of the learners, or more adequately put, the mathematical *ethos* of the learners. This means we should consider the conceptualisation of mathematics held by then, but also which mathematical concepts and objects "belong" to that *ethos*, and how they are organised and articulated. *As with individuals in relation to their mathematical ethos*, in history we have knowledge being exchanged by or imported into a given mathematical culture, and those elements are directly absorbed, reinterpreted, or rejected, depending on how they relate to the culture into which they are being inserted. But, in the same way that we are forced to abandon the search for the "line of progress" in history, such an approach to educational research forces us to abandon the notion that "recapitulating" history offers a sensible approach to teaching; a child living in the urban area of a modern city, will, *by no means*, "recapitulate" the mathematical culture of Babylonia. Moreover, which is the line the child has to take in order to achieve this supposed "recapitulation"? Does it start at Babylonia Station or before? Is it an express line from Greece to Europe—as some would like to have it—or does it take detours through China, Islam, the Hindus, the Maias, the Navajos? The almost comic—but, in fact, tragic—character of the analogy, only makes clear that in investigating the learning of mathematics as a cultural

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[128] "La rédaction historique du mathématicien est ... significative: entre la préhistoire grecque de la géometrie algébrique et Descartes, Dieudonné ne trouve qu'un vide qui, loin de faire peur, est idéologiquement rassurant." (Rashed, 1984, p309)

process, we *must* examine and take into consideration the *ethos* with which we are interacting. Our investigation of the historical development of algebra makes the examples examined in Chapter 1 (*Introduction*)—for example Freudenthal's teaching experiment and Luria's interviews—more significant, and at the same time it provides an extended and mathematically specific illustration of the general points we approached there.

Examining the *ethos* of the learners differs in one essential way from identifying their *misconceptions*. In the former, we try and establish not the weak points, but, on the contrary, the strong points in the learners' mathematical *ethos*, those points around which their knowledge is organised. The objective of such examination is not only to inform a corrective teaching, but also to prevent the mistaken attribution of *intentions* and *conceptualisations* to the learners, where they do not exist.

A second consequence is that our claim that it is adequate to distinguish *algebraic thinking* from the contents of algebra is shown to be correct. We learned from history that algebra can present itself in many different forms and mathematical contexts, some of which are evidently less complex than others, some of which are theoretical, some not, and some of which intend solving problems, while in others it is the process of solving problems which is highlighted; in each instance, however, that knowledge was or could be produced or justified through *algebraic thinking* as we defined it, and those cases in which it was not, provide useful illustrations of *non-algebraic algebra*—the term being as adequate as speaking of Algebraic Geometry. In relation to some parts of the algebraic body of knowledge, for example the extension of the number system to include complex numbers, *algebraic thinking* proved—as in Cardano and Bombelli—irreplaceable, showing that the distinction is not only useful, but also essential, pointing out to the need of making *algebraic thinking*, and not only the content of algebra, an aim of teaching.

Another key result of our historical investigation, is that we could identify the role of *algebraic thinking* as an *intention* that drives—at the same time rendering it meaningful—the development of a body of algebraic knowledge, particularly in Islamic and Western mathematics. In al-Khwarizmi, the algebra itself is technically poor, but the novelty of the approach points towards *algebraic thinking*, and it is by the intention of producing an *algebraic* algebra that the development of an algebraic knowledge is guided. In a different mathematical context, but in similar fashion, we see *algebraic thinking* driving the development of Peacock's Symbolical Algebra and Hamilton's Quaternions.

We will now examine two foci of tension within algebraic activity, and around which the remaining findings of our historical investigation will be organised.

MEANING IN THE ALGEBRAIC ACTIVITY

In relation to meaning in algebraic activity, tension builds between an ONTOLOGY OF THE ELEMENTS[129] and the PROPERTIES OF THE OPERATIONS. The strict ontological commitment of Greek mathematics, largely precludes, as we saw, a numerical interpretation of the results of geometry; on the other extreme of the spectrum, in Abstract Algebra, the only meaning possible is that provided by the properties of the operations.

The distinction used by Novy, between *intrasystemic* and *extrasystemic* meaning, is useful, but requires some refinements.

There is, first, the *extrasystemic* meaning produced by an ontological determination, in which case the element's *essence and mode of being* is determined, and not only the operations are derived from this determination, but, also, those operations intend exactly those elements; it is in this respect that Jacob Klein says that in Greek mathematics the general applicability of the method depends on the generality of the object. There is also the *extrasystemic* meaning produced by a foundational model, which is intended to lend intelligibility to the elements and operations, but not to determine *essence*. A foundational model is, of course, built on the basis of objects which are considered as intelligible—as in the case of reducing fractions to integers, or complex numbers to points on the plane. Taken in its stricter sense, the notion of foundation in mathematics—for example, providing a model of irrational numbers within the structure of the rational numbers—does not appear until quite recently in history, and we should certainly not expect to find it in our students, so we prefer to understand it in a more flexible sense, namely, as a familiar model in which we can "see" the original elements being represented and we can formulate operations that "behave" as the original ones, thus enabling us to deal indirectly—and more safely—with the original system by dealing with the model instead. In this "intuitive" sense, we will call foundational models simply, *models* or *interpretations*.[130] An ontology says "*what* it *is*," and a model shows "*how* it is."

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[129] By "elements," here and in the rest of this section, we mean "the elements of the base set of an algebraic system."

[130] In his *Treatise on Algebra*, Peacock (1845, p448ff) said that, "To *define*, is to assign beforehand the meaning or conditions of a term or operation; to *interpret*, is to determine the meaning of a term or operation conformably to definitions or conditions previously given or assigned. It is for this reason that we *define* operations in arithmetical algebra conformably to their popular meaning, and we *interpret* them in symbolical algebra conformably to the symbolical conditions to which they are

On the other hand, the importance of introducing the notion of *intrasystemic* meaning, is to make clear that the notion of "meaningless" elements in an algebraic system is not adequate. First, because it is beyond doubt that an absolute lack of meaning would be identical with the impossibility of algebraic activity. Second, because an algebraic treatment of an algebraic system, pressupposes precisely the *internalism* which renders all extrasystemic interpretations irrelevant, as meaning is an *internal* meaning, derived only from the properties of the operations and of the equality.

If it is the case that we want to say that ***extrasystemic meaning*** has been abandoned, this should be made absolutely clear, but we must also make clear that abandoning *extrasystemic meaning* is only possible because there is a shift in referential, a shift to a distinct *Semantical Field.*

Another important issue directly related to that of meaning, is about the ways in which the procedures of algebra are justified. In our historical investigation we found three basic models used for justifying those procedures: geometric models, combinatorial models, and algebraic models; and, of course, models that combine aspects of those three.

To prove that[131] $(a+b)^2 \overrightarrow{=} a^2+2ab+b^2$ using a square cut into four parts, is simply to *make evident* the fact that the four parts identically correspond to the whole, and it can be said to be a simple geometric proof; it does not *prove*, of course, that *in all cases*, ie, *for any arrangement of the parts*, $a^2+2ab+b^2 \overrightarrow{=} (a+b)^2$, but by showing that those parts can be always *combined* to restore the square, would do the trick. The latter is an example of a combinatorial proof supported by geometric objects.

As we have pointed out in Section 2 of this chapter, the rule for the multiplication of the "wanting" components of two binomials can be justified in a purely combinatorial manner; whole-part models are used essentially in a combinatorial way.

As to purely algebraic models, the solution of bi-quadratic equations by radicals provides a typical instance. In purely algebraic models, operations are *objects*, in the sense that they provide the information which guides the algebraic activity, they provide information on "what can be done" and on "what should be done."

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subject." Seen in those terms, an ontology *defines*, and a model makes the *intrasystemic* meaning intelligible.

[131] The symbol "$\overrightarrow{=}$" indicates a unidirectional transformation, in which an equality is produced.

There is a subtle interplay between using a model to *justify* an algebraic procedure, and to use a model to *guide* the algebraic activity. In the former case, radically taken, the use of the model intends to make the procedure intelligible, whereas in the latter the model actually provides information on "what can be done," for example, to manipulate an equation. The subtlety resides precisely in the fact that a *guiding* model can remain hidden throughout the solution of a problem, and the justification of the solving procedure, using that model, would not represent the interpretation of the procedure in another model, so to make it intelligible, but rather the procedure itself being explicated; it is necessary to understand which objects the procedure *intends*, and also how those objects are perceived as relating to the objects dealt with in the justification model.

A failure to take this distinction into account can, as we have seen, lead to erroneous interpretation of historically situated mathematical texts, but also, and of great importance to our overall argument, it can lead to erroneous didactic readings. The main objective of the Experimental Study, the results of which are presented later in this dissertation, is to investigate the *guiding* models used by those students when solving the problems we have proposed to them.

There is one last aspect of algebraic activity we want to examine in this sub-section. We saw that in Chinese mathematics, mathematical objects are "confined," to a great extent, to the methods in which they appear; it is possible, then, to characterise each method as a *mathematical context*. Each of those methods are used to solve problems arising from various "concrete" contexts, and in this precise sense, they can be said to be *abstract in relation to the extrasystemic meanings of its objects*.

Allowing mathematical objects belonging to one mathematical context to become part of another mathematical context, we termed *horizontal development*; the refinement or extension of a method—eg, the generalisation of the method for extraction of square root to allow the solution of quadratic equations, or the extension of the method to powers of degree higher than three, or the adoption of the cross-multiplication in the *fang chen*—those developments which expand only internally a mathematical context, we termed *vertical developments*. Seen from this point of view, in Chinese mathematics we have a strong vertical development but almost no horizontal development.

The result of our historical investigation suggests that a notion essential to promote horizontal development is that of *theoretical mathematical knowledge*, in the precise sense of a body of knowledge that is organised around mathematical objects, and not around procedures for solving problems. In Greek mathematics, although this notion was available, the horizontal development is severely hindered by the existence of strict ontological commitments, and the notion of irrational or negative numbers, for example, cannot be developed; on the other hand, *within the limits imposed by the ontologically determined objects*, there is a strong horizontal development, as the content of the *Elements* of Euclid clearly indicates.

In Islamic mathematics we find a theoretical treatment of the algebraic knowledge, and greater horizontal development, to the benefit of algebra[132]. In European mathematics, but also—and decisively—in all branches of science, horizontal development, represented as the generalisation of the methods, is a driving force; the project of a world described by numbers is part of this effort. (See, for example, Davis and Hersh, 1988)

Vertical development is closer to "solving problems"; horizontal development is closer to "investigating methods."

OBJECTS IN THE ALGEBRAIC ACTIVITY

The central object in the algebraic activity is the operation, which is in all cases to be understood as a composition law[133].

Around the concept of operation, a tension exists, between OPERATION AS CALCULATION, as in for example,

"$5+3 \overrightarrow{=} 8$" or "$(3a-5b)-(a-3b) \overrightarrow{=} 2a-2b$,"

and OPERATION AS PRODUCER OF *ARITHMETICAL ARTICULATION* , as in

"$x^2+5x+6 = (x+2)(x+3)$" or "$2n+1$ is an odd number"

In the former, it is the *result* that is intended, whereas in the latter, it is the properties of the expression—derived from the properties of the operations—which are intended. In the definition of even number as "an integer number that divided by two gives an exact result," the division is used in its first aspect, but in "an even number is a number of the form $2a$, where a is an integer number," the multiplication is used in its second aspect.

Another essential element of the algebraic activity is the equality relationship. Understood in relation to the two aspects of operations just examined, the equality can be seen as: (i) a unidirectional relation, where the right-hand side is the *result* of the *calculations* on the left-hand side; (ii) a bi-directional relation, meaning that if the

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[132]Rashed (1984) explores the use and development of algebra in relation to the theory of algebraic equations, the development of decimal fractions, number theory, and combinatorial analysis.
[133]See, for example, the entry "algebraic operations" in Daindith and Nelson (1989, p13).

calculations on both left- and right-hand sides are carried out, the *results* will be the same; and, (iii) as a bi-directional relation, meaning that the expressions on both sides can *substitute* each other in any other algebraic expression where one of them appears.

We will reproduce again, for its preciseness, the definition of equality presented in Robinson (1951, p4), and already quoted on Section 6:

> "1.5. *Definition of Equality.* We shall say that a relation of equality is defined in a given system of axioms if it includes a relation $E(x,y)$ which is symmetrical, reflexive and transitive, and such that every relation $F(x_1,\ldots,x_n)$ included in the system, it can be proved that equal objects can be substituted for one another as arguments,
>
> $$(x_1)\ldots(x_n)(y_1)\ldots(y_n)\ [[E(x_1,y_1) \wedge E(x_2,y_2) \wedge \ldots \wedge E(x_n,y_n)] \supset [F(x_1,\ldots,x_n) \supset F(y_1,\ldots,y_n)]$$
>
> where $\wedge$ and $\supset$ denote conjunction and implication respectively."

The notion of *result* as in a calculation is not in evidence, and a property such as $a=b \Rightarrow a+c=b+c$ also acquires a meaning independent of that of calculation.

In the process of dealing with equations, the three "types" of equality produce different situations.

With (i), an equation like $100=25+15x$ makes little sense, and even less does $100+2x=25+13x$, while with (ii) they do. In both cases, solving the equation is seen as determining a number such that if x is replaced by it, the calculations will come out correct, ie, the equality will be preserved.

With (iii), solving the equation is seen rather as transforming the equation until one reaches an equation of the form $x=\ldots$.

The tension between those modes can be seen in the fact that students who are taught to solve equations as "isolating x on one side," often do not "check" the answer obtained: the task of reaching the desired form is not clearly linked to the task of finding a number which satisfies the given relation.

When equality is seen as in (iii), the very notion of "unknown" becomes, to a great extent, irrelevant, and the focus of attention in manipulating equalities is in *expressing the arithmetical articulation of any of the expressions in the equality in terms of the other expressions in the equality.*

In the practice of solving equations with specific coefficients, the possibility of actually performing the numerical calculations obscures the aspect of expressing the *arithmetical articulation*, and emphasises the aspect of operation as *calculation*. On the other hand, if the *arithmetical articulation* is emphasised, instead, carrying out the actual calculations can be seen as a particular way of manipulating the *arithmetical articulation*. This suggests that a more efficient approach to teaching the manipulation of equalities in algebra might be to begin with generic expressions, and not numerically specific ones.

It also indicates that a calculating practice, in the context of solving problems, does not lead *by itself*, to the notion of *arithmetical articulation*. First, because, as we said, the *arithmetical articulation* is absorbed in the course of the actual calculations, but also because the procedures involved in solving problems are not usually, and specially in "practical," everyday use, related to the arithmo-algebraic structure of the problem. The case of money change is typical: I spend £A, and pay with a £B note. What is the change I should get? One strategy is to "count up" from A to B. Another is to subtract A from B. In the first case, it is taken into consideration that A plus the change must give B, but this does not imply that a subtraction is involved, serving only to control the "count up"; in the second case, the general scheme is that I have to take, from what I gave, the money I spent, and see what is left, ie, A *and* the change make B, not A *plus* the change. In both cases the underlying, *guiding*, model is a whole-part model. In the chapter on the Experimental Study we will examine several similar examples.

In the last paragraphs of the Conclusions to Section 6, we think enough was said about the way in which *analiticity* has to be understood in the context of operations as producer of *arithmetical articulation*, and differently from the understanding in the context of "unknowns." We should also add that although *analysis* and *synthesis* are complementary processes, in such uses of *analysis* as that we have examined in Section 2, in relation to Euclid, there is always an attempt at *avoiding the unknown*, although, of course, it has to be considered in the process. This observation is of importance for us, because in many cases in the Experimental Study, this use of *analysis* is visible, and will be distinguished from the *deliberate* and *dominant* use of *analysis* as we have in *algebraic thinking*.

ALGEBRAIC STRUCTURE AND ORDER STRUCTURE

We think it is important to examine, yet briefly, this aspect, in relation to number systems.

In Section 6 we presented an objection, raised by Artaud (17th century) against negative numbers: "How can it be that -1:+1=+1:-1, that is, a smaller is to a greater as a greater is to a smaller?" We have also shown that this objection is clearly raised because the notion of order is not properly understood, and in fact, it is, in this specific case, a notion of order derived from the idea that *smaller than* can only mean *a part of*. The problems caused by statements like $-5>-10$ are well known.

Schematically, the source of such misconception could be this. First, positive numbers are defined as "number of something," and then as "a number of (fractional) parts" or as "measure". Negative numbers, then, are defined as "bellow zero"—be it in the context of bank accounts or temperature; in both cases the negative indicates "*less* than zero," but the meaning of *less* is only casually examined. The fact that "$(-2)+(+2)=0$" is *derived* from those "pseudo-ontologies," as in "if I have a debt of 2 (the -2) and deposit (add) 2 (the +2), I end up with nothing (zero) in my account," or in "if the temperature is 2 degrees bellow zero and it raises by 2 degrees, it will become zero degree."

Mathematically, there is a problem here: given the set of numbers ≥ 0, and an order structure for those numbers, *together with the intention to preserve the properties of the order structure in relation to the operations*, it is possible to deduce, from the fact that $(-1)+(+1)=0$—the equation that *defines* -1—the fact that $-1<0$:

(i) for $a, b \geq 0$, one has that $a>b \Rightarrow a+c>b+c$

(ii) but $+1>0$, and thus, $+1+(-1)>0+(-1)$, ie, $0>(-1)$

That if (i) holds for $c>0$ it also holds for $c<0$ is thus proved:

given $a, b>0$, and $c>0$, so $c+(-c)=0$, then

(iii) $a>b \Rightarrow a+(-c)>b+(-c)$ $[\alpha]$ <u>or</u>

$a>b \Rightarrow a+(-c)=b+(-c)$ $[\beta]$ <u>or</u>

$a>b \Rightarrow a+(-c)<b+(-c)$ $[\gamma]$

That $[\beta]$ cannot hold is evident.

If $[\gamma]$ holds, then, $a+(-c)+c<b+(-c)+c$ also holds, as $c>0$

But in this case, $a<b$, once $c+(-c)=0$, and this is impossible.

As a consequence, $[\alpha]$ holds.

It is obvious that from $(-1)<0$ one cannot deduce $(-1)+(+1)=0$.

The objective of this little mathematical exercise is threefold. First, to highlight the fact that the traditional approach outlined above is only possible because addition is *redefined*, not any longer as *conjoining*, but as a vectorial, directed, addition, ie, the whole algebraic structure is substituted, while the "new" addition is made to seem simply an extension of the "old" one, which it is not. Second, to support the suggestion that it might be profitable to consider looking for interesting ways of introducing negative numbers *first* in relation to the algebraic structure. Third, and most important, to indicate that teaching should aim at showing that the order structure and the algebraic structure are distinct aspects of the number system—although articulated by the properties of the operations in relation to the order structure—and that the notion of order based on "part of" does not apply in the case of negative numbers.

Chapter 4
Experimental Study

"Batatinha quando nasce,
esparrama pelo chão.
Menininha quando dorme,
põe a mão no coração."

Brazilian nursery rhyme

4.1 INTRODUCTION

As we have indicated in Chapter 1, the main objectives of our experimental study are two:

(i) to investigate to what extent our characterisation of *algebraic thinking* enables us to distinguish between different types of solutions for "algebraic verbal problems," and,

(ii) to ascertain the nature of the non-algebraic models used to solve those problems.

The choice of "algebraic verbal problems" as the basic type of problem to be used, is due, first, to our interest in examining the extent to which the *situational context* of a problem may suggest a model or impose unnecessary restrains on the chosen models. Second, *algebraic thinking* involves a shift towards "modelling in numbers," and by using contextualised problems we would be able to discern more shades of the solution process, as the amplitude of the shift would be greater than if we used "pure number" problems. Third, "algebraic verbal problems" are material typically used in the later series of primary school and early series of secondary school, a period of schooling in which we have particular interest; by using our framework to examine that material, we would be, at the same time we conducted the research more closely connected with the thesis's objectives, furthering our understanding of that specific type of problems.

We decided to include "secret number" problems in order to investigate whether the absence of a *situational context* would lead the students to use an algebraic, or at least a purely numerical model, or whether they would try to model the problems by interpreting them "back" into some *situational context* or into some non-numerical *Semantic Field* (eg, whole-part models or geometric models); by using a syncopated notation—abbreviations for the variable names and the conventional symbols for the arithmetical operations and the equality—we would be able to examine how the non-algebraic solvers would make sense of the "arithmetical" context[1], and understand some of the difficulties involved in making sense of a problem presented in that form. This is an issue of particular interest for research on the learning of algebra, and by avoiding the use of "letters" we would be able to focus on the value of the "arithmetical" expressions as *informative articulations*, ie, (local) structures which inform the solution process.

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[1] We use quotes in order to emphasise that we are only referring to a form of *presentation*—as opposed to a form of *representation*. Whether or not the solver will deal *arithmetically*, ie, *in numbers only*, with the problem, is something which cannot be predicted *a priori*.

THE EXPLORATORY STUDY

The object of this small scale investigation was to study the strategies used to solve "algebraic verbal problems" by subjects with little or no instruction in school algebra. Its aim was to understand to what extent the strategies of school algebra are compatible with or similar to those informal solutions, and what kind of obstacles would have to be overcome if one wanted to build a knowledge of school algebra from those informal strategies.

The exploratory study was carried out with three groups. Two third-year groups, 3T and 3A (19 students in each) were from Fernwood Comprehensive School; a younger group, on the last year of primary school, J (21 students), was from Fernwood Junior School. Both schools are in Nottingham, England.

Group 3T was rated as top-ability by the school; group 3A was rated as low- to average-ability.

The test presented to J and 3T consisted of five "algebraic verbal problems," plus two questions about "making change". The test presented to 3A consisted of different versions of four of those five problems, plus the remaining problem with the same text, plus five short questions about solving problems.

Each problem corresponded to a different "algebraic structure," ie, it would correspond to a different type of equation.

Both sets of problems are presented in Annex A.

Of the five main problems used in this study, only one, the "Consecutive Numbers" problem, was not used in the main study, primarily because its investigative nature required more time for it to be solved. The specific results of the exploratory are in complete agreement with those obtained in the main study—which are presented in the subsequent sections—and for this reason will not be discussed here.

The only remark which is worth making is related to the "Consecutive Numbers" problem, which was not, as we said, used in the main study. Unexpectedly, the primary school students performed equally well as, if not slightly better than, the secondary school students. Given the very small size of the samples, this information cannot be taken as indicative of any general phenomenon, but we were led to believe that the students in J dealt more freely with the problem, ie, apparently they had less expectations about how this type of problem "should" be solved, both because the problem was completely new for them, but also because their experience with solving problems was much less related to the use of specific methods, and as a consequence they were more able to explore the situation and make sense of it in their own terms. The importance of making this remark, is that we believe this may prove to be an area worth further investigation by researchers interested in the effects of teaching in the problem-solving attitude of students.

The main result of the exploratory study concerns the methodological aspect of our research into students' modelling of the problems.

In many cases it was clearly impossible to determine, from the scripts only, which model had been used to solve the problems. The use of algebraic notation cannot guarantee, alone, that *algebraic thinking* was involved in the solution process, as we have shown in Chapter 1 [2], and in the case of simple problems, it is perfectly possible that a "calculations only" solution is in fact the result of solving an equation "in the head." The difficulty for us was, then, to identify the underlying model in the absence of further written explanations or justifications, particularly because of our commitment to using written tests only.

Those difficulties led us to redesign the tests, keeping the use of "algebraic verbal problems," but instead of having only one problem with each "algebraic structure" examined, ie, "corresponding" to a type of equation, we decided that we would have a group of problems "corresponding" to each equation. This way, we could compare the effect of different numerical values and *situational contexts* for the same type of problem in the choice and efficiency of the models used, and we could also include "purely numerical" problems and study the effect of suppressing the *situational context.* The resulting tests are described in the following sections, in which the results of the main study are analysed.

The second consequence of having to face those difficulties in identifying the underlying models, was to lead us to develop a much finer perception of the details in the scripts than that we had when we began examining them, an improvement which was certainly crucial in allowing us to perceive many of the subtle points "hidden" in the scripts. Dr Bell's experience with this kind of analysis proved to be invaluable for us, and we are particularly thankful for his help in this part of our work.

THE MAIN STUDY

For the main study we had developed six new test papers, which are presented in full in Annex B, although the problems analysed for the purpose of this dissertation are again presented at the beginning of each section concerning a group of problems.

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[2] Where we discuss the use of a whole-part model in the solution of an equation set in algebraic notation.

The six test papers composed, in fact, three pairs of test papers; each pupil was presented with one of the pairs, each test paper presented in a session, never on the same day, and never more than a week later. Each paper was solved in a 50 minutes session.

An important aspect of the testing conditions, was that the students were allowed to use calculators whenever they were available, as well as being told, **in all cases**, that the calculations could be just indicated if the student thought it was "too hard" to perform. They were told, moreover, that they could solve the problems using whichever method they wished, and the word "algebra" was carefully avoided in the introductions, in order to prevent induction to a specific method, but also to prevent causing anxiety in those students who knew little or nothing of "algebra."

The particular aspects of each group of problems examined in this dissertation are presented in the relevant sections on the data analysis.

For the main study we contacted two schools in Brazil—*Escola de Aplicação da USP* and *Colégio Hugo Sarmento*, both in the city of São Paulo—and two schools in England—*Friesland Comprehensive School* and *Margaret Glen-Bott Secondary School*—both in Nottingham. We decided to work both with Brazilian and English groups for two reasons. First because the marked differences in the teaching of mathematics in the two countries—in method as well as in content[3]—suggested that we would have a much more varied sample in terms of approaches and models used, a suggestion which proved to be correct. Second, because we would have the opportunity to carry out a preliminary investigation into the effect of different teaching approaches in the development of an algebraic mode of thinking, an aspect which we intend to further examine in the future.

Two Brazilian 7th grade groups (age 13-14 years, 56 students), two Brazilian 8th grade groups (age 14-15 years, 53 students), three English 2nd year groups (age 13-14 years, 53 students) and three English 3rd year groups (age 14-15 years, 66 students), form the sample of the main study. The number of students and the average age for each group, are given in Annex C.

As a consequence of the test papers structure, each question was solved by roughly one-third of all students in the sample (total of 228 students).

[3]The teaching of mathematics—particularly the teaching of algebra—in Brazilian schools is, almost invariably, content-driven and quite formal; investigative activities are very rare in Brazilian mathematics classrooms. One may safely say that quite the opposite is true in English schools. This general picture applies very well in the case of the four schools where our experimental research was conducted.

Five categories were used to classify the solutions:

1) correct solutions in which the problem is solved by setting and solving a numerical equation in a recognisable form (**OKEQT**);
2) correct solutions that did not use any recognisable form of equation; the calculations used to produce the answer are presented, with or without an explanation or a diagram supporting the choices of calculations to be performed (**OKCALC**)
3) incorrect solutions where there was an attempt at using an equation (**WEQT**);
4) incorrect solutions where equations are not used; calculations are presented, with or without an explanation or a diagram supporting the choices of calculations to be performed (**WCALC**);
5) trial-and-error solutions (**T&E**);

Calculations wrongly performed did *not* characterise a solution as "incorrect": if the overall procedure would lead to a correct answer had the calculations been performed correctly, the solution was classified as "correct"; also, there were cases in which a complete answer involved the determination of two values and only one of them was given by the student: the correctness of the solution in those cases was assessed in relation to the potential of the method employed to produce the second value, and in relation to the student's awareness of the existence of two values to be determined, as shown in the establishment and manipulation of the chosen model.

The categories above are intended to describe only the form of ***presentation*** *of the solutions, not the underlying model*; an **OKEQT** solution, for example, does *not* imply the presence of *algebraic thinking*. We consider this set of categories to be suitable for two reasons: (i) on the one hand, it is *standard*, providing categories which are easily understood and applied by other people; and, (ii) precisely because it is based on the perceived proximity of a solution to "standard algebraic solutions"—notationwise—the analysis of scripts belonging to a same category allows us to highlight the importance of understanding the underlying model in the process of investigating the nature of the thinking involved in producing a given solution.

In this sense, the categories above provide a general "background" framework, which is not supposed to correspond to the much finer understanding which is produced by the analysis of the scripts. Moreover, in the examination of the scripts, we have not characterised them according to the polarities produced in Chapter 3, from the historical study. The analysis conducted in Chapter 3 has a much more dynamic nature than that conducted in the context of the experimental study, mainly because in Chapter 3 we not only elicit the models accepted by a given mathematical culture, but also relate the acceptance of those models to the more general conceptual framework of the mathematical culture in question; in the case of the experimental study, the application of a similar type of analysis would necessarily involve examining the mathematical *ethos* of those students—a line of research which seems to belong naturally to future extensions of our present work. Attempting to use the polarities from Chapter 3 to produce some sort of justification of the choice of models we had identified, seemed, thus, an artificial and inadequate approach.

Although recognising the importance of providing a more complete and "actual" framework for characterising the non-algebraic solutions, we think that it would not be possible to produce such a framework in the context of this dissertation, above all because it would depend on a much deeper study of modes of thinking other than the algebraic one.

* * *

For the purpose of our analysis, four groups were considered: **AH7**, which comprises all the Brazilian 7th grade groups; **AH8**, which comprises all the Brazilian 8th grade groups; **FM2**, the English 2nd year groups; and **FM3**, the English 3rd year groups.

All the percentage results of each problem examined in the analysis of the experimental study, given for each of the four groups above, is in Annex D; nevertheless, those percentages which suggest relevant or interesting aspects of the overall solving activity, are quoted again in the the section corresponding to the group of problems to which they refer.

The methodological approach of our analysis of the data gathered in the main study is thoroughly *qualitative*; this means that no strong claim is made exclusively on the basis of the percentage results, but also that no statistical treatment was applied to the percentage data. In our analysis, the percentage data only *suggests* underlying modelling trends, and any claim is supported by instances to be found in the scripts.

4.2 Ticket and Driving Problems

The Problems

Sam and George bought tickets to a concert.
Because Sam wanted a better seat, his ticket cost four times as much as George's ticket.
Altogether they spent 74 pounds on the tickets.

What was the cost of each ticket?
(Explain how you solved the problem and why you dit it that way)

Tickets 4x

Sam and George bought tickets to a concert.
Because Sam wanted a better seat, his ticket cost 2.7 times as much as George's ticket.
Altogether they spent 74 pounds on the tickets.

What was the cost of each ticket?
(Explain how you solved the problem and why you dit it that way)

Tickets 2.7

Mr Sweetmann and his family have to drive 261 miles to get from London to Leeds.
At a certain point they decided to stop for lunch.
After lunch they still had to drive four times as much as they had already driven.

How much did they drive before lunch? And after lunch?
(Explain how you solved the problem and how you knew what to do)

Driving 4x

Mr Sweetmann and his family have to drive 261 miles to get from London to Leeds.
At a certain point they decided to stop for lunch.
After lunch they still had to drive 2.7 times as much as they had already driven.

How much did they drive before lunch? And after lunch?
(Explain how you solved the problem and why you did it that way)

Driving 2.7

General Description

This is the only pair of problems to appear on all three sets of questions, with the pair Tickets [4 times] **(T4)** / Driving [2.7 times] **(D2.7)** appearing in the Blue-Gray and Green-Beige tests, and the pair Tickets [2.7 times] **(T2.7)** / Driving [4 times] **(D4)** appearing on the Yellow tests.

The questions were designed to investigate to what extent different kinds of numbers — namely, counting numbers vs. decimal non-integer numbers — would affect the choice of models used to solve problems with the same "algebraic" structure, and which models would result. The [4] problems have the structure "this is 4 times as much as that, and altogether...", and the [2.7] problems have the same structure with 2.7 replacing 4.

In order to have some control over possible effects of the context in which the problems were set, we used two contexts with different characteristics. In the "Driving" problems the objects are portions of a road with different lengths, which can be sectioned (for example, to be compared) and still maintain their characteristic as a portion of a road. In the "Tickets" problems the objects are tickets with different values; there is no real meaning in "sectioning" one of the tickets, and any direct contextualised comparison would have to be made on the basis of the exchange values. It is clear that in both cases a comparison is possible using respectively the lengths and the values.

Discussion of Possible Solutions

The simplest algebraic model that fits into those problems is a linear equation in one unknown . A direct "translation" from the problems would in fact produce a set of two linear equations in two unknowns. In *Tickets and Driving* , however, this representation was never used; instead, *direct substitutions* were used, which we will comment a few paragraphs ahead.

Depending on whether the unknown (here represented by **x**) is taken as the cheaper ticket or the distance travelled before lunch, or as the more expensive ticket or the distance travelled after lunch, we would have one of the following equations:

(E1) $x + ax = b$

(E2) $x + x/a = b$

(E3) $b - x = ax$

(E4) $b - x/4 = x$

with the corresponding values of **a** and **b**.

Equations E2 and E4 were never used by any student. Equation E3 was used by one student only.

Setting the equation can be done in two very distinct ways, either by directly representing a **numerical** relationship ("a number plus **a** times this number is equal to **b**") or by representing instead a **whole-part** relationship. On the former situation, the model applies equally both to [4] and to [2.7] problems, because only a knowledge of operating with decimal numbers is required (to multiply, to add — very much as it has to be done with the [4] problems where only counting numbers are involved) and for the students in our study this knowledge was sufficiently developed. On the latter situation, however, producing meaning for "**4x**" and for "**2.7x**" are processes that involve different degrees of difficulty, even if calculating aspects of decimal numbers are well understood.

A **whole-part** model is quite simply produced for [4] problems: "1 (lot of) **x** plus 4 (lots of) **x** is equal to ..."; the 1 and the 4 play their natural role of "counting numbers". When the same model is applied to [2.7] problems, the need to interpret 2.7 as a "counting number" becomes an obstacle because it requires — at least — the additional step of decomposing the "2.7 lots" into "2 lots and 7 tenths of a lot" for the "counting" to become visible.

Alternatively, an analogy could be drawn with "2.7 pounds of beans" (and one would reasonably expect the students in our study to have no difficulty in concluding that "if one buys 1 pound of black beans and 2.7 pounds of chilli beans, one has 3.7 pounds of beans altogether", indicating a willingness to accept decimals as quantifier). However, to successfully apply this analogy to [2.7] problems one has to take the smaller of the two quantities (cheaper ticket or shorter portion of journey) as a *unit*[5].

No matter which model is used to set the equation, an Algebraic solution of the equation is one that is based on properties of the arithmetical operations and of the equality involved in the equation.

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[5]A step not easily seen by those students, as the analysis of the data will show.

A property like $\mathbf{ab = c \Rightarrow b = \frac{c}{a}}$ can be easily <u>justified</u> in terms of "sharing" if **a** is a positive integer ("if **a** lots of **b** is equal to **c**, then sharing **c** into **a** parts will give the value of **b**"), but not otherwise. If however this property is seen as a property of the **numerical** relationship, and thus also applicable when **a** is not a positive integer, we will consider that an algebraic understanding exists, and if the "explanation" is maintained it will be seen as a particular illustration of the property.

A straightforward solution to E1 would be,

(D2.7)

$$\mathbf{x + 2.7x = 261}$$

$$\mathbf{3.7x = 261}$$

$$\mathbf{x = \frac{261}{3.7} \approx 70.5 \text{ miles, etc..}}$$

It is important to observe that the operations performed with **D2.7** would be:

(i) $1 + 2.7$; (ii) $261 \div 3.7$; (iii) 70.5×2.7 ;

and with **D4**,

(i) $1+4$; (ii) $261 \div 5$; (iii) 52.2×4 .

Non-algebraic models that fit into those problems' context would almost certainly be of the type "1 lot and **a** lots, giving...", be they supported by or derived from a line diagram, a Venn diagram, or a block diagram, ie, a **whole-part** model (Figure T&D 1). As we saw above, the structure produced by such models can be reinterpreted as a numerical relationship and manipulated algebraically, to produce an algebraic solution. But such structures can also be directly manipulated, with calculations performed only to achieve required evaluations of parts.

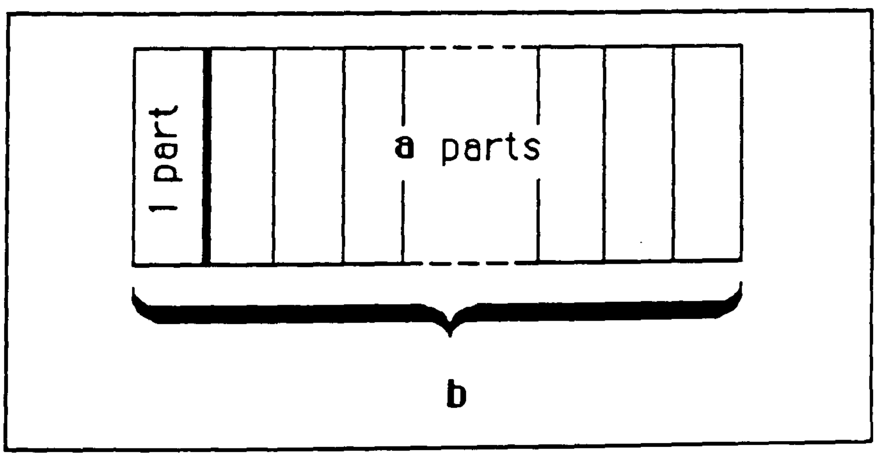

fig T&D 1

With **T4** the manipulation of the whole-part structure would proceed like this:

(i) one of the tickets is 4 times more expensive then the other one; this is the same as saying it "is" 4 tickets;

(ii) 1 ticket and 4 tickets cost **b** pounds, ie, 5 tickets cost **b** pounds;

(iii) now, to know how much 1 ticket costs, I <u>share</u> the **b** pounds into 5 tickets.

With **D4** we would have the same general procedure, with "parts" or "sections" replacing "tickets". It is clear that "lots" would work well with both .

Operations are used to <u>evaluate</u> parts as necessary. Thus,

(ii') 1+4 corresponds to <u>evaluating</u> the total number of tickets, and,

(iii') **b** ÷ 5 corresponds to <u>evaluating</u> how much goes to each of the 5 tickets **through the sharing**.

When the same model is applied to [2.7] problems, two difficulties arise. One is the reinterpretation of "2.7 times more" as "2.7 tickets" or as "2.7 sections". Although the problem is concerned with the <u>value</u> of the tickets, the non-algebraic models deal with this by associating "the **value** of one ticket" to "one ticket", the image of the ticket working as an icon for the value. It is from this point-of-view that the 2.7 should have to "count" tickets in the way the 4 naturally does, with the consequences pointed out a few paragraphs above.

The second difficulty is in fact twofold. On the one hand, there is a problem with step (iii) above. In our description of the *non-algebraic* solution for **T4** we used the word "share" — underlined for emphasis — because we wanted to stress that the main aspect of the manipulation is the sharing, the result of which is eventually made actual either by performing the division by 5, a build-up calculation or by a trial-and-error process. In the case of [2.7] problems, obtaining the value of "1 lot" by "sharing" the total into "3.7 lots (?)" is certainly a difficult and "unnatural" step.[6]

On the other hand, it is difficult to see why anyone would want to step into (ii) without being aware that this is an intermediate step leading to (iii); step (ii) corresponds to "finding how many altogether so I can share between them" instead of "collecting the various occurrences of the unknown". Although in procedural terms step (ii) is processed

[6]"unnatural" to the extent that experts would use such metaphor only to try and make a verbal link with some situation where only "true" counting numbers appear.

before step (iii), both steps are engendered in conjunction: the two aspects are composed to produce a larger obstacle that has to be overcome in one go[7].

One important point in relation to this group of questions is that it is clear here that the use of algebraic symbolism (standard or not) is not enough to guarantee that algebraic processes are involved in the solution of [4] problems. Algebraic notation could be used as a concise notation for a non-algebraic solution, a complete correspondence existing with the steps of an algebraic solution (figure T&D 2), as much as a "calculations only" solution could have been guided algebraically (the problem being simple enough to allow that).

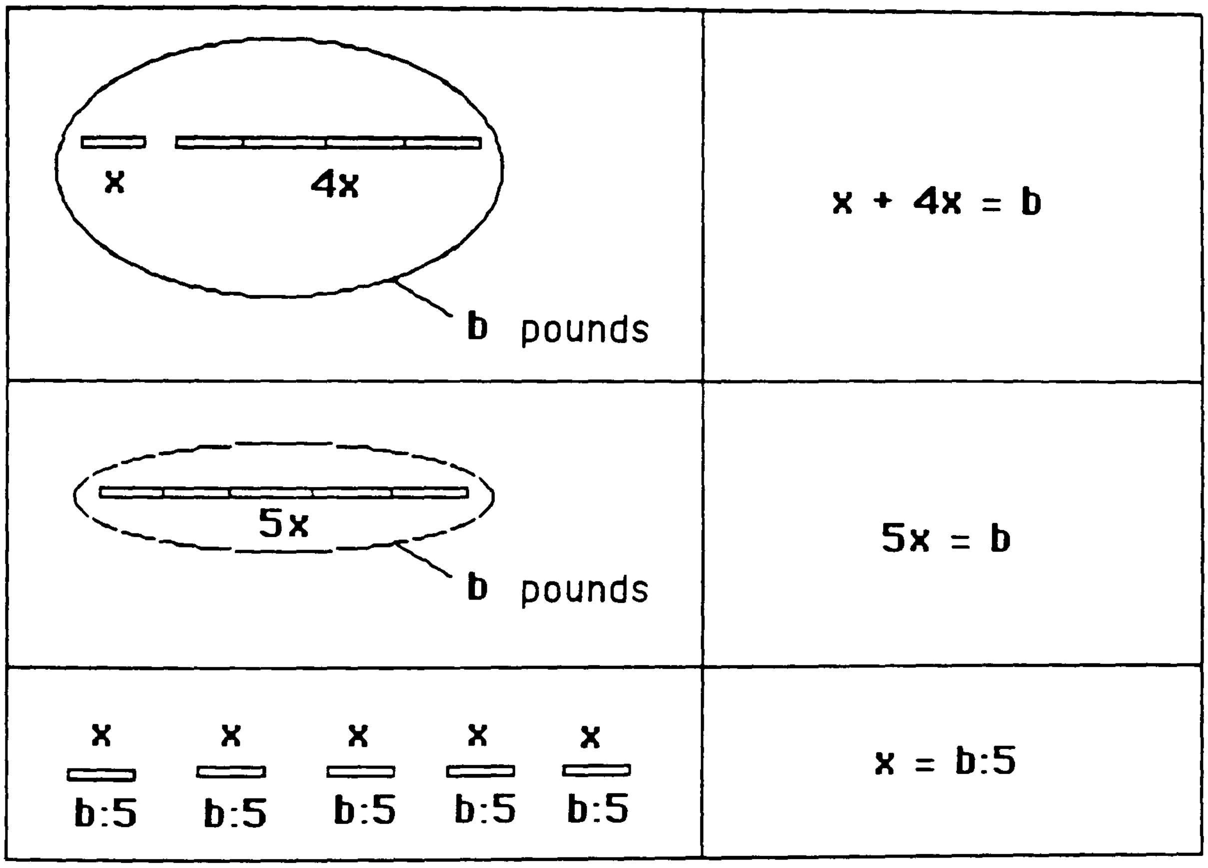

fig T&D 2

Nevertheless, our analysis also indicates that no matter the notation employed, the greater the use of an algebraic model by a group of students would produce a smaller difference between the facility levels for [4] and [2.7] problems.

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[7]The analogy with "buying x and y pounds of ..." would not be enough to overcome alone this double difficulty: the "anticipation" problem would remain.

Previous research on the solution of multiplicative problems has pointed out that the operations of arithmetic (multiplication and division being of interest for us in this section) might remain linked to "primitive behavioural models that influence tacitly the choice of operations [to be used to solve problems] even after the learner has had a solid formal-algorithmic training" (Fischbein et al., 1985, p.3). According to Fischbein, the preferred model for multiplication would be one of *repeated addition*, and the preferred models for division would be those of *partitive or sharing division* and of *quotative or measurement division*. It is clear that "under such an interpretation ...a multiplication in which the operator is 0.22 or 5/3 has no intuitive meaning." (op. cit., p.4)

Our identification of the difficulties that might arise from applying a whole-part model to [2.7] problems is in resonance with the interpretation provided by Fischbein and his colleagues to the difficulties they identified. Moreover, it is an integral part of their interpretation that the "...Identification of the operation needed to solve a problem with two items of numerical data takes place not directly but as mediated by the model" (ibid.), which means that the phenomenon they identified can be examined as an instance of non-algebraic thinking. From this viewpoint, the fact that "...the enactive prototype of an arithmetical operation may remain rigidly attached to the concept long after the concept has acquired a formal status" (ibid., pp. 5-6) is reinterpreted in two ways[8]:

- that the enactive prototype remains attached to the concept (at least in relation to contextualised problems) is seen as <u>a consequence of rather than a cause to the preferential use of non-algebraic models</u>; the properties of the operations that will be reinforced — and will thus remain characteristic of the use of the operations in such situations — are those that correspond well to, for example, whole-part models: Fischbein's repeated sum corresponding to our "counting multiplication"; and division as "sharing";
- if what is meant by "acquiring a formal status" is understanding the reversibility of operations, then it is clear that the use of non-algebraic models would account for the observed effect, once something that would be meaningful in the *Semantical Field* of numbers and arithmetical operations has to be blatantly overlooked for the [2.7] problems to have a higher degree of difficulty; if on the other hand it simply corresponds to "...the learner has had solid formal-algorithmic training" as quoted before, it then means that the

[8]The primary aim of reinterpreting Fischbein's findings in terms of our framework is not to add directly to them—although we think we do, but part of our effort to bring together several research findings of interest for the research on Algebraic Thinking, providing a common explanation in terms of our framework.

operations are not used in the problems with this same generality because the models used do not have the required generality, and we have shown that this is the case with whole-part models.

Bell et al. (1989a, p. 438) criticized Fischbein's *Theory of Intuitive Models*, saying that

> "...First, although its basis is the children's assumed perceptions of the structural properties **of the operations**, it can only be made consistent with experimental results by adding an extraneous hypothesis; second, numerical perceptions involving the ignoring of decimal points cause conflict with its predictions. These considerations suggest that the theory gives insufficient weight to pupils' numerical, rather than structural, perceptions" (our emphasis)

and developed a *Theory of Competing Claims* that takes Numerical Preferences as the most significant factor in determining the choice of operation. By considering four possible aspects of solving the problems, rather then focusing in only one as the *Theory of Intuitive Models* does, the *Theory of Competing Claims* produces a much finer analysis, with a much more precise adjustment to the experimental data. It is true, however, that the difference between the results of the two analysis is one of degree of precision rather then one of major conflict[9]. Moreover, the Numerical Preferences hypothesized in Bell et al. (1989a, p. 438) — "...preferences for dividing the larger by the smaller number and for multiplying or dividing by an integer..." — can be put, at least partially, into correspondence with Fischbein's preferred models[10].

There is an important point to be examined here. Both Fischbein's and Bell's models consider only the case where *the operations* have a "structure" (Bell) or "model" (Fischbein) associated to them. But if we are examining the choice of operation, then one of the following cases must apply: (i) the subject solving the problem simply "scans" the list of all calculations — arrangements of numerical data and arithmetical operations — until one is found that seems to be a correct choice, or (ii) the subject produces a model of the

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[9]Where Bell's analysis produced four clearly distinct levels of difficulty, Fischbein's analysis produced only two, without however any major inversion on predicted levels of facility, ie, if question A is at a lower level than question B according to Fischbein, it is never the case that B is at a higher level than A according to Bell's analysis. (see Bell et al., 1989a, pp. 441-442)

[10]Preference for multiplication by an integer corresponding to the repeated addition model, and preference for division by an integer corresponding to the sharing model.

situation given — in many cases a partial model only — and **on the basis of the model** decides which operations could and should be used; it is only then that this or that operation will be seen as suitable or not. On the first case, numerical aspects — which account directly for three of the four aspects examined by Bell — would certainly constitute a strong factor.

In the second case, we argue that there are two layers of behaviour. At the first level, the subject tries to make sense of the situation and to produce a model that seems adequate. If she or he considers to have found a suitable model, the solution proceeds by manipulation of the chosen model; the use of an operation is suitable or not only in relation to this model, ie, it depends on whether or not using it makes sense in the context of the semantic framework of the model. The solution process might be eventually blocked if the model can not be purposefully manipulated by the subject any further. At a second level, if and when the subject does not produce a model that works in a satisfactory way for her or him, then other aspects come into direct consideration to guide the choice of operation (for example the fact that buying 0.75 pounds of flour must cost less than buying one pound together with the belief that "division makes smaller", makes division a natural choice). This is not to say that such factors play no role in the elaboration of the model, but only that their influence is direct or indirect — and thus more or less diluted — depending on the level one is working at.

This formulation of the process shifts the focus of the analysis from limitations intrinsic to the operations to limitations to their use created by the purpose with which they are used. With non-algebraic models, the purpose would be to evaluate parts as required by the manipulation of the model; with algebraic models, the purpose would be to produce new numerical relationships of required forms, by transforming previously produced relationships; when a structure fails to be produced, operations are chosen as to produce (psychological) contentment in relation to the expected outcome of the problem. It is clear that the last of the three situations is the one where Numerical Preferences — in Bell's sense — are bound to predominate.

Moreover, this approach enables us to understand beyond "arithmetical ability" (performing the operations with different kinds of numbers) the difficulties here examined.[11]

[11]This is a very adequate outcome of our approach. Fischbein (1985, p.4) reminds us that "To say that multiplication by 0.22 or 5/3 has no intuitive meaning is not to say that it has no mathematical meaning. Children may know very well that 1.20 x 0.22 and 9 x 5/3 are legitimate mathematical expressions", and Bell's study (1989a, in particular figure 1, p. 440) shows that although performance improves with age (which most certainly means, in the

The results of a second study presented on the same paper (op. cit., pp. 444-447) also offer some support to our interpretation[12],

> "The making of a correct estimate depends on a correct perception of the operational structure of the problem. This does not necessarily require identification of the numerical operation needed to calculate the exact result. We know from the numerical misconception MMBDS that pupils must have an awareness of the size of the expected answer before making a choice of operation. We suggest that in division problems and problems involving multiplication by numbers less than 1, the estimate is made directly by a semiqualitative ratio comparison, without explicit identification of the division operation".

suggesting that modelling happens prior to the choice of operations.

On the basis of our analysis a local hierarchy can be established for the Tickets and Driving problems:

- if the model used is totally algebraic, with respect to both setting and solving the equation, then the degree of difficulty is the same for all four problems;
- if the model used consists of setting the equation as a description of a non-algebraic structuring, and then solving it algebraically, then [4] problems are easier than [2.7] problems;
- if the model used is purely non-algebraic, then [4] problems are significantly easier than [2.7] problems.

It is against this local hierarchy that we will examined the preferred models used by the students.

