The determination of solutions of linear differential equations with entire coefficients from their zeros

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Dedicated to my father, my mother, my brothers, my sisters, my wife and my sons (Moath and Anas).

Abstract

This thesis starts from the following observation; if v, w are solutions of

$$y'' + Py = 0$$

where P is entire, and v and w are both 0 at $z_0 \in \mathbb{C}$, then $W(v, w) = vw' - v'w \equiv 0$ and v, w are linearly dependent. It is then natural to ask what happens if v, w solve different equations, but have (mostly) the same zeros. The case where the first equation is of the second order and has a polynomial coefficient while the second equation is of order greater than one with entire coefficients was investigated first, and some relations between the solutions and between the coefficients were proved. We next obtained approximately the same results when a transcendental coefficient was considered instead of a polynomial in the first equation, but with some amendments to the conditions. We then examined the case where the equations are non-homogeneous of the first order and determined what the solutions have to be. We also could determine the solutions in the case where the equations are a combination of homogeneous and non-homogeneous equations. Finally, the case where the solutions take the value 0 and a non-zero value at mostly the same points was studied, and again the solutions were determined. In order to prove our results, we used some background from Nevanlinna theory and some of its applications.

List of published and submitted papers

1. A. Asiri. Common zeros of the solutions of two differential equations. *Comput. Methods Funct. Theory*, 12(1):67-85, 2012.

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3. A. Asiri. Common zeros of the solutions of two non-homogeneous first order differential equations. *Results in Mathematics*, pages 1-10, Published online: 17 November 2011. Available at `http://dx.doi.org/10.1007/s00025-011-0213-y`. DOI: 10.1007/s00025-011-0213-y.

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Contents

CHAPTER 1

Introduction

This chapter contains some important preliminary definitions and concepts that are needed to state our results in the following chapters. In addition, some well-known results that will be used frequently will be stated.

1.1 Analytic and meromorphic functions

In this section, some types of functions occurring throughout function theory will be defined.

Definition 1.1.1 (Differentiable function [19])**.**
Let U be an open set in the complex plane, and let z be a point of U. Let f be a complex-valued function on U. We say that f is complex differentiable at z if the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists and is finite. Also, we say that f is differentiable on U if f is differentiable at every point of U.

Definition 1.1.2 (Analytic and entire functions [26])**.**
A complex function $f(z)$ is said to be analytic at a point z_0 if f is differentiable at z_0 and at every point in some neighbourhood of z_0. A function f is analytic in a domain D

if it is analytic at every point in D. Moreover, a function that is analytic at every point z in the complex plane is called an entire function.

Examples 1.1.1. The functions e^z and $\sin z$ are entire functions. Also, all polynomials in z are entire functions.

Definition 1.1.3 (Meromorphic functions)**.**
A function f on a domain in $\mathbb{C}$ is said to be meromorphic if it is analytic apart from isolated poles.

Examples 1.1.2. The functions $\frac{e^z}{z}$ and $\frac{1}{\cos \pi z}$ are meromorphic functions. Also, all rational functions are meromorphic functions in the complex plane.

Definition 1.1.4 (O and o notation [20])**.**
Let $f(r), g(r)$ be functions defined on $[a, \infty)$, with $f(r)$ complex-valued and $g(r)$ real and positive. We say that $f(r) = O(g(r))$ as $r \to \infty$ if there exist constants K, L such that

$$|f(r)| \leq Kg(r) \quad \text{for all} \quad r \geq L.$$

Also, we say that $f(r) = o(g(r))$ if

$$\frac{f(r)}{g(r)} \to 0 \quad \text{as} \quad r \to \infty.$$

Examples 1.1.3.

- For any polynomial P of degree n, we have $P(r) = O(r^n)$ as $r \to \infty$.
- Since $\frac{\log r}{r} \to 0$ as $r \to \infty$, we have $\log r = o(r)$.

1.2 Nevanlinna theory

Throughout the 1920s, the value distribution theory underwent remarkable development led by Rolf Nevanlinna. Nevanlinna theory plays an essential rule in our research area.

So, we will introduce some important definitions and theorems related to Nevanlinna Theory. For extra details we refer the reader to [13, 18, 20].

Assume henceforth that f is a meromorphic function on the whole complex plane, unless clearly stated otherwise.

Definition 1.2.1 (Proximity function)**.**
Suppose that f is a meromorphic function on $|z| \leq r$, then

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta.$$

Here, $\log^+ x = \max\{\log x, 0\}$.

Definition 1.2.2 (Counting function)**.**
We define $n(r, f)$ to be the number of poles of f counting with multiplicity in $|z| \leq r$. Also we define $\overline{n}(r, f)$ to be the number of poles of f counting each pole just once.

Definition 1.2.3 (Counting function)**.**
For $f \not\equiv 0$, we define $n(r, \frac{1}{f})$ to be the number of zeros of f counting with multiplicity in $|z| \leq r$. Also we define $\overline{n}(r, \frac{1}{f})$ to be the number of zeros of f counting each zero just once.

Example 1.2.1. Suppose that

$$f(z) = \frac{(z-3)^2(z-5)^4}{(z-2)^7}.$$

Then we have, for $r \geq 5$,

$$n(r, f) = 7, \quad \overline{n}(r, f) = 1, \quad n\left(r, \frac{1}{f}\right) = 6, \quad \overline{n}\left(r, \frac{1}{f}\right) = 2.$$

Definition 1.2.4 (Integrated counting function)**.**
We set

$$N(r, f) = \int_0^r [n(t, f) - n(0, f)] \frac{dt}{t} + n(0, f) \log r.$$

Definition 1.2.5 (Integrated counting function)**.**
We also set

$$\overline{N}(r, f) = \int_0^r [\overline{n}(t, f) - \overline{n}(0, f)] \frac{dt}{t} + \overline{n}(0, f) \log r.$$

Definition 1.2.6 (Nevanlinna characteristic function)**.**

We set

$$T(r, f) = m(r, f) + N(r, f).$$

Some properties of the proximity function, the integrated counting function and the Nevanlinna characteristic function will be stated in the following proposition.

Proposition 1.2.1 ([2])**.**

Assume that f_k are meromorphic functions. Then we have the following properties:

1. $m(r, \sum_{k=1}^{n} f_k(z)) \leq \sum_{k=1}^{n} m(r, f_k(z)) + O(1)$.
2. $m(r, \prod_{k=1}^{n} f_k(z)) \leq \sum_{k=1}^{n} m(r, f_k(z))$.
3. $N(r, \sum_{k=1}^{n} f_k(z)) \leq \sum_{k=1}^{n} N(r, f_k(z))$.
4. $N(r, \prod_{k=1}^{n} f_k(z)) \leq \sum_{k=1}^{n} N(r, f_k(z))$.
5. $T(r, \sum_{k=1}^{n} f_k(z)) \leq \sum_{k=1}^{n} T(r, f_k(z)) + O(1)$.
6. $T(r, \prod_{k=1}^{n} f_k(z)) \leq \sum_{k=1}^{n} T(r, f_k(z))$.

Theorem 1.2.1 (The first fundamental Nevanlinna theorem)**.**

Suppose that f is a non-constant meromorphic function in the plane, and suppose that $a \in \mathbb{C}$. Then:

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1) \quad \textit{as} \quad r \to \infty.$$

Theorem 1.2.2 (The second fundamental Nevanlinna theorem, Version I)**.**

Suppose that f is a non-constant meromorphic function in the plane, and suppose that $q \geq 2$. Suppose that $a_1, a_2, ..., a_q$ are distinct complex numbers. Then:

$$m(r, f) + \sum_{n=1}^{q} m(r, \frac{1}{f-a_n}) \leq 2T(r, f) + S(r, f),$$

where $S(r, f)$ means some quantity such that $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set of finite measure.

The terminology $S(r, f)$ will be used throughout the thesis.

Exceptional sets as in Theorem 1.2.2 are very common in the Nevanlinna theory, but can sometimes be avoided by using the following lemma.

Lemma 1.2.1 ([18])**.**
Let $g : (0, +\infty) \to \mathbb{R}$, $h : (0, +\infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

If we take $q = 2, a_1 = 0, a_2 = 1$ in Theorem 1.2.2, then a stronger version of the second Nevanlinna theorem leads to the following theorem.

Theorem 1.2.3 (The second fundamental Nevanlinna theorem, Version II)**.**
Suppose that f is a non-constant meromorphic function. Then

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-1}) + S(r, f).$$

This result implies Picard's theorem, because if f omits $0, 1$ and ∞ then we get $T(r, f) = S(r, f)$, a contradiction.

The following lemma is essential to prove the second fundamental Nevanlinna theorem.

Lemma 1.2.2 (Lemma of the Logarithmic Derivative)**.**
Suppose that f is a transcendental meromorphic function in the plane and suppose that k is a positive integer. Then

$$m(r, \frac{f^{(k)}}{f}) = S(r, f).$$

If $\log T(r, f) = O(\log r)$, we have

$$m(r, \frac{f^{(k)}}{f}) = O(\log r).$$

Proposition 1.2.2. *Suppose that f is a meromorphic function in the plane. Then*

$$T(r, f) = O(\log r) \quad as \quad r \to \infty$$

if and only if f is a rational function.

For functions over the complex numbers, we often need to know a rough measure of how fast a function grows. The order of growth is used to measure how fast a function grows.

Definition 1.2.7 (Order of growth for meromorphic function)**.**
Let f be a meromorphic function in the plane. Then

$$\rho(f) = \limsup_{r \to \infty} \frac{\log^{+} T(r, f)}{\log r},$$

is called the order of growth for the function f.

We now state some properties of the order of growth.

Proposition 1.2.3 ([2])**.**
Let f and g be meromorphic functions. Then

1. $\rho(f + g) \leq \max\{\rho(f), \rho(g)\}$.
2. $\rho(fg) \leq \max\{\rho(f), \rho(g)\}$.
3. *If $g(z) = f(az + b)$, where $a, b \in \mathbb{C}$, $a \neq 0$, then $\rho(g) = \rho(f)$.*
4. *If $g(z) = f(z^k)$, where k is a positive integer, then $\rho(g) = k\rho(f)$.*
5. *If f is a polynomial, then $\rho(f) = 0$.*
6. *If P is a polynomial of degree n, then $\rho\left(e^{P(z)}\right) = n$.*
7. *If f is a transcendental entire function, then $\rho\left(e^{f(z)}\right) = \infty$.*

Examples 1.2.1.

1. $\rho(\sin z^2) = \rho(\cos z^2) = 2$.
2. $\rho(e^{z^k}) = k$.
3. $\rho(e^{\sin z}) = \rho(e^{e^z}) = \infty$.

Definition 1.2.8 (Exponent of convergence)**.**

Let f be a meromorphic function in the plane and assume that f is not identically 0. The exponent of convergence $\lambda(f) = \lambda(f, 0)$ of the zeros of f is defined by

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log^+ N(r, \frac{1}{f})}{\log r}.$$

Examples 1.2.2.

1. $\lambda(e^z) = \lambda(P) = 0$, where P is a polynomial.

2. $\lambda(e^z + 1) = \lambda(\sin z) = \lambda(\cos z) = 1$.

Proposition 1.2.4 (Relation between ρ and λ [18])**.**

For all meromorphic functions $f \not\equiv 0$, we have $\lambda(f) \leq \rho(f)$.

Definition 1.2.9 (Nevanlinna deficiency [12])**.**

The Nevanlinna deficiency $\delta(a) = \delta(a, f)$ of the value a is defined by

$$\delta(a) = \liminf_{r\to\infty} \frac{m(r, a)}{T(r, f)} = 1 - \limsup_{r\to\infty} \frac{N(r, a)}{T(r, f)}.$$

In particular, $\delta(a) = 0$ except for at most countably many a . If f omits the value a then $\delta(a) = 1$.

The following theorem was proved by Rolf Nevanlinna that shows under what circumstances can two distinct functions take the same value at the same points.

Theorem 1.2.4 (Nevanlinna's 5-values theorem [13])**.**

Suppose that $f_1(z), f_2(z)$ are meromorphic in the plane and let $E_j(a)$ be the set of points z such that $f_j(z) = a \quad (j = 1, 2)$. Then if $E_1(a) = E_2(a)$ for five distinct values of a, we have $f_1(z) \equiv f_2(z)$, or f_1, f_2 are both constant.

The following example shows that the five values in the last theorem cannot be replaced by four values.

Example 1.2.2. The functions $f_1(z) = e^z, f_2(z) = e^{-z}$ take the values $a = 0, 1, -1, \infty$ at the same points, but $f_1(z) \neq f_2(z)$.

We are interested in the particular case where v and w are entire functions which solve linear differential equations and which take one value (usually 0) at the same points.

1.3 Wiman-Valiron theory

The Wiman-Valiron theory is mainly used to study the local behaviour of an entire function from its power series. It will be used to estimate the growth of the solutions in some of our results. The content of this section was mostly taken from [14, 20].

Definition 1.3.1 (Maximum modulus [20])**.**
Suppose that f is an entire function and let $r > 0$. The maximum modulus $M(r, f)$ is defined by

$$M(r, f) = \max\{|f(z)| : |z| \leq r\},$$

which is non-decreasing.

Initially, let

$$P(z) = a_n z^n + \cdots + a_0, \quad a_n \neq 0,$$

be a polynomial of degree n, and assume that z and z_0 are large. Then we have

$$P(z) \sim \left(\frac{z}{z_0}\right)^n P(z_0) \quad \text{and} \quad \frac{P'(z)}{P(z)} \sim \frac{n}{z}.$$

However, if P is a non-polynomial entire function, then by Picard's theorem we can see that no such asymptotic relation can hold for all large z and z_0. The aim of Wiman-Valiron theory is to find a similar relation when z is close to z_0 and $|f(z_0)|$ is close to $M(|z_0|, f)$.

Now, we set

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{1.3.1}$$

to be a transcendental entire function ($a_k \neq 0$ for infinitely many k). We need the following definitions in order to state the main Wiman-Valiron theorem. Proofs can be found in [14, 20].

Definition 1.3.2 (Maximum term [14, 20])**.**

For each $r \geq 0$, we define the maximum term $\mu(r, f)$ as follows.

$$\mu(r) = \mu(r, f) = \max\{|a_k|r^k : k = 0, 1, 2, \ldots\}. \qquad (1.3.2)$$

Lemma 1.3.1. *For f a transcendental entire function, we have*

- $\lim_{r\to\infty} \mu(r, f) = \infty.$
- *For $r > 0$, we have $\mu(r, f) \leq M(r, f) \leq 2\mu(2r, f)$.*
- *$\mu(r, f)$ is continuous and non-decreasing on $[0, \infty)$, and there exists $R \geq 0$ such that $\mu(r)$ is strictly increasing on $[R, \infty)$.*

Definition 1.3.3 (Central index [20])**.**

For $r > 0$ and $\mu(r)$ as above, we define the central index $\nu(r) = \nu(r, f)$ (also called $N(r)$) to be the largest k for which $|a_k|r^k = \mu(r, f)$. Note that if $a_0 = 0$ then $\nu(0)$ is not defined, whereas if $a_0 \neq 0$ then $\nu(0) = 0$.

Lemma 1.3.2. *For f a transcendental entire function, we have*

- *The central index $\nu(r)$ is non-decreasing on $(0, \infty)$, and $\nu(r) \to \infty$ as $r \to \infty$. Also, $\nu(r)$ is continuous from the right, i.e., for each $s > 0$,*

$$\lim_{r\to s^+} \nu(r) = \nu(s).$$

- *Let $\epsilon > 0$. Then*

$$N(r) = \nu(r) \leq (\log \mu(r))^{1+\epsilon} \leq (\log M(r, f))^{1+\epsilon}$$

  for all $r \geq 1$ outside a set E of finite logarithmic measure, i.e. $\int_{[1,\infty)\cap E} \frac{dt}{t} < \infty$.

