Diffeomorphism Invariant Gauge Theories

Alexander Torres Gomez

Thesis submitted to The University of Nottingham
for the degree of Doctor of Philosophy

September 2012

Abstract

A class of diffeomorphism invariant gauge theories is studied. The action for this class of theories can be formulated as a generalisation of the well known topological BF-theories with a potential for the B-field or in a pure connection formulation. When the gauge group is chosen to be $SU(2)$ the theory describes gravity. For a larger gauge group G one gets a unified model of gravity and Yang-Mills fields. A background for the theory is chosen which breaks the gauge group G by selecting in it a preferred $SU(2)$ subgroup which describes the gravitational sector. The Yang-Mills sector is described by the part of the gauge group that commutes with this $SU(2)$. Thus, when the action is expanded around this background the spectrum of the linearised theory consists of the usual gravitons plus Yang-Mills fields. In addition, there is a set of massive scalar fields that are charged both under the gravitational and Yang-Mills subgroups. The latter sector is described by the part of the gauge group that does not commute with $SU(2)$. A fermionic Lagrangian is also proposed which can be coupled to the BF plus potential formulation.

Acknowledgements

I would like to thank my supervisor Dr. Kirill Krasnov for his guidance through my PhD studies, this thesis would not had been possible without him. I also wish to acknowledge the support of my friends in the School of Mathematical Sciences at Nottingham, and in particular to the people in the Quantum Gravity group. Finally, I am extremely appreciative of the support of my wife, my parents, my mother in law and my mother's family. I certainly could not have accomplished all that I have without them.

Contents

CHAPTER 1

Introduction

There have been numerous attempts to unify Einstein's theory of gravity with gauge fields describing other interactions. One such unification proposal is that of Kaluza-Klein, where the metric and gauge fields arise from a higher-dimensional metric tensor upon compactification of extra dimensions. This scenario has become an indispensable part of string theory, which also provides another unifying perspective by viewing gravity and Yang-Mills as excitations of closed and open strings respectively. For more details on string-inspired unification schemes see a recent exposition [1].

There have also been attempts to unify gravity with Yang-Mills theories without introducing extra dimensions but instead trying to extend the methods used in Grand Unified Theories [2] to include gravity. There is, however, a very strong no-go theorem [3] that shows that at least one type of such unification is impossible. The theorem states that the symmetry group of the S-matrix of a consistent quantum field theory (in Minkowski spacetime) is the product of Poincare and internal gauge group. In other words, the spacetime and internal symmetries do not mix. Now, since gravity can be (at least loosely) viewed as a gauge theory for the diffeomorphism group, and the latter contains Poincare group as that of rigid global transformations, the Coleman-Mandula theorem [3] is sometimes interpreted as saying that no unification of gravity and gauge theory that puts together diffeomorphisms and gauge transformations is possible. In this discussion, however, one must be careful to distinguish between local gauge invariances of a theory and global symmetries whose presence or absence depends on a particular state one works with, see [4] that emphasises this point. While it may be difficult or impossible to "unify" diffeomorphisms and gauge transformations into a single gauge group, this is not the only possible way to approach the unification problem. To understand how a different type of unification might be possible, let us recall that in the so-called first-order formalism gravity becomes a theory of a tetrad as well as a Lorentz group spin connection. The "internal" Lorentz group acts by rotating the tetrad and has no effect on the metric defined by this tetrad. Thus, the physical dynamical variable is still the metric, one simply added some gauge variables and enlarged the gauge group, which in this formulation is a (semi-) direct product of the diffeomorphism group and $SO(3,1)$. Further, in the Hamiltonian formulation this theory can be easily cast into one on the Yang-Mills phase space. This is done by adding to the action a term that vanishes on-shell [5]. The phase space is then that of pairs $SU(2)$ connection plus the canonically conjugate "electric" field. Thus, after the trick of adding an on-shell unimportant term, gravity becomes a generally covariant theory of an $SU(2)$ connection. The tetrad (spacetime metric) is still a dynamical variable but in this formulation it receives the interpretation of the canonically conjugate field to the connection. Yang-Mills theory, on the other hand, after it is written for a general spacetime metric, also becomes a generally covariant theory of a connection and spacetime metric. One could then attempt to put the two generally covariant gauge theories together in some way that combines the "internal" gauge groups, while leaving the total gauge group to be a (semi-) direct product of diffeomorphisms and "internal" symmetries. This would not be in any conflict with the no-go theorem [3] for what is unified is not the Poincare and internal symmetry groups.

As far as we are aware, the first proposal of this type was put forward in [6, 7], with the idea being precisely to extend the gauge group of gravity formulated in tetrad first-order formalism as a theory of the Lorentz connection. This proposal was later pushed forward in [8, 9], see also [10] for the most recent development. The key point of this proposal is that it is a non-degenerate metric that breaks the gauge symmetry of the unified theory down to a smaller group consisting of $SO(3,1)$ for gravity and some "internal" group for Yang-Mills fields. A similar in spirit, but very different in the realisation idea was proposed in [11], and further developed in [12–14]. This approach stems from the fact that Einstein's general relativity (GR) can be reformulated as a theory on the Yang-Mills phase space. At the time of writing [11] it was achieved in Ashtekar's Hamiltonian formulation of GR [15] that interprets gravity as a special generally covariant (complexified) $SU(2)$ gauge theory. The fact that gravity in this formulation becomes a theory of connection suggests that a gauge group larger than $SU(2)$ can be considered. This is what was attempted in [11–14], with the main result of [14] being that Yang-Mills theory arises in an expansion of the theory around the de Sitter background.

The unification by enlarging the internal gauge group proposal was recently revisited in [16], where the new action principle [17] for a class of modified gravity theories [18], extended to a larger gauge group was used. This work extended the gauge group of an explicitly real formulation of gravity that works with the Lorentz, not with the complexified rotation group. Specifically, it was suggested in [16] that the action of the type proposed in [17] considered for a general Lie group G describes gravity in its $SO(4)$ part plus Yang-Mills fields in the remaining quotient $G/SO(4)$. As in [14], the Yang-Mills coupling constant is related in [16] to the cosmological constant. As in the approach [6, 7], in [16] it is a non-degenerate metric that breaks the symmetry down to a smaller gauge group. The approach of [16] is also similar to that of [6, 7] in that many new bosonic degrees of freedom are introduced. Thus, it was shown in [19] that the BF-type action of [17] for $G = SO(4)$ does not describe anymore a pure gravity theory in that it describes six new DOF.

This thesis describes a new framework for unification of gravity and Yang-Mills fields starting from a general diffeomorphism invariant gauge theory. The action for the theory can be presented in a BF plus potential formulation, where the field variables are a connection one-form and a Lie algebra-valued two-form, or in a pure connection formulation, where the only variable is a connection one-form. These actions are naturally constructed as the "most general" ones with those field content which are diffeomorphism and gauge invariant and lead to second order in derivatives field equations. The pure connection formulation of the theory can be thought to be obtained from the BF plus potential formulation after the Lie algebra-value two-form field has been integrated out. The general procedure to obtain gravity and Yang-Mills is similar to what is done in the unification proposals by enlarging the internal gauge group which have been briefly explained above. However, unlike in [16], we interpret only a (complexified) $SU(2)$ subgroup of the gauge group G as that corresponding to gravity. The part of the gauge group that commutes with this gravitational $SU(2)$ is then seen to describe Yang-Mills fields, and the part that does not commute with $SU(2)$ describes massive scalar fields.

Although, in our opinion, the model studied in this thesis has achieved the desirable basic facts that any unification scheme should have without any trivial contradiction with the known physical models, there is still a long path until we can claim that we have a realistic model and be able to make some prediction. We could say that we have just set up the basic ingredients for a new unification model and that it is now the time to start constructing the realistic model we are looking for based on the first steps we have done. We think an important quality of our unification framework is that we have not used extra dimensions to achieved unification, so we always work in the usual 4 dimensional spacetime. Extra dimensions is a beautiful idea that opens a universe of new possibilities but unfortunately there is not proof of this yet. It could also be said that our proposal is very conservative in the sense that we are not changing a big paradigm, as is done for example in string theory where the concept of particles is replaced by that of strings, but we have found a new action which accomplished unification with an approach similar to the one adopted in Grand Unified Theories but this time including gravity as well. The cornerstone of our model is the description of gravity as a diffeomorphism invariant gauge theory. The action functional for this new description of gravity uses $SU(2)$ as gauge group and it is remarkable that when a bigger Lie group is utilised, for the same action, we obtain gravity and Yang-Mills fields as different sectors in the Lie algebra.

This thesis is divided in two parts, i.e., the BF plus Potential Formulation and the Pure Connection Formulation approach for this model. In chapter 2 we define the class of diffeomorphism invariant gauge theories in its BF formulation. This chapter contains a general discussion on the problem of linearisation. In chapter 3 we consider the case of pure gravity corresponding to $G = SU(2)$. The Minkowski space background that we expand around is described here. Chapter 4 studies a unified gravity non-linear electrodynamics model. We switch off the gravitational sector and study the resulting non-linear electrodynamics theory. Then, we switch the gravitational force back on and study the spherically-symmetric solution of the theory. Chapter 5 deals with an example of a non-trivial group for which we take $G = SU(3)$. Here we obtain a Lagrangian describing gravity, a gauge field and some massive scalar fields. A fermionic Lagrangian which can be coupled to the BF plus potential formulation is studied in chapter 6. In chapter 7 the pure connection formalism of the theory is explained and the perturbation theory is studied. Chapter 8 shows how to describe gravity using a gauge potential as the only field variable and how the usual propagating degrees of freedom appears. Finally, chapter 9 explores the unification of gravity and Yang-Mills fields in this pure connection formalism studying the resulting Lagrangian for the different sectors found.

The material found in this thesis is based on research done between September 2008 and December 2011. Chapters 2 and 3 contains calculations and results taken from [20]. Chapter 4 is taken from [21]. Chapter 5 improves sections 7 to 9 that appears in [20]. Chapter 6 it is new and has not been reported anywhere else. In section 8.1 the same results as in [22] are found, but using a different procedure. Section 8.2 and chapter 9 are heavily based on [23] and [24], respectively, but we have used some different notations and conventions.

Part I

BF plus Potential Formulation

CHAPTER 2

A Class of Diffeomorphism Invariant Gauge Theories

Consider a principal G-bundle over the spacetime manifold M with Lie group G. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. The Lie algebra $\mathfrak{g}$ is assumed to be a general semisimple complex one. As is usual in physics literature, the bundle is assumed to be trivial, so the connection can be viewed as a Lie-algebra-valued one-form on M.
The action that we would like to consider is of BF-type and is given by

$$S[A,B] = \int_M g_{IJ} B^I \wedge F^J - \frac{1}{2} V(B^I \wedge B^J)\,, \tag{2.0.1}$$

where g_{IJ} is an inner product on the Lie-algebra $\mathfrak{g}$; B^I is a $\mathfrak{g}$-valued two-form; F^I is the curvature of the connection[1] A^I; and $V(\cdot)$ is a G-invariant, holomorphic and homogeneous order one function of symmetric $n \times n$ matrices. The indices $I, J, K, L, \dots = 1, 2, \dots, n$ with $n = \dim(\mathfrak{g})$.
The potential term $V(B^I \wedge B^J)$ deserve a more detail explanation. Consider the 4-form $B^I \wedge B^J$. This is a 4-form valued in the space of symmetric bilinear forms in $\mathfrak{g}$. Choosing an arbitrary volume 4-form (vol) we can write $B^I \wedge B^J = (vol)\, h^{IJ}$, where now h^{IJ} is a symmetric $n \times n$ matrix. Since (vol) is defined only modulo rescalings, $(vol) \to \alpha(vol)$, so is the matrix h^{IJ} that under such rescalings transforms as $h^{IJ} \to (1/\alpha) h^{IJ}$. Let us now introduce a function $V(h)$ of symmetric $n \times n$ matrices h^{IJ} with the following properties. First, the function is gauge-invariant: $V(\mathrm{ad}_g h) = V(h)$, where ad_g is the adjoint action of the gauge group on the space of symmetric bilinear forms on the Lie algebra. Second, the function is holomorphic (we work with complex-valued quantities). Third,

[1] The field strength of the connection one-form A^I is defined as usual by

$$F^I = dA^I + \frac{1}{2}[A,A]^I = dA^I + \frac{1}{2} C^I_{JK} A^J \wedge A^K\,,$$

where C^I_{JK} stands for the structure constants of the Lie algebra $\mathfrak{g}$.

and most important, the function is homogeneous of degree one $V(\alpha h) = \alpha V(h)$, for $\alpha \in \mathbb{C}$. Indeed, we have $V(B^I \wedge B^J) = (vol)V(h^{IJ})$, and it is easy to see that due to the homogeneity of $V(\cdot)$, the resulting 4-form does not depend on which particular volume form (vol) is chosen. Thus, the quantity $V(B^I \wedge B^J)$ is an invariantly defined 4-form, and it can be integrated over spacetime.

2.1 A Generalisation of BF theory

A way to arrive at (2.0.1) is considering possible directions to generalise a topological BF theory. For the case of $\mathfrak{g} = \mathfrak{su}(2)$ this was done in [25], and here we generalise this analysis to a general semisimple Lie algebra $\mathfrak{g}$.
Following this reference we begin with the action

$$S[A, B] = \int g_{IJ}\, B^I \wedge F^J - \frac{1}{2}\Phi_{IJ}\, B^I \wedge B^J\,, \tag{2.1.1}$$

where B^I is a two-form valued in $\mathfrak{g}$, F^I is the curvature of a connection A^I and Φ^{IJ} is a function (zero-form) valued in the symmetric product of two copies of $\mathfrak{g}$. At this stage this quantity is undetermined. But we should say already now that it is not to be thought of as an independent field to be varied with respect to, for it will later be fixed by Bianchi identities. Note that only the symmetric part of Φ^{IJ} enters the action, this is why it is assumed symmetric from the beginning. We raise and lower internal indices $I, J, \ldots$ with the inner product g_{IJ} and its inverse g^{IJ}. We also note that for a semisimple Lie algebra we can always find a basis in which this inner product is diagonal, i.e. $g_{IJ} = \delta_{IJ}$, where δ_{IJ} is the Kronecker delta.

Varying (2.1.1) with respect to the connection A and the field B we get, respectively,

$$DB^I \equiv dB^I + C^I_{JK}\, A^J \wedge B^K = 0\,, \tag{2.1.2}$$

$$F^I = \Phi^I_J\, B^J\,. \tag{2.1.3}$$

We see that the idea of the above action ansatz is to generalise the BF theory in such a way that the equation (2.1.2) relating B and A is unchanged, while we now allow for a non-zero curvature. As we have already said, we do not consider a variation with respect to Φ^{IJ} because we will later show that the Bianchi identities fix this quantity in terms of certain components of the two-form field B^I.
Let us now take the covariant exterior derivative D of (2.1.3) and use (2.1.2) together with the Bianchi identity $DF^I = 0$. We obtain

$$D\Phi^I_J \wedge B^J = 0\,. \tag{2.1.4}$$

Now, the covariant exterior derivative of DB^I is

$$D(DB^I) = C^I_{JK} dA^J \wedge B^K + C^I_{JK} C^K_{LM}\, A^J \wedge A^L \wedge B^M\,. \tag{2.1.5}$$

Using the Jacobi identity $C^N_{IJ}C^L_{NK} + C^N_{JK}C^L_{NI} + C^N_{KI}C^L_{NJ} = 0$, the equation above can be rewritten as

$$D(DB^I) = C^I_{JL}\, F^J \wedge B^L\,, \tag{2.1.6}$$

and using equation (2.1.2) and equation (2.1.3) we get

$$C^I_{JL}\, \Phi^J_K\, B^K \wedge B^L = 0\,. \tag{2.1.7}$$

Let us denote the interior product between an arbitrary vector field ξ and the two form B^I as $\xi \lrcorner B^I$. Now computing the wedge product between (2.1.4) and the one-form $\xi \lrcorner B^I$, which has components $(\xi \lrcorner B^I)_\mu = \xi^\alpha B^I_{\alpha\mu}$, we get

$$D\Phi_{IJ} \wedge \xi \lrcorner B^{(I} \wedge B^{J)} = 0\,. \tag{2.1.8}$$

Using $\xi \lrcorner B^{(I} \wedge B^{J)} = \frac{1}{2}\xi \lrcorner (B^I \wedge B^J)$, we can rewrite this as

$$D\Phi_{IJ} \wedge \xi \lrcorner (B^I \wedge B^J) = 0\,. \tag{2.1.9}$$

Let us now define the "internal" metric h^{IJ} by means of the following relation

$$B^I \wedge B^J = h^{IJ}\,(vol), \tag{2.1.10}$$

where (vol) is an arbitrary volume 4-form. We can then rewrite (2.1.9) as

$$h_{IJ}\, D\Phi^{IJ} \wedge \xi \lrcorner (vol) = 0\,. \tag{2.1.11}$$

Using the definition of h^{IJ}, we can also rewrite (2.1.7) as

$$C^I_{JK}\, \Phi^J_L\, h^{LK} = 0\,. \tag{2.1.12}$$

Now, computing $h_{IJ}\, D\Phi^{IJ}$,

$$h_{IL}\, D\Phi^{IL} = h_{IL}\,(d\Phi^{IL} + 2\, C^I_{JK}\, A^J\, \Phi^{KL})\,, \tag{2.1.13}$$

we can see that the second term in the right hand side vanishes because of (2.1.12) and the condition that the Lie algebra is semisimple. The latter is used because for a semisimple Lie algebra it is possible to define an inner product, in our case δ_{IJ}, with respect to which the object $C_{IJK} = \delta_{IL}\, C^L_{JK}$ is completely anti-symmetric.
Our final result is

$$h_{IJ}\, \partial_\mu \Phi^{IJ}\, \xi^\mu = 0\,, \tag{2.1.14}$$

which implies

$$h_{IJ}\,\partial_\mu \Phi^{IJ} = 0\,, \tag{2.1.15}$$

since ξ is an arbitrary vector.

The above equation implies that the quantities h^{IJ} and Φ^{IJ} are not independent. Let us define the "potential function" $V \equiv h^{IJ}\,\Phi_{IJ}$. Then,

$$dV = \Phi_{IJ}\,dh^{IJ} + h_{IJ}\,d\Phi^{IJ} = \Phi_{IJ}\,dh^{IJ}\,, \tag{2.1.16}$$

where we have used (2.1.15). This means that: a) the potential V is only a function of h^{IJ}, i.e., $V = V(h^{IJ})$; b) the quantities Φ^{IJ} are given by

$$\Phi_{IJ} = \frac{\partial V}{\partial h^{IJ}}\,; \tag{2.1.17}$$

and c) the potential V is a homogeneous function of order one in h^{IJ} since

$$V = h^{IJ}\frac{\partial V}{\partial h^{IJ}}. \tag{2.1.18}$$

Thus, using the above definition of h^{IJ}, and the fact that $V(\cdot)$ is homogeneous, we can rewrite the action (2.1.1) as

$$S = \int g_{IJ}\,B^I \wedge F^J - \frac{1}{2}\,V(B^I \wedge B^J)\,, \tag{2.1.19}$$

which is exactly the action (2.0.1).

2.1.1 Parametrisation of the potential

The potential term defined as

$$V(B^I \wedge B^J) = (vol)V(h^{IJ})\,, \tag{2.1.20}$$

still have an arbitrariness because of the freedom of rescaling of (vol). A possible way to avoid this arbitrariness is as follows. With our choice of conventions[2], $dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = -\tilde{\epsilon}^{\mu\nu\rho\sigma}d^4x$, we have

$$B^I \wedge B^J = \frac{1}{4}B^I_{\mu\nu}B^J_{\rho\sigma}dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = -\frac{1}{4}\tilde{\epsilon}^{\mu\nu\rho\sigma}B^I_{\mu\nu}B^J_{\rho\sigma}d^4x, \tag{2.1.21}$$

[2]Our convention is that the Levi-Civita symbol $\tilde{\epsilon}^{\mu\nu\lambda\rho}$ has components

$$\tilde{\epsilon}^{\mu\nu\lambda\rho} = \begin{cases} -1, & \text{if } \mu\nu\lambda\rho \text{ is an even permutation of 0123,} \\ 1, & \text{if } \mu\nu\lambda\rho \text{ is an odd permutation of 0123,} \\ 0, & \text{if } \mu=\nu \text{ or } \nu=\lambda \text{ or } \lambda=\rho \text{ or } \rho=\mu, \end{cases}$$

in any coordinate system. Note that the Levi-civita symbol $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is a tensor density of weight minus one which does not require a metric for its definition.

Thus, if we now define a densitised "internal metric"

$$\tilde{h}^{IJ} = \frac{1}{4} B^I_{\mu\nu} B^J_{\rho\sigma} \tilde{\epsilon}^{\mu\nu\rho\sigma} \,, \tag{2.1.22}$$

we can write the action as

$$S[A,B] = \int g_{IJ}\, B^I \wedge F^J + \frac{1}{2} \int d^4x \; V(\tilde{h}) \;. \tag{2.1.23}$$

Then, the argument of the potential function $V(\cdot)$ is the $n \times n$ matrix $\tilde{h}^{IJ}$, and its derivatives can be computed via the usual partial differentiation.
For example, the first variation of this action can be seen to be given by

$$\delta S = \int \delta B^I \wedge \left(g_{IJ} F^J - \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} B^J \right) - g_{IJ} D B^I \wedge \delta A^J \,. \tag{2.1.24}$$

Indeed, the variation of the last, potential term is given by

$$\frac{1}{2} \int \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} \frac{1}{2} \delta B^I_{\mu\nu} B^J_{\rho\sigma} \tilde{\epsilon}^{\mu\nu\rho\sigma} d^4x = - \int \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} \delta B^I \wedge B^J, \tag{2.1.25}$$

where the matrix of first derivatives $(\partial V(\tilde{h}) / \partial \tilde{h}^{IJ})$ is a density of weight zero. Thus, the field equations of our theory can be written as

$$F_I = \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} B^J \,, \tag{2.1.26}$$

$$D B^I \equiv d B^I + [A, B]^I = 0 \,. \tag{2.1.27}$$

In the literature on this class of theories a different parameterisation of the potential is sometimes used, see e.g. the original paper [17], and also the unification paper [16]. Then, to avoid having to work with a homogeneous function, one can parameterise the potential so that an ordinary function of one less variable arises. This can be done via a Legendre transform trick. Thus, we introduce a new variable Ψ^{IJ} that is required to be tracefree $g_{IJ}\Psi^{IJ} = 0$. The idea is that the matrix Ψ^{IJ} is the tracefree part of the matrix of first derivatives $\Phi^{IJ} = (\partial V / \partial \tilde{h}^{IJ})$. In other words, we write

$$\Phi_{IJ} = \Psi_{IJ} - \frac{\Lambda}{n} g_{IJ} \,, \tag{2.1.28}$$

where Ψ_{IJ} is traceless. With Φ^{IJ} being a function of $\tilde{h}^{IJ}$, so is the trace part Λ. However, we can also declare Λ to be a function of Ψ^{IJ}, make Ψ^{IJ} and independent variable and write the action in the form

$$S[B, A, \Psi] = \int g_{IJ}\, B^I \wedge F^J - \frac{1}{2} \left(\Psi_{IJ} - \frac{\Lambda(\Psi)}{n} g_{IJ} \right) B^I \wedge B^J. \tag{2.1.29}$$

Varying the action with respect to Ψ^{IJ} one gets an equation for this matrix, which, after being solved and substituted into the action gives back (2.1.23) with $V(\cdot)$ being an appropriate Legendre transform of $\Lambda(\Psi)$. In the formulation (2.1.29) the function $\Lambda(\Psi)$ is an arbitrary function of the tracefree matrix Ψ^{IJ}, so there is no complication of having to require $V(\cdot)$ to be homogeneous. This formulation was used in the first papers on this class of theories, but it was later realised that the formulation that works solely with the two-form field B^I is more convenient.

2.2 Hamiltonian analysis

To exhibit the physical content of the above theory it is useful to perform the canonical analysis. After the 3+1 decomposition the action reads, up to an unimportant overall numerical factor,

$$S = \int dt \int_\Sigma d^3x \left(\tilde{P}^{aI} \dot{A}^I_a - H \right), \tag{2.2.1}$$

where

$$\tilde{P}^{aI} \equiv \tilde{\epsilon}^{abc} B^I_{bc}, \tag{2.2.2}$$

and the Hamiltonian H is

$$-\tilde{H} = A^I_0 D_a \tilde{P}^{aI} + B^I_{0a} \tilde{\epsilon}^{abc} F^I_{bc} - V\left(B^{(I}_{0a} \tilde{P}^{aJ)} \right). \tag{2.2.3}$$

If we dealt with the pure BF theory the last "potential" term would be absent and all the quantities B^I_{0a} would be Lagrange multipliers. However, now the Lagrangian is not linear in B^I_{0a}, and, as we shall see, all but 4 of these quantities are no longer Lagrange multipliers and should be solved for. The equations one obtains by varying the Lagrangian with respect to B^I_{0a} are

$$\tilde{\epsilon}^{abc} F^I_{bc} = \frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} \tilde{P}^{aJ}. \tag{2.2.4}$$

The equations (2.2.4) can be solved in quite a generality by finding a convenient basis in the Lie algebra. Thus, consider the canonically conjugate field $\tilde{P}^{aI}$. There are at least $n-3$ vectors N^I_α, with $\alpha = 1, \ldots, n-3$, that are orthogonal to this field, i.e.,

$$\tilde{P}^{aI} N^I_\alpha = 0, \quad \forall a, \alpha. \tag{2.2.5}$$

These vectors can be chosen (uniquely up to SO$(n-3)$ rotations) by requiring

$$N^I_\alpha N^I_\beta = \delta_{\alpha\beta}. \tag{2.2.6}$$

We can then use the quantities $\tilde{P}^{aI}$, with $a = 1, 2, 3$, and N^I_α as a basis in the Lie algebra.

We can now decompose the quantity B^I_{0a} as

$$B^I_{0a} = \tilde{P}^{bI} \underset{\sim}{B}_{ab} + N^I_\alpha B^\alpha_a, \tag{2.2.7}$$

where $\underset{\sim}{B}_{ab}, B^\alpha_a$ are components of B^I_{0a} in this basis. There are in total $3n$ components of B^I_{0a} and they are represented here as 9 quantities $\underset{\sim}{B}_{ab}$ as well as $3(n-3)$ quantities B^α_a. The argument of the function $V(\cdot)$ is now given by

$$B^{(I}_{0a} \tilde{P}^{a|J)} = \tilde{P}^{b(I} \tilde{P}^{a|J)} \underset{\sim}{B}_{ab} + N^{(I}_\alpha B^\alpha_a \tilde{P}^{a|J)}. \tag{2.2.8}$$

It is clear that this depends only on the symmetric part $\underset{\sim}{B}_{ab}$ of the components $\underset{\sim}{B}_{ab}$. Thus, the anti-symmetric part of this 3×3 matrix cannot be determined from the equations (2.2.4) and the N^a in $\underset{\sim}{B}_{[ab]} \equiv (1/2)\epsilon_{abc} N^c$ remain Lagrange multipliers. It is also clear that due to the homogeneity of $V(\cdot)$ one more component of B^I_{0a} cannot be solved for. This can be chosen, for example, to be the trace part $B^I_{0a}\tilde{P}^{aI}$, which will then play the role of the lapse function. All other $6 + 3(n-3) - 1$ components of B^I_{0a} can be solved for a generic function $V(\cdot)$, i.e., under the condition that the matrix of second derivatives of $V(\cdot)$ is non-degenerate. We are not going to demonstrate this in full generality, but we will verify it in the linearised theory below.
After the quantities B^I_{0a} are solved for we substitute them into (2.2.3) and obtain the following Hamiltonian:

$$-\tilde{H} = A^I_0 D_a \tilde{P}^{aI} + N^a \tilde{P}^{bI} F^I_{ab} + \tilde{N}\Lambda(F, P), \tag{2.2.9}$$

where $\tilde{N}$ is the lapse function and $\Lambda(F, P)$ is an appropriate Legendre transform of $V(\cdot)$ that now becomes a function of the curvature F^I_{ab} and the field $\tilde{P}^{aI}$. Thus, there are n Gauss as well as 4 diffeomorphism constraints in the theory. It should be possible to check by an explicit computation that they are first class, as was done, for example for the case of $\mathfrak{g} = \mathfrak{su}(2)$ in [26], but we shall not attempt this here, postponing such an analysis till the linearised case considerations. The above arguments allow a simple count of the degrees of freedom described by the theory: we have $3n$ configurational degrees of freedom minus n Gauss constraints minus 4 diffeomorphisms, thus leading to $2n-4$ DOF. Thus, when $\mathfrak{g} = \mathfrak{h} \otimes \mathfrak{su}(2)$ the above count of DOF gives the right number for a gravity (describe by the $\mathfrak{su}(2)$ part) plus Yang-Mills theory[3] (describe by the $\mathfrak{h}$ part). For a general $\mathfrak{g}$ one might suspect that the centraliser of the gravitational $\mathfrak{su}(2)$ describes Yang-Mills, while the rest of the Lie algebra corresponds to some new kind of fields. Below we will unravel their nature by considering the linearised theory. We also note that the above count of degrees of freedom agrees with the one presented

[3]When we refer to Yang-Mills fields here we are thinking about abelian and non-abelian gauge fields, e.g., $\mathfrak{h}$ could be $\mathfrak{u}(1)$ or $\mathfrak{su}(n)$.

in [19] for the case $G = \mathrm{SO}(4)$. Thus, it was seen there that the theory describes in total $2 \cdot 6 - 4 = 8$ DOF, which were interpreted as those corresponding to 2 graviton polarisations plus six new DOF.

2.3 Linearised theory: general considerations

As we have seen in the previous section, the mechanism that selects the gravitational $\mathfrak{su}(2)$ in $\mathfrak{g}$ is that the conjugate variable $\tilde{P}^{aI}$ provides a map from the (co-) tangent space to the spatial slice into $\mathfrak{g}$. This selects a 3-dimensional subspace in $\mathfrak{g}$ that plays the role of the gravitational gauge group. Below we are going to see this mechanism at play at the level of the Lagrangian formulation, by studying the linearisation of the action (2.0.1). In this section it will be convenient to introduce a certain numerical prefactor in front of this action so that the normalisation of the graviton kinetic term in the case of gravity will come out right.

Thus, we shall from now on consider the following action

$$S[A, B] = 4\mathrm{i} \int_M g_{IJ} B^I \wedge F^J - \frac{1}{2} V(B^I \wedge B^J)\,, \tag{2.3.1}$$

where $\mathrm{i} = \sqrt{-1}$.

The following general considerations apply to any background. We specialise to the Minkowski spacetime background in the next chapters.

2.3.1 Kinetic term

Let us call the first term in (2.3.1) S_{BF} and the second "potential" term S_{BB}. Then, the second variation of S_{BF} is given by

$$\delta^2 S_{\mathrm{BF}} = 4\mathrm{i} \int 2\delta B_I \wedge D\delta A^I + B_I \wedge [\delta A, \delta A]^I, \tag{2.3.2}$$

and the action linearised around B_o, A_o is obtained by evaluating this on B_o, A_o.

We are going to view our theory as that of the two-form field B, with the connection A to be eliminated (whenever possible, see below) by solving its field equations.

Let us assume that we are given a background two-form B_o. The linearised connection A_o is then to be determined from the linearised equation (2.1.2), that reads

$$D_o \delta B^I + [\delta A, B_o]^I = 0, \tag{2.3.3}$$

where D_o is the covariant derivative with respect to the background connection A_o.

Now the background two-form B_o^I is a map from the six-dimensional space of bivectors to $\mathfrak{g}$, and thus selects in $\mathfrak{g}$ at most a 6-dimensional preferred subspace. Let us denote this subspace by $\mathfrak{k}$. This subspace may or may not be closed under Lie brackets, but for simplicity, in this paper we shall assume that our background B^I_o is such that $\mathfrak{k}$ is a Lie subalgebra (below we shall make an even stronger assumption about $\mathfrak{k}$). It is then clear that the part of δA^I that lies in the centraliser of $\mathfrak{k}$ in $\mathfrak{g}$ drops from the equation (2.3.3) and cannot be solved for. As we shall see later, this will be the part of the Lie algebra that is to describe Yang-Mills fields. The other part of δA^I can in general be found. For this part of the connection both terms in (2.3.2) are of the same form due to (2.3.3), and the linearised action can be written compactly as

$$\delta^2 S_{\rm BF} = 4{\rm i} \int \delta B_I \wedge D_o \delta A^I, \tag{2.3.4}$$

where δA^I has to be solved for from (2.3.3). On the other hand, for the subalgebra of $\mathfrak{g}$ that centralises $\mathfrak{k}$ the last term in (2.3.2) is absent, $B_I \wedge [\delta A, \delta A]^I = \delta A_I \wedge [\delta A, B]^I$, and we have

$$\delta^2 S_{\rm BF} = 8{\rm i} \int \delta B_I \wedge D_o \delta A^I. \tag{2.3.5}$$

Thus, the analysis of the "kinetic" term is going to be different for different parts of the Lie algebra.

2.3.2 Potential term

The second variation of the potential term $S_{\rm BB}$ is

$$\delta^2 S_{\rm BB} = 4{\rm i} \int 2 \frac{\partial^2 V}{\partial \tilde{h}^{IJ} \partial \tilde{h}^{KL}} (B_o \delta B)^{IJ} (B_o \delta B)^{KL} + \frac{\partial V}{\partial \tilde{h}^{IJ}} (\delta B \delta B)^{IJ} , \tag{2.3.6}$$

where the integration measure d^4x is implied, and we have introduced notations

$$(B_o \delta B)^{IJ} = \frac{1}{4} \tilde{\epsilon}^{\mu\nu\rho\sigma} B^{(I}_{0\,\mu\nu} \delta B^{J)}_{\rho\sigma}, \qquad (\delta B \delta B)^{IJ} = \frac{1}{4} \tilde{\epsilon}^{\mu\nu\rho\sigma} \delta B^I_{\mu\nu} \delta B^J_{\rho\sigma} . \tag{2.3.7}$$

Note that the matrix of second derivatives is a density of weight one ($\tilde{h}$ is a scalar density of weigh minus one).

In general, with the potential function $V(\tilde{h})$ being a homogeneous order one function of the $n \times n$ matrix $\tilde{h}$, it can be reduced to a function of ratios of its invariants. A subset of invariants is obtained by considering traces of powers of $\tilde{h}^{IJ}$. Another class of invariants can also involve the structure constants of the Lie algebra. Below we will see different examples for the invariants that define this potential function V.

CHAPTER 3

Gravity

The case $\mathfrak{g} = \mathfrak{su}(2)$ describes (complexified) gravity theory. A particular choice of the potential function, see below, gives general relativity, while a general potential corresponds to a family of deformations of GR. In this chapter we shall study the corresponding $\mathfrak{su}(2)$ linearised theory. A similar analysis appeared in [27]. However, our method and goals here differ significantly from that reference.

3.1 The metric

To understand how the $\mathfrak{g} = \mathfrak{su}(2)$ case can describe gravity we need to see how the spacetime metric described by the theory is encoded. The answer to this is very simple: there is a unique (conformal) metric that makes the triple B^i, where i is the $\mathfrak{su}(2)$ index, into a set of self-dual two-forms. This is the so-called Urbantke metric [28]

$$\sqrt{-g}g_{\mu\nu} \sim \epsilon^{ijk} B^i_{\mu\alpha} B^j_{\nu\beta} B^k_{\rho\sigma} \tilde{\epsilon}^{\alpha\beta\rho\sigma} , \tag{3.1.1}$$

that is defined modulo an overall factor[1]. We remind the reader that at this stage all our fields are complex, and later reality conditions will be imposed to select physical real Lorentzian signature metrics.

Alternatively, given a metric $g_{\mu\nu}$ one can easily construct a "canonical" triple of self-dual two-forms that encode all information about $g_{\mu\nu}$. This proceeds via introducing tetrad one-forms $\theta^{\mathcal{I}}$, with $\mathcal{I} = 0, 1, 2, 3$ a vector Lorentz index. One then constructs the

[1]Note that the internal Levi-civita tensor ϵ^{ijk} is defined as

$$\epsilon^{ijk} = \begin{cases} 1, & \text{if } ijk \text{ is an even permutation of 123,} \\ -1, & \text{if } ijk \text{ is an odd permutation of 123,} \\ 0, & \text{if } i = j \text{ or } j = k \text{ or } k = i\,. \end{cases}$$

two-forms $\Sigma^{\mathcal{I}\mathcal{J}} \equiv \theta^{\mathcal{I}} \wedge \theta^{\mathcal{J}}$ and takes the self-dual part of $\Sigma^{\mathcal{I}\mathcal{J}}$ with respect to $\mathcal{I}\mathcal{J}$. The resulting two-forms are automatically self-dual. They can be explicitly constructed by decomposing $\mathcal{I} = (0, a)$ and then writing

$$\Sigma^a_\theta = \mathrm{i}\,\theta^0 \wedge \theta^a - \frac{1}{2}\epsilon^a{}_{bc}\,\theta^b \wedge \theta^c\,, \tag{3.1.2}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit and ϵ^{abc} is the three-dimensional Levi-Civita tensor. The presence of the imaginary unit in this formula has to do with the fact that self-dual quantities in a spacetime of Lorentzian signature are necessarily complex. Thus, even though at this stage there is no well defined signature (all quantities are complex), it is convenient to introduce this “i" here so that later appropriate reality conditions are easily imposed. We note that “internal” Lorentz rotations of the tetrad $\theta^{\mathcal{I}}$ at the level of Σ^a_θ boil down to (complexified) $SU(2)$ rotations of Σ^a_θ.

A general $\mathfrak{su}(2)$-valued two-form field B^i carries more information than just that about a metric. Indeed, one needs 3×6 numbers to specify it, while only 10 are necessary to specify a metric. A very convenient description of the other components is obtained by introducing a metric defined by B^i via (3.1.1) and then using the “metric” self-dual two-forms (3.1.2) as a basis and decomposing

$$B^i = b^i_a \Sigma^a_\theta\,. \tag{3.1.3}$$

The quantities b^i_a give 9 components, the metric gives 10, and the choice of “internal” frame for Σ^a_θ adds 3 more components. There is also a freedom of rescalings $b^i_a \to \Omega^{-2} b^i_a$, $\Sigma^a_\theta \to \Omega^2 \Sigma^a_\theta$, as well as freedom of $SO(3)$ rotations acting simultaneously on Σ^a_θ and b^i_a, overall producing 18 independent components of B^i.

When one substitutes the parameterisation (3.1.3) into the action (2.0.1) one finds that the fields b^i_a are non-propagating and should be integrated out. Once this is done one obtains an “effective” Lagrangian for the metric described by Σ^a_θ [29]. Below we shall see how this works in the linearised theory. However, we first need to choose a background.

3.2 Minkowski background

The Minkowski background is described in our framework by a collection of metric two-forms (3.1.2) constructed from the Minkowski tetrad. Thus, we choose an arbitrary time plus space split and write

$$\Sigma^a_{dx} \equiv \Sigma^a = \mathrm{i}dt \wedge dx^a - \frac{1}{2}\epsilon^a{}_{bc} dx^b \wedge dx^c, \tag{3.2.1}$$

where dt, dx^a (with $a = 1,2,3$) form a tetrad for the Minkowski metric $ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. Our two-form field background is then chosen to be

$$B^i_o = \delta^i_a \Sigma^a, \tag{3.2.2}$$

where δ^i_a is an arbitrary $SO(3)$ matrix that for simplicity can be chosen to be the identity matrix.

In what follows we will also need a triple of anti-self-dual metric two-forms that together with (3.1.2) form a basis in the space of two-forms. A convenient choice is given by

$$\overline{\Sigma}^a = \mathrm{i} dt \wedge dx^a + \frac{1}{2}\epsilon^a{}_{bc} dx^b \wedge dx^c. \tag{3.2.3}$$

The following formulas, which can be shown to follow directly from definition (3.2.1), are going to be very useful

$$\Sigma^a_{\mu\sigma}\Sigma^{b\sigma}{}_\nu = -\delta^{ab}\,\eta_{\mu\nu} + \epsilon^{abc}\,\Sigma^c_{\mu\nu}\,, \tag{3.2.4}$$

$$\Sigma^{a\mu\nu}\Sigma^b_{\mu\nu} = 4\,\delta^{ab}\,, \tag{3.2.5}$$

$$\epsilon^{abc}\,\Sigma^a_{\mu\sigma}\Sigma^{b\sigma}{}_\lambda\Sigma^{c\lambda\mu} = -4!\,, \tag{3.2.6}$$

$$\epsilon^{abc}\,\Sigma^a_{\mu\nu}\Sigma^b_{\rho\sigma}\Sigma^{d\nu\sigma} = -2\delta^{cd}\,\eta_{\mu\rho}\,, \tag{3.2.7}$$

$$\Sigma^a_{\mu\nu}\Sigma^a_{\rho\sigma} = \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho} - \mathrm{i}\epsilon_{\mu\nu\rho\sigma}\,, \tag{3.2.8}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. We are going to refer to them as the algebra of Σ's. The first of the relations above, namely (3.2.4), is central, for all others (apart from (3.2.8)) can be derived from it. It is useful to develop some basis-independent understanding of this relation. We are working with the Lie algebra $\mathfrak{su}(2)$ and considering a basis X^a in it in which the structure constants read $[X^a, X^b] = \epsilon^{ab}{}_c X^c$. This is the basis given by $X^a = -(\mathrm{i}/2)\tau^a$, where τ^a are Pauli matrices. The inner product $g^{ab} = \delta^{ab}$ on the Lie algebra can be obtained as $g^{ab} = -2\mathrm{Tr}(X^a X^b)$. Then (3.2.4) can be understood as follows: the product of two Σ's is given by minus the metric plus the structure constants times Σ. We will see that in this form the relations (3.2.4) persist to any basis in $\mathfrak{su}(2)$.

3.3 The potential function V

Let us consider a special class of potentials that only depend on the invariants obtained as the traces of powers of $\tilde{h}^{ij}$. Many aspects of our theory can be seen already for this special choice.