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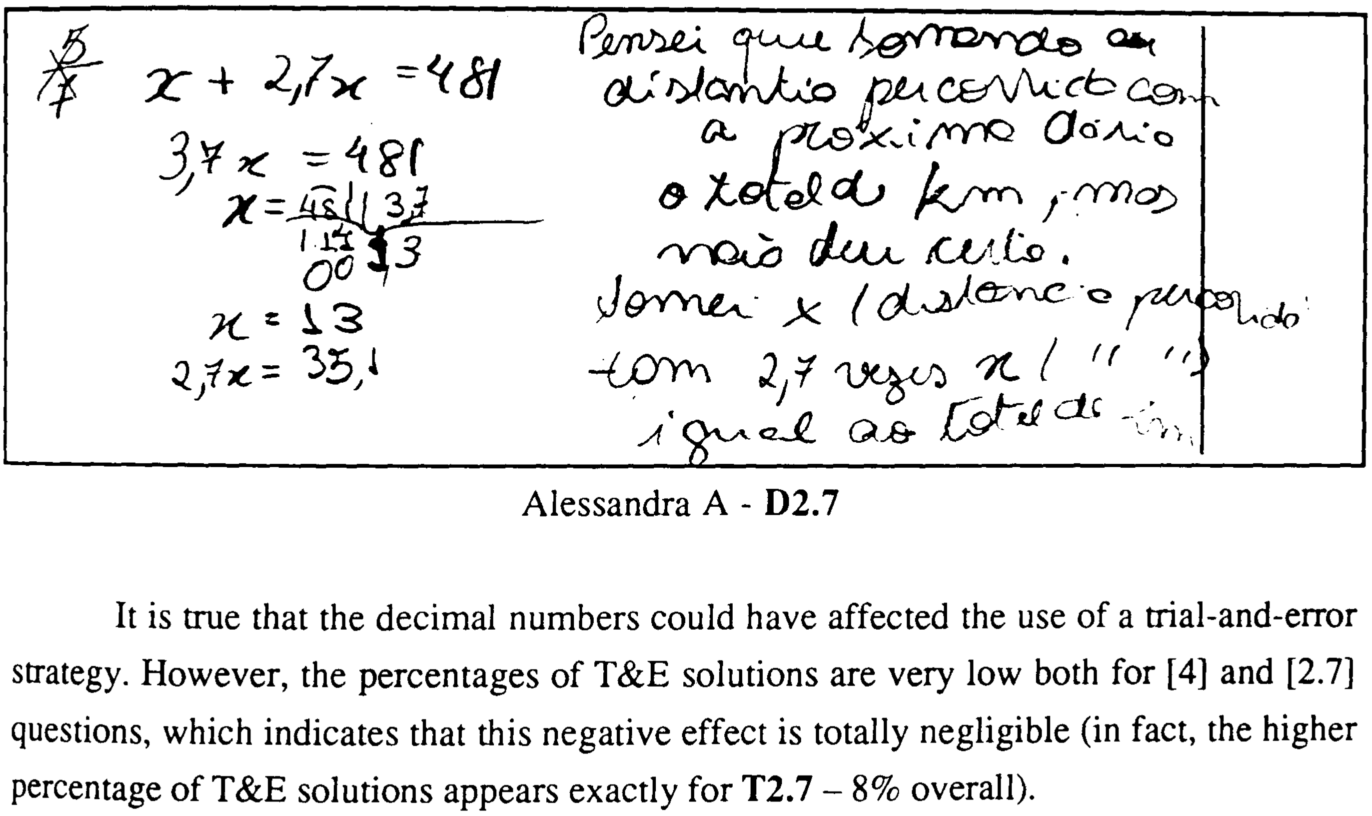
case of the study's sample — all engaged in formal education — improved "arithmetical ability"), similar difficulties occur throughout the whole range of age groups.

[12]This becomes even more clear if one substitutes "... a correct perception of **the** operational structure of the problem" by " ... the perception of **an adequate** operational structure for the problem."

As it is clear from the data, the [4] problems were much more accessible to the students than the [2.7] problems. This is true not only for the overall numbers, but also for each of the four groups.

A possible explanation for such a difference in the facility levels would be that the decimal numbers introduced difficulties with the actual calculations. This is not the case, however, because: (i) errors in the calculations were not considered as errors when the overall procedure would lead to a correct answer were the calculations correctly performed (Alessandra A, A8I), and (ii) the students either used calculators or were told that calculations could be just indicated if they felt it was too "hard" to do. There is also the fact that 32% of all wrong answers to **T2.7** and 45% of all wrong answers to **D2.7** resulted from dividing the total by 2.7 instead of 3.7 .



Alessandra A - **D2.7**

It is true that the decimal numbers could have affected the use of a trial-and-error strategy. However, the percentages of T&E solutions are very low both for [4] and [2.7] questions, which indicates that this negative effect is totally negligible (in fact, the higher percentage of T&E solutions appears exactly for **T2.7** – 8% overall).

In all four groups, solutions for the [4] problems depended **less** on an algebraic model being used for a correct answer to be achieved, as it is indicated by the fact that the percentages of correct algebraic solutions in relation to the total of correct answers is smaller for the [4] problems than for the [2.7] problems (41% for **T4**, 21% for **D4**, 71% for **D2.7** and 53% for **T2.7**). In **FM2** this is not strictly true because the percentage of correct algebraic solutions for **T2.7** is zero, but given that the level of correct answers is so low (6%) — and all of them obtained through T&E — the dependence on an algebraic model — or to put it another way, the inefficiency of other models — is also established. The same observation is valid for **FM3** in relation to **D2.7**, but not in relation to **T2.7**.

The distinctive aspect in **FM3-T2.7** is that the percentage of T&E correct solutions is much higher than in the other three groups, accounting for 56% of the correct answers. The same group produced no T&E solutions for **D2.7** and one explanation is that the numbers in **T2.7** are far more "triable" than those in **D2.7** . However — and from the viewpoint of our research this is more relevant — the percentage of "÷3.7" (correct) solutions is only 16%, with no correct algebraic solutions, which would produce, were it not for the T&E answers, a very low level of correct answers.

Central in respect to this group of problems, the percentages of correct answers are significantly higher for [4] problems than for the corresponding [2.7] problems, which indicates, in the light of our previous analysis, a clear tendency towards *non-algebraic* models.

This finding is supported in a more direct way by the fact that:

- differences in percentages of "÷3.7 or 5" (correct) solutions for corresponding [4] and [2.7] problems are also very significant (below 25% only for **AH8-T4** and **T2.7**; to **AH8**, however, corresponds the highest percentage of correct algebraic solutions for **T4**, 73%), and
- whenever there is a significant difference in the percentages of correct algebraic solutions to corresponding [4] and [2.7] problems, the balance leans towards the [4] side.

STUDENTS' SOLUTIONS

A number of solutions involved the whole-part models examined in the previous sub-section. With *Tickets* problems this meant for example, stating that "there are the equivalent of 5 tickets in the sum" (David W, F3A; Sergio R, HS8I),

Sam's ticket was £14·80, and Georges was £59·20. As there are the equivalent to 5 tickets in the sum, I divided 74 by 5 to get Sam's figure. To get Georges, I multiplied that by 4.

David W - **T4**

222 = 5 entradas

222 |5
44.4

Cada entrada cuesta 44.4

Sergio R - **T4**

and with *Driving* problems, "splitting" the journey into 5 *sections* or *parts* (Elizabeth W, F3B; Clare B, F3B; Jack D, F3B; Jacob B, F3A).

261 ÷ 5 = 52·2 before lunch

261 - 52·2 = 208·8 after lunch

there is 4 more sectons to do after lunch you have done 1 there are 5 all together 8 261 ÷ 5 = one secton take the one secton from 261 = 208·8 give you the rest

Clare B - **T4**

261 miles

261 ÷ 5 = 52.2 miles before lunch

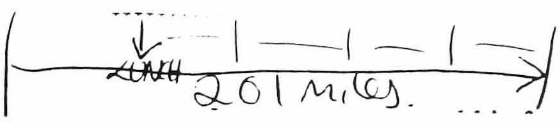
52.2 × 4 = 208.8 miles after lunch

Then I add the miles together 208.8
+52.2
261 miles

Jack D - **T4**

I took 261 (miles) and divided by 5, giving 52.2, I then took 52.2 off 261, giving 208.8, now 4 x 52.2 = 208.8, 208.8 + 52.2 = 261. so they drove 52.2 miles before lunch, and 208.8 miles after lunch. I knew this because it is obvious that the Journey is 'split' into 5 'parts' (1 before lunch, 4 after) so all I do is divide 261 by 5, and take the result of 261 ÷ 5 (52.2) off 261, giving the last 4 parts of the journey as 52.2 x 4 = 208.8, 208.8 + 52.2 = 261

Jacob B - **T4**



First I decided ~~if~~ it was 4 times as much after lunch, and 1 time before hand, that would equal 5 sections. So I divided 261 by 5 which come to 52.2 miles..

So they drove 52.2 miles before lunch and 208.8 miles after lunch.

Elizabeth W - **T4**

The use of diagrams not only shows how parts and sections themselves are taken as objects, but also emphasize how difficult it would be to use this model in a [2.7] problem.

One "calculations only" solution to **T2.7** shows, on the other hand, how close it may be to an algebraic solution that does not employ algebraic symbolism (Nick P, F3B).

$2.7 + 1 = 3.7$
$74 \div 3.7 = 20.$
$20 \times 2.7 = 54$
$54 + 20 = 74.$
Sam's ticket cost £54
george's ticket cost £20.

Nick P - **T2.7**

This is a particularly interesting instance: Nick's solutions to a "secret number" problem corresponding to **6x + 165 = 63** shows his awareness of treating numerical relationships in purely numerical terms, but nevertheless, his scripts also show that he never spontaneously produced numerical relationships to model problems that had not one already given in some explicit form (the "secret number" problems, for example). Another script, however, shows us the opposite case: Jenny G (F3B) writes down an arithmetical sentence that correctly models the problem, but fails to go any further (supposedly for not knowing how to derive the value of the question mark from that expression).

$? + (? \times 2.7) = 261$

Jenny G - **D2.7**

Each of those students' cases illustrate an aspect of embryonic algebraic thinking: Jenny's awareness of the numerical model; Nick's awareness of the purely numerical treatment of numerical relationships. It is the fusion of those two aspects that produces the algebraic solution in Vanessa J's (F3A) script.

$x + (x \times 4) = 74.$
$5x = 74$
$1x = 14.8$
EACH TICKET COSTS £14·80

Vanessa J - **T4**

Flavia C (A7I) and Alex K. (A8I) correctly set and solved equations, as did Carolina R (HS8I). It is important to notice, however, that Carolina's equation derives from an initial representation of the problem that is different from Flavia and Ernesto's. While they thought in terms of "what composes the total", she thought in terms of "what is left after the first part of the journey". However derived from different initial readings of a whole-part scheme, the three solutions converge as they reach a point from where they are only concerned with operating within the realm of numbers.

$x \rightarrow$ Entrada de Samuel
$4x \rightarrow$ Entrada de Jorge (4 vezes mais que a de Samuel).

$4x + x = 220{,}00 \rightarrow$ equação
$5x = 220$
$x = \frac{220}{5}$
$x = 44$

– x é o preço de 1 entrada.
$4x$ é 4 vezes o preço da entrada.
$5x$ é o que todos gastaram. 220 é o que gastaram
$x = 44$, uma entrada custa Cr\$ 44,00.

R: A de Samuel custou Cr\$ 44,00. A de Jorge, Cr\$ 176,00

Flávia C - **T4**

x = antes do almoço (substituindo o nº de quilômetros
$2{,}7x$ = depois do almoço (pela incógnita x)
$x + 2{,}7x = 481$ (somando "antes" e "depois" ~~para~~ resulta no nº total de quilômetros)
$3{,}7x = 481$
$x = 481 : 3{,}7$ (passando a operação para o outro lado da igualdade para se isolar a incógnita)
$x = 130$ (total de Km rodados antes

$2{,}7x = 2{,}7 \cdot 130 = 351$ (Total de Km rodados depois)

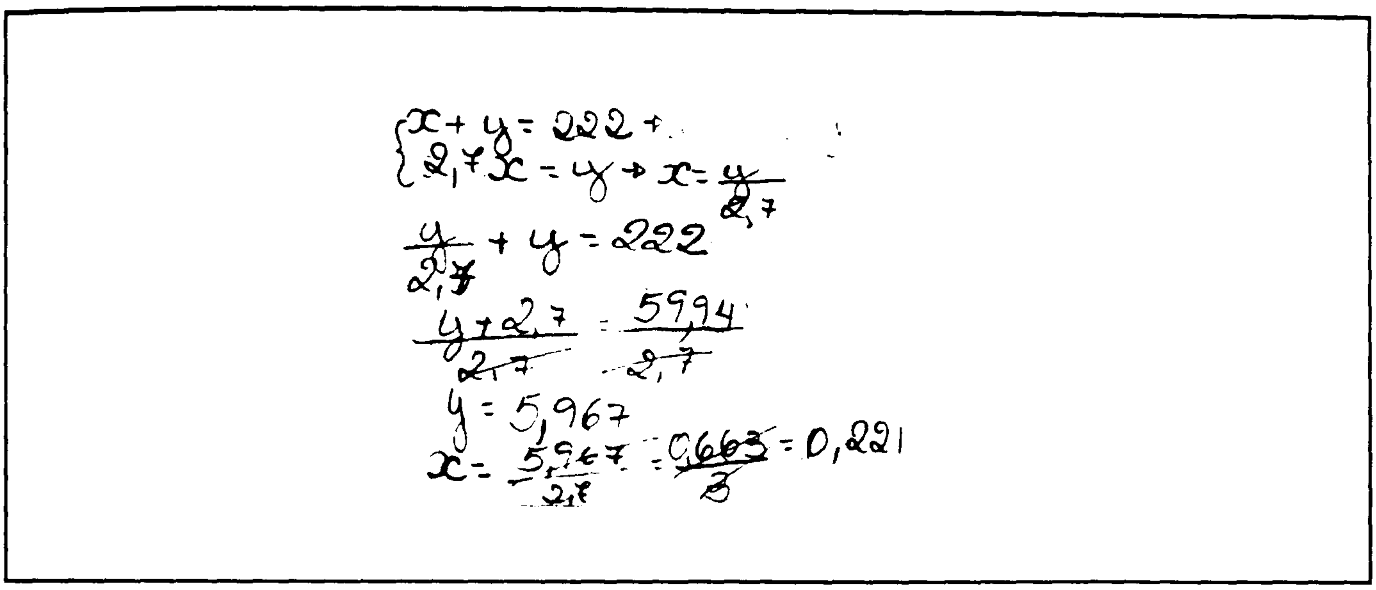
Alex K. - **D2.7**

$481 - x = 2{,}7x$
$3{,}7x = 481$
$x = \frac{481}{3{,}7}$
$x = 130$

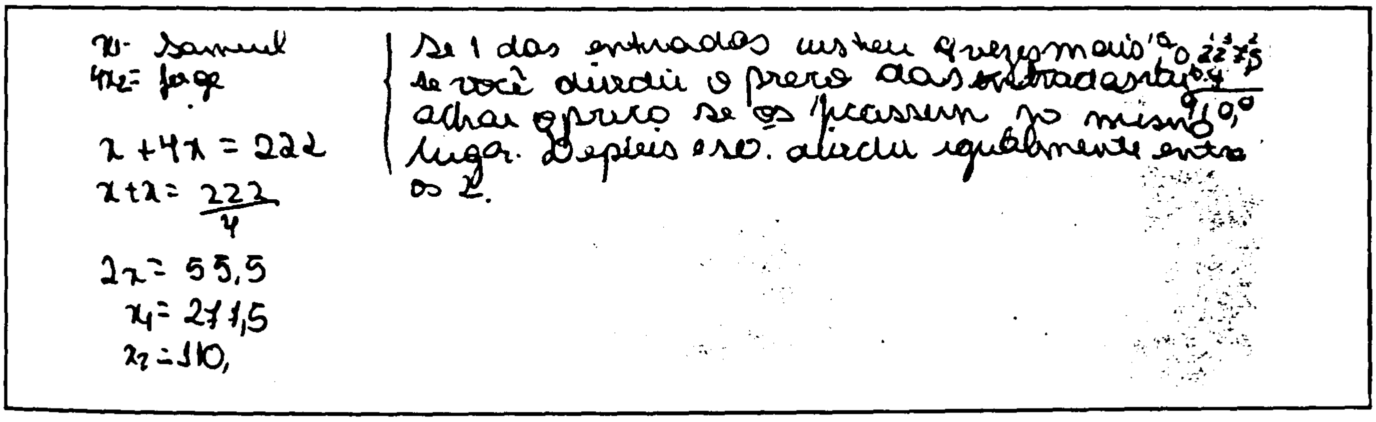
antes do almoço = 130
depois = 5.06,9.

Carolina R - **D2.7**

Another group worth examining is that of wrong solutions in which standard algebraic notation is employed. In two of our examples (Adriana V, A8I; Ana C, A8I), the initial equations correctly model the problem's situation, but they are dealt with in an incorrect way: there are *technical* errors.



Adriana V - **T2.7**



Ana C - **T4**

On the other two examples (Vinícius G, A8I; Adriano I, A8I), the initial equations do not model the problem correctly, but this time they are correctly solved: there are *modelling* errors.

$$x(4+x) = 222$$
$$4x + x^2 = 222$$
$$x^2 + 4x - 222 = 0$$
$$x = \frac{-4 \pm \sqrt{16+888}}{2}$$
$$x = \frac{-4 \pm \sqrt{904}}{2}$$

Só entendi deste jeito e dá errado

Vinícius G - **T4**

$$2{,}7x \cdot x = 481$$
$$2{,}7x^2 = 481$$
$$x^2 = 481 \div 2{,}7$$
$$x^2 = 17{,}81$$
$$x: 4{,}13$$
$$x: 11{,}151$$

Adriano I - **D2.7**

What is common to all the four solutions is the assumption that by modelling the problem with a numerical relationship and then numerically manipulating it is an acceptable method for solving the problem.

Summary of Findings and Conclusion

We think that the most important aspect in relation to this group of problems, is that it provides direct and clear illustration of different ways of modelling an "algebraic verbal problem," both algebraic and non-algebraic, particularly throwing light in the use of whole-part models, the superficial similarities and the deep differences between those models and algebraic ones.

It became clear that the choice of operations used in the solution process was mostly subordinated to the modelling of the problem. In the case of algebraic solutions, it is the *arithmetical articulation*, as discussed in chapter 3, that informs the solution; in the case of whole-part solutions, it is the composition of the whole in terms of its parts—the *whole-part* articulation.

It was important to see, in Ticket[4] problems, the transformation of the more expensive ticket into "four tickets," ie, the application of the whole-part model independently from a "geometric" representation, indicating that those models are not simply a direct representation of the objects of the context; this suggests the possibility of the existence of a more general underlying model, in which case we would have a bigger obstacle to the development of an algebraic mode of thinking than if it were simply the case of totally contextualised solution, as an already established general model—even if not explicitly stated—would "compete" with the newly offered algebraic one. On the other hand, the teacher may take this to her or his advantage, by making the underlying *whole-part* model explicit, so it can be compared with algebraic models and the differences clearly established.

The fact that [2.7] problems are so more difficult if a *whole-part* model is used, can be understood in relation to the way in which the numbers involved are understood. Used with T&D problems, *whole-part* models impose a distinction between "the numbers that count the number of parts" and "the numbers that correspond to each part." Because the "unknown" parts are never dealt directly with, the notion of number that dominates in the model is that of *counting* number, and this clearly makes *whole-part* models not applicable at all to [2.7] situations. It is likely that teaching aiming at developing an awareness of the fact that, say,

2.7 x price per pound=price of 2.7 pounds

would significantly enhance the performance in [2.7] problems, but, as we have already indicated, the justification of such knowledge in terms of a decomposition of the decimal "coefficient" is far from immediately visible, so this seems to be an area to which anyone developing a teaching approach for the teaching of algebra has to pay careful attention.

Finally, the scripts in this section show ways in which, as we had indicated in the theoretical analysis of possible solutions, equations of the type

$$\mathbf{ax + bx = c}, \ \mathbf{a} \text{ and } \mathbf{b} \text{ positive integers}$$

can be modelled back into a *whole-part* model, but not if **a** or **b** are not integers; for the teacher or researcher, the fact that the model used can be completely hidden behind the use of "algebraic notation," indicates that it is not enough to *suppose* that the ability to solve equations of the type above imply the ability to solve the case with at least one of **a** and **b** non-integer.

We think that this is an extremely important result of our study, as it clarifies the inadequacy of "starting with examples with simple numbers" approach in the specific case of the types of equation involved in the solution of the problems in this section, but at the same time pointing out that a general problem exists in this respect, and that the underlying model has to be examined if we are to understand students' difficulties in learning algebra and in developing an algebraic mode of thinking.

4.3 SEESAW-SALE-SECRET NUMBER PROBLEMS

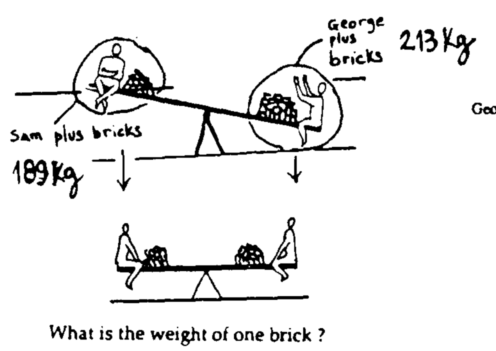
THE PROBLEMS

I am thinking of a "secret" number.
I will only tell you that ...

181 - (12 x secret no.) = 128 - (7 x secret no.)

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

SN1 Problem

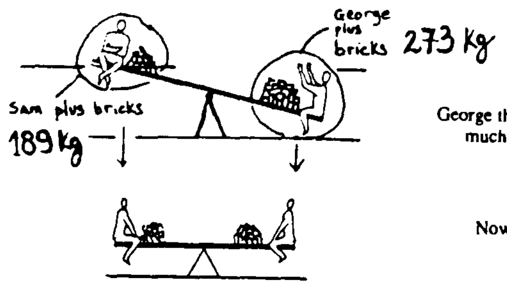


George throws away 11 bricks and Sam throws away 5 bricks.

Now they are balanced.

What is the weight of one brick ?
(Explain how you solved the problem and why you did it that way)

Seesaw 11-5 Problem



George throws away four times as much weight as Sam does.

Now they are balanced.

How many kilograms did George throw away? And Sam?
(Explain how you solved the problem and why you did it that way)

Seesaw 4x Problem

Maggie and Sandra went to a records sale.

Maggie took 67 pounds with her, and Sandra took 85 ponds with her (a lot of money!!).

Sandra bought 11 Lp's, and Maggie bought 5 Lp's.

As a result, when they left the shop both of them had the same amount of money.

What is the price of an Lp?

(Explain how you solved the problem and why you did it that way)

Sale 11-5 Problem

Maggie and Sandra went to a records sale.

Maggie took 67 pounds with her, and Sandra took 85 pounds with her (a lot of money!!).

Sandra spent four times as much money as Maggie spent.

As a result, when they left the shop both of them had the same amount of money.

How much did each of them spend in the sale?
(Explain how you solved the problem and why you did it that way)

Sale 4x Problem

GENERAL DESCRIPTION

This group of problems consisted of five problems, four of them contextualised (two contexts, Seesaw and Sale) and one "secret number" problem, where the problem condition is given in the form of a "syncopated" numerical equation.

Both Seesaw (**E**) and Sale (**A**) problems were presented in two distinct ways.

The first one gives the relationship between how much each of the two persons involved "threw away" (for **E** problems) or "spent" (for **A** problems) in terms of *number of pieces* ([11-5] problems). The second one gives that relationship in terms of *ratio* ([4x] problems).

Giving the relationship in terms of *number of pieces* sets the number of unknowns in the problems to only one, namely the weight of a brick or the price of an Lp (or a T-shirt, in the case of the Brazilian tests).

On the case of [4x] problems, on the other hand, they *primarily* involve two unknown quantities, linked by the given *ratio*, and the reduction into a problem with one unknown is a necessary step towards a correct solution of the problem, a step that involves a *substitution*.

The **SN1** problem was included in this group for the reasons already discussed in the introduction to this chapter.

On the Brazilian tests, **Sale** problems had numbers significantly larger than those on the English version, due to the necessity of adjusting the context to Brazilian prices. This may have discouraged trial-and-error solutions, but in any case trial-and-error solutions are not common in Brazilian classrooms, being in general explicitly characterised by the teachers as a "non-solution", and are not accepted by most teachers as a valid answer in a test. Although we insisted with the students that **any** method would be accepted, we expected a very low level of trial-and-error answers from the Brazilian groups — what actually happened — so the effect of larger numbers would be insignificant. We also chose to use "T-shirts" instead of "Lp's" because buying the former is a more usual activity for those students.

DISCUSSION OF POSSIBLE SOLUTIONS

Strictly speaking, [4x] problems are modelled algebraically by the set of equations

$$\begin{cases} a - x = b - y \\ y = 4x \end{cases}$$

while [11-5] problems are modelled algebraically by

$$a - 11x = b - 5x$$

From this point of view, [4x] problems are intrinsically more difficult than [11-5] problems.

However, it is possible that the given *ratio* is used to produce a *direct parts substitution* ("one lot and four lots") or a *direct numerical substitution* ("a number, four times a number"), thus reducing [4x] problems to the algebraic form

$$a - x = b - 4x$$

without going through the set of equations. From then on, both problems would be equally difficult from the algebraic point-of-view.

We expected non-algebraic solutions to fall into one of two main categories:

(i) a *qualitative analysis of the situation*, for example,

> " If George's side was heavier but now they are the same, it must be because the amount George threw away in excess of what Sam did corresponded to the original difference between the two sides."

In this case, two subtractions would be performed in order to *evaluate* the original difference in weight and the number of *units* put away in excess, and then a division, in order to *evaluate* how much of the original difference corresponds to each *unit* thrown away in excess.

(ii) a *comparison of wholes* strategy, supported or not by a diagram (fig SSE 1)

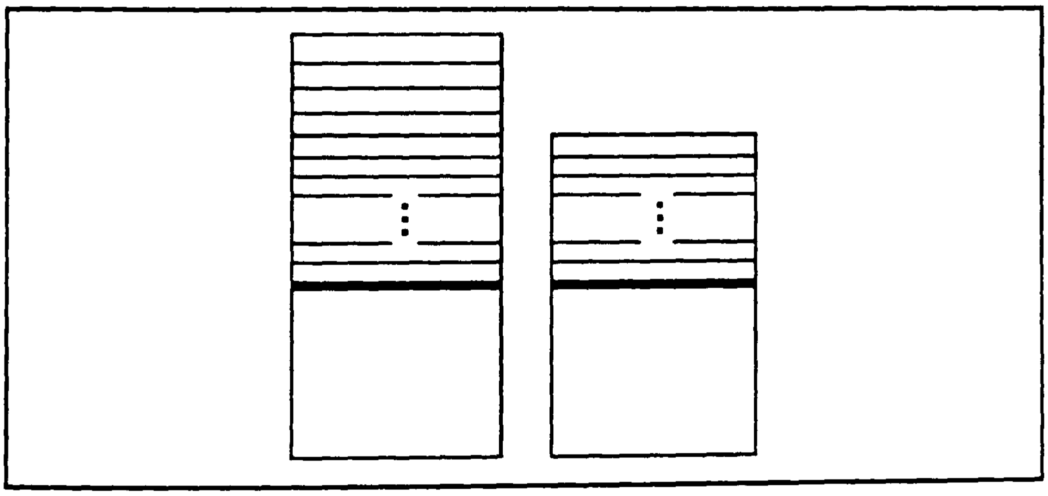


Fig SSE 1

Here two subtractions would also be performed, this time in order to *evaluate* the difference between the two wholes and the number of *units* "missing" on the smaller of the two wholes, and then a division, in order to *evaluate* how much of the difference corresponds to each *unit*.

The Secret Number (**SN1**) problem can be seen in three very distinct ways.

1) as *an equation in syncopated form*, in which case the numerical relationship could either be (1a) manipulated algebraically, or (1b) *modelled back* (for example, a scale-balance situation) and the resulting model manipulated to produce the answer .

2) as a *template*, providing a condition that has to be satisfied by the secret number but no information as to how to find it;

3) as *a compact description of a whole-part model situation*—eg, the one described some paragraphs above—that can be manipulated to find the required number. It is important to emphasise that this **does not** mean *modelling back* a numerical problem, but **actually seeing it that way from the beginning**. The subtraction signs are *literally* interpreted as "separating" or "removing" from the unequal wholes, an *action* that produces two new, equal, wholes.

There is a subtle but important difference between (1b) and (3). In (1b) the numerical relationship is recognised *as such*, although as a "by-product" of modelling a situation, and an effort is made to *model it back* into a setting where manipulation is possible; in (3), however, the arithmetical symbolism is never seen as such, once the expression involves an unknown number that cannot be used in calculations, and even worse, this number appears on both sides of the equality sign, completely removing any sight of a "result", and thus, any sight of "calculations". Instead, adding is seen as *joining*, subtraction as *disjointing or separating or taking away*, and multiplication as *grouping that many lots or parts*.

A study by John Mason (1982) reveals not only that symbols for arithmetical operations are easily used with this interpretation by young students, but also that when used in this way they might evoke properties different from those evoked by the arithmetical use, as in, for example, when trying to symbolise the Cuisinaire rods configuration in fig. SSE 2, where

3 x 3blacks and 2whites

can be consistently interpreted as

3(3blacks+2whites)

even in the absence of the original configuration (a correct interpretation in the context of the activity), but

3 x 3blacks + 2whites

might be interpreted, in the absence of the original configuration, as

(3 x 3blacks) + 2whites

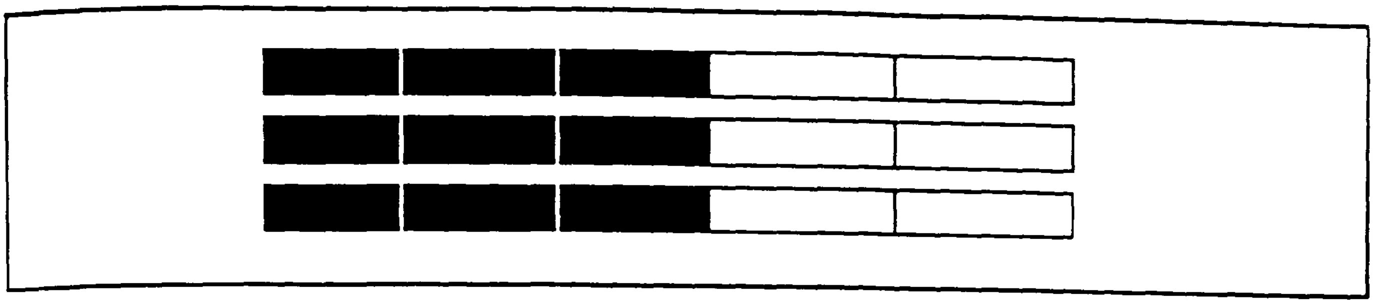


Fig SSE 2: configuration of rods to be described

The stronger bond produced by "and" is in correspondence to its use in normal speech, where in a phrase like "Sam and George's excellent performance!" the judgement is immediately seen as applying to both.

The use of non-algebraic models is bound by the necessity of maintaining a *dimensional homogeneity* when using addition and subtraction, ie, as far as the operations are used to *evaluate a total or a difference in measures*, the two operands must be seen as having the same *dimensional type*, once they are seen as *measures*. Algebraic models, on the other hand, avoids this concern *by introducing a homogeneity in numbers that can be sustained throughout exactly because of the* ***internalism*** *characteristic to thinking algebraically*. Dimensionality does not belong to the scope of *algebraic thinking*. This characteristic of the manipulation of non-algebraic models can serve, for example, to indicate the inadequacy of performing certain calculations (for example, on **E11-5** problems, the inadequacy of subtracting 11 (the number of bricks Sam threw away) from 273 (the initial weight on Sam's side)).

One aspect of algebraic and non-algebraic solutions is of special interest in relation to this group, because it is well recognisable in the range of different solutions to this group of problems.

In the general characterisation of our framework we have indicated that algebraic solutions are *analytical*. Moreover, we have seen that all the problems in this group can be correctly modelled by a numerical equation of the form

$$\mathbf{a - bx = c - dx}$$

Because the unknown appears on both sides of the equality sign, an algebraic solution to this equation cannot avoid *manipulating the unknown*, ie, adding or subtracting terms involving the unknown. But this is not an intrinsic characteristic of the relationship, it is rather a consequence of the *analytical* character of the algebraic method, of the need — so to speak — to express the unknown (required) number in terms of known numbers and operations on them.

We have also shown that the problems in this group, including **SN1** — and very similarly the above equation when **b** and **d** are whole numbers — can be modelled into a whole-part model, and that the manipulation of such model to produce the required number or measure completely avoids *manipulating the unknown* by producing successive *evaluations* of unknown measures from known ones, until one finally reaches a step where the unknown (required) measure is *evaluated.* Again, this is not a characteristic of the whole-part model itself, but of the *synthetical* character of *non-algebraic* methods.

Research on the solution of equations has indicated that there is a "didactic cut" in the passage from manipulating equations where the unknown appears on one side only of the equal sign to manipulating those where it appears on both sides, and that this cut corresponds to the "...need to operate on the unknown in the solution of [such] linear equations" (Gallardo, 1987).

Our analysis above indicates that the root of the difficulty with unknowns on both sides might lie on the fact that non-algebraic thinkers operate *synthetically* thus not operating with unknown values, ie, an important part of the strategy required to solve algebraically those equations does not fit into their normal, general framework. Also, it could be that the process of *translating back* a numerical equation with unknowns on both sides of the equal sign into a non-algebraic model is too difficult because of the complexity of the required models, and building some expertise on the process depends on a reasonable amount of experience. Nevertheless, students can be taught *translating back* skills (Gallardo, 1990).

Gallardo's example on page 44 (op. cit.) is particularly insightful, and we will examine it in some detail. It is about a student that had been taught to solve equations of the type

$$\mathbf{ax + b = cx + d \ , \ a>c \ , \ b<d, \ a,b,c,d>0}$$

by "...translating the equation's elements into a geometrical situation, where figures with equivalent areas were involved" (ibid.) (fig SSE 3).

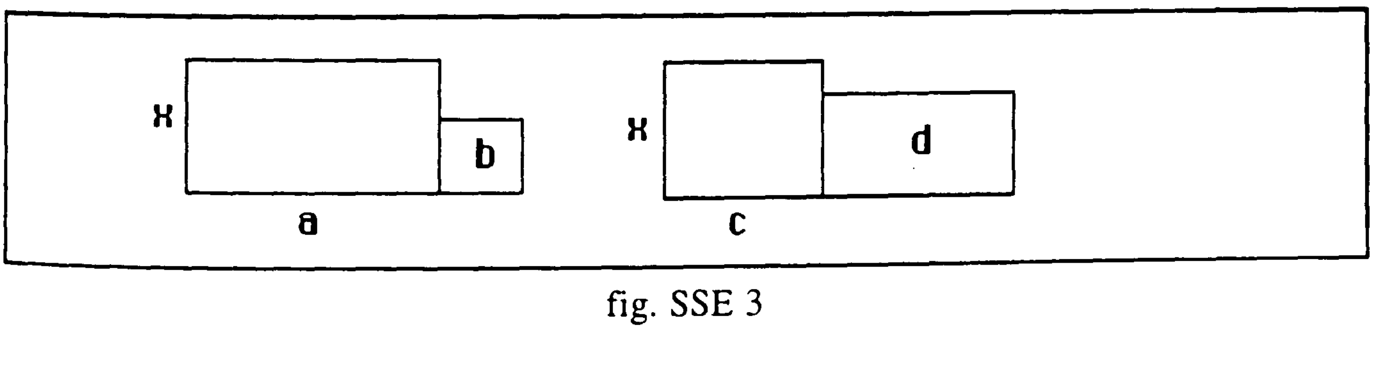


fig. SSE 3

When she had understood this model, she was then given the equation

$$9x + 33 = 5x - 17$$

which she modelled using the model taught with an "invention of her own": the subtraction of 17 was taken as meaning the *removal of a piece of the area equivalent to 5x*. (fig SSE 4)

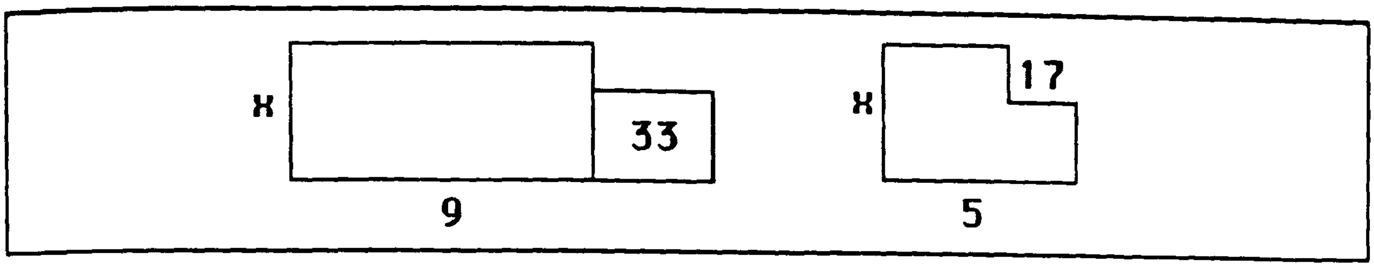


fig. SSE 4

The student manipulates this model to arrive at

$$4x + 33 + 17 = 0$$

corresponding to fig. SSE 5, and then a block occurs, because she is not willing to accept the negative solution.

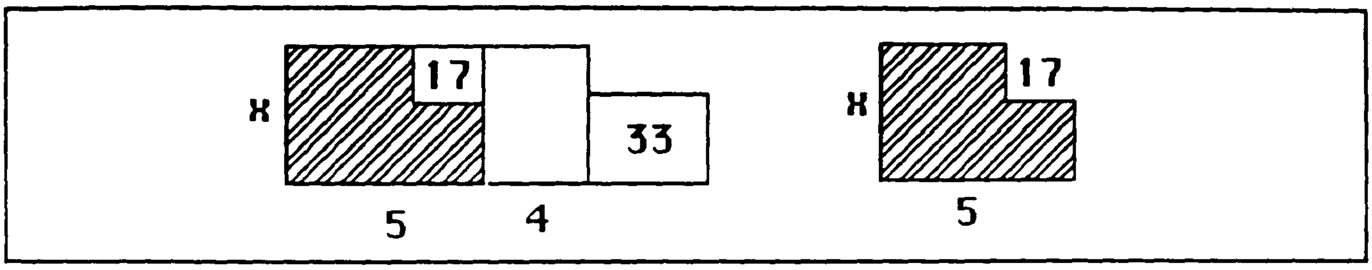


fig. SSE 5

This example is insightful, in the first place, because it suggests that the refusal to accept a negative answer is due to the fact that the "x" is representing the *measure* of a side in the figures, and thus can be but a positive number. In the second place, it shows the extent to which such solution is dependent on properties of the geometrical configuration, ie, the geometrical configuration is not just a *support diagram* to help to keep track of a reasoning that is "in essence" identical to the one behind an algebraic solution. Finally, this example supports our suggestion that the process of *translating back* is far from simple and straightforward, as finding a similar geometrical configuration to model and solve an equation like

$$173 - 5x = 265 - 11x$$

would certainly involve either a reasonable amount of experience with such models, having being taught the configuration as a "solution formula", or a high degree of ingenuity[13].

* * *

On the basis of our analysis of the problems, we hypothesized that:

A) [4x] problems might be more difficult to solve than [11-5] problems for a student using a non-algebraic approach, because [11-5] problems provide *objects* (bricks or Lp's) that can be immediately seen as *parts*, while on the case of [4x] problems one has first to establish a *unit* (more easily, how much Sam threw away or how much Maggie spent) to be then manipulated as a *part* and to represent the "4 times" as "4 parts" or "4 lots";

B) [4x] problems might be easier to solve if an algebraic approach is used rather than a non-algebraic one, because the "4 times as much" statement would suggest within a *Numerical Semantical Field* — by suggesting a multiplication — the correct "unknown, 4 times the unknown" structure; this approach reduces the difficulty of having to establish a *unit*, once seeing the "4 times as much" — *times* indicating a *ratio* — as meaning "4 times the other amount" — *times* indicating multiplication — immediately entails the "other amount" that is to be multiplied as an *object* (multiplication requiring two numbers to be performed). The predominant use of an algebraic approach within a group of students would thus reduce the difference between the facility levels for [11-5] and [4x] corresponding problems.

C) **SN1** problems would be extremely difficult to solve using a non-algebraic approach.

GENERAL DATA ANALYSIS

One aspect of the data is helpful in understanding other aspects on the data, so we examine it first.

For both Brazilian groups the **SN1** problem had the highest level of facility among the problems in this group (43% for AH7 and 88% for AH8), all but one of the correct solutions employing equations. On the other hand, for both English groups the **SN1**

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[13] The degree to which this is true can be easily verified by trying to produce such configuration and to solve the equation using it. It was not immediately that I found a way out of it myself.

problem had the lowest facility level among the problems in this group (4% for FM2 and 15% for FM3); four of the seven correct answers employed equations.

Those numbers are a direct indication of the extent to which Brazilian pupils dealt better with equations than their English counterparts, once eventual difficulties with modelling the problem onto an equation are almost reduced to none. More important here, however, is the fact that solving **SN1** problems depended so heavily on the use of equations.

Only 4 students on the combined FM2-FM3 group (75 students solving **SN1** altogether) tried to use an equation with **SN1** and failed to solve it correctly. Together with the very low level of success on **SN1** that suggests that students on the FM2-FM3 group were predominantly trying to use non-algebraic methods to solve **SN1** problems.

Another aspect of interest arising from the data is the use of equations on corresponding [11-5] and [4x] problems. In almost all cases — the exception being **A11-5** and **A4x** for FM3, where the use of equations was nil for both problems — the percentage of correct solutions using equations is higher for [4x] than for [11-5] problems[14]. This indicates that algebraic solutions do belong to a *Semantical Field* where numerical relationships are meaningful by themselves, as the suggestion of the multiplication seems to be the factor that triggered the choice of an algebraic solution.

More support for this interpretation can be drawn from the fact that on the AH7 group the bulk of the correct answers to [11-5] problems came from non-equation solutions but **all** the correct solutions to [4x] problems used equations. Algebra is systematically introduced only on the 7th grade of Brazilian schools, usually later on the first half of the academic year; thus, seventh graders can be considered *well informed* and somewhat *skilful* in solving equations, but not yet deeply committed to using equations whenever they are given a verbal "algebraic" problem. This can be also seen in the fact that in all of the four contextualised problems, most of the incorrect solutions on the AH7 group do not attempt to use an equation and most of the incorrect solutions on the AH8 group **do** represent a mistaken use of equations. This suggests that for the Brazilian 7th graders the "default" approach is non-algebraic, and for the 8th graders it is an algebraic one, namely the use of equations.

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[14]This difference is significant on the Brazilian groups, although it is not significant on the English groups due to the very low level of correct answers using equations.

The use of algebraic methods resulted — as we have predicted — in very similar facility levels for three out of four pairs of corresponding [11-5] and [4x] problems on the Brazilian groups, while on the English groups [11-5] problems were always significantly easier than the corresponding [4x] problems.

On the Brazilian groups **SN1** has a high facility level, and the lower levels of correct answers to the four contextualised problems indicate *difficulties with modelling them with an equation*, ie, with establishing a correct arithmetical relationship; this is even more evident as we look at the percentages of **incorrect** solutions involving equations at AH8, that "by design" (curriculum) is bound to use equations more than AH7. On the other hand, on the English groups **SN1** has a low facility level, and the differences between corresponding contextualised problems reflect *difficulties in seeing meaningful relationships between the elements in the context of the problems.*

The former difficulty might be seen as having a greater degree of complexity, as one would have to make sense of the structure of the given situation and *then* transform it into a numerical-arithmetical problem. However, the *mode of thinking* in which one is operating is of substantial importance in determining for a given problem the degree of difficulty in understanding the structure of a problem. The fact that a person is aiming at transforming a contextualised problem into a numerical-arithmetical one may be, as we saw in relation to [4x] problems, of great help in making sense of a structure for the problem, which shows that difficulties with the algebraic approach do not represent the simple accumulation of the *numerical* difficulties on the top of other difficulties in understanding the structure of the problem.

STUDENTS' SOLUTIONS

The SN1 problem

All of the 43 **OKEQT** solutions by Brazilian students (of a total of 71 students presented with the question) used standard algebraic symbolism while the three **OKEQT** solutions by English students (out of 75) employed "secret no", "sn" or "?". In itself this suggests that the use of a special form of symbolism, rather than syncopation or the "iconic" interrogation mark might become a significant factor in establishing equations as recognisable—and thus acceptable and capable of being manipulated—mathematical objects. This suggestion is supported by a number of explanations presented with the solutions (Bartira G, AH7; Ana B, AH8; Eurico G, AH8):

nº secreto = x

$181 - (12 \times x) = 128 - (7 \times x)$

$181 - 12x = 128 - 7x$

$-12x + 7x = 128 - 181$

$(-1).\ -5x = -53.(-1)$

$5x = 53$

$x = \frac{53}{5}$ $\qquad x = \frac{53}{5}$

Quando eu falo que o nº secreto é x. É porque x pode ser um nº qualquer É incógnita.

Bartira G, AH7: "When I say that the secret number is x, it is because x can be any number. It is [the] unknown."

$181 - (12x) = 128 - (7x)$

$181 - 12x = 128 - 7x$

$181 - 128 - 12x + 7x = 0$

$53 - 5x = 0 \rightarrow -5x = -53$

$5x = 53$

$\boxed{x = \frac{53}{5} \rightarrow x = 10{,}6}$

Eu substitui o "nº secreto" que está na dica, por x e depois transformei a dica em uma equação e resolvi-a. Até ~~[illegible]~~ descobrir o x.

O nº secreto é 10,6

Ana B, AH8: "I replaced the "secret no." that is in the *hint* by x and then *transformed the hint into an equation* and solved it until I found out the x." (our italics)

$181 - 12x = 128 - 7x$

$5x = 181 - 128$

$x = \frac{53}{5}$

R: O número secreto $= \frac{53}{5}$

Eu peguei a fórmula dada e substitui ~~[illegible]~~ o nº secreto por uma incógnita, após fazer isso passei as incógnitas p/ um lado e os números p/ o outro, aí foi só concluir

Eurico G, AH8: "I took the given formula and replaced the secret no. by an unknown, after this I moved the unknowns to one side and the numbers to the other, then it was just a matter of completing [the solution]."

In 19 out of the 43 **OKEQT** solutions by Brazilian students, an intermediate form is produced between the problem's statement and the equation in its standard form, putting **12x** and **7x** or **12** x **x** and **7** x **x** in brackets (as Bartira G, AH7, script already shown, did), an aspect that also supports that suggestion.

In 23 **OKEQT** solutions by Brazilian students, the following line appeared:

$$-5x = -53$$

instances showing that in algebraic solutions the meaningfulness of each expression produced is related only to the perceived correctness of the process that produced it, ie, the *internalism* of thinking algebraically.

A variety of algebraic techniques appeared on the **OKEQT** scripts:

(i) multiplying both sides by (-1) to get rid of the negative signs (Cláudia F, AH7) or to transform the side of the equation containing terms in the unknown into a more appropriate form (Andrea M, AH8);

$181 - (12 \times x) = 128 - (7 \times x)$ $181 - 12x = 128 - 7x$ $-12x + 7x = 128 - 181$ $-5x = -53$ $-1(-5x) = -1.(-53)$ $5x = 53$ $x = \frac{53}{5}$ $\quad x = 10,6$	Resposta. O número "secreto" é 10,6 EXPLICAÇÃO: Tanto você fazendo a operação ① ou a operação ②, o resultado será o mesmo

Claudia F, AH7

$181 - 12n = 128 - 7n$ $-12n + 7n = 128 - 181 \; (x-1)$ $12n - 7n = -128 + 181$ $5n = 53$ $n = 10,6$	como no enunciado já existia uma expressão era só trocar a palavra número secreto por uma variável qualquer e separar parte literal e parte numérica.

Andrea M, AH8

(ii) directly performing the division (-53)÷(-5), without first performing the step described on the previous item (Ernesto K, AH7);

$181 - 12x = 128 - 7x$

$-12x + 7x = -181 + 128$

$-5x = -53$

$x = -53 : (-5)$

$x = 10{,}6$

$$\begin{array}{r} 181 \\ -\;128 \\ \hline 053 \end{array} \qquad \begin{array}{r|l} 53 & 5 \\ 030 & 10{,}6 \\ 0 & \end{array}$$

Ernesto K, AH7

(iii) transforming the equation into a standard form (ax + b = 0, Ana B, AH8, script already shown on this section), (ax + b = cx + d, Robert M, FM3);

181 - (12 x secret no = 128 - (7x secret no)

181 = 128 - (7x secret no) + 12 x sn)

181 + (7 x sn) = 128 + (12 x sn)

53 + (7 x sn) = 12 x sn.

53 = 5 x sn

$\frac{53}{5}$ = sn.

sn = 10.5

Robert M, FM3

(iv) expressing the answer both as a fraction or as a decimal number;

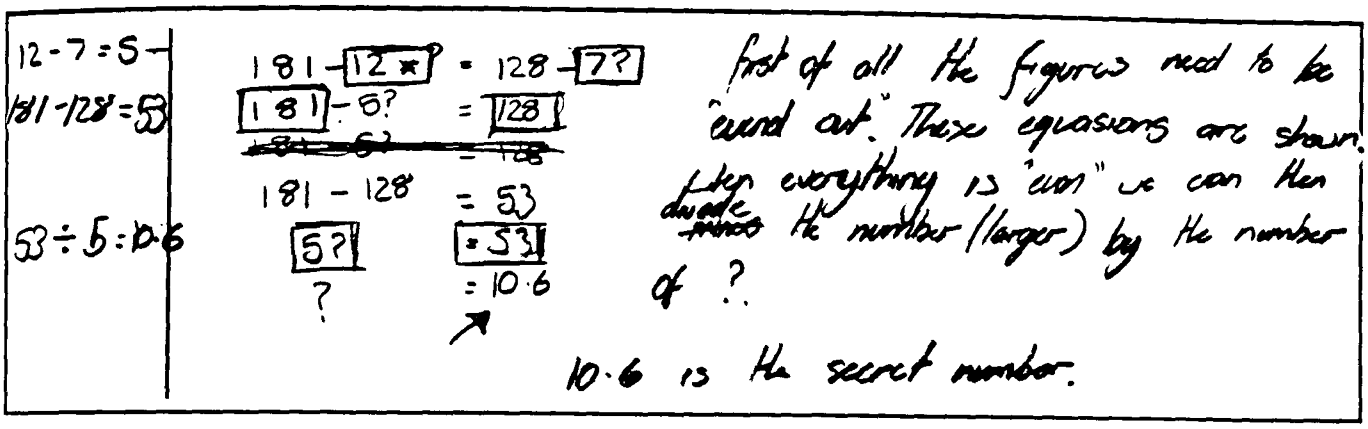
One solution is of particular interest (Nick A, FM3). Apart from the use of "?" for the unknown, it seems to present us with a mixed solution. The first step,

$$\begin{aligned} 181 - 12x? &= 128 - 7? \\ 181 - 5? &= 128 \end{aligned}$$

could be seen as the result of an algebraic manipulation. The second step, however,

$$\begin{aligned} 181 - 5? &= 128 \\ 181 - 128 &= 53 \\ 5? &= 53 \end{aligned}$$

seems to be based on a *whole-part* modelling of 181 - 5? = 128, once no intermediate step is provided except the evaluation of 181-128, and the transformation seems to be a direct one. Whether the first step was also based on a non-algebraic model, nothing can be concluded.



Nick A, FM3

From all four groups (a total of 146 students presented with the question) there were only five **OKCALC** solutions to **SN1**. This immediately indicates that to model **SN1** into a non-algebraic model was a very hard task for those not able to use an algebraic one for whatever reason.

Of the five **OKCALC** solutions, Elizabeth W's (FM3) was certainly the most peculiar. First, because she **does** produce the right number, using the most direct calculations possible, only to "conclude" that — for some unexplained reason — 10.6 is **not** the secret number. Second, for the *rationale* to her choices of subtractions ("181 is bigger than 121 and 12 is bigger than 7"). However, it is difficult to see why she chose to divide 53 by 5, and not to perform some other operation. The numerical preference "divide the bigger by the smaller" cannot provide a justification for the choice of a division itself, and we are led to believe that she did have the insight of an underlying non-algebraic model, and she so expressed herself because she was not able to make the model explicit — even to herself. Another interesting aspect is that she never thought of **trying** the 10.6 she thus obtained to see if it "worked", saying instead that she would use a **trial**-and-error approach.

At first I thought I would answer this with trial & error," but I have had a better idea (if it works!). 181 is bigger than 128, and 12 is bigger than 7, so I got this idea: 181 - 128 = $\boxed{53}$ 12 - 7 = $\boxed{5}$. 53 ÷ 5 = 10.6. No, that was a bad idea - 10.6 is not the secret number. I will use trial + error. 12 × 5 = 60

Elizabeth W, FM3

Two of the remaining four **OKCALC** solutions (Fabiana M, AH7; Gareth A, FM2) do not provide us with information enough to decide whether they represent non-symbolic solutions of an equation. Even if they are not, this is probably as close to it as we will get, once Gareth actually produces a standard equation (replacing "secret no." by "**x**") and Fabiana says "to know the difference between known numbers and between unknown numbers and divide them". Another possibility would be, as we have already seen, to reason in a manner similar to that described as possible non-algebraic solutions to the contextualised problems, only this time reasoning with the numbers themselves:

> "The amount of secret nos. that is taken in excess from the left-hand side must be the difference between 181 and 128", etc..

and this seems to be exactly the model used by Joe V (FM3) and Jacob B (FM3).

(181 - 128) : (12 - 7)

53 : 5

$10\frac{3}{5}$

181
128
053

12
7
5

saber a diferença dos nºs conhecidos e dos nºs desconhecidos e dividi-los.