- $\nu(r, f') \sim \nu(r, f)$ *as $r \to \infty$ with $r \notin E$, where $\int_E \frac{dt}{t} < \infty$.*

Theorem 1.3.1 (The main theorem of the Wiman-Valiron theory)**.**

Let f be defined by (1.3.1)*, and let $\frac{1}{2} < \gamma < 1$ and $0 < k \leq 1$. Let q be a positive* *integer. Then there exists a set $E_2 \subseteq [1,\infty)$, of finite logarithmic measure, such that, if $|z_0| = r \in [1,\infty) \setminus E_2$ and $|f(z_0)| \geq kM(r,f)$ then*

$$f(z) \sim \left(\frac{z}{z_0}\right)^N f(z_0) \quad \textit{and} \quad \frac{f^{(j)}(z)}{f(z)} \sim \frac{N^j}{z^j} \quad \textit{for } |\log(z/z_0)| \leq N^{-\gamma}$$

and $j = 1,\ldots,q$, where $N = \nu(r,f)$. Furthermore, for $|\log(\rho/r)| \leq N^{-\gamma}$, we have

$$M(\rho, f^{(j)}) \sim \frac{N^j}{\rho^j} M(\rho, f), \quad M(\rho,f) \sim \left(\frac{\rho}{r}\right)^N M(r,f)$$

for $j = 1,\ldots,q$.

1.4 Complex differential equations

In this section, we will see how Nevanlinna theory can be used to study the solutions of complex differential equations. Moreover, we will state some useful theorems needed to prove our results.

Definition 1.4.1. By a complex differential equation we mean a differential equation whose coefficients are meromorphic functions of a complex variable.

In particular, we will take the equation

$$w'' + Pw = 0, \tag{1.4.1}$$

where P is a polynomial of degree n as an example of our study.

An important theorem in differential equations is the existence-uniqueness theorem.

Theorem 1.4.1 (Existence-uniqueness theorem [20])**.**
Let $k \geq 1$, let D be a simply connected domain in $\mathbb{C}$, and let $a_0(z),\ldots,a_{k-1}(z)$ be analytic in D. Let $a \in D$ and let $c_0,\ldots,c_{k-1} \in \mathbb{C}$. Then there exists a unique solution f of the equation

$$w^{(k)} + \sum_{j=0}^{k-1} a_j w^{(j)} = 0,$$

such that f is analytic in D and $f^{(j)}(a) = c_j$, $0 \leq j \leq k-1$.

In 1955, Wittich [25] proved the following theorem.

Theorem 1.4.2. *If f is a non-trivial solution of $w'' + Aw = 0$, i.e. $f \not\equiv 0$, and $A \not\equiv 0$ is entire, then we have:*

(i) $T(r, A) = S(r, f)$.

(ii) *If f has finite order, then A is a polynomial.*

(iii) *If a is a non-zero complex number, then f takes the value a infinitely often, and in fact, outside a set of finite measure,*

$$N\left(r, \frac{1}{f-a}\right) \backsim T(r, f).$$

Cauchy [17] proved that all solutions of (1.4.1) are entire functions. Furthermore, the 1982 paper of Bank and Laine [6] contained the following theorems.

Theorem 1.4.3. *Let P be a polynomial of degree n, and let w be a non-trivial solution of the equation* (1.4.1). *Then, w has order of growth equal to $\frac{n+2}{2}$.*

Theorem 1.4.4. *If w is a non-trivial solution of* (1.4.1) *which has infinitely many zeros, then we have*

$$\liminf_{r\to\infty} \frac{N(r, \frac{1}{w})}{r^{(n+2)/2}} > 0. \tag{1.4.2}$$

In 1962, Clunie proved a lemma [11, Lemma 1] which has frequently been used in applications to complex differential equations. The following lemma is an alternative form of Clunie's lemma which has been proved by Laine in [18, Lemma 2.4.2].

Lemma 1.4.1 (Clunie's lemma [18])**.**
Let f be a transcendental meromorphic solution of

$$f^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in f and its derivatives with meromorphic coefficients, say $\{a_\lambda | \lambda \in I\}$, such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the total degree of $Q(z, f)$ as a polynomial in f and its derivatives is $\leq n$, then

$$m(r, P(z, f)) = S(r, f).$$

Definition 1.4.2 (The Wronskian determinant [18])**.**

Let $f_1, \cdots, f_n$ be meromorphic functions in the complex plane. The Wronskian determinant $W(f_1, \cdots, f_n)$ is given by

$$W(f_1, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}. \tag{1.4.3}$$

We will only use this for $n = 2$, in which case

$$W(f_1, f_2) = f_1 f_2' - f_1' f_2.$$

Proposition 1.4.1 ([18])**.**

Let $f_1, \cdots, f_n$ be meromorphic functions in the complex plane. Then $W(f_1, \cdots, f_n) \equiv 0$ if and only if $f_1, \cdots, f_n$ are linearly dependent over $\mathbb{C}$.

1.5 Bank-Laine functions

This section contains some essential tools that will be used throughout the proofs of our results. In particular, the Bank-Laine product formula will be used sometimes to determine the solutions.

Suppose that A is an entire function and suppose that we have the equation

$$y'' + A(z)y = 0. \tag{1.5.1}$$

Definition 1.5.1 (Bank-Laine Function [2])**.**

A Bank-Laine function E is an entire function such that $E'(z_0) = \pm 1$ at every zero z_0 of E.

Example 1.5.1. $E(z) = e^z - 1$ is a Bank-Laine function because if z_0 is a zero of E then $E'(z_0) = +1$ since $E'(z) = e^z$.

The following theorem gives another way to characterise Bank-Laine functions.

Theorem 1.5.1 (Bank-Laine Function [2])**.**
An entire function E is a Bank-Laine function if and only if E is the product $f_1 f_2$, where f_1 and f_2 are linearly independent normalized solutions of the equation (1.5.1) *such that A is an entire function. Here "normalized" means that $W(f_1, f_2) = f_1 f_2' - f_1' f_2 = 1$.*

Definition 1.5.2 (Bank-Laine product formula [12])**.**
If A is an entire function and $E = f_1 f_2$ where f_1 and f_2 are linearly independent solutions of the equation (1.5.1), such that $W(f_1, f_2) = 1$, then a simple calculation shows that

$$4A = \left(\frac{E'}{E}\right)^2 - 2\frac{E''}{E} - \frac{1}{E^2}. \tag{1.5.2}$$

This is called the Bank-Laine product formula. This relation between A and E works in the opposite direction also.

Theorem 1.5.2 ([12])**.**
Let E be an entire function such that $E'(z) = \pm 1$ at every zero of E. Define A by (1.5.2)*. Then A is an entire function and E has a representation $E = f_1 f_2$, where f_1 and f_2 are normalized linearly independent solutions of* (1.5.1).

Example 1.5.2. If we take $E = e^z - 1$, then we can calculate A from the Bank-Laine product formula (1.5.2). So,

$$\begin{aligned}
4A &= \left(\frac{E'}{E}\right)^2 - 2\frac{E''}{E} - \frac{1}{E^2} \\
&= \left(\frac{e^z}{e^z - 1}\right)^2 - \frac{2e^z}{e^z - 1} - \frac{1}{(e^z - 1)^2} \\
&= \frac{e^{2z}}{(e^z - 1)^2} - \frac{2e^z(e^z - 1)}{(e^z - 1)^2} - \frac{1}{(e^z - 1)^2} \\
&= \frac{e^{2z}}{(e^z - 1)^2} - \frac{2e^{2z} - 2e^z}{(e^z - 1)^2} - \frac{1}{(e^z - 1)^2} \\
&= \frac{-e^{2z} + 2e^z - 1}{e^{2z} - 2e^z + 1} \\
&= -1.
\end{aligned}$$

Then, $A = -1/4$ and we can write $E = f_1 f_2$ where f_1, f_2 are linearly independent solutions of the equation

$$w'' - \frac{1}{4}w = 0$$

and $W(f_1, f_2) = 1$.

We now outline to find f_1 and f_2 given E. We have

$$1 = W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = f_1 f_2' - f_1' f_2. \tag{1.5.3}$$

Also, since $E = f_1 f_2$ and by using equation (1.5.3), we get

$$\frac{1}{E} = \frac{1}{f_1 f_2} = \frac{f_2'}{f_2} - \frac{f_1'}{f_1}$$

and

$$\frac{E'}{E} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}.$$

Therefore,

$$\frac{E'}{E} - \frac{1}{E} = 2\frac{f_1'}{f_1}$$

and so

$$\frac{f_1'}{f_1} = \frac{1}{2}\left(\frac{E' - 1}{E}\right) = \frac{1}{2}\left(\frac{e^z - 1}{e^z - 1}\right) = \frac{1}{2}.$$

Hence, we can take

$$f_1(z) = e^{\frac{z}{2}}$$

and

$$f_2(z) = \frac{E(z)}{f_1(z)} = \frac{e^z - 1}{e^{\frac{z}{2}}} = e^{\frac{z}{2}} - e^{-\frac{z}{2}}$$

and it is easy to check that $W(f_1, f_2) = 1$.

The following lemma states the relation between the order of A and the order of E.

Lemma 1.5.1 ([12]).

Suppose that A and E are entire functions satisfying the Bank-Laine product formula (1.5.2). *Then*

(i) *If E has finite order, so has A.*

(ii) *If $\rho(A) < \infty$ and $\lambda(E) < \infty$ then $\rho(E) < \infty$.*

Conjecture (Bank-Laine conjecture [6])**.**
If A is a transcendental entire function and the equation (1.5.1) *has linearly independent solutions f_1, f_2 with $\max\{\lambda(f_1), \lambda(f_2)\} < \infty$, then the order of A is either ∞ or a positive integer.*

Definition 1.5.3 (Maximal order [22])**.**
Let $A_0, \ldots, A_{k-1}$ be entire functions and consider the equation

$$y^{(k)}(z) + \sum_{j=0}^{k-1} A_j y^{(j)}(z) = 0. \tag{1.5.4}$$

We say that the coefficient A_s in (1.5.4), $0 \leq s \leq k-1$, has maximal order relative to (1.5.4) if A_s is transcendental of finite order ρ and all other coefficients A_j with $j \neq s$ either are polynomials or have order less than ρ.

The following theorem is a comprehensive theorem which was stated by Langley [22] to summarize some known facts and results in this field from [6, 10, 15, 16, 21, 23, 24].

Theorem 1.5.3 ([22])**.**

(i) *Suppose that f is a solution of* (1.5.4)*, and that all of the coefficients A_j have finite order and $A_0 \not\equiv 0$. If f has infinite order, then the Nevanlinna deficiency $\delta(c, f) = 0$ for any $c \in \mathbb{C} \setminus \{0\}$.*

(ii) *Assume again that all coefficients A_j have finite order, and suppose further that $A_{k-1} \equiv 0$ and that* (1.5.4) *has a fundamental set of solutions $f_1, \ldots, f_k$ each with $\lambda(f_j) < \infty$. Then the product $E = f_1 \cdots f_k$ has finite order.*

(iii) *Suppose that A_0 has maximal order relative to* (1.5.4)*. Then every non-trivial solution of* (1.5.4) *has infinite order, as has the quotient of any two linearly independent solutions.*

(iv) *If A_0 has maximal order relative to* (1.5.4)*, and has order at most $1/2$, then the equation* (1.5.4) *cannot have linearly independent solutions f_1, f_2 each with $\lambda(f_j)$ finite.*

(v) *Suppose that A_s has maximal order relative to* (1.5.4)*, where $1 \leq s \leq k-1$, and that A_s has order at most $1/2$. Then every transcendental solution of* (1.5.4) *has infinite order.*

(vi) *Suppose that A_s has maximal order relative to* (1.5.4)*, where $1 \leq s \leq k-1$, and suppose that $\rho(A_s) \leq 1/2$ and that $A_{k-1} \equiv 0$. Then* (1.5.4) *cannot have k linearly independent solutions $f_1, \ldots, f_k$ each with $\lambda(f_j)$ finite.*

(vii) *Suppose that A_{k-1} has maximal order relative to* (1.5.4)*, and $\rho(A_{k-1}) \leq 1/2$. Then every transcendental solution f of* (1.5.4) *has $\lambda(f) = \infty$.*

(viii) *Suppose that A_0 has maximal order relative to* (1.5.4) *and has order ρ, and that $A_{k-1} \equiv 0$, and further that* (1.5.4) *has a fundamental set of solutions $f_1, \ldots, f_k$ each with $\lambda(f_j) < \rho$. Then ρ is an integer, and the product $E = f_1 \cdots f_k$ has order ρ.*

1.6 Asymptotics for solutions of differential equations with rational and polynomial coefficients

There has been extensive research describing the behaviour of the solutions of differential equations with polynomial coefficients [17, 18]. This section describes how to prove Theorem 1.4.4 which plays an important role in some of our results.

First of all, we would like to focus on the equation

$$w'' + b(z)w = 0, \tag{1.6.1}$$

where $b(z)$ is a rational function with

$$b(z) = cz^n(1 + o(1)) \quad \text{as} \quad z \to \infty \tag{1.6.2}$$

where $n \geq -1$ and $c \neq 0$.

Definition 1.6.1 (Critical rays [20]).

The critical rays are those rays $\arg z = \theta \in \mathbb{R}$ for which

$$\arg c + (n+2)\theta = 0 \pmod{2\pi}.$$

Assume that $\arg z = \theta_0$ is a critical ray, let R_0 be large and positive, and define

$$Z = \int_{2R_0 e^{i\theta_0}}^{z} b(t)^{1/2} dt = \frac{2c^{1/2}}{n+2} z^{(n+2)/2}(1+o(1)), \quad |\arg z - \theta_0| \leq \frac{2\pi}{n+2}. \quad (1.6.3)$$

Example 1.6.1. We calculate Z and find the critical rays when $b(z) = z^4$ in (1.6.1). Here

$$Z = \int^{z} \left(t^4\right)^{1/2} dt = \frac{z^3}{3}.$$

To find the critical rays we have

$$b(z) = cz^n, \qquad c = 1, \qquad n = 4.$$

The critical rays are $\arg z = \theta$ where

$$\arg c + (n+2)\theta = 0 \pmod{2\pi}.$$

Then,

$$6\theta = 0 \pmod{2\pi}.$$

Hence, $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$. [See Figure 1.1 on page 18].

Now, we consider the equation (1.6.1) where $b(z)$ is as in (1.6.2). We have the following theorem.

Theorem 1.6.1 ([17, 20]).

Let $b(z)$ be as in (1.6.2) *and let $\arg z = \theta_0$ be a critical ray. Let $\epsilon > 0$. Then the equation* (1.6.1) *has solutions*

$$u_j(z) = b(z)^{-1/4} \exp((-1)^j iZ + o(1)) \qquad (j = 1, 2),$$

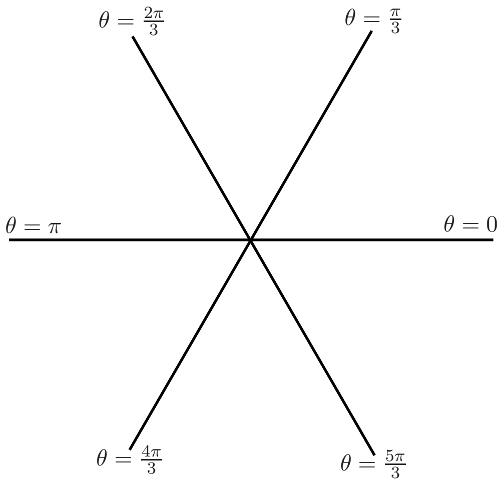

Figure 1.1: Critical rays for Example 1.6.1

in the sector S_ϵ given by

$$|z| > R_0, \qquad |\arg z - \theta_0| \le \frac{2\pi}{n+2} - \epsilon,$$

where Z is defined in (1.6.3). [See Figure 1.2 on page 19].