Thus, consider the potential of the form

$$V = \frac{Tr\,\tilde{h}}{3}\,\mathcal{F}\left(\frac{Tr\,\tilde{h}^2}{(Tr\,\tilde{h})^2}, \frac{Tr\,\tilde{h}^3}{(Tr\,\tilde{h})^3}\right) . \tag{3.3.1}$$

where $\mathcal{F}$ is now an arbitrary function of its 2 arguments, $Tr\,\tilde{h} = g_{ij}\,\tilde{h}^{ij}$ and

$$Tr\,\tilde{h}^2 = \tilde{h}^i{}_j\,\tilde{h}^j{}_i\,, \tag{3.3.2}$$

$$Tr\,\tilde{h}^3 = \tilde{h}^i{}_j\,\tilde{h}^j{}_k\tilde{h}^k{}_i\,, \tag{3.3.3}$$

and in general

$$Tr\,\tilde{h}^p = \tilde{h}^i_{m_1}\,\tilde{h}^{m_1}{}_{m_2}\cdots\cdots\tilde{h}^{m_{p-1}}{}_i\,. \tag{3.3.4}$$

The parameterisation given above allows derivatives to be computed easily. Then, the first derivative of the potential function with respect to $\tilde{h}^{ij}$ is

$$\frac{\partial V(\tilde{h})}{\partial\tilde{h}^{ij}} = \frac{g_{ij}}{3}\,\mathcal{F} + \frac{Tr\,\tilde{h}}{3}\,\frac{\partial\mathcal{F}}{\partial\tilde{h}^{ij}}\,, \tag{3.3.5}$$

with $(\partial\mathcal{F}/\partial\tilde{h}^{ij})$ given by

$$\begin{aligned}\frac{\partial\mathcal{F}}{\partial\tilde{h}^{ij}} =& \mathcal{F}_2'\,\frac{\partial}{\partial\tilde{h}^{ij}}\left(\frac{Tr\,\tilde{h}^2}{(Tr\,\tilde{h})^2}\right) + \mathcal{F}_3'\,\frac{\partial}{\partial\tilde{h}^{ij}}\left(\frac{Tr\,\tilde{h}^3}{(Tr\,\tilde{h})^3}\right)\,,\\ =& 2\mathcal{F}_2'\,\left(\frac{\tilde{h}_{ij}}{(Tr\,\tilde{h})^2} - \frac{Tr\,\tilde{h}^2}{(Tr\,\tilde{h})^3}\,g_{ij}\right) + 3\mathcal{F}_3'\,\left(\frac{\tilde{h}^2_{ij}}{(Tr\,\tilde{h})^3} - \frac{Tr\,\tilde{h}^3}{(Tr\,\tilde{h})^4}\,g_{ij}\right)\,,\end{aligned} \tag{3.3.6}$$

where $\mathcal{F}_2'$ is the derivative of $\mathcal{F}$ with respect to its argument $(Tr\,\tilde{h}^2/(Tr\,\tilde{h})^2)$ and similar for $\mathcal{F}_3'$. The second derivative of $V(\tilde{h})$ is

$$\frac{\partial^2 V(\tilde{h})}{\partial\tilde{h}^{kl}\partial\tilde{h}^{ij}} = \frac{g_{ij}}{3}\,\frac{\partial\mathcal{F}}{\partial\tilde{h}^{kl}} + \frac{g_{kl}}{3}\,\frac{\partial\mathcal{F}}{\partial\tilde{h}^{ij}} + \frac{Tr\,\tilde{h}}{3}\,\frac{\partial^2\mathcal{F}}{\partial\tilde{h}^{kl}\partial\tilde{h}^{ij}}\,, \tag{3.3.7}$$

with $(\partial^2\mathcal{F}/\partial\tilde{h}^{kl}\partial\tilde{h}^{ij})$ given by

$$\frac{\partial^2\mathcal{F}}{\partial\tilde{h}^{kl}\partial\tilde{h}^{ij}} = \sum_{p=2}^{3}\sum_{q=2}^{3}\mathcal{F}''_{pq}\,\frac{\partial}{\partial\tilde{h}^{ij}}\left(\frac{Tr\,\tilde{h}^p}{(Tr\,\tilde{h})^p}\right)\,\frac{\partial}{\partial\tilde{h}^{kl}}\left(\frac{Tr\,\tilde{h}^q}{(Tr\,\tilde{h})^q}\right) + \sum_{p=2}^{3}\mathcal{F}'_p\,\frac{\partial^2}{\partial\tilde{h}^{kl}\partial\tilde{h}^{ij}}\left(\frac{Tr\,\tilde{h}^p}{(Tr\,\tilde{h})^p}\right)\,, \tag{3.3.8}$$

where $\mathcal{F}''_{pq}$ stands for the derivative of $\mathcal{F}'_p$ with respect to its q argument and

$$\frac{\partial^2}{\partial\tilde{h}^{kl}\partial\tilde{h}^{ij}}\left(\frac{Tr\,\tilde{h}^2}{(Tr\,\tilde{h})^2}\right) = \frac{2}{(Tr\,\tilde{h})^2}\,\frac{\partial\tilde{h}_{ij}}{\partial\tilde{h}^{kl}} - \frac{4\,\tilde{h}_{ij}}{(Tr\,\tilde{h})^3}\,g_{kl} - \frac{4\,\tilde{h}_{kl}}{(Tr\,\tilde{h})^3}\,g_{ij} + \frac{6\,Tr\,\tilde{h}^2}{(Tr\,\tilde{h})^4}\,g_{ij}g_{kl}\,, \tag{3.3.9}$$

$$\frac{\partial^2}{\partial\tilde{h}^{kl}\partial\tilde{h}^{ij}}\left(\frac{Tr\,\tilde{h}^3}{(Tr\,\tilde{h})^3}\right) = \frac{3}{(Tr\,\tilde{h})^3}\,\frac{\partial\tilde{h}^2_{ij}}{\partial\tilde{h}^{kl}} - \frac{9\,\tilde{h}^2_{ij}}{(Tr\,\tilde{h})^4}\,g_{kl} - \frac{9\,\tilde{h}^2_{kl}}{(Tr\,\tilde{h})^4}\,g_{ij} + \frac{12\,Tr\,\tilde{h}^3}{(Tr\,\tilde{h})^5}\,g_{ij}g_{kl}\,, \tag{3.3.10}$$

with

$$\frac{\partial\tilde{h}_{ij}}{\partial\tilde{h}^{kl}} = g_{i(k}\,g_{l)j}\,, \tag{3.3.11}$$

$$\frac{\partial\tilde{h}^2_{ij}}{\partial\tilde{h}^{kl}} = g_{i(k}\,\tilde{h}_{l)j} + \tilde{h}_{i(k}\,g_{l)j}\,. \tag{3.3.12}$$

3.4 Linearised action

We are now going to linearise the $\mathfrak{g} = \mathfrak{su}(2)$ theory around the background (3.2.2). Thus, we take

$$B^i = B^i_o + b^i\,. \tag{3.4.1}$$

As we have already discussed, to linearise the kinetic BF term of the action we need to solve for the linearised connection if we can. This is certainly possible for the case at hand, as we shall now see.
If we denote the linearised connection by a^i, we have to solve the following system of equations

$$db^i + \epsilon^i{}_{jk}\, a^j \wedge B^k_o = 0\,, \tag{3.4.2}$$

where we have used the fact that the background connection is zero. It is convenient at this stage to replace all i-indices by a-ones, which we can do using the background object δ^i_a that provides such an identification. We can now use the self-duality $\epsilon^{\mu\nu\rho\sigma}\Sigma^a_{\mu\nu} = 2\mathrm{i}\Sigma^{a\,\mu\nu}$ of the background to rewrite this equation as

$$\frac{1}{2\mathrm{i}}\epsilon^{\mu\nu\rho\sigma}\partial_\nu b^a_{\rho\sigma} + \epsilon^{abc} a^b_\nu \Sigma^{c\,\mu\nu} = 0\,. \tag{3.4.3}$$

We now multiply this equation by $\Sigma^{a\,\alpha\beta}\Sigma^d_{\alpha\mu}$, and use the identity (3.2.7) to get

$$a^a_\beta = \frac{1}{2}\Sigma^{b\;\alpha}_{\beta}\,\Sigma^a_{\alpha\mu}\frac{1}{2\mathrm{i}}\epsilon^{\mu\nu\rho\sigma}\partial_\nu b^b_{\rho\sigma}, \qquad \text{or} \qquad a^a_\beta = \frac{1}{4\mathrm{i}}\Sigma^{b\;\alpha}_{\beta}\,\Sigma^a_{\alpha\mu}(\partial b^b)^\mu, \tag{3.4.4}$$

where we have introduced a compact notation

$$(\partial b^b)^\mu \equiv \epsilon^{\mu\nu\rho\sigma}\partial_\nu b^b_{\rho\sigma}\,, \tag{3.4.5}$$

for a multiple of the Hodge dual of the exterior derivative of the perturbation two-form b^i.

The BF part of the linearised action was obtained in (2.3.4). We need to divide the second variation given in this formula by 2 to get the correct action quadratic in the perturbation. Thus, we have

$$S^{(2)}_{\mathrm{BF}} = 2\mathrm{i}\int b^a \wedge da^a = -\mathrm{i}\int a^a_\mu (\partial b^a)^\mu, \tag{3.4.6}$$

where we have written everything in index notations and integrated by parts to put the derivative on $b^a_{\mu\nu}$, and used the definition (3.4.5). Now substituting (3.4.4) we get

$$S^{(2)}_{\mathrm{BF}} = \frac{1}{4}\int \eta^{\alpha\beta}\Sigma^a_{\alpha\mu}(\partial b^b)^\mu \Sigma^b_{\beta\nu}(\partial b^a)^\nu. \tag{3.4.7}$$

Let us now linearise the potential term. For this we need to know the value of $\tilde{h}^{ij}$ at the background as well as the matrices of first and second derivatives for the background.

Using (3.2.1) is easy to see that $\tilde{h}_o^{ij} = 2\mathrm{i}\delta^{ij}$. Since the background we are working with is just Minkowski we can now safely remove the density weight symbol from the matrix $\tilde{h}_o^{ij}$. Also, as before, let us replace all i-indices by a-indices using δ_a^i. Using (3.3.5) and the fact that the first derivatives $(\partial\mathcal{F}/\partial h^{ab})$ vanish on this background we immediately get

$$\left.\frac{\partial V}{\partial h^{ab}}\right|_o = \frac{\delta_{ab}}{3}\,\mathcal{F}_o, \tag{3.4.8}$$

where $\mathcal{F}_o$ is the background value of the function $\mathcal{F}$ in the parametrisation (3.3.1). It is not hard to see that this value plays the role of the cosmological constant of the theory, so in our Minkowski background it is necessarily zero by the background field equations. The matrix of second derivatives of the potential is easily evaluated using (3.3.7) and we find

$$\left.\frac{\partial^2 V}{\partial h^{cd}\partial h^{ab}}\right|_o = \frac{g_{\mathrm{gr}}}{2\mathrm{i}}\left(\delta_{a(c}\delta_{d)b} - \frac{1}{3}\delta_{ab}\delta_{cd}\right), \tag{3.4.9}$$

where we have introduced

$$g_{\mathrm{gr}} \equiv \sum_{p=2,3} \frac{(\mathcal{F}'_p)_o\, p(p-1)}{3^p}. \tag{3.4.10}$$

This is a constant of dimension of the cosmological constant, i.e., $1/L^2$. It is going to play a role of a parameter determining the strength of gravity modifications.

We can now write the linearised potential term (2.3.6). We must divide it by two to get the correct action for the perturbation. This gives

$$S^{(2)}_{\mathrm{BB}} = -\frac{g_{\mathrm{gr}}}{2}\int\left(\delta_{a(c}\delta_{d)b} - \frac{1}{3}\delta_{ab}\delta_{cd}\right)\left(\Sigma^{a\,\mu\nu}b^b_{\mu\nu}\right)\left(\Sigma^{c\,\rho\sigma}b^d_{\rho\sigma}\right). \tag{3.4.11}$$

Note that the tensor in brackets here is just the projector on the tracefree part. This fact will be important in our Hamiltonian analysis below. Our total linearised action is thus (3.4.7) plus (3.4.11).

3.5 Symmetries

The quadratic form obtained above is degenerate, and its degenerate directions correspond to the symmetries of the theory. These are not hard to write down. An obvious symmetry is that under (complexified) $SO(3)$ rotations of the fields. Considering an infinitesimal gauge transformation of the background $\Sigma^a_{\mu\nu}$, we find that the action must be invariant under the following set of transformations

$$\delta_\omega b^a_{\mu\nu} = \epsilon^{abc}\omega^b\Sigma^c_{\mu\nu}, \tag{3.5.1}$$

where ω^a are infinitesimal generators of the transformation. It is clear that (3.4.11) is invariant since it involves only the ab-symmetric part of $(\Sigma^{a\,\mu\nu} b^b_{\mu\nu})$, and the transformation (3.5.1) affects the anti-symmetric part.
Let us check the invariance of the kinetic term (3.4.7). We have the following expression for the variation

$$\frac{1}{2}\int \eta^{\alpha\beta}\Sigma^a_{\alpha\mu}(\partial\delta_\omega b^b)^\mu \Sigma^b_{\beta\nu}(\partial b^a)^\nu. \tag{3.5.2}$$

Substituting here the expression (3.5.1) for the variation we find

$$\eta^{\alpha\beta}\Sigma^a_{\alpha\mu}(\partial\delta_\omega b^b)^\mu \Sigma^b_{\beta\nu} = 2\mathrm{i}\eta^{\alpha\beta}\Sigma^a_{\alpha\mu}\epsilon^{bcd}\partial_\rho\omega^c\Sigma^{d\,\mu\rho}\Sigma^b_{\beta\nu} = 4\mathrm{i}\partial_\nu\omega^i, \tag{3.5.3}$$

where we have used the self-duality of $\Sigma^a_{\mu\nu}$ and applied the identity (3.2.7) once. Substituting this to (3.5.2) and integrating by parts to move the derivative from ω^a to b^a we get under the integral $\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu b^a_{\rho\sigma} = 0$, since the partial derivatives commute. This proves the invariance under gauge transformations.

Another set of symmetries of the action is that of diffeomorphisms. These are given by

$$\delta_\xi b^a = d(\xi \lrcorner \Sigma^a)\,, \tag{3.5.4}$$

where $\lrcorner$ is the operator of interior product. It is not hard to compute this explicitly in terms of derivatives of the components of the vector field. However, we do not need all the details of this two-form. Indeed, let us first note that the first "kinetic" term of the action is in fact invariant under a larger symmetry, i.e.,

$$\delta_\eta b^a = d\eta^a, \tag{3.5.5}$$

where η^a is an arbitrary Lie-algebra valued one-form. Indeed, this is obvious given that the kinetic term is constructed from the components of the 3-form db^a given by the exterior derivative of the perturbation two-form. Thus, (3.5.5) indeed leaves the kinetic term invariant. Then, since (3.5.4) is of the form (3.5.5) with $\eta^a = \xi \lrcorner \Sigma^a$ we have the invariance of the first term.
To see that the potential term (3.4.11) is invariant we should simply show that the symmetric tracefree part of the matrix $(\Sigma\,\delta_\xi b)^{ab}$ is zero. Let us compute the symmetric part explicitly. We have

$$\Sigma^{(a|\,\mu\nu}\partial_\mu\xi^\rho\Sigma^{|b)}_{\rho\nu} = \delta^{ab}\partial_\rho\xi^\rho, \tag{3.5.6}$$

where we have used (3.2.4). Thus, there is only the trace symmetric part, so the part that enters into the variation of the action (3.5.2) is zero. This proves the invariance under diffeomorphisms. Note that the second "potential" term is not invariant under all transformations (3.5.5), since for such a transformation that is not a diffeomorphism the matrix $(\Sigma\, \delta_\eta b)^{ab}$ contains a non-trivial symmetric tracefree part, as can be explicitly checked.

We will see that these are the only symmetries when we perform the Hamiltonian analysis. However, before we do this, let us show how the usual linearised GR appears from our theory.

3.6 Relation to GR

In this section we would like to describe how general relativity (linearised) with its usual gravitons appears from the linearised Lagrangian described above. We shall see that to get GR we must take the limit when the "mass" parameter $g_{\rm gr}$ for the components $(\Sigma\, b)^{ab}_{\rm tf}$, where "tf" stands for the tracefree part, is sent to infinity. Indeed, the potential part (3.4.11) depends precisely on these components, and when the parameter $g_{\rm gr}$ is sent to infinity these components are effectively set to zero. We shall now see that this gives GR.

It is not hard to show that in general the tracefree part $h^{\rm tf}_{\mu\nu} \equiv h_{\mu\nu} - (1/4)\eta_{\mu\nu}h^\rho_\rho$ of the metric perturbation $h_{\mu\nu}$, defined via $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, corresponds in our language of two-forms to the anti-self-dual part of the two-form perturbation [29], i.e.,

$$(b^a_{\mu\nu})_{\rm asd} = \Sigma^{a\ \ \rho}_{[\mu} h^{\rm tf}_{\nu]\rho}. \tag{3.6.1}$$

The fact that this two-form is anti-self-dual can be easily checked by contracting it with $\Sigma^{b\,\mu\nu}$ and using the algebra (3.2.4). The result is zero, as appropriate for an anti-self-dual two-form. In addition to (3.6.1) there is in general also the self-dual part of the two-form perturbation. However, in the limit $g_{\rm gr} \to \infty$ all but the trace part of this gets set to zero by the potential term. The trace part, on the other hand, is proportional to the trace part $\eta^{\mu\nu}h_{\mu\nu}$ of the metric perturbation. To simplify the analysis it is convenient to set this to zero $\eta^{\mu\nu}h_{\mu\nu} = 0$. This is allowed since in pure gravity the trace of the perturbation does not propagate. Then (3.6.1) is the complete two-form perturbation, and we can drop the "tf" symbol.

To simplify the analysis further, instead of deriving the full linearised action for the metric perturbation $h_{\mu\nu}$, let us work in the gauge where the perturbation is transverse $\partial^\mu h_{\mu\nu} = 0$. Let us then compute the quantity $(\partial b^a)^\mu$ in this gauge. Using anti-self-duality of $b^a_{\mu\nu}$ given by (3.6.1) we have

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu b^a_{\rho\sigma} = -2{\rm i}\partial_\nu b^{a\,\mu\nu}. \tag{3.6.2}$$

Substituting here the explicit expression (3.6.1) and using the transverse gauge condition we find

$$(\partial b^a)^\mu = \mathrm{i}\Sigma^{a\,\nu\rho}\,\partial_\nu h^\mu_\rho. \tag{3.6.3}$$

We can now substitute this into the action (3.4.7) to get

$$\begin{aligned} S^{(2)} &= -\frac{1}{4}\int \eta^{\alpha\beta}\Sigma^a_{\alpha\mu}\,\Sigma^{b\,\rho\sigma}\,\partial_\rho h^\mu_\sigma\,\Sigma^b_{\beta\nu}\,\Sigma^{a\,\gamma\delta}\,\partial_\gamma h^\nu_\delta \\ &= -\frac{1}{4}\int \eta^{\alpha\beta}(\delta^\gamma_\alpha\delta^\delta_\mu - \delta^\delta_\alpha\delta^\gamma_\mu - \mathrm{i}\epsilon_{\alpha\mu}{}^{\gamma\delta})(\delta^\rho_\beta\delta^\sigma_\nu - \delta^\sigma_\beta\delta^\rho_\nu - \mathrm{i}\epsilon_{\beta\nu}{}^{\rho\sigma})\partial_\rho h^\mu_\sigma\partial_\gamma h^\nu_\delta, \end{aligned} \tag{3.6.4}$$

where we have used (3.2.8) to get the second line. We can now contract the indices and take into account the tracefree as well as the transverse condition on $h_{\mu\nu}$. We get the following simple action as the result:

$$S^{(2)} = -\frac{1}{2}\int \partial_\mu h_{\rho\sigma}\partial^\mu h^{\rho\sigma}, \tag{3.6.5}$$

which is the correctly normalised transverse traceless graviton action. Note that in the passage to GR we have secretly assumed that $h_{\mu\nu}$ in (3.6.1) is a real metric perturbation. Below we will see how to impose the reality conditions on our theory that this comes out. Also note that the sign in front of (3.6.5) is correct for our choice of the signature being $(-,+,+,+)$.

3.7 Hamiltonian analysis of the linearised theory

For a finite parameter $g_{\rm gr}$ our theory describes a deformation of GR. Since not all components of the two-form perturbation $b^a_{\mu\nu}$ are dynamical, the nature of this deformation is most clearly seen in the Hamiltonian framework. This is what this section is about.

We note that the outcome of this subsection is that at "low" energies, $E^2 \ll g_{\rm gr}$, the modification can be ignored and one can safely work with the usual linearised GR. Let us start by analysing the kinetic BF part.

3.7.1 Kinetic term

Expanding the product of two Σ's in (3.4.7) using (3.2.4) we can rewrite the linearised Lagrangian density for the BF part as

$$\mathcal{L}_{\rm BF} = \frac{1}{4}\,(\partial b^a)^\mu(\partial b^b)^\nu\,\left(\epsilon^{abc}\,\Sigma^c_{\mu\nu} + \delta^{ab}\,\eta_{\mu\nu}\right). \tag{3.7.1}$$

Let us now perform the space plus time decomposition. Thus, we split the spacetime index as $\mu = (0, a)$, where $a = 1, 2, 3$. Note that we have denoted the spatial index by the same lower case Latin letter from the beginning of the alphabet that we are already using to denote the internal $\mathfrak{su}(2)$ index. This is allowed since we can use spatial projection of the $\Sigma^a_{\mu\nu}$ two-form to provide such an identification. Thus, from (3.2.1) we have

$$\Sigma^a_{bc} = -\epsilon^a{}_{bc}\,, \tag{3.7.2}$$

and

$$\Sigma^a_{0b} = \mathrm{i}\delta^a_b\,. \tag{3.7.3}$$

Let us now use these simple relations to obtain the space plus time decomposition of the Lagrangian. First, we need to know components of the $(\partial b^a)^\mu$ vector. The time component is given by

$$(\partial b^a)^0 = \epsilon^{0bcd}\partial_b b^a_{cd} = -\partial_b t^{ab}, \tag{3.7.4}$$

where our conventions are $\epsilon^{0abc} = -\epsilon^{abc}$ and we have introduced

$$t^{ab} \equiv \epsilon^{bcd} b^a_{cd}\,. \tag{3.7.5}$$

The spatial component of $(\partial b^a)^\mu$ is given by

$$(\partial b^a)^b = \epsilon^{b0cd}\partial_0 b^a_{cd} + 2\epsilon^{bc0d}\partial_c b^a_{0d} = \partial_0 t^{ab} - 2\epsilon^{bcd}\partial_c b^a_{0d}\,. \tag{3.7.6}$$

Now, the Lagrangian (3.7.1) is given by

$$\mathcal{L}_{\mathrm{BF}} = -\frac{1}{4}(\partial b^a)^0(\partial b^a)^0 + \frac{1}{2}(\partial b^a)^0(\partial b^b)^d\epsilon^{abc}\Sigma^c_{0d} + \frac{1}{4}(\partial b^a)^e(\partial b^b)^f(\epsilon^{abc}\Sigma^c_{ef} + \delta^{ab}\delta_{ef})\,. \tag{3.7.7}$$

Substituting the above expressions we get

$$\begin{aligned}\mathcal{L}_{\mathrm{BF}} = &-\frac{1}{4}\partial_b t^{ab}\partial_c t^{ac} - \frac{\mathrm{i}}{2}\partial_d t^{ad}(\partial_0 t^{bc} - 2\epsilon^{cef}\partial_e b^b_{0f})\epsilon^{abc} \\ &-\frac{1}{4}(\partial_0 t^{ae} - 2\epsilon^{emn}\partial_m b^a_{0n})(\partial_0 t^{bf} - 2\epsilon^{fpq}\partial_p b^b_{0q})(\epsilon^{abc}\epsilon^c{}_{ef} - \delta^{ab}\delta_{ef})\,.\end{aligned} \tag{3.7.8}$$

Our fields are now therefore b^a_{0b} and t^{ab}. There will also be another, potential part to this Lagrangian, but it does not contain time derivatives, so the canonically conjugate field can be determined already at this stage. Thus, it is clear that the field b^a_{0b} is non-dynamical since the Lagrangian does not depend on its time derivatives. The canonically conjugate field to t^{ab}, on the other hand, is given by

$$\pi^{ab} \equiv \frac{\partial\mathcal{L}_{\mathrm{BF}}}{\partial(\partial_0 t^{ab})} = -\frac{\mathrm{i}}{2}\epsilon^{abc}\partial_d t^{cd} - \frac{1}{2}(\partial_0 t^{ef} - 2\epsilon^{fpq}\partial_p b^e_{0q})(\epsilon^{aec}\epsilon^{cbf} - \delta^{ae}\delta^{bf})\,. \tag{3.7.9}$$

It is not hard to check that the canonically conjugate variable is simply related to the spatial projection of the connection (3.4.4) as

$$\pi^a_b = -2\mathrm{i}a^a_b. \tag{3.7.10}$$

To rewrite the Lagrangian in the Hamiltonian form one must solve for the velocities $\partial_0 t^{ab}$ in terms of the conjugate field π^{ab}. However, it is clear that not all the velocities can be solved for - there are constraints. A subset of these constraints is given by the $\mu = 0$ component of the (3.4.3) equation that, when written in terms of π^{ab}, becomes

$$\mathcal{G}^a \equiv \epsilon^{abc}\pi^{bc} + \mathrm{i}\partial_b t^{ab} = 0\,. \tag{3.7.11}$$

These are primary constraints that must be added to the Hamiltonian with Lagrange multipliers.

Thus, the expression for velocities in terms of the canonically conjugate field will contain undetermined functions. These functions are simply the a^a_0 components of the connection, as well as (at this stage undetermined) b^a_{0b} components of the two-form field. The expression for velocities is given by the spatial components of equation (3.4.3). After some algebra it gives

$$\partial_0 t^{ab} = 2\epsilon^{bef}\partial_e b^a_{0f} - 2\epsilon^{abc}a^c_0 - \epsilon^{aed}\epsilon^{dbf}\pi^{ef}\,. \tag{3.7.12}$$

Let us now obtain a slightly more convenient expression for the Lagrangian. Indeed, recall that using the compatibility equation between the connection and the perturbation two-form (3.4.2), we could have chosen to write our linearised action (3.4.6) as

$$S^{(2)}_{\mathrm{BF}} = -2\mathrm{i}\int \epsilon^{abc}\Sigma^a \wedge a^b \wedge a^c = -2\int \Sigma^{a\,\mu\nu}\epsilon^{abc}a^b_\mu a^c_\nu\,. \tag{3.7.13}$$

Introducing the time plus space split and writing the result in terms of the conjugate variable (3.7.10), we get the following Lagrangian

$$\mathcal{L}_{\mathrm{BF}} = -2\epsilon^{abc}\pi^{ab}a^c_0 - \frac{1}{2}\epsilon^{aef}\epsilon^{abc}\pi^{be}\pi^{cf}\,. \tag{3.7.14}$$

We can now easily find the BF part of the Hamiltonian, i.e.,

$$\mathcal{H}_{\mathrm{BF}} = \pi^{ab}\partial_0 t^{ab} - \mathcal{L}_{\mathrm{BF}} = 2\pi^{ab}\epsilon^{bef}\partial_e b^a_{0f} - \frac{1}{2}\epsilon^{aef}\epsilon^{abc}\pi^{be}\pi^{cf}\,. \tag{3.7.15}$$

We need to add to this the primary constraints (3.7.11) with Lagrange multipliers. Thus, the total Hamiltonian coming from the BF part of the action is

$$\mathcal{H}^{\mathrm{total}}_{\mathrm{BF}} = 2\pi^{ab}\epsilon^{bef}\partial_e b^a_{0f} - \frac{1}{2}\epsilon^{aef}\epsilon^{abc}\pi^{be}\pi^{cf} + \omega^a\mathcal{G}^a. \tag{3.7.16}$$

This is, of course, the standard result for the linearised BF Hamiltonian. If not for the potential term, the Hamiltonian would be a sum of terms generating the topological constraint $\partial_{[b}\pi^a_{c]} = 0$ and the Gauss constraint (3.7.11). Let us now consider the other BB part of the Lagrangian.

3.7.2 Potential term

We can rewrite the linearised Lagrangian density for the BB part (3.4.11) as

$$\mathcal{L}_{\rm BB} = -\frac{g_{\rm gr}}{2}\left(b^{(a}_{\mu\nu}\Sigma^{b)\mu\nu}\right)_{\rm tf}\left(b^{(a}_{\rho\sigma}\Sigma^{b)\rho\sigma}\right)_{\rm tf}, \tag{3.7.17}$$

where "tf" stands for the tracefree parts of the matrices. Splitting the space and time indices gives

$$\left(b^{(a}_{\mu\nu}\Sigma^{b)\mu\nu}\right)_{\rm tf} = -\left(2{\rm i}b_0^{(ab)} + t^{(ab)}\right)_{\rm tf}, \tag{3.7.18}$$

and so we have

$$\mathcal{L}_{\rm BB} = -\frac{g_{\rm gr}}{2}\left(2{\rm i}b_0^{(ab)} + t^{(ab)}\right)_{\rm tf}\left(2{\rm i}b_0^{(ab)} + t^{(ab)}\right)_{\rm tf}. \tag{3.7.19}$$

3.7.3 Analysis of the constraints

Thus, the total linearised Hamiltonian density $\mathcal{H} = \mathcal{H}^{\rm total}_{\rm BF} - \mathcal{L}_{\rm BB}$ is given by

$$\mathcal{H} = 2\pi^{ab}\epsilon^{bef}\partial_e b_0^{af} - \frac{1}{2}\epsilon^{aef}\epsilon^{abc}\pi^{be}\pi^{cf} + \omega^a\mathcal{G}^a + \frac{g_{\rm gr}}{2}\left(2{\rm i}b_0^{(ab)} + t^{(ab)}\right)_{\rm tf}\left(2{\rm i}b_0^{(ab)} + t^{(ab)}\right)_{\rm tf}. \tag{3.7.20}$$

It is now clear that only the anti-symmetric part and trace parts of b_0^{ab} remain Lagrange multipliers in the full theory. These are the generators of the diffeomorphisms. The other part of b_0^{ab}, namely the symmetric traceless is clearly non-dynamical and should be solved for from its field equations. Varying the Hamiltonian with respect to this symmetric tracefree part we get

$$\left(2{\rm i}b_0^{(ab)} + t^{(ab)}\right)_{\rm tf} = \frac{\rm i}{g_{\rm gr}}\left(\epsilon^{ef(a}\partial_e\pi_f^{b)}\right)_{\rm tf}. \tag{3.7.21}$$

Now writing

$$b_0^{ab} = {\rm i}N\delta^{ab} + \frac{1}{2}\epsilon^{abc}N^c + (b_0^{(ab)})_{\rm tf} \tag{3.7.22}$$

and substituting the symmetric tracefree part from (3.7.21) we get the following Hamiltonian

$$\begin{aligned}\mathcal{H} &= -2N{\rm i}\epsilon^{abc}\partial_a\pi_{bc} - 2\partial_{[a}\pi^a_{b]}N^b + \omega^a\mathcal{G}^a \\ &\quad - \frac{1}{2}\epsilon^{aef}\epsilon^{abc}\pi^{be}\pi^{cf} + {\rm i}\left(\epsilon^{ef(a}\partial_e\pi_f^{b)}\right)_{\rm tf}(t^{(ab)})_{\rm tf} + \frac{1}{2g_{\rm gr}}\left(\epsilon^{ef(a}\partial_e\pi_f^{b)}\right)_{\rm tf}\left(\epsilon^{pq(a}\partial_p\pi_q^{b)}\right)_{\rm tf}.\end{aligned} \tag{3.7.23}$$

The reason why we introduced a factor of i in front of the lapse function will become clear below. One can recognise in the first line the usual Hamiltonian, diffeomorphism and Gauss linearised constraints of Ashtekar's Hamiltonian formulation of general relativity [15]. The first two terms in the second line comprise the Hamiltonian. Finally, the last term is due to the modification and goes away in the limit $g_{\rm gr} \to \infty$.

It is not hard to show that the reduced phase space for the above system is obtained by considering π^{ab}, t^{ab} that are symmetric, traceless and transverse $\partial_a \pi^{ab} = 0, \partial_a t^{ab} = 0$. On such configurations the matrix $\epsilon^{efa}\partial^e \pi^{fb}$ is automatically symmetric traceless and transverse. The reduced phase space Hamiltonian density is then given by

$$\mathcal{H}^{\rm phys} = \frac{1}{2}(\pi^{ab})^2 + {\rm i}\epsilon^{efa}\partial^e t^{fb}\pi^{ab} + \frac{1}{2g_{\rm gr}}(\partial^a \pi^{bc})^2, \tag{3.7.24}$$

where we have integrated by parts and put the derivative on t^{ab} in the second term. This Hamiltonian is complex, so we need to discuss the reality conditions.

3.7.4 Reality conditions

So far our discussion was in terms of complex-valued fields. Thus, the reduced phase space obtained above after imposing the constraints and quotienting by their action was complex dimension $2+2$. Reality conditions need to be imposed to select the physical phase space corresponding to Lorentzian signature gravity.

In the case of GR, that corresponds to $g_{\rm gr} \to \infty$, the reality condition could be guessed from the form of the Hamiltonian (3.7.24). Indeed, we can write it as

$$\mathcal{H}^{\rm phys}_{\rm GR} = \frac{1}{2}\left(\pi^{ab} + {\rm i}\epsilon^{efa}\partial^e t^{fb}\right)^2 + \frac{1}{2}(\partial^a t^{bc})^2. \tag{3.7.25}$$

Thus, it is clear that we just need to require t^{ab} and $\pi^{ab} + {\rm i}\epsilon^{efa}\partial^e t^{fb}$ to be real. This procedure, however, does not work for the full Hamiltonian because of the last term in (3.7.24).

Let us now note that the last term in (3.7.24), when written in momentum space behaves as E^2/M^2, where E is the energy and $M^2 = g_{\rm gr}$ is the modification parameter. Thus, for energies $E \ll M$ the modification term is much smaller than the term π^2 and can be dropped. It is natural to expect that gravity is only modified close to the Planck scale, so it is natural to expect $M^2 \approx M_p^2$, where M_p is the Planck mass. With this assumption the last term in (3.7.24) is unimportant for "ordinary" energies and can be dropped. Thus, if we are to work at energies much smaller than the Planck scales ones then we do not need to go beyond GR described by the first two terms in (3.7.24).

The above discussion shows that a discussion of the reality conditions for the full Hamiltonian (3.7.24), even though possible and necessary if one is interested in the behaviour of the theory close to the Planck scale, is not needed if one only wants to work for with much smaller energies.

The "correct" reality conditions for the full modified gravity theory can be worked out from the condition $B^i \wedge (B^j)^* = 0$. In the linearised theory this becomes

$$\Sigma^a \wedge (b^b)^* = \overline{\Sigma}^b \wedge b^a, \qquad \text{or} \qquad \Sigma^{a\,\mu\nu}(b^b_{\mu\nu})^* + \overline{\Sigma}^{b\,\mu\nu} b^a_{\mu\nu} = 0, \tag{3.7.26}$$

where $(b^a)^*$ is the complex conjugate two-form perturbation and $\overline{\Sigma}$ is given by (3.2.3). We now rewrite this reality condition using the space plus time split. We get

$$\mathrm{i}(t^{ab} - (t^{ba})^*) + 2(b^{ab}_0 + (b^{ba}_0)^*) = 0. \tag{3.7.27}$$

To get this condition we have used $\overline{\Sigma}^a_{bc} = \epsilon^a{}_{bc}$, $\overline{\Sigma}^a_{0b} = \mathrm{i}\delta^a_b$ and recalled the definition (3.7.5) of the configurational variable. We should now analyse this condition together with the already known solution (3.7.22), (3.7.21) for the components b^{ab}_0.

Let us first consider the trace and anti-symmetric parts of (3.7.27). Then, in the tracefree symmetric gauge for t^{ab} these conditions simply state that the lapse and shift functions N, N^a are real. This explains why the factor of i was introduced in (3.7.22) in front of the lapse function N.

Consider now the symmetric tracefree part of (3.7.27). The corresponding components of b^{ab}_0 are known from (3.7.21) and we arrive at the following condition on the phase space variables

$$\frac{1}{2g_{\rm gr}} \mathrm{Re}\left(\epsilon^{ef(a}\partial_e \pi^{b)}_f\right)_{\rm tf} = \mathrm{Im}(t^{ab})_{\rm tf}. \tag{3.7.28}$$

In the case $g_{\rm gr} \to \infty$ that corresponds to GR this implies that $(t^{ab})_{\rm tf}$ is real, but in the modified case the situation is more interesting.

In addition to (3.7.28) there is another condition that is obtained by requiring that (3.7.28) is preserved under the evolution. Thus, we need to compute the Poisson bracket of (3.7.28) with the Hamiltonian and impose the resulting condition as well. Indeed, even in the case of GR it is clear from the form of the Hamiltonian (3.7.23) that the relevant condition cannot be that the canonically conjugate field is real, for the Hamiltonian would be complex due to the presence of the second term in the second line. The computation of the Poisson bracket can be done as follows. First, we introduce the real and imaginary parts of the phase space variables, i.e.,

$$t^{ab} = t^{ab}_1 + \mathrm{i}t^{ab}_2, \qquad \pi^{ab} = \pi^{ab}_1 + \mathrm{i}\pi^{ab}_2. \tag{3.7.29}$$

Second, we substitute this decomposition into the action written in the Hamiltonian form. The resulting action has real and imaginary parts. It is not hard to convince oneself that any one of these two parts can be used as an action for the system, the resulting equations are the same due to Riemann-Cauchy equations that follow from the fact that the original action was holomorphic. We choose to work with the real part of the action. The relevant Poisson brackets are easily seen to be

$$\{\pi_1^{ab}(x), t_{1\,cd}(y)\} = \delta_c^{(a}\delta_d^{b)}\delta^3(x-y)\,, \quad \{\pi_2^{ab},(x)t_{2\,cd}(y)\} = -\,\delta_c^{(a}\delta_d^{b)}\delta^3(x-y)\,, \tag{3.7.30}$$

with all the other ones being zero. The real part of the Hamiltonian (with the constraint part already imposed and dropped) reads

$$\mathcal{H}^{\text{real}} = \frac{1}{2}(\pi_1^{ab})^2 - \frac{1}{2}(\pi_2^{ab})^2 - \epsilon^{efa}\partial^e\pi_1^{bf}t_2^{ab} - \epsilon^{efa}\partial^e\pi_2^{bf}t_1^{ab} + \frac{1}{2g_{\text{gr}}}(\partial^a\pi_1^{bc})^2 - \frac{1}{2g_{\text{gr}}}(\partial^a\pi_2^{bc})^2\,. \tag{3.7.31}$$

We can now compute the Poisson bracket with the reality condition (3.7.28) that becomes

$$\frac{1}{2g_{\text{gr}}}\epsilon^{efa}\partial_e\pi_1^{fb} = t_2^{ab}\,. \tag{3.7.32}$$

The Poisson bracket with the left-hand-side is

$$\{\mathcal{H}^{\text{real}}, \frac{1}{2g_{\text{gr}}}\epsilon^{efa}\partial^e\pi_1^{bf}\} = -\frac{1}{2g_{\text{gr}}}\Delta\pi_2^{ab}\,, \tag{3.7.33}$$

where $\Delta = \partial_a\partial^a$ is the Laplacian. The Poisson bracket with the right-hand-side is

$$\{\mathcal{H}^{\text{real}}, t_2^{ab}\} = \pi_2^{ab} + \epsilon^{efa}\partial^e t_1^{bf} - \frac{1}{g_{\text{gr}}}\Delta\pi_2^{ab}\,. \tag{3.7.34}$$

Thus, the sought conditions that guarantees the consistency of (3.7.32) is

$$\pi_2^{ab} + \epsilon^{efa}\partial^e t_1^{bf} - \frac{1}{2g_{\text{gr}}}\Delta\pi_2^{ab} = 0\,. \tag{3.7.35}$$

We now need to solve this for π_2^{ab}, which gives

$$\pi_2^{ab} = -\frac{\epsilon^{efa}\partial^e t_1^{bf}}{1-\Delta/(2g_{\text{gr}})}\,, \tag{3.7.36}$$

where the denominator should be understood as a formal series in powers of Δ/g_{gr}. When $g_{\text{gr}} \to \infty$ we reproduce the GR result reviewed in the beginning of this subsection.

We now have to substitute this, as well as the expression (3.7.32) for t_2^{ab} into the action. This is a simple exercise with the result being

$$S^{\text{real}} = \int dt\, d^3x\, \left[\pi_{\text{GR}}^{ab}\partial_0 t_{\text{GR}}^{ab} - \frac{1}{2}\left((\pi_{\text{GR}}^{ab})^2 + (\partial^a t_{\text{GR}}^{bc})^2\right)\right], \tag{3.7.37}$$

where we have defined

$$\pi^{ab}_{\mathrm{GR}} = \pi^{ab}_1 , \qquad t^{ab}_{\mathrm{GR}} = \frac{t^{ab}_1}{1 - \Delta/(2g_{\mathrm{gr}})} . \tag{3.7.38}$$

These are the phase space variables in terms of which the Hamiltonian takes the standard GR form. This shows how an explicitly real formulation with a positive definite Hamiltonian can be obtained. We also see that for any finite value of g_{gr} the graviton is unmodified.

Now that we understood how the simple case $\mathfrak{g} = \mathfrak{su}(2)$ gives rise to gravity we can apply the same procedure to more interesting cases of larger Lie algebras.

CHAPTER 4

Gravity-Non-Linear Electrodynamics Unification

In this chapter we are going to study the case $G = \mathrm{GL}(2, \mathbb{C})$. We obtain a simple unified gravi-electro-magnetic theory, and our aim is to shed some light on its properties. To this end we first look at the pure electromagnetic sector of the model. This is obtained when the gravitational interactions are switched off by setting the gravitational fields to their Minkowski spacetime values. The resulting theory turns out to be just the most general non-linear electrodynamics with the Lagrangian being an arbitrary function of two invariants $E^2 - B^2, EB$. Such models have been studied in the literature in the past, see in particular [30] and works by Plebanski and co-authors, including [31]. The usual Maxwell Lagrangian can be obtained in a limit when some parameters of the defining potential function are sent to zero.

The general count of the degrees of freedom of this unified gravi-electro-magnetic theory establishes that it is a deformation of both Einstein and Maxwell theory with the key property that the number of propagating DOF described by this model is unchanged as compared to Einstein-Maxwell. The deformation is controlled by the potential function, see below, and if one so wishes can be switched off in a continuous fashion. Moreover, as we shall explain below, the deformation is only of significance at Planckian energies, while for low energies the theory with any generic choice of the defining potential is indistinguishable from Einstein-Maxwell. Another possible way to think about this unified model for $G = \mathrm{GL}(2, \mathbb{C})$ is that they arise by replacing the constraint term of the theory studied by Robinson[1] [32] by a potential term.

[1]In this paper, Robinson shows how a Plebanski type Lagrangian for the group $\mathrm{U}_{\mathbb{C}}(2) = \mathrm{GL}(2, \mathbb{C})$ describes (complexified) unified Einstein-Maxwell theory.

4.1 The action

The Lie algebra in this case is 4 (complex) dimensional and splits as $\mathfrak{g} = \mathfrak{su}_{\mathbb{C}}(2) \oplus \mathfrak{u}_{\mathbb{C}}(1)$. Up to rescalings, there is a unique invariant bilinear form in each factor. Thus, if we split $I = (i, 4)$, where $i = 1, 2, 3$ is a $\mathfrak{su}_{\mathbb{C}}(2)$ Lie algebra index, then the most general bilinear form is

$$\langle X, Y \rangle = \kappa_1 \delta_{ij} X^i Y^j + \kappa_2 X^4 Y^4, \tag{4.1.1}$$

where X^i, Y^i, X^4, Y^4 are components of X^I, Y^I and δ_{ij} is the usual invariant form on $\mathfrak{su}_{\mathbb{C}}(2)$. The curvature components are

$$F^i = dA^i + \frac{1}{2}\epsilon^i{}_{jk} A^j \wedge A^k, \qquad F^4 = dA^4, \tag{4.1.2}$$

where $\epsilon^i{}_{jk}$ are the $\mathfrak{su}_{\mathbb{C}}(2)$ structure constants.
The first BF term of the action then takes the following form

$$\mathrm{i}\kappa_1 \int \delta_{ij} B^i \wedge F^j + \mathrm{i}\kappa_2 \int B^4 \wedge dA^4. \tag{4.1.3}$$

Since the normalisations of the two-form fields are not yet fixed we can freely absorb the constants $\kappa_{1,2}$ into the fields, and we shall do so.

Let us now discuss the potential term. Let us introduce the following quantities:

$$\tilde{h}^{ij} \equiv \frac{1}{4}\tilde{\epsilon}^{\mu\nu\rho\sigma} B^i_{\mu\nu} B^j_{\rho\sigma}, \qquad \tilde{\phi}^i \equiv \frac{1}{4}\tilde{\epsilon}^{\mu\nu\rho\sigma} B^i_{\mu\nu} B^4_{\rho\sigma}, \qquad \tilde{\psi} \equiv \frac{1}{4}\tilde{\epsilon}^{\mu\nu\rho\sigma} B^4_{\mu\nu} B^4_{\rho\sigma}. \tag{4.1.4}$$

The matrix $\tilde{h}^{IJ}$ is then

$$\tilde{h}^{IJ} = \begin{pmatrix} \tilde{h}^{ij} & \tilde{\phi}^j \\ \tilde{\phi}^i & \tilde{\psi} \end{pmatrix}. \tag{4.1.5}$$

The G-invariants of this matrix are

$$\mathrm{Tr}(\tilde{h}^{ij}), \quad \mathrm{Tr}((\tilde{h}^{ij})^2), \quad \mathrm{Tr}((\tilde{h}^{ij})^3), \quad (\tilde{\phi})^2, \quad \tilde{\psi}, \tag{4.1.6}$$

where the traces of powers of the matrix $\tilde{h}^{ij}$ are computed using the invariant metric δ_{ij}, and $(\tilde{\phi})^2 = \delta_{ij}\tilde{\phi}^i\tilde{\phi}^j$. We can take any of these quantities as the basic one, and construct ratios of the other quantities and powers of the basic one to form quantities invariant under rescalings of B^I. It is convenient to choose as the basic quantity $\mathrm{Tr}(\tilde{h}^{ij})$. The potential function can then be written as

$$V(\tilde{h}^{IJ}) = \frac{\mathrm{Tr}(\tilde{h}^{ij})}{3} \mathcal{F}\left(\frac{\mathrm{Tr}((\tilde{h}^{ij})^2)}{(\mathrm{Tr}(\tilde{h}^{ij}))^2}, \frac{\mathrm{Tr}((\tilde{h}^{ij})^3)}{(\mathrm{Tr}(\tilde{h}^{ij}))^3}, \frac{(\tilde{\phi})^2}{(\mathrm{Tr}(\tilde{h}^{ij}))^2}, \frac{\tilde{\psi}}{\mathrm{Tr}(\tilde{h}^{ij})} \right), \tag{4.1.7}$$

where $\mathcal{F}$ is an arbitrary function of its 4 arguments.