$10\frac{3}{5}$

Logo o nº é

Fabiana M, AH7

$181 - (12 \times X) = 128 (7 \times X)$

$181 - 128$
$= 53$

$12 \div 7$
$= 5$

$\frac{5}{53} = 10\frac{3}{5}$

Gareth A, FM2

106
5)53
5
30

$12 \times 10.6 = 127.2$
$7 \times 10.6 = 74.2$

181.0 − 127.2 = 53.8
128.0 − 74.2 = 53.8

first I need to find what 12 x secret, take 128 from 181.

$181 - 128 = 53 \div 12 - 7$ (numbers that you times secret

$53 \div 5 = \boxed{10.6}$

now I check that this is right

sums $181 - 10.6 \times 12 = 53.8 \longrightarrow 128 - 7$...

Joe V, FM3

The difference between 128 and 181 is 53, and as *12x is 5 more than *7x, I must find out how many 5's go into 53, the answer is 10.6, this is the secret no. secret no. = 10.6

$181 - (12 \times 10.6) = 128 - (7 \times 10.6)$

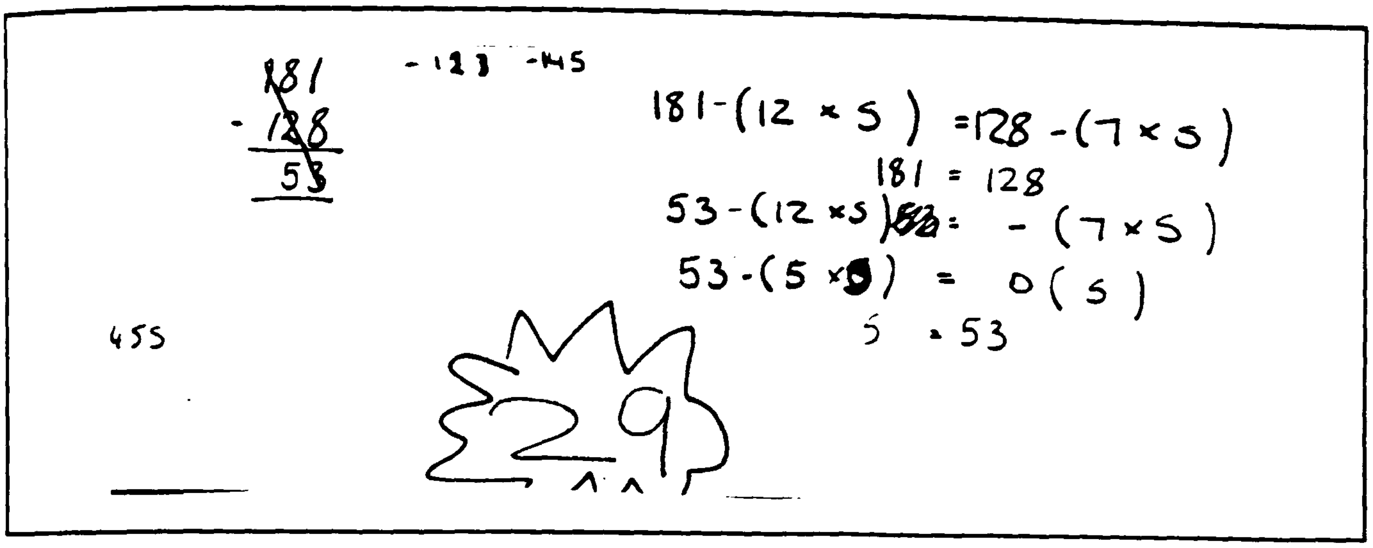
$181 - (127.2) = 128 - (74.2)$
$53.8 = 53.8$

Jacob B, FM3

There were altogether 11 **WEQT** solutions. In three of them the original equation was correctly manipulated up to a point, and then the solution process was halted. In one case (Russell P, FM3) the difficulty came when he reached the equation

$$53 - (5 \times s) = 0 \ (s)$$

to conclude that s=53. It appears that the difficulty lied in perceiving that 0s=0.



Russel P, FM3

One student, Shelley S (FM2, script not shown), replaced "secret no." by "x" but failed to go any further.

Jack D (FM3) tried to apply a *scale-balance analogy*. It is interesting that he stopped (and crossed out his previous efforts) when he reached (through a sequence of mistaken steps) the equation

$$53 \ - \ (5 \times \mathrm{SN1}) \ = \ 0$$

but it is equally interesting to observe that the use of such model produced two mistakes that are clearly associated with treating the problem using the *scale-balance analogy*:

(i) the analogy treats the unknown number as the unknown weight of an object; although the minus sign is kept on the left-hand side, probably meaning "removal", a "negative" amount of objects or "removing 7 objects from nothing" does not make sense in the *Semantical Field* of the *scale-balance analogy*. Thus, the minus sign is simply dropped.

(ii) on the second step, he says "**take off** 7 from each side", where the correct algebraic strategy would be "**add** 7 [xSN] to each side" or at least — given the equation on which he was operating — "**add** 12 [xSN] to each side". That by using this incorrect strategy he produces the transformation

$$53 - (12 \times \mathrm{SN1}) = (7 \times \mathrm{SN1})$$

to $$53 - (5 \times \mathrm{SN1}) = 0$$

is enough evidence that the subtractions were thoroughly ignored by being meaningless in this *Semantical Field.*

Balance the scales.

181 - 128 = 53.

So 53 (12 x 5N) $\overset{7}{=}$ (7 x 5N)

Then take off 7 from each side

So 53 - (5 x 5N) = 0.

Jack D, FM3

There is an important point to be discussed here. The *scale-balance analogy* has been one of the most popular didactic artifacts used to teach the solution of linear equations. Let us analyse the use of such analogy to model equations of the form

$$\mathbf{a + bx = c + dx \quad , abcd \neq 0}$$

for various sets of conditions for the parameters **a, b, c, d**.

- **a>c, b<d, b and d positive integers** (eg, 100 + 10x = 80 + 15x)

  On such cases, the analogy thoroughly applies; the plus sign is understood as *conjoining*, and thus there is a definite correspondence between the "taking off weights" strategy on the scale-balance model and the "subtracting a quantity of x's" on the algebraic model, and also division corresponding to *evaluating* a sharing action.

- **a>c, b>d, b and d positive numbers** (eg, 100 + 15x = 80 + 10x)

  On this case the analogy simply does not apply: it is not possible to put more objects on the side that is already heavier and make it balanced. Unless, of course, that the objects have *negative weight*, an impossibility within the *Semantical Field* of the scale-balance.

- **a>c, b<d, b and d positive non-integers** (eg,.......................... 100 + 3.4x = 80 + 7.8x)

  The difficulties arising here because of the decimal numbers were analysed in depth when we discussed the *Ticket and Driving* problems. The meaning of "3.4 objects" is not at all natural within the *Semantical Field* of the scale-balance, and an extension that makes it meaningful is not easy to grasp.

- **a>c, b>d, b and d negative integers** (eg, 100 - 15x = 80 - 10x)

  As analysed with Jack D's script.

It is not necessary to go any further. One obvious problem with the *scale-balance analogy* is the limitation imposed on the coefficients of the unknown and on the sign of the unknown itself. Certainly more important, the variety of strategies required to use this analogy across equations with different sets of conditions for the parameters is in clear contrast with the fairly reduced set of principles and strategies used with an algebraic model. As a consequence, the *scale-balance analogy* is inadequate not only for very quickly becoming a complex net of what are in effect different models, but also for not fostering a frame of mind adequate for the development of an *algebraic mode of thinking*.

In the remaining 6 **WEQT** solutions, the errors are always in the manipulation of the equations, as in Lilian P's (AH8) script. Those types of errors are well documented by research and in teaching practice.

$$181 - (12 \cdot x) = 128 - (7 \times x)$$
$$181 - 12x = 128 - 7x$$
$$53 - 19x = 0$$
$$-19x = -53$$
$$x = \frac{-53}{19} \rightarrow x = \frac{53}{19}$$

Lilian P, AH8

The 27 **WCALC** attempts divide naturally into two groups. In one of the groups (21 scripts), a subtraction **181-128** was always attempted. It is not possible to decide from the scripts whether those students were producing a first step in the solution of an equation of the type

$$\mathbf{181 - 12x = 128}$$

temporarily putting away the **-7x** term, or just "taking away the smaller from the greater". In any case, it is clear that manipulating the unknown or even its coefficients in a meaningful way presented a much greater degree of difficulty. Some attempts proceeded by dividing 53 — the result of the subtraction — by 12, which again appears to be the result of dealing with the incomplete equation above; some others multiplied 53 by 12 or by 7, clearly for not grasping the structure of the equation. Two students in this group (one of them Ian C, FM3) produced the subtraction **12-7** but failed to use this information correctly, which again shows a lack of grasp of the structure of the equation.

~~181 - 128 = 53~~

181 - 128 = 53

12 - 7 = 5 53 - 5 = 48

Ian C, FM3

All but one of the remaining students in the **WCALC** category seem to be merely attempting to produce a "sensible answer" by trying different combinations of operations with the numbers given. Alessandra S's (AH8) attempt, however, exhibits some intention to manipulate numerical equalities but no sense of how to do it; it is interesting that she takes the **7=7** equality as signaling the end of the process, clearly of formal meaning only.

181 - 7 = 128 - 7
174 = 121 O nº secreto é 7

181 - 174 = 128 - 121
7 = 7

Alessandra O, AH8

The Seesaw 11-5 problem

Only 5 out of 77 students presented with this problem correctly used an equation to solve the problem (**OKEQT** solutions); one of them had to be categorised as an incorrect answer once he simply erased his correct solution (which, of course, still remained visible). Those solutions do not provide much additional information on the solution of equations. However, in one script (Andrea M, AH8) we have a quite clear description of her process of solution.

$189 - 5x = 273 - 11x$
$189 - 5x - 273 + 11x = 0$
$6x = 273 - 189$
$6x = 84$
$x = 14$

o peso do tijolo é x, e para formar uma igualdade precisava estar os dois peso equivalentes, como essa equivalência foi dada foi só montar as duas contas de subtração.
O resto é o mesmo processo do isolamento do x, fazendo a operação inversa.

Andrea M, AH8

(i) the brick' weight is x...

(ii) and to form an equality we would have to have both weights equivalent...

(iii) as this equivalence was given...

(iv) I only had to assemble the two subtraction sums.

(v) the rest is just the process of isolating x, doing the inverse operation.

From considerations involving characteristics particular to the problem's context—namely, that seesaws are balanced only when the weight on both sides are the same—she moves into a numerical-arithmetical context, and then solves the equation. This is, thus, an exemplary case of *algebraic thinking* "in action."

The **OKCALC** solutions are roughly equally divided between two solving strategies:

i) *qualitative analysis of the situation*, as we have already described at the beginning of the section on this group of problems (Tarek S, AH7, provides a clear written explanation)

273 -
189
084

Jogando-se 11 tijolos de um lado e 5 do outro, a diferença fica = a diferença do peso. Então é só dividir o peso pelo nº da diferença de tijolos

84 | 6
24 | 14

14
cada tijolo pesa quilos

Tarek S, AH7: "Throwing away 11 bricks from one side and 5 from the other, the difference becomes [equal] to the difference in weight. Then, one has only to divide the weight by the number of the difference of bricks"

(ii) *hypothetical manipulation of the context* (Bridget S, FM3). This strategy is different from (i), as it actually transforms the problem into another one. The fact that the subtraction **11-5** still had to be performed is not as relevant here as the importance — in finding a solution — of the new image generated.

213 - 189 = 24kg ÷ 6 = 4kg   1 brick = 4kg
If George throws away only 6 then he is equal
as Sam if Sam doesn't throw away any so
you subtract 189kg from 213kg and divide by 6 bricks

Bridget S, FM3

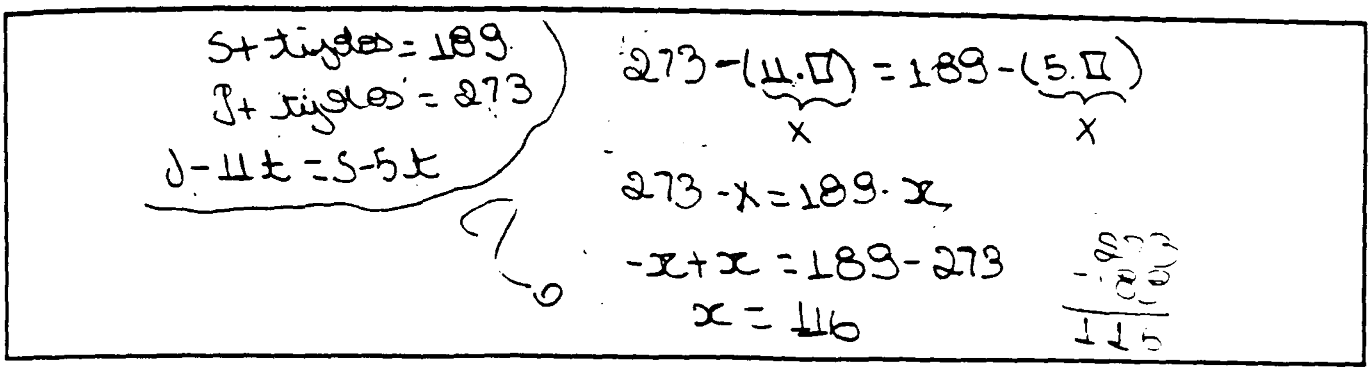
In no solution a diagram like the one we provided with the *comparison of wholes* strategy was produced, and the fact that all **OKCALC** solutions mention "weight" or "bricks" or both in association with the numbers produced strongly indicates that it was not used "in the background" either[15].

In all **WEQT** attempts we could identify mistakes deriving from a very loose use of the algebraic notation.

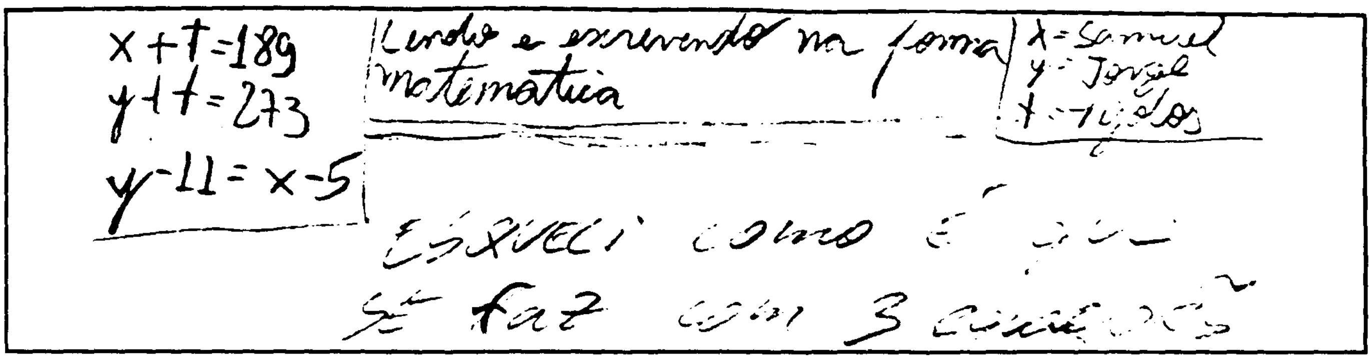
One student (Fabiola, AH7), first produced a syncopated translation of the problem (left upper corner), that apparently served as the basis for writing the (correct) equation on the first line — using a box for the weight of a brick. She then replaces the two occurrences of the box with their coefficients, by **x**. The reason is not clear at all, and this is the step that produces the critical mistake. This script is interesting for bringing together three different uses of notation: descriptive and both standard and non-standard algebraic and the urge to use **x** to make the expression on the first line into a recognisable equation is certainly related to the same aspects we discussed in relation to **OKEQT** solutions to

[15]We want to emphasise that we have already commented on page ... on the distinction between "there is in any case a whole-part structure manipulation" and "a comparison of wholes strategy is used".
We think it would not be an useful approach here, to consider that some form of abstract comparison of wholes structure was "actually" used "in the background". The crucial distinction between the *comparison of wholes* strategy as we described it, and the two strategies used by the students, is that the problem is transposed to another — in this case, more general — embodiment, one where the notion of measure is used in a different way.

SN1. Another good example of a descriptive use of literal notation is found in Marcel S's (AH8) script, who also adds: "Reading and writing in *mathematical form*" (top, our emphasis) and "I forgot how to do it with 3 *equations* [sic]" (bottom)[16,17].

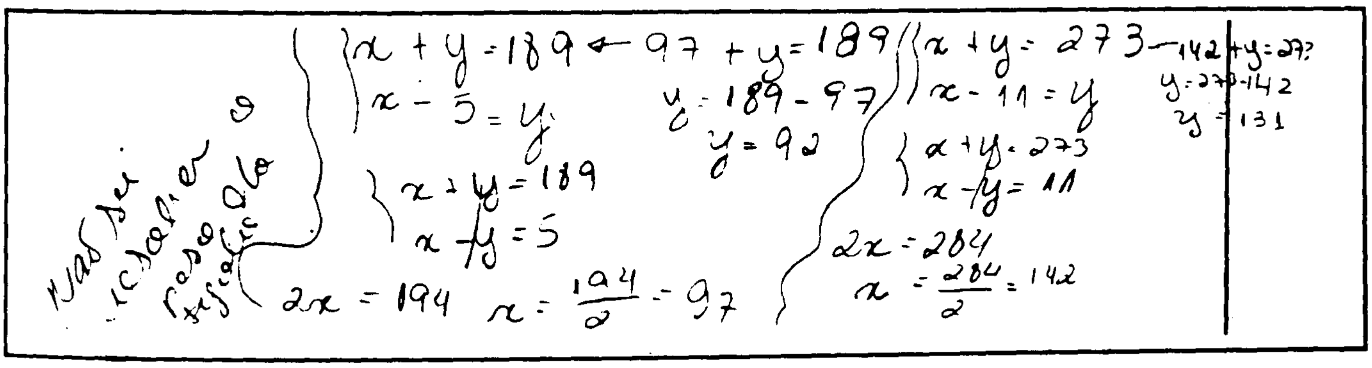


Fabiola, AH7



Marcel S, AH8

Other mistaken solutions show a combination of loose and incorrect use of notation with poor understanding of the elements and structure of the problem (Marina F, AH8).



Marina F, AH8

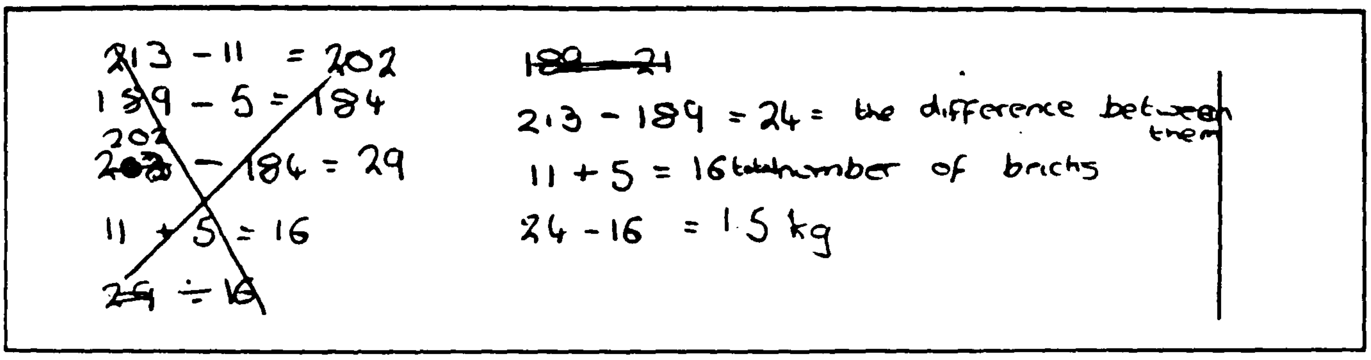
[16]In Portuguese, "tijolos" stands for "bricks".

[17]Although the expressions are clearly descriptive — for example, by the use of t ("tijolos") for both amounts — the literal notation leads the student to see them as *equations*. The usual Brazilian teaching practice puts much emphasis on "doing with letters" on the one side and "algebra" and "equations" on the other.

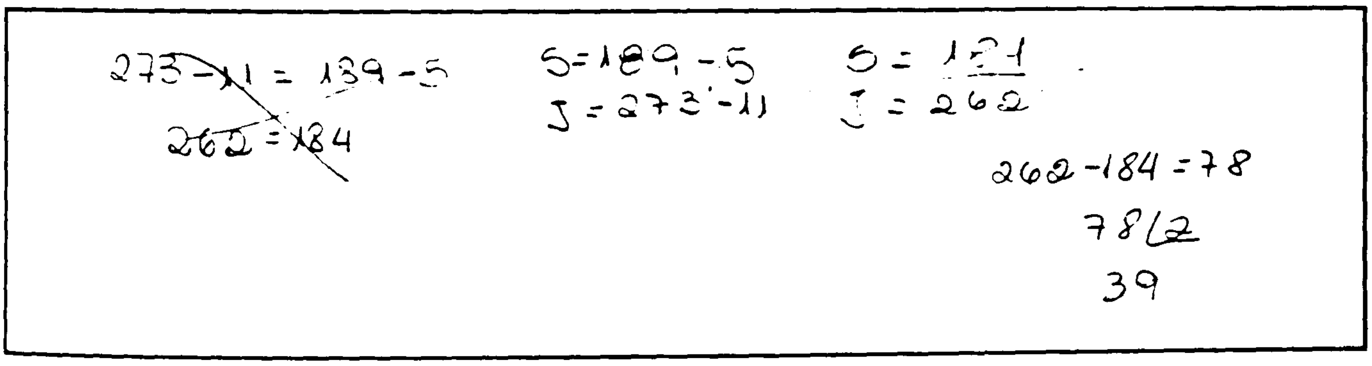
Many of the **WCALC** solutions (9 out of 16) are *contextwise homogeneous*, ie, the calculations produced always involve pairs of numbers that measure the same kind of thing (eg, weight). Those solutions were either incomplete (simply subtracted the smaller weight from the greater), considered that the difference in weight had to be shared between the **total** number of bricks involved (Clare B, FM3, a script that illustrates well *contextwise homogeneous* solutions), or considered that the **total weight** had to be shared between the total number of bricks. Of the remaining **WCALC** solutions, three used the representation

$$\mathbf{189 - 5 = 273 - 11}$$

which seems to be a mere (incomplete) syncopation of the problem's statement. In two of the cases it resulted in the focus of solution being totally diverted to the calculations involved, with no regard for the structure of the problem (Ana F, AH8). The other student did not go any further, and this suggests that she kept the awareness that it was only an incomplete syncopation.



Clare B, FM3



Ana F, AH8

The only aspect of interest on **T&E** solutions, is that none of the students actually wrote down numerical-arithmetical expressions involving the variable to be tested that would serve as a *template* for testing the "guesses". As we said before, **T&E** solutions are in a sense closer to algebraic solutions than non-algebraic solutions, both because the original problem is transformed into a numerical-arithmetical one and because the notion of *variable* is involved, even if in a rudimentary form; nevertheless, the lack of a representation of the *template* makes it difficult for the students to go beyond the trial-and-error process and to perceive the numerical-arithmetical equality as an *object* that could be directly manipulated to produce the required number[18]. That those students in our study had the *template* represented in some internal form, is out of doubt; Sanjay (FM3) actually writes down an "algebraic" version of the *template* to illustrate the condition that his guess would have to satisfy, and immediately substitutes a value to show it is the correct answer. The fact that both the *template* and the "confirmation" calculations have in fact the subtractions inverted — but to produce correct results — shows the extent to which the notation is merely descriptive.

Each brick weighs 4 kg   I found answer by ~~finding~~ estimating a number for the brick & carry out the calculation

$11a - 213 \quad 5a - 189 = 11 \times 4 - 213 \quad 5 \times 4 - 189$

$= 169 \qquad = 169$

Sanjay, FM3

<u>The Seesaw 4x problem</u>

The **OKEQT** solutions to the **E4x** problem do not add much to what we have already said about **OKEQT** solutions in the analysis of the previous two problems in this group. One aspect only is worth mention, that of the three **OKEQT** solutions coming from English groups, in only one the use of symbolism is totally standard[19]. The other two solutions use algebraic notation in much less standard ways. Sukhpal (FM3) uses an extra — descriptive — **x** to reaffirm to herself that both sides will come to a same total, while Keith W (FM3) keeps the multiplication sign with the coefficients of the unknown and mixes lines with an equation with lines with numerical calculations only; his solution does

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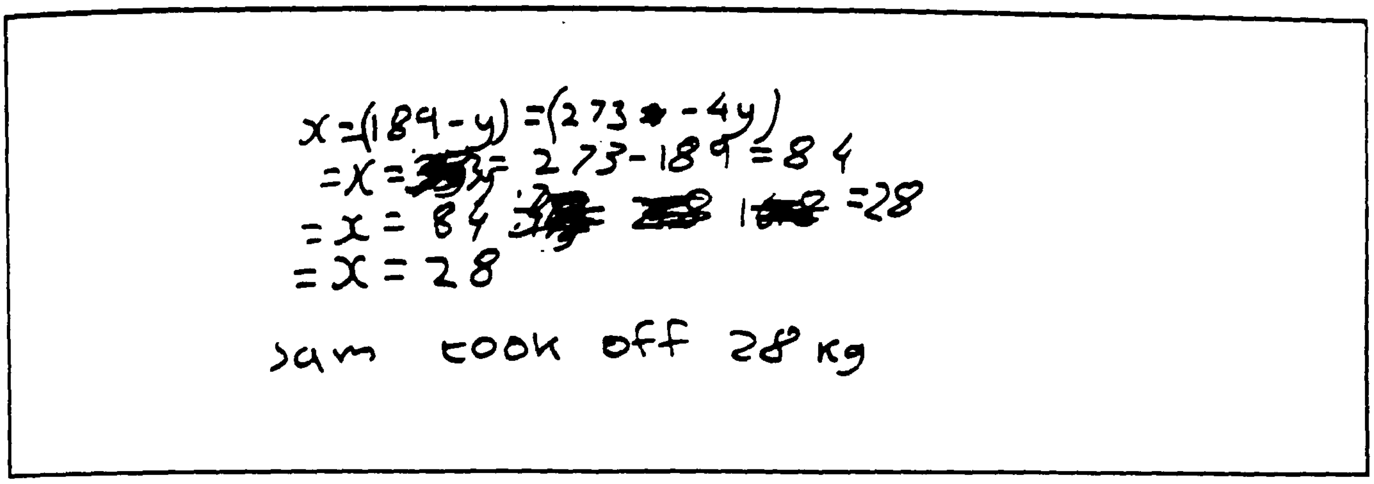
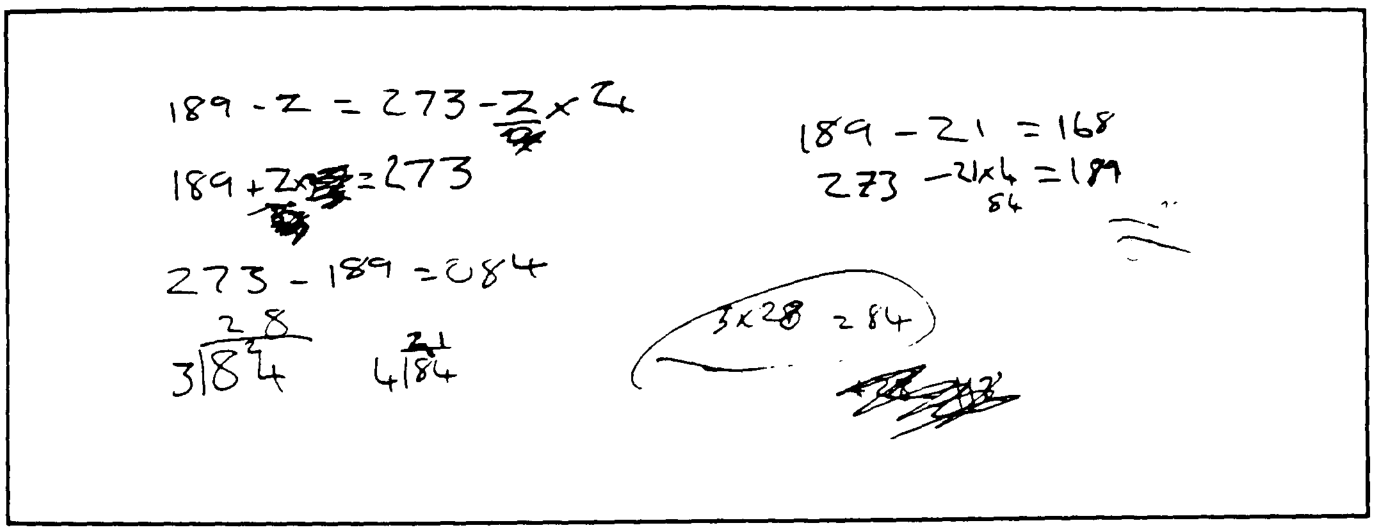
[18] In a study by C. Kieran (mentioned in Kieran, 1988), "those [pupils] who preferred substitution viewed the letter in an equation as representing a number in a balanced equality relationship; those who preferred inversing viewed the letter as having no meaning until its value was found by means of certain transposing operations."

[19] Actually, this student was a visitor from Bulgaria, where, judging by the tradition of the pedagogy of Eastern Europe countries, much attention is paid to the formal aspect of algebraic symbolism.

not reach a formal end, and one has to assume its correctness from the encircled **3** x **28** = **84** expression at the bottom.



Sukhpal FM3



Keith W, FM3

All **WEQT** solutions come from Brazilian students, and there is always an initial mistake in setting the equation. The one worth noting is Celia R's (AH7), because her main mistake (reversing the written form of the subtractions) is also seen on purely arithmetical contexts[20].

[20]In this case, **x-189** could be representing "take x from [-] **189**", a literal, non-mathematical translation of the textual structure of the problem. From this and other examples, one should be aware that the using the notion of *translation* to describe the process of transforming a contextualised problem into a numerical-arithmetical equations might be a didactic mistake, as much as it involves the false notion that "it is the same thing, only said in a different *language*". Of course, the notion that "algebra is a *language*", itself mistaken, is in the **root** of such misleading statement.

S = x - 179 = 92,4   5x = 462
J = 4x - 27...   x = 462 : 5
x = 92,4

462 : 5   92,4
12
2   179
...
462

m 5u

Celia R, AH7

Significantly, only two **OKCALC** solution (out of 77 scripts) were produced, confirming our prediction that establishing a *unit* that could be manipulated as a *part* would be a major difficulty for students not using an algebraic model. The two scripts show only the calculations, and present no verbal explanation of the process of establishing the *unit.*

WCALC solutions provide an even stronger confirmation of our prediction. 20 out of 24 **WCALC** attempts simply ignored that there was 1 part (Sam's) to be considered. In 9 of those solutions the students gave the difference between the weights as the answer (James O, FM2) and in 10 of them the **4** is used to *divide* or *share* the difference between the weights (Helen C, FM2). Four students did considered Sam's one part, but in three of those cases they also considered that the amount to be shared into **5** was the **total weight**, and not the difference (Fabio P, AH7). It seems that because they were thinking of **total weight** the **total amount put away** had to be considered, and this led them to the **5** divisor.

George threw away 84 kg
I did it that way because it was the first one that I thought of.

James O, FM2

George threw away 84 kg and sam threw 21 kg away
I subtracted George's weight with Sam's weight and I got 84
so I divided 84 by 4 and Got 21
Thus I did it this way because of the way the problem was set up

Helen C, FM2

2,7,3
1,8,9
0,8,4

84 | 5
34 16,8
40 4
67,2

Jorge = 67,2
Samuel = 16,8 k
Com a diferença entre os dois peguei quantas vezes eles tiraram
e dividi pela diferença e resulta o quanto Samuel tirou e do Jorge multipliquei × 4.

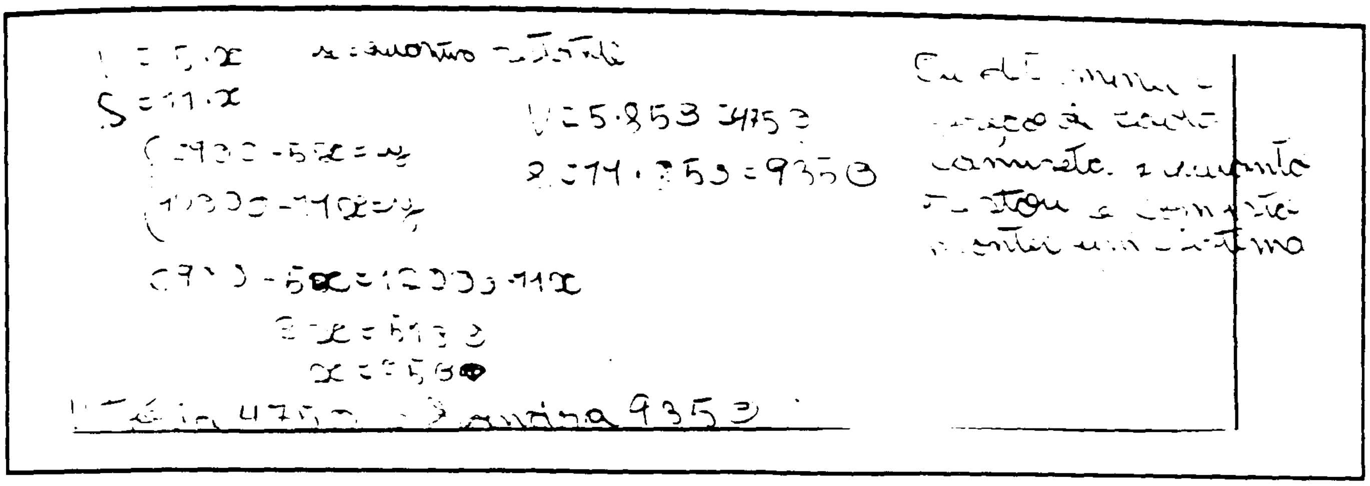
Fabio P,AH7

"With the difference between the two, I took how many times they took away and divided by the difference [sic] and the result [is] how much Samuel took away and for Jorge multiply by four."

It is clear that the **E4x** statement did not easily provide *parts which can be manipulated* for the weights wasted by Sam and by George, and the fact that this caused major difficulties for those students strongly suggests that the models they were using depended heavily on that kind of *object*.

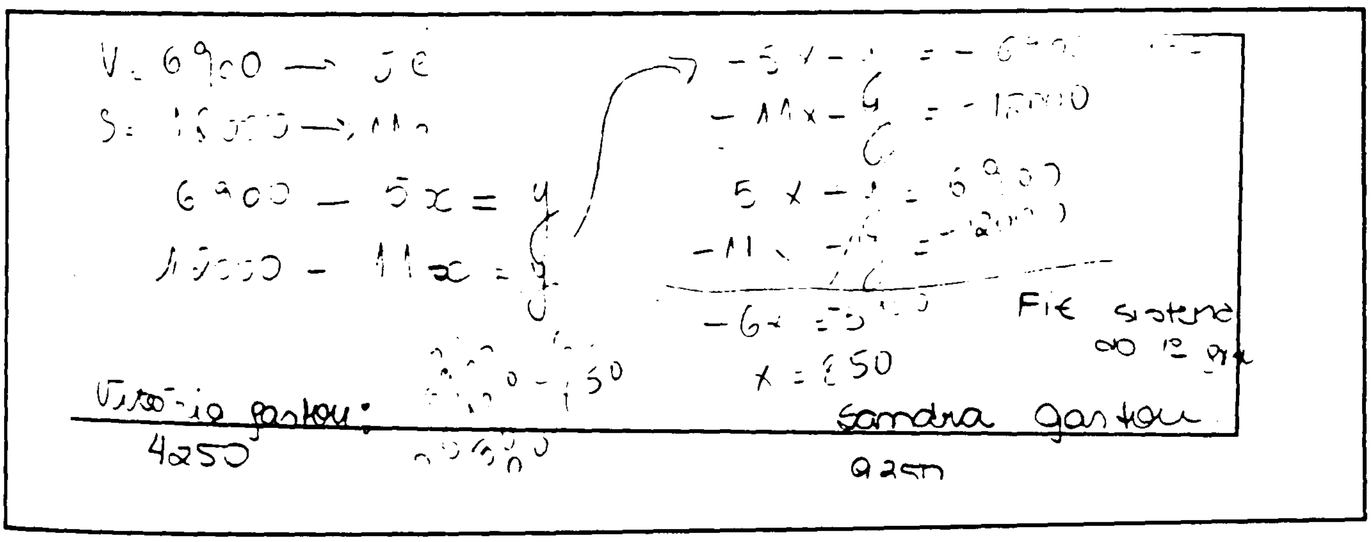
The Sale 11-5 problem

One characteristic aspect of the algebraic method appears in three of the **OKEQT** solutions to this problem, the introduction of an auxiliary unknown, as in Mateus C's (AH8) solution. The **y** he used to represent the amount of money left is not an essential *element* of the problem, once it can be totally avoided by the immediate use of the equality. Mateus's solution does not deal directly with this auxiliary unknown; rather, it plays a more *descriptive* role, although being clearly seen as a number (by belonging to the numerical-arithmetical context of the expressions). Whether he saw the two expressions on the left hand side of the two equalities as representing "calculations" or as true "complex" algebraic *objects*, one cannot infer from the script alone, but the notation certainly provides an environment where the latter is made easier.

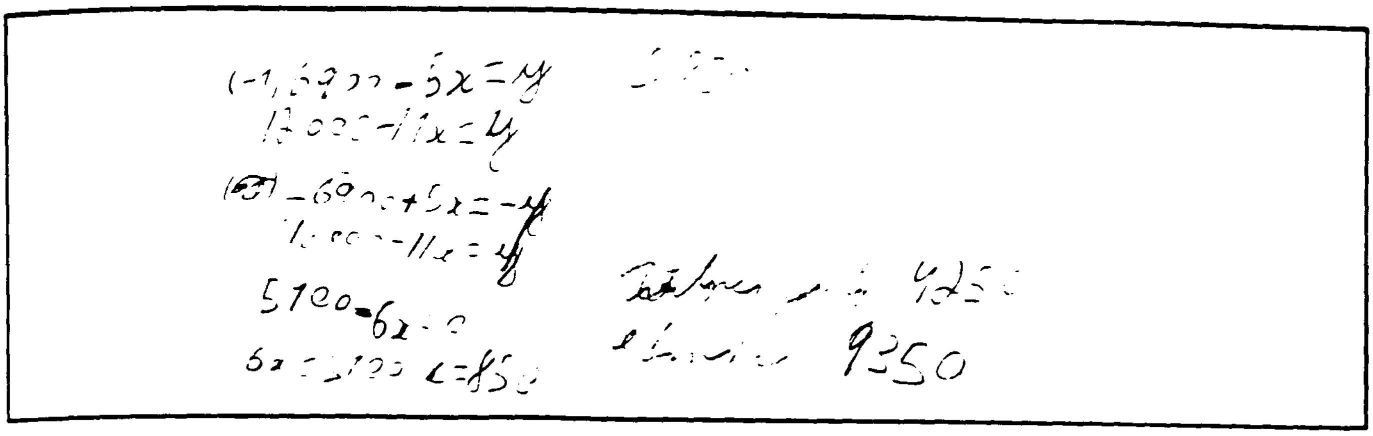


Mateus C,AH8

On the other two solutions that employed an auxiliary unknown (again a **y**), the algebraic processing included its direct manipulation (Tathy G, AH8; Silvio S, AH8), once the two equations were primarily seen as a set of equations in two unknowns; Tathy says: "I did a *system of the 1st degree* [*=linear*]". Although not being the simplest solution — from the technical point-of-view — their approach shows exactly the *internalism* that is characteristic of *algebraic thinking*: the quantity represented by **y** was not required in the problem to be evaluated nor necessary to the continuation of the solution, and that those students were aware of that can be seen on the fact that they did not substitute the **x** back to determine **y**". Their solutions are quite characteristic examples of *thinking algebraically*.



Tathy G, AH8



Silvio S, AH8

One **WEQT** solution is of interest. Sergio P (AH7) writes down an equation that does not model the problem correctly, clearly for not understanding the problem's statement; he never bothered with the fact that **x** representing the price of a T-shirt, it would not be possible to begin with less money, to "add" less T-shirts and to end up with the same amount of money as the other person that had begun with more money and "added" more T-shirts. Then — and this makes the previous "disregard for the context" even more striking — he wrongly manipulates the equations (between the third and fourth lines) to produce a value for **x** that is positive, once *he knows it represents a price and thus has to be a positive number.*

$$6900 + 5x = 12000 + 11x$$
$$+5x - 11x = 12000 - 6900$$
$$-6x = +5100$$
$$(-1)\ \cancel{-6x} = 5100 / 6 = 850$$
$$x = 850$$

Sergio P, AH7

On the previous subsection (**Seesaw 4x problems**), we pointed out the importance of having a representation of the **T&E** *templates* in order to foster the process of transforming them into *objects*. Kelly L's (FM3) script shows, however, that there is a significant difference between the two types of representation, once the *equation* form might not convey the order of operations— as it indeed does not in the type of problem we are examining. Obviously, this problem can be overcome if the student has a good grasp of the process of evaluating numerical expressions.

Had 85 pounds - Bought 11 Lp's — (11 x ? = ?)   85 - ? = price
67 pounds - 5 Lp's
5 x ? = ? 67 - ? = price   5 x 1 = 5   67 - 5 = 62
11 x 1 = 11   85 - 11 = 74 } not the same
11 x 3 = 33   85 - 33 = 52 } the same
5 x 3 = 15   67 - 15 = 52
Came out with £52 at the
Price of each LP is £3.00 each

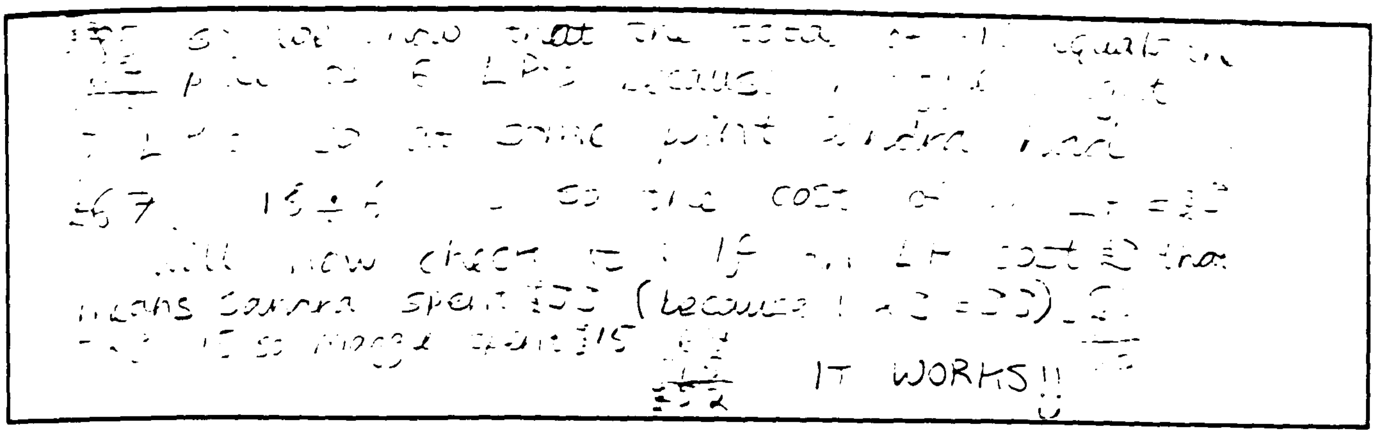
Kelly L, FM3

Of all **OKCALC** solutions to this problem, only one does not correspond to the scheme "the extra money Sandra had corresponds to the extra Lp's she bought, etc." (David W, FM3). Esther F (FM3) instead, reasoned in a manner similar to the "if George throws away 6 bricks and Sam does no throw away any..." described on the **Seesaw 11-5 problem** subsection. That only one solution employed such reasoning with **A11-5** problems, while a significant number of them appeared with **E11-5** problems, suggests that "objects" of the context of the problem become in fact *objects* in the model used to solve the problems, as the "balancing process" property is immediately associated with the Seesaw context but not with the Sale situation[21].

The answer is £3.00. First, I found the difference between the amounts that the pupils took. Then, I divided that by 6 (the difference between the amount of LP's they bought). That gave me the answer.

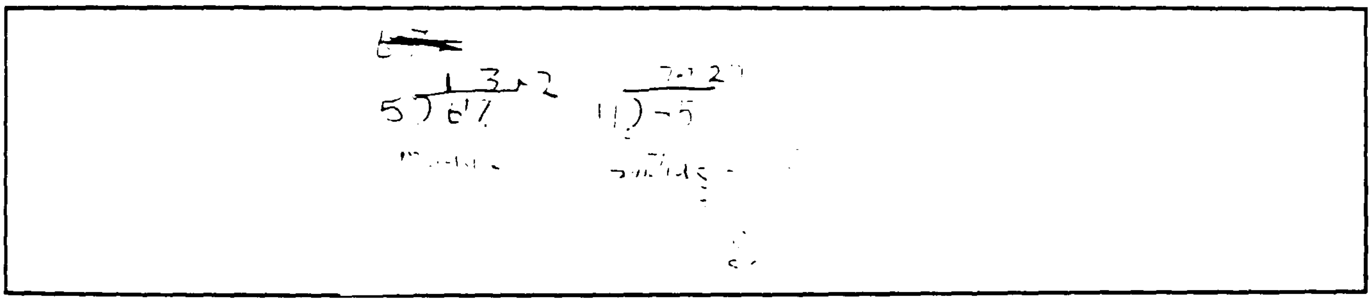
David W, FM3

[21]This "balancing process" property consists in the possibility of a gradual qualitative change in the balance state of the situation: the two sides of the seesaw being more or less near a balanced state or the difference between the money the two friends have being greater or smaller.



Esther F, FM3

Seven of the **WCALC** solutions take us in the same direction. In those solutions (eg, Shelley S, FM2) the students treat the problem as if both friends had spent all their money, and try to divide Sandra's money by the number of Lp's she bought and the same for Maggie to see if both divisions come to the same result. This type of solution did not appear on any **Seesaw 11-5** problems, most probably because it is quite obvious that the two friends will still be sitting on the seesaw when it is balanced, and this means that not all the weight will have been thrown away.



Shelley S, FM2

Of the remaining **WCALC** solutions, in four of them the **total money** is divided by the **total number of Lp's** — a strategy similar to dividing each friends' money by the number of Lp's she bought, but avoiding the possibility of having different priced Lp's for each friend — and the rest are attempts to produce a sensible answer from the numbers involved, some of them not very clear at all.

The Sale 4x problem

The most remarkable fact in relation to the solutions to this problem is that there is only one **OKCALC** solution (Keith W, FM3) out of a total of 82 students attempting it.

Keith's solution is unique in that he divided by 3 not because he modelled the problem with "1 lot, 4 lots" and concluded that "there is 3 lots more to Sandra", as one would expect, but instead he saw that Sandra would have to spend the difference between them (so they would be equal) and also some more money to allow for Maggie's expenditure; this means that the difference consists of three parts that will make four together with the extra part, that Maggie also gets.

[illegible] = 18. Sandra spent £24 Maggie spent £6.
I found the difference and ÷ it by 3 because
Sandra would have to spend that and
£6 more (which is ⅓ of the difference) then
Maggie would spend £6 which takes
them to the same amount.

Keith W, FM3

This finding shows that it was very difficult, if not impossible for those students to establish the necessary *unit* that would allow them to use the "1part, 4 parts" strategy; the same situation was found with **Seesaw 4x** problems, indicating the extent to which non-algebraic solutions depended on the existence of *parts and wholes which can be manipulated.*

The mistakes found on **WCALC** solutions to this problem represent mainly two aspects:

(i) not considering at all the relationship between what each of the two friends spent, thus focusing only on the difference between what they initially had (Joanna J, FM2),

67 - 85 = 18 Sandra spent £18
3×6 = 18 , Maggie spent £0
They bouth came out with £67

Joanna J, FM2

(ii) ignoring the fact that Maggie also spent one "lot" and dealing only with the 4 parts of Sandra (William C, AH7).

12000
- 6900
5100

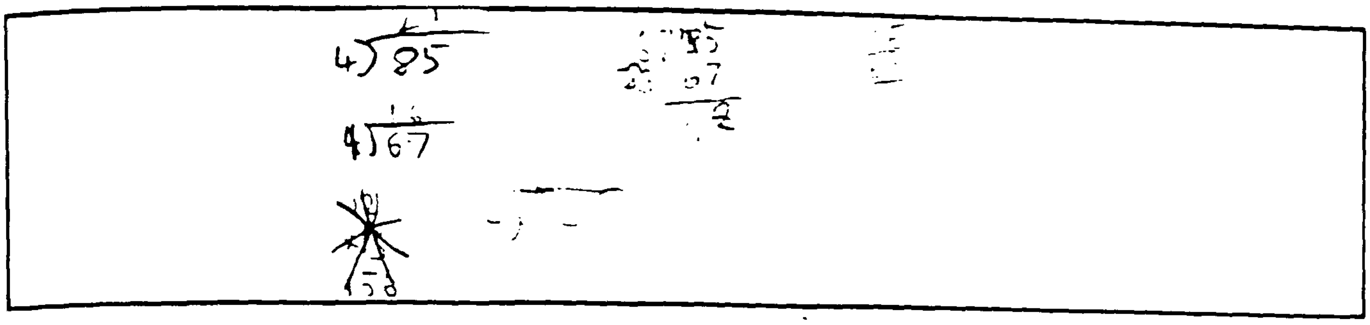
5100 |4
1275

Sandra gastou 5100 a mais que Vitoria
Sandra 6,375
Vitoria 1,275

Tentativa

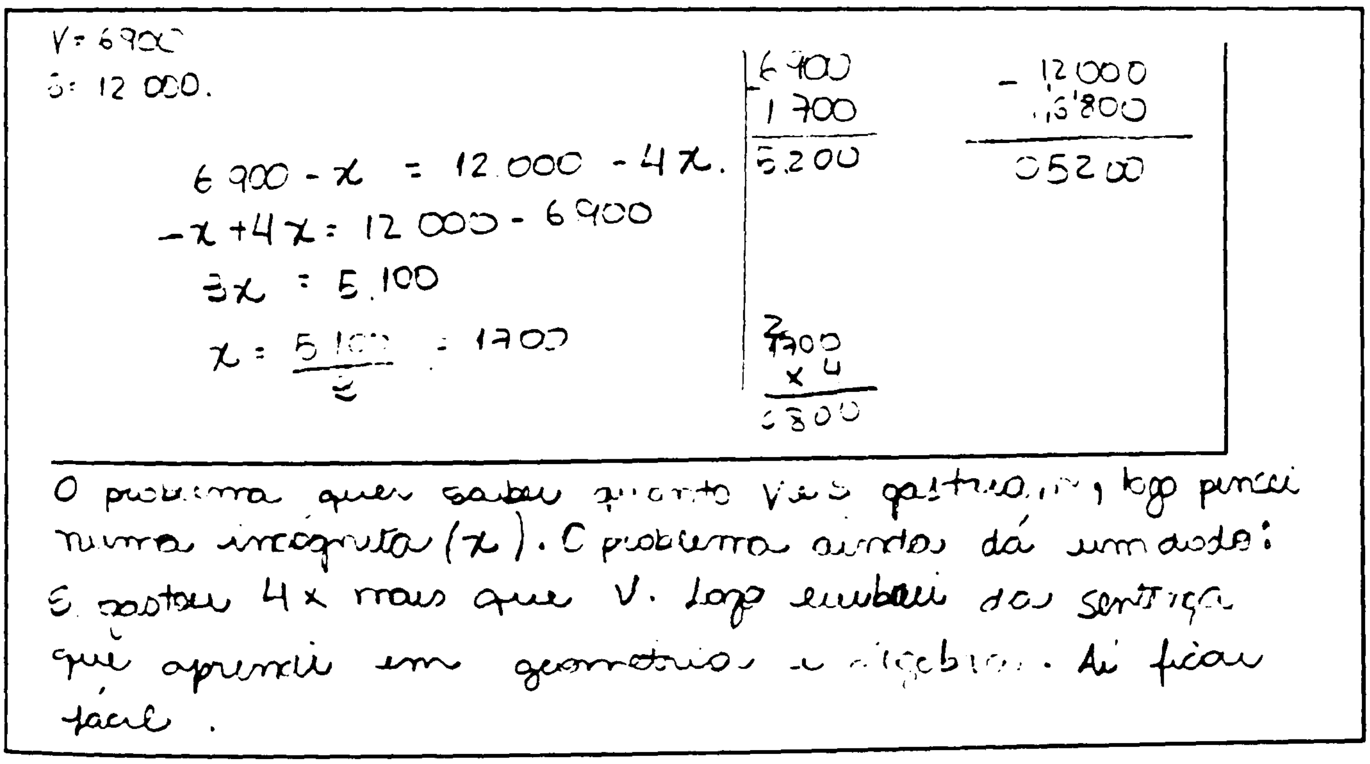
William C, AH7
"Sandra spent 5100 more than Vitoria" and at the bottom line, "Attempt"
(meaning probably that he was not sure of his solution)

As it had happened with **Sale 11-5** problems, there were a number of attempts to divide the **total money** by the **total number of parts** (Brian H, FM3), this being again a consequence of the possibility of the friends having spent all their money; only this time those attempts use only divisions by **4**, for the reasons explained above. In only two cases a division of one of the friends' money by **5** was used, in both cases taking the bigger initial amount (Sandra's). It might be that those students interpreted the "4 times as much" statement as meaning "4 parts more than" and this produced the need to consider one extra part.