Now let w be any solution of (1.6.1) in S_ϵ. If w/u_1 or w/u_2 is constant, then w has no zeros in S_ϵ. Now suppose that

$$w = A_1u_1 - A_2u_2, \qquad A_1, A_2 \in \mathbb{C} \setminus \{0\}.$$

Then w has zeros where

$$\frac{A_1}{A_2} = \frac{u_2}{u_1} = \exp(2iZ + o(1)).$$

The equation

$$e^{2iZ} = \frac{A_1}{A_2}$$

has solutions

$$2iZ = \log\frac{A_1}{A_2} + k2\pi i \qquad (k \in \mathbb{Z})$$

and Rouché's theorem gives solutions of $u_2/u_1 = A_1/A_2$, and so zeros of w, near to these points. The proof may be found in [20], and this is how theorem 1.4.4 is proved.

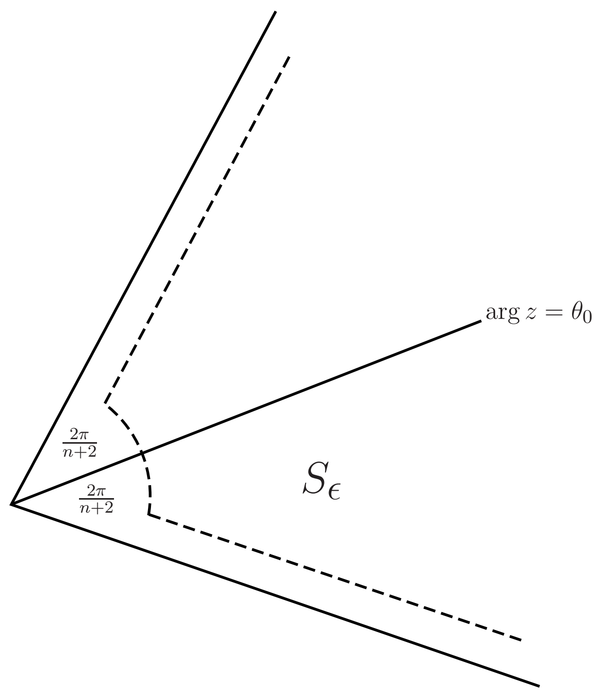

Figure 1.2: Asymptotic regions for solutions of differential equations with rational and polynomial coefficients

CHAPTER 2

Equations with polynomial coefficients

After giving an introduction about Nevanlinna theory and some of its applications in complex differential equation, we now start investigating some new results. First of all, we start by studying the behaviour of zeros of the solutions of differential equations. The results of this chapter were published in [4].

2.1 Introduction

We consider two homogeneous linear differential equations and we use Nevanlinna theory to determine when the solutions of these differential equations can have the same zeros or (mostly) the same zeros.

In this chapter, we will see some ways in which Nevanlinna theory can be used to study the solutions of complex differential equations. First of all, we are going to state some useful theorems needed to prove our new results.

In particular, we will take the equation

$$w'' + Pw = 0, \tag{2.1.1}$$

where P is a polynomial of degree n, as an object of study. It is well known that every solution of (2.1.1) is an entire function.

From Theorem 1.4.3 and Theorem 1.4.4 in Chapter 1, we recall the following facts which follow from the asymptotic representation for solutions of (2.1.1); we refer the reader to [6, 17] for details.

Theorem 2.1.1. *Let P be a polynomial of degree n, and let w be a non-trivial solution of the equation* (2.1.1)*. Then, w has order of growth equal to $\frac{n+2}{2}$ and $T(r,w) = O\left(r^{(n+2)/2}\right)$. Moreover, if w is a solution of* (2.1.1) *which has infinitely many zeros, then we have*

$$\liminf_{r\to\infty} \frac{N(r,\frac{1}{w})}{r^{(n+2)/2}} > 0. \tag{2.1.2}$$

We refer the reader to the book of Laine [18], the influential paper [6], and to [1, 7–10, 21, 22, 24] for extensive results on the zeros of solutions of linear differential equations with entire coefficients.

In the next section, we consider two differential equations with solutions having the same or (mostly) the same zeros.

2.2 A theorem for the general case

To state our main result we need the following lemma.

Lemma 2.2.1. *Suppose $w'' = -Pw$ where P is a polynomial. Then for $j \geq 0$, there exist polynomials Q_j and R_j such that*

$$w^{(j)} = Q_j w + R_j w'. \tag{2.2.1}$$

Proof. In fact, we have the following initial cases:

$$\begin{aligned}
j = 0 \ &\Rightarrow\ Q_0 = 1, \quad R_0 = 0;\\
j = 1 \ &\Rightarrow\ Q_1 = 0, \quad R_1 = 1;\\
j = 2 \ &\Rightarrow\ Q_2 = -P, \quad R_2 = 0;\\
j = 3 \ &\Rightarrow\ Q_3 = -P', \quad R_3 = -P;\\
j = 4 \ &\Rightarrow\ Q_4 = P^2 - P'', \quad R_4 = -2P'.
\end{aligned} \tag{2.2.2}$$

Now we proceed by induction; assume that $j \geq 0$ and that (2.2.1) is true. Then we have

$$\begin{aligned} w^{(j+1)} &= Q_j' w + Q_j w' + R_j' w' + R_j w'' \\ &= Q_j' w + Q_j w' + R_j' w' - R_j P w \\ &= (Q_j' - R_j P) w + (Q_j + R_j') w' \\ &= (Q_{j+1}) w + (R_{j+1}) w', \end{aligned}$$

where $Q_{j+1} = Q_j' - R_j P$ and $R_{j+1} = Q_j + R_j'$.

Since Q_{j+1} and R_{j+1} are polynomials, the induction is complete. □

Theorem 2.2.1. *Let $P \not\equiv 0$ be a polynomial of degree n. Let $w \not\equiv 0$ be a solution of* (2.1.1). *Assume that w has infinitely many zeros. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v^{(k)} + \sum_{1 \leq j \leq k-2} B_j v^{(j)} + Av = 0, \quad k \geq 2, \tag{2.2.3}$$

such that A and B_j are entire functions with

$$\sum_{1 \leq j \leq k-2} T(r, B_j) + T(r, A) = o(r^{(n+2)/2}), \qquad \text{as } r \to \infty, \tag{2.2.4}$$

where the sum $\sum_{1 \leq j \leq k-2}$ should be interpreted as zero when $k = 2$. Assume that $N(r) = o(r^{(n+2)/2})$ where $N(r)$ counts both zeros and poles of $\frac{v}{w}$. Let

$$v = Lw. \tag{2.2.5}$$

Then one of the following possibilities holds.

(a) *L is constant, and*

$$A = -Q_k - \sum_{j=1}^{k-2} B_j Q_j, \tag{2.2.6}$$

where Q_k and Q_j are defined by Lemma 2.2.1.

(b) *L is not constant, but L satisfies*

$$\sum_{m=0}^{k} \left[\binom{k}{m} L^{(m)} R_{k-m} + \sum_{j=1}^{k-2} \binom{j}{m} B_j L^{(m)} R_{j-m} \right] = 0, \tag{2.2.7}$$

and A satisfies

$$A = -\left(\sum_{m=0}^{k} \left[\binom{k}{m} \frac{L^{(m)}}{L} Q_{k-m} + \sum_{j=1}^{k-2} \binom{j}{m} B_j \frac{L^{(m)}}{L} Q_{j-m} \right] \right), \tag{2.2.8}$$

where R_{k-m}, R_{j-m}, Q_{k-m} and Q_{j-m} are also defined by Lemma 2.2.1.

(c) *If $B_1, B_2, \cdots, B_{k-2}$ are polynomials, and case* (b) *holds, then A is a polynomial.*

There is no loss of generality in assuming that there is no term in w' in (2.1.1) and that there is no term B_{k-1} in (2.2.3). This is because for any equation

$$y^{(m)} + A_{m-1}y^{(m-1)} + \cdots + A_0 y = 0,$$

with entire coefficients A_j, the change of variables $y = UY$, where $mU'/U = -A_{m-1}$, gives an equation

$$Y^{(m)} + B_{m-2}Y^{(m-2)} + \cdots + B_0 Y = 0$$

and Y has the same zeros as y.

From Theorem 2.2.1 we will deduce the following results for the cases $k = 2$, $k = 3$ and $k = 4$.

Theorem 2.2.2. *Let $P \not\equiv 0$ be a polynomial of degree n. Let $w \not\equiv 0$ be a solution of* (2.1.1)*. Assume that w has infinitely many zeros. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v'' + Av = 0, \tag{2.2.9}$$

such that A is an entire function and $N(r)$ counts zeros of v which are not zeros of w and zeros of w which are not zeros of v. Assume that

$$N(r) + T(r, A) = o(r^{(n+2)/2}).$$

Then $\frac{v}{w}$ is a constant and $A = P$.

Example 2.2.1. Obviously we may take $v = w$ and $A = P$.

Example 2.2.2. We give an example to show that $T(r, A) = o(r^{(n+2)/2})$ is necessary in Theorem 2.2.2. To show this put $v = we^g$ where g is an entire function. Then we have

$$\begin{aligned}\frac{v''}{v} &= \frac{w''}{w} + 2g'\frac{w'}{w} + g'' + g'^2\\ &= -P + g'' + g'^2 + 2g'\frac{w'}{w}\\ &= -A.\end{aligned}$$

Now, if we put $g' = w$ then we get

$$-A = -P + w' + w^2 + 2w'.$$

So A is entire and Theorem 2.1.1 gives

$$T(r, A) = O(r^{(n+2)/2}).$$

But, $\frac{v}{w} = e^g = e^{\int w}$ is not a constant. So, Theorem 2.2.2 is not true for $T(r, A) \neq o(r^{(n+2)/2})$.

Theorem 2.2.3. *Let $P \not\equiv 0$ be a polynomial of degree n. Let $w \not\equiv 0$ be a solution of* (2.1.1). *Assume that w has infinitely many zeros. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v''' + Av = 0, \tag{2.2.10}$$

such that A is an entire function with $T(r, A) = o(r^{(n+2)/2})$ and $N(r) = o(r^{(n+2)/2})$, where $N(r)$ counts both zeros and poles of $\frac{v}{w}$. Then $v = Lw$, where $L'' = \frac{P}{3}L$ and $A = \frac{2}{3}P' + \frac{8}{3}P\frac{L'}{L}$ is a polynomial.

Example 2.2.3. The exceptional case in the conclusion can occur in Theorem 2.2.3. For example, take $L = e^Q$ where Q is a polynomial, and set

$$\frac{P}{3} = \frac{L''}{L} = Q'^2 + Q'',$$

so that P is a polynomial. Then

$$L'' = \frac{P}{3}L, \qquad L''' = \frac{P}{3}L' + \frac{P'}{3}L.$$

If w solves (2.1.1) then $v = Lw$ satisfies

$$v''' = L(-P'w - Pw') + 3L'(-Pw) + 3\frac{P}{3}Lw' + \left(\frac{P}{3}L' + \frac{P'}{3}L\right)w$$

and so v solves (2.2.10) with

$$\begin{aligned} A &= \frac{2}{3}P' + \frac{8}{3}P\frac{L'}{L} \\ &= \frac{2}{3}P' + \frac{8}{3}PQ', \end{aligned}$$

which is also a polynomial.

Theorem 2.2.4. *Let $P \not\equiv 0$ be a polynomial of degree n. Let $w \not\equiv 0$ be a solution of* (2.1.1)*. Assume that w has infinitely many zeros. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v''' + Bv' + Av = 0, \qquad (2.2.11)$$

such that A and B are entire functions with $T(r, A) + T(r, B) = o(r^{(n+2)/2})$. Assume that $N(r) = o(r^{(n+2)/2})$, where $N(r)$ counts both zeros and poles of $\frac{v}{w}$. Then $v = Lw$ and one of the following holds.

(a) *L is constant and $A = P'$, $B = P$.*

(b) *L is non-constant and $L'' = \frac{1}{3}PL - \frac{1}{3}BL$ and*

$$A = \frac{8}{3}P\frac{L'}{L} + \frac{2}{3}P' + \frac{1}{3}B' - \frac{2}{3}B\frac{L'}{L}.$$

Example 2.2.4. To show that case (b) can occur in Theorem 2.2.4, we can use Example 2.2.3, with $B = 0$.

Example 2.2.5. We give an example to show that $T(r, A) + T(r, B) = o(r^{(n+2)/2})$ is necessary in Theorem 2.2.4. To show this put $v = we^g$ where g is an entire function. Then we need

$$\begin{aligned} \frac{v'''}{v} &= \frac{w'''}{w} + 3\frac{w''}{w}g' + 3\frac{w'}{w}\left(g'' + g'^2\right) + g'^3 + 3g'g'' + g''' \\ &= -P' - P\frac{w'}{w} - 3Pg' + 3\frac{w'}{w}\left(g'' + g'^2\right) + g'^3 + 3g'g'' + g''' \\ &= -B\frac{v'}{v} - A. \end{aligned}$$

Now, if we put $g' = w$ then we get

$$-B\frac{v'}{v} - A = -P' - P\frac{w'}{w} - 3Pw + 3\frac{w'}{w}\left(w' + w^2\right) + w^3 + 3ww' + w''.$$

So

$$-B\left(\frac{w'}{w} + g'\right) - A = -P' - P\frac{w'}{w} - 3Pw + 3\frac{w'^2}{w} + 3w'w + w^3 + 3ww' + w''.$$

Then

$$-B\left(\frac{w'}{w} + w\right) - A = -P' - P\frac{w'}{w} - 3Pw + 3\frac{w'^2}{w} + 3ww' + w^3 + 3ww' + w''.$$

We want A to be entire. Put $-B = -P + 3w'$, then

$$-A = -P' - 2Pw + 3ww' + w^3 + w''$$

and since $w'' = -Pw$, then we get

$$A = P' + 3Pw - 3ww' - w^3. \tag{2.2.12}$$

Then A and B are entire functions, and

$$T(r, A) + T(r, B) = O(r^{(n+2)/2}).$$

But, $\frac{v}{w} = e^g = e^{\int w}$ is not a constant. This shows that case (a) does not hold.

Now, we check whether case (b) holds or not. If case (b) holds we have

$$\begin{aligned} A &= \frac{8}{3}P\frac{L'}{L} + \frac{2}{3}P' + \frac{1}{3}B' - \frac{2}{3}B\frac{L'}{L} \\ &= \frac{8}{3}Pw + \frac{2}{3}P' + \frac{1}{3}(P' - 3w'') - \frac{2}{3}(P - 3w')w \\ &= \frac{8}{3}Pw + \frac{2}{3}P' + \frac{P'}{3} + Pw - \frac{2}{3}Pw + 2ww' \\ &= 3Pw + P' + 2ww'. \end{aligned}$$

But this and (2.2.12) are not the same, because if they are, then

$$P' + 3Pw - 3ww' - w^3 = 3Pw + P' + 2ww'$$

and so

$$w^3 = -5ww'.$$

Dividing by w^3 gives

$$1 = -5\frac{w'}{w^2},$$

and by integrating we can write

$$z + c = \frac{5}{w}, \qquad \text{where } c \text{ is a constant,}$$

which is impossible since w is transcendental entire. So case (b) also does not hold.

Therefore, Theorem 2.2.4 is not true for $T(r, A) + T(r, B) \neq o(r^{(n+2)/2})$.