The full action, written in terms of components of forms, is

$$-\,\mathrm{i}\int d^4x \left(\frac{1}{4}\tilde{\epsilon}^{\mu\nu\rho\sigma}(\delta_{ij}B^i_{\mu\nu}F^j_{\rho\sigma}+B^4_{\mu\nu}F^4_{\rho\sigma})-\frac{1}{2}V(\tilde{h}^{IJ})\right). \tag{4.1.8}$$

Varying this with respect to the two-form field components one can easily obtain the field equations. It is most compact to write them using the form notations, i.e.,

$$F^i=\frac{\partial V}{\partial\tilde{h}^{ij}}B^j+\frac{\partial V}{\partial(\tilde{\phi})^2}\tilde{\phi}^iB^4, \tag{4.1.9}$$

$$dA^4=\frac{\partial V}{\partial(\tilde{\phi})^2}\tilde{\phi}^iB^i+\frac{\partial V}{\partial\tilde{\psi}}B^4, \tag{4.1.10}$$

where all partial derivatives of the potential function can be obtained in an elementary way from (4.1.7). We note that it might appear that a factor of two is missing from the last term on the right-hand-side of the first equation, and the first term of the second. However, let us carefully compute the variation. We have, dropping unessential constant factors and the integral sign,

$$\frac{1}{2}\tilde{\epsilon}^{\mu\nu\rho\sigma}(\delta B^i_{\mu\nu}F^i_{\rho\sigma}+\delta B^4_{\mu\nu}F^4_{\rho\sigma})=\frac{\partial V}{\partial\tilde{h}^{ij}}\delta\tilde{h}^{ij}+\frac{\partial V}{\partial(\tilde{\phi})^2}2\tilde{\phi}^i\delta\tilde{\phi}^i+\frac{\partial V}{\partial\tilde{\psi}}\delta\tilde{\psi}\,. \tag{4.1.11}$$

Now, computing the variations on the right-hand-side from the definitions (4.1.4) we have

$$\begin{aligned}\delta\tilde{h}^{ij}&=\frac{1}{2}\tilde{\epsilon}^{\mu\nu\rho\sigma}\delta B^{(i}_{\mu\nu}B^{j)}_{\rho\sigma}\,,\\ \delta\tilde{\phi}^i&=\frac{1}{4}\tilde{\epsilon}^{\mu\nu\rho\sigma}\left(\delta B^i_{\mu\nu}B^4_{\rho\sigma}+B^i_{\mu\nu}\delta B^4_{\rho\sigma}\right),\\ \delta\tilde{\psi}&=\frac{1}{2}\tilde{\epsilon}^{\mu\nu\rho\sigma}\delta B^4_{\mu\nu}B^4_{\rho\sigma}\,.\end{aligned}$$

We now substitute these into (4.1.11) and equate to zero the coefficients in front of independent variations $\delta B^i_{\mu\nu}, \delta B^4_{\mu\nu}$. We get precisely (4.1.9), (4.1.10).

The equations obtained by varying the action with respect to the connection components are

$$dB^i+\epsilon^i{}_{jk}A^j\wedge B^k=0, \qquad dB^4=0. \tag{4.1.12}$$

The first equation here can be solved for the components of A^i in terms of the derivatives of B^i. One then substitutes the solution into (4.1.9) and obtains a second-order differential equation for B^i involving also B^4. The latter is found by integrating $dB^4=0$, and then the connection A^4 is found from (4.1.10). Below we shall see how this procedure works explicitly by working out the spherically-symmetric solution of our theory.

We also note that the equations of our theory are very similar to those of the unified theory [32], with the main difference being that the constraints $B^i\wedge B^j\sim\delta^{ij}$ and $B^i\wedge B^4=0$

of [32] are absent in our case. Related to this is the absence on the right-hand-side of the Lagrange multipliers that imposed those constraints. Their role is now played by the derivatives of the potential function. This is precisely analogous to what happens in the case of deformations of pure gravity, where the constraint term in the action is replaced by a potential term, and the Lagrange multipliers on the right-hand-side of field equations for B^i get replaced by $\partial V/\partial \tilde{h}^{ij}$. Thus, the theory that we are considering is a deformation of the Einstein-Maxwell theory of precisely the same type as the $\mathrm{SL}(2,\mathbb{C})$-based theory with a potential is a deformation of Einstein's GR. Similarly to the case of pure gravity, we shall see that it is possible to send some of the parameters of the potential to infinity to recover the usual Einstein-Maxwell theory. To understand how this happens, it is useful to first switch off the gravitational force, and consider what the theory under consideration becomes as a purely electromagnetic theory.

4.2 Non-linear electrodynamics

4.2.1 A version of non-linear electrodynamics

In this section we switch off the gravitational part of the theory by fixing the $\mathfrak{su}_{\mathbb{C}}(2)$ part of the 2-form field to be given by

$$B^i = \Sigma^i = \mathrm{i}dt \wedge dx^i - \frac{1}{2}\epsilon^i{}_{jk} dx^j \wedge dx^k \,, \tag{4.2.1}$$

which corresponds to the Minkowski spacetime background. We further expand the B^4 field into the basis of self- and anti-self-dual two-forms

$$B^4 = \Phi^i \Sigma^i + \Psi^i \overline{\Sigma}^i = (\Psi^i + \Phi^i)\mathrm{i}dt \wedge dx^i + (\Psi^i - \Phi^i)\frac{1}{2}\epsilon^{ijk} dx^j \wedge dx^k \,, \tag{4.2.2}$$

where Φ^i and Ψ^i are complex functions, and $\overline{\Sigma}^i$ are anti-self-dual two-forms

$$\overline{\Sigma}^i = \mathrm{i}dt \wedge dx^i + \frac{1}{2}\epsilon^i{}_{jk} dx^j \wedge dx^k \,. \tag{4.2.3}$$

We now compute the action (4.1.8) on this field configuration. Using $\Sigma^i \wedge \Sigma^j = -2\mathrm{i}\delta^{ij} d^4x$ we get

$$S[\Phi, \Psi, A^4] = \int d^4x \left((\Phi^i \Sigma^{i\,\mu\nu} - \Psi^i \overline{\Sigma}^{i\,\mu\nu})\partial_{[\mu} A^4_{\nu]} + \mathcal{F}(\Phi^2, \Phi^2 - \Psi^2) \right) \,, \tag{4.2.4}$$

where $\mathcal{F}$ is an arbitrary function of its two arguments. The action depends on fields Φ, Ψ, A^4 that are at this stage all complex. In anticipation of the reality conditions to be imposed on the connection A^4, let us rewrite the Lagrangian in terms of a new connection $\mathbf{A}$,

$$A^4 = \mathrm{i}\mathbf{A} \,. \tag{4.2.5}$$

We will later require this connection to be real, with the original A^4 thus being an U(1) connection. Using the explicit form of (4.2.1), (4.2.3) we have

$$S[\Phi,\Psi,\mathbf{A}] = \int d^4x \left((\Psi^i - \Phi^i)(\partial_0 \mathbf{A}_i - \partial_i \mathbf{A}_0) - \mathrm{i}(\Psi^i + \Phi^i)\epsilon_i{}^{jk}\partial_j \mathbf{A}_k + \mathcal{F}(\Phi^2, \Phi^2 - \Psi^2) \right) . \tag{4.2.6}$$

It is now clear that the combination $\Psi^i - \Phi^i$ plays the role of the canonically conjugate field to the spatial projection of the connection $\mathbf{A}$,

$$\mathbf{E}^i \equiv \Psi^i - \Phi^i \,, \tag{4.2.7}$$

and the combination

$$\mathbf{Q}^i \equiv \mathrm{i}(\Psi^i + \Phi^i) \tag{4.2.8}$$

is non-dynamical, to be eliminated via its field equation.
The action in the Hamiltonian form thus becomes

$$S[\mathbf{E},\mathbf{Q},\mathbf{A}] = \int d^4x \left(\mathbf{E}^i\partial_0 \mathbf{A}_i + \mathbf{A}_0 \partial_i \mathbf{E}^i - \mathbf{Q}^i \mathbf{B}_i + \mathcal{F}((\mathbf{E}^2 - \mathbf{Q}^2)/4 + (\mathrm{i}/2)\mathbf{EQ}, \mathrm{i}\mathbf{EQ}) \right) , \tag{4.2.9}$$

where we have introduced the magnetic field

$$\mathbf{B}_i \equiv \epsilon_i{}^{jk}\partial_j \mathbf{A}_k. \tag{4.2.10}$$

Once the field $\mathbf{Q}^i$ is eliminated by solving its field equation, we get the non-linear electrodynamics action in the Hamiltonian form, i.e.,

$$S[\mathbf{E},\mathbf{A}] = \int d^4x \left(\mathbf{E}^i\partial_0 \mathbf{A}_i + \mathbf{A}_0 \partial_i \mathbf{E}^i - H(\mathbf{E}, \mathrm{i}\mathbf{B}) \right) , \tag{4.2.11}$$

where H is the Legendre transform of the original potential function $\mathcal{F}$ with respect to the $\mathbf{Q}$ variable. Below we will see how this procedure works explicitly by working out the Lagrangian for the function $\mathcal{F}$ expanded in powers of its arguments. With the Hamiltonian density H being a Legendre transform of an arbitrary Lorentz-invariant function, this is the most general non-linear electrodynamics Lagrangian, see e.g. [30, 31]. The only difference with the Lagrangians typically considered in the literature is that in our case the dependence on the invariant $\mathbf{EB}$ is with a factor of $\mathrm{i} = \sqrt{-1}$ in front, and so it is in general complex even after the reality condition $\mathbf{A}, \mathbf{E} \in \mathbb{R}$ is imposed. The presence of this extra imaginary unit in the action makes the action invariant under a simultaneous operation of parity inversion and complex conjugation, similar to what happens in the case of the pure gravitational modified theory, see [29]. This is a very interesting feature of the class of theories considered, whose interpretation is still to be understood. In contrast, the non-linear electrodynamics real Hamiltonians containing odd powers of $\mathbf{EB}$ are, in general, parity violating (if the coefficients in front of these terms are taken to be usual scalars). It is however clear that the same constraint that is imposed on the Hamiltonian of the usual non-linear electrodynamics to have a parity-even theory in our case will produce a real Lagrangian. This will be our strategy for dealing with reality conditions below. We leave the more interesting case of non-Hermitian Hamiltonians containing odd powers of **EB** (and its physical interpretation) to further research.

Now, to get a better insight into this theory let us consider its linearisation, in which only terms quadratic in the fields are kept.

4.2.2 Linearised theory

Unlike considerations of the previous subsection where we have derived the action in the Hamiltonian form and kept $\mathbf{E}^i$ as an independent field, we will now integrate out all fields apart from the connection and produce a more familiar Lagrangian that depends only on the field strength. At the linearised level we should only keep the terms

$$\mathcal{F}^{(2)}(\Phi^2, \Phi^2 - \Psi^2) = \frac{\alpha}{2}\Phi^2 + \frac{\gamma}{2}(\Phi^2 - \Psi^2) \tag{4.2.12}$$

in the expansion of the function $\mathcal{F}$ in Taylor series, where α and γ are constant parameters. Once this is done, we can integrate out the fields Φ^i, Ψ^i from the action. The solutions for Φ^i, Ψ^i are given by

$$\Phi^i = -\frac{1}{\alpha+\gamma}\Sigma^{i\,\mu\nu}\partial_{[\mu}A^4_{\nu]}, \qquad \Psi^i = -\frac{1}{\gamma}\overline{\Sigma}^{i\,\mu\nu}\partial_{[\mu}A^4_{\nu]}\,, \tag{4.2.13}$$

and the resulting action is

$$S[A^4] = -\frac{1}{2}\int d^4x\left(\frac{1}{\alpha+\gamma}(\Sigma^{i\,\mu\nu}\partial_\mu A^4_\nu)^2 - \frac{1}{\gamma}(\overline{\Sigma}^{i\,\mu\nu}\partial_\mu A^4_\nu)^2\right). \tag{4.2.14}$$

Using the identities

$$\Sigma^{i\mu\nu}\Sigma^{i\rho\sigma} = 2\eta^{\mu[\rho}\eta^{\sigma]\nu} - \mathrm{i}\epsilon^{\mu\nu\rho\sigma}, \quad \overline{\Sigma}^{i\mu\nu}\overline{\Sigma}^{i\rho\sigma} = 2\eta^{\mu[\rho}\eta^{\sigma]\nu} + \mathrm{i}\epsilon^{\mu\nu\rho\sigma}\,, \tag{4.2.15}$$

we get

$$S[A^4] = \frac{1}{4}\left(\frac{1}{\gamma} - \frac{1}{\alpha+\gamma}\right)\int d^4x\, F^{4\,\mu\nu}F^4_{\mu\nu} + \frac{\mathrm{i}}{8}\left(\frac{1}{\gamma} + \frac{1}{\alpha+\gamma}\right)\int d^4x\, \epsilon^{\mu\nu\rho\sigma}F^4_{\mu\nu}F^4_{\rho\sigma}\,, \tag{4.2.16}$$

where $F^4_{\mu\nu} = \partial_\mu A^4_\nu - \partial_\nu A^4_\mu$. Thus, modulo the (purely imaginary) second term that is a total derivative, we get the following action

$$S[A^4] = \frac{\alpha}{4\gamma(\alpha+\gamma)}\int d^4x\, F^{4\,\mu\nu}F^4_{\mu\nu}\,. \tag{4.2.17}$$

Let us note that very little in the above analysis depends on the fact that the gravitational part of the two-form field was chosen to be (4.2.1). One can see that the procedure of integrating out the B^4 two-form field can be carried out in the same way whenever $B^i \wedge B^j \sim \delta^{ij}$. Thus, whenever the gravitational background is chosen to be "metric", in the sense that the Plebanski constraint $B^i \wedge B^j \sim \delta^{ij}$ is satisfied, it can be seen that the linearised electromagnetic Lagrangian is just the Maxwell one, with the metric being the one defined by declaring the two-forms B^i to span the space of self-dual two forms. This means that the linearised electromagnetic theory is the usual Maxwell electrodynamics not only when considered around the Minkowski spacetime, but for any fixed metric background. On the other hand, when the condition $B^i \wedge B^j \sim \delta^{ij}$ is not satisfied (non-metric case using the terminology of [17]), the linearised electromagnetic Lagrangian is different from that of Maxwell theory. This means that on a non-metric background light no longer has to follow geodesics of the metric defined by B^i. Of course, such non-metric backgrounds are only of significance in the high-energy regime (small distances). So, we can safely ignore them for low energies. Still, it would be interesting to study the effects of non-metricity on light propagation; we leave this to further research.

4.2.3 Linearised reality conditions

Assuming (for simplicity) that both α, γ are real and positive, we easily deduce the linearised level reality conditions that must be imposed on our fields. Thus, the condition that A^4 is purely imaginary, which is appropriate if we want to think of A^4 as the $\mathfrak{u}(1)$ component of a connection field, gives the correct Lorentzian signature action. Thus, for

$$A^4 = \mathrm{i}\mathbf{A}, \quad \mathbf{A} \in \mathbb{R}, \tag{4.2.18}$$

we get

$$S[\mathbf{A}] = -\frac{1}{4g^2_{\mathfrak{u}(1)}} \int d^4x\, F^{\mu\nu} F_{\mu\nu}, \tag{4.2.19}$$

where $F_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu$ is the field strength and the coupling constant is

$$g^2_{\mathfrak{u}(1)} = \frac{\gamma(\alpha+\gamma)}{\alpha}. \tag{4.2.20}$$

We now note that, if desired, we can obtain the [32] version of the electrodynamics in which the field B^4 is purely anti-self-dual by sending $\alpha \to \infty$. Indeed, as clear from (4.2.13), in this limit Φ^i that describes the self-dual part of B^4 goes to zero. In this limit the coupling constant of our Maxwell theory becomes $g^2_{\mathrm{u}(1)} = \gamma$. Thus, the theory considered in [32] is easily recovered.

When $\alpha \to \infty$ the two-form field B^4 becomes (proportional to) the anti-self-dual part of the real field strength $F_{\mu\nu}$. In the case of finite α the reality conditions that B^4 satisfies are much more involved. We find

$$\mathrm{i}B^4_{\mu\nu} = \frac{\alpha + 2\gamma}{\gamma(\alpha+\gamma)} F_{\mu\nu} + \frac{\alpha}{\gamma(\alpha+\gamma)} \frac{\mathrm{i}}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}. \tag{4.2.21}$$

The structure arising is typical for the theories under consideration in that the part of the expression that carries the $\epsilon_{\mu\nu\rho\sigma}$ tensor contains an additional factor of i as compared to the part that does not contain $\epsilon_{\mu\nu\rho\sigma}$.

4.2.4 Non-linear electrodynamics

Above we have analysed the theory with the potential function f truncated to its quadratic terms in the quantities Φ^2, Ψ^2. To understand the structure of the full non-linear theory we expand the potential and keep higher powers of Φ^2, Ψ^2. Thus, let us see what happens at the next order, which is quartic (Lorentz invariance prevents us from having any cubic terms). The quartic order part of the potential can be parametrised as

$$\mathcal{F}^{(4)}(\Phi^2, \Phi^2 - \Psi^2) = \frac{\delta_1}{4}(\Phi^2)^2 + \frac{\delta_2}{2}\Phi^2\Psi^2 + \frac{\delta_3}{4}(\Psi^2)^2, \tag{4.2.22}$$

where δ_1, δ_2 and δ_3 are constant parameters.
One can now vary the action (4.2.4) with respect to Φ^i, Ψ^i and solve for these fields perturbatively in powers of A^4. We get for the cubic order terms

$$\begin{aligned}
\Phi^{(3)\,i} &= \frac{1}{(\alpha+\gamma)^2} \Sigma^{i\,\mu\nu} \partial_\mu A^4_\nu \left(\frac{\delta_1}{(\alpha+\gamma)^2} (\Sigma\partial A^4)^2 + \frac{\delta_2}{\gamma^2} (\overline{\Sigma}\partial A^4)^2 \right), \\
\Psi^{(3)\,i} &= -\frac{1}{\gamma^2} \overline{\Sigma}^{i\,\mu\nu} \partial_\mu A^4_\nu \left(\frac{\delta_2}{(\alpha+\gamma)^2} (\Sigma\partial A^4)^2 + \frac{\delta_3}{\gamma^2} (\overline{\Sigma}\partial A^4)^2 \right),
\end{aligned} \tag{4.2.23}$$

where we have introduced a compact notation

$$(\Sigma\partial A^4)^2 \equiv \Sigma^{i\,\mu\nu} \partial_\mu A^4_\nu \Sigma^{i\,\rho\sigma} \partial_\rho A^4_\sigma = \frac{1}{2} F^{4\,\mu\nu} F^4_{\mu\nu} - \frac{\mathrm{i}}{4} \epsilon^{\mu\nu\rho\sigma} F^4_{\mu\nu} F^4_{\rho\sigma}, \tag{4.2.24}$$

and similarly for $(\overline{\Sigma}\partial A^4)^2$. Now we can compute the quartic order Lagrangian, with the result being

$$\mathcal{L}^{(4)} = \frac{\delta_1}{4(\alpha+\gamma)^4} ((\Sigma\partial A^4)^2)^2 + \frac{\delta_2}{2\gamma^2(\alpha+\gamma)^2} (\Sigma\partial A^4)^2 (\overline{\Sigma}\partial A^4)^2 + \frac{\delta_3}{4\gamma^4} ((\overline{\Sigma}\partial A^4)^2)^2. \tag{4.2.25}$$

This can be expanded in terms of the usual field strength invariants. Thus, using

$$(\epsilon^{\mu\nu\rho\sigma}F^4_{\mu\nu}F^4_{\rho\sigma})^2 = -8(F^{4\,\mu\nu}F^4_{\mu\nu})^2 + 16F^{4\nu}_{\mu}F^{4\rho}_{\nu}F^{4\sigma}_{\rho}F^{4\mu}_{\sigma}\,, \tag{4.2.26}$$

we get the following Lagrangian:

$$\begin{aligned}\mathcal{L}^{(4)} =& \frac{1}{16}(F^{4\,\mu\nu}F^4_{\mu\nu})^2\left(\frac{3\delta_1}{(\alpha+\gamma)^4} - \frac{2\delta_2}{\gamma^2(\alpha+\gamma)^2} + \frac{3\delta_3}{\gamma^4}\right) \\ &-\frac{1}{4}F^{4\nu}_{\mu}F^{4\rho}_{\nu}F^{4\sigma}_{\rho}F^{4\mu}_{\sigma}\left(\frac{\delta_1}{(\alpha+\gamma)^4} - \frac{2\delta_2}{\gamma^2(\alpha+\gamma)^2} + \frac{\delta_3}{\gamma^4}\right) \\ &-\frac{\mathrm{i}}{4}(F^{4\,\mu\nu}F^4_{\mu\nu})(\epsilon^{\alpha\beta\gamma\sigma}F^4_{\alpha\beta}F^4_{\gamma\sigma})\left(\frac{\delta_1}{(\alpha+\gamma)^4} - \frac{\delta_3}{\gamma^4}\right)\,.\end{aligned} \tag{4.2.27}$$

We can now substitute here the linearised reality conditions (4.2.18) and obtain the Lagrangian for the real-valued connection. However, we note that now, unlike what happened in the quadratic order of the theory, the imaginary term in the Lagrangian is no longer a total derivative. Thus, as we have already discussed above, the non-linear action for the real connection (4.2.18) is, in general, complex. This is precisely similar to what happens in the case of the effective gravitational Lagrangian, see [29]. There the metric Lagrangian that one gets from a similar BF-type theory with a potential (but in the case of $G = \mathrm{SL}(2,\mathbb{C})$) at cubic order in the curvature in general contains an imaginary term that is not a total derivative. Similar to what we are seeing here, in the purely gravitational case it is also the higher-order interaction term that is in general complex, while the theory linearised around the Minkowski background does not exhibit any complexity issues. We also note that, similar to what happens in the case [29] of pure gravity, the imaginary term is odd under parity. Thus, the full Lagrangian is invariant under the operation of complex conjugation accompanied by parity inversion.

4.2.5 Reality conditions

There are several strategies that one could follow when facing such a non-real Lagrangian. One, advocated in [29], is to impose the linearised reality conditions and take the real part of the full non-linear action. As was, however, realised more recently in the context of work [33] on the purely gravitational theory linearised around the expanding FRW background of relevance for cosmology, this real part of the action prescription does not in general produce a consistent theory. In the case of the FRW background the problem arises when one considers the gravitational waves (tensor perturbations).

Let us describe what the problem is in some more details. As in our electromagnetic considerations above, in the purely gravitational case one starts from the complex action and "integrates out" the non-dynamical fields to obtain an action that depends only on the physical metric. The action one gets is a functional depending holomorphically on the complex “physical" fields. One has to impose some reality conditions to extract the real action. Since the action depends on the complex fields holomorphically, one can instead consider the, say, real part of the action, and vary it with respect to real and imaginary parts of the fields. The arising Euler-Lagrange equations are the same as the real and imaginary parts of the complex Euler-Lagrange equations one obtains from the holomorphic action (this follows from Cauchy-Riemann equations). Thus, the real part of the holomorphic action considered as a functional of real and imaginary parts of all the fields carries exactly the same information as the original holomorphic action and can be taken as the action for the theory. However, this action depends on twice the number of physical fields, and, moreover, kinetic terms for the “imaginary" parts of the fields are typically negative-definite. Thus, this action describes twice the number of propagating modes of the physical theory, and is badly unstable. Reality conditions are needed to select a good physical sector of the theory, which describes half the modes of the complex sector and is void of any instability problems. It is natural to require that the reality conditions one imposes are some second-class constraints that cut the dimension of the phase space by half. However, as a consideration of simple examples shows, for an action that is obtained as the real part of a holomorphic action, it is in general not consistent to impose the constraint that the field is real. The reason for this is that the condition that this constraint is preserved in time generates a secondary constraint, and the condition that the secondary constraint is preserved in time in general produces a new constraint that is not equivalent to the original constraint of the reality of the field. Thus, in general, requiring the field to be real imposes more constraints than one would want. So, in general the dynamics of the Lagrangian such as (4.2.27) (or the real part of this Lagrangian with all fields complexified) is not consistent with the reality condition that the physical field is real. If one imposes this condition at some instant of time (and arranges the canonically conjugate field to be real as well), the dynamics will in general generate an imaginary part of the field. So, the strategy of dealing with the problem of reality conditions for non-real Lagrangians should be more sophisticated. We will leave any attempt at such to further research.

We note that, at least in our case of non-linear electrodynamics, we can restrict the potential function defining the theory so that the arising Lagrangian is real (for real connections). This is precisely what is usually done in the context of non-linear electrodynamics theories studied in the literature, where there is typically a restriction on the class of defining functions so that the theory is parity-invariant. Thus, in the case of our Lagrangian (4.2.27) we can arrange the coefficients in the expansion of the function $\mathcal{F}(\Phi^2, \Phi^2 - \Psi^2)$ in such a way that the coefficient in the last term in (4.2.27) is zero. We

can arrange things so that no imaginary terms arise in the higher orders of the expansion either. This will produce a consistent theory with a real Lagrangian as far as the electromagnetic sector is concerned.
In the next section we shall see that in the gravitational sector at least in the spherically-symmetric situation the reality condition that the metric is real is completely consistent.

4.3 Spherically-symmetric solution

In this section we obtain and analyse the spherically-symmetric solution of the gravi-electro-magnetic theory described above. As we have already noted above, on non-metric backgrounds where $B^i \wedge B^j \neq \delta^{ij}$ the coupling of electromagnetism to gravity as prescribed by our theory is different from that in the Maxwell case. Thus, there are two different ways that electromagnetism can be coupled to deformations of GR [17]: one way is to couple the electromagnetic potential to the metric defined by declaring the gravitational sector two-forms B^i to be self-dual. Such a coupling has been studied in [34], where also the spherically-symmetric solution was analysed. A different coupling is given by our theory. Thus, the spherically-symmetric case field equations that we shall analyse are distinct from those in [34]. We emphasise that this difference is only of relevance for very small scales (or Planckian curvatures). For low energies (large distances) the sperically-symmetric solutions of both [34] and this section become indistinguishable from Reissner-Nordström.

4.3.1 The spherically symmetric ansatz

We start by making an ansatz for all the fields as dictated by the symmetry. The gravitational $\mathfrak{su}(2)$ sector B-fields can be selected as in the purely gravitational case first studied in [35]. This reference has worked in spinor notations and used a complex null tetrad. However, it is not hard to repeat the analysis for a real tetrad for the usual spherically-symmetric metric asatz

$$ds^2 = -f^2 dt^2 + g^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \tag{4.3.1}$$

where as usual f, g are (real) functions of the radial coordinate r only. The starting point of the analysis is to construct the self-dual two-forms for this metric, see e.g. [36] for a description of this procedure for the case of Einstein's GR. The modified theory ansatz is then obtained by allowing for an extra functions of the radial coordinate multiplying the metric B-field ansatz of the GR case. Using the available coordinate freedom one can put the B-field in the following convenient form:

$$\begin{aligned} B^1 &= b(\mathrm{i} frdt \wedge d\theta - gr\sin\theta d\phi \wedge dr), \\ B^2 &= b(\mathrm{i} fr\sin\theta dt \wedge d\phi - grdr \wedge d\theta), \\ B^3 &= \mathrm{i} fgdt \wedge dr - r^2 \sin\theta d\theta \wedge d\phi\,, \end{aligned} \tag{4.3.2}$$

where b is a function of the radial coordinate. When $b = 1$ one gets the usual metric self-dual two-form ansatz of relevance for Einstein's GR, see [36]. As we already mentioned in the previous section, when the parameters of the potential of the electromagnetic sector are chosen so that the purely electromagnetic Lagrangian is real, the metric also turns out to be real. Thus, in the spherically-symmetric case one can assume that the metric is real from the start. We therefore assume the functions f, g and b, as well as the coordinate functions t, r, θ, ϕ to be real.

The ansatz that we make for the B^4 two-form field is a general combination of the "electric" and "magnetic" two-forms, i.e.,

$$B^4 = -2cr^2 \sin\theta d\theta \wedge d\phi + 2\mathrm{i} fgm\, dt \wedge dr\,, \tag{4.3.3}$$

where c, m are functions of r only, and the numerical constants are introduced for future convenience. No reality conditions on c, m are assumed at this stage.

4.3.2 B-compatible $\mathrm{GL}(2,\mathbb{C})$-connection

We now solve

$$DB^I = dB^I + C^I_{JK} A^J \wedge B^K = 0$$

for the connection. The gravitational $\mathfrak{su}(2)$ part of this "compatibility" equation reads

$$DB^i = dB^i + \epsilon^i_{jk} A^j \wedge B^k = 0, \tag{4.3.4}$$

which gives

$$\begin{aligned} A^1 &= -\frac{1}{bg}\sin\theta d\phi, \\ A^2 &= \frac{1}{bg} d\theta, \\ A^3 &= \frac{\mathrm{i} f}{g}\left[\frac{(bfr)'}{bfr} - \frac{1}{b^2 r}\right] dt + \cos\theta d\phi\,. \end{aligned} \tag{4.3.5}$$

For the $\mathfrak{u}(1)$ part of the compatibility equation we have $DB^4 = dB^4 = 0$. This implies

$$(cr^2)' = 0\,. \tag{4.3.6}$$

Note that we cannot solve this equation for A^4, so we will need to find the electromagnetic connection from another equation. For now, we make the following spherically-symmetric ansatz for it:

$$A^4 = \mathrm{i}a dt + \mathrm{i}p\cos\theta d\phi\,, \tag{4.3.7}$$

where a is, at this stage, arbitrary functions of r, and the imaginary unit is introduced in the expectation that later the reality condition will be imposed requiring the connection to be purely imaginary, as appropriate for a $\mathfrak{u}(1)$ connection. The spherical symmetry requires p to be a constant (proportional to the magnetic charge of our system).

We can now compute the curvature

$$F^I = dA^I + \frac{1}{2}C^I_{JK}A^J\wedge A^K \tag{4.3.8}$$

of the connection that we found (or made an ansatz for) above. We have for the gravitational sector $I = 1,2,3$,

$$\begin{aligned}
F^1 &= -\frac{1}{bg}\left\{\frac{\mathrm{i}f}{g}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]dt\wedge d\theta + bg\left(\frac{1}{bg}\right)'\sin\theta dr\wedge d\phi\right\},\\
F^2 &= -\frac{1}{bg}\left\{\frac{\mathrm{i}f}{g}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]\sin\theta dt\wedge d\phi - bg\left(\frac{1}{bg}\right)' dr\wedge d\theta\right\},\\
F^3 &= -\left\{\frac{\mathrm{i}f}{g}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]\right\}' dt\wedge dr - \left(1-\frac{1}{b^2g^2}\right)\sin\theta d\theta\wedge d\phi,
\end{aligned} \tag{4.3.9}$$

and for the electromagnetic field strength $I = 4$,

$$F^4 = dA^4 = -\mathrm{i}a' dt\wedge dr - \mathrm{i}p\sin(\theta)d\theta\wedge d\phi\,. \tag{4.3.10}$$

4.3.3 Field Equations

The remaining field equations to consider are given in (4.1.9) above. Defining the matrix h^{IJ} via

$$B^I\wedge B^J = h^{IJ}(-2\mathrm{i}fgr^2\sin\theta\, dt\wedge dr\wedge d\theta\wedge d\phi)\,, \tag{4.3.11}$$

we get

$$h^{IJ} = \begin{bmatrix} b^2 & 0 & 0 & 0\\ 0 & b^2 & 0 & 0\\ 0 & 0 & 1 & c+m\\ 0 & 0 & c+m & 4cm \end{bmatrix}. \tag{4.3.12}$$

We can now compute the derivatives of the potential (4.1.7) needed in (4.1.9). We find

$$\begin{aligned}\frac{\partial V}{\partial h^{ij}} =&\delta_{ij}\left(\frac{\mathcal{F}}{3}-\mathcal{F}'_1\frac{2(2b^4+1)}{3(2b^2+1)^2}-\mathcal{F}'_2\frac{2b^6+1}{(2b^2+1)^3}-\mathcal{F}'_3\frac{2(c+m)^2}{3(2b^2+1)^2}-\mathcal{F}'_4\frac{4cm}{3(2b^2+1)}\right)\\ &+\mathcal{F}'_1\frac{2h_{ij}}{3(2b^2+1)}+\mathcal{F}'_2\frac{(h^2)_{ij}}{(2b^2+1)^2}, \\ \frac{\partial V}{\partial \phi^2} =&\frac{\mathcal{F}'_3}{3(2b^2+1)}, \qquad\qquad \frac{\partial V}{\partial \psi}=\frac{\mathcal{F}'_4}{3}.\end{aligned} \tag{4.3.13}$$

Here $\mathcal{F}'_n$ is the derivative of the function $\mathcal{F}$ with respect to n-th argument evaluated at $h^{ij}=\mathrm{diag}(b^2,b^2,1), \phi^2=(c+m)^2, \psi=4cm$.

It turns out to be very convenient to separate the trace and the tracefree parts in the gravitational part, and introduce a separate notation for the electromagnetic part potential first derivatives. Thus, let us write the matrix of the first derivatives of the potential as

$$\frac{\partial V}{\partial h^{IJ}}=\begin{bmatrix}\Lambda-\beta & 0 & 0 & 0\\ 0 & \Lambda-\beta & 0 & 0\\ 0 & 0 & \Lambda+2\beta & \sigma\\ 0 & 0 & \sigma & \rho\end{bmatrix}, \tag{4.3.14}$$

where $\Lambda,\beta,\rho,\sigma$ are functions of b,c,m given by

$$\begin{aligned}\Lambda =&\frac{\mathcal{F}}{3}-\mathcal{F}'_1\frac{4(b^2-1)^2}{9(2b^2+1)^2}-\mathcal{F}'_2\frac{2(b^2-1)(b^4-1)}{3(2b^2+1)^3}-\mathcal{F}'_3\frac{2(c+m)^2}{3(2b^2+1)^2}-\mathcal{F}'_4\frac{4cm}{3(2b^2+1)},\\ \beta =&\mathcal{F}'_1\frac{2(1-b^2)}{9(2b^2+1)}+\mathcal{F}'_2\frac{(1-b^4)}{3(2b^2+1)^2},\\ \sigma =&\mathcal{F}'_3\frac{c+m}{3(2b^2+1)}, \qquad\qquad \rho=\frac{\mathcal{F}'_4}{3}.\end{aligned} \tag{4.3.15}$$

The field equations (4.1.9) then read, in the gravitational sector,

$$-\frac{1}{bgr}\left(\frac{1}{bg}\right)'=-\frac{1}{b^2g^2r}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]=\Lambda-\beta\,, \tag{4.3.16}$$

$$\frac{1}{r^2}\left(1-\frac{1}{b^2g^2}\right)=\Lambda+2\beta+2c\sigma\,, \tag{4.3.17}$$

$$-\frac{1}{fg}\left\{\frac{f}{g}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]\right\}'=\Lambda+2\beta+2m\sigma\,. \tag{4.3.18}$$

The electromagnetic sector equation gives the following two equations:

$$a'=-fg(\sigma+2\rho m), \qquad \mathrm{i}p=r^2(\sigma+2\rho c). \tag{4.3.19}$$

Before we analyse these equations let us describe a convenient change of independent functions that will eventually allow us to integrate the system.

4.3.4 Legendre Transformation

As was done in the purely gravitational spherically-symmetric case treated in [35], we can think of Λ as the Legendre transform of the function $\mathcal{F}$. In fact, we have

$$\Lambda = \frac{\mathcal{F}}{3} - x\beta - y\sigma - z\rho, \tag{4.3.20}$$

where

$$x = 2\frac{1-b^2}{2b^2+1}, \qquad y = 2\frac{c+m}{2b^2+1}, \qquad z = \frac{4cm}{2b^2+1}. \tag{4.3.21}$$

Thus, we get

$$\Lambda_\beta \equiv \frac{\partial \Lambda}{\partial \beta} = -x, \qquad \Lambda_\sigma \equiv \frac{\partial \Lambda}{\partial \sigma} = -y, \qquad \Lambda_\rho \equiv \frac{\partial \Lambda}{\partial \rho} = -z. \tag{4.3.22}$$

We can use these relations to express the original functions b, c, m appearing in our two-form field ansatz in terms of derivatives of the new function $\Lambda = \Lambda(\beta, \sigma, \rho)$. We find

$$b^2 = \frac{2+\Lambda_\beta}{2(1-\Lambda_\beta)}, \qquad c+m = -\frac{3\Lambda_\sigma}{2(1-\Lambda_\beta)}, \qquad 2cm = -\frac{3\Lambda_\rho}{2(1-\Lambda_\beta)}. \tag{4.3.23}$$

This gives, for c and m

$$\begin{aligned} c &= -\frac{3}{4}\left(\frac{\Lambda_\sigma}{1-\Lambda_\beta} + \sqrt{\frac{\Lambda_\sigma^2}{(1-\Lambda_\beta)^2} + \frac{4\Lambda_\rho}{3(1-\Lambda_\beta)}}\right), \\ m &= -\frac{3}{4}\left(\frac{\Lambda_\sigma}{1-\Lambda_\beta} - \sqrt{\frac{\Lambda_\sigma^2}{(1-\Lambda_\beta)^2} + \frac{4\Lambda_\rho}{3(1-\Lambda_\beta)}}\right). \end{aligned} \tag{4.3.24}$$

We have chosen the solution such that $m = 0$ for $\Lambda_\rho = 0$. Thus, one can now take the viewpoint that the theory is parametrised by the function $\Lambda = \Lambda(\beta, \sigma, \rho)$, and that the above relations give us the functions b, c, m once the quantities β, σ, ρ are solved for. This change of viewpoint will allow us to integrate the field equations.

4.3.5 Bianchi identities

A very powerful method for analysing the system of equations that we have obtained is by rewriting them as differential equations for the functions β, σ, ρ. These are nothing but the Bianchi identities obtained from the equation $DF^I = 0$. Alternatively, these equations can be obtained directly from the field equations (4.3.16)-(4.3.18). Thus, differentiating the equation (4.3.17), and using one of the equations in (4.3.16), as well as (4.3.6) we get

$$\Lambda' + 2\beta' + 2c\sigma' = -\frac{6\beta}{r}. \tag{4.3.25}$$

Another Bianchi identity is obtained by differentiating the second equation in (4.3.16). We have

$$\frac{1}{b^2g^2r}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]\left(\frac{2(bg)'}{bg}+\frac{1}{r}\right)-\frac{1}{b^2g^2r}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]'=\Lambda'-\beta'. \quad (4.3.26)$$

We now rewrite the equation (4.3.18) expanding the terms on the left and dividing the whole equation by b^2r. We find

$$\frac{1}{b^2g^2r}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]\left(-\frac{f'}{f}+\frac{g'}{g}\right)-\frac{1}{b^2g^2r}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]'=\frac{\Lambda-\beta+3\beta+2m\sigma}{b^2r}. \quad (4.3.27)$$

Using the second equation in (4.3.16) we express $\Lambda-\beta$ in terms of other quantities and then take this term to the left-hand-side of the equation. We get

$$\frac{1}{b^2g^2r}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]\left(\frac{1}{b^2r}-\frac{f'}{f}+\frac{g'}{g}\right)-\frac{1}{b^2g^2r}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]'=\frac{3\beta+2m\sigma}{b^2r}. \quad (4.3.28)$$

We now subtract (4.3.28) from (4.3.26). We obtain

$$\frac{1}{b^2g^2r}\left[\frac{(bfr)'}{bfr}-\frac{1}{b^2r}\right]\left(\frac{(b^2fg)'}{b^2fg}+\frac{b^2-1}{b^2r}\right)=\Lambda'-\beta'-\frac{3\beta+2m\sigma}{b^2r}. \quad (4.3.29)$$

We should now note that the following equation is true:

$$\frac{(b^2fg)'}{b^2fg}=\frac{1-b^2}{b^2r}. \quad (4.3.30)$$

Indeed, this is just a rewrite of the first equality in (4.3.16). Therefore, the quantity in the second brackets on the left-hand-side of (4.3.29) is zero and we get

$$\Lambda'-\beta'=\frac{3\beta+2m\sigma}{b^2r}. \quad (4.3.31)$$

Equations (4.3.25), (4.3.31), together with (4.3.6), after the functions b, c, m are expressed in terms of β, σ, ρ via (4.3.23), (4.3.24), become 3 first order differential equations for the 3 unknown functions β, σ, ρ. Once these are found, the electromagnetic connection is found from (4.3.19), and the metric functions g is found from

$$\frac{1}{(bg)^2}=1-(\Lambda+2\beta+2c\sigma)r^2, \quad (4.3.32)$$

which is easily obtained from (4.3.17). The metric function f is then found from (4.3.30).

4.3.6 Consistency

Yet another Bianchi identity can be obtained from $dF^4=0$, and is equivalent to the statement that the magnetic charge $p=const$. On the other hand, the second equation in (4.3.19) expresses the magnetic charge in terms of other functions. Differentiating this equation and using the, known from (4.3.6), derivative of c we get

$$\sigma' + 2\rho' c + \frac{2\sigma}{r} = 0\,. \tag{4.3.33}$$

This equation can be shown to follow from the two Bianchi identities (4.3.25), (4.3.31) and the relations (4.3.23). Thus, let us show that (4.3.33) together with (4.3.31) imply (4.3.25). We multiply (4.3.33) by $2m$ and use the last two identities in (4.3.23) to write the result as

$$\frac{4m\sigma}{r} = \frac{3\rho'\Lambda_\rho + 3\sigma'\Lambda_\sigma}{1-\Lambda_\beta} + 2\sigma' c\,. \tag{4.3.34}$$

We now use this, as well as the first identity in (4.3.23), to write $-2b^2$ times (4.3.31) as

$$(\beta' - \Lambda')\frac{2+\Lambda_\beta}{1-\Lambda_\beta} + \frac{3\rho'\Lambda_\rho + 3\sigma'\Lambda_\sigma}{1-\Lambda_\beta} + 2\sigma' c = -\frac{6\beta}{r}\,. \tag{4.3.35}$$

The first two terms on the left-hand-side combine to

$$\frac{2\beta' - 2\beta'\Lambda_\beta + \Lambda' - \Lambda'\Lambda_\beta}{1-\Lambda_\beta} = \Lambda' + 2\beta'\,. \tag{4.3.36}$$

Thus, (4.3.35) is just (4.3.25) and the obtained system of equations is consistent.

4.3.7 Non-metric gravity

In the limit $\Lambda_\sigma = \Lambda_\rho = 0$ the electromagnetic part of the theory is switched off and we recover the spherically symmetric solution [35] of non-metric gravity. The two Bianchi identities (4.3.25), (4.3.31) in this case coincide and give the following equation

$$(\Lambda_\beta + 2)\beta' = -\frac{6\beta}{r}\,, \tag{4.3.37}$$

for β. After this is solved the metric functions f, g are determined from (4.3.32), (4.3.30). For more details on the pure gravity sector solution see [35].

4.3.8 Reissner-Nordström solution

Let us now see how the usual Reissner-Nordström solution of GR coupled to Maxwell can be recovered. First, we should switch off the gravity modifications, which is done by putting $\Lambda_\beta = 0$ which gives $b^2 = 1$ and the gravitational part of the two-form field becomes the usual spherically-symmetric triple of metric two-forms. The simplest way to get the RN solution is to set $\Lambda_\sigma = 0$ so that $m = -c$ and the B^4 field (4.3.3) is anti-self-dual. However, let us see the appearance of the charged solution in full generality.

We will also allow the magnetic charge to be present, to illustrate how the issues of complexity should be dealt with.

First, we need to perform the Legendre transform of the original defining function $\mathcal{F}$. In a previous section we have seen that in the absence of gravity modifications, and in the case which gives the usual Maxwell theory, this function is given by (4.2.12). Inspection of (4.3.12) reveals that we have to replace $\Phi^2 \to (c+m)^2, \Phi^2 - \Psi^2 \to 4cm$. Thus, in the case that gives Maxwell theory our defining function is

$$\mathcal{F}(c,m) = \frac{\alpha}{2}(c+m)^2 + \frac{\gamma}{2}(4cm). \tag{4.3.38}$$

We then easily find σ, ρ from (4.3.15), i.e.,

$$\sigma = \frac{\partial \mathcal{F}}{\partial (c+m)^2}(c+m) = \frac{\alpha(c+m)}{2}, \qquad \rho = \frac{\partial \mathcal{F}}{\partial (4cm)} = \frac{\gamma}{2}. \tag{4.3.39}$$

The Legendre transform (4.3.20) now gives

$$\Lambda = -\frac{2}{3\alpha}\sigma^2. \tag{4.3.40}$$

Note that this is independent of ρ, as the original function was linear in ρ. However, the derivative Λ_ρ cannot be considered to be zero because it must satisfy the last equation in (4.3.23). Thus, in this case the parametrisation by Λ is somewhat degenerate. This can be dealt with by declaring the last equation in (4.3.23) to be satisfied by definition. This degeneracy is removed when one considers more complicated, non-linear dependence on cm.

We can now proceed to solving the equations. We first find σ from the second equation in (4.3.19). Using the value of ρ given by (4.3.39) we have

$$\sigma = \frac{\mathrm{i}p}{r^2} - \gamma c\,. \tag{4.3.41}$$

We now find m from the second equation in (4.3.23) and get

$$m = \frac{2\mathrm{i}p}{\alpha r^2} - c\frac{2\gamma+\alpha}{\alpha}\,, \tag{4.3.42}$$

where we have used $\Lambda_\beta = 0$. We now use the first equation in (4.3.19) and, in anticipation that no modification to the electromagnetic potential will be introduced, put $a' = -q/r^2$, where q is the usual (real) electric charge. This allows us to express the quantity c in terms of q, p. Using $fg = 1$ (which follows from (4.3.30)) we obtain

$$c = \frac{-\alpha q + (2\gamma+\alpha)\mathrm{i}p}{2\gamma r^2(\gamma+\alpha)}\,. \tag{4.3.43}$$

Note that this does have the required $1/r^2$ dependence on the radial coordinate. The above expression for c gives the following expression for σ, m

$$\sigma = \frac{\alpha(q+\mathrm{i}p)}{2r^2(\gamma+\alpha)}\,, \qquad m = \frac{(2\gamma+\alpha)q - \alpha\mathrm{i}p}{2\gamma r^2(\gamma+\alpha)}\,. \tag{4.3.44}$$

We note that all quantities σ, m, c became complex, so it is by no means obvious that one will arrive at a real metric at the end. Note also that, interestingly, the quantities c, m can be obtained one from another by exchanging $\mathrm{i}p \leftrightarrow q$.