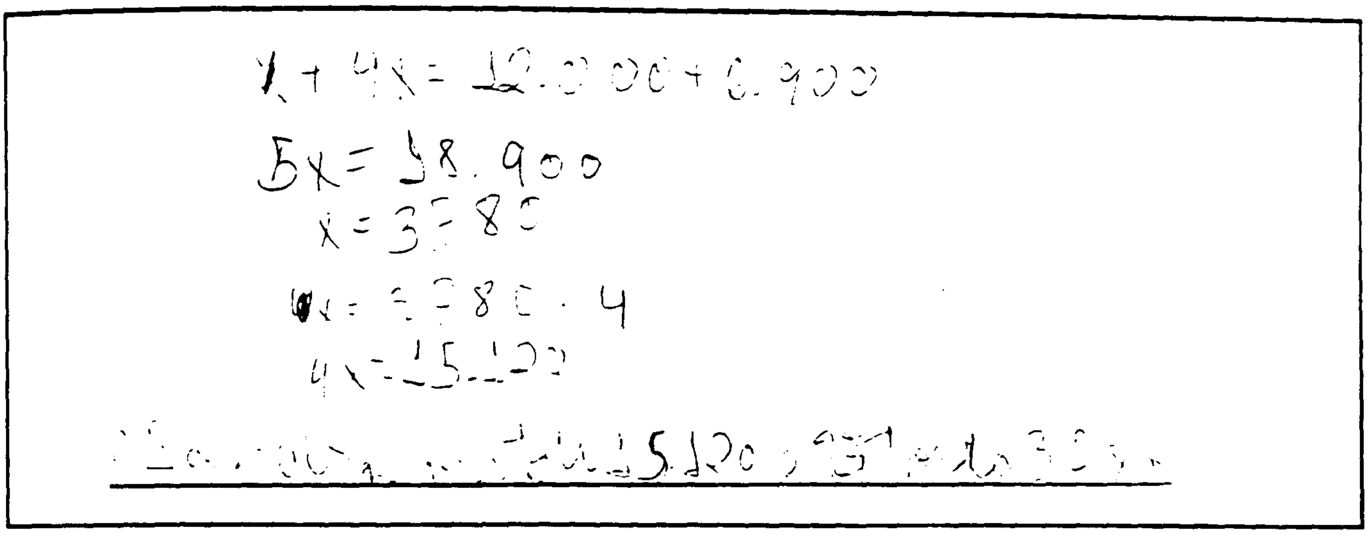
Brian H, FM3

One of the **OKEQT** scripts (Fabiana M,AH7) provides an important insight on how the ability to solve "algebraic word problems" in general can benefit from the ability to think algebraically, and we do not mean, of course, the possibility of developing "automatic" solution procedures. In Fabiana's script it is immediately clear that she thought first of all of the existence of an unknown quantity — most probably a habit developed through the use of equations; we have already seen that in a problem like the [4x] problems this comes to be an essential step to reach a correct solution. Although the availability of a special notation certainly promotes a better grasp of that notion (Fabiana: "…I thought of an unknown (**x**)…"), we must keep in mind that it is the *analytical* character of the algebraic method that produces the need to make the unknown into an *object.*

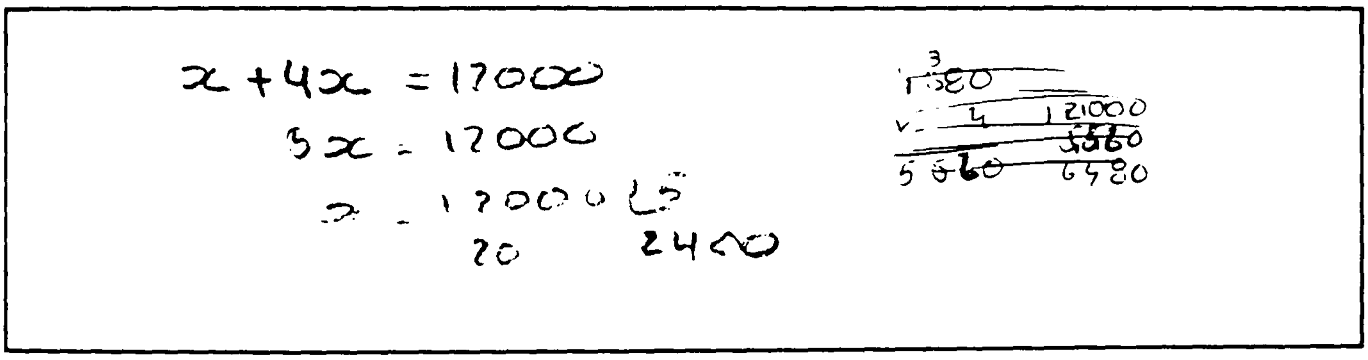


Fabiana M, AH7: "The problem wants to know how much V and S spent, thus I thought of an unknown (x). The problem also gives an information: S spent 4 x more than V. Then I remembered the sentence that I learned in geometry and algebra. It then became easy."

Four of the **WEQT** solutions reproduce in the wrong setting of the equations, some of the mistakes we observed with **WCALC** solutions. Fernando C (AH8), for example, equalises the **total number of parts** to the **total money**, and correctly solves the equation and Sidnei A (AH7) attributes **5** parts to Sandra (the "1 and 4" mistake we discussed 3 paragraphs above).



Fernando C, AH8

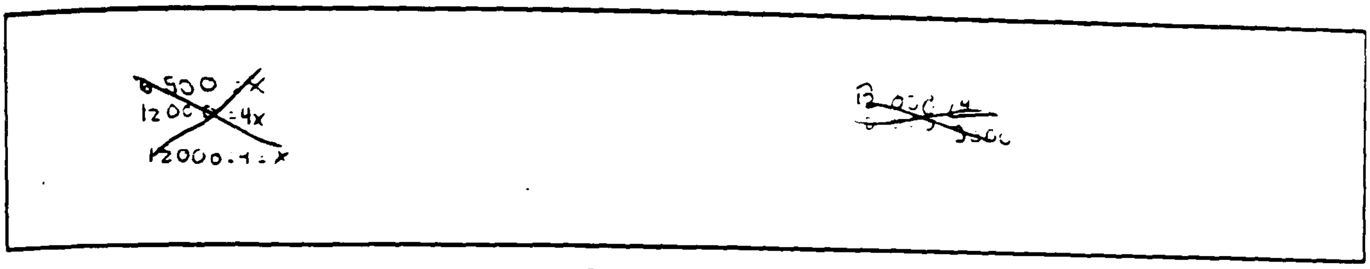


Sidnei A, AH7

One has to be amazed by Luis N's (AH7) attempt, as he writes <u>on the first line</u>

$$6500 = x$$

without immediately concluding that the solution to had been found. We think that he had in fact structured the problem by attributing one part to Vitoria's total money and 4 parts to Sandra's total money, as some students did with the **Sale 11-5** problem, and that the algebraic notation was not being seen by him—at that point—as representing true equations to be solved. He then seems to move away from this initial interpretation and "solves" the second equation, and that is when he realizes that the two values for x do not agree, and something must be wrong.



Luis N, AH7

This *shift of interpretation,* so dramatically illustrated by this script, is certainly at the core of using algebra to solve contextualised problems: the equation is set by transforming series of calculations — *analogically* associated with the problem's "story" or context — into arithmetical expressions[22], and then those expressions are linked by equalities —again, *analogically* associated with the context. It is only then that it is treated *internally,* as an equation, and this shift, by marking the transition to a different *Semantical Field* marks also the passage to a distinct *mode of thinking.*

SUMMARY OF FINDINGS AND CONCLUSION

An aspect of the non-algebraic models used by the students emerged clearly from the analysis of this group of scripts: their *synthetical* nature, with the process of solution always proceeding from the known values to the required unknown one through a series of *evaluations.* The few exceptions would be those solutions to **E11-5** where there is a hypothetical manipulation of the situation that leads to the "only 6 bricks need to be removed from George's side and none from Sam's side" structure.

Another conclusion to be drawn from the analysis of this group of answers is that many students did not see numerical-arithmetical expressions and equalities as *objects* that could be manipulated on themselves to produce further useful information in the process of solving the problem. This aspect was particularly crucial in relation to the **SN1** problem, that is, as we saw, very difficult to be modelled into a *geometrical* or *comparison of wholes* model, and thus the inability to see numerical-arithmetical expressions as informative led to very low facility levels among the English students. That those same students did significantly better on the contextualised problems, shows that *the non-algebraic methods used by them is based to a great extent in the perception of **parts** which can be manipulated, and that the choice of arithmetical operations to be performed is almost completely dependent on the manipulation of non-numerical objects*; the numbers in the

[22]**At this stage** those expressions are in fact *arithmetical,* once the unknown numbers are treated as if they were known, as we have already seen, and they are seen as calculations to be carried out.

problems were rather seen as *measures*. The greater difficulty with [4x] problems, in comparison with [11-5] problems also provides a clear support to this conclusion. To put it in terms of our framework, ***those students that failed to solve the SN1 problem but could handle the contextualised problems were unable to operate within the Semantical Field of numbers and arithmetical operations***. Moreover, it was difficult for many students — probably most of those not using an algebraic approach — to move away from the *Semantical Field* where the problems were originally set, eg, to model a contextualised problem with a *comparison of wholes* model. They kept strongly attached to the original "icons" provided with the problems' statements and consequently limited their perception of the problems' structures to what is more ordinarily associated with those contexts.

Moreover, the non-algebraic solutions, correct or not, were characterised by their *contextwise homogeneity* in relation to addition and subtraction of measures. This is an important aspect for two reasons. First, because it points out to a possible important source of information used by those students on what can or has to be done to solve a given problem. Second, because if this is indeed a deeply rooted informative pointer in a person's problem solving schemes, it would certainly be difficult to operate on a *Numerical Semantical Field*, where such pointers are truly meaningless. As a consequence, it might be that teaching "intuitive", "contextualised" or "localised" strategies for solving algebra word problems builds in fact a huge obstacle to be overcome when the "algebra time" arrives, and this suggests that an early start with the algebraic approach might be of great help to reduce the difficulties with the learning of algebra, not because of the "extra time to practice", but because of the earlier development of a degree of independence from such pointers[23].

Still in relation to the influence of schooling in the development of an algebraic mode of thinking, we found it very significant that the "default" approach for Brazilian 7th graders was non-algebraic — although they were able to use an algebraic one — while for the 8th graders the "default" approach was an algebraic one ; that the same was not found in relation to the corresponding English groups, and that a considerable similarity of ages existed, strongly suggests that the development of *algebraic thinking* is a process

[23]Obviously, those pointers are not useless in all situations, and they may even be of great help when one is trying to make sense of the relationships involved in a more complex task or problem. What we imply here, is that **both** "homogeneity bound" and "not-homogeneity bound" strategies should be made available and equally developed. Once much of everyday activity is indeed "homogeneity bound", we suggest that schooling could avoid the development of a too strong primacy — eventually a pernicious one — by offering an early alternative way of thinking.

much more akin to ***cultural processes*** than to age-related stages of intellectual development.

The analysis of the scripts for this group of questions threw much light on different uses of algebraic notation and on possible consequences of resorting to the notion that setting up an equation to model a problem is a *translation* process. Students used letters both in a truly algebraic way — to denote numbers — and in syncopated forms of the verbal statement. The latter use caused two types of difficulty:

(i) as letters were used as an abbreviation of the verbal text, and there was a context to support this usage, different quantities — different at least in principle — ended up being represented by the same letter; also, this usage sometimes introduced new "unknowns" (as, for example the individual weights of each friend on the seesaw);

(ii) as one "describes" a sequence of things happening, no care has to be taken to match the order of the verbal syncopation with the conventions of numerical-arithmetical expressions — which are not necessarily useful if one is simply trying to make the statement more comprehensible by breaking and syncopating it, and both conventions are very distinct in most cases. Also, the objects involved are not numbers, but objects of the context (as we said, numbers are seen as *measures* and *operators*), and one should reasonably expect the subject to manipulate the letters — in fact *icons* of those objects — according to the properties he or she sees as relating to the objects those *icons* refer to; there is no *shift of referential*, no passage to another *Semantical Field*.

It seems, on the other hand, that the use of **standard** algebraic notation—instead of more iconic forms like boxes and question marks—might be of use to promote a more immediate ***transformation*** of a contextualised problem into an algebraic one, for example through the association between "**x**" and "**the unknown**", one immediate advantage being, as we saw with the [4x] problems, to make easier to overcome the difficulty of having to establish *units* that do not correspond to objects of the context.

Another important aspect to emerge from the algebraic solutions offered, is that we could distinguish levels of *sophistication* in the processing of the algebraic models used to model the problems. The introduction of auxiliary unknowns, the use or not of "standard forms" of equations in the process of solution, a more or less restricted use of negative numbers, "one step-one line" solutions and more flexible ones, and above all, some solutions that treated the equation as a whole (eg, multiplying a whole equation by -1)[24], instead of the more limited perception of thinking only in terms of "chunks" (eg, breaking the equation down into 273, -11x, =, 181, -5x, and seeing those as the blocks to be dealt with). In all cases, however, the same basic characteristics that our theoretical characterisation of algebraic thinking established can be identified: *internalism*, *arithmeticism*, and *analiticity*.

4.4 Carpenter-Chocolate-Sets of Equations Problems

The Problems

I am thinking of two secret numbers.
I will only tell you that...

(first no.) + (second no.) = 185
and
(first no.) - (second no.) = 47

Now, which are the secret numbers?
(Explain how you solved the problem out and why you did it that way)

Sets 1-1

I am thinking of two secret numbers.
I will only tell you that...

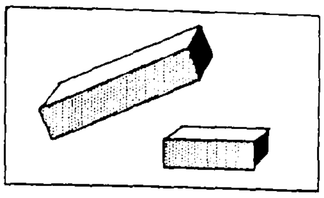
(first no.) + (3 x second no.) = 185
and
(first no.) - (3 x second no.) = 47

Now, which are the secret numbers?
(Explain how you solved the problem and why you did it that way)

Sets 1-3

[24]Of course this corresponds *formally* to multiplying each side of the equation by -1, but we are dealing here with the perception of algebraic objects and their properties, and not with a strict formal justification.

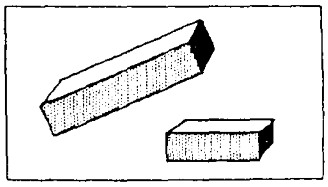
At the right you have a sketch of wooden blocks.
A long block and a short block measure 162 cm altogether.
A short blocks measures 28 cm less than a long block.



What is the lenght of each individual block?
(Explain how you solved the problem and why you did it that way)

Carp 1-1

At the right you have a sketch of wooden blocks.
A long block put together with two of the short blocks measure 162 cm altogether.
If two short blocks are put together, they still measure 28 cm less than a long block.



What is the lenght of each individual block?
(Explain how you solved the problem and why you did it that way)

Carp1-2

At Celia's shop you can buy boxes of chocolate bars or you can buy spare bars as well.

A box and three spare bars cost £8.85.
A box with three bars missing cost £5.31

What is the price of a box of chocolate bars in Celia's shop? What is the price of a single bar?
(Explain how you solved the problem and why you dit it that way)

Choc

GENERAL DESCRIPTION

This group of problems was developed with the objective of:

(i) examining students' strategies to solve "secret number" problems involving two secret numbers and to compare those strategies with the ones used with the corresponding contextualised problems; each of the secret number problems in this group corresponds to one or two contextualised problems and the relationship between the models employed on a secret number problem and its correspondent contextualised problem(s) will be closely examined. Both secret number problems were set in a *normal form* of sets of simultaneous equations, given in a syncopated, rather than literal, notation; the use of symbols for arithmetical operations and for equality — as opposed to the traditional verbal formulation[25] — was intended to keep the problem as close as possible to the *Numerical Semantic Field* and to allow us to examine to what extent those *numerical-arithmetical* statements made sense to the students.

(ii) examining the effects of an increase in the structural complexity of a problem in the strategies used;

As we will show, it was easier with this group of problems than with the previous ones to distinguish *algebraic* and *non-algebraic thinking* even in the context of a solution using algebraic symbolism to describe and control a non-algebraic process, once the students were more generous with the explanations provided with their answers , and those explanations were in general of a much better quality, this being particularly true for the contextualised problems.

DISCUSSION OF POSSIBLE SOLUTIONS

Chocolate Box problem (**Choc**)

This problem seems to inevitably involve two unknowns.

An algebraic model is

$$\begin{cases} x + 3y = 8.85 \\ x - 3y = 5.51 \end{cases}$$

where **x** is the price of a box of chocolate bars and **y** is the price of a single bar. The most likely solution to this set of equations is to add the two equations to produce

$$2x = 14.36$$

and to solve it from there.

[25] Eg, "I am thinking of two numbers. If I add the two of them the result is ...," and so on.

Two non-algebraic models seem possible here:

(i) *"The first box has 6 bars more than the second, so, if I work out the difference between the two values* [8.85 and 5.51] *I will have the price of 6 bars"*, etc.

(ii) *"If I put together the two boxes* [the one with extra bars and the one with bars missing] *the three extra bars on the first box can be transferred to the second box, making two complete boxes. So, if I add the two prices I will have the price of two boxes"*, etc.

It is central that with the non-algebraic models, the choice of operations to perform is totally subordinated to the manipulation of the image of the boxes and the bars. Also, on those models one thinks of **two boxes and three bars** and not of **the price of a box and the price of a bar used in different places**. Moreover, the divisions that would follow (by 6 or by 2, respectively) would certainly be a way of *evaluating* the sharing of an amount of money into the corresponding number of parts.

Another possible *analogical reasoning* would be,

(iii) *"If one box with 3 bars missing cost 5.51, then a box costs 5.51 plus 3 bars"* and proceed to *"then, 5.51 plus 3 bars with the extra 3 bars cost 8.85"*, etc.. This reasoning could both produce a direct solution, through the manipulation of the *whole-part* relationship, or lead to the single equation

$$(5.51 + 3y) + 3y = 8.85$$

This approach is substantially different from both (i) and (ii), as the meaning of the "plus" in "5.51 *plus* 3 bars" can only be understood in the context of *prices* ("3 bars" ≈ "the price of three bars", while in (i) and (ii) "bars" stand for bars, as we saw. If one writes

1 box - 3 bars = 5.51

the "=" sign reads "cost" and means that the object on the left is *labelled with the price 5.51*. On the other hand, if one writes

1 box = 5.51 + 3 bars

the equality has to be interpreted as meaning an equality between prices, if not pure numbers. Reading the "=" sign as "costs" produces a somewhat puzzling phrase, very similar to the one in the well-known riddle "a fish's weight is 10 pounds plus half a fish...".

If the *shift* in the interpretation of the equal sign in the two *written* sentences can be made bearable by the ambiguous use of the equal sign, it corresponds in fact to a change in the type of relationship that is being considered, and it seems to offer a substantial obstacle to be overcome within the *Semantical Field* of chocolate boxes and bars in which the problem is set, and one has to remember that it is within this *Semantical Field* that the manipulation producing "1 box = 5.51 + 3 bars" from "a box with 3 bars missing costs 5.51" would have to happen, ie, the manipulation would have to occur *before* the sentence being written.

The substitution of the resulting sentence into the first line of the problem's statement, to produce "(5.51 + 3 bars) + 3 bars = 8.85" would also be problematic, as the substitution of the "actual" box by its price would require a strong shift in the understanding of the original statement (with the added difficulty that the price replacing the object is stated in terms of another object's price).

The importance of analysing possibility (iii) in some detail is that within the *Semantic Field of numbers and arithmetical operations* the manipulation

$$\begin{cases} x + 3y = 8.85 \\ x - 3y = 5.51 \end{cases}$$

$$x - 3y = 5.51 \Rightarrow x = 5.51 + 3y$$

$$\therefore \quad (5.51 + 3y) + 3y = 8.85$$

presents none of the difficulties discussed above, which is a clear indication that (a) within the *Semantic Field* of the chocolate boxes and bars the *objects* one deals with are completely distinct from those one deals with within the *Semantic Field of numbers and arithmetical operations* — and thus the types of relationship involved and the requirements on a notational system — and (b) *arithmetical internalism*, a most central characteristic of thinking algebraically, allows one to operate continuously without having to consider shifts such as those we have just discussed. We have here a very fine example of the fact that a compact notation is possible if one is thinking algebraically, exactly because of the *homogeneity* produced by the *arithmetical internalism.*

Solutions (i) and (ii) above, resemble very much the strategy of adding or subtracting the two equations in a set of equations. Nevertheless there is a fundamental difference between the two processes. In solution (i) the full boxes are thoroughly ignored, and the conclusion that the first box has six bars more than the second box comes from a "counting up"[26] strategy, rather than from "subtracting" the second line from the first, once it is obvious that the "taking away" meaning of the subtraction would make no sense in this situation because of the need to "take away what is already missing". In solution (ii), what is done in fact is a *transfer* of the three extra bars in the first box to fill up the second box;

[26]Evaluated, of course, with an addition. The full box works, in fact, as a form of "zero level."

the extra bars in the first box are never operated with the missing bars in the second box. Finally, in the additive solution of the set of equations **-3y** is ***numerically*** added to 3y and *the terms cancel each other out because the result is zero.* Similarly for subtracting the second line from the first. The point to be made here is that although solution (ii) "written" using algebraic notation is actually indistinguishable from a true algebraic additive solution of a set of equations modelling the problem, the two solutions are essentially distinct, each one being the result of operating within a different *Semantic Field.*

Carpenter 1-1 problem (**Carp1-1**)

Two algebraic models seem more likely to be used to model this problem. One is the set of equations (L stands for the length of the longer block, S for the length of the shorter block)

$$\begin{cases} \mathbf{L + S = 162} \\ \mathbf{L = S + 28} \end{cases}$$

and the other is the single equation

$$\mathbf{(S + 28) + S = 162}$$

It is obvious that by a substitution, one will arrive from the set of equations at the same single equation, but by separating the two models we want to emphasise that the substitution can be made within the *Semantic Field of numbers and arithmetical operations* (from the set of equations to the single equation) or within the *Semantic Field of the Wooden Blocks* (the longer block being *represented* as a short block with an extra bit added to it). It is clear that in the latter case the "+" sign means "conjoining" and not the arithmetical operation.

From the results obtained on the exploratory study we expected non-algebraic solutions to this problem to be of one of two types[27] (figure CCS 1, for (i), a similar diagram for (ii)):

(i) "*if I cut 28 out of the longer block I will have 2 equal [short] blocks, so if I take 28 from the total, I will be left with the length of two short blocks...,*" etc.

(ii) "*I cut the total in two, take away 14 from one half and add it to the other half, thus making the difference 28.*"

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[27]The original problem in the Exploratory Investigation had a slightly different form from this one, but we still expected the solutions to follow the same pattern.

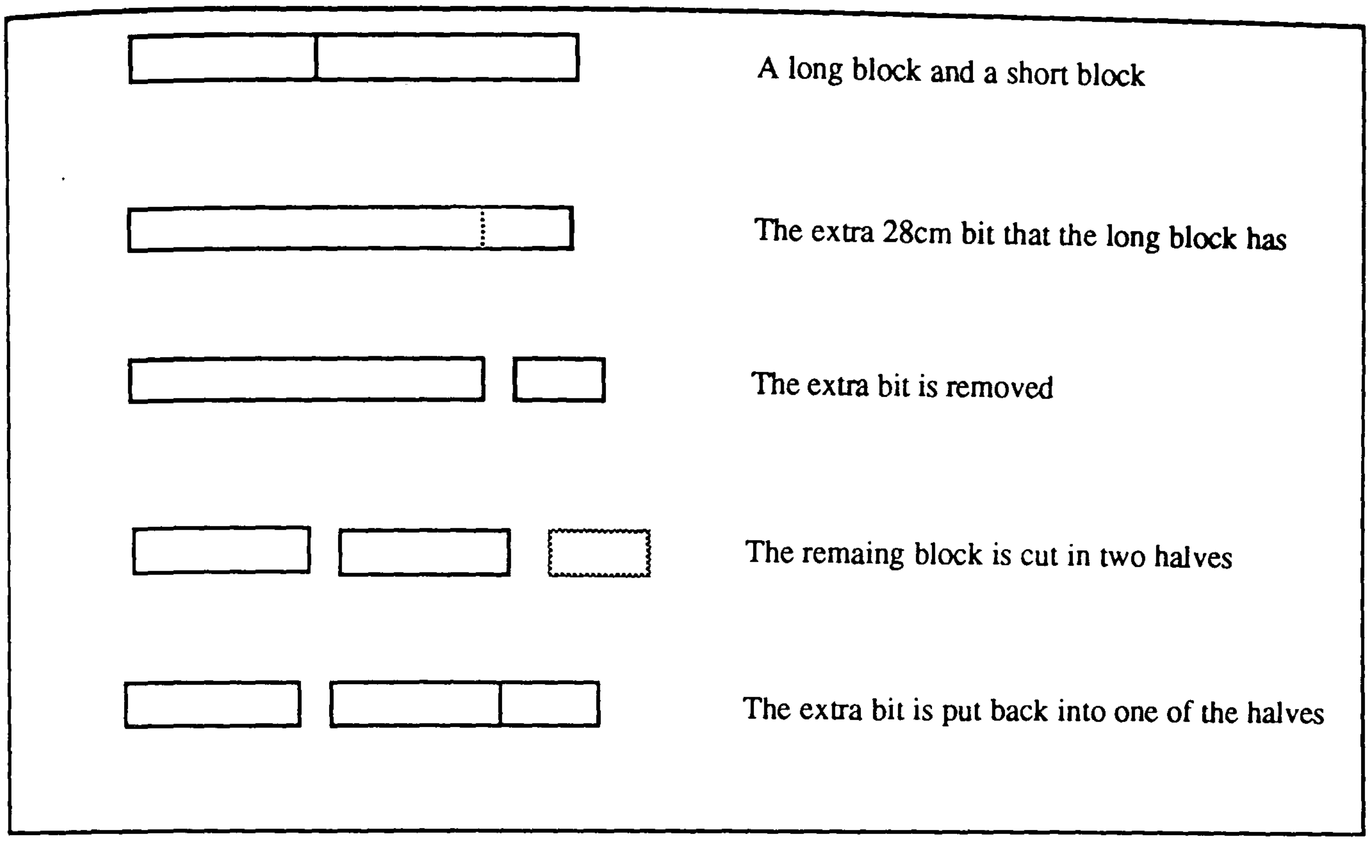


fig. CCS 1

Again, in those non-algebraic solutions the choice of operations to be used would be totally guided by the manipulation of the objects of the context, eg, a subtraction to *evaluate* how much is left after a bit 28cm long is cut from the total.

From a script containing only equation(s) without any other explanation, it would be virtually impossible to distinguish solution (i) above from an algebraic solution using a single equation.

Carpenter 1-2 problem (**Carp1-2**)

As for the Carpenter 1-1 problem, the two likely algebraic models would be a set of equations

$$\begin{cases} L + 2S = 162 \\ L = 2S + 28 \end{cases}$$

or a single equation

$$(2S + 28) + 2S = 162$$

Also, the same non-algebraic procedures could be used, with the additional step of "slicing" the shorter block in **Carp1-1** into the two required smaller blocks. The additional difficulty that appears in **Carp1-2** is that non-algebraic solutions similar to those presented a few paragraphs above for **Carp 1-1** would have to deal with the "complex" object "two short bars" replacing the "short bar" in **Carp1-1**.

Secret Number problems (**Sets1-1** and **Sets1-3**)

Those two problems could be represented by the sets of equations

$$\begin{cases} x + y = 185 \\ x - y = 47 \end{cases}$$

and

$$\begin{cases} x + 3y = 185 \\ x - 3y = 47 \end{cases}$$

presented in a more "syncopated" form.

The standard algebraic solutions would be:

(i) adding the two equations and solving the resulting equation for **x**, etc., and

(ii) isolating one of the variables from one of the equations and substituting in the other, etc..

As with the **SN1** problem in SSE, non-algebraic solutions to those problems would involve modelling the problem's statement into a non-numerical *Semantic Field*, for example for **Sets1-1**:

> "Altogether they are 185, and the second number is 47 less than the first one.
> So, if I take 47 from the 185 it is like having two of the second numbers...,"
> etc.

which of course corresponds to a structure similar to the one depicted on figure CCS 1. The specific model described above involves the additional difficulty of interpreting

(first secret no) - (second secret no) = 47

as meaning

(first secret no) = (second secret no) + 47

Seen within the *Semantical Field of numbers and arithmetical operations*, it is a simple equivalence, but when seen as a transformation of *whole-part* relationships — where the subtraction means "removal" and the addition means "conjoining" — the equivalence is not as direct as before, because each expression involve a subtle but significantly different representation; the main difference would be that on the first expression the difference is the *result* (or *final state*) *of an action*, while on the second expression it is either the *initial state* or *first operand*, or the *operator parameter* or *second operand*, depending on which model is used. As we will see in the analysis of the problems in the **Buckets** group of problems, students can easily produce the transformation

$$x + a = b \quad \Rightarrow \quad x = b - a$$

in the context of a secret number problem <u>if **a** and **b** are known and **b**>**a**</u>, which suggests that this difficulty is strongly linked to the fact that the required transformation does not produce or permit any *evaluation*.

GENERAL DATA ANALYSIS

The performance of the Brazilian group AH7 is much superior than that of the age-corresponding English group, FM2, and in fact it is comparable to that of the older FM3 group. In relation to the last group of problems, we saw that FM2 performed better than AH7 on the contextualised [11-5] problems, where the context objects were more readily available and performed worse on [4x] problems, where the meaningfulness of an arithmetical relationship (derived from the 1 to 4 ratio) was shown to be a crucial factor in successfully solving those problems. Here this should not be a relevant factor, because all the parts and relationships in the three contextualised problems are explicitly given and only *conjoining*, *taking away* and *sharing* are sufficient to model these problems non-algebraically.

Another interesting aspect of AH7 students' performance is that their approach is clearly non-algebraic on the contextualised problems (which can be seen on both correct and incorrect answers), but on the **Sets** problems the preferential approach shifts to an algebraic one, a feature more clearly seen on the choice of strategies used in incorrect solutions (for the contextualised problems, all the incorrect solutions are **WCALC**; for **Sys1-1** the incorrect solutions are almost equally divided between **WCALC** and **WEQT**, and for **Sys1-3** most of them are **WEQT**). This behaviour corresponds well to a similar behaviour observed on the SSE group problems, and it suggests that those AH7 students had a more selective approach to the choice of strategies than the students on the AH8 group.

That almost no **OKCALC** solution for the sets of equations appeared, offers further support to our conclusion that it was extremely hard for those students to *model-back* the numerical-arithmetical statements into a non-numerical *Semantical Field*, as we had observed with the Secret Number problem on the Seesaw-Sale group. Although the complexity of the problems' statement is certainly an issue here, we think that it is not a crucial one, once the facility level for the contextualised problems is significantly higher than on the **Sets** problems on AH7 and on FM3, showing that they could to some extent cope with the complexity offered by those problems. We think that two factors have to be taken into consideration. First, the difficulty in extracting information from the numerical-arithmetical relationships on what can and should be done to solve those problems, ie, the *lack of meaning* of those expressions, which would indicate that those students could not operate on a *Semantical Field* where those expressions were numerically meaningful by themselves. Second, the fact that "the first number" was greater than the "second number" or "three times the second number" was expressed by a subtraction, and our results suggest that a non-numerical interpretation of such a subtraction is much harder than a non-numerical interpretation of addition in the context of comparing measures.

Two points arise the from analysis of the use of equations and sets of simultaneous equations by students on AH8 to solve the contextualised problems[28]:

(i) on **Choc** all **OKEQT** solutions (47%) used sets of equations. The form in which **Choc** was introduced, with two "conditions" or "statements" clearly distinguishable, two unknowns clearly distinguishable, and a visual presentation strongly resembling sets of equations (eg, the two conditions written on bellow the other) strongly suggested the "sets of equations" approach, at the same time it discouraged the direct modelling into one single equation; in fact 12% of those **OKEQT** solutions to **Choc** proceeded from the set of equations by a substitution, but this procedure was never used *before* the statement had been represented in algebraic notation. This shows that what was not seen as *meaningful* in the *Semantical Field* of the chocolate boxes became visible in the *Numerical Semantical Field* (as we had indicated in the analysis of possible models).

(ii) the greater complexity of the conditions in **Carpl-2** made a *direct non-algebraic substitution* leading to a model with a single equation much more difficult; as a result, the separate representation of the two relationships usually preceded their manipulation. This is absolutely clear from the fact that one has, for **Carpl-1**, 47% of

[28]We restrict our analysis here to AH8 because this was the only group to consistently use this approach.

solutions from a single equation and 32% of solutions from a set of simultaneous equations, but for **Carp1-2** the percentages change to only 5% of single equation solutions and 42% of sets of simultaneous equations solutions

A possibly relevant mistake was made when producing the Brazilian version of **Carp1-2**, as the original phrase "If two short blocks are put together, they still measure 28cm ***less*** than a long block" ended up as the equivalent of "The long block is 28cm ***longer*** than two short blocks put together." In **Carp1-1** both Brazilian and English versions used the former form. Nevertheless, this difference in the statement did not seem to produce significant effects on the results, as in **Carp1-2** AH7 kept at a substantially higher level than FM2, and AH8 kept at a higher level than FM3 — as it happens for both pairs of corresponding groups in **Carp1-1**.

The biggest fall in the facility level from **Carp1-1** to **Carp1-2** is for AH8 (from 90% to 52%), and it is associated with a much greater difficulty in producing a single equation by a *direct non-algebraic substitution*; this failure to directly reduce the problem was not compensated by an increase in the proportion of non-algebraic solutions, but only by a moderate increase in the number of solutions using a set of equations. This shows again the lack of flexibility on the problem-solving behaviour of AH8[29]. In AH7, the fall in the facility level is smaller but still significant (from 69% to 44%), and it corresponds mainly to a smaller proportion of **OKCALC** solutions. In FM3 the facility levels are more similar (64% to 52%), and in FM2 practically nil (6% in both cases, for a sample of 17 students, ie, one correct solution for each of the two problems).

STUDENTS' SOLUTIONS

The Sets1-1 problem

All but two **OKEQT** solutions to this problem were produced by solving the set of equations directly suggested by the problem's statement. One of those two solutions employing a single equation, however, provides a good example of a *direct non-algebraic substitution*, with the added relevance of the descriptive use of literal notation (Mairê M, AH8).

[29]Also, the proportion of **WCALC** solutions remains the same and that of **WEQT** increases dramatically.

$x + y = 185$ — $x - y = 47$
$x + x + 47 = 185$
$2x = 185 - 47$
$x = 138 : 2$
$x_1 = 69$
$x_2 = x + 47 = 69 + 47 = 116$
$\cancel{113} + \cancel{66} = 1\cancel{79}$
$116 + 69 = 185$

Se a diferença entre eles é de 47, um tem 47 a mais que o outro, portanto se um é x o outro é x+47 e a soma dos dois é 185.

Mairê M, AH8

"If the difference between them is of 47, one has 47 more than the other, thus one is **x** and the other is **x+47** and their sum is 185."

Normally, from the script alone it would not be possible to decide whether the *direct substitution* was *non-algebraic* or *algebraic*, ie, whether it was respectively based on *modelling back* the second expression into, for example, a two sticks situation, one longer than the other, or a non-written manipulation of the second "equation". At first sight it seems the second is the case, as Mairê wrote down the two equations first (top left) and solved the problem algebraically before writing down the explanation (which is to the right of the algebraic solution). One detail of the solution, however, clearly suggests that she was not dealing directly with the equations she had written: her second equation (first line, after the m-dash) says that "the difference between the two numbers is 47" <u>but it also implies that "**x** is the greater of the two"</u>. Nevertheless, on the second line she writes

$$\mathbf{x + x + 47 = 185}$$

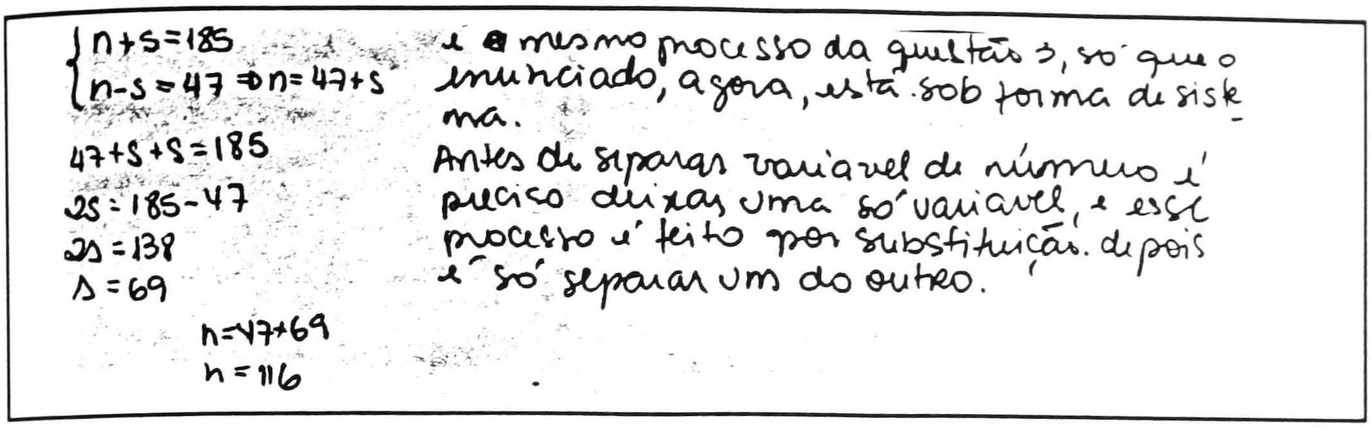
and not

$$\mathbf{y + 47 + y = 185}$$

as it should be the case were she actually dealing with the equations written on the first line as objects being manipulated[30]. Although it is truly possible that the property she evoked to substantiate the substitution was seen by her purely as a property of numbers, we are led to the conclusion that in fact she was using a non-algebraic model, as it took her away enough from the equations' context to allow a complete shift in the meaning of the symbols used.

Andrea M's (AH8) solution, on the other hand, clearly exemplifies the *algebraic* substitution, done within the context of the algebraic model, ie, *after she had produced the algebraic model, and the substitution being meaningful within that* **Semantical Field**.

[30] In this case it is obvious that this procedure did not affect the correctness of the solution, once in fact the actual algebraic solution begins at the second line, and not at the first, as it would seem to begin.



$\begin{cases} n+s=185 \\ n-s=47 \Rightarrow n=47+s \end{cases}$

$47+s+s=185$

$2s=185-47$

$2s=138$

$s=69$

$n=47+69$

$n=116$

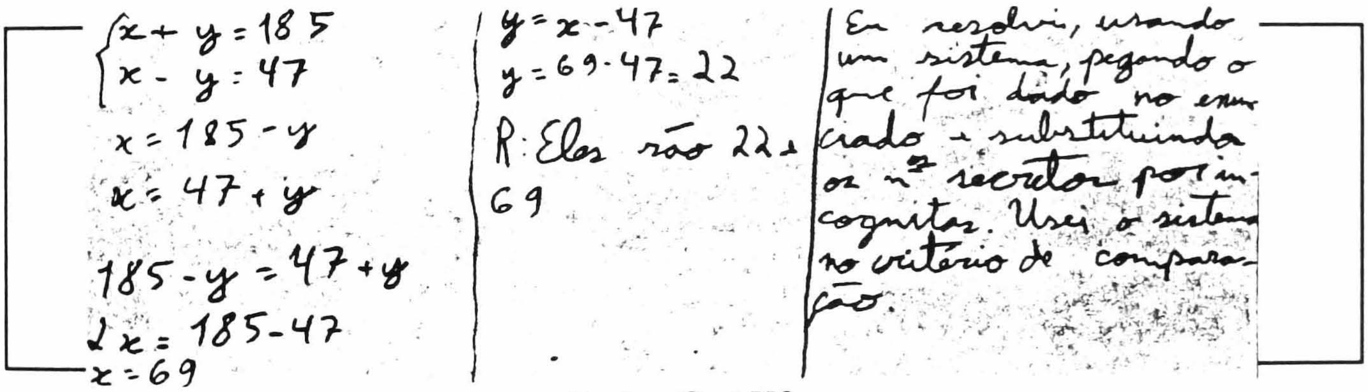
é o mesmo processo da questão 3, só que o enunciado, agora, está sob forma de sistema.
Antes de separar variável de número é preciso deixar uma só variável, e esse processo é feito por substituição. depois é só separar um do outro.

Andrea M, AH8

"it's the same process as in question 3 [31], but only this time the statement is on the form of a *system*[32].

Before separating the variables one has to leave only one variable, and this process is done by *substitution* then it is only separating one from the other." (our emphasis)

Eurico G's (AH8) solution shows another procedure to reduce the set of equations into a single equation with one unknown, using "...the criteria of *comparison*." [33]



$\begin{cases} x+y=185 \\ x-y=47 \end{cases}$

$x=185-y$

$x=47+y$

$185-y=47+y$

$2x=185-47$

$x=69$

$y=x-47$

$y=69-47=22$

R: Eles são 22 e 69

Eu resolvi, usando um sistema, pegando o que foi dado no enunciado e substituindo os nº secretos por incógnitas. Usei o sistema no critério de comparação.

Eurico G, AH8

Moreover, it shows that he directly attached an *arithmetical* meaning to the "+", "-" and "=" signs, as it is indicated by him saying that "I solved using a system, *taking what was given in the statement and substituting the secret numbers by unknowns*" (our emphasis). On his solution one can also see the importance of *internalism* in thinking algebraically, once the <u>production</u> of the expressions

$$x = 185 - y \quad \text{and} \quad x = 47 + y$$

is meaningful only in the context of the *method of solution*.

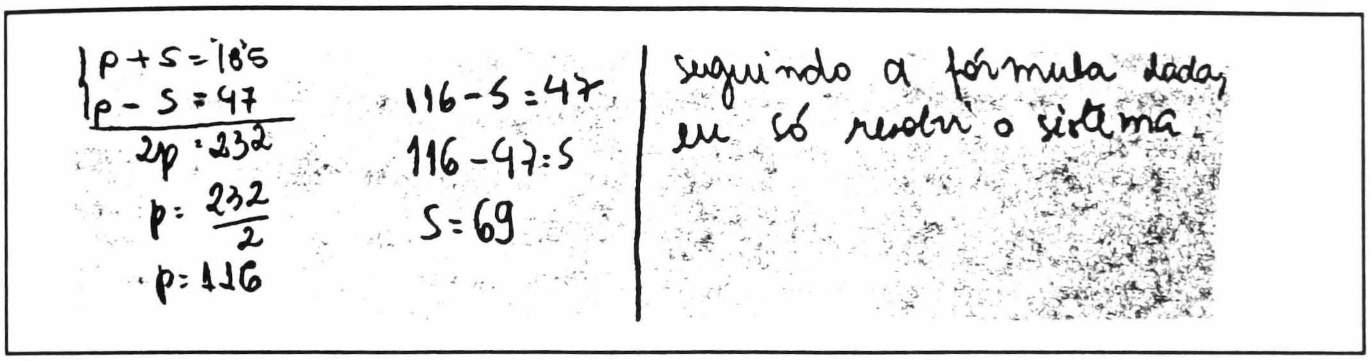
---

[31] We believe that she mistakenly referred to question 3 (**SN1**), having in fact intended to refer to question 2 (**Carp1-2**), which she solved using a set of equations.

[32] In Portuguese, *system of equations* stands for *set of equations*.

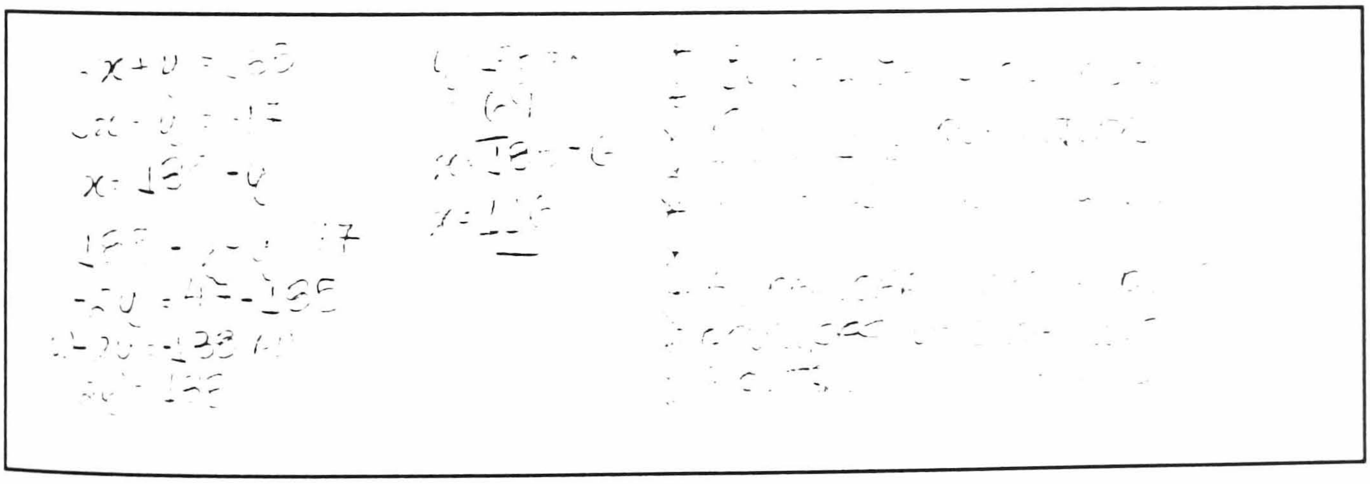
[33] *Comparison* being the "official" name for that strategy according to Brazilian textbooks.

Eurico's was the only **OKEQT** solution to use the *comparison* strategy. All the others used either *addition of equations* (eg, Erika M, AH8) or *substitution* (Andrea M, AH8, script already shown) strategies, with twice as many *substitution* solutions as *addition of equation* ones. *Formally*, the *addition of equations* strategy involves a more sophisticate algebraic perception than the *substitution* strategy, as one would have to perceive the equations as an *objects* that can be operated with. Nevertheless, one can actually perform the addition of the two equations *term by term*, with the correctness of the procedure being guaranteed by a trust in its algorithmic side rather than a deeper understanding of the procedure's roots.


p + s = 185
p - s = 47
2p = 232
p = 232/2
p = 116
116 - s = 47
116 - 47 = s
s = 69
seguindo a fórmula dada, eu só resolvi o sistema.

Erika M, AH8

The solutions by Bruno N (AH8) and Alberto SA (AH8) also throw light into how students might identify the adequacy of using an algebraic strategy — in this case solving a set of equations. In Bruno's case it is the structural aspect that provides the hint (identifying equations, operations involved and variables), and in Alberto's case it is the direct recognition of equations in the problem's statement (as in Eurico's case, analysed above) together with the visual aspect ("...2 equations one bellow the other.").



Bruno N, AH8

$$\begin{cases} x + y = 185 \\ x - y = 47 \end{cases}$$

$$\begin{aligned}
x &= 185 - y \rightarrow 185 - 69 = 116 \\
185 - y - y &= 47 \\
-2y &= 47 - 185 \\
-2y &= -138 \\
y &= 69
\end{aligned}$$

Alberto SA, AH8

From the six **WEQT** solutions, three are of greater interest.

Ricardo G's (AH8) makes an almost careless mistake by "forgetting" to include the second **y** when he substitutes into the first equation the expression for **x** obtained from the second equation. Apart from that his solution is neat and correct, and had he checked his answer, he would have probably spotted the mistake and corrected it.

$$\begin{cases} x + y = 185 \\ x - y = 47 \end{cases} \Rightarrow x = 47 + y$$

$$\begin{aligned}
47 + y &= 185 \\
y &= 185 - 47 \\
y &= 138
\end{aligned}
\qquad
\begin{aligned}
x + 138 &= 185 \\
x &= 185 - 138 \\
x &= 47
\end{aligned}$$

Ricardo G, AH8

In Nicola D's (FM3) solution, the derivation of the three expressions

$$A = 185 - B \;,\; B = 185 - A \;\text{ and }\; B = A + 47$$

is technically correct, but she never gets any further. In a sense it seems that she was trying to put the expressions in a form in which she could see how to proceed, being unaware that from any of the expressions involving two unknowns alone she could not get "the" answer. It did not occur to her a *substitution* or a *comparison*, although she had already produced the necessary steps to use any of the two strategies.

$$A + B = 185 \qquad A = 185 - B$$
$$A - B = 47 \qquad B = 185 - A$$
$$185 - 47 = 148 / 47$$
$$B + 47 = A$$

Nicola D, FM3

Finally, we have Adriana C's (AH7) solution, in which she fails to perceive that letting the same letter to stand for both secret numbers is the main cause of her attempt not working.

$$x + x = 185$$
$$x - x = 47$$
$$2x = 185$$
$$-2x = 47$$
$$x = \frac{185}{2} = 92{,}5$$

Não deu certo..!

Adriana C, AH7

As she was writing the first two lines she might well have been aware that the two secret numbers could be different, and was making use of a heavily context-dependent notation (thinking of "a number" and "a[nother] number"), but then she shifts her attention to the written expression and looses control of the process. It is also interesting to notice how she tried to make sense of the second equation

$$x - x = 47$$

by producing

$$-2x = 47$$

instead of accepting the obviously "puzzling"

$$0 = 47$$

Although so evidently distinct in terms of the level of knowledge and technical competence, in those last three scripts one can see the unknown numbers (or parts) being part of the solution process, ie, being assumed as *objects* in the model, as having the same properties of the known ones[34] (***Analyticity***). Also present in all three is a willingness to manipulate *numerical-arithmetical* expressions in order to produce the answer, this manipulation developing within the *Semantical Field of Numerical-arithmetical* expressions[35].

Only three **OKCALC** solutions were produced, two of them of interest to us.

First we have Laura W's (FM3) solution. Her solution to this problem is exactly the same she gave to **Carp1-1** and **Carp1-2** (scripts also shown bellow), and we are led to believe that she actually *modelled back* the set of equations into wooden blocks as in the **Carp** context.

$\frac{185}{2}$ · 92½ . $\frac{47}{2}$ 23½   23½ + 92½ 116.
92½ − 23½ 89.
~~NGGE89~~ =

Laura W, FM3 — **Sets1-1**

102 ÷ 2 = 51cm
51 − 14 = 67
51 + 14 = 95

I divided 162 by 2 then added half of 28 (14 onto it then took 14 off the other half.

Small block 67cm
Big block 95cm.