Theorem 2.2.5. *Let $P \not\equiv 0$ be a polynomial of degree n. Let $w \not\equiv 0$ be a solution of* (2.1.1). *Assume that w has infinitely many zeros. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v^{(4)} + Av = 0, \tag{2.2.13}$$

such that A is an entire function with $T(r, A) = o(r^{(n+2)/2})$ and $N(r) = o(r^{(n+2)/2})$, where $N(r)$ counts both zeros and poles of $\frac{v}{w}$. Then $v = Lw$ and one of the following holds.

(a) *L is constant and so are P and A.*

(b) *L is non-constant, $S = \frac{L'}{L}$ is a rational function, and $P = S^2 + 2S'$, while*

$$A = 5P\frac{L''}{L} + \frac{5}{2}P'\frac{L'}{L} + \frac{1}{2}P'' - P^2$$

and A is a polynomial.

Example 2.2.6.

- To show that case (a) can occur in Theorem 2.2.5, let $v = w = \sin z$, and $P = 1$, $A = -1$.

- To show that case (b) can occur in Theorem 2.2.5, take $L = Y^2 = e^Q$ where Q is a polynomial and set

$$Q' = S = 2y = 2\frac{Y'}{Y}, \qquad P = S^2 + 2S' = 4(y^2 + y')$$

so that P is a polynomial. Then

$$\begin{aligned} L' &= 2YY', \\ L'' &= 2Y'^2 + 2YY'' = 2Y'^2 + \frac{P}{2}L, \\ L''' &= PL' + \frac{P'}{2}L, \\ L^{(4)} &= PL'' + \frac{3}{2}P'L' + \frac{P''}{2}L. \end{aligned}$$

If w solves (2.1.1) then $v = Lw$ satisfies

$$\begin{aligned} v^{(4)} = L\left((P^2 - P'')w - 2P'w'\right) + 4L'(-Pw' - P'w) + 6L''(-Pw) \\ + 4\left(PL' + \frac{P'}{2}L\right)w' + \left(PL'' + \frac{3}{2}P'L' + \frac{P''}{2}L\right)w \end{aligned}$$

and so v solves (2.2.13) with

$$\begin{aligned} A &= 5P\frac{L''}{L} + \frac{5}{2}P'\frac{L'}{L} + \frac{1}{2}P'' - P^2 \\ &= 5P(Q'^2 + Q'') + \frac{5}{2}P'Q' + \frac{1}{2}P'' - P^2, \end{aligned}$$

which is also a polynomial.

2.3 Proof of Theorem 2.2.1

Let w and v be as in the hypotheses. Since w has infinitely many zeros, then by Theorem 2.1.1 we have (2.1.2).

Claim 2.1. We claim that w has simple zeros and

$$N\left(r, \frac{1}{w}\right) = N\left(r, \frac{w'}{w}\right).$$

This holds by the existence-uniqueness theorem [17].

From equation (2.1.1) and Lemma 2.2.1, we have (2.2.1).

From (2.2.1), (2.2.5) and by using Leibniz' rule, we get, for $1 \le j \le k$,

$$\begin{aligned} v^{(j)} &= \sum_{m=0}^{j} \binom{j}{m} L^{(m)} w^{(j-m)} \\ &= \sum_{m=0}^{j} \binom{j}{m} L^{(m)} \left(Q_{j-m} w + R_{j-m} w'\right) \qquad \text{(by Lemma 2.2.1)} \\ &= \left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} Q_{j-m} \right) w + \left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} R_{j-m} \right) w'. \end{aligned} \tag{2.3.1}$$

From (2.2.3) and (2.3.1), we find that

$$\begin{aligned} -Av &= -ALw \\ &= v^{(k)} + \sum_{1 \le j \le k-2} B_j v^{(j)} \\ &= \left(\sum_{m=0}^{k} \binom{k}{m} L^{(m)} Q_{k-m} \right) w + \left(\sum_{m=0}^{k} \binom{k}{m} L^{(m)} R_{k-m} \right) w' \\ &\quad + \sum_{j=1}^{k-2} B_j \left[\left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} Q_{j-m} \right) w + \left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} R_{j-m} \right) w' \right]. \end{aligned}$$

Now, we can write, for $1 \le j \le k-2$,

$$\sum_{m=0}^{j} \binom{j}{m} L^{(m)} Q_{j-m} = \sum_{m=0}^{k} \binom{j}{m} L^{(m)} Q_{j-m},$$

and

$$\sum_{m=0}^{j} \binom{j}{m} L^{(m)} R_{j-m} = \sum_{m=0}^{k} \binom{j}{m} L^{(m)} R_{j-m},$$

because $\binom{j}{m} = 0$ when $j < m \le k$.

So,

$$\begin{aligned}
-ALw &= \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}B_j\left(\sum_{m=0}^{k}\binom{j}{m}L^{(m)}Q_{j-m}\right)\right]w \\
&\quad + \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}B_j\left(\sum_{m=0}^{k}\binom{j}{m}L^{(m)}R_{j-m}\right)\right]w' \\
&= \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{m=0}^{k}\sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right]w \\
&\quad + \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}R_{k-m} + \sum_{m=0}^{k}\sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]w' \\
&= \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right]\right)w \\
&\quad + \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\right)w'.
\end{aligned}$$

Then we get

$$\begin{aligned}
0 &= \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right] + AL\right)w \\
&\quad + \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\right)w',
\end{aligned}$$

and so

$$\begin{aligned}
0 &= \sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right] + AL \\
&\quad + \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\right)\frac{w'}{w}. \qquad (2.3.2)
\end{aligned}$$

Now, we have three cases.

Case (I): If L is a constant, then w solves (2.2.3) and, by using Lemma 2.2.1, we get the following equations

$$\begin{cases} w^{(k)} + \sum_{1\le j\le k-2} B_j w^{(j)} + Aw = 0, \\ -w^{(k)} + Q_k w + R_k w' = 0. \end{cases}$$

By adding these two equations and using (2.2.1) again, we obtain

$$
\begin{aligned}
0 &= \sum_{1\le j\le k-2} B_j\left(Q_j w + R_j w'\right) + Aw + Q_k w + R_k w' \\
&= \left(A + Q_k + \sum_{1\le j\le k-2} B_j Q_j\right) w + \left(R_k + \sum_{1\le j\le k-2} B_j R_j\right) w'.
\end{aligned}
$$

Then, we get

$$
\left(R_k + \sum_{1\le j\le k-2} B_j R_j\right)\frac{w'}{w} + A + Q_k + \sum_{1\le j\le k-2} B_j Q_j = 0.
$$

Now, if $R_k + \sum_{1\le j\le k-2} B_j R_j \equiv 0$, then $A + Q_k + \sum_{1\le j\le k-2} B_j Q_j \equiv 0$ and so we have (2.2.6) and conclusion (a).

Suppose next that $R_k + \sum_{1\le j\le k-2} B_j R_j \not\equiv 0$; then

$$
w = 0 \Rightarrow \frac{w'}{w} = \infty \Rightarrow R_k + \sum_{1\le j\le k-2} B_j R_j = 0.
$$

Recall that all zeros of w are simple. We deduce that

$$
\begin{aligned}
N\left(r, \frac{1}{w}\right) &\le N\left(r, \frac{1}{R_k + \sum_{1\le j\le k-2} B_j R_j}\right) \\
&\le T\left(r, R_k + \sum_{1\le j\le k-2} B_j R_j\right) + O(1) \\
&= o\left(r^{(n+2)/2}\right).
\end{aligned}
$$

But this contradicts (2.1.2).

Case (II): Suppose that L is not constant and (2.2.7) holds. Then from (2.3.2) we get (2.2.8) and conclusion (b) of the theorem.

Suppose in addition that $B_1, B_2, \cdots, B_{k-2}$ are polynomials. Since $R_0 = 0$ and $R_1 = 1$ in (2.2.2) we see from (2.2.7) that L satisfies a homogenous linear differential equation of order $k-1$ with polynomial coefficients, and so L has finite order. Furthermore, (2.2.8) and the lemma of the logarithmic derivative now give

$T(r, A) = m(r, A) = O(\log r)$, so that A is a polynomial. This completes the discussion of Case (II) and the proof of part (c) of the theorem.

It remains only to show that the following case is impossible.

Case (III): Supposed that L is not constant and (2.2.7) does not hold, that is

$$\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m}+\sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\not\equiv 0.$$

Let $S = L'/L$. We first compare $N(r, S)$ with $N(r)$. Recall that all zeros of w are simple. On the other hand, v solves a linear differential equation of order k. So, zeros of v have multiplicities less than or equal to $k-1$.

So, $L = \frac{v}{w}$ has zeros with multiplicities at most $k-1$, and has simple poles. Then, we have

$$\begin{aligned} N(r,S) &= \overline{N}\left(r,\frac{1}{L}\right)+\overline{N}(r,L)\\ &\le N\left(r,\frac{1}{L}\right)+N(r,L)\\ &= N(r) \\ &\le (k-1)\overline{N}\left(r,\frac{1}{L}\right)+\overline{N}(r,L)\\ &\le (k-1)N(r,S). \end{aligned} \tag{2.3.3}$$

Claim 2.2. We claim that

$$T(r,S)\le o\left(r^{(n+2)/2}\right)$$

for r outside a set E of finite linear measure.

To prove this, we use the fact that $Q_0 = 1$ and $R_0 = 0$ in Lemma 2.2.1 to write (2.3.2) in the form

$$\begin{aligned} 0 &= \frac{L^{(k)}}{L}+A\\ &\quad+\sum_{m=0}^{k-1}\frac{L^{(m)}}{L}\left[\binom{k}{m}\left(Q_{k-m}+R_{k-m}\frac{w'}{w}\right)+\sum_{j=1}^{k-2}\binom{j}{m}B_j\left(Q_{j-m}+R_{j-m}\frac{w'}{w}\right)\right]. \end{aligned} \tag{2.3.4}$$

We can write, for $1 \leq m \leq k$,

$$\frac{L^{(m)}}{L} = S^m + U_{m-1}(S),$$

where $U_{m-1}(S)$ is a polynomial in $S, S', S'', \cdots, S^{(k)}$ with constant coefficients and total degree at most $m-1$. This follows immediately from Lemma 3.5 in [13, p.73], and is easily proved by induction.

This gives us an integer $q > 0$ such that (2.3.4) may be written as

$$S^k = \sum_{j=0}^{q} \left(a_j + b_j \frac{w'}{w} \right) S^{i_{0,j}} \left(S'\right)^{i_{1,j}} \left(S''\right)^{i_{2,j}} \cdots \left(S^{(k)}\right)^{i_{k,j}}, \tag{2.3.5}$$

where $i_{\mu,j} \geq 0$ are integers and

$$\sum_{\mu=0}^{k} i_{\mu,j} \leq k-1$$

for each j. Here a_j and b_j are polynomials in A, B_μ, Q_μ and R_μ, and so satisfy

$$\log^+ |a_j(z)| + \log^+ |b_j(z)| = o\left(|z|^{(n+2)/2}\right) \qquad \text{as } z \to \infty.$$

Let z be large with $|S(z)| \geq 1$.Then dividing (2.3.5) by S^{k-1} gives

$$\begin{aligned} |S(z)| &\leq \sum_{j=0}^{q} \left(|a_j(z)| + \left| b_j(z) \frac{w'(z)}{w(z)} \right| \right) \left| \frac{S'(z)}{S(z)} \right|^{i_{1,j}} \cdots \left| \frac{S^{(k)}(z)}{S(z)} \right|^{i_{k,j}} \\ &\leq \sum_{j=0}^{q} \left(|a_j(z)| + \left| b_j(z) \frac{w'(z)}{w(z)} \right| \right) \left(\max\left\{ 1, \left| \frac{S'(z)}{S(z)} \right| \right\} \right)^k \cdots \\ &\qquad \cdots \left(\max\left\{ 1, \left| \frac{S^{(k)}(z)}{S(z)} \right| \right\} \right)^k . \end{aligned}$$

Therefore,

$$\begin{aligned} m(r,S) &\leq \sum_{j=0}^{q} \left(m(r,a_j) + m(r,b_j) + m\left(r, \frac{w'}{w}\right) \right) \\ &\qquad + \sum_{j=1}^{k} km\left(r, \frac{S^{(j)}}{S}\right) + O(1) \\ &\leq o\left(r^{(n+2)/2}\right) + O(\log^+ T(r,S)) \end{aligned} \tag{2.3.6}$$

for r outside a set E of finite linear measure.

We also can use Clunie's lemma [18, p. 39] and the fact that

$$m(r, a_j) + m(r, b_j) = o(r^{(n+2)/2}) \qquad \text{as } r \to \infty$$

to obtain (2.3.6) for r outside a set E of finite linear measure directly, but the previous proof is self-contained and so might be easier for the reader.

Now, we use (2.3.3) and (2.3.6) to get

$$\begin{aligned} T(r, S) &= N(r, S) + m(r, S) \\ &\leq N(r) + m(r, S) \\ &\leq o\left(r^{(n+2)/2}\right) + O\left(\log^+ T(r, S)\right) \end{aligned}$$

and so

$$T(r, S) = o\left(r^{(n+2)/2}\right)$$

for r outside a set E of finite linear measure. This proves Claim 2.2.

Claim 2.3. We claim that

$$T(r, S) \leq o\left(r^{(n+2)/2}\right) \qquad \text{for all large } r.$$

To prove this, take any large r and $r_1 \in [r, 2r]$ with $r_1 \notin E$; then

$$\begin{aligned} T(r, S) &\leq T(r_1, S) \\ &= o\left(r_1^{(n+2)/2}\right) \\ &\leq o\left((2r)^{(n+2)/2}\right) \\ &\leq 2^{(n+2)/2} o\left(r^{(n+2)/2}\right) \\ &\leq o\left(r^{(n+2)/2}\right). \end{aligned}$$

Thus,

$$T(r,S) \le o\left(r^{(n+2)/2}\right) \qquad \text{for all large } r.$$

This proves Claim 2.3.

Also, Claim 2.3 can be proved by using Claim 2.2 and [18, Lemma 1.1.1], but we used a self-contained method for the reader.

Now, dividing (2.3.2) by L shows that if $\frac{w'}{w}$ has a pole at z then either

$$A_2 = \sum_{m=0}^{k}\left[\binom{k}{m}\frac{L^{(m)}}{L}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_j\frac{L^{(m)}}{L}R_{j-m}\right] = 0$$

at z or

$$A_1 = \sum_{m=0}^{k}\left[\binom{k}{m}\frac{L^{(m)}}{L}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_j\frac{L^{(m)}}{L}Q_{j-m}\right] + A = \infty$$

at z.

In view of Claim 2.3, we can write (2.3.2) as

$$A_1 + A_2\frac{w'}{w} = 0, \tag{2.3.7}$$

where $T(r,A_j) = o\left(r^{(n+2)/2}\right)$, $j = 1,2$ and $A_2 \not\equiv 0$ by the assumption of Case (III).

Now, by using Claim 2.1 and (2.3.7), we get

$$\begin{aligned} N\left(r,\frac{1}{w}\right) &= N\left(r,\frac{w'}{w}\right) \\ &\le N\left(r,\frac{1}{A_2}\right) + N(r,A_1) \\ &\le T(r,A_2) + T(r,A_1) + O(1) \\ &\equiv o\left(r^{(n+2)/2}\right). \end{aligned}$$

Hence,

$$\lim_{r\to\infty}\frac{N\left(r,\frac{1}{w}\right)}{r^{(n+2)/2}} = 0.$$

But this contradicts (2.1.2). Therefore, Case (III) cannot occur. This completes the proof of Theorem 2.2.1 □

2.4 Proof of Theorem 2.2.2

Assume the hypotheses of Theorem 2.2.2. Taking $k = 2$ in Theorem 2.2.1, we have two cases to consider.