We can now solve for the unknown function β using for example (4.3.25). Using $\Lambda' = -(4/3\alpha)\sigma\sigma'$ and putting all the terms not containing β to the right-hand-side we get

$$\beta' = -\frac{3\beta}{r} - \frac{\alpha(q+\mathrm{i}p)((2\gamma+3\alpha)q - (4\gamma+3\alpha)\mathrm{i}p)}{6\gamma r^5(\alpha+\gamma)^2}\,. \tag{4.3.45}$$

The solution with correct behaviour at infinity is

$$\beta = \frac{r_s}{2r^3} + \frac{\alpha(q+\mathrm{i}p)((2\gamma+3\alpha)q - (4\gamma+3\alpha)\mathrm{i}p)}{6\gamma r^4(\alpha+\gamma)^2}\,. \tag{4.3.46}$$

Note that this quantity is, when $p \neq 0$, complex even when the reality conditions are imposed.

We can finally find the metric functions from (4.3.32). The above analysis does not seem to make it plausible that the arising function g can be real. However, once we substitute all the quantities we have found above into (4.3.32) we obtain

$$g^{-2} = 1 - \frac{r_s}{r} + \frac{\alpha(q^2+p^2)}{2\gamma r^2(\alpha+\gamma)}\,. \tag{4.3.47}$$

Thus, the metric is the usual real Reissner-Nordström black hole with electric and magnetic charges provided we choose α, γ so that:

$$\frac{\alpha}{\gamma(\alpha+\gamma)} = 2\,, \tag{4.3.48}$$

which is exactly the condition expected from the formula (4.2.20) for the coupling constant.

To summarise, the analysis of this subsection confirms that there exist a two-parameter family of potentials giving rise to unmodified Einstein-Maxwell system. It also illustrates how non-trivial can the issue of reality become. Indeed, we have worked with complex quantities at intermediate stages of the computation, but at the end all the complexity disappeared to give rise to the real metric functions. This could have been expected from general considerations, since we have switched off the gravitational and Maxwell sector modifications. However, it is reassuring to see this happening explicitly.

Any departure from the simple choice of the defining potential considered in this subsection produces a modified theory, where one can either modify the gravitational sector, or electromagnetic, or both. In this subsection we have only covered the usual

Reissner-Nordström solution but black-hole geometries which take the modified theory into account may very well exist classically, and outside regimes of strong curvature. Although such solutions have not yet been provided, it is conceivable that they may exist, possibly offering observable consequences even in the domain of validity of a classical theory. At the same time it is gratifying to know that the theory is simple enough that the problem of determining such a solution for a general defining potential reduces to three first order ODE's for the functions β, σ, ρ.

CHAPTER 5

Gravity-$U(1)$-Gauge Field Unification

In this chapter we perform an analysis analogous to that in the previous chapters but taking a larger Lie algebra. We illustrate the general G case by considering the simplest non-trivial example of $G = SU(3)$. This example is rather generic, and the same technology that we develop for $G = SU(3)$ can be used for any Lie group. We could have presented a general semisimple case treatment phrased in terms of the root basis in the Lie algebra. However, we decided to keep our discussion as simple as possible and treat one example that, if necessary, is easily extendible to the general situation.

We start from the diffeomorphism invariant gauge theory (2.0.1) with Lie group complexified $SU(3)$, with certain reality conditions later imposed to select real physical configurations. A particularly simple solution of the theory describes Minkowski spacetime. This solution breaks $SU(3)$ down to a (complexified) $SU(2)$ times the centraliser of $SU(2)$ in $SU(3)$, i.e., $U(1)$. The spectrum of linearised theory around the Minkowski background is then shown to consist of the usual gravitons with their two propagating DOF, a gauge boson charged under the centraliser of $SU(2)$ in $SU(3)$, and a set of massive scalar fields. The mass of the scalar fields is related to a certain parameter of the potential defining the theory. After the reality conditions are imposed all sectors of the theory have a positive-definite Hamiltonian. We also work out the gravity $U(1)$-gauge field interactions to cubic order and show that they are precisely as expected, i.e., the $U(1)$-gauge field interacts with gravity via their stress-energy tensor. Thus, our unification scheme passes the zeroth order test of being not in any obvious contradiction with observations. However, to obtain a truly realistic unification model many problems have to be solved. Thus, our results provide only one of the first steps along this potentially interesting research direction.

For a general complex semisimple Lie group G, bigger than $SU(3)$, we will find the same structure as the one explained above but this time instead of getting a $U(1)$-gauge field we will get Yang-Mills fields in the centraliser of $SU(2)$ in G and therefore gauge bosons charged under this centraliser. The formalism is developed for the general case G but we restrict ourselves to the $SU(3)$ case in the analysis.

5.1 More general potentials

Up to now we have considered a very special class of potentials that depend only on the invariants constructed from the "internal" metric $\tilde{h}^{IJ}$ using the inner product g_{IJ}, i.e., traces of powers of $\tilde{h}^{IJ}$. However, it is clear that these are not the only possible invariants. Indeed, a more general gauge-invariant function of $\tilde{h}^{IJ}$ can also involve invariants constructed using the structure constants C^I_{JK} of the Lie algebra $\mathfrak{g}$ of G. For instance, let us consider

$$CC\tilde{h}\tilde{h}\tilde{h} \equiv C^{PQR}C^{STU}\tilde{h}_{PS}\tilde{h}_{QT}\tilde{h}_{RU}\,, \tag{5.1.1}$$

where the indices on the structure constants are raised using g^{IJ} (the inverse of g_{IJ}). More generally, one can construct a matrix

$$(CC\tilde{h}\tilde{h})^{IJ} \equiv C^{IQR}C^{JTU}\tilde{h}_{QT}\tilde{h}_{RU} \tag{5.1.2}$$

and build more complicated invariants from traces of powers of $\tilde{h}^{IJ}$ and $(CC\tilde{h}\tilde{h})^{IJ}$. This leads to a much more general set of gauge-invariant functions. In this chapter we shall study implications of such more general potentials. Our main point is that these more general potential functions lead naturally to "extra" massive fields. This is very important for phenomenology, for massless "extra" fields interacting with the "visible" Yang-Mills (or $U(1)$) sector in the standard way is obviously inconsistent with observations.

5.1.1 Potential with an extra invariant

For simplicity, we shall consider only one additional invariant given by (5.1.1). We shall see that such a potential is sufficient to generate masses for the "extra" sector particles. It is not hard to consider even more general potentials.

Thus, let us consider the potential, depending on one more invariant,

$$V(\tilde{h}) = \frac{Tr\,\tilde{h}}{n}\,\mathcal{F}\left(\frac{Tr\,\tilde{h}^2}{(Tr\,\tilde{h})^2}, \ldots, \frac{Tr\,\tilde{h}^n}{(Tr\,\tilde{h})^n}, \frac{CC\tilde{h}\tilde{h}\tilde{h}}{(Tr\,\tilde{h})^3}\right)\,, \tag{5.1.3}$$

where n is the dimension of the Lie algebra $\mathfrak{g}$ and we have divided (5.1.1) by $(Tr\,\tilde{h})^3$ to make the potential homogeneous degree one. Then, the first derivative with respect to $\tilde{h}$ is

$$\frac{\partial V(\tilde{h})}{\partial \tilde{h}^{IJ}} = \frac{g_{IJ}}{n}\,\mathcal{F} + \frac{Tr\,\tilde{h}}{n}\,\frac{\partial \mathcal{F}}{\partial \tilde{h}^{IJ}}\,, \tag{5.1.4}$$

with $(\partial\mathcal{F}/\partial\tilde{h}^{IJ})$ given by

$$\frac{\partial \mathcal{F}}{\partial \tilde{h}^{IJ}} = \sum_{p=2}^{n} \mathcal{F}'_p\,\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{Tr\,\tilde{h}^p}{(Tr\,\tilde{h})^p}\right) + \mathcal{F}'_{n+1}\,\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{CC\tilde{h}\tilde{h}\tilde{h}}{(Tr\,\tilde{h})^3}\right)\,, \tag{5.1.5}$$

where $\mathcal{F}'_p$ is the derivative of $\mathcal{F}$ with respect to its argument $(Tr\,\tilde{h}^p/(Tr\,\tilde{h})^p)$, $\mathcal{F}'_{n+1}$ is the derivative of $\mathcal{F}$ with respect to its last argument and

$$\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{Tr\,\tilde{h}^p}{(Tr\,\tilde{h})^p}\right) = p\,\left(\frac{\tilde{h}^{p-1}_{IJ}}{(Tr\,\tilde{h})^p} - \frac{Tr\,\tilde{h}^p}{(Tr\,\tilde{h})^{p+1}}\,g_{IJ}\right) \tag{5.1.6}$$

and

$$\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{CC\tilde{h}\tilde{h}\tilde{h}}{(Tr\,\tilde{h})^3}\right) = \frac{3\,C^{PQ}{}_{(I}C_{J)}{}^{RS}\,\tilde{h}_{PR}\tilde{h}_{QS}}{(Tr\,\tilde{h})^3} - \frac{3\,CC\tilde{h}\tilde{h}\tilde{h}}{(Tr\,\tilde{h})^4}\,g_{IJ}\,. \tag{5.1.7}$$

What we mean by $\tilde{h}^p_{IJ}$ above is

$$\tilde{h}^p_{IJ} = \tilde{h}_{IM_1}\,\tilde{h}^{M_1}{}_{M_2}\cdot\cdots\cdot\tilde{h}^{M_{p-1}}{}_J\,. \tag{5.1.8}$$

Now, let us compute the second derivative of V with respect to $\tilde{h}$. We get

$$\frac{\partial^2 V}{\partial \tilde{h}^{IJ}\partial \tilde{h}^{KL}} = \frac{g_{IJ}}{n}\,\frac{\partial \mathcal{F}}{\partial \tilde{h}^{KL}} + \frac{g_{KL}}{n}\,\frac{\partial \mathcal{F}}{\partial \tilde{h}^{IJ}} + \frac{Tr\,\tilde{h}}{n}\,\frac{\partial^2 \mathcal{F}}{\partial \tilde{h}^{IJ}\partial \tilde{h}^{KL}}\,, \tag{5.1.9}$$

with $(\partial^2\mathcal{F}/\partial\tilde{h}^{IJ}\partial\tilde{h}^{KL})$ given by

$$\begin{aligned}\frac{\partial^2 \mathcal{F}}{\partial \tilde{h}^{IJ}\partial \tilde{h}^{KL}} &= \sum_{p=2}^{n} \mathcal{F}'_p\,\frac{\partial^2}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}}\left(\frac{Tr\,\tilde{h}^p}{(Tr\,\tilde{h})^p}\right) + \mathcal{F}'_{n+1}\,\frac{\partial^2}{\partial \tilde{h}^{KL}\partial \tilde{h}^{IJ}}\left(\frac{CC\tilde{h}\tilde{h}\tilde{h}}{(Tr\,\tilde{h})^3}\right)\\ &+ \sum_{p=2}^{n}\sum_{q=2}^{n}\left(\mathcal{F}''_{pq}\frac{\partial}{\partial \tilde{h}^{KL}}\left(\frac{Tr\,\tilde{h}^q}{(Tr\,\tilde{h})^q}\right) + \mathcal{F}''_{p(n+1)}\frac{\partial}{\partial \tilde{h}^{KL}}\left(\frac{CC\tilde{h}\tilde{h}\tilde{h}}{(Tr\,\tilde{h})^3}\right)\right)\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{Tr\,\tilde{h}^p}{(Tr\,\tilde{h})^p}\right)\\ &+ \sum_{p=2}^{n}\left(\mathcal{F}''_{p(n+1)}\frac{\partial}{\partial \tilde{h}^{KL}}\left(\frac{Tr\,\tilde{h}^p}{(Tr\,\tilde{h})^p}\right) + \mathcal{F}''_{(n+1)(n+1)}\frac{\partial}{\partial \tilde{h}^{KL}}\left(\frac{CC\tilde{h}\tilde{h}\tilde{h}}{(Tr\,\tilde{h})^3}\right)\right)\frac{\partial}{\partial \tilde{h}^{IJ}}\left(\frac{CC\tilde{h}\tilde{h}\tilde{h}}{(Tr\,\tilde{h})^3}\right)\,,\end{aligned} \tag{5.1.10}$$

where $\mathcal{F}''_{pq}$ stands for the derivative of $\mathcal{F}'_p$ with respect to its q argument and similar for $\mathcal{F}''_{p(n+1)}$ and $\mathcal{F}''_{(n+1)(n+1)}$. Moreover, we have

$$\frac{\partial^2}{\partial \tilde{h}^{IJ}\partial \tilde{h}^{KL}}\left(\frac{Tr\tilde{h}^p}{(Tr\tilde{h})^p}\right) = \frac{p}{(Tr\tilde{h})^p}\frac{\partial \tilde{h}^{p-1}_{IJ}}{\partial \tilde{h}^{KL}} - \frac{p^2\,\tilde{h}^{p-1}_{IJ}}{(Tr\tilde{h})^{p+1}}g_{KL} - \frac{p^2\,\tilde{h}^{p-1}_{KL}}{(Tr\tilde{h})^{p+1}}g_{IJ} + \frac{p(p+1)\,Tr\,\tilde{h}^p}{(Tr\tilde{h})^{p+2}}g_{IJ}g_{KL}, \tag{5.1.11}$$

with

$$\frac{\partial \tilde{h}^p_{IJ}}{\partial \tilde{h}^{KL}} = g_{I(K}\,\tilde{h}^{p-1}_{L)J} + \tilde{h}_{I(K}\,\tilde{h}^{p-2}_{L)J} + \tilde{h}_{IM_1}\,\tilde{h}^{M_1}{}_{(K}\,\tilde{h}^{p-3}_{L)J} + \cdots\cdots + \tilde{h}^{p-2}_{I(K}\,\tilde{h}_{L)J} + \tilde{h}^{p-1}_{I(K}\,g_{L)J}\,, \tag{5.1.12}$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial\tilde{h}^{IJ}\partial\tilde{h}^{KL}}\left(\frac{CC\tilde{h}\tilde{h}\tilde{h}}{(Tr\,\tilde{h})^3}\right) = &-\frac{3}{(Tr\,\tilde{h})^3}\,\tilde{h}_{PQ}\,\left(C^P{}_{K(I}C_{J)L}{}^Q + C^P{}_{L(I}C_{J)K}{}^Q\right)\\
&-\frac{3^2}{(Tr\,\tilde{h})^4}\,\tilde{h}_{PQ}\,\tilde{h}_{RS}\,\left(C^{PR}{}_{(K}C_{L)}{}^{QS}g_{IJ} + C^{PR}{}_{(I}C_{J)}{}^{QS}g_{KL}\right)\\
&+\frac{3\cdot 4}{(Tr\,\tilde{h})^5}\,(CC\tilde{h}\tilde{h}\tilde{h})\,g_{IJ}\,g_{KL}\,. \qquad (5.1.13)
\end{aligned}
$$

With the above formulas for the first and second derivatives of this kind of potential it is relatively easy to find the linearised action for any semisimple Lie algebra. However, from now on in this chapter we will specialise to the $\mathfrak{g} = \mathfrak{su}(3)$ case. Let us start by reviewing some basic facts about the $\mathfrak{su}(3)$ Lie algebra.

5.2 Lie algebra of $SU(3)$

The defining matrix representation of the Lie algebra of $SU(3)$ consists of all traceless anti-hermitian 3 x 3 complex matrices. The standard basis for $\mathfrak{su}(3)$ space is given by the imaginary unit times a generalisation of Pauli matrices, known as Gell-Mann matrices. These hermitian matrices are given by

$$
\begin{aligned}
&\lambda_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix}0&-\mathrm{i}&0\\\mathrm{i}&0&0\\0&0&0\end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\\
&\lambda_4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix}0&0&-\mathrm{i}\\0&0&0\\\mathrm{i}&0&0\end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\\
&\lambda_7 = \begin{pmatrix}0&0&0\\0&0&-\mathrm{i}\\0&\mathrm{i}&0\end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}. \qquad (5.2.1)
\end{aligned}
$$

However, in our computations the Cartan-Weyl basis is going to be more convenient. Let us recall that in the Cartan-Weyl formalism one starts with the maximally commuting Cartan subalgebra, which in our case is spanned by two elements λ_3, λ_8. One then selects basis vectors that are eigenstates of the elements of Cartan under the adjoint action. This leads to the following basis, see [37, 38],

$$
\begin{aligned}
T_\pm =&\frac{1}{\sqrt{2}}(T_x \pm \mathrm{i}\,T_y)\,, \qquad V_\pm =\frac{1}{\sqrt{2}}(V_x \pm \mathrm{i}\,V_y)\,, \qquad W_\pm =\frac{1}{\sqrt{2}}(W_x \pm \mathrm{i}\,W_y)\,,\\
T_z =&\frac{1}{2}\lambda_3\,, \qquad Y =\frac{1}{2}\lambda_8\,, \qquad (5.2.2)
\end{aligned}
$$

$[\downarrow,\rightarrow]$	T_+	T_-	T_z	V_+	V_-	W_+	W_-	Y
T_+	0	T_z	$-T_+$	0	$-\frac{1}{\sqrt{2}}W_-$	$\frac{1}{\sqrt{2}}V_+$	0	0
T_-	$-T_z$	0	T_-	$\frac{1}{\sqrt{2}}W_+$	0	0	$-\frac{1}{\sqrt{2}}V_-$	0
T_z	T_+	$-T_-$	0	$\frac{1}{2}V_+$	$-\frac{1}{2}V_-$	$-\frac{1}{2}W_+$	$\frac{1}{2}W_-$	0
V_+	0	$-\frac{1}{\sqrt{2}}W_+$	$-\frac{1}{2}V_+$	0	$\frac{1}{2}(\sqrt{3}Y+T_z)$	0	$\frac{1}{\sqrt{2}}T_+$	$-\frac{\sqrt{3}}{2}V_+$
V_-	$\frac{1}{\sqrt{2}}W_-$	0	$\frac{1}{2}V_-$	$-\frac{1}{2}(\sqrt{3}Y+T_z)$	0	$-\frac{1}{\sqrt{2}}T_-$	0	$\frac{\sqrt{3}}{2}V_-$
W_+	$-\frac{1}{\sqrt{2}}V_+$	0	$\frac{1}{2}W_+$	0	$\frac{1}{\sqrt{2}}T_-$	0	$\frac{1}{2}(\sqrt{3}Y-T_z)$	$-\frac{\sqrt{3}}{2}W_+$
W_-	0	$\frac{1}{\sqrt{2}}V_-$	$-\frac{1}{2}W_-$	$-\frac{1}{\sqrt{2}}T_+$	0	$-\frac{1}{2}(\sqrt{3}Y-T_z)$	0	$\frac{\sqrt{3}}{2}W_-$
Y	0	0	0	$\frac{\sqrt{3}}{2}V_+$	$-\frac{\sqrt{3}}{2}V_-$	$\frac{\sqrt{3}}{2}W_+$	$-\frac{\sqrt{3}}{2}W_-$	0

Table 5.1: Commutators between $T_+, T_-, T_z, V_+, V_-, W_+, W_-, Y$.

where $T_x = \frac{1}{2}\lambda_1$, $T_y = \frac{1}{2}\lambda_2$, $V_x = \frac{1}{2}\lambda_4$, $V_y = \frac{1}{2}\lambda_5$, $W_x = \frac{1}{2}\lambda_6$ and $W_y = \frac{1}{2}\lambda_7$. Then the Cartan subalgebra is $H_i = \text{Span}(T_z, Y)$, and the commutator between any of the H_i's and the rest of the elements of the basis E_α, $E_\alpha = \{T_+, T_-, T_z, V_+, V_-, W_+, W_-\}$, is a multiple of E_α, i.e. $[H_i, E_\alpha] = \alpha_i\, E_\alpha$. One considers the α_i's, for $i = 1, 2$, as the components of a vector, called a root of the system. In this case we have six roots, i.e. $\{1, 0\}$, $\{-1, 0\}$, $\{\frac{1}{2}, \frac{\sqrt{3}}{2}\}$, $\{-\frac{1}{2}, -\frac{\sqrt{3}}{2}\}$, $\{-\frac{1}{2}, \frac{\sqrt{3}}{2}\}$, $\{\frac{1}{2}, -\frac{\sqrt{3}}{2}\}$. The Lie brackets between elements of this basis are given in Table 5.1. We also need to know the metric $g_{IJ} = -2\text{Tr}(T_I T_J)$ in this basis. It is given in Table 5.2.

5.3 Background

Let us now discuss how a background to expand around can be chosen. A background two-form field B^I_o is a map from the space of bivectors, which is 6-dimensional, to the Lie algebra in question. Thus, its image is at most a 6-dimensional subspace in $\mathfrak{su}(3)$. There are many different subspaces one can consider. Here we study the simplest possibility. Thus, we choose B^I_o such that the image of the space of bivectors that it produces in $\mathfrak{su}(3)$ is 3 dimensional. Moreover, we choose this image to be an $\mathfrak{su}(2)$ Lie subalgebra. Even further, we choose this subalgebra to be that spanned by $\{T_+, T_-, T_z\}$. Clearly, this is not the only $\mathfrak{su}(2)$ subalgebra in $\mathfrak{su}(3)$. Other possibilities include $\left\{V_+, V_-, \frac{1}{2}\left(\sqrt{3}Y + T_z\right)\right\}$ and $\left\{W_+, W_-, \frac{1}{2}\left(\sqrt{3}Y - T_z\right)\right\}$. Here we do not study these different possibilities. We believe that the example we choose to study is sufficiently illustrating.

$\langle\downarrow \mid \rightarrow\rangle$	T_+	T_-	T_z	V_+	V_-	W_+	W_-	Y
T_+	0	-1	0	0	0	0	0	0
T_-	-1	0	0	0	0	0	0	0
T_z	0	0	-1	0	0	0	0	0
V_+	0	0	0	0	-1	0	0	0
V_-	0	0	0	-1	0	0	0	0
W_+	0	0	0	0	0	0	-1	0
W_-	0	0	0	0	0	-1	0	0
Y	0	0	0	0	0	0	0	-1

Table 5.2: Components for the internal metric in the base $\{T_+, T_-, T_z, V_+, V_-, W_+, W_-, Y\}$.

Thus, our background is essentially the same as the one we considered in the gravity case. This is motivated by our desire to have the usual gravity theory arising as the part of the larger theory we are now considering. Since in the general Lie algebra context it is convenient to work with the Cartan-Weyl basis, we need to change the basis of the basic two-forms (3.2.1) as well. This can be worked out as follows. Before we were using a basis in the $\mathfrak{su}(2)$ Lie algebra in which the structure constants were given by ϵ_{abc}. If we denote the corresponding generators by X_a then $[X_a, X_b] = \epsilon_{ab}{}^c X_c$. On the other hand, for generators T_a used in (5.2.2) we have $[T_a, T_b] = \mathrm{i}\epsilon_{ab}{}^c T_c$. The relation between these two basis is $X_a = -\mathrm{i}T_a$. We can then define a new set of self-dual two-forms $\Sigma^\pm, \Sigma^z$ via

$$\Sigma \equiv \sum_{a=1,2,3} \Sigma^a X_a = \Sigma^+ T_+ + \Sigma^- T_- + \Sigma^z T_z \,. \tag{5.3.1}$$

This gives

$$\Sigma^+ = \frac{-\mathrm{i}}{\sqrt{2}}\left(\Sigma^1 - \mathrm{i}\,\Sigma^2\right) \qquad \Sigma^- = \frac{-\mathrm{i}}{\sqrt{2}}\left(\Sigma^1 + \mathrm{i}\,\Sigma^2\right) \qquad \Sigma^z = -\mathrm{i}\,\Sigma^3 \,. \tag{5.3.2}$$

This $\mathfrak{su}(2)$-valued two-form Σ is our background to expand about.

5.4 Linearisation: Kinetic term

As in the gravity case, the first step of the linearisation procedure is to solve for those components of the connection for which this is possible. As we have discussed in section 2.3, this is in general possible for the components of the connection in the directions in the Lie algebra that do not commute with the directions spanned by the background two-forms. In our case these are the directions spanned by $T_{\pm}, T_z$ and $V_{\pm}, W_{\pm}$. We already know how to solve for the connection components in the directions $T_{\pm}, T_z$. Indeed, the solution is given by (3.4.4) which we just have to rewrite in the different basis. It is, however, more practical to solve the equations once more by working in the different basis from the very beginning.

5.4.1 The $\mathfrak{su}(2)$ part

The $\mathfrak{su}(2)$ linearised compatibility equations sector in the Cartan-Weyl basis are

$$\begin{aligned} &db^+ + a^z \wedge \Sigma^+ - a^+ \wedge \Sigma^z = 0\,, \\ &db^- + a^- \wedge \Sigma^z - a^z \wedge \Sigma^- = 0\,, \\ &db^z + a^+ \wedge \Sigma^- - a^- \wedge \Sigma^+ = 0\,. \end{aligned} \tag{5.4.1}$$

We rewrite them in spacetime notations, take the Hodge dual, and use the self-duality of the $\Sigma^{\pm}, \Sigma^z$ to get

$$\begin{aligned} &\frac{1}{2\mathrm{i}}(\partial b^+)^\mu + a^z_\nu \Sigma^{+\,\mu\nu} - a^+_\nu \Sigma^{z\,\mu\nu} = 0\,, \\ &\frac{1}{2\mathrm{i}}(\partial b^-)^\mu + a^-_\nu \Sigma^{z\,\mu\nu} - a^z_\nu \Sigma^{-\,\mu\nu} = 0\,, \\ &\frac{1}{2\mathrm{i}}(\partial b^z)^\mu + a^+_\nu \Sigma^{-\,\mu\nu} - a^-_\nu \Sigma^{+\,\mu\nu} = 0\,, \end{aligned} \tag{5.4.2}$$

where the notation is, as before, $(\partial b)^\mu = \epsilon^{\mu\nu\rho\sigma}\partial_\nu b_{\rho\sigma}$. We now need the algebra of the new Σ's. It can be worked out from the relations (5.3.2) and the algebra (3.2.4). We get

$$\begin{aligned} \Sigma^+_{\mu\sigma}\,\Sigma^{-\sigma}{}_\nu &= \eta_{\mu\nu} + \Sigma^z_{\mu\nu}\,, & \Sigma^z_{\mu\sigma}\,\Sigma^{+\sigma}{}_\nu &= \Sigma^+_{\mu\nu}\,, & \Sigma^z_{\mu\sigma}\,\Sigma^{-\sigma}{}_\nu &= -\Sigma^-_{\mu\nu}\,, \\ \Sigma^z_{\mu\sigma}\,\Sigma^{z\sigma}{}_\nu &= \eta_{\mu\nu}\,, & \Sigma^+_{\mu\sigma}\,\Sigma^{+\sigma}{}_\nu &= 0\,, & \Sigma^-_{\mu\sigma}\,\Sigma^{-\sigma}{}_\nu &= 0\,. \end{aligned} \tag{5.4.3}$$

For purposes of the calculation it is very convenient to rewrite these relations in a schematic form, by viewing them as matrix algebra. Our matrix multiplication convention for the two-forms is $(XY)_\mu{}^\nu = X_\mu^\rho Y_\rho{}^\nu$. Then, we have

$$\begin{aligned} \Sigma^+\Sigma^- &= \eta + \Sigma^z\,, & \Sigma^z\Sigma^+ &= \Sigma^+\,, & \Sigma^z\Sigma^- &= -\Sigma^-\,, \\ \Sigma^z\Sigma^z &= \eta\,, & \Sigma^+\Sigma^+ &= 0\,, & \Sigma^-\Sigma^- &= 0\,. \end{aligned} \tag{5.4.4}$$

This is precisely the relations (3.2.4), just written in terms of metric and the structure constants of $\mathfrak{su}(2)$ for a different basis.

In our matrix product conventions, the equations (5.4.2) take the following transparent form:

$$\begin{aligned} &\frac{1}{2\mathrm{i}}(\partial b^+) + \Sigma^+ a^z - \Sigma^z a^+ = 0\,, \\ &\frac{1}{2\mathrm{i}}(\partial b^-) + \Sigma^z a^- - \Sigma^- a^z = 0\,, \\ &\frac{1}{2\mathrm{i}}(\partial b^z) + \Sigma^- a^+ - \Sigma^+ a^- = 0\,, \end{aligned} \tag{5.4.5}$$

where the convention is that the second spacetime index of Σ is contracted with the spacetime index of a.

We can now solve (5.4.5) by using the algebra (5.4.4). To this end we multiply the first equation by Σ^+ and the second one by Σ^-. This leads to two equations involving only $a^\pm$ but not a^z. We can obtain another two equations of the same sort by multiplying the last equation in (5.4.5) by $\Sigma^\pm$. Then adding-subtracting the resulting equations we get

$$a^+ = -\frac{1}{4\mathrm{i}}\left(\Sigma^-\Sigma^+(\partial b^+) + \Sigma^+(\partial b^z)\right), \qquad a^- = -\frac{1}{4\mathrm{i}}\left(\Sigma^+\Sigma^-(\partial b^-) - \Sigma^-(\partial b^z)\right). \tag{5.4.6}$$

To obtain the last component of the connection we multiply the first equation in (5.4.5) by Σ^- and second by Σ^+, and then subtract the resulting equations. We find $\Sigma^- a^+ - \Sigma^+ a^- = -(1/2\mathrm{i})(\partial b^z)$ using (5.4.6). Then, we have

$$a^z = -\frac{1}{4\mathrm{i}}\left((\partial b^z) + \Sigma^-(\partial b^+) - \Sigma^+(\partial b^-)\right). \tag{5.4.7}$$

It is now easy to write the $\mathfrak{su}(2)$ part of the linearised BF part of the action. Using the metric components given in Table 5.2, from (3.4.6) we have

$$\begin{aligned} S^{\rm grav}_{\rm BF} = -\frac{1}{4}\int &(\partial b^+)\left(\Sigma^+\Sigma^-(\partial b^-) - \Sigma^-(\partial b^z)\right) + (\partial b^-)\left(\Sigma^-\Sigma^+(\partial b^+) + \Sigma^+(\partial b^z)\right) \\ &+ (\partial b^z)\left((\partial b^z) + \Sigma^-(\partial b^+) - \Sigma^+(\partial b^-)\right), \end{aligned} \tag{5.4.8}$$

where again our convenient schematic form of the notation is used. This is simplified to give

$$S^{\rm grav}_{\rm BF} = -\frac{1}{2}\int (\partial b^+)(\eta + \Sigma^z)(\partial b^-) + (\partial b^-)\Sigma^+(\partial b^z) - (\partial b^+)\Sigma^-(\partial b^z) + \frac{1}{2}(\partial b^z)(\partial b^z). \tag{5.4.9}$$

We could now use this as the starting point of the Hamiltonian analysis similar to the one in the chapter about gravity. However, it is clear that its results are basis-independent, so we do not need to repeat it. Still, the above considerations are quite useful as a warm-up for the more involved analysis that now follows.

5.4.2 The part that does not commute with $\mathfrak{su}(2)$

Let us denote the four directions $V_\pm, W_\pm$ collectively by index $\alpha = 4,5,6,7$. We have to solve the following system of equations:

$$db^\alpha + C^\alpha{}_{\beta a}\, a^\beta \wedge \Sigma^a = 0\,, \tag{5.4.10}$$

where the terms $C^\alpha{}_{ab}\, a^a \wedge \Sigma^b$ are absent since the corresponding structure constants are zero. Explicitly, using table 5.1 we have

$$db^4 - \frac{1}{\sqrt{2}}\, a^6 \wedge \Sigma^+ - \frac{1}{2}\, a^4 \wedge \Sigma^z = 0\,, \tag{5.4.11}$$
$$db^5 + \frac{1}{\sqrt{2}}\, a^7 \wedge \Sigma^- + \frac{1}{2}\, a^5 \wedge \Sigma^z = 0\,, \tag{5.4.12}$$
$$db^6 - \frac{1}{\sqrt{2}}\, a^4 \wedge \Sigma^- + \frac{1}{2}\, a^6 \wedge \Sigma^z = 0\,, \tag{5.4.13}$$
$$db^7 + \frac{1}{\sqrt{2}}\, a^5 \wedge \Sigma^+ - \frac{1}{2}\, a^7 \wedge \Sigma^z = 0\,. \tag{5.4.14}$$

We can solve this system using the same technology that we used above for the $\mathfrak{su}(2)$ sector. Thus, we take the Hodge dual of the above equations, use the self-duality of the Σ's, and rewrite everything in the schematic matrix form. We get

$$\begin{aligned}
&\frac{1}{2\mathrm{i}}(\partial b^4) - \frac{1}{\sqrt{2}}\,\Sigma^+ a^6 - \frac{1}{2}\,\Sigma^z a^4 = 0\,,\\
&\frac{1}{2\mathrm{i}}(\partial b^5) + \frac{1}{\sqrt{2}}\,\Sigma^- a^7 + \frac{1}{2}\,\Sigma^z a^5 = 0\,,\\
&\frac{1}{2\mathrm{i}}(\partial b^6) - \frac{1}{\sqrt{2}}\,\Sigma^- a^4 + \frac{1}{2}\,\Sigma^z a^6 = 0\,,\\
&\frac{1}{2\mathrm{i}}(\partial b^7) + \frac{1}{\sqrt{2}}\,\Sigma^+ a^5 - \frac{1}{2}\,\Sigma^z a^7 = 0\,.
\end{aligned} \tag{5.4.15}$$

We can now manipulate these equations using the algebra (5.4.4). Thus, let us multiply the third equation by $\sqrt{2}\Sigma^+$ and subtract the result from the first equation. This gives

$$\frac{1}{2\mathrm{i}}(\partial b^4) - \frac{\sqrt{2}}{2\mathrm{i}}\Sigma^+(\partial b^6) + (\eta + \frac{1}{2}\Sigma^z)a^4 = 0\,. \tag{5.4.16}$$

It is now easy to find a^4 by noting that $(\eta + (1/2)\Sigma^z)^{-1} = (4/3)(\eta - (1/2)\Sigma^z)$. Thus, we have

$$a^4 = \frac{1}{3\mathrm{i}}\left(\sqrt{2}\Sigma^+(\partial b^6) - (2\eta - \Sigma^z)(\partial b^4)\right). \tag{5.4.17}$$

Similarly, we multiply the last equation by $\sqrt{2}\Sigma^-$ and add it to the second equation. Multiplying then by the inverse of $(\eta - (1/2)\Sigma^z)$ we get

$$a^5 = -\frac{1}{3\mathrm{i}}\left(\sqrt{2}\Sigma^-(\partial b^7) + (2\eta + \Sigma^z)(\partial b^5)\right). \tag{5.4.18}$$

To find a^6 we multiply the first equation by $\sqrt{2}\Sigma^-$ and subtract the result from the third equation. We then multiply the result by the inverse of $(\eta - (1/2)\Sigma^z)$. We obtain

$$a^6 = \frac{1}{3\mathrm{i}} \left(\sqrt{2}\Sigma^-(\partial b^4) - (2\eta + \Sigma^z)(\partial b^6) \right). \tag{5.4.19}$$

Finally, to find a^7 we multiply the second equation by $\sqrt{2}\Sigma^+$ and add the result to the last equation. Multiplying the result by the inverse of $(\eta + (1/2)\Sigma^z)$ we get

$$a^7 = -\frac{1}{3\mathrm{i}} \left(\sqrt{2}\Sigma^+(\partial b^5) + (2\eta - \Sigma^z)(\partial b^7) \right). \tag{5.4.20}$$

We should now substitute the above results into the relevant part of the action. We shall refer to this part of the action as "extra" because these are the extra fields that appear in addition to the gravitational and $U(1)$-gauge field. We have

$$S_{\mathrm{BF}}^{\mathrm{extra}} = \mathrm{i} \int a^4(\partial b^5) + a^5(\partial b^4) + a^6(\partial b^7) + a^7(\partial b^6), \tag{5.4.21}$$

where we took into account and extra minus sign that comes from the metric. Substituting here the above connections, we get, after some simple algebra,

$$\begin{aligned} S_{\mathrm{BF}}^{\mathrm{extra}} = \frac{2}{3} \int & \sqrt{2}(\partial b^5)\Sigma^+(\partial b^6) - \sqrt{2}(\partial b^4)\Sigma^-(\partial b^7) \\ & -(\partial b^4)(2\eta + \Sigma^z)(\partial b^5) - (\partial b^6)(2\eta - \Sigma^z)(\partial b^7). \end{aligned} \tag{5.4.22}$$

A more illuminating way to write this action is by introducing two two-component fields

$$\begin{pmatrix} b^4 \\ b^6 \end{pmatrix}, \qquad \begin{pmatrix} b^5 \\ b^7 \end{pmatrix}. \tag{5.4.23}$$

It is not hard to see that this split of the "extra" sector part of the Lie algebra is just the split into two irreducible representation spaces with respect to the action of the gravitational $\mathfrak{su}(2)$. In terms of these columns the above action takes the following form:

$$S_{\mathrm{BF}}^{\mathrm{extra}} = \frac{2}{3} \int \left((\partial b^5)\,(\partial b^7) \right) \begin{pmatrix} -2\eta + \Sigma^z & \sqrt{2}\Sigma^+ \\ \sqrt{2}\Sigma^- & -2\eta - \Sigma^z \end{pmatrix} \begin{pmatrix} (\partial b^4) \\ (\partial b^6) \end{pmatrix}. \tag{5.4.24}$$

Below we will use this action as the starting point for an analysis that will eventually exhibit the physical DOF propagating in this sector.

5.4.3 Centraliser $U(1)$ part

We cannot solve for the components of the connection in the part that commutes with $\mathfrak{su}(2)$. In our case this is the direction Y of the Lie algebra. We shall refer to this part of the action as "U(1)". Thus, the action remains of BF type, i.e.,

$$S_{\mathrm{BF}}^{\mathrm{U(1)}} = -4\mathrm{i} \int b^8 \wedge da^8, \tag{5.4.25}$$

where the extra minus sign is the one in the metric.

5.5 Linearisation: Potential term

As in the $\mathfrak{su}(2)$ case, our background internal metric $\tilde{h}_o^{IJ}$ is $2\mathrm{i}g^{ab}$ in the $\mathfrak{su}(2)$ directions and zero in all other directions. Since the background metric is flat we shall drop the tilde from $\tilde{h}^{IJ}$ in this section. We compute the matrix of first derivatives of the potential using (5.1.4). We get

$$\left.\frac{\partial V}{\partial h^{ab}}\right|_o = \frac{\mathcal{F}_o}{8}\, g_{ab}\,, \tag{5.5.1}$$

$$\left.\frac{\partial V}{\partial h^{a\alpha}}\right|_o = 0\,, \tag{5.5.2}$$

$$\left.\frac{\partial V}{\partial h^{\alpha\beta}}\right|_o = \left(\frac{\mathcal{F}_o}{8} - \frac{1}{8}\sum_{p=2}^{6}(\mathcal{F}'_p)_o\,\frac{p}{3^{p-1}} + \frac{2}{3}\frac{(\mathcal{F}'_{n+1})_o}{8}\right) g_{\alpha\beta}\,. \tag{5.5.3}$$

Here $\mathcal{F}_o$, $(\mathcal{F}'_p)_o$ are the value of the function and its derivatives at the background, and index α stands for all directions in the Lie algebra that are not in $\mathfrak{su}(2)$. The quantity $\mathcal{F}_o$ can be identified with a multiple of the cosmological constant. More specifically

$$\Lambda = -\frac{3}{8}\mathcal{F}_o\,. \tag{5.5.4}$$

Let us also define another two constants of dimensions $1/L^2$, i.e.,

$$\kappa \equiv \frac{1}{8}\sum_{p=2}^{6}(\mathcal{F}'_p)_o\,\frac{p}{3^{p-1}}. \tag{5.5.5}$$

and

$$\lambda \equiv \frac{(\mathcal{F}'_{n+1})_o}{8}\quad . \tag{5.5.6}$$

Then, we have

$$\left.\frac{\partial V}{\partial h^{\alpha\beta}}\right|_o = -\left(\frac{\Lambda}{3} + \kappa + \frac{2}{3}\lambda\right) g_{\alpha\beta}\,. \tag{5.5.7}$$

The sum in the previous formula is taken over $p = 2, \ldots, 6$, because the function $\mathcal{F}$ can at most depend on 5 ratios of 6 invariants of the matrix h^{IJ}. It has at most only 6 independent invariants since it is constructed from the map $B^I_{\mu\nu}$ that has the rank at most six. Since we want to work with the Minkowski spacetime background we should set $\Lambda = 0$, which we do in what follows.

We now need to compute the matrix of second derivatives. Let us first obtain its $\mathfrak{su}(2)$ part. Using (5.1.9) we obtain

$$\left.\frac{\partial^2 V}{\partial h^{ab}\partial h^{cd}}\right|_o = \frac{1}{2\mathrm{i}}\left(g - \frac{2}{3}\lambda\right)\left(g_{a(c}g_{d)b} - \frac{1}{3}g_{ab}g_{cd}\right)\,, \tag{5.5.8}$$

where we have defined

$$g = \frac{1}{8}\sum_{p=2}^{6}(\mathcal{F}'_p)_o\,\frac{p(p-1)}{3^{p-1}}. \tag{5.5.9}$$

We can define a new constant $\tilde{g} = g - 2/3\lambda$. As in the $\mathfrak{su}(2)$ case this constant $\tilde{g}$ is going to measure strength of gravity modifications. The κ, g and λ constants have dimensions of $1/L^2$ and are, in general, independent parameters of our linearised theory, related to first derivatives $(\mathcal{F}'_p)_o$ and $(\mathcal{F}'_{n+1})_o$ of the function $\mathcal{F}$ of the ratios.

Let us now compute the matrix of second derivatives in its part not in $\mathfrak{su}(2)$. We only need its mixed components $a\alpha$ and $b\beta$. The computation is easy and using (5.1.9) we get

$$\left.\frac{\partial^2 V}{\partial h^{a\alpha}\partial h^{b\beta}}\right|_o = \frac{\kappa}{4\mathrm{i}}\,g_{ab}g_{\alpha\beta} + \frac{\lambda}{6\mathrm{i}}\,g_{cd}\,C^c{}_{ab}\,C^d{}_{\alpha\beta}\,. \tag{5.5.10}$$

We can now compute all the potential parts. We use (2.3.6) which we have to divide by two to get the correct quadratic action. For the $\mathfrak{su}(2)$ gravitational part the result is unchanged from that in the gravity chapter with the only difference that instead of g_{gr} we have $\tilde{g}$, i.e.,

$$S^{\mathrm{grav}}_{\mathrm{BB}} = -\frac{\tilde{g}}{2}\int\left(g_{a(c}g_{d)b} - \frac{1}{3}g_{ab}g_{cd}\right)\left(\Sigma^{a\,\mu\nu}b^b_{\mu\nu}\right)\left(\Sigma^{c\,\rho\sigma}b^d_{\rho\sigma}\right). \tag{5.5.11}$$

The “extra" and “U(1)" parts of the potential term are both given by

$$\begin{aligned} S^{\mathrm{extra-U(1)}}_{\mathrm{BB}} =& -\frac{\kappa}{4}\int g_{ab}g_{\alpha\beta}(\Sigma^{a\,\mu\nu}b^\alpha_{\mu\nu})(\Sigma^{b\,\rho\sigma}b^\beta_{\rho\sigma}) + 2\mathrm{i}g_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma}b^\alpha_{\mu\nu}b^\beta_{\rho\sigma} \\ &-\frac{2\lambda}{3}\int\frac{1}{4}g_{cd}C^c_{ab}C^d_{\alpha\beta}(\Sigma^{a\,\mu\nu}b^\alpha_{\mu\nu})(\Sigma^{b\,\mu\nu}b^\beta_{\mu\nu}) + \frac{\mathrm{i}}{2}\,g_{\alpha\beta}\,\epsilon^{\mu\nu\rho\sigma}b^\alpha_{\mu\nu}b^\beta_{\rho\sigma}\,. \end{aligned} \tag{5.5.12}$$

so the indices α,β here take values $4,5,6,7,8$. We can further simplify this using (3.2.8). We get

$$\begin{aligned} S^{\mathrm{extra-U(1)}}_{\mathrm{BB}} =& -\kappa\int g_{\alpha\beta}b^{\alpha\,\mu\nu}b^{\beta\,\rho\sigma}P^-_{\mu\nu\rho\sigma} \\ &-\frac{2\lambda}{3}\int\frac{1}{4}g_{cd}C^c_{ab}C^d_{\alpha\beta}(\Sigma^{a\,\mu\nu}b^\alpha_{\mu\nu})(\Sigma^{b\,\mu\nu}b^\beta_{\mu\nu}) + \frac{\mathrm{i}}{2}\,g_{\alpha\beta}\,\epsilon^{\mu\nu\rho\sigma}b^\alpha_{\mu\nu}b^\beta_{\rho\sigma}\,, \end{aligned} \tag{5.5.13}$$

where

$$P^- = \frac{1}{2}\left(\eta_{\mu[\rho}\eta_{\sigma]\nu} + \frac{\mathrm{i}}{2}\epsilon_{\mu\nu\rho\sigma}\right) \tag{5.5.14}$$

is the anti-self-dual projector.

5.6 Symmetries

We have seen that the $\mathfrak{su}(2)$ sector of the theory is completely unchanged from what we have obtained in the chapter about gravity. One can moreover see that diffeomorphisms still act only within this sector. Indeed, the action of a diffeomorphism in the direction of a vector field ξ^μ is still given by (3.5.4) and only changes the $\mathfrak{su}(2)$ part of the two-form field. Similarly, the $SU(2)$ gauge transformations act only on the gravitational $\mathfrak{su}(2)$ sector. Thus, the gravity story that we have considered in the chapter above is unchanged.

