Some results associated with random walks

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Abstract

In this thesis we treat three problems from the theory and applications of random walks.

The first question we tackle is from the theory of the optimal stopping of random walks. We solve the infinite-horizon optimal stopping problem for a class of reward functions admitting a representation introduced in Boyarchenko and Levendorskii [1], and obtain closed expressions for the expected reward and optimal stopping time. Our methodology is a generalization of an early paper by Darling et al. [2] and is based on probabilistic techniques; in particular a path decomposition related to the Wiener-Hopf factorization. Examples from the literature and perturbations are treated to demonstrate the flexibility of our approach.

The second question is related to the path structure of lattice random walks. We obtain the exact asymptotics of the variance of the self-intersection local time V_n, which counts the number of times the paths of a random walk intersect themselves. Our approach extends and improves upon that of Bolthausen [3], by making use of complex power series. In particular we state and prove a complex Tauberian lemma, which avoids the assumption of monotonicity present in the classical Tauberian theorem. While a bound of order $O(n^2)$ has previously been claimed in the literature ([3], [4]), we argue that existing methods only show the upper bound $O(n^2 \log n)$, unless extra conditions are imposed to ensure monotonicity of the underlying sequence. Using the complex Tauberian approach we show that $\mathrm{Var}(V_n) \sim Cn^2$, thus settling a long-standing misunderstanding.

Finally, in the last chapter, we prove a functional central limit theorem for one-dimensional random walk in random scenery, a result conjectured in 1979 by Kesten and Spitzer [5]. Essentially random walk in random scenery is the process defined by the partial sums of a collection of random variables (the random scenery), sampled by a random walk. We show that for $\mathbb{Z}$-valued random walk attracted to the symmetric Cauchy law, and centered random scenery with second moments, a functional central limit theorem holds, thus proving the Kesten and Spitzer [5] conjecture which had remained open since 1979. Our proof makes use of the asymptotic results obtained in the Chapter 3.

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*"The philosophers have only interpreted the world, in various ways;
the point, however, is to change it.*

Karl Marx.

Τι συμφορά, ενώ είσαι καμωμένος
για τα ωραία και μεγάλα έργα
η άδικη αυτή σου η τύχη πάντα
ενθάρρυνσι κ' επιτυχία να σε αρνείται·
να σ' εμποδίζουν ευτελείς συνήθειες,
και μικροπρέπειες, κι αδιαφορίες.
Και τι φρικτή η μέρα που ενδίδεις,
(η μέρα που αφέθηκες κ' ενδίδεις),
και φεύγεις οδοιπόρος για τα Σούσα,
και πιαίνεις στον μονάρχην Αρταξέρξη
που ευνοϊκά σε βάζει στην αυλή του,
και σε προσφέρει σατραπείες και τέτοια.
Και συ τα δέχεσαι με απελπισία
αυτά τα πράγματα που δεν τα θέλεις.
Άλλα ζητεί η ψυχή σου, γι' άλλα κλαίει·
τον έπαινο του Δήμου και των Σοφιστών,
τα δύσκολα και τ' ανεκτίμητα Εύγε·
την Αγορά, το Θέατρο, και τους Στεφάνους.
Αυτά πού θα σ' τα δώσει ο Αρταξέρξης,
αυτά πού θα τα βρεις στη σατραπεία·
και τι ζωή χωρίς αυτά θα κάμεις.

Η Σατραπεία,
Κωνσταντίνος Π . Καβάφης (1910).

Dedicated to my parents.

Contents

CHAPTER 1

Introduction

A random walk is defined as the partial sums process $\{S_n\}_{n\geqslant 0}$ of a collection of independent and identically distributed random variables $X_1, X_2, \dots,$

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad \text{for } n \geqslant 1. \tag{1.0.1}$$

Random walk is a fundamental example in Markov processes, processes with stationary independent increments and in certain cases of martingales thus bringing together methods from both analysis and probability.

There are two obvious reasons for the widespread study of random walks. First, the rules that govern it are simple and, as is often the case in mathematics, simple rules allow for very rich behaviour with far reaching consequences. Second, it allows for a great variety of applications. Any system that evolves through small random changes can be modelled by a random walk, or by its continuous time analogues, Brownian motion, Lévy processes and diffusions.

Random walk is a central theme for this thesis. We shall consider three open problems from the theory and applications of random walks. The methods used for the proofs vary from probabilistic to analytical. However a unifying theme is the use of characteristic functions and their probabilistic interpretation. Let us now give a brief description of the questions we shall attempt to answer.

1.1 Optimal stopping

The first question treated in this thesis is from the theory of optimal stopping of one-dimensional random walks.

Intuitively an *optimal stopping problem* consists in choosing a time to take a particular action, in order to maximize an expected reward, or to minimize an expected cost. To give a precise description of the situation we need the concept of *stopping times*. Assume we are given a stochastic process $\{S_t : t \geqslant 0\}$ and its natural filtration $\{\mathcal{F}_t : t \geqslant 0\} = \sigma(S_s : s \leqslant t)$. Then an $\{\mathcal{F}_t\}$-stopping time is a random variable $\tau \in [0, \infty]$ such that the events $\{\tau \leqslant t\}$ are $\mathcal{F}_t$-

measurable for every $t \in [0, \infty)$. This technical condition captures the important feature that the decision to stop at the time t must be based only on prior information. In other words, we must be able to decide whether or not to stop at time t having only observed the history of the process until time t, $\mathcal{F}_t = \sigma(S_s : s \leqslant t)$, without having to "look into the future".

In this setting an optimal stopping problem can be mathematically formulated as the optimization problem

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[\beta^{\tau} G(x + S_{\tau})], \tag{1.1.1}$$

where the supremum is taken over the set $\mathcal{T}$ of permissible $\mathcal{F}_t$-stopping times. Let us now have a look at the other quantities involved in the above equation. The function G is the *reward function*, β is a discount factor, and V is the *value function*. The reward function captures the essential features of a cost or reward, while the discount factor models the cost of waiting; in financial applications it is often interpreted as the interest rate. The value function is the optimal expected reward. A *solution* to (1.1.1) consists in finding the stopping time τ^* where the supremum is attained and the value function $V(x)$.

Optimal stopping problems can be divided in two classes, *finite-horizon* problems, and *infinite-horizon*, or perpetual problems. In the first case we require $\tau \leqslant T$ for some fixed $T \geqslant 0$, while in the second case stopping times can take values in $[0, \infty)$. Finite-horizon problems are closely related with *free-boundary problems*, which are boundary value problems, where the boundary is an unknown function of time. While these rarely admit explicit solutions, this is not the case for the perpetual problem. For this reason, in this thesis we shall restrict our attention to the infinite-horizon case.

Optimal stopping problems first appeared in the sequential analysis of statistical observations with Wald's theory of sequential probability ratio tests in his seminal paper [6]. In essence this represented a method of statistical inference where the number of observations was not predetermined, but was instead decided during the experiment. Snell [7] was the first to formulate a general optimal stopping problem for discrete time processes, and also obtained an elegant characterization of the solution as the *minimal* supermartingale dominating the process $\{\beta^n G(S_n)\}_{n \geqslant 0}$, well known as *Snell's envelope*. Snell's envelope is part of a large collection of optimal stopping techniques which take advantage of the probabilistic structure of the process and its unconditional finite-dimensional distributions, known as the *martingale approach*.

On the other hand, the *Markovian approach* exploits the analytical structure of the conditional *transition functions*, which take into account the starting point of the process. This approach is closely related with the *Wald-Bellman equation*, satisfied by the value function

$$V(x) = \max(G(x), \beta V(x + S_1)).$$

This equation was introduced by Arrow et al. [8] and Wald [6] and is based on the principles of dynamic programming and the concept of "backwards induction". The Markovian method has proved to be very effective since, it brings in powerful analytical tools from the theory of differential and integral equations through the infinitesimal generator of the process. The intimate relation between optimal stopping and free-boundary problems was explored in the 60's by Mikhalevich [9] and a host of other authors. Of particular importance is the work of McKean [10] who found the price of the *American call option*, a well known problem from mathematical finance, as the solution to the optimal stopping problem (1.1.1) with

$$G(x) = (x - K)^+ \stackrel{\text{def}}{=} \max(x - K, 0),$$

for geometric Brownian motion, which he then formulated as a free-boundary problem.

Apart from the vast theoretical progress achieved over the past fifty years (see Peskir and Shiryaev [11] for an excellent overview), optimal stopping has found numerous applications in economics, operational research and mathematical finance. The huge success of mathematical finance, attracted further attention to optimal stopping as an invaluable tool for pricing financial options.

Despite the progress made explicit expressions for the value function and the optimal stopping time are rarely available. The finite-horizon case corresponds to free-boundary problems, which in general do not admit explicit solutions. However, in the infinite-horizon case, particular cases have been successfully treated in the literature. Darling et al. [2] solved (1.1.1), in the context of random walks, for the reward functions $G(x) = x^+$, and $G(x) = (e^x - 1)^+$, and obtained closed formulas for the value function and optimal stopping time. The authors proposed candidates for V and τ, in terms of the extrema of the random walk, and then used a *supermartingale verification lemma* (Lemma 2.2.1) to prove that the candidates are indeed optimal. The same approach was more recently used by Mordecki [12] to solve the case $G(x) = (K - e^x)^+$, corresponding to the *perpetual American put option*, while for the case $G(x) = (x^+)^\nu$, with $\nu > 0$, explicit solutions were given in Novikov and Shiryaev [13, 14] and Kyprianou and Surya [15] for random walks and Lévy processes.

In all of the above, candidates are proposed in terms of the extrema of the process, and are then shown to be optimal using a supermartingale lemma, and a decomposition of the paths of the process related to the *Wiener-Hopf factorization*. These similarities suggest that general reward functions can be treated using the same techniques. However no method was given for finding the candidates, and the particular choices were not justified, making generalizations impossible.

In a different direction Boyarchenko and Levendorskii [16, 17] proposed a representation of the reward function as a transformation

$$G(x) = \sum_{n=0}^{\infty} \beta^n \, \mathbb{E}\left[g(x + S_n)\right] \tag{1.1.2}$$

of some auxiliary function g, which we shall call the *payoff function*. This representation was introduced and developed in a series of papers by Boyarchenko and Levendorskii [1, 16, 17, 18] in the context of random walks and Lévy processes, where the authors develop a systematic method for choosing candidates for the optimal stopping time and the value function, for any reward function that satisfies (1.1.2). In sharp contrast to [2, 12–15], Boyarchenko and Levendorskii [16] used the Markovian approach to prove optimality by showing that the proposed value function solves the boundary value problem associated with (1.1.1), using an analytical version of the Wiener-Hopf factorization for pseudo-differential operators. In this way the authors solved the optimal stopping problem (1.1.1) for a fairly wide class of reward functions. In the discrete case, they considered random walks whose increments admit a density and obtained an explicit solution for the optimal stopping problem in the class of *hitting times* of semi-infinite intervals. Recall that a hitting time of a set is the first time the process enters the set. Global optimality in the class of all stopping times was obtained for monotone payoffs, under some regularity conditions on the density of the increments. In the continuous case, they obtained an explicit solution, once again in the class of hitting-times of semi-infinite intervals, with global optimality in the class of all stopping times established for monotone payoffs. The approach is analytical in flavour, and requires a lengthy proof and fairly strong assumptions.

Our approach is a combination of the two approaches described above and is based on two key features, the *Wiener-Hopf factorization* (WHf) and the representation of the reward function adapted from Boyarchenko and Levendorskii [16].

The term "Wiener-Hopf factorization" refers to a collection of results concerning the Laplace transform of the characteristic function of a random walk, and is the culmination of a number of works which include Spitzer [19, 20], Feller [21], Greenwood and Pitman [22], while the relation with Lévy processes was developed in Pecherski and Rogozin [23], Greenwood and Pitman [24](see also Bingham [25] for an overview).

The roots of the WHf can be traced back in time to analytical work by Paley and Wiener [26] and Hopf [27] on the solutions of the integral equation

$$Q(x) = \int_0^\infty Q(y) f(x-y)\,\mathrm{d}y, \quad x > 0, \tag{1.1.3}$$

where f is a given kernel. The complex analytic techniques used to analyze (1.1.3) include the factorization of Fourier transforms and other operators, which in our setting is interpreted as a factorization of the characteristic function. In the context of random walks and Lévy processes, the probabilistic interpretation of this factorization, is a decomposition of the path in terms of two independent parts.

The formulation we shall use is closely related with this decomposition, and was introduced in Greenwood and Pitman [22] in terms of an independent geometric time. This geometric time fits in very nicely with the representation (1.1.2) which can also be written as

$$G(x) = \sum_{n=0}^{\infty} \beta^n\,\mathbb{E}\left[g(x+S_n)\right] = \frac{1}{1-\beta}\mathbf{E}g(x+S_T), \tag{1.1.4}$$

where T is a geometric random time with parameter β, independent of the process. The path of the random walk up to this geometric time is decomposed in terms of two independent parts, equal in distribution to the infimum and supremum respectively. This decomposition is fundamental and rooted in key properties of random walks, namely stationary and independent increments, and the memoryless property of the geometric distribution.

To be more precise, let $\{S_n\}_{n\in\mathbb{Z}^+}$ be the random walk defined in (1.0.1), β the discount factor, and assume $T = T_\beta$ (*geometric time*) has distribution $\mathbb{P}(T \geqslant k) = \beta^k$ for all $k \geqslant 0$. We denote the extrema of the path stopped at the geometric time T by

$$I \equiv I_\beta = \inf_{0\leqslant n\leqslant T} S_n \quad \text{and} \quad M \equiv M_\beta = \sup_{0\leqslant n\leqslant T} S_n.$$

The key fact of the Wiener-Hopf factorization that we shall use is that $S_T - M$ is independent of M and equal in distribution to I,

$$S_T - M \perp\!\!\!\perp M, \quad S_T - M \overset{\mathrm{D}}{=} I.$$

In terms of characteristic functions this is written as

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}tS_T}\right] = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}t(S_T-M)}\right]\mathbb{E}\left[\mathrm{e}^{\mathrm{i}tM}\right] = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}tI}\right]\mathbb{E}\left[\mathrm{e}^{\mathrm{i}tM}\right].$$

The Wiener-Hopf decomposition provides us with an invaluable tool for verifying optimality of given candidates for the value function and the stopping time. However, the major shortcoming of this method is that it does not provide us with a consistent method for choosing these candidates. This brings us to the second key feature of our approach, which is the representation (1.1.2) of the reward function introduced in Boyarchenko and Levendorskii [16]. This representation provides us with explicit expressions for the candidate value function and stopping time. In addition, the fact that the candidates are given in terms of the extrema up to the geometric time T, makes the Wiener-Hopf decomposition a natural choice, and thus creates an important link between the two approaches.

Overall our methodology brings together the probabilistic approach developed in [2, 12–15] and the analytical approach in [16]. We use the representation (1.1.2) to obtain the candidates for the optimal stopping time and value function in terms of the payoff function g, and the extrema of the random walk. Then rather than using boundary value problem techniques, we employ the verification lemma used in Darling et al. [2], and the Wiener-Hopf path decomposition of the random walk to prove that indeed the candidates are optimal. Global optimality, as opposed to optimality in the class of hitting times, of the proposed stopping time is shown for general random walks thus improving upon Boyarchenko and Levendorskii [16]. On the other hand, the representation (1.1.2) allows us to treat a fairly general class of reward functions, and gives a precise method for choosing the candidates, thus avoiding the major shortcoming of the probabilistic approach proposed in Darling et al. [2], Novikov and Shiryaev [13], Kyprianou and Surya [15].

We first solve (1.1.1) in the setting of random walks since they are simpler than continuous time processes while still allowing for rich behaviour. On the other hand random walks share a key property with Brownian motion and Lévy processes, namely stationary and independent increments. From our analysis, it turns out that this property is essential for obtaining closed formulas, allowing us to extend our results to continuous time and Lévy processes. Finally several examples are given to demonstrate the flexibility of our method. Various classes of reward functions are treated, including linear, exponential and power functions, and some perturbations. In a more applied direction, we also obtain the price of a *Canadian option*, a generalization of perpetual American options, which features in a numerical scheme for approximating American options(finite-horizon) introduced by Carr [28]. Our solution is for a general random walk and can be extended to Lévy processes with jumps of both signs, improving upon the existing literature where spectrally one-sided Lévy processes were treated (see for example [29]).

The results presented in this chapter are joint work with H. Le and S. Utev and have been published at the *Journal of Applied Probability* [30].

1.2 Asymptotics for self-intersections of random walks

The second object treated in this thesis is the asymptotic variance of the *self-intersection local time* of random walks on the integer lattice $\mathbb{Z}^d$, for $d = 1, 2$. Given a random walk $\{S_n\}_{n\geqslant 0}$ in the integer lattice $\mathbb{Z}^d$, the self-intersection local time V_n counts the number of times the paths of the random walk intersect themselves,

$$V_n = \sum_{i,j=0}^{n} \mathbf{1}_{\{S_i=S_j\}} = \sum_x N_n^2(x),$$

where $N_n(x) = \sum_{i=0}^{n} \mathbf{1}_{\{S_i=x\}}$ is the *local time* at point $x \in \mathbb{Z}^d$ up to time n.

The self-intersection characteristic V_n features prominently in the study of the *self-avoiding walk* (SAW), a path that never visits the same site more than once. The SAW originated in statistical physics and the theory of critical phenomena and it is the simplest model for linear polymer molecules; these are long chains of smaller molecules called monomers, which tend to spread out as much as possible demonstrating the *excluded-volume effect*. Despite their simple definition, SAWs have raised many interesting questions (see Lawler [31] for an excellent account) which are notoriously difficult to solve. The reason for this difficulty is the fact that the SAW cannot be defined as a stochastic process in terms of transition probabilities, and thus it cannot be treated by standard Markovian techniques.

A natural way to avoid this problem is by looking instead at a simple random walk whose paths have been reweighed according to the number of intersections V_n. The resulting model is known as a *weakly self-avoiding walk* (WSAW) or the *Domb-Joyce model*. The most common *polymer measure* used to suppress self-intersections is defined in terms of the characteristic V_n

$$\frac{\mathrm{d}\mathbf{Q}_n}{\mathrm{d}\mathbf{Q}} = c_n e^{\zeta V_n},$$

where $\mathbf{Q}$ is the random walk measure which assigns every path equal probability and c_n is a normalizing constant. The *temperature constant* ζ controls the self-avoiding effect; negative values result in self-avoiding walks by imposing a penalty on paths with high values of V_n, while positive values produce a self-attracting effect. For an overview of such models see [32]. A different approach was recently given in Mörters and Sidorova [33] where rather than using a change of measure, the self-intersection characteristic was conditioned to stay relatively small, and the distribution of the resulting paths was studied through the asymptotics of V_n.

Apart from SAW's, the asymptotics of V_n are also essential in the study of *random walk in random scenery*(RWRS). RWRS is defined as the partial sums of a collection of iid variables sampled by a random walk, and was first introduced in Kesten and Spitzer [5]. To be more precise, if S_n is a random walk in $\mathbb{Z}^d$ and $\{\xi_x\}_{x\in\mathbb{Z}^d}$ an independent collection of iid real random variables (*random scenery*), then by RWRS we shall mean the process

$$Z_0 = 0, \quad Z_n = \sum_{i=1}^{n} \xi(S_i), n \geqslant 1.$$

The connection between RWRS and local times becomes apparent through the expression

$$Z_n = \sum_{i=1}^{n} \xi(S_i) = \sum_{x} N_n(x)\xi(x),$$

To illustrate the relation with the self-intersection characteristic V_n we now give a simple example taken from Kesten and Spitzer [5].

Example 1.2.1. Assume S_n is a one-dimensional simple random walk, and let the random scenery take the values ± 1, each with probability $1/2$. Then we have

$$\begin{aligned}
\mathrm{Var}(Z_n) &= \mathbb{E}(\sum_{x\in\mathbb{Z}} \xi(x)N_n(x))^2 \\
&= \mathbb{E}\sum_{x\in\mathbb{Z}} N_n^2(x) = \mathbb{E}\sum_{x\in\mathbb{Z}}\Big(\sum_{k=0}^{n} 1_{\{S_k=x\}}\Big)^2 \\
&= \mathbb{E}\sum_{x\in\mathbb{Z}}\sum_{k=0}^{n}\sum_{j=0}^{n} 1_{\{S_k=S_j=x\}} = \mathbb{E}\sum_{k=0}^{n}\sum_{j=0}^{n} 1_{\{S_k=S_j\}} = \mathbb{E}V_n.
\end{aligned}$$

It should by now be obvious that the statistical behaviour of V_n features in many calculations concerning the limiting distribution of RWRS. Asymptotic results for the mean and variance of V_n have been used to prove functional limit theorems for Z_n in the literature (see [3, 5, 34, 35]). In fact we shall make use of such results in Chapter 4 to prove that, when appropriately normalized, Z_n satisfies a functional central limit theorem, a result conjectured by Kesten and Spitzer [5].

In this thesis we shall be primarily concerned with $\mathrm{Var}(V_n)$, for which we shall obtain an exact asymptotic for one and two-dimensional random walks. The relation with RWRS will be explored further in the last chapter, where we shall prove a functional central limit theorem for one-dimensional RWRS making heavy use of the results presented in this chapter.

The origins of the study of the path structure of random walks can be traced back to Pólya [36] who proved the remarkable fact that a simple random walk is recurrent in dimensions one and two, and transient in higher dimensions. The topic truly flourished during the 50's and 60's attracting much attention from the mathematical community. It is worth noting the significant contributions of P. Erdös, who published a series of papers on the simple random walk and its path structure (see for example Dvoretzky and Erdos [37], Erdős and Taylor [38, 39], Erdős and Révész [40]). Despite the vast research conducted, the topic is by no means saturated still hosting a number of open questions. One such case, the Erdös-Taylor conjecture which states that the largest local time is of the order of $\log(n)^2$, was formulated in 1960 (Erdős and Taylor [38]) but was not proved until recently by Dembo et al. [41] using techniques from fractal measures.

The study of the structure of random walk paths brings together techniques from both analysis and probability and allows a delicate interplay between the theory of Markov processes, harmonic analysis, ergodic and potential theory. It is one of the most elegant topics in the theory of random walks featuring a number of beautiful results and a vast literature, which would be impossible to list in its entirety. Thus, for the remaining of this section, we shall restrict our attention to those results which are most relevant to self-intersections and local time asymptotics.

In 1979, in a paper which introduced random walk in random scenery, Kesten and Spitzer [5] considered the one-dimensional random walk with increments X_i

$$\mathbb{E}X_i = 0, \quad \mathbb{P}[n^{-1/\alpha}S_n \leqslant x] \to F_\alpha(x),$$

attracted to the stable distribution F_α with parameter $1 < \alpha \leqslant 2$, and obtained asymptotics of the form

$$\mathbb{E}V_n = \sum_{x\in\mathbb{Z}} \mathbb{E}N_n^2(x) \sim Cn^{2-\frac{1}{\alpha}}, \quad \mathbb{E}[N_n^\nu(0)] \sim C_\nu n^{\nu-\frac{\nu}{\alpha}}, \quad n\to\infty, \nu = 1,2,3,\dots.$$

Their proofs were based on a class of results known as *local limit theorems* which describe the asymptotic behaviour of the probability of return to the origin. For a random walk in the domain of attraction of α-stable law ($1 < \alpha \leqslant 2$), Stone [42] showed that

$$\mathbb{P}[S_n = 0] \sim Cn^{-\frac{1}{\alpha}}, \quad n \to \infty,$$

while for mean zero random walk with second moments Spitzer [43] showed that

$$\mathbb{P}[S_n = 0] \sim C(2\pi n)^{-\frac{d}{2}}, \quad n \to \infty.$$

Local limit theorems for simple random walks were given in Lawler [31, Section 1.2], where they were then used (see Lawler [31, Chapter 6]) to obtain asymptotic bounds on V_n in the context of the *Edwards model* and self-avoiding walks

$$\mathbb{E}V_n \sim \begin{cases} Cn^{\frac{3}{2}}, & d=1 \\ Cn\log n, & d=2 \\ Cn, & d\geqslant 3, \end{cases}$$

while for $d = 2$ we have $\mathrm{Var}(V_n) = O(n^2)$.

Using a similar approach, Bass et al. [44] proved that for two-dimensional random walk with finite variance we have

$$\mathbb{E}V_n = Cn\log n + o(n\log n),$$

while under slightly stronger moment conditions the error term was shown to be of the order $O(n)$. These estimates were then used to obtain a law of the iterated logarithm, and a *large deviations principle* for the self-intersection local time V_n. Using a significantly different methodology, Chen [45] and Chen and Rosen [46] proved results of a similar nature. The self-intersection local times of random walks were approximated by those of Brownian motion through the use of the invariance principle, and the asymptotics were then derived from equivalent results for continuous-time processes. More recently, Mörters and Sidorova [33] gave large and moderate deviation results for the slightly more general p-fold intersection local times

$$\Lambda_n(p) = \sum_x N_n^p(x),$$

and introduced a new class of weakly self-avoiding walks, resulting from simple random walk with $\Lambda_n(p)$ conditioned to stay small.

While local limit theorems have been very successful in proving bounds for the statistics of V_n, these usually require strong moment conditions to provide good convergence estimates. Bolthausen [3], used a completely different approach, based on Karamata's Tauberian theory, which avoids the use of local limit theorems. More specifically Bolthausen [3] considered the power series

$$g(\lambda) = \sum_{i=0}^{\infty} \mathrm{Var}(V_i)\lambda^i, \quad \lambda \in (0,1),$$

and its asymptotic behaviour as $\lambda \to 1-$. The author then used Karamata's Tauberian theorem for power series ([21, Theorem XIII 5.5]) in order to deduce the asymptotic behaviour of $\sum_{i=0}^{n} \mathrm{Var}(V_i)$, and $\mathrm{Var}(V_n)$, as $n \to \infty$, and proved that the centered random walk in $\mathbb{Z}^2$ with finite variance satisfies

$$\mathbb{E}V_n \sim Cn\log n, \quad \mathrm{Var}(V_n) = O(n^2\log n),$$

for some constant C. This approach was recently extended Černý [4] to treat the asymptotics of the variance of $\Lambda_n(p)$, obtaining the result of [3] as a special case. The improved bound $\mathrm{Var}(V_n) = O(n^2)$ is shown under the additional condition that the random walk distribution is symmetrized.

Our methodology is based on and extends that of Bolthausen [3] in that we also consider the power series

$$g(\lambda) = \sum_{i=0}^{\infty} \lambda^i \,\mathrm{Var}(V_i),$$

avoiding the use of local limit theorems. From that point on though, our approach deviates significantly from that of Bolthausen [3]. In particular rather than appealing to Karamata's Tauberian theorem, we will state and prove a much different complex Tauberian result, which considers the behaviour of the power series on the open unit disc, rather than on $(0, 1]$. Related results (see Flajolet and Odlyzko [47, Theorem 4]) have been previously used in the context of combinatorial analysis, and are closely related with Darboux's lemma (see Knuth and Wilf [48]), which deduces the asymptotics of the Taylor coefficients of an analytic function from its asymptotic expansion around its singularity nearest to the origin. The reason for this detour is one of the basic assumptions of Karamata's Tauberian theorem, namely the monotonicity of the underlying sequence. Specifically, the Tauberian theorem can deduce the asymptotics of the sequence a_n from it's z-transform $\sum_{i=0}^{\infty} a_n \lambda^n$, but only if a_n is a monotone sequence. In the general case the underlying sequences that come up in the study of $\mathrm{Var}(V_n)$ are essentially non-monotone, thus prohibiting the use of classical Tauberian results.

Bolthausen [3] avoided this technical difficulty by bounding the variance above by a monotone sequence. While this allowed him to invoke the Tauberian theorem, the best bound possible for $\mathrm{Var}(V_n)$ is of the order of $n^2 \log n$. On the other hand Černý [4] did not consider an upper bound, but instead attempted to obtain an exact asymptotic, overlooking the monotonicity condition. The result is true for symmetrized distributions, a condition which guarantees the monotonicity of the underlying sequence. Even though for the general case the best bound available is $O(n^2 \log n)$, the $O(n^2)$ bound given in Lawler [31] for simple random walk, suggests that a tighter bound should be available. As we shall see in Chapter 3 this is indeed the case.

In terms of methodology, the main contribution of Chapter 3 is the application of the complex Tauberian Lemma 3.1.2. At the extra cost of having to bound certain integrals involving characteristic functions, our approach improves upon existing techniques in two aspects. First we completely avoid the monotonicity assumption, which allows us to treat situations inaccessible to the classical Tauberian theory. Second, our complex Tauberian approach keeps track of smaller order terms, which turn out to be of vital importance for obtaining the correct asymptotic for the variance of V_n. By repeated use of an expansion of the characteristic function we derive an asymptotic expansion for $\mathrm{Var}(V_n)$, and we use Lemma 3.1.2 in order to control the lower order terms. From the resulting expression the exact asymptotic is immediately available. It is very important to stress that our approach is flexible and with only slight modifications can be applied to a range of similar situations, such as for example local limit and renewal theorems (see Deligiannidis and Utev [49]).

In terms of results, the main contribution of Chapter 3 is the exact asymptotic for the variance of V_n for one and two dimensional random walks. In two dimensions, our assumptions coincide with those of Bolthausen [3], Černý [4]. The proof in Bolthausen [3] is completed, and extended by showing that $\mathrm{Var}(V_n) \sim Cn^2$, and thus that $O(n^2)$ is the best possible bound. The exact constant is also calculated. For the one-dimensional case, we obtain the same result for random walks in the domain of attraction of the symmetric Cauchy distribution, which corresponds to the α-stable law with $\alpha = 1$. This case has not been treated before, and the resulting asymptotics are applied in Chapter 4 to prove a functional central limit for one-dimensional RWRS, thus proving a conjecture by Kesten and Spitzer [5].

The results of this chapter are joint work with S. Utev and have been prepared as a preprint

([50]). They will soon be submitted for publication.

1.3 A central limit theorem for random walk in random scenery

The last result presented in this thesis is a central limit theorem for one-dimensional random walk in random scenery (RWRS). Let $\{S_n\}_{n\geqslant 0}$ be a random walk in $\mathbb{Z}^d$ and $\{\xi_x\}_{x\in\mathbb{Z}^d}$ a collection of iid real random variables, independent of S_n, which we shall call the *random scenery*. Then by RWRS we shall mean the process

$$Z_0 = 0, \quad Z_n = \sum_{i=1}^{n} \xi(S_i),$$

ie the partial sums of the random scenery indexed by the random walk.

The idea of random variables sampled by a random walk is certainly not new. Spitzer [43] first proved that for arbitrary transient random walk, $n^{-1/2}Z_n$ is asymptotically normal. However, the term RWRS didn't appear until 1979 in Kesten and Spitzer [5], where a more general one-dimensional random walk was treated. Kesten and Spitzer [5] proved that the scaled process $n^{-1/\delta}Z_{[nt]}$ satisfies a functional limit theorem, where the resulting limits were then used to obtain a new class of self-similar processes. The two-dimensional case was treated by Bolthausen [3] obtaining a Gaussian limit.

The dependence of the sampled scenery introduced by the random walk, implies that RWRS can be considered as part of the limit theory of dependent variables. In the transient case the effect of resampling is not significant, and hence the normalization for the central limit theorem, $n^{1/2}$, is the same as in the independent case (see Spitzer [43, p. 52]). In the recurrent case however, the resampling effect kicks in, requiring normalization of the order of $\sqrt{n\log n}$. In the direction of dependent variables, Guillotin-Plantard and Prieur [35, 51] treated a more general situation by dropping the assumption of independence from the random scenery; weakly dependent sceneries were treated for both transient and recurrent random walks. Finally it is worth noting that RWRS is also closely related with random ergodic theorems which deal with ergodic averages of measure preserving flows sampled by random walks (see Lacey et al. [52]).

Since 1979 and the introduction of RWRS by Kesten and Spitzer [5] several limit theorems have appeared in the literature, usually dealing with the weak convergence of $c(n)Z_{[nt]}$, where $c(n)$ is some normalizing sequence.

Kesten and Spitzer [5] considered the one-dimensional case where X_i belongs to the domain of attraction of a stable distribution F_α with parameter $1 < \alpha \leqslant 2$,

$$\mathbb{E}X_i = 0, \quad \mathbb{P}[n^{-1/\alpha}S_n \leqslant x] \to F_\alpha(x),$$

and ξ belongs to the domain of attraction of a stable distribution G_β with parameter $0 < \beta \leqslant 2$,

$$\mathbb{E}\xi(x) = 0, \quad \mathbb{P}[n^{-1/\beta} \sum_{k=1}^{n} \xi(k) \leqslant x] \to G_\beta(x).$$

The authors proved that the process $n^{-\delta} Z_{nt}$, where $\delta = 1 - 1/\alpha + 1/(\alpha\beta)$, converges weakly in $C[0, \infty)$, the space of continuous functions, to the non-Gaussian limit

$$\Delta_t = \int_0^\infty L_t(x)\,\mathrm{d}Z_+(x) + \int_0^\infty L_t(x)\,\mathrm{d}Z_-(x),$$

where $\{Z_+(t); t \geqslant 0\}$ and $\{Z_-(t); t \geqslant 0\}$ are two right-continuous stable processes of index β, and $L_t(x)$ is the local time at x of a right-continuous stable process of index α, independent of $Z_\pm$.

The simpler case with $0 < \alpha < 1$, corresponds to transient random walks and was treated in Spitzer [43], for the random scenery $\xi(x)$ taking the values ± 1 with equal probability. The more general case where β is arbitrary is shortly discussed in Kesten and Spitzer [5] and shown to converge, when appropriately scaled, to a stable process of index β.

The two dimensional random walk with finite non-singular covariance matrix Σ, and centered random scenery with finite positive variance σ^2, was treated in Bolthausen [3] where it was shown that

$$Y_n(t) = \sqrt{\pi}|\Sigma|^{1/4} Z_{[nt]}/\sigma\sqrt{n \log n}$$

converges weakly in $D[0, \infty)$, the space of right-continuous functions, to standard Brownian motion.

More recently, Guillotin-Plantard and Prieur [35, 51] treated the case where the random scenery is assumed to be a family of weakly dependent random variables. When the sampling process is a transient one-dimensional random walk the limiting process was Gaussian, while when the random walk was recurrent the limiting process had a similar form to the one found in [5],

$$\int_0^\infty L_t(x)\,\mathrm{d}Z_+(x) + \int_0^\infty L_t(x)\,\mathrm{d}Z_-(x)$$

where in this case $\{Z_+(t); t \geqslant 0\}$ and $\{Z_-(t); t \geqslant 0\}$ are independent Brownian motions, while L_t is the local time of yet another independent Brownian motion.

In Chapter 4 we shall treat one-dimensional random walk in random scenery. In particular we shall assume that the random walk has characteristic function satisfying the expansion

$$f(t) = \mathbb{E}e^{itX_i} = 1 - \gamma|t| + o(|t|),$$

as $t \to \infty$, while the random scenery is independent with mean zero and finite variance σ^2. Our conditions on the random walk imply that the random walk is recurrent and its increments lie in the domain of attraction of the symmetric Cauchy law, $\alpha = 1$. Even though this case has not been treated in the literature, it was conjectured by Kesten and Spitzer [5] that the normalized sums should converge weakly to standard Brownian motion. We show that indeed the laws of

$$Y_n(t) = \sqrt{\pi\gamma} Z_{[nt]}/\sigma\sqrt{2n\log n}, \quad t \in [0,1],$$

converge weakly to the Wiener measure in $D[0,1]$, thus proving the Kesten-Spitzer conjecture, which has remained open since 1979.

In proving weak convergence we shall repeatedly use the results of Chapter 3 on the variance of the self-intersections characteristic V_n. It is important to note that the asymptotic bound of order n^2, proved in Chapter 3, is essential for the central limit theorem to hold for almost every path of the random walk, in the following sense: if $\mathcal{A} = \sigma(S_k : k \in \mathbb{Z})$ encodes the full history of the random walk then

$$\mathbb{P}\Big(\sum_{i=1}^{n} \xi(S_i)/c\sqrt{n\log n} \leqslant x \Big| \mathcal{A}\Big) \to \int_{-\infty}^{x} \mathrm{e}^{-x^2/2}, \quad \mathbb{P}\text{-almost surely.}$$

In other words for almost every realization of the random walk $\omega = (\omega_1, \omega_2, \dots)$ the partial sums of the sampled scenery

$$\frac{c}{\sqrt{n\log n}} \sum_{i=1}^{n} \xi(\omega_i)$$

satisfy a central limit theorem.

This is in sharp contrast to the results in Bolthausen [3] where the best bound proved was of the order of $n^2 \log n$. The weaker bound implies that the limit $V_n/\mathbb{E}V_n \to 1$, used in the convergence of the finite dimensional distributions, holds only in probability. Our improved bounds allow us to show almost sure convergence, which in turn means that the central limit theorem can be applied for almost every path of the random walk. It is also worth noting, that the bound n^2 claimed in Bolthausen [3] was subsequently used in Cabus and Guillotin-Plantard [34]. Our results of Chapter 3 and their application in Chapter 4, indeed complete the proofs in [3, 34] thus finally settling a long-standing misunderstanding.

The results of this chapter are based on earlier work with S. Utev and have been presented at the *33rd Conference on Stochastic Processes and Their Applications*, held in Berlin in July 2009, and at the *Conference in Memory of Walter Philipp* held in Graz in June 2009. They can also be found in Deligiannidis and Utev [50].

CHAPTER 2

Optimal stopping

2.1 Introduction

Let $\{S_t\}_{t\in\mathcal{I}}$ be a process with stationary independent increments, where the time parameter is either discrete, $t \in \mathcal{I} = \mathbb{Z}^+ = \{0,1,2,\dots\}$, or continuous $t \in \mathcal{I} = \mathbb{R}^+$.

For a given reward function G and discount factor $\beta = \mathrm{e}^{-r}, r \geqslant 0$, we consider the problem of finding a pair (V^*, τ^*) such that $\tau^* \in \mathcal{T}$ and

$$V^*(x) = V(x,\tau^*) = \mathbb{E}\left[\beta^{\tau^*} G^+(x+S_{\tau^*})\right] = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\beta^{\tau} G^+(x+S_{\tau})\right], \tag{2.1.1}$$

where $G^+(x) = \max\{G(x), 0\}$ and $\mathcal{T}$ is the set of all $\mathbf{F}$-stopping times. In the discrete case we define $\mathbf{F} = \{\mathcal{F}_n\}_{n\geqslant 0}$ to be the natural filtration of the random walk, while in the continuous case $\mathbf{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ is the *augmented natural filtration* of $\{S_t\}_{t\in\mathbb{R}^+}$that is *right-continuous* and contains all $\mathbb{P}$-null sets (see Rogers and Williams [53]). Recall that a filtration is right-continuous if for all $t \geqslant 0$ we have $\mathcal{F}_t = \mathcal{F}_{t+}$, where

$$\mathcal{F}_{t+} \stackrel{\text{def}}{=} \bigcap_{h>0} \mathcal{F}_{t+h}.$$

We shall refer to the function $V^*(x)$ as the *value function* of the optimal stopping problem (2.1.1).

We combine the two approaches described in section 1.1 to give an explicit solution, in terms of the extrema of the process, to (2.1.1) for payoff functions which admit the representation (2.2.1)(or (2.2.11) in the continuous case), proving global optimality in the class of all stopping times. This indeed contains all examples treated in the literature as special cases. In section 2.2 we present the main results for random walks. We also give explicit solutions in the continuous time case for Lévy processes, the continuous time analogue of random walks. The proof is very similar and is based on the key properties of stationary and independent increments. We also consider the case when the representation (2.2.11) holds only on the half-line. Examples are then provided in section 2.3 which cover cases from the literature and some perturbations which demonstrate the flexibility of our approach.

2.2 Main results

In this section we give an explicit solution to (2.1.1) for the general class of reward functions which admit the representation (2.2.11) introduced in [1, 17], and we prove that the optimal stopping time is the hitting time of a semi-infinite interval. We propose a candidate function and a stopping time following closely the approach of [1, 17]. However rather than considering the boundary value problem of (2.1.1), we prove optimality using Lemma 2.2.1, thus combining the two approaches mentioned in the introduction. The dependence of the proposed candidate function on the extrema of the process up to an independent geometric time allows us to use results from the Wiener-Hopf factorization of random walks and Lévy processes (c.f. [22, 24]). Using this approach, we rederive the solution to the optimal stopping problem obtained in [1, 17], independently, with a new and significantly simpler proof. We also manage to weaken the assumptions as we do not require monotonicity of the payoff function in order to show global optimality of the stopping time. The results of [2, 13–15, 54] are then shown to be particular cases.

First we treat the problem in discrete time in subsection 2.2.1, and then the continuous time case in subsection 2.2.2.

2.2.1 Optimal stopping in discrete time

Consider the random walk $S_0 = 0$, and $S_n = \sum_{k=1}^{n} X_k$, for $n \geqslant 1$. Let $\mathbf{F} = \{\mathcal{F}_n\}_{n\geqslant 0}$ be the natural filtration of the random walk, ie $\mathcal{F}_n = \sigma(S_k, k \leqslant n)$. We shall in the following assume that the discount factor satisfies $\beta = \mathrm{e}^{-r} < 1$, unless otherwise stated.

The reward functions G that we are interested in are those that have the representation

$$G(x) = \sum_{n=0}^{\infty} \beta^n \,\mathbb{E}\left[g(x + S_n)\right] \tag{2.2.1}$$

for some payoff function g.

Our result is closely linked with a geometric random time $T = T_\beta$ which is independent of the random walk $\{S_n\}_{n\in\mathbb{Z}^+}$ and whose distribution is given by $\mathbb{P}(T \geqslant k) = \beta^k$ for $k \in \mathbb{Z}^+$. Note that, in terms of T, we can rewrite G in (2.2.1) as

$$G(x) = \frac{1}{1-\beta}\,\mathbb{E}\left[g(x + S_T)\right].$$

Let further the random variables I and M be defined as follows

$$I \equiv I_\beta = \inf_{0\leqslant n\leqslant T} S_n \quad \text{and} \quad M \equiv M_\beta = \sup_{0\leqslant n\leqslant T} S_n,$$

the infimum and supremum of the process up to the geometric time T. We emphasize the dependence on β of the objects introduced above when it is convenient.

We state the main result for the discrete time case in Theorem 2.2.2 and we deal with two distinct cases. In the first case we assume that there is an x^* such that for all $x < x^*$, $\mathbb{E}[g(x+M)] > 0$ and non-increasing, while for $x > x^*$, $\mathbb{E}[g(x+M)] \leqslant 0$, where $M \equiv \sup_{n \leqslant T} S_n$. Note that this condition is automatic for any decreasing g which crosses zero once. The second class we consider is defined similarly and contains increasing payoffs which cross zero once. We prove the result using probabilistic arguments and most importantly the Wiener-Hopf factorization of random walks.

Wiener-Hopf factorization for random walks. Greenwood and Pitman [22] have shown that $S_T - M$ is independent of M and equal in distribution to I (see also [55]). Thus in particular,

$$\begin{aligned} \mathbb{E}\left[e^{S_T}\right] &= \mathbb{E}\left[e^{S_T - M}\right]\mathbb{E}\left[e^{M}\right] \\ &= \mathbb{E}[e^{\tilde{I}}]\mathbb{E}\left[e^{M}\right], \quad \text{implying that} \\ S_T &\stackrel{\text{D}}{=} M + \tilde{I}, \end{aligned} \tag{2.2.2}$$

where $\tilde{I}$ is a copy of I independent of the random walk and of T. Note that by considering the reflected random walk $\{\hat{S}_n\}_{n\in\mathbb{Z}^+} = \{-S_n\}_{n\in\mathbb{Z}^+}$, we also have that $S_T - I$ is independent of I and equal in distribution to M.

The starting point of the proof of our main result is the following lemma obtained in [2].

Lemma 2.2.1. *Define the random walk $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, where $X, X_1, X_2, \ldots$ are iid random variables. Let r and f be nonnegative functions and β a constant satisfying $0 \leqslant \beta \leqslant 1$. If for all x*

$$f(x) \geqslant r(x) \quad \textit{and} \quad f(x) \geqslant \mathbb{E}\left[\beta f(x+X)\right],$$

then

$$f(x) \geqslant \mathbb{E}\left[\beta^{\tau} r(x + S_{\tau})\right]$$

for all x and stopping times τ.

In other words in the context of the optimal stopping problem (2.1.1), for a given reward function G and a discount factor $\beta \in [0,1]$, if $f \geqslant 0$ satisfies the conditions of Lemma 2.2.1, then f is an upper bound for the value function of the problem, ie

$$f \geqslant V^*(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\beta^{\tau} G^{+}(x + S_{\tau})\right].$$

When the reward function G has the representation (2.2.1) for some decreasing payoff function g, Boyarchenko and Levendorskii [17] propose the following expression for the value function of (2.1.1)

$$(1-\beta)^{-1}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}} g(x+S_T)\right], \tag{2.2.3}$$

where x^* is such that for all $x < x^*$, we have that $\mathbb{E}[g(x+I)] \leqslant 0$, while for all $x > x^*$, $\mathbb{E}[g(x+I)] \geqslant 0$.

In the next section we intend to use Lemma 2.2.1 and the results of [22] on the Wiener-Hopf factorization of random walks, to prove that (2.2.3) is indeed the value function of (2.1.1). The conditions we impose on the payoff function g and the random walk, are the weakest possible for our method, and weaker than those required in [17].

Theorem 2.2.2. *Assume that the function $G(x)$ can be written in the form (2.2.1) for some function g.*

(i) *If there is an $x^* > -\infty$ such that*

$$\begin{aligned} &\mathbb{E}\left[g(x+M)\right] \text{ is positive and non-increasing,} && \forall x < x^*; \\ &\mathbb{E}\left[g(x+M)\right] \leqslant 0, && \forall x > x^*, \end{aligned} \tag{2.2.4}$$

then the solution to (2.1.1) is given by the stopping time

$$\tau^* = \inf\{n \geqslant 0 : x + S_n \leqslant x^*\} \tag{2.2.5}$$

and can be presented as

$$V^*(x) = \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}g(x+S_T)\right].$$

(ii) *If there is an $x^* < \infty$ such that*

$$\begin{aligned} &\mathbb{E}\left[g(x+I)\right] \leqslant 0, && \forall x < x^*; \\ &\mathbb{E}\left[g(x+I)\right] \text{ is positive and non-decreasing,} && \forall x > x^*, \end{aligned} \tag{2.2.6}$$

then the solution to (2.1.1) is given by the stopping time $\tau^* = \inf\{n \geqslant 0 : x + S_n \geqslant x^*\}$ and can be presented as

$$V^*(x) = \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{x+M\geqslant x^*\}}g(x+S_T)\right].$$

Proof. (i) Writing

$$W(x) = (1-\beta)^{-1}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}g(x+S_T)\right],$$

we would like to show that $W(x) = V^*(x)$. We proceed by first showing that $W(x) \geqslant V^*(x)$. To do this we apply Lemma 2.2.1 and thus we need to show that its conditions are satisfied for the particular choices $r = G^+$ and $f = W$.

By writing $\tilde{M}$ for a copy of M which is independent of the random walk and of the geometric time T, we have

$$
\begin{aligned}
W(x) &= \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}g(x+S_T)\right] \\
&= \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}g(x+(S_T-I)+I)\right] \qquad (2.2.7)\\
&= \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}g(x+\tilde{M}+I)\right],
\end{aligned}
$$

since from (2.2.2) we know that $S_T - I$ is independent of I and equal in distribution to M. However, we have assumed that for all $x > x^*$, $\mathbb{E}[g(x+M)] \leqslant 0$. Thus

$$
\mathbb{E}\left[\mathbf{1}_{\{x+I>x^*\}}g(x+I+\tilde{M})\middle|I\right] \leqslant 0,
$$

from which it follows that

$$
\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{x+I>x^*\}}g(x+I+\tilde{M})\,|I\right]\right] \leqslant 0.
$$

Continuing from (2.2.7) and using the above inequality we have

$$
\begin{aligned}
W(x) &= \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}g(x+\tilde{M}+I)\right] \\
&\geqslant \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}g(x+\tilde{M}+I)\right] \\
&\qquad + \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{x+I>x^*\}}g(x+\tilde{M}+I)\right] \\
&= \frac{1}{1-\beta}\mathbb{E}\left[g(x+I+\tilde{M})\right] \\
&= \frac{1}{1-\beta}\mathbb{E}\left[g(x+S_T)\right] = G(x).
\end{aligned}
$$

The choice of x^* also implies that $W(x) \geqslant 0$ and so what we have actually shown is that $W(x) \geqslant G^+(x)$.