Case (a): L is a constant and

$$A = -Q_2 = -(-P) = P \qquad \text{(by using (2.2.6) and Lemma 2.2.1).}$$

Case (b): L is not constant, but

$$\begin{aligned} 0 &= \sum_{m=0}^{2} \binom{2}{m} L^{(m)} R_{2-m} \\ &= LR_2 + 2L'R_1 + L''R_0 \\ &= 2L' \qquad \text{(by using Lemma 2.2.1).} \end{aligned}$$

But this implies that L is constant, a contradiction. □

2.5 Proof of Theorem 2.2.4

Assume the hypotheses of Theorem 2.2.4. Taking $k = 3$ and $B_1 = B$ in Theorem 2.2.1, we have two cases to consider.

Case (a): L is a constant and

$$A = -Q_3 - B_1 Q_1 = -Q_3 = P' \qquad \text{(by using (2.2.6) and Lemma 2.2.1).}$$

But, since w solves (2.2.11) we have

$$w''' + Bw' + Aw = 0$$

and so

$$w''' + Bw' + P'w = 0. \tag{2.5.1}$$

Also, by differentiating (2.1.1) we get

$$w''' + P'w + Pw' = 0. \tag{2.5.2}$$

Now, (2.5.1) and (2.5.2) give $P = B$.

Case (b): L is not constant and

$$\begin{aligned}
0 &= \sum_{m=0}^{3} \left[\binom{3}{m} L^{(m)} R_{3-m} + \binom{1}{m} B_1 L^{(m)} R_{1-m} \right] \\
&= LR_3 + B_1 L + 3L'R_2 + 0 + 3L''R_1 + 0 + L'''R_0 + 0 \\
&= -PL + BL + 3L''
\end{aligned}$$

and so

$$L'' = \frac{1}{3}PL - \frac{1}{3}BL. \tag{2.5.3}$$

Differentiating (2.5.3) we get

$$L''' = \frac{1}{3}P'L + \frac{1}{3}PL' - \frac{1}{3}B'L - \frac{1}{3}BL'. \tag{2.5.4}$$

Also, we have

$$\begin{aligned}
-A &= \sum_{m=0}^{3} \left[\binom{3}{m} \frac{L^{(m)}}{L} Q_{3-m} + \binom{1}{m} B_1 \frac{L^{(m)}}{L} Q_{1-m} \right] \\
&= [Q_3 + 0] + \left[3\frac{L'}{L}Q_2 + B_1\frac{L'}{L} \right] + \left[3\frac{L''}{L}Q_1 + 0 \right] + \left[\frac{L'''}{L}Q_0 \right] \\
&= -P' - 3P\frac{L'}{L} + B\frac{L'}{L} + \frac{L'''}{L}.
\end{aligned}$$

Thus,

$$A = P' + 3P\frac{L'}{L} - B\frac{L'}{L} - \frac{L'''}{L}.$$

Using (2.5.4) we get

$$A = P' + 3P\frac{L'}{L} - B\frac{L'}{L} - \frac{1}{3}P' - \frac{1}{3}P\frac{L'}{L} + \frac{1}{3}B' + \frac{1}{3}B\frac{L'}{L}$$

and so

$$A = \frac{8}{3}P\frac{L'}{L} + \frac{2}{3}P' + \frac{1}{3}B' - \frac{2}{3}B\frac{L'}{L}.$$

□

2.6 Proof of Theorem 2.2.3

We will deduce Theorem 2.2.3 from Theorem 2.2.4, since (2.2.10) is just (2.2.11) with $B = 0$.

Assume the hypotheses of Theorem 2.2.3. Then v and w satisfy conclusion (a) or (b) of Theorem 2.2.4 with $B = 0$. But conclusion (a) gives $P = B = 0$, which is impossible, and so we must have conclusion (b). Since $B = 0$ this gives $L'' = \frac{P}{3}L$ and $A = \frac{2}{3}P' + \frac{8}{3}P\frac{L'}{L}$ as asserted, and A is a polynomial by part (c) of Theorem 2.2.1. □

2.7 Proof of Theorem 2.2.5

Assume the hypotheses of Theorem 2.2.5. Taking $k = 4$ and $B_1 = B_2 = 0$ in Theorem 2.2.1, we have two cases to consider.

Case (a): L is a constant and

$$A = -Q_4 = -P^2 + P'' \qquad \text{by using (2.2.2).}$$

But, differentiating (2.1.1) two times gives

$$\begin{aligned} 0 &= w^{(4)} + P''w + 2P'w' + Pw'' \\ &= w^{(4)} + (P'' - P^2)w + 2P'w' \end{aligned}$$

Since we also have $w^{(4)} + Aw = 0$, this gives

$$0 = 2P'w'$$

and so P must be constant and so must A.

Case (b): L is non-constant and L satisfies, using (2.2.2) and (2.2.7),

$$\begin{aligned} 0 &= \sum_{m=0}^{4} \binom{4}{m} L^{(m)} R_{4-m} \\ &= LR_4 + 4L'R_3 + 6L''R_2 + 4L'''R_1 + L^{(4)}R_0 \\ &= -2LP' - 4L'P + 0 + 4L''' + 0 \\ &= 4L''' - 4L'P - 2LP' \end{aligned}$$

and so

$$L''' = L'P + \frac{1}{2}LP'. \tag{2.7.1}$$

Since this is a linear differential equation and P is a polynomial it follows that L is an entire function.

We write (2.7.1) in the form

$$P' + 2\frac{L'}{L}P = 2\frac{L'''}{L}. \tag{2.7.2}$$

It is then elementary to show that

$$P = 2\frac{L''}{L} - \left(\frac{L'}{L}\right)^2 + \frac{c}{L^2} \tag{2.7.3}$$

with c a constant. Also L is not a polynomial, since L is non-constant and $P(\infty) \neq 0$.

Claim 2.4. We claim that $\rho(L) = (n+2)/2$ and

$$\liminf_{r\to\infty} \frac{T(r,L)}{r^{(n+2)/2}} > 0. \tag{2.7.4}$$

To prove this we use Wiman-Valiron theory [14]. Take z_0 such that $|z_0| = r$, $|L(z_0)| = M(r,L)$ and $r \notin E$, where E is the exceptional set of finite logarithmic measure. Then,

$$\frac{L'}{L}(z_0) \sim \frac{\nu(r,L)}{z_0},$$

where $\nu(r, L)$ is the central index, and

$$\frac{L''}{L}(z_0) \sim \frac{\nu(r,L)^2}{z_0^2}, \qquad \frac{c}{L(z_0)^2} = \frac{o(1)}{z_0^2}.$$

Now, we have, for some $c_1 \neq 0$,

$$\begin{aligned} c_1 z_0^n \sim P(z_0) &= 2\frac{\nu(r,L)^2}{z_0^2}(1+o(1)) - \frac{\nu(r,L)^2}{z_0^2}(1+o(1)) + \frac{o(1)}{z_0^2} \\ &= (1+o(1))\frac{\nu(r,L)^2}{z_0^2}. \end{aligned}$$

So,

$$\nu(r,L)^2 \sim c_1 z_0^{(n+2)}$$

and so using c_j to denote non-zero constants,

$$\nu(r,L) \sim c_2 r^{(n+2)/2} \qquad (r \notin E).$$

Therefore,

$$\nu(r,L) \sim c_2 r^{(n+2)/2} \qquad (\text{ for all } r \to \infty),$$

since we may choose $r' < r < r''$ with $r' \sim r \sim r''$ and $r', r'' \notin E$. So the maximum term $\mu(r, L)$ satisfies [14]

$$\begin{aligned} \log \mu(r,L) &= c_3 + \int_{c_4}^{r} \nu(t,L)\frac{dt}{t} \\ &\sim c_5 r^{(n+2)/2}. \end{aligned}$$

Also,

$$\mu(r,L) \leq M(r,L) \leq 2\mu(2r,L).$$

This gives

$$\begin{aligned} c_5 r^{(n+2)/2} \sim \log \mu(r,L) &\leq \log M(r,L) \\ &\leq \log \mu(2r,L) + \log 2 \\ &\leq c_6 r^{(n+2)/2}. \end{aligned}$$

Similarly, we have

$$T(r, L) \leq \log M(r, L) \leq 3T(2r, L).$$

So

$$T(r, L) \leq c_6 r^{(n+2)/2}$$

and

$$T(r, L) \geq \frac{1}{3} \log M(\frac{r}{2}, L) \geq c_7 r^{(n+2)/2}.$$

This leads to $\rho(L) = (n+2)/2$ and (2.7.4) which completes the proof of Claim 2.4.

Claim 2.5. We claim that $c = 0$ in (2.7.3).

If this is not the case then (2.7.3) and the lemma of the logarithmic derivative give

$$m\left(r, \frac{1}{L}\right) = O(\log r).$$

But then

$$\begin{aligned} T(r, L) &= T\left(r, \frac{1}{L}\right) + O(1) \\ &\leq m\left(r, \frac{1}{L}\right) + N\left(r, \frac{1}{L}\right) + O(1) \\ &= o(r^{(n+2)/2}), \end{aligned}$$

which contradicts (2.7.4).

Hence, $c = 0$ in (2.7.3) as asserted and this completes the proof of Claim 2.5.

Now L is entire, and we write, locally, $L = Y^2$ and $S = \frac{L'}{L} = 2\frac{Y'}{Y} = 2y$.

Then (2.7.3) gives

$$\begin{aligned} P &= 2(S^2 + S') - S^2 \\ &= S^2 + 2S' \\ &= 4(y^2 + y') \end{aligned}$$

and so

$$Y'' = \frac{P}{4}Y.$$

Hence, Y is an entire function. But

$$N\left(r, \frac{1}{Y}\right) = \frac{1}{2}N\left(r, \frac{1}{L}\right) = o\left(r^{(n+2)/2}\right)$$

and so Y has finitely many zeros, by Theorem 2.1.1. Hence y and S are rational functions and A satisfies, using (2.2.2) and (2.2.8),

$$\begin{aligned}
-A &= \sum_{m=0}^{4} \binom{4}{m} \frac{L^{(m)}}{L} Q_{4-m} \\
&= Q_4 + 4\frac{L'}{L}Q_3 + 6\frac{L''}{L}Q_2 + 4\frac{L'''}{L}Q_1 + \frac{L^{(4)}}{L}Q_0 \\
&= P^2 - P'' - 4P'\frac{L'}{L} - 6P\frac{L''}{L} + \frac{L^{(4)}}{L}
\end{aligned}$$

and so

$$A = P'' - P^2 + 4P'\frac{L'}{L} + 6P\frac{L''}{L} - \frac{L^{(4)}}{L}. \tag{2.7.5}$$

Differentiating (2.7.1) and dividing by L gives

$$\frac{L^{(4)}}{L} = P\frac{L''}{L} + \frac{3}{2}P'\frac{L'}{L} + \frac{1}{2}P''.$$

By substituting this in (2.7.5), we get

$$A = P'' - P^2 + 4P'\frac{L'}{L} + 6P\frac{L''}{L} - P\frac{L''}{L} - \frac{3}{2}P'\frac{L'}{L} - \frac{1}{2}P''$$

and so

$$A = 5P\frac{L''}{L} + \frac{5}{2}P'\frac{L'}{L} + \frac{1}{2}P'' - P^2.$$

Then we can write

$$\begin{aligned}
A &= 5P(S' + S^2) + \frac{5}{2}P'S + \frac{1}{2}P'' - P^2 \\
&= 5P\left(\frac{P - S^2}{2} + S^2\right) + \frac{5}{2}P'S + \frac{1}{2}P'' - P^2 \\
&= \frac{5}{2}P^2 - \frac{5}{2}PS^2 + 5PS^2 + \frac{5}{2}P'S + \frac{1}{2}P'' - P^2 \\
&= \frac{3}{2}P^2 + \frac{5}{2}PS^2 + \frac{1}{2}P'' + \frac{5}{2}P'S.
\end{aligned}$$

Finally, A is a rational function and so a polynomial. □

Example 2.7.1. Suppose that $w'' = -Pw$ and P is a constant. Then

$$w^{(4)} = -Pw'' = P^2 w,$$
$$w^{(6)} = P^2 w'' = -P^3 w,$$
$$w^{(8)} = -P^3 w'' = P^4 w.$$

So, $v^{(k)} + Av = 0$, $v = w$, is possible for all even k, where

$$A = (-1)^{1+k/2} P^{k/2}$$

is also a constant.

CHAPTER 3

The case of transcendental coefficients

In this chapter, we consider a pair of homogeneous linear differential equations with transcendental coefficients. We apply Nevanlinna theory to determine when solutions of these equations can have the same zeros or (mostly) the same zeros. The results of this chapter were published in [3].

3.1 Introduction

In particular, we will take the equation

$$w'' + Pw = 0, \tag{3.1.1}$$

where P is a transcendental entire function. It is well known that every solution of (3.1.1) is an entire function.

This chapter continues our study from Chapter 2 of the question of when two linear differential equations in the complex domain can have solutions with (mostly) the same zeros. When one of the equations has order 2 and a polynomial coefficient, this question was investigated in Chapter 2, and in this chapter we consider the transcendental case.

We use the same notation as in Chapter 2, but several aspects of the proofs will be different.

3.2 A theorem for the general case

To state our main result we need the following lemma.

Lemma 3.2.1. *Suppose $w'' = -Pw$ where P is an entire function with order of growth $\rho(P) < \infty$. Then for $j \geq 0$, there exist entire functions Q_j and R_j of finite order such that*

$$w^{(j)} = Q_j w + R_j w'. \tag{3.2.1}$$

Proof. This is elementary, and just uses induction. For details see Chapter 2, but we recall for convenience later that

$$\begin{aligned} j = 0 &\Rightarrow Q_0 = 1, \quad R_0 = 0; \\ j = 1 &\Rightarrow Q_1 = 0, \quad R_1 = 1; \\ j = 2 &\Rightarrow Q_2 = -P, \quad R_2 = 0; \\ j = 3 &\Rightarrow Q_3 = -P', \quad R_3 = -P; \\ j = 4 &\Rightarrow Q_4 = P^2 - P'', \quad R_4 = -2P'. \end{aligned} \tag{3.2.2}$$

□

The following theorem is a counterpart of Theorem 2.2.1.

Theorem 3.2.1. *Let P be a transcendental entire function with $\rho(P) < \infty$. Let $w \not\equiv 0$ be a solution of* (3.1.1)*. Assume that the zeros of w have infinite exponent of convergence, i.e.*

$$\lambda(w) = \limsup_{r\to\infty} \frac{\log^+ N\left(r, \frac{1}{w}\right)}{\log r} = \infty. \tag{3.2.3}$$

Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation

$$v^{(k)} + \sum_{1\leq j\leq k-2} B_j v^{(j)} + Av = 0, \quad k \geq 2, \tag{3.2.4}$$

such that A and B_j are entire functions and $\rho(A) < \infty$ and $\rho(B_j) < \infty$. Assume that $N(r)$ has finite order, where $N(r)$ counts both zeros and poles of $\frac{v}{w}$. Let

$$v = Lw. \tag{3.2.5}$$

Then one of the following two possibilities holds.

(a) *L is constant, and*

$$A = -Q_k - \sum_{j=1}^{k-2} B_j Q_j, \tag{3.2.6}$$

where Q_k and Q_j are defined by Lemma 3.2.1.