Let us now consider what happens in directions not in $\mathfrak{su}(2)$. Let us first consider the "extra" sector spanned by $V_\pm, W_\pm$. A gauge transformation with the gauge parameter ω valued in this sector acts as $\delta_\omega b^\alpha = [\omega, \Sigma]^\alpha$, with $\alpha = 4, 5, 6, 7$. In components this reads

$$
\begin{aligned}
\delta_\omega b^4 &= -\frac{1}{\sqrt{2}}\,\omega^6\Sigma^+ - \frac{1}{2}\,\omega^4\Sigma^z\,,\\
\delta_\omega b^5 &= \frac{1}{\sqrt{2}}\,\omega^7\Sigma^- + \frac{1}{2}\,\omega^5\Sigma^z\,,\\
\delta_\omega b^6 &= -\frac{1}{\sqrt{2}}\,\omega^4\Sigma^- + \frac{1}{2}\,\omega^6\Sigma^z\,,\\
\delta_\omega b^7 &= \frac{1}{\sqrt{2}}\,\omega^5\Sigma^+ - \frac{1}{2}\,\omega^7\Sigma^z\,,
\end{aligned}
\tag{5.6.1}
$$

where we have used table 5.1. The remaining part of the Lie algebra is that spanned by Y. The corresponding gauge transformation has no effect on the two-form field b^8 (nor on $b^\alpha, \alpha = 4, 5, 6, 7$) since it commutes with the background. However this gauge transformation does act on the connection a^8 by the usual $U(1)$ gauge transformation $a^8 \to a^8 + d\omega^8$. The kinetic part (5.4.25) clearly remains invariant, and the potential part is invariant since it only depends on b^8 that does not transform.

5.7 U(1) sector

In this section we work out the Lagrangian for the sector of the theory which lives in the part of the Lie algebra that commutes with the background $\mathfrak{su}(2)$. The total Lagrangian we start with is a sum of kinetic term (5.4.25) and the potential term (5.5.13), with an extra sign in the potential term coming from the metric component $g_{88} = -1$. This gives

$$
\mathcal{L}^{\mathrm{U}(1)} = \mathrm{i}\,\epsilon^{\mu\nu\rho\sigma} b^8_{\mu\nu}\, F_{\rho\sigma} + \kappa\, P^{-\,\mu\nu\rho\sigma}\, b^8_{\mu\nu} b^8_{\rho\sigma} + \frac{\mathrm{i}\lambda}{3}\,\epsilon^{\mu\nu\rho\sigma} b^8_{\mu\nu} b^8_{\rho\sigma}\,,
\tag{5.7.1}
$$

where we have defined $F_{\mu\nu} = 2\partial_{[\mu}a^8_{\nu]}$. Taking a variation with respect to $b^8_{\mu\nu}$ we learn that

$$
\begin{aligned}
b^8_{\mu\nu} &= -\frac{1}{4(k+\frac{2\lambda}{3})}\left(\frac{6\kappa}{\lambda}\,P^{+\,\rho\sigma}{}_{\mu\nu} + 4\,\delta^{[\rho}_{\mu}\delta^{\sigma]}_{\nu}\right)F_{\rho\sigma} \\
&= -\frac{\left(\kappa+\frac{4\lambda}{3}\right)}{\frac{4\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,F_{\mu\nu} + \frac{\kappa}{\frac{4\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,\frac{\mathrm{i}}{2}\,\epsilon_{\mu\nu}{}^{\rho\sigma}\,F_{\rho\sigma}\,.
\end{aligned}
\tag{5.7.2}
$$

To get the first line of the above expression we have used the identities $P^{-\,\mu\nu\rho\sigma}P^{+}_{\rho\sigma\lambda\tau} = 0$, $P^{+}_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\lambda\tau} = 2\mathrm{i}\,P^{+\lambda\tau}{}_{\mu\nu}$ and $P^{-}_{\mu\nu\rho\sigma}\epsilon^{\rho\sigma\lambda\tau} = -2\mathrm{i}\,P^{+\lambda\tau}{}_{\mu\nu}$.

Substituting the result back into the Lagrangian we get

$$
\mathcal{L}^{\mathrm{U}(1)} = \frac{\kappa}{\frac{4\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{\left(\kappa+\frac{4\lambda}{3}\right)}{\frac{4\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,\frac{\mathrm{i}}{2}\,\epsilon^{\mu\nu\rho\sigma}\,F_{\mu\nu}\,F_{\rho\sigma}\,, \tag{5.7.3}
$$

This Lagrangian looks like the standard electrodynamics Lagrangian (first term) plus a total derivative (second term) or topological term. Now, if we want to think of a^8 as a $U(1)$ connection field we can define a new real connection $\tilde{a}^8$ as $a^8 = \mathrm{i}\,\tilde{a}^8$. Thus, in terms of $\tilde{a}$ the Lagrangian is written as

$$
\mathcal{L}^{\mathrm{U}(1)} = -\frac{\kappa}{\frac{4\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,\tilde{F}_{\mu\nu}\,\tilde{F}^{\mu\nu} + \frac{\left(\kappa+\frac{4\lambda}{3}\right)}{\frac{4\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,\frac{\mathrm{i}}{2}\,\epsilon^{\mu\nu\rho\sigma}\,\tilde{F}_{\mu\nu}\,\tilde{F}_{\rho\sigma}\,. \tag{5.7.4}
$$

The constant in from of $(\tilde{F}_{\mu\nu})^2$ has to be proportional to the coupling constant $g^2_{\mathrm{U}(1)}$. However, to convert this into a physical coupling constant we recall that we need to multiply the Lagrangian by $1/32\pi G$, with G the Newton constant, as this is exactly the prefactor that converts the canonically-normalised graviton Lagrangian (3.6.5) into the Einstein-Hilbert one. Thus, the physical coupling constant in our arising $U(1)$ gauge theory is given by

$$
g^2_{\mathrm{U}(1)} = \frac{32\pi G\lambda}{3\kappa}\left(\kappa+\frac{2\lambda}{3}\right)\,. \tag{5.7.5}
$$

5.8 Low energy limit of the “extra" sector

Note that the Planck mass M_p is the only scale in our theory, so all dimensionful quantities must be of the Planck size. This immediately implies that κ, λ, g must be taken to be of the order M_p^2. If this is the case then the role of the first term in (5.5.13), i.e, the one proportional to κ, is to make the anti-self-dual components of the two-forms $b^{\alpha}_{\mu\nu}$ “infinitely energetic" and thus effectively set them to zero[1]. This is quite similar to what happened in the gravitational sector in the limit $g_{\mathrm{gr}} \to \infty$ with the b^{ab}_{tf} components. Thus, we see that in the low energy limit $E^2 \ll \kappa$ the two-forms $b^{\alpha}_{\mu\nu}$ can be effectively

[1]For the interpretation of the term proportional to λ in (5.5.13) see the next section.

assumed to be self-dual. As such they can be expanded in the background self-dual two-forms $\Sigma^a_{\mu\nu}$. After such an ansatz is substituted into the action (5.4.24) the result simplifies considerably. However, in order to exhibit the physical modes we need to introduce some convenient gauge-fixing. Inspecting (5.6.1) we see that it is possible to set to zero the following components of the $b^{\alpha a}$:

$$b^4_+ = 0, \qquad b^5_- = 0, \qquad b^6_- = 0, \qquad b^7_+ = 0\,. \tag{5.8.1}$$

This gauge turns out to be very convenient. We now write the gauge-fixed self-dual two-forms $b^\alpha_{\mu\nu}$ as follows:

$$\begin{aligned}
b^4_{\mu\nu} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}b^4_-\,\Sigma^-_{\mu\nu} + \frac{\sqrt{3}}{2}b^4_z\,\Sigma^z_{\mu\nu}\right),\\
b^5_{\mu\nu} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}b^5_+\,\Sigma^+_{\mu\nu} + \frac{\sqrt{3}}{2}b^5_z\,\Sigma^z_{\mu\nu}\right),\\
b^6_{\mu\nu} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}b^6_+\,\Sigma^+_{\mu\nu} + \frac{\sqrt{3}}{2}b^6_z\,\Sigma^z_{\mu\nu}\right),\\
b^7_{\mu\nu} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}b^7_-\,\Sigma^-_{\mu\nu} + \frac{\sqrt{3}}{2}b^7_z\,\Sigma^z_{\mu\nu}\right),
\end{aligned} \tag{5.8.2}$$

where the independent fields are now b^4_-, b^5_+, b^6_+, b^7_-, b^α_z and the “strange" normalisation coefficients are chosen in order for the Lagrangian to be obtained to have the canonical form.

Substituting (5.8.2) into (5.4.24) and using the algebra of Σ's we get the following simple effective kinetic low-energy action[2]

$$S^{\text{extra}}_{\text{kin, eff}} = -\int \partial^\mu b^5_+\partial_\mu b^4_- + \partial^\mu b^7_-\partial_\mu b^6_+ + \partial^\mu b^5_z\partial_\mu b^4_z + \partial^\mu b^7_z\partial_\mu b^6_z\,. \tag{5.8.3}$$

This form of the Lagrangian makes the reality conditions necessary to get a real theory obvious. Indeed, it is clear that the reality conditions are

$$(b^5_+)^* = b^4_-\,, \qquad (b^7_-)^* = b^6_+\,, \qquad (b^5_z)^* = b^4_z\,, \qquad (b^7_z)^* = b^6_z\,. \tag{5.8.4}$$

These conditions can be compactly stated by introducing the following $\mathfrak{su}(2)\otimes\mathfrak{g}$ valued object:

$$\begin{aligned}
\mathbf{b} \equiv{}& \left(b^4_- T_+ + b^4_z T_z\right)\otimes V_+ + \left(b^5_+ T_- + b^5_z T_z\right)\otimes V_- \\
&+ \left(b^6_+ T_- + b^6_z T_z\right)\otimes W_+ + \left(b^7_- T_+ + b^7_z T_z\right)\otimes W_-
\end{aligned} \tag{5.8.5}$$

and requiring it to be hermitian, i.e.,

$$\mathbf{b}^\dagger = \mathbf{b}. \tag{5.8.6}$$

[2]The part of the action proportional to λ in (5.5.13) will be analyse in the next section.

The action can also be written quite compactly in terms of $\mathbf{b}$. Indeed, using the pairing given by the inner product $\langle\cdot,\cdot\rangle$ in the Lie algebra we get

$$\mathcal{L}^{\text{extra}}_{\text{kin, eff}} = -\langle\partial^\mu \mathbf{b}^\dagger, \partial_\mu \mathbf{b}\rangle\,, \tag{5.8.7}$$

for the low-energy $E^2 \ll \kappa$ kinetic effective "extra" sector Lagrangian. It is thus clear that, at least in the low energy regime, the "extra" sector of our theory consists just of 4 complex scalar fields with the usual Lagrangian. It is not hard to show that in the finite κ limit the content of this sector does not change.

5.9 "Extra" sector masses

In this section we show that the new parameter λ introduced above receives the interpretation of mass squared of the "extra" sector scalar fields. To this end, let us work out the quadratic part of the action that comes from the potential, concentrating only on the λ-dependent part, see (5.5.13). The κ-dependent part was already taken care of by setting the "extra" sector perturbation two-forms $b^\alpha_{\mu\nu}$ to be self-dual. Thus, the part of the action we want to analyse is

$$S^{(2)}_\lambda = -\frac{2\lambda}{3}\int \frac{1}{4} g_{cd} C^c_{ab} C^d_{\alpha\beta} (\Sigma^{a\,\mu\nu} b^\alpha_{\mu\nu})(\Sigma^{b\,\rho\sigma} b^\beta_{\rho\sigma}) - g_{\alpha\beta} b^{\alpha\,\mu\nu} b^\beta_{\mu\nu}\,, \tag{5.9.1}$$

where we have used the self-duality of $b^\alpha_{\mu\nu}$ in the second term. We now substitute in this expression the expansions (5.8.2) for our two-forms (in a specific gauge). It is not hard to see that only the term $C^z_{ab} C^z_{\alpha\beta}$ contributes, see table (5.1) and (5.2), and we find

$$S^{(2)}_\lambda = -\lambda \int b^4_- b^5_+ + b^6_+ b^7_- + b^4_z b^5_z + b^6_z b^7_z = -m^2_{\text{extra}} \langle \mathbf{b}^\dagger, \mathbf{b}\rangle\,, \tag{5.9.2}$$

where

$$m^2_{\text{extra}} = \lambda\,. \tag{5.9.3}$$

Thus, as all other physical parameters arising in our theory, the mass of the "extra" sector particles also comes from the defining potential.

5.10 Interactions

In this section we work out some of the cubic order interactions for our theory. Our main goal is to verify that the U(1) sector interacts with the gravitational sector in the usual way. We start with general considerations on the cubic order expansion of our theory. As it was done for the quadratic action, we first develop the formalism for a general Lie group G and then we specialise to the $G = SU(3)$ case.

5.10.1 General considerations

The third variation of the BF term is

$$\delta^3 S_{\rm BF} = 4\mathrm{i} \int 3\, g_{IJ}\, \delta B^I \wedge [\delta A, \delta A]^J \,, \tag{5.10.1}$$

and the third variation of the BB term is

$$\delta^3 S_{\rm BB} = 4\mathrm{i} \int d^4x \left(4 \frac{\partial^3 V}{\partial \tilde{h}^{IJ} \partial \tilde{h}^{KL} \partial \tilde{h}^{MN}} \, (B_0 \delta B)^{IJ} (B_0 \delta B)^{KL} (B_0 \delta B)^{MN} \right. \\ \left. + 6 \, \frac{\partial^2 V}{\partial \tilde{h}^{IJ} \partial \tilde{h}^{KL}} \, (B_0 \delta B)^{IJ} (\delta B \delta B)^{KL} \right) . \tag{5.10.2}$$

To compute the third derivatives of the potential that are needed to get the gravitational-gauge fields interaction Lagrangian we are going to use the property of the potential function $V(h)$ of being homogeneous of degree one. Thus, $V(h)$ being homogeneous of degree one, it can be written as

$$V = \tilde{h}^{PQ} \, \frac{\partial V}{\partial \tilde{h}^{PQ}} \,. \tag{5.10.3}$$

Taking the derivative of the above equation with respect to $\tilde{h}^{IJ}$ and again with respect to $\tilde{h}^{KL}$ we get

$$\frac{\partial^2 V}{\partial \tilde{h}^{IJ} \partial \tilde{h}^{PQ}} \, \tilde{h}^{PQ} = 0 \,, \tag{5.10.4}$$

and

$$\frac{\partial^3 V}{\partial \tilde{h}^{IJ} \partial \tilde{h}^{KL} \partial \tilde{h}^{MN}} \, \tilde{h}^{MN} = - \frac{\partial^2 V}{\partial \tilde{h}^{IJ} \partial \tilde{h}^{KL}} \,. \tag{5.10.5}$$

Evaluating both equations above at the background we obtain

$$\left. \frac{\partial^2 V}{\partial h^{IJ} \partial h^{ab}} \right|_o g^{ab} = 0 \,, \tag{5.10.6}$$

and

$$\left. \frac{\partial^3 V}{\partial h^{IJ} \partial h^{KL} \partial h^{ab}} \right|_o 2\mathrm{i}\, g^{ab} = - \left. \frac{\partial^2 V}{\partial h^{IJ} \partial h^{KL}} \right|_o \,, \tag{5.10.7}$$

where we have used the values of h^{IJ} at the Minkowski background, i.e., $h^{ab}\big|_o = 2\mathrm{i} g^{ab}$, $h^{a\alpha}\big|_o = 0$ and $h^{\alpha\beta}\big|_o = 0$.

5.10.2 Interactions with gravity

We shall not consider here gravitational sector self-interactions. They are easily computable, but since the main emphasis of this work is on unification, it is of much more interest to compute the interactions of other fields with gravity. In this subsection we consider the coupling of the $U(1)$-gauge field to gravity.

Thus, at least one of the perturbation fields δB^I is to be taken to lie in the gravitational sector. The third variation of the Lagrangian for the interaction between the GR sector (label by the indices $a, b, \ldots$) and the extra-U(1) sector (label by the indices $\alpha, \beta, \ldots$) is

$$\begin{aligned}\frac{1}{4\mathrm{i}}\delta^3 \mathcal{L}_{\mathrm{GR-(extra-U(1))}} =& 12\left\{\frac{\partial^3 V}{\partial h^{ab}\partial h^{cd}\partial h^{e\alpha}}\,(\Sigma\, b)^{ab}(\Sigma\, b)^{cd}(\Sigma\, b)^{e\alpha}\right.\\ &\left.+\frac{\partial^3 V}{\partial h^{ab}\partial h^{c\alpha}\partial h^{d\beta}}(\Sigma\, b)^{ab}(\Sigma\, b)^{c\alpha}(\Sigma\, b)^{d\beta}\right\}\\ &+12\,\frac{\partial^2 V}{\partial h^{a\alpha}\partial h^{b\beta}}\,(\Sigma\, b)^{a\alpha}(b\, b)^{b\beta}+6\,\frac{\partial^2 V}{\partial h^{ab}\partial h^{\alpha\beta}}\,(\Sigma\, b)^{ab}(b\, b)^{\alpha\beta}\,,\end{aligned} \tag{5.10.8}$$

where we used the notation

$$(\Sigma\, b)^{ab} = \frac{1}{4}\,\epsilon^{\mu\nu\rho\sigma}\,\Sigma^{(a}_{\mu\nu} b^{b)}_{\rho\sigma}\,, \qquad (\Sigma\, b)^{a\alpha} = \frac{1}{4}\,\epsilon^{\mu\nu\rho\sigma}\,\Sigma^{a}_{\mu\nu} b^{\alpha}_{\rho\sigma}\,, \tag{5.10.9}$$

and

$$(b\, b)^{a\alpha} = \frac{1}{4}\,\epsilon^{\mu\nu\rho\sigma}\,b^{a}_{\mu\nu} b^{\alpha}_{\rho\sigma}\,, \qquad (b\, b)^{\alpha\beta} = \frac{1}{4}\,\epsilon^{\mu\nu\rho\sigma}\,b^{\alpha}_{\mu\nu} b^{\beta}_{\rho\sigma}\,. \tag{5.10.10}$$

Now, as the U(1) sector is described by $\alpha, \beta = 8$, we get

$$\begin{aligned}\frac{1}{4\mathrm{i}}\delta^3 \mathcal{L}_{\mathrm{GR-U(1)}} =& 12\left\{\frac{\partial^3 V}{\partial h^{ab}\partial h^{cd}\partial h^{e8}}(\Sigma b)^{ab}(\Sigma b)^{cd}(\Sigma b)^{e8}+\frac{\partial^3 V}{\partial h^{ab}\partial h^{c8}\partial h^{d8}}(\Sigma b)^{ab}(\Sigma b)^{c8}(\Sigma b)^{d8}\right\}\\ &+12\,\frac{\partial^2 V}{\partial h^{a8}\partial h^{b8}}\,(\Sigma\, b)^{a8}(b\, b)^{b8}+6\,\frac{\partial^2 V}{\partial h^{ab}\partial h^{88}}\,(\Sigma\, b)^{ab}(b\, b)^{88}\,.\end{aligned} \tag{5.10.11}$$

Let us see how is the coupling to the trace of the metric perturbation[3] $h = h^{\mu}_{\mu}$. In this case the gravitational part of the b field is given by [29]

$$b^{a}_{\mu\nu} = \frac{h}{4}\,\Sigma^{a}_{\mu\nu}\,. \tag{5.10.12}$$

Then,

$$(\Sigma\, b)^{ab} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\Sigma^{(a}_{\mu\nu} b^{b)}_{\rho\sigma} = \frac{ih}{8}\Sigma^{a\mu\nu}\Sigma^{b}_{\mu\nu} = \frac{ih}{2}g^{ab}\,, \tag{5.10.13}$$

where we have used the self-duality of Σ and the identity $\Sigma^{a\mu\nu}\Sigma^{b}_{\mu\nu} = 4g^{ab}$. Thus, using (5.10.6) and (5.10.7), we have

$$\left.\frac{\partial^2 V}{\partial h^{ab}\partial h^{88}}\right|_o (\Sigma\, b)^{ab} \sim \left.\frac{\partial^2 V}{\partial h^{ab}\partial h^{88}}\right|_o g^{ab} = 0\,, \tag{5.10.14}$$

$$\left.\frac{\partial^3 V}{\partial h^{ab}\partial h^{cd}\partial h^{e8}}\right|_o (\Sigma\, b)^{ab} \sim \left.\frac{\partial^3 V}{\partial h^{ab}\partial h^{cd}\partial h^{e8}}\right|_o g^{ab} \sim \left.\frac{\partial^2 V}{\partial h^{cd}\partial h^{e8}}\right|_o = 0\,, \tag{5.10.15}$$

and

$$\left.\frac{\partial^3 V}{\partial h^{ab}\partial h^{c8}\partial h^{d8}}\right|_o (\Sigma\, b)^{ab} = \frac{\mathrm{i}h}{2}\left.\frac{\partial^3 V}{\partial h^{ab}\partial h^{c8}\partial h^{d8}}\right|_o g^{ab} = -\frac{h}{4}\left.\frac{\partial^2 V}{\partial h^{c8}\partial h^{d8}}\right|_o\,. \tag{5.10.16}$$

[3] Recall that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $\eta_{\mu\nu}$ the Minkowski metric and $h_{\mu\nu}$ the metric perturbation.

Then, in (5.10.11), we get

$$\frac{1}{4\mathrm{i}}\delta^3\mathcal{L}^{h}_{\mathrm{GR-U(1)}} = -3\ \frac{\partial^2 V}{\partial h^{a8}\partial h^{b8}}\bigg|_o\ (\Sigma\, b)^{a8}\left(h\,(\Sigma\, b)^{b8} - 4\,(b\, b)^{b8}\right). \tag{5.10.17}$$

Moreover, we have

$$(\Sigma\, b)^{b\beta} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\Sigma^{b}_{\mu\nu}b^{\beta}_{\rho\sigma} = \frac{\mathrm{i}}{2}\Sigma^{b\mu\nu}b^{\beta}_{\mu\nu}\,, \tag{5.10.18}$$

and

$$(b\, b)^{b\beta} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}b^{b}_{\mu\nu}b^{\beta}_{\rho\sigma} = \frac{\mathrm{i}h}{8}\Sigma^{b\mu\nu}b^{\beta}_{\mu\nu}\,, \tag{5.10.19}$$

where we have used the self-duality of Σ. Using the above two equations we find

$$\delta^3\mathcal{L}^{h}_{\mathrm{GR-U(1)}} = 0\,. \tag{5.10.20}$$

Then, the U(1) sector vanishes on such gravitational perturbations. This is, of course, as expected, for the U(1) sector is expected to be conformally-invariant (classically). Indeed, this is standard for $U(1)$-gauge fields (and also Yang-Mills fields). Note that this also provides quite a non-trivial check of our scheme, for the whole scheme would be invalidated if we had found that our $U(1)$-gauge field couples to the trace of the metric.

We now confirm that the coupling to the trace-free part of the metric perturbation is also as expected. For the coupling to the trace-free part of the metric perturbation $h^{\mathrm{tf}}_{\mu\nu}$ we have the gravitational b field

$$b^{a}_{\mu\nu} = \Sigma^{a\ \ \rho}_{[\mu}h^{\mathrm{tf}}_{\nu]\rho}\,. \tag{5.10.21}$$

Contracting the above expression with $\Sigma^{b\,\mu\nu}$ it is easy to see that $b^{a}_{\mu\nu}$ is anti-self-dual. Then,

$$(\Sigma\, b)^{ab} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\Sigma^{(a}_{\mu\nu}b^{b)}_{\rho\sigma} = 0 \tag{5.10.22}$$

because Σ is self-dual,

$$(\Sigma\, b)^{a8} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\Sigma^{a}_{\mu\nu}b^{8}_{\rho\sigma} = \frac{\mathrm{i}}{2}\Sigma^{a\mu\nu}b^{8}_{\mu\nu}\,, \tag{5.10.23}$$

where we have used the self-duality of Σ, and

$$(b\, b)^{b8} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}b^{b}_{\mu\nu}b^{8}_{\rho\sigma} = -\frac{\mathrm{i}}{2}\Sigma^{b\mu}{}_{\rho}h^{\mathrm{tf}\,\nu\rho}b^{8}_{\mu\nu}\,, \tag{5.10.24}$$

where we have used the anti-self-duality of b^a. Thus, this time we find the third order variation

$$\frac{1}{4\mathrm{i}}\delta^3\mathcal{L}^{h^{\mathrm{tf}}}_{\mathrm{GR-U(1)}} = 3\ \frac{\partial^2 V}{\partial h^{a8}\partial h^{b8}}\bigg|_o\ \Sigma^{a\mu\nu}\Sigma^{b\rho\lambda}\, b^{8}_{\mu\nu}b^{8}_{\rho\sigma}\, h^{\mathrm{tf}\,\sigma}{}_{\lambda}\,. \tag{5.10.25}$$

As this is the third variation of the Lagrangian we have to divide this result by 3!. Then, the interaction of the trace-free part of the metric perturbation $h^{\rm tf}_{\mu\nu}$ with U(1) is

$$\frac{1}{2\mathrm{i}}\mathcal{L}^{\rm h^{tf}}_{\rm GR-U(1)} = \left.\frac{\partial^2 V}{\partial h^{a8}\partial h^{b8}}\right|_o \Sigma^{a\mu\nu}\Sigma^{b\rho\lambda}\, b^8_{\mu\nu} b^8_{\rho\sigma}\, h^{\rm tf\,\sigma}{}_{\lambda}\,. \tag{5.10.26}$$

We know

$$\left.\frac{\partial^2 V}{\partial h^{a\alpha}\partial h^{b\beta}}\right|_o = \frac{\kappa}{4\mathrm{i}} g_{ab} g_{\alpha\beta} + \frac{\lambda}{6\mathrm{i}} C_{abc}\, C^c_{\alpha\beta}\,. \tag{5.10.27}$$

Then,

$$\left.\frac{\partial^2 V}{\partial h^{a8}\partial h^{b8}}\right|_o = -\frac{\kappa}{4\mathrm{i}} g_{ab}, \tag{5.10.28}$$

where we used $g_{88} = -1$. Thus, we obtain

$$\mathcal{L}^{\rm h^{tf}}_{\rm GR-U(1)} = -2\kappa\, P^{+\,\mu\nu\rho\lambda}\, b^8_{\mu\nu} b^8_{\rho\sigma}\, h^{\rm tf\,\sigma}{}_{\lambda}\,, \tag{5.10.29}$$

where we have used the identity $g_{ab}\Sigma^{a\mu\nu}\Sigma^{b\rho\lambda} = 4\,P^{+\,\mu\nu\rho\lambda}$.

Now, the second order BB Lagrangian and second order BF kinetic term for the U(1) sector, see equation (5.7.1), are

$$\mathcal{L}^{(2)\,\rm U(1)}_{\rm BB} = \kappa\, P^{-\,\mu\nu\rho\sigma}\, b^8_{\mu\nu} b^8_{\rho\sigma} + \frac{\mathrm{i}\lambda}{3}\,\epsilon^{\mu\nu\rho\sigma} b^8_{\mu\nu} b^8_{\rho\sigma}\,, \tag{5.10.30}$$

and

$$\mathcal{L}^{(2)\,\rm U(1)}_{\rm BF} = \mathrm{i}\,\epsilon^{\mu\nu\rho\sigma}\, b^8_{\mu} F_{\rho\sigma}\,, \tag{5.10.31}$$

where $F_{\mu\nu} = 2\partial_{[\mu} a^8_{\nu]}$. Moreover, the third order BF kinetic term for the U(1) sector, see (5.10.1), vanishes, i.e.,

$$\mathcal{L}^{(3)\,\rm U(1)}_{\rm BF} = 0\,. \tag{5.10.32}$$

Then, the U(1) sector Lagrangian up to third order is given by

$$\mathcal{L}^{\rm U(1)} = \mathcal{L}^{(2)}_{\rm BF} + \mathcal{L}^{(3)\,\rm U(1)}_{\rm BF} + \mathcal{L}^{(2)\,\rm U(1)}_{BB} + \mathcal{L}^{\rm h^{tf}}_{\rm GR-U(1)}\,, \tag{5.10.33}$$

or, written explicitly,

$$\mathcal{L}^{U(1)} = \mathrm{i}\,\epsilon^{\mu\nu\rho\sigma} b^8_{\mu\nu}\, F_{\rho\sigma} + \kappa\, P^{-\,\mu\nu\rho\sigma}\, b^8_{\mu\nu} b^8_{\rho\sigma} + \frac{\mathrm{i}\lambda}{3}\,\epsilon^{\mu\nu\rho\sigma} b^8_{\mu\nu} b^8_{\rho\sigma} - 2\kappa P^{+\,\mu\nu\rho\lambda}\, b^8_{\mu\nu} b^8_{\rho\sigma}\, h^{\rm tf\,\sigma}_{\lambda}\,, \tag{5.10.34}$$

where P^- is the anti-self-dual projector. Recall that

$$P^{+\mu\nu}{}_{\rho\sigma} = \frac{1}{2}\left(\delta^{[\mu}_{\rho}\delta^{\nu]}_{\sigma} - \frac{\mathrm{i}}{2}\,\epsilon^{\mu\nu}{}_{\rho\sigma}\right), \qquad P^{-\mu\nu}{}_{\rho\sigma} = \frac{1}{2}\left(\delta^{[\mu}_{\rho}\delta^{\nu]}_{\sigma} + \frac{\mathrm{i}}{2}\,\epsilon^{\mu\nu}{}_{\rho\sigma}\right). \tag{5.10.35}$$

For this part we shall drop the index 8 and the label "tf". Let us denote the self-dual and anti-self-dual parts of b by ${}^+b$ and ${}^-b$ and similar for F. Then, with $b = {}^+b + {}^-b$ and $F = {}^+F + {}^-F$, the action can be written as

$$\begin{aligned}\mathcal{L}^{\rm U(1)} =& -2\,{}^+b^{\mu\nu}\,{}^+F_{\mu\nu} - \frac{2\lambda}{3}\,{}^+b^{\mu\nu}\,{}^+b_{\mu\nu} - 2\kappa\,{}^+b^{\mu\sigma}\,{}^+b_{\mu\nu}\, h^{\nu}_{\sigma} \\ &+ 2\,{}^-b^{\mu\nu}\,{}^-F_{\mu\nu} + \left(\kappa + \frac{2\lambda}{3}\right){}^-b^{\mu\nu}\,{}^-b_{\mu\nu} - 2\kappa\,{}^+b^{\mu\sigma}\,{}^-b_{\mu\nu}\, h^{\nu}_{\sigma}\,,\end{aligned} \tag{5.10.36}$$

where we have used the fact that the total contraction of a self-dual and anti-self-dual two forms vanishes. Varying this Lagrangian with respect to ${}^+b$ and ${}^-b$ we get, respectively,

$$-{}^+F_{\mu\nu} - \frac{2\lambda}{3}\,{}^+b_{\mu\nu} + 2\kappa\,{}^+b_{\sigma[\mu}h^\sigma_{\nu]} + \kappa\,{}^-b_{\sigma[\mu}\,h^\sigma_{\nu]} = 0\,, \tag{5.10.37}$$

and

$$-F_{\mu\nu} + \left(\kappa + \frac{2\lambda}{3}\right)\,{}^-b_{\mu\nu} + \kappa\,{}^+b_{\sigma[\mu}\,h^\sigma_{\nu]} = 0\,. \tag{5.10.38}$$

The solution to these equations to first order in the perturbation $h_{\mu\nu}$ are

$${}^+b_{\mu\nu} = -\frac{1}{\frac{2\lambda}{3}}\,{}^+F_{\mu\nu} - \frac{2\kappa}{\left(\frac{2\lambda}{3}\right)^2}\,{}^+F_{\sigma[\mu}h^\sigma_{\nu]} - \frac{\kappa}{\frac{2\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,{}^-F_{\sigma[\mu}h^\sigma_{\nu]}\,, \tag{5.10.39}$$

and

$${}^-b_{\mu\nu} = -\frac{1}{\left(\kappa+\frac{2\lambda}{3}\right)}\,{}^-F_{\mu\nu} + \frac{\kappa}{\frac{2\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,{}^+F_{\sigma[\mu}\,h^\sigma_{\nu]}\,. \tag{5.10.40}$$

Replacing these expressions for ${}^+b$ and ${}^-b$ in the Lagrangian, we find

$$\mathcal{L}^{\mathrm{U}(1)} = \frac{1}{\frac{2\lambda}{3}}\,{}^+F^2 - \frac{1}{\kappa+\frac{2\lambda}{3}}\,{}^-F^2 - \frac{2\kappa}{\left(\frac{2\lambda}{3}\right)^2}\,{}^+F_{\mu\nu}\,{}^+F^{\mu\sigma}\,h^\nu_\sigma - \frac{2\kappa}{\frac{2\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,{}^-F_{\mu\nu}\,{}^+F^{\mu\sigma}\,h^\nu_\sigma\,. \tag{5.10.41}$$

We know that

$${}^+F_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} - \frac{\mathrm{i}}{2}\,\epsilon_{\mu\nu}{}^{\rho\sigma}\,F_{\rho\sigma}\right)\,, \qquad {}^-F_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} + \frac{\mathrm{i}}{2}\,\epsilon_{\mu\nu}{}^{\rho\sigma}\,F_{\rho\sigma}\right)\,. \tag{5.10.42}$$

Thus, the Lagrangian can be written as

$$\begin{aligned}\mathcal{L}^{\mathrm{U}(1)} =& \frac{\kappa}{\frac{4\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{\left(\kappa+\frac{4\lambda}{3}\right)}{\frac{4\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,\frac{\mathrm{i}}{2}\,\epsilon^{\mu\nu\rho\sigma}\,F_{\mu\nu}\,F_{\rho\sigma} \\ &- \frac{\kappa}{\frac{2\lambda}{3}\left(\kappa+\frac{2\lambda}{3}\right)}\,F_{\mu\nu}\,F^{\mu\sigma}\,h^{\nu\sigma} - \frac{\mathrm{i}\kappa}{2\left(\frac{2\lambda}{3}\right)^2}\,\epsilon^{\mu\nu\rho\lambda}\,F_{\mu\nu}\,F_{\sigma\rho}\,h^\sigma_\lambda\,.\end{aligned} \tag{5.10.43}$$

To get this result we used the trace-free property of the metric perturbation $h_{\mu\nu}$. Now, we are going to show that the last term in this Lagrangian is zero. Let us expand F in terms of the self-dual Σ's and the anti-self-dual $\overline{\Sigma}$'s as

$$F_{\mu\nu} = F^+_a\,\Sigma^a_{\mu\nu} + F^-_a\,\overline{\Sigma}^a_{\mu\nu}\,. \tag{5.10.44}$$

Using the self-duality of Σ, we find

$$\epsilon^{\mu\nu\rho\lambda}\,F_{\mu\nu} = 2\mathrm{i}\left(F^+_a\,\Sigma^{a\,\rho\lambda} - F^-_a\,\overline{\Sigma}^{a\,\rho\lambda}\right)\,. \tag{5.10.45}$$

Then, we have

$$\begin{aligned}\epsilon^{\mu\nu\rho\lambda}\,F_{\mu\nu}F_{\sigma\rho}h^\sigma_\lambda &= 2\mathrm{i}\left(F^+_a\,\Sigma^{a\,\rho\lambda} - F^-_a\,\overline{\Sigma}^{a\,\rho\lambda}\right)\left(F^+_b\,\Sigma^b_{\sigma[\rho}h^\sigma_{\lambda]} - F^-_b\,\overline{\Sigma}^b_{\sigma[\rho}h^\sigma_{\lambda]}\right) \\ &= F^+_a F^-_b\,\Sigma^{a\,\rho\lambda}\overline{\Sigma}^b_{\sigma\rho}h^\sigma_\lambda - F^+_b F^-_a\,\overline{\Sigma}^{a\,\rho\lambda}\Sigma^b_{\sigma\rho}h^\sigma_\lambda \\ &= 0\,,\end{aligned} \tag{5.10.46}$$

where in the second line we used the fact that $\Sigma^b_{\sigma[\rho} h^\sigma_{\lambda]}$ is anti-self-dual and $\overline{\Sigma}^b_{\sigma[\rho} h^\sigma_{\lambda]}$ is self-dual, and in the third line we reorganised the indices and realised that these two terms cancel. Then, we finally get the Lagrangian

$$\mathcal{L}^{\mathrm{U}(1)} = \frac{\kappa}{\frac{4\lambda}{3}\left(\kappa + \frac{2\lambda}{3}\right)} F_{\mu\nu} F^{\mu\nu} - \frac{\left(\kappa + \frac{4\lambda}{3}\right)}{\frac{4\lambda}{3}\left(\kappa + \frac{2\lambda}{3}\right)} \frac{\mathrm{i}}{2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} - \frac{\kappa}{\frac{2\lambda}{3}\left(\kappa + \frac{2\lambda}{3}\right)} F_{\mu\nu} F^{\mu\sigma} h^{\nu\sigma} \, . \tag{5.10.47}$$

In the third term we recognise precisely the coupling to the stress-energy tensor that arises from the quadratic U(1) sector Lagrangian (i.e., the first two terms).

Thus, the arising coupling of the $U(1)$-gauge field to the gravitational sector is as expected.

CHAPTER 6

Fermions

In this chapter we propose and study a fermionic Lagrangian compatible with the field variable content of our BF plus potential formulation of the theory described in the chapters before.

In the usual fermionic Lagrangian the metric plays a principal role and when we go from flat spacetime to curved spacetime the only way we know how to couple gravity with fermions is using the tetrad formulation. However, in [39] there is a proposal of how to couple fermions with gravity using the two-form field of the Plebanski formulation. Unfortunately, this proposal in not valid anymore in our formulation where the simplicity constraint has been removed.

The field content of the new fermionic Lagrangian is an anti-commuting spinor one-form and an anti-commuting spinor. This new Lagrangian can be transformed into the usual fermionic Lagrangian. We work in Minkowski spacetime but the calculations can be easily generalised to curved spacetime.

6.1 Usual fermion formulation

For completeness, in this section we remind ourself how the Weyl, Majorana and Dirac fermions are described using two-component spinors. Such a description is now part of at least some quantum field theory treatments, see for example [40, 41]. Here we give a brief description of fermions using this language.

6.1.1 Preliminaries

We work with two kind of spinors: the left-handed or $(1/2, 0)$ representation of the Lorentz group and the right-handed or $(0, 1/2)$ representation. These representations are related by complex conjugation. These two types of spinors are represented by symbols with an unprimed index for left-handed spinors and a primed index for right-handed spinors. Then, λ_A stands for a left-handed spinor and $\overline{\lambda}_{A'} = \overline{(\lambda_A)}$ for the right-handed one. The overline means complex conjugation. Unless otherwise noted, all our two-components spinors are Grassmann-odd-valued objects, or in other words anti-commuting spinors.

The anti-symmetric rank 2 spinor ϵ_{AB} and its inverse ϵ^{AB}, with $\epsilon_{AC}\epsilon^{AC} = \epsilon_A{}^B = \delta_A^B$, provide an isomorphism between unprimed spinors and their dual[1]. Then, we can define a contravariant spinor λ^A in terms of the covariant spinor λ_A as

$$\lambda^A = \epsilon^{AB}\lambda_B \,, \tag{6.1.2}$$

and a covariant spinor in terms of a contravariant one as

$$\lambda_B = \lambda^A \epsilon_{AB} \,. \tag{6.1.3}$$

All these equations are similar for primed spinors, just change unprimed indices by primed ones and use an overline for the primed spinors. The spinor conventions that we use are the ones in [42].

As is usual in the 2-component spinor literature, we shall often use an index-free notation. Our conventions, are

$$\lambda\xi \equiv \lambda^A \xi_A \,, \qquad\qquad \overline{\lambda}\,\overline{\xi} \equiv \overline{\lambda}_{A'}\,\overline{\xi}^{A'} \,. \tag{6.1.4}$$

This is a natural convention because

$$\begin{aligned} \overline{(\lambda\xi)} =& \overline{(\lambda^A \xi_A)} \,, \\ =& \overline{(\xi_A)}\;\overline{(\lambda^A)} \,, \\ =& \overline{\xi}_{A'}\overline{\lambda}^{A'} = \overline{\lambda}\,\overline{\xi} \,, \end{aligned} \tag{6.1.5}$$

where in the second line we have used the fact that the complex conjugation of two anti-commuting numbers reverse its order, i.e., for two anti-commuting numbers z and w we obtain $\overline{(zw)} = \overline{w}\,\overline{z}$.

It is useful to introduce a spin-frame $\{o^A, \iota^A\}$ such that any unprimed spinor can be written in this base as

$$\lambda^A = \lambda^1\, o^A + \lambda^2\, \iota^A \,, \tag{6.1.6}$$

[1]In components we have

$$\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,, \qquad\qquad \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,, \tag{6.1.1}$$

where the first index label the row of the matrix and the second the column.

and for the covariant spinor

$$\lambda_A = -\lambda_1 \iota_A + \lambda_2 o_A \,, \tag{6.1.7}$$

where $\lambda^1, \lambda^2, \lambda_1, \lambda_2$ are anti-commuting fields and $o^A, \iota^A, o_A, \iota_A$ are commuting spinors[2]. This spin-frame satisfies the conditions

$$o_A \iota^A = 1 \,, \qquad \iota_A o^A = -1 \,, \qquad o_A o^A = 0 \,, \qquad \iota_A \iota^A = 0 \,. \tag{6.1.9}$$

In terms of this spin-frame the anti-symmetric spinor ϵ_{AB} is written as

$$\epsilon_{AB} = o_A \iota_B - \iota_A o_B \,. \tag{6.1.10}$$

As before the equations for primed spinors are similar to these ones but with unprimed indices.

Now, we use the Minkowski tetrad $\{t_\mu, x_\mu, y_\mu, z_\mu\}$ to write the Minkowski metric as[3]

$$\eta_{\mu\nu} = -t_\mu t_\nu + x_\mu x_\nu + y_\mu y_\nu + z_\mu z_\nu \,. \tag{6.1.12}$$

In components $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. The Minkowski tetrad satisfies the conditions

$$t_\mu t^\mu = -1 \,, \qquad x_\mu x^\mu = 1 \,, \qquad y_\mu y^\mu = 1 \,, \qquad z_\mu z^\mu = 1 \,, \tag{6.1.13}$$

with the other contractions being zero. A null tetrad $\{l_\mu, n_\mu, m_\mu, \overline{m}_\mu\}$ can be defined in terms of the Minkowski tetrad as

$$l^\mu = \frac{1}{\sqrt{2}} \left(t^\mu + z^\mu\right) \,, \qquad n^\mu = \frac{1}{\sqrt{2}} \left(t^\mu - z^\mu\right) \,, \tag{6.1.14}$$

$$m^\mu = \frac{1}{\sqrt{2}} \left(x^\mu + i y^\mu\right) \,, \qquad \overline{m}^\mu = \frac{1}{\sqrt{2}} \left(x^\mu - i y^\mu\right) \,. \tag{6.1.15}$$

This null tetrad satisfies the conditions

$$l^\mu n_\mu = -1 \,, \qquad m^\mu \overline{m}_\mu = 1 \,, \tag{6.1.16}$$

with all the other contractions equal to zero and allows the Minkowski metric to be written as

$$\eta_{\mu\nu} = -2 l_{(\mu} n_{\nu)} + m_{(\mu} \overline{m}_{\nu)} \,. \tag{6.1.17}$$

[2]In components

$$o^A = (1, 0) \,, \qquad \iota^A = (0, 1) \,, \qquad o_A = (0, 1) \,, \qquad \iota_A = (-1, 0) \,. \tag{6.1.8}$$

[3]In components we have

$$t_\mu = (1, 0, 0, 0) \,, \qquad x_\mu = (0, 1, 0, 0) \,, \qquad y_\mu = (0, 0, 1, 0) \,, \qquad z_\mu = (0, 1, 0, 0) \,. \tag{6.1.11}$$

Now, we define a one-form valued in the tensor product of primed and unprimed commuting spinors as

$$\sigma_\mu^{A'A} = l_\mu\, o^{A'} o^A + n_\mu\, \iota^{A'} \iota^A + m_\mu\, \iota^{A'} o^A + \overline{m}_\mu\, o^{A'} \iota^A\,. \tag{6.1.18}$$

In component, we have

$$\begin{aligned}
\sigma_0^{A'A} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \\
\sigma_1^{A'A} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_2^{A'A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\,, \quad \sigma_3^{A'A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,
\end{aligned} \tag{6.1.19}$$

or in a compact notation $\sigma_\mu^{A'A} = 1/\sqrt{2}(1_{2x2}, \tau_i)$, where 1_{2x2} stands for the 2x2 identity matrix and τ_i for the Pauli matrices. It is not hard to check, using (6.1.18), that the Minkowski metric can be written as the "square" of $\sigma_\mu^{A'A}$, i.e.,

$$\eta_{\mu\nu} = -\sigma_\mu^{A'A}\, \sigma_{AA'\,\nu}\,. \tag{6.1.20}$$

Now, let us define a spatial one-form valued in the product of two unprimed spinors as

$$\widetilde{\sigma}_{i\,A}{}^{B} = \sqrt{2}\, \sigma_{0\,AA'}\, \sigma^{i\,A'B}\,. \tag{6.1.21}$$

Explicitly, in terms of the spin-frame and the null tetrad, we get

$$\widetilde{\sigma}_{i\,A}{}^{B} = m_i\, o_A o^B - \overline{m}_i\, \iota_A \iota^B - \frac{z_i}{\sqrt{2}} \left(o_A \iota^B + \iota_A o^B \right)\,. \tag{6.1.22}$$

In components, we have

$$\widetilde{\sigma}_{i\,A}{}^{B} = \frac{1}{\sqrt{2}}\, \tau_i\,, \tag{6.1.23}$$

with τ_i being the Pauli matrices. Note that the component expressions which we have shown for $\sigma^{i\,A'A}$ and $\widetilde{\sigma}_{i\,A}{}^{B}$ in (6.1.19) and (6.1.23), respectively, are for this specific position of the indices. If the position of the indices are changed in general the component expressions will change.