To prove that $W(x)$ satisfies the second condition of Lemma 2.2.1, write

$$
F(x) \equiv \mathbf{1}_{(-\infty,x^*]}(x)\,\mathbb{E}[g(x+\tilde{M})].
$$

We express W in terms of F

$$
W(x) = \frac{1}{1-\beta}\mathbb{E}\left[F(x+I)\right].
$$

Let J be an independent Bernoulli random variable with parameter β. Mordecki [54] has shown that $I \stackrel{\mathrm{D}}{=} -J(X+I)^-$, where x^- is defined to be $\max(-x,0)$ and X is an independent increment of the random walk. Thus

$$
\begin{aligned}
\mathbb{E}\left[F(x+I)\right] &= \mathbb{E}\left[F\left(x-J(X+I)^-\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{J=0\}}F\left(x-J(X+I)^-\right)\right] \\
&\qquad + \mathbb{E}\left[\mathbf{1}_{\{J=1\}}F\left(x-J(X+I)^-\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}\left[\mathbf{1}_{\{J=0\}}F(x)\right] + \mathbb{E}\left[\mathbf{1}_{\{J=1\}}F\left(x-(X+I)^-\right)\right] \\
&= \mathbb{P}(J=0)F(x) + \mathbb{P}(J=1)\,\mathbb{E}\left[F\left(x-(X+I)^-\right)\right] \\
&= (1-\beta)F(x) + \beta\,\mathbb{E}\left[F\left(x-(X+I)^-\right)\right] \\
&\geqslant \beta\,\mathbb{E}\left[F\left(x-(X+I)^-\right)\right] \\
&\geqslant \beta\,\mathbb{E}\left[F(x+X+I)\right],
\end{aligned}
$$

where the fifth equality holds since we have assumed that J is independent of I,X. The first inequality follows from the non-negativity of F, and the second from the facts that $-x^- \leqslant x$ and that F is non-increasing. We have shown that $W(x) \geqslant \mathbb{E}\left[\beta\, W(x+X)\right]$. It now follows from Lemma 2.2.1 that, for all stopping times τ,

$$
W(x) \geqslant \mathbb{E}\left[\beta^{\tau} G^{+}(x+S_\tau)\right],
$$

so that $W(x) \geqslant V^*(x)$.

It remains to show that $W(x) \leqslant V^*(x)$ and that the stopping time τ^* defined in (2.2.5) is indeed optimal. For these, it suffices to show that

$$
W(x) \leqslant \mathbb{E}\left[\beta^{\tau^*} G^{+}(x+S_{\tau^*})\right].
$$

If this holds then we have shown that the supremum in (2.1.1) is attained at τ^* which is thus optimal.

Observe that $x + I \leqslant x^*$ if and only if the process has hit the interval $(-\infty, x^*]$ before or at the time T. Since τ^* is by definition the hitting time of $(-\infty, x^*]$, it must be true that $\{\tau^* \leqslant T\} = \{x + I \leqslant x^*\}$. Thus we have

$$
\begin{aligned}
W(x) &= \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}\, g\Big(x+I+\tilde{M}\Big)\right] \\
&= \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{T\geqslant\tau^*\}}\, g\Big(x+I+\tilde{M}\Big)\right] \\
&= \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{T\geqslant\tau^*\}}\, g\Big(x+S_{\tau^*}+(I-S_{\tau^*})+\tilde{M}\Big)\right] \\
&= \frac{1}{1-\beta}\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{T\geqslant\tau^*\}}\, g\Big(x+S_{\tau^*}+(I-S_{\tau^*})+\tilde{M}\Big)\,|\mathcal{F}_{\tau^*}\right]\right].
\end{aligned}
\tag{2.2.8}
$$

It is obvious from its definition that at the stopping time τ^*, the random walk attains a new minimum value, and it is the first time the process drops below x^*. In other words $S_n \geqslant S_{\tau^*}$, for all $n \leqslant \tau^*$, and thus on the event $\{T \geqslant \tau^*\}$,

$$
\begin{aligned}
I - S_{\tau^*} &= \inf_{0\leqslant n\leqslant T} S_n - S_{\tau^*} \\
&= \inf_{0\leqslant n\leqslant T} (S_n - S_{\tau^*}) \\
&= \inf_{\tau^*\leqslant n\leqslant T} (S_n - S_{\tau^*}).
\end{aligned}
$$

By the strong Markov property and the above equality, the quantity $I - S_{\tau^*}$ is independent of

$\mathcal{F}_{\tau^*}$, since it depends only on the sequence $\{S_n - S_{\tau^*} : n = \tau^*, \dots, T\}$. Also by the memoryless property of the geometric distribution conditional on $\mathcal{F}_{\tau^*}$, on the event $\{T \geqslant \tau^*\}$, the length $T - \tau^*$ of the path $\{S_n - S_{\tau^*} : n = \tau^*, \dots, T\}$ is independent of $\{T \geqslant \tau^*\}$ and $\mathcal{F}_{\tau^*}$, and is geometrically distributed with parameter β. These two facts together imply that in the last expression of (2.2.8) we can replace $I - S_{\tau^*}$ by a copy of I, denoted by $\tilde{I}$, which is independent of both the random walk $\{S_n\}_{n\in\mathbb{Z}^+}$ and $\tilde{M}$ so that

$$\mathbb{E}\left[\mathbf{1}_{\{T\geqslant\tau^*\}}\, g\Big(x + S_{\tau^*} + (I - S_{\tau^*}) + \tilde{M}\Big)\,|\mathcal{F}_{\tau^*}\right] \\ = \mathbb{E}\left[\mathbf{1}_{\{T\geqslant\tau^*\}}\, g\Big(x + S_{\tau^*} + \tilde{I} + \tilde{M}\Big)\,|\mathcal{F}_{\tau^*}\right]. \quad (2.2.9)$$

In this way we get

$$\begin{aligned} W(x) &= \frac{1}{1-\beta}\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{T\geqslant\tau^*\}} g(x + S_{\tau^*} + \tilde{I} + \tilde{M})\,|\mathcal{F}_{\tau^*}\right]\right] \\ &= \frac{1}{1-\beta}\mathbb{E}\left[\mathbb{E}[\mathbf{1}_{\{T\geqslant\tau^*\}} g(x + S_{\tau^*} + \hat{S}_T)|\mathcal{F}_{\tau^*}]\right] \\ &= \frac{1}{1-\beta}\mathbb{E}[\mathbb{P}(T \geqslant \tau^*\,|\,\mathcal{F}_{\tau^*})\,\mathbb{E}[g(x + S_{\tau^*} + \hat{S}_T)\,|\,\mathcal{F}_{\tau^*}]] \\ &= \mathbb{E}\left[\beta^{\tau^*}\mathbb{E}[g(x + S_{\tau^*} + \hat{S}_T)|\mathcal{F}_{\tau^*}]\right]. \end{aligned}$$

Since $\tilde{I}$,$\tilde{M}$ are independent of each other, of $\mathcal{F}_{\tau^*}$ and of T, their sum can be replaced by an independent copy of S_T which we denote by $\hat{S}_T$. Note that this is also independent of T and of $\mathcal{F}_{\tau^*}$. Further using the representation (2.2.1) we can now show that

$$\begin{aligned} W(x) &= \mathbb{E}\left[\beta^{\tau^*}\mathbb{E}[G(x + S_{\tau^*})|\mathcal{F}_{\tau^*}]\right] \\ &= \mathbb{E}\left[\beta^{\tau^*} G(x + S_{\tau^*})\right] \leqslant \mathbb{E}\left[\beta^{\tau^*} G^+(x + S_{\tau^*})\right] \end{aligned} \quad (2.2.10)$$

Since it has been established that for any stopping time τ, $W(x) \leqslant \mathbb{E}\left[\beta^{\tau} G^+(x + S_\tau)\right]$, we have

$$W(x) = \mathbb{E}\left[\beta^{\tau^*} G^+(x + S_{\tau^*})\right] = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\beta^{\tau} G^+(x + S_\tau)\right].$$

(ii) This can be reduced to case (i) by the transformation $(x, S_n) \mapsto (-x, -S_n)$. □

Remark 2.2.1. The last inequality in (2.2.10) is actually an equality. To see this, we only need to show that $G(x + S_{\tau^*}) \geqslant 0$. It follows from the definition of τ^* that $x + S_{\tau^*} \leqslant x^*$. Thus, for $\tilde{I} \overset{\mathcal{D}}{=} I$ and $\tilde{M} \overset{\mathcal{D}}{=} M$, independent of each other and both also independent of $\{S_t\}_{t\in\mathbb{R}^+}$, we have that $x + S_{\tau^*} + \tilde{I} \leqslant x^*$, and so it follows that

$$\mathbb{E}\left[g(x + S_{\tau^*} + \tilde{I} + \tilde{M})\;|\; S_{\tau^*}, \tilde{I}\right] \geqslant 0,$$

implying that $G(x + S_{\tau^*}) \geqslant 0$.

Remark 2.2.2. (a) The results of Theorem 2.2.2 hold even when $W(x)$ is infinite. In that case, our proof establishes that the solution $V^*(x)$ of the optimal stopping problem must also be

infinite.
(b) It is clear from the proof that the solution of (2.1.1) is the same as the solution of (2.1.1) with $G^+(x)$ replaced by $G(x)$.
(c) If g is decreasing, then the monotonicity assumption on $\mathbb{E}[g(x+M)]$ in Theorem 2.2.2(i) is satisfied. Similarly, if g is increasing, then the monotonicity assumption on $\mathbb{E}[g(x+I)]$ in Theorem 2.2.2(ii) is satisfied.

2.2.2 Optimal stopping in continuous time

We now treat the continuous time case. We state and prove the main result for reward functions G of the form

$$G(x) = \int_0^\infty e^{-rt}\,\mathbb{E}[g(x+S_t)]\,dt = r^{-1}\mathbb{E}[g(x+S_T)], \tag{2.2.11}$$

where g is the payoff stream corresponding to G, and T an independent exponential time with parameter $r > 0$. We conclude section 2.2 by proving that our results are still true even if the reward function G has the desired representation only on the set where G is positive, allowing us to treat power reward functions such as $(x^+)^2$, which do not have the representation on the whole of the real line.

We consider a Lévy process $\{S_t\}_{t\in\mathbb{R}^+}$ starting from the origin, with its augmented natural filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$, that is right-continuous and contains all $\mathbb{P}$-null sets (see Rogers and Williams [53]). We assume that the sample paths of $\{S_t\}_{t\in\mathbb{R}^+}$ are a.s. right-continuous with left limits. The discount factor is given by $\beta = e^{-r}$ with $r > 0$. We also introduce an independent exponential random variable $T = T_r$, with parameter $r > 0$ and define

$$I \equiv I_r = \inf_{0\leqslant t\leqslant T} S_t \quad \text{and} \quad M \equiv M_r = \sup_{0\leqslant t\leqslant T} S_t.$$

Then, the results on the Wiener-Hopf factorization of Lévy processes given in [24] show that

$$S_T - M \text{ is independent of } M \quad \text{and} \quad S_T - M \overset{\mathrm{D}}{=} I.$$

We consider only reward functions G, which have the representation

$$G(x) = \int_0^\infty e^{-rt}\,\mathbb{E}[g(x+S_t)]\,dt \tag{2.2.12}$$

for some payoff function g. This representation was introduced by Boyarchenko and Levendorskii [1] and the optimal stopping problem was solved for monotone g. The method employed by the authors is analytical and the proof is very extensive. One shortcoming of the analytical approach is that if g is non-monotone optimality is only obtained in the class of hitting times of semi-infinite intervals.

We shall follow an approach similar to the discrete case scenario presented in Theorem 2.2.2. We use Wiener-Hopf factorization and a continuous time analogue of Lemma 2.2.1 to weaken the assumptions on g. As we shall see the optimal stopping problem can be solved even for non-monotone g, as long as some weaker monotonicity condition is satisfied.

In this section we present the main result for optimal stopping in continuous time.

Theorem 2.2.3. *Assume that the function $G(x)$ has the representation (2.2.12), where g is continuous.*

(i) *If there is $x^* > -\infty$ such that*

$$\begin{aligned} &\mathbb{E}\left[g(x+M)\right] \text{ is positive and non-increasing}, && \forall x < x^*; \\ &\mathbb{E}\left[g(x+M)\right] \leqslant 0, && \forall x > x^*, \end{aligned} \tag{2.2.13}$$

then the solution of (2.1.1) is given by the optimal stopping time $\tau^* = \inf\{t \geqslant 0 : x+S_t \leqslant x^*\}$ and can be written as

$$V^*(x) = r^{-1}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}\, g(x+S_T)\right].$$

(ii) *If there is $x^* < \infty$ such that*

$$\begin{aligned} &\mathbb{E}\left[g(x+I)\right] \leqslant 0, && \forall x < x^*; \\ &\mathbb{E}\left[g(x+I)\right] \text{ is positive and non-decreasing}, && \forall x > x^*, \end{aligned} \tag{2.2.14}$$

then the solution of (2.1.1) is given by the optimal stopping time $\tau^* = \inf\{t \geqslant 0 : x+S_t \geqslant x^*\}$ and can be written as

$$V^*(x) = r^{-1}\mathbb{E}\left[\mathbf{1}_{\{x+M\geqslant x^*\}}\, g(x+S_T)\right].$$

Proof. The proof is similar to that for the discrete case and so, in the following, we only outline that for case (i).

Writing $W(x) = r^{-1}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}\, g(x+S_T)\right]$, we intend to show that

(i) $W(x) \geqslant G^+(x)$, and

(ii) $\{e^{-rt}W(S_t)\}_{t\in\mathbb{R}^+}$ is a right-continuous supermartingale.

By Doob's optional stopping theorem this will show that $W(x)$ is an upper bound for the value function of the continuous time optimal stopping problem

$$V^*(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r\tau}G^+(x+S_\tau)]. \tag{2.2.15}$$

Finally we will show that the stopping time τ^* defined in the statement of Theorem 2.2.3 achieves a value at least as good as $W(x)$ thus proving that τ^* is optimal.

Using the result on the Wiener-Hopf factorization of Lévy processes, we can write

$$\begin{aligned} W(x) &= r^{-1}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}\, g\left(x+(S_T-I)+I\right)\right] \\ &= r^{-1}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}} g(x+\tilde{M}+I)\right], \end{aligned}$$

where $\tilde{M}$ is a copy of M independent of S_t and of I. By our assumptions on g and x^*, on the event $\{x+I>x^*\}$, we have that

$$\mathbb{E}[\mathbf{1}_{\{x+I>x^*\}}g(x+I+\tilde{M})\,|\,I]\leqslant 0$$

and thus

$$\begin{aligned} W(x) &\geqslant r^{-1}\mathbb{E}\left[\mathbf{1}_{\{x+I\leqslant x^*\}}g(x+\tilde{M}+I)\right] \\ &\quad + r^{-1}\mathbb{E}\left[\mathbf{1}_{\{x+I>x^*\}}g(x+\tilde{M}+I)\right] \\ &= r^{-1}\mathbb{E}\left[g(x+I+\tilde{M})\right] \\ &= r^{-1}\mathbb{E}\left[g(x+S_T)\right] = G(x). \end{aligned}$$

Since by choice of x^* one can show that $W(x)\geqslant 0$, we actually have $W(x)\geqslant G^+(x)$.

Next, we show that $\{e^{-rt}W(x+S_t)\}_{t\in\mathbb{R}^+}$ is a right-continuous supermartingale. For this, we define

$$F(x)=\mathbf{1}_{(-\infty,x^*]}(x)\mathbb{E}\left[g(x+M)\right],$$

and we write $W(x)=r^{-1}\mathbb{E}\left[F(x+I)\right]$. Continuing, it has been shown in [55] that if $\tilde{I}$ is a copy of I, independent of the Lévy process and of T, then on the event $\{T>t\}$, I is equal in distribution to $(S_t+\tilde{I})\wedge I_t$, where $I_t=\min\limits_{0\leqslant s\leqslant t}S_s$, giving us the new expression

$$W(x)=r^{-1}\mathbb{E}\left[F\left(x+\min\{(S_t+\tilde{I}),I_t\}\right)\right].$$

By choice of x^*, $F(x)$ is non-negative and non-increasing, giving

$$\begin{aligned} W(x) &\geqslant r^{-1}\mathbb{E}\left[\mathbf{1}_{\{T>t\}}F\left(x+\min\{(S_t+\tilde{I}),I_t\}\right)\right] \\ &\geqslant r^{-1}\mathbb{E}\left[\mathbf{1}_{\{T>t\}}F(x+S_t+\tilde{I})\right]. \end{aligned}$$

Since T is independent of the process and of $\tilde{I}$ we have that

$$\mathbb{E}\left[\mathbf{1}_{\{T>t\}}F(x+S_t+\tilde{I})\right]=\mathbb{P}(T>t)\mathbb{E}[F(x+S_t+\tilde{I})]$$

and since by definition of T, $\mathbb{P}(T>t)=e^{-rt}$ we have

$$W(x)=r^{-1}\mathbb{E}\left[e^{-rt}F(x+S_t+\tilde{I})\right]=\mathbb{E}\left[e^{-rt}W(x+S_t)\right].$$

By continuity of g we have that $\{e^{-rt}W(S_t)\}_{t\in\mathbb{R}^+}$ is a right-continuous supermartingale. Because $e^{-rt}W(S_t)\geqslant 0$ for all $t\geqslant 0$, we can apply Doob's Optional Stopping Theorem (see Rogers and Williams [53]) without requiring uniform integrability in order to get

$$W(x)\geqslant\mathbb{E}\left[e^{-r\tau}W(x+S_\tau)\right]\geqslant\mathbb{E}\left[e^{-r\tau}G^+(x+S_\tau)\right].$$

The arbitrary nature of τ implies that $W(x) \geqslant V^*(x)$.

Similar arguments to the discrete case show that $\{x + I \leqslant x^*\} = \{T \geqslant \tau^*\}$, and that on the event $\{T \geqslant \tau^*\}$, $S_T - S_{\tau^*}$ is independent of $\mathcal{F}_{\tau^*}$ and equal in distribution to S_T. Then, if $\hat{S}_T$ is a copy of S_T independent of S, T and of $\mathcal{F}_{\tau^*}$, it follows that

$$\begin{aligned} W(x) &= r^{-1}\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{T \geqslant \tau^*\}} g(x + (S_T - S_{\tau^*}) + S_{\tau^*}) \,|\mathcal{F}_{\tau^*}\right]\right] \\ &= r^{-1}\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{T \geqslant \tau^*\}} g(x + \hat{S}_T + S_{\tau^*}) \,|\mathcal{F}_{\tau^*}\right]\right] \\ &= r^{-1}\mathbb{E}\left[\mathbb{P}(T \geqslant \tau^* \mid \mathcal{F}_{\tau^*})\mathbb{E}\left[g(x + \hat{S}_T + S_{\tau^*}) \,|\mathcal{F}_{\tau^*}\right]\right]. \end{aligned}$$

By definition of T we have that $\mathbb{P}(T \geqslant \tau^* \mid \mathcal{F}_{\tau^*}) = \mathrm{e}^{-r\tau^*}$, and by the Strong Markov property it can be deduced that

$$\begin{aligned} \mathbb{E}\left[g(x + \hat{S}_T + S_{\tau^*}) \,|\mathcal{F}_{\tau^*}\right] &= \mathbb{E}\left[g(x + \hat{S}_T + S_{\tau^*}) \,|\, \sigma(S_{\tau^*})\right] \\ &= rG(x + S_{\tau^*}). \end{aligned}$$

With the above in mind we find that

$$\begin{aligned} W(x) &= \mathbb{E}\left[\mathrm{e}^{-r\tau^*}\, \mathbb{E}[G(x + S_{\tau^*}) \mid \mathcal{F}_{\tau^*}]\right] \\ &= \mathbb{E}\left[\mathrm{e}^{-r\tau^*}\, G(x + S_{\tau^*})\right] \\ &\leqslant \mathbb{E}\left[\mathrm{e}^{-r\tau^*}\, G^+(x + S_{\tau^*})\right]. \end{aligned}$$

Hence τ^* is indeed optimal and $W(x)$ is the value function of the continuous time optimal stopping problem (2.2.15). □

Representation on the half-line. We note that the requirement of Theorem 2.2.2 (respectively Theorem 2.2.3) that $G(x)$ has the representation (2.2.1) in the discrete time setting (respectively (2.2.12) in the continuous time setting) on the whole of the real line can be modified to allow for the representation to hold only on the set $D = \{x : G(x) > 0\}$, under the restriction that D is a semi-infinite interval. We make this precise in the following proposition.

Proposition 2.2.4. *Let $D = \{x : G(x) > 0\}$ and assume that on this set G has the representation* (2.2.1) *in the discrete time setting (respectively* (2.2.12) *in the continuous time setting) for some g.*

(i) *If D has the form $(-\infty, h)$ for some h, and if there is an $x^* < h$ such that* (2.2.4) *(resp.* (2.2.13)*) holds for all $x \in D$, then the result of Theorem* 2.2.2(i) *(resp. Theorem* 2.2.3(i)*) is still valid.*

(ii) *If D has the form (h, ∞) for some h, and if there is an $x^* > h$ such that* (2.2.6) *(resp.* (2.2.14)*) holds for all $x \in D$, then the result of Theorem* 2.2.2(ii) *(resp. Theorem* 2.2.3(ii)*) is still valid.*

Proof. We outline the proof for the first case in the discrete time setting. Most of the argument in the proof for Theorem 2.2.2(i) follows, except that we need to show that, under the current assumptions, (a) it is still true that $W(x) \geqslant G^+(x)$ and (b) the representation (2.2.1) holds at $x + S_{\tau^*}$. For (a), the inequality $W(x) \geqslant G^+(x)$ is true on D^c, the complement of D, since $W(x) \geqslant 0 \geqslant G(x)$ for $x \in D^c$. When x lies in D, our assumptions imply that $x + I > x^*$ if and only if $h > x + I > x^*$ and so

$$\mathbb{E}\left[\mathbf{1}_{\{x+I>x^*\}} g(x + S_T)\right] \leqslant 0.$$

Thus, $W(x) \geqslant (1 - \beta)^{-1}\mathbb{E}\left[g(x + S_T)\right]$ and the representation (2.2.1) leads to $W(x) \geqslant G^+(x)$ as in the proof for Theorem 2.2.2(i). For (b), it follows from the definition of τ^* that $x + S_{\tau^*} \in D$, implying that the representation (2.2.1) holds at $x + S_{\tau^*}$. □

2.3 Examples and applications

In this section, we apply our method to three classes of reward functions, specifically linear, exponential and power functions. We examine specific examples from each case as well as some perturbations which result in non-monotone payoff streams, thus demonstrating the weakening of the restrictions imposed by Boyarchenko and Levendorskii [1].

In this section, we demonstrate how the results described in section 2.2 can be applied to a wide class of functions, recalling that T, I and M are those defined at the beginning of subsection 2.2.1 in the discrete time setting, and those defined at the beginning of subsection 2.2.2 in the continuous time setting.

2.3.1 Linear reward functions

As a first example, we treat the linear reward function $G(x) = x$ in discrete time. We show that $G(x)$ has the representation (2.2.1) and we derive the corresponding payoff stream. By applying Theorem 2.2.2 we obtain the explicit solution to the problem

$$\sup_{\tau} \mathbb{E}\left[\beta^{\tau}(x + S_{\tau})^+\right],$$

given in [2]. We then restrict ourselves to a symmetric random walk on the integers in order to consider the perturbed reward function $G(x) = x + (-1)^x c$, for some constant c. We show that $G(x)$ has the required representation in terms of the payoff function $g(x)$, and that the restrictions imposed by Theorem 2.2.2 allow us to consider cases where g is non-monotone. By restricting the processes under consideration even further, we treat simple symmetric random walk and obtain a simple condition on g. More specifically we only require g to be monotone on the interval $(-\infty, x^*)$, for some x^*. On the rest of the real line we require $g(x) + g(x + 1)$ to be monotone. In continuous time the case $G(x) = x + c_1 + c_2 \cos(x) + c_3 \sin(x)$ is treated. We show in particular that the non-monotone payoff function $g(x) = x + 1 + \sin(x) + \cos(x)$ satisfies our assumptions for Brownian motion and spectrally positive Lévy processes.

(a) Let $G(x) = x$ and recall that $\beta < 1$. Note that the case $\beta = 1$ has already been treated in [2]. Assume for now that $\mathbb{E}[X] = \mu$. It is easy to see that, if we take $g(x) = (1-\beta)x - \beta\mu$, then $G(x) = (1-\beta)^{-1}\mathbb{E}[g(x+S_T)]$ since

$$\begin{aligned}(1-\beta)^{-1}\mathbb{E}[g(x+S_T)] &= \sum_{n\geqslant 0}\beta^n\mathbb{E}[g(x+S_n)] \\ &= \sum_{n\geqslant 0}\beta^n[(1-\beta)(x+n\mu)-\beta\mu] \\ &= x + \mu(1-\beta)\sum_{n\geqslant 0}n\beta^n - \mu\sum_{n\geqslant 1}\beta^n \\ &= x + \frac{(1-\beta)\mu\beta}{(1-\beta)^2} - \frac{\mu\beta}{1-\beta} = x.\end{aligned}$$

Clearly, $g(x)$ is an increasing function of x. To find the optimal barrier, we solve the following equation for x^*

$$\mathbb{E}[g(x+I)] = (1-\beta)x^* + (1-\beta)\mathbb{E}[I] - \mu\beta = 0. \tag{2.3.1}$$

Since $\mathbb{E}[S_{T_\beta}] = \mu\beta/(1-\beta)$, by direct calculation, and $\mathbb{E}[S_{T_\beta}] = \mathbb{E}[I] + \mathbb{E}[M]$, by (2.2.2), we have

$$\mathbb{E}[I] = \frac{\mu\beta}{1-\beta} - \mathbb{E}[M].$$

Expressing $\mathbb{E}[I]$ in (2.3.1) in terms of $\mathbb{E}[M]$ shows that $x^* = \mathbb{E}[M]$. Then by Theorem 2.2.2(ii), the optimal stopping time is given by

$$\tau^* = \inf\{n \geqslant 0 \ : \ x + S_n \geqslant \mathbb{E}[M]\}.$$

Denoting the indicator function $\mathbf{1}_{\{x+M\geqslant\mathbb{E}[M]\}}$ by $\mathbf{I}$ to simplify notation, the solution is given by

$$\begin{aligned}(1-\beta)V^*(x) &= \mathbb{E}[\mathbf{I}\{(1-\beta)(x+M+I)-\mu\beta\}] \\ &= \mathbb{E}\left[\mathbf{I}\{(1-\beta)(x+M)-\mu\beta\} + \mathbf{I}(1-\beta)\left\{\frac{\mu\beta}{1-\beta} - \mathbb{E}[M]\right\}\right]\end{aligned}$$

since I is independent of M and hence of $\mathbf{I}$. This gives

$$(1-\beta)V^*(x) = \mathbb{E}[\mathbf{I}(1-\beta)(x+M-\mathbb{E}[M])],$$

so that

$$V^*(x) = \mathbb{E}[\mathbf{I}(x+M-\mathbb{E}[M])] = \mathbb{E}\left[(x+M-\mathbb{E}[M])^+\right].$$

In particular, by letting $\beta \uparrow 1$, we recover the solution obtained in [2] for the same reward function but with $\beta = 1$.

(b) We now consider a non-degenerate symmetric integer-valued random walk and the perturbed reward function $G(x) = x + (-1)^x c$, where $x \in \mathbb{Z}$ and c is a constant. Direct calculation shows that $G(x)$ has the representation (2.2.1) with $g(x) = (1-\beta)\{x + (-1)^x c\delta\}$, where $\delta = (1-q\beta)/(1-\beta) > 1$ and $q = \mathbb{E}[(-1)^X]$. We have that $\mathbb{E}[(-1)^I]\mathbb{E}[(-1)^M] = \mathbb{E}[(-1)^{S_T}]$, by the

Wiener-Hopf factorization, and that $\mathbb{E}[(-1)^I] = \mathbb{E}[(-1)^M]$, by symmetry. Hence,

$$\mathbb{E}[g(x+I)] = (1-\beta)\{x + \mathbb{E}[I] + (-1)^x c\sqrt{\delta}\},$$

which is non-decreasing provided that $1 \geqslant 2|c\sqrt{\delta}|$. This condition for $\mathbb{E}[g(x+I)]$ to be non-decreasing is clearly weaker than requiring that g is non-decreasing, i.e. $1 \geqslant 2|c\delta|$.

(c) By restricting our attention to the simple symmetric random walk, we are able to consider a much broader class of functions. Let $\{S_n\}_{n\in\mathbb{Z}^+}$ be a simple symmetric random walk so that the iid increments X are such that $X = 1$ with probability $1/2$, and $X = -1$ with probability $1/2$. Let $M_n = \max_{0\leqslant i\leqslant n} S_i$ be the maximum of the random walk up to time n. Then, for $k > 0$, since by the Reflection Principle

$$\mathbb{P}(M_n \geqslant k, S_n \leqslant k) = \mathbb{P}(S_n \geqslant k),$$

it is true that,

$$\mathbb{P}(M_n = k) = \mathbb{P}(S_n = k) + \mathbb{P}(S_n = k+1),$$

which obviously also holds for $k = 0$ by symmetry. Hence,

$$\mathbb{P}(M = k) = \mathbb{P}(S_T = k) + \mathbb{P}(S_T = k+1), \qquad k \in \mathbb{Z}^+,$$

and so

$$\mathbb{E}[g(x+M)] = \sum_{k\geqslant 1}\{g(x+k) + g(x+k-1)\}\,\mathbb{P}(S_T = k) + g(x)\,\mathbb{P}(S_T = 0).$$

For $\mathbb{E}[g(x+M)]$ to be a non-increasing function of $x \in \mathbb{Z}$, it is sufficient to require $g(x)+g(x+1)$ to be non-increasing for integer-valued x, and $g(x)$ non-increasing for $x < x^*$, which is clearly weaker than the assumption that g is globally decreasing.

As for the discrete time setting, Theorem 2.2.3 does not hold just for monotone functions, as the following demonstrates.

(d) Assume that $\{S_t\}_{t\in\mathbb{R}^+}$ is a Lévy process starting from the origin such that the negative of its infimum at the exponential time T has exponential distribution with parameter λ. Note that Brownian motion and spectrally positive Lévy processes have this property. For simplicity, we assume that $\lambda = 1$. Consider the perturbed reward function $G(x) = x + c_1 + c_2\cos(x) + c_3\sin(x)$. The function $G(x)$ has the representation (2.2.1) with $g(x) = x + \tilde{c}_1 + \tilde{c}_2\cos(x) + \tilde{c}_3\sin(x)$, where the relations between the coefficients can easily be obtained. A particular case is $g(x) = x + 1 + \sin(x) + \cos(x)$ which is clearly not monotone. However, $\mathbb{E}g(x+I) = x + \sin(x)$ is monotone.

This can be extended to a broader class of functions. Observe that

$$\mathbb{E}\left[g(x+I)\right] = \int_0^\infty g(x-y)\,\mathrm{e}^{-y}\,\mathrm{d}y$$

and so, if g is differentiable,

$$\frac{d}{dx}\mathbb{E}\left[g(x+I)\right] = \int_{-\infty}^{x} g'(z)\,\mathrm{e}^{z-x}\,\mathrm{d}z.$$

Thus, if $\int_{-\infty}^{x} g'(z)\,\mathrm{e}^{z}\,\mathrm{d}z > 0$ almost everywhere then, if $\mathbb{E}\left[g(x+I)\right] = 0$ has a root x^*, it must be unique. Also $\mathbb{E}\left[g(x+I)\right]$ is non-decreasing in x, so that Theorem 2.2.3(ii) can be applied to G.

2.3.2 Exponential reward functions

We now move on to exponential reward functions. We treat perpetual American calls and puts rederiving the solutions presented earlier in [54], as well as in [2] as a special case. We also consider the perturbed function $G(x) = K\mathrm{e}^x + c_1 + c_2\sin(x) + \cos(x)$ and show that under certain conditions the non-monotone payoff $g(x) = \mathrm{e}^x - 2 + \sin(x) - \cos(x)$ can be treated. We write $a = \mathbb{E}\left[\mathrm{e}^X\right]$ and assume that $a\beta < 1$.

(a) *Perpetual American call.* The price of a Perpetual American call option with strike K, under the random walk model, is the solution of the optimal stopping problem

$$V^*(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\beta^\tau(\mathrm{e}^{x+S_\tau} - K)^+\right]$$

so that it corresponds to the case $G(x) = \mathrm{e}^x - K$. Note that the solution for this problem with $K = 1$ was obtained in [2] using a different method.

Observe that, if $g(x) = (1-a\beta)\mathrm{e}^x - (1-\beta)K$, where recall that $a \equiv \mathbb{E}[\mathrm{e}^X]$, then

$$\begin{aligned}\sum_{n\geqslant 0}\beta^n\mathbb{E}\left[g(x+S_n)\right] &= \sum_{n\geqslant 0}\beta^n\mathbb{E}\left[(1-a\beta)\mathrm{e}^{x+S_n}\right] - (1-\beta)K\\ &= (1-a\beta)\mathrm{e}^x\sum_{n\geqslant 0}(a\beta)^n - K = \mathrm{e}^x - K = G(x).\end{aligned}$$

Since g is an increasing function and

$$\mathbb{E}\left[\exp(S_T)\right] = \frac{1-\beta}{1-a\beta}, \tag{2.3.2}$$

we apply Theorem 2.2.2(ii) to deduce that the optimal stopping barrier is the solution x^* of

$$\mathbb{E}\left[(1-a\beta)\mathrm{e}^{x^*+I} - (1-\beta)K\right] = 0.$$

Hence,

$$e^{x^*} = \frac{1-\beta}{1-\alpha\beta}\frac{K}{\mathbb{E}\left[e^I\right]}, \qquad \text{i.e. } e^{x^*} = K\,\mathbb{E}\left[e^M\right]$$

by (2.3.2) and ((2.2.2)) and so the optimal stopping time is

$$\tau^* = \inf\{n \geqslant 0 \ : \ x + S_n \geqslant \ln(K\,\mathbb{E}[e^M])\}.$$

The value function is

$$V^*(x) = \frac{1}{1-\beta}\mathbb{E}\left[\mathbf{1}_{\{x+M\geqslant x^*\}}\left((1-\alpha\beta)e^{x+S_T} - (1-\beta)K\right)\right]$$

and so, writing $\mathbf{I}$ for the indicator function $\mathbf{1}_{\{x+M\geqslant x^*\}}$, we get

$$\begin{aligned} V^*(x) &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{I}\left(\frac{1-\alpha\beta}{1-\beta}e^{x+I+M} - K\right) \,\Big|\, M\right]\right] \\ &= \mathbb{E}\left[\mathbf{I}\left(\frac{e^{x+M}}{\mathbb{E}[e^M]} - K\right)\right] \\ &= \frac{\mathbb{E}\left[\left(e^{x+M} - K\,\mathbb{E}\left[e^M\right]\right)^+\right]}{\mathbb{E}\left[e^M\right]}, \end{aligned}$$

agreeing with the solution given in [54].

(b) (*Perpetual American put*) Similarly, for a Perpetual American put, the reward function can be expressed in terms of the function $g = K(1-\beta) - (1-\alpha\beta)e^x$ as

$$K - e^x = \sum_{n\geqslant 0}\beta^n\,\mathbb{E}\left[g(x+S_n)\right].$$

The function g is decreasing and thus Theorem 2.2.2(i) gives the optimal stopping time as

$$\tau^* = \inf\{n \geqslant 0 \ : \ x + S_n \leqslant \ln(K\,\mathbb{E}[e^I])\}$$

and the price as

$$V^*(x) = \frac{\mathbb{E}\left[\left(K\mathbb{E}[e^I] - e^{x+I}\right)^+\right]}{\mathbb{E}[e^I]},$$

again agreeing with the solution given in [54].

(c) Let $\{S_n\}_{n\in\mathbb{Z}^+}$ be a random walk such that the distribution of the negative of its infimum at an independent geometric time T is a mixture of an atom at zero and an exponential distribution with parameter 1. To see that such a random walk exists, consider the descending ladder process of the random walk. The number of descending ladder points, up to an independent geometric time, is also geometric (cf. [22]), say of parameter p, and is independent of the ladder height process. Thus the distribution of the negative of the infimum is the geometric compounding of the distribution of the descending ladder heights. If the descending ladder heights are exponentially distributed, then conditional on the geometric time not being zero, the infimum of the random walk will also have exponential distribution. An example of a random walk with exponentially distributed descending ladder heights is given in [21] (p. 193), where the increments are distributed as the difference of two independent exponential random variables.

We note that, as in subsection 2.3.1(d), if $G(x) = Ke^x + c_1 + c_2\cos(x) + c_3\sin(x)$ then $G(x)$ has the representation (2.2.1) with $g(x) = Ke^x + \tilde{c}_1 + \tilde{c}_2\cos(x) + \tilde{c}_3\sin(x)$. For example, if

$$\begin{aligned} G(x) &= \frac{1-\beta}{1-a\beta}e^x - 2 + \{\mathbb{E}[\sin(S_T)] - \mathbb{E}[\cos(S_T)]\}\cos(x) \\ &\quad + \{\mathbb{E}[\cos(S_T)] + \mathbb{E}[\sin(S_T)]\}\sin(x), \end{aligned}$$

then $g(x) = e^x - 2 + \sin(x) - \cos(x)$, which is non-monotone. For such a g,

$$\mathbb{E}\left[g(x+I)\right] = (1-p)g(x) + p\left\{\frac{e^x}{2} - 2 - \cos(x)\right\}.$$

It can be checked that, for all $p > 0.4$, $\mathbb{E}\left[g(x+I)\right] = 0$ has a unique solution x^*. The derivative of $F(x) = \mathbf{1}_{[x^*,\infty)}(x)\,\mathbb{E}\left[g(x+I)\right]$ is given by

$$F'(x) = (1-p)\{e^x - \sin(x) + \cos(x)\} + p\left\{\frac{e^x}{2} + \sin(x)\right\}$$

for $x > x^*$, which again can be checked to be non-negative for $x > x^*$ when $p > 0.4$. Thus, $F(x)$ is non-decreasing and Theorem 2.2.2(ii) can be applied to G.

2.3.3 Canadian options

We now move on to Canadian options, a problem arising from the finance industry, in particular from numerical schemes for the pricing of finite expiry American options. Under certain conditions we get an explicit formula for the price of the Canadian put option with arbitrary *final payoff*. So far this problem has been solved for Brownian motion and spectrally one-sided Lévy processes(c.f. [28, 29]), while our result is for a general random walk and can be easily extended to general Lévy processes with jumps of both signs, as long as we can compute its Wiener-Hopf factors.

Canadian options have two rewards, a *boundary payoff* and a *final payoff*. The option, with strike K, can be exercised at any time before maturity to receive the boundary payoff $(K - e^x)^+$, or at maturity to receive the final payoff $f(x) \geqslant 0$. Under the random walk model, the maturity $\tilde{T}$ of the Canadian option is a random variable which has a geometric distribution and is independent of the random walk. We continue to write $a = \mathbb{E}\left[e^X\right]$ and assume that the parameter for the distribution of $\tilde{T}$ is α. We further assume that the following condition on the derivative of f holds:

$$f'(x) \geqslant -\frac{1-a\gamma}{1-\alpha}e^x,$$

where $\gamma = \alpha\beta$, and assume that $a\gamma < 1$.

The price for the Canadian option is then given by

$$\tilde{V}^*(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\beta^\tau (K - e^{x+S_\tau})^+ \mathbf{1}_{\{\tau \leqslant \tilde{T}\}} + \beta^{\tilde{T}} f(x + S_{\tilde{T}}) \mathbf{1}_{\{\tau > \tilde{T}\}}\right]. \tag{2.3.3}$$

Observe that

$$\begin{aligned}\tilde{V}^*(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\beta^\tau (K - e^{x+S_\tau})^+ \mathbf{1}_{\{\tau \leqslant \tilde{T}\}} + \beta^{\tilde{T}} f(x + S_{\tilde{T}}) \mathbf{1}_{\{\tau > \tilde{T}\}}\right] \\ &= \mathbb{E}\left[\beta^{\tilde{T}} f(x + S_{\tilde{T}})\right] \\ &\quad + \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\{\gamma^\tau (K - e^{x+S_\tau})^+ - \beta^{\tilde{T}} f(x + S_{\tilde{T}})\} \mathbf{1}_{\{\tau \leqslant \tilde{T}\}}\right].\end{aligned}$$

Thus, solving (2.3.3) is equivalent to solving the optimal stopping problem

$$\bar{V}^*(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\{\gamma^\tau (K - e^{x+S_\tau})^+ - \beta^{\tilde{T}} f(x + S_{\tilde{T}})\} \mathbf{1}_{\{\tau \leqslant \tilde{T}\}}\right]. \tag{2.3.4}$$

For this, we first re-express the second term in the above equation as follows:

$$\begin{aligned}&\mathbb{E}\left[\beta^{\tilde{T}} f(x + S_{\tilde{T}}) \mathbf{1}_{\{\tau \leqslant \tilde{T}\}}\right] \\ &= \mathbb{E}\left[\beta^\tau \sum_{n=0}^{\infty} \mathbb{E}\left[\beta^n f(x + S_\tau + \tilde{S}_n) \,\middle|\, \mathcal{F}_\tau\right] \mathbb{E}\left[\mathbf{1}_{\{\tilde{T} = \tau + n\}} \,\middle|\, \mathcal{F}_\tau\right]\right] \\ &= \mathbb{E}\left[\gamma^\tau \sum_{n=0}^{\infty} \mathbb{E}\left[(1-\alpha)\gamma^n f(x + S_\tau + \tilde{S}_n) \,\middle|\, \mathcal{F}_\tau\right]\right] \\ &= \frac{1-\alpha}{1-\gamma} \mathbb{E}\left[\gamma^\tau \mathbb{E}\left[f(x + S_\tau + \tilde{S}_{\bar{T}}) \,\middle|\, \mathcal{F}_\tau\right]\right],\end{aligned}$$

where $\{\tilde{S}_n\}_{n \in \mathbb{Z}^+}$ is an iid copy of $\{S_n\}_{n \in \mathbb{Z}^+}$ and $\bar{T}$ is a geometric random variable, with parameter γ independent of the random walk.

Write $H(x) = \frac{1-\alpha}{1-\gamma} \mathbb{E}\left[f(x + S_{\bar{T}})\right]$. Then, (2.3.4) becomes

$$\bar{V}^*(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\gamma^\tau \left\{(K - e^{x+S_\tau})^+ - H(x + S_\tau)\right\}\right]. \tag{2.3.5}$$

Since $H \geqslant 0$, $((K - e^x)^+ - H(x))^+ = (K - e^x - H(x))^+$. However, as noted in Remark 2.2.2, the solution for (2.3.5) is identical with the solution for the optimal stopping problem

$$\begin{aligned}V^*(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\gamma^\tau \left((K - e^{x+S_\tau})^+ - H(x + S_\tau)\right)^+\right] \\ &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\gamma^\tau \left(K - e^{x+S_\tau} - H(x + S_\tau)\right)^+\right].\end{aligned} \tag{2.3.6}$$

Hence, if we write $g(x) = K(1-\gamma) - (1-\alpha\gamma) e^x - (1-\alpha) f(x)$, we have

$$K - e^x - H(x) = \sum_{n=0}^{\infty} \gamma^n \mathbb{E}\left[g(x + S_n)\right].$$

Assume that there is an $x^* \in \mathbb{R}$ such that $g(x)$ satisfies the assumptions of Theorem 2.2.2(i).

Then, it follows from Theorem 2.2.2 that, if I_γ is defined in a similar manner to that for I with β replaced by γ, the function

$$V^*(x) = \frac{1}{1-\gamma}\mathbb{E}\left[\mathbf{1}_{\{x+I_\gamma \leqslant x^*\}}\, g(x+S_{\tilde{T}})\right]$$

is the solution for (2.3.6), and so also for (2.3.5). In other words,

$$\tilde{V}^*(x) = V^*(x) + \mathbb{E}\left[\beta^{\tilde{T}} f(x+S_{\tilde{T}})\right]$$

is the solution for (2.3.3), while the optimal stopping time is $\tau^* = \inf\{n \geqslant 0 \;:\; x+S_n \leqslant x^*\}$.

2.3.4 Power reward functions

As a last example we treat power functions. We treat the quadratic case $G(x) = (x^+)^2$ using Proposition 2.2.4 which allows the representation to exist on a subset of the real line. We obtain the explicit solution which has also appeared in [15]. The methodology can easily be used to treat $(x^+)^n$ for any integer n. Finally we consider power functions with non-integer exponent and under certain restrictions we rederive the solution given in [14]. In the following let $\{S_t\}_{t\in\mathbb{R}^+}$ be a Lévy process starting from the origin.

(a) Assume that $\mathbb{E}\left[S_t^2\right] < \infty$ and let $G(x) = (x^+)^2$. Writing m_1, m_2 for the first two moments of S_1 respectively, one can check that the function $g(x) = rx^2 - 2m_1 x - (m_2 - m_1^2)$ satisfies

$$x^2 = \int_0^\infty \mathrm{e}^{-rt}\,\mathbb{E}\left[g(x+S_t)\right]\,\mathrm{d}t.$$

Thus, with this choice of g, G admits the representation (2.2.12) on the set $D = \{x : G(x) > 0\} = (0,\infty)$ and so Proposition 2.2.4(ii) applies subject to its conditions. Using the identity $S_T \overset{\mathrm{D}}{=} I + \tilde{M}$ and by direct calculation of $\mathbb{E}\left[S_T\right]$ and $\mathbb{E}\left[S_T^2\right]$ in terms of the moments of I and M, we get

$$r^{-1}\mathbb{E}\left[g(x+I)\right] = x^2 - 2\mathbb{E}\left[M\right]x + 2\mathbb{E}\left[M\right]^2 - \mathbb{E}\left[M^2\right] = (x-\kappa_1)^2 - \kappa_2 = Q(x),$$

where $\kappa_1 = \mathbb{E}\left[M\right]$ and $\kappa_2 = \mathbb{E}\left[M^2\right] - \mathbb{E}\left[M\right]^2$. It is shown in [15] that $Q(x)$ has a unique positive root x^*, such that $Q(x)$ is positive increasing for all $x > x^*$, and $Q(x) \leqslant 0$ for all $0 < x < x^*$, thus proving that the conditions for Proposition 2.2.4(ii) are satisfied. It then follows that the solution of the corresponding optimal stopping problem (2.1.1) is

$$\begin{aligned} V^*(x) &= r^{-1}\mathbb{E}[\mathbf{1}_{\{x+M \geqslant x^*\}} g(x+S_T)] \\ \tau^* &= \inf\{t : x+S_t \geqslant x^*\}, \end{aligned}$$

which is precisely the solution given in [15].

(b) Let non-integer $\nu > 0$ and assume that $\mathbb{E}|S_t|^\nu < \infty$. Consider $G(x) = (x^+)^\nu$. For this, we first define

$$f_\nu(u) = \frac{\nu e^{-(\nu+1)\ln(iu)}}{\Gamma(1-\nu)}$$

and $P_k(x) = \sum_{i=0}^{k} x^i/i!$, for $k \in \mathbb{Z}^+$. Then for positive x it can be shown using Cauchy's integral theorem that

$$\int_0^\infty f_\nu(u)\left\{P_{[\nu]}(-iux) - e^{-iux}\right\} d(iu) = x^\nu.$$

For each k, we further define a k-degree polynomial $R_{k,u}$ such that

$$\mathbb{E}[R_{k,u}(x + S_T)] = P_k(-ux).$$

Using this polynomial, we define

$$g_\nu(x) = r\,\Re\left\{\int_0^\infty f_\nu(u)\left(R_{[\nu],iu}(x) - \frac{e^{-iux}}{\mathbb{E}[e^{-iuS_T}]}\right) d(iu)\right\},$$

when the right-hand side is defined. It is now easy to see that

$$\begin{aligned}\int_0^\infty \mathbb{E}[g_\nu(x+S_t)]\,dt &= r^{-1}\mathbb{E}\left[g_\nu(x+S_T)\right]\\ &= \Re\left\{\int_0^\infty f_\nu(u)\left(P_{[\nu]}(-iux) - e^{-iux}\right) d(iu)\right\} = x^\nu,\end{aligned}$$

so that G has the representation (2.2.12) in terms of g on $(0,\infty)$. Now

$$\frac{\nu e^{-(\nu+1)\ln(z)}}{\Gamma(1-\nu)}\left\{\mathbb{E}\left[R_{[\nu],z}(x+I)\right] - \frac{e^{-zx}}{\mathbb{E}\left[e^{-zM}\right]}\right\},$$

is analytic for $\Re(z) > 0$ and continuous for $\Re(z) \geqslant 0$ (cf. Theorem 1, [25]). If

$$\sup_{\substack{|z|=R\\ \Re(z)>0\\ \Im(z)>0}} \left|\frac{z^{-[\nu]}e^{-zx}}{\mathbb{E}\left[e^{-zM}\right]}\right| \tag{2.3.7}$$

remains bounded for all R, then by Cauchy's rule

$$\begin{aligned} & r^{-1}\mathbb{E}\left[g_\nu(x+I)\right]\\ = \;& \Re\left\{\mathbb{E}\left[\int_0^\infty f_\nu(u)\left(R_{[\nu],iu}(x+I) - \frac{e^{-iu(x+I)}}{\mathbb{E}\left[e^{-iuS_T}\right]}\right) d(iu)\right]\right\}\\ = \;& \Re\left\{\int_0^\infty f_\nu(u)\left(\mathbb{E}\left[R_{[\nu],iu}(x+I)\right] - \frac{e^{-iux}}{\mathbb{E}\left[e^{-iuM}\right]}\right) d(iu)\right\}\\ = \;& \int_0^\infty \nu\frac{e^{-(\nu+1)\ln(u)}}{\Gamma(1-\nu)}\left\{\mathbb{E}\left[R_{[\nu],u}(x+I)\right] - \frac{e^{-ux}}{\mathbb{E}\left[e^{-uM}\right]}\right\} du = Q_\nu(x).\end{aligned}$$

It has been shown in [14] that $Q_\nu(x)$ has a unique positive root x^* such that $Q_\nu(x)$ is positive increasing for all $x > x^*$, and $Q_\nu(x) \leqslant 0$ for all $0 < x < x^*$. Thus, by Proposition 2.2.1, the solution of the optimal stopping problem with $G(x) = (x^+)^\nu$ is given by

$$r^{-1}\mathbb{E}\left[\mathbf{1}_{\{x+M\geqslant x^*\}}g_\nu(x+S_T)\right]$$

which is the same as the solution given in [14].