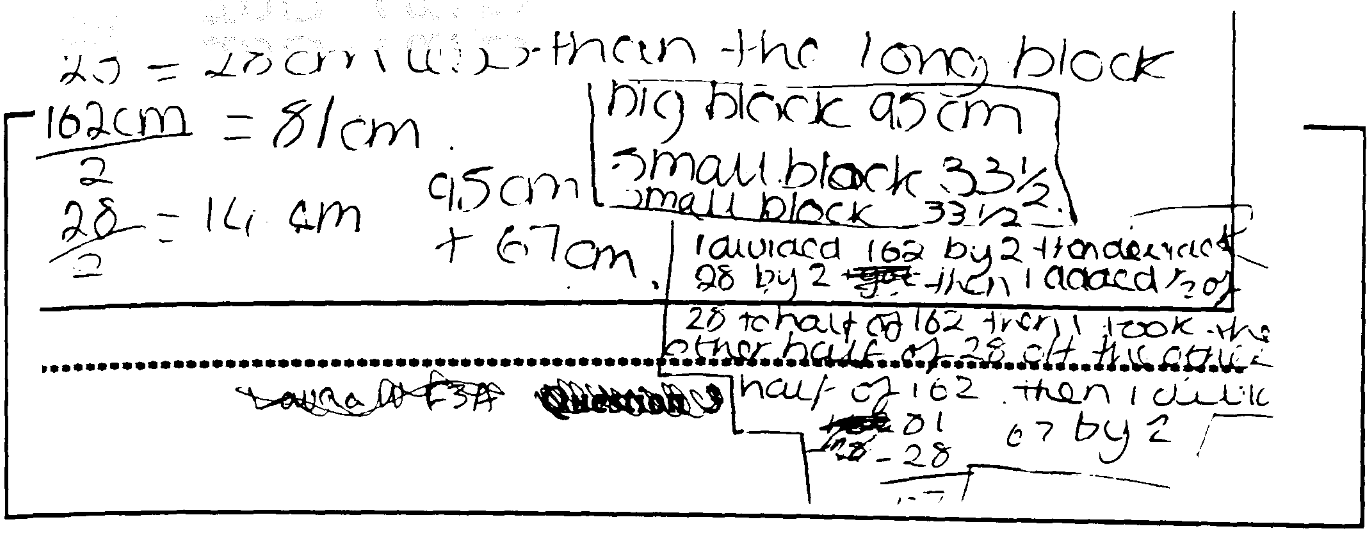
Laura W, FM3 — **Carp1-1**

[34] At least at a manipulative level.

[35] Actually, Ricardo's and Nicola's solution *could* be entirely justified in terms of *whole-part* and *sharing* — which nevertheless does not seem to be the case, specially in Ricardo's case. In Adriana's solution, however, we have the expression

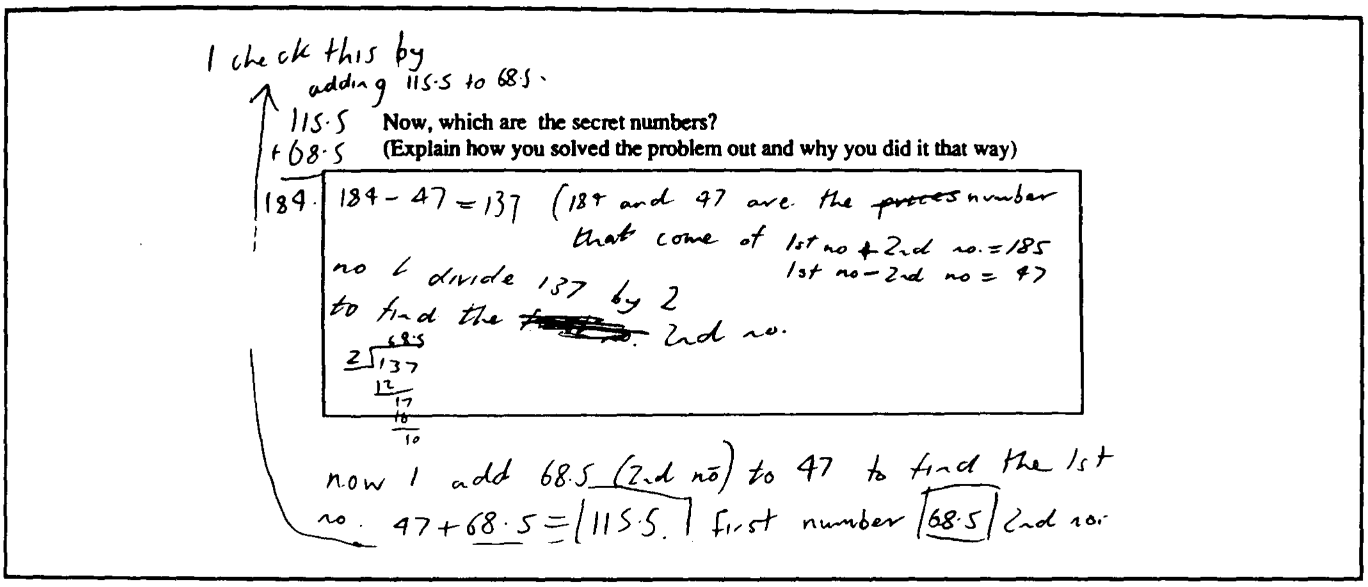
$$-2x = 47$$

which indicates some degree of — if not conscious — *numerical internalism*.



Laura W, FM3 — **Carp1-2**

Second, we have Joe V's (FM3) solution[36].



Joe V, FM3

A few points indicate that his is an *non-algebraic* solution and not a non-symbolised *algebraic* solution: *he begins by subtracting 47 from 185*; if the intent of this step was to work out the resulting right-hand side that would result from subtracting the second equation from the first, one has to assume that he did it in order to eliminate the first secret number from the resulting expression. But if this was his intention, why not simply add the two equations, a much simpler procedure by all means? On the other hand, we may see this subtraction as an *evaluation* of the result of taking the excess 47 from the total, so to produce two equal parts, and that he perceived the 47 as an excess of the first number over the second is clear from the fact that near the end of the solution (right before checking his answers up) he says "...I add ...[the] (2nd no) to 47 to find the 1st no.".

[36] As we said before, the fact that he made a numerical mistake was of no importance to us, once the process would lead to a correct answer.

The third solution offers only the calculations and no explanation as to why those steps were chosen.

What emerges clearly from the **WCALC** solutions is that the lack of some kind of written representation seriously hindered the solution process, as those students were trying produce a chain of calculations that made sense and produced an answer. One script is particularly illustrative (Ian C, FM3), who seems to be doing well, only to make a mistake on the last calculation, most probably by judging 69 to be the first and not the second secret number.

$185 - 47 = 138$

$138 - 2 = 69.$

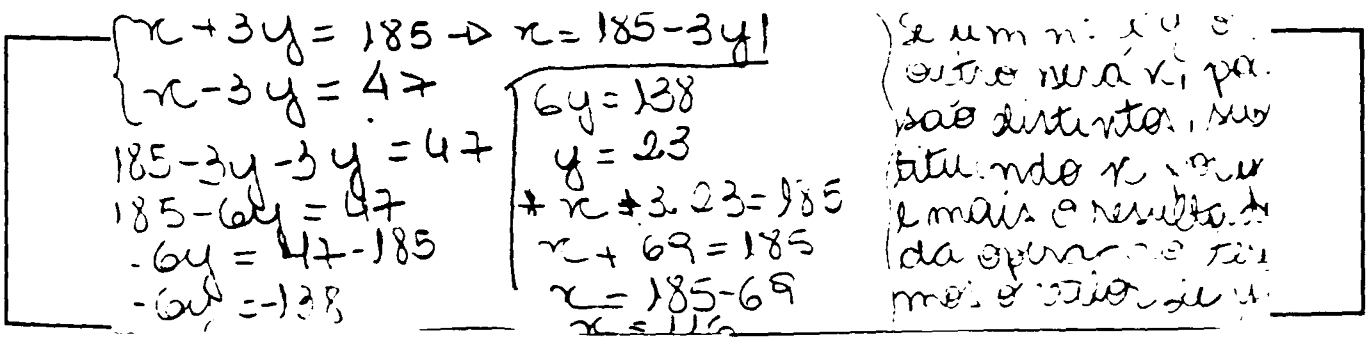
$69 - 47 = 22$

Ian C, FM3

The Sets 1-3 problem

All the **OKEQT** solutions to this problem used a set of simultaneous equations.

Three of them were solved by a *substitution* method, eg, Daniela V (AH8), in which script we find explicated a very important characteristic of the algebraic method, the need to distinguish different unknowns and parameters from the outset, to assure that the correctness of the derived relationships is kept.



Daniela V (AH8)

"If one number is y the other will be x, because they are distinct..." (beginning of text)

Ten of the sets of simultaneous equations were solved by the *addition* method, and three of those solutions present us with characteristic aspects of *algebraic thinking*.

In Ricardo M's (AH7) solution, the addition of the two equations is justified as he writes down "-3m+3m" and only then simplifies it. This procedure shows the *arithmetical internalism* characteristic of thinking algebraically as it gives the reason for adding the two equations **and** a justification for the addition producing an equation in only one unknown that is completely based on a property of numbers[37].

$$n+3m = 185$$
$$n-3m = 47$$
$$2n-3m+3m = 185+47$$
$$2n = 232$$
$$n = \frac{232}{2} = 116$$

1º nº = 116
2º nº = 23

Ricardo M, AH7

Walter R's (AH8) solution exhibits the *method driven internalism* characteristic of thinking algebraically. For no "good" reason he first multiplies the first equation by minus one and only the performs the addition of equations[38]. Nevertheless, the objective of such step is to prepare the set of equations for a subsequent transformation, ie, it is *meaningful* within the *Semantical Field of numbers and arithmetical operations*.

$$\begin{cases} x+3y = 185 \\ x-3y = 47 \end{cases}$$
$$(-1)\begin{cases} -x-3y = -185 \\ x-3y = 47 \end{cases}$$
$$-6y = -138$$
$$(-1)\quad 6y = 138$$

$$y = \frac{138}{6} \qquad y = 23$$

$$x + 69 = 185 \qquad x = 116$$

1º 116 2º 23 (sistema)

Walter R, AH8

---

[37]We think that the particular detail of Ricardo writing "2n - 3m + 3m" instead of "2n + 3m - 3m" (the "natural" order, following the order of the equations) shows that he was thinking of the *addition of opposites* property and not of "take away and put back" or "complementing" strategies, the former corresponding to a way of avoiding to write "+3m + (-3m)", a mere symbolic convenience.

[38]The quotes mean that he could have obviously applied the *addition* strategy without this extra step.

Finally, Giuliano G (AH8) sees the generality of the *method of addition* in enabling him to find either of the unknowns from the same set of simultaneous equations by applying the same strategy, and it shows that:

(i) it is the *addition of opposites* that is the centre of his attention (an *arithmetical* property), and,

(ii) although dealing with a numerically specific instance, the generality of the method is clearly expressed even if no "generalised numbers" ("letters") are used for parameters.

$$\begin{array}{l} x+3y=185 \\ x-3y=47 \\ \hline \quad 232 \\ 2x=232 \\ x=116 \end{array} \qquad \begin{array}{l} x+3y=185 \\ -x+3y=-47 \\ 6y=138 \\ y=23 \end{array} \qquad \begin{array}{l} x+3y=185 \\ 116+69=185 \end{array} \qquad \begin{array}{l} x=116 \\ y=23 \end{array}$$

$$\begin{array}{r} 185 \\ -47 \\ \hline 138 \end{array} \qquad 138 \,|\, 6 = 23 \qquad \begin{array}{r} 23 \\ \times 3 \\ \hline 69 \end{array} \qquad \begin{array}{r} 116 \\ +69 \\ \hline 185 \end{array} \qquad V=\{116, 23\}$$

Giuliano G, AH8

One of the **WEQT** solutions (Juliana B, AH7) shows one of the possible effects of not distinguishing the two unknowns.

$$\begin{array}{ll} (x-3x)\;(x+3x) & = 185+47 \\ 2x & = 232 \\ x & = \frac{232}{2} = 116 \end{array}$$

esse tipo de conta eu fiz porque $(x-3x)(x+3x)=185+47$, essa foi a única forma que achei tentei outros jeito não deu só deu desse jeito

Juliana B, AH7

The result for the first secret number is <u>incidentally</u> correct, given the "friendliness" of the set of simultaneous equations, but she fails to perceive that the second secret number had not yet been determined (also because she does not check the answer against the problem's statement)[39]. It is also interesting that she does not use a "+" sign between the two bracketed expressions on the left-hand side of the equation on the first line, but operates correctly on it, which suggests that the *conjoining* meaning of addition was used

[39] In fact it is not possible to firmly determine whether she did not distinguish the two unknowns at the level of the problem's statement or at a symbolic level, the latter being carried through the remaining steps of her solution process to end with her giving the answer "The number is 116" (bottom line at the left).

in "putting the two equations together", rather than a purely *numerical-arithmetical* one. Nevertheless, she was aware that both "conditions" (equations) had to be taken into account, and did not simply substituted **x** for both numbers in one or both equations and proceeded from that to produce the answer, as did Bartira (AH7).

Exemplo:

1° n$^{\circ}$ = $x = 25$

2° n$^{\circ}$ = $x + 1 = 26$

$$x + (3.(x+1) \ 185$$
$$x + 3x + 3 = 185$$
$$4x = 185 - 3$$
$$4x = 182$$
$$x = \frac{182}{4}$$

$$x - (3.\ x+1) = 47$$
$$x - (3x+3) = 47$$
$$x - 3x - 3 = 47$$
$$-2x = 47 + 3$$
$$(-1).-2x = 50.(-1)$$
$$2x = -50 \quad x = \frac{-50}{2} = 25$$

Bartira, AH7

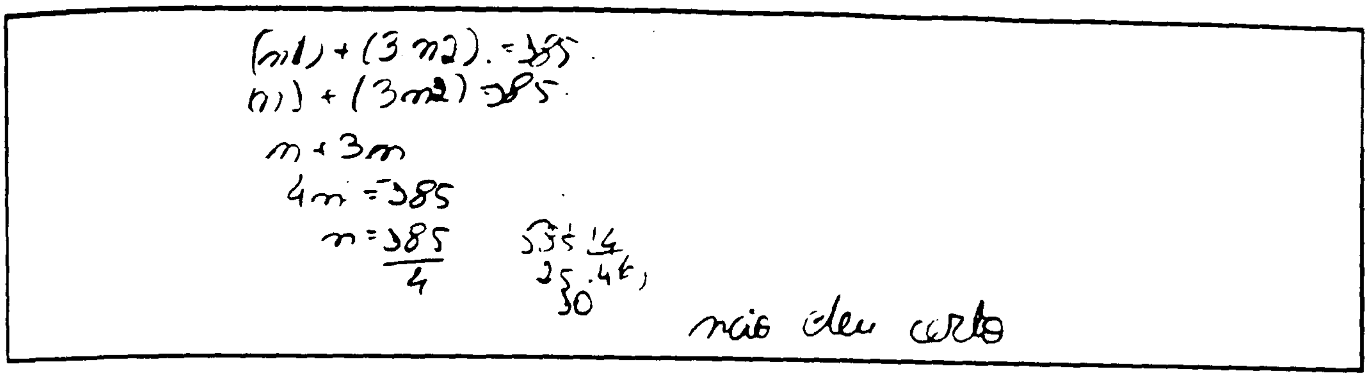
Bartira added the extra condition

$$\begin{cases} \textbf{1st number} = \textbf{x} \\ \textbf{2nd number} = \textbf{x + 1} \end{cases}$$

reducing the problem to one in one unknown only and correctly manipulated the two resulting equations[40]; we want to emphasise that she correctly handled the distribution of **3** over **x+1** even if the latter was not indicated by brackets, and this shows that she was being guided by properties of numbers and also that she was keeping control of the structure of the expressions she was manipulating, even if the notation did not suggest so. Bartira's mistake was at the level of understanding the relationships implied by the problem's statement (a *modelling* mistake), and not at the level of thinking algebraically.

Another **WEQT** solution (Rubens K, AH7) presents the case of manipulation of algebraic expressions being deformed by considerations *external* to the *Semantical Field of numbers and arithmetical operations*.

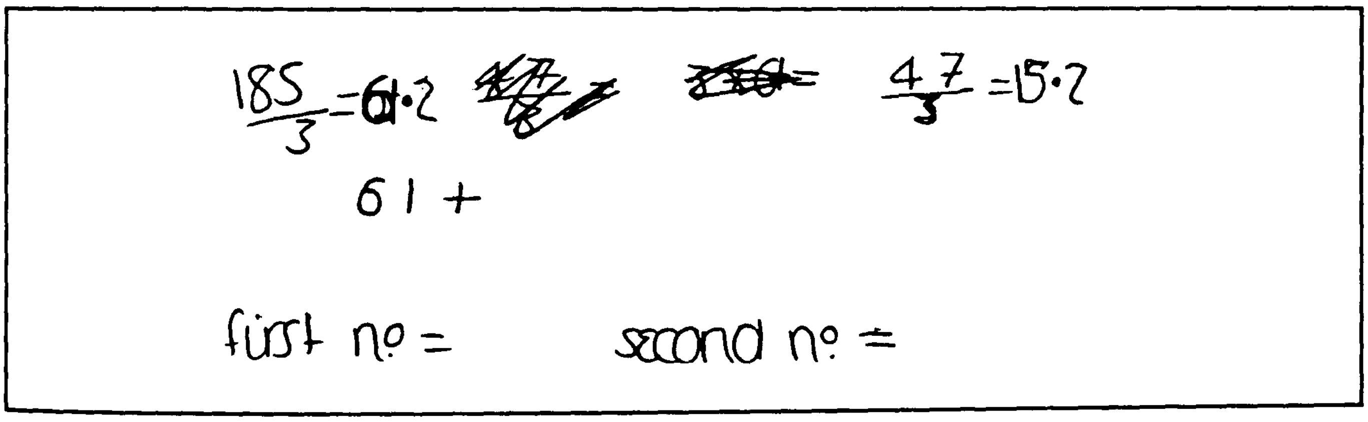
[40]This is not entirely true, as she makes a mistake on the very last calculation, putting (-50)/2 = 25. However, as she did not make any other mistakes in calculations with directed numbers, it might well be that this was not a true error, being instead a deliberate subversion of the usual rules in order to make the result to fit her expectations (for example, that the numbers were positive, an expectation which could have come, for example, from the fact that the answer resulting from the first equation was positive).



Rubens K, AH7

Rubens begins by deciding to deal with the first equation separately, and correctly identifies two unknowns (**n1** and **n2**). Being unable to proceed from there, he wipes out the distinction in order to reduce the equation to one in one unknown, correctly solves the resulting equation, but fails to go any further, apparently because he could not see how to "revert" the process and go back to the two distinct unknowns.

On the **WCALC** group, the most common error was to take the two conditions given in the problem's statement separately. As one cannot "solve" any of the two equations separately[41], usually this error was followed by the additional error of trying to produce an "answer" by dividing the independent term by 3, the only other "visible" number in the expressions (Nicola B, FM3).



Nicola B, FM3

Gurdeep S (FM3), however, goes further, producing a series of calculations that actually result in the correct second secret number.

[41]One could obviously treat each of them as an indeterminate equation in two variables and find some solutions or express a dependence condition explicitly, but it is clear that this procedure was far too sophisticate for those students.

~~185 ÷ 3 = 61.6666667 = secret number~~

15.666667

185 ÷ 3 = 61.666667   61.666667

47 ÷ 3 = 15.666667   − 15.666667

46.000000

÷ 2 = 23 =

Gurdeep S, FM3

His procedure could be seen as corresponding to the algebraic procedure

$$\begin{cases} x + 3y = 185 & (\div 3) \\ x - 3y = 47 & (\div 3) \end{cases}$$

$$\begin{cases} \dfrac{x}{3} + y = \dfrac{185}{3} & (\mathrm{I}) \\[2ex] \dfrac{x}{3} - y = \dfrac{47}{3} & (\mathrm{II}) \end{cases}$$

$$2y = \frac{185}{3} - \frac{47}{3} \qquad (\mathrm{I}) - (\mathrm{II})$$

$$y = \frac{\frac{185}{3} - \frac{47}{3}}{2} = 23$$

Although possible, this interpretation is highly unlikely to be correct because:

(i) to keep control of the solution process is not simple even with the help of algebraic notation; without it, it seems to be at least very hard;

(ii) if Gurdeep had in mind the *subtraction of equations* strategy, he would have probably applied it directly, without going through the step of dividing both equations by 3.

We offer the following alternative interpretation. Gurdeep begins by dealing with the two relationships separately, and "ignoring" the first secret number he produces the second secret number from each equation[42]. Realizing that he had produced two distinct values, he then tries to make sense of and to coordinate the two pieces of information. We believe that he tried to do so by "averaging" the two values he had obtained.

[42]This initial part of our interpretation is supported by the fact that on the first line of his script he wrote "**185/3 = 61.6666667** = secret number"

Only one **OKCALC** solution was produced (David W, FM3), and it is clearly *non-algebraic*, most probably supported by the imagery of a number line (see fig. CCS 2).

The first number is 116, the second is 23. To do this, I found the middle number in between 185 and 47. To do this, I found the difference between the two and halved it. This gave me the first number. To get the second, I found the difference between the first no. and either 185 and 47. This was 69. I then divided this by 3, to get the second number.

David W,FM3

The text in David's script has to be in a sense "decoded", because it does not literally correspond to his solution.

- He first says that he "...found the middle number in between 185 and 47. *"To do this I found the difference between 185 and 47.* This gave me the first number." It is clear that it is not the difference between 185 and 47 that produced the middle number, which he correctly gives as 116. Rather, he found the difference between 185 and 47 (138), divided it by two (69) and added the result to 47 (all three calculations at the left of the script). In relation to the diagram in fig CCS 2, this corresponds to finding the distance between the two extremes **A** and **B**, halving it and adding this to **A** to produce the point **M**.
- He the says that "...To get the second, I found the difference between the first number and **either 185 and 47** [our emphasis]...", a step that clearly corresponds to finding the <u>distance</u> between **A** and **M** or between **M** and **B**.
- Finally, he divides the result by **3** to find the second number, as the distance between the first number and either 185 or 47 corresponds to three times the second number.

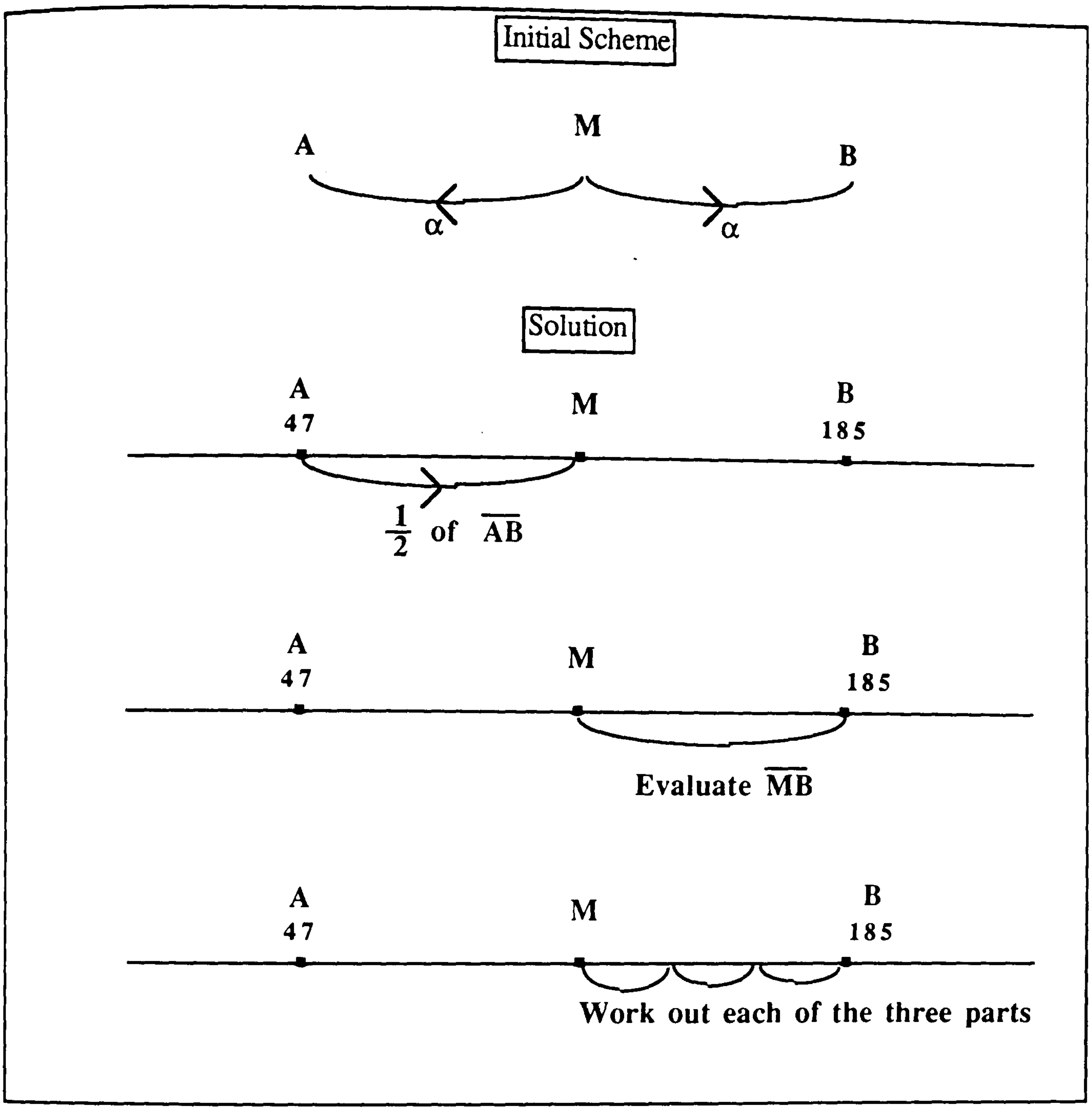


fig. CCS 2

David's solution is *synthetical.* It always proceeds by using the known values to calculate new values until he finally reaches the required answer. It is reasonable to suppose — although no explicit indication exists in the script — that the structuring of the problem itself never involved assuming the unknown values as known in order to guide the process of solution. Given David's description of his solution process, we believe he began by reasoning that the first number was a sort of "centre" from which the same amount was taken from and added to (or, in the context of the geometrical imagery, two points taken, to the right and left of the "centre", and at equal distances — see the "Initial Scheme" on fig CCS 2); from this model it is possible to envisage the necessary steps to produce the answer without any *analytic* reasoning being involved[43]. A second point of interest is that he did not realise that he had already worked out the difference between the middle and extreme points, and recalculates it as **185-116**; the relevance of this point is that it suggests that at each step a new model was produced and then manipulated according to what was seen as relevant in *that* model, and that previous *evaluations* and manipulations were not necessarily seen as "belonging to" the most recent model. Finally, it is worth to remark that he produce a literal representation of the problem's statement (upper left corner of script), that although incorrect — it uses x for both unknowns — might have been important in suggesting the geometrical model by compacting the problem's statement.

The Carp 1-1 problem

WCALC solutions were mostly of two types.

Five students misread the problem's statement and assumed that the length of the shorter block was 28cm, consequently getting the length of the longer block by simply subtracting 28cm from the total 162cm. It is almost certain that this type of mistake arose from a poor reading of the problem's statement, but it has to be pointed out that it was favoured by the actual typing of the questions, which in both Brazilian and English versions — especially the latter — might suggest the mistaken interpretation to a reader more inclined to "quickly inferring."

Twelve students, however, used a more complete — although incorrect — approach (eg, Fabiola AH7). Those students used a "÷2, +28, -28" strategy that many students had used in the exploratory study. This mistaken procedure is certainly due to a failure to perceive that taking 28cm from one of the halves automatically makes the difference between the two measures to be 28cm, but while satisfying the "difference of lengths" requirement, it alters the total length. Those students perceived this unwanted effect and corrected it by adding to the other half the 28cm that had been taken away to produce the shorter block. This step, in its turn, if adjusts the values to satisfy the "total length" requirement, alters the difference between the blocks, thus producing incorrect answers.

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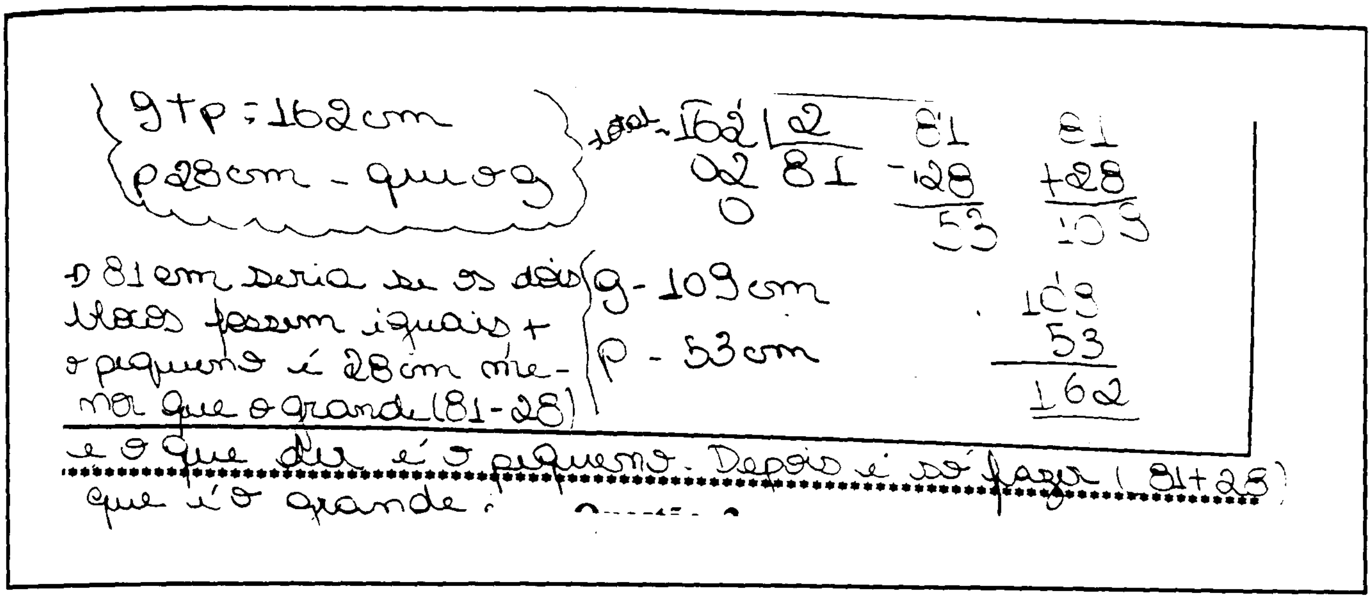
[43]The only relevant property used is that the middle point is at equal distances from the extremes.



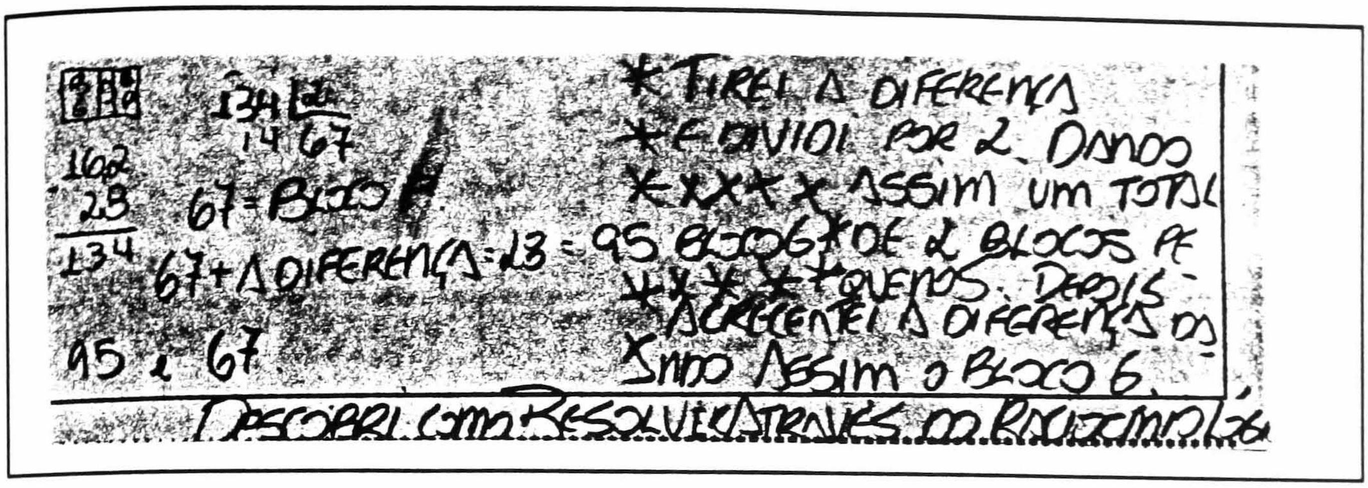
Fabiola, AH7

"81cm would be if both blocks were equal, but the small is 28cm smaller than the big one (81-28) and what you get is the small. Then it is only to do (81+28) and that's the big [block]."

At the root of this kind of mistake is a characteristic of many of the *non-algebraic* solutions presented, and that we have already examined on the last paragraph of the last sub-section, namely the fact that at each step of the solution process a new model is produced — representing or not a correct derivation from the previous models — and it is the most recent model that is manipulated *according to what is perceived as relevant and required in relation to **this** model*; *each step is **locally** meaningful.* The result is a step-by-step solution in the sense that the goals and the means to achieve them might be constantly changing, sometimes resulting in a loss of overall control of the solution process or in a deterioration of the original conditions and requirements through overall inadequate transformations of the intervening models.

The **OKEQT** solutions offer a variety of approaches.

The most common strategy was to *take away* 28cm from the total, so to produce two short blocks, and divide the result of the subtraction by two to obtain the length of the short block; then add 28cm to the length of the short block to obtain the long one (eg, Bruno N, AH8).



Bruno N, AH8

"I removed the difference and divided by 2, resulting in **a total of two short blocks** [our emphasis]. Then I appended the difference [,] resulting in the big block. I found out how to solve it by logical reasoning."

Bruno's solution is a very clear and well explained instance of the use of this strategy, including a diagram that is enough to guide the whole solution process. Some aspects of his solution are of extreme interest to us. The presence of the diagram assures us that the word "tirei", that in Portuguese could also mean "subtracted", is used in the sense of "removed". Moreover, he says that the division resulted in "...a total of two short blocks...", clearly corresponding to a "cut" followed by a division to evaluate the lengths of the two resulting halves. Finally, the word "acrescentar", that in Portuguese might also be interpreted as "adding", has to be interpreted here as meaning "appending", in agreement with the clear-cut indications of the rest of the script. The *objects* being manipulated in Bruno's solution are objects of the context, and the choice of operations is subordinated to the need to *evaluate* measures; moreover, his solution is totally *synthetical*, working from known *objects* to produce other *objects* that are shown to satisfy the required conditions. As in David W's solution to **Sets1-3**, Bruno's solutions never deals directly with as yet unknown *parts*.

Hannah G's (FM3) solution is very similar to Bruno's, but instead of "cutting" the difference to make two short bars, she adds the difference to the total, *pretending there were two long blocks*, showing that hypothetical manipulation of the context of the problem can become a key element in non-algebraic solutions. In Hannah's script one can also see the extent to which the choice of operations is subordinated to the manipulation of the non-numerical model ("I did this **to *find out how much they measured if they were the same length*.**")

162, 28

$$\begin{array}{r} 162 \\ -\ 28 \\ \hline 134 \end{array}$$

162 ← I did this to find out
\+ 28 how much they measured
190 if they were the same length.

$2\overline{)190}$ = 095 ← is how ~~long~~ long they'd be each

one block = 95   95 − 28 = 67   2nd block = 67
(95+67=162)

Hannah G, FM3

Two other **OKCALC** solutions are worth examining, both using a "÷2, +14, -14" strategy.

We think that Joe V (FM3) decided that he had to add and subtract 14, and not 28, based on his perception — probably due to the expression on the second line — that the 28cm "in excess" on the long block had also been divided in two, an interpretation that is supported by him writing

$$81 + \frac{28}{2}$$

before writing

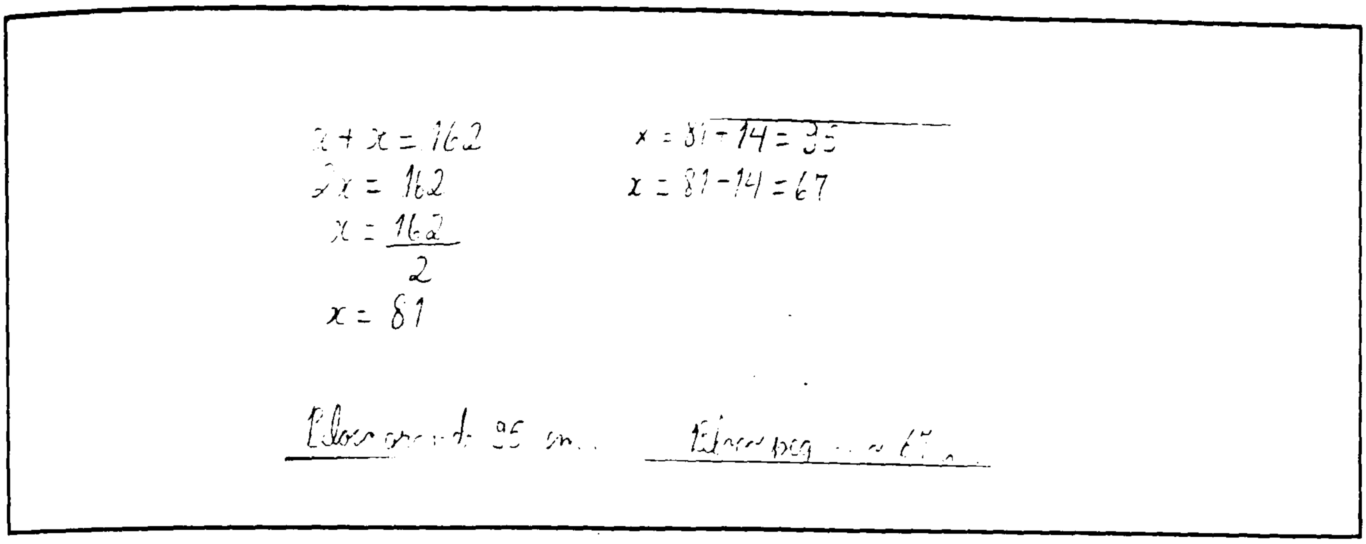
$$81 + 14$$

which indicates that the former expression carried with it something important enough to be made explicit.

first of all you divide 162 by 2
162÷2 = 81 this is the $\frac{\text{long blo.} + \text{short blo.}}{2}$
now as a short block measures 28cm less than a long block
81 + $\frac{28}{2}$ = must equal the length of the long block
81 + 14 = $\boxed{\text{95cm}}$ long block
therefor a short blocks length is – $\boxed{67.}$ short block
Check 95+67=162

Joe V, FM3

On Ricardo G's (AH8) script, on the other hand, there is no clue to how he decided to add and subtract 14, but it is his peculiar way of using algebra that we want to examine.



Ricardo G, AH8

He clearly begins with the assumption of the blocks being of the same size, and writes down and solves an equation that reflects that; just by looking at the equation one cannot decide whether he was dealing with a *numerical* relationship or simply using the literal notation to describe an *non-algebraic* process. In any case one has to notice that he explicitly deals with the unknown number-measure, ie, this part of the solution process has an *analytic* character. At the following step, where he adds and subtracts 14, it becomes clear that he saw the division by two as producing **two halves** instead of producing **one value**, as each of the two lines begin with **x** (one of the halves) and represent in fact *the transformation of each half (x) into the required blocks.* His is a *non-algebraic* solution "dressed" in algebraic notation[44].

Tatiane R's (AH7) solution is another instance of a *non-algebraic* solution "dressed" in algebraic notation, but it seems much closer to a true algebraic solution than Ricardo's, as the model used to set the equation takes aboard — as unknowns —the lengths to be determined, as opposed to Ricardo's solution (see note 20), and she produces an equation that directly and simply represents the problem's statement.

[44]Although it is obvious that one cannot be totally sure that the equation was not seen as a *numerical* expression, and that subsequently a shift in the meaning of **x** occurred, we think that in the face of the model he used to set the equation — with **x** representing none of the unknown lengths — *together with the use of x in the remaining two lines*, we must conclude for the "non-algebraic" interpretation.

bloco pequeno = x
$2x + 28 = 162$
$2x = 162 - 28$
$2x = 134$
$x = 67$ cm
bloco grande = 67 + 28 = 95

Os dois blocos juntos = 162
Só que se eu tirar o pedaço do bloco [illegible] que [illegible] pequeno [illegible] mesma coisa [illegible] dois [illegible] pequenos mais [illegible]

Tatiane R (AH7)

"The two blocks together = 162cm

But if I remove the bit of block that is in excess in relation to the small block, then it is the same as two small blocks plus the extra bit."

Her explanation however, fully reveals that throughout the process of solving the equation she was being guided by — or at least constantly checking for meaning against — the manipulation of a model that took the objects of the context as *objects*, an *non-algebraic* model. The decisive detail in the text is when she says that "it is the same as two small blocks ***plus the extra bit***," showing that the solution process was in fact guided by a *composition-decomposition of parts* process.

In the **OKEQT** group of solutions, a number of points arise.

Alessandra O (AH8) produces a substitution in the context of the set of equations, while Andrea M (AH8) produces a *direct non-algebraic substitution*, to solve the problem from a single equation.

$$\begin{cases} x + y = 162 \\ x - 28 = y \end{cases} \rightarrow 95 + y = 162, \quad y = 67$$

$x + x - 28 = 162$
$2x = 190$
$x = 95$

R: o Bloco grande tem 95 cm de comprimento e o pequeno 67 cm. Resolvi através de um sistema

Alessandra O, AH8

$x+x+28=162$
$2x=162-28$
$2x=134$
$x=67$

$67+28=95$

→ o x será o número do bloco pequeno já que eu não sei as medidas completas, mas sei o número de "comparação" de um para o outro. Eu faço o mesmo processo se eu tivesse as medidas completas: somo. A soma é feita normalmente somo separado os x's e os números. Depois separo o x de um lado e os números no outro. se tiver ainda algum número junto ao x passo para o outro lado com a operação inversa.

Andrea M (AH8)

"x will be the *number* of the small block, as I don't know the complete measures but known the number of "comparison" of one to the other. I do the same process as if I had the complete measures: add. The sum is done normally [,] I add separately the numbers and the x's. Then I separate x to one side and the numbers to the other. If there still is some number with x, I move it to the other side [,] with the inverse operation." (our emphasis)

Andrea's solution, moreover, provides a clear statement of:

(i) the *analiticity* of her reasoning, by saying "I do the same process as if I had the complete measures: add.";

(ii) the *arithmeticity* of her reasoning, by saying that "x will be the ***number*** of the small block..." and treating numerically the setting of the equation.

Marília M's (AH8) and Rogério C's (AH8) solutions exhibit an important feature of thinking algebraically, the use of *normal forms* of numerical-arithmetical expressions.

→ Através de um Sistema

$\begin{cases} x+y=162 \\ x-28=y \end{cases} \Rightarrow 95+y=162$

$y=162-95$

$y=67$

$\begin{cases} x+y=162 \\ x-y=28 \end{cases}$

$2x=190$

$x=95$

$162 + 28 = 190$

$162 - 95 = 067$

$190 \div 2 = 95$

Bloco grande = 95 cm

Bloco pequeno = 67 cm

Marília M, AH8

$$\begin{cases} x + y = 162 \\ x - y = 28 \end{cases} \Rightarrow 95 + y = 162$$

$$+y = 162 - 95$$

$$2x = +190 \qquad +y = +67\text{cm}$$

$$x = \frac{+190}{2}$$

$$x = 95\text{cm}$$

R: O comprimento do bloco grande é 95 cm o comprimento do bloco pequeno é 67 cm

Indicação: resolvi pelo processo do sistema

Rogério C, AH8

In Marília's case, the *normal form* is produced at the *algebraic* level, by manipulating the second equation

$$\mathbf{x - 28 = y}$$

to produce

$$\mathbf{x - y = 28}$$

while in Rogério's case the *normal form* is directly produced by interpreting — and representing — the fact that one of the blocks is 28cm longer as meaning that the difference of their lengths is 28cm[45].

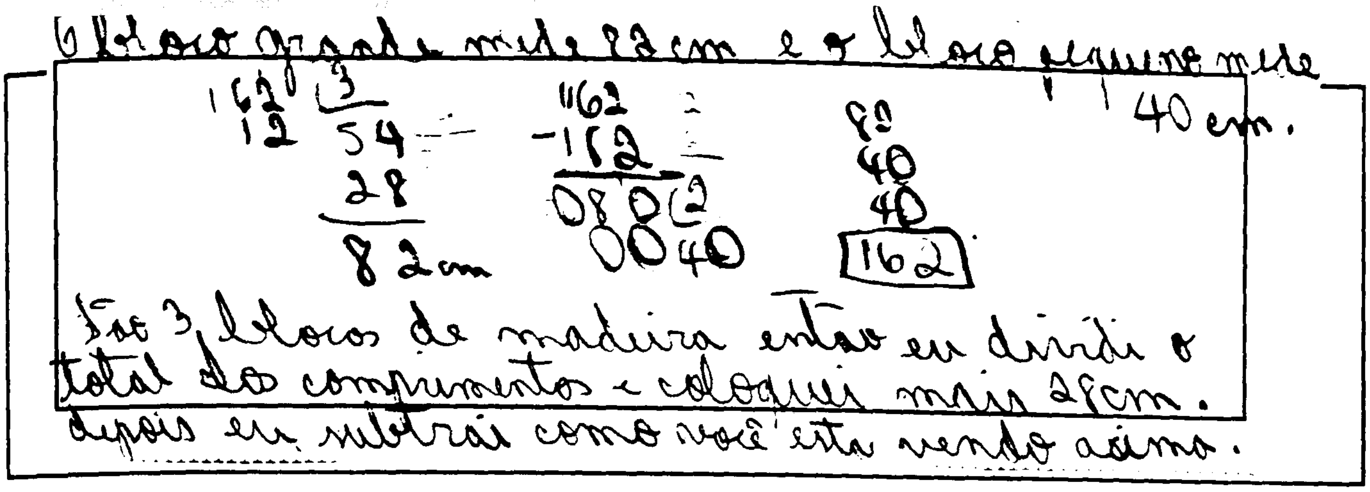
The Carp 1-2 problem

An undesirable and unexpected effect appeared in relation to this problem, with nine students solving **Carp1-2** as if it were **Carp1-1**, ie, only one short block had been mentioned in the problem's statement. We are led to believe that those students had already been presented with **Carp1-1** on the first session, and when they saw **Carp1-2** they did not bother to read the statement, as both the drawing and the first sentence are the same in both problems' statements, a flaw in the design of the tests[46]. Also, five students solved the problem assuming that 28cm was the length of two short blocks; this mistake had already been identified in the solutions to **Carp1-1**, and here again it might have been urged by the unfortunate choice of line break for the text.

Other **WCALC** solutions reveal some difficulties caused by the increase in complexity in relation to **Carp1-1**.

Ricardo B (AH7) applies a "generalised" version of the "÷2, -28, +28" that was examined in relation to **Carp1-1**.

[45]This type interpretation was in fact very rare in all the problems in all groups.
[46]Our original intention was to cause the two problems to be seen as much as possible as very similar.



Ricardo B (AH7)

"There are three wooden blocks, so I divided the total length and put another 28cm. then I subtracted as you can see above."

As a result of the increased complexity, Ricardo fails to perceive that the 28cm he adds to one of the parts produced by the *cut-division* makes the long block 28cm longer than **each** of the other ones, but at this stage the two short blocks put together are in fact 26cm <u>longer</u> than the long block[47]. A very odd shift now takes place, as to work out the length of the short blocks he subtracts the now known length of the long block from the total length, and divides the result by two to obtain the length of each short block; it should be immediately clear, as he obtains 80cm for two short blocks that something went wrong, as the difference is only 2cm. We think that this fact was not enough to trigger a revision of the previous working exactly because at that point the model he was working with included only the "total" and the "two short blocks" conditions, but not the "difference" condition; as it had happened with the solutions to **Carp1-1** mentioned earlier in this paragraph, each step resulted in a new model that was then manipulated anew, with the product of previous manipulations not always being taken into consideration[48].

Helen R (FM3) produces a very good diagrammatic representation of the problem (except that the diagram on the right is not correct because it includes the "extra" 28cm in the total as a separate bit), a representation that would almost certainly lead to a correct solution in **Carp1-1**, but fails to draw further information from it and fails to manipulate it into a more *informative* diagram, which suggests that the need to deal with the two short blocks as one single *object* functioned as an obstacle that was not overcome by her.

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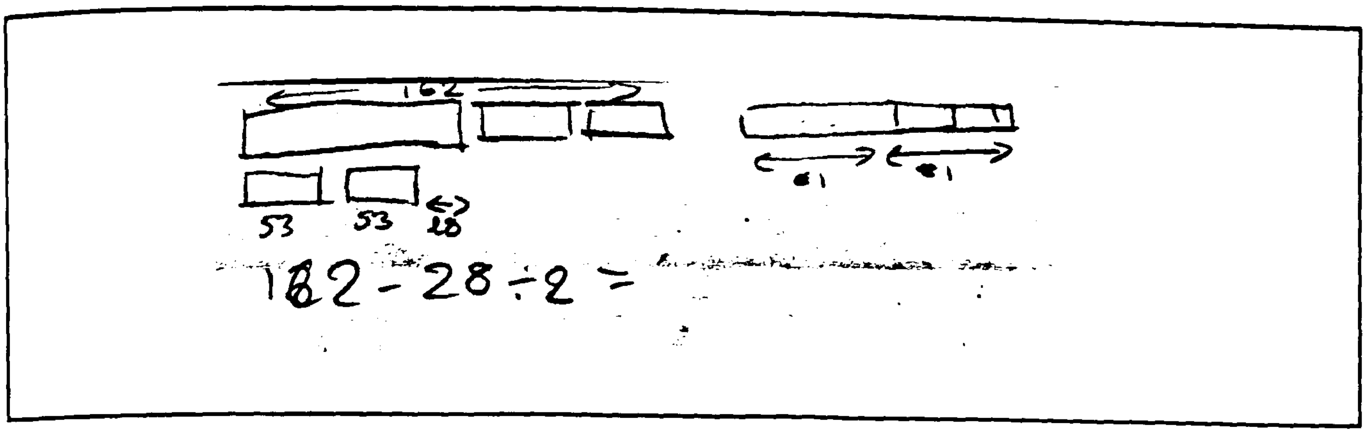
[47]It is legitimate **at this point** to assume that the two remaining blocks are the two short blocks, as Ricardo's rationale for dividing by 3 is that there are three blocks.

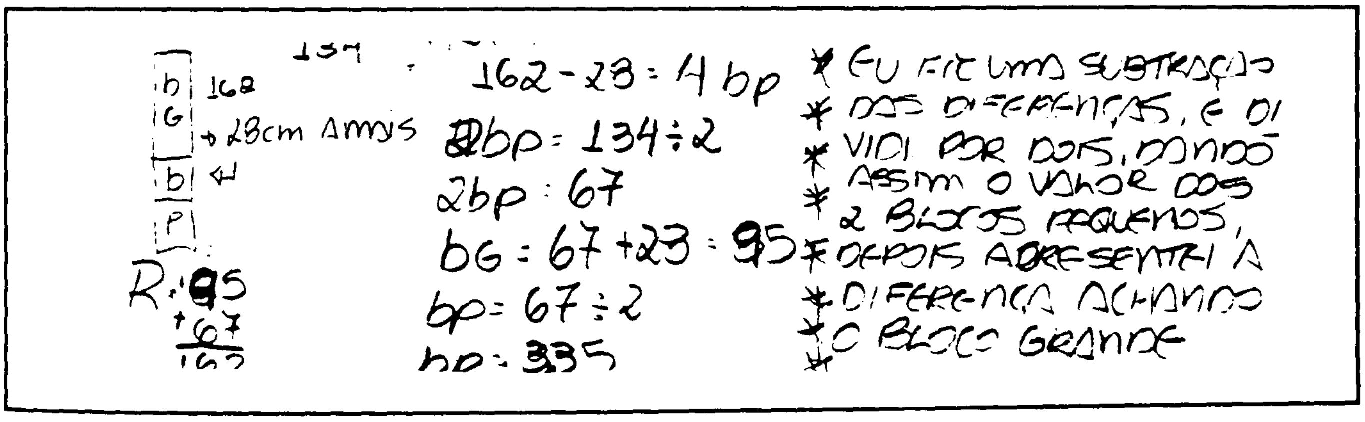
[48]As in the total disregard for the two 54cm bits that *ought* to correspond to the two short blocks — if not immediately, after some possible adjusting steps. Instead he shifts to the model "I know the total length of a long plus two short blocks, and I know the length of the long one, so..."



Helen R, FM3

The **OKCALC** solutions to this problem underline and clarify several relevant aspects of *non-algebraic* solutions.