As the Pauli matrices satisfy the identity $\tau_i \tau_j = \delta_{ij}\, 1_{2x2} + \mathrm{i}\, \epsilon_{ij}{}^{k} \tau_k$, we find

$$\widetilde{\sigma}_{i\,A}{}^{C}\, \widetilde{\sigma}_{j\,C}{}^{B} = \frac{1}{2} \delta_{ij}\, \epsilon_A{}^{B} + \frac{\mathrm{i}}{\sqrt{2}} \epsilon_{ij}{}^{k}\, \widetilde{\sigma}_{k\,A}{}^{B}\,. \tag{6.1.24}$$

The above identity can also be found directly from (6.1.22).

6.1.2 Weyl fermion

The Lagrangian density for a Weyl fermion is

$$\mathcal{L}_{\text{Weyl}} = \mathrm{i}\sqrt{2}\, \overline{\lambda}_{A'}\, \sigma^{\mu A'A}\, \partial_\mu \lambda_A = \mathrm{i}\sqrt{2}\, \overline{\lambda}\, \sigma^\mu\, \partial_\mu \lambda\,, \tag{6.1.25}$$

where λ_A is a left-handed anti-commuting two-component spinor, $\overline{\lambda}_{A'}$ is a right-handed anti-commuting two-component spinor, and we have also rewritten the Lagrangian in an index-free way. Splitting into space and time indices we find

$$\mathcal{L}_{\text{Weyl}} = \mathrm{i}\sqrt{2}\,\overline{\lambda}_{A'}\,\sigma^{0A'A}\,\partial_0\lambda_A + \mathrm{i}\sqrt{2}\,\overline{\lambda}_{A'}\,\sigma^{iA'A}\,\partial_i\lambda_A\,. \tag{6.1.26}$$

Thus, the canonically conjugate field to λ_A is

$$\Pi^A = \mathcal{L}_{\text{Weyl}}\frac{\overleftarrow{\partial}}{\partial(\partial_0\lambda_A)} = \mathrm{i}\sqrt{2}\,\overline{\lambda}_{A'}\,\sigma^{oA'A}\,, \tag{6.1.27}$$

where the partial derivative with a left pointing arrow over it stands for partial derivation from the right. Multiplying the above equation by $\sigma_{0\,AB'}$ we obtain

$$\overline{\lambda}_{B'} = \mathrm{i}\sqrt{2}\,\Pi^A\,\sigma_{0\,AB'}\,, \tag{6.1.28}$$

where we have used the identity

$$\sigma^{0\,A'A}\,\sigma_{0\,AB'} = -\frac{1}{2}\,\epsilon_{B'}{}^{A'}\,, \tag{6.1.29}$$

which can be easily shown from (6.1.18). Then, the Lagrangian $\mathcal{L}_{\text{Weyl}}$ can be rewritten in terms of the Hamiltonian variables $\{\lambda_A, \Pi^A\}$ as

$$\mathcal{L}_{\text{Weyl}} = \Pi\,\partial_0\lambda - \sqrt{2}\,\Pi\,\widetilde{\sigma}^i\,\partial_i\lambda\,, \tag{6.1.30}$$

where we have used (6.1.21). Let us find the field equations that follow from the above Lagrangian. Treating the fermionic fields λ_A, Π^A as independent and varying the Lagrangian with respect to them, we find

$$\partial_0\Pi^B - \sqrt{2}\,\partial_i\Pi^A\,\widetilde{\sigma}^i{}_A{}^B = 0\,, \tag{6.1.31}$$

and

$$\partial_0\lambda_B - \sqrt{2}\,\widetilde{\sigma}^i{}_B{}^C\,\partial_i\lambda_C = 0\,. \tag{6.1.32}$$

Applying the operator $\sqrt{2}\,\widetilde{\sigma}^i{}_A{}^B\,\partial_j$ to the above equation and using (6.1.24), we obtain

$$\sqrt{2}\,\widetilde{\sigma}^i{}_A{}^B\,\partial_0\partial_j\lambda_B - \Delta\lambda_A = 0\,, \tag{6.1.33}$$

where $\Delta = \partial_i\partial^i$ stands for the Laplacian. Now, differentiating (6.1.32) with respect to time, relabelling the indices and adding the result to the above equation, we get

$$\begin{aligned}(\partial_0\partial_0 - \Delta)\lambda_A &= 0\,,\\ \partial^\mu\partial_\mu\lambda_A &= 0\,,\end{aligned} \tag{6.1.34}$$

which is the relativistic wave equation for a massless two-component spinor λ.

6.1.3 The Majorana mass term

Let us now briefly consider the massive case. Since our fermions are anti-commuting it is possible to write a Majorana mass term. Thus, adding a mass term to (6.1.25) we now consider the Lagrangian

$$\mathcal{L}_{\text{Majorana}} = \mathrm{i}\sqrt{2}\,\overline{\lambda}\,\sigma^{\mu}\,\partial_{\mu}\lambda - \frac{M}{2}\,\lambda\lambda - \frac{M}{2}\,\overline{\lambda}\,\overline{\lambda}\,, \tag{6.1.35}$$

where we have used the index-free notation. Note that we need to add both mass terms in order for the Lagrangian to be real. The canonically conjugate field to λ is again given by (6.1.27). Then, the last term of $\mathcal{L}_{\text{Majorana}}$ can be written in terms of Π as

$$\frac{M}{2}\,\overline{\lambda}\,\overline{\lambda} = \frac{M}{2}\,\Pi\,\Pi\,, \tag{6.1.36}$$

where we have used (6.1.28) and the identity (6.1.29). So the Majorana Lagrangian can be written in the Hamiltonian form as

$$\mathcal{L}_{\text{Majorana}} = \Pi\,\partial_0\lambda - \sqrt{2}\,\Pi\,\widetilde{\sigma}^{i}\,\partial_i\lambda - \frac{M}{2}\,\lambda\lambda - \frac{M}{2}\,\Pi\,\Pi\,. \tag{6.1.37}$$

Varying this Lagrangian with respect to λ and Π, respectively, we find the field equations

$$\partial_0\Pi^{B} - \sqrt{2}\,\partial_i\Pi^{A}\,\widetilde{\sigma}^{i}{}_{A}{}^{B} + M\,\lambda^{B} = 0\,, \tag{6.1.38}$$

and

$$\partial_0\lambda_{B} - \sqrt{2}\,\widetilde{\sigma}^{i}{}_{B}{}^{C}\,\partial_i\lambda_{C} - M\,\Pi_{B} = 0\,. \tag{6.1.39}$$

Solving for Π from equation (6.1.39) and substituting the result in (6.1.38), we find

$$(\partial_{\mu}\partial^{\mu} - M^2)\lambda = 0\,, \tag{6.1.40}$$

where we have used the (6.1.24). The second order differential equation above is the desired massive wave equation for a two-component fermion.

6.1.4 Dirac fermion

The Dirac Lagrangian is obtained by adding two uncoupled Majorana Lagrangians of equal mass. Thus, we have the Lagrangian

$$\mathcal{L}_{\text{Dirac}} = \mathrm{i}\sqrt{2}\,\overline{\lambda}_{\mathbf{1}}\,\sigma^{\mu}\,\partial_{\mu}\lambda_{\mathbf{1}} + \mathrm{i}\sqrt{2}\,\overline{\lambda}_{\mathbf{2}}\,\sigma^{\mu}\,\partial_{\mu}\lambda_{\mathbf{2}} - \frac{M}{2}(\lambda_{\mathbf{1}}\lambda_{\mathbf{1}} + \lambda_{\mathbf{2}}\lambda_{\mathbf{2}}) - \frac{M}{2}(\overline{\lambda}_{\mathbf{1}}\,\overline{\lambda}_{\mathbf{1}} + \overline{\lambda}_{\mathbf{2}}\,\overline{\lambda}_{\mathbf{2}})\,, \tag{6.1.41}$$

where we have used an index-free notation. The boldface lower indices **1** and **2** label the two uncoupled spinors and it has nothing to do with the components of every two-component spinor. This Lagrangian possesses a global internal $SO(2)$ symmetry of the form

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \to \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \sin\theta\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}. \tag{6.1.42}$$

Since $SO(2) \sim U(1)$, complex linear combinations of fermions can be defined to make the Lagrangian explicitly $U(1)$ invariant. Then, let us introduce the two-component spinor fields ξ and χ such that

$$\begin{aligned}\lambda_1 =& \frac{1}{\sqrt{2}}(\xi + \chi)\,,\\ \lambda_2 =& \frac{\mathrm{i}}{\sqrt{2}}(\xi - \chi)\,.\end{aligned} \tag{6.1.43}$$

In terms of these fields the Dirac Lagrangian is

$$\mathcal{L}_{\text{Dirac}} = \mathrm{i}\sqrt{2}\,\overline{\xi}\,\sigma^\mu\,\partial_\mu\xi + \mathrm{i}\sqrt{2}\,\overline{\chi}\,\sigma^\mu\,\partial_\mu\chi - M(\chi\xi + \overline{\xi}\,\overline{\chi})\,. \tag{6.1.44}$$

A Dirac spinor is described by a pair $(\chi, \overline{\xi})$ of two two-components spinors.
This Lagrangian has a $U(1)$ global symmetry, i.e., it is invariant under the transformation

$$\xi \to e^{i\varphi}\,\xi\,, \qquad\qquad \chi \to e^{-i\varphi}\,\chi\,, \tag{6.1.45}$$

with φ a parameter.

Splitting in space and time indices the Dirac Lagrangian can be written in the Hamiltonian form as

$$\mathcal{L}_{\text{Dirac}} = {}^{\xi}\Pi\,\partial_0\xi + {}^{\chi}\Pi\,\partial_0\chi - \sqrt{2}\,{}^{\xi}\Pi\,\widetilde{\sigma}^i\,\partial_i\xi - \sqrt{2}\,{}^{\chi}\Pi\,\widetilde{\sigma}^i\,\partial_i\chi - M\left(\chi\xi + {}^{\xi}\Pi\,{}^{\chi}\Pi\right), \tag{6.1.46}$$

where ${}^{\xi}\Pi^A = \mathrm{i}\sqrt{2}\,\overline{\xi}_{A'}\,\sigma^{0\,A'A}$ is the canonically conjugate field to ξ and ${}^{\chi}\Pi^A = \mathrm{i}\sqrt{2}\,\overline{\chi}_{A'}\,\sigma^{0\,A'A}$ is the canonically conjugate field to χ.

6.2 New formulation

In this section we propose a new first-order fermion Lagrangian whose basic field variable is, in addition to the usual spinors, a spinor-valued one-form. The spatial components of this spinor-valued one form can be decomposed into irreducible representations of $SL(2,\mathbb{C})$ as the direct sum of a spin-1/2 and a spin-3/2 parts. The Lagrangian is designed in such a way that only the spin-1/2 part propagates.

6.2.1 Massless fermion

The Lagrangian density which will be used as basic building block for the rest of this chapter is

$$\begin{aligned}\mathcal{L}_{\text{Massless}} =& -\sqrt{2}\,\rho_\mu^A\,\Sigma_A{}^{B\,\mu\nu}\,\partial_\nu\lambda_B + \alpha\,\rho_\mu^A\,\Sigma_A{}^{B\,\mu\nu}\,\rho_{B\,\nu} + \beta\,\rho^{A\,\mu}\,\rho_{A\,\mu}\,,\\ =& -\sqrt{2}\,\rho_\mu\,\Sigma^{\mu\nu}\,\partial_\nu\lambda + \alpha\,\rho_\mu\,\Sigma^{\mu\nu}\,\rho_\nu + \beta\,\rho^\mu\,\rho_\mu\,,\end{aligned} \tag{6.2.1}$$

where ρ_μ^A is an anti-commuting spinor one-form; $\Sigma_{\mu\nu}^{AB}$ is a self-dual two-form with values on the symmetric product of two unprimed spinors, which will be defined below; λ_A is an anti-commuting spinor; and α and β are two parameters. Note that the Lagrangian above uses only unprimed spinor, i.e., it is a chiral formulation.
The self-dual two-form $\Sigma_{\mu\nu}^{AB}$ is defined in terms of $\sigma_{\mu\nu}^{A'A}$ as

$$\Sigma_{\mu\nu}^{AB} = \sigma^A_{A'\,[\mu}\sigma_{\nu]}^{A'B}\,. \tag{6.2.2}$$

Using (6.1.18) we can write an explicit expression for $\Sigma_{\mu\nu}^{AB}$ in terms of the spin-frame $\{o^A, \iota^A\}$ and the null tetrad $\{l, n, m, \overline{m}\}$ as

$$\Sigma_{\mu\nu}^{AB} = 2l_{[\mu}m_{\nu]}\,o^A o^B + 2\overline{m}_{[\mu}n_{\nu]}\,\iota^A\iota^B + 2\left(l_{[\mu}n_{\nu]} - m_{[\mu}\overline{m}_{\nu]}\right)\,o^{(A}\iota^{B)}\,. \tag{6.2.3}$$

Therefore, the temporal-spatial component of this two-form is

$$\begin{aligned}\Sigma_{oi}^{AB} =& \frac{1}{\sqrt{2}}\,m_i\,o^A o^B - \frac{1}{\sqrt{2}}\,\overline{m}_i\,\iota^A\iota^B - z_i\,o^{(A}\iota^{B)}\,,\\ =& \frac{1}{\sqrt{2}}\,\widetilde{\sigma}_i{}^{AB}\,,\end{aligned} \tag{6.2.4}$$

and the spatial-spatial component is

$$\begin{aligned}\Sigma_{ij}^{AB} =& \sqrt{2}\,z_{[i}m_{j]}\,o^A o^B - \sqrt{2}\,\overline{m}_{[i}z_{j]}\,\iota^A\iota^B - 2\,m_{[i}\overline{m}_{j]}\,o^{(A}\iota^{B)}\,,\\ =& -\frac{\mathrm{i}}{\sqrt{2}}\,\epsilon_{ijk}\,\widetilde{\sigma}^{k\,AB}\,,\end{aligned} \tag{6.2.5}$$

where in the second line of the above equation we have used the easily derivable identities

$$\epsilon^{ijk}\,m_i z_j = \mathrm{i}m^k\,, \qquad \epsilon^{ijk}\,\overline{m}_i z_j = -\,i\,\overline{m}^k\,, \qquad \epsilon^{ijk}\,m_i\overline{m}_j = -\,i\,z^k\,. \tag{6.2.6}$$

In particular we have

$$-\,2\mathrm{i}\,\Sigma_{0i}^{AB} = \epsilon_i{}^{jk}\,\Sigma_{jk}^{AB}\,, \tag{6.2.7}$$

which is the condition of self-duality with the convention $\epsilon^{0123} = 1$.

The space and time splitting of the Lagrangian (6.2.1) is given by

$$
\begin{aligned}
\mathcal{L}_{\text{Massless}} =& -\rho_i\,\widetilde{\sigma}^i\,\partial_0\lambda + \rho_0\,\widetilde{\sigma}^i\,\partial_i\lambda - \mathrm{i}\,\epsilon^{ijk}\,\rho_i\,\widetilde{\sigma}_j\,\partial_k\lambda - \sqrt{2}\alpha\,\rho_0\,\widetilde{\sigma}^i\,\rho_i + \frac{\mathrm{i}\alpha}{\sqrt{2}}\,\epsilon^{ijk}\,\rho_i\,\widetilde{\sigma}_j\,\rho_k \\
&- \beta\,\rho_0\,\rho_0 + \beta\,\rho^i\,\rho_i\,,
\end{aligned} \tag{6.2.8}
$$

where we have used the index-free notation convention. Now, the spatial one-form $\rho_{i\,A}$ is a $1\otimes 1/2$ reducible representation of the Lie algebra of $SL(2,\mathbb{C})$. Since the tensor representation $1\otimes 1/2$ can be decomposed into the irreducible spin-1/2 and spin-3/2 representations, i.e., $1\otimes 1/2 = 1/2\oplus 3/2$, we can write

$$
\begin{aligned}
\rho_{iA} =& \rho^{(1/2)}_{iA} + \rho^{(3/2)}_{iA}\,, \\
=& \left(P^{(1/2)}{}_{i\ A}^{\ j\ B} + P^{(3/2)}{}_{i\ A}^{\ j\ B} \right)\,\rho_{jB}\,,
\end{aligned} \tag{6.2.9}
$$

where

$$
\begin{aligned}
P^{(1/2)}{}_{i\ A}^{\ j\ B} =& \frac{1}{3}\left(\delta_i^j\,\epsilon_A{}^B + \mathrm{i}\sqrt{2}\,\epsilon_i{}^{jk}\,\widetilde{\sigma}_{k\,A}{}^B \right)\,, \\
P^{(3/2)}{}_{i\ A}^{\ j\ B} =& \frac{1}{3}\left(2\,\delta_i^j\,\epsilon_A{}^B - \mathrm{i}\sqrt{2}\,\epsilon_i{}^{jk}\,\widetilde{\sigma}_{k\,A}{}^B \right)\,,
\end{aligned} \tag{6.2.10}
$$

are the spin-1/2 and spin-3/2 projectors, respectively. It is not hard to check that the spin-1/2 component of ρ_{iA} is of the form

$$
\rho^{(1/2)}_{iA} = \frac{2}{3}\,\widetilde{\sigma}_{i\,A}{}^B\,\zeta_B\,, \tag{6.2.11}
$$

for some spinor ζ_B. The prefactor is introduced so that

$$
\zeta_A = \widetilde{\sigma}^i{}_A{}^B\,\rho_{iB}\,, \tag{6.2.12}
$$

where we have utilised (6.1.24). Using again the identity (6.1.24) it is easy to see that the spin-1/2 and spin-3/2 part of ρ_{iA} are eigenvectors of the operator $\epsilon_i{}^{jk}\,\widetilde{\sigma}_{j\,A}{}^B$ with eigenvalues $\mathrm{i}\sqrt{2}$ and $-\mathrm{i}\sqrt{2}/2$, i.e.,

$$
\begin{aligned}
\epsilon_i{}^{jk}\,\widetilde{\sigma}_{j\,A}{}^B\,\rho^{(1/2)}_{kB} &= \mathrm{i}\sqrt{2}\,\rho^{(1/2)}_{iA}\,, \\
\epsilon_i{}^{jk}\,\widetilde{\sigma}_{j\,A}{}^B\,\rho^{(3/2)}_{kB} &= -\frac{\mathrm{i}\sqrt{2}}{2}\,\rho^{(3/2)}_{iA}\,.
\end{aligned} \tag{6.2.13}
$$

Then, we have

$$
\rho_{iA} = \frac{2}{3}\,\widetilde{\sigma}_{i\,A}{}^B\,\zeta_B + \rho^{(3/2)}_{iA}\,. \tag{6.2.14}
$$

Now, we can write (6.2.8) as

$$
\begin{aligned}
\mathcal{L}_{\text{Massless}} =& \zeta\,\partial_0\lambda - \frac{2\sqrt{2}}{3}\,\zeta\,\widetilde{\sigma}^i\,\partial_i\lambda + \frac{2}{3}(\alpha-\beta)\,\zeta\,\zeta \\
&+ \rho^{(3/2)\,i}\left[\left(\frac{\alpha}{2}+\beta\right)\,\rho^{(3/2)}_i - \frac{1}{\sqrt{2}}\,\partial_i\lambda \right] - \rho_0\left(\beta\,\rho_0 + \sqrt{2}\alpha\,\zeta - \widetilde{\sigma}^i\,\partial_i\lambda \right)\,,
\end{aligned} \tag{6.2.15}
$$

where we have used (6.2.14), (6.2.13) and (6.1.24). From the above Lagrangian we immediately realised that the canonically conjugate field to λ_A is ζ^A, i.e.,

$$\pi^A = \mathcal{L}\frac{\overleftarrow{\partial}}{\partial(\partial_0\lambda_A)} = \zeta^A\,, \tag{6.2.16}$$

and that the fields $\rho^{(3/2)}_{iA}$ and ρ^A_0 are not propagating and therefore can be integrated out. The field equation for $\rho^{(3/2)}_{iA}$ gives

$$(\alpha+2\beta)\,\rho^{(3/2)}_{iA} = \frac{1}{\sqrt{2}}\left(P^{(3/2)}\,\partial\lambda\right)_{iA}\,, \tag{6.2.17}$$

where the projection on the spin-3/2 part is taken on the right. This can be solved when $\alpha \neq -2\beta$. Substituting this solution back we obtain

$$\begin{aligned}\mathcal{L}_{\text{Massless}} =&\pi\,\partial_0\lambda + \frac{2}{3}(\alpha-\beta)\,\pi\,\pi - \frac{2\sqrt{2}}{3}\,\pi\widetilde{\sigma}^i\,\partial_i\lambda - \frac{1}{4(\alpha+2\beta)}\,\left(P^{(3/2)}\partial\lambda\right)^i\,\partial_i\lambda\\ &-\rho_0\left(\beta\,\rho_0 + \sqrt{2}\alpha\,\zeta - \widetilde{\sigma}^i\,\partial_i\lambda\right)\,.\end{aligned} \tag{6.2.18}$$

The field ρ^A_0 can also be eliminated using its field equations. We get

$$\rho^A_0 = \frac{1}{2\beta}\left(\widetilde{\sigma}^{iAB}\,\partial_i\lambda_B - \sqrt{2}\alpha\,\pi^A\right)\,, \tag{6.2.19}$$

for $\beta \neq 0$. Substituting this back and putting similar terms together we find

$$\begin{aligned}\mathcal{L}_{\text{Massless}} =&\pi\,\partial_0\lambda + \frac{1}{6\beta}(\alpha+2\beta)(3\alpha-2\beta)\;\pi\,\pi - \frac{\sqrt{2}}{6\beta}(3\alpha+4\beta)\;\pi\,\widetilde{\sigma}^i\,\partial_i\lambda\\ &-\frac{1}{6\beta}\,\frac{3\alpha+10\beta}{4(\alpha+2\beta)}\,\partial^i\lambda\,\partial_i\lambda\,,\end{aligned} \tag{6.2.20}$$

where we have used the explicit form of the $P^{(3/2)}$ projector, equation (6.2.10), and the identity (6.1.24). We have also dropped, after integrating by parts, a term proportional to $\epsilon^{ijk}\,\partial_i\lambda\,\widetilde{\sigma}_j\,\partial_k\lambda$ because it is equal to a surface term.

Rescaling the fields λ and π in the following form

$$\lambda \to \left[\frac{4\cdot 6\beta(\alpha+2\beta)}{3\alpha+10\beta}\right]^{1/2}\lambda\,,\qquad\qquad \pi \to \left[\frac{3\alpha+10\beta}{4\cdot 6\beta(\alpha+2\beta)}\right]^{1/2}\pi\,, \tag{6.2.21}$$

we can rewrite (6.2.20) as

$$\mathcal{L}_{\text{Massless}} = \pi\,\partial_0\lambda + \frac{1}{4}(C^2-1)\,\pi\,\pi - \sqrt{2}\,C\,\pi\,\widetilde{\sigma}^i\,\partial_i\,\lambda - \partial_i\lambda\,\partial_i\lambda\,, \tag{6.2.22}$$

where

$$C = \frac{3\alpha+4\beta}{6\beta}\,. \tag{6.2.23}$$

Now, to confirm we are actually studying a massless particle, let us find out which field equation satisfies λ. Firstly, we compute the equations of motion that follows from (6.2.22). Varying (6.2.22) with respect to π and λ, respectively, we get

$$\partial_0 \lambda_A + \frac{1}{2}(C^2-1)\,\pi_A + \sqrt{2}C\,\widetilde{\sigma}^i{}_A{}^B\,\partial_i\lambda_B = 0\,, \tag{6.2.24}$$

$$\partial_0 \pi^A - \sqrt{2}C\,\partial_i\pi^B\,\widetilde{\sigma}^i{}_B{}^A - 2\,\Delta\lambda^A = 0\,. \tag{6.2.25}$$

Recall that $\Delta = \partial^i\partial_i$ is the Laplacian operator. Solving for π in the first equation, substituting the result in the second one and using the identity (6.1.24), we find

$$\begin{aligned}(\partial_0\partial_0 - \Delta)\lambda &= 0\,,\\ \partial^\mu\partial_\mu\,\lambda &= 0\,.\end{aligned} \tag{6.2.26}$$

Then, the relativistic wave field equation is as it should be for a massless particle.

Now we have checked we are working with a fermion Lagrangian for a massless particle let us try to find a field redefinition that maps this action to the usual Weyl Lagrangian. The non-local field redefinition that does the job is

$$\begin{aligned}\pi^A &\to a\,\pi^A + b\,\widetilde{\sigma}^{i\,AB}\,\partial_i\lambda_B\,,\\ \lambda_A &\to 2c\,\widetilde{\sigma}^i{}_A{}^B\,\frac{\partial_i}{\Delta}\pi_B + d\,\lambda_A\,,\end{aligned} \tag{6.2.27}$$

where a, b, c, d are constant parameters. Every term in the Lagrangian transforms in the following manner under this field redefinition

$$\begin{aligned}\pi\,\partial_0\lambda &\to (ad-bc)\,\pi\,\partial_0\lambda\,,\\ \pi\,\pi &\to a^2\,\pi\,\pi + 2ab\,\pi\,\widetilde{\sigma}^i\,\partial_i\lambda - \frac{b^2}{2}\,\partial^i\lambda\,\partial_i\lambda\,,\\ \pi\,\widetilde{\sigma}^i\,\partial_i\lambda &\to (ad+bc)\,\pi\,\widetilde{\sigma}^i\,\partial_i\lambda + ac\,\pi\,\pi + \frac{bd}{2}\,\partial^i\lambda\,\partial_i\lambda\,,\\ \partial^i\lambda\,\partial_i\lambda &\to d^2\,\partial^i\lambda\,\partial_i\lambda - 2c^2\,\pi\,\pi - 4cd\,\pi\,\widetilde{\sigma}^i\,\partial_i\lambda\,,\end{aligned} \tag{6.2.28}$$

where we have used the identity (6.1.24). Thus, (6.2.22) is transformed under this field redefinition to the form

$$\begin{aligned}\mathcal{L}_{\text{Massless}} = {}&(ad-bc)\,\pi\,\partial_0\lambda + \left[\frac{ab}{2}(C^2-1) - \sqrt{2}(ad+bc)C + 4cd\right]\,\pi\,\widetilde{\sigma}^i\,\partial_i\lambda\\ &+ \left[\frac{a^2}{4}(C^2-1) - \sqrt{2}acC + 2c^2\right]\,\pi\,\pi + \left[-\frac{b^2}{8}(C^2-1) + \frac{bd}{\sqrt{2}}C - d^2\right]\partial^i\lambda\,\partial_i\lambda\,.\end{aligned} \tag{6.2.29}$$

For the parameter values

$$b = \frac{\sqrt{2}}{a}\,, \qquad c = \frac{a}{2\sqrt{2}}(C-1)\,, \qquad d = \frac{1}{2a}(C+1)\,, \tag{6.2.30}$$

this Lagrangian density is map to

$$\mathcal{L}_{\text{Massless}}|_{b,c,d} = \pi\,\partial_0\lambda - \sqrt{2}\,\pi\,\widetilde{\sigma}^i\,\partial_i\lambda\,, \tag{6.2.31}$$

which is the Lagrangian for a Weyl fermion, see equation (6.1.30).

6.2.2 Massive fermion

Let us add a term of the type $\lambda\lambda$ to the Lagrangian studied in the section before. Thus, we want to study the Lagrangian

$$\mathcal{L}_{\text{Massive}} = -\sqrt{2}\,\rho_\mu\,\Sigma^{\mu\nu}\,\partial_\nu\lambda + \alpha\,\rho_\mu\,\Sigma^{\mu\nu}\,\rho_\nu + \beta\,\rho^\mu\,\rho_\mu - \frac{\mu}{\beta}\,\lambda\,\lambda\,, \tag{6.2.32}$$

with μ a parameter. As it is not hard to see, the effective Lagrangian (i.e., the resulting Lagrangian after the $\rho^{(3/2)}_{iA}$ and ρ^A_0 fields has been integrated out) for the massive case is basically the same as the massless case with the only difference being the addition of the term $\mu/\beta\,\lambda\lambda$. Then, the effective Lagrangian in this case, see equation (6.2.20), is

$$\begin{aligned}\mathcal{L}_{\text{Massless}} =&\pi\,\partial_0\lambda + \frac{1}{6\beta}(\alpha+2\beta)(3\alpha-2\beta)\;\pi\,\pi - \frac{\sqrt{2}}{6\beta}(3\alpha+4\beta)\;\pi\,\widetilde{\sigma}^i\,\partial_i\lambda\\ &- \frac{1}{6\beta}\,\frac{3\alpha+10\beta}{4(\alpha+2\beta)}\,\partial^i\lambda\,\partial_i\lambda - \frac{\mu}{\beta}\,\lambda\,\lambda\,.\end{aligned} \tag{6.2.33}$$

Rescaling the fields π and λ as in (6.2.21) we obtain

$$\mathcal{L}_{\text{Massless}} = \pi\,\partial_0\lambda + \frac{1}{4}(C^2-1)\,\pi\,\pi - \sqrt{2}\,C\,\pi\,\widetilde{\sigma}^i\,\partial_i\,\lambda - \partial_i\lambda\,\partial_i\lambda - \frac{8(3C+1)}{3(C+1)}\mu\,\lambda\,\lambda\,, \tag{6.2.34}$$

where again

$$C = \frac{3\alpha+4\beta}{6\beta}\,. \tag{6.2.35}$$

Now, let us find the field equations that follows from (6.2.34). Varying (6.2.34) with respect to π and λ, respectively, we obtain

$$\partial_0\lambda_A + \frac{1}{2}(C^2-1)\,\pi_A + \sqrt{2}C\,\widetilde{\sigma}^i_A{}^B\,\partial_i\lambda_B =0\,, \tag{6.2.36}$$

$$\partial_0\pi^A - \sqrt{2}C\,\partial_i\pi^B\,\widetilde{\sigma}^i_B{}^A - 2\,\Delta\lambda^A + \frac{16(3C+1)}{3(C+1)}\mu\,\lambda^A =0\,. \tag{6.2.37}$$

Solving for π in the first equation, substituting the result in the second one and using the identity (6.1.24) we find

$$\begin{aligned}(\partial_0\partial_0 - \Delta + m^2)\lambda =&0\,,\\ (\partial^\mu\partial_\mu - m^2)\,\lambda =&0\,,\end{aligned} \tag{6.2.38}$$

where

$$m^2 = \frac{8}{3}(1-C)(1+3C)\mu\,. \tag{6.2.39}$$

Thus, we have checked that the Lagrangian $\mathcal{L}_{\text{Massive}}$ describes a massive fermion.

6.2.3 Dirac-type fermion

Here we will follow the same recipe that it is used in the construction of Dirac fermions from two-components spinor, subsection 6.1.4. Then, the Lagrangian of two uncoupled massive fermions of equal mass of the type (6.2.32) is given by

$$\begin{aligned}\mathcal{L}_{\text{Dira-type}} =& -\sqrt{2}\left(\rho_{\mathbf{1}\,\mu}\,\Sigma^{\mu\nu}\,\partial_\nu\lambda_{\mathbf{1}} + \rho_{\mathbf{2}\,\mu}\,\Sigma^{\mu\nu}\,\partial_\nu\lambda_{\mathbf{2}}\right) + \alpha\left(\rho_{\mathbf{1}\,\mu}\,\Sigma^{\mu\nu}\,\rho_{\mathbf{1}\,\nu} + \rho_{\mathbf{2}\,\mu}\,\Sigma^{\mu\nu}\,\rho_{\mathbf{2}\,\nu}\right)\\ &+ \beta\left(\rho_{\mathbf{1}}^{\mu}\,\rho_{\mathbf{1}\,\mu} + \rho_{\mathbf{2}}^{\mu}\,\rho_{\mathbf{2}\,\mu}\right) - \frac{\mu}{\beta}\left(\lambda_{\mathbf{1}}\,\lambda_{\mathbf{1}} + \lambda_{\mathbf{2}}\,\lambda_{\mathbf{2}}\right)\,, \end{aligned} \tag{6.2.40}$$

where the boldface lower case indices **1** and **2** label the two uncoupled spinors.

Now, making the following complex field transformation

$$\begin{aligned}\rho^A_{\mathbf{1}\,\mu} &= \frac{1}{\sqrt{2}}\left(\omega^A_\mu + \upsilon^A_\mu\right)\,, & \lambda_{\mathbf{1}\,A} &= \frac{1}{\sqrt{2}}\left(\xi_A + \chi_A\right)\,,\\ \rho^A_{\mathbf{2}\,\mu} &= \frac{\mathrm{i}}{\sqrt{2}}\left(\omega^A_\mu - \upsilon^A_\mu\right)\,, & \lambda_{\mathbf{2}\,A} &= \frac{\mathrm{i}}{\sqrt{2}}\left(\xi_A - \chi_A\right)\,, \end{aligned} \tag{6.2.41}$$

we get

$$\mathcal{L}_{\text{Dira-type}} = -\sqrt{2}\left(\omega_\mu\,\Sigma^{\mu\nu}\,\partial_\nu\chi + \upsilon_\mu\,\Sigma^{\mu\nu}\,\partial_\nu\xi\right) + 2\alpha\,\omega_\mu\,\Sigma^{\mu\nu}\,\upsilon_\nu + 2\beta\,\omega^\mu\,\upsilon_\mu - \frac{2\mu}{\beta}\,\xi\,\chi\,. \tag{6.2.42}$$

It is now obvious that this Lagrangian is invariant under the global $U(1)$ symmetry

$$\begin{aligned}\omega_\mu &\to e^{-i\varphi}\,\omega_\mu\,, & \upsilon_\mu &\to e^{i\varphi}\,\upsilon_\mu\,,\\ \chi &\to e^{i\varphi}\,\chi\,, & \xi &\to e^{-i\varphi}\,\xi\,, \end{aligned} \tag{6.2.43}$$

To get the effective Lagrangian in this case we can proceed in two ways. In the same manner as we did above, we can split the Lagrangian (6.2.42) into space and time indices and integrate out the non-propagating fields. The second way, it is to start from (6.2.40), use the effective Lagrangian we already computed for the massive case and then make a field transformation that resembles (6.2.41) but instead of transforming the spinor-valued one-forms we will transform the canonically conjugate fields. Computing the effective Lagrangian, in any of these two forms, has to give the same result. Let us show how it works for the second one.

The effective Lagrangian for two uncouple fermions of equal mass (6.2.40) is, see equation (6.2.34),

$$\begin{aligned}\mathcal{L}_{\text{Dirac-type}} =& \pi^{\mathbf{1}}\,\partial_0\lambda_{\mathbf{1}} + \pi^{\mathbf{2}}\,\partial_0\lambda_{\mathbf{2}} + \frac{1}{4}(C^2-1)\left(\pi^{\mathbf{1}}\,\pi^{\mathbf{1}} + \pi^{\mathbf{2}}\,\pi^{\mathbf{2}}\right)\\ &- \sqrt{2}\,C\left(\pi^{\mathbf{1}}\,\widetilde{\sigma}^i\,\partial_i\,\lambda_{\mathbf{1}} + \pi^{\mathbf{2}}\,\widetilde{\sigma}^i\,\partial_i\,\lambda_{\mathbf{2}}\right) - \left(\partial_i\lambda_{\mathbf{1}}\,\partial_i\lambda_{\mathbf{1}} + \partial_i\lambda_{\mathbf{2}}\,\partial_i\lambda_{\mathbf{2}}\right)\\ &- \frac{8(3C+1)}{3(C+1)}\mu\left(\lambda_{\mathbf{1}}\,\lambda_{\mathbf{1}} + \lambda_{\mathbf{2}}\,\lambda_{\mathbf{2}}\right)\,, \end{aligned} \tag{6.2.44}$$

where the fields have been rescaled, and as the notation remarks $\pi^{\mathbf{1}}$ is the conjugate field to $\lambda_{\mathbf{1}}$ and $\pi^{\mathbf{2}}$ the conjugate field to $\lambda_{\mathbf{2}}$. Now, let

$$\pi^{\mathbf{1}\,A} = \frac{1}{\sqrt{2}}\left({}^{\chi}\pi^{A} + {}^{\xi}\pi^{A}\right) , \qquad \lambda_{\mathbf{1}\,A} = \frac{1}{\sqrt{2}}\left(\xi_A + \chi_A\right) , \tag{6.2.45}$$

$$\pi^{\mathbf{2}\,A} = \frac{\mathrm{i}}{\sqrt{2}}\left({}^{\chi}\pi^{A} - {}^{\xi}\pi^{A}\right) , \qquad \lambda_{\mathbf{2}\,A} = \frac{\mathrm{i}}{\sqrt{2}}\left(\xi_A - \chi_A\right) . \tag{6.2.46}$$

Then, the effective Lagrangian above is written as

$$\begin{aligned}\mathcal{L}_{\mathrm{Dirac-type}} = {}& {}^{\chi}\pi\,\partial_0\chi + {}^{\xi}\pi\,\partial_0\xi + \frac{1}{2}(C^2-1)\,{}^{\chi}\pi\,{}^{\xi}\pi - \sqrt{2}C\left({}^{\chi}\pi\,\widetilde{\sigma}^i\partial_i\chi + {}^{\xi}\pi\,\widetilde{\sigma}^i\,\partial_i\xi\right) \\ & - 2\,\partial^i\chi\,\partial_i\xi - \frac{16(3C+1)}{3(C+1)}\mu\,\chi\,\xi\,.\end{aligned} \tag{6.2.47}$$

Note that ${}^{\chi}\pi$ and ${}^{\xi}\pi$ are the canonical conjugate fields to χ and ξ, respectively.

In all this chapter we have been working in Minkowski spacetime. When gravity is introduced this formalism has to be generalised to more general non-flat metrics. This can be done using the tetrad formalism which makes the spinor fields feel like they are in a Minkowski spacetime at every point of the curved manifold. For example, in a curved manifold, where general coordinate system indices are denoted by greek letters $\mu, \nu, \ldots$ and Lorentz indices by $\mathcal{I}, \mathcal{J}, \cdots$, the Lorentz one-form $\sigma_{\mathcal{I}}^{A'A}$ that was used to define the two-form $\Sigma_{\mathcal{I}\mathcal{J}}^{AB}$ will be generalised to a "soldering form" $\theta_{\mu}^{A'A} = \sigma_{\mathcal{I}}^{A'A}\,\theta_{\mu}^{\mathcal{I}}$, where $\theta_{\mu}^{\mathcal{I}}$ represents a tetrad.

Part II

Pure Connection Formulation

CHAPTER 7

A Class of Diffeomorphism Invariant Gauge Theories

Let us consider a general diffeomorphism and gauge invariant action of a connection one-form with values in the Lie algebra $\mathfrak{g}$ of a complex semisimple Lie group G, i.e.,

$$S[A] = \frac{1}{\mathrm{i}} \int \mathcal{F}(F \wedge F) \equiv \mathrm{i} \int d^4x \, \mathcal{F}(X^{IJ}) \,, \tag{7.0.1}$$

where $\mathcal{F}$ is a complex-valued gauge invariant, holomorphic and homogeneous of degree one function of the matrices[1] X^{IJ}. The indices I, J run form 1 to the dimension of the Lie group. These matrices are defined in terms of the field strength of the connection one-form A^I as

$$X^{IJ} = \frac{1}{4} \tilde{\epsilon}^{\mu\nu\lambda\rho} F^I_{\mu\nu} F^J_{\lambda\rho} \equiv * \left(F^I \wedge F^J \right) \,, \tag{7.0.2}$$

where $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol[2] and we have defined the asterisk operator $*$ as

$$* \left(dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho \right) = \tilde{\epsilon}^{\mu\nu\lambda\rho} \,. \tag{7.0.3}$$

Note that the asterisk operator is defined only on 4-forms.

[1] The holomorphicity condition implies that the function $\mathcal{F}$ can be expanded in a Taylor series. To be gauge invariant means that satisfy $\mathcal{F}(gXg^{-1}) = \mathcal{F}(X)$ for any Lie group element g. To be homogeneous of degree one in the matrix X means $\mathcal{F}(\alpha X) = \alpha \mathcal{F}(X)$, for any $\alpha \in \mathbb{C}$.

[2] We use the convention that the Levi-Civita symbol has component

$$\tilde{\epsilon}^{\mu\nu\lambda\rho} = \begin{cases} -1, & \text{if } \mu\nu\lambda\rho \text{ is an even permutation of 0123,} \\ 1, & \text{if } \mu\nu\lambda\rho \text{ is an odd permutation of 0123,} \\ 0, & \text{if } \mu = \nu \text{ or } \nu = \lambda \text{ or } \lambda = \rho \text{ or } \rho = \mu, \end{cases}$$

in any coordinate system. Recall that we don't need a metric to define the Levi-Civita symbol. Moreover, with this convention we have $dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho = -\tilde{\epsilon}^{\mu\nu\lambda\rho} \, d^4x$.

7.1 Symmetries

Let us show by explicitly computation that our action (7.0.1) is invariant under general coordinate transformation, diffeomorphisms and gauge transformations.
Under a general coordinate transformation the Levi-Civita symbol $\tilde{\epsilon}$, the FF term and the volume element d^4x transform as

$$\begin{aligned}
\tilde{\epsilon}^{\mu\nu\lambda\rho} &\to \left|\frac{\partial x'}{\partial x}\right| \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\rho}}{\partial x^{\rho'}}\,\tilde{\epsilon}^{\mu'\nu'\lambda'\rho'}\,, \\
F_{\mu\nu}F_{\lambda\rho} &\to \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\frac{\partial x^{\rho'}}{\partial x^{\rho}}\,F_{\mu'\nu'}F_{\lambda'\rho'}\,, \\
d^4x &\to \left|\frac{\partial x}{\partial x'}\right|\, d^4x'\,,
\end{aligned} \tag{7.1.1}$$

where $|\partial x'/\partial x|$ is the Jacobian determinant of the transformation $x \to x'$ and $|\partial x/\partial x'|$ its inverse.
From these expressions is easy to check the Lagrangian density is invariant under a general coordinates transformation, i.e.,

$$d^4x\;\mathcal{F}(X^{IJ}) = d^4x\;\frac{1}{4}\tilde{\epsilon}^{\mu\nu\lambda\rho}\,\mathcal{F}\left(F^I_{\mu\nu}F^J_{\lambda\rho}\right) \to d^4x'\;\frac{1}{4}\tilde{\epsilon}^{\mu'\nu'\lambda'\rho'}\,\mathcal{F}\left(F^I_{\mu'\nu'}F^J_{\lambda'\rho'}\right) = d^4x'\;\mathcal{F}(X'^{IJ})\,, \tag{7.1.2}$$

where we have used the homogeneity of the function $\mathcal{F}$.
Now, under an infinitesimal gauge transformation with parameter ω^I, the connection one-form A^I transform as

$$\begin{aligned}
\delta_\omega A^I =& D\omega^I \equiv d\omega^I + [A,\omega]^I\,, \\
=& d\omega^I + C^I_{JK}A^J\,\omega^K\,.
\end{aligned} \tag{7.1.3}$$

Moreover, the action of diffeomorphisms on the connection are given by

$$\delta_\xi A = \xi \lrcorner F^I\,, \tag{7.1.4}$$

where ξ is a general vector and the symbol $\lrcorner$ stands for the interior product (contraction between a vector and a form). In components $(\xi\lrcorner F^I)_\mu = \xi^\alpha F^I_{\alpha\mu}$. Note that the diffeomorphism has been corrected by a gauge transformation with parameter $\xi\lrcorner A^I$.[3]
Let us prove first the invariance of the action with respect to infinitesimal gauge trans-

[3]In general, the connection A^I transform under a diffeomorphism as

$$\delta_\xi A^I = \mathfrak{L}_\xi A^I = \xi\lrcorner dA^I + d(\xi\lrcorner A^I)\,,$$

where the symbol $\mathfrak{L}$ denotes the Lie derivative. We are free to subtract to this expression an infinitesimal gauge transformation $D(\xi\lrcorner A^I)$ to get $\delta_\xi A = \xi\lrcorner F^I$.

formations. Taking the variation of the action, we have

$$
\begin{aligned}
\delta_\omega S =& \mathrm{i}\int d^4x\, \frac{\partial \mathcal{F}}{\partial X^{IJ}}\delta X^{IJ} = 2\mathrm{i}\int d^4x\, *\left(\frac{\partial \mathcal{F}}{\partial X^{IJ}}\, D\delta_\omega A^I \wedge F^J\right), \\
=& -2\mathrm{i}\int \frac{\partial \mathcal{F}}{\partial X^{IJ}}\, D\delta_\omega A^I \wedge F^J = -2\mathrm{i}\int \frac{\partial \mathcal{F}}{\partial X^{IJ}}\, D^2\omega^I \wedge F^J, \\
=& -2\mathrm{i}\int \frac{\partial \mathcal{F}}{\partial X^{IJ}}\, [F,\omega]^I \wedge F^J = 2\mathrm{i}\int \frac{\partial \mathcal{F}}{\partial X^{IJ}}\, C^I_{KL}\omega^K F^L \wedge F^J, \\
=& 2\mathrm{i}\int d^4x\, \frac{\partial \mathcal{F}}{\partial X^{IJ}}\, C^I_{KL}\omega^K X^{JL},
\end{aligned}
\tag{7.1.5}
$$

where in the first line we used $\delta X^{IJ} = 2*(D\delta A^{(I} \wedge F^{J)})$; in the second line we used the identity $d^4x\, *(\text{“4-form"}) = (\text{“4-form"}) * (d^4x) = -\text{“4-form"}$ [4]; in the third line $D^2\omega^I = [F,\omega]^I$; and in the fourth line the definition of X^{JL}.