One case, in which (2.3.7) is satisfied and g_ν is well defined, is when the characteristic function of the supremum is of the order $(a+|z|)^{-k}$, for positive integer k, and $\nu > 2k$. For example, when S_t is Brownian motion or any spectrally negative Lévy process that is not a subordinator, the supremum is exponentially distributed with some parameter $\lambda > 0$. Another situation is when the positive jumps of the process have a phase-type distribution. In this case, it has been shown (cf. [12]) that the supremum also has a phase-type distribution. Phase-type distributions have rational transforms and, if the process is not a subordinator, then its characteristic function is of form $P(z)/Q(z)$, where the degree of the polynomial Q is one higher than that of the polynomial $P(z)$ (cf. [56]).

Remark 2.3.1. The author would like to thank Prof. A. Kyprianou for bringing to his attention a paper by Surya [57].

CHAPTER 3

Asymptotics for the intersections of random walks

In what follows we write C for a generic positive constant whose value is of no importance to the work presented here. For fixed positive ε, we also write $C(\varepsilon)$, and $D(\varepsilon)$ for generic positive constants, depending on ε where $C(\varepsilon) \to 0$, while $D(\varepsilon)$ may be unbounded as $\varepsilon \to 0$.

3.1 Introduction and main results

Let X_i, $i \in \mathbb{N}$ be an i.i.d. sequence of $\mathbb{Z}^d$-valued random variables. We shall only consider the cases $d = 1, 2$. We define the random walk

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad \text{for } n \geqslant 1. \tag{3.1.1}$$

We further assume that all random walks considered are *strongly aperiodic* in the following sense.

Definition 3.1.1 (Strongly aperiodic random walk). A random walk in $\mathbb{Z}^d$ is *strongly aperiodic* if there is no proper subgroup L of $(\mathbb{Z}^d, +)$ such that for some $x \in \mathbb{Z}^d$ with $\mathbb{P}(X_i = x) > 0$, one has $\mathbb{P}(X_i - x \in L) = 1$.

The objects of interest to us are the *local time* and the *self-intersection local time* of the random walk.

Definition 3.1.2 (Local time and Self-intersection local time). Given a random walk $(S_n)_{n \geqslant 0}$ we define its local time at point x up to time n by

$$N_n(x) = \sum_{i=0} \mathbf{1}_{S_i = x},$$

and the self-intersection local time up to time n to be

$$V_n = \sum_{i,j=0}^{n} \mathbf{1}_{S_i=S_j}. \tag{3.1.2}$$

In this chapter we shall obtain exact asymptotics for the variance of the self-intersection characteristic V_n of one and two-dimensional random walks. The remaining of this chapter is structured as follows. In subsection 3.1.1 we discuss the limitations of the existing methods, while in subsection 3.1.2 we state and prove the Tauberian Lemma 3.1.2 for complex power series mentioned in the introduction. The main results of this chapter are then summarized in Theorem 3.1.3 given in subsection 3.1.3. The proof of Theorem 3.1.3(i), corresponding to the one-dimensional case is given in section 3.2, while the proof of Theorem 3.1.3(ii), corresponding to two-dimensions, is given in section 3.3.

3.1.1 Limitations of existing methods

As discussed in section 1.2 of the introduction, our methodology is an extension of the Tauberian approach introduced in Bolthausen [3], where asymptotics for the mean and variance of V_n for centered, planar random walks with second moments first appeared. In particular Bolthausen [3] showed that

$$\mathbb{E}V_n \sim n \log n / 2\pi\sqrt{|\Sigma|}, \quad \mathrm{Var}(V_n) = O(n^2 \log n),$$

where Σ is the finite, non-singular covariance matrix. It was actually claimed by the author that $\mathrm{Var}(V_n) = O(n^2)$. However, as we shall see in what follows, the methodology used only obtains the weaker bound $O(n^2 \log(n))$.

The same approach, which relies on characteristic functions and the Tauberian Theorem for power series (see Theorem 3.1.1 below), was used by Černý [4] to treat more general local time asymptotics which include the result on the variance of V_n obtained in [3]. Once again it is claimed that the variance is of the order of $O(n^2)$, however a vital assumption of Theorem 3.1.1 was overlooked. Thus the best, rigorously proven bound so far in the literature for the general case remains that of $O(n^2 \log n)$.

The approach used in [3, 4] relies on the following theorem which we quote from Feller [21, Theorem XIII 5.5].

Theorem 3.1.1. *Let $q_n \geqslant 0$ and suppose that*

$$Q(s) = \sum_{n=0}^{\infty} q_n s^n$$

converges for $0 \leqslant s < 1$. If L varies slowly at infinity and $0 \leqslant \rho < \infty$, then the following are equivalent

$$Q(s) \sim \frac{1}{(1-s)^{\rho}} L\left(\frac{1}{1-s}\right), \qquad s \to 1-, \tag{3.1.3}$$

$$q_0 + q_1 + \cdots + q_{n-1} \sim \frac{1}{\Gamma(\rho+1)} n^\rho L(n), \qquad n \to \infty. \tag{3.1.4}$$

Furthermore, if the sequence $\{q_n\}$ *is monotone and* $0 < \rho < \infty$, *then* (3.1.3) *is equivalent to*

$$q_n \sim \frac{1}{\Gamma(\rho)} n^{\rho-1} L(n), n \to \infty. \tag{3.1.5}$$

It is obvious from the above theorem, that in order to obtain the asymptotic order of $a(n)$ from that of the power series $\sum \lambda^n a(n)$, the sequence needs to be monotone. As we shall soon see this is not necessarily the case.

Bolthausen [3] avoided this technical difficulty by aiming for an upper bound, rather than an exact asymptotic in the following manner

$$\begin{aligned}
\mathrm{Var}(V_n) &= 4 \sum_{0 \leqslant i_1 < j_1 \leqslant n} \sum_{0 \leqslant i_2 < j_2 \leqslant n} \mathbb{P}(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) - \mathbb{P}(S_{i_1} = S_{j_1})\mathbb{P}(S_{i_2} = S_{j_2}) \\
&\leqslant 8 \sum_{I_1} \mathbb{P}(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) + 8 \sum_{I_2} \mathbb{P}(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) \\
&\quad + 8 \sum_{I_1} \mathbb{P}(S_{i_1} = S_{j_1})\mathbb{P}(S_{i_2} = S_{j_2}) + 8 \sum_{I_2} \mathbb{P}(S_{i_1} = S_{j_1})\mathbb{P}(S_{i_2} = S_{j_2}) \\
&\quad + 4 \sum_{0 \leqslant i < j \leqslant n} \left(\mathbb{P}(S_i = S_j) - \mathbb{P}(S_i = S_j)^2\right) \\
&= 8\big(\tilde{a}_1(n) + \tilde{a}_2(n) + \tilde{a}_3(n) + \tilde{a}_4(n)\big) + 4\tilde{a}_5(n),
\end{aligned}$$

where I_1, I_2 are the sets of 4-tuples

$$\begin{aligned}
I_1 &= \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1 \leqslant i_2 < j_1 < j_2 \leqslant n\}, \\
I_2 &= \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1 < i_2 < j_2 \leqslant j_1 \leqslant n\}.
\end{aligned}$$

It was then shown that $\tilde{a}_1(n) \sim Cn^2$, $\tilde{a}_3(n) \sim Cn^2$ and $\tilde{a}_5(n) = O(n^2)$. However as we show next $\tilde{a}_4(n) \sim Cn^2 \log(n)$.

To see why let us consider the power series $\sigma(\lambda) = \sum_{k=0}^n \tilde{a}_4(n)\lambda^n$. Let M_n be the set of 5-tuples

$$M_n = \{(m_1, m_2, m_3, m_4, m_5) : m_1, m_4, m_5 \geqslant 0, m_2, m_3 > 0, m_1 + \cdots + m_5 = n\}.$$

By a simple change of variables in the summation we have

$$\begin{aligned}
\sigma(\lambda) &= \sum_{n=0}^{\infty} \sum_{M_n} \mathbb{P}(S_{m_1} = S_{m_1+m_2+m_3+m_4})\mathbb{P}(S_{m_1+m_2} = S_{m_1+m_2+m_3}) \\
&= (1-\lambda)^{-2} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \sum_{m_4=0}^{\infty} \lambda^{m_2+m_3+m_4} \mathbb{P}(S_{m_2+m_3+m_4} = 0)\mathbb{P}(S_{m_3} = 0).
\end{aligned}$$

Then using the formula

$$\mathbb{P}(S_k = 0) = (2\pi)^{-2} \int_J f^k(t)\, \mathrm{d}t,$$

where $J = [-\pi, \pi)^2$ and f is the characteristic function of the X_i, we have

$$\begin{aligned}
&\sigma(\lambda) \\
&= (1-\lambda)^{-2}(2\pi)^{-4} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \sum_{m_4=0}^{\infty} \lambda^{m_2+m_3+m_4} \iint_{J^2} f^{m_2+m_3+m_4}(t) f^{m_3}(s) \, ds \, dt \\
&= (1-\lambda)^{-2}\lambda^2(2\pi)^{-4} \iint_{J^2} \frac{f^2(t)f(s)\, dt\, ds}{(1-\lambda f(t)f(s))(1-\lambda f(t))^2}.
\end{aligned}$$

Fix $\varepsilon > 0$. By strong aperiodicity for $|t| \geqslant \varepsilon$, $t \in J$ we have $|f(t)| < 1 - C(\varepsilon) \leqslant 1$ and thus

$$|1 - \lambda f(t)| \geqslant 1 - \lambda |f(t)| > C(\varepsilon) > 0,$$

where $C(\varepsilon)$ is a generic constant depending on the choice of ε. Therefore, as explained in [3], as $\lambda \to 1$, the most significant term must arise from integrating on the set

$$U_\varepsilon = \{(t, s) \in J^2 : |t| < \varepsilon, |s| < \varepsilon\}.$$

Then on this set we have, writing $|t|_\Sigma$ for $\langle \Sigma t | t \rangle$ and $\varepsilon' = \varepsilon^2/2$

$$\begin{aligned}
&\iint_{U_\varepsilon} \frac{f^2(t)f(s)\, dt\, ds}{(1-\lambda f(t)f(s))(1-\lambda f(t))^2} \\
&\sim C \iint_{U_\varepsilon} \frac{dt\, ds}{(1-\lambda+\frac{1}{2}(|t|_\Sigma + |s|_\Sigma))(1-\lambda+\frac{1}{2}|t|_\Sigma)^2} \\
&\sim C(1-\lambda)^{-1} \int_0^{\varepsilon'/(1-\lambda)} \frac{\log\left(1+r+\frac{\varepsilon'}{1-\lambda}\right) dr}{(1+r)^2} \\
&\sim C(1-\lambda)^{-1} \log\left(\frac{1}{1-\lambda}\right).
\end{aligned}$$

By Theorem 3.1.1 it follows that $\tilde{a}_5(n) \sim Cn^2 \log n$.

On the other hand Černý [4] considered differences of the form

$$\sum_{i_1, i_2, j_1, j_2} \mathbb{P}(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) - \mathbb{P}(S_{i_1} = S_{j_1})\mathbb{P}(S_{i_2} = S_{j_2}),$$

which does actually give the correct order n^2, however not in the general case. The reason is that Theorem 3.1.1 gives the order of a_n only if the sequence is monotone. This is not always the case with sequences such as the above. Under the additional restriction that the random walk has symmetrized increments, which forces the characteristic function to be real and non-negative, then this result holds.

In the rest of this chapter we shall use the approach used in [3, 4] to obtain an exact asymptotic for the variance of V_n for one and two dimensional random walks. Unlike [3] and its modification in [4], we allow the parameter λ to be complex in order to make use of the complex Tauberian Lemma 3.1.2. This enables us to remove the assumption of monotonicity from the characteristic associated with V_n. In particular, this completes the proof in [3], since we do not need additional symmetry restrictions required for the classical Tauberian Theorem to be applied as in [3, 4], nor extra moment assumptions required for application of the local limit theorems as in [31].

3.1.2 A complex Tauberian Lemma

In subsection 3.1.1 we saw how Karamata's Tauberian Theorem for power series fails to give the correct asymptotic order for the variance of the self-intersection local time for general random walk, because of its restriction to monotone sequences. The key step for the results presented in this chapter is the introduction of complex power series and the use of Lemma 3.1.2.

Various complex Tauberian results in the context of power series have appeared in the literature. Of notable importance is Wiener's Tauberian theory (see [58]). We refer the reader to [59] for an overview of the field. Results similar to Lemma 3.1.2 have been used in the past in the context of combinatorial analysis, and in fact Lemma 3.1.2 generalizes Flajolet and Odlyzko [47, Theorem 4], which only treats algebraic singularities. Even though this approach has been well known in the combinatorial community for some time now, it has not received the due attention in the context of random walk intersections. In fact as we shall see in the remaining of this chapter, the complex Tauberian approach, which finds its origins in early work by Wiener and Darboux (see Knuth and Wilf [48] for Darboux's lemma), is the key ingredient needed for the estimation of the asymptotics of the self-intersections of random walks.

Lemma 3.1.2. *Assume that $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is analytic for $|z|<1$. Suppose that there exist $\alpha\in(0,1)$, a constant $K>0$, such that $|f(z)|\leqslant K$, for $\Re(z)\leqslant\alpha$, a sequence of non-negative constants $A_m>0$, $\gamma_m>1$, and non-negative monotone increasing functions l_m such that*

$$|f(z)|\leqslant \sum_m A_m|1-z|^{-\gamma_m} l_m(|1-z|^{-1}),\quad \textit{for } \Re(z)>\alpha.$$

Then

$$|a_n|\leqslant 4K+\sum_m A_m C(\gamma_m) n^{\gamma_m-1} l_m(n),$$

where $C(\gamma)=4\pi^{-1/2}\Gamma(\frac{\gamma-1}{2})/\Gamma(\frac{\gamma}{2})$.

Proof. Let S be a circle around the origin of radius $R=1-1/n$, for $n\geqslant 2$ and $R=1/2$ for $n=1$. We split S in two arcs, $\mathrm{S}_1\equiv\{z\in\mathrm{S}:\Re(z)\leqslant\alpha\}$, and $\mathrm{S}_2\equiv\{z\in\mathrm{S}:\Re(z)>\alpha\}$. Next we apply the Cauchy inversion formula which states that if S is any closed curve around the origin contained in the region of convergence of $\sum_n a_n z^n$, then

$$a_n=\frac{1}{2\pi \mathrm{i}}\oint_{\mathrm{S}}\Big(\sum_n a_n z^n\Big) z^{-n-1}\,\mathrm{d}z.$$

Thus we have the bound

$$|a_n|=\left|\frac{1}{2\pi i}\int_{\mathrm{S}} f(z) z^{-n-1}\,\mathrm{d}z\right|\leqslant\frac{1}{2\pi}\left|\int_{\mathrm{S}_1} f(z)z^{-n-1}\,\mathrm{d}z\right|+\frac{1}{2\pi}\left|\int_{\mathrm{S}_2} f(z)z^{-n-1}\,\mathrm{d}z\right|.$$

By our assumptions, $|f(z)| \leqslant K$ when $\Re(z) \leqslant \alpha$, and since $R^{-n} \leqslant 4$, for $n \geqslant 1$,

$$\left|\int_{S_1} f(z) z^{-n-1}\,\mathrm{d}z\right| \leqslant \int_0^{2\pi} K R^{-n}\,\mathrm{d}t \leqslant 8\pi K\ .$$

On the other hand for the integral on S_2,

$$\left|\int_{S_2} f(z) z^{-n-1}\,\mathrm{d}z\right| \leqslant \sum_m R^{-n} A_m \int_{-\pi/2}^{\pi/2} |1-Re^{\mathrm{i}t}|^{-\gamma_m} l_m(|1-Re^{\mathrm{i}t}|^{-1})\,\mathrm{d}t,$$

Fix m in the sum. Let the summand be denoted by I and to simplify notation let us ignore the dependence on m. It remains to prove that

$$I \leqslant 2\pi C(\gamma) A n^{\gamma-1} l(n).$$

Then since $|1-Re^{\mathrm{i}t}| = [(1-R)^2 + 2R(1-\cos(t))]^{1/2}$ and l is monotone increasing, observe that for all t and n

$$l(|1-Re^{\mathrm{i}t}|^{-1}) = l\left([n^{-2} + 2R(1-\cos(t))]^{-1/2}\right) \leqslant l(n)$$

which together with $R^{-n} \leqslant 4$ leads to the bound

$$I \leqslant 4 l(n) A \int_{-\pi/2}^{\pi/2} |1-Re^{\mathrm{i}t}|^{-\gamma}\,\mathrm{d}t.$$

From $\cos(t) \leqslant 1 - t^2/4$ for $t \in [-\pi/2, \pi/2]$, it follows that

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} |1-Re^{\mathrm{i}t}|^{-\gamma}\,\mathrm{d}t &\leqslant \int_{-\pi/2}^{\pi/2} \left[(1-R)^2 + \frac{Rt^2}{2}\right]^{-\gamma/2} \mathrm{d}t \\ &\leqslant 4n^{\gamma-1} \int_0^{\infty} \left[1+t^2\right]^{-\gamma/2} \mathrm{d}t = 2\sqrt{\pi}\frac{\Gamma(\frac{\gamma-1}{2})}{\Gamma(\frac{\gamma}{2})} n^{\gamma-1},\end{aligned}$$

for all $\gamma > 1$, and therefore

$$I \leqslant 8\sqrt{\pi}\frac{\Gamma(\frac{\gamma-1}{2})}{\Gamma(\frac{\gamma}{2})} A n^{\gamma-1} l(n) = 2\pi C(\gamma) A n^{\gamma-1} l(n). \qquad \square$$

The ability to treat non-monotone sequences, as well as to keep track of smaller order terms which turn out to be significant in this situation, is the vital step that allows us to correctly estimate the order of the variance of V_n and actually calculate the constant of convergence.

3.1.3 Main results

Let X_i, $i \in \mathbb{N}$ be an i.i.d. sequence $\mathbb{Z}^d$-valued random variables and define the random walk $(S_n)_{n\geqslant 0}$ by $S_0 = 0$, $S_n = \sum_{i=1}^n X_i$, for $d = 1, 2$. We assume the random walk is *strongly aperiodic*.

We write $f(t)$, $t \in J = [-\pi, \pi)^d$, for the characteristic function of the X_i. Our assumptions on the X_i imply that for all $t \in J$,

$$f(t) = 1 \text{ if and only if } t = 0. \tag{3.1.6}$$

We also make the following assumption.

Assumption 1. (i) For $d = 1$ we assume that $f(t)$ has the following expansion around $t = 0$:

$$f(t) = 1 - \gamma|t| + R(t), t \in J = [-\pi, \pi), \quad \text{where } R(t) = o(|t|), \text{ as } t \to 0. \tag{3.1.7}$$

(ii) For $d = 2$ we assume that the X_i have a non-singular covariance matrix Σ, which implies that the characteristic function has the following expansion around $t = 0$:

$$f(t) = 1 - \tfrac{1}{2}|t|_\Sigma + R(t), t \in J = [-\pi, \pi)^2, \quad \text{where } R(t) = o(|t|^2), \text{ as } t \to 0. \tag{3.1.8}$$

and $|t|_\Sigma := \langle \Sigma t | t \rangle$, where $\langle \cdot | \cdot \rangle$ denotes the dot product on $\mathbb{R}^2$.

We have the following result on the variance of V_n.

Theorem 3.1.3. *Let X_i, S_n be defined as in (3.1.1), $f(t)$ the characteristic function of X_i, and V_n the self-intersection local time of S_n up to time n defined in (3.1.2)*

(i) For $d = 1$ and $f(t)$ satisfying (3.1.7)

$$\mathrm{Var}(V_n) \sim 4\left(\frac{1}{12\gamma^2} + \frac{1}{\pi^2\gamma^2}\right) n^2.$$

(ii) For $d = 2$ and X_i with a non-singular, finite covariance matrix Σ, $f(t)$ satisfies (3.1.8) and we have

$$\mathrm{Var}(V_n) \sim 4(2\pi)^{-2}|\Sigma|^{-1/2}(1+\kappa)n^2, \text{ where}$$

$$\kappa \equiv \int_0^\infty \int_0^\infty \frac{dr\, ds}{(1+r)(1+s)\sqrt{(1+r+s)^2 - 4rs}} - \frac{\pi^2}{6}.$$

In section 3.2 we prove Theorem 3.1.3(i) and in section 3.3 Theorem 3.1.3(ii).

3.2 Proof of Theorem 3.1.3(i)

Power series involving characteristic functions involve integrals of rational terms of the form $(1 - \lambda f(t))^{-1}$. In order to make use of the full strength of Lemma 3.1.2 we will need bounds for these quantities. We obtain these bounds in the following subsection.

3.2.1 Preliminary calculations

In our computations we shall be constantly dealing with integrals involving terms of the form $1 - \lambda f(t)$ and $1 - \lambda f(t) f(s)$ for complex λ with $|\lambda| < 1$. In order to control the behaviour of the integrals we shall need bounds on these quantities, which we derive now using the expansion of the characteristic function.

Lemma 3.2.1. *Let $t, s \in J = [-\pi, \pi)$, $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and fix $\alpha \in (0,1)$ and $\varepsilon > 0$ small enough.*

Let $C(\varepsilon) > 0$ be a generic constant such that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

(i)For all $|t| \geqslant \varepsilon$

$$\begin{aligned} |1 - \lambda f(t)| &\geqslant C(\varepsilon) > 0, \\ |1 - \lambda f(t) f(s)| &\geqslant C(\varepsilon) > 0. \end{aligned} \tag{B1}$$

(ii) For all $|t| < \varepsilon$ and $\Re(\lambda) \leqslant \alpha$

$$\begin{aligned} |1 - \lambda f(t)| &\geqslant C > 0, \\ |1 - \lambda f(t) f(s)| &\geqslant C > 0. \end{aligned} \tag{B2}$$

(iii) For $|t|, |s| < \varepsilon$ and $\Re(\lambda) > \alpha$,

$$|1 - \lambda f(t)| \geqslant \big|1 - \lambda + \lambda\gamma|t|\big| - \theta_\varepsilon |t| \geqslant C|t|, \tag{B3a}$$

$$|1 - \lambda f(t) f(s)| \geqslant |1 - \lambda + \lambda\gamma(|t| + |s|)| - \Delta_\varepsilon(|t| + |s|) \geqslant C(|t| + |s|), \tag{B3b}$$

$$|1 - f(t)| \leqslant C|t|. \tag{B4}$$

(iv) For $\Re(\lambda) > \alpha$, $z_1 \equiv (1 - \lambda)/|1 - \lambda|$, $z_2 = \lambda\gamma$, and $\delta > 0$ small enough

$$\big|z_1 + z_2|t|\big| - \theta_\varepsilon |t| \geqslant C > 0, \qquad \textit{for } |t| < \delta, \tag{B5a}$$

$$\big|z_1 + z_2|t|\big| - \theta_\varepsilon |t| \geqslant C|t|, \qquad \textit{for all } t, \tag{B5b}$$

$$|z_1 + z_2(|t| + |s|)| - \theta_\varepsilon(|t| + |s|) \geqslant C > 0, \qquad \textit{for all } |t|, |s| < \delta, \tag{B6a}$$

$$|z_1 + z_2(|t| + |s|)| - \theta_\varepsilon(|t| + |s|) \geqslant C(|t| + |s|), \qquad \textit{for all } t, s. \tag{B6b}$$

Remark 3.2.1. Note that from the proof we also obtain the following bounds for $|t|$, $|s| < \varepsilon$ and some $C > 0$

$$\big|1 - \lambda + \lambda\gamma|t|\big| \geqslant C|t|, \tag{B7}$$

$$|1 - \lambda + \lambda\gamma(|t| + |s|)| \geqslant C(|t| + |s|), \tag{B8}$$

for all ε small enough.

Proof. First observe that since $R(t) = o(|t|)$, for each $\varepsilon > 0$ there exists θ_ε such that for all $|t| < \varepsilon$ we have $|R(t)| \leqslant \theta_\varepsilon |t|$, where $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$. In the following we fix $\varepsilon > 0$, and we let θ_ε be the corresponding constants. The bounds we derive hold for all $\varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$.

(i) By strong aperiodicity $|t| \geqslant \varepsilon$ implies $|f(t)| \leqslant 1 - C(\varepsilon) < 1$. Then using the triangle inequality

$$\begin{aligned} |1 - \lambda f(t)| &\geqslant 1 - |\lambda||f(t)| \\ &\geqslant 1 - |f(t)| \geqslant C(\varepsilon) > 0. \end{aligned}$$

Similarly for $|t| \geqslant \varepsilon$ and arbitrary $s \in J$

$$\begin{aligned} |1 - \lambda f(t) f(s)| &\geqslant 1 - |\lambda||f(t)||f(s)| \\ &\geqslant 1 - |f(t)| \geqslant C(\varepsilon) > 0. \end{aligned}$$

(ii) Suppose the real part of λ is bounded above, $\Re(\lambda) \leqslant \alpha$. Then using (3.1.8) and the triangle inequality

$$\begin{aligned} |1 - \lambda f(t)| &= |1 - \lambda + \lambda\gamma|t| - \lambda R(t)| \\ &\geqslant |1 - \lambda| - |\lambda|\gamma|t| - \theta_\varepsilon|t| \\ &\geqslant \Re(1 - \lambda) - (\gamma + \theta_\varepsilon)|t| \\ &\geqslant 1 - \alpha - (\gamma + \theta_\varepsilon)|t| \\ &\geqslant 1 - \alpha - (\gamma + \theta_\varepsilon)\varepsilon \geqslant C > 0 \end{aligned}$$

for all $\varepsilon < \varepsilon_1$, for some $\varepsilon_1 > 0$.

Similarly

$$\begin{aligned} |1 - \lambda f(t) f(s)| &\geqslant |1 - \lambda(1 - \gamma|t| + R(t))(1 - \gamma|s| + R(s))| \\ &\geqslant |1 - \lambda| - 2\gamma\varepsilon - 2\theta_\varepsilon\varepsilon - \gamma^2\varepsilon^2 - 2\gamma\theta_\varepsilon\varepsilon^2 - 2\theta_\varepsilon^2\varepsilon^2 \\ &\geqslant 1 - \alpha - C(\varepsilon) \geqslant C > 0, \end{aligned}$$

for all $\varepsilon < \varepsilon_2$, for some $\varepsilon_2 > 0$.

(iii) Suppose now that $|t|, |s| < \varepsilon$ and $\Re(\lambda) > \alpha$. Then using (3.1.7) and the triangle inequality

$$\begin{aligned} |1 - \lambda f(t)| &\geqslant \big|1 - \lambda + \lambda\gamma|t|\big| - |R(t)| \\ &\geqslant \big|1 - \lambda + \lambda\gamma|t|\big| - \theta_\varepsilon|t| \end{aligned}$$

and noting that $\Re(1 - \lambda) \geqslant 0$ and $\Re(\lambda) \geqslant \alpha$ we have

$$\begin{aligned} \big|1 - \lambda + \lambda\gamma|t|\big| - \theta_\varepsilon|t| &\geqslant \Re(1 - \lambda + \lambda\gamma|t|) - \theta_\varepsilon|t| \\ &\geqslant \Re(1 - \lambda) + \Re(\lambda)\gamma|t| - \theta_\varepsilon|t| \\ &\geqslant \alpha\gamma|t| - \theta_\varepsilon|t| = (\alpha\gamma - \theta_\varepsilon)|t| \geqslant C|t| \end{aligned}$$

for all $\varepsilon < \varepsilon_3$, for some $\varepsilon_3 > 0$. Inequality (B3a) follows.

Similarly

$$\begin{aligned}
&|1-\lambda f(t)f(s)| \\
&= |1-\lambda(1-\gamma|t|+R(t))(1-\gamma|s|+R(s))| \\
&= \Big|1-\lambda\Big(1-\gamma|t|+R(t)-\gamma|s|+\gamma^2|t||s| \\
&\qquad -\gamma|s|R(t)+R(s)-\gamma|t|R(s)+R(t)R(s)\Big)\Big| \\
&\geqslant |1-\lambda+\lambda\gamma(|t|+|s|)|-\gamma^2|t||s|-\gamma(|t|R(s)+|s|R(t))-R(t)R(s) \\
&\geqslant |1-\lambda+\lambda\gamma(|t|+|s|)|-\gamma^2\varepsilon(|t|+|s|)-\gamma\theta_\varepsilon(|t|+|s|)-\theta_\varepsilon^2(|t|+|s|) \\
&\geqslant |1-\lambda+\lambda\gamma(|t|+|s|)|-\Delta_\varepsilon(|t|+|s|) \\
&\geqslant (\alpha-\Delta_\varepsilon)(|t|+|s|)
\end{aligned}$$

where $\Delta_\varepsilon=\gamma^2\varepsilon+\gamma\theta_\varepsilon+\theta_\varepsilon^2$. Since $\Delta_\varepsilon\to 0$ as $\varepsilon\to 0$, there exists $\varepsilon_4>0$ such that for all $\varepsilon<\varepsilon_4$

$$|1-\lambda f(t)f(s)|\geqslant C(|t|+|s|)>0,$$

and Inequality (B3b) follows.

Finally for $|t|<\varepsilon$ we have

$$|1-f(t)|=|1-1+\gamma|t|-R(t)|\leqslant\gamma|t|+\theta_\varepsilon|t|\leqslant C|t|.$$

proving Inequality (B4).

(iv) Now let $z_1\equiv(1-\lambda)/|1-\lambda|$ and $z_2\equiv\lambda\gamma$, with $\Re(\lambda)>\alpha$. Then $|z_1|=1$, $\Re(z_1)\geqslant 0$, $|z_2|\leqslant\gamma$, and $\Re(z_2)\geqslant\alpha\gamma$. For $|t|<\delta$ we have by the triangle inequality,

$$\begin{aligned}
|z_1+z_2|t||-\theta_\varepsilon|t| &\geqslant |z_1|-|z_2||t|-\theta_\varepsilon|t| \\
&\geqslant 1-(\gamma+\theta_\varepsilon)|t|\geqslant 1-(\gamma+\theta_\varepsilon)\delta>0
\end{aligned}$$

for δ small enough and all $\varepsilon<\varepsilon_5$ for some $\varepsilon_5>0$, proving Inequality (B5a).

Also for all $t\in J$

$$\begin{aligned}
|z_1+z_2|t||-\theta_\varepsilon|t| &\geqslant \Re(z_1+z_2|t|)-\theta_\varepsilon|t| \\
&\geqslant 0+\Re(\lambda)\gamma|t|-\theta|t|\geqslant(\alpha\gamma-\theta_\varepsilon)|t|\geqslant 0
\end{aligned}$$

since for $\varepsilon<\varepsilon_6$, for some $\varepsilon_6>0$, it is true that $\alpha\gamma-\theta_\varepsilon>0$, showing Inequality (B5b). Inequalities (B6a) and (B6b) follow similarly. The Lemma then holds with $\varepsilon<\varepsilon_0=\min(\varepsilon_1,\ldots,\varepsilon_6)$. □

We are now ready to proceed with the proof of Theorem 3.1.3(i). We first expand $\mathrm{Var}(V_n)$ as a sum

$$\mathrm{Var}(V_n)=4\sum_{0\leqslant i_1<j_1\leqslant n}\ \sum_{0\leqslant i_2<j_2\leqslant n}\Big[\mathbb{P}(S_{i_1}=S_{j_1},S_{i_2}=S_{j_2})-\mathbb{P}(S_{i_1}=S_{j_1})\mathbb{P}(S_{i_2}=S_{j_2})\Big].$$

We are thus considering the sum over the set of 4-tuples

$$H = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1, j_1, i_2, j_2 \leqslant n, i_1 < j_1, i_2 < j_2\},$$

which we partition by intersecting it with the sets $\{i_1 \leqslant i_2\}$ and $\{i_1 > i_2\}$ to get the sets

$$A = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1, j_1, i_2, j_2 \leqslant n, i_1 < j_1, i_2 < j_2, i_1 \leqslant i_2\}, \text{ and}$$
$$B = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1, j_1, i_2, j_2 \leqslant n, i_1 < j_1, i_2 < j_2, i_2 < i_1\}.$$

We further partition A in the following sets

$$A^1 = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1 < j_1 \leqslant i_2 < j_2 \leqslant n\},$$
$$A^2 = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1 \leqslant i_2 < j_1 < j_2 \leqslant n\},$$
$$A^3 = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1 \leqslant i_2 < j_2 \leqslant j_1 \leqslant n\},$$

and B into the sets

$$B^1 = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_2 < j_2 \leqslant i_1 < j_1 \leqslant n\},$$
$$B^2 = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_2 < i_1 < j_1 \leqslant j_2 \leqslant n\},$$
$$B^3 = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_2 < i_1 < j_2 < j_1 \leqslant n\}.$$

By independence it is obvious that the sums over the sets A^1, B^1 are zero, and therefore

$$\mathrm{Var}(V_n) = 4(a_2(n) + a_3(n) + b_2(n) + b_3(n)), \tag{3.2.1}$$

where $a_2(n)$, $a_3(n)$ are the sums over A^2,A^3 respectively, and $b_2(n)$,$b_3(n)$ the sums over B^2 and B^3 respectively. We treat each one of these terms separately by considering the power series

$$\rho(\lambda) = \sum_{n=0}^{\infty} a(n)\lambda^n.$$

3.2.2 First term

We first consider the sum over A^3,

$$A^3 = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1 \leqslant i_2 < j_2 \leqslant j_1 \leqslant n\}.$$

Then we have

$$\begin{aligned}
a_3(n) &= \sum_{A^3} [\mathbb{P}(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) - \mathbb{P}(S_{i_1} = S_{j_1})\mathbb{P}(S_{i_2} = S_{j_2})] \\
&= \sum_{\mathbf{m}\in M_n} \Big[\mathbb{P}(S_{m_1} = S_{m_1+m_2+m_3+m_4}, S_{m_1+m_2} = S_{m_1+m_2+m_3}) \\
&\qquad - \mathbb{P}(S_{m_1} = S_{m_1+m_2+m_3+m_4})\mathbb{P}(S_{m_1+m_2} = S_{m_1+m_2+m_3})\Big]
\end{aligned}$$

$$= \sum_{\mathbf{m}\in M_n} \left[\mathbb{P}(S_{m_2+m_3+m_4} = 0, S_{m_3} = 0) - \mathbb{P}(S_{m_2+m_3+m_4} = 0)\mathbb{P}(S_{m_3} = 0)\right],$$

where M_n is the set of 5-tuples $(m_1, \ldots, m_5)$ with sum n, such that $m_1, m_2, m_4, m_5 \geqslant 0$ and $m_3 > 0$. Using the Fourier inversion formula for the probability of return to the origin

$$\mathbb{P}(S_n = 0) = (2\pi)^{-1} \int_J f^n(t)\,\mathrm{d}t, \tag{3.2.2}$$

we calculate

$$\begin{aligned}
\rho_3(\lambda) &= \sum_{n\geqslant 0} a_3(n)\lambda^n \\
&= (1-\lambda)^{-2} \\
&\quad \times \sum_{m_2,m_3,m_4} \lambda^{m_2+m_3+m_4}\mathbb{P}(S_{m_3} = 0)\Big[\mathbb{P}(S_{m_2+m_4} = 0) - \mathbb{P}(S_{m_2+m_3+m_4} = 0)\Big] \\
&= (1-\lambda)^{-2}(2\pi)^{-2} \\
&\quad \times \sum_{m_2,m_3,m_4} \lambda^{m_2+m_3+m_4} \iint_{J^2} f^{m_3}(y)\Big(f^{m_2+m_4}(x) - f^{m_2+m_3+m_4}(x)\Big)\,\mathrm{d}x\,\mathrm{d}y \\
&= (1-\lambda)^{-2}(2\pi)^{-2} \\
&\quad \times \iint_{J^2} \frac{\lambda f(y)}{(1-\lambda f(x))^2}\left(\frac{1}{(1-\lambda f(y))} - \frac{f(x)}{1-\lambda f(x) f(y)}\right)\mathrm{d}x\,\mathrm{d}y \\
&= (1-\lambda)^{-2}(2\pi)^{-2} \iint_{J^2} \frac{\lambda f(y)(1-f(x))\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))},
\end{aligned}$$

where $J = [-\pi, \pi)$.

By Inequality (B2), when $\Re(\lambda) \leqslant \alpha$, for some $\alpha \in (0,1)$, we have the bound

$$\left|\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{\lambda f(y)(1-f(x))\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))}\right| \leqslant C < \infty.$$

This together with Inequality (B1), which bounds the integral away from zero, imply that

$$|\rho_3(\lambda)| \leqslant C < \infty, \quad \text{for } \Re(\lambda) \leqslant \alpha, \tag{3.2.3}$$

From now on we shall assume that $\Re(\lambda) > \alpha$.

Fix a small $\varepsilon > 0$ and let

$$U_\varepsilon = \{(t,s) \in J^2 : |t| < \varepsilon, |s| < \varepsilon\}.$$

Integral away from zero. First we bound the integral when at least one of the variables is bounded away from zero, i.e. on the complement of the set U_ε.

$$F(\lambda) = \iint_{J^2/U_\varepsilon} \frac{\lambda f(y)(1-f(x))\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))},$$

which we decompose as follows,

$$F_1(\lambda) = \int_J \int_{J/B_\varepsilon(0)} \frac{\lambda f(y)(1-f(x))\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))}$$
$$F_2(\lambda) = \int_{J/B_\varepsilon(0)} \int_J \frac{\lambda f(y)(1-f(x))\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))}$$
$$F_3(\lambda) = \int_{J/B_\varepsilon(0)} \int_{J/B_\varepsilon(0)} \frac{\lambda f(y)(1-f(x))\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))}.$$

We first treat $I_1(\lambda)$ where $|x| > \varepsilon$. Using the facts that $|f(y)| \leqslant 1$ and $|1-f(x)| \leqslant 2$ and Inequality (B1)

$$\begin{aligned}
|F_1(\lambda)| &= \left| \int_J \int_{J/B_\varepsilon(0)} \frac{\lambda f(y)(1-f(x))\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))} \right| \\
&\leqslant \int_J \int_{J/B_\varepsilon(0)} \frac{|f(y)||1-f(x)|\,\mathrm{d}x\,\mathrm{d}y}{|1-\lambda f(x)|^2|1-\lambda f(y)||1-\lambda f(x)f(y)|} \\
&\leqslant C(\varepsilon)^{-1} \int_J \int_{J/B_\varepsilon(0)} \frac{\mathrm{d}x\,\mathrm{d}y}{|1-\lambda f(y)|} \\
&= D(\varepsilon) \int_J \frac{\mathrm{d}y}{|1-\lambda f(y)|} \\
&\leqslant D(\varepsilon)\left(D(\varepsilon) + \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{d}y}{|1-\lambda f(y)|} \right)
\end{aligned}$$

where $C(\varepsilon)$,$D(\varepsilon)$, are the generic positive constants introduced at the beginning of this chapter. Since $\Re(\lambda) > \alpha$, by Inequality (B3a) and a change of variables, $x = y/|1-\lambda|$, and arbitrary $0 < \delta < K$, where $K \equiv \varepsilon/|1-\lambda|$,

$$\begin{aligned}
|F_1(\lambda)| &\leqslant D(\varepsilon)\Big(\int_0^{\varepsilon} \frac{\mathrm{d}y}{|1-\lambda+\lambda\gamma y| - \theta_\varepsilon y} + D(\varepsilon) \Big) \\
&= D(\varepsilon)\Big(\int_0^{\varepsilon} \frac{\mathrm{d}y}{\big||1-\lambda+\lambda\gamma y| - \theta_\varepsilon y\big|} + D(\varepsilon) \Big)
\end{aligned}$$

where the absolute value is justified since for $|y| < \varepsilon$ by Inequality (B3a) we have $|1-\lambda+\lambda\gamma|t|| - \theta_\varepsilon|t| \geqslant C|t| \geqslant 0$

$$\begin{aligned}
&\leqslant D(\varepsilon)\Big(\int_0^{K} \frac{|1-\lambda|\,\mathrm{d}x}{|1-\lambda|\,\big||1+\lambda\gamma x| - \theta_\varepsilon x\big|} + D(\varepsilon) \Big) \\
&= D(\varepsilon)\Big(\int_0^{K} \frac{\mathrm{d}x}{\big||z_1+z_2x| - \theta_\varepsilon x\big|} + D(\varepsilon) \Big) \\
&= D(\varepsilon)\Big(\int_0^{\delta} \frac{\mathrm{d}x}{\big||z_1+z_2x| - \theta_\varepsilon x\big|} + \int_\delta^{K} \frac{\mathrm{d}x}{\big||z_1+z_2x| - \theta_\varepsilon x\big|} + D(\varepsilon) \Big)
\end{aligned}$$

and by Inequalities (B5a) and (B5b)

$$\leqslant D(\varepsilon)\Big(D(\varepsilon) + C\int_0^{\delta} \mathrm{d}x + C\int_\delta^{K} \frac{\mathrm{d}x}{x} \Big)$$

$$= D(\varepsilon)\Big(D(\varepsilon) + C\log\Big(\frac{1}{|1-\lambda|}\Big)\Big). \tag{3.2.4}$$

By Inequalities (B1), (B3a) and (B4) and the usual change of variables, we have

$$\begin{aligned}
|F_2(\lambda)| &\leqslant \int_{J/B_\varepsilon(0)}\int_J \frac{|f(y)||1-f(x)|\,\mathrm{d}x\,\mathrm{d}y}{|1-\lambda f(x)|^2|1-\lambda f(y)||1-\lambda f(x)f(y)|}\\
&\leqslant D(\varepsilon)\int_J \frac{|x|\,\mathrm{d}x}{|1-\lambda f(x)|^2}\\
&\leqslant D(\varepsilon)\left(D(\varepsilon) + \int_{-\varepsilon}^{\varepsilon}\frac{|x|\,\mathrm{d}x}{|1-\lambda f(x)|^2}\right)\\
&\leqslant D(\varepsilon)\left(D(\varepsilon) + C\int_0^{\varepsilon}\frac{\mathrm{d}x}{|1-\lambda+\lambda\gamma x| - \theta_\varepsilon x}\right)\\
&\leqslant D(\varepsilon)\left(D(\varepsilon) + C\log\Big(\frac{1}{|1-\lambda|}\Big)\right),
\end{aligned}$$

where the last inequality follows from our calculations for Inequality (3.2.4).

Finally when both x and y are bounded away from zero the integral is bounded above uniformly in λ by Inequality (B1), and thus $|F_3(\lambda)| \leqslant D(\varepsilon)$.

Thus so far we have that for complex λ with $|\lambda| < 1$

$$\rho_3(\lambda) = (2\pi)^{-2}(1-\lambda)^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\lambda f(y)(1-f(x))\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))} + F(\lambda), \tag{3.2.5}$$

where $F(\lambda) = F_1 + F_2 + F_3$ is the integral away from zero and satisfies the bound

$$|F(\lambda)| \leqslant D(\varepsilon)|1-\lambda|^{-2}\left(D(\varepsilon) + C\log\Big(\frac{1}{|1-\lambda|}\Big)\right), \quad \text{for } \Re(\lambda) > \alpha.$$

Remark 3.2.2. Note that in the bound for $F(\lambda)$, $D(\varepsilon)$ may be unbounded as $\varepsilon \to 0$. This has no effect though, since for now ε is fixed. Later on we shall allow $\varepsilon \to 0$ but the behaviour of $D(\varepsilon)$ will not concern us since we first take limits with respect to n and the effect of $F(\lambda)$ disappears due to the higher order terms.

Error from use of expansion. The next step is to replace the characteristic function in the integral term of $\rho_3(\lambda)$ by its expansion (3.1.8). This will make explicit calculations possible. This will introduce an error term, which we have to bound.

We write

$$\begin{aligned}
\rho_3(\lambda) = \frac{1}{\pi^2(1-\lambda)^2}&\int_0^{\varepsilon}\int_0^{\varepsilon}\frac{\lambda(\gamma x)\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda+\lambda\gamma x)^2(1-\lambda+\lambda\gamma y)(1-\lambda+\lambda\gamma(x+y))}\\
&+ F(\lambda) + E(\lambda),
\end{aligned}$$

where E is the error from using the expansion (3.1.7)

$$E = \frac{1}{\pi^2(1-\lambda)^2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\lambda f(y)(1-f(x))\,dx\,dy}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))}$$
$$-\frac{1}{\pi^2(1-\lambda)^2}\int_0^{\varepsilon}\int_0^{\varepsilon}\frac{\lambda(\gamma x)\,dx\,dy}{(1-\lambda+\lambda\gamma x)^2(1-\lambda+\lambda\gamma y)(1-\lambda+\lambda\gamma(x+y))}.$$

and satisfies the bound

$$|E(\lambda)| \leqslant C(\varepsilon)|1-\lambda|^{-3} \quad \text{for } \Re(\lambda) > \alpha.$$

The proof of the above bound involves lengthy calculations and is given in full detail in Appendix A.

Expansion of the integral. We are now ready to calculate an expansion for $\rho_3(\lambda)$.

$$\rho_3(\lambda) = \frac{1}{\pi^2(1-\lambda)^2}\int_0^{\varepsilon}\int_0^{\varepsilon}\frac{\lambda\gamma x\,dx\,dy}{(1-\lambda+\lambda\gamma x)^2(1-\lambda+\lambda\gamma y)(1-\lambda+\lambda\gamma(x+y))}$$
$$+F(\lambda)+E(\lambda),$$

and by Appendix A

$$|E(\lambda)| \leqslant C(\varepsilon)|1-\lambda|^{-3}, \quad \text{for } \Re(\lambda) > \alpha.$$

To make explicit calculations easier we extend the region of integration to $[0,\infty)^2$, and introduce a new error term $H(\lambda)$,

$$\rho_3(\lambda) = \frac{1}{\pi^2(1-\lambda)^2}\int_0^{\infty}\int_0^{\infty}\frac{\lambda\gamma x\,dx\,dy}{(1-\lambda+\lambda\gamma x)^2(1-\lambda+\lambda\gamma y)(1-\lambda+\lambda\gamma(x+y))}$$
$$+F(\lambda)+E(\lambda)-H(\lambda)$$

where we define, $H(\lambda) = H_1(\lambda)+H_2(\lambda)+H_3(\lambda)$, and

$$H_1(\lambda) = \frac{1}{\pi^2(1-\lambda)^2}\int_{\varepsilon}^{\infty}\int_0^{\varepsilon}\frac{\lambda\gamma x\,dx\,dy}{(1-\lambda+\lambda\gamma x)^2(1-\lambda+\lambda\gamma y)(1-\lambda+\lambda\gamma(x+y))}$$
$$H_2(\lambda) = \frac{1}{\pi^2(1-\lambda)^2}\int_0^{\varepsilon}\int_{\varepsilon}^{\infty}\frac{\lambda\gamma x\,dx\,dy}{(1-\lambda+\lambda\gamma x)^2(1-\lambda+\lambda\gamma y)(1-\lambda+\lambda\gamma(x+y))}$$
$$H_3(\lambda) = \frac{1}{\pi^2(1-\lambda)^2}\int_{\varepsilon}^{\infty}\int_{\varepsilon}^{\infty}\frac{\lambda\gamma x\,dx\,dy}{(1-\lambda+\lambda\gamma x)^2(1-\lambda+\lambda\gamma y)(1-\lambda+\lambda\gamma(x+y))}.$$

We now proceed to bound the integrals H_i, $i = 1,2,3$ for $\lambda \in \mathbb{C}$, with $\Re(\lambda) > \alpha$ and $|\lambda| < 1$. Once again we write K for $\varepsilon|1-\lambda|^{-1}$.