(b) *L is not constant, but L satisfies*

$$\sum_{m=0}^{k} \left[\binom{k}{m} L^{(m)} R_{k-m} + \sum_{j=1}^{k-2} \binom{j}{m} B_j L^{(m)} R_{j-m} \right] = 0, \tag{3.2.7}$$

and A satisfies

$$A = -\left(\sum_{m=0}^{k} \left[\binom{k}{m} \frac{L^{(m)}}{L} Q_{k-m} + \sum_{j=1}^{k-2} \binom{j}{m} B_j \frac{L^{(m)}}{L} Q_{j-m} \right] \right), \tag{3.2.8}$$

where R_{k-m}, R_{j-m}, Q_{k-m} and Q_{j-m} are also defined by Lemma 3.2.1.

The conclusions of Theorem 3.2.1 are the same as for Theorem 2.2.1, and the proof will follow a similar structure, but will require different estimates. The full proofs are included for the convenience of the reader.

We recall that there is no loss of generality in assuming that there is no term in w' in (3.1.1) and that there is no term B_{k-1} in (3.2.4). This can be showed by changing variables as in Chapter 2.

We also note that the hypothesis (3.2.3) is not redundant: to see this, let $w = e^B$ and $v = e^C$ where B and C are any entire functions of finite order. Then w and v solve

$$w'' + Pw = 0, \qquad v^{(k)} + Qv = 0,$$

where $P = -(B'' + B'^2)$ and $Q = -v^{(k)}/v = -(C')^k + \cdots$ are entire of finite order, but since B and C are arbitrary, there is no relationship between P and Q.

From Theorem 3.2.1 we will deduce the following results for the cases $k = 2$, $k = 3$ and $k = 4$, which are counterparts of results from Chapter 2.

Theorem 3.2.2. *Let P be a transcendental entire function and $\rho(P) < \infty$. Let $w \not\equiv 0$ be a solution of* (3.1.1). *Assume that the zeros of w have infinite exponent of convergence, i.e.* (3.2.3) *holds. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v'' + Av = 0, \tag{3.2.9}$$

such that A is an entire function and $\rho(A) < \infty$. Assume that $N(r)$ has finite order, where $N(r)$ counts zeros and poles of $\frac{v}{w}$. Then $L = \frac{v}{w}$ is a constant and $A = P$.

Example 3.2.1. Obviously we may take $v = w$ and $A = P$.

Example 3.2.2. We give an example to show that $\rho(A) < \infty$ is necessary in Theorem 3.2.2. To show this put $v = we^g$ where g is an entire function. Then we have

$$\begin{aligned}\frac{v''}{v} &= \frac{w''}{w} + 2g'\frac{w'}{w} + g'' + g'^2 \\ &= -P + g'' + g'^2 + 2g'\frac{w'}{w} \\ &= -A.\end{aligned}$$

Now, if we put $g' = w$ then we get

$$-A = -P + w' + w^2 + 2w'.$$

So A is entire, $\frac{v}{w}$ is non-constant and v has the same zeros as w.

Theorem 3.2.3. *Let P be a transcendental entire function and $\rho(P) < \infty$. Let $w \not\equiv 0$ be a solution of* (3.1.1). *Assume that the zeros of w have infinite exponent of convergence, i.e.* (3.2.3) *holds. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v''' + Bv' + Av = 0, \tag{3.2.10}$$

such that A and B are entire functions with $\rho(A) < \infty$ and $\rho(B) < \infty$. Assume that $N(r)$ has finite order, where $N(r)$ counts zeros and poles of $\frac{v}{w}$. Then $v = Lw$ and one of the following holds.

(a) *L is constant and $A = P'$, $B = P$.*

(b) *L is non-constant and* $L'' = \frac{1}{3}PL - \frac{1}{3}BL$ *and*

$$A = \frac{8}{3}P\frac{L'}{L} + \frac{2}{3}P' + \frac{1}{3}B' - \frac{2}{3}B\frac{L'}{L}.$$

Remark. If A or B is a polynomial in Theorem 3.2.3 then case (a) cannot hold since P is transcendental.

Example 3.2.3. To show that case (b) can occur in Theorem 3.2.3, we can use Example 2.2.3 from Chapter 2 with $B = 0$ and Q a transcendental entire function of finite order.

Take $L = e^Q$ and set

$$\frac{P}{3} = \frac{L''}{L} = Q'^2 + Q''.$$

Then

$$L'' = \frac{P}{3}L, \qquad L''' = \frac{P}{3}L' + \frac{P'}{3}L.$$

If w solves (3.1.1) then $v = Lw$ satisfies

$$v''' = L(-P'w - Pw') + 3L'(-Pw) + 3\frac{P}{3}Lw' + \left(\frac{P}{3}L' + \frac{P'}{3}L\right)w$$

and so v solves $v''' + Av = 0$ with

$$\begin{aligned} A &= \frac{2}{3}P' + \frac{8}{3}P\frac{L'}{L} \\ &= \frac{2}{3}P' + \frac{8}{3}PQ'. \end{aligned}$$

Note that P is transcendental since Q is transcendental and in fact writing

$$Q' = \frac{P}{3Q'} - \frac{Q''}{Q'}$$

shows that

$$m(r, Q') \leq m(r, P) + S(r, Q'),$$

so that

$$\rho(P) = \rho(Q') = \rho(Q).$$

Also, P and A have finite order.

Theorem 3.2.4. *Let P be a transcendental entire function and $\rho(P) < \infty$. Let $w \not\equiv 0$ be a solution of* (3.1.1). *Assume that the zeros of w have infinite exponent of convergence, i.e.* (3.2.3) *holds. Suppose that we have an entire solution $v \not\equiv 0$ of the differential equation*

$$v^{(4)} + Av = 0, \qquad (3.2.11)$$

such that A is an entire function with $\rho(A) < \infty$. Assume that $N(r)$ has finite order, where $N(r)$ counts zeros and poles of $\frac{v}{w}$. Then $v = Lw$ where L is non-constant,

$$L''' = L'P + \frac{1}{2}LP', \qquad A = 5P\frac{L''}{L} + \frac{5}{2}P'\frac{L'}{L} + \frac{1}{2}P'' - P^2 \qquad (3.2.12)$$

and $L = y_1 y_2$ where y_1, y_2 are solutions of

$$y'' - \frac{P}{4}y = 0.$$

In particular, if v and w have the same zeros with the same multiplicities, then L is entire with no zeros and so are y_1 and y_2. In addition, when v and w have the same zeros:

(i) *if y_1, y_2 are linearly dependent then $L = e^{2C}$ with C an entire function, $P = 4(C'' + C'^2)$ and A is a differential polynomial in C';*

(ii) *if y_1, y_2 are linearly independent then $L = e^{C}$ with C an entire function, $P = 2C'' + C'^2 + k^2 e^{-2C}$ where k is a constant, and A is a differential polynomial in e^{-C} and C'.*

Example 3.2.4. To show that (3.2.12) can occur, we modify Example 2.2.6 as follows. Take $L = Y^2 = e^Q$ where Q is a transcendental entire function of finite order and set

$$Q' = S = 2y = 2\frac{Y'}{Y}, \qquad P = S^2 + 2S' = 4(y^2 + y')$$

so that P is an entire function of finite order, and the same argument as in Example

3.2.3 shows that P is transcendental. Then

$$\begin{aligned} L' &= 2YY', \\ L'' &= 2Y'^2 + 2YY'' = 2Y'^2 + \frac{P}{2}L, \\ L''' &= PL' + \frac{P'}{2}L, \\ L^{(4)} &= PL'' + \frac{3}{2}P'L' + \frac{P''}{2}L. \end{aligned}$$

If w solves (3.1.1) then $v = Lw$ satisfies

$$\begin{aligned} v^{(4)} = L\left((P^2 - P'')w - 2P'w'\right) + 4L'(-Pw' - P'w) + 6L''(-Pw) \\ + 4\left(PL' + \frac{P'}{2}L\right)w' + \left(PL'' + \frac{3}{2}P'L' + \frac{P''}{2}L\right)w \end{aligned}$$

and so v solves (3.2.11) with

$$\begin{aligned} A &= 5P\frac{L''}{L} + \frac{5}{2}P'\frac{L'}{L} + \frac{1}{2}P'' - P^2 \\ &= 5P(Q'^2 + Q'') + \frac{5}{2}P'Q' + \frac{1}{2}P'' - P^2, \end{aligned}$$

which is also entire of finite order.

3.3 Proof of Theorem 3.2.1

In this proof we use $M_1, M_2, \ldots$ to denote positive constants.

Claim 3.1. We claim that w has simple zeros and

$$N\left(r, \frac{1}{w}\right) = N\left(r, \frac{w'}{w}\right).$$

This holds by the existence-uniqueness theorem [17].

Lemma 3.3.1. *There exists $N > 0$ such that $m(r, \frac{w'}{w}) < r^N$ as $r \to \infty$.*

Proof. We can get this as follows. Use $N_1, N_2, \ldots$ to denote positive constants. Since P has finite order,

$$M(r,P) \leq \exp(r^{N_1}) \qquad \text{as } r \to \infty.$$

So for r outside a set E_0 of finite logarithmic measure we get from Wiman-Valiron theory

$$\left(\frac{\nu(r,w)}{r}\right)^2 \leq (1+o(1))M(r,P),$$

where $\nu(r,w)$ denotes the central index, and so

$$\nu(r,w) \leq \exp(r^{N_2}) \qquad\qquad (r \to \infty,\ r \notin E_0).$$

Hence,

$$\nu(r,w) \leq \exp(r^{N_3}) \qquad \text{as } r \to \infty, \text{ for all } r.$$

So the maximum term $\mu(r,w)$ satisfies

$$\log \mu(r,w) \leq \exp(r^{N_4})$$

and so

$$T(r,w) \leq \log M(r,w) \leq \exp(r^{N_5}).$$

Now we can use the lemma of logarithmic derivative in the form given in Lemma 2.3 in [13, p. 36] with $R = 2r$ to get

$$\begin{aligned} m(r, \frac{w'}{w}) &\leq O(\log^+ T(R,w) + \log r) \\ &\leq r^{N_6}. \end{aligned}$$

This completes the proof of this lemma. □

From equation (3.1.1) and Lemma 3.2.1, we have (3.2.1).

We will now deduce an equation (3.3.2) connecting L and A and the B_j, in the same way as we deduced (2.3.2) in Chapter 2.

From (3.2.1), (3.2.5) and by using Leibniz' rule, we get, for $1 \leq j \leq k$,

$$
\begin{aligned}
v^{(j)} &= \sum_{m=0}^{j} \binom{j}{m} L^{(m)} w^{(j-m)} \\
&= \sum_{m=0}^{j} \binom{j}{m} L^{(m)} \left(Q_{j-m} w + R_{j-m} w'\right) \qquad \text{(by Lemma 3.2.1)} \\
&= \left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} Q_{j-m} \right) w + \left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} R_{j-m} \right) w'. \qquad (3.3.1)
\end{aligned}
$$

From (3.2.4) and (3.3.1), we find that

$$
\begin{aligned}
-Av &= -ALw \\
&= v^{(k)} + \sum_{1 \leq j \leq k-2} B_j v^{(j)} \\
&= \left(\sum_{m=0}^{k} \binom{k}{m} L^{(m)} Q_{k-m} \right) w + \left(\sum_{m=0}^{k} \binom{k}{m} L^{(m)} R_{k-m} \right) w' \\
&\quad + \sum_{j=1}^{k-2} B_j \left[\left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} Q_{j-m} \right) w + \left(\sum_{m=0}^{j} \binom{j}{m} L^{(m)} R_{j-m} \right) w' \right].
\end{aligned}
$$

Now, we can write, for $1 \leq j \leq k-2$,

$$
\sum_{m=0}^{j} \binom{j}{m} L^{(m)} Q_{j-m} = \sum_{m=0}^{k} \binom{j}{m} L^{(m)} Q_{j-m},
$$

and

$$
\sum_{m=0}^{j} \binom{j}{m} L^{(m)} R_{j-m} = \sum_{m=0}^{k} \binom{j}{m} L^{(m)} R_{j-m},
$$

because $\binom{j}{m} = 0$ when $j < m \leq k$.

So,

$$
\begin{aligned}
-ALw &= \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}B_j\left(\sum_{m=0}^{k}\binom{j}{m}L^{(m)}Q_{j-m}\right)\right]w \\
&\quad + \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}B_j\left(\sum_{m=0}^{k}\binom{j}{m}L^{(m)}R_{j-m}\right)\right]w' \\
&= \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{m=0}^{k}\sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right]w \\
&\quad + \left[\sum_{m=0}^{k}\binom{k}{m}L^{(m)}R_{k-m} + \sum_{m=0}^{k}\sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]w' \\
&= \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right]\right)w \\
&\quad + \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\right)w'.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
0 &= \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right] + AL\right)w \\
&\quad + \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\right)w',
\end{aligned}
$$

and so

$$
\begin{aligned}
0 &= \sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}Q_{j-m}\right] + AL \\
&\quad + \left(\sum_{m=0}^{k}\left[\binom{k}{m}L^{(m)}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_jL^{(m)}R_{j-m}\right]\right)\frac{w'}{w}. \qquad (3.3.2)
\end{aligned}
$$

Now, we have three cases.

Case (I): If L is a constant, then w solves (3.2.4) and, by using Lemma 3.2.1, we get the following equations

$$
\begin{cases}
w^{(k)} + \sum_{1\le j\le k-2} B_j w^{(j)} + Aw = 0, \\
-w^{(k)} + Q_k w + R_k w' = 0.
\end{cases}
$$

By adding these two equations and using (3.2.1) again, we obtain

$$\begin{aligned} 0 &= \sum_{1\le j\le k-2} B_j\left(Q_j w + R_j w'\right) + Aw + Q_k w + R_k w' \\ &= \left(A + Q_k + \sum_{1\le j\le k-2} B_j Q_j\right) w + \left(R_k + \sum_{1\le j\le k-2} B_j R_j\right) w'. \end{aligned}$$

Then, we get

$$\left(R_k + \sum_{1\le j\le k-2} B_j R_j\right)\frac{w'}{w} + A + Q_k + \sum_{1\le j\le k-2} B_j Q_j = 0.$$

Now, if $R_k + \sum_{1\le j\le k-2} B_j R_j \equiv 0$, then $A + Q_k + \sum_{1\le j\le k-2} B_j Q_j \equiv 0$ and so we have (3.2.6) and conclusion (a).

Suppose next that $R_k + \sum_{1\le j\le k-2} B_j R_j \not\equiv 0$; then

$$w = 0 \Rightarrow \frac{w'}{w} = \infty \Rightarrow R_k + \sum_{1\le j\le k-2} B_j R_j = 0.$$

Recall that all zeros of w are simple. We deduce that

$$\begin{aligned} N\left(r, \frac{1}{w}\right) &\le N\left(r, \frac{1}{R_k + \sum_{1\le j\le k-2} B_j R_j}\right) \\ &\le T\left(r, R_k + \sum_{1\le j\le k-2} B_j R_j\right) + O(1) \\ &= O\left(r^{M_1}\right). \end{aligned}$$

But this contradicts (3.2.3).

Case (II): Suppose that L is not constant and (3.2.7) holds. Then from (3.3.2) we get (3.2.8) and conclusion (b) of the theorem.

It remains only to show that the following case is impossible.

Case (III): Suppose that L is not constant and (3.2.7) does not hold, that is

$$\sum_{m=0}^{k}\left[\binom{k}{m} L^{(m)} R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m} B_j L^{(m)} R_{j-m}\right] \not\equiv 0.$$

Let $S = L'/L$. We first compare $N(r, S)$ with $N(r)$. Recall that all zeros of w are simple. On the other hand, v solves a differential equation of order k. So, zeros of v have multiplicities less than or equal to $k-1$.