As $\mathcal{F}(X)$ is gauge invariant, we have

$$
\delta_\omega \mathcal{F} = 0 = \frac{\partial \mathcal{F}}{\partial X^{IJ}}\delta_\omega X^{IJ}, \tag{7.1.6}
$$

where

$$
\delta_\omega X^{IJ} = C^I_{KL}\omega^K X^{JL} + C^J_{KL}\omega^K X^{IL}. \tag{7.1.7}
$$

Then, we find the identity

$$
\frac{\partial \mathcal{F}}{\partial X^{IJ}} C^I_{KL} X^{JL} = 0. \tag{7.1.8}
$$

Using the above identity we get

$$
\delta_\omega S = 0, \tag{7.1.9}
$$

that is what we wanted to show.

Now, let us prove the invariance of the action under diffeomorphisms. Taking the variation of the action again, but this time using the action of the diffeomorphisms on the connection, we get

$$
\begin{aligned}
\delta_\xi S =& 2\mathrm{i}\int \frac{\partial \mathcal{F}}{\partial X^{IJ}} D\delta_\xi A^I \wedge F^J, \\
=& -2\mathrm{i}\int D\left(\frac{\partial \mathcal{F}}{\partial X^{IJ}}\right) \wedge \delta_\xi A^I \wedge F^J = -2\mathrm{i}\int D\left(\frac{\partial \mathcal{F}}{\partial X^{IJ}}\right) \wedge \left(\xi \lrcorner F^I\right) \wedge F^J, \\
=& -\mathrm{i}\int D\left(\frac{\partial \mathcal{F}}{\partial X^{IJ}}\right) \wedge \xi \lrcorner \left(F^I \wedge F^J\right) = -\mathrm{i}\int D\left(\frac{\partial \mathcal{F}}{\partial X^{IJ}}\right) X^{IJ} \wedge \xi \wedge d^4x,
\end{aligned}
\tag{7.1.10}
$$

where in the second line we have integrated by parts, used the Bianchi identity $DF = 0$ and dropped the surface term; in the third line we have used the property of the interior product $\xi \lrcorner (F^I \wedge F^J) = (\xi \lrcorner F^I) \wedge F^J + (-1)^2 F^I \wedge (\xi \lrcorner F^J) = 2(\xi \lrcorner F^I) \wedge F^J$ and the definition of X^{IJ}.

As $\mathcal{F}$ is an homogeneous function in X, it satisfies

$$
\mathcal{F}(X) = \frac{\partial \mathcal{F}}{\partial X^{IJ}} X^{IJ}. \tag{7.1.11}
$$

[4] As $d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, we have $*(d^4x) = \tilde{\epsilon}^{0123} = -1$.

Applying the exterior derivative at both sides of this equation, we find

$$\begin{aligned} d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial X^{IJ}} dX^{IJ} =& d\left(\frac{\partial \mathcal{F}}{\partial X^{IJ}}\right) X^{IJ} + \frac{\partial \mathcal{F}}{\partial X^{IJ}} dX^{IJ} \,, \\ & d\left(\frac{\partial \mathcal{F}}{\partial X^{IJ}}\right) X^{IJ} = 0 \,. \end{aligned} \tag{7.1.12}$$

Using the above identity and (7.1.8), we have

$$D\left(\frac{\partial \mathcal{F}}{\partial X^{IJ}}\right) X^{IJ} = 0 \,. \tag{7.1.13}$$

Thus, we get for the variation of the action

$$\delta_{\xi} S = 0 \,, \tag{7.1.14}$$

and we have showed that the action is invariant under diffeomorphisms as well.

7.2 Field equations

The first variation of (7.0.1) is

$$\mathcal{L}^{(1)} \equiv \delta \mathcal{L} = \mathrm{i}\, \frac{\partial \mathcal{F}}{\partial X^{IJ}} \delta X^{IJ} \,, \tag{7.2.1}$$

where

$$\delta X^{IJ} = 2 * \left(D\delta A^{(I} \wedge F^{J)} \right) \,. \tag{7.2.2}$$

Then, the equations of motion for the connection A^I that results form (7.0.1) are

$$D\left(\frac{\partial \mathcal{F}}{\partial X^{IJ}} F^J\right) = 0 \,. \tag{7.2.3}$$

Note that because of the Bianchi identity $DF = 0$, the curvature F^J can be taken outside the brackets and leave the covariant derivative acting only on the partial derivative of the the function $\mathcal{F}$.

7.3 Perturbation theory

Taking the second variation of the action (7.0.1), we get

$$2\mathcal{L}^{(2)} \equiv \delta^2 \mathcal{L} = \mathrm{i}\, \frac{\partial^2 \mathcal{F}}{\partial X^{IJ} \partial X^{KL}} \delta X^{IJ} \delta X^{KL} + \mathrm{i}\, \frac{\partial \mathcal{F}}{\partial X^{IJ}} \delta^2 X^{IJ} \,, \tag{7.3.1}$$

with $\delta^2 X^{IJ}$ given by

$$\begin{aligned} \delta^2 X^{IJ} =& 2 * \left([\delta A, \delta A]^{(I} \wedge F^{J)} + D\delta A^{(I} \wedge D\delta A^{J)} \right) \,, \\ =& 2 * \left([\delta A, \delta A]^{(I} \wedge F^{J)} + \delta A^{(I} \wedge [F, \delta A]^{J)} + D(\delta A^{(I} \wedge \delta A^{J)}) \right) \,, \end{aligned} \tag{7.3.2}$$

where we have used $D^2\delta A^J = [F,\delta A]^J$. In the same way we can compute the nth-variation and have and explicit expression for $\mathcal{L}^{(n)}$ in terms of the perturbation of the connection δA^I.

Expanding the commutators in $\delta^2 X$ and reorganising we find for $\mathcal{L}^{(2)}$

$$\begin{aligned}-2\mathrm{i}\,\mathcal{L}^{(2)} =& 4\,\frac{\partial^2\mathcal{F}}{\partial X^{IJ}\partial X^{KL}}\ *(F^I\wedge Da^J)\ *(F^K\wedge Da^L)\\ &+2\,\frac{\partial\mathcal{F}}{\partial X^{IJ}}\,C^{(I|}_{KL}\ *\Big(F^{|I)}\wedge a^K\wedge a^L+F^K\wedge a^{|J)}\wedge a^L\Big)+\text{total derivatives}\,,\end{aligned} \tag{7.3.3}$$

where we have defined $\delta A = a$. The field a^I is now our field variable.

CHAPTER 8

Gravity

As in the BF plus potential formulation of this theory the $\mathfrak{su}(2)$ case describes gravity. A specific form of the defining function $\mathcal{F}$ which represent GR is found starting from the Plebanski formulation of GR. Then, GR can be cast in a pure connection form where the only field variable is a $SU(2)$ gauge potential. For a general function $\mathcal{F}$ we obtain a gravity theory with two polarisations of the graviton.
Let us first describe how General Relativity fits in this formalism.

8.1 General Relativity

The Plebanski formulation of General Relativity (GR) is described by the action [39, 43]

$$S[A,B,\psi] = \frac{1}{8\pi G}\int B_i \wedge F^i - \frac{1}{2}\Phi_{ij}\, B^i \wedge B^j\,, \tag{8.1.1}$$

where B^i is an $\mathfrak{su}(2)$-valued two-form, and

$$\Phi_{ij} = \tilde{\psi}_{ij} + \frac{\Lambda}{3}\delta_{ij}\,. \tag{8.1.2}$$

The scalar field $\tilde{\psi}_{ij}$ is symmetric and traceless, i.e., $\tilde{\psi}_{ij} = \psi_{ij} - \delta^{kl}\psi_{kl}/3\,\delta_{ij} = \psi_{ij} - \mathrm{tr}\psi/3\,\delta_{ij}$ and Λ is the cosmological constat.
The field equations for B, A and ψ are, respectively,

$$F^i = \Phi^i_j\, B^j\,, \tag{8.1.3}$$

$$DB^i = 0\,, \tag{8.1.4}$$

$$B^i \wedge B^j = \frac{\delta^{ij}}{3}\delta_{kl}B^k \wedge B^l\,. \tag{8.1.5}$$

Assuming that Φ_{ij} is invertible, i.e., $\Phi^{-1}_{ik}\Phi^{kj} = \delta^j_i$, the first equation above can be rewritten as

$$B^i = \left(\Phi^{-1}\right)^i_j F^j\,. \tag{8.1.6}$$

Then, integrating out the B-field we will get the action

$$S[A,\psi] = \frac{1}{16\pi G}\int \Phi^{-1}_{ij}\, F^i \wedge F^j \,. \tag{8.1.7}$$

The next step is to integrate out the ψ-field to get an action functional of the connection only. Let us compute the variation of Φ with respect to ψ. Taking the variation, with respect to ψ, of the identity $\Phi^{-1}_{ik}\Phi^{kj} = \delta^j_i$, we get

$$\delta_\psi \Phi^{-1}_{ij} = -\delta\psi^{mn}\left(\Phi^{-1}_{m(i}\Phi^{-1}_{j)n} - \frac{1}{3}\delta_{mn}\,\Phi^{-2}_{ij}\right), \tag{8.1.8}$$

where we have used

$$\delta_\psi \Phi^{ij} = \delta\psi^{kl}\left(\delta^i_{(k}\delta^j_{l)} - \frac{1}{3}\delta_{kl}\delta^{ij}\right). \tag{8.1.9}$$

The equation of motion for ψ, varying the action (8.1.7), is

$$\Phi^{-1}_{ik}X^{kj} = \frac{1}{3}\mathrm{tr}\left(\Phi^{-2}X\right)\Phi^j_i \,. \tag{8.1.10}$$

Recall that $X^{ij} = 1/4\, F^i_{\mu\nu}F^j_{\lambda\rho}\tilde{\epsilon}^{\mu\nu\lambda\rho}$. The above equation can be written in matrix notation as

$$\Phi^{-1}X = \frac{1}{3}\mathrm{tr}\left(\Phi^{-2}X\right)\Phi \,. \tag{8.1.11}$$

Or solving for Φ

$$\Phi = \sqrt{\frac{3}{\mathrm{tr}\left(\Phi^{-2}X\right)}}\,\sqrt{X}\,, \tag{8.1.12}$$

where we have assumed that the square root of the matrix X exists. We can make this assumption for positive definite matrix.
Taking the trace of the equation (8.1.11), we find

$$\mathrm{tr}\left(\Phi^{-1}X\right) = \frac{1}{3}\mathrm{tr}\left(\Phi^{-2}X\right)\,\mathrm{tr}\Phi\,, \tag{8.1.13}$$

$$\mathrm{tr}\left(\Phi^{-1}X\right) = \frac{1}{\sqrt{3}}\sqrt{\mathrm{tr}\left(\Phi^{-2}X\right)}\,\mathrm{tr}\left(\sqrt{X}\right), \tag{8.1.14}$$

where in the second line we used, see (8.1.12),

$$\mathrm{tr}\Phi = \sqrt{\frac{3}{\mathrm{tr}\left(\Phi^{-2}X\right)}}\,\mathrm{tr}\left(\sqrt{X}\right). \tag{8.1.15}$$

In the other hand, from the definition of Φ, see (8.1.2), we have $\mathrm{tr}\Phi = \Lambda$. Then, we can also write

$$\mathrm{tr}\left(\Phi^{-1}X\right) = \frac{\Lambda}{3}\mathrm{tr}\left(\Phi^{-2}X\right). \tag{8.1.16}$$

Thus, equation (8.1.14) can be rewritten as

$$\mathrm{tr}\left(\Phi^{-1}X\right) = \frac{1}{\Lambda}\left(\mathrm{tr}\sqrt{X}\right)^2, \tag{8.1.17}$$

where we used (8.1.16).

Finally, we can rewrite the action (8.1.7),

$$S[A,\psi] = \frac{1}{16\pi G}\int \Phi^{-1}_{ij}\, F^i \wedge F^j = -\frac{1}{16\pi G}\int d^4x \,\mathrm{tr}\left(\Phi^{-1}X\right)\,,$$

as

$$S[A] = -\frac{1}{16\pi G\Lambda}\int d^4x \,\left(\mathrm{tr}\sqrt{X}\right)^2\,, \tag{8.1.18}$$

or in a compact notation

$$S[A] = \frac{1}{16\pi G\Lambda}\int \left(\mathrm{tr}\sqrt{F\wedge F}\right)^2\,. \tag{8.1.19}$$

The above action shows a formulation of GR where the only field variable is a connection one-form. This action was first proposed in [22].

In the next section we shall describe a class of modified gravity theories with just two propagating degrees of freedom.

8.2 Modified gravity theory

Let us considerate the action (7.0.1) with the connection an $\mathfrak{su}(2)$-valued one-form A^i, with $i = 1,2,3$, and the function $\mathcal{F}$ defined as

$$\mathcal{F} = TrX\,\chi\left(\frac{Tr\,X^2}{(Tr\,X)^2}, \frac{Tr\,X^3}{(Tr\,X)^3}\right)\,, \tag{8.2.1}$$

where χ is an arbitrary holomorphic function of is two arguments and the traces are computed using the inner product on the Lie algebra δ_{ij}.

8.2.1 Background

The background which we are going to choose to expand around is the constant curvature one. A constant curvature spacetime is described by the line element

$$ds^2 = c^2(t)\left(-dt^2 + \delta_{ij}dx^i dx^j\right)\,, \tag{8.2.2}$$

where t is the conformal time and x^i the spatial coordinates. The orthonormal tetrad associated to the above line element is $\theta^0 = c(t)\,dt$ and $\theta^i = c(t)\,dx^i$. From this tetrad we can define a triple of two-forms Σ^i as

$$\Sigma^i = c^2\left(idt\wedge dx^i - \frac{1}{2}\epsilon^i_{jk}dx^j\wedge dx^k\right)\,. \tag{8.2.3}$$

Now, we define the constant curvature connection A^i_o as the solution of the equation

$$D\Sigma^i = d\Sigma^i + \epsilon^i_{jk}A^j_o\wedge\Sigma^k = 0\,. \tag{8.2.4}$$

As A_o is the solution of the equation above, the condition that our background has constant curvature can be compactly written as

$$F_o^i = M^2\,\Sigma^i\,, \tag{8.2.5}$$

where M is a dimensionful parameter. Explicitly, the constant curvature connection is

$$A_o^i = iMcdx^i\,. \tag{8.2.6}$$

Recall that we can define the Σ's for a general manifold as

$$\Sigma^i = i\theta^0 \wedge \theta^i - \frac{1}{2}\epsilon^i_{kj}\theta^j \wedge \theta^k\,, \tag{8.2.7}$$

where $\theta^{\mathcal{I}} = \{\theta^0, \theta^i\}$ is an orthonormal tetrad with line element given by $ds^2 = \eta_{\mathcal{I}\mathcal{J}}\theta^{\mathcal{I}}\theta^{\mathcal{J}}$, with $\eta_{\mathcal{I}\mathcal{J}} = \mathrm{diag}(-1,1,1,1)$ the Minkowski metric.
It is easy to show the general two-forms (8.2.7) are selfdual (with respect to the Hodge operation defined by the metric $\eta_{\mathcal{I}\mathcal{J}}$), i.e., $\epsilon_{\mu\nu\lambda\rho}\Sigma^{i\lambda\rho} = 2\mathrm{i}\,\Sigma^i_{\mu\nu}$. These general two-forms satisfy the identities

$$\Sigma^i_{\mu\sigma}\Sigma^{j\sigma}{}_{\nu} = -\,\delta^{ij}g_{\mu\nu} + \epsilon^i_{jk}\Sigma^k_{\mu\nu}\,, \tag{8.2.8}$$

$$\Sigma_{i\mu\nu}\Sigma^i_{\lambda\rho} = g_{\mu\lambda}g_{\rho\mu} - g_{\mu\rho}g_{\lambda\mu} - i\epsilon_{\mu\nu\lambda\rho}\,, \tag{8.2.9}$$

for a general metric $g_{\mu\nu}$. We will be able to go to a flat background taking the limit $M \to 0$. However, as we will see below we have to be careful taking this limit.

8.2.2 Action evaluated on the background

Using (8.2.7), it is easy to check that

$$\Sigma^i \wedge \Sigma^j = -2\mathrm{i}\sqrt{-g}\,\delta^{ij}\,d^4x\,, \tag{8.2.10}$$

where g is the determinant of the metric defined by the tetrad $\theta^{\mathcal{I}}$. Then, because the curvature F^i at the background is $F_o^i = M^2\,\Sigma^i$, we obtain that the matrix $\tilde{X}^{ij}$ evaluated at the background is proportional to the identity, i.e.,

$$\tilde{X}_o^{ij} = *(F_o^i \wedge F_o^j) = 2\mathrm{i}M^4\sqrt{-g}\,\delta^{ij}\,, \tag{8.2.11}$$

where we have used $*d^4x = -1$. Then, the the action (7.0.1) at the background A_o^i is

$$S[A_o^i] = -2M^4\mathcal{F}_o\int d^4x\,\sqrt{-g}\,, \tag{8.2.12}$$

where $\mathcal{F}_o = \mathcal{F}(\delta^{ij})$, i.e., the value of the defining function $\mathcal{F}$ at the identity matrix.
On the other hand, on a constant curvature background we have that the Ricci scalar R

is proportional to the cosmological constant Λ, i.e., $R = 4\Lambda$. Then, the Einstein-Hilbert action

$$S[g]_{\mathrm{E-H}} = -\frac{1}{16\pi G}\int d^4x\,(R - 2\Lambda)\,, \tag{8.2.13}$$

evaluated on a constant curvature background is given by

$$S^0_{\mathrm{E-H}} = -\frac{\Lambda}{8\pi G}\int d^4x\,\sqrt{-g}\,. \tag{8.2.14}$$

We expect the two action (8.2.12) and (8.2.14) be equivalent when the defining function $\mathcal{F}$ is such that it describes general relativity. Then, we find that for the general relativity case

$$2M^4\mathcal{F}_o = \frac{\Lambda}{8\pi G}\,. \tag{8.2.15}$$

8.2.3 Linearised action

The partial derivative of $\mathcal{F}$ with respect to $\tilde{X}^{ij}$ is

$$\frac{\partial\mathcal{F}}{\partial\tilde{X}^{ij}} = \chi\,\delta_{ij} + 2\chi_1'\,Tr\,\tilde{X}\left(\frac{\tilde{X}_{ij}}{(Tr\,\tilde{X})^2} - \frac{Tr\,\tilde{X}^2}{(Tr\,\tilde{X})^3}\delta_{ij}\right) + 3\chi_2'\,Tr\,\tilde{X}\left(\frac{\tilde{X}^2_{ij}}{(Tr\,\tilde{X})^3} - \frac{Tr\,\tilde{X}^3}{(Tr\,\tilde{X})^4}\delta_{ij}\right)\,, \tag{8.2.16}$$

where χ_1' and χ_2' are the derivatives of χ with respect to its first and second arguments. It is easy to show that the parenthesis on the second and third term are zero when $\tilde{X}$ is evaluated at the background. Then, we obtain for the expression above evaluated at the background

$$\left.\frac{\partial\mathcal{F}}{\partial\tilde{X}^{ij}}\right|_o = \frac{\mathcal{F}_o}{3}\delta_{ij}\,. \tag{8.2.17}$$

As you can see this expression does not depend on the parameter M.
In the same way, we find for the partial derivative of $\mathcal{F}$ with respect to $\tilde{X}^{ij}$ and $\tilde{X}^{kl}$ at the background

$$\left.\frac{\partial^2\mathcal{F}}{\partial\tilde{X}^{ij}\partial\tilde{X}^{kl}}\right|_o = 2\frac{(\chi_1')_o + (\chi_2')_o}{Tr\,\tilde{X}_o}\,P_{ij|kl} \tag{8.2.18}$$

where $P_{ij|kl}$ is the projector on the symmetric traceless part, i.e.,

$$P_{ij|kl} = \delta_{i(k}\delta_{l)j} - \frac{1}{3}\delta_{ij}\delta_{kl}\,. \tag{8.2.19}$$

This projector satisfy $P_{ij|kl}\delta^{ij} = 0 = P_{ij|kl}\delta^{kl}$.
The linearised Lagrangian evaluated at the background A^i_o is given by, see (7.3.3),

$$\begin{aligned}-2\mathrm{i}\,\mathcal{L}^{(2)}_o = 4\left.\frac{\partial^2\mathcal{F}}{\partial X^{ij}\partial X^{kl}}\right|_o &* (F^i_o\wedge D_o a^j)\ * (F^k_o\wedge D_o a^l)\\ &+ 2\left.\frac{\partial\mathcal{F}}{\partial X^{ij}}\right|_o \epsilon^{(i}_{kl} * (F^{|j)}_o\wedge a^k\wedge a^l + F^k_o\wedge a^{|j)}\wedge a^l)\,,\end{aligned} \tag{8.2.20}$$

where ϵ^i_{jk} are the structure constants of $\mathfrak{su}(2)$. As $(\partial\mathcal{F}/\partial\tilde{X}^{ij})_o$ is proportional to δ_{ij} the second term vanishes. Then, we finally get for the linearised Lagrangian

$$\mathcal{L}^{(2)}_o = -\frac{g_{\rm gr}}{2}\sqrt{-g}\,P_{ij|kl}(\Sigma^{i\mu\nu}D_{o\,\mu}a^j_\nu)(\Sigma^{k\rho\sigma}D_{o\,\rho}a^l_\sigma)\,, \tag{8.2.21}$$

where we have used the selfduality of the Σ's and defined

$$g_{\rm gr} \equiv \frac{4}{3}\left((\chi'_1)_o + (\chi'_2)_o\right)\,. \tag{8.2.22}$$

Note that the factors of M have been cancelled for the linearised Lagrangian.

8.2.4 High energy limit

We are interested in energies $E \gg M$. We can arrived at this regime by taking the limit $M \to 0$ and then we can replace the covariant derivatives by partial derivatives and effectively work in a flat Minkowski background. In Minkowski spacetime the Σ's are given by

$$\Sigma^i = \mathrm{i}\,dt\wedge dx^i - \frac{1}{2}\epsilon^i_{jk}\,dx^j\wedge dx^k\,, \tag{8.2.23}$$

i.e., the Σ's are built from the Minkowski tetrad $\{dt, dx^i\}$.

The linearised Lagrangian (8.2.21) is not canonically normalised because its kinetic term has a factor $g_{\rm gr}$ in front of it. To put it in a canonical form we rescales the field variable a^i as

$$a^i \to \frac{1}{\sqrt{g_{\rm gr}}}a^i\,. \tag{8.2.24}$$

Then, replacing the covariant derivative by partial derivative and rescaling the field variable in (8.2.21) we get for our high energy linearised Lagrangian

$$\mathcal{L}_{\rm lin} = -\frac{1}{2}\,P_{ij|kl}(\Sigma^{i\mu\nu}\partial_\mu a^j_\nu)(\Sigma^{k\rho\sigma}\partial_\rho a^l_\sigma)\,. \tag{8.2.25}$$

8.2.5 Hamiltonian analysis

Let us split in space and time indices the quantity $\Sigma^{i\mu\nu}\partial_\mu a^j_\nu$ which is the one that appears as the main building block of the linearised action (8.2.25). We have

$$\Sigma^{\mu\nu}_i\partial_\mu a^j_\nu = -\mathrm{i}\,\dot{a}^j_i + \mathrm{i}\,\partial_i a^j_0 - \epsilon_i^{kl}\partial_k a^j_l\,, \tag{8.2.26}$$

where the dot over a^j stands for time derivative and we have identified the spatial and internal indices using the time component of the background two-forms, i.e., $\Sigma^i_{0a} = \mathrm{i}\,\delta^i_a$. We also used $\Sigma^i_{jk} = -\epsilon^i_{jk}$ and we raise and lower indices using δ^{ij} and δ_{ij}.

The canonically conjugate field π^{ij} of the field variable a_{ij} is given by

$$\pi^{ij} = P^{ij|kl}(\dot{a}_{kl} - \partial_k a_{0l} - \mathrm{i}\,\epsilon_k^{mn}\partial_m a_{nl})\,. \tag{8.2.27}$$

Using the identity $P^{ij|mn}P_{mn|kl} = P^{ij}{}_{kl}$, we find that

$$\pi^2 = -P_{ij|kl}\,(\Sigma^{i\mu\nu}\partial_\mu a^j_\nu)(\Sigma^{k\rho\sigma}\partial_\rho a^l_\sigma)\,. \tag{8.2.28}$$

Now, let us decompose a_{ij} in irreducible components as the sum of its symmetric trace-free, trace and anti-symmetric parts, i.e.,

$$a_{ij} = a^{\mathrm{s}}_{ij} + b\delta_{ij} + \epsilon^k_{ij}c_k\,. \tag{8.2.29}$$

In the above expression a^{s}_{ij} stands for the symmetric trace-free part, and b and c_k parameterise the trace and anti-symmetric parts, respectively. Then, using this decomposition of a_{ij} in irreducible components the canonically conjugate field can be rewritten as

$$\pi^{ij} = \dot{a}^{\mathrm{s}\,ij} - \mathrm{i}\,\epsilon^{(i|kl}\partial_k a^{\mathrm{s}\,j)}_l + P^{ij|kl}\partial_k(\mathrm{i}\,c_l - a_{0l})\,. \tag{8.2.30}$$

It is easy to see that this expression for π^{ij} is symmetric and trace-free. Now, the Lagrangian density can be written as

$$\mathcal{L} = \frac{1}{2}(\pi^{ij})^2\,, \tag{8.2.31}$$

where π^{ij} is given by the expression (8.2.30). Then, the Hamiltonian density $\mathcal{H} = \pi^{ij}\dot{a}_{ij} - \mathcal{L}$ is given by

$$\mathcal{H} = \frac{1}{2}\pi^2 + \mathrm{i}\,\epsilon^{kl}_i\pi^{ij}\partial_k a_{lj} + \phi_i\partial_j\pi^{ij}\,, \tag{8.2.32}$$

where we have dropped the index s from a^{s}_{ij} for brevity and have defined $\phi_i = \mathrm{i}\,c_i - a_{oi}$. The Lagrange multiplier ϕ_i serves as a parameter for the $SU(2)$ gauge transformation on the connection a_{ij}, i.e.,

$$\delta_\phi a_{ij} = \partial_{(i}\phi_{j)}\,. \tag{8.2.33}$$

Now, we can gauge fix the connection to make it transverse, i.e.,

$$\partial^i a_{ij} = 0\,. \tag{8.2.34}$$

Moreover, the canonically conjugate field π^{ij} is made transverse by the condition obtained varying the action with respect to the Lagrange multiplier ϕ_i. Thus, as it was expected the reduced phase space of our linearised theory is parameterised by two symmetric, trace-free and transverse field a_{ij} and π^{ij} which correspond to two propagating degrees of freedom. Thus, the reduced Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\pi^{ij} + \mathrm{i}\,\epsilon^{ikl}\partial_k a^j_l\,, \tag{8.2.35}$$

and the field equation that follows from it is

$$\begin{aligned}\ddot{a}_{ij} - \partial^k\partial_k a_{ij} &= 0\,,\\ \partial^\mu\partial_\mu a_{ij} &= 0\,,\end{aligned} \tag{8.2.36}$$

which is just the relativistic wave equation for the field a_{ij}.

CHAPTER 9

Gravity-Yang-Mills Unification

In this chapter we show how starting from the diffeomorphism invariant gauge theory (7.0.1), for a Lie algebra larger than $\mathfrak{su}(2)$, we are able to describe gravity and Yang-Mills in a unified framework. Similar to what happens in the BF plus potential formulation what breaks the gauge group into its gravitational, Yang-Mills and "extra" sectors is the apparition of a spacetime metric. The part of the Lie algebra corresponding to the "extra" sector splits into irreducible representations of the $\mathfrak{su}(2)$ gravitational part. A simple example shows explicitly the different representations that appear and the kind of massive fields that it describes.

9.1 Background

The background connection is basically the same one that we used in the $\mathfrak{s}u(2)$ case but this time adapted to be used in a general semisimple Lie algebra $\mathfrak{g}$. Then, we introduce a coordinate system $\{\eta, x^i\}$ such that the connection is given by

$$A_o^I = \mathrm{i}\, \mathcal{A}_i^I \, dx^i \,, \tag{9.1.1}$$

where $\mathcal{A}_i^I$ is only a function of the coordinate η. Moreover, the field $\mathcal{A}_i^I$ has the interpretation of an embedding of $\mathfrak{su}(2)$ into $\mathfrak{g}$, i.e., $\mathcal{A}_i^I : \mathfrak{su}(2) \to \mathfrak{g}$. In simple words $\mathcal{A}_i^I$ tells us how the $\mathfrak{su}(2)$ Lie algebra is formed inside $\mathfrak{g}$.
As $\mathcal{A}_i^I$ is an embedding of $\mathfrak{su}(2)$ into $\mathfrak{g}$ we have

$$[\mathcal{A}_j, \mathcal{A}_k]^I = C^I_{JK}\, \mathcal{A}_j^J \mathcal{A}_k^K \sim \epsilon^i_{jk}\, \mathcal{A}_i^I \,. \tag{9.1.2}$$

Now, let us defined a normalised constant embedding e_i^I that is independent of the coordinates $\{\eta, x^i\}$ as

$$\mathcal{A}_i^I = \mathcal{A}\, e_i^I \,, \tag{9.1.3}$$

with $\mathcal{A} = \mathcal{A}(\eta)$, such that

$$C^I_{JK}\, e^J_j e^K_k = \epsilon^i_{jk}\, e^I_i\,. \tag{9.1.4}$$

Thus, the curvature two-form associated to the connection one-form $\mathcal{A}^I$ is

$$F^I_o = \mathcal{A}^2 e^I_i \left(\mathrm{i}\, \frac{\mathcal{A}'}{\mathcal{A}^2} d\eta \wedge dx^i - \frac{1}{2}\, \epsilon^i_{jk}\, dx^j \wedge dx^k \right), \tag{9.1.5}$$

where the prime in $\mathcal{A}'$ stands for the derivative of $\mathcal{A}$ with respect to η. Let us define a new coordinate t such that

$$\frac{\mathcal{A}'}{\mathcal{A}^2} d\eta = dt\,. \tag{9.1.6}$$

From the above equation we can integrate $\mathcal{A}$ and find an expression for it as a function of t, i.e.,

$$\mathcal{A}(t) = \frac{1}{t_o - t}, \tag{9.1.7}$$

where t_o is the integration constant. Now we can define a dimensionless function $c(t)$ as

$$c(t) = \frac{1}{M(t_o - t)}, \tag{9.1.8}$$

where M is a mass dimension one parameter. Finally, the curvature can be written as

$$F^I_o = M^2\, e^I_i\, \Sigma^i\,, \tag{9.1.9}$$

where

$$\Sigma^i = c^2 \left(\mathrm{i}\, dt \wedge dx^i - \frac{1}{2} \epsilon^i_{jk}\, dx^j \wedge dx^k \right). \tag{9.1.10}$$

We have already found these Σ^i's when we studied the gravitational case. They represent a constant curvature background. Let us define Σ^I as

$$\Sigma^I \equiv e^I_i\, \Sigma^i\,. \tag{9.1.11}$$

Then, the curvature two-form F^I at the background can be rewritten as

$$F^I_o = M^2\, \Sigma^I\,. \tag{9.1.12}$$

Recall that these Σ^i's are self-dual, i.e., $\tilde{\epsilon}^{\mu\nu\lambda\rho}\Sigma^i_{\lambda\rho} = 2\mathrm{i}\, \Sigma^{i\mu\nu}$. The relations between the contravariant and covariant Levi-civita tensor $\epsilon^{\mu\nu\lambda\rho}$ and Levi-civita symbol $\tilde{\epsilon}^{\mu\nu\lambda\rho}$ are

$$\epsilon^{\mu\nu\lambda\rho} = \frac{1}{\sqrt{-g}}\, \tilde{\epsilon}^{\mu\nu\lambda\rho}\,, \qquad \epsilon_{\mu\nu\lambda\rho} = \sqrt{-g}\, \tilde{\epsilon}_{\mu\nu\lambda\rho}\,. \tag{9.1.13}$$

In Minkowski spacetime they are the same because the determinant of the metric is -1.

9.2 Linearisation

The matrix X^{IJ} evaluated at the background connection A^I_o is

$$X^{IJ}_0 = *(F^I_o \wedge F^J_o) = 2\mathrm{i}M^4 \sqrt{-g}\, e^I_i e^J_j \delta^{ij}\,, \tag{9.2.1}$$

where we have used $\epsilon^{0ijk} = -\epsilon^{ijk}$. Let us define the normalised matrix $\hat{X}^{IJ}$ as

$$\hat{X}^{IJ} = \frac{1}{2\mathrm{i}M^4\sqrt{-g}}\, X^{IJ}\,. \tag{9.2.2}$$

Then, its value at the background is

$$\hat{X}^{IJ}_o = e^I_i e^J_j \delta^{ij}\,. \tag{9.2.3}$$

As the defining function of our theory $\mathcal{F}$ is homogeneous of degree one we have

$$\mathcal{F}(X) = (2\mathrm{i}M^4\sqrt{-g})\,\mathcal{F}(\hat{X})\,. \tag{9.2.4}$$

Moreover,

$$\frac{\partial\mathcal{F}(X)}{\partial X^{IJ}} = \frac{\partial\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}}\,, \tag{9.2.5}$$

and

$$\frac{\partial^2\mathcal{F}(X)}{\partial X^{IJ}\partial X^{KL}} = \frac{1}{2\mathrm{i}M^4\sqrt{-g}}\,\frac{\partial^2\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}\partial\hat{X}^{KL}}\,. \tag{9.2.6}$$

Thus, we obtain for the linearised Lagrangian (7.3.3) evaluated at the background

$$\begin{aligned} -2\mathrm{i}\,\mathcal{L} = \frac{4}{2\mathrm{i}\sqrt{-g}}\,\frac{\partial^2\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}\partial\hat{X}^{KL}}\bigg|_o * \left(\Sigma^{(I}\wedge D_o a^{J)}\right) * \left(\Sigma^{(K}\wedge D_o a^{L)}\right) \\ + 2M^2\,\frac{\partial\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}}\bigg|_o C^{(I|}_{KL} * \left(\Sigma^{|J)}\wedge a^K\wedge a^L + \Sigma^K\wedge a^{|J)}\wedge a^L\right)\,. \end{aligned} \tag{9.2.7}$$

Or in components

$$\begin{aligned} \mathcal{L} = -\sqrt{-g}\,\frac{\partial^2\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}\partial\hat{X}^{KL}}\bigg|_o \left(\Sigma^{(I|\,\mu\nu}D_{o\,\mu}a^{|J)}_\nu\right)\left(\Sigma^{(K|\,\lambda\rho}D_{o\,\lambda}a^{|L)}_\rho\right) \\ -\sqrt{-g}M^2\,\frac{\partial\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}}\bigg|_o C^{(I|}_{KL}\left(\Sigma^{|J)\mu\nu}a^K_\mu a^L_\nu + \Sigma^{K\,\mu\nu}a^{|J)}_\mu a^L_\nu\right)\,, \end{aligned} \tag{9.2.8}$$

where $D_{o\,\mu}$ stands for the components of the covariant derivative with respect to the background connection $A^I_{o\,\mu}$ and we have used the self-duality property of Σ^i and the relation $\tilde{\epsilon}^{\mu\nu\lambda\rho} = \sqrt{-g}\,\epsilon^{\mu\nu\lambda\rho}$.

9.3 Gravitational sector

In this section we show that the components of the linearised connection a^I charged under the embedded $\mathfrak{su}(2)$ Lie algebra describe gravitons. Let us decompose the linearised connection one-form as

$$a^I = \mathrm{i}\, e^I_i\, a^i + a^I_\perp\,, \tag{9.3.1}$$

where

$$g_{IJ}\, a^I_\perp e^I_i = 0\,, \tag{9.3.2}$$

for every i. The above expression means that the connection parts $(e^I_i\, a^i)$ and $a^I_\perp$ are perpendicular with respect to the inner product on the Lie algebra g_{IJ}.

9.3.1 Derivatives of the defining function $\mathcal{F}$ at the background

As $\partial\mathcal{F}(\hat{X})/\partial\hat{X}^{IJ}$ must be invariant under the action of the embedded $\mathfrak{su}(2)$ we have

$$\left.\frac{\partial\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}}\right|_o e^J_j \sim g_{IJ} e^J_j\,. \tag{9.3.3}$$

Moreover, as $g_{IJ}e^I_i e^J_j \sim \delta_{ij}$, where δ_{ij} is the usual inner product on $\mathfrak{su}(2)$, we get

$$\left.\frac{\partial\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}}\right|_o e^I_i e^J_j = \frac{\mathcal{F}(\hat{X}_o)}{3}\,\delta_{ij}\,. \tag{9.3.4}$$

Now, let us consider the second derivative of the defining function. For this let us define the tensor $\mathcal{F}_{ij|kl}$ by

$$\mathcal{F}_{ij|kl} \equiv \left.\frac{\partial^2\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}\partial\hat{X}^{KL}}\right|_o e^I_i e^J_j e^K_k e^L_l\,. \tag{9.3.5}$$

This tensor is symmetric in its first two indices, $\mathcal{F}_{ij|kl} = \mathcal{F}_{ji|kl}$; it is symmetric in its last two indices, $\mathcal{F}_{ij|kl} = \mathcal{F}_{ij|lk}$; and it is invariant under the interchange of the first pair of indices with the second, $\mathcal{F}_{ij|kl} = \mathcal{F}_{kl|ij}$. Now, as $\mathcal{F}(\hat{X})$ is homogeneous of degree one in $\hat{X}$ it satisfies

$$\mathcal{F}(\hat{X}) = \frac{\partial\mathcal{F}(\hat{X})}{\partial\hat{X}^{KL}}\,\hat{X}^{KL}\,. \tag{9.3.6}$$

Differentiating the above expression with respect to $\hat{X}^{IJ}$ we obtain

$$\frac{\partial^2\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}\partial\hat{X}^{KL}}\,\hat{X}^{KL} = 0\,, \tag{9.3.7}$$

and evaluating at the background, $\hat{X}^{KL}_o = e^K_k e^L_l\,\delta^{kl}$,

$$\left.\frac{\partial^2\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}\partial\hat{X}^{KL}}\right|_o e^K_k e^L_l\,\delta^{kl} = 0\,. \tag{9.3.8}$$

Thus, we find that

$$\mathcal{F}_{ij|kl}\,\delta^{kl} = 0\,, \tag{9.3.9}$$

and

$$\mathcal{F}_{ij|kl}\,\delta^{ij} = 0\,. \tag{9.3.10}$$

Then, from the above two equations and the symmetries of $\mathcal{F}_{ij|kl}$ we conclude that this tensor has to be of the form

$$\mathcal{F}_{ij|kl} = \frac{g_{\rm gr}}{2}\,P_{ij|kl}\,, \tag{9.3.11}$$

where $g_{\rm gr}$ is a parameter that depends on the defining function $\mathcal{F}$ and the embedding e^I_i and

$$P_{ij|kl} = \delta_{i(k}\delta_{l)j} - \frac{1}{3}\delta_{ij}\delta_{kl}\,, \tag{9.3.12}$$

is the projector on symmetric traceless tensors.

9.3.2 Linearised gravity Lagrangian

The linearised Lagrangian that describes gravity is then, see equation (9.2.8),

$$\begin{aligned}\mathcal{L}_{\rm GR} = -\frac{g_{\rm gr}}{2}\sqrt{-g}\,P_{ij|kl}\left(\Sigma^{i\,\mu\nu}D_{o\,\mu}a^j_\nu\right)\left(\Sigma^{k\,\lambda\rho}D_{o\,\lambda}a^l_\rho\right)\\ -\sqrt{-g}M^2\frac{\mathcal{F}(\hat{X}_o)}{3}\delta_{ij}\,\epsilon^{(i|}_{kl}\left(\Sigma^{|j)\mu\nu}a^k_\mu a^l_\nu + \Sigma^{k\,\mu\nu}a^{|j)}_\mu a^l_\nu\right)\,,\end{aligned} \tag{9.3.13}$$

where we have used the identity $C^I_{JK}e^J_j e^K_k = \epsilon^i_{jk}\,e^I_i$ and (9.3.4). It is easy to check that the second term vanishes. Then, we are left with

$$\mathcal{L}_{\rm GR} = -\frac{g_{\rm gr}}{2}\sqrt{-g}\,P_{ij|kl}\left(\Sigma^{i\,\mu\nu}D_{o\,\mu}a^j_\nu\right)\left(\Sigma^{k\,\lambda\rho}D_{o\,\lambda}a^l_\rho\right)\,. \tag{9.3.14}$$

This is exactly the same Lagrangian that we found when we studied the $\mathfrak{su}(2)$ case, see (8.2.21).

9.4 Yang-Mills sector

In this section we study the sector of the theory which is described by the perturbation of the connection in the direction of the Lie algebra $\mathfrak{g}$ that commutes with the embedded $\mathfrak{su}(2)$, $e(\mathfrak{su}(2))$. To do this we need to split $a^I_\perp$ into those directions in the Lie algebra which commute with $e(\mathfrak{su}(2))$ and those which do not commute. Then, we write $a^I_\perp$ as

$$a^I_\perp = e^I_a\,a^a + e^I_\alpha\,a^\alpha\,, \tag{9.4.1}$$

where e^I_a form a basis for the centraliser of $e(\mathfrak{su}(2))$ in $\mathfrak{g}$ and e^I_α spans the remainder of the Lie algebra. Thus, we have

$$C^I_{JK}\, e^J_j e^K_a = [e_j, e_a]^I = 0\,, \tag{9.4.2}$$

for every j and a. Being this centraliser a Lie algebra itself, we can choose the basis e^I_a such that

$$C^I_{JK}\, e^J_a e^K_b = C^c_{ab}\, e^I_c\,, \tag{9.4.3}$$

where C^a_{bc} are the structure constants of the centraliser under discussion.

9.4.1 Linearised Yang-Mills Lagrangian

The linearised Lagrangian for the a^a part of the connection is, see (9.2.8),

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} = &-\sqrt{-g}\,\frac{\partial^2\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}\partial\hat{X}^{KL}}\bigg|_o\; e^I_i e^J_a e^K_k e^I_b\left(\Sigma^{i\,\mu\nu}D_{o\,\mu}a^a_\nu\right)\left(\Sigma^{k\,\lambda\rho}D_{o\,\lambda}a^b_\rho\right)\\ &-\sqrt{-g}M^2\,\frac{\partial\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}}\bigg|_o\left(C^I_{KL}e^K_a e^L_b e^J_j\,\Sigma^{j\mu\nu}\,a^a_\mu a^b_\nu + C^I_{KL}e^J_a e^K_k e^L_b\,\Sigma^{k\,\mu\nu}\,a^a_\mu a^b_\nu\right)\,.\end{aligned} \tag{9.4.4}$$

Using (9.4.2) we see that the second term of the second line vanishes. Moreover, from (9.3.3) we obtain that the first term of the second line is proportional to

$$\begin{aligned}g_{IJ}e^J_j\, C^I_{KL}e^K_a e^L_b &= C_{JKL}e^J_j e^K_a e^L_b\\ &= g_{LI}\, C^I_{JK}e^J_j e^K_a e^L_b = 0\,,\end{aligned} \tag{9.4.5}$$

where we have used the fact that the covariant structure constants C_{IJK} are completely anti-symmetric and (9.4.2). This shows the first term of the second line also vanishes and we are left only with the first line of this expression.

Now, let us define the tensor $\mathcal{F}_{ij|ab}$ by

$$\mathcal{F}_{ij|ab} \equiv \frac{\partial^2\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}\partial\hat{X}^{KL}}\bigg|_o\; e^I_i e^K_j e^J_a e^L_b\,. \tag{9.4.6}$$

This tensor must be invariant under the independent action of $\mathfrak{su}(2)$ on the indices i, j and the action of the centraliser $\mathfrak{h}$, of $e(\mathfrak{su}(2))$ in $\mathfrak{g}$, on the indices a, b. Then, it must be proportional to the inner product δ_{ij} in $\mathfrak{su}(2)$ and the inner product g_{ab} in $\mathfrak{h}$, where we have assumed that $\mathfrak{h}$ is semisimple. Thus, we have

$$\mathcal{F}_{ij|ab} = \frac{1}{2g_{\mathrm{ym}}}\,\delta_{ij}\, g_{ab}\,, \tag{9.4.7}$$

where g_{ym} is an arbitrary parameter that depends on $\mathcal{F}(X_o)$ and the embedding.