By changing variables and using Inequality (B.5b) we have the following bound for H_1

$$|H_1(\lambda)| \leqslant C|1-\lambda|^{-2}\int_{\varepsilon}^{\infty}\int_0^{\varepsilon}\frac{\gamma|x|\,dx\,dy}{|1-\lambda+\lambda\gamma x|^2|1-\lambda+\lambda\gamma y||1-\lambda+\lambda\gamma(x+y)|}$$
$$\leqslant C|1-\lambda|^{-3}\int_K^{\infty}\int_0^{K}\frac{\gamma|x|\,dx\,dy}{|z_1+z_2x|^2|z_1+z_2y||z_1+z_2(x+y)|}$$

$$\begin{aligned}
&\leqslant C|1-\lambda|^{-3}\int_K^\infty\int_0^K \frac{\gamma\,dx\,dy}{|z_1+z_2x||z_1+z_2y||z_1+z_2(x+y)|}\\
&\leqslant C|1-\lambda|^{-3}\int_K^\infty\int_0^K \frac{\gamma\,dx\,dy}{|z_1+z_2x|y(x+y)}
\end{aligned}$$

and since $y \geqslant x$ implies $2y \geqslant x+y$

$$\begin{aligned}
&\leqslant C|1-\lambda|^{-3}\int_K^\infty y^{-2}\,dy\int_0^K |z_1+z_2x|^{-1}\,dx\\
&\leqslant C|1-\lambda|^{-3}K^{-1}(C+\ln|K|) = C|1-\lambda|^{-2}\varepsilon^{-1}(C+\ln|K|)\\
&= D(\varepsilon)|1-\lambda|^{-2}(C+\ln|K|)
\end{aligned}$$

where we have used Inequality (B5b) and the fact that for some fixed small $\delta > 0$ by Inequality (B5a), it follows that

$$\begin{aligned}
\int_0^K |z_1+z_2x|^{-1}\,dx &= \int_0^\delta |z_1+z_2x|^{-1}\,dx + \int_\delta^K |z_1+z_2x|^{-1}\,dx\\
&\leqslant C + \int_\delta^K x^{-1}\,dx \leqslant C + C\ln|K|. \qquad (3.2.6)
\end{aligned}$$

Thus we have

$$|H_1(\lambda)| \leqslant D(\varepsilon)|1-\lambda|^{-2}\left[\log\left(\frac{1}{|1-\lambda|}\right)+C\right].$$

We continue with the second integral. By a change of variables and Inequalities (B5b) and (B6b)

$$\begin{aligned}
|H_2(\lambda)| &\leqslant C|1-\lambda|^{-2}\int_0^\varepsilon\int_\varepsilon^\infty \frac{|x|\,dx\,dy}{|1-\lambda+\lambda\gamma x|^2|1-\lambda+\lambda\gamma y||1-\lambda+\lambda\gamma(x+y)|}\\
&\leqslant C|1-\lambda|^{-3}\int_0^K\int_K^\infty \frac{|x|\,dx\,dy}{|z_1+z_2x|^2|z_1+z_2y||z_1+z_2(x+y)|}\\
&\leqslant C|1-\lambda|^{-3}\int_0^K\int_K^\infty \frac{dx\,dy}{x|z_1+z_2y|(x+y)}\\
&\leqslant C|1-\lambda|^{-3}\int_0^K\int_K^\infty \frac{dx\,dy}{x^2|z_1+z_2y|}\\
&\leqslant C|1-\lambda|^{-3}K^{-1}(C+\ln|K|)
\end{aligned}$$

by Inequality (3.2.6). Thus

$$|H_2(\lambda)| \leqslant D(\varepsilon)|1-\lambda|^{-2}\left[\log\left(\frac{1}{|1-\lambda|}\right)+C\right].$$

Finally Inequalities (B5b), (B6b) and (3.2.6) and the fact that for $x, y > 0$, $x+y \geqslant \sqrt{xy}$ imply that

$$\begin{aligned}
|H_3(\lambda)| &\leqslant C|1-\lambda|^{-2}\int_\varepsilon^\infty\int_\varepsilon^\infty \frac{x\,dx\,dy}{|1-\lambda+\lambda\gamma x|^2|1-\lambda+\lambda\gamma y||1-\lambda+\lambda\gamma(x+y)|}\\
&\leqslant C|1-\lambda|^{-3}\int_K^\infty\int_K^\infty \frac{x\,dx\,dy}{|z_1+z_2x|^2|z_1+z_2y||z_1+z_2(x+y)|}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C|1-\lambda|^{-3}\int_K^\infty\int_K^\infty \frac{dx\,dy}{xy(x+y)} \\
&\leqslant C|1-\lambda|^{-3}\int_K^\infty\int_K^\infty \frac{dx\,dy}{x^{3/2}y^{3/2}} \\
&\leqslant C|1-\lambda|^{-3}K^{-1} \leqslant D(\varepsilon)|1-\lambda|^{-2}.
\end{aligned}
$$

Assume for now that $\lambda \in (1/2, 1)$. It is straightforward to calculate that

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{\lambda\gamma x\,dx\,dy}{(1-\lambda+\lambda\gamma x)^2(1-\lambda+\lambda\gamma y)(1-\lambda+\lambda\gamma(x+y))} \\
&\quad = (1-\lambda)^{-1}\lambda\gamma\int_0^\infty\int_0^\infty \frac{x\,dx\,dy}{(1+\lambda\gamma x)^2(1+\lambda\gamma y)(1+\lambda\gamma(x+y))} \\
&\quad = (1-\lambda)^{-1}(\lambda\gamma)^{-2}\int_0^\infty\int_0^\infty \frac{x\,dx\,dy}{(1+x)^2(1+y)(1+x+y)} \\
&\quad = (1-\lambda)^{-1}(\lambda\gamma)^{-2},
\end{aligned}
$$

thus arriving at

$$
\int_0^\infty\int_0^\infty \frac{\lambda\gamma x\,dx\,dy}{(1-\lambda+\lambda\gamma x)^2(1-\lambda+\lambda\gamma y)(1-\lambda+\lambda\gamma(x+y))} = (1-\lambda)^{-1}(\lambda\gamma)^{-2}.
$$

Both sides of this equality are analytic in the open unit disc when considered as functions of $\lambda \in \mathbb{C}$, $|\lambda| < 1$, and equal on the set $(1/2, 1)$. Thus by analytic continuation (see for example Kaplan [60, pg. 49]), the equality must hold on all of the open unit disc, proving that

$$
\rho_3(\lambda) = 4(2\pi)^{-2}(1-\lambda)^{-3}(\lambda\gamma)^{-2} + F(\lambda) + E(\lambda) - H(\lambda),
$$

for all complex λ with $|\lambda| < 1$.

Now we have bounds for all the error terms and we obtain the following expansion for $\rho_3(\lambda)$,

$$
\rho_3(\lambda) = 4(2\pi)^{-2}\gamma^{-2}(1-\lambda)^{-3} + \mathcal{E}(\lambda),
$$

where $\mathcal{E}$ is the total error and satisfies

$$
|\mathcal{E}| \leqslant D(\varepsilon)|1-\lambda|^{-2}\log|1-\lambda|^{-1} + D(\varepsilon)|1-\lambda|^{-2} + C(\varepsilon)|1-\lambda|^{-3},
$$

for $\Re(\lambda) > \alpha$.

Remark 3.2.3. Note that factors involving powers of λ can be included in the error term since for example

$$
\left|C(1-\lambda)^{-3}\left(1-\frac{1}{\lambda^2}\right)\right| \leqslant C|(1-\lambda)|^{-3}\frac{|\lambda-1|}{|\lambda|^2} \leqslant C|1-\lambda|^{-2}|\lambda|^{-2} \leqslant C|1-\lambda|^{-2},
$$

for $|\lambda| \geqslant \Re(\lambda) > \alpha$.

Define the function

$$
f(\lambda) = \rho_3(\lambda) - (\pi\gamma)^{-2}(1-\lambda)^{-3} = \mathcal{E}(\lambda).
$$

For $\Re(\lambda) \leqslant \alpha$, $\rho_3(\lambda)$ is bounded above by a constant $K > 0$ by Inequality (3.2.3) and the second term is also obviously bounded since

$$|1 - \lambda| \geqslant \Re(1 - \lambda) \geqslant 1 - \alpha > 0.$$

Thus for $\Re(\lambda) \leqslant \alpha$ we have that $f(\lambda)$ is indeed bounded above by a constant. On the other hand, for $\Re(\lambda) > \alpha$, we have by our previous calculations the bound

$$|f(\lambda)| \leqslant D(\varepsilon)|1 - \lambda|^{-2} \log |1 - \lambda|^{-1} + D(\varepsilon)|1 - \lambda|^{-2} + C(\varepsilon)|1 - \lambda|^{-3},$$

and thus $f(\lambda)$ satisfies the conditions of Lemma 3.1.2.

Recall that $\rho_3(\lambda) = \sum_{n=0}^{\infty} a_3(n) z^n$ and the fact that

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}n^2 + \frac{3}{2}n + 1\right) \lambda^n = (1 - \lambda)^{-3}.$$

Then by Lemma 3.1.2 applied to the function $f(\lambda)$ we obtain

$$\left|a_3(n) - (\pi\gamma)^{-2}\left(\frac{1}{2}n^2 + \frac{3}{2}n + 1\right)\right| \leqslant 4K + C(\varepsilon)n^2 + D(\varepsilon)n \log n + D(\varepsilon)n.$$

Therefore diving both sides by n^2 and taking limits as $n \to \infty$ we have

$$\limsup_{n\to\infty} \left|2\pi^2\gamma^2 \frac{a_3(n)}{n^2} - 1\right| \leqslant C(\varepsilon),$$

and since this holds for all ε small enough, and $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, it must be that

$$\lim_{n\to\infty} \left|2\pi^2\gamma^2 \frac{a_3(n)}{n^2} - 1\right| = \limsup_{n\to\infty} \left|2\pi^2\gamma^2 \frac{a_3(n)}{n^2} - 1\right| = 0,$$

and we conclude that

$$a_3(n) \sim n^2/2\pi^2\gamma^2.$$

3.2.3 Second term

Next we consider the sum over the set A^2,

$$A^2 = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1 \leqslant i_2 < j_1 < j_2 \leqslant n\}.$$

$$\begin{aligned} a_2(n) &= \sum_{A^2} [\mathbb{P}(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) - \mathbb{P}(S_{i_1} = S_{j_1})\mathbb{P}(S_{i_2} = S_{j_2})] \\ &= \sum_{m \in M_n} \Big[\mathbb{P}(S_{m_1} = S_{m_1+m_2+m_3}, S_{m_1+m_2} = S_{m_1+m_2+m_3+m_4}) \\ &\quad - \mathbb{P}(S_{m_1} = S_{m_1+m_2+m_3})\mathbb{P}(S_{m_1+m_2} = S_{m_1+m_2+m_3+m_4})\Big] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathbf{m}\in M_n} \left[\mathbb{P}(S_{m_2+m_3}=0, S_{m_3+m_4}=0) - \mathbb{P}(S_{m_2+m_3}=0)\mathbb{P}(S_{m_3+m_4}=0)\right], \\
&= \sum_{\mathbf{m}\in M_n} \Big[\sum_{x\in\mathbb{Z}^2} \mathbb{P}(S_{m_2}=x)\mathbb{P}(S_{m_3}=-x)\mathbb{P}(S_{m_4}=x) \\
&\quad - \mathbb{P}(S_{m_2+m_3}=0)\mathbb{P}(S_{m_3+m_4}=0)\Big],
\end{aligned}
$$

where M_n is the set of 5-tuples $(m_1,\dots,m_5)$ such that $m_1, m_2, m_5 \geqslant 0$, $m_3, m_4 > 0$ and $m_1+\dots+m_5 = n$. We define $\rho_2(\lambda) = \sum_{n=0}^{\infty} a_2(n)\lambda^n$, and following the same approach as for the first term we calculate

$$
\begin{aligned}
\rho_2(\lambda) &= (1-\lambda)^{-2}\lambda^2(2\pi)^{-2} \\
&\times \iint_{J^2} \frac{f(x)\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))(1-\lambda f(y))}\left[\frac{f(x+y)}{1-\lambda f(x+y)} - \frac{f(y)^2}{1-\lambda f(x)f(y)}\right]
\end{aligned}
$$

where we have made repeated use of the identity (3.2.2).

For $\Re(\lambda) \leqslant \alpha \in (0,1)$, by considerations similar to those for the first term we have that

$$|\rho_2(\lambda)| \leqslant K,$$

for some positive constant K.

From now on assume that $\Re(\lambda) > \alpha$.

Fix an $\varepsilon > 0$ and let $U_\varepsilon = \{(x,y) \in J\times J : |x| < \varepsilon, |y| < \varepsilon\}$. We claim that all significant contributions to the integral $Y(\lambda)$ come from this region.

Integral away from zero. Let us consider the integral on the complement of U_ε, that is when at least one of x, y is bounded away from zero. Obviously if both are bounded away from zero the integrand is bounded above by a constant since the numerator is obviously bounded above, and the denominator is bounded below by strong aperiodicity and Inequality (B1). Therefore for

$$F_1(\lambda) = \int\limits_{J/B_\varepsilon(0)}\int\limits_{J/B_\varepsilon(0)} \frac{f(x)}{(1-\lambda f(x))(1-\lambda f(y))}\left[\frac{f(x+y)}{1-\lambda f(x+y)} - \frac{f(y)^2}{1-\lambda f(x)f(y)}\right]\mathrm{d}x\,\mathrm{d}y$$

we have the bound

$$
\begin{aligned}
|F_1(\lambda)| &\leqslant \int_{J/B_\varepsilon(0)}\int_{J/B_\varepsilon(0)} \frac{|f(x)|}{|1-\lambda f(x)||1-\lambda f(y)|} \\
&\quad\times\left[\frac{|f(x+y)|}{|1-\lambda f(x+y)|} + \frac{|f(y)^2|}{|1-\lambda f(x)f(y)|}\right]\mathrm{d}x\,\mathrm{d}y \\
&\leqslant C(\varepsilon)^{-1} = D(\varepsilon),
\end{aligned}
$$

by Inequality (B1). Recall that $C(\varepsilon)$, $D(\varepsilon)$ denote generic positive constants such that $C(\varepsilon)\to 0$ as $\varepsilon\to 0$, but $D(\varepsilon)$ may be unbounded.

Next consider the region $V_1 = \{(x,y) \in J \times J : |x| \geqslant \varepsilon, |y| < \varepsilon/2\}$ where only x is bounded away from zero. It is straightforward to show, using the triangle inequality, that in V_1,

$$|x+y| \geqslant |x| - |y| > \varepsilon/2$$

and thus $|x+y|$ is also bounded away from zero. Thus using Inequality (B1) we have that all terms in the denominator are bounded below by $C(\varepsilon)$, apart from $|1-\lambda f(y)|$ for which we use Inequality (B3a). Then the integral

$$F_2(\lambda) = \iint_{V_1} \frac{f(x)}{(1-\lambda f(x))(1-\lambda f(y))} \left[\frac{f(x+y)}{1-\lambda f(x+y)} - \frac{f(y)^2}{1-\lambda f(x)f(y)}\right] \mathrm{d}x\,\mathrm{d}y,$$

satisfies by Inequality (B3a)

$$\begin{aligned}|F_2(\lambda)| &\leqslant \left|\iint_{V_1} \frac{f(x)}{(1-\lambda f(x))(1-\lambda f(y))} \left[\frac{f(x+y)}{1-\lambda f(x+y)} - \frac{f(y)^2}{1-\lambda f(x)f(y)}\right] \mathrm{d}x\,\mathrm{d}y\right| \\ &\leqslant D(\varepsilon) \int_{|y|<\varepsilon} \frac{\mathrm{d}y}{|1-\lambda f(y)|}.\end{aligned}$$

Since $\Re(\lambda) > \alpha$, by Inequality (B3a) and a change of variables

$$\begin{aligned}|F_2(\lambda)| &\leqslant D(\varepsilon) \int_0^{\varepsilon} \frac{\mathrm{d}y}{|1-\lambda+\lambda\gamma|y|| - \theta_\varepsilon|y|} \\ &\leqslant D(\varepsilon) \int_0^{K} \frac{\mathrm{d}y}{|z_1+z_2|y|| - \theta_\varepsilon|y|} \\ &\leqslant D(\varepsilon)\left(C + C\log\left(\frac{\varepsilon}{|1-\lambda|}\right)\right),\end{aligned}$$

by Inequality (3.2.4), where $z_1 = (1-\lambda)/|1-\lambda|$ and $z_1 = \lambda\gamma$. The last integral F_3 with y near zero and x away, follows similarly giving the same order.

We can now write

$$\rho_2(\lambda) = Y(\lambda) + F,$$

where

$$Y(\lambda) = (1-\lambda)^{-2}\lambda^2(2\pi)^{-2} \iint\limits_{U_\varepsilon} \frac{f(x)\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))(1-\lambda f(y))} \left[\frac{f(x+y)}{1-\lambda f(x+y)} - \frac{f(y)^2}{1-\lambda f(x)f(y)}\right]$$

and $F = F_1 + F_2 + F_3$ is the error from considering the integral on the smaller region U_ε, and satisfies

$$|F| \leqslant D(\varepsilon)(C + \log|1-\lambda|^{-1}).$$

Error from expansion. The next step is to replace the characteristic function by its expansion (3.1.7) under the integral sign to allow for explicit calculations. This will introduce an additional error term $E(\lambda)$ which satisfies the bound

$$|E(\lambda)| \leqslant C(\varepsilon)|1-\lambda|^{-3}, \quad \text{for } \Re(\lambda) > \alpha.$$

The proof of this involves lengthy calculations and is given in full detail in appendix A.

Expansion of the integral. Having estimated the errors arising from using the expansion (3.1.7) we now have the following expression for $\rho_2(\lambda)$,

$$\begin{aligned}\rho_2(\lambda) = \frac{\lambda^2}{4\pi^2(1-\lambda)^2}\Big(&\iint_{U_\epsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma|x+y|)} \\ &- \iint_{U_\epsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma(|x|+|y|))} \\ &+ E + F,\end{aligned}$$

where for $\Re(\lambda) > \alpha$

$$\begin{aligned}|E| &\leqslant C(\epsilon)|1-\lambda|^{-3}, \\ |F| &\leqslant D(\epsilon)|1-\lambda|^{-2}\log\left(|1-\lambda|^{-1}\right).\end{aligned}$$

We will consider $\rho_2(\lambda)$ as the difference of two integrals which we shall treat one by one. The first integral is given by

$$\iint_{U_\epsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma|x+y|)}.$$

To simplify the calculations we first enlarge the region of integration to $[0,\infty)^2$ introducing a new error term $H(\lambda)$,

$$\begin{aligned}&\iint_{U_\epsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma|x+y|)} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma|x+y|)} - H(\lambda),\end{aligned}$$

where $H(\lambda) = H_1(\lambda) + H_2(\lambda) + H_3(\lambda)$ and

$$\begin{aligned}H_1 &= \int_{|y|>\epsilon}\int_{-\epsilon}^{\epsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma|x+y|)} \\ H_2 &= \int_{-\epsilon}^{\epsilon}\int_{|x|>\epsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma|x+y|)} \\ H_3 &= \int_{|y|>\epsilon}\int_{|x|>\epsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma|x+y|)}.\end{aligned}$$

The calculations involved in bounding these terms are fairly lengthy and thus they are given in full detail in the appendix. For H_1, by Lemma A.2.3, we have the uniform bound

$$\begin{aligned}|H_1| &\leqslant \int_{|y|>\epsilon}\int_{-\epsilon}^{\epsilon} \frac{dx\,dy}{|1-\lambda+\lambda\gamma|x||\,|1-\lambda+\lambda\gamma|y||\,|1-\lambda+\lambda\gamma|x+y||} \\ &\leqslant |1-\lambda|^{-1}\int_{|y|>K}\int_{-K}^{K} \frac{dx\,dy}{|z_1+z_2|x||\,|z_1+z_2|y||\,|z_1+z_2|x+y||}\end{aligned}$$

$$\leqslant D(\varepsilon)\log\left(\frac{\varepsilon}{|1-\lambda|}\right)+C.$$

Similarly for H_2, by symmetry and the bound for H_1, we have that

$$|H_2| \leqslant C\varepsilon^{-1}\log|1-\lambda|^{-1}.$$

Finally we have $|H_3| \leqslant C + D(\varepsilon)\log|1-\lambda|^{-1}$ by Lemma A.2.4.

Let for now λ be real and $\lambda \in (1/2, 1)$. A straightforward calculation gives

$$\iint_{\mathbf{R}^2} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma|x+y|)} = (1-\lambda)^{-1}(\lambda\gamma)^{-2}\pi^2.$$

Since both sides of the above equality are analytic on the open unit disc, and equality holds for all real $\lambda \in (1/2,1)$, by analytic continuation it must be true for all $\lambda \in \{z \in \mathbb{C} : |z| < 1\}$.

Now we can move on to consider the next integral

$$\begin{aligned}\iint_{U_\varepsilon} &\frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma(|x|+|y|))}\\ &= \iint_{\mathbf{R}^2} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma(|x|+|y|))} - H_1' - H_2' - H_3',\end{aligned}$$

where

$$\begin{aligned}H_1' &= \int_{|y|\geqslant\varepsilon}\int_{|x|<\varepsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma(|x|+|y|))}\\ H_2' &= \int_{|y|<\varepsilon}\int_{|x|\geqslant\varepsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma(|x|+|y|))}\\ H_3' &= \int_{|y|\geqslant\varepsilon}\int_{|x|\geqslant\varepsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma(|x|+|y|))}.\end{aligned}$$

Using Inequality (3.5b) and the fact that on the region of integration $|y| \geqslant |x|$, we have the upper bound

$$\begin{aligned}|H_1'| &\leqslant |1-\lambda|^{-1}\int_{|y|\geqslant K}\int_{|x|<K} \frac{dx\,dy}{|z_1+z_2|x||\,|z_1+z_2|x||\,|z_1+z_2(|x|+|y|)|}\\ &\leqslant C|1-\lambda|^{-1}\int_{|y|\geqslant K}\int_{|x|<K} \frac{dx\,dy}{|y|(|x|+|y|)|z_1+z_2|x||}\\ &\leqslant C|1-\lambda|^{-1}\int_K^\infty\int_0^K \frac{dx\,dy}{y^2|z_1+z_2|x||}\\ &\leqslant C|1-\lambda|^{-1}\int_K^\infty \frac{dy}{y^2}\int_0^K \frac{dx}{|z_1+z_2|x||} \leqslant C|1-\lambda|^{-1}K^{-1}\log(|1-\lambda|^{-1}),\end{aligned}$$

where the last inequality follows from Inequality (3.2.6). Then by the definition of K

$$|H_1'| \leqslant D(\varepsilon)\log(|1-\lambda|^{-1}).$$

Similarly by symmetry, the same bound holds for H_2'

$$H_2' \leqslant D(\varepsilon)\log(|1-\lambda|^{-1}).$$

Finally by Inequality (B5b) and the fact that for $x, y > 0$ we have $x + y \geqslant \sqrt{xy}$

$$\begin{aligned}
|H_3'| &\leqslant C|1-\lambda|^{-1}\int_K^\infty\int_K^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{|z_1+z_2|x|||z_1+z_2|x|||z_1+z_2(|x|+|y|)|} \\
&\leqslant C|1-\lambda|^{-1}\int_K^\infty\int_K^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{xy(x+y)} \\
&\leqslant C|1-\lambda|^{-1}\int_K^\infty\int_K^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{xy\sqrt{xy}} \leqslant C\varepsilon^{-1}|1-\lambda| \leqslant D(\varepsilon).
\end{aligned}$$

It is straightforward to calculate for real $\lambda \in (1/2, 1)$

$$\begin{aligned}
&\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma(|x|+|y|))} \\
&= (1-\lambda)^{-1}(\lambda\gamma)^{-2}\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{(1+|x|)(1+|y|)(1+(|x|+|y|))} \\
&= (1-\lambda)^{-1}(\lambda\gamma)^{-2}2\pi^2/3.
\end{aligned}$$

By analytic continuation this equality remains true for all complex λ with $|\lambda| < 1$.

Using all of our estimates, we have the following expansion for $\rho_2(\lambda)$

$$\rho_2(\lambda) = \sum_{n=0}^\infty a_2(n)\lambda^n = \frac{1}{12\gamma^2}(1-\lambda)^{-3} + \mathcal{E}$$

where $\mathcal{E}$ is the sum of all the errors and for $\Re(\lambda) > \alpha$

$$|\mathcal{E}| \leqslant C(\varepsilon)|1-\lambda|^{-3} + D(\varepsilon)|1-\lambda|^{-2} + D(\varepsilon)|1-\lambda|^{-2}\log\left(\frac{1}{|1-\lambda|}\right),$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, while $D(\varepsilon)$ may be unbounded as $\varepsilon \to 0$.

We define the function

$$f(\lambda) = \mathcal{E}(\lambda) = \rho_2(\lambda) - \frac{1}{12\gamma^2}(1-\lambda)^{-3}.$$

For $\Re(\lambda) > \alpha$ we have by our previous calculations

$$|f(\lambda)| \leqslant C(\varepsilon)|1-\lambda|^{-3} + D(\varepsilon)|1-\lambda|^{-2} + D(\varepsilon)|1-\lambda|^{-2}\log\left(\frac{1}{|1-\lambda|}\right),$$

while for $\Re(\lambda) \leqslant \alpha$ it follows from the arguments given for the $a_3(n)$ that $|f(\lambda)| \leqslant K$, for some positive constant K. Thus f satisfies the conditions of Lemma 3.1.2, and using the fact

$$\sum_{n=0}^\infty \left(\frac{1}{2}n^2 + \frac{3}{2}n + 1\right)\lambda^n = (1-\lambda)^{-3},$$

we have

$$\left|a_2(n) - \frac{1}{12\gamma^2}\left(\frac{n^2}{2} + \frac{3n}{2} + 1\right)\right| \leqslant 4K + C(\varepsilon)n^2 + D(\varepsilon)n + D(\varepsilon)n\log(n).$$

After we divide by n^2 and take limits as $n \to \infty$ we have

$$\limsup_{n\to\infty}\left|\frac{24\gamma^2 a_2(n)}{n^2} - 1\right| \leqslant C(\varepsilon).$$

The arbitrary nature of ε and the fact that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$ imply that

$$\lim_{n\to\infty}\left|\frac{24\gamma^2 a_2(n)}{n^2} - 1\right| = 0,$$

and therefore

$$a_2(n) \sim \frac{1}{24\gamma^2}n^2.$$

3.2.4 Third and fourth terms

We now consider the sum over B^2.

$$\begin{aligned}
b_2(n) &= \sum_{B^2}\left[\mathbb{P}(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) - \mathbb{P}(S_{i_1} = S_{j_1})\mathbb{P}(S_{i_2} = S_{j_2})\right] \\
&= \sum_{\mathbf{m}\in M_n}\Big[\mathbb{P}(S_{m_1+m_2} = S_{m_1+m_2+m_3}, S_{m_1} = S_{m_1+m_2+m_3+m_4}) \\
&\qquad - \mathbb{P}(S_{m_1+m_2} = S_{m_1+m_2+m_3})\mathbb{P}(S_{m_1} = S_{m_1+m_2+m_3+m_4})\Big] \\
&= \sum_{\mathbf{m}\in M_n}\left[\mathbb{P}(S_{m_2+m_4} = 0)\mathbb{P}(S_{m_3} = 0) - \mathbb{P}(S_{m_2+m_3+m_4} = 0)\mathbb{P}(S_{m_3} = 0)\right],
\end{aligned}$$

where M_n is the set of 5-tuples $(m_1, \ldots, m_5)$ such that $m_1, m_4, m_5 \geqslant 0$ and $m_2, m_3 > 0$ such that $m_1 + \cdots + m_5 = n$. For $J = [-\pi, \pi)$ we have

$$\begin{aligned}
\sigma_2(\lambda) &= \sum_{n\geqslant 0} b_2(n)\lambda^n \\
&= (1-\lambda)^{-2}\sum_{m_3\geqslant 1}\sum_{m_2\geqslant 1}\sum_{m_4\geqslant 0}\lambda^{m_2+m_3+m_4}\Big[\mathbb{P}(S_{m_2+m_4} = 0)\mathbb{P}(S_{m_3} = 0) \\
&\qquad - \mathbb{P}(S_{m_2+m_3+m_4} = 0)\mathbb{P}(S_{m_3} = 0)\Big] \\
&= (1-\lambda)^{-2}(2\pi)^{-2}\sum_{m_3\geqslant 1}\sum_{m_2\geqslant 1}\sum_{m_4\geqslant 0}\lambda^{m_2+m_3+m_4}\iint_{J^2}\Big(f^{m_2+m_4}(x)f^{m_3}(y) \\
&\qquad - f^{m_2+m_3+m_4}(x)f^{m_3}(y)\Big)\,\mathrm{d}x\,\mathrm{d}y \\
&= (1-\lambda)^{-2}(2\pi)^{-2}\iint_{J^2}\frac{\lambda^2 f(x)f(y)}{(1-\lambda f(x))^2}\frac{1-f(x)}{(1-\lambda f(y))(1-\lambda f(x)f(y))}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

It is trivial to see that this is of the same order as $\rho_3(\lambda)$ and thus we have that

$$b_2(n) \sim (2(2\pi)^{-2}\gamma^{-2} + C(\varepsilon))n^2.$$

Now consider the sum over B^3.

$$B^3 = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_2 < i_1 < j_2 < j_1 \leqslant n\}.$$

$$\begin{aligned}
b_3(n) &= \sum_{A^2} [\mathbb{P}(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) - \mathbb{P}(S_{i_1} = S_{j_1})\mathbb{P}(S_{i_2} = S_{j_2})] \\
&= \sum_{\mathbf{m}\in M_n} \Big[\mathbb{P}(S_{m_1+m_2} = S_{m_1+m_2+m_3+m_4}, S_{m_1} = S_{m_1+m_2+m_3}) \\
&\qquad - \mathbb{P}(S_{m_1+m_2} = S_{m_1+m_2+m_3+m_4})\mathbb{P}(S_{m_1} = S_{m_1+m_2+m_3})\Big] \\
&= \sum_{\mathbf{m}\in M_n} [\mathbb{P}(S_{m_2+m_3} = 0, S_{m_3+m_4} = 0) - \mathbb{P}(S_{m_2+m_3} = 0)\mathbb{P}(S_{m_3+m_4} = 0)], \\
&= \sum_{\mathbf{m}\in M_n} \Big[\sum_{x\in\mathbf{Z}^2} \mathbb{P}(S_{m_2} = x)\mathbb{P}(S_{m_3} = -x)\mathbb{P}(S_{m_4} = x) \\
&\qquad - \mathbb{P}(S_{m_2+m_3} = 0)\mathbb{P}(S_{m_3+m_4} = 0)\Big],
\end{aligned}$$

where M_n is the set of 5-tuples $(m_1, \ldots, m_5)$ such that $m_1, m_5 \geqslant 0$ and $m_2, m_3, m_4 > 0$ and $m_1 + \cdots + m_5 = n$.

Then we have

$$\begin{aligned}
\sigma_3(\lambda) &= (1-\lambda)^{-2}\lambda^3(2\pi)^{-2} \\
&\times \int_J\int_J \frac{f(x)f(y)}{(1-\lambda f(x))(1-\lambda f(y))} \times \left[\frac{f(x+y)}{1-\lambda f(x+y)} - \frac{f(x)f(y)}{1-\lambda f(x)f(y)}\right] dx\,dy.
\end{aligned}$$

It is straightforward to show that this has similar expansion as $\rho_2(\lambda)$ and thus we conclude that

$$b_3(n) \sim a_2(n) \sim \frac{1}{24\gamma^2}n^2.$$

Conclusion. Recall that the variance of the self-intersections is given by

$$\mathrm{Var}(V_n) = 4(a_2(n) + a_3(n) + b_2(n) + b_3(n)). \tag{3.2.7}$$

Our calculations have shown that

$$\begin{aligned}
a_2(n) &\sim b_3(n) \sim n^2/24\gamma^2 \ . \\
a_3(n) &\sim b_2(n) \sim n^2/2\pi^2\gamma^2.
\end{aligned}$$

Thus we can conclude that

$$\mathrm{Var}(V_n) \sim 4\left(\frac{1}{\pi^2\gamma^2} + \frac{1}{12\gamma^2}\right)n^2.$$

3.3 Proof of Theorem 3.1.3(ii)

We begin with the decomposition given in the proof of Theorem 3.1.3(i), and we write

$$\mathrm{Var}(V_n) = 4(a_2(n) + a_3(n) + b_2(n) + b_3(n)). \tag{3.3.1}$$

We first obtain bounds for quantities of the form $|1 - \lambda f(t)|$, $|1 - \lambda f(t)f(s)|$.

3.3.1 Preliminary Calculations

It is now shown that Lemma 3.2.1 can be extended to two-dimensional characteristic functions satisfying (3.1.8). We do not give a proof as it follows from the calculations for the one-dimensional case and the following two facts.

First note that $R(t) = o(|t|^2)$ as $t \to 0$ and thus for each $\varepsilon > 0$, there exists a positive θ_ε such that

$$|R(t)| \leqslant \theta_\varepsilon |t|^2,$$

where $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$. Second there are constants $C_1, C_2 > 0$ such that

$$C_1|t|^2 \leqslant |\Sigma^{1/2}t|^2 \leqslant C_2|t|^2.$$

To avoid cumbersome expressions we write

$$g(t,s) := |t|_\Sigma + |s|_\Sigma.$$

The proof of the following lemma is identical to the proof of Lemma 3.2.1, a direct analogue for the one-dimensional case.

Lemma 3.3.1. *Let $t, s \in J = [-\pi,\pi)^2$, $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and fix $\alpha \in (0,1)$. Suppose that $\varepsilon > 0$ is small enough, and $C(\varepsilon) > 0$ is a generic constant such that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

(i)For all $|t| \geqslant \varepsilon$

$$\begin{aligned} |1 - \lambda f(t)| &\geqslant C(\varepsilon) > 0, \\ |1 - \lambda f(t)f(s)| &\geqslant C(\varepsilon) > 0. \end{aligned} \tag{3.3.2}$$

(ii) For all $|t| < \varepsilon$ and $\Re(\lambda) \leqslant \alpha$

$$\begin{aligned} |1 - \lambda f(t)| &\geqslant C > 0, \\ |1 - \lambda f(t)f(s)| &\geqslant C > 0. \end{aligned} \tag{3.3.3}$$

(iii) For $|t|, |s| < \varepsilon$ and $\Re(\lambda) > \alpha$, and some $\Delta_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$|1 - \lambda f(t)| \geqslant \left|1 - \lambda + \frac{\lambda}{2}|t|_\Sigma\right| - \theta_\varepsilon |t|_\Sigma \geqslant C|t|_\Sigma \geqslant C|t|^2, \tag{3.3.4}$$

$$|1-\lambda f(t)f(s)| \geqslant \left|1-\lambda+\frac{\lambda}{2}g(t,s)\right| - \Delta_\varepsilon g(t,s) \geqslant C(|t|^2+|s|^2), \tag{3.3.5}$$

$$|1-f(t)| \leqslant C|t|^2. \tag{3.3.6}$$

(iv) For $\Re(\lambda) > \alpha$ *and* $z_1 \equiv (1-\lambda)/|1-\lambda|$, $z_2 = \lambda/2$,

$$|z_1+z_2r| - \theta_\varepsilon r \geqslant C > 0, \qquad \text{for } 0 < r < \delta, \tag{3.3.7}$$

$$|z_1+z_2r| - \theta_\varepsilon r \geqslant Cr, \qquad \text{for all } r > 0, \tag{3.3.8}$$

$$|z_1+z_2(r+s)| - \theta_\varepsilon(r+s) \geqslant C > 0, \qquad \text{for all } 0 < r,s < \delta, \tag{3.3.9}$$

$$|z_1+z_2(r+s)| - \theta_\varepsilon(r+s) \geqslant C(r+s), \qquad \text{for all } r,s > 0. \tag{3.3.10}$$

Remark 3.3.1. Note that from the proof we also obtain the following bounds for $|t|,|s| < \varepsilon$ and some $C > 0$

$$\left|1-\lambda+\frac{\lambda}{2}|t|_\Sigma\right| \geqslant C|t|^2, \tag{3.3.11}$$

$$\left|1-\lambda+\frac{\lambda}{2}g(t,s)\right| \geqslant C(|t|^2+|s|^2), \tag{3.3.12}$$

for all ε small enough.

3.3.2 The first term

Let us now consider the sum over A^3,

$$A^3 = \{(i_1,j_1,i_2,j_2) : 0 \leqslant i_1 \leqslant i_2 < j_2 \leqslant j_1 \leqslant n\}.$$

Then we have

$$\begin{aligned}
a_3(n) &= \sum_{A^3} [\mathbb{P}(S_{i_1}=S_{j_1}, S_{i_2}=S_{j_2}) - \mathbb{P}(S_{i_1}=S_{j_1})\mathbb{P}(S_{i_2}=S_{j_2})] \\
&= \sum_{\mathbf{m}\in M_n} \Big[\mathbb{P}(S_{m_1}=S_{m_1+m_2+m_3+m_4}, S_{m_1+m_2}=S_{m_1+m_2+m_3}) \\
&\qquad - \mathbb{P}(S_{m_1}=S_{m_1+m_2+m_3+m_4})\mathbb{P}(S_{m_1+m_2}=S_{m_1+m_2+m_3})\Big] \\
&= \sum_{\mathbf{m}\in M_n} \Big[\mathbb{P}(S_{m_2+m_3+m_4}=0, S_{m_3}=0) - \mathbb{P}(S_{m_2+m_3+m_4}=0)\mathbb{P}(S_{m_3}=0)\Big],
\end{aligned}$$

where M_n is the set of 5-tuples $(m_1,\ldots,m_5)$ such that $m_1,m_2,m_4,m_5 \geqslant 0$, $m_3 > 0$ and $m_1+\cdots+m_5 = n$. Using the Fourier inversion formula for the probability of return to the origin for two-dimensional random walk

$$\mathbb{P}(S_n=0) = (2\pi)^{-2}\int_J f^n(t)\,\mathrm{d}t,$$

where $J = [-\pi, \pi)^2$, and by calculations similar to those for the one-dimensional case

$$\begin{aligned}\rho_3(\lambda) &= \sum_{n\geqslant 0} a_3(n)\lambda^n \\ &= (1-\lambda)^{-2}(2\pi)^{-4} \iint_{J^2} \frac{\lambda f(s)(1-f(t))\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))}.\end{aligned}$$

By calculations similar to those in section 3.2 we have that

$$|\rho_3(\lambda)| \leqslant K, \quad \text{for } \Re(\lambda) \leqslant \alpha,$$

for some $\alpha \in (0,1)$.

From now on assume that $\Re(\lambda) > \alpha$.

Fix a small $\varepsilon > 0$ and let

$$U_\varepsilon = \{(t,s) \in J^2 : |t| < \varepsilon, |s| < \varepsilon\}.$$

Integral away from zero. We claim that all significant contributions in the integral above come from the region U_ε. To see this we bound the integral when at least one of the variables is bounded away from zero,

$$F(\lambda) = \iint_{J^2\setminus U_\varepsilon} \frac{\lambda f(s)(1-f(t))\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))},$$

which we decompose as follows,

$$\begin{aligned}F_1(\lambda) &= \int_{J\setminus B_\varepsilon(0)}\int_J \frac{\lambda f(s)(1-f(t))\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))} \\ F_2(\lambda) &= \int_J\int_{J\setminus B_\varepsilon(0)} \frac{\lambda f(s)(1-f(t))\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))} \\ F_3(\lambda) &= \int_{J\setminus B_\varepsilon(0)}\int_{J\setminus B_\varepsilon(0)} \frac{\lambda f(s)(1-f(t))\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))},\end{aligned}$$

where $B_\varepsilon(0) \subset J$ is an open ball of radius ε around 0.

Let us consider I_1 first. Then since $|s| \geqslant \varepsilon$, by Inequality (3.3.2)

$$\begin{aligned}|F_1(\lambda)| &\leqslant \int_{J\setminus B_\varepsilon(0)}\int_J \frac{|1-f(t)|\,\mathrm{d}t\,\mathrm{d}s}{|1-\lambda f(t)|^2|1-\lambda f(s)||1-\lambda f(t)f(s)|} \\ &\leqslant D(\varepsilon)\int_J \frac{|1-f(t)|\,\mathrm{d}t}{|1-\lambda f(t)|^2} \\ &\leqslant D(\varepsilon)\Big(C + \int_{B_\varepsilon(0)} \frac{|1-f(t)|\,\mathrm{d}t}{|1-\lambda f(t)|^2}\Big),\end{aligned}$$

where recall that $D(\varepsilon)$ stands for a generic positive constant that may be unbounded as $\varepsilon \to 0$.

Since $\Re(\lambda) > \alpha$, by Inequality (3.3.4), the change of variables $\Sigma^{1/2}t \mapsto t$, and polar coordinates

$$
\begin{aligned}
|F_1(\lambda)| &\leqslant D(\varepsilon) \int_{B_\varepsilon(0)} \frac{|t|_\Sigma \, dt}{\left(|1-\lambda+\frac{\lambda}{2}|t|_\Sigma| - \theta_\varepsilon |t|_\Sigma\right)^2} \\
&\leqslant D(\varepsilon) \int_0^\varepsilon \frac{r^2 r \, dr}{\left(|1-\lambda+\frac{\lambda}{2}r^2| - \theta_\varepsilon r^2\right)^2} \\
&\leqslant D(\varepsilon) \int_0^\varepsilon \frac{r \, dr}{|1-\lambda+\frac{\lambda}{2}r^2| - \theta_\varepsilon r^2} \\
&\leqslant D(\varepsilon) \int_0^{\varepsilon'} \frac{dr}{|1-\lambda+\frac{\lambda}{2}r| - \theta_\varepsilon r} \\
&\leqslant D(\varepsilon) \int_0^{\varepsilon'/|1-\lambda|} \frac{dr}{|z_1+z_2 r| - \theta_\varepsilon r}
\end{aligned}
$$

and for some δ small enough, by Inequalities (3.3.7) and (3.3.8)

$$
\begin{aligned}
&\leqslant D(\varepsilon)\Big(C + \int_\delta^{\varepsilon'/|1-\lambda|} \frac{dr}{r}\Big) \\
&\leqslant D(\varepsilon)(C + \log(1/|1-\lambda|)). \qquad (3.3.13)
\end{aligned}
$$

On the other hand, for $F_2(\lambda)$ we have

$$
\begin{aligned}
|I_2(\lambda)| &\leqslant \int_J \int_{J \setminus B_\varepsilon(0)} \frac{|1-f(t)| \, dt \, ds}{|1-\lambda f(t)|^2 |1-\lambda f(s)| |1-\lambda f(t) f(s)|} \\
&\leqslant D(\varepsilon) \int_J \frac{ds}{|1-\lambda f(s)|} \\
&\leqslant D(\varepsilon)\Big(C + \int_{B_\varepsilon(0)} \frac{ds}{|1-\lambda+\frac{\lambda}{2}|s|^2| - \theta_\varepsilon |s|^2}\Big) \\
&\leqslant D(\varepsilon)\Big(C + \int_0^\varepsilon \frac{r \, dr}{|1-\lambda+\frac{\lambda}{2}r^2| - \theta_\varepsilon r^2}\Big) \\
&\leqslant D(\varepsilon)(C + \log(|1-\lambda|^{-1}))
\end{aligned}
$$

by the same calculations as for $F_1(\lambda)$.

The case when both t and s are bounded away from zero is trivial and the integral in this region is bounded above by the constant $D(\varepsilon)$ by Inequality (3.3.2).

These facts imply that

$$
\rho_3(\lambda) = (1-\lambda)^{-2}(2\pi)^{-4} \iint_{U_\varepsilon} \frac{\lambda f(s)(1-f(t))}{(1-\lambda f(t))^2 (1-\lambda f(s))(1-\lambda f(t) f(s))} + F,
$$

where

$$
F(\lambda) = F_1(\lambda) + F_2(\lambda) + F_3(\lambda),
$$

is the error from integrating only over U_ε and not over the whole region J^2. By the above calculations we have the bound

$$|F| \leqslant D(\varepsilon)|1-\lambda|^{-2}\log(1/|1-\lambda|) + D(\varepsilon)|1-\lambda|^{-2}, \quad \text{when } \Re(\lambda) > \alpha.$$

Error from using the expansion. The next step is to use the expansion (3.1.8) inside the integral sign in order to make exact calculations possible. When we replace the characteristic function by its expansion, an additional error term E will appear given by

$$E = C(1-\lambda)^{-2}(2\pi)^{-4}\Bigg(\iint_{U_\varepsilon} \frac{\lambda f(s)(1-f(t)\,dt\,ds)}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))} \\ - \iint_{U_\varepsilon} \frac{\lambda|t|_\Sigma/2\,dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)^2(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}(|t|_\Sigma+|s|_\Sigma))}\Bigg).$$

In Appendix A it is shown that when $\Re(\lambda) > \alpha$

$$|E| \leqslant C(\varepsilon)|1-\lambda|^{-3}.$$

Error from region of integration. Having estimated the error from the use of the expansion we now use polar coordinates to obtain the following expression for $\rho_3(\lambda)$,

$$\rho_3(\lambda) = \frac{\lambda}{(1-\lambda)^2(2\pi)^2|\Sigma|}\int_0^\varepsilon\int_0^\varepsilon \frac{r^3 s\,dr\,ds}{(1-\lambda+\frac{\lambda}{2}r^2)^2(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))} \\ + E + F.$$

To simplify calculations we extend the region of integration to the whole of $[0,\infty)^2$ to get the expression

$$\rho_3(\lambda) = \frac{\lambda}{(1-\lambda)^2(2\pi)^2|\Sigma|}\int_0^\infty\int_0^\infty \frac{r^3 s\,dr\,ds}{(1-\lambda+\frac{\lambda}{2}r^2)^2(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))} \\ + E + F - H.$$

where $H = H(\lambda)$ is the error from integrating over the larger region $[0,\infty)^2$. We split this error into a sum of three terms

$$H(\lambda) = C(1-\lambda)^{-2}(H_1(\lambda) + H_2(\lambda) + H_3(\lambda)),$$

given by

$$H_1 \equiv C\int_\varepsilon^\infty\int_0^\varepsilon \frac{r^3 s\,dr\,ds}{(1-\lambda+\frac{\lambda}{2}r^2)^2(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))},$$

$$H_2 \equiv C\int_0^\varepsilon\int_\varepsilon^\infty \frac{r^3 s\,dr\,ds}{(1-\lambda+\frac{\lambda}{2}r^2)^2(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))}$$

$$H_3 \equiv C \int_\varepsilon^\infty \int_\varepsilon^\infty \frac{r^3 s \, dr \, ds}{(1-\lambda+\frac{\lambda}{2}r^2)^2(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))}.$$

We proceed by bounding these integrals for $\Re(\lambda) > \alpha \in (0,1)$. For H_1, by Inequalities (3.3.7) and (3.3.8)

$$\begin{aligned}
|H_1| &\leqslant C \int_\varepsilon^\infty \int_0^\varepsilon \frac{r^3 s \, dr \, ds}{|1-\lambda+\frac{\lambda}{2}r^2|^2|1-\lambda+\frac{\lambda}{2}s^2||1-\lambda+\frac{\lambda}{2}(r^2+s^2)|}, \\
&\leqslant C|1-\lambda|^{-1} \int_K^\infty \int_0^K \frac{r^3 s \, dr \, ds}{|z_1+z_2r^2|^2|z_1+z_2s^2||z_1+z_2(r^2+s^2)|} \\
&\leqslant C|1-\lambda|^{-1} \int_K^\infty \int_0^K \frac{rs \, dr \, ds}{|z_1+z_2r^2|s^3r} \leqslant C|1-\lambda|^{-1}K^{-1} \int_0^K \frac{dr}{|z_1+z_2r^2|}.
\end{aligned}$$

Fix now a small $\delta > 0$. Then since for $r < \delta$ by the triangle inequality $|z_1 + z_2r^2| \geqslant C > 0$,

$$\begin{aligned}
\int_0^K \frac{dr}{|z_1+z_2r^2|} &= \int_0^\delta \frac{dr}{|z_1+z_2r^2|} + \int_\delta^K \frac{dr}{|z_1+z_2r^2|} \\
&\leqslant C + \int_\delta^\infty \frac{dr}{r^2} \leqslant C.
\end{aligned}$$

Putting these two last inequalities together we have

$$|H_1(\lambda)| \leqslant C|1-\lambda|^{-1}\frac{|1-\lambda|}{\varepsilon} = D(\varepsilon),$$

where $D(\varepsilon)$ is unbounded as $\varepsilon \to 0$.