Bruno N's (AH8) solution[49] shows the way in which a diagram is used to provide a simplified representation of the problem's statement, mixing a *whole-part* figure to represent the first condition, with an added verbal remark ("28cm more") to represent the second condition. It is clear that this diagram guides the solution process, as the labels used in it for the long and short blocks are used throughout, and the first line in the sequence of equalities indicates — by having the numerical calculation on the left-hand side and the part that its result measures on the right-hand side— that the numerical calculations are used to *evaluate* the measures of parts according to the manipulation of the *whole-part* model.



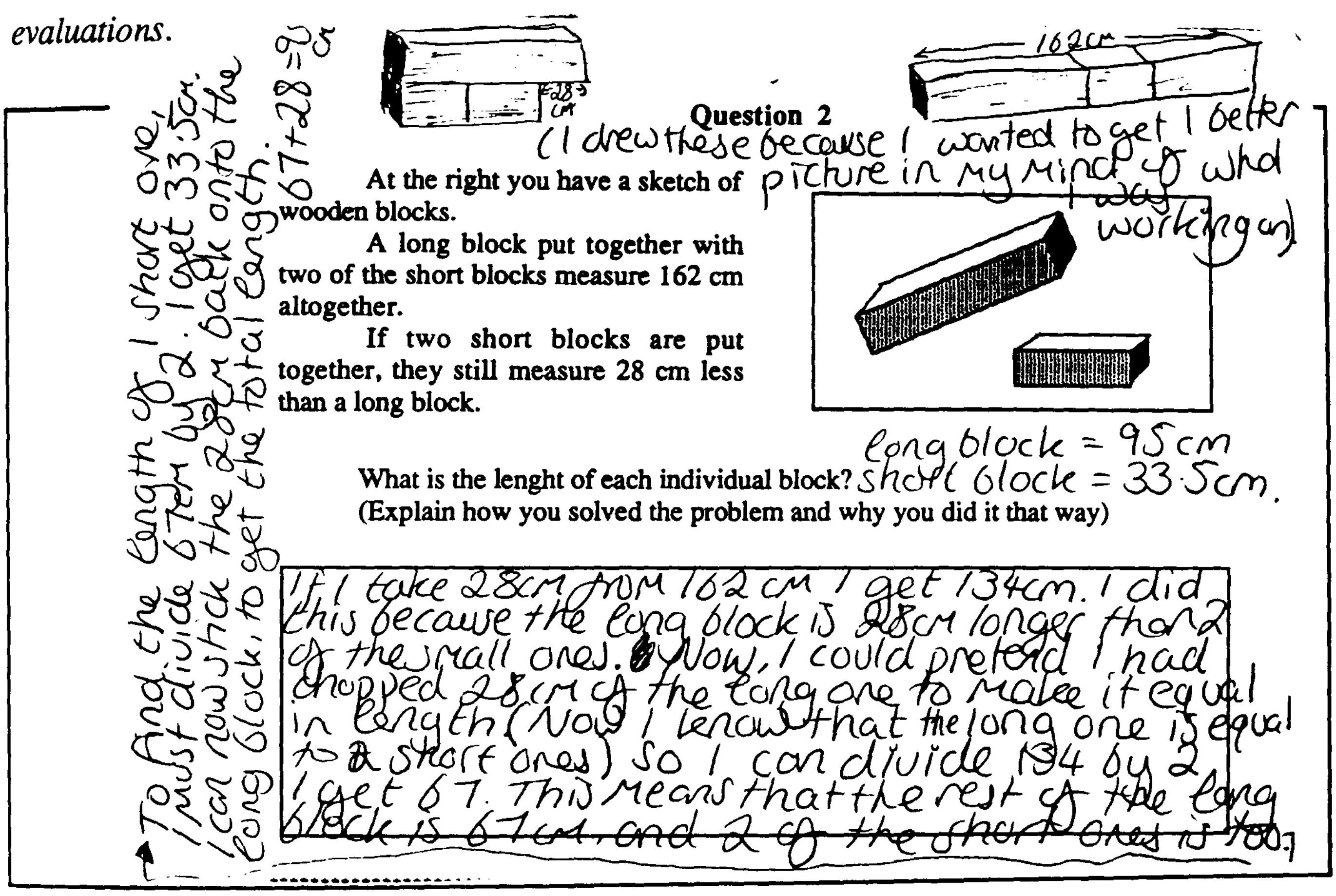
Bruno N, AH8

Elizabeth W (FM3) provided us with what is probably the clearest example of an *non-algebraic* solution among all scripts we examined.

First, because she makes it explicit that the figures she draws at the top are used to guide the solution process. Second, because she always describe the manipulative steps

[49]The text to the right does not add anything that is not already evident in the rest of the script, and for this reason is not translated.

that justify the choice of operations to be performed on the measures to *evaluate* other parts, eg, "...I could pretend I had ***chopped 28cm from the long one...***", and "I can now ***stick the 28cm back into the long block...***". Moreover, in her solution there is a *transformation* of the problem when she reduces it to one where a long block measures the same as two short blocks. This strategy is different from taking the difference away to be left with four short blocks, as it actually establishes a new variable and a new relationship, the shortened long block becoming "the" long block. Her solution is throughout well controlled and *synthetical*, and above all it shows that *verbal language* is totally adequate to describe the hypothetical assumptions and the transformations that support the choice of operations, while standard written arithmetical statements take care of describing the *evaluations*.



Elizabeth W, FM3

In Matthew K's (FM3) script also we find a solution process that is typically *non-algebraic*, with the 28cm taken as a separate bit that can be *appended* to the combination of one long and two short blocks, the arithmetical operations being performed to *evaluate* lengths. It is also distinctively *synthetical*.

long block = 95 cm Short block = 33.5

To work it out I added 28 cm to 162
to find the 2x long ÷ by two to
find 1x long − 28 to find 2x Short
÷ 2 to find 1x short

Matthew K, FM3

Finally, we examine Joe V's (FM3) solution , which uses literal notation ("...a little formula...," as he calls it) but is guided by the manipulation os a *whole-part* model.

First I do a little formula

n = short block, x = long block

n + n + x = 162

now I take the 28cm from 162cm so that the
answer is n4

162 − 28 = 134

to find out n I need to divide 134 by
4 4)134 = 33.5 a small block is 33.5cm long

Joe V, FM3

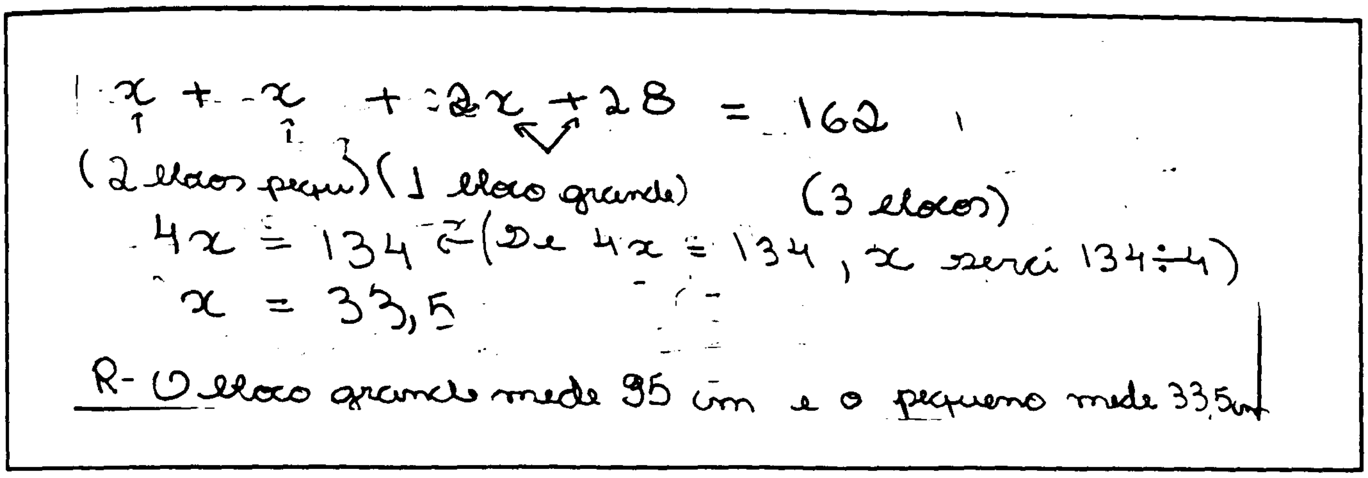
On the second line he writes

$$\mathbf{n + n + x = 162}$$

his "formula", but it is not a *numerical* one, as one gathers from the subsequent manipulation of the model it is intended to represent. Instead, the "+" sign means the *conjoining* "and", and the "=" sign denotes "measures" — acting as a *value label,* as we saw on page.... This interpretation becomes more clear when Joe "...take[s] the 28cm from 162cm so that the answer is n^4 " — in which he obviously meant 4n; the subtraction 162-28 (an *evaluation*) is different in nature from the action that produces the "4n" (a *decomposition*) corresponding to its result[50]. Although apparently it is an *analytic* model, in fact it is not, because the parts of unknown measure are not there to be directly manipulated, but to provide the *whole-part* structure and allow him to visualise a sequence of decompositions, compositions and correspondent *evaluations* that will lead to the answer.

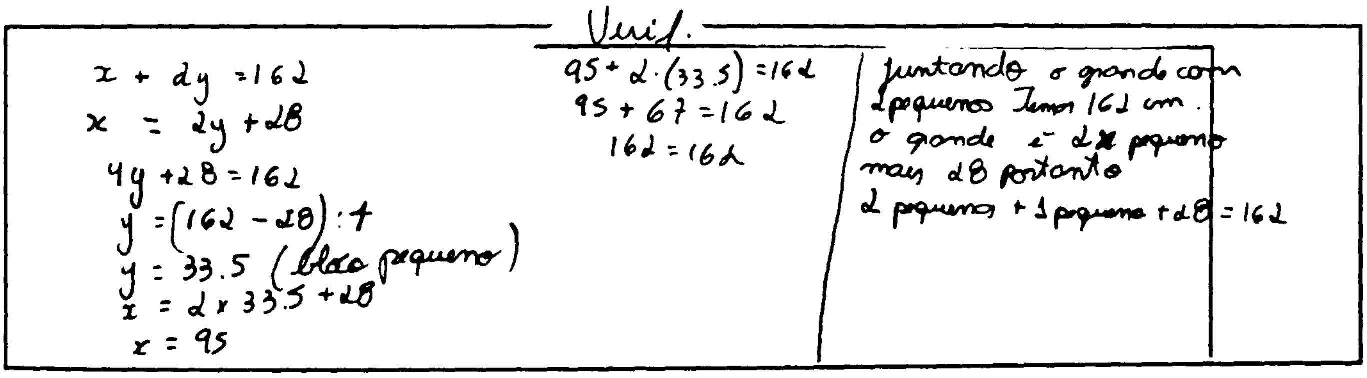
[50]We think it is telling that Joe states the *decomposition* — with its outcome — as a separate and prior step from the actual calculation.

As with the **OKEQT** solutions to **Carp1-1**, we had for **Carp1-2** both cases of a model with a single equation in one variable being produced through a *direct non-algebraic substitution* (eg, Laura G, AH7) and of a model with a set of simultaneous equations being initially produced and from there a substitution that reduces the set of equations to a single equation in one variable (Mairê M, AH8[51]).



Laura G, AH7

"(2 short blocks) (1 big block) (3 blocks)"



Mairê M, AH8

One last **OKEQT** deserves examination. Tatiane R (AH7) first solves the problem with equations (left), with a peculiar use of indexed **x**'s, possibly meaning that she saw the two short blocks in the second line as distinct[52] from those in the first line; the distinction is finally blurred on the fourth line, and the solution correctly completed[53]. On the verbal explanation, however, she shows an understanding of the *back-interpretation* of the

[51]The text at the right of the script is a restatement of the problem's statement, and thus was not translated.
[52]Physically distinct; some *other* blocks.
[53]Although there is a mistake in the subtraction, the solution is considered correct, following our criteria of prioritising the overall correctness of the procedure over the actual calculations.

algebraic procedure in terms of the problem's context that is mistaken ("...when the three [blocks] are equal one has only to divide by the sum that made the three equal"). Had she followed the image of three equal blocks, she would have made a mistake, and this strongly highlights that by focusing the solution process on the *method* and by keeping it *internal*, algebraic thinking provides a powerful way of keeping correct control of it.

Bloco grande = 2x + 28 = 100 2 Blocos pequenos x₁+x₂ = 72 2x₁+28+x₁+x₂ = 162 4x = 162-28 4x = 144 x = 36	1 Bloco grande é igual a 2 blocos pequenos + 28 cm então nós temos que somar os 3 blocos, que é igual 162 desses 162 nós temos que tirar a diferença entre os três blocos. Chegando os três ficarem iguais é só dividir pela soma que deu os três iguais

Tatiane R, AH7

Two **WEQT** solutions present two distinct — but both critical — aspects of using algebraic models to solve problems.

Mariana O (AH8) starts by setting a correct single equation in one variable — a *direct substitution* — and correctly solves it for **x** to determine the length of the short block.

(2x+28)+2x = 162 4x = 134 $\boxed{x = 33,5}$ x+14 = 47,5 o grande 47,5 o pequeno 33,5	Eu coloquei o nome de x no pequeno, e se o maior é 14 cm a mais ele é x+14.

Mariana O, AH8

"I put the name of x on the small, and if the larger is 14cm more, it is x+14"

Having already correctly recognised and used the relationship between the lengths of the long and short blocks, she then shifts to another model and this produces the error. The model she shifts to seems to be related to the "÷2, +14, -14, ÷2" approach[54], which

[54] An extension of the approach of dividing the total in two parts and then adding 14cm to one of them and subtracting 14cm from the other one to produce the required lengths.

nevertheless is not correctly interpreted by her, producing the misunderstanding that the longer block is 14 cm longer than each of the short ones[55]. Mariana correctly solved **Carp1-1** using an equation, and we are led to think that the increase in complexity was at least partially responsible for the lack of appropriate control. The crucial point, however, is that the shift to a distinct — although potentially correct — model produced an error, and this indicates the extent to which an algebraic approach depend on keeping the solution within the boundaries of the initially set equations, as the *arithmetical internalism* characteristic of *algebraic thinking* involves a shift away from the *Semantical Field of the Wooden Blocks*, and any new relationship introduced during the process of solution would have to be double checked, first within that *Semantical Field* — to assure that it correctly models the problem's statement — but also against the initial algebraic model, to guarantee, for example, that the unknowns used are in correct correspondence. Marina's lack of perception that the resulting length of the long block is not 28cm greater than the length of the short ones — let alone 28cm longer than **two** of them put together — is remarkable.

The second **WEQT** script we want to examine is Marcel S's (AH8).

$$\begin{cases} x = 2x + 28 \\ x + 2x + 28 = 162 \end{cases}$$

$2x + 28 + 2(2x + 28) = 162$

$2x + 28 + 4x + 56 = 162$

$6x = 162 - 28 - 56$

$6x = 78$

$x = \frac{78}{6}$

$x = 13$

$2x + 28$

$2 \cdot 13 + 28$

$26 + 28$

54

Eu resolvi por ter aprendido o sistema. Eu descobri porque quando eu estava ~~tentando~~ aprendendo o sistema eu fiz varios exercicios parecidos com este

O pequeno mede 13 e o grande 54

Marcel S, AH8

This script shows how deeply an algebraic solution can be guided by the meaningfulness of transformation strategies rather than by any other considerations, ie, how strong a factor the method can become. Marcel's solution has several errors. The first is the failure to distinguish the two unknowns *notationally*, a mistake that we have already examined. Also, the second equation of the bracketed set (top-left) does not model the problem's statement correctly, not even allowing for the interpretation — derived from the first equation — that **x** alone represents the long bar and **x** in "**2x**" represents a short bar. Finally, when he "substitutes" in the second equation the "value" of the left-hand side **x**, he

[55]She might have reasoned that if the long block is 28cm longer than **two** short blocks, it is 14cm longer than **one** short block.

"omits" the 28 that is immediately to the left of the equal sign on the second equation. Nevertheless, he **does** produce a substitution, one that might seem absurd as he had not one, but two equations in one variable that he could easily solve — as he does with the equation resulting from the faulty substitution — and this indicates that although he did not distinguish the two unknowns *notationally*, he apparently did it *semantically*. Moreover, it might be that the 28 was "missed" because in the *Semantical Field* within which Marcel was operating, it was meaningful only when added to the "2x".

The Choc problem

In previous passages, we have already analysed some of the difficulties caused by the use of context-dependent or loose notation. Two attempted solutions to this problem suffer from such shortcomings, but the outcome — although incorrect in both cases — is quite different. Both Tathy G (AH8) and Daniela V (AH8) use the notation "x + 3" for a box and three spare bars, and "x - 3" for a box with three bars missing.

$$\begin{array}{r} 966 \\ -\ 714 \\ \hline 252 \end{array}$$

$(x+3) = 966$
$(x-3) = 714$
$x + 3 = 966$
$x = 966 - 3$
$x = 963$

$x - 3 = 714$
$x = 714 + 3$
$x = 717$

Tathy G, AH8

se uma caixa $x+3$ (mais três chocolates [illegible]) = (custam) 966, o chocolate custa o preço de todos ÷ por 3, ou seja $x = \frac{966}{3} = 322$
$x - 3 = 714$
$x = 717$
Pois ajuntamos os três chocolates que faltavam.

Daniela V, AH8

"if one box $x + 3$ (plus three spare bars) = (cost) 966, a bar costs the price of all of them ÷ by 3, that is, $x = \frac{966}{3} = 322$.

Because we add the three bars that were missing

Tathy treats the two resulting equations separately, and abandons the attempt when she gets different values for x, both equations being correctly solved. On Tathy's solution there is a shift into a *Numerical Semantical Field* immediately after the equations being produced, and this results in the variable "chocolate bar" being simply overlooked and not considered at all after that.

Daniela, on the other hand, stays within the *Semantical Field of the Chocolate Boxes* even after writing — and carefully explaining — the expression "x + 3". She then interprets the situation as meaning that the total price corresponds to the 3 spare bars — disregarding the full box — and divides 966 by 3 to obtain the price of a single bar[56]. However, when she uses the same kind of notation to express the second combination, the strategy does not apply any longer, because it makes no sense to think of sharing the total by what is not. It is only then that she tries to make a new sense of the expression and shifts into a numerical-arithmetical interpretation and correctly solves the equation — as meaningless as it can be in regard to the problem's statement. When she tries to justify the shifted procedure, she says "Because we add the three bars that were missing"; there is a clear disturbance in the meaning of the 714.

Nine students produced a value for the price of a chocolate bar by dividing the difference between the two combinations of box and bars by 3, **WCALC** solutions. The root of this mistake is probably similar to what caused the shift in Daniela's solution: those students knew that the difference in price corresponded to a difference in the number of bars, but considered only the spare bars in the first combination, the bars that "actually" existed. Claire B's (FM3) script is quite clear about this, as she labels the 3 as "...(the number of bars in question)..." Also in Claire's script, we find a forceful example of the subordination of the use of the arithmetical operations to the manipulation of a non-numerical model, as she takes away "...£5.31 from £8.85 to get £3.54..." and from there produces the price of a bar, but "...To check ***this*** [that the price of £1.18 for a bar is correct] I took £3.54 away from £8.85 to get £5.31." (our emphasis)

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[56]We believe that Daniela's flow of thought passed through the feeling that the 3 corresponded to the only thing being actually "counted", "the number of chocolate bars" — forget the "spare" — as the number of bars in a box is unknown and is not mentioned as an element of the problem's statement or question.

First I took £5·31 away from £8·85 to get £3·54. I then devided £3·54 by 3 (the number of bars in the question) to get £1·18. To check this I took £3·54 away from £8·85 to get £5·31. So the cost of a chocolate bar is £1·18.

Claire B, FM3

All but one of the **OKCAL** solutions were of one of two types: (i) putting together the two combinations, with the three spare bars in the first combination "compensating" for the missing ones in the second combination (eg, Clare F, FM3[57]), or (ii) proceeding from the fact the the extra price corresponds to 6 extra bars (eg, Cláudia F, AH7).

£8·85
− 5·31
£3·54

£8·85
5·31
£3·54

8·85
+ 5·31
14·16

2 | 14·16 £7·80

£1·05

3 | 1·05 35

The price for 1 box of chocolates is £7·80
The price for 1 bar is 35p

To solve the problem I added the price of both boxes of chocolates together. This would give you the price of 2 boxes of chocolates. This is obvious. If you take the three spare bars and put them in the box with 3 missing it gives you two boxes

(script restored)

Clare F, FM3

[57]In Clare F's solution we have the "compensation" strategy explained in terms of a possible *physical* action, but most students in the **OKCALC** category did not mention this kind of *rationale* explicitly.

caixa = x   x + 3 = Cr$966,00 e x - 3 = Cr$714,00

Essa caixa com +3 chocolates separados, no final terá 6 chocolates a mais que a outra, porque na outra faltam 3 chocolates e na caixa com +6, ela é completa e ainda tem +3 chocolates.

Cr$966,00
Cr$714,00 -
Cr$252,00 } Preço dos 6 chocolates = diferença entre as caixas

Cr$252,00 | 6
12   Cr$42,00
000

Cláudia F, AH7

"box=x

This box with +3 separate bars, in the end will have 6 bars more than the other one, because in the other 3 bars are missing and the box with +6 is full and has +3 bars.

Price of 6 bars = difference between boxes."

Cláudia uses literal notation, but the intention is clearly descriptive only, as those written expressions are never directly manipulated; instead, the *objects* manipulated are objects of the context, and the model based on which the problem is solved is made up of those objects of the context and and relationships involving them, and perceived properties of both the *objects* and relationships.

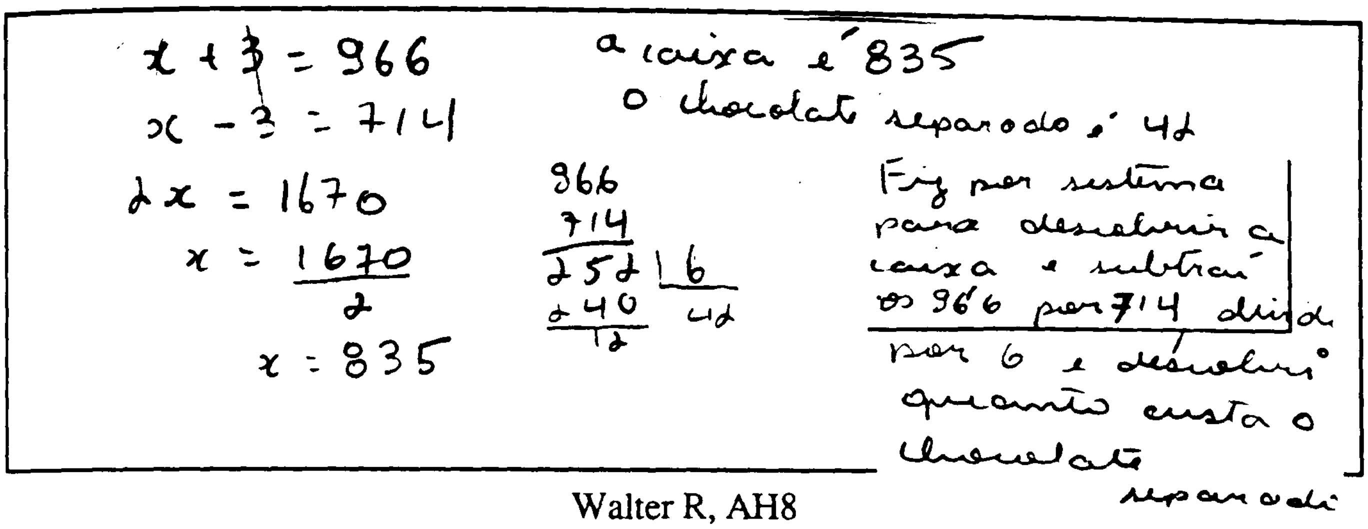
The one **OKCALC** solution that does not conform to types (i) and (ii) above is David W's (FM3).

A box of chocolate bars costs £7.08. The price of a single bar is 59p.
First, I worked out the difference between 885 & 531, and halved it. I then added that figure, (177) to 531. This was to find the price of a box of chocolates. To find the price of a single bar, I just divided 177 by 3.

David W, FM3

His solution to **Choc** is absolutely similar to his solution to **Sets1-3**, and as we argued before on page 254, it seems to be based on a model involving points in a number line (as in figure CCS 2). David is one of the very few students that produced solutions that are clearly non-algebraic but using a model that is not built based on the objects of the context. Moreover, the model he employed here and at **Sets1-3** is perfectly general for this class of problems.

One solution stands halfway between algebraic and non-algebraic. Walter R (AH8) says that he "...solved with a *system* [58] to find out the box [sic] and subtracted the 966 by 714 and divided by 6 and found out how much is the bar."


$x + 3 = 966$
$x - 3 = 714$
$2x = 1670$
$x = \frac{1670}{2}$
$x = 835$

966
714
252 | 6
240   42
12

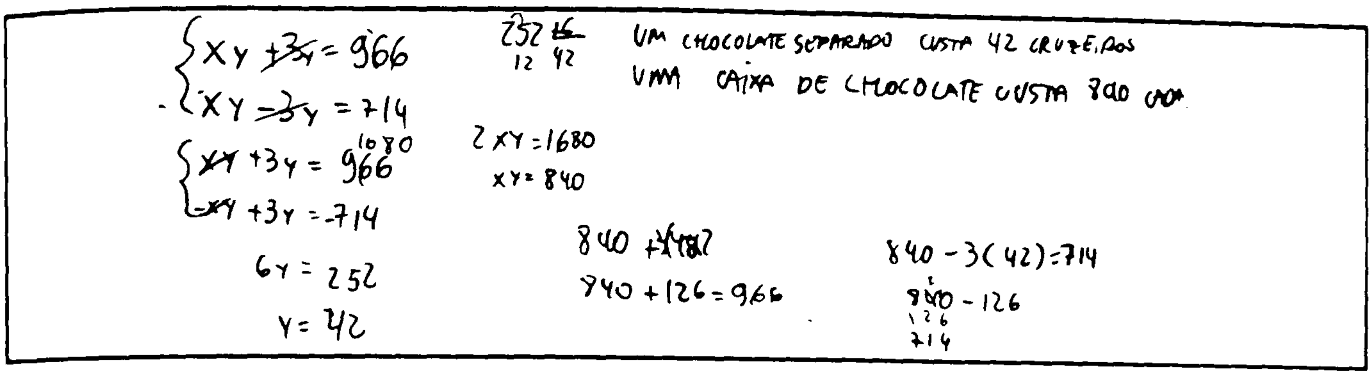
a caixa é 835
o chocolate separado é 42
Fiz por sistema para desolver a caixa e subtraí o 966 por 714 dividi por 6 e desolvi quanto custa o chocolate separado

Walter R, AH8

When he says that used a set of equations, one has an indication of how he classified what he was dealing with, but at the same time the notation is incomplete and one wonders how he would deal with a problem like "a box and three spare bars,..., a box with two bars missing." The fact that he starts afresh to determine the price of a bar, suggests that the he did not perceived the "system" as composed by expressions linking the price of a box and the price of the bars, and we are thus led to believe that he was very much influenced by the *form* of the literal expressions in his choice of method of attack to this first part of the problem.

Only one script actually adds to what we have said so far about **OKEQT** solutions. Giuliano G (AH8) uses absolutely the same method of solution — unique in this group of students — he uses with **Sets1-3**, namely, solving the set of equations twice, once for each unknown, and both by the *addition* method. Moreover, his maturity and confidence with algebraic solutions shows in his use of symbolism: if **y** stands for "(the price of) a bar" **xy** stands naturally for "(the price of) a box of y's", or **x of y**. He is never troubled by this potentially ambiguous notation. Finally, we think it is very significant that from a mature algebraic thinker comes the only script in the whole of this group of problems where the answers are checked against both conditions.

[58]See note 32, p242.



Giuliano G, AH8

SUMMARY OF FINDINGS AND CONCLUSIONS

The analysis of the responses to the problems in this group threw light on many characteristic aspects of both *algebraic* and *non-algebraic thinking*, but also on the ways in which the two modes interact, and on the modelling processes that develop on the border between *algebraic* and other modes.

The issue around which all the others can be organised, is that of *meaning*. Seen in its broader sense — and we think this is the correct approach here — *meaning* is related to the stipulation of which *elements* are to belong to a model and in which way, ie, how they will relate to other *objects* of the model and how those *objects* can be manipulated, or what *properties* they have; *meaning* is related to the constitution of *objects* from *elements*, and inevitably linked to the perception — by the solver — of what could and should be done in order to solve a problem.

In relation to this group of problems, the clearest instance of different ways of producing *meaning* from the *elements* of a problem comes from the **Choc** problem. While a *substitution* strategy involves a strong shift in *meaning* when performed within the *Semantical Field of the Boxes*, it does not when performed within a *Semantical Field of numbers and arithmetical operations*, as we have already seen. Another very important indication of the effect of the types of *objects* that are constituted — and, of course, of the effect of what the solver sees as meaningful in a problem's statement — is in the fact that many students simply could not make sense of the **Sets** problems; taken as arithmetical relationships, they did not provide them with information on how to solve the problem because *to them* arithmetical relationships cannot be constituted into *objects* and manipulated, being rather a form of descriptive, static statement. The other possibility for solving **Sets** problems, modelling them back into another context, ie, interpreting the numbers as measures and the arithmetical operations as *whole-part* operations (conjoining and separating, for example) was thoroughly ignored by the students (only 12% of FM3 did that in **Setsl-1**; no-one else did it in **Setsl-1**, and no student did it in **Setsl-3**). The fact that many students were able to handle — *non-algebraically* — problems with the same *whole-part* structure, shows that the difficulty was in interpreting the arithmetical statements in *whole-part* terms.

Another key element in the direct manipulation of those relationships in **Sets**, the willingness to incorporate unknown numbers or parts into the model and deal with them as if they were known (ie, a willingness to operate *analytically*), was present in none of the non-algebraic solutions. From the examination of the scripts to the contextualised questions, we learned that the lack of *analiticity* is a **consequence of**, rather than a cause to the use of non-algebraic models. Non-algebraic models involved a separation between the objects to be manipulated and the measures involved in the *evaluation* steps; the transformation of a relationship involving two parts of unknown measure can only be meaningful if it enables an immediate or almost immediate *evaluation*. For example, if one knows that "a long block put together with two short blocks measure 162cm altogether", one can derive that "if from the total one removes the long block one is left with two short blocks". Although in terms of *whole-part* manipulation this is an easy step, it does not entail the immediate *evaluation* of any as yet unknown part and is thus, in itself, meaningless in the context of an *synthetic* solutions[59].

Only one student used a non-algebraic, "decontextualised" model[60]. David W's model is clearly *geometrical*. In many instances we could positively identify *non-algebraic* models through their use of objects of the context as *objects* (eg, "cut the extra bit", "move the extra bars to the other box" or "3 bars, the ones that count"), but even on those non-algebraic solutions where this positive identification was not possible — leaving open the possibility of them using a more general *whole-part* scheme, based on a line-diagram, for example — we almost always found that the models used were constrained by limitations very similar to those in a model based on objects of the context (for example, to take 28cm corresponding to cutting the extra bit, but not add 28cm in a hypothetical move), and this characterises a *non-algebraic* model.

Diagrams were used only with **Carpl-1** and **Carpl-2** problems, supporting our conclusion that non-algebraic solutions were almost always *context-based*, as in those contexts bar and line diagrams belong naturally as schematic representations of block combinations. Also, there were more diagrams with **Carpl-2** than with **Carpl-1**, and we think it was so because the greater complexity of the former made it more difficult to be

[59]There would also be another difficulty, in this specific case, that the 162cm is a *measure* to the combination of blocks, and only meaningful in this respect.
[60]That means, out of the original context of the problem

handled without the aid of a representation on paper. The lack of written representation resulted many times in the solver loosing track of the unknowns or of the solution process[61].

In most of the solutions using equations we could reasonably establish that the reference to the problems' context was abandoned, in particular through the generation of expressions where the minus sign could not be given an immediate *non-algebraic* interpretation, but also through a process of manipulation of expressions that could only be meaningful in the context of the algebraic method of solution (not enabling, as we said before, an immediate *evaluation*). The *internalism* of those solutions imply their *arithmeticity*, and as it is reasonable to expect that most of those students would not justify their manipulation of equations on the basis of properties of numbers, this *arithmeticity* means instead a focus of attention on the arithmetical operations as a source of information on what could and should be done to solve the equations, thus the problems.

Much more frequently than not, algebraic solutions were *method-driven*, with the overall control and meaning of the process being related to the process of producing transformations leading to the special form

$$\mathbf{x = f(data)}$$

while non-algebraic solutions were frequently constituted of a sequence of models, each one produced through the evaluation of a part or partial whole and manipulated locally, which in many cases led the students to disregard initial conditions or to introduce new ones. This is not, however, a necessary characteristic of non-algebraic models.

The relevant aspect we could detect in relation to the effect of teaching, is the greater flexibility of AH7 when compared to AH8. The younger AH7 group used mainly non-algebraic approaches where the problems were amenable to them, but were inclined to switch to an algebraic approach whenever they were not, **even when they did not have the necessary technique to deal with the resulting algebraic model readily available**. This effect had already been detected in the previous two sections, but the greater complexity of the questions in this group made it even more clear.

[61]Loosing track of the variables means not being able to correctly associate the result of a series of *evaluations* with the parts or partial wholes it corresponds to; loosing track of the process of solution means disregarding one or more of the initial conditions of the problem.

4.5 THE BUCKETS-SECRET NUMBER PROBLEMS

THE PROBLEMS

From a tank filled with 745 litres of water, 17 buckets of water were taken. Now there are only 626 litres of water in the tank.

How many litres does a bucket hold?
(Explain how you solved the problem and why you did it that way)

Buckets

Question 1

I am thinking of a "secret" number.
I will only tell you that ...

181 - (12 x secret no.) = 97

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Sec+

I am thinking of a secret number.
I will only tell you that

120 - (13 x secret no.) =315

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Sec-

GENERAL DESCRIPTION

The problems in this group were designed mainly in order to check the extent to which a *whole-part* model — the most natural model to use with the **Buckets** problem —

would be used to *model back* **Sec+** and **Sec-**. We expected **Buckets** to be easier than both **Sec+** and **Sec-**, and **Sec+** to be easier than **Sec-**.

The complexity of the problems was kept low in order that issues relating to the choice of model could be highlighted.

DISCUSSION OF POSSIBLE SOLUTIONS

All three problems could be modelled algebraically either directly, with an equation like

$$\mathbf{745 - 17x = 626} \qquad \text{(I)}$$

or first producing a reformation of the problem's situation to produce an equation like

$$\mathbf{17x + 626 = 745} \qquad \text{(II)}$$

corresponding in **Buckets** to the fact that the water taken, together with the water that was left, corresponded to the initial amount of water, and then solving it algebraically. Nevertheless, setting the equation could serve only to make the problem's statement more compact, with the solution proceeding from there non-algebraically.

Non-algebraic solutions for **Buckets** and **Sec+** would probably involve the same model, relying on the perception of a *whole-part* relationship, namely the one leading to equation (II), and solved on the basis that if one removes from the *whole* the *part* that remained, what is left is the *part* that was taken, and this resulting *part* would be *shared* between the 17 buckets or into 13 *parts*. In relation to **Buckets**, the procedure is very much analogical and requires no further modelling or interpretation; in relation to **Sec+**, there has to be an interpretation of the subtraction as "removal" and from there the *whole-part* relationship is established.

This model, however, is obviously inadequate to **Sec-**, and because it is impossible to avoid the acceptance of negative numbers at some point, this problem is naturally closer to the *Semantical Field of numbers and arithmetical operations*. This inadequacy accounts, in fact, for much of the importance of this group of problems in relation to the whole set of test problems; the low level of complexity allows us to better examine the effect of the "push" towards the *Semantical Field of numbers and arithmetical operations*. The two subtraction items involving negative numbers (**25-37** and **20-(-10)**) were designed to provide supporting information to the analysis of the responses to these problems and those in the group analysed on section 4.6, one of which also involves a negative number as the answer.

As we expected, a clear hierarchy emerged, with **Buckets** being the easiest problem, then **Sec+**, and **Sec-** being the most difficult. The differences in the facility levels were significant in all cases but between **Buckets** and **Sec+** in AH8 and in FM2, a fact that we will closer examine ahead. AH8 was the only group where the level of facility for **Sec-** was high (71%, against 14%, 15% and 17% for AH7, FM2 and FM3 respectively), and it is very significant that all those correct answers in AH8 were produced by solving an equation. As with all the previous problems we have analysed, the level of use of equations by FM2 and FM3 was very low.

The flexibility in the choice of approach previously shown by AH7 is also present here in a very clear manner. Although the facility level falls from **Buckets** to **Sec+**, the huge fall in the number of **OKCALC** solutions is compensated by an increase in the number of **OKEQT** solutions; moreover, on **Sec+** two-thirds of the incorrect answers are **WCALC**, but on **Sec-** this situation is more than reversed, with three-fourths of the incorrect answers being **WEQT**, and this is a good indication of their willingness to switch to an algebraic model when the non-algebraic models are not enabling them to solve the problem. AH8 also show some flexibility here, with almost two-thirds of their correct answers to **Buckets** being **OKCALC** solutions. On the **Sec** problems however, **all** their correct and incorrect solutions use an equation; the use of an algebraic approach is certainly responsible for the high level of facility for **Sec-** in AH8, indicating that in the case of this problem it represents indeed a more powerful tool for solving it than non-algebraic approaches. This will be examined more closely on the students' solutions.

Because **Buckets** and **Sec+** have an identical *whole-part* structure, the difference in the facility levels strongly suggests that many students could not interpret the arithmetical subtraction as a *removal* to produce a situation similar to the one in **Buckets**. Given that many students correctly used in those and previous problems a subtraction to *evaluate* the result of a *removal*, a subordination of the use of the arithmetical operation to the perception of the a *whole-part* model is established in this case, as opposed to some form of more or less symmetrical correspondence between *subtraction* and *removal*.

The Buckets problem

By far, the typical correct solution to this problem was an **OKCALC** solution. In most of those (38 out of 59 **OKCALC** instances) some explanation was given, making reference to the fact that to know how much had been taken on the buckets one had to subtract what was left from the initial amount of water (eg,Fabiana M, AH7; Sidnei A, AH7; Alexander P, FM2; Rebecca H, FM3).

$$\begin{array}{r} 745 \\ -\,626 \\ \hline 119 \end{array} \qquad \begin{array}{l|l} 119 & 17 \\ \hline 00 & 7 \end{array}$$

Pensei... se tinha 745 e agora tem 626, quer dizer que tiraram 119 l de água em 17 baldes.
l ÷ quantidade = l.

Fabiana M, AH7

"I thought... if there were 745 and now there are 626, it means that 119 l. of water were taken on 17 buckets."

$$\begin{array}{r} 745 \\ 626 \\ \hline 119 \\ 7 \end{array} \qquad \begin{array}{l|l} 119 & 17 \\ \hline 00 & 7 \\ & 7 \end{array}$$

Fiz esse conta p/ saber quantos litros sai da caixa d'água

Eu usei esse conta porque se saiu 119 litros no total o lógico que tenho que dividir p/ saber quanto litros entram em cada balde

Sidnei A, AH7

"I did this sum to know how many litres were taken from the tank. [at the left of script]

I did this sum because if 119 litres were taken altogether [,] the ***logical*** thing [is] that one would have to divide to know how many litres go into each bucket."

To find how many litres does a bucket hold we first have to find how many litres were taken from the tank
745 - 626 = 119 litres
119 litres were taken. There were 17 buckets. So each bucket hold $\frac{119}{17}$ litres of water. The answer is 7 l.

Alexander P, FM2

I took 626 away from 745 which left me with 119 divide 119 by 17 and got 7. I did this because if you take 626 away from 745 you get the amount of water taken then divide by 17 because there are 17 buckets.

Rebecca H, FM3

Sidnei's reference to "the *logical* thing to do" seems to be his way of saying that no explanation is necessary as to why it is so. In all four scripts the subtraction part of the procedure is taken as self-evident; in no case an explanation is provided as to why this subtraction correctly provides the amount taken, not in verbal terms nor using some kind of diagram. Also, in none of the solutions the intermediate step of saying or showing that the amount taken plus the amount left corresponded to the initial total amount was taken. Altogether, this is an exceptionally strong indication that the direct procedure was perceived as an intrinsic property of the situation and the explanation would only have to indicate which numbers corresponded to which "roles." Similarly, no explanation was ever provided as to why the division by 17 produced the amount taken on each bucket.

Only six solutions used equations, five correctly solved and one incorrectly solved. Flávia C (AH7)[62] first makes a mistake by writing 75 instead of 745 on the initial equation; then, instead of the correct — in that context — 75-626 subtraction, she does 626-75. This "corrective" manipulation probably corresponded to the perceived need to produce a positive number as the answer or to a pre-equation perception of the calculations required to solve the problem. The latter seems to be a better interpretation, as hers is the only of the six solutions using the equation

[62]The text on Flavia's script simply explains that "17x...means...17 times x."

$$a - bx = c$$

where the first step of the solution leads to

$$bx = d$$

and not to

$$-bx = d$$

strongly suggesting that her solution uses algebraic notation but is guided by a *whole-part* model as in the **OKCALC** solutions examined above, and the 626-75 subtraction simply corresponds to "initial total minus remaining water", where the smaller of the two numbers obviously had to play the role of "remaining water".

$75-(17.x)=626$

$17x = 551$

$x = \frac{551}{17}$

$x = 32,4$

Você fazendo 17x, quer dizer que foram tirados 17 vezes x de água de um tanque de 745 litros, que foi para 626 litros.

R. Cabem 32.4 litros em cada balde.

Flávia C, AH7

In only one of those six solutions using equations, Andrea T's (AH8), the initial equation does not correspond literally to the problem's statement, corresponding instead to the statement "the water in the buckets together with the water that remained is the water one had originally" — obviously derived from the problem's statement.

$17x + 626 = 745$

$17x = 119$

$x = 7$

R: cabem 7 litros em cada balde

explicação - eu somei 17 baldes multiplicado por x, porque eu não sei a quantidade de água que contém em cada balde, com a água que restou e dei como resultado a água que tinha antes

Andrea T, AH8

"explanation- I added the 17 buckets multiplied by x, because I don't know the amount of water in each bucket, with the water that was left, and [I] gave as the result the water that was there before."

Andrea's procedure displays a characteristic similar to the *direct non-algebraic substitution* procedure we examined in relation to the problems in the Choc-Carp group of problems, by manipulating a non-algebraic model first, and then producing an equation from there. All other four **OKEQT** solutions proceeded without going through the equation

$$17x + 626 = 745$$

preferring instead to operate directly with the negative coefficient of **x** (eg, Ana RW, AH8). In Andrea's script we also find a clear example of the *analiticity* and *arithmeticity* of algebraic solutions.

$745 - 17x = 626$ $\qquad$ $745 - 17(7) = 626$
$-17x = 626 - 745$ $\qquad$ $745 - 119 = 626$
$(-1)\ -17x = -119$
$\boxed{1x = 7}$

Cabem 7 litros em cada balde.
Encontrei o valor através de uma equação.

Ana RW, AH8

The seven **WCALC** solutions do not provide any interesting insight or instance.

<u>The Sec+ problem</u>

Characteristic of the **OKEQT** solutions is that here — as before with the **OKEQT** solutions for **Buckets** — in all cases but one the equation initially set corresponds directly to the problem's statement. Also — and more important, given that the problem's statement directly suggest a specific equation — in all instances, the solvers accepted and dealt with a negative coefficient for **x**, rather than first producing the transformation into

$$181 = 12x + 97$$

In two **OKEQT** scripts are displayed peculiar aspects of thinking algebraically. First, in Fabio C's (AH7) solution, one sees the constitution of a new *object* (**12x**), meaning more than a syncopated notation for the multiplication — even if slightly more; in his solution Fabio *operates arithmetically with the unknown.*

resolução

$$181 - (12 \cdot x) = 97$$
$$181 - 12x = 97$$
$$-12x = 97 - 181$$
$$-12x = -84$$
$$x = 7$$

Eu primeiro resolvi a operação que estava entre parênteses ($12 \cdot x = 12x$) depois eu resolvi o resto do problema como se fosse uma equação normal.

Fabio C, AH7

"First I solved the operation in brackets ($12 \cdot x = 12x$) then I solved the rest of the problem as if it were an ordinary equation."

Christiane T's (AH8) script is a fine example of the *method-driven* aspect of algebraic solutions, as she multiplies the second equation by -1 even before performing the calculation on the right-hand side of the equation, in a sense treating the known numbers as unknown ones, but actually showing the extent to which the distinction between known and unknown numbers has faded.

$$181 - 12x = 97 \qquad x = 7$$
$$(-1)\ -12x = -181 + 97$$
$$12x = 181 - 97$$
$$12x = 84$$
$$x = \frac{84}{12}$$

Eu resolvi com a forma de equação.

Christiane T, AH8

In three scripts algebraic notation is used but the solution process is not algebraic. Célia R (AH7) solves the problem by first restructuring into the equivalent of "the amount that was taken corresponds to the difference between the initial and final amounts"; from there she writes and solves an equation, and one cannot positively decide whether there was a shift into the *Numerical Semantical Field* or whether she was using literal notation to describe a non-algebraic solution. In any case, the main step that allows her to *evaluate* **x** — the manipulation that led to the first equation — was most likely based on the perception of the *whole-part* relationship. In the other script the situation is much more clear, as Marcelle D (AH7) writes down the equation directly derived from the problem's statement, but the rest of the solution is void of further use of literal notation, and the solution process corresponds directly to one guided by the *whole-part* relationship. Finally, Gil S (AH7) uses literal notation only to express the general form of the procedure he used, possibly as a way of justifying it; we think that on the light of what we have said so far, there should be little doubt that his solution was guided by a *whole-part* relationship.

~~Então~~ 181 − (12 × 7) = 97
181 − 84 = 97

$12x = 181 - 97$
$12x = 84$
$x = 84 : 12$
$x = 7$

Célia R, AH7

nº secreto = x
181 − 12 · x = 97
181 − 97 = 84
84 ÷ 12 = 7 (resto 0)
181 − 84 = 97

Marcelle D, AH7

181 = Nº
Nº secr = X
97 = R
12 × Nº sec = S
Nº − (12 · x) = R
Nº − R = S ; S : 12 = nº secreto X
O NÚMERO SECRETO É 7

181 − 97 = 084
84 ÷ 12 = 7 (resto 0)

Gil S, AH7

In most of the **OKCALC** solutions the explanation provided indicates that the *whole-part* relationship was on the basis of the solution process (Simon J, FM3; Sarah G, FM3; Marcelo A, AH7; Leandro F, AH7; Jennifer J, FM3).

The secret number is 7
I took away 97 from 181 = 84. This leaves me with
the amount of A, A is 12 × something so
if I divide 84 by 12, it will give
me the number

Simon J, FM3

$181 - 97 = 84$
$12 \times ? = 84$
$12 \times (7) = 84$
$181 - (12 \times 7) = 97$

Sarah G, FM3;

181
−97
084 ÷ 12
07 número secreto

Eu subtraí 97 de 181 para saber qual era o outro fator, e posteriormente dividi este número por 12 para descobrir o número secreto.

Marcelo A, AH7

"I subtracted 97 from 181 to know which was the other factor..."

$181 - 97 = 84 \div 12$
07

Eu resolvi porque se o resultado = 97 [illegible] e então [illegible]

Leandro F, AH7

"I solved [it like this] because if the result=97 then 181-97 will give the result of the multiplication..."

181 − 97 = 84 thats the sum for the secret number. 84 ÷ 12 = 7 so that must be the secret number.

Jennifer J, FM3

It is central that the form in which it is expressed is of no importance, as the *decomposition* process is always clearly visible. The use of a letter (the "A" in Simon's script), a verbal specialised term ("factor", in Marcelo's), or a more or less standard, non-literal notation (the question mark in Sarah's) do not make the solution essentially distinct from those using verbal, relatively neutral references ("the multiplication" in Leandro's or "the sum" in Jennifer's). As with the **OKCALC** solutions to **Buckets**, there was never any explanation as to why the subtraction would produce the remaining the value of the remaining *part*.

In some of the incorrect solutions the source of the errors can be traced back to the use of loose and incorrectly generalised, verbally formulated rules like "undo it using the inverse operation" or the rules for the manipulation of algebraic expressions (Rebecca H, FM3; Sukhpal S, FM3; Ana Lúcia E, AH7). Nevertheless, in this kind of behaviour one can identify the focus of attention being at the arithmetical operations — even if it does not result in correct procedures — and this evidences at least a willingness to limit one's attention to the arithmetical context, a necessary aspect if one is to operate within the *Semantical Field of numbers and arithmetical operations*.

Secret no

i took the 97 and + it by 181 because it is the opposite of ~~-~~ and with that number ÷ 12 because it is the opposite of ×.

Rebecca H, FM3

~~\$~~181 − (12 × x) = 97

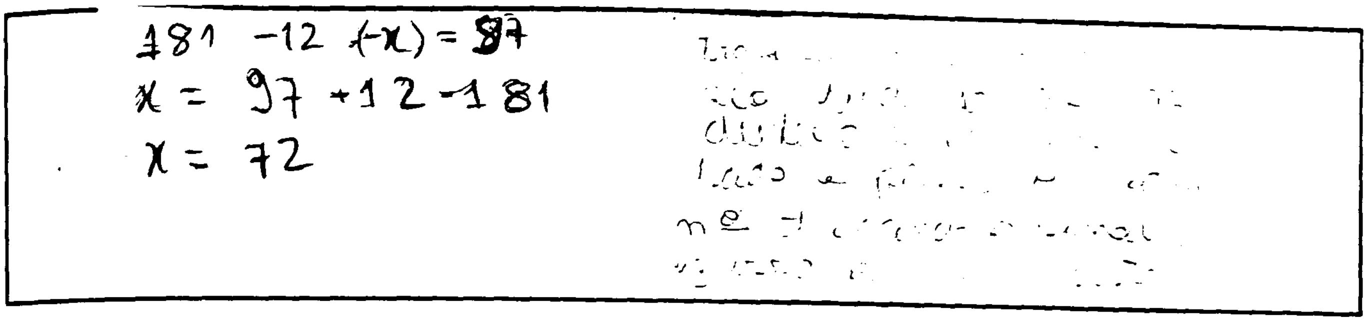
= 97 ÷ 12 − 18 = x

= 12)97 = 8 r1 + 181 = 207r1

= x = 26r1

= secret No. = 207r1

Sukhpal S, FM3

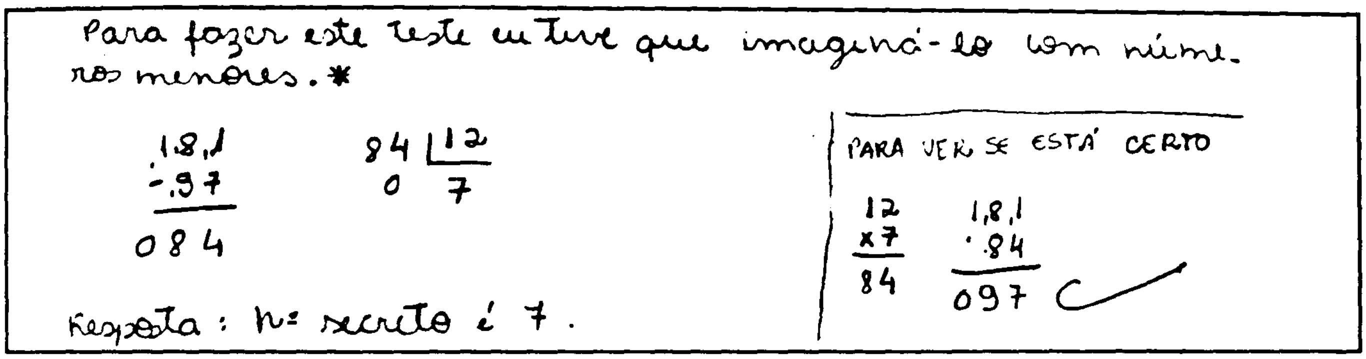


$181 - 12(-x) = 97$
$x = 97 + 12 - 181$
$x = 72$

Ana Lúcia E, AH7

"I changed the sign of the parenthesis..." [as if it were an addition or subtraction instead of a multiplication]

One script in this group is of interest to us, because it employs a unique approach (Cecília B, AH7).