Then, the linearised Lagrangian for this sector is

$$\mathcal{L}_{\mathrm{YM}} = -\frac{\sqrt{-g}}{2g_{\mathrm{ym}}}\,\delta_{ij}\, g_{ab}\left(\Sigma^{i\,\mu\nu}\partial_\mu a^a_\nu\right)\left(\Sigma^{k\,\lambda\rho}\partial_\lambda a^b_\rho\right)\,, \tag{9.4.8}$$

where we have replaced the covariant derivatives by partial derivatives because we now are in the part of the Lie algebra described by the centraliser $\mathfrak{h}$ and this one commutes with the embedded $e(\mathfrak{su}(2))$ sector.
Using the identity

$$\delta_{ij}\,\Sigma^{i\,\mu\nu}\Sigma^{j\,\lambda\rho} = 2\,g^{\mu[\lambda}g^{\rho]\nu} - \mathrm{i}\,\epsilon^{\mu\nu\lambda\rho}$$

we can rewrite $\mathcal{L}_{\mathrm{YM}}$ as

$$\mathcal{L}_{\mathrm{YM}} = -\frac{\sqrt{-g}}{4g_{\mathrm{ym}}}\,g_{ab}\,F^{a\,\mu\nu}F^{b}_{\mu\nu} + \text{ surface term}\,, \tag{9.4.9}$$

where $F^a_{\mu\nu} = 2\partial_{[\mu}a^a_{\nu]}$ is the linearised curvature and the "surface term" is proportional to $\epsilon^{\mu\nu\lambda\rho}\partial_\mu a^a_\nu \partial_\lambda a^b_\rho$.

Thus, as we have already announced, this sector describes linearised Yang-Mills fields.

9.5 "Extra" sector

The linearised Lagrangian for the a^α part of the connection is, see (9.2.8),

$$\begin{aligned}\mathcal{L}_{\mathrm{extra}} = &-\sqrt{-g}\,\frac{\partial^2\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}\partial\hat{X}^{KL}}\bigg|_o\, e^I_i e^J_\alpha e^K_k e^L_\beta \left(\Sigma^{i\,\mu\nu}D_{o\,\mu}a^\alpha_\nu\right)\left(\Sigma^{k\,\lambda\rho}D_{o\,\lambda}a^\beta_\rho\right)\\ &-\sqrt{-g}M^2\,\frac{\partial\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}}\bigg|_o\left(C^I_{KL}e^K_\alpha e^L_\beta e^J_j\,\Sigma^{j\mu\nu}\,a^\alpha_\mu a^\beta_\nu + C^I_{KL}e^J_\alpha e^K_k e^L_\beta\,\Sigma^{k\,\mu\nu}\,a^\alpha_\mu a^\beta_\nu\right)\,.\end{aligned} \tag{9.5.1}$$

Let us first analyse the massive term (the second line) and then the kinetic term (the first line).

9.5.1 Mass term

In the gravity and Yang-Mills sectors the massive term vanishes, as we have seen above, but here it will not as we will show below.
The partial derivative of the defining function with respect to $\hat{X}^{IJ}$ projected on the α-part of the Lie algebra must be proportional to the inner product g_{IJ}, then

$$\frac{\partial\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}}e^J_\alpha\bigg|_o = \kappa\, g_{IJ}e^J_\alpha\,, \tag{9.5.2}$$

where κ is a parameter related to the defining function $\mathcal{F}$ at the background. From the analysis in the gravitational sector we also know

$$\frac{\partial\mathcal{F}(\hat{X})}{\partial\hat{X}^{IJ}}e^J_j\bigg|_o = \frac{\mathcal{F}(\hat{X}_o)}{Tr\,\hat{X}_o}\,g_{IJ}e^J_j\,, \tag{9.5.3}$$

where $Tr\,\hat{X}_o = g_{IJ} X_o^{IJ}$. Now, let us define the tensor $\mathcal{F}_{i\alpha\beta}$ by

$$\frac{\mathcal{F}(\hat{X}_o)}{Tr\,\hat{X}_o}\,\mathcal{F}_{i\alpha\beta} \equiv \left.\frac{\partial \mathcal{F}(\hat{X})}{\partial \hat{X}^{IJ}}\right|_o C^I_{KL}\, e^K_\alpha e^L_\beta e^J_j\,. \tag{9.5.4}$$

Then, this tensor can be written as

$$\begin{aligned}\mathcal{F}_{j\alpha\beta} =& g_{IJ} C^I_{KL}\, e^J_j e^K_\alpha e^L_\beta = C_{JKL}\, e^J_j e^K_\alpha e^L_\beta \\ =& -g_{IJ} C^I_{KL}\, e^J_\alpha e^K_j e^L_\beta\,.\end{aligned} \tag{9.5.5}$$

Thus, using the equations above we get for the massive term, second line in (9.5.1),

$$\sqrt{-g}M^2 \left.\frac{\partial \mathcal{F}(\hat{X})}{\partial \hat{X}^{IJ}}\right|_o (\quad)^{IJ} = \sqrt{-g}M^2 \left(\frac{\mathcal{F}(\hat{X}_o)}{Tr\,\hat{X}_o} - \kappa\right) \mathcal{F}_{i\alpha\beta}\, \Sigma^{i\,\mu\nu} a^\alpha_\mu a^\beta_\nu\,. \tag{9.5.6}$$

9.5.2 Kinetic term

Now, we take a look at the kinetic term, first line in (9.5.1). Let us define the tensor $\mathcal{F}_{i\alpha|j\beta}$ as

$$\mathcal{F}_{i\alpha|j\beta} \equiv \left.\frac{\partial^2 \mathcal{F}(\hat{X})}{\partial \hat{X}^{IJ}\partial \hat{X}^{KL}}\right|_o e^I_i e^K_j e^J_\alpha e^L_\beta\,. \tag{9.5.7}$$

As the $\mathfrak{su}(2)$ embedded subalgebra $e(\mathfrak{su}(2))$ and the α-part of the Lie algebra $\mathfrak{g}$ do not commute, it can be built different invariant tensors. The explicit expression for this tensor depends on the Lie algebra $\mathfrak{g}$ and the embedding e^I used.

The α-part of the Lie algebra can be used as the vector space for the representation of $e(\mathfrak{su}(2))$, the embedded $\mathfrak{su}(2)$ subalgebra of $\mathfrak{g}$. In general, this representation is reducible. Then, the α-part (i.e., the part of the Lie algebra $\mathfrak{g}$ that does not commute with the $e(\mathfrak{su}(2))$ subalgebra) splits into irreducible representations of $\mathfrak{su}(2)$. Thus, we have that the vector space V^α that it is spanned by the α-part of the Lie algebra can be written as

$$V^\alpha = V^{J_1} \oplus V^{J_2} \oplus \cdots \oplus V^{J_n}\,, \tag{9.5.8}$$

where V^J is the irreducible representation of dimension $2J+1$ and $J_1, J_2, \ldots, J_n$ are the irreducible representations that appear. There can be more than one copy of a representation of the same spin in this decomposition. This occurs when the centraliser of $e(\mathfrak{su}(2))$ in $\mathfrak{g}$ is non-trivial.

We can think about the tensor $\mathcal{F}_{i\alpha|j\beta}$ as a map from the tensor product $V^1 \otimes V^\alpha$ to itself. This map must be invariant under both the action of $\mathfrak{su}(2)$ and the action of the centraliser. Moreover, the representation $V^1 \otimes V^\alpha$ splits as

$$V^1 \otimes V^\alpha = \oplus_{i=1}^k \left(V^{|J_i-1|} \oplus \cdots \oplus V^{J_i+1}\right)\,. \tag{9.5.9}$$

Then, $\mathcal{F}_{i\alpha|j\beta}$ must be given by a linear combination of projectors on all representations that appear in (9.5.9) with multiplicity one, as well as combinations of invariant maps between different copies of the same representation in case of multiplicity higher that one. For a detail explanation of this see [24].

9.6 Linearised "extra" Lagrangian

For a general semisimple Lie algebra $\mathfrak{g}$ and embedding $e(\mathfrak{su}(2))$ we obtain the linearised Lagrangian

$$\mathcal{L}_{\text{extra}} = -\sqrt{-g}\,\mathcal{F}_{i\alpha|j\beta}\left(\Sigma^{i\,\mu\nu}D_{o\,\mu}a^{\alpha}_{\nu}\right)\left(\Sigma^{j\,\lambda\rho}D_{o\,\lambda}a^{\beta}_{\rho}\right) - \sqrt{-g}\,M^2\left(\frac{\mathcal{F}(\hat{X}_o)}{Tr\,\hat{X}_o} - \kappa\right)\mathcal{F}_{i\alpha\beta}\,\Sigma^{i\,\mu\nu}a^{\alpha}_{\mu}a^{\beta}_{\nu}\,. \tag{9.6.1}$$

We know that the quantity $\mathcal{F}(\hat{X}_o) \sim M_p^2/M^2$ is very large [24]. In the other hand, there is no a priori reason why the parameter κ should be large. Then, we can assume that $\mathcal{F}(\hat{X}_o) \gg \kappa$ and dropped κ from the above Lagrangian. Thus, we can rewrite $\mathcal{L}_{\text{extra}}$ as

$$\frac{1}{\sqrt{-g}}\mathcal{L}_{\text{extra}} = -\mathcal{F}_{i\alpha|j\beta}\left(\Sigma^{i\,\mu\nu}D_{o\,\mu}a^{\alpha}_{\nu}\right)\left(\Sigma^{j\,\lambda\rho}D_{o\,\lambda}a^{\beta}_{\rho}\right) - M^2\,\mathcal{F}^1\mathcal{F}_{i\alpha\beta}\,\Sigma^{i\,\mu\nu}a^{\alpha}_{\mu}a^{\beta}_{\nu}\,, \tag{9.6.2}$$

where $\mathcal{F}^1 = \mathcal{F}(\hat{X}_o)/Tr\,\hat{X}_o \sim M_p^2/M^2$.

Now, to give some taste of how the expressions in the "extra" sector look like for a specific example, let us analyse the oversimplify case where $V^\alpha = V^1$, i.e., the α-part of the Lie algebra is a spin-1 representation of $\mathfrak{su}(2)$. In this case $\mathcal{F}_{i\alpha|j\beta}$ is a map from $V^1 \otimes V^1 = V^0 \oplus V^1 \oplus V^2$ to itself[1]. As the α part of the Lie algebra is V^1, i.e. the adjoint representation, we can relabel the greek indices $\alpha, \beta, \ldots$ in this sector by latin indices $i, j, k, l, \ldots$. Then, we rewrite $\mathcal{L}_{\text{extra}}$ as

$$\frac{1}{\sqrt{-g}}\mathcal{L}_{\text{extra}} = -\frac{1}{2}\mathcal{F}_{ij|kl}\left(\Sigma^{i\,\mu\nu}D_{o\,\mu}a^{j}_{\nu}\right)\left(\Sigma^{k\,\lambda\rho}D_{o\,\lambda}a^{l}_{\rho}\right) - M^2\,\mathcal{F}^1\,\epsilon_{ijk}\,\Sigma^{i\,\mu\nu}a^{j}_{\mu}a^{k}_{\nu}\,. \tag{9.6.3}$$

The spin-2, spin-1 and spin-0 representation projectors are given by

$$\text{spin}-2\ \ \text{Projector} \rightarrow P_{ijkl} = \delta_{i(k}\delta_{l)j} - \frac{1}{3}\delta_{ij}\delta_{kl}\,, \tag{9.6.4}$$

$$\text{spin}-1\ \ \text{Projector} \rightarrow \epsilon_{ijm}\epsilon^{m}_{kl}\,, \tag{9.6.5}$$

$$\text{spin}-0\ \ \text{Projector} \rightarrow \delta_{ij}\delta_{kl}\,. \tag{9.6.6}$$

Then, $\mathcal{F}_{i\alpha|k\beta}$ now written as $\mathcal{F}_{ij|kl}$ is given by

$$\mathcal{F}_{ij|kl} = \kappa_2 P_{ijkl} + \frac{\kappa_1}{2}\epsilon_{ijm}\epsilon^{m}_{kl} + \frac{\kappa_0}{3}\delta_{ij}\delta_{kl}\,, \tag{9.6.7}$$

[1]To see the analysis for other examples and the general case see [24]

where $\kappa_2, \kappa_1, \kappa_0$ are parameters related to the derivatives of the defining function $\mathcal{F}$. Now, $\mathcal{L}_{\text{extra}}$ has to be invariant under infinitesimal gauge transformations, or in other words the infinitesimal gauge transformation of $\mathcal{L}_{\text{extra}}$ vanishes. Then, taking the variation δ_ϕ of $\mathcal{L}_{\text{extra}}$ and equating the result to zero we find

$$\mathcal{F}^1 = \kappa_1 \,, \tag{9.6.8}$$

where we have used $\delta_\phi a^i_\mu = D_{o\,\mu}\phi^i$ and

$$\delta_\phi(\Sigma^{i\mu\nu} D_{o\,\mu} a^j_\nu) = \Sigma^{i\mu\nu} D_{o\,\mu} D_{o\,\nu}\phi^j = \frac{1}{2}\Sigma^{i\mu\nu}\epsilon^j_{\ kl} F^k_{o\,\mu\nu}\phi^l = -2M^2\epsilon^{ij}{}_k\phi^k \,. \tag{9.6.9}$$

Thus, the linearised Lagrangian for the "extra" sector can be written as

$$\frac{1}{\sqrt{-g}}\mathcal{L}_{\text{extra}} = -\frac{1}{2}\left(\kappa_2 P_{ijkl} + \frac{\kappa_1}{2}\epsilon_{ijm}\epsilon^m_{kl} + \frac{\kappa_0}{3}\delta_{ij}\delta_{kl}\right)\left(\Sigma^{i\,\mu\nu} D_{o\,\mu} a^j_\nu\right)\left(\Sigma^{k\,\lambda\rho} D_{o\,\lambda} a^l_\rho\right) \\ - \kappa_1\, M^2 \epsilon_{ijk}\,\Sigma^{i\,\mu\nu} a^j_\mu a^k_\nu \,, \tag{9.6.10}$$

where we have replaced $\mathcal{F}_{ijk}$ by ϵ_{ijk}. As $\kappa_1 = \mathcal{F}^1 \sim M_p^2/M^2$ we have

$$\kappa_1 M^2 \sim M_p^2 \,. \tag{9.6.11}$$

To have a better understanding of $\mathcal{L}_{\text{extra}}$ as given by (9.6.10) let us make the Hamiltonian analysis of this one.

9.6.1 Hamiltonian analysis

At this point we are going to take the limit $M \to 0$, i.e., we will work in Minkowski spacetime. Recall that the quantity $\kappa_1 M^2 \sim M_p^2$ stays constant in this limit. When taking this limit in the Lagrangian $\mathcal{L}_{\text{extra}}$ we are basically only replacing covariant derivative $D_{o\,\mu}$ by usual partial derivative ∂_μ. Then, we have the Lagrangian

$$\mathcal{L}_{\text{extra}} = -\frac{1}{2}\left(\kappa_2 P_{ijkl} + \frac{\kappa_1}{2}\epsilon_{ijm}\epsilon^m_{kl} + \frac{\kappa_0}{3}\delta_{ij}\delta_{kl}\right)\left(\Sigma^{i\,\mu\nu}\partial_\mu a^j_\nu\right)\left(\Sigma^{k\,\lambda\rho}\partial_\lambda a^l_\rho\right) \\ - \left(\kappa_1 M^2\right)\epsilon_{ijk}\,\Sigma^{i\,\mu\nu} a^j_\mu a^k_\nu \,. \tag{9.6.12}$$

The kinetic term of the above Lagrangian it is basically the "square" of $\Sigma^{i\,\mu\nu}\partial_\mu a^j_\nu$. The space plus time decomposition of this expression is

$$\Sigma^{i\mu\nu}\partial_\mu a^j_\nu = -\mathrm{i}\,\partial_0 a^{ij} + \mathrm{i}\,\partial^i a^j_0 - \epsilon^{ikl}\partial_k a^j_l \,, \tag{9.6.13}$$

where we have used $\Sigma^i_{0j} = \mathrm{i}\,\delta^i_j$ and $\Sigma^i_{jk} = -\epsilon^i_{jk}$.
The spatial part a_{ij} of the connection perturbation can be split into its irreducible spin-2, spin-1 and spin-0 parts as

$$a_{ij} = a^{(2)}_{ij} + \frac{1}{2}\epsilon^k_{ij}\, a^{(1)}_k + \frac{1}{3}\delta_{ij}\, a^{(0)} \,, \tag{9.6.14}$$

where $a^{(2)}_{ij}$ is symmetric and tracefree and the numbers in the parenthesis denote the irreducible representations. The canonically conjugate fields to $a^{(2)}_{ij}, a^{(1)}_i, a^{(0)}$ are

$$\pi^{(2)\,ij} \equiv \frac{\partial \mathcal{L}_{\text{extra}}}{\partial(\partial_0 a^{(2)}_{ij})} = \kappa_2\,\partial_0 a^{(2)\,ij} - \kappa_2 P^{ijkl}\left(\partial_k a_{0l} + \mathrm{i}\,\epsilon_k^{mn}\partial_m a_{nl}\right)\,, \tag{9.6.15}$$

$$\pi^{(1)\,i} \equiv \frac{\partial \mathcal{L}_{\text{extra}}}{\partial(\partial_0 a^{(1)}_{i})} = \frac{\kappa_1}{2}\,\partial_0 a^{(1)\,i} - \frac{\kappa_1}{2}\epsilon^{ikl}\left(\partial_k a_{0l} + \mathrm{i}\,\epsilon_k^{mn}\partial_m a_{nl}\right)\,, \tag{9.6.16}$$

$$\pi^{(0)} \equiv \frac{\partial \mathcal{L}_{\text{extra}}}{\partial(\partial_0 a^{(0)})} = \frac{\kappa_0}{3}\,\partial_0 a^{(0)} - \frac{\kappa_0}{3}\delta^{kl}\left(\partial_k a_{0l} + \mathrm{i}\,\epsilon_k^{mn}\partial_m a_{nl}\right)\,. \tag{9.6.17}$$

From these expression for $\pi^{(2)\,ij}$, $\pi^{(1)\,i}$ and $\pi^{(0)}$, it is easy to check that

$$-\,\mathrm{i}\left(\frac{\pi^{(2)\,ij}}{\kappa_2} + \epsilon^{ij}{}_k\frac{\pi^{(1)\,i}}{\kappa_1} + \delta^{ij}\frac{\pi^{(0)}}{\kappa_0}\right) = \Sigma^{i\mu\nu}\partial_\mu a^j_\nu\,. \tag{9.6.18}$$

Now, using the decomposition (9.6.14), we find that the contraction of $\epsilon_k^{mn}\partial_m a_{nl}$ with P^{ijkl}, ϵ^{ikl} and δ^{kl} is

$$\begin{aligned}
P^{ijkl}\epsilon_k^{mn}\partial_m a_{nl} &= P^{ijkl}\left(\epsilon_k^{mn}\partial_m a^{(2)}_{nl} - \frac{1}{2}\partial_k a^{(1)}_l\right)\,,\\
\epsilon_i^{kl}\epsilon_k^{mn}\partial_m a_{nl} &= \partial^m a^{(2)}_{mi} + \frac{1}{2}\epsilon_i^{kl}\partial_k a^{(1)}_l - \frac{2}{3}\partial_i a^{(0)}\,,\\
\delta^{kl}\epsilon_k^{mn}\partial_m a_{nl} &= \partial^m a^{(1)}_m\,.
\end{aligned} \tag{9.6.19}$$

Then, utilising the above expressions the velocities $\partial_0 a^{(2)\,ij}, \partial_0 a^{(1)\,i}, \partial_0 a^{(0)}$ can be written in terms of the canonically conjugate fields $\pi^{(2)\,ij}, \pi^{(1)\,i}, \pi^{(0)}$ as

$$\partial_0 a^{(2)\,ij} = \frac{\pi^{(2)\,ij}}{\kappa_2} + P^{ijkl}\left[\partial_k\left(a_{0l} - \frac{\mathrm{i}}{2}a^{(1)}_l\right) + \mathrm{i}\,\epsilon_k^{mn}\partial_m a^{(2)}_{nl}\right]\,, \tag{9.6.20}$$

$$\partial_0 a^{(1)\,i} = \frac{2\pi^{(1)\,i}}{\kappa_1} + \epsilon^{ikl}\partial_k\left(a_{0l} + \frac{\mathrm{i}}{2}a^{(1)}_l\right) + \mathrm{i}\,\partial_l a^{(2)\,li} - \frac{2}{3}\mathrm{i}\,\partial_i a^{(0)}\,, \tag{9.6.21}$$

$$\partial_0 a^{(0)} = \frac{3\pi^{(0)}}{\kappa_0} + \partial^i\left(a_{0i} + \mathrm{i}\,a^{(1)}_i\right)\,. \tag{9.6.22}$$

From (9.6.18) we find that the kinetic term of $\mathcal{L}_{\text{extra}}$ is given by

$$\frac{1}{2}\left[\frac{\left(\pi^{(2)}\right)^2}{\kappa_2} + 2\frac{\left(\pi^{(1)}\right)^2}{\kappa_1} + 3\frac{\left(\pi^{(0)}\right)^2}{\kappa_0}\right]\,, \tag{9.6.23}$$

where $\left(\pi^{(2)}\right)^2 = \pi^{(2)\,ij}\pi^{(2)}_{ij}$ and $\left(\pi^{(1)}\right)^2 = \pi^{(1)\,i}\pi^{(1)}_i$.

Now, splitting in space and time indices and using (9.6.14), we obtain for the mass term (second line in $\mathcal{L}_{\text{extra}}$)

$$-\,(\kappa_1 M^2)\left[2\mathrm{i}a_{0i}\,a^{(1)\,i} - \frac{2}{3}\left(a^{(0)}\right)^2 - \frac{1}{2}\left(a^{(1)}\right)^2 + \left(a^{(2)}\right)^2\right]\,. \tag{9.6.24}$$

Thus, the linearised Hamiltonian for the "extra" sector $\mathcal{H}_{\text{extra}}$, where

$$\mathcal{H}_{\text{extra}} = \pi^{(2)\,ij}\,\partial_0 a^{(2)}_{ij} + \pi^{(1)\,i}\,\partial_0 a^{(1)}_i + \pi^{(0)}\partial_0 a^{(0)} - \mathcal{L}_{\text{extra}}\,, \tag{9.6.25}$$

is given by

$$\begin{aligned}\mathcal{H}_{\text{extra}} = \frac{1}{2}\left[3\frac{\left(\pi^{(0)}\right)^2}{\kappa_0} + 2\frac{\left(\pi^{(1)}\right)^2}{\kappa_1} + \frac{\left(\pi^{(2)}\right)^2}{\kappa_2}\right] - (\kappa_1 M^2)\left[\frac{2}{3}\left(a^{(0)}\right)^2 + \frac{1}{2}\left(a^{(1)}\right)^2 - \left(a^{(2)}\right)^2\right] \\ + \mathrm{i}\,\pi^{(2)\,ij}\left(\epsilon_i^{\;kl}\partial_k a^{(2)}_{lj} - \frac{1}{2}\partial_i a^{(1)}_j\right) + \mathrm{i}\,\pi^{(1)\,i}\left(\partial^j a^{(2)}_{ji} + \frac{1}{2}\epsilon_i^{\;jk}\partial_j a^{(1)}_k - \frac{2}{3}\partial_i a^{(0)}\right) + \mathrm{i}\,\pi^{(0)}\,\partial^i a^{(1)}_i \\ - a^i_0\,\mathcal{C}_i\,,\end{aligned} \tag{9.6.26}$$

with

$$\mathcal{C}_i = -2\mathrm{i}(\kappa_1 M^2)\,a^{(1)}_i + \partial^j \pi^{(2)}_{ji} - \epsilon_i^{\;jk}\,\partial_j \pi^{(1)}_k + \partial_i \pi^{(0)} \tag{9.6.27}$$

the Gauss constraint.

9.6.2 Gauge-fixing

The canonical variables in our Hamiltonian $\mathcal{H}_{\text{extra}}$ are $(a^{(2)}, a^{(1)}, a^{(0)}; \pi^{(2)}, \pi^{(1)}, \pi^{(0)})$. The Poisson bracket between them are

$$\begin{aligned}&\left\{a^{(2)}_{ij}(t,x), \pi^{(2)\,kl}(t,y)\right\} = \delta^k_i \delta^l_j\,\delta^3(x-y)\,, \quad \left\{a^{(1)}_i(t,x), \pi^{(1)\,j}(t,y)\right\} = \delta^j_i\,\delta^3(x-y)\,, \\ &\left\{a^{(0)}(t,x), \pi^{(0)}(t,y)\right\} = \delta^3(x-y)\,,\end{aligned} \tag{9.6.28}$$

with all the other brackets vanishing. Now, let us define the constraint function C_ω as

$$C_\omega(a,\pi) = \int d^3y\;\omega^i\,\mathcal{C}_i\,, \tag{9.6.29}$$

where ω^i is a local parameter, i.e., $\omega^i = \omega^i(t,y)$. The constraint function C_ω generates the following transformations on the field variables $a^{(2)}, a^{(1)}, a^{(0)}$:

$$\begin{aligned}\delta_\omega a^{(2)}_{ij} &= \left\{C_\omega, a^{(2)}_{ij}\right\} = \partial_i \omega_j\,, \\ \delta_\omega a^{(1)}_i &= \left\{C_\omega, a^{(1)}_i\right\} = \epsilon_i^{\;jk}\partial_j \omega_k\,, \\ \delta_\omega a^{(0)} &= \left\{C_\omega, a^{(0)}\right\} = \partial_i \omega^i\,,\end{aligned} \tag{9.6.30}$$

i.e., it generates gauge transformations of the spatial connection irreducible components. Moreover, for the canonically conjugate fields $\pi^{(2)}, \pi^{(1)}, \pi^{(0)}$ we obtain

$$\begin{aligned}\delta_\omega \pi^{(2)\,ij} &= \left\{C_\omega, a^{(2)}_{ij}\right\} = 0\,, \\ \delta_\omega \pi^{(1)\,i} &= \left\{C_\omega, a^{(1)}_i\right\} = -2\mathrm{i}(\kappa_1 M^2)\,\omega^i\,, \\ \delta_\omega \pi^{(0)} &= \left\{C_\omega, a^{(0)}\right\} = 0\,.\end{aligned} \tag{9.6.31}$$

Then, we can fix the gauge completely by requiring that the field $\pi^{(1)}$ vanishes, i.e.,

$$\pi^{(1)\,i} = 0\,. \tag{9.6.32}$$

After imposing this gauge, the Gauss constraint can be solved for $a^{(1)}$ in terms of the fields $\pi^{(2)}$ and $\pi^{(0)}$, i.e.,

$$a_i^{(1)} = \frac{1}{2\mathrm{i}(\kappa_1 M^2)} \left(\partial^j \pi_{ji}^{(2)} + \partial_i \pi^{(0)}\right)\,. \tag{9.6.33}$$

Thus, the spin-1 part of the spatial connection is not propagating and completely determined but the spin-2 and spin-0 canonically conjugate fields.
Using (9.6.32) and (9.6.33), we are able to rewrite the Hamiltonian $\mathcal{H}_{\text{extra}}$ as

$$\mathcal{H}_{\text{extra}} = \mathcal{H}_{\text{extra}}^{(0)} + \mathcal{H}_{\text{extra}}^{(2)}\,, \tag{9.6.34}$$

where

$$\mathcal{H}_{\text{extra}}^{(0)} = \frac{3}{2\kappa_0}\left(\pi^{(0)}\right)^2 - \frac{2}{3}(\kappa_1 M^2)\left(a^{(0)}\right)^2 - \frac{3}{8(\kappa_1 M^2)}\left(\partial_i \pi^{(0)}\right)^2\,, \tag{9.6.35}$$

and

$$\mathcal{H}_{\text{extra}}^{(2)} = \frac{1}{2\kappa_2}\left(\pi^{(2)}\right)^2 + (\kappa_1 M^2)\left(a^{(2)}\right)^2 + \mathrm{i}\,\epsilon_i^{kl}\,\pi^{(2)\,ij}\partial_k a_{lj}^{(2)} + \frac{3}{8(\kappa_1 M^2)}\left(\partial^j \pi_{ji}^{(2)}\right)^2\,. \tag{9.6.36}$$

Thus, the gauge-fixed Hamiltonian for the “extra" sector decouples into the sum of one Hamiltonian for the spin-0 fields and one Hamiltonian for the spin-2 fields.

9.6.3 Evolution equations

The Hamiltonians H is defined as the space integral of the Hamiltonian density. Then, we have

$$H_{\text{extra}} = \int d^3y\, \mathcal{H}_{\text{extra}}\,. \tag{9.6.37}$$

The Hamiltonian equations for $a_{ij}^{(2)}$ and $\pi^{(2)\,ij}$ are

$$\begin{aligned}
\dot{a}_{ij}^{(2)} &= \left\{a_{ij}^{(2)}, H_{\text{extra}}^{(2)}\right\}\,,\\
\dot{a}_{ij}^{(2)} &= \frac{1}{\kappa_2}\pi_{ij}^{(2)} - \frac{3}{4(\kappa_1 M^2)}\,\partial_i\partial^k \pi_{kj}^{(2)} + \mathrm{i}\,\epsilon_i^{kl}\,\partial_k a_{lj}^{(2)}\,,
\end{aligned} \tag{9.6.38}$$

and

$$\begin{aligned}
\dot{\pi}^{(2)\,ij} &= \left\{\pi^{(2)\,ij}, H_{\text{extra}}^{(2)}\right\}\,,\\
\dot{\pi}^{(2)\,ij} &= -\,2(\kappa_1 M^2)\,a^{(2)\,ij} - \mathrm{i}\,\epsilon^{ik}{}_{l}\,\partial_k \pi^{(2)\,lj}\,.
\end{aligned} \tag{9.6.39}$$

Taking the time derivative of (9.6.38) and using (9.6.39) we get

$$\ddot{a}^{(2)}_{ij} = -\frac{2(\kappa_1 M^2)}{\kappa_2}\, a^{(2)}_{ij} - \frac{\mathrm{i}}{\kappa_2}\, \epsilon_i{}^{kl} \partial_k \pi^{(2)}_{lj} + \frac{3}{2}\, \partial_i \partial^k a^{(2)}_{kj} + \mathrm{i}\, \epsilon_i{}^{kl} \partial_k \dot{a}^{(2)}_{lj}\,. \tag{9.6.40}$$

Utilising (9.6.38) in the equation above, we find

$$\ddot{a}^{(2)}_{ij} = -\frac{2(\kappa_1 M^2)}{\kappa_2}\, a^{(2)}_{ij} + \Delta a_{ij} + \frac{1}{2} \partial_i \partial^k a^{(2)}_{kj}\,, \tag{9.6.41}$$

where $\Delta = \partial^i \partial_i$ is the Laplacian. Then, from this equation we can see that the spin-2 part describes particles with mass $m_{(2)}$ given by

$$m^2_{(2)} = \frac{2(\kappa_1 M^2)}{\kappa_2}\,. \tag{9.6.42}$$

Now, the Hamiltonian equations for $a^{(0)}$ and $\pi^{(0)}$ are

$$\begin{aligned} \dot{a}^{(0)} &= \left\{ a^{(0)}, H^{(2)}_{\text{extra}} \right\}\,, \\ \dot{a}^{(0)} &= \frac{3}{\kappa_0} \pi^{(0)} + \frac{3}{4(\kappa_1 M^2)} \Delta \pi^{(0)}\,, \end{aligned} \tag{9.6.43}$$

and

$$\begin{aligned} \dot{\pi}^{(0)} &= \left\{ \pi^{(0)}, H^{(2)}_{\text{extra}} \right\}\,, \\ \dot{\pi}^{(0)} &= \frac{4}{3} (\kappa_1 M^2)\, a^{(0)}\,. \end{aligned} \tag{9.6.44}$$

Using the same recipe that we used in the spin-2 case, i.e., taking the time derivative of (9.6.43) and utilising (9.6.44) and (9.6.43), we obtain

$$\ddot{a}^{(0)} = \frac{4(\kappa_1 M^2)}{\kappa_0}\, a^{(0)} + \Delta a^{(0)}\,. \tag{9.6.45}$$

Thus, the spin-0 part describe particles with mass

$$m^2_{(0)} = -\frac{4(\kappa_1 M^2)}{\kappa_0}\,. \tag{9.6.46}$$

CHAPTER 10

Conclusions

In this thesis we have proposed and studied a class of diffeomorphism invariant gauge theories which describe gravity and Yang-Mills fields[1] in a unified framework. The action for the theory, both in its BF-formulation and its pure connection formulation, can be argued to be the most general functional on its field variables subject to the conditions of gauge and diffeomorphism invariance and which lead to second order in derivatives field equations. The principal role in our formulation is played by the potential defining function, $V(B \wedge B)$ in the BF-formulation and $\mathcal{F}(F \wedge F)$ in the pure connection formulation. All the parameters in the theory are the result of evaluating the defining potential function at the background. An important point to remark is that the appearance of a spacetime metric, which breaks the gauge invariance of the theory and reduces it to a smaller one, is responsible for the separation in different sectors one of which describes gravity and another one Yang-Mills. Note that to be more accurate what we are actually describing in our model is a unification of a generalise gravity and Yang-Mills theories. The modifications of gravity and Yang-Mills are found at high energies and are characterised by values of the defining potential function at the background.
It is worth noting that the starting Lagrangian of our model is complex and we have only discussed the reality conditions at the linearised level where they are obvious. However, one should in general add some reality conditions for the field variables which make the non-perturbative Lagrangian real. This is still an open problem on which we are working in.

Now, let us summarise what has been done in the different chapters and the results which have been found:

- Chapter 2: the model in its BF-formulation is introduced, it is sketched its Hamil-

[1]When we refer to Yang-Mills fields here we are thinking about abelian and non-abelian gauge fields.

tonian analysis and counted the number of degrees of freedom for a general gauge group, i.e., $2n - 4$ where n is the dimension of the Lie algebra. It is also discussed the linearisation of the kinetic and potential terms of the action.

- Chapter 3: the gauge group is taken to be $SU(2)$ and it is shown that this case describes a generalised gravity theory. When some parameters related to the defining potential function are sent to infinity we recover general relativity. The action can be seen as a deformation of the Pleblanski formulation of GR. The Hamiltonian analysis is performed and it is found that the reduced phase space consists of the usual symmetric, transverse and traceless fields.

- Chapter 4: here the fields of the model are valued in the $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ Lie algebra. It is found that this case describes gravity in the $\mathfrak{su}(2)$ part and non-linear electrodynamics for the $\mathfrak{u}(1)$ part. The usual Einstein-Maxwell system is recovered when some parameters specified by the defining potential function are sent to zero. The spherically-symmetric solution is studied and specifically it is shown how the Reissner-Nordström solution appears inside this formalism. The departure of our theory from Einstein-Maxwell would only be visible at high energies.

- Chapter 5: the $SU(3)$ case is studied as a generic example of the general case. Here a broader defining potential with an extra invariant is used. Three sectors are found, i.e., the gravitational one described by the Lie subalgebra $\mathfrak{su}(2)$ of $\mathfrak{su}(3)$, the $U(1)$ gauge field sector described by the centraliser of the $\mathfrak{su}(2)$ subalgebra in $\mathfrak{su}(3)$ and a set of massive scalars described by the part of the Lie algebra that does not commute with the $\mathfrak{su}(2)$ subalgebra. The $g_{\mathrm{U}(1)}$ coupling constant and the mass of the set of scalar fields are determined from the defining potential function evaluated at the background. The interaction between linearised gravity and the $U(1)$-gauge field is analysed and it is found that it is through the stress-energy tensor, as it should be.

- Chapter 6: a new fermionic Lagrangian using anti-commuting spinors-valued one-forms is proposed and studied. The Hamiltonian analysis for the massless, massive and Dirac-type new fermionic Lagrangian is carried out. The relativistic wave equations are found and it is confirmed that indeed these Lagrangians represent massless and massive particles. A non-local field redefinition is presented which maps the new massless Lagrangian formulation to the usual Weyl one.

- Chapter 7: it is described the theory in its pure connection formulation, it is shown explicitly the invariance of the action under gauge transformations and diffeomorphism, and it is discussed the linearisation.

- Chapter 8: a new pure connection formulation of GR is studied which is obtained integrating out the B-field and the Lagrange multiplier function of the Plebanski formulation of GR. Moreover, the pure connection formulation of our theory for the $\mathfrak{su}(2)$ case is analysed. The Lagrangian is expanded around a constant curvature background and it is found to describe a generalised gravity theory with two propagating degrees of freedom.

- Chapter 9: here we studied the linearisation of our theory, in the pure connection formulation, for the connection field variable valued in a general semisimple Lie algebra $\mathfrak{g}$. We have three different sectors, i.e., the gravitational one, the Yang-Mills one and what we have called "the rest" which consists in a bunch of massive scalar fields. The vacuum solution on which we decided to expand the theory around gives rise to an embedding of the $\mathfrak{su}(2)$ algebra into $\mathfrak{g}$ and thus breaks $\mathfrak{g}$ down to $\mathfrak{su}(2) \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the centraliser of $\mathfrak{su}(2)$ in $\mathfrak{g}$. The part of the connection valued on $\mathfrak{su}(2)$ describes gravitons, the one valued on $\mathfrak{h}$ describes Yang-Mills bosons, and the remaining components of the connection describe massive particles of generally non-zero spin.

There are several missing parts in this unification scheme which can be the subject of possible future directions of research. One of them is the study of interactions. Although, we have given the first steps in section 5.10, this is a topic that has to be further explored and which is of vital importance for the success of our model. Another one is the study of fermions in a unified framework. In chapter 6, we studied a fermionic Lagrangian which gives some hints of how this can be done. Here a possible approach could be the use of super-algebras with a super-connection which embeds the bosonic and fermionic fields. One more point that this model has to be able to describe is the spontaneous symmetry breaking of the standard model. In this direction there have been some advances using the pure connection formulation [24] but we are still far away from a realistic model. Finally, the most important open problem of the whole approach is to study the quantum mechanical behaviour of this model and see if it continues making sense also as a quantum theory.

References

[1] C. Vafa, *Geometry of Grand Unification*, arXiv:0911.3008 [math-ph].

[2] G. G. Ross, *Grand Unified Theories*, The Benjamin/Cummings Publishing Company, U.S., 1984.

[3] S. R. Coleman and J. Mandula, *All Possible Symmetries of the S Matrix*, Phys. Rev. **159**, 1251 (1967).

[4] R. Percacci, *Mixing internal and spacetime transformations: some examples and counterexamples*, J. Phys. A **41**, 335403 (2008) [arXiv:0803.0303 [hep-th]].

[5] S. Holst, *Barbero's Hamiltonian derived from a generalised Hilbert-Palatini action*, Phys. Rev. D **53**, 5966 (1996) [arXiv:gr-qc/9511026]

[6] R. Percacci, *Spontaneous Soldering*, Phys. Lett. B **144**, 37 (1984).

[7] R. Percacci, *The Higgs Phenomenon in Quantum Gravity*, Nucl. Phys. B **353**, 271 (1991) [arXiv:0712.3545 [hep-th]].

[8] F. Nesti, *Standard Model and Gravity from Spinors*, Eur. Phys. J. C **59**, 723 (2009) [arXiv:0706.3304 [hep-th]].

[9] F. Nesti and R. Percacci, *Graviweak Unification*, J. Phys. A **41**, 075405 (2008) [arXiv:0706.3307 [hep-th]].

[10] F. Nesti and R. Percacci, *Chirality in unified theories of gravity*, Phys. Rev. D **81**,025010 (2010) [arXiv:0909.4537 [hep- th]].

[11] P. Peldan, *Ashtekar's variables for arbitrary gauge group*, Phys. Rev. D **46**, 2279 (1992) [arXiv:hep-th/9204069].

[12] P. Peldan, *Unification of gravity and Yang-Mills theory in (2+1)-dimensions*, Nucl. Phys. B **395**, 239 (1993) [arXiv:gr-qc/9211014].

[13] S. Chakraborty and P. Peldan, *Towards a unification of gravity and Yang-Mills theory*, Phys. Rev. Lett. **73**, 1195 (1994) [arXiv:gr-qc/9401028].

[14] S. Chakraborty and P. Peldan, *Gravity and Yang-Mills theory: Two faces of the same theory?*, Int. J. Mod. Phys. D **3**, 695 (1994) [arXiv:gr-qc/9403002].

[15] A. Ashtekar, *New Hamiltonian Formulation of General Relativity*, Phys. Rev. D **36**, 1587 (1987).

[16] L. Smolin, *The Plebanski action extended to a unification of gravity and Yang-Mills theory*, Phys. Rev. D **80**, 124017 (2009) [arXiv: hep-th/0712.0977].

[17] K. Krasnov, *Renormalizable Non-Metric Quantum Gravity?*, arXiv: hep-th/0611182.

[18] I. Bengtsson, *The Cosmological constants*, Phys. Lett. B **254**, 55 (1991).

[19] S. Alexandrov and K. Krasnov, *Hamiltonian Analysis of non-chiral Plebanski Theory and its Generalizations*, Class. Quant. Grav. **26**, 055005 (2009) [arXiv: gr-qc/0809.4763].

[20] A. Torres-Gomez and K. Krasnov, *Gravity-Yang-Mills-Higgs unification by enlarging the gauge group*, Phys. Rev. D **81**, 085003 (2010) [arXiv:0911.3793].

[21] A. Torres-Gomez, K. Krasnov, C. Scarinci, *A Unified Theory of Non-Linear Electrodynamics and Gravity*, Phys. Rev. D83, 025023 (2011). [arXiv:1011.3641 [gr-qc]].

[22] K. Krasnov, *Pure Connection Action Principle for General Relativity*, Phys. Rev. Lett. **106**, 251103 (2011) [arXiv:1103.4498 [gr-qc]].

[23] K. Krasnov, *Gravity as a diffeomorphism gauge theory*, Phys. Rev. **D84** (2011) 024034 [arXiv:hep-th/1101.4788]

[24] K. Krasnov, *Spontaneous symmetry breaking and Gravity*, arXiv:hep-th/1112.5097.

[25] K. Krasnov, *Plebanski gravity without the simplicity constraints*, Class. Quant. Grav. **26**, 055002 (2009) [arXiv:0811.3147 [gr-qc]].

[26] K. Krasnov, *On deformations of Ashtekar's constraint algebra*, Phys. Rev. Lett. **100**, 081102 (2008) [arXiv:0711.0090 [gr-qc]].

[27] L. Freidel, *Modified gravity without new degrees of freedom*, arXiv: gr-qc/0812.3200.

[28] H. Urbantke, *On Integrability Properties Of SU(2) Yang-Mills Fields. I. Infinitesimal Part*, J. Math. Phys. **25**, 2321 (1984).

[29] K. Krasnov, *Effective metric Lagrangians from an underlying theory with two propagating degrees of freedom*, Phys. Rev. D **81**, 084026 (2010) [arXiv: hep-th/0911.4903].

[30] G. Boillat, *Nonlinear Electrodynamics: Lagrangians and Equations of Motion*, J. Math. Phys. **11**, 941 (1970).

[31] S. A. Gutierrez, A. L. Dudley and J. F. Plebanski, *Signals And Discontinuities In General Relativistic Nonlinear Electrodynamics*, J. Math. Phys. **22**, 2835 (1981).

[32] D. C. Robinson, *A GL(2,C) formulation of Einstein-Maxwell theory*, Class. Quant. Grav. **11**, L157 (1994).

[33] K. Krasnov and Y. Shtanov, *Cosmological perturbations in a family of deformations of general relativity*, JCAP 1006, **006** (2010) [arXiv: 1002.1210].

[34] A. Ishibashi and S. Speziale, *Spherically symmetric black holes in minimally modified self-dual gravity*, Class. Quant. Grav. **26**, 175005 (2009) [arXiv: gr-qc/0904.3914].

[35] K. Krasnov and Y. Shtanov, *Non-Metric Gravity II: Spherically Symmetric Solution, Missing Mass and Redshifts of Quasars*, Class. Quant. Grav. **25**, 025002 (2008) [arXiv: gr-qc/0705.2047].

[36] K. Krasnov, *Plebanski Formulation of General Relativity: A Practical Introduction*, Gen. Rel. Grav. **43**, 1 (2011) [arXiv: gr-qc/0904.0423].

[37] H. Georgi, *Lie Algebras in Particle Physics, 2nd Edition Westview Press*, U.S., 1991.

[38] R. Cahn, *Semi-Simple Lie Algebras and Their Representation*, The Benjamin/ Cummings Publishing Company, U.S., 1984.

[39] R. Capovilla, J. Dell and T. Jacobson, *Self-dual two forms and gravity*, Class. Quant. Grav. **8**, 89 (1991).

[40] M. Srednicki, Quantum Field Theory, Cambridge University Press, UK, 2007.

[41] H. K. Dreiner, H. E. Haber and S. P. Martin, *Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry*, Phys. Rept. **494**, 1 (2010) [arXiv: hep-ph/0812.1594v4].

[42] R. Penrose and W. Rindler, *Spinors and space-time Vol. 1: Two-spinor calculus and relativistic fields*, Cambridge University Press, Great Britain, 1986.

[43] F. Plebanski, *On the separation of Einsteinian substructures*, J. Math. Phys. **18**, 2511 (1977).