By similar calculations, for H_2 we have

$$\begin{aligned}
|H_2| &\leqslant C \int_0^\varepsilon \int_\varepsilon^\infty \frac{r^3 s \, dr \, ds}{|1-\lambda+\frac{\lambda}{2}r^2|^2|1-\lambda+\frac{\lambda}{2}s^2||1-\lambda+\frac{\lambda}{2}(r^2+s^2)|}, \\
&\leqslant C|1-\lambda|^{-1} \int_0^K \int_K^\infty \frac{r^3 s \, dr \, ds}{|z_1+z_2r^2|^2|z_1+z_2s^2||z_1+z_2(r^2+s^2)|} \\
&\leqslant C|1-\lambda|^{-1} \int_0^K \int_K^\infty \frac{rs \, dr \, ds}{r^2|z_1+z_2s^2|rs} \leqslant C|1-\lambda|^{-1} \int_0^K \int_K^\infty \frac{dr \, ds}{r^2|z_1+z_2s^2|} \\
&\leqslant C|1-\lambda|^{-1}K^{-1} \int_0^K \frac{ds}{|z_1+z_2s^2|} \leqslant D(\varepsilon).
\end{aligned}$$

And finally concerning H_3 we have

$$|H_3| = C\int_\varepsilon^\infty\int_\varepsilon^\infty \frac{r^3 s\,\mathrm{d}r\,\mathrm{d}s}{r^4 s^2 rs} \leqslant \int_\varepsilon^\infty\int_\varepsilon^\infty \frac{\mathrm{d}r\,\mathrm{d}s}{r^2 s^2} \leqslant D(\varepsilon).$$

Expansion of the integral. Overall, having bound all the error terms we have the expression

$$\rho_3(\lambda)$$
$$= \frac{\lambda}{(1-\lambda)^2(2\pi)^2|\Sigma|}\int_0^\infty\int_0^\infty \frac{r^3 s\,\mathrm{d}r\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}r^2)^2(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))} + \mathcal{E},$$

where $\mathcal{E}$ is the total error and for $\Re(\lambda) > \alpha$ is bounded by

$$|\mathcal{E}| \leqslant C(\varepsilon)|1-\lambda|^{-3} + D(\varepsilon)|1-\lambda|^{-2} + D(\varepsilon)|1-\lambda|^{-2}\log(1/|1-\lambda|).$$

Assume for the moment that λ is real and $\lambda \in (1/2, 1)$. We can then calculate

$$\int_0^\infty\int_0^\infty \frac{r^3 s\,\mathrm{d}r\,\mathrm{d}s}{(1+\frac{\lambda}{2}r^2)^2(1+\frac{\lambda}{2}s^2)(1+\frac{\lambda}{2}(r^2+s^2))} = 2\lambda^{-3}.$$

Since both sides of the equality are analytic on the open unit disc and equal on the set $(1/2, 1)$ by analytic continuation they are equal on the whole of the open unit disc. Finally, we have the expansion

$$\begin{aligned}\rho_3(\lambda) &= (2\pi)^{-2}|\Sigma|^{-1}\lambda^{-2}(1-\lambda)^{-3} + \mathcal{E}(\lambda)\\ &= (2\pi)^{-2}|\Sigma|^{-1}(1-\lambda)^{-3} + \mathcal{E}(\lambda),\end{aligned}$$

Note that the λ^{-2} factor in the leading term has been moved inside the error term (see Remark 3.2.3). Now we are ready to use the expansion of $\rho_3(\lambda)$ and Lemma 3.1.2 to calculate the exact order of $a_3(n)$.

It is easy to see that if

$$c(n) = \frac{1}{2}(n^2 + 3n + 2),$$

then

$$g(\lambda) \equiv \sum_{n=0}^{\infty} c(n)\lambda^n = (1-\lambda)^{-3}.$$

Recall that $\rho_3(\lambda) := \sum_{n=0}^{\infty} a_3(n)\lambda^n$. Define the function

$$f(\lambda) := \mathcal{E}(\lambda) = \rho_3(\lambda) - g(\lambda).$$

The function f satisfies the conditions of Lemma 3.1.2 since by our previous calculations we have that

$$|f| \leqslant \begin{cases} C(\varepsilon)|1-\lambda|^{-3} + D(\varepsilon)|1-\lambda|^{-2} + D(\varepsilon)|1-\lambda|^{-2}\log(1/|1-\lambda|), & \text{if } \Re(\lambda) > \alpha, \\ K, & \text{if } \Re(\lambda) \leqslant \alpha. \end{cases}$$

Then Lemma 3.1.2 applied to f implies that for all n,

$$|a_3(n) - c(n)| \leqslant 4K + C(\varepsilon)n^2 + D(\varepsilon)(n\log n + n),$$

and thus dividing by n^2 and taking limits as $n \to \infty$ we have the inequality

$$\limsup_{n\to\infty} \frac{1}{n^2}\left|a_3(n) - \frac{1}{8\pi^2|\Sigma|}n^2\right| \leqslant C(\varepsilon).$$

The arbitrary nature of ε along with the fact that $\lim_{\varepsilon\to 0} C(\varepsilon) = 0$, imply that

$$\lim_{n\to\infty} \frac{1}{n^2}\left|a_3(n) - \frac{1}{8\pi^2|\Sigma|}n^2\right| = 0,$$

and thus $a_3(n) \sim n^2/8\pi^2|\Sigma|$ as $n \to \infty$.

3.3.3 Second term

Now we consider the sum over the set A^2,

$$A^2 = \{(i_1, j_1, i_2, j_2) : 0 \leqslant i_1 \leqslant i_2 < j_1 < j_2 \leqslant n\}.$$

$$\begin{aligned} a_2(n) &= \sum_{A^2} [\mathbb{P}(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) - \mathbb{P}(S_{i_1} = S_{j_1})\mathbb{P}(S_{i_2} = S_{j_2})] \\ &= \sum_{m\in M_n} \Big[\mathbb{P}(S_{m_1} = S_{m_1+m_2+m_3}, S_{m_1+m_2} = S_{m_1+m_2+m_3+m_4}) \\ &\quad - \mathbb{P}(S_{m_1} = S_{m_1+m_2+m_3})\mathbb{P}(S_{m_1+m_2} = S_{m_1+m_2+m_3+m_4})\Big] \\ &= \sum_{m\in M_n} [\mathbb{P}(S_{m_2+m_3} = 0, S_{m_3+m_4} = 0) - \mathbb{P}(S_{m_2+m_3} = 0)\mathbb{P}(S_{m_3+m_4} = 0)], \\ &= \sum_{m\in M_n} \Big[\sum_{x\in\mathbb{Z}^2} \mathbb{P}(S_{m_2} = x)\mathbb{P}(S_{m_3} = -x)\mathbb{P}(S_{m_4} = x) \\ &\quad - \mathbb{P}(S_{m_2+m_3} = 0)\mathbb{P}(S_{m_3+m_4} = 0)\Big], \end{aligned}$$

where M_n is the set of 5-tuples $(m_1, \ldots, m_5)$ such that $m_1, m_2, m_5 \geqslant 0$ and $m_3, m_4 > 0$. Then we have

$$\begin{aligned} \rho_2(\lambda) &= (1-\lambda)^{-2}\lambda^2(2\pi)^{-4} \\ &\quad \times \int_J\int_J \frac{f(t)\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))(1-\lambda f(s))}\left[\frac{f(t+s)}{1-\lambda f(t+s)} - \frac{f(s)^2}{1-\lambda f(t)f(s)}\right] \end{aligned}$$

$$
\begin{aligned}
&= \int_J \int_J \frac{(1-\lambda)^{-2}\lambda^2(2\pi)^{-4} f(t)}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t+s))(1-\lambda f(t)f(s))} \\
&\qquad \times \left[f(t+s)\,(1-\lambda f(t)f(s)) - f(s)^2\,(1-\lambda f(t+s))\right] dt\,ds \\
&= (1-\lambda)^{-2}\lambda^2(2\pi)^{-4} \int_J \int_J X(t,s,\lambda)\,dt\,ds.
\end{aligned}
$$

By calculations similar to those in section 3.2 we have that for some constants $K > 0$ and $\alpha \in (0,1)$

$$
|\rho_2(\lambda)| \leqslant K, \quad \text{for } \Re(\lambda) \leqslant \alpha.
$$

From now on assume that $\Re(\lambda) > \alpha$.

Fix a small $\varepsilon > 0$ and let

$$
U_\varepsilon = \{(t,s) \in J \times J : |t| < \varepsilon, |s| < \varepsilon\}.
$$

Integral away from the origin. We claim that all significant contributions to the integral $Y(\lambda)$ come from this region, so let us consider the integral when t or s is bounded away from zero. Obviously if both are bounded away from zero the integral is bounded above by a constant. Note also that when $\Re(\lambda) \leqslant \alpha \in (0,1)$ the integral is bounded above by a constant. Assume from now on that $\Re(\lambda) > \alpha$.

Consider for example the region $V = \{(t,s) \in J \times J : |t| \geqslant \varepsilon, |s| < \varepsilon/2\}$. It is trivial to show that in V, $|t+s|$ is also bounded away from zero. By similar calculations as for the first term

$$
\begin{aligned}
\left|\iint_V X(t,s,\lambda)\,dt\,ds\right| &\leqslant D(\varepsilon) \int_{|s|<\varepsilon} \frac{ds}{|1-\lambda f(s)|} \\
&\leqslant D(\varepsilon) \int_{|s|<\varepsilon} \frac{ds}{\left||1-\lambda+\frac{\lambda}{2}|s|^2| - \Theta_\varepsilon |s|^2\right|} \leqslant D(\varepsilon) \log\left(\frac{1}{|1-\lambda|}\right),
\end{aligned}
$$

where once again $D(\varepsilon)$ is a generic positive constant, which may be unbounded as $\varepsilon \to 0$. The other cases follow similarly giving the same or smaller order.

We can now write

$$
\begin{aligned}
\rho_3(\lambda) &= (1-\lambda)^{-2}\lambda^2(2\pi)^{-4} \\
&\quad \times \iint_{U_\varepsilon} \frac{f(t)\,dt\,ds}{(1-\lambda f(t))(1-\lambda f(s))} \left[\frac{f(t+s)}{1-\lambda f(t+s)} - \frac{f(s)^2}{1-\lambda f(t)f(s)}\right] + F(\lambda),
\end{aligned}
$$

where $F(\lambda)$ is the error term from integrating over U_ε and satisfies the bound

$$
|F(\lambda)| \leqslant D(\varepsilon)|1-\lambda|^{-2}\log(1/|1-\lambda|).
$$

Error from the expansion. We now only have to consider the integral on a small neighborhood of the origin. Near the origin we can use the Taylor expansion (3.1.8) of the characteristic function under the integral sign, to make exact calculations possible. This will introduce an error term $E = E(\lambda)$ for which we have the bound

$$|E(\lambda)| \leqslant C(\varepsilon)|1-\lambda|^{-3}, \quad \text{for } \Re(\lambda) > \alpha.$$

The proof of this bound is given in full detail in Appendix A.

So far we have the expression

$$\begin{aligned}\rho_3(\lambda) = {} & \frac{\lambda^2}{(1-\lambda)^2(2\pi)^4} \times \iint_{U_\varepsilon} \frac{dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}|t+s|_\Sigma)} \\ & - \frac{\lambda^2}{(1-\lambda)^2(2\pi)^4} \iint_{U_\varepsilon} \frac{dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}(|t|_\Sigma+|s|_\Sigma))}\Bigg) \\ & + F + E = Y_1(\lambda) - Y_2(\lambda) + F + E.\end{aligned}$$

The above expression gives $\rho_3(\lambda)$ as the difference of two integrals plus some error terms. We will treat these integrals one by one.

Expansion of first integral. For the first integral we have

$$\begin{aligned}Y_1(\lambda) &= (1-\lambda)^{-2}\lambda^2(2\pi)^{-4} \\ &\times \iint_{U_\varepsilon} \frac{dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}|t+s|_\Sigma)}.\end{aligned}$$

To make exact calculations possible we extend the region of integration. Assume for now that $\lambda \in (1/2, 1)$ is real. For such λ we have

$$\begin{aligned}&\iint_{U_\varepsilon} \frac{dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}|t+s|_\Sigma)} \\ &= |\Sigma|^{-1} \int_0^\infty\int_0^\infty\int_{-\pi}^{\pi} \frac{rs\,dt\,dr\,ds}{(1-\lambda+\frac{\lambda}{2}r^2)(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2-2rs\cos(t)))} \\ &\quad - I_1 - I_2 - I_3,\end{aligned}$$

where we define

$$\begin{aligned}I_1 &= |\Sigma|^{-1}|1-\lambda|^{-1} \iint_{|k_1|<K\leqslant|k_2|} \frac{dk_1\,dk_2}{|z_1+z_2|k_1|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1+k_2|^2|}, \\ I_2 &= |\Sigma|^{-1}|1-\lambda|^{-1} \iint_{|k_2|<K\leqslant|k_1|} \frac{dk_1\,dk_2}{|z_1+z_2|k_1|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1+k_2|^2|}, \\ I_3 &= |\Sigma|^{-1}|1-\lambda|^{-1} \iint_{K\leqslant|k_1|,|k_2|} \frac{dk_1\,dk_2}{|z_1+z_2|k_1|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1+k_2|^2|}.\end{aligned}$$

By analytic continuation this representation holds for all complex λ with $|\lambda| < 1$. For such λ we now bound the errors I_i for $i = 1, 2, 3$ and $\Re(\lambda) > \alpha$. By Lemma A.4.4 and symmetry

$$|I_1|, |I_2| \leqslant D(\varepsilon) + D(\varepsilon) \log_+ |1-\lambda|^{-1},$$

while by Lemma A.4.5 we have $|I_3| \leqslant D(\varepsilon)$, where as before $D(\varepsilon) > 0$ may be unbounded as $\varepsilon \to 0$.

Therefore, from the above, the usual change of variables and Lemma A.4.3 we have that

$$Y_1(\lambda) = \frac{|\Sigma|^{-1}\lambda^2}{4\pi^2(1-\lambda)^3} \int_0^\infty\!\!\int_0^\infty \frac{rs \, \mathrm{d}r \, \mathrm{d}s}{(1+\frac{\lambda}{2}r^2)(1+\frac{\lambda}{2}s^2)\sqrt{(1+\frac{\lambda}{2}(r^2+s^2))^2 - \lambda^2 r^2 s^2}} + \mathcal{E},$$

where $\mathcal{E}$ is the total error and satisfies

$$|\mathcal{E}| \leqslant D(\varepsilon)|1-\lambda|^{-2} + D(\varepsilon)|1-\lambda|^{-2}\log_+|1-\lambda|^{-1} + C(\varepsilon)|1-\lambda|^{-3}.$$

Expansion of the second integral. Thus we now have the following expansion

$$\begin{aligned} Y_2(\lambda) &= (1-\lambda)^{-2}(2\pi)^{-4}\lambda^2 \\ &\quad \times \iint_{U_\varepsilon} \frac{\mathrm{d}t\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}\langle \Sigma k_| s\rangle)(1-\lambda+\frac{\lambda}{2}(|t|_\Sigma+|s|_\Sigma))} + \mathcal{E}, \end{aligned}$$

where $\mathcal{E}$ is the error from the Taylor expansion, and satisfies the bound $|\mathcal{E}| \leqslant C(\varepsilon)|1-\lambda|^{-3}$.

Let us now consider the integral above. Assume for now that λ is real and lies in the interval $(1/2, 1)$. Then we have

$$\begin{aligned} &\iint_{U_\varepsilon} \frac{\mathrm{d}t\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}(|t|_\Sigma+|s|_\Sigma))} \\ &\quad = |\Sigma|^{-1} \iint_{U_\varepsilon} \frac{\mathrm{d}t\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}|t|^2)(1-\lambda+\frac{\lambda}{2}|s|^2)(1-\lambda+\frac{\lambda}{2}(|t|^2+|s|^2))} \\ &\quad = |\Sigma|^{-1}(2\pi)^2 \int_0^\varepsilon\!\!\int_0^\varepsilon \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}r^2)(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))} \\ &\quad = |\Sigma|^{-1}(2\pi)^2 \int_0^\infty\!\!\int_0^\infty \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}r^2)(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))} \\ &\quad - I_1 - I_2 - I_3, \end{aligned}$$

where we define

$$I_1 = |\Sigma|^{-1}(2\pi)^2 \int_\varepsilon^\infty\!\!\int_0^\varepsilon \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}r^2)(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))}$$

$$I_2 = |\Sigma|^{-1}(2\pi)^2 \int_0^{\varepsilon}\int_{\varepsilon}^{\infty} \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}r^2)(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))}$$

$$I_3 = |\Sigma|^{-1}(2\pi)^2 \int_{\varepsilon}^{\infty}\int_{\varepsilon}^{\infty} \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}r^2)(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))}.$$

By analytic continuation the expansion also holds for complex λ with $|\lambda| < 1$. For such λ we now bound the terms $I_i, i = 1,2,3$. By Inequalities (3.3.7) and (3.3.8) and a change of variables

$$\begin{aligned}
I_1 &\leqslant C|1-\lambda|^{-1} \int_K^{\infty}\int_0^{K} \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{|z_1+z_2r^2||z_1+z_2s^2||z_1+z_2(r^2+s^2)|} \\
&\leqslant C|1-\lambda|^{-1} \int_K^{\infty}\int_0^{K} \frac{\mathrm{d}r\,\mathrm{d}s}{|z_1+z_2r^2|s^2} \leqslant C|1-\lambda|^{-1}K^{-1} \leqslant C.
\end{aligned}$$

The bound for I_2 is also bounded above by symmetry. So now let us consider the integral I_3.

$$\begin{aligned}
I_3 &\leqslant C|1-\lambda|^{-1} \int_K^{\infty}\int_K^{\infty} \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{|z_1+z_2r^2||z_1+z_2s^2||z_1+z_2(r^2+s^2)|} \\
&\leqslant C|1-\lambda|^{-1} \int_K^{\infty}\int_K^{\infty} \frac{\mathrm{d}r\,\mathrm{d}s}{r^2s^2} \\
&= C|1-\lambda|^{-1}K^{-2} = C|1-\lambda| \leqslant C,
\end{aligned}$$

since $|1-\lambda| \leqslant 1+|\lambda| \leqslant 2$.

Now we can calculate for $\lambda \in (1/2,1)$ the integral

$$\begin{aligned}
&\int_0^{\infty}\int_0^{\infty} \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}r^2)(1-\lambda+\frac{\lambda}{2}s^2)(1-\lambda+\frac{\lambda}{2}(r^2+s^2))} \\
&\quad= (1-\lambda)^{-1} \int_0^{\infty}\int_0^{\infty} \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{(1+\frac{\lambda}{2}r^2)(1+\frac{\lambda}{2}s^2)(1+\frac{\lambda}{2}(r^2+s^2))} \\
&\quad= (1-\lambda)^{-1} \int_0^{\infty}\int_0^{\infty} \frac{\mathrm{d}r\,\mathrm{d}s}{(1+\lambda r)(1+\lambda s)(1+\lambda(r+s))} \\
&\quad= (1-\lambda)^{-1}\lambda^{-2} \int_0^{\infty}\int_0^{\infty} \frac{\mathrm{d}r\,\mathrm{d}s}{(1+r)(1+s)(1+(r+s))} = \frac{\pi^2}{6}\lambda^{-2}(1-\lambda)^{-1}.
\end{aligned}$$

All these facts together imply that we have the expansion for $Y_2(\lambda)$

$$\begin{aligned}
Y_2(\lambda) &= (1-\lambda)^{-2}(2\pi)^{-4}\lambda^2(2\pi)^2|\Sigma|^{-1/2}(1-\lambda)^{-1}\frac{\pi^2}{6}\lambda^{-2} + \mathcal{E} + I \\
&= (2\pi)^{-2}|\Sigma|^{-1/2}\frac{\pi^2}{6}(1-\lambda)^{-3} + \mathcal{E} + I,
\end{aligned}$$

where $\mathcal{E}$ is the error from using Taylor under the integral sign, and satisfies $|\mathcal{E}| \leqslant C(\varepsilon)|1-\lambda|^{-3}$ while $|I| \leqslant C|1-\lambda|^{-2}$.

Calculation of the order. We are now ready to calculate the exact order of $a_2(n)$. We have seen that

$$\begin{aligned} &Y_1(\lambda) - Y_2(\lambda) \\ &\quad = (2\pi)^{-2}|\Sigma|^{-1/2}(1-\lambda)^{-3}\left(\int_0^\infty\!\!\int_0^\infty \frac{dr\,ds}{(1+r)(1+s)\sqrt{(1+r+s)^2-4rs}} - \frac{\pi^2}{6}\right) + \mathcal{E} \end{aligned}$$

where for $\Re(\lambda) > \alpha$ we have

$$|\mathcal{E}| \leqslant C(\varepsilon)|1-\lambda|^{-3} + D(\varepsilon)|1-\lambda|^{-2}\log(|1-\lambda|^{-1}) + D(\varepsilon)|1-\lambda|^{-2}.$$

Now putting summing the error terms from the two integrals we have

$$\rho_2(\lambda) = (2\pi)^{-2}\varkappa|\Sigma|^{-1/2}(1-\lambda)^{-3} + \mathcal{E}$$

where

$$\varkappa \equiv \int_0^\infty\!\!\int_0^\infty \frac{dr\,ds}{(1+r)(1+s)\sqrt{(1+r+s)^2-4rs}} - \frac{\pi^2}{6},$$

and thus by application of Lemma 3.1.2 and calculations similar to the one-dimensional case we have

$$\left|a_2(n) - \frac{1}{2}(2\pi)^{-2}|\Sigma|^{-1}\varkappa^2\right| \leqslant C(\varepsilon)n^2 + D(\varepsilon)(n\log n + n)$$

from which we obtain

$$\limsup_{n\to\infty} \frac{1}{n^2}\left|a_2(n) - \frac{1}{2}(2\pi)^{-2}|\Sigma|^{-1}\varkappa n^2\right| \leqslant C(\varepsilon),$$

and by the arbitrary nature of ε and the fact that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, we finally have

$$a_2(n) \sim \frac{1}{2}(2\pi)^{-2}|\Sigma|^{-1}\varkappa n^2.$$

3.3.4 Third and fourth terms

We now consider the sum over B^2.

$$\begin{aligned} b_2(n) &= \sum_{B^2} [\mathbb{P}(S_{i_1} = S_{j_1}, S_{i_2} = S_{j_2}) - \mathbb{P}(S_{i_1} = S_{j_1})\mathbb{P}(S_{i_2} = S_{j_2})] \\ &= \sum_{\mathbf{m}\in M_n} \Big[\mathbb{P}(S_{m_1+m_2} = S_{m_1+m_2+m_3}, S_{m_1} = S_{m_1+m_2+m_3+m_4}) \\ &\qquad - \mathbb{P}(S_{m_1+m_2} = S_{m_1+m_2+m_3})\mathbb{P}(S_{m_1} = S_{m_1+m_2+m_3+m_4})\Big] \end{aligned}$$

$$= \sum_{\mathbf{m}\in M_n} [\mathbb{P}(S_{m_2+m_4}=0)\mathbb{P}(S_{m_3}=0) - \mathbb{P}(S_{m_2+m_3+m_4}=0)\mathbb{P}(S_{m_3}=0)],$$

where M_n is the set of 5-tuples $(m_1,\dots,m_5)$ such that $m_1,m_4,m_5 \geqslant 0$ and $m_2,m_3>0$ such that $m_1+\cdots+m_5=n$.

$$\begin{aligned}
\sigma_2(\lambda) &= \sum_{n\geqslant 0} b_2(n)\lambda^n \\
&= (1-\lambda)^{-2}\sum_{m_3\geqslant 1}\sum_{m_2\geqslant 1}\sum_{m_4\geqslant 0}\lambda^{m_2+m_3+m_4}\Big[\mathbb{P}(S_{m_2+m_4}=0)\mathbb{P}(S_{m_3}=0) \\
&\quad - \mathbb{P}(S_{m_2+m_3+m_4}=0)\mathbb{P}(S_{m_3}=0)\Big] \\
&= (1-\lambda)^{-2}(2\pi)^{-4}\sum_{m_3\geqslant 1}\sum_{m_2\geqslant 1}\sum_{m_4\geqslant 0}\lambda^{m_2+m_3+m_4}\iint_{J^2}\Big(f^{m_2+m_4}(t)f^{m_3}(s) \\
&\quad - f^{m_2+m_3+m_4}(t)f^{m_3}(s)\Big)\,\mathrm{d}t\,\mathrm{d}s \\
&= (1-\lambda)^{-2}(2\pi)^{-4}\iint_{J^2}\frac{\lambda^2 f(t)f(s)}{(1-\lambda f(t))^2}\frac{1-f(t)}{(1-\lambda f(s))(1-\lambda f(t)f(s))}\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

It is trivial to see that this is of the same order as $\rho_3(\lambda)$ and thus we have that

$$b_2(n) \sim \frac{1}{2}(2\pi)^{-2}|\Sigma|^{-1}n^2.$$

Now consider the sum over B^3.

$$B^3 = \{(i_1,j_1,i_2,j_2): 0\leqslant i_2 < i_1 < j_2 < j_1 \leqslant n\}.$$

$$\begin{aligned}
b_3(n) &= \sum_{A^3}[\mathbb{P}(S_{i_1}=S_{j_1},S_{i_2}=S_{j_2}) - \mathbb{P}(S_{i_1}=S_{j_1})\mathbb{P}(S_{i_2}=S_{j_2})] \\
&= \sum_{\mathbf{m}\in M_n}\Big[\mathbb{P}(S_{m_1+m_2}=S_{m_1+m_2+m_3+m_4}, S_{m_1}=S_{m_1+m_2+m_3}) \\
&\quad - \mathbb{P}(S_{m_1+m_2}=S_{m_1+m_2+m_3+m_4})\mathbb{P}(S_{m_1}=S_{m_1+m_2+m_3})\Big] \\
&= \sum_{\mathbf{m}\in M_n}[\mathbb{P}(S_{m_2+m_3}=0, S_{m_3+m_4}=0) - \mathbb{P}(S_{m_2+m_3}=0)\mathbb{P}(S_{m_3+m_4}=0)], \\
&= \sum_{\mathbf{m}\in M_n}\Big[\sum_{x\in\mathbf{Z}^2}\mathbb{P}(S_{m_2}=x)\mathbb{P}(S_{m_3}=-x)\mathbb{P}(S_{m_4}=x) \\
&\quad - \mathbb{P}(S_{m_2+m_3}=0)\mathbb{P}(S_{m_3+m_4}=0)\Big],
\end{aligned}$$

where M_n is the set of 5-tuples $(m_1,\dots,m_5)$ such that $m_1,m_5\geqslant 0$ and $m_2,m_3,m_4>0$ and $m_1+\cdots+m_5=n$.

Then we have

$$\sigma_3(\lambda) = (1-\lambda)^{-2}\lambda^3(2\pi)^{-4}\int_J\int_J\frac{f(t)f(s)}{(1-\lambda f(t))(1-\lambda f(s))}\times\left[\frac{f(t+s)}{1-\lambda f(t+s)} - \frac{f(t)f(s)}{1-\lambda f(t)f(s)}\right]\mathrm{d}t\,\mathrm{d}s$$

This has similar expansion as $\rho_2(\lambda)$ and thus we conclude that

$$b_3(n) \sim a_2(n) \sim \frac{1}{2}(2\pi)^{-2}|\Sigma|^{-1}\varkappa n^2.$$

Conclusion. Recall that the variance of the self-intersections is given by

$$\mathrm{Var}(V_n) = 4(a_2(n) + a_3(n) + b_2(n) + b_3(n)). \tag{3.3.14}$$

Our calculations have shown that

$$\begin{aligned} a_2(n) \sim b_3(n) &\sim \frac{1}{2}(2\pi)^{-2}|\Sigma|^{-1}\varkappa n^2 \\ a_3(n) \sim b_2(n) &\sim \frac{1}{2}(2\pi)^{-2}|\Sigma|^{-1}n^2. \end{aligned}$$

Thus overall we can conclude that

$$\mathrm{Var}(V_n) \sim 4(2\pi)^{-2}|\Sigma|^{-1/2}(1+\kappa)n^2,$$

where

$$\kappa \equiv \int_0^\infty\int_0^\infty \frac{dr\,ds}{(1+r)(1+s)\sqrt{(1+r+s)^2-4rs}} - \frac{\pi^2}{6}.$$

CHAPTER 4

A central limit theorem for random walk on random scenery

4.1 Introduction and main result

Let $S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$, for $n \geqslant 1$, be the random walk defined by the partial sums of the iid sequence of $\mathbf{Z}$-valued random variables $X_1, X_2, \ldots$. Suppose further, that $\xi(\alpha)$, indexed by $\alpha \in \mathbf{Z}$, are iid, real valued random variables and independent of the X_i. Then by *random walk in random scenery* we shall mean the process

$$Z_0 = 0, \quad Z_n = \sum_{k=1}^{n} \xi(S_k), n \geqslant 1.$$

In this chapter we shall consider the one dimensional random walk S_n, such that the characteristic function $f(t)$ of the increments satisfies (3.1.7), and an independent random scenery $\{\xi(i)\}_{i \in \mathbf{Z}}$ with mean zero and finite positive variance σ^2. Then we define a random variable in $D[0,1]$, the space of right-continuous functions with left limits,

$$Y_n(t) = \sqrt{\pi\gamma} Z_{[nt]} / \sigma\sqrt{2n \log n}, \quad t \in [0,1], \tag{4.1.1}$$

where $[x]$ denotes the integer part of x and c_n is a normalization constant that depends on n.

The main result of this chapter is a functional central limit theorem for one dimensional random walk in random scenery. We shall show that the laws of Y_n defined above converge weakly in $D[0,1]$ to the Wiener measure. This answers a question raised by Kesten and Spitzer [5] which has remained open since 1979. The main result is summarized in Theorem 4.1.1 and is given in the next subsection. The proof of Theorem 4.1.1 is then given in section 4.2. In the proof of weak convergence we shall use the asymptotics for the variance of the self-intersections obtained in the previous chapter. In particular the upper bound of order n^2 we obtained for the variance of the self-intersection local time is particularly important since it allows us to obtain the central limit theorem almost surely. In other words conditionally on the full history of the random walk

$\mathcal{A} = \sigma(S_n, n \geqslant 0)$, the observed random scenery $(\xi(S_0), \xi(S_1), \dots)$ satisfies the classical central limit theorem, for almost every path of the random walk.

4.1.1 Main result

We now state the main result of this chapter which answers a conjecture stated in Kesten and Spitzer [5] and covers the case of random walk with increments in the domain of attraction of the α-stable law with $\alpha = 1$. We have the following limit theorem.

Theorem 4.1.1. *Let $S_0 = 0$, $S_n = \sum_{k=1}^n X_k, n \geqslant 1$ be a strongly aperiodic random walk satisfying (3.1.7). Suppose further, that $\{\xi(\alpha)\}_{\alpha \in \mathbf{Z}}$, are iid, real valued random variables and independent of the X_i with mean zero and variance $\sigma^2 > 0$. Define $Z_0 = 0$, $Z_n = \sum_{k=1}^n \xi(S_k), n \geqslant 1$ and the processes*

$$Y_n(t) = \sqrt{\pi\gamma} Z_{[nt]} / \sigma\sqrt{2n \log n}, \quad t \in [0, 1].$$

The laws of $(Y_n)_{n \geqslant 0}$ converge weakly in $D[0,1]$ to the Wiener measure.

We prove the theorem in the next section. First we show convergence of the finite-dimensional distributions and then we prove tightness.

Remark 4.1.1. Kesten and Spitzer [5] conjectured that if the $\{X_i\}$ are i.i.d. random variables in the domain of attraction of the symmetric Cauchy law, and $\{\xi_i\}$ are normal, then $\{Y_n(t)\}_{t \geqslant 0}$ converges weakly to standard Brownian motion. Theorem 4.1.1 actually strengthens this conjecture, since variables attracted to the Cauchy law can be shown to satisfy (3.1.7). To see this observe that

$$\mathbb{E}\Big[\exp\Big(\mathrm{i}t \sum_{i=1}^n X_i/n\Big)\Big] = f(t/n)^n = \Big(1 - \gamma|t|/n + R(t/n)\Big)^n.$$

But then this can be expressed as

$$(1 - \gamma|t|/n)^n + (1 - \gamma|t|/n)^{n-1} nR(t/n) + (1 - \gamma|t|/n)^{n-2} n^2 R(t/n)^2 + \cdots.$$

The first term converges to $\mathrm{e}^{-\gamma|t|}$, while the rest all converge to zero, because

$$(1 - \gamma|t|/n)^{n-k} n^k R(t/n)^k \leqslant C n^k R(t/n)^k \to 0$$

for $k \leqslant n$ since $R(t) = o(|t|)$. It is well known that $\mathrm{e}^{-\gamma|t|}$ is the characteristic function of the symmetric Cauchy law, corresponding to the symmetric α-stable distribution with $\alpha = 1$.

As an example it can be shown that a symmetric random variable with density

$$\mathbb{P}(X = k) = \frac{C}{1 + k^2},$$

where C is an appropriate normalizing constant to make this a probability measure, satisfies (3.1.7).

4.2 Proof of Theorem 4.1.1

We prove weak convergence of the laws of $Y_n(t)$ in $D[0,1]$ by first showing convergence of the finite dimensional distributions, and then showing tightness, in a manner similar to [3].

Weak convergence in D. $D = D[0,1]$ is the space of functions x on $[0,1]$ that are right-continuous with left-hand limits, and $\mathcal{D}$ the Borel σ-algebra relative to the Skorokhod topology (see Billingsley [61]). For $t_1, \dots, t_k \in [0,1]$, define the natural projection $\pi_{t_1,\dots,t_k}$ from D to $\mathbb{R}^k$ by:

$$\pi_{t_1,\dots,t_k}(x) = \big(x(t_1), \dots, x(t_k)\big). \tag{4.2.1}$$

Then Billingsley [61] states that π_0, π_1 are everywhere continuous in the Skorokhod topology, while for $0 < t < 1$, *π_t is continuous at $x \in D$ if and only if x is continuous at t.* For a probability measure $\mathbb{P}$ on $(D, \mathcal{D})$, let $T_{\mathbb{P}}$ consist of those t in $[0,1]$ for which the natural projection π_t is continuous except at a $\mathbb{P}$-null set. We shall also need the concept of *tightness* for probability measures.

Definition 4.2.1. A sequence $\{\mathbb{P}_n\}_{n\geqslant 0}$ of probability measures on a metric space S is said to be *tight* if for every positive ε, there exists a compact set K such that $\mathbb{P}_n(K) > 1 - \varepsilon$ for all n.

To prove weak convergence we shall be using the following theorem which we quote from [61].

Theorem 4.2.1. *If $\{\mathbb{P}_n\}$ is tight and if $\mathbb{P}_n\pi^{-1}_{t_1,\dots,t_k} \Rightarrow \mathbb{P}\pi^{-1}_{t_1,\dots,t_k}$, holds whenever $t_1, \dots, t_k$ all lie in $T_{\mathbb{P}}$, then $\mathbb{P}_n \Rightarrow \mathbb{P}$.*

The Wiener measure assigns zero probability to all discontinuous paths and thus the corresponding $T_{\mathbb{P}}$ is the whole unit interval. Thus the second condition of Theorem 4.2.1 reduces to showing weak convergence of the finite dimensional distributions. To prove tightness we shall use [61, Theorem 15.5] which we quote below. We define for positive δ, $w_x(\delta) := \sup_{|t-s|<\delta} |x(s) - x(t)|$.

Theorem 4.2.2. *Suppose that, for each positive η, there exists an a such that*

$$\mathbb{P}_n\{x : |x(0)| > a\} \leqslant \eta, \quad n \geqslant 1. \tag{i}$$

Suppose further that, for each positive ε, η, there exist a δ, $0 < \delta < 1$, and an integer n_0, such that

$$\mathbb{P}_n\{x : w_x(\delta) \geqslant \varepsilon\} \leqslant \eta, \quad n \geqslant n_0. \tag{ii}$$

Then $\{\mathbb{P}_n\}$ is tight.

Note that since $Y_n(0) \equiv 0$, identically, (i) is automatically satisfied. We shall prove the second condition by proving a slightly stronger one. If we can show that:

for each positive ε, η, there exist a δ, $0 < \delta < 1$, and an integer n_0, such that

$$\frac{1}{\delta}\mathbb{P}_n\Big\{x : \sup_{t\leqslant s\leqslant t+\delta} |x(s) - x(t)| \geqslant \varepsilon\Big\} \leqslant \eta, \quad n \geqslant n_0, \tag{ii$'$}$$

then (ii) follows.

To see why we follow the proof of [61, Theorem 8.3]. Let $\delta > 0$ be fixed and define

$$A_t = \{x : \sup_{t \leqslant s \leqslant t+\delta} |x(s) - x(t)| \geqslant \varepsilon\}.$$

We split $[0,1]$ into intervals of the form $[i\delta, (i+1)\delta]$. If $|s-t| < \delta$, then s, t lie in the same or in adjacent intervals. If for some $x \in D$ we have $w_x(\delta) \geqslant 3\varepsilon$, then there exist s, t with $|s-t| < \delta$ with $|x(s) - x(t)| \geqslant 2\varepsilon$. Assume $s, t \in [i\delta, (i+1)\delta]$. Then by the triangle inequality

$$|x(s) - x(i\delta)| + |x(i\delta) - x(t)| \geqslant 2\varepsilon,$$

and thus at least one of the terms in the left hand side must be greater than ε. Similarly, if $s \in [(i-1)\delta, i\delta]$ and $t \in [i\delta, (i+1)\delta]$. This proves that

$$\{x : w_x(\delta) \geqslant 3\varepsilon\} \subset \bigcup_{i<\delta^{-1}} A_{i\delta},$$

from which it follows that

$$\mathbb{P}_n\{x : w_x(\delta) \geqslant 3\varepsilon\} \leqslant \mathbb{P}_n\Big(\bigcup_{i<\delta^{-1}} A_{i\delta}\Big) \leqslant \sum_{i<\delta^{-1}} \mathbb{P}_n(A_{i\delta}),$$

and thus $\mathbb{P}_n\{x : w_x(\delta) \geqslant 3\varepsilon\} \leqslant (1 + [1/\delta])\delta\eta < 2\eta$. This proves (ii) with ε, η replaced by $3\varepsilon, 2\eta$.

By the definition of $Y_n(t)$ it is clear that if $t = k/n$ and $t + \delta = j/n$, $k < j \leqslant n$, then (ii′) becomes

$$\frac{1}{\delta}\mathbb{P}\{\sup_{k/n \leqslant s \leqslant j/n} |Y_n(s) - Y_n(k/n)| \geqslant \varepsilon\} < \eta,$$

or equivalently

$$\frac{1}{\delta}\mathbb{P}\{\sup_{i \leqslant n\delta} |Z_{k+i} - Z_k| \geqslant \varepsilon\sigma\sqrt{n \log n}\} < \eta.$$

If t, δ are not integral multiples of $1/n$ then we can find k, j such that

$$\frac{k}{n} \leqslant t < \frac{k+1}{n}, \quad \frac{j-1}{n} \leqslant t + \frac{\delta}{2} < \frac{j}{n}.$$

Then

$$\begin{aligned}
\sup_{t \leqslant s \leqslant t+\delta} |Y_n(s) - Y_n(t)| &\leqslant \sup_{k/n \leqslant s < j/n} |Y_n(s) - Y_n(t)| \\
&\leqslant \sup_{k/n \leqslant s < j/n} \frac{C}{\sigma\sqrt{n \log n}} |Z_{[ns]} - Z_k| \\
&\leqslant \max_{i \leqslant j-k} \frac{C}{\sigma\sqrt{n \log n}} |Z_{k+i} - Z_k| \\
&\leqslant \max_{i \leqslant n\delta} \frac{C}{\sigma\sqrt{n \log n}} |Z_{k+i} - Z_k|,
\end{aligned}$$

where the last inequality follows since if n is large enough we have $j - k \leqslant n\delta$.

We now show that the above condition reduces to:

for every positive $\varepsilon > 0$ there exist a $\lambda > 1$, and an integer n_0 such that, if $n \geqslant n_0$, then

$$\mathbb{P}\{\max_{i\leqslant n}|Z_{k+i} - Z_k| \geqslant \lambda\sigma\sqrt{n\log n}\} \leqslant \frac{\varepsilon}{\lambda^2}. \tag{ii''}$$

By the above, with $\eta\varepsilon^2$ instead of ε, there exist $\lambda > 1$ and an integer n_1 such that

$$\mathbb{P}\{\max_{i\leqslant n}|Z_{k+i} - Z_k| \geqslant \lambda\sigma\sqrt{n\log n}\} \leqslant \frac{\eta\varepsilon^2}{\lambda^2},$$

for $n \geqslant n_1$ and $k \geqslant 1$. Let $\delta = \varepsilon^2/\lambda^2 \in (0,1)$, and n_0 an integer exceeding n_1/δ.

If $n \geqslant n_0$, then $[n\delta] \geqslant n_1$ and by (ii'')

$$\mathbb{P}\{\max_{i\leqslant [n\delta]}|Z_{k+i} - Z_k| \geqslant \lambda\sigma\sqrt{n\log n}\} \leqslant \frac{\eta\varepsilon^2}{\lambda^2} = \eta\delta,$$

and thus (ii') follows. Finally if, as in our situation, $\{Z_i\}_{i\geqslant 0}$ is defined as the partial sums of a stationary sequence then (ii'') becomes

$$\mathbb{P}\{\max_{i\leqslant n}|Z_i| \geqslant \lambda\sigma\sqrt{n\log n}\} \leqslant \frac{\varepsilon}{\lambda^2}.$$

4.2.1 Convergence of finite-dimensional distributions

We first obtain a few results which will be used in the proof.

Lemma 4.2.3. *Suppose that $S_n, n \geqslant 0$ satisfies the assumptions of Theorem 4.1.1 and let V_n be its self-intersection local time as defined in* (3.1.2). *Then*

$$\mathbb{E}(V_n) \sim 2n\log n/\pi\gamma, \quad \mathrm{Var}(V_n) = O(n^2).$$

Proof. For $\lambda \in [0,1)$, let $\rho(\lambda) = \sum_{m=0}^{\infty}\lambda^m\mathbb{P}(S_m = 0)$. Using (3.2.2) we express $\rho(\lambda)$ in terms of the characteristic function $f(t)$,

$$\rho(\lambda) = (2\pi)^{-1}\int_J (1 - \lambda f(t))^{-1}\,\mathrm{d}t.$$

Fix a small $\varepsilon > 0$. By strong aperiodicity for $|t| > \varepsilon$, $|1 - \lambda f(t)| \geqslant C(\varepsilon)$ and thus

$$\int_{|t|>\varepsilon}(1 - \lambda f(t))^{-1}\,\mathrm{d}t \leqslant D(\varepsilon),$$

where as before $C(\varepsilon)$, $D(\varepsilon)$ denote generic positive constants such that as $\varepsilon \to 0$, $C(\varepsilon) \to 0$ but

$D(\varepsilon)$ may be unbounded. Thus we have

$$\begin{aligned}
\rho(\lambda) &= (2\pi)^{-1} \int_{-\pi}^{\pi} (1 - \lambda f(t))^{-1}\,\mathrm{d}t \\
&= (2\pi)^{-1} \int_{-\varepsilon}^{\varepsilon} (1 - \lambda f(t))^{-1}\,\mathrm{d}t + (2\pi)^{-1} \int_{\varepsilon<|t|<\pi} (1 - \lambda f(t))^{-1}\,\mathrm{d}t \\
&=: (2\pi)^{-1} \int_{-\varepsilon}^{\varepsilon} (1 - \lambda f(t))^{-1}\,\mathrm{d}t + J_1(\varepsilon),
\end{aligned}$$

where $|J_1(\varepsilon)| \leqslant D(\varepsilon)$. On the other hand, for $|t| < \varepsilon$ we use (3.1.7) to show that

$$\begin{aligned}
(2\pi)^{-1} \int_{-\varepsilon}^{\varepsilon} (1 - \lambda f(t))^{-1}\,\mathrm{d}t &= (2\pi)^{-1} \int_{-\varepsilon}^{\varepsilon} (1 - \lambda + \lambda\gamma|t|)^{-1}\,\mathrm{d}t + J_2(\varepsilon) \\
&= 2(2\pi)^{-1} \int_{0}^{\varepsilon/(1-\lambda)} (1 + \lambda\gamma t)^{-1}\,\mathrm{d}t + J_2(\varepsilon) \\
&= \frac{1}{\pi\gamma} \log\Big(\frac{1}{1-\lambda}\Big) + J_2(\varepsilon),
\end{aligned}$$

where $|J_2(\varepsilon)| \leqslant C(\varepsilon)\log\big(1/|1-\lambda|\big)$ is the error from using the expansion and can be bounded using the techniques from Chapter 3. Overall we have that

$$\rho(\lambda) = \frac{1}{\pi\gamma} \log\Big(\frac{1}{1-\lambda}\Big) + J_1(\varepsilon) + J_2(\varepsilon).$$

Since ε is arbitrarily small, we let first $\lambda \to 1$ and then $\varepsilon \to 0$ to prove that

$$\rho(\lambda) \sim \frac{1}{\pi\gamma} \log\Big(\frac{1}{1-\lambda}\Big).$$

Since by definition $\rho(\lambda) = \sum_{m=0}^{\infty} \lambda^m \mathbb{P}(S_m = 0)$, from the last asymptotic and Karamata's Tauberian theorem we have

$$\sum_{j=0}^{n} \mathbb{P}(S_j = 0) \sim \log n/\pi\gamma, \quad \text{as } n \to \infty.$$

Finally

$$\begin{aligned}
\mathbb{E}(V_n) &= \mathbb{E}\Big(\sum_{i,j=0}^{n} 1_{S_i=S_j}\Big) \\
&= n + 1 + 2\mathbb{E}\Big(\sum_{i<j} 1_{S_i=S_j}\Big) \\
&= n + 1 + 2\sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \mathbb{P}(S_i = S_j) \\
&= n + 1 + 2\sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \mathbb{P}(S_{j-i} = 0) \\
&= n + 1 + 2\sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \mathbb{P}(S_j = 0)
\end{aligned}$$

$$
\begin{aligned}
&= n+1+2\sum_{j=1}^{n}\sum_{i=0}^{n-j}\mathbb{P}(S_j=0)\\
&= n+1+2\sum_{j=1}^{n}(n-j+1)\mathbb{P}(S_j=0)\\
&= n+1+(2n+1)\sum_{j=1}^{n}\mathbb{P}(S_j=0)-2\sum_{j=1}^{n}j\mathbb{P}(S_j=0).
\end{aligned}
$$

It is a quick calculation to see that for $\lambda \in [0,1)$ we have

$$
\begin{aligned}
\sum_{j=1}^{\infty} j\mathbb{P}(S_j=0)\lambda^j &= C\int_{-\pi}^{\pi}\sum_{j=1}^{\infty} j\lambda^j f^j(t)\,\mathrm{d}t\\
&= C\int_{-\pi}^{\pi}\sum_{j=1}^{\infty}\frac{\lambda f(t)\,\mathrm{d}t}{(1-\lambda f(t))}\sim C(1-\lambda)^{-1}
\end{aligned}
$$

by the usual calculations. By the Tauberian Theorem 3.1.1 we conclude that

$$
\sum_{j=1}^{n} j\mathbb{P}(S_j=0)\sim Cn.
$$

Overall we have that $\mathbb{E}(V_n)=2n\sum_{j=1}^{n}\mathbb{P}(S_j=0)+O(n)$. Therefore by our previous calculation $\mathbb{E}(V_n)\sim 2n\log n/\pi\gamma+O(n)$ and the lemma follows. For the variance estimate see Chapter 2. □

This gives the following important corollary.

Corollary 1.
$$
\frac{V_n}{\mathbb{E}V_n}\to 1, \quad a.s.\ as\ n\to\infty.
$$

Proof. Let $\varepsilon>0$ be given. Then

$$
\begin{aligned}
\mathbb{P}\left(\left|\frac{V_n}{\mathbb{E}V_n}-1\right|\geqslant\varepsilon\right) &= \mathbb{P}\left(\left|V_n-\mathbb{E}V_n\right|\geqslant\varepsilon\mathbb{E}V_n\right)\\
&\leqslant \frac{\operatorname{var}(V_n)}{\varepsilon^2(\mathbb{E}V_n)^2}\\
&= \frac{\operatorname{var}(V_n)}{\varepsilon^2(2n\log n/\pi\gamma)^2}\frac{(2n\log n/\pi\gamma)^2}{(\mathbb{E}V_n)^2}\\
&\leqslant \frac{C}{\varepsilon^2(\log n)^2},
\end{aligned}
$$

for all n large enough, since by the first lemma $\mathbb{E}V_n\sim Cn\log n$. This already proves convergence in probability.