So, $L = \frac{v}{w}$ has zeros with multiplicities at most $k-1$, and has simple poles. Then, we have

$$\begin{aligned} N(r,S) &= \overline{N}\left(r, \frac{1}{L}\right) + \overline{N}(r,L) \\ &\leq N\left(r, \frac{1}{L}\right) + N(r,L) \\ &= N(r) \qquad\qquad (3.3.3) \\ &\leq (k-1)\overline{N}\left(r, \frac{1}{L}\right) + \overline{N}(r,L) \\ &\leq (k-1)N(r,S). \end{aligned}$$

Claim 3.2. We claim that

$$T(r,S) \leq O\left(r^{M_2}\right)$$

for r outside a set E of finite linear measure.

To prove this, we proceed as in Chapter 2 and use the fact that $Q_0 = 1$ and $R_0 = 0$ in Lemma 3.2.1 to write (3.3.2) in the form

$$\begin{aligned} 0 &= \frac{L^{(k)}}{L} + A \\ &+ \sum_{m=0}^{k-1} \frac{L^{(m)}}{L} \left[\binom{k}{m} \left(Q_{k-m} + R_{k-m}\frac{w'}{w} \right) + \sum_{j=1}^{k-2} \binom{j}{m} B_j \left(Q_{j-m} + R_{j-m}\frac{w'}{w} \right) \right]. \end{aligned} \qquad (3.3.4)$$

We can write, for $1 \leq m \leq k$,

$$\frac{L^{(m)}}{L} = S^m + U_{m-1}(S),$$

where $U_{m-1}(S)$ is a polynomial in $S, S', S'', \ldots, S^{(k)}$ with constant coefficients and total degree at most $m-1$. This follows immediately from Lemma 3.5 in [13], and is easily proved by induction.

This gives us an integer $q > 0$ such that (3.3.4) may be written as

$$S^k = \sum_{j=0}^{q} \left(a_j + b_j \frac{w'}{w} \right) S^{i_{0,j}} \left(S'\right)^{i_{1,j}} \left(S''\right)^{i_{2,j}} \cdots \left(S^{(k)}\right)^{i_{k,j}}, \qquad (3.3.5)$$

where $i_{\mu,j} \geq 0$ are integers and

$$\sum_{\mu=0}^{k} i_{\mu,j} \leq k - 1$$

for each j.

The next steps are different to Chapter 2.

Lemma 3.3.1 gives $m(r, \frac{w'}{w}) < r^{M_3}$. Also a_j and b_j are polynomials in A, B_μ, Q_μ and R_μ, and so satisfy

$$m(r, a_j) + m(r, b_j) = O(r^{M_4}) \qquad \text{as } r \to \infty.$$

By Clunie's lemma [18, p. 39] we obtain

$$m(r, S) \leq O\left(r^{M_4}\right) + O(\log^+ T(r, S)) \qquad (3.3.6)$$

for r outside a set E of finite linear measure. The detailed method that was used in Chapter 2 can also be used to obtain (3.3.6).

Now, we use (3.3.3) and (3.3.6) to get

$$\begin{aligned} T(r, S) &= N(r, S) + m(r, S) \\ &\leq N(r) + m(r, S) \\ &\leq O\left(r^{M_5}\right) + O(\log^+ T(r, S)) \end{aligned}$$

and so

$$T(r, S) = O\left(r^{M_6}\right)$$

for r outside a set E of finite linear measure. This proves Claim 3.2.

Claim 3.3. We claim that

$$T(r,S) \le O\left(r^{M_6}\right), \qquad \text{for all large } r.$$

This follows from Claim 3.2 and [18, Lemma 1.1.1]. We can also use the detailed method that was used in Chapter 2 to prove Claim 2.3.

Now, dividing (3.3.2) by L shows that if $\frac{w'}{w}$ has a pole at z then either

$$A_2 = \sum_{m=0}^{k}\left[\binom{k}{m}\frac{L^{(m)}}{L}R_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_j\frac{L^{(m)}}{L}R_{j-m}\right] = 0$$

at z or

$$A_1 = \sum_{m=0}^{k}\left[\binom{k}{m}\frac{L^{(m)}}{L}Q_{k-m} + \sum_{j=1}^{k-2}\binom{j}{m}B_j\frac{L^{(m)}}{L}Q_{j-m}\right] + A = \infty$$

at z.

Because of Claim 3.3, we can write (3.3.2) as

$$A_1 + A_2\frac{w'}{w} = 0, \tag{3.3.7}$$

where $T(r,A_j) = O\left(r^{M_7}\right)$, $j = 1,2$ and $A_2 \not\equiv 0$ by the assumption of Case (III).

Now, by using Claim 3.1 and (3.3.7), we get

$$\begin{aligned} N\left(r,\frac{1}{w}\right) &= N\left(r,\frac{w'}{w}\right) \\ &\le N\left(r,\frac{1}{A_2}\right) + N(r,A_1) \\ &\le T(r,A_2) + T(r,A_1) + O(1) \\ &\equiv O\left(r^{M_8}\right). \end{aligned}$$

So, $N(r,\frac{1}{w})$ has finite order. But this contradicts (3.2.3). Therefore, Case (III) cannot occur. □

3.4 Proof of Theorem 3.2.2

Assume the hypotheses of Theorem 3.2.2. We proceed as in Theorem 2.2.2. Taking $k = 2$ in Theorem 3.2.1, we have two cases to consider.

Case (a): L is a constant and by using (3.2.6) and Lemma 3.2.1, we get

$$A = -Q_2 = -(-P) = P.$$

Case (b): L is not constant, but

$$\begin{aligned} 0 &= \sum_{m=0}^{2} \binom{2}{m} L^{(m)} R_{2-m} \\ &= LR_2 + 2L'R_1 + L''R_0 \\ &= 2L' \qquad\qquad \text{(by using Lemma 3.2.1).} \end{aligned}$$

But this implies that L is constant, a contradiction. □

3.5 Proof of Theorem 3.2.3

Assume the hypotheses of Theorem 3.2.3. The calculations in this proof are the same as in Theorem 2.2.4. Taking $k = 3$ and $B_1 = B$ in Theorem 3.2.1, we have two cases to consider.

Case (a): L is a constant and

$$A = -Q_3 - B_1Q_1 = -Q_3 = P' \qquad \text{(by using (3.2.6) and Lemma 3.2.1).}$$

But, since w solves (3.2.10) we have

$$w''' + Bw' + Aw = 0$$

and so

$$w''' + Bw' + P'w = 0. \tag{3.5.1}$$

Also, by differentiating (3.1.1) we get

$$w''' + P'w + Pw' = 0. \tag{3.5.2}$$

Now, (3.5.1) and (3.5.2) give $P = B$.

Case (b): L is not constant and

$$\begin{aligned} 0 &= \sum_{m=0}^{3}\left[\binom{3}{m}L^{(m)}R_{3-m} + \binom{1}{m}B_1 L^{(m)}R_{1-m}\right] \\ &= LR_3 + B_1 L + 3L'R_2 + 0 + 3L''R_1 + 0 + L'''R_0 + 0 \\ &= -PL + BL + 3L'' \end{aligned}$$

and so

$$L'' = \frac{1}{3}PL - \frac{1}{3}BL. \tag{3.5.3}$$

Differentiating (3.5.3) we get

$$L''' = \frac{1}{3}P'L + \frac{1}{3}PL' - \frac{1}{3}B'L - \frac{1}{3}BL'. \tag{3.5.4}$$

Also, we have

$$\begin{aligned} -A &= \sum_{m=0}^{3}\left[\binom{3}{m}\frac{L^{(m)}}{L}Q_{3-m} + \binom{1}{m}B_1\frac{L^{(m)}}{L}Q_{1-m}\right] \\ &= [Q_3 + 0] + \left[3\frac{L'}{L}Q_2 + B_1\frac{L'}{L}\right] + \left[3\frac{L''}{L}Q_1 + 0\right] + \left[\frac{L'''}{L}Q_0\right] \\ &= -P' - 3P\frac{L'}{L} + B\frac{L'}{L} + \frac{L'''}{L}. \end{aligned}$$

Thus,

$$A = P' + 3P\frac{L'}{L} - B\frac{L'}{L} - \frac{L'''}{L}.$$

Using (3.5.4) we get

$$A = P' + 3P\frac{L'}{L} - B\frac{L'}{L} - \frac{1}{3}P' - \frac{1}{3}P\frac{L'}{L} + \frac{1}{3}B' + \frac{1}{3}B\frac{L'}{L}$$

and so

$$A = \frac{8}{3}P\frac{L'}{L} + \frac{2}{3}P' + \frac{1}{3}B' - \frac{2}{3}B\frac{L'}{L}.$$

□

3.6 Proof of Theorem 3.2.4

The proof is different to that of Theorem 2.2.5.

We will need the following lemma, which is due to Bank and Laine [6, 7].

Lemma 3.6.1. *Let B an entire function. Then every solution of the equation*

$$u''' + 4Bu' + 2B'u = 0 \tag{3.6.1}$$

is of the form $u = y_1 y_2$, where y_1, y_2 are solutions (possibly linearly dependent) of

$$y'' + By = 0. \tag{3.6.2}$$

- *If y_1, y_2 are linearly dependent, then $u = y^2$ with y a solution of* (3.6.2) *and*

$$B = \frac{-y''}{y} = \frac{1}{4}\left(\frac{u'}{u}\right)^2 - \frac{1}{2}\frac{u''}{u}.$$

- *If y_1, y_2 are linearly independent, then*

$$4B = \left(\frac{u'}{u}\right)^2 - 2\frac{u''}{u} - \frac{k^2}{u^2}, \tag{3.6.3}$$

  where $k = W(y_1, y_2)$.

We remark that (3.6.3) is the well known Bank-Laine product formula [6].

Now, assume the hypotheses of Theorem 3.2.4. Taking $k = 4$ and $B_1 = B_2 = 0$ in Theorem 3.2.1, we have two cases to consider.

Case (a): L is a constant and

$$A = -Q_4 = -P^2 + P'' \qquad \text{by using (3.2.2).}$$

But, differentiating (3.1.1) two times gives

$$\begin{aligned} 0 &= w^{(4)} + P''w + 2P'w' + Pw'' \\ &= w^{(4)} + (P'' - P^2)w + 2P'w' \end{aligned}$$

Since we also have $w^{(4)} + Aw = 0$, this gives

$$0 = 2P'w'.$$

So P must be constant, but this is a contradiction since P is transcendental.

Hence, Case (a) cannot occur.

Case (b): L is non-constant and L satisfies, using (3.2.2),

$$\begin{aligned} 0 &= \sum_{m=0}^{4} \binom{4}{m} L^{(m)} R_{4-m} \\ &= LR_4 + 4L'R_3 + 6L''R_2 + 4L'''R_1 + L^{(4)}R_0 \\ &= -2LP' - 4L'P + 0 + 4L''' + 0 \\ &= 4L''' - 4L'P - 2LP' \end{aligned}$$

and so

$$L''' = L'P + \frac{1}{2}LP'. \tag{3.6.4}$$

Since this is a linear differential equation and P is an entire function it follows that L is an entire function.

By using (3.2.2) and (3.2.8), we get

$$\begin{aligned} -A &= \sum_{m=0}^{4} \binom{4}{m} \frac{L^{(m)}}{L} Q_{4-m} \\ &= Q_4 + 4\frac{L'}{L}Q_3 + 6\frac{L''}{L}Q_2 + 4\frac{L'''}{L}Q_1 + \frac{L^{(4)}}{L}Q_0 \\ &= P^2 - P'' - 4P'\frac{L'}{L} - 6P\frac{L''}{L} + \frac{L^{(4)}}{L} \end{aligned}$$

and so

$$A = P'' - P^2 + 4P'\frac{L'}{L} + 6P\frac{L''}{L} - \frac{L^{(4)}}{L}. \tag{3.6.5}$$

Differentiating (3.6.4) and dividing by L gives

$$\frac{L^{(4)}}{L} = P\frac{L''}{L} + \frac{3}{2}P'\frac{L'}{L} + \frac{1}{2}P''.$$

By substituting this in (3.6.5), we get

$$A = P'' - P^2 + 4P'\frac{L'}{L} + 6P\frac{L''}{L} - P\frac{L''}{L} - \frac{3}{2}P'\frac{L'}{L} - \frac{1}{2}P''$$

and so

$$A = 5P\frac{L''}{L} + \frac{5}{2}P'\frac{L'}{L} + \frac{1}{2}P'' - P^2. \tag{3.6.6}$$

Thus (3.6.4) and (3.6.6) prove (3.2.12).

Now, we set $B = \frac{-P}{4}$ in (3.6.4) and apply Lemma 3.6.1. Then $L = y_1y_2$ where y_1, y_2 are solutions of (3.6.2).

Suppose that w, v have the same zeros. Then L has no zeros and poles.

Therefore, we have the following two cases:

Case 1: If y_1, y_2 are linearly dependent, then $L = y^2$ with y a solution of (3.6.2) and

$$B = \frac{-P}{4} = \frac{1}{4}\left(\frac{L'}{L}\right)^2 - \frac{1}{2}\frac{L''}{L}.$$

So,

$$P = -\left(\frac{L'}{L}\right)^2 + 2\frac{L''}{L}.$$

Let $L = e^{2C}$ with C an entire function. Then

$$\begin{aligned} P &= -(2C')^2 + 2\left(2C'' + 4C'^2\right) \\ &= -4C'^2 + 4C'' + 8C'^2 \\ &= 4\left(C'^2 + C''\right). \end{aligned}$$

Substituting these in (3.6.6) shows that A is a differential polynomial in C'.

Case 2: If y_1, y_2 are linearly independent, then

$$P = -\left(\frac{L'}{L}\right)^2 + 2\frac{L''}{L} + \frac{k^2}{L^2}, \tag{3.6.7}$$

where $k = W(y_1, y_2)$. Also L is not a polynomial, since P is transcendental.

Let $L = e^C$ with C an entire function. Then

$$\begin{aligned} P &= -C'^2 + 2C'' + 2C'^2 + k^2 e^{-2C} \\ &= 2C'' + C'^2 + k^2 e^{-2C}. \end{aligned}$$

Substituting these in (3.6.6) shows that A is a differential polynomial in e^{-C} and C'. □

Remark. The reader will observe that the proofs in Chapter 2 and Chapter 3 have similar structures. However, in Chapter 2 the error terms are $o\left(r^{(n+2)/2}\right)$, while in Chapter 3 they are $O\left(r^M\right)$ for any $M > 0$. Also the Bank-Laine method is used in Chapter 3 but is not needed in Chapter 2. Note finally that (3.6.7) is the same equation as (2.7.3) but, unlike in Chapter 2, the coefficient k^2 is non-zero.

CHAPTER 4

First order non-homogeneous equations

In this chapter, we consider two non-homogeneous first order differential equations and we use Nevanlinna theory to determine when the solutions of these differential equations can have the same zeros or (mostly) the same zeros. The results of this chapter were published in [5].

4.1 Introduction

Chapter 2 and Chapter 3 studied homogeneous linear differential equations having solutions with mostly the same zeros, where one of the equations is

$$w'' + Pw = 0. \tag{4.1.1}$$

Chapter 2 has some results for homogeneous linear differential equations where P is a polynomial, and the corresponding problem where P is a transcendental entire function of finite order is studied in Chapter 3.

In this chapter we study a similar problem, but for non-homogeneous first order differential equations. We will prove the following.

Theorem 4.1.1. *Assume that $v' = Av + B$ and $w' = Cw + D$, where A, B, C and D are entire functions of order less than 1 and v, w are transcendental functions. Assume that $v = Lw$, where L has finitely many zeros and poles, and*

$$\begin{aligned} T(r, A) + T(r, B) &= S(r, v), \\ T(r, C) + T(r, D) &= S(r, w). \end{aligned} \tag{4.1.2}$$

Then the following conclusions hold.