Para fazer este teste eu tive que imaginá-lo com números menores. *

$181 - 97 = 084$ $\quad 84 \div 12 = 7$, resto 0

PARA VER SE ESTÁ CERTO

$12 \times 7 = 84$ $\quad 181 - 84 = 097$

Resposta: nº secreto é 7.

Cecília B, AH7 (solution to Sec+)

"To do this test I had to imagine it with smaller numbers"

on the left, parallel to the margin: **36 - (2 · secret no.) = 20**

on the right-hand corner: "to see if it's correct"

From the simpler example, Cecília works out the string a calculations that leads to the solution of the equation, and simply applies it to the original numbers. On one hand, her solution seems to rely completely on insights emerging from the simpler example; the solution is thoroughly *synthetical*. On the other hand, she easily accepts that the "algorithm" can be applied to a problem from which she did not feel able to derive the solving steps, ie, that the range of numbers to which it applies is not dependent on properties of the small numbers on the "exemplary" case and the relevant factor is the *numerical-arithmetical* structure. Even more striking, Cecília applies exactly the same method to solve **Sec-** (script also shown bellow), and the "simpler problem" she uses with **Sec-** is not, as one might have expected, in direct correspondence with its statement, where the "result" (ie, the number on the left-hand side) is greater then the "starting number" (ie, the number from which a multiple of the secret number is subtracted). The "simpler problem" she invents is

$$20 - (4 \times \text{secret no.}) = 12$$

from which, knowing that the secret number is 2, she correctly derives the solving algorithm as

$$\text{secret no.} = \frac{20-12}{4}$$

The crucial step, thus, is that she puts in correspondence the numbers in this model with the numbers in the problem's statement, *regardless of the fact that in Sec- the "result" is greater than the "starting number,"* and correctly applies the algorithm, paying attention to the order of the terms in the subtraction and of the sign of the final answer. It is clear that the process is carried out completely within the *Semantical Field of numbers and arithmetical operations*, as control of the operations depends totally on the *arithmetical articulation* of the paradigmatic expression. Hers, however, is not an *algebraic* solution, as it is *synthetical* by the very nature of the solving technique.

Para descobrir, eu inventei este outro problema:
20 - (4 x nº secreto) = 12. Sei que o nº é 2. Então
eu vi como se pode, com esses números chegar em 2.
ENTÃO:
120 - 315 = -195
-195 : 13 = -15
Resposta: o nº secreto é -15.

Cecília B, AH7 (solution to Sec-)

"To find out, I invented this other problem:

20- (4 x secret no) =12. I know that the secret number is 2. So I saw how one

can, with those numbers, to get to 2.]

Then..."

Finally, we have Melissa R (FM3). The first step of her solution — *evaluating* "what is between the brackets" — seems clearly based on the *whole-part* relationship. The second step, however, instead of representing an *evaluation* of the *sharing* is explicitly a manipulation of the newly established relationship, namely **12·x=84**, based on syntactical transformation. We would not go so far as to say that she was fully aware that the "reversing of the multiplication **sign**" stands in fact for a property of the operation, but the source of information on what to do next was certainly the *numerical-arithmetical* expression, in particular the multiplication operation. We have thus a mixed solution. When she solves **Sec-** (script also shown bellow, together with the script for **Sec+**), she first concludes for the answer being 15 and only then adjusts the answer to -15 in order for it to fit the problem's statement (15 is encircled at the top-left corner of the script, and the minus sign at the end of the string of calculations on the first line was certainly inserted afterwards, looking "squeezed" between the equal sign and the number); the adjustment is made by assuming that the 195 had to be negative (and she puts a minus sign to the left of 195 on the first line, which is later obliterated). Her solution does not proceed through successive transformation of equations, but much of it is clearly performed within the *Semantical Field of numbers and arithmetical operations*; again, Melissa shows flexibility in mixing different models, but she is successful only due to the extreme care taken in seeing that the overall result was adequate in relation to the original condition set on the problem's statement.

The secret no. is 7. Take 94 from 151 because you want to get a no. to help you with what is between the brackets. The no. you get is 84. You then reverse the multiplication ... division and divide 84 by 12 to get the answer

Melissa R, FM3 (solution to Sec+)

15   315 − 120 = -195 ÷ 13 = -15

Do the first to get the multiplication number

reverse the multiplication sign and divide by 13 to get the secret number.

13 × -15 = -195   120 − -195 = 315!

Melissa R, FM3 (solution to Sec-)

The Sec- problem

The main difficulty in dealing with this problem using non-algebraic models is that the *whole-part* model that worked so smoothly with **Buckets** and **Sec+** simply does not make sense in this case, as Daniel S (FM2) puts it.

This is impossible because 315 is already higher that 120, so you can't take 13 x secret no. because it will end up even smaller
But it could be a minus number.

Daniel S, FM2

The observation at the bottom line might indeed serve as the seed of a corrective approach that can be used to make a *whole-part* useful. By assuming the secret number to be negative, one immediately has that the subtraction notationally indicated is not "in fact" a subtraction, but an addition, and the problem is reduced to

$$\mathbf{120 + (13 \times secret\ no) = 315} \qquad \text{(equation I)}$$

which can be easily solved with the help of a *whole-part* model. In Mi P's (FM3) solution the minus sign is added to the answer only after the "amount" is found; Sophie W (FM3) on the other hand, worked out the value of **13xsecret no** to be **-195** and proceeded from there by dividing it by **13**, as also did Jennifer J (FM3, script not shown). In both Mi P's and Sophie's solutions the main step relies on a property of numbers, but the use of the *whole-part* relationship is also crucial. The perception that the secret number is negative expresses not only the *numerical* treatment of the problem, but also some degree of *analiticity* in the approach, as the secret number — yet unknown — is taken as having a property, which means it has been made into an *object*.

120 - (13 x -15) = 315
I thought the number had to be a
minus so it could equal 315.
All I had to do was find the difference
between 315 - 120 = 195.
Then I had to find 13 times whatever the number
was to equal 195 (15)

Mi P, FM3

-195 13 x -15
15
13)195
315
~~-120~~
~~195~~
13x15=195
+35
~~195~~
+ ~~120~~
~~13~~
-15

I worked out that
the secret no needed
to be a negative
number then I worked
Secret number out that is the secret no
had to equal -15 so
[illegible]

Sophie W, FM3

Attempts to use a *whole-part* model lacking the perception that the secret number is negative, led to two types of error. In eight cases the solver simply assumed that **315** corresponds to the *whole* and that **120** and **13·secret no** correspond to the *parts* (eg, Marcelo A, AH7), as if the problem said

315 - (13 x secret no) =120

and the problem is solved as **Sec+** would be using a *whole-part* model.

$$\begin{array}{r} 315 \\ -120 \\ \hline 195 \end{array} \qquad 195 \,|\underline{13} \quad 65 \quad 15$$

Primeiro eu subtraí 120 de 315 para saber qual era o número dos parêntesis e depois dividi este número por treze.

número secreto

Marcelo A, AH7

"First I subtracted 120 from 315 to know which was the number in the brackets and then divided this number by thirteen."

He encircles 15 and writes "secret number"

We can safely conclude that this inversion is caused by the "meaninglessness" of the original statement in terms of wholes and parts, as expressed by Daniel S two paragraphs above, representing an attempt to make sense of the situation, as **all** eight student who produced this type of solution had solved **Sec+** using a *whole-part* model. Another inversion produced by students in the problem's statement was to take the subtraction

120 - (13 x secret no)

as actually indicating

(13 x secret no) - 120

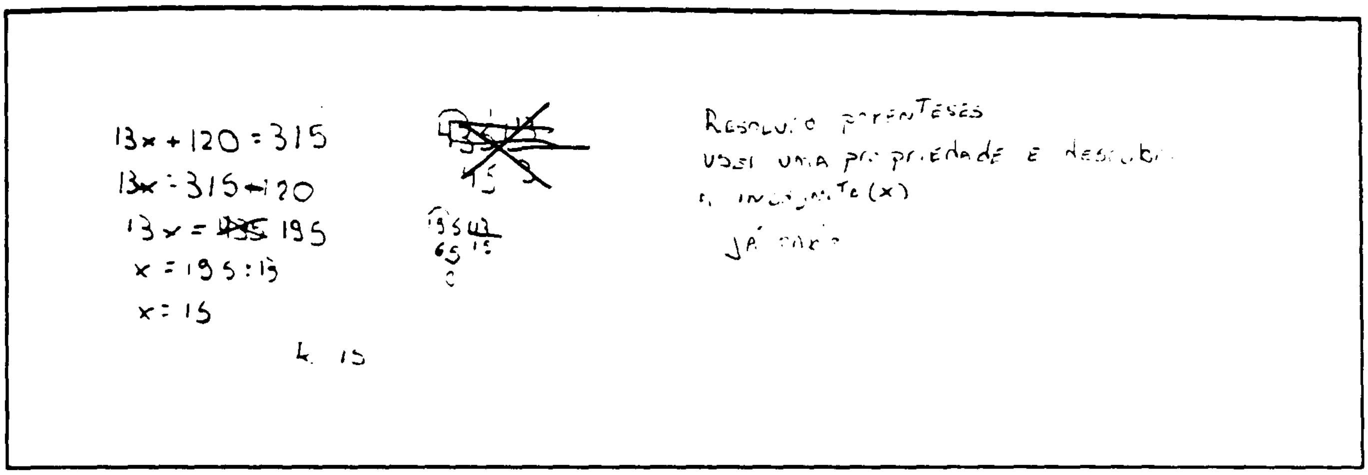
which also restores the meaning in terms of wholes and parts (David B, FM3).

315 + 120 = 435
435 - 13 = (secret number) 33.46
I added 120 onto 315 because in the sum it has to be taken away. Then I divided it by 13 as the secret number is timesed by 13.

David B, FM3

Five students produced this type of solution; only two of them had correctly solved Sec+, both **OKCALC** solutions, one was a **T&E** solution, one was **NATT**, and in one case a similar error was made there as here. If one thinks in terms of a hierarchy, it seems that incorrectly reversing the terms of the subtraction (second type of error) represents a cruder error than adjusting the roles of the numbers involved (first type of error), as the students doing the latter error seemed to be operating much closer to a consistent model for dealing with problems of this kind[63].

To one of the students, Luís N (AH7), the drive to make sense of the problem's statement in the context of whole and parts was so strong that he simply "corrects" the statement, to produce equation I we showed a few paragraphs above, without realizing that the number coming from the new equation would have to be adjusted to fit the problem's condition[64].



Luís N, AH7

"I solved the brackets

used a property and found out the unknown (x)

I already knew it [how to do it]"

Marcelle D (AH7) uses algebraic notation; at first sight it might seem as if she simply misapplied rules for the manipulation of equations[65]. On the light of the analysis of the previous few paragraphs, however, we are led to conclude that in fact she made sense of the equation by producing the same reversion of the subtraction as David B above. Her solution to **Sec+** also begins with an equation, but proceeds with calculations only.

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[63]Disregarding the order of the terms in a simple subtraction is a mistake that has been identified by several researchers, and it might have contributed to making the mistaken reversing more acceptable to those students.

[64]It is impossible to decide from the script only whether he solved the resulting equation by thinking algebraically or whether he stayed with the whole-part model, but because of the seemingly cause for the "correction", we would — more as a matter of exercising interpretation than as a matter of this decision being relevant — prefer the latter interpretation.

[65]Namely, "change sides, change sign", with the "-" sign seen as "belonging" to **120**.

$$
\begin{aligned}
120-13x&=315\\
13x&=315+120\\
13x&=435\\
13x&=435\div 13\\
x&=33
\end{aligned}
$$

Marcelle D, AH7

As it happened on **Sec+**, almost all **OKEQT** solutions reached at some point the equations

$$\mathbf{-13x = 315 - 120 \text{ or } -13x = 195} \;;$$

in most of them the solver multiplied both sides by -1 (Flávia C, AH8) to obtain

$$\mathbf{13x = -195}$$

and in a few cases the solver carried on with **-x**, dividing first by 13 and only at the end reversing the signs on both sides. Fábio C (AH7) directly reaches an equation of the form **13x=...**, but this step is justified in terms of the process of solving the equation, and not in terms of a relationship derived from the initially given *whole-part* relationship. It is significant that this form of control of the process results in a correct derivation, while Marcelle — even with the support of literal notation — and other students whose solutions were guided by a *whole-part* model failed. By shifting the *meaning* of the process into one closely related to the *method of manipulation of the expressions*, away from the context of *evaluation of measure of parts*, Fábio's approach overcomes the difficulties involved in making sense of this problem within a *whole-part* semantic.

$$
\begin{aligned}
120-(13x)&=315\\
120-13x&=315\\
-13x&=315-120\\
-13x&=195\\
(-1)\quad 13x&=-195\\
x&=-\frac{195}{13}\\
\boxed{x}&\boxed{=-15}
\end{aligned}
$$

R. Eu substituí o número secreto pelo "x" e resolvi a equação de 1º grau.

Flávia C, AH8

$120 - (13 \times \overset{\text{nº secreto}}{x}) = 315$

$120 - 13x = 315$

$120 - 315 = 13x$

$-195 = 13x$

$x = \frac{-195}{13}$

Resposta → $\boxed{x = -15}$

Eu resolvi como se fosse uma equação.
Eu, primeiro resolvi o parêntesis, depois eu passei o nº secreto (x) para um lado e os números para o outro, aí é só resolver a equação.

Fábio C, AH7

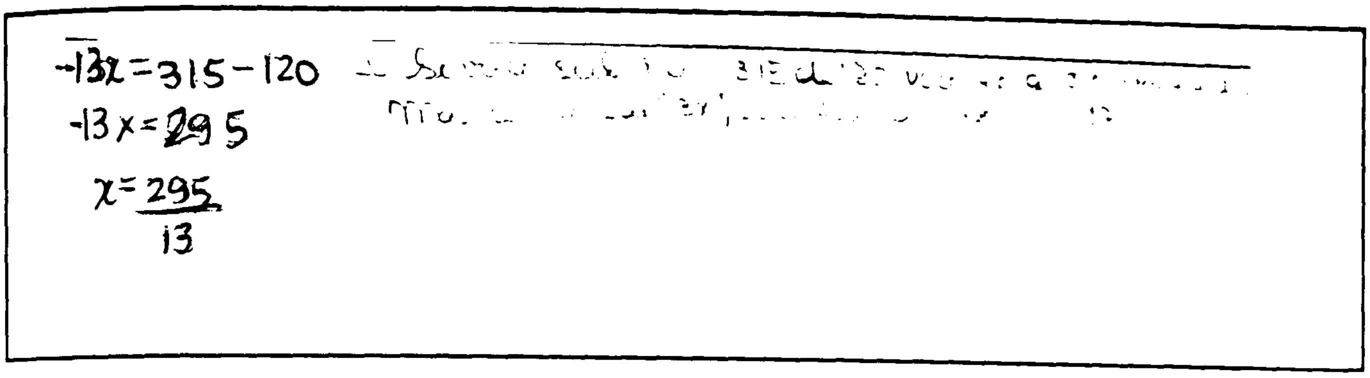
"I solved as if it were an equation.
First I solved the brackets, then I moved the secret number (x) to one side and the numbers to the other, then it's only to solve the equation."

In several **WEQT** solutions, the solver arrives at either

$$\mathbf{-13x = 195} \quad \text{or} \quad \mathbf{13x = -195}$$

only to produce **15** (instead of **-15**) as the answer. Difficulties with the division involving a negative number could certainly be responsible for the incorrect result, but one script suggests another possible source for it (Ana C, AH8). Although keeping the algebraic correctness at a syntactical level — in this case, keeping the coefficient of x negative — it is possible that the model underlying the reasoning was in fact based on the perception of a *whole-part* relationship; in Ana's script this is indicated by the fact that she refers to "the number '**x**'" — probably a reference to the amount taken — and also to it being "'13**x**'", but she never refers to the negative coefficient or to the fact that her reasoning would have to be complemented by something like "but in fact each **x** is negative". The perception that the result had to be a negative number did not come from the awareness that "I subtracted something and it got bigger" nor from the recognition that the coefficient was in fact **-13** and not **13** — and thus the divisor would have to be **-13** were she "reversing" the multiplication. Both aspects being essentially *numerical-arithmetical*, this lack of understanding supports the case that the model underlying her solution process was indeed a non-algebraic one. Ana's solution to **Sec+** (script bellow) is similar in this respect to the solution to **Sec-**, as she correctly keeps the minus sign but does not deal directly with it (when most **OKEQT** solutions did), and the process produces a correct result only by virtue of the "friendliness" of the problem; the written explanation certainly corresponds to a solution guided by a *whole-part* model[66].

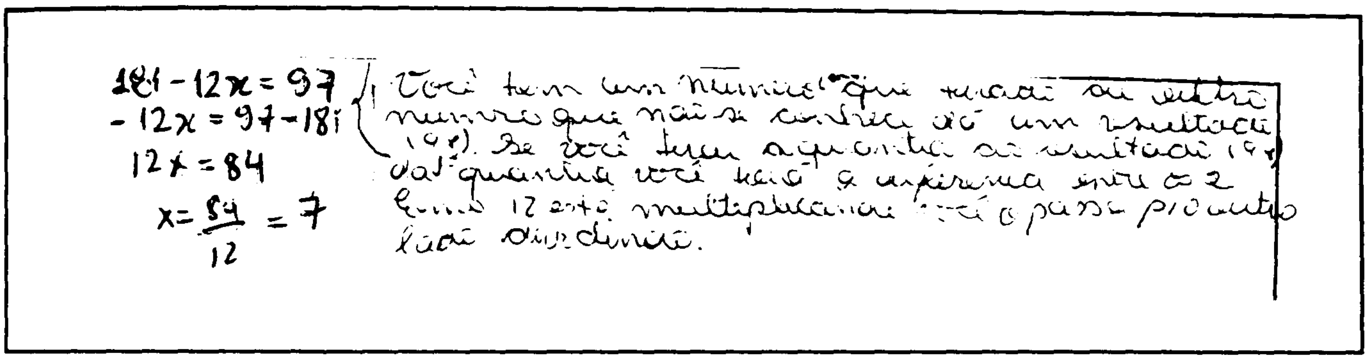
[66]One might argue that she justifies the division as reversing the multiplication and this brings the solution closer to an algebraic one, but we think the crucial and characteristic step here is deriving **12x=84** from the initial statement, as in algebraic terms this would involve directly manipulating the unknown.



$-13x=315-120$
$-13x=295$
$x=\frac{295}{13}$

Ana C, AH8 (solution to Sec-)

"If you subtract 315 from 120 [sic] you'll have the number "x". But as there "13x", you have to divide by 13."

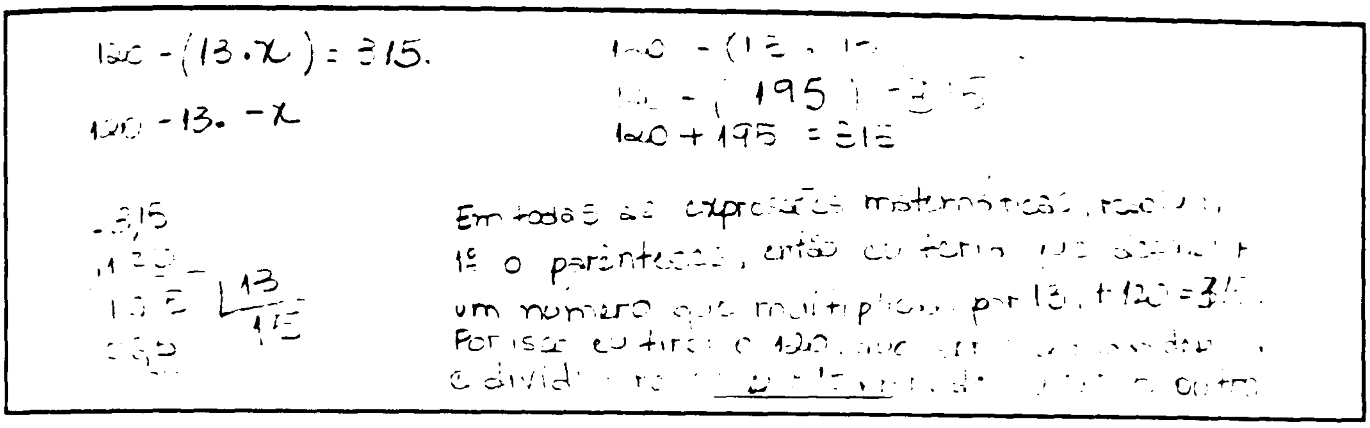


$181-12x=97$
$-12x=97-181$
$12x=84$
$x=\frac{84}{12}=7$

Ana C, AH8 (solution to Sec+)

"You have a number (181) that taken <u>from</u> the unknown number [our emphasis] gives a result (97). If you take the amount of the result (97) from the 1st amount, you'll have the difference between the two. As 12 is multiplying, you move it to the other side dividing."

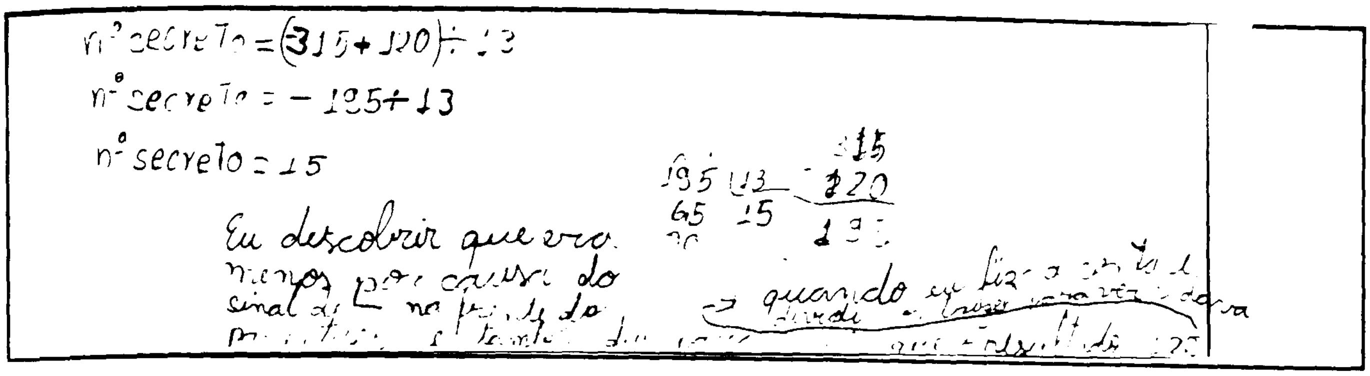
Fabiana M's (AH7) script is very interesting for several reasons. At first she tries setting and solving an equation, and it seems that she tries to "distribute" the minus sign over **13x** (top-left corner); as the resulting expression is not *meaningful* to her, ie, she cannot get information on how to proceed with the solution from it, she shifts to another model, which is clearly based on a perceived *whole-part* relationship. From the verbal explanation we learn that she had already transformed the problem — inadequately — into one equivalent to the additive equation I some paragraphs above ("...a number that multiplied by 13, +[!]120=315..."). We think it is extremely significant that the model takes control of the solution process to the extent that the simple arithmetic rules are subordinated to its *semantic*; it is enough to observe that on the three lines of expressions (top, center-right), the subtraction notationally indicated is never meant to be one, as it is revealed on the third line. Fabiana had solved the item **25-37** correctly, which indicates that the disregard for the rules of arithmetic were not a mistake but part of *operating in another Semantic Field.*



Fabiana M, AH7

"In all mathematical expressions we first solve the brackets, then I would have to find out a number that multiplied by 13, +120=315. That's why I took the 120, that would be added later, and divided the rest by 13 to find out the other number."

Leandro F's (AH7) solution offers us a rare instance of algebraic thinking without manipulation of literal notation or algebraic expressions. The expression he derives for the secret number is correct, and it takes into account that if the secret number is to have a positive coefficient — or, as he would possibly put it, "for the secret number to be 'positive'" — the correct subtraction is **120-315**, and he also uses the brackets correctly. We think Leandro's solution is substantially different from those in which an awareness that the secret number was negative existed but the solution process proceeded within the context of the additive equation, and this difference is clearly shown by the fact that from the beginning the terms involved in the calculations he indicates are correctly signed; there is no transformation of the problem with an adjustment *a posteriori* to fit the original condition of the problem. His verbal explanation is very confuse, and almost nothing more can be gathered from it; we produced a very literal, almost word-by-word translation in order to convey this state of things. For all we said above, the fact that his final answer is **15** and not the correct **-15** only supports our interpretation, once it indicates that he was not aware beforehand that the answer had to be negative, and produced the necessary transformations on the basis of his perception of the numerical structure of the problem's statement.



Leandro F, AH7

"I found out it was minus because of the - sign in front of the brackets and also it was possible to know that the result-120 and when I did the calculation and divided by thirteen to see if it would be possible."

Finally we examine Vicky H's (FM3) script. There are two points of interest. First she rewrites the problem's statement using letters not only for the unknown, but also for known numbers. According to our traditional usage, she is not distinguishing the known numbers substituted from the unknown one, as the choice of letters seems to indicate a mere sequential A-B-C from left to right. On the other hand, she distinguishes A and C as having a different role than **13**, which she left as a definite number. We think that she was trying to put the problem's statement in a general form from which she could derive a pattern and a solution procedure. The generalised form she attained appears to bring three things into consideration:

(i) a possible *whole-part* model, which does not fit back into the problem's statement, as C<A (and she crosses out the generalised expression)

(ii) the perception that the subtraction had in fact to represent an increase, and thus an addition (and she concludes that "275 are needed"), and

(iii) the perception that the secret number had to be negative in order for the subtraction to result in an addition (and she gives as the answer -2.5).

There is no reference as to how she found those numbers, which are thoroughly incorrect. Nevertheless, her solution exemplifies the process of trying to make sense of the problem, and the successive changes in the understanding of the problem through this effort. The conflict between the general *whole-part* scheme and the situation posed by the problem is clear, as also are the necessary intervention of a knowledge of how numbers behave and the disadvantage of having to search through different new models when an algebraic model would be equally adequate for A>C and A<C.

~~A = (B × B) = C~~ . 275 needed

Secret number is -2·5 (I think)

Vicky H, FM3

SUMMARY OF FINDINGS AND CONCLUSIONS

As we expected, a hierarchy appeared in relation to the facility levels of the three problems, with **Buckets** being the easiest and **Sec-** the most difficult; although the difference between **Buckets** and **Sec+** is not significant for AH8 and FM2, in AH8 there is a definite shift towards solutions using equations in **Sec+**.

Of all students, 83% correctly solved the item **25-37**, and 56% correctly solved the item **20-(-30)**, which strongly suggests that the inability to produce correct solutions to **Sec-** without using equations is due to the students' lack of willingness to operate *numerically*, ie, within the *Semantical Field of numbers and arithmetical operations*; this behaviour had been observed on the analysis of the previous groups of problems, but what makes it particularly significant here is the fact that **Sec+** and **Sec-** are not only identical in terms of their *arithmetical articulation*, but also all the one-step strategies that are available to reduce **Sec-** into a problem that can be modelled by a *whole-part* model — eg, presuming that the subtraction "is in fact" an addition", or simply considering the solution to **Sec+** and applying it as an algorithm to **Sec-** — depend in varying degrees on operating *numerically*, and the low level of complexity of the problems only highlights this aspect of the students' difficulties.

The percentages quoted at the beginning of the previous paragraph also accentuate the significance of the fact that many students reconstructed the problem in order to make it meaningful within the context of wholes and parts, showing that for many students the first-choice model is a non-algebraic one, in particular, a non-numerical one. Cecília's script establishes with great exactness that an analogy can be built between **Sec+** and **Sec-** in a way to engender a method to correctly solve **Sec-**, but this analogy is only possible within the *Semantical Field of numbers and arithmetical operations*.

Fabiana's solution, on the other hand, shows that the meaning of arithmetical operations can be adjusted to one's use according to the model being employed when one is operating in a *Non-numerical Semantical Field*. The important insight here is that many "mistakes" that have been used by researchers to characterise *misconceptions* might in fact be *conceptions within a Semantical Field other than the one intended by the researcher*, ie, it might be truly useful to consider that those students are not in fact thinking of what the researchers thought they were.

One important aspect related to the use of algebraic notation emerged. We had seen in solutions to previous "secret number" problems that employed equations that the substitution of specific symbols for "secret number" — usually **x** — was taken by many students as making the problems' statements into equations. In the explanations to their handling of **Sec+** and **Sec-**, a number of students referred to "**13x**" being the <u>result</u> of "**13·x**", revealing that the notion of *representation* was not readily available to them; this is a central part of *meaning* in *algebraic thinking*, and we think the lack of such understanding might represent a substantial obstacle in dealing, for example, with substitution solutions to sets of simultaneous equations. Also, the lack of the notion of *representation* might constitute an obstacle to the development of an understanding of thinking algebraically as proceeding within the *Numerical Semantical Field*, and thus, an obstacle to the constitution of the notion of *numerical-arithmetical structure*.

Finally, a few scripts—in particular Sophie's and Mi's—threw light into the use of algebraic and non-algebraic approaches on different stages of the same solution process, highlighting the possibility of usefully combining algebraic and non-algebraic models, and at the same time emphasising the dissimilarities between them.

4.6 Pattern-Salesperson-Secret Problems

The Problems

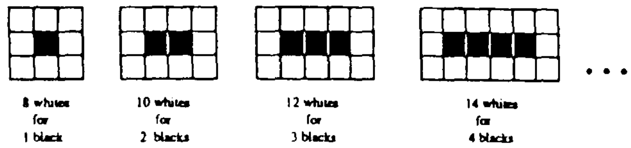
Charles sells cars, and he is paid weekly.
He earns a fixed £185 per week, plus £35 for each car he sells.

This week he was paid a total of £360

How many cars did Charles sell this week?
(Explain how you solved the problem and why you did it that way)

Salesperson

Her you have a pattern of tiles:



One possible formula that gives the number of white tiles that go with a cert
number of black tiles is:

no. of whites = (2 x no. of blacks) + 6

How many black tiles are needed, if I want to use 988 white tiles?
(Explain how you solved the problem and why you did it that way)

Pattern

I am thinking of a "secret number".
I will only tell you that

(6 x secret no.) + 165 = 63

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Secret

GENERAL DESCRIPTION

(i) **Patt**, is a problem where both the generation of a pattern of black and white tiles and a formula relating the number of tiles of each colour on any composition respecting the pattern are given; the central objective was to investigate whether students would prefer to solve the problem reasoning directly from the spatial configuration or would use the formula given, and how they would manipulate those referents;

(ii) **Salesp**, is a very elementary problem about a salesperson who earns a fixed salary plus commission for each item sold; we never expected this problem to offer any difficulty to our students. It was included with the main objective of verifying how the students would justify the choice of arithmetical operations employed — would any justification at all be offered; we expected students to *explain* the use of the operations (eg, an addition used to know...) but not to *justify* the choice in terms of a more general scheme, numerical or otherwise, the reason for our expectation being the great familiarity with the type of situation[67]. The Brazilian version uses fridges instead of cars to make the numbers in the problem smaller.

(iii) A "secret number" problem, **Secret**, is stated in a syncopated form, rather than the usual verbal one; in this problem the solution is a negative number, and we expected it to be significantly more difficult than the other two. It was included in this group to allow us to examine the models produced in a situation where a *whole-part* model is not immediately available.

DISCUSSION OF POSSIBLE SOLUTIONS

All three problems in this group can be solved with an equation in one unknown,

$$\mathbf{b + ax = c}$$

If this approach is used, the three problems would present a very similar facility level, as the only one where an equation is not immediately given, **Salesp**, is very straightforward in verbal structure.

Patt offered the alternative of working on the basis of perceiving, for example, that if the three white tiles at each end of the pattern are removed, one is left with a simple 2 whites to 1 black ratio. From this point of view, the formula provided with the problem's statement would be an unfortunate choice, as the *non-algebraic* procedure we have just described would use the same calculations as algebraic solutions employing the formula, and this makes the more difficult to distinguish between approaches. However, the alternative would be to give, instead, a formula such as

$$\textbf{no. of whites} = 2 \times (\textbf{no. of blacks} + 2) + 2$$

which is obviously more complex than the one we decided to use, making a direct comparison with **Secret** — an important point — more difficult.

Secret could also be solved through the perception that the answer had to be a negative number, leading to the transformation of the problem into

[67]Another situation equally typical and familiar would be, for example, a problem involving change and the buying of several of the same items.

$$165 - 6n = 63$$

which would be solved as **Sec+** in the previous group of problems, possibly based on the whole-part relationship.

The obvious solution to **Salesp** would be to consider that the total income is composed by the fixed part together with the commission for sold items, so to know how much came from selling, it is only necessary to take the fixed part from the total income, a procedure based on the perception of a whole-part relationship.

GENERAL DATA ANALYSIS

Two unexpected results emerged. First, the overall facility level for **Secret** was 56%, much higher than we expected, specially if one considers that the other "secret number" problem with a negative answer (**Sec-**) had one of the lowest facility levels of all problems (27%)[68]. Second, in the Brazilian groups **Patt** was more difficult than **Secret**, while in the English groups this is not the case; this fact is surprising given that **Patt** offers not only the equation but also the support of a diagram, and even more so if one considers that AH8 proved to be very proficient in solving equations. One likely explanation is that the context of a pattern of tiles might have confused the Brazilian students, as this is a very unlikely context for a problem in Brazilian schools, while it is a very common one in English schools. A close examination of the students' solutions will provide further insight on the reasons for this result.

Also unexpected was the very low level of facility for **Patt** in FM2 (18%), as this problem should be familiar to them and offers no difficulty with the numbers. Nevertheless, while for **Secret** 71% of the scripts were **NATT**, 53% of the students in FM2 attempted a solution to **Patt** and failed, suggesting that they at least felt the possibility of producing a correct solution.

In agreement with the result of the previous groups of problems, the Brazilian groups preferred to use equations whenever they were suggested (**Pattern** and **Secret**), while in the English groups equations were used by only one student in **Secret**.

Salesp was the easiest problem in all four groups, with an overall facility level of 84%, identical to that of **Buckets**, in the last group of problems we analysed. As the

[68] Although the facility level in AH8 is very high (89%), forcing the overall result up, one has to observe that the percentages for AH7 and FM3 are very much in agreement with the overall result.

scripts will further demonstrate, those two problems were treated in very much the same way, with the choice of operations being taken as "logical" and never justified.

STUDENTS' SOLUTIONS

<u>The Patt problem</u>

All but one correct solutions to **Patt** from the English students — most of them on the third year group — were **OKCALC**, and many of them were justified by appeal to "reversing the formula", "reversing the procedure", etc..(Ian C, FM3; Joe V, FM3; Katy S, FM3).

Formula = ×2 +6 = Answer

988 ÷ 6 = 164.6	988 − 6 = 982.
164.6 − 2 = 162.6	982 ÷ 2 = 491.
"incorrect answer."	"you will need 491 black tiles."
"now I will reverse the formula"	

Ian C, FM3

To work out the number of black tiles from 988 white tiles you have to reverse the formula so —

no of whites = (2 × no. of blacks) + 6.

becomes

no of blacks = (no. of whites − 6) ÷ 2

notice that '+' becomes '−' and that '×' becomes '÷'.

Therefore the answer to the problem is

no of blacks = 988 − 6 = 982   982 ÷ 2 = $\boxed{491}$ the answer.

Joe V, FM3

To work out the number of black tiles, the formula needs to be reversed: e.g
Instead of x 2 + 6. It is -6 ÷ 2. I can check this on an example 14 whites - 6 ÷ 2 = 4 blacks. This is correct.
Sum: 988 wt - 6 ÷ 2 = 491 black tiles. I can check this back on the example given W = 2x491 + 6 = 988, so it's right.
Answer: 491 black tiles

Katy S, FM3

Although this type of justification was given to other problems, what is remarkable here is the high proportion of students producing it, together with the specific notation used by some students, suggesting a strong influence of taught models. No student actually used a "boxes and arrows" diagram (figure Patt 1), but the treatment of ÷**2** and **+6** as operators, rather than treating **2** and **6** as operands, is clear. Those solutions are *numerical-arithmetical*, as they are guided by properties related to the arithmetical operations only (as it is made clear in Joe's solution), but they are not *analytical*; the secret number is perceived as an initial state and never directly manipulated. Also, the solution process concentrates only in producing "the way back", so to speak, and the transformation of the arithmetical operations into their inverses never involves the manipulation of a *numerical-arithmetical* relationship.

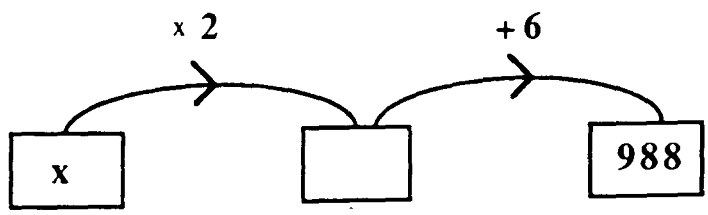


fig. Patt 1

In the Brazilian groups, on the other hand, all but three of the correct solutions are **OKEQT**. In most cases the solution of the equation is

$$
\begin{gathered}
\mathbf{988 = 2x + 6} \\
\mathbf{988 - 6 = 2x} \\
\mathbf{982 = 2x} \\
\mathbf{x = \frac{982}{2} = 491}
\end{gathered}
$$

or very similar. As we pointed out before, it is impossible to decide—in the absence of further explanation about the underlying model—whether this solution is guided by the "undo" perception linked to the "machine" model, by the perception of the whole-part relationship, or by a *numerical-arithmetical* model. In some cases, however, the solution of the equation involved steps that clearly characterise them as *numerical-arithmetical*, and the manipulation of the term involving the unknown characterises the *analiticity* of the solution, so those solutions are truly *algebraic* (Maurício N, AH8, who uses a *normal form* of the equation; Rogério C, AH8); in Maurício's explanation we have a further characterisation of the *analiticity* of his solution, as the unknown is treated explicitly as a number.

$$
\begin{gathered}
988 = 2x + 6 \\
-2x - 6 + 988 = 0 \\
-2x + 982 = 0 \\
2x - 982 = 0 \\
2x = 982 \\
x = \frac{982}{2} \\
\boxed{x = 491}
\end{gathered}
$$

SÃO 988 BRANCOS E EU MULTIPLIQUEI POR 2 O Nº DE PRETOS QUE É x. ~~[illegible]~~ E SOMEI 6. O RESULTADO É O Nº DE PRETOS.

Maurício N, AH8

"There are 988 whites and I multiplied by 2 the no. of blacks and that is **x**. And added 6. The result is the number of blacks [sic]" (there should be no doubt from his script that he actually meant "the number of whites")

$988 = 2 \times x + 6$

$988 = 2x + 6$

$-2x = +6 - 988$

$-2x = -982 \quad (-1)$

$-2x = +982$

$x = \frac{-982}{-2} \qquad x = 491$

R: serão necessários 491 azulejos

Rogério C, AH8

In another **OKEQT** solution (Andrea M, AH8), the evidence for an *algebraic* solution is direct from the explanation.

$b = 2p + 6$

$988 = 2p + 6$

$2p + 6 = 988$

$2p = 988 - 6$

$2p = 982$

$p = 491$

no enunciado tinha a fórmula. E também tinha o nº de azulejos brancos, então, foi só substituir na fórmula a variável (nº de brancos) pelo número dado. E depois ~~falta~~ separar variável de número.

Andrea M, AH8

"in the statement there was the formula. And also the no. of white tiles, so, it was only a matter of substituting into the formula the variable (no. of whites) by the number given. And then to separate variable from number."

Three solutions — all coming from Brazilian students — treated the problem as one directly involving proportion, most probably suggested by the "8 whites for 1 black, etc." subtitles to the illustration[69]. Both Mariana O's (AH8) and Mairê M's (AH8) solutions are incorrect due to a mistaken perception of the relationship between the number of white and black tiles. Mariana's is clearly based on an algebraic model for solving the proportion: it is

[69]That no English student made this type of mistake suggests that the unfamiliarity of the Brazilian students with the problem also played a role.

numerical-arithmetical and *analytical*, with the focus being in determining the number of black tiles.

$\frac{10}{2} = \frac{988}{x}$ $\quad 10x = 1976 \quad x = 197,6$

Eu descobri porque a 1ª fração tem que ser proporcional à segunda, a segunda à terceira e assim vai...

Mariana O (AH8)

"I found out because the first fraction has to be proportional to the second, the third and so on..."

Mairê's solution, however, is *synthetical*, as the focus of the solution process is in determining the multiplier that multiplied by the number of black tiles in the simpler ratio (in this case, by 1) will produce the number of black tiles corresponding to 988 white tiles.

988 BRANCOS para ? PRETOS
8 BRANCOS PARA 1 PRETO
988 : 8 = 123.5 x 1
988 brancos para 123.5 pretos, mas não podemos cortar o azulejo então:
988 brancos para 124 pretos.

Se para 1 preto são 8 brancos (8 vezes a mais) é só saber quantos "8" tem 988 e multiplicar por 1 pois é 1 preto para 8 brancos

Mairê M (AH8)

Left: "988 whites for ? blacks(...)8 whites for 1 black (...)988 whites for 123.5 blacks, but we can't split the tile, so: 988 whites for 124 blacks."

Right: "If for 1 black there are 8 whites (8 times more), then it's only a matter of knowing how many '8' there are in 988 and multiply by 1, because it is 1 black for 8 whites"

Around a third of all **WCALC** mistaken solutions resulted from the incorrect use of the "reverse the formula" approach (Dawn H, FM3)[70].

~~988÷2=494-6=488~~

~~488×2+6=~~ 6

988 ÷ 2 = 494 − 6 = 488

488 + 6 = 494 × 2 = 988

Dawn H, FM3

In Laura G (AH7) we have a behaviour that is as close as one can get to a pure syntactical "shuffle": "white" and "black" are swapped, and the operations "reversed" without any regard for the *arithmetical articulation* or to the *meaning* of the resulting transformation within the *Semantical Field of numbers and arithmetic operations.* Nevertheless — and this is an important point in relation to *meaning* — from Laura's point of view not only the procedure enabled her to find out the answer in an acceptable way, but she was also able to correctly distinguish the symbols for the operations and associate them correctly with the symbols for the corresponding reverse operations; however, she has certainly not grasped the *intended meaning* that the teacher tried to convey.

nº de pretos = (2 ÷ nº brancos) − 6 (operação inversa)

(988)

(divisão)

nº pretos = 494 − 6

nº pretos = 488

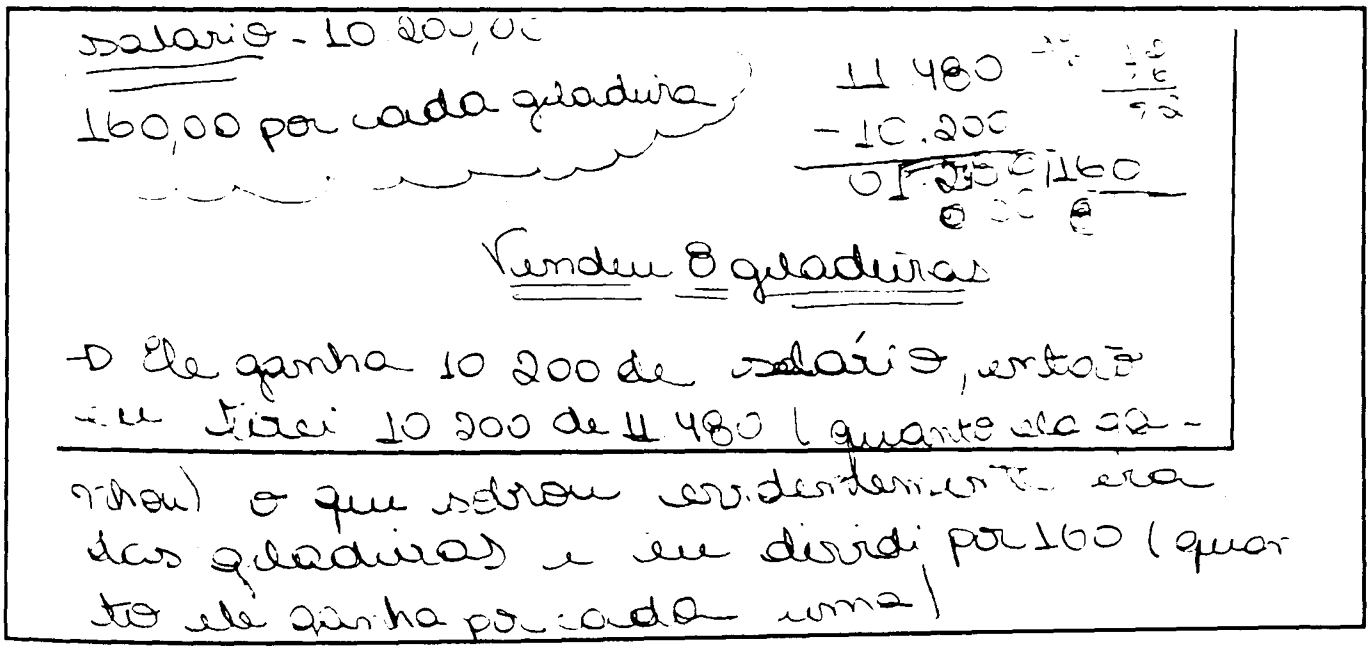
Laura G, AH7

[70] It is interesting that at first she incorrectly applies the "reverse the formula" approach, not regarding the order in which the operations would be performed were the formula being used. When she tries to check the result against the original formula, it naturally does not work, but instead of rethinking the solution process, she alters the checking "template" to fit the mistaken solution procedure.

It is interesting that although the preferential approach to produce correct answers in AH7 was to solve the formula as an equation, more than three-quarters of the mistakes come from **WCALC** solutions, suggesting that even those solutions "by equation" might well have been guided by a contextualised model, as a failure to produce an algebraic model is strongly associated with a failure to produce a contextualised one.

<u>The Salesp problem</u>

As we expected, all the explanations provided with **OKCALC** solutions (which account for 77% of **all** answers) corresponded to the model "take away the fixed part from the total and see how many cars (or fridges) it corresponds to". The "explanation" for the initial subtraction is always a non-explanation (ie, "that's what you do"), and there was never any attempt to relate it explicitly to a whole-part relationship, the procedure being considered as self-justified (Fabíola, AH7); in a few scripts only there is a slight hint that the perception of a whole-part relationship might have guided those solutions (Aluízio A, AH7; Jacob B, FM3; Tarek S, AH7). Both Aluízio and Jacob seem to use a *comparison of wholes* strategy, while Tarek uses a *whole-part decomposition* model.


salario - 10 200,00
160,00 por cada geladeira

11 480
- 10.200
01 280 | 160
0 00 8

Vendeu 8 geladeiras

-> Ele ganha 10 200 de salário, então eu tirei 10 200 de 11 480 (quanto ela ganhou) o que sobrou evidentemente era das geladeiras e eu dividi por 160 (quanto ela ganha por cada uma)

Fabíola, AH7

She gets a 10,200 salary, so I took 10,200 from 11,480 (the money she earned) what is left is **evidently** [the money earned] because of the fridges..." (our emphasis)

11.480
- 10.200
1 280

1280 |160
000 8 geladeiras

R. [illegible]

Explicação: se ela recebe 10.200 cruzeiros + 160 por cada geladeira (salário fixo) e este mês ela recebeu 11.480 eu tenho que tirar a diferença entre os dois salários para descobrir o que ela recebeu a mais, e dividir [illegible] preço de cada [illegible] para saber o nº de geladeiras

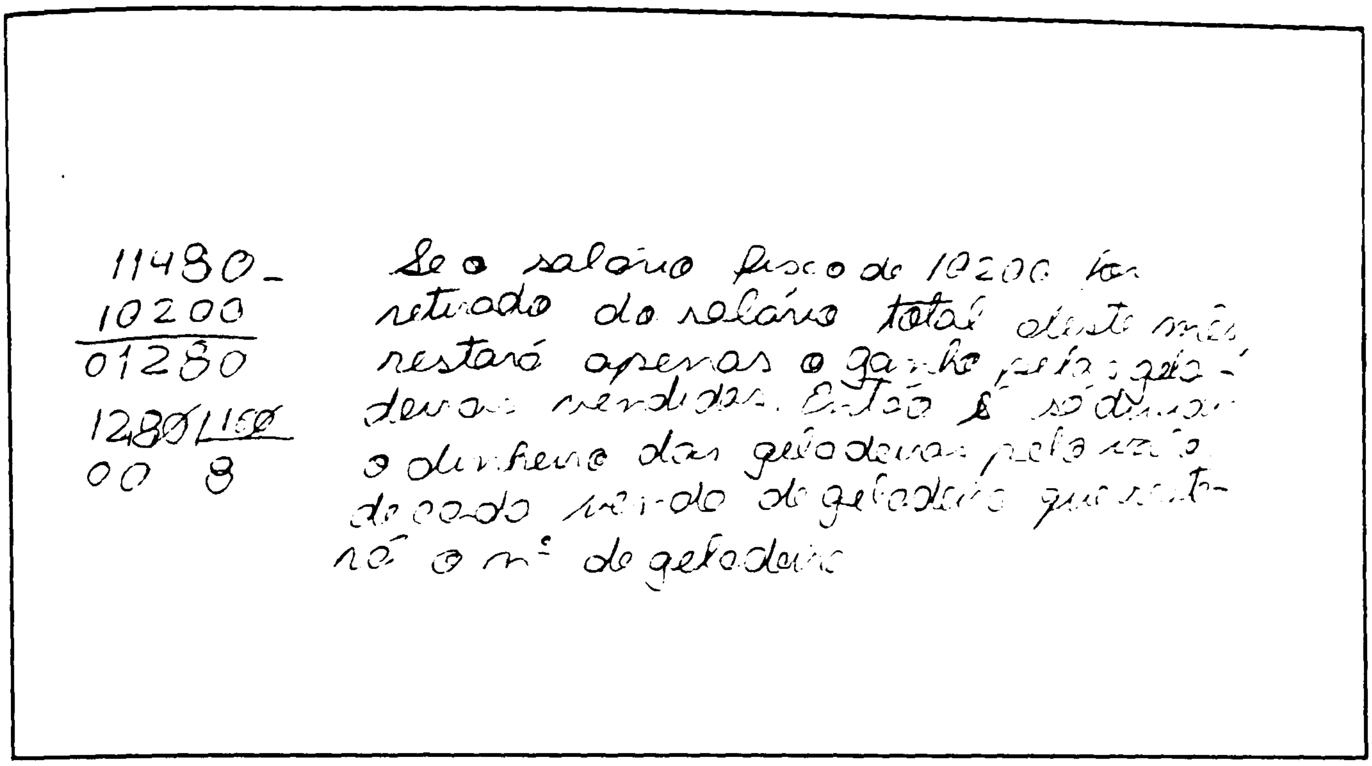
Aluízio A, AH7

"Explanation: if she got 10,200 + 160 for each fridge (fixed salary) and this month she got 11,480, then I have to calculate the difference between the two salaries to know how much she got **in excess** ..." (our emphasis)

I would take £185 off £360, giving £175, now all I have to do is find out how many £35 there are in £175, 175 ÷ 35 = 5, so Charles sold 5 cars that week.
I did it that way because I know he earns £185 fixed per week, anything over that is for cars he sells, so I took £185 off £360, and the answer (£175) must be divided by £35 to get the no. of cars, which is 5

he sold

Jacob B, FM3



Tarek S, AH7

"If the fixed salary of 10,200 is taken from the total income there will be left only the [money] earned from the fridges..."

The focal point here is that in all three cases, the choice of subtraction is not informed by the *arithmetical articulation of an equation*, but by the need to evaluate parts produced through a decomposition of the whole, ie, the arithmetical operations are *tools* used to produce a required *evaluation*, and not informative *objects*. Nevertheless, a distinction between the approaches may be made, as the whole-part based model apparently guiding Aluízio's and Jacob's and Tarek's solutions is certainly more general.

Another illuminating aspect of the scripts, is that in 29% of the **OKCALC** solutions, the determination of the number of cars (fridges) sold is done using a number of different build-up and "build-down" strategies (Helen R, FM3; Derek G, FM2), and in those cases the *evaluation* of the "extra" money is not even considered, as the "fixed salary" (£185 in the English tests) is the target or the starting point, showing conclusively that those procedures are not "disguised" or "primitive" forms of division or multiplication.