Consider now the limits along the geometric subsequence $n_m=[\rho^m]$, where $\rho>1$ is arbitrary. Then we have

$$
\mathbb{P}\left(\left|\frac{V_{n_m}}{\mathbb{E}V_{n_m}}-1\right|\geqslant\varepsilon\right)\leqslant\frac{C}{\varepsilon^2 m^2},
$$

which is summable over m. By the Borel-Cantelli lemma we now have that

$$\mathbb{P}\left(\limsup_{m\to\infty}\left\{\left|\frac{V_{n_m}}{\mathbb{E}V_{n_m}}-1\right|\geqslant\varepsilon\right\}\right)=0,$$

and thus for all $\varepsilon>0$

$$\mathbb{P}\left(\bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty}\left\{\left|\frac{V_{n_k}}{\mathbb{E}V_{n_k}}-1\right|<\varepsilon\right\}\right)=1.$$

This holds for arbitrary $\varepsilon>0$ and thus

$$\mathbb{P}\left(\bigcap_{l=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty}\left\{\omega:\left|\frac{V_{n_k}}{\mathbb{E}V_{n_k}}-1\right|<\frac{1}{l}\right\}\right)=1,$$

but if ω is in the above set, then for all $l\in\mathbb{N}$, there is an M such that

$$|V_{n_m}(\omega)/\mathbb{E}V_{n_m}-1|<\varepsilon,\quad\text{for all } m\geqslant M$$

and thus $V_{n_m}(\omega)/\mathbb{E}V_{n_m}\to 1$. It follows that $V_{n_m}/\mathbb{E}V_{n_m}\to 1$ a.s. as $m\to\infty$. Observe now that V_n is monotone increasing in n. For each n we can find $m=m(n)$ such that $[\rho^m]\leqslant n<[\rho^{m+1}]$. By monotonicity we have

$$\frac{V_{n_m}}{\mathbb{E}V_{n_{m+1}}}\leqslant\frac{V_n}{\mathbb{E}V_n}\leqslant\frac{V_{n_{m+1}}}{\mathbb{E}V_{n_m}}.$$

It is obvious that as $n\to\infty$ we also have that $m=m(n)\to\infty$. Then letting $n\to\infty$ we have

$$\lim_{m\to\infty}\frac{V_{n_m}}{\mathbb{E}V_{n_m}}\frac{\mathbb{E}V_{n_m}}{\mathbb{E}V_{n_{m+1}}}\leqslant\liminf_{n\to\infty}\frac{V_n}{\mathbb{E}V_n}\leqslant\limsup_{n\to\infty}\frac{V_n}{\mathbb{E}V_n}\leqslant\lim_{m\to\infty}\frac{V_{n_{m+1}}}{\mathbb{E}V_{n_{m+1}}}\frac{\mathbb{E}V_{n_{m+1}}}{\mathbb{E}V_{n_m}},$$

and thus we have

$$\frac{1}{\rho}\leqslant\liminf_{n\to\infty}\frac{V_n}{\mathbb{E}V_n}\leqslant\limsup_{n\to\infty}\frac{V_n}{\mathbb{E}V_n}\leqslant\rho.$$

Since $\rho>1$ is arbitrary we may allow $\rho\to 1$ and thus it must be that

$$\lim_{n\to\infty}\frac{V_n}{\mathbb{E}V_n}=1,\quad\text{a.s.}$$

□

Note that by the last two results we also have that

$$\pi\gamma V_n/2n\log n\to 1,\quad\text{a.s.}$$

Proposition 4.2.4. *Suppose that as $t\to 0$, for $f(t)=\mathbb{E}[e^{itX}]$, we have*

$$f(t)-1=O(|t|^{\alpha}),$$

where $0<\alpha\leqslant 2$. Then as $A\to\infty$,

$$\mathbb{P}(|X|>A)=O(A^{-\alpha}).$$

Proof. Let F be the distribution function of X, and f its characteristic function. By our assumptions there exist positive C and t_0 such that for all $t \leqslant t_0$

$$|1 - f(t)| \leqslant C|t|^\alpha.$$

Then since

$$|1 - f(t)| \geqslant \Re(1 - f(t)) = \int (1 - \cos tx)\, \mathrm{d}F(x) \geqslant \int\limits_{|x|>A} (1 - \cos tx)\, \mathrm{d}F(x),$$

for $A > 1/t_0$ we have

$$\int_0^{1/A} \int\limits_{|x|>A} (1 - \cos tx)\, \mathrm{d}F(x)\, \mathrm{d}t \leqslant C \int_0^{1/A} t^\alpha\, \mathrm{d}t \leqslant CA^{-1-\alpha}$$

and bringing the cosine term over to the right hand side

$$\begin{aligned} \mathbb{P}(|X| > A) = \int\limits_{|x|>A} \mathrm{d}F(x)\, \mathrm{d}t &\leqslant CA^{-\alpha} + A \int_0^{1/A} \int\limits_{|x|>A} \cos tx\, \mathrm{d}F(x)\, \mathrm{d}t \\ &\leqslant CA^{-\alpha} + \int\limits_{|x|>A} \frac{\sin(x/A)}{x/A}\, \mathrm{d}F(x) \end{aligned}$$

and since $|x| > 1$ implies $\sin(x)/x \leqslant \sin(1) < 1$ we have

$$\mathbb{P}(|X| > A)(1 - \sin(1)) \leqslant CA^{-\alpha},$$

and thus

$$\int\limits_{|x|>A} \mathrm{d}F(x) \leqslant CA^{-\alpha},$$

for all $A > 1/t_0$. □

For $\alpha \in \mathbb{Z}$, recall that

$$N_\alpha(n) = \sum_{j=0}^{n} \mathbf{1}_{S_j=\alpha}.$$

Lemma 4.2.5.

$$\sup_{\alpha \in \mathbb{Z}} N_\alpha(n) = o(n^\varepsilon), \quad \textit{a.s. for each } \varepsilon > 0.$$

Proof. If $m \in \mathbb{N}$, then

$$\begin{aligned} N_0(n)^m &= \sum_{j_1,\ldots,j_m=0}^{n} \mathbf{1}(S_{j_1} = \cdots = S_{j_m} = 0) \\ &\leqslant m! \sum_{j_1 \leqslant \cdots \leqslant j_m} \mathbf{1}(S_{j_1} = \cdots = S_{j_m} = 0) \\ &= m! \sum_{j_1 \leqslant \cdots \leqslant j_m} \mathbf{1}(S_{j_1} = S_{j_2 - j_1} = \cdots = S_{j_m - j_{m-1}} = 0). \end{aligned}$$

Changing variables and writing M for the set of indices

$$M \equiv \{(i_1, \ldots, i_m) : 0 \leqslant i_k \leqslant n, \text{for all } k, \text{ and } i_1 + \cdots + i_k \leqslant n\},$$

we have that

$$\begin{aligned}
\mathbb{E}(N_0(n)^m) &\leqslant m! \sum_M \mathbb{P}(S_{i_1} = 0)\mathbb{P}(S_{i_2} = 0) \cdots \mathbb{P}(S_{i_m} = 0) \\
&= m! \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} \cdots \sum_{i_m=0}^{n-\sum_{k=1}^{m-1} i_k} \mathbb{P}(S_{i_1} = 0)\mathbb{P}(S_{i_2} = 0) \cdots \mathbb{P}(S_{i_m} = 0) \\
&\leqslant m! \Big(\sum_{i=0}^{n} \mathbb{P}(S_i = 0) \Big)^m .
\end{aligned}$$

By our earlier calculations we know that

$$\sum_{i=0}^{n} \mathbb{P}(S_i = 0) \sim C \log n,$$

and thus $\mathbb{E}(N_0(n)^m) = O(\log(n)^m)$, which further implies that for all positive ε, we have

$$\mathbb{E}(N_0(n)^m) = o(n^\varepsilon).$$

$N_0(n)$ is stochastically the largest among the $N_\alpha(n)$ in the sense that

$$\mathbb{P}(N_0(n) \geqslant t) \geqslant \sup_{\alpha \in \mathbf{Z}} \mathbb{P}(N_\alpha(n) \geqslant t),$$

by stationary, independent increments since the walk begins at 0. Also note that by the triangle inequality

$$\begin{aligned}
&\mathbb{P}(\sup\{N_\alpha(n) : |\alpha| > n^2\} \geqslant t) \\
&\quad\leqslant \mathbb{P}(\sup\nolimits_{i \leqslant n} |S_i| > n^2) \\
&\quad\leqslant \mathbb{P}(\sup\nolimits_{i \leqslant n} \textstyle\sum_{k=0}^{i} |X_k| > n^2) \\
&\quad\leqslant \mathbb{P}(\textstyle\sum_{k=0}^{n} |X_k| > n^2) \\
&\quad\leqslant \mathbb{P}(|X_i| > n) = O(n^{-1})
\end{aligned}$$

by Proposition 4.2.4.

Continuing, since $N_0(n)$ is stochastically the largest and by the Markov inequality we have

$$\begin{aligned}
&\mathbb{P}\Big(\sup_\alpha N_\alpha(n) \geqslant t\Big) - O(n^{-1}) \\
&\quad= \mathbb{P}(\sup\{N_\alpha(n) : |\alpha| \leqslant n^2\} \geqslant t) \\
&\quad\leqslant (2n^2 + 1) \sup_\alpha \mathbb{P}(N_\alpha(n) \geqslant t) \leqslant (2n^2 + 1)\mathbb{P}(N_0(n) \geqslant t) \\
&\quad\leqslant (2n^2 + 1)t^{-m}\mathbb{E}(N_0(n)^m) = (2n^2 + 1)t^{-m}o(n^\varepsilon) \quad \text{for any } m \in \mathbb{N}, \varepsilon > 0.
\end{aligned}$$

Letting $t = n^{\varepsilon}$ we have

$$\mathbb{P}(\sup_{\alpha} N_{\alpha}(n) \geqslant n^{\varepsilon}) \leqslant (2n^2+1)n^{-m\varepsilon}o(n^{\varepsilon}) + O(n^{-1}),$$

for all $m \in \mathbb{N}$ and thus the lemma follows. □

Lemma 4.2.6. *If* $0 < a < b$, *then*

$$\sum_{j=1}^{[an]} \sum_{i=[an]+1}^{[bn]} \mathbf{1}(S_i = S_j) = o(n \log n), \quad a.s.$$

Proof. Observe that

$$\begin{aligned} V_{[bn]} &= \sum_{j=0}^{[bn]}\sum_{i=0}^{[an]} \mathbf{1}(S_i = S_j) + \sum_{j=0}^{[bn]} \sum_{i=[an]+1}^{[bn]} \mathbf{1}(S_i = S_j) \\ &= V_{[an]} + \sum_{i,j=[an]+1}^{[bn]} \mathbf{1}(S_i = S_j) + 2\sum_{j=0}^{[an]} \sum_{i=[an]+1}^{[bn]} \mathbf{1}(S_i = S_j). \end{aligned}$$

Note that

$$\begin{aligned} \sum_{i,j=[an]+1}^{[bn]} \mathbf{1}(S_i = S_j) &= \sum_{i,j=[an]+1}^{[bn]} \mathbf{1}(S_i - S_{[an]+1} = S_j - S_{[an]+1}) \\ &= \sum_{i,j=[an]+1}^{[bn]} \mathbf{1}(S_i - S_{[an]+1} = S_j - S_{[an]+1}) = \sum_{i,j=0}^{[bn]-[an]-1} \mathbf{1}(\tilde{S}_i = \tilde{S}_j) \\ &= \sum_{i,j=0}^{[bn]-[an]-1} \mathbf{1}(S_i = S_j) = \tilde{V}_{[bn]-[an]-1} \end{aligned}$$

where $\tilde{S}_i$ is an independent copy of the random walk started at zero. Thus we have

$$V_{[bn]} = V_{[an]} + \tilde{V}_{[bn]-[an]-1} + 2\sum_{j=0}^{[an]} \sum_{i=[an]+1}^{[bn]} \mathbf{1}(S_i = S_j).$$

We divide both sides by $2n\log(n)/\pi\gamma$ and take the limit as $n \to \infty$. Using Lemma 4.2.3 we have

$$b = a + (b-a) + C \lim_{n\to\infty} \frac{1}{n\log(n)} \sum_{j=0}^{[an]} \sum_{i=[an]+1}^{[bn]} \mathbf{1}(S_i = S_j),$$

and the lemma follows. □

We are now ready to show convergence of finite dimensional distributions.

Let $a_1, \ldots, a_m \in \mathbf{R}$, $0 = t_0 < t_1 < \cdots < t_m$ be given.

$$\sum_{j=1}^{m} a_j(Y_n(t_j) - Y_n(t_{j-1}))$$

$$= \sum_{j=1}^{m} \sum_{\alpha \in \mathbb{Z}} a_j (N_\alpha([nt_j]) - N_\alpha([nt_{j-1}])) \xi(\alpha)/d_n, \tag{4.2.2}$$

where $d_n = \sigma\sqrt{2n \log n}/\sqrt{\pi\gamma}$.

First note that since by the finite time $[nt_j]$ the random walk will have attained a finite number of distinct values, only finitely many terms in the double sum in (4.2.2) are non-zero and thus we can interchange the order of summation so that

$$\begin{aligned}
&\sum_{j=1}^{m} a_j (Y_n(t_j) - Y_n(t_{j-1})) \\
&= \sum_{\alpha \in \mathbb{Z}} \sum_{j=1}^{m} a_j (N_\alpha([nt_j]) - N_\alpha([nt_{j-1}])) \xi(\alpha)/d_n.
\end{aligned}$$

Let $\mathcal{A} = \sigma(X_1, X_2, \dots)$, the σ-algebra generated by the random walk increments. Conditional on $\mathcal{A}$, the above expression is a sum of independent random variables with non-identical distributions.

We would like to apply the Lindeberg version of the central limit theorem for triangular arrays, which we quote below from Billingsley [61, Theorem 7.2]. For each n, let $\xi_{n1}, \dots, \xi_{nk_n}$, be independent random variables with mean 0 and finite variance $\sigma^2_{nk_n}$. Let

$$S_n = \xi_1 + \cdots + \xi_{nk_n}$$

and suppose its variance

$$s_n^2 = \sigma_{n1}^2 + \cdots + \sigma_{nk_n}^2$$

is positive, and let N be a standard normal variable.

Theorem 4.2.7 (Central Limit Theorem). *If*

$$\frac{1}{s_n^2} \sum_{k=1}^{k_n} \int_{|\xi_{nk}| \geqslant \varepsilon s_n} \xi_{nk}^2 \, d\mathbb{P} \to 0 \tag{4.2.3}$$

as $n \to \infty$, for each positive ε, then $S_n / s_n \xrightarrow{\text{D}} N$.

We proceed by checking if the Lindeberg condition (4.2.3) is satisfied. Let

$$s_n^2 = d_n^{-2} \sigma^2 \sum_{\alpha \in \mathbb{Z}} \left(\sum_{j=1}^{m} a_j (N_\alpha([nt_j]) - N_\alpha([nt_{j-1}])) \right)^2.$$

To simplify notation we write

$$C(n, \alpha) = d_n^{-1} \sum_{j=1}^{m} a_j (N_\alpha([nt_j]) - N_\alpha([nt_{j-1}])).$$

Then we have to check that for all $\varepsilon > 0$

$$s_n^{-2}\sum_{\alpha\in\mathbf{Z}}\mathbb{E}\Big[C(n,\alpha)^2\xi(\alpha)^2\mathbf{1}\{C(n,\alpha)\xi(\alpha)\geqslant\varepsilon s_n\}\Big|\mathcal{A}\Big]\to 0,$$

as $n\to\infty$.

We start with a lemma.

Lemma 4.2.8.

$$d_n^{-2}\sum_{\alpha\in\mathbf{Z}}\Big(\sum_{j=1}^m a_j\big(N_\alpha([nt_j])-N_\alpha([nt_{j-1}])\big)\Big)^2\to\sigma^{-2}\sum_{j=1}^m a_j^2(t_j-t_{j-1}),$$

a.s. as $n\to\infty$.

Proof. To show this we proceed as follows,

$$\begin{aligned}
&d_n^{-2}\sum_{\alpha\in\mathbf{Z}}\Big(\sum_{j=1}^m a_j\big(N_\alpha([nt_j])-N_\alpha([nt_{j-1}])\big)\Big)^2\\
&=d_n^{-2}\sum_{\alpha\in\mathbf{Z}}\sum_{j=1}^m a_j^2\big(N_\alpha([nt_j])-N_\alpha([nt_{j-1}])\big)^2\\
&\quad+2d_n^{-2}\sum_{\alpha\in\mathbf{Z}}\sum_{i<j}a_ia_j\big(N_\alpha([nt_j])-N_\alpha([nt_{j-1}])\big)\big(N_\alpha([nt_i])-N_\alpha([nt_{i-1}])\big)\\
&=d_n^{-2}\sum_{\alpha\in\mathbf{Z}}\sum_{j=1}^m a_j^2\Big(\sum_{k,l=0}^{[nt_j]}\mathbf{1}_{\{S_k=S_l=\alpha\}}-2\sum_{k=0}^{[nt_j]}\sum_{l=0}^{[nt_{j-1}]}\mathbf{1}_{\{S_k=S_l=\alpha\}}+\sum_{k,l=0}^{[nt_{j-1}]}\mathbf{1}_{\{S_k=S_l=\alpha\}}\Big)\\
&\quad+2d_n^{-2}\sum_{\alpha\in\mathbf{Z}}\sum_{i<j}a_ia_j\sum_{k=[nt_{j-1}]+1}^{[nt_j]}\sum_{k=[nt_{i-1}]+1}^{[nt_i]}\mathbf{1}_{\{S_k=S_l=\alpha\}}\\
&=d_n^{-2}\sum_{\alpha\in\mathbf{Z}}\sum_{j=1}^m a_j^2\Big(\sum_{k,l=0}^{[nt_j]}\mathbf{1}_{\{S_k=S_l=\alpha\}}-\sum_{k,l=0}^{[nt_{j-1}]}\mathbf{1}_{\{S_k=S_l=\alpha\}}-2\sum_{k=[nt_{j-1}]+1}^{[nt_j]}\sum_{l=0}^{[nt_{j-1}]}\mathbf{1}_{\{S_k=S_l=\alpha\}}\Big)\\
&\quad+d_n^{-2}\sum_{\alpha\in\mathbf{Z}}\sum_{i<j}a_ia_j\sum_{k=[nt_{j-1}]+1}^{[nt_j]}\sum_{k=[nt_{i-1}]+1}^{[nt_i]}\mathbf{1}_{\{S_k=S_l=\alpha\}}
\end{aligned}$$

and after we change the order of summation

$$\begin{aligned}
&=d_n^{-2}\sum_{j=1}^m a_j^2\Big(V_{[nt_j]}-V_{[nt_{j-1}]}-2\sum_{k=[nt_{j-1}]+1}^{[nt_j]}\sum_{l=0}^{[nt_{j-1}]}\mathbf{1}_{\{S_k=S_l\}}\Big)\\
&\quad+d_n^{-2}\sum_{i<j}a_ia_j\sum_{k=[nt_{j-1}]+1}^{[nt_j]}\sum_{k=[nt_{i-1}]+1}^{[nt_i]}\mathbf{1}_{\{S_k=S_l\}}.
\end{aligned}$$

The proof now follows by applying Corollary 1 and Lemma 4.2.6. □

By Lemma 4.2.5 we have

$$\sum_{j=1}^{m} a_j(N_\alpha([nt_j]) - N_\alpha([nt_{j-1}])) = o(n^\delta) \quad a.s.$$

as $n \to \infty$ for any $\delta > 0$. Therefore for any positive δ, there exists a positive constant C' such that

$$\begin{aligned}\frac{s_n^2}{C(n,\alpha)^2} &\geqslant \sigma^2 C' \sum_{\alpha\in\mathbf{Z}} \Big(\sum_{j=1}^{m} a_j(N_\alpha([nt_j]) - N_\alpha([nt_{j-1}]))\Big)^2 n^{-\delta} \\ &= C\frac{\sum_{\alpha\in\mathbf{Z}}\left(\sum_{j=1}^{m} a_j(N_\alpha([nt_j]) - N_\alpha([nt_{j-1}]))\right)^2}{n\log n} \times \frac{n\log n}{n^\delta} \to \infty,\end{aligned}$$

as $n \to \infty$, for $\delta < 1$, since the first fraction converges to a constant in probability. Note that by Lemma 4.2.5, we have that $s_n/C(n,\alpha) \to \infty$ uniformly in α. Now let $M > 0$ be arbitrary. We can find $N(M)$ such that for all $n \geqslant N(M)$ and all α we have $s_n/C(n,\alpha) \geqslant M$. Therefore conditional on $\mathcal{A}$ we have for arbitrary $\varepsilon > 0$, $M > 0$, and for all $n \geqslant N(M)$ large enough that

$$\mathbb{E}\Big(\xi(\alpha)^2 \mathbf{1}\{\xi(\alpha)^2 \geqslant \varepsilon s_n^2/C(n,\alpha)^2\}\Big|\mathcal{A}\Big) \leqslant \mathbb{E}\Big(\xi(\alpha)^2\mathbf{1}\{\xi(\alpha)^2 \geqslant \varepsilon M\}\Big|\mathcal{A}\Big).$$

Since this holds for arbitrary M and all $n \geqslant N(M)$ and since $\xi(\alpha)$ is square integrable, it is now clear that as $n \to \infty$

$$\mathbb{E}\Big(\xi(\alpha)^2\mathbf{1}\{\xi(\alpha)^2 \geqslant \varepsilon s_n^2/C(n,\alpha)^2\}\Big) \to 0.$$

Finally we have conditional on $\mathcal{A}$

$$\begin{aligned}&s_n^{-2}\sum_{\alpha\in\mathbf{Z}} \mathbb{E}\Big(C(n,\alpha)^2\xi(\alpha)^2\mathbf{1}\{\xi(\alpha)^2 \geqslant \varepsilon s_n^2/C(n,\alpha)^2\}\Big|\mathcal{A}\Big) \\ &= s_n^{-2}\sum_{\alpha\in\mathbf{Z}} C(n,\alpha)^2\mathbb{E}\Big(\xi(\alpha)^2\mathbf{1}\{\xi(\alpha)^2 \geqslant \varepsilon s_n^2/C(n,\alpha)^2\}\Big|\mathcal{A}\Big) \\ &= C\mathbb{E}\Big(\xi(\alpha)^2\mathbf{1}\{\xi(\alpha)^2 \geqslant \varepsilon s_n^2/C(n,\alpha)^2\}\Big|\mathcal{A}\Big) \to 0,\end{aligned}$$

as $n \to \infty$. We have shown that the central limit theorem applies to our case giving

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}\left(\frac{d_n\sum_{j=1}^{m} a_j(Y_n(t_j) - Y_n(t_{j-1}))}{\left\{\sigma^2\sum_{\alpha\in\mathbf{Z}}\left(\sum_{j=1}^{m} a_j(N_\alpha([nt_j]) - N_\alpha([nt_{j-1}]))\right)^2\right\}^{1/2}} \leqslant x \middle| \mathcal{A}\right) \\ &= (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-s^2/2}\,ds, \qquad \text{a.s.}\end{aligned}$$

Since we have already shown that

$$d_n^{-2}\sigma^2\sum_{\alpha\in\mathbf{Z}}\Big(\sum_{j=1}^{m} a_j(N_\alpha([nt_j]) - N_\alpha([nt_{j-1}]))\Big)^2$$

converges a.s. to

$$\sum_{j=1}^{m} a_j^2(t_j - t_{j-1}),$$

we have that

$$\sum_{j=1}^{m} a_j(Y_n(t_j) - Y_n(t_{j-1})) \xrightarrow{\mathrm{D}} N(0, \textstyle\sum_{j=1}^{m} a_j^2(t_j - t_{j-1})).$$

By the Cramér-Wold theorem (see [61], Theorem 7.7) $(Y_n(t_j))_{j=1,\dots,m}$ is asymptotically normally distributed with mean 0 and covariance matrix $(\min(t_i, t_j))_{i,j=1,\dots,m}$. We have shown convergence of the finite dimensional distributions and now it remains to show tightness.

4.2.2 Tightness

Following the discussion in the beginning of this section, it suffices to show that for any $\varepsilon > 0$ there exists arbitrarily large $\lambda > 0$ such that for all $n \in \mathbb{N}$ large enough we have

$$\mathbb{P}\left(\sup_{i \leqslant n} |Z_i| \geqslant \lambda\sqrt{n \log n}\right) \leqslant \frac{\varepsilon}{\lambda^2}.$$

Let $Z_m^* = \max_{0 \leqslant i \leqslant m} Z_i$. Now it is true that for $\rho > \sqrt{2}$

$$\begin{aligned}
&\mathbb{P}\left(Z_m^* \geqslant \rho\sigma\sqrt{V_m} | \mathcal{A}\right) \\
&\quad \leqslant \mathbb{P}\left(Z_m^* \geqslant \rho\sigma\sqrt{V_m}, Z_m \geqslant (\rho - \sqrt{2})\sigma\sqrt{V_m} \middle| \mathcal{A}\right) \\
&\qquad + \mathbb{P}\left(Z_m^* \geqslant \rho\sigma\sqrt{V_m}, Z_m < (\rho - \sqrt{2})\sigma\sqrt{V_m} \middle| \mathcal{A}\right) \\
&\quad \leqslant \mathbb{P}\left(Z_m \geqslant (\rho - \sqrt{2})\sigma\sqrt{V_m} \middle| \mathcal{A}\right) \\
&\qquad + \mathbb{P}\left(Z_m^* \geqslant \rho\sigma\sqrt{V_m}, Z_m < (\rho - \sqrt{2})\sigma\sqrt{V_m} \middle| \mathcal{A}\right) \\
&\quad = \mathbb{P}\left(Z_m \geqslant (\rho - \sqrt{2})\sigma\sqrt{V_m} \middle| \mathcal{A}\right) \\
&\qquad + \mathbb{P}\left(Z_{m-1}^* \geqslant \rho\sigma\sqrt{V_m}, -Z_m > -(\rho - \sqrt{2})\sigma\sqrt{V_m} \middle| \mathcal{A}\right) \\
&\quad = \mathbb{P}\left(Z_m \geqslant (\rho - \sqrt{2})\sigma\sqrt{V_m} \middle| \mathcal{A}\right) \\
&\qquad + \mathbb{P}\left(Z_{m-1}^* \geqslant \rho\sigma\sqrt{V_m}, Z_{m-1}^* - Z_m > \sqrt{2}\sigma\sqrt{V_m} \middle| \mathcal{A}\right).
\end{aligned}$$

To proceed we need the concept of *associated variables* which we take from [62].

Definition 4.2.2. A finite collection of random variables, $X_1, \dots, X_m$ is said to be associated if for any two coordinatewise nondecreasing functions f_1, f_2 on $\mathbb{R}^m$ such that $\tilde{f}_j \equiv f_j(X_1, \dots, X_m)$ has finite variance for $j = 1,2$, $\mathrm{Cov}(\tilde{f}_1, \tilde{f}_2) \geqslant 0$. An infinite collection is said to be associated if every finite subcollection is associated.

To determine whether the collection $\xi(\alpha)_{\alpha \in \mathbf{Z}}$ is associated, we use Kemperman [63, Corollary 2] which we quote below.

Corollary 2. *Let $X = X_1 \times \cdots \times X_n$ be the direct product of totally ordered measure spaces $(X_i, \mathcal{F}_j, \lambda_j)$ such that $\{(x_j, y_j) \in X_j \times X_j : x_j \leqslant y_j\}$ is jointly measurable with respect to*

$\mathcal{F}_j$. Let further μ be a probability measure on X which is absolutely continuous with respect to $\lambda = \lambda_1 \otimes \cdots \otimes \lambda_n$ and possesses a density $\phi = \mathrm{d}\mu/\mathrm{d}\lambda$ which satisfies

$$\phi(x)\phi(y) \leqslant \phi(x \vee y)\phi(x \wedge y). \tag{4.2.4}$$

Then

$$\int f(x)g(x)\mu(\mathrm{d}x) \geqslant \int f(x)\mu(\mathrm{d}x) \cdot \int g(x)\mu(\mathrm{d}x), \tag{4.2.5}$$

whenever f and g are increasing measurable functions on X, such that all the integrals above exist.

This corollary implies that all collections of independent variables are associated. To see why in the case of $\{\xi(\alpha)\}_{\alpha\in\mathbf{Z}}$ let $X_i = \mathbb{R}$ and let λ_i be the law of $\xi(\alpha)$, i.e. $\lambda_i(A) = \mathbb{P}(\xi(\alpha) \in A)$ for all $A \in \beta(\mathbb{R})$. Let $\mu = \lambda = \lambda_1 \otimes \cdots \otimes \lambda_n$ so that $\phi = 1$ and thus (4.2.4) is trivially satisfied. Then (4.2.5) shows that the collection $\{\xi(\alpha)\}_{\alpha\in\mathbf{Z}}$ is associated.

It is clear that conditional on $\mathcal{A}$, Z^*_{m-1} is non-decreasing in all components of the random field $\{\xi(\alpha)\}_{\alpha\in\mathbf{Z}}$. Also observe that

$$Z_m - Z^*_{m-1} = -\max_{i\leqslant m-1}\Big(-\sum_{j=i+1}^{m}\xi(S_j)\Big) = \min_{i\leqslant m-1}\sum_{j=i+1}^{m}\xi(S_j)$$

which is also non-decreasing in all components of the random field conditional on $\mathcal{A}$. It is now easy to see for all constants c_1, c_2, that $\mathbf{1}_{\{Z^*_{m-1}\geqslant c_1\}}$ and $\mathbf{1}_{\{Z_m - Z^*_{m-1}\geqslant c_2\}}$ are also non-decreasing in all components of the random field and since the variables are associated we have

$$\mathbb{E}(\mathbf{1}_{\{Z^*_{m-1}\geqslant c_1\}}\mathbf{1}_{\{Z_m-Z^*_{m-1}\geqslant c_2\}}|\mathcal{A}) \geqslant \mathbb{E}(\mathbf{1}_{\{Z^*_{m-1}\geqslant c_1\}}|\mathcal{A})\mathbb{E}(\mathbf{1}_{\{Z_m-Z^*_{m-1}\geqslant c_2\}}|\mathcal{A}),$$

which implies that

$$\mathbb{P}(Z^*_{m-1} \geqslant c_1, Z_m - Z^*_{m-1} \geqslant c_2|\mathcal{A}) \geqslant \mathbb{P}(Z^*_{m-1} \geqslant c_2|\mathcal{A})\mathbb{P}(Z_m - Z^*_{m-1} \geqslant c_2|\mathcal{A}).$$

It is now obvious that

$$\begin{aligned}\mathbb{P}\left(Z^*_{m-1} \geqslant \rho\sigma\sqrt{V_m}, Z^*_{m-1} - Z_m > \sqrt{2}\sigma\sqrt{V_m}\middle|\mathcal{A}\right)\\ \leqslant \mathbb{P}\left(Z^*_{m-1} \geqslant \rho\sigma\sqrt{V_m}\middle|\mathcal{A}\right)\mathbb{P}\left(Z^*_{m-1} - Z_m \geqslant \sqrt{2}\sigma\sqrt{V_m}\middle|\mathcal{A}\right)\end{aligned}$$

and using Chebyshev's inequality

$$\leqslant \mathbb{P}\left(Z^*_m \geqslant \rho\sigma\sqrt{V_m}\middle|\mathcal{A}\right)(2\sigma^2V_m)^{-1}\mathbb{E}\left((Z_{m-1} - Z_m)^2\middle|\mathcal{A}\right).$$

We would now like to show that conditional on $\mathcal{A}$ the collection $\{\xi(S_i)\}_{i\leqslant n}$ is also associated. Conditioned on $\mathcal{A}$, the random walk S_i will have attained a finite number $m(n) \leqslant n$ of distinct values which we denote by $a_0, \ldots, a_{m(n)}$. Let f, g be coordinatewise non-decreasing functions defined on $\mathbb{R}^n$. Then there are coordinatewise non-decreasing functions $\hat{f}, \hat{g}$ on $\mathbb{R}^{m(n)}$ such that conditional on $\mathcal{A}$,

$$f(\xi(S_0),\dots,\xi(S_n)) = \hat{f}(\xi(\alpha_0),\dots,\xi(\alpha_{m(n)})), \quad g(\xi(S_0),\dots,\xi(S_n)) = \hat{g}(\xi(\alpha_0),\dots,\xi(\alpha_{m(n)})).$$

Since the collection $\{\xi(\alpha)\}_{\alpha\in\mathbf{Z}}$ is associated we have

$$\begin{aligned}\mathbb{E}&\left(\hat{f}(\xi(\alpha_0),\dots,\xi(\alpha_{m(n)}))\hat{g}(\xi(\alpha_0),\dots,\xi(\alpha_{m(n)}))\right)\\ &\geqslant \mathbb{E}\left(\hat{f}(\xi(\alpha_0),\dots,\xi(\alpha_{m(n)}))\right)\mathbb{E}\left(\hat{g}(\xi(\alpha_0),\dots,\xi(\alpha_{m(n)}))\right).\end{aligned}$$

This implies that

$$\begin{aligned}&\mathbb{E}\left(f(\xi(S_0),\dots,\xi(S_n))g(\xi(S_0),\dots,\xi(S_n))\,|\mathcal{A}\right)\\ &= \mathbb{E}\left(\hat{f}(\xi(\alpha_0),\dots,\xi(\alpha_{m(n)}))\hat{g}(\xi(\alpha_0),\dots,\xi(\alpha_{m(n)}))\right)\\ &\geqslant \mathbb{E}\left(\hat{f}(\xi(\alpha_0),\dots,\xi(\alpha_{m(n)}))\right)\mathbb{E}\left(\hat{g}(\xi(\alpha_0),\dots,\xi(\alpha_{m(n)}))\right)\\ &= \mathbb{E}\left(f(\xi(S_0),\dots,\xi(S_n))\,|\mathcal{A}\right)\mathbb{E}\left(g(\xi(S_0),\dots,\xi(S_n))\,|\mathcal{A}\right)\end{aligned}$$

proving that conditional on $\mathcal{A}$ the collection $\{\xi(S_j)\}_{j\leqslant n}$ is associated.

Then by Newman and Wright [62], Theorems 2,3

$$\mathbb{E}\left((Z^*_{m-1} - Z_m)^2\,|\mathcal{A}\right) \leqslant \mathbb{E}\left(Z_m^2\,|\mathcal{A}\right) = \sigma^2 V_m,$$

and thus

$$\mathbb{P}(Z^*_m \geqslant \rho\sigma\sqrt{V_m}) \leqslant 2\mathbb{P}(|Z_m| \geqslant (\rho - \sqrt{2})\sigma\sqrt{V_m}).$$

By the same inequality for the reflected random field $(-\xi(\alpha))_{\alpha\in\mathbf{Z}}$ we have

$$\mathbb{P}\left(\max_{j\leqslant m}|Z_j| \geqslant \rho\sigma\sqrt{V_m}\right) \leqslant 2\mathbb{P}\left(|Z_m| \geqslant (\rho - \sqrt{2})\sigma\sqrt{V_m}\right).$$

By Lemma 4.2.3, $V_m/m\log m$ converges in probability to $2(\pi\gamma)^{-1}$, for each $\delta > 0$ there exists a $m_1(\delta)$ such that for $m \geqslant m_1(\delta)$ and $\rho \geqslant \sqrt{2}$ we have

$$\begin{aligned}&\mathbb{P}\left(\max_{j\leqslant m}|Z_j| \geqslant C\rho\sigma\sqrt{m\log m}\right)\\ &\qquad 2\mathbb{P}\left(|Z_m| \geqslant C(\rho - \sqrt{2})\sigma\sqrt{m\log m}\right) + \delta.\end{aligned}$$

We write $\lambda = C\rho\sigma$ and let $\varepsilon > 0$.

We have already seen that $Y_n(t)$ is asymptotically normally distributed with mean zero and variance t. Letting $t = 1$ and using the definition of $Y_n(t)$ we have that $CZ_m/\sigma\sqrt{m\log}$ is asymptotically normally distributed. Thus we have for $\mathcal{N}$ a standard normal variable

$$\mathbb{P}\left(C|Z_m|/\sigma\sqrt{m\log m} \geqslant C'\lambda\right) \to \mathbb{P}(\mathcal{N} \geqslant C'\lambda) \leqslant \frac{C}{\lambda^3}.$$

Thus we can find a number m_2 such that for all $m \geqslant m_2$ we have

$$\mathbb{P}\left(C|Z_m|/\sigma\sqrt{m\log m} \geqslant C'\lambda\right) \leqslant \frac{C}{\lambda^3},$$

and finally we can pick λ large enough so that $1/\lambda \leqslant \varepsilon$ and thus

$$2\mathbb{P}\left(C|Z_m|/\sigma\sqrt{m\log m} \geqslant C'\lambda\right) \leqslant C\frac{\varepsilon}{\lambda^2}.$$

Finally for $m \geqslant \max(m_1(\varepsilon/2\lambda^2), m_2)$ we have

$$\mathbb{P}(\max_{j\leqslant m}|Z_j| \leqslant \lambda\sqrt{m\log m}) \leqslant \varepsilon/\lambda^2,$$

thus proving tightness of the sequence of laws of $Y_n()$. Along with the convergence of the finite dimensional distributions Theorem 4.1.1 follows.

CHAPTER 5

Conclusions and further research

In this thesis we treated three problems from the theory and applications of random walks. In all of the following $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geqslant 1$, where $X, X_1, X_2, \ldots$ is an iid sequence of random variables.

5.1 Optimal stopping

The first problem we treated is the infinite-horizon optimal stopping problem (1.1.1) for the class of reward functions G which admit the representation

$$G(x) = \sum_{n=0}^{\infty} \beta^n \, \mathbb{E}\left[g(x + S_n)\right] \tag{5.1.1}$$

in terms of a payoff function g; this representation was introduced and developed by Boyarchenko and Levendorskii [1, 16, 17, 18].

Using the Wiener-Hopf factorization we obtained explicit expressions for the value function and optimal stopping time in terms of the extrema of the random walk. Our approach is probabilistic in flavour, and a combination of the analytical methodology in [1, 16–18] and the probabilistic techniques in [2, 12–15]. In fact, this is one of our main contributions since we obtain the full generality of [1, 17] with a much simpler and shorter proof, while we treat general reward functions as opposed to the results in [1, 16–18] where particular cases were treated. In addition, we weaken the assumptions in [1, 17] since we show global optimality of the stopping time without requiring monotonicity of the payoff function g. For general random walks, the proposed stopping time is shown to be *globally optimal* while in [17, 18] optimality was only obtained in the smaller class of hitting times of semi-infinite intervals. Also, we only require that the representation (5.1.1) holds on a semi-infinite interval rather than on the whole of the real line. Finally, our methodology is slightly modified and used to obtain the same results in continuous time in the case of Lévy processes.

The generality of our method is clearly demonstrated since we obtain the results of [2, 12–15] as particular cases; non-monotone perturbations of the payoff function g are also treated to illustrate the weaker monotonicity assumptions imposed on g.

As a new application, we obtain the price of a *Canadian option*, a problem arising in finance and the numerical pricing of American options. Our solution is for general random walks and can be extended to Lévy processes with jumps of both signs –assuming we can compute its Wiener-Hopf factors– while in the existing literature only spectrally one-sided Lévy processes have been treated (see for example [29]).

Further research. Extensions of our results and further research are possible in several directions. First of all, the monotonicity assumption on g may be dropped completely. The resulting problem will be significantly different since for example in that case the optimal stopping time may be the entry time of a finite rather than a semi-infinite interval. The lack of monotonicity also implies that the value function will have a different form. Apart from considering different reward functions, one may modify the process under consideration. For example Ruschendorf and Urusov [64] treated diffusions. Even though our approach is not directly applicable to this case, since the increments are no longer stationary, it would be very interesting to study possible connections and whether our methodology can be adapted to this case.

In a different direction, optimal stopping games are a relatively new and lively topic bringing together ideas from game theory and optimal stopping. The general theory in terms of Markov processes has been extensively studied (see for example [65]), but explicit solutions are rarely available (see [66]). Once again, the monotonicity structure will not be present introducing several complications. However ideas from our research are still applicable and explicit solutions may be available, albeit of a different form.

For practical applications it is important to note that although the solutions obtained in this thesis are in a closed-form, their dependence on the extrema of the process implies that in order to obtain numerical results we need to be able to calculate the Wiener-Hopf factors. These are rarely available explicitly but they can be computed numerically. In addition, a general Lévy process can be very well approximated by one with phase-type jumps in which case the Wiener-Hopf factors are well known(see [12]). Also, numerical approximations to the finite-horizon problem are possible using a sequence of simpler perpetual problems (see [18, 28]). This problem is closely related with the pricing of American options and is therefore fundamental to mathematical finance.

5.2 Asymptotics for the intersections of random walks

The second problem we treated is from the path structure of $\mathbb{Z}^d$-valued random walks. In particular we studied the moments of the self-intersection local time $V_n = \sum_{i,j=0}^{n} 1_{S_i=S_j}$ and obtained exact asymptotics of its variance as $n \to \infty$ for one and two-dimensional recurrent random walks.

The two dimensional case has already been studied by Bolthausen [3] and Černý [4] where the bound $O(n^2)$ was claimed. As explained in the introduction and Chapter 3, [3, 4] prove the weaker bound $O(n^2 \log n)$ for the general case while the claimed $O(n^2)$ bound was only obtained under additional assumptions. The approach used relies on generating functions and the classical Tauberian theorem for power series. In Chapter 3 we show rigorously that this approach breaks down in the general case and can only be used to prove the $O(n^2 \log n)$ due to the monotonicity assumption of the Tauberian theorem. To avoid this restriction we state and prove the Darboux-Wiener complex Tauberian Lemma 3.1.2 which completely removes the monotonicity restriction at the extra cost of having to prove certain bounds in the complex plane. This approach turns out to be the key ingredient needed to revive the generating function approach in [3] and to obtain the correct asymptotics. Using Lemma 3.1.2 we complete the proof in [3, 4] and strengthen it by showing that $O(n^2)$ is the best possible upper bound. In the one-dimensional case, we treat random walks with increments attracted to the symmetric Cauchy law, which are related with a conjecture in Kesten and Spitzer [5].

Apart from completing the proof in [3, 4] and thus settling a long-standing question, another major contribution is the application of Darboux-Wiener type results which allow one to treat non-monotone sequences inaccessible to classical Tauberian theory. This is a powerful and flexible method with many possible applications.

Further research. Further research is possible in several directions. Exact asymptotics can be obtained for the p-fold self-intersection local times defined in section 1.2, which can then be used to study the properties of weakly self-avoiding walks. Apart from V_n, our approach can be used to treat quantities such as $\sum_x N_n(x)N_n(x+y)$ which capture the covariance between the local time at different points. In further research conducted with S. Utev and M. Peligrad, for the one dimensional random walk treated in Chapter 3, we have shown that $\mathbb{E}(\sum_x N_n(x)N_n(x+y)) \sim 2n \log n/\pi\gamma$ and $\mathrm{Var}(\sum_x N_n(x)N_n(x+y)) = O(n^2)$. These asymptotics are useful for limit theorems for random walk in random scenery, when the random scenery is stationary. In that case quantities of the form $\sum_X N_n(x)N_n(x+y)$ appear as coefficients in the covariance terms, and hence their asymptotics are essential for proving limit theorems.

Finally extensions of the complex Tauberian Lemma 3.1.2 are of great theoretical interest. A continuous version of Lemma 3.1.2 for Laplace transforms(instead of z-transforms) would be a powerful alternative to the classical Tauberian theory as it would be applicable to many cases where monotonicity cannot be verified.

5.3 A central limit theorem for random walk on random scenery

The third and last result obtained in this thesis is a functional central limit theorem for one-dimensional random walk in random scenery, where the random walk has increments attracted to the symmetric Cauchy law, thus proving a conjecture by Kesten and Spitzer [5]. The proof makes heavy use of the asymptotics of Chapter 3, and in fact the tight $O(n^2)$ bound allows us to prove a stronger version of the limit theorem in [3], where the partial sums of the sampled scenery satisfy a central limit theorem for almost every path of the random walk.

Further research. One popular direction for further research is to remove the independence assumption from the random scenery. Several results have appeared in this direction (see for example [35, 51]). In fact in further research with S. Utev and M. Peligrad we have obtained a functional central limit theorem for stationary random sceneries with dependence structures such as negative association (see for example [67]) and projective type criteria (see [68]). When the scenery is positively associated the dependence structure is preserved under sampling by a recurrent random walk, and thus similar results can be obtained. Finally, we have recently considered the case where the random walk is replaced by a Markov chain, and under certain assumptions it is also possible to obtain functional central limit theorems.

Appendix A

Appendix to Chapter 3

A.1 Error analysis for Theorem 3.1.3(i)

A.1.1 First term error analysis

We assume that $\Re(\lambda) > \alpha$.

Whenever we analyse an error term arising from the use of the expansion we will telescope the difference by replacing one factor at a time by its expansion. Thus we have

$$E = E_1 + E_2 + E_3 + E_4,$$

where

$$E_1 = (2\pi)^{-2}(1-\lambda)^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{f(y)(1-f(x))-\gamma|x|}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))}\,\mathrm{d}x\,\mathrm{d}y,$$

$$\begin{aligned}E_2 = (2\pi)^{-2}(1-\lambda)^{-2}\Bigg(&\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{f(y)(1-f(x))-\gamma|x|}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))}\,\mathrm{d}x\,\mathrm{d}y\\ &-\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{f(y)(1-f(x))-\gamma|x|}{(1-\lambda+\lambda\gamma|x|)^2(1-\lambda f(y))(1-\lambda f(x)f(y))}\,\mathrm{d}x\,\mathrm{d}y\Bigg)\end{aligned}$$

$$\begin{aligned}E_3 = (2\pi)^{-2}(1-\lambda)^{-2}\Bigg(&\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{f(y)(1-f(x))-\gamma|x|}{(1-\lambda+\lambda\gamma|x|)^2(1-\lambda f(y))(1-\lambda f(x)f(y))}\,\mathrm{d}x\,\mathrm{d}y\\ &-\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{f(y)(1-f(x))-\gamma|x|}{(1-\lambda+\lambda\gamma|x|)^2(1-\lambda+\lambda\gamma|y|)(1-\lambda f(x)f(y))}\,\mathrm{d}x\,\mathrm{d}y\Bigg)\end{aligned}$$

$$\begin{aligned}E_4 = (2\pi)^{-2}(1-\lambda)^{-2}\Bigg(&\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{f(y)(1-f(x))-\gamma|x|}{(1-\lambda+\lambda\gamma|x|)^2(1-\lambda+\lambda\gamma|y|)(1-\lambda f(x)f(y))}\,\mathrm{d}x\,\mathrm{d}y\\ &-\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{f(y)(1-f(x))-\gamma|x|}{(1-\lambda+\lambda\gamma|x|)^2(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma(|x|+|y|))}\,\mathrm{d}x\,\mathrm{d}y\Bigg)\end{aligned}$$

First error. We first replace the numerator by $\gamma|x|$. Note that for $|x|,|y|<\varepsilon$

$$|f(y)(1-f(x))-\gamma|x||=|(1-\gamma|y|+R(y))(\gamma|x|-R(x))-\gamma|x||\leqslant C(\varepsilon)|x|,$$

where $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then by the above and Inequalities (B3a) and (B3b)

$$
\begin{aligned}
&|E_1| \\
&= C|1-\lambda|^{-2}\left|\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{(f(y)(1-f(x))-\gamma|x|)\,dx\,dy}{(1-\lambda f(x))^2(1-\lambda f(y))(1-\lambda f(x)f(y))}\right| \\
&\leqslant \int_0^{\varepsilon}\int_0^{\varepsilon}\frac{C(\varepsilon)|1-\lambda|^{-2}x\,dx\,dy}{(|1-\lambda+\lambda\gamma x|-\theta_\varepsilon x)^2(|1-\lambda+\lambda\gamma y|-\theta_\varepsilon y)(|1-\lambda+\lambda\gamma(x+y)|-\Delta_\varepsilon(x+y))} \\
&\leqslant \int_0^{K}\int_0^{K}\frac{C(\varepsilon)|1-\lambda|^{-3}x\,dx\,dy}{(|z_1+z_2x|-\theta_\varepsilon x)^2(|z_1+z_2y|-\theta_\varepsilon y)(|z_1+z_2(x+y)|-\Delta_\varepsilon(x+y))}
\end{aligned}
$$

and since by Inequality (B5b)

$$
\frac{x}{|1-\lambda+\lambda\gamma x|-\theta_\varepsilon x} \leqslant \frac{x}{Cx} \leqslant C,
$$

we have

$$
\begin{aligned}
|E_1| &\leqslant \int_0^{K}\int_0^{K}\frac{C(\varepsilon)|1-\lambda|^{-3}\,dx\,dy}{(|z_1+z_2x|-\theta_\varepsilon x)(|z_1+z_2y|-\theta_\varepsilon y)(|z_1+z_2(x+y)|-\Delta_\varepsilon(x+y))} \\
&= C(\varepsilon)|1-\lambda|^{-3}F_1(\lambda) < C(\varepsilon)|1-\lambda|^{-3},
\end{aligned}
$$

where $F_1(\lambda)$ is uniformly bounded for all λ by Lemma A.2.1 given in section A.2.

Second error. We now bound the error from replacing $1-\lambda f(x)$ in the denominator by $1-\lambda+\lambda\gamma|x|$.

$$
\begin{aligned}
E_2 = C(1-\lambda)^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} &\frac{\gamma|x|}{(1-\lambda f(y))(1-\lambda f(x)f(y))} \\
&\times\left[\frac{1}{(1-\lambda f(x))^2}-\frac{1}{(1-\lambda+\lambda\gamma|x|)^2}\right]dx\,dy.
\end{aligned}
$$

Using (3.1.7) we calculate

$$
(1-\lambda f(x))^2-(1-\lambda+\lambda\gamma|x|)^2 \leqslant R(x)^2+2R(x)|1-\lambda+\lambda\gamma|x||,
$$

and therefore by Inequalities (B3a) and (B3b)

$$
\begin{aligned}
|E_2| &\leqslant C|1-\lambda|^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\gamma|x|(\theta_\varepsilon|x||1-\lambda+\lambda\gamma|x||+\theta_\varepsilon^2|x|^2)\,dx\,dy}{|1-\lambda f(x)|^2|1-\lambda+\lambda\gamma|x||^2|1-\lambda f(y)||1-\lambda f(x)f(y)|} \\
&\leqslant C|1-\lambda|^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\gamma|x|(\theta_\varepsilon|x||1-\lambda+\lambda\gamma|x||+\theta_\varepsilon^2|x|^2)}{(|1-\lambda+\lambda\gamma|x||-\theta_\varepsilon|x|)^2|1-\lambda+\lambda\gamma|x||^2} \\
&\quad\times\frac{dx\,dy}{(|1-\lambda+\lambda\gamma|y||-\theta_\varepsilon|y|)(|1-\lambda+\lambda\gamma(|x|+|y|)|-\Delta_\varepsilon(|x|+|y|))} \\
&= I_1(\lambda)+I_2(\lambda).
\end{aligned}
$$

For the first integral we cancel out

$$\begin{aligned}|I_1(\lambda)| &\leqslant C|1-\lambda|^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{\gamma\theta_\varepsilon|x|^2|1-\lambda+\lambda\gamma|x||}{(|1-\lambda+\lambda\gamma|x||-\theta_\varepsilon|x|)^2|1-\lambda+\lambda\gamma|x||^2}\\&\quad\times\frac{dx\,dy}{(|1-\lambda+\lambda\gamma|y||-\theta_\varepsilon|y|)(|1-\lambda+\lambda\gamma(|x|+|y|)|-\Delta_\varepsilon(|x|+|y|))}\\&\leqslant C|1-\lambda|^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{|x|^2\theta_\varepsilon}{(|1-\lambda+\lambda\gamma|x||-\theta_\varepsilon|x|)^2|1-\lambda+\lambda\gamma|x||}\\&\quad\times\frac{dx\,dy}{(|1-\lambda+\lambda\gamma|y||-\theta_\varepsilon|y|)(|1-\lambda+\lambda\gamma(|x|+|y|)|-\Delta_\varepsilon(|x|+|y|))},\end{aligned}$$

and since by Inequality (B3a)

$$\frac{|x|^2}{\big||1-\lambda+\lambda\gamma|x||-\theta_\varepsilon|x|\big|\,|1-\lambda+\lambda\gamma|x||}\leqslant C,$$

the following upper bound is obtained

$$\begin{aligned}I_1(\lambda) &\leqslant C\theta_\varepsilon|1-\lambda|^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\frac{1}{(|1-\lambda+\lambda\gamma|x||-\theta_\varepsilon|x|)(|1-\lambda+\lambda\gamma|y||-\theta_\varepsilon|y|)}\\&\quad\times\frac{dx\,dy}{(|1-\lambda+\lambda\gamma(|x|+|y|)|-\Delta_\varepsilon(|x|+|y|))}\\&= C\theta_\varepsilon|1-\lambda|^{-2}\int_0^{\varepsilon}\int_0^{\varepsilon}\frac{1}{(|1-\lambda+\lambda\gamma x|-\theta_\varepsilon x)(|1-\lambda+\lambda\gamma y|-\theta_\varepsilon y)}\\&\quad\times\frac{dx\,dy}{(|1-\lambda+\lambda\gamma(x+y)|-\Delta_\varepsilon(x+y))}\\&= C\theta_\varepsilon|1-\lambda|^{-3}\int_0^{K}\int_0^{K}\frac{1}{(|z_1+z_2x|-\theta_\varepsilon x)(|z_1+z_2y|-\theta_\varepsilon y)}\\&\quad\times\frac{dx\,dy}{(|z_1+z_2(x+y)|-\Delta_\varepsilon(x+y))}\\&\leqslant C(\epsilon)|1-\lambda|^{-3},\end{aligned}$$

uniformly in λ by Lemma A.2.1.

For I_2, again using Inequality (B3a) we cancel out the x^3 term in the numerator to get

$$\begin{aligned}&|I_2(\lambda)|\\&\leqslant C|1-\lambda|^{-2}\int_0^{\varepsilon}\int_0^{\varepsilon}\frac{\theta_\varepsilon^2x^3}{(|1-\lambda+\lambda\gamma x|-\theta_\varepsilon x)^2|1-\lambda+\lambda\gamma x|^2}\\&\times\frac{dx\,dy}{(|1-\lambda+\lambda\gamma y|-\theta_\varepsilon y)(|1-\lambda+\lambda\gamma(x+y)|-\Delta_\varepsilon(x+y))}\\&\leqslant C(\epsilon)|1-\lambda|^{-2}\int_0^{\varepsilon}\int_0^{\varepsilon}\frac{1}{(|1-\lambda+\lambda\gamma x|-\theta_\varepsilon x)(|1-\lambda+\lambda\gamma y|-\theta_\varepsilon y)}\\&\times\frac{dx\,dy}{(|1-\lambda+\lambda\gamma(x+y)|-\Delta_\varepsilon(x+y))}\\&\leqslant C(\epsilon)|1-\lambda|^{-3}\\&\times\int_0^{\varepsilon}\int_0^{\varepsilon}\frac{dx\,dy}{(|z_1+z_2x|-\theta_\varepsilon x)(|z_1+z_2y|-\theta_\varepsilon y)(|z_1+z_2(x+y)|-\Delta_\varepsilon(x+y))},\end{aligned}$$

and thus

$$|I_2(\lambda)| \leqslant C(\epsilon)|1-\lambda|^{-3},$$

uniformly in λ by Lemma A.2.1.

Third error. The third error appears from replacing the y-factor in the denominator by its expansion and is given by

$$\begin{aligned}E_3 &= C(1-\lambda)^{-2}\\ &\times \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} \frac{\gamma|x|}{(1-\lambda+\lambda\gamma|x|)^2(1-\lambda f(x)f(y))}\left[\frac{1}{1-\lambda f(y)} - \frac{1}{1-\lambda+\lambda\gamma|y|}\right] \mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Using (3.1.7) it is easy to show that

$$|1-\lambda f(y) - (1-\lambda+\lambda\gamma|y|)| \leqslant \theta_\epsilon|x||y|,$$

which along with Inequalities (B3a), (B3b) and (B3a) and a change of variables gives

$$\begin{aligned}&|E_3|\\ &\leqslant C|1-\lambda|^{-2}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\frac{\theta_\epsilon|x||y|\,\mathrm{d}x\,\mathrm{d}y}{|1-\lambda+\lambda\gamma|x||^2|1-\lambda f(x)f(y)||1-\lambda f(y)||1-\lambda+\lambda\gamma|y||}\\ &\leqslant C|1-\lambda|^{-2}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\frac{\theta_\epsilon\,\mathrm{d}x\,\mathrm{d}y}{|1-\lambda+\lambda\gamma|x|||1-\lambda f(x)f(y)||1-\lambda f(y)|}\\ &\leqslant C(\epsilon)|1-\lambda|^{-2}\\ &\quad\times\int_0^{\epsilon}\int_0^{\epsilon}\frac{\mathrm{d}x\,\mathrm{d}y}{|1-\lambda+\lambda\gamma|x||(|1-\lambda+\lambda\gamma y|-\theta_\epsilon y)(|1-\lambda+\lambda\gamma(x+y)|-\Delta_\epsilon(x+y))}\\ &\leqslant C(\epsilon)|1-\lambda|^{-3}\int_0^K\int_0^K\frac{\mathrm{d}x\,\mathrm{d}y}{|z_1+z_2|x||(|z_1+z_2y|-\theta_\epsilon y)(|z_1+z_2\gamma(x+y)|-\Delta_\epsilon(x+y))}\\ &\leqslant C(\epsilon)|1-\lambda|^{-3},\end{aligned}$$

by a minor modification to Lemma A.2.1.

Fourth error. We can now consider the last error term given by

$$\begin{aligned}E_4 = C(1-\lambda)^{-2}\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}&\frac{\gamma|x|}{(1-\lambda+\lambda\gamma|x|))^2(1-\lambda+\lambda\gamma|y|)}\\ &\times\left[\frac{1}{1-\lambda f(x)f(y)} - \frac{1}{1-\lambda+\lambda\gamma(|x|+|y|)}\right]\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

(3.1.7) and some algebra gives for $|x|, |y| < \epsilon$

$$|1-\lambda f(x)f(y) - (1-\lambda+\lambda\gamma(|x|+|y|))| \leqslant C(\epsilon)(|x|+|y|).$$

By Inequalities (B3a), (B3b), (B7) and (B8)

$$
\begin{aligned}
&|E_4| \\
&\leqslant C(\varepsilon)|1-\lambda|^{-2} \\
&\times \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{\gamma|x|(|x|+|y|)\,\mathrm{d}x\,\mathrm{d}y}{|1-\lambda+\lambda\gamma|x||^2|1-\lambda+\lambda\gamma|y|||1-\lambda f(x)f(y)||1-\lambda+\lambda\gamma(|x|+|y|)|} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{d}x\,\mathrm{d}y}{|1-\lambda+\lambda\gamma|x|||1-\lambda+\lambda\gamma|y|||1-\lambda f(x)f(y)|} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-2}\int_{0}^{\varepsilon}\int_{0}^{\varepsilon} \frac{\mathrm{d}x\,\mathrm{d}y}{|1-\lambda+\lambda\gamma x||1-\lambda+\lambda\gamma y|(|1-\lambda+\lambda\gamma(x+y)|-\Delta_\varepsilon(x+y))} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-3}\int_{0}^{\varepsilon}\int_{0}^{\varepsilon} \frac{\mathrm{d}x\,\mathrm{d}y}{|z_1+z_2x||z_1+z_2y|(|z_1+z_2(x+y)|-\Delta_\varepsilon(x+y))} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-3},
\end{aligned}
$$

uniformly in λ by a minor modification of Lemma A.2.1.

A.1.2 Second term error analysis

We split the integral in two parts and perform error analysis on each part separately.

Errors in the first integral.

For the first integral the error is given by

$$
\begin{aligned}
|E^{(1)}| \leqslant C|1-\lambda|^{-2}\Bigg| &\iint_{U_\varepsilon} \frac{f(x)f(x+y)\,\mathrm{d}x\,\mathrm{d}y}{(1-\lambda f(x))(1-\lambda f(y))(1-\lambda f(x+y))} \\
&- \iint_{U_\varepsilon} \frac{\mathrm{d}x\,\mathrm{d}y}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma|x+y|)}\Bigg|.
\end{aligned}
$$

We telescope the difference by replacing each factor consecutively by its expansion, and we bound the resulting differences.

First error. First replace the numerator by 1. It is straightforward using (3.1.7) to see that for $|x|, |y| < \varepsilon$ and the fact that $R(x) \leqslant \theta_\varepsilon|x|$, where $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$
\begin{aligned}
|f(x)f(x+y)-1| &= |(1-\gamma|x|+R(x))(1-\gamma|x+y|+R(x+y))-1| \\
&\leqslant C(\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

Then using Inequality (B3a)

$$
\begin{aligned}
|E_1^{(1)}| &\leqslant C(\varepsilon)|1-\lambda|^{-2}\iint_{U_\varepsilon} \frac{|f(x)f(x+y)-1|\,\mathrm{d}x\,\mathrm{d}y}{|1-\lambda f(x)||1-\lambda f(y)||1-\lambda f(x+y)|} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-2}\iint_{U_\varepsilon} \frac{1}{(|1-\lambda+\lambda\gamma|x||-\theta_\varepsilon|x|)(|1-\lambda+\lambda\gamma|y||-\theta_\varepsilon|y|)} \\
&\quad\times \frac{\mathrm{d}x\,\mathrm{d}y}{(|1-\lambda+\lambda\gamma(|x+y|)|-\theta_\varepsilon|x+y|)}
\end{aligned}
$$

$$\leqslant C(\epsilon)|1-\lambda|^{-3}\int_{-K}^{K}\int_{-K}^{K}\frac{1}{(|z_1+z_2|x||-\theta_\epsilon|x|)(|z_1+z_2|y||-\theta_\epsilon|y|)}$$
$$\times\frac{dx\,dy}{(|z_1+z_2(|x+y|)|-\theta_\epsilon|x+y|)}$$
$$\leqslant C(\epsilon)|1-\lambda|^{-3},$$

uniformly in λ by Lemma A.2.2.

Second error. Next we replace the x-term in the denominator by its expansion,

$$E_2^{(1)}=C(1-\lambda)^{-2}\iint_{U_\epsilon}\frac{dx\,dy}{(1-\lambda f(y))(1-\lambda f(x+y))}\left[\frac{1}{1-\lambda f(x)}-\frac{1}{1-\lambda+\lambda\gamma|x|}\right].$$

Once again we obtain the bound

$$|1-\lambda f(x)-(1-\lambda+\lambda\gamma|x|)|\leqslant\theta_\epsilon|x|.$$

Using the above bound, the inequalities Inequalities (B7) and (B3a) and the change of variables $r=x/|1-\lambda|$, $s=y/|1-\lambda|$ we obtain

$$|E_2^{(1)}|$$
$$\leqslant C(\epsilon)|1-\lambda|^{-2}\iint_{U_\epsilon}\frac{|x|\,dx\,dy}{|1-\lambda f(y)||1-\lambda f(x+y)||1-\lambda f(x)||1-\lambda+\lambda\gamma|x||}$$
$$\leqslant C(\epsilon)|1-\lambda|^{-2}\iint_{U_\epsilon}\frac{dx\,dy}{|1-\lambda f(y)||1-\lambda f(x+y)||1-\lambda f(x)|}$$
$$\leqslant C(\epsilon)|1-\lambda|^{-2}\iint_{U_\epsilon}\frac{1}{(|1-\lambda+\lambda\gamma|x||-\theta_\epsilon|x|)(|1-\lambda+\lambda\gamma|y||-\theta_\epsilon|y|)}$$
$$\times\frac{1}{(|1-\lambda+\lambda\gamma|x+y||-\Delta_\epsilon|x+y|)}\,dx\,dy$$
$$\leqslant C(\epsilon)|1-\lambda|^{-3}$$
$$\times\int_{-K}^{K}\int_{-K}^{K}\frac{dx\,dy}{(|z_1+z_2|x||-\theta_\epsilon|x|)(|z_1+z_2|y|-\theta_\epsilon|y|)(||z_1+z_2(|x+y|)|-\Delta_\epsilon|x+y|)}$$
$$\leqslant C(\epsilon)|1-\lambda|^{-3},$$

uniformly in λ by Lemma A.2.2.

Third error. The next error arises from replacing the y-term by its expansion and is given by

$$E_3^{(1)}=C(1-\lambda)^{-2}\iint_{U_\epsilon}\frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda f(x+y))}\left[\frac{1}{1-\lambda f(y)}-\frac{1}{1-\lambda+\lambda\gamma|y|}\right].$$

Once again the bound

$$|1-\lambda f(y)-(1-\lambda+\lambda|y|)|\leqslant\theta_\epsilon|y|,$$

and Inequalities (B3a) and (B7) give

$$
\begin{aligned}
|E_3^{(1)}| &= C(\epsilon)|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{|y|\,dx\,dy}{|1-\lambda+\lambda\gamma|x||\,|1-\lambda f(y)||1-\lambda f(x+y)||1-\lambda+\lambda\gamma|y||} \\
&\leqslant C(\epsilon)|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{1}{|1-\lambda+\lambda\gamma|x||\,||1-\lambda+\lambda\gamma|y|-\theta_\epsilon|y||} \\
&\quad \times \frac{dx\,dy}{||1-\lambda+\lambda\gamma(|x+y|)|-\Delta_\epsilon|x+y||} \\
&\leqslant C(\epsilon)|1-\lambda|^{-3} \int_{-K}^{K}\int_{-K}^{K} \frac{1}{|z_1+z_2|x||(|z_1+z_2|y|-\theta_\epsilon|y|)} \\
&\quad \times \frac{dx\,dy}{(|z_1+z_2|x+y||-\Delta_\epsilon|x+y|)} \\
&\leqslant C(\epsilon)|1-\lambda|^{-3},
\end{aligned}
$$

uniformly in λ by a minor modification of Lemma A.2.2.

Fourth error. Finally we replace the x, y-term by its expansion and we get the fourth error,

$$
E_4^{(1)} = C(1-\lambda)^{-2} \iint_{U_\epsilon} \frac{1}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)} \times \left[\frac{1}{1-\lambda f(x+y)} - \frac{1}{1-\lambda+\lambda\gamma(|x+y|)}\right] dx\,dy.
$$

It is straightforward to get the bound

$$
|1-\lambda f(x+y)-(1-\lambda+\lambda\gamma|x+y|)| \leqslant \theta_\epsilon|x+y|.
$$

The above along with a change of variables and Inequalities (B3a), (B7) and (3.2.6) imply that

$$
\begin{aligned}
|E_4^{(1)}| &= C(\epsilon)|1-\lambda|^{-2} \\
&\quad \times \iint_{U_\epsilon} \frac{|x+y|\,dx\,dy}{|1-\lambda+\lambda\gamma|x||\,|1-\lambda+\lambda\gamma|y||\,|1-\lambda f(x+y)||1-\lambda+\lambda\gamma|x+y||} \\
&\leqslant C(\epsilon)|1-\lambda|^{-2} \\
&\quad \times \iint_{U_\epsilon} \frac{dx\,dy}{|1-\lambda+\lambda\gamma|x||\,|1-\lambda+\lambda\gamma|y||(|1-\lambda+\lambda\gamma|x+y||-\theta_\epsilon|x+y|)} \\
&\leqslant C(\epsilon)|1-\lambda|^{-3} \int_{-K}^{K}\int_{-K}^{K} \frac{dx\,dy}{|z_1+z_2|x|||z_1+z_2|xy|(|z_1+z_2|x+y||-\theta_\epsilon|x+y|)} \\
&\leqslant C(\epsilon)|1-\lambda|^{-3},
\end{aligned}
$$

uniformly in λ.

Errors in the second integral.

The total error in the second integral is given by

$$E^{(2)}(\lambda) = C(1-\lambda)^{-2} \iint_{U_\epsilon} \frac{f(x)f(y)^2}{(1-\lambda f(x))(1-\lambda f(y))(1-\lambda f(x)f(y))} - \frac{1}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)(1-\lambda+\lambda\gamma(|x|+|y|))}\,dx\,dy.$$

First error. We replace the numerator with 1. It is straightforward to obtain the bound

$$|f(x)f(y)^2 - 1| \leqslant C(\epsilon),$$

which implies along with a change of variables, Inequalities (B3a) and (B3b) that

$$\begin{aligned}
|E_1^{(2)}(\lambda)| &\leqslant |1-\lambda|^{-2} \iint_{U_\epsilon} \frac{|f(x)f(y)^2-1|\,dx\,dy}{|1-\lambda f(x)||1-\lambda f(y)||1-\lambda f(x)f(y)|} \\
&\leqslant C(\epsilon)|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{1}{(|1-\lambda+\lambda\gamma|x||-\theta_\epsilon|x|)(|1-\lambda+\lambda\gamma|y||-\theta_\epsilon|y|)} \\
&\quad\times \frac{dx\,dy}{(|1-\lambda+\lambda\gamma(|x|+|y|)|-\theta_\epsilon(|x|+|y|))} \\
&= C(\epsilon)|1-\lambda|^{-3} \int_{-K}^{K}\int_{-K}^{K} \frac{1}{(|z_1+z_2|x||-\theta_\epsilon|x|)(|z_1+z_2|y||-\theta_\epsilon|y|)} \\
&\quad\times \frac{dx\,dy}{(|z_1+z_2(|x|+|y|)|-\Delta_\epsilon(|x|+|y|))} \\
&\leqslant C(\epsilon)|1-\lambda|^{-3}
\end{aligned}$$

uniformly in λ, by Lemma A.2.1.

Second error. We replace the x-term in the denominator to obtain the error

$$E_2^{(2)}(\lambda) = C(1-\lambda)^{-2} \iint_{U_\epsilon} \frac{dx\,dy}{(1-\lambda f(y))(1-\lambda f(x)f(y))}\left[\frac{1}{1-\lambda f(x)} - \frac{1}{1-\lambda+\lambda\gamma|x|}\right].$$

From a change of variables, Inequalities (B3a), (B3b) and (B7) and Lemma A.2.1 it follows that

$$\begin{aligned}
|E_2^{(2)}(\lambda)| &\leqslant C(\epsilon)|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{|x|\,dx\,dy}{|1-\lambda f(y)||1-\lambda f(x)f(y)||1-\lambda f(x)||1-\lambda+\lambda\gamma|x||} \\
&\leqslant C(\epsilon)|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{dx\,dy}{|1-\lambda f(y)||1-\lambda f(x)f(y)||1-\lambda f(x)|} \\
&\leqslant C(\epsilon)|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{1}{(|1-\lambda+\lambda\gamma|x||-\theta_\epsilon|x|)(|1-\lambda+\lambda\gamma|y||-\theta_\epsilon|y|)} \\
&\quad\times \frac{dx\,dy}{(|1-\lambda+\lambda\gamma(|x|+|y|)|-\theta_\epsilon(|x|+|y|))} \\
&\leqslant C(\epsilon)|1-\lambda|^{-3} \int_{-K}^{K}\int_{-K}^{K} \frac{1}{(|z_1+z_2|x||-\theta_\epsilon|x|)(|z_1+z_2|y||-\theta_\epsilon|y|)} \\
&\quad\times \frac{dx\,dy}{(|z_1+z_2(|x|+|y|)|-\Delta_\epsilon(|x|+|y|))} \\
&\leqslant C(\epsilon)|1-\lambda|^{-3},
\end{aligned}$$

uniformly in λ.

Third error. The expansion of the y-term gives the third error

$$E_3^{(2)}(\lambda) = C(1-\lambda)^{-2} \times \iint_{U_\epsilon} \frac{1}{(1-\lambda+\lambda\gamma|x|)(1-\lambda f(x)f(y))} \left[\frac{1}{1-\lambda f(y)} - \frac{1}{1-\lambda+\lambda\gamma|y|}\right] dx\,dy.$$

Then a change of variables, Inequalities (B3a), (B3b) and (B7) and a minor modification of Lemma A.2.1

$$\begin{aligned}
|E_3^{(2)}(\lambda)| &\leqslant C|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{\theta_\epsilon|y|\,dx\,dy}{|1-\lambda+\lambda\gamma|x|||1-\lambda+\lambda\gamma|y|||1-\lambda f(y)||1-\lambda f(x)f(y)|} \\
&\leqslant C|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{\theta_\epsilon\,dx\,dy}{|1-\lambda+\lambda\gamma|x|||1-\lambda f(y)||1-\lambda f(x)f(y)|} \\
&\leqslant C|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{1}{|1-\lambda+\lambda\gamma|x||(|1-\lambda+\lambda\gamma|y||-\theta_\epsilon|y|)} \\
&\quad \times \frac{dx\,dy}{(|1-\lambda+\lambda\gamma(|x|+|y|)|-\theta_\epsilon(|x|+|y|))} \\
&\leqslant C(\epsilon)|1-\lambda|^{-3} \int_{-K}^{K}\int_{-K}^{K} \frac{dx\,dy}{|z_1+z_2|x||(|z_1+z_2|y||-\theta_\epsilon|y|)} \\
&\quad \times \frac{dx\,dy}{|z_1+z_2(|x|+|y|)|-\Delta_\epsilon(|x|+|y|)} \\
&\leqslant C(\epsilon)|1-\lambda|^{-3}.
\end{aligned}$$

uniformly in λ .

Fourth error. Finally replacing the term involving both x and y gives the error

$$E_4^{(2)}(\lambda) = C(1-\lambda)^{-2} \times \iint_{U_\epsilon} \frac{dx\,dy}{(1-\lambda+\lambda\gamma|x|)(1-\lambda+\lambda\gamma|y|)} \left[\frac{1}{1-\lambda f(x)f(y)} - \frac{1}{1-\lambda+\lambda\gamma(|x|+|y|)}\right].$$

Once again Inequalities (B3a), (B3b) and (B7) and a minor modification of Lemma A.2.1 imply that

$$\begin{aligned}
|E_4^{(2)}(\lambda)| &\leqslant C(\epsilon)|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{1}{|1-\lambda+\lambda\gamma|x|||1-\lambda+\lambda\gamma|y||} \\
&\quad \times \frac{\theta_\epsilon(|x|+|y|)}{|1-\lambda f(x)f(y)||1-\lambda+\lambda\gamma(|x|+|y|)|}\,dx\,dy \\
&\leqslant C(\epsilon)|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{dx\,dy}{|1-\lambda+\lambda\gamma|x|||1-\lambda+\lambda\gamma|y|||1-\lambda f(x)f(y)|} \\
&\leqslant C(\epsilon)|1-\lambda|^{-2} \iint_{U_\epsilon} \frac{1}{|1-\lambda+\lambda\gamma|x|||1-\lambda+\lambda\gamma|y||} \\
&\quad \times \frac{1}{(|1-\lambda+\lambda\gamma(|x|+|y|)|-\Delta_\epsilon(|x|+|y|))}\,dx\,dy \\
&\leqslant C(\epsilon)|1-\lambda|^{-3} \\
&\quad \times \int_{-K}^{K}\int_{-K}^{K} \frac{dx\,dy}{|z_1+z_2|x|||z_1+z_2|y||(|z_1+z_2(|x|+|y|)|-\Delta_\epsilon(|x|+|y|))},
\end{aligned}$$

and therefore

$$|E_4^{(2)}(\lambda)| \leqslant C(\epsilon)|1-\lambda|^{-3}$$

uniformly in λ.

A.2 Integral calculations for Theorem 3.1.3(i)

Let $K \equiv \epsilon|1-\lambda|^{-1}$, $z_1 \equiv (1-\lambda)|1-\lambda|^{-1}$, $z_2 \equiv \lambda\gamma$.

Lemma A.2.1. *For small $\epsilon > 0$, complex λ with $|\lambda| < 1$, $\gamma > 0$ and K, z_1, z_2 as defined above the integral*

$$F_1(\lambda) = \int_0^K\int_0^K \frac{\mathrm{d}x\,\mathrm{d}y}{(|z_1+z_2x|-\theta_\epsilon x)(|z_1+z_2y|-\theta_\epsilon y)(|z_1+z_2(x+y)|-\Delta_\epsilon(x+y))} \tag{A.2.1}$$

is finite.

Proof. Let $\alpha > 0$ be fixed and very small. Then we split the region of integration into four parts. The integral is then split into

$$F_1(\lambda) = F_1^{(1)}(\lambda) + F_1^{(2)}(\lambda) + F_1^{(3)}(\lambda) + F_1^{(4)}(\lambda),$$

where

$$F_1^{(1)}(\lambda) = \int_0^\alpha\int_0^\alpha \frac{\mathrm{d}x\,\mathrm{d}y}{(|z_1+z_2x|-\theta_\epsilon x)(|z_1+z_2y|-\theta_\epsilon y)(|z_1+z_2(x+y)|-\Delta_\epsilon(x+y))}$$

$$F_1^{(2)}(\lambda) = \int_0^\alpha\int_\alpha^K \frac{\mathrm{d}x\,\mathrm{d}y}{(|z_1+z_2x|-\theta_\epsilon x)(|z_1+z_2y|-\theta_\epsilon y)(|z_1+z_2(x+y)|-\Delta_\epsilon(x+y))}$$

$$F_1^{(3)}(\lambda) = \int_\alpha^K\int_0^\alpha \frac{\mathrm{d}x\,\mathrm{d}y}{(|z_1+z_2x|-\theta_\epsilon x)(|z_1+z_2y|-\theta_\epsilon y)(|z_1+z_2(x+y)|-\Delta_\epsilon(x+y))}$$

$$F_1^{(4)}(\lambda) = \int_\alpha^K\int_\alpha^K \frac{\mathrm{d}x\,\mathrm{d}y}{(|z_1+z_2x|-\theta_\epsilon x)(|z_1+z_2y|-\theta_\epsilon y)(|z_1+z_2(x+y)|-\Delta_\epsilon(x+y))}.$$

We treat each one separately.

For the first integral we use Inequality (B5a) to get that all terms in the denominator are bounded above by constants and thus $F_1^{(1)}(\lambda) < C < \infty$.

For the second integral we use Inequality (B5a) for terms involving just y, and Inequality (B5b) for the terms involving x. We then have

$$F_1^{(2)}(\lambda) \leqslant C \int_0^\alpha\int_\alpha^K \frac{\mathrm{d}x\,\mathrm{d}y}{x(x+y)} \leqslant C\int_\alpha^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{x^2} \leqslant C < \infty.$$

The third integral is finite by symmetry and the above calculations.

Finally for the fourth integral we use Inequality (B5b) for all terms in the denominator to get

$$F_1^{(4)}(\lambda) \leqslant C \int_\alpha^K\int_\alpha^K \frac{\mathrm{d}x\,\mathrm{d}y}{xy(x+y)} \leqslant \int_\alpha^K\int_\alpha^K \frac{\mathrm{d}x\,\mathrm{d}y}{xy\sqrt{xy}} \leqslant \int_\alpha^\infty\int_\alpha^\infty \frac{\mathrm{d}x\,\mathrm{d}y}{(xy)^{3/2}} < C < \infty \qquad \square$$

Lemma A.2.2. *For small $\varepsilon > 0$, $\Re(\lambda) > \alpha$, $|\lambda| < 1$, $\gamma > 0$ and K, z_1, z_2 as defined above the integral*

$$F_2(\lambda) = \int_{-K}^K\int_{-K}^K \frac{\mathrm{d}x\,\mathrm{d}y}{(|z_1+z_2|x|| - \theta_\varepsilon|x|)(|z_1+z_2|y|| - \theta_\varepsilon|y|)(|z_1+z_2|x+y|| - \Delta_\varepsilon|x+y|)} \tag{A.2.2}$$

is bounded uniformly in λ.

Proof. To simplify notation we write

$$X(x,y,\lambda) \equiv \frac{1}{(|z_1+z_2|x|| - \theta_\varepsilon|x|)(|z_1+z_2|y|| - \theta_\varepsilon|y|)(|z_1+z_2|x+y|| - \Delta_\varepsilon|x+y|)}.$$

Then we have for some fixed positive α

$$\begin{aligned} F_2(\lambda) &= \iint\limits_{\substack{B_K(0)\times B_K(0)\\ |x+y|<\alpha}} X(x,y,\lambda)\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{\substack{B_K(0)\times B_K(0)\\ |x+y|\geqslant\alpha}} X(x,y,\lambda)\,\mathrm{d}x\,\mathrm{d}y \\ &= F_2^{(1)}(\lambda) + F_2^{(2)}(\lambda). \end{aligned}$$

Let us first treat $F_2^{(1)}(\lambda)$.

$$\begin{aligned} F_2^{(1)}(\lambda) &\leqslant \int_{B_K(0)}\int_{B_\alpha(-y)} X(x,y,\lambda)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{B_K(0)}\int_{B_\alpha(0)} X(x-y,y,\lambda)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \int_{-2\alpha}^{2\alpha}\int_{-\alpha}^{\alpha} X(x-y,y,\lambda)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{|y|>2\alpha}\int_{-\alpha}^{\alpha} X(x-y,y,\lambda)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

For the first of these integrals we use Inequality (B5a), which implies that for α, small enough then $|z_1+z_2|x|| - \theta_\varepsilon|x|$ is bounded below by a constant independent of λ. Also since $|x-y| \leqslant |y|+|x| \leqslant 3\alpha$ we also have that $|z_1+z_2|x-y|| - \theta_\varepsilon|x-y|$ is also bounded below by a constant. This implies that the first integral is bounded above uniformly in λ.

Let us now consider the second integral. In this case note that $|x-y| \geqslant |y|-|x| \geqslant \alpha > 0$. And thus we have using Inequality (B5b)

$$2(|z_1 + z_2|x-y|| - \theta_\varepsilon|x-y|) \geqslant 2C|x-y| \geqslant C\alpha + C|x-y|.$$

Using this, Inequality (B5b) for the y-term, and Inequality (B5a) for the x-term, we get

$$\begin{aligned}
&\int\limits_{|y|>2\alpha}\int\limits_{-\alpha}^{\alpha} X(x-y,y,\lambda)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\leqslant \int\limits_{|y|>2\alpha}\int\limits_{-\alpha}^{\alpha} \frac{\mathrm{d}x\,\mathrm{d}y}{(|z_1+z_2|x-y||-\theta_\varepsilon|x-y|)(|z_1+z_2|y||-\theta_\varepsilon|y|)(|z_1+z_2|x||-\Delta_\varepsilon|x|)} \\
&\quad\leqslant C\int\limits_{|y|>2\alpha}\int\limits_{-\alpha}^{\alpha} \frac{\mathrm{d}x\,\mathrm{d}y}{(\alpha+|x-y|)|y|} \leqslant C\int\limits_{|y|>2\alpha}\int\limits_{-\alpha}^{\alpha} \frac{\mathrm{d}x\,\mathrm{d}y}{|y|^2} = C\alpha\frac{1}{\alpha} \leqslant C
\end{aligned}$$

uniformly in λ. Note that in the last inequality we have used the fact that

$$\alpha + |x-y| \geqslant \alpha + |y| - |x| \geqslant |y|.$$

Now we can consider $F_2^{(2)}(\lambda)$. Using Inequality (B5b) and $|x+y| \geqslant \alpha > 0$ we have

$$2(|z_1 + z_2|x+y|| - \Delta_\varepsilon|x+y|) \geqslant 2C|x+y| \geqslant C\alpha + C|x+y|.$$

Therefore

$$\begin{aligned}
F_2^{(2)}(\lambda) &= \iint\limits_{\substack{B_K(0)\times B_K(0)\\ |x+y|\geqslant\alpha}} X(x,y,\lambda)\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant C \iint\limits_{B_K(0)\times B_K(0)} \frac{\mathrm{d}x\,\mathrm{d}y}{(|z_1+z_2|x||-\theta_\varepsilon|x|)(|z_1+z_2|y||-\theta_\varepsilon|y|)(\alpha+|x+y|)}.
\end{aligned}$$

Now observe that

$$\int\limits_{-\alpha}^{\alpha}\int\limits_{-\alpha}^{\alpha} \frac{\mathrm{d}x\,\mathrm{d}y}{(|z_1+z_2|x||-\theta_\varepsilon|x|)(|z_1+z_2|y||-\theta_\varepsilon|y|)(\alpha+|x+y|)} \leqslant C < \infty$$

uniformly in λ by Inequality (B5a). Using Inequalities (B5a) and (B5b) and since if $|x| < \alpha$ we have that $\alpha + |x+y| \geqslant |y| + \alpha - |x| \geqslant |y|$, we get

$$\begin{aligned}
&\int\limits_{\alpha<|y|<K}\int\limits_{-\alpha}^{\alpha} \frac{\mathrm{d}x\,\mathrm{d}y}{(|z_1+z_2|x||-\theta_\varepsilon|x|)(|z_1+z_2|y||-\theta_\varepsilon|y|)(\alpha+|x+y|)} \\
&\quad\leqslant \int\limits_{\alpha<|y|<K}\int\limits_{-\alpha}^{\alpha} \frac{\mathrm{d}x\,\mathrm{d}y}{|y|(\alpha+|x+y|)} \leqslant C\int\limits_{\alpha<|y|<K}\int\limits_{-\alpha}^{\alpha} \frac{\mathrm{d}x\,\mathrm{d}y}{|y|^2} \leqslant C\alpha\int\limits_{\alpha}^{\infty}\frac{\mathrm{d}y}{y^2} \leqslant C < \infty,
\end{aligned}$$

uniformly in λ.

By symmetry we also have

$$\int_{-\alpha}^{\alpha}\int_{\alpha<|y|<K}\frac{dx\,dy}{(|z_1+z_2|x||-\theta_\varepsilon|x|)(|z_1+z_2|y||-\theta_\varepsilon|y|)(\alpha+|x+y|)}\leqslant C<\infty.$$

uniformly in λ by symmetry and the previous integral.

Since $|-x| = |x|$ and $|-x-y| = |x+y|$, we have that the integral over the regions $(\alpha, K)^2$ and $(-K,-\alpha)^2$ must be equal. The same holds by symmetry for the integrals over the regions $(-K,-\alpha)\times(\alpha,K)$ and $(\alpha,K)\times(-K,-\alpha)$. Therefore

$$\begin{aligned}
&\int_{\alpha<|y|<K}\int_{\alpha<|x|<K}\frac{dx\,dy}{(|z_1+z_2|x||-\theta_\varepsilon|x|)(|z_1+z_2|y||-\theta_\varepsilon|y|)(\alpha+|x+y|)}\\
&=2\int_\alpha^K\int_\alpha^K\frac{dx\,dy}{(|z_1+z_2|x||-\theta_\varepsilon|x|)(|z_1+z_2|y||-\theta_\varepsilon|y|)(\alpha+|x+y|)}\\
&\quad+2\int_{-K}^{-\alpha}\int_\alpha^K\frac{dx\,dy}{(|z_1+z_2|x||-\theta_\varepsilon|x|)(|z_1+z_2|y||-\theta_\varepsilon|y|)(\alpha+|x+y|)}\\
&\equiv 2I_1(\lambda)+2I_2(\lambda).
\end{aligned}$$

Then for I_1 we have

$$\begin{aligned}
I_1(\lambda)&\leqslant C\int_\alpha^K\int_\alpha^K\frac{dx\,dy}{(|x||y|(\alpha+|x+y|)}\leqslant\int_\alpha^K\int_\alpha^K\frac{dx\,dy}{xy(x+y)}\\
&\leqslant C\int_\alpha^K\int_\alpha^K\frac{dx\,dy}{xy\sqrt{xy}}\leqslant C\int_\alpha^\infty\int_\alpha^\infty\frac{dx\,dy}{xy\sqrt{xy}}\\
&\leqslant C\alpha^{-1}\leqslant C<\infty.
\end{aligned}$$

For the second integral we calculate

$$\begin{aligned}
I_2(\lambda)&\leqslant C\int_{-K}^{-\alpha}\int_\alpha^K\frac{dx\,dy}{|x||y|(\alpha+|x+y|)}\leqslant C\int_{-K}^{-\alpha}\int_\alpha^K\frac{dx\,dy}{x(-y)(\alpha+|x+y|)}\\
&=C\int_\alpha^K\int_\alpha^K\frac{dx\,dy}{xy(\alpha+|x-y|)}\\
&=C\int_\alpha^K\int_\alpha^y\frac{dx\,dy}{xy(\alpha+y-x)}+C\int_\alpha^K\int_y^K\frac{dx\,dy}{xy(x+\alpha-y)}\\
&\leqslant C\int_\alpha^K\frac{\log(y/\alpha)\,dy}{y(y+\alpha)}+C\int_\alpha^K\frac{\log(y/\alpha)\,dy}{y(y-\alpha)}\leqslant C<\infty.
\end{aligned}$$

These two facts together imply that $F_2(\lambda) < C < \infty$, uniformly in λ. □

Lemma A.2.3.

$$\iint_{|x|<K<|y|} \frac{dx\,dy}{|z_1 + z_2|x|||z_1 + z_2|y|||z_1 + z_2|x+y||} \leqslant CK^{-1}(C + \log(K)).$$

Proof. Let $\alpha > 0$ be a fixed constant. We can choose this to be as small as we need.

$$\begin{aligned}
&\iint_{|x|<K<|y|} \frac{dx\,dy}{|z_1 + z_2|x|||z_1 + z_2|y|||z_1 + z_2|x+y||} = \\
&\quad \iint_{\substack{|x|<K<|y|\\|x+y|<\alpha}} \frac{dx\,dy}{|z_1 + z_2|x|||z_1 + z_2|y|||z_1 + z_2|x+y||} \\
&\quad\quad + \iint_{\substack{|x|<K<|y|\\|x+y|\geqslant\alpha}} \frac{dx\,dy}{|z_1 + z_2|x|||z_1 + z_2|y|||z_1 + z_2|x+y||} \\
&\quad = H_1 + H_2.
\end{aligned}$$

We first consider H_1. Then observe that

$$\begin{aligned}
|H_1| &\leqslant C \iint_{\substack{|x|<K<|y|\\|x+y|<\alpha}} \frac{dx\,dy}{|z_1 + z_2|x|||y||z_1 + z_2|x+y||} \\
&\leqslant C \int_{|y|>K} \int_{x\in B_\alpha(-y)} \frac{dx\,dy}{|z_1 + z_2|x|||y||z_1 + z_2|x+y||} \\
&\leqslant C \int_{|y|>K} \int_{x\in B_\alpha(0)} \frac{dx\,dy}{|z_1 + z_2|x-y|||y||z_1 + z_2|x||} \\
&\leqslant C \int_{|y|>K} \int_{x\in B_\alpha(0)} \frac{dx\,dy}{|y-x||y|} \\
&\leqslant C \int_{|y|>K} \int_{x\in B_\alpha(0)} \frac{dx\,dy}{(|y|-|x|)|y|} \\
&\leqslant C \int_{|y|>K} \int_{x\in B_\alpha(0)} \frac{dx\,dy}{(|y|-\alpha)|y|} \\
&\leqslant C \int_K^\infty \frac{dy}{(y-\alpha)y} \\
&\leqslant C \int_{K-\alpha}^\infty \frac{dy}{(y+\alpha)y} \\
&\leqslant C \int_{K-\alpha}^\infty \frac{dy}{y^2} \leqslant C(K-\alpha)^{-1} \leqslant CK^{-1}.
\end{aligned}$$

Now let us move on to H_2.

$$
\begin{aligned}
|H_2| &\leqslant C \iint\limits_{\substack{|x|<K<|y| \\ |x+y|\geqslant\alpha}} \frac{\mathrm{d}x\,\mathrm{d}y}{|z_1+z_2|x|||y||z_1+z_2|x+y||} \\
&\leqslant C \iint\limits_{\substack{|x|<K<|y| \\ |x+y|\geqslant\alpha}} \frac{\mathrm{d}x\,\mathrm{d}y}{|z_1+z_2|x|||y|(\alpha+|y+x|)} \\
&\leqslant C \int\limits_{|y|>K} \int\limits_{|x|<\alpha} \frac{\mathrm{d}x\,\mathrm{d}y}{|z_1+z_2|x|||y|(\alpha+|y+x|)} \\
&\quad + C \int\limits_{|y|>K} \int\limits_{\alpha\leqslant|x|<K} \frac{\mathrm{d}x\,\mathrm{d}y}{|z_1+z_2|x|||y|(\alpha+|y+x|)} \\
&= H_2^{(1)} + H_2^{(2)}.
\end{aligned}
$$

For the first integral we have

$$
\begin{aligned}
H_2^{(1)} &\leqslant C \int\limits_{|y|>K} \int\limits_{|x|<\alpha} \frac{\mathrm{d}x\,\mathrm{d}y}{|y|(\alpha+|y|-|x|)} \\
&\leqslant C \int\limits_{|y|>K} \int\limits_{|x|<\alpha} \frac{\mathrm{d}x\,\mathrm{d}y}{|y|(\alpha+|y|-\alpha)} \\
&\leqslant C \int_K^\infty y^{-2}\,\mathrm{d}y \leqslant CK^{-1}.
\end{aligned}
$$

For the second integral we have

$$
\begin{aligned}
H_2^{(2)} &\leqslant C \int\limits_{|y|>K} \int\limits_{\alpha\leqslant|x|<K} \frac{\mathrm{d}x\,\mathrm{d}y}{|x||y|(\alpha+|y|-|x|)} \\
&\leqslant C \int_K^\infty \int_\alpha^K \frac{\mathrm{d}x\,\mathrm{d}y}{xy(\alpha+y-x)} \\
&\leqslant C \int_K^\infty \frac{1}{y^2} \log\left(\frac{Ky}{a(a+y-K)}\right) \mathrm{d}y
\end{aligned}
$$

and since $y > K$

$$
\leqslant C \int_K^\infty \frac{1}{y^2} \log\left(\frac{y^2}{a^2}\right) \mathrm{d}y \leqslant CK^{-1}\log(K). \qquad \square
$$

Lemma A.2.4.

$$
\int\limits_{|y|>K} \int\limits_{|x|>K} \frac{\mathrm{d}x\,\mathrm{d}y}{|z_1+z_2|x|||z_1+z_2|y|||z_1+z_2|x+y||} \leqslant CK^{-1}(C+\log(K)).
$$

Proof. We first split it into two parts

$$
\int\limits_{|y|>K} \int\limits_{|x|>K} \frac{\mathrm{d}x\,\mathrm{d}y}{|z_1+z_2|x|||z_1+z_2|y|||z_1+z_2|x+y||}
$$

$$\leqslant \iint\limits_{\substack{|y|,|x|>K \\ |x+y|<\alpha}} \frac{dx\,dy}{|z_1+z_2|x|||z_1+z_2|y|||z_1+z_2|x+y||}$$

$$+ \iint\limits_{\substack{|y|,|x|>K \\ |x+y|\geqslant\alpha}} \frac{dx\,dy}{|z_1+z_2|x|||z_1+z_2|y|||z_1+z_2|x+y||} \equiv H_1 + H_2.$$

Let us first consider H_1.

$$\begin{aligned}
H_1 &\leqslant C \int\limits_{|y|>K} \int\limits_{x\in B_\alpha(-y)} \frac{dx\,dy}{|z_1+z_2|x|||z_1+z_2|y|||z_1+z_2|x+y||} \\
&\leqslant C \int\limits_{|y|>K} \int\limits_{x\in B_\alpha(0)} \frac{dx\,dy}{|z_1+z_2|x-y|||z_1+z_2|y|||z_1+z_2|x||} \\
&\leqslant C \int\limits_{|y|>K} \int\limits_{x\in B_\alpha(0)} \frac{dx\,dy}{|y||y-x|} \leqslant C \int\limits_{|y|>K} \int\limits_{x\in B_\alpha(0)} \frac{dx\,dy}{|y|(|y|-|x|)} \\
&\leqslant C \int\limits_{|y|>K} \int\limits_{x\in B_\alpha(0)} \frac{dx\,dy}{|y|(|y|-\alpha)} \\
&\leqslant C \int_{K-\alpha}^{\infty} \frac{dx\,dy}{y(y+\alpha)} \leqslant CK^{-1} = C|1-\lambda|.
\end{aligned}$$

Next we consider H_2,

$$\begin{aligned}
H_2 \leqslant C \int_K^\infty \int_K^\infty \frac{dx\,dy}{|x||y|(\alpha+|x+y|)} &+ \int_{-\infty}^{-K} \int_{-\infty}^{-K} \frac{dx\,dy}{|x||y|(\alpha+|x+y|)} \\
+ C \int_K^\infty \int_{-\infty}^{-K} \frac{dx\,dy}{|x||y|(\alpha+|x+y|)} &+ C \int_{-\infty}^{-K} \int_K^\infty \frac{dx\,dy}{|x||y|(\alpha+|x+y|)}.
\end{aligned}$$

Then integral over $(K,\infty)^2$ is the same. Finally observe that

$$\int_{-\infty}^{-K} \int_{-\infty}^{-K} \frac{dx\,dy}{|x||y|(\alpha+|x+y|)} = \int_K^\infty \int_K^\infty \frac{dx\,dy}{xy(\alpha+x+y)} = \int_K^\infty \int_K^\infty \frac{dx\,dy}{xy\sqrt{xy}} \leqslant CK^{-1}.$$

Now

$$\begin{aligned}
&\int_K^\infty \int_{-\infty}^{-K} \frac{dx\,dy}{|x||y|(\alpha+|x+y|)} \\
&= \int_K^\infty \int_{-\infty}^{-K} \frac{dx\,dy}{(-x)y(\alpha+|x+y|)} = \int_K^\infty \int_K^\infty \frac{dx\,dy}{xy(\alpha+|y-x|)} \\
&= \int_K^\infty \int_K^y \frac{dx\,dy}{xy(\alpha+y-x)} + \int_K^\infty \int_y^\infty \frac{dx\,dy}{xy(\alpha+x-y)} \\
&\leqslant \int_K^\infty \log\left(\frac{a+y-K}{aK}\right) \frac{dy}{y^2} + \int_K^\infty y^{-2} \log(\tfrac{y}{a})\,dy \\
&\leqslant CK^{-1}(C+\log(K)).
\end{aligned}$$

□

A.3 Error analysis for Theorem 3.1.3(ii)

A.3.1 First term error analysis

We now estimate the error arising from replacing the factors in the integral

$$\iint_{U_\epsilon} \frac{\lambda f(s)(1-f(t))\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))} \tag{A.3.1}$$

by their Taylor expansions. In other words we would like to find an upper bound for the quantity

$$|E(\lambda)| = C|1-\lambda|^{-2}\left|\iint_{U_\epsilon} \frac{\lambda f(s)(1-f(t))}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))}\right.$$
$$\left. - \iint_{U_\epsilon} \frac{\frac{\lambda}{2}|t|_\Sigma\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)^2(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}(|t|_\Sigma+|s|_\Sigma))}\right|.$$

To simplify calculations we telescope the difference into a sum of errors arising from replacing each factor consecutively and we use the fact that

$$|E| \leqslant C|1-\lambda|^{-2}\left(|E_1|+|E_2|+|E_3|+|E_4|\right),$$

where the E_j are defined as follows

$$E_1 = \iint_{U_\epsilon} \frac{\lambda f(s)(1-f(t))\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))} - \iint_{U_\epsilon} \frac{\frac{\lambda}{2}|t|_\Sigma\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))},$$

$$E_2 = \iint_{U_\epsilon} \frac{\frac{\lambda}{2}|t|_\Sigma\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))} - \iint_{U_\epsilon} \frac{\frac{\lambda}{2}|t|_\Sigma\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)^2(1-\lambda f(s))(1-\lambda f(t)f(s))},$$

$$E_3 = \iint_{U_\epsilon} \frac{\frac{\lambda}{2}|t|_\Sigma\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)^2(1-\lambda f(s))(1-\lambda f(t)f(s))} - \iint_{U_\epsilon} \frac{\frac{\lambda}{2}|t|_\Sigma\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)^2(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda f(t)f(s))},$$

$$E_4 = \iint_{U_\epsilon} \frac{\frac{\lambda}{2}|t|_\Sigma\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)^2(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda f(t)f(s))} - \iint_{U_\epsilon} \frac{\frac{\lambda}{2}|t|_\Sigma\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)^2(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}(|t|_\Sigma+|s|_\Sigma))}.$$

First error In the first error we replace the numerator by $\lambda|t|_\Sigma/2$ and thus the error is

$$|E_1| = \left|\iint_{U_\varepsilon} \frac{[\lambda f(s)(1-f(t)) - \frac{\lambda}{2}|t|_\Sigma]\,dt\,ds}{(1-\lambda f(t))^2(1-\lambda f(s))(1-\lambda f(t)f(s))}\right|$$
$$\leqslant \iint_{U_\varepsilon} \frac{|\lambda f(s)(1-f(t)) - \frac{\lambda}{2}|t|_\Sigma|\,dt\,ds}{|1-\lambda f(t)|^2|1-\lambda f(s)||1-\lambda f(t)f(s)|}.$$

Let us consider the numerator first. It can be shown that

$$\left|\lambda f(s)(1-f(t)) - \tfrac{\lambda}{2}|t|_\Sigma\right| \leqslant \theta_\varepsilon|t|_\Sigma + C|t|_\Sigma|s|_\Sigma \leqslant C(\varepsilon)|t|_\Sigma,$$

where $C(\varepsilon) > 0$ tends to 0 as $\varepsilon \to 0$.