1. 360
- 35
325
2. - 35
290
3. - 35
255
4. - 35
220

5. 220
- 35
185

Charles sold 5 cars

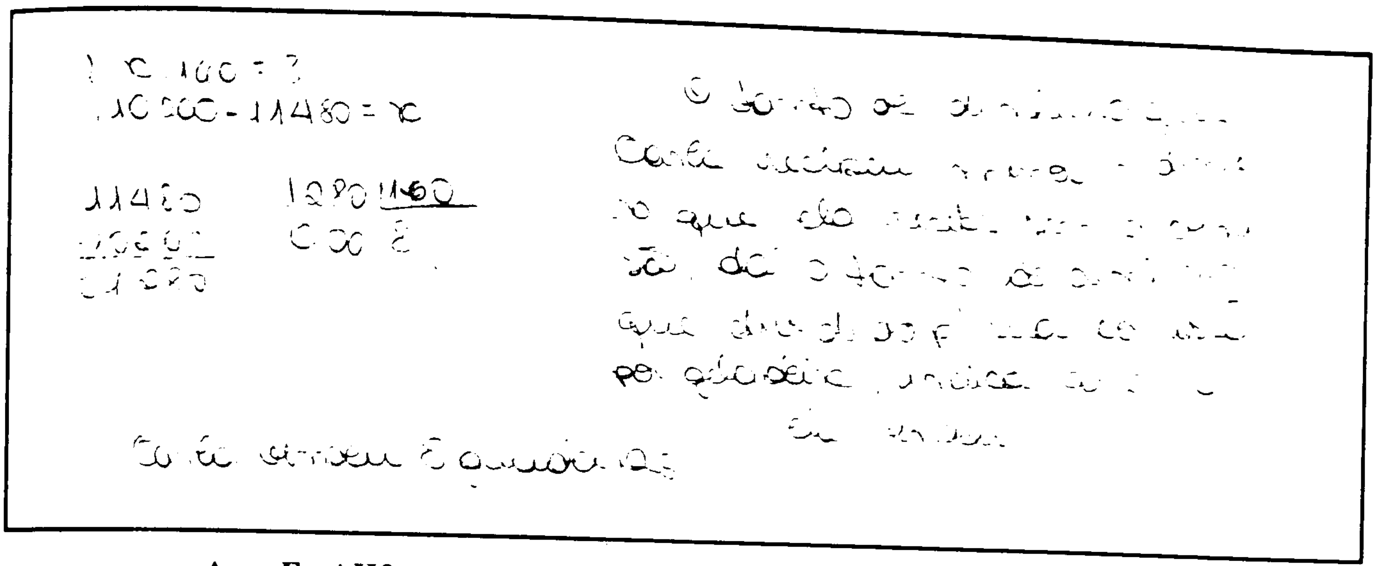
Helen R, FM3

He Sold 5 cars

I Put 185 on the calculator and added 35 5 times to the 155

Derek G, FM

Ana F (AH8) uses an "x", but her solution is clearly guided by the "selling" context, as the accompanying explanation shows; the "x" is used only to represent a value that can be immediately determined and is never manipulated before it is *evaluated*. It is suggested in the script that the focus of attention of the solution process seems to be the amount the saleswoman got for selling fridges, as Ana first writes "x÷140=?," and this may be linked to the fact that as many students she saw the *evaluation* of the "extra" money as nothing more than evident and immediately possible.



Ana F, AH8

"The amount of money Carla got, minus the money she gets without the commission, gives the amount of money that divided by her commission by fridge indicates how many she sold."

The Secret problem

As we saw before, one relevant aspect in relation to this question was the unexpectedly high facility level, with the exception of FM2, which performed very badly.

The **OKEQT** solutions were in all cases solved by following the very standard

$$\begin{aligned} 6x + 165 &= 63 \\ 6x = 63 - 165 &= -102 \\ x &= \frac{-102}{6} \\ x &= -17 \end{aligned}$$

The one aspect of interest is that of all solutions employing equations, in only one case the solver correctly reached the third line then to produce an incorrect result (**+17**). When we compare this with the fact that many more similar mistakes were made in **Sec-** (analysed in the previous section), there is an indication that using a positive integer as a divisor makes more sense than using a negative one, possibly because the positive integer corresponds better to a "sharing" model of division, even if the amount being shared is negative; a further implication of this would be that the preference for *non-numerical* models (in this case the *analiticity* does not seem to be relevant) might be on the basis of some obstacles to the learning of the arithmetic of directed numbers.

In some of the **OKCALC** solutions (Elizabeth W, FM3, for example), the student considers that the answer **has to be** a negative number; however, as opposed to similar situations in solutions to **Sec-** (see previous group of problems), this consideration was never central to the process of solution, ie, it did not result in the transformation of the original problem into an auxiliary one.

To find the secret number I must do the same to each side. Really, the answer is going to be a minus, because on its own 63 is less than the extra 165. So, I take 165 from 63 and get -102. So $\boxed{6 \times \text{secret no.}}$ = -102. So, -102 divided by 6 is -17.
I cannot be sure if this is correct!!!

Elizabeth W, FM3

In one case the student concluded that the problem could not be done because adding would make always more than 165 (Jayne H, FM2).

I don't think that this can be done because 6 x n + 165 would be more than 63 e.g 6x2.
12+165 = 177 would it does not equal 63.

Jayne H, FM2

Jayne, however, failed to solve both **25-37** and **20-(-10)**, showing that her understanding — and possibly perception — of negative numbers was very weak. As a consequence, the distinction between using a *whole-part* model or a *numerical-arithmetical* one becomes somewhat blurred, as the *objects* in each of the two *Semantical Fields* have properties that are easily put into correspondence, or, put in a more precise way, it is easy to establish a much stronger *isomorphism* between the two *Semantical Fields* than in the general case. Nevertheless, and this is a central point in respect to the overall argument of our research work, it would be incorrect to characterise under those circumstances and on the basis of the possibility of the *isomorphism*, solutions using a *whole-part* model as involving *algebraic thinking*. The crucial point to produce the distinction is that arithmetical operations will still be used as tools only, while operations on the wholes and parts (joining, separating, etc) will be the *object operations*.

From the remaining **OKCALC** solutions, in all but two cases of an explanation being provided beyond a restatement of the calculations performed, they refer explicitly to "doing it backwards" or "reversing the process" (Camila A, AH7; Clare B, FM3; Hannah G, FM3; Shazia A, FM3).

$$\begin{array}{r} 63 \\ -165 \\ \hline -102 \end{array} \quad \begin{array}{r} 165 \\ -\ 63 \\ \hline 102 \end{array} \quad \begin{array}{r|l} -102 & 6 \\ \hline 42 & -17 \\ 0 & \end{array} \quad \begin{array}{r} -17 \\ \times 6 \\ \hline -102 \end{array}$$

$6 + x + 165 = 63$
$6 + 165 - 63 = x$
$108 =$
→ 'inverti o processo.

Camila A, AH7
"I reversed the process"

$63 - 165 = -102$
$-102 \div 6 = -17$
$(6 \times \underline{-17}) + 165 = 63$

eg + instead of -
do the sum backwards using the opposite sines

Clare B, FM3

63
-165
-102

-102 ÷ 6 = -17

on this I worked backwords so first I found out what secret number was before adding 165. That was 63 - 165 = -102. Now I need something x 6 = -102. so I divided -102 by 6 = -17. now -17 x 6 = -102
-102 + 165 = 63. secret number = -17

Hannah G, FM3

~~165 - 63 = 102~~

63 - 165 ÷ 6 = -17   -17 is the secret number

Because you have the final answer you can find the missing answer by doing the sum the opposite way round

Shazia A, FM3

It is clear from those scripts that the resemblance with the "reverse the formula" procedure used by many students to solve **Patt** is strong. In Camila's script we have no further explanation, but Clare makes a distinction between "doing the sum backwards" — which seems to refer to the process of "going back" — and "using the opposite signs" — referring to the "undoing" of the effect of the operators, while Hannah specifically mentions that she "found out what secret number was **before adding 165**" (our emphasis), showing the "undo" intention. In Shazia's script the indication is even more complete, as she speaks of "the **final** number" (our emphasis), again a clear reference to a chain of calculations.

Given the reasonably high level of facility for this problem, and that, as we saw in respect to **Sec-** (see previous group of problems), the use of *whole-part* models with problems involving negative numbers is troublesome, we are led to think that most of the **OKCALC** solutions to this problem were guided by a *state-operator machine* model, as the one depicted in figure Patt 1. As we have already shown, this model develops within a *Numerical Semantical Field*, although it is not an *algebraic* model in this case for the lack of *analiticity*. The important implication of this result is that around 50% of all students answering this question were willing to operate within the *Semantical Field of numbers and arithmetical operations*. Moreover, it shows that this willingness is not the expression of a general, conscious, *conception*, but rather an implicit component of the procedure — either taught or developed — to deal with this specific type of problem.

Two other aspects are worth mentioning. First, that a *state-operator machine* model could be made to work with a problem like **Sec-** if *analiticity* becomes a part of the mode of thinking in which one is operating (see figure Patt 2)

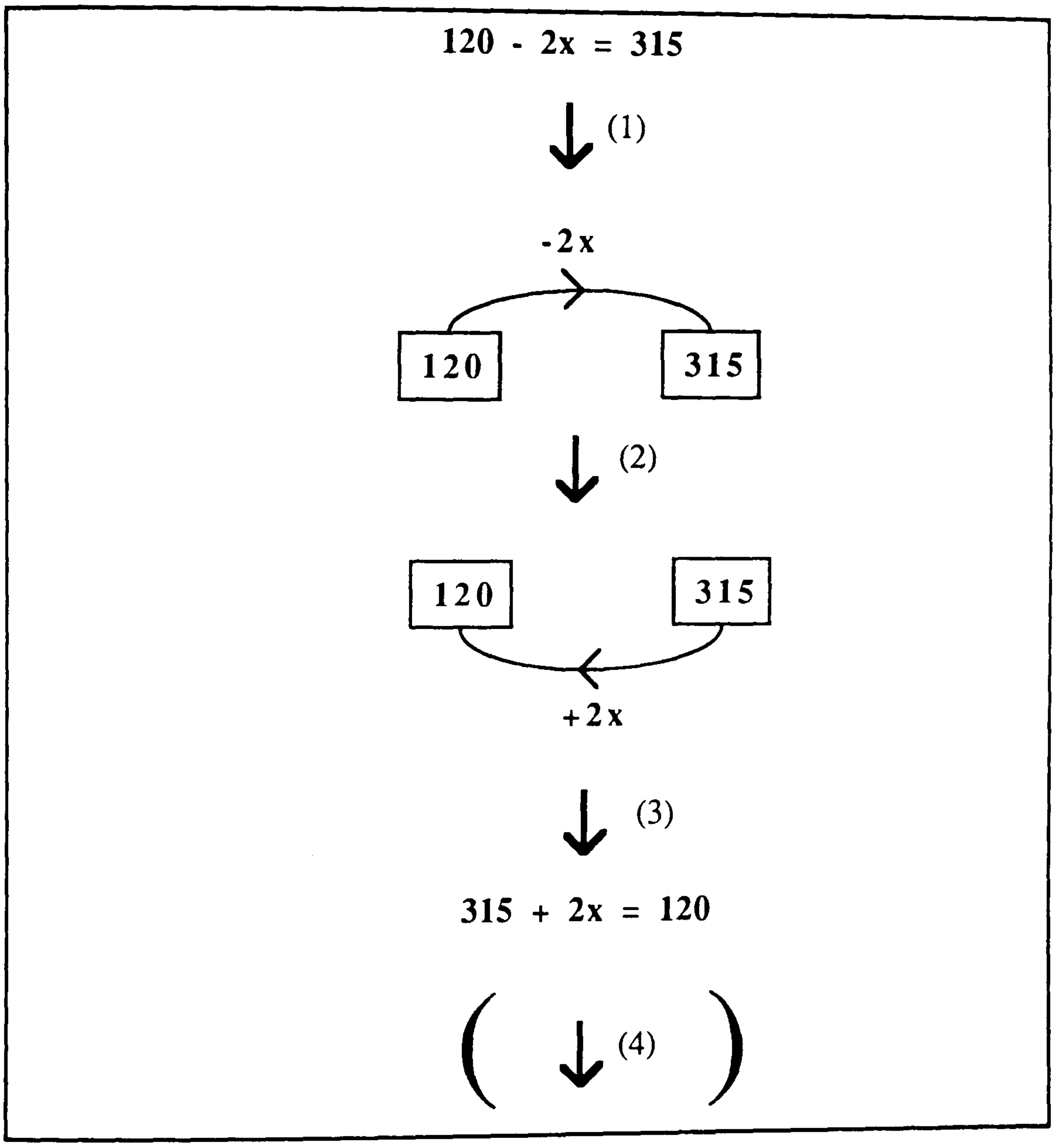


fig. Patt 2

Such approach has two merits: (i) it can be built entirely within the *Semantical Field of numbers and arithmetical operations*, from much simpler cases, and (ii) it introduces the notion of unknown with an *analytical* characteristic. A further advantage would be to strengthen the links between two useful forms of representation of *arithmetical articulation*, namely, the *state-operator diagram* and the standard algebraic notation. Step (4) in fig. Patt 2 could either be a return to a *state-operator model*, which would be similar to that used with **Secret**, or an algebraic solution of the equation, if the solver sees it as *meaningful*. In any case, steps (1), (2) and (3) alone might well serve as an alternative to a justification based on DSBS, for the transformation

$$\mathbf{120 - 2x = 315}$$
$$\mathbf{120 = 315 + 2x}$$

It must be clearly understood that we are not advocating this approach as a *panacea* that would provide the solution for all the problems involved in developing an *algebraic mode of thinking*, but it certainly is a strong and helpful paradigm from which other approaches may be developed.

SUMMARY OF FINDINGS AND CONCLUSION

The main point illustrated by the scripts to this group of problems is the possibility of a model that is clearly *numerical-arithmetical* but not *analytical*. Some solutions to problems in the previous groups had already presented this characteristic (for example, using a paradigmatic simpler example), but the use of a *state-operator machine model* highlighted the fact that it is possible for children in the age group we studied to accept a mode of thought that involves operating totally within the *Semantical Field of numbers and arithmetical operations*; this is particularly relevant because **Patt** is a problem where a spatial configuration is present, making clear that the problem is about numbers of tiles and not "pure" numbers, and yet many students used the *numerical-arithmetical model*. The use of a *state-operator machine model* also offers a singular illustration of the following points:

- arithmetical operators as *objects*, informing the manipulation process;
- the possibility of achieving some degree of *analiticity* in the process, by using generic or unknown parameters in the arithmetical operators (as in figure Patt 2)
- both *structure*—in the form of the *arithmetical articulation*—and *process*—in the form, for example, of the actual inversion of an operator, or of the actual chain of calculations—are indissoluble aspects of the manipulation of the model;

Structure in relation to the establishment and manipulation of a model is a notion that has to accommodate the possibility that there are *objects* that are not "formally" distinguished (eg, both the unknown and the parameters **are** seen as numbers) but neither there exists in the model a super-class containing both *objects* nor all properties applying to one such *object* applies to all of them (eg, in the "meaninglessness" of operating on or with the unknown). The *structure* of a model is, then, a *net of meanings*, necessarily local, and not an abstract and "clean" construction. Even when the establishment of a model is consciously informed by the knowledge of a more generic, general or abstract knowledge, it is only in the local sense of a *net of meanings* that the *structure* of the model is realised, and it is precisely in this sense that the term *arithmetical articulation* expresses the *structure of an algebraic expression as given by its composition in terms of numbers and arithmetical operations.*.

Also, a solution to, say, **Patt**, using a *state-operator machine model* is structurally distinct from one using a *whole-part model* to model the "formula", and both are structurally distinct from the *analogical* solution that is based on a perception of the spatial configuration, and they are all structurally distinct from an *algebraic* solution employing an equation, although the *procedural* aspects may be similar.

4.7 Conclusions to the Chapter

The main result of the experimental study was to confirm that there are different models underlying students' solutions; moreover, it has also shown that our distinction between *algebraic* and *non-algebraic* solutions, based on our characterisation of *algebraic thinking*, offers a clear and useful framework for distinguishing and characterising those solutions.

From the point of view of the methodology adopted—using groups of related problems, instead of "isolated" items—proved to be a correct and very useful choice, as many of the aspects of the models that were identified could only be clearly understood by comparing its use in problems with different contexts and with different numerical parameters. The decision of not using interviews meant we could not probe in depth some aspects of the underlying models, but, on the other hand, it reassured us that it is indeed possible to understand much of those underlying models by examining only pupils' written work, an important feature of the methodology, both because of the possibility of carrying out studies with a larger number of pupils, but also for the teacher who, many times, does not have the necessary time to accompany closely the discussion that goes on on each group during classroom activity.

The most problematic aspect for the students in our study, was that for those unable to deal algebraically with the *secret number* problems, the process of modelling them into a non-algebraic model proved to be an impossible, or at least, very difficult, task. The fact that most of those students could cope with the "contextualised version" of those *secret number* problems, led us to conclude that two are the probable sources of difficulties in the case of those *secret number* problems: (i) difficulties in interpreting the elements of the *arithmetical expressions* in terms of other models; particularly in the case of *whole-part* models, expressions of the type

$$\mathbf{ax + b = c} \quad \text{and} \quad \mathbf{b + ax = c}$$

were easier to interpret than expressions of the type

$$\mathbf{b - ax = c}$$

We suggest that this was the case because the former provide a much more direct representation of "a whole and its parts," while in the case of the latter, the elements have to be separately identified, and the *whole-part articulation* constructed; and (ii) this difficulty is only enhanced by the fact that the notion of a general *whole-part* model seems to be to a great extent alien to what those students see as knowledge applicable to those problems; as a consequence, making sense of the "decontextualised" *secret number* problems implied, in each case, looking for an adequate interpretation, possibly in terms of another problem with a "story," possibly in terms of experience with "plain calculations."

Another relevant aspect we were able to identify, was the importance of what we called *pointers*, in the manipulation of non-algebraic models, for example the fact that one should not add a weight with a length, or that a seesaw will be balanced only if equal weights are put on each side. As we have already pointed out, but wish to stress, this aspect suggests that the use of non-algebraic models to facilitate the learning of specific aspects of algebra—for example the scale balance—has to be carefully examined, in order to avoid the association of the algebraic procedures learned with those *pointers*, an association which may, and probably will, constitute a huge obstacle for the development of an algebraic mode of thinking, particularly in the case of "concrete" models.

From a more general point of view, it became clear that the central notion being examined in our study was that of *meaning*. In this sense, the distinction we used between *elements* of the problem and *objects* of the model, proved very helpful in highlighting the choice and interpretation of the *elements* of the problem which is involved in the process of establishing and manipulating a model.

The non-algebraic models we have identified in the scripts almost always involved an underlying *whole-part articulation*. Hypothetical manipulation of the context of the problem and geometric models appeared only in very few scripts.

The *state-operator machine* model, which appeared only in the Pattern group of problems, represents a special case, as it is clearly a *numerical* but *non-algebraic* model, as it lacks *analiticity*. The fact it was used by so many students, suggest that operating within a purely *numerical* environment, and using the arithmetical operations as *objects*, ie, manipulating a model *informed* by them, is not beyond the grasp of those students, supporting our claim that the development of an algebraic mode of thinking has to be understood as the process of cultural immersion from which the development of an *intention* is produced, and a process that is very much dependent on the exposure to that mode of thinking. The fact that among Brazilian students we were able to find many more instances of algebraic models being used than among English students, also supports this claim, given the distinct emphasis on the teaching of algebra—much greater in Brazil—in the grades in question.

Chapter 5
General Discussion

Both the evidence from the historical study and from the experimental study showed that our characterisation of algebraic thinking—*arithmeticity*, *internalism*, and *analiticity*—provides an adequate framework for distinguishing different ways of modelling problems and of manipulating those models. Moreover, we have also shown that by distinguishing those different modes of thinking, we were able to identify the tensions underlying the production of an algebraic knowledge, as well as the sources of the difficulties faced by the students in our experimental investigation and the constraints acting upon the development of an algebraic knowledge in historically situated mathematical cultures.

The central issue which provided the thread followed in our investigation is that of *meaning*. We identified two ways in which the issue of *meaning* is related to our study of *algebraic thinking*.

First, an "algebraic verbal problem" can be seen either as the problem of determining the required measure(s) or as the problem of determining a number or numbers which satisfy some given arithmetical conditions; in the case of "purely numerical problems," interpreting it as the problem of determining a measure requires the extra step of interpreting the elements in the "arithmetical" statements—as, for example, in the *secret number* problems in our test papers—as representing or describing some contextualised problem[1]. The fact that *secret number* problems were consistently more difficult than the corresponding contextualised problems—apart from the case of the older Brazilian students, who had had a somewhat thorough experience with using equations to solve problems—indicates that for the students in our experimental study, interpreting the "arithmetical" statements into another *Semantical Field* was not an easy task; both the lack of the pointers we have mentioned in Chapter 4—eg, "weights can only be added to or subtracted from, other weights"—and the lack of taught *whole-part* models, which could provide a more or less standard *Semantical Field* for interpreting the "arithmetical" statements, seem to account for the failure of so many students to make sense of those statements.

The second way in which *meaning* is related to *algebraic thinking*, is through the process of manipulating the model used with a problem. Even if a problem is seen as the problem of determining a number or numbers which satisfy given conditions, the conceptions involved in the determination of the concept of *number* play a central role

[1]The quotes in *arithmetical* are necessary for this precise reason: as the solver makes sense of the statements by interpreting them in a *Semantical Field* other than that of numbers and arithmetical operations, we may safely assume that those statements are not seen *primarily* as arithmetical statements; this does not imply, however, that the solver is *intelectually incompetent* to do so, but only that within his or her mathematical culture that is not the preferential mode of thinking.

in determining what can and should be done to manipulate relationships involving *number*; the historical study provided precisely the evidence about how conceptualisations of number are central if we are to understand the mathematical activity within a mathematical culture—or of an individual. We have clearly shown that *algebraic thinking* depends on a *symbolic* understanding of numbers, but also that such a *symbolic* understanding of numbers have to compete with other—quite acceptable—conceptions, such as "number as measure." The tension between a *symbolic* understanding of number, which implies that numerical-arithmetical relations are treated *arithmetically*, *internally*, and *analytically*, ie, algebraically, and an *ontology* of *number*, which says what number *is* and only from there one determines how it can be dealt with, is a central issue in the process of developing an algebraic mode of thinking; our experimental study did not intend to probe into the students' mathematical conceptions underlying their mathematical activity, but nonetheless, it provided evidence that the models underlying their solutions to the proposed problems did not present—in many cases—the generality *as a method* that Jacob Klein indicates as the central aspect distinguishing Vieta's conceptualisation of algebra from that of Diophantus, and which is a central characteristic of what he calls the "modern" conceptualisation of the mathematical activity.

Those two aspects of the relationship between *meaning* and *algebraic thinking* suggest a focus of tension in the development of an algebraic mode of thinking. The acceptance of the "arithmetical" statements as informative in themselves, ie, as true arithmetical statements, certainly depends on the possibility of treating them *algebraically*, at the same time thinking algebraically depends on the ability to recognise arithmetical statements as informative in their own right. Our approach to this question was to consider *algebraic thinking* as an *intention*, more precisely, the *intention* to treat problems which involve the determination of a number or numbers *algebraically*, according to our characterisation of *algebraic thinking*; the *intention* to think algebraically can certainly evolve from very simple algebraic situations, such as solving simple equations, but precisely because this *intention* is not algebra, only a way of dealing with algebra, *the production of an algebraic knowledge, eg, "how to solve equations of a certain type," does not depend on or involves* ***by itself*** *algebraic thinking*. It is only by making that *intention* explicit, and by contrasting *algebraic thinking* with other modes of thinking which can be used to produce algebra, that the *intention* of thinking algebraically can be consciously acquired. Moreover, it is only when such *intention* is in place that the requirement of a treating arithmetical statements in a way which is *arithmetical*, *internal*, and *analytical*, can be meaningful.

In the course of our investigation of the nature of *algebraic thinking*, two important distinctions were elicited: (i) that between *intrasystemic* and *extrasystemic meaning*; and, (ii) that between *situational* and *mathematical context*.

The former allows us to account for the possibility of an *algebraic* algebraic activity (as opposed to a *non-algebraic* one), by making clear that, far from being *meaningless*, or *semantically weak*, the elements involved in *algebraic thinking* are *meaningful* and *semantically full*, but only when interpreted within the *Semantical Field* of numbers and arithmetical operations, ie, there is a shift of referential which makes the *algebraic* algebraic activity meaningful. In the historical study we had the opportunity to refer to the *syntactical meaning* of the elements in *algebraic thinking*. This notion, which might seem paradoxical at first, is essential for one to understand what *algebraic thinking* is, and must be accepted not as a linguistic detour to indicate the usually accepted notion of "rule manipulation," be it in a poorly or in a highly skilful manner, but as indicating that there is nothing "outside" the statements being manipulated which are required to make their elements "meaningful."

The second of the two distinctions allows us to understand the importance of one's willingness to shift into a new *Semantical Field* in the process of thinking algebraically. It is the shift from the *situational context* of a problem—or from its *local context* in the case of "purely numerical problems"—into a *mathematical context*, representing also the transition from the problem to a method for solving the problems of a class to which the specific problem in question belongs, or seems to belong, that makes *algebraic thinking* possible; moreover, the very *intention* of producing that shift—and, thus, its acceptance—is that which characterises mathematics as an accepted cultural object. The refusal by Luria's and by Freudenthal's subjects to operate within a "context-free" environment strongly indicates that the development of a given mathematical mode of thinking depends on the acceptance of the fact that certain ways of organising the world are adequate and useful, ie, that they produce insights which conform to one's cultural needs. It is exactly in this sense that *algebraic thinking* has to be understood as an *intention*: it represents the affirmation of the need to use numerical-arithmetical models and to treat those models *arithmetically*, *internally*, and *analitically*, and it is by affirming this need that it drives the development of an algebraic knowledge.

By understanding *algebraic thinking* as a cultural component, rather than a developmental one, we opened a line of research into the difficulties faced by children in the learning of algebra; we have shown that non-algebraic models used as primary ways of dealing with problems involving the determination of a number or numbers do constitute an obstacle to the development of an algebraic mode of thinking, and we have elicited some of those models and their main characteristics. By also showing that *algebraic thinking* is better understood as an *intention*, we demonstrated that the process of developing an algebraic mode of thinking is one of *cultural immersion*, and by doing so, we open the possibility of explaining the "failure" of individuals in "naturally" developing the ability to think algebraically—as Piaget's theory, for example, would predict—in terms of a *lack of a cultural component*. In a similar way, we think that it is possible to explain, for example, the "failure" of individuals in "naturally" developing *proportional reasoning*.

At a deeper level, this aspect of our investigation shows, in particular in relation to the historical study, that asserting a parallel between the historical development of algebra and *algebraic thinking* and the development, by individuals, of an algebraic mode of thinking, cannot be understood in the context of searching for similar "stages of development." The cultural factors are, we believe, too complex to be "read through," and it thus seems to be the case that even if an underlying, inevitable, cognitive engine exists—as Garcia and Piaget say—we are unlike ever to reach it. The culturalistic approach, on the other hand, highlights knowledge as the result of trying to make sense of the world, and as the world is presented to us largely through the culture we live in, and as cultures are in perpetual recreation, the culturalistic approach to the nature of *algebraic thinking* provides an immediate understanding of the cultural process of being initiated to it.

Although our research has been thoroughly concerned with characterising *algebraic thinking*, one of its clearest results was to reveal the interplay between algebraic and non-algebraic modes of thinking. First, because non-algebraic models can provide, as in Davydov's teaching programme, the raw material which is to be examined algebraically; second, and more important, because the deep distinction between algebraic and non-algebraic modes of thinking point out to the impossibility of reducing one to the other, ie, it points out to the inadequacy of substituting algebraic for non-algebraic "whenever possible"; *algebraic thinking* can only be understood in the context of all different modes of thinking, and, thus, the development of non-algebraic modes of thinking has to be kept as a central objective of teaching. The possibility of interpreting a problem or situation within different *Semantical Fields*, certainly offers a richer perspective for organising one's world and for producing knowledge.

The results of our investigation point out, although in a provisional manner, that an early introduction of children to *algebraic thinking* should be carried out. First, because it provides a unifying and powerful *mathematical context*, one in which a deeper understanding of the structure of large classes of problems is possible. Second, because it allows the development of an understanding of numbers and of the arithmetical operations which is algebraic—and, thus, symbolic—from very early stages of learning, resulting in a much sounder mathematical foundation to those aspects of the children's mathematical knowledge. Third, because *situational* models and abstract non-algebraic models (eg, whole-part models) are a much more present part of everyone's life, and opportunities for refining and discussing them are much more abundant; emphasising the importance of algebraic models, particularly to the teacher and curriculum developer, is a proper way of restoring a balance which is necessary. Fourth, and finally, the traditionally accepted view of "algebra as generalised arithmetic"—under the guise of "numbers first and then algebra"—leads in fact to the formation of sometimes insuperable obstacles to learning, and an early start with *algebraic thinking* would address this difficulty.

There are two natural directions to follow after the research presented in this dissertation, both of which we will pursue.

The first is to extend our research into the history of mathematics, by examining other historically situated cultures and by considering the non-mathematical characteristics of the cultures examined. This last aspect is particularly important to provide a more comprehensive view of the place of the mathematical cultures in their "parent" cultures.

Second, we will study, this time making extensive use of interviews, students' conceptualisations in mathematics, particularly in relation to elements related to *algebraic thinking*. At the same time, we will engage in developing a teaching approach for the development of *algebraic thinking* in the later years of primary school and early years of secondary school; some of the exploratory work in this respect has already been conducted, both in Brazil and in England, and will be reported elsewhere.

Annex A

Problems used in the exploratory experimental study

1) Two friends, Maggie and Sandra, went to the Goose Fair.
Maggie brought £12 with her and Sandra brought £18.
During the afternoon, Sandra spent twice as much as Maggie, and when they left the fair, both of them had the same amount of money.
How much didi each of them spend?

2) A car salesman earns, per week, a fixed £200 <u>plus</u> £35 for each car sold.
This week his total income was £375.
How many cars did he sell this week?

3) A carpenter wants to cut a 73 cm long stick in two, but he wants one of the pieces to be 17 cm longer than the other.
How long will the pieces be?

4) I have a 'secret' number in my mind.
If I multiply it by three, and take the result away from 210, I'm left with 156.
Now, which is my 'secret' number?

5) Pick up any three consecutive numbers and write them down inside the squares.
Now add them up and put the result inside the circle.
Finally, divide the number in the circle by three and put this last result in the triangle.

An example:

$\boxed{12} + \boxed{13} + \boxed{14} = 39$ $\qquad$ $39 \div 3 = 13$

Now try with other successive numbers.

(a) will the number in the triangle always be equal to the middle number in the squares?

(b) Please explain how do you know that your answer to (a) is correct.

6) Johanne bought some bottles of milk and paid for it with a £5 note.

(a) can you work oput the change she received?

(b) If not, what else should you know to be able to work out the change?

7) Suppose you buy two chocolate bars, you pay for it and you get the change.

Then you decide to buy a can of cola.

When you are o pay, the clerk says: "Give me back your cahnge and I'll give you back your money. Now I add up the prices for the chocolates and the cola and you pay for the whole sum."

Is this the same as just paying, from the cahnge, for the cola?

Please explain your answer.

Annex B

Problems used in the main experimental study

Question 1

I am thinking of a secret number.
I will only tell you that

120 - (13 x secret no.) =315

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Question 2

To know the number of oranges that will be in a box, one has to divide the total ımber of oranges by the number of boxes, that is,

(oranges per box) = (number of oranges) + (number of boxes)

a) There are 1715 oranges and we want to have 49 oranges per box.
How many boxes are needed?

b) If you are told the number of oranges per box and the number of boxes, how would you work out the total number of oranges?

Question 3

From a tank filled with 745 litres of water, 17 buckets of water were taken. Now there are only 626 litres of water in the tank.

How many litres does a bucket hold?
(Explain how you solved the problem and why you did it that way)

Question 4

Maggie and Sandra went to a records sale.
Maggie took 67 pounds with her, and Sandra took 85 pounds with her (a lot of money!!).

Sandra spent four times as much money as Maggie spent.
As a result, when they left the shop both of them had the same amount of money.

How much did each of them spend in the sale?
(Explain how you solved the problem and why you did it that way)

Question 5

Mr Sweetmann and his family have to drive 261 miles to get from London to Leeds.
At a certain point they decided to stop for lunch
After lunch they still had to drive 2.7 times as much as they had already driven.

How much did they drive after lunch? And before?
(Explain how you solved the problem and why you did it that way)

Question 6

a) 25 – 37 = ...

b) 20 - (-10) = ...

Test paper A1

Question 1

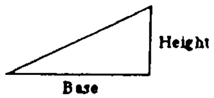
I am thinking of a "secret" number.
I will only tell you that ...

181 - (12 x secret no.) = 97

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Question 2

The slope of a ramp is calculated by dividing the height of the ramp by the lenght of its base. That is,

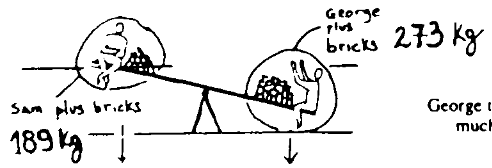


slope = height ÷ base

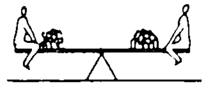
a) If the slope of a ramp is 1.2 and its base measures 15 metres, what is the height of this ramp?

b) If you are given the slope and the height of a ramp, how would you work out the base of this ramp?

Question 3



George throws away four times as much weight as Sam does.



Now they are balanced.

How many kilograms did George throw away? And Sam?
(Explain how you solved the problem and why you did it that way)

Question 4

On a TV show...
"Well, Mrs Swetmann! You have so far won 731 pounds in our show...
Now I have an offer for you:

CHOICE A: We multiply your prize by 1.2 and then we multiply the result by ... (and the presenter whispered a number in Mrs Sweetman's ear) ... or...
CHOICE B: the other way around; we first multiply your prize by the number I have just whispered to you, and then we multiply the result by 1.2 ... "

What would your choice be? (Justify your answer)

Question 5

John is organizing a big party for children.
He bought a 8 big boxes of candies, each one containing 250 candies.

If 250 children show up to the party, how many candies will each of them get? (Evereybody gets the same number of candies, of course!)

Explain very clearly how you solved this problem.

Question 6

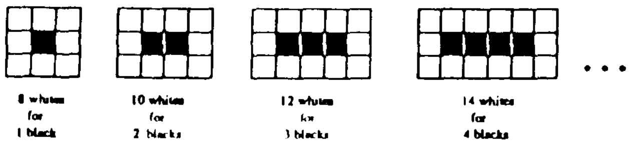
Sam and George bought tickets to a concert.
Because Sam wanted a better seat, his ticket cost four times as much as George's ticket.
Altogether they spent 74 pounds on the tickets.

What was the cost of each ticket?
(Explain how you solved the problem and why you dit it that way)

Test paper A2

Question 1

Her you have a pattern of tiles:



One possible formula that gives the number of white tiles that go with a cert: nber of black tiles is:

no. of whites = (2 x no. of blacks) + 6

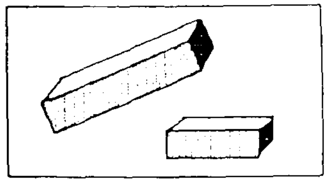
How many black tiles are needed, if I want to use 988 white tiles?
(Explain how you solved the problem and why you did it that way)

Question 2

At the right you have a sketch of wooden blocks.

A long block put together with two of the short blocks measure 162 cm altogether.

If two short blocks are put together, they still measure 28 cm less than a long block.



What is the lenght of each individual block?
(Explain how you solved the problem and why you did it that way)

Question 3

I am thinking of a "secret" number.
I will only tell you that ...

181 - (12 x secret no.) = 128 - (7 x secret no.)

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Question 4

Sam and George bought tickets to a concert.
Because Sam wanted a better seat, his ticket cost 2.7 times as much as George's ticket.

Altogether they spent 74 pounds on the tickets.

What was the cost of each ticket?
(Explain how you solved the problem and why you dit it that way)

Question 5

I am thinking of two secret numbers.
I will only tell you that...

(first no.) + (second no.) = 185
and
(first no.) - (second no.) = 47

Now, which are the secret numbers?
(Explain how you solved the problem out and why you did it that way)

Question 6

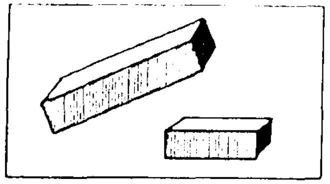
a) 25 – 37 = ...

b) 20 - (-10) = ...

Test paper B1

Question 1

At the right you have a sketch of ooden blocks.
A long block and a short block easure 162 cm altogether.
A short blocks measures 28 cm ss than a long block.



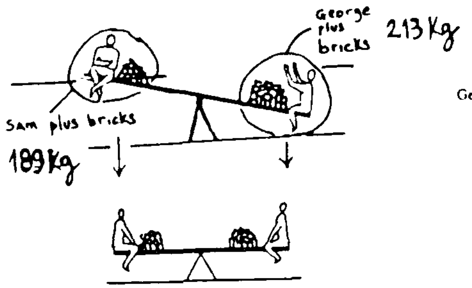
What is the lenght of each individual block?
(Explain how you solved the problem and why you did it that way)

Question 2

Mr Sweetmann and his family have to drive 261 miles to get from London to eeds.
At a certain point they decided to stop for lunch.
After lunch they still had to drive four times as much as they had already iven.

How much did they drive before lunch? And after lunch?
(Explain how you solved the problem and how you knew what to do)

Question 3



George throws away 11 bricks and Sam throws away 5 bricks.

Now they are balanced.

What is the weight of one brick ?
(Explain how you solved the problem and why you did it that way)

Question 4

I am thinking of a "secret number".
I will only tell you that

(6 x secret no.) + 165 = 63

The question is: Which is my secret number?
(Explain how you solved the problem and why you did it that way)

Question 5

Charles sells cars, and he is paid weekly.
He earns a fixed £185 per week, plus £35 for each car he sells.

This week he was paid a total of £360

How many cars did Charles sell this week?
(Explain how you solved the problem and why you did it that way)

Test paper B2

Question 1

I am thinking of a "secret" number.
I will only tell you that ...

181 - (12 x secret no.) = 128 - (7 x secret no.)

The question is: Which is my secret number?
(Explain how you solved the problem and why you dit it that way)

Question 2

Sam and George bought tickets to a concert.
Because Sam wanted a better seat, his ticket cost four times as much as George's ticket.
Altogether they spent 74 pounds on the tickets.

What was the cost of each ticket?
(Explain how you solved the problem and why you dit it that way)

Question 3

To know the number of oranges that will be in a box, one has to divide the total number of oranges by the number of boxes, that is,

(oranges per box) = (number of oranges) ÷ (number of boxes)

a) If there are 17 oranges per box and we have 49 boxes, how many oranges there are altogether?

b) If you are told the number of oranges per box and the total number of oranges, how would you work out the number of boxes needed?

Question 4

At Celia's shop you can buy boxes of chocolate bars or you can buy spare bars as well.

A box and three spare bars cost £8.85.
A box with three bars missing cost £5.31

What is the price of a box of chocolate bars in Celia's shop? What is the price of a single bar?
(Explain how you solved the problem and why you dit it that way)

Question 5

Abigail is having a hard time to decide what to dress.
She has socks of 6 different colours, skirts of 5 different colours, and T-shirts of 7 different colours.

In how many different ways can she dress? (Explain how you solved the problem and why you did it that way)

Test paper C1

Question 1

Maggie and Sandra went to a records sale.
Maggie took 67 pounds with her, and Sandra took 85 ponds with her (a lot of money!!).

Sandra bought 11 Lp's, and Maggie bought 5 Lp's.
As a result, when they left the shop both of them had the same amount of money.
What is the price of an Lp?

(Explain how you solved the problem and why you did it that way)

Question 2

Mr Sweetmann and his family have to drive 261 miles to get from London to Leeds.
At a certain point they decided to stop for lunch.
After lunch they still had to drive 2.7 times as much as they had already driven.

How much did they drive before lunch? And after lunch?
(Explain how you solved the problem and why you did it that way)

Question 3

I am thinking of two secret numbers.
I will only tell you that...

(first no.) + (3 x second no.) = 185
and
(first no.) - (3 x second no.) = 47

Now, which are the secret numbers?
(Explain how you solved the problem and why you did it that way)

Question 4

The speed of a car can be calculated by dividing the distance covered by the time spent to do it. That is,

speed = distance ÷ time

a) If one has to travel 351 kilometres at a speed of 110 kilometres per hour, how much time will it take?

b) If you are told the speed of a car and the amount of time it ran, how would you work out the distance it covered?

Question 5

Joe's Cafe offers a number of choices of bread, fillings and sauces. There are 84 different combinations altogether.

A customer counted 14 different sauces on the menu.

If one wants only bread and filling, how many choices are available?
(Explain how you solved the problem and why you did it that way)

Test paper C2

Annex C
Data on the groups in the main experimental study

Group: AH7 (Brazilian 7th graders)

Total no. of students: 56

Average age (yrs.mths): 13.11

Standard deviation (yrs.mths): 0.9

Group: AH8 (Brazilian 8th graders)

Total no. of students: 53

Average age (yrs.mths): 15.0

Standard deviation (yrs.mths): 1.0

Group: FM2 (English 2nd year)

Total no. of students: 53

Average age (yrs.mths): 13.2

Standard deviation (yrs.mths): 0.4

Group: FM3 (English 3rd year)

Total no. of students: 66

Average age (yrs.mths): 14.3

Standard deviation (yrs.mths): 0.3

Group: ALL

Total no. of students: 228

Average age (yrs.mths): 14.1

Standard deviation (yrs.mths): 0.11

Observation: In Brazilian groups, the much greater standard deviation is due to the fact that students can actually fail a whole year, which does not happen in English schools.

Annex D

Tables of frequencies for the problems in the main experimental study

TICKET AND DRIVING

	AH7					AH8					FM2					FM3			
	T4	D2.7	D4	T2.7		T4	D2.7	D4	T2.7		T4	D2.7	D4	T2.7		T4	D2.7	D4	T2.7
	40	40	16	16		34	34	19	19		36	36	17	17		41	41	25	25
OKEQT	0.42	0.15	0.13	0.13		0.73	0.29	0.42	0.53		0.03	0.05	0.00	0.00		0.02	0.00	0.00	0.00
OK ÷3.7 or 5	0.43	0.00	0.32	0.06		0.15	0.03	0.31	0.00		0.33	0.06	0.41	0.00		0.68	0.10	0.80	0.16
OKT&E	0.00	0.00	0.00	0.00		0.00	0.00	0.00	0.00		0.00	0.00	0.00	0.06		0.02	0.00	0.04	0.20
W ÷2.7 or 4	0.02	0.23	0.13	0.25		0.06	0.09	0.00	0.11		0.20	0.14	0.06	0.00		0.07	0.32	0.08	0.20
WOTH	0.05	0.20	0.19	0.38		0.05	0.39	0.11	0.26		0.20	0.14	0.29	0.29		0.12	0.27	0.04	0.28
NATT	0.08	0.43	0.25	0.19		0.03	0.21	0.16	0.11		0.25	0.61	0.24	0.65		0.07	0.31	0.04	0.16
OK	0.85	0.15	0.44	0.19		0.88	0.32	0.74	0.53		0.36	0.11	0.41	0.06		0.72	0.10	0.84	0.36
WRONG	0.07	0.43	0.31	0.63		0.11	0.48	0.11	0.37		0.40	0.28	0.35	0.29		0.19	0.59	0.12	0.48
NATT	0.08	0.43	0.25	0.19		0.03	0.21	0.16	0.11		0.25	0.61	0.24	0.65		0.07	0.31	0.04	0.16

SEESAW-SALE-SECRET NUMBER ("E"="Seesaw"; "A"="Sale.")

	A7						AH8						FM2						FM3				
	E11-5	E4x	A11-5	A4x	SecNo		E11-5	E4x	A11-5	A4x	SecNo		E11-5	E4x	A11-5	A4x	SecNo		E11-5	E4x	A11-5	A4x	SecNo
	16	21	19	21	35		19	17	17	17	36		17	20	16	20	33		25	24	17	24	42
OKEQT	0.06	0.14	0.05	0.24	0.40		0.16	0.47	0.35	0.47	0.88		0.00	0.05	0.00	0.10	0.00		0.00	0.04	0.00	0.00	0.10
OKCALC	0.13	0.00	0.16	0.00	0.03		0.00	0.00	0.06	0.00	0.00		0.06	0.00	0.00	0.00	0.03		0.44	0.08	0.35	0.04	0.05
OKT&E	0.00	0.00	0.00	0.00	0.00		0.00	0.00	0.00	0.00	0.00		0.12	0.00	0.38	0.15	0.00		0.20	0.13	0.24	0.33	0.00
WEQT	0.13	0.10	0.05	0.05	0.09		0.32	0.12	0.18	0.24	0.06		0.00	0.00	0.00	0.00	0.00		0.00	0.00	0.00	0.04	0.10
WCALC	0.25	0.43	0.31	0.43	0.08		0.10	0.12	0.12	0.06	0.03		0.41	0.60	0.38	0.20	0.37		0.24	0.42	0.24	0.37	0.43
NATT	0.44	0.33	0.42	0.29	0.40		0.42	0.29	0.29	0.24	0.03		0.41	0.35	0.25	0.55	0.58		0.12	0.33	0.18	0.21	0.31
OK	0.19	0.14	0.21	0.24	0.43		0.16	0.47	0.41	0.47	0.88		0.18	0.05	0.38	0.25	0.04		0.64	0.25	0.59	0.38	0.15
WRONG	0.38	0.52	0.37	0.48	0.17		0.42	0.24	0.29	0.29	0.09		0.41	0.60	0.38	0.20	0.37		0.24	0.42	0.24	0.42	0.53
NATT	0.44	0.33	0.42	0.29	0.40		0.42	0.29	0.29	0.24	0.03		0.41	0.35	0.25	0.55	0.58		0.12	0.33	0.24	0.21	0.31

CARP-CHOC-SECRET NUMBER

	AH7					AH8					FM2					FM3				
	Choc	Carp1-1	Carp1-2	Sys1-1	Sys1-3	Choc	Carp1-1	Carp1-2	Sys1-1	Sys1-3	Choc	Carp1-1	Carp1-2	Sys1-1	Sys1-3	Choc	Carp1-1	Carp1-2	Sys1-1	Sys1-3
	19	16	16	16	16	17	19	19	19	19	16	17	17	17	17	17	25	25	25	25
OKEQT	0.05	0.19	0.13	0.06	0.11	0.47	0.79	0.47	0.79	0.82	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.04	0.00
OKCALC	0.74	0.50	0.31	0.00	0.00	0.18	0.11	0.05	0.00	0.00	0.13	0.00	0.00	0.00	0.06	0.29	0.56	0.40	0.12	0.06
OKT&E	0.00	0.00	0.00	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.06	0.06	0.06	0.00	0.00	0.08	0.08	0.20	0.00
WEQT	0.00	0.00	0.00	0.13	0.42	0.12	0.05	0.42	0.16	0.18	0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.04	0.06
WCALC	0.16	0.31	0.44	0.19	0.11	0.12	0.05	0.05	0.00	0.00	0.44	0.65	0.71	0.41	0.38	0.47	0.24	0.24	0.12	0.41
NATT	0.05	0.00	0.13	0.56	0.37	0.12	0.00	0.00	0.05	0.00	0.38	0.29	0.24	0.53	0.56	0.24	0.12	0.20	0.48	0.47
OK	0.79	0.69	0.44	0.12	0.11	0.65	0.90	0.52	0.79	0.82	0.13	0.06	0.06	0.06	0.06	0.29	0.64	0.52	0.36	0.06
WRONG	0.16	0.31	0.44	0.32	0.53	0.24	0.10	0.47	0.16	0.18	0.50	0.65	0.71	0.41	0.38	0.47	0.24	0.28	0.16	0.47
NATT	0.05	0.00	0.13	0.56	0.37	0.12	0.00	0.00	0.05	0.00	0.38	0.29	0.24	0.53	0.56	0.24	0.12	0.20	0.48	0.47

BUCKETS

	AH7			AH8			FM2			FM3		
	Buckets	**Sec+**	**Sec-**	**Buckets**	**Sec+**	**Sec-**	**Buckets**	**Sec+**	**Sec-**	**Buckets**	**Sec+**	**Sec-**
	21	21	21	17	17	17	20	20	20	24	24	24
OKEQT	0.05	0.24	0.10	0.29	1.00	0.71	0.00	0.05	0.05	0.00	0.00	0.00
OKCALC	0.90	0.29	0.05	0.59	0.00	0.00	0.60	0.45	0.10	0.88	0.50	0.17
OKT&E	0.00	0.00	0.00	0.06	0.00	0.00	0.00	0.05	0.00	0.00	0.17	0.00
WEQT		0.14	0.48		0.00	0.29		0.00	0.00		0.04	0.00
WCALC		0.24	0.19		0.00	0.00		0.15	0.40		0.13	0.71
NATT	0.00	0.10	0.19	0.06	0.00	0.00	0.20	0.30	0.45	0.04	0.17	0.13
OK	0.95	0.52	0.14	0.94	1.00	0.71	0.60	0.55	0.15	0.88	0.67	0.17
WRONG	0.05	0.38	0.67	0.00	0.00	0.29	0.20	0.15	0.40	0.08	0.17	0.71
NATT	0.00	0.10	0.19	0.06	0.00	0.00	0.20	0.30	0.45	0.04	0.17	0.13

PATTERN-SALESPERSON-SECRET NUMBER

	AH7	AH7	AH7	AH8	AH8	AH8	FM2	FM2	FM2	FM3	FM3	FM3
	Pattern	Salesp.	SecNo	Pattern	Salesp.	SecNo	Pattern	Salesp.	SecNo	Pattern	Salesp.	SecNo
	16	16	16	19	19	19	17	17	17	25	25	25
OKEQT	0.31	0.00	0.38	0.58	0.16	0.79	0.00	0.00	0.00	0.00	0.00	0.04
OKCALC	0.06	0.75	0.19	0.00	0.79	0.11	0.18	0.65	0.00	0.60	0.84	0.56
OKT&E	0.06	0.00	0.00	0.05	0.00	0.00	0.00	0.12	0.06	0.04	0.04	0.04
WEQT	0.13	0.00	0.19	0.11	0.05	0.11	0.00	0.00	0.00	0.00	0.00	0.00
WCALC	0.43	0.19	0.19	0.21	0.00	0.00	0.53	0.12	0.24	0.32	0.04	0.16
NATT	0.00	0.06	0.06	0.05	0.00	0.00	0.29	0.12	0.71	0.04	0.08	0.20
OK	0.44	0.75	0.56	0.63	0.95	0.89	0.18	0.76	0.06	0.64	0.88	0.64
WRONG	0.56	0.19	0.38	0.32	0.05	0.11	0.53	0.12	0.24	0.32	0.04	0.16
NATT	0.00	0.06	0.06	0.05	0.00	0.00	0.29	0.12	0.71	0.04	0.08	0.20

Annex E
Overall facility levels for all problems in the main experimental study

Overall facility levels

Questions		Location in test papers	% of correct answers
Sandwiches			12
Driving [2.7]	✓	A1	16
Seesaw [4x]	✓	A2	22
Slope B		A2	24
x+3y, x-3y	✓	C2	26
120-13n=315	✓	A1	27
Tickets [2.7]	✓	B1	30
Seesaw [11-5]	✓	B2	32
Sale [x, 4x]	✓	A1	33
x+y, x-y	✓	B1	35
181-12n=128-7n	✓	B1	37
Sale [11-5]	✓	C2	39
Clothes combin.		C1	39
Carp [1-2]	✓	B1	40
Speed B		C2	46
Choc [a+3b, a-3b]	✓	C1	48
Pattern	✓	B1	49
Slope A		A2	50
TV [commutativ.]		A2	50
20-(-10)		A1	56
6n+165=63	✓	B2	56
Speed A		C2	57
Carp [1-1]	✓	B2	58
Driving [4]	✓	B2	64
181-12n=97	✓	A2	67
Tickets [4]	✓	A2	71
Oranges B		A1	78
25-37		A1	83
Salesperson	✓	B2	84
Buckets	✓	A1	84
Oranges A		A1	88
Candies		A2	94

(Not all locations provided; marked items are analaysed in the dissertation)

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