We change variables $\Sigma^{1/2}t \mapsto t$. Then by Inequalities (3.3.4) and (3.3.5) and scaling t and s by $1/|1-\lambda|$ we get

$$|E_1| \leqslant \frac{C(\varepsilon)}{|\Sigma||1-\lambda|}\times$$
$$\iint_{U_K} \frac{|t|^2\,dt\,ds}{\left||z_1+z_2|t|^2| - \theta_\varepsilon|t|^2\right|^2\left||z_1+z_2|s|^2| - \theta_\varepsilon|s|^2\right|\left||z_1+z_2(|t|^2+|s|^2)| - \Delta_\varepsilon(|t|^2+|s|^2)\right|},$$

where $z_1 = (1-\lambda)/|1-\lambda|$, $z_2 = \lambda/2$, and $K = K(\lambda,\varepsilon) = \varepsilon/|1-\lambda|$.

From Inequality (3.3.8) we get

$$\frac{|t|^2}{\left||z_1+z_2|t|^2| - \theta_\varepsilon|t|^2\right|} \leqslant C,$$

and thus after changing to polar coordinates we have

$$|E_1| \leqslant C(\varepsilon)|1-\lambda|^{-1}$$
$$\times \int_0^K\!\!\int_0^K \frac{rs\,dr\,ds}{\left||z_1+z_2r^2| - \theta_\varepsilon r^2\right|\left||z_1+z_2s^2| - \theta_\varepsilon s^2\right|\left||z_1+z_2(r^2+s^2)| - \Delta_\varepsilon(r^2+s^2)\right|}$$
$$= C(\varepsilon)|1-\lambda|^{-1}F(\lambda) \leqslant C(\varepsilon)|1-\lambda|^{-1},$$

where $F(\lambda)$, as defined in (A.4.9), is uniformly bounded by Lemma A.4.2.

Second error. We now replace the factor $1-\lambda f(t)$ in the denominator. Observe that

$$\left|(1-\lambda f(t))^2 - (1-\lambda+\frac{\lambda}{2}|t|_\Sigma)^2\right|$$
$$\leqslant \theta_\varepsilon|t|_\Sigma\left|1-\lambda+\tfrac{\lambda}{2}|t|_\Sigma\right| + \theta_\varepsilon^2|t|_\Sigma^2.$$

Using this, Inequalities (3.3.4) and (3.3.5) and the usual change of variables

$$|E_2| \leqslant C(\varepsilon) \iint\limits_{U_\varepsilon} \frac{|t|_\Sigma^2 |1-\lambda+\frac{\lambda}{2}|t|_\Sigma| + |t|_\Sigma^3 \, dt \, ds}{|1-\lambda+\frac{\lambda}{2}|t|_\Sigma|^2 |1-\lambda f(s)||1-\lambda f(t) f(s)||1-\lambda f(t)|^2}$$

$$\leqslant C(\varepsilon) \iint\limits_{U_\varepsilon} \frac{|t|^4 |1-\lambda+\frac{\lambda}{2}|t|^2| + |t|^6}{\left||1-\lambda+\frac{\lambda}{2}|s|^2| - \theta_\varepsilon |s|^2\right| \left||1-\lambda+\frac{\lambda}{2}|t|^2| - \theta_\varepsilon |t|^2\right|^2}$$

$$\times \frac{dt \, ds}{\left|1-\lambda+\frac{\lambda}{2}|t|^2\right|^2 \left||1-\lambda+\frac{\lambda}{2}(|t|^2+|s|^2)| - \Delta_\varepsilon(|t|^2+|s|^2)\right|}.$$

Then using Inequality (3.3.8) for each summand in the numerator and the inequality

$$\left|1-\lambda+\frac{\lambda}{2}|t|^2\right| \geqslant C|t|^2,$$

which follows from using the real part as a lower bound, we have

$$\frac{|t|^2 |1-\lambda+\frac{\lambda}{2}|t|^2| + |t|^4}{\left|1-\lambda+\frac{\lambda}{2}|t|^2\right|^2} \leqslant C,$$

and therefore

$$|E_2| \leqslant C(\varepsilon) \iint\limits_{U_\varepsilon} \frac{|t|^2}{\left||1-\lambda+\frac{\lambda}{2}|s|^2| - \theta_\varepsilon |s|^2\right| \left||1-\lambda+\frac{\lambda}{2}|t|^2| - \theta_\varepsilon |t|^2\right|^2}$$

$$\times \frac{dt \, ds}{\left||1-\lambda+\frac{\lambda}{2}(|t|^2+|s|^2)| - \Delta_\varepsilon(|t|^2+|s|^2)\right|}.$$

Then by Inequality (3.3.4), polar coordinates and scaling t and s by $|1-\lambda|$

$$|E_2| \leqslant C(\varepsilon)|1-\lambda|^{-1} \int_0^K \int_0^K \frac{rs}{\left||z_1+z_2 r^2| - \theta_\varepsilon r^2\right| \left||z_1+z_2 s^2| - \theta_\varepsilon s^2\right|}$$

$$\times \frac{dt \, ds}{\left||z_1+z_2(r^2+s^2)| - \Delta_\varepsilon(r^2+s^2)\right|},$$

where $K = \varepsilon/|1-\lambda|$. Finally by Lemma A.4.2 we have $|E_2| \leqslant C(\varepsilon)|1-\lambda|^{-1}$.

Third error. The next error comes from replacing $1-\lambda f(s)$ in the denominator. It is an easy calculation to check that for $|t| < \varepsilon$

$$\left|1-\lambda f(s) - (1-\lambda+\frac{\lambda}{2}|s|_\Sigma)\right| \leqslant C(\varepsilon)|s|_\Sigma.$$

Using the above, the usual changes of variables and Inequality (3.3.8)

$$|E_3| \leqslant C \iint\limits_{U_\varepsilon} \frac{|s|_\Sigma |t|_\Sigma \, dt \, ds}{\left|1-\lambda+\frac{\lambda}{2}|t|_\Sigma\right|^2 |1-\lambda+\frac{\lambda}{2}|s|_\Sigma||1-\lambda f(t)f(s)||1-\lambda f(s)|}$$

$$\leqslant C(\varepsilon)|1-\lambda|^{-1} \times \int_0^K \int_0^K \frac{rs \, dr \, ds}{|z_1+z_2 r^2| \left||z_1+z_2(r^2+s^2)| - \Delta_\varepsilon(r^2+s^2)\right| \left||z_1+z_2 s^2| - \theta_\varepsilon s^2\right|},$$

and finally by a minor modification to Lemma A.4.2, $|E_3| \leqslant C(\varepsilon)|1-\lambda|^{-1}$.

Fourth Error. Finally the last error arises from replacing the term $1-\lambda f(t)f(s)$ in the denominator by its expansion. Using (3.1.8) we obtain the following bound

$$\left|1-\lambda f(t)f(s)-\left[1-\lambda+\frac{\lambda}{2}(|t|_\Sigma+|s|_\Sigma)\right]\right| \leqslant C(\varepsilon)(|t|_\Sigma+|s|_\Sigma).$$

Then we have the following bound using the usual change of variables $\Sigma^{1/2}t \mapsto t$,

$$\begin{aligned}
|E_4| \leqslant C(\varepsilon) &\iint\limits_{U_\varepsilon} \frac{1}{\left|1-\lambda+\frac{\lambda}{2}|t|_\Sigma\right|^2 \left|1-\lambda+\frac{\lambda}{2}|s|_\Sigma\right|} \\
&\times \frac{|t|_\Sigma(|t|_\Sigma+|t|_\Sigma)\,\mathrm{d}t\,\mathrm{d}s}{\left|1-\lambda+\frac{\lambda}{2}(|t|_\Sigma+|s|_\Sigma)\right|\,|1-\lambda f(t)f(s)|} \\
\leqslant C(\varepsilon) &\iint\limits_{U_\varepsilon} \frac{1}{\left|1-\lambda+\frac{\lambda}{2}|t|^2\right|^2 \left|1-\lambda+\frac{\lambda}{2}|s|^2\right|} \\
&\times \frac{|t|^2(|t|^2+|s|^2)\,\mathrm{d}t\,\mathrm{d}s}{\left|1-\lambda+\frac{\lambda}{2}(|t|^2+|s|^2)\right|\,\left|\left|1-\lambda+\frac{\lambda}{2}(|t|^2+|s|^2)\right|-\Delta_\varepsilon(|t|^2+|s|^2)\right|}.
\end{aligned}$$

Using the above bounds, Inequalities (3.3.4) and (3.3.5), changing to polar coordinates and scaling by $|1-\lambda|$ we have

$$\begin{aligned}
&|E_4| \\
&\leqslant C(\varepsilon) \iint\limits_{U_\varepsilon} \frac{\mathrm{d}t\,\mathrm{d}s}{\left|1-\lambda+\frac{\lambda}{2}|t|^2\right|\left|1-\lambda+\frac{\lambda}{2}|s|^2\right|\left|\left|1-\lambda+\frac{\lambda}{2}(|t|^2+|s|^2)\right|-\Delta_\varepsilon(|t|^2+|s|^2)\right|} \\
&= C(\varepsilon)|1-\lambda|^{-1} \int_0^K\!\!\int_0^K \frac{\Delta_\varepsilon r s\,\mathrm{d}r\,\mathrm{d}s}{|z_1+z_2r^2|\,|z_1+z_2s^2|\,\left||z_1+z_2(r^2+s^2)|-\Delta_\varepsilon(r^2+s^2)\right|} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-1}
\end{aligned}$$

by a minor modification of Lemma A.4.2.

Thus all the errors are bounded above by $C(\varepsilon)|1-\lambda|^{-1}$ with $C(\varepsilon)\to 0$ as $\varepsilon\to 0$ and therefore

$$|E(\lambda)| \leqslant |1-\lambda|^{-2}(|E_1|+|E_2|+|E_3|+|E_4|) \leqslant C(\varepsilon)|1-\lambda|^{-3}.$$

A.3.2 Second term error analysis

Errors in first integral

We estimate the error arising from using Taylor in the integral $Y_1(\lambda)$. That is we would like to have an upper bound for the quantity

$$|E| \leqslant |1-\lambda|^{-2}\left|\iint\limits_{U_\varepsilon} \frac{f(t)f(t+s)\,\mathrm{d}t\,\mathrm{d}s}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t+s))}\right.$$

$$- \frac{dt\, ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}|t+s|_\Sigma)}\Bigg|.$$

Similarly to the first term we telescope the difference,

$$|E| \leqslant |1-\lambda|^{-2}(|E_1|+|E_2|+|E_3|+|E_4|),$$

where we define

$$\begin{aligned}
E_1 &= \Bigg|\iint_{U_\varepsilon} \frac{f(t)f(t+s)\, dt\, ds}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t+s))} \\
&\qquad - \frac{dt\, ds}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t+s))}\Bigg|, \\
E_2 &= \Bigg|\iint_{U_\varepsilon} \frac{dt\, ds}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t+s))} \\
&\qquad - \frac{dt\, ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda f(s))(1-\lambda f(t+s))}\Bigg|, \\
E_3 &= \Bigg|\iint_{U_\varepsilon} \frac{dt\, ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda f(s))(1-\lambda f(t+s))} \\
&\qquad - \frac{dt\, ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda f(t+s))}\Bigg|, \\
E_4 &= \Bigg|\iint_{U_\varepsilon} \frac{dt\, ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda f(t+s))} \\
&\qquad - \frac{dt\, ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}|t+s|_\Sigma)}\Bigg|.
\end{aligned}$$

First error. We first consider the error from replacing the denominator $f(t)f(t+s)$ by 1.

$$|E_1| \leqslant \iint_{U_\varepsilon} \frac{|f(t)f(t+s)-1|\, dt\, ds}{|1-\lambda f(t)|\,|1-\lambda f(s)|\,|1-\lambda f(t+s)|}.$$

Using (3.1.8) and the fact that $R(t) = o(|t|^2)$ as $t \to 0$, we bound the numerator

$$\begin{aligned}
|f(t)f(t+s)-1| &= \left|\left(1-\frac{1}{2}|t|_\Sigma + R(t)\right)\left(1-\frac{1}{2}|t+s|_\Sigma + R(t+s)\right) - 1\right| \\
&\leqslant C(\varepsilon) \to 0, \text{ as } \varepsilon \to 0.
\end{aligned}$$

Using the above and a change of variables,

$$\begin{aligned}
|E_1| &\leqslant C(\varepsilon)|1-\lambda|^{-1}|\Sigma|^{-1} \\
&\quad\times \iint_{U_K} \frac{dt\, ds}{\big||z_1+z_2|t|^2| - \Theta_\varepsilon|t|^2\big|\,\big||z_1+z_2|s|^2| - \Theta_\varepsilon|s|^2\big|\,\big||z_1+z_2|t+s|^2| - \Theta_\varepsilon|t+s|^2\big|} \\
&= |1-\lambda|^{-1}C(\varepsilon)G(\lambda) \leqslant C(\varepsilon)|1-\lambda|^{-1},
\end{aligned}$$

where $G(\lambda)$ is shown to be uniformly bounded in Lemma A.4.1.

Second error. We proceed by replacing the t term in the denominator by its expansion. By the same calculations as above,

$$
\begin{aligned}
|E_2| &\leqslant C(\varepsilon) \iint_{U_\varepsilon} \frac{|t|_\Sigma \, dt\, ds}{|1-\lambda f(t)|\,|1-\lambda f(s)|\,|1-\lambda f(t+s)|\,\left|1-\lambda+\frac{\lambda}{2}|t|_\Sigma\right|} \\
&\leqslant C(\varepsilon) \iint_{U_\varepsilon} \frac{1}{\left|\left|1-\lambda+\frac{\lambda}{2}|t|^2\right|-\theta_\varepsilon|t|^2\right|\left|\left|1-\lambda+\frac{\lambda}{2}|s|^2\right|-\theta_\varepsilon|s|^2\right|} \\
&\quad\times \frac{dt\,ds}{\left|\left|1-\lambda+\frac{\lambda}{2}|t+s|^2\right|-\theta_\varepsilon|t+s|^2\right|} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-1} \\
&\quad\times \iint_{U_K} \frac{dt\,ds}{\left||z_1+z_2|t|^2|-\theta_\varepsilon|t|^2\right|\left||z_1+z_2|s|^2|-\theta_\varepsilon|s|^2\right|\left||z_1+z_2|t+s|^2|-\theta_\varepsilon|t+s|^2\right|} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-1}
\end{aligned}
$$

by Lemma A.4.1.

Third error. By similar calculations we have

$$
\begin{aligned}
|E_3| &\leqslant C(\varepsilon) \iint_{U_\varepsilon} \frac{|s|^2\, dt\, ds}{\left|1-\lambda+\frac{\lambda}{2}|t|^2\right|\left|1-\lambda+\frac{\lambda}{2}|s|^2\right|\left|\left|1-\lambda+\frac{\lambda}{2}|s|^2\right|-\Theta_\varepsilon|s|^2\right|\left|1-\lambda f(t+s)\right|} \\
&= C(\varepsilon)|1-\lambda|^{-1} \\
&\quad\times \iint_{U_K} \frac{dt\,ds}{|z_1+z_2|t|^2|\,\left||z_1+z_2|s|^2|-\Theta_\varepsilon|s|^2\right|\left||z_1+z_2|t+s|^2|-\Theta_\varepsilon|t+s|^2\right|} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-1}G(\lambda) \leqslant C(\varepsilon)|1-\lambda|^{-1}
\end{aligned}
$$

by Lemma A.4.1.

Fourth error. Finally we replace the $t+s$-term in the denominator.

$$
\begin{aligned}
|E_4| &\leqslant C(\varepsilon) \iint_{U_\varepsilon} \frac{|t+s|_\Sigma\, dt\, ds}{\left|1-\lambda+\frac{\lambda}{2}|t|_\Sigma\right|\left|1-\lambda+\frac{\lambda}{2}|s|_\Sigma\right|\left|1-\lambda+\frac{\lambda}{2}|t+s|_\Sigma\right|\left|1-\lambda f(t+s)\right|} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-1} \iint_{U_K} \frac{dt\,ds}{|z_1+z_2|t|^2|\,|z_1+z_2|s|^2|\,\left||z_1+z_2|t+s|^2|-\Theta_\varepsilon|t+s|^2\right|} \\
&\leqslant C(\varepsilon)|1-\lambda|^{-1},
\end{aligned}
$$

by minor modification to Lemma A.4.1.

Errors in the second integral

We now estimate the error arising from using Taylor in the integral $Y_2(\lambda)$. In other words we would like to bound

$$
\begin{aligned}
|E| = \iint_{U_\epsilon} & \frac{f(t)f(s)^2 \,dt\,ds}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t)f(s))} \\
& - \iint_{U_\epsilon} \frac{1 \,dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}(|t|_\Sigma+|s|_\Sigma))}.
\end{aligned}
$$

Once again we telescope the difference by replacing each factor one by one

$$|E| \leqslant |E_1| + |E_2| + |E_3| + |E_4|,$$

where we define

$$
\begin{aligned}
|E_1| = \iint_{U_\epsilon} & \frac{f(t)f(s)^2 \,dt\,ds}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t)f(s))} \\
& - \iint_{U_\epsilon} \frac{dt\,ds}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t)f(s))}, \\
|E_2| = \iint_{U_\epsilon} & \frac{dt\,ds}{(1-\lambda f(t))(1-\lambda f(s))(1-\lambda f(t)f(s))} \\
& - \iint_{U_\epsilon} \frac{1 \,dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda f(s))(1-\lambda f(t)f(s))}, \\
|E_3| = \iint_{U_\epsilon} & \frac{dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda f(s))(1-\lambda f(t)f(s))} \\
& - \iint_{U_\epsilon} \frac{1 \,dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda f(t)f(s))}, \\
|E_4| = \iint_{U_\epsilon} & \frac{dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda f(t)f(s))} \\
& - \iint_{U_\epsilon} \frac{dt\,ds}{(1-\lambda+\frac{\lambda}{2}|t|_\Sigma)(1-\lambda+\frac{\lambda}{2}|s|_\Sigma)(1-\lambda+\frac{\lambda}{2}(|t|_\Sigma+|s|_\Sigma))}.
\end{aligned}
$$

First error. By a trivial calculation we find that $|f(t)f(s)^2 - 1| \leqslant C(\varepsilon)$ and therefore by Inequalities (3.3.4) to (3.3.8) and changing to polar coordinates

$$
\begin{aligned}
|E_1| & \leqslant C(\epsilon)|1-\lambda|^{-1} \\
& \times \int_0^K\!\!\int_0^K \frac{rs \,dr\,ds}{\big||z_1+z_2r^2|-\Theta_\epsilon r^2\big|\big||z_1+z_2s^2|-\Theta_\epsilon s^2\big|\big||z_1+z_2(r^2+s^2)|-\Delta_\epsilon(r^2+s^2)\big|} \\
& \leqslant C(\epsilon)|1-\lambda|^{-1},
\end{aligned}
$$

by Lemma A.4.2.

Second error. This arises from replacing $1-\lambda f(t)$ in the denominator. Using (3.1.8) and the fact that $R(t)=o(|t|^2)$ one obtains the bound

$$1-\lambda f(t)-\left(1-\lambda+\frac{\lambda}{2}|t|^2\right)\leqslant C(\varepsilon)|t|^2,$$

and therefore by Inequalities (3.3.4) to (3.3.8)

$$\begin{aligned}|E_2| &\leqslant C(\varepsilon)|1-\lambda|^{-1}\iint\limits_{U_K}\frac{1}{\big||z_1+z_2|t|^2|-\theta_\varepsilon|t|^2\big|\,\big||z_1+z_2|s|^2|-\theta_\varepsilon|s|^2\big|}\\ &\quad\times\frac{dt\,ds}{\big||z_1+z_2\left(|t|^2+|s|^2\right)|-\Delta_\varepsilon\left(|t|^2+|s|^2\right)\big|}\\ &\leqslant C(\varepsilon)|1-\lambda|^{-1}\end{aligned}$$

by Lemma A.4.2.

Third error We replace the $1-\lambda f(s)$ term in the denominator. By the same calculations as above we obtain

$$\begin{aligned}|E_3| &\leqslant C(\varepsilon)|1-\lambda|^{-1}\\ &\quad\times\iint\limits_{U_\varepsilon}\frac{dt\,ds}{|z_1+z_2|t|^2|\,\big||z_1+z_2|s|^2|-\theta_\varepsilon|s|^2\big|\,\big||z_1+z_2(|t|^2+|s|^2)|-\theta_\varepsilon(|t|^2+|s|^2)\big|}\\ &\leqslant C(\varepsilon)|1-\lambda|^{-1}\end{aligned}$$

once again by Inequalities (3.3.4) to (3.3.8) and Lemma A.4.2.

Fourth error The last error term arises from replacing $1-\lambda f(t)f(s)$ in the denominator. By similar calculations and application of Inequalities (3.3.4) to (3.3.8) we have

$$\begin{aligned}|E_4| &\leqslant C(\varepsilon)|1-\lambda|^{-1}\iint\limits_{U_K}\frac{1}{|z_1+z_2|s|^2|\,\big||z_1+z_2(|t|^2+|s|^2)|-\theta_\varepsilon(|t|^2+|s|^2)\big|}\\ &\quad\times\frac{(|t|^2+|s|^2)\,dt\,ds}{|z_1+z_2|t|^2|\,|z_1+z_2(|t|^2+|s|^2)|}\\ &\leqslant C(\varepsilon)|1-\lambda|^{-1},\end{aligned}$$

by a slight modification of Lemma A.4.2.

A.4 Integral Calculations for Theorem 3.1.3(ii)

Lemma A.4.1. *For* $z_1 = (1-\lambda)/|1-\lambda|$, $z_2 = \lambda/2$, $\Re(\lambda) > \alpha$ *for some* $\alpha \in (0,1)$

$$G(\lambda) = \iint\limits_{U_K} \frac{1}{\big||z_1 + z_2|x|^2| - \theta_\varepsilon|x|^2\big|\big||z_1 + z_2|y|^2| - \theta_\varepsilon|y|^2\big|} \times \frac{\mathrm{d}x\,\mathrm{d}y}{\big||z_1 + z_2|x+y|^2| - \theta_\varepsilon|x+y|^2\big|} \leqslant C \tag{A.4.1}$$

uniformly in λ.

Proof. Fix some small $\beta > 0$. We split the region of integration in two parts. We first consider the integral in the region $|x+y| < \beta$ for some fixed positive $\beta > 0$, which we can choose to be as small as we desire. On this region we have, changing to polar coordinates,

$$\begin{aligned}
&\iint\limits_{\substack{U_K\\ |x+y|<\beta}} \frac{\mathrm{d}y\,\mathrm{d}x}{\big||z_1 + z_2|x|^2| - \theta_\varepsilon|x|^2\big|\,\big||z_1 + z_2|y|^2| - \theta_\varepsilon|y|^2\big|\,\big||z_1 + z_2|x+y|^2| - \theta_\varepsilon|x+y|^2\big|}\\
&= \iint\limits_{B_K\times B_\beta(-x)} \frac{\mathrm{d}y\,\mathrm{d}x}{\big||z_1 + z_2|x|^2| - \theta_\varepsilon|x|^2\big|\,\big||z_1 + z_2|y|^2| - \theta_\varepsilon|y|^2\big|\,\big||z_1 + z_2|x+y|^2| - \theta_\varepsilon|x+y|^2\big|}\\
&= \iint\limits_{B_K\times B_\beta} \frac{\mathrm{d}y\,\mathrm{d}x}{\big||z_1 + z_2|x|^2| - \theta_\varepsilon|x|^2\big|\,\big||z_1 + z_2|y|^2| - \theta_\varepsilon|y|^2\big|\,\big||z_1 + z_2|y-x|^2| - \theta_\varepsilon|y-x|^2\big|}\\
&\leqslant C + C\int_\beta^K\int_0^\beta\int_0^\pi \frac{rs\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}r}{\big||z_1 + z_2r^2| - \theta_\varepsilon r^2\big|\,\big||z_1 + z_2s^2| - \theta_\varepsilon s^2\big|\,\big||z_1 + z_2f(r,s,t))| - \theta_\varepsilon f(r,s,t)\big|}
\end{aligned}$$

where

$$f(r,s,t) = r^2 + s^2 + 2rs\cos(t) = (r-s)^2 + 2rs(1-\cos(t)).$$

Now let $\delta < \beta$ be a small fixed positive constant. We can choose this to be as small as we want. We split the integral

$$\begin{aligned}
&\int_\beta^K\int_0^\beta\int_0^{2\pi} \frac{rs\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}r}{\big||z_1 + z_2r^2| - \theta_\varepsilon r^2\big|\,\big||z_1 + z_2s^2| - \theta_\varepsilon s^2\big|\,\big||z_1 + z_2f(r,s,t))| - \theta_\varepsilon f(r,s,t)\big|}\\
&= \int_{\beta+\delta}^K\int_0^\beta\int_0^{2\pi} \frac{rs\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}r}{\big||z_1 + z_2r^2| - \theta_\varepsilon r^2\big|\,\big||z_1 + z_2s^2| - \theta_\varepsilon s^2\big|\,\big||z_1 + z_2f(r,s,t))| - \theta_\varepsilon f(r,s,t)\big|}\\
&+ \int_0^{\beta+\delta}\int_0^\beta\int_0^{2\pi} \frac{rs\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}r}{\big||z_1 + z_2r^2| - \theta_\varepsilon r^2\big|\,\big||z_1 + z_2s^2| - \theta_\varepsilon s^2\big|\,\big||z_1 + z_2f(r,s,t))| - \theta_\varepsilon f(r,s,t)\big|}\\
&= H_1 + H_2.
\end{aligned}$$

For H_2, since $r, s < \beta + \delta$, we can find lower bounds for all the factors of the denominator using

Inequality (3.3.7). For the term including both r and s we have

$$\begin{aligned}\big||z_1 + z_2 f(r,s,t))| - \theta_\varepsilon f(r,s,t)\big| &\geqslant |z_1 + z_2 f(r,s,t))| - \theta_\varepsilon f(r,s,t) &\text{(A.4.2)}\\ &\geqslant 1 - \left(\frac{1}{2} + \theta_\varepsilon\right)(r^2 + s^2 + 2rs\cos(t)) &\text{(A.4.3)}\\ &\geqslant 1 - \left(\frac{1}{2} + \theta_\varepsilon\right)(r+s)^2 &\text{(A.4.4)}\\ &\geqslant 1 - 4\left(\frac{1}{2} + \theta_\varepsilon\right)(\beta+\delta)^2 \geqslant C > 0, &\text{(A.4.5)}\end{aligned}$$

and therefore $|H_2| \leqslant C < \infty$.

For H_1 we use Inequalities (3.3.7) and (3.3.8)

$$\begin{aligned}\big||z_1 + z_2 f(r,s,t))| - \theta_\varepsilon f(r,s,t)\big| &\geqslant |z_1 + z_2 f(r,s,t))| - \theta_\varepsilon f(r,s,t) &\text{(A.4.6)}\\ &\geqslant C\left((r-s)^2 + 2rs(1-\cos(t))\right) &\text{(A.4.7)}\\ &\geqslant C(r-s)^2. &\text{(A.4.8)}\end{aligned}$$

Thus we have

$$|H_1| \leqslant C \int_{\beta+\delta}^{K}\int_{0}^{\beta} \frac{rs\,\mathrm{d}s\,\mathrm{d}r}{r^2(r-s)^2} \leqslant C \int_{\beta+\delta}^{K}\int_{0}^{\beta} \frac{s\,\mathrm{d}s\,\mathrm{d}r}{r(r-s)^2},$$

which is finite by Fubini's Theorem and the fact that

$$C\int_{0}^{\beta}\int_{\beta+\delta}^{K} \frac{s\,\mathrm{d}r\,\mathrm{d}s}{r(r-s)^2} \leqslant C\int_{0}^{\beta}\int_{\delta}^{\infty} \frac{s\,\mathrm{d}r\,\mathrm{d}s}{(r+s)r^2} \leqslant C\int_{0}^{\beta}\int_{\delta}^{\infty} \frac{s\,\mathrm{d}r\,\mathrm{d}s}{r^3} < \infty.$$

We are left to show that the integral on the region $|k_1 + k_2| \geqslant \beta$ is finite.

Let

$$\begin{aligned} D_1 &= \left\{(x,y) \in \mathbf{R}^2 \times \mathbf{R}^2 : |x| < \beta, |y| < \beta, |x+y| \geqslant \beta\right\},\\ D_2 &= \left\{(x,y) \in \mathbf{R}^2 \times \mathbf{R}^2 : |x| < \beta, \beta \leqslant |y| \leqslant K, |x+y| \geqslant \beta\right\},\\ D_3 &= \left\{(x,y) \in \mathbf{R}^2 \times \mathbf{R}^2 : \beta \leqslant |x| \leqslant K, |y| < \beta, |x+y| \geqslant \beta\right\},\\ D_4 &= \left\{(x,y) \in \mathbf{R}^2 \times \mathbf{R}^2 : \beta \leqslant |x| \leqslant K, \beta \leqslant |y| \leqslant K, |x+y| \geqslant \beta\right\}.\end{aligned}$$

Over D_1 all the factors of the denominator are bounded below by Inequality (3.3.7); therefore the integral is finite. Note that since $|x+y| \geqslant \beta$, we have that

$$|x+y|^2 \geqslant C + C|x+y|^2.$$

By Inequality (3.3.8)

$$\begin{aligned}
&\iint_{D_2} \frac{dy\,dx}{\left||z_1+z_2|x|^2|-\theta_\varepsilon|x|^2\right|\left||z_1+z_2|y|^2|-\theta_\varepsilon|y|^2\right|\left||z_1+z_2|x+y|^2|-\theta_\varepsilon|x+y|^2\right|}\\
&\leqslant C\iint_{D_2} \frac{dy\,dx}{\left||z_1+z_2|x|^2|-\theta_\varepsilon|x|^2\right|\left||z_1+z_2|y|^2|-\theta_\varepsilon|y|^2\right|(\alpha+|x+y|^2)}\\
&\leqslant C\int_0^\beta\int_\beta^K\int_0^{2\pi} \frac{rs\,dt\,ds\,dr}{s^2\,(\beta+r^2+s^2+2rs\cos(t))}\\
&\leqslant C\int_0^\beta\int_\beta^K\int_0^{2\pi} \frac{rs\,dt\,ds\,dr}{s^2\,(\alpha+r^2+s^2+2rs\cos(t))}\\
&\leqslant C\int_0^\beta\int_\beta^K \frac{r\,ds\,dr}{s\sqrt{r^2+s^2}} \leqslant C\int_0^\beta\int_\beta^\infty \frac{r\,ds\,dr}{s^2} \leqslant C<\infty.
\end{aligned}$$

where we have used Lemma A.4.3 and the inequality

$$(C+r^2+s^2)^2-4r^2s^2 \geqslant C(r^2+s^2).$$

Similarly over the region D_3 by symmetry.

Finally over the region D_4 we have by Inequality (3.3.8) and Lemma A.4.3 and the last inequality

$$\begin{aligned}
&\iint_{D_4} \frac{dy\,dx}{\left||z_1+z_2|x|^2|-\theta_\varepsilon|x|^2\right|\left||z_1+z_2|y|^2|-\theta_\varepsilon|y|^2\right|\left||z_1+z_2|x+y|^2|-\theta_\varepsilon|x+y|^2\right|}\\
&\leqslant C\int_\beta^\infty\int_\beta^\infty \frac{ds\,dr}{rs\sqrt{r^2+s^2}} \leqslant C<\infty.
\end{aligned}$$

□

Lemma A.4.2. *For $z_1 := (1-\lambda)/|1-\lambda|$, $z_2 := \lambda/2$, $K=\varepsilon/|1-\lambda|$, and $\Re(\lambda)>\alpha\in(0,1)$*

$$F(\lambda) = \int_0^K\int_0^K \frac{rs\,dr\,ds}{\left||z_1+z_2r^2|-\theta_\varepsilon r^2\right|\left||z_1+z_2s^2|-\theta_\varepsilon s^2\right|\left||z_1+z_2(r^2+s^2)|-\Delta_\varepsilon(r^2+s^2)\right|}, \tag{A.4.9}$$

is bounded above uniformly in λ.

Proof. Fix a small positive $\beta>0$, and split the integral

$$F(\lambda)=F_1(\lambda)+F_2(\lambda)+F_3(\lambda)+F_4(\lambda),$$

where F_1,F_2,F_3, and F_4, are the integrals over the regions $[0,\beta]^2$, $[0,\beta]\times[\beta,K]$, $[\beta,K]\times[0,\beta]$, and $[\beta,K]\times[\beta,K]$ respectively.

When $r,s<\beta$, by Inequalities (3.3.7) and (3.3.9) the denominator is bounded below and thus $|F_1(\lambda)|\leqslant C<\infty$ uniformly in λ.

By Inequalities (3.3.7), (3.3.8) and (3.3.10)

$$|F_2(\lambda)| \leqslant \int_0^\beta \int_\beta^K \frac{r s \, \mathrm{d}r \, \mathrm{d}s}{r^2(r^2+s^2)} \leqslant \int_0^\beta \int_\beta^\infty \frac{s \, \mathrm{d}r \, \mathrm{d}s}{r^3} < \infty.$$

and similarly for F_3 by symmetry.

By Inequalities (3.3.8) and (3.3.10)

$$|F_4(\lambda)| \leqslant \int_\beta^K \int_\beta^K \frac{r s \, \mathrm{d}r \, \mathrm{d}s}{r^2 s^2 (r^2+s^2)} \leqslant C \int_\beta^\infty \int_\beta^\infty \frac{\mathrm{d}r \, \mathrm{d}s}{r^2 s^2 (r+s)} \leqslant C < \infty \tag{A.4.10}$$

uniformly in λ. □

Lemma A.4.3. *For $a > b > 0$*

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}t}{a \pm b\cos(t)} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

Proof. We bring the integral in contour form,

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}t}{a + b\cos(t)} = \oint_\Gamma \frac{1}{a + \frac{b}{2}\left(z+\frac{1}{z}\right)} \frac{\mathrm{d}z}{\mathrm{i}z}$$

where Γ is the circle of radius 1 around the origin. continuing we have

$$\begin{aligned}
\oint_\Gamma \frac{1}{a + \frac{b}{2}\left(z+\frac{1}{z}\right)} \frac{\mathrm{d}z}{\mathrm{i}z} &= \frac{1}{\mathrm{i}} \oint_\Gamma \frac{\mathrm{d}z}{\frac{b}{2}z^2 + az + \frac{b}{2}} \\
&= \frac{2}{\mathrm{i}b} \oint_\Gamma \frac{\mathrm{d}z}{z^2 + \frac{2a}{b}z + 1} \\
&= \frac{2}{\mathrm{i}b} \oint_\Gamma \frac{\mathrm{d}z}{z^2 + \frac{2a}{b}z + 1} \\
&= \frac{2}{\mathrm{i}b} \oint_\Gamma \frac{\mathrm{d}z}{\left(z + \frac{a}{b} + \sqrt{\frac{a^2}{b^2}-1}\right)\left(z + \frac{a}{b} - \sqrt{\frac{a^2}{b^2}-1}\right)}.
\end{aligned}$$

Recall that $a > b > 0$. Then it is trivial to see that the larger root

$$\left|\tfrac{a}{b} + \sqrt{\tfrac{a^2}{b^2}-1}\right| > 1,$$

and thus lies outside the unit disc. It requires a little more work, but it is also easy to show that

$$\left|\tfrac{a}{b} - \sqrt{\tfrac{a^2}{b^2}-1}\right| < 1,$$

and thus that this root lies within the unit disc. Thus the integrand has one simple singularity within the unit disc and applying Cauchy's residue theorem we obtain

$$\oint_\Gamma \frac{1}{a + b\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{\mathrm{d}z}{\mathrm{i}z} = \frac{2\pi \mathrm{i}}{\mathrm{i}b} \frac{1}{\sqrt{\frac{a^2}{b^2}-1}} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

□

Lemma A.4.4. *If for some $\alpha \in (0,1)$, we have $|\lambda| < 1$ and $\Re(\lambda) > \alpha$ then*

$$\iint_{|k_1|<\varepsilon\leqslant|k_2|} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|1-\lambda+\frac{\lambda}{2}|k_1|^2||1-\lambda+\frac{\lambda}{2}|k_2|^2||1-\lambda+\frac{\lambda}{2}|k_1+k_2|^2|} \leqslant D(\varepsilon)\log_+|1-\lambda|^{-1}.$$

Proof. We first change variables, and with z_1, z_2 as defined previously we have

$$\begin{aligned}&\iint_{|k_1|<\varepsilon\leqslant|k_2|} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|1-\lambda+\frac{\lambda}{2}|k_1|^2||1-\lambda+\frac{\lambda}{2}|k_2|^2||1-\lambda+\frac{\lambda}{2}|k_1+k_2|^2|}\\ &= |1-\lambda|^{-1}\iint_{|k_1|<K\leqslant|k_2|} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1+z_2|k_1|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1+k_2|^2|}.\end{aligned}$$

For some fixed, small $\delta > 0$ we split the region of integration

$$\begin{aligned}&\iint\limits_{|k_1|<K\leqslant|k_2|} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1+z_2|k_1|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1+k_2|^2|}\\ &\leqslant \iint\limits_{\substack{|k_1|<K\leqslant|k_2|\\ |k_1+k_2|<\delta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1+z_2|k_1|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1+k_2|^2|}\\ &\quad+\iint\limits_{\substack{|k_1|<K\leqslant|k_2|\\ |k_1+k_2|\geqslant\delta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1+z_2|k_1|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1+k_2|^2|}.\end{aligned}$$

For the first integral we have

$$\begin{aligned}&\iint\limits_{\substack{|k_1|<K\leqslant|k_2|\\ |k_1+k_2|<\delta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1+z_2|k_1|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1+k_2|^2|}\\ &\leqslant C\iint\limits_{\substack{|k_2|\geqslant K\\ k_1\in B_\delta(-k_2)}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1+z_2|k_1|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1+k_2|^2|}\\ &\leqslant C\iint\limits_{\substack{|k_2|\geqslant K\\ |k_1|<\delta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1+z_2|k_1-k_2|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1|^2|}\\ &\leqslant C\int_K^\infty\int_0^\delta\int_{-\pi}^{\pi} \frac{rs\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s}{|z_1+z_2(r^2+s^2-2rs\cos(t))||z_1+z_2s^2||z_1+z_2r^2|}\\ &\leqslant C\int_K^\infty\int_0^\delta\int_{-\pi}^{\pi} \frac{rs\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s}{(r^2+s^2-2rs\cos(t))s^2|z_1+z_2r^2|}\end{aligned}$$

and by Lemma A.4.3 and Inequality (3.3.7)

$$\begin{aligned}&\leqslant C\int_K^\infty\int_0^\delta \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{(s-r)(s+r)s^2|z_1+z_2r^2|}\\ &\leqslant C\int_{CK}^\infty \frac{\mathrm{d}s}{s^3} \leqslant CK^{-2} \leqslant D(\varepsilon)|1-\lambda|.\end{aligned}$$

Thus we have that

$$\iint\limits_{\substack{|k_1|<\epsilon\leqslant|k_2|\\|k_1+k_2|<\delta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|1-\lambda+\frac{\lambda}{2}|k_1|^2||1-\lambda+\frac{\lambda}{2}|k_2|^2||1-\lambda+\frac{\lambda}{2}|k_1+k_2|^2|} \leqslant D(\epsilon).$$

Let us now consider the integral over the region $|k_1+k_2| \geqslant \delta$. Then we use the following trick:

$$2|k_1+k_2|^2 \geqslant D+|k_1+k_2|^2,$$

for some positive constant D, since $|k_1+k_2| \geqslant \delta$. Then by Inequalities (3.3.7), (3.3.8) and (3.3.13) and Lemma A.4.3

$$\begin{aligned}
&\iint\limits_{\substack{|k_1|<K\leqslant|k_2|\\|k_1+k_2|\geqslant\delta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1+z_2|k_1|^2||z_1+z_2|k_2|^2||z_1+z_2|k_1+k_2|^2|}\\
&\leqslant \iint\limits_{\substack{|k_1|<K\leqslant|k_2|\\|k_1+k_2|\geqslant\delta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1+z_2|k_1|^2||k_2|^2(D+|k_1+k_2|^2)}\\
&\leqslant \int_K^\infty\int_0^K\int_{-\pi}^{\pi} \frac{rs\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s}{|z_1+z_2r^2|s^2(D+r^2+s^2+2rs\cos(t))}\\
&\leqslant C\int_K^\infty\int_0^K \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{|z_1+z_2r^2|s^2\sqrt{(D+r^2+s^2)^2-4r^2s^2}}\\
&\leqslant C\int_K^\infty\int_0^K \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{|z_1+z_2r^2|s^2\sqrt{r^2+s^2}}\\
&\leqslant C\int_K^\infty\int_0^K \frac{r\,\mathrm{d}r\,\mathrm{d}s}{|z_1+z_2r^2|s^2}\\
&\leqslant C\int_K^\infty s^{-2}\,\mathrm{d}s\int_0^{K^2} \frac{\mathrm{d}r}{|z_1+z_2r^2|} \leqslant D(\epsilon)|1-\lambda|\log_+|1-\lambda|^{-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\iint\limits_{|k_1|<\epsilon\leqslant|k_2|} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|1-\lambda+\frac{\lambda}{2}|k_1|^2||1-\lambda+\frac{\lambda}{2}|k_2|^2||1-\lambda+\frac{\lambda}{2}|k_1+k_2|^2|}\\
&\leqslant D(\epsilon)\log_+|1-\lambda|^{-1}\\
&\leqslant D(\epsilon)+D(\epsilon)\log_+|1-\lambda|^{-1}.
\end{aligned}$$

□

Lemma A.4.5. *For z_1, z_2 as defined above and $\Re(\lambda) > \alpha$*

$$|I_3| = \left|\iint\limits_{\epsilon\leqslant|k_1|,|k_2|} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|1-\lambda+\frac{\lambda}{2}|k_1|^2||1-\lambda+\frac{\lambda}{2}|k_2|^2||1-\lambda+\frac{\lambda}{2}|k_1+k_2|^2|}\right| \leqslant C < \infty,$$

uniformly in λ.

Proof. We first change variables scaling by $|1-\lambda|$ and then split the region of integration for some fixed, small $\beta > 0$

$$\begin{aligned}
|I_3| &\leqslant |1-\lambda|^{-1} C \iint\limits_{K \leqslant |k_1|, |k_2|} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1 + z_2|k_1|^2||z_1 + z_2|k_1|^2||z_1 + z_2|k_1 + k_2|^2|} \\
&\leqslant |1-\lambda|^{-1} C \iint\limits_{\substack{K \leqslant |k_1|, |k_2| \\ |k_1 + k_2| < \beta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1 + z_2|k_1|^2||z_1 + z_2|k_1|^2||z_1 + z_2|k_1 + k_2|^2|} \\
&\quad + |1-\lambda|^{-1} C \iint\limits_{\substack{K \leqslant |k_1|, |k_2| \\ |k_1 + k_2| \geqslant \beta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1 + z_2|k_1|^2||z_1 + z_2|k_1|^2||z_1 + z_2|k_1 + k_2|^2|}.
\end{aligned}$$

For the first integral we have

$$\begin{aligned}
&\iint\limits_{\substack{K \leqslant |k_1|, |k_2| \\ |k_1 + k_2| < \beta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1 + z_2|k_1|^2||z_1 + z_2|k_1|^2||z_1 + z_2|k_1 + k_2|^2|} \\
&= \iint\limits_{\substack{K \leqslant |k_1|, |k_2| \\ k_1 \in B_\beta(-k_2)}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1 + z_2|k_1|^2||z_1 + z_2|k_1|^2||z_1 + z_2|k_1 + k_2|^2|} \\
&\leqslant C \iint\limits_{\substack{K \leqslant |k_2| \\ k_1 \in B_\beta(-k_2)}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1 + z_2|k_1|^2||z_1 + z_2|k_1|^2||z_1 + z_2|k_1 + k_2|^2|} \\
&\leqslant C \iint\limits_{\substack{K \leqslant |k_2| \\ k_1 \in B_\beta(0)}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1 + z_2|k_1 - k_2|^2||z_1 + z_2|k_2|^2||z_1 + z_2|k_1|^2|} \leqslant C,
\end{aligned}$$

by the calculations in the proof of Lemma A.4.4.

For the other integral we have Inequality (3.3.8) and Lemma A.4.3

$$\begin{aligned}
&\iint\limits_{\substack{K \leqslant |k_1|, |k_2| \\ |k_1 + k_2| \geqslant \beta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|z_1 + z_2|k_1|^2||z_1 + z_2|k_1|^2||z_1 + z_2|k_1 + k_2|^2|} \\
&\leqslant C \iint\limits_{\substack{K \leqslant |k_1|, |k_2| \\ |k_1 + k_2| \geqslant \beta}} \frac{\mathrm{d}k_1\,\mathrm{d}k_2}{|k_1|^2|k_2|^2(D + |k_1 + k_2|^2)} \\
&\leqslant C \int_K^\infty \int_K^\infty \int_{-\pi}^{\pi} \frac{rs\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}s}{r^2 s^2 (D + r^2 + s^2 + 2rs\cos(t))} \\
&C \int_K^\infty C \int_K^\infty \frac{rs\,\mathrm{d}r\,\mathrm{d}s}{r^2 s^2 \sqrt{(D + r^2 + s^2)^2 - 4r^2 s^2}} \\
&C \int_K^\infty C \int_K^\infty \frac{\mathrm{d}r\,\mathrm{d}s}{rs\sqrt{rs}} = CK^{-1} \leqslant D(\epsilon)|1-\lambda|,
\end{aligned}$$

and the lemma follows. □

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