(I) *If L is a rational function, then $A \equiv C$, L is a constant and $B = LD$.*

(II) *If L is a transcendental function, then one of the following cases holds:*

(i) *$B \equiv D \equiv 0$ and v, w have no zeros.*

(ii) *$A = -C$ and B/A, D/C are non-zero constants, and*

$$v = c_1 + c_2 e^{A_1},$$

$$w = c_3 + c_4 e^{-A_1},$$

where $c_j \in \mathbb{C}$, $A_1' = A$ and $L = (constant)\, e^{A_1}$.

If, in addition, L has finite order in case (ii)*, then A, B, C, D are polynomials and so is A_1.*

Example 4.1.1. We give an example to show that the order of growth of A, B, C and D must be less than 1 in Theorem 4.1.1. In this example,

$$\max\{\rho(A), \rho(B), \rho(C), \rho(D)\} = 1.$$

Suppose that $w = e^{z^2} + 1$, then $w' = 2ze^{z^2} = 2z(w-1)$ and so

$$w' = 2zw - 2z = Cw + D, \qquad C = 2z, \qquad D = -2z.$$

Also, suppose that $L = e^z$, and set

$$v = e^z w = e^{z^2+z} + e^z,$$

and so

$$v' = (2z+1)e^{z^2+z} + e^z = (2z+1)(v - e^z) + e^z = (2z+1)v - 2ze^z = Av + B.$$

Hence, $A = 2z + 1$ and $B = -2ze^z$ and (4.1.2) is satisfied.

We see that D/C is constant but B/A is not constant and $A \neq -C$.

4.2 Proof of Theorem 4.1.1

(I) Assume that L is a rational function. Since $v' = Av + B$ and $v = Lw$, we can write

$$L'w + Lw' = ALw + B. \tag{4.2.1}$$

Also we have $w' = Cw + D$. So, multiplying by L gives

$$Lw' = CLw + DL. \tag{4.2.2}$$

Subtracting (4.2.2) from (4.2.1) gives

$$(L' - AL + CL)w = B - DL. \tag{4.2.3}$$

We must have $L' - AL + CL \equiv 0$ because if not

$$w = \frac{B - DL}{L' - AL + CL},$$

from which we get, using (4.1.2),

$$T(r, w) \leq S(r, v) + S(r, w) = S(r, w),$$

which is a contradiction.

So, $L'/L = A - C$.

Let $\phi_1 = L'/L$. Then ϕ_1 is rational and $\phi_1(\infty) = 0$. Also, $A - C$ is entire and $(A - C)(\infty) = 0$.

So, $A - C \equiv 0$ by Liouville's theorem. Therefore, $A = C$ and $L'/L \equiv 0$.

Hence, L is a constant and so $L' = 0$. We put $A = C$ and $L' = 0$ in (4.2.3) to get $B = LD$.

(II) Here we assume that L is a transcendental function. Since L has finitely many zeros and poles, the lower order of L is at least 1, and

$$T(r, A) + T(r, B) + T(r, C) + T(r, D) = S(r, L),$$

because A, B, C and D have order less than 1. We start with

Claim 4.1. If $B \equiv 0$ or $D \equiv 0$ then $B \equiv D \equiv 0$ and v, w have no zeros.

To prove this, assume that $B \equiv 0$. Then $\frac{v'}{v} = A$ is entire. So v has no zeros and w has finitely many zeros. We can write

$$\left(\frac{w'}{w} - C\right) w = D.$$

So $T(r, U_1) = S(r, w)$, where $U_1 = \frac{w'}{w} - C$, because w is transcendental and

$$T\left(r, \frac{w'}{w}\right) \leq m\left(r, \frac{w'}{w}\right) + \overline{N}\left(r, \frac{1}{w}\right) \leq S(r, w) + O(\log r) = S(r, w).$$

Therefore, $U_1 \equiv D \equiv 0$ because otherwise $w = D/U_1$ and so $T(r, w) = S(r, w)$ which is a contradiction.

Hence, $w'/w = C$, which is entire, so w has no zeros.

Similarly, we can show that if $D \equiv 0$ then $B \equiv 0$, and w and v have no zeros.

This completes the proof of Claim 4.1 and we have case (i).

Assume henceforth that $BD \not\equiv 0$.

Claim 4.2. We have $B - LD \not\equiv 0$.

This is because if $B - LD \equiv 0$, then $L = B/D$ and so $T(r, L) = S(r, L)$ which is a contradiction. This completes the proof of Claim 4.2.

Now, we write

$$v' = L'w + Lw' = Av + B = ALw + B,$$

and so

$$L'w + L(Cw + D) = ALw + B,$$

which gives

$$Sw = B - LD, \qquad S = L' + LE, \qquad E = C - A. \tag{4.2.4}$$

Claim 4.3. We have $S \not\equiv 0$.

This follows from (4.2.4) and Claim 4.2.

We can now write

$$S = UL, \qquad U = \frac{L'}{L} + E \not\equiv 0, \qquad T(r, U) = S(r, L). \tag{4.2.5}$$

Now, (4.2.4) gives

$$w = \frac{B - LD}{S} = \frac{M}{L} + N, \qquad M = \frac{B}{U}, \qquad N = \frac{-D}{U} \tag{4.2.6}$$

and

$$T(r, M) + T(r, N) = S(r, L). \tag{4.2.7}$$

Differentiation gives

$$\frac{M'}{L} - \frac{ML'}{L^2} + N' = w' = Cw + D = \frac{CM}{L} + CN + D.$$

Hence, we can write

$$\frac{P}{L} = Q, \qquad P = M' - M\frac{L'}{L} - CM, \qquad Q = CN + D - N' \tag{4.2.8}$$

and, using (4.2.7),

$$T(r, P) + T(r, Q) = S(r, L). \tag{4.2.9}$$

It follows from (4.2.9) that we must have

$$P \equiv Q \equiv 0, \tag{4.2.10}$$

because otherwise we get $T(r, L) = S(r, L)$, which is a contradiction.

Now (4.2.3), (4.2.4), (4.2.5), (4.2.6), (4.2.8) and (4.2.10) give

$$\begin{aligned} 0 = C + \frac{D}{N} - \frac{N'}{N} &= C - U - \frac{N'}{N} = C - E - \frac{L'}{L} - \frac{N'}{N} \\ &= A - \frac{L'}{L} - \frac{N'}{N} = A - \frac{L'}{L} - \frac{D'}{D} + \frac{U'}{U}, \end{aligned}$$

and so

$$0 = A - \frac{D'}{D} + \frac{S'}{S} - 2\frac{L'}{L}. \tag{4.2.11}$$

We also have, using the fact that $M = B/U$,

$$0 = \frac{M'}{M} - \frac{L'}{L} - C = \frac{B'}{B} - \frac{U'}{U} - \frac{L'}{L} - C,$$

and by using the fact that $S = UL$, we get

$$0 = \frac{B'}{B} - \frac{S'}{S} - C. \tag{4.2.12}$$

Combining (4.2.11) and (4.2.12) gives

$$0 = A - \frac{D'}{D} + \frac{B'}{B} - C - 2\frac{L'}{L},$$

and so

$$0 = -E - \frac{D'}{D} + \frac{B'}{B} - 2\frac{L'}{L}. \tag{4.2.13}$$

Claim 4.4. B/D is a rational function.

Since L has finitely many zeros and poles, L'/L has finitely many poles. So if z_0 is large and is a zero of B, then (4.2.13) shows that z_0 is a pole of B'/B and a pole of D'/D of the same residue, and a zero of D of the same multiplicity. The same argument applies if $D(z_0) = 0$, and so B/D has finitely many zeros and poles.

Since $\rho(B/D) < 1$ and B/D has finitely many zeros and poles, Claim 4.4 is proved.

Also, in (4.2.12), suppose that z_0 is a zero of B of multiplicity m. Then B'/B has a simple pole at z_0 of residue m and so has S'/S since C is entire.

Similarly, if z_0 is a zero or pole of S then S'/S has a pole at z_0 and B'/B has a pole at z_0 with the same residue, which must be a positive integer since B is entire. So S is entire because a pole of S gives a pole of S'/S with negative residue, and B and S have the same zeros and multiplicities.

Since S is entire and $S = L' + LE$ with E entire, L is also entire.

Since B and S have the same zeros and multiplicities, we can write

$$S = L' + LE = e^h B, \tag{4.2.14}$$

where h is an entire function.

Let $A_1' = A$. We then write $v' = Av + B$ in the form

$$\left(ve^{-A_1}\right)' = e^{-A_1}B.$$

Then, there exists $c \in \mathbb{C}$ such that

$$v = e^{A_1}\left(c + \int_0^z e^{-A_1(t)}B(t)dt\right).$$

We have $\rho(A) < 1$ and $\rho(B) < 1$, and so $\rho(A_1) < 1$.

So, using α_j to denote positive constants,

$$\log M(r, e^{-A_1}) < \exp(\alpha_1 r^{1-\epsilon}) \qquad \text{and} \qquad \log M(r, B) = o(r^{1-\epsilon})$$

where $\epsilon > 0$ is small.

Then

$$|e^{-A_1(t)}B(t)| \leq \exp\exp(\alpha_2 r^{1-\epsilon})$$

for $|t| \leq r$ and r large.

Therefore,

$$\left|\int_0^z e^{-A_1(t)}B(t)dt\right| \leq |z|\, M(|z|, e^{-A_1}B) \leq r\exp\exp(\alpha_3 r^{1-\epsilon}) \qquad (|z| < r).$$

Similar estimates hold for e^{A_1}, and we have

$$M(r, v) \leq \exp\exp(c' r^{1-\epsilon}), \qquad T(r, v) \leq \log M(r, v) \leq \exp(c'' r^{1-\epsilon}),$$

where c' and $c'' > 0$.

In the same way, we can show that, with $c''' > 0$,

$$T(r, w) \leq \exp(c''' r^{1-\epsilon}).$$

It follows that

$$T(r, L) = T\left(r, \frac{v}{w}\right) \leq \exp(\alpha_4 r^{1-\epsilon}) \quad \text{for some} \quad \alpha_4 > 0. \tag{4.2.15}$$

Since L is an entire function with finitely many zeros, we can write

$$L = Pe^{\phi}, \tag{4.2.16}$$

where P is a polynomial and ϕ is an entire function.

Claim 4.5. We claim that P is a non-zero constant.

To show this, we have $L = Pe^{\phi}$. Then, outside a set of finite measure, using (4.2.15),

$$T(r, \phi') = m(r, \phi') \leq m\left(r, \frac{L'}{L}\right) + m\left(r, \frac{P'}{P}\right) \leq O(\log r + \log T(r, L)) \leq O(r^{1-\epsilon}).$$

Hence, $\rho(\phi') = \rho(\phi) < 1$.

Now, (4.2.14) gives

$$S = L' + LE = (P' + P\phi' + PE)e^{\phi} = e^{h}B.$$

Then we can write

$$\frac{P' + P\phi' + PE}{B} = e^{h-\phi}. \tag{4.2.17}$$

So $e^{h-\phi}$ has no zeros and poles but has order less than 1, so is a non zero constant. So

$$P' + P\phi' + PE = d_1 B, \tag{4.2.18}$$

where $d_1, d_2, \ldots$ will denote non-zero constants.

Also, since $e^{h-\phi}$ is a constant, we have $h' = \phi'$. Then (4.2.12) and (4.2.14) yield

$$\phi' = h' = \frac{S'}{S} - \frac{B'}{B} = -C. \tag{4.2.19}$$

Substituting this in (4.2.18) and recalling that $E = C - A$, we get

$$P' = AP + d_1 B. \tag{4.2.20}$$

Recall next from Claim 4.4 that $D = RB$ where R is a rational function. So, (4.2.13) and (4.2.16) give

$$E = -\left(\frac{R'}{R} + \frac{B'}{B}\right) + \frac{B'}{B} - 2\frac{L'}{L} = -\frac{R'}{R} - 2\frac{L'}{L} = -\frac{R'}{R} - 2\frac{P'}{P} - 2\phi',$$

and so

$$E + 2\phi' = -\frac{R'}{R} - 2\frac{P'}{P}.$$

But $E + 2\phi'$ is entire and $-R'/R - 2P'/P$ is a rational function which is equal to 0 at infinity. Therefore, by Liouville's theorem, we have

$$E + 2\phi' \equiv 0, \qquad \phi' = \frac{-E}{2}. \tag{4.2.21}$$

Also,

$$\frac{R'}{R} + 2\frac{P'}{P} = 0,$$

and so

$$D = d_2 P^{-2} B. \tag{4.2.22}$$

Also, (4.2.18) and (4.2.21) give

$$P' + P\left(\frac{-E}{2}\right) + PE = d_1 B = d_3 P^2 D,$$

and so

$$2\frac{P'}{P} = d_4 PD - E.$$

Since $d_4 PD - E$ is entire, P'/P is also entire. Then, P is a polynomial and has no zeros, and so P is a non-zero constant. This completes the proof of Claim 4.5.

Now, $L = Pe^{\phi}$ and P is constant. Without loss of generality, let $P = 1$ and so $L = e^{\phi}$. Thus B/D is a non-zero constant by (4.2.22) and so is A/B by (4.2.20). Now (4.2.19) and (4.2.21) give

$$C = -\phi' = \frac{E}{2} = \frac{C - A}{2}, \qquad C = -A. \tag{4.2.23}$$

Now we can solve for v and w by writing $B = -c_1 A$ where $c_1 \neq 0$ is a constant, and the first order differential equation

$$v' = Av + B = Av - c_1 A$$

solves to give

$$v = c_1 + c_2 e^{A_1}, \qquad A_1' = A,$$

with c_2 constant. Similarly, since $C = -A$ and $D/C = (D/B)(B/A)(A/C)$ is a non-zero constant, we can solve to obtain

$$w = c_3 + c_4 e^{-A_1}$$

with c_3, c_4 constants.

Since (4.2.16) and (4.2.23) give

$$\frac{L'}{L} = \phi' = -C = A,$$

we also have

$$L = d_5 e^{A_1}.$$

If, in addition, L has finite order in case (ii), then $A = L'/L$ is a polynomial and so are B, C and D. □

4.3 A corollary

For completeness we include the following corollary of Theorem 4.1.1, which does not appear in [5].

Corollary 4.3.1. *Suppose that we have*

$$v' = Av + B, \qquad w' = Cw + D,$$

with v, w, A, B, C, D all as in Theorem 4.1.1.

Suppose that λ, μ are entire functions of order less than 1, with

$$T(r, \lambda) = S(r, v), \qquad T(r, \mu) = S(r, w),$$

and suppose

$$v_1 = v - \lambda, \qquad w_1 = w - \mu \tag{4.3.1}$$

have the same zeros with finitely many exceptions. Then

$$\begin{aligned} v_1' &= v' - \lambda' = Av + B - \lambda' = A(v_1 + \lambda) + B - \lambda' \\ &= Av_1 + B_1, \qquad B_1 = A\lambda + B - \lambda'. \end{aligned} \tag{4.3.2}$$

Also

$$\begin{aligned} w_1' &= w' - \mu' = Cw + D - \mu' = C(w_1 + \mu) + D - \mu' \\ &= Cw_1 + D_1, \qquad D_1 = C\mu + D - \mu'. \end{aligned} \tag{4.3.3}$$

We have that A, B_1, C, D_1 are entire functions of order less than 1. Also we can write $v_1 = Lw_1$, where L has finitely many zeros and poles, and we have

$$\begin{aligned} T(r, A) + T(r, B_1) &= S(r, v_1), \\ T(r, C) + T(r, D_1) &= S(r, w_1). \end{aligned} \tag{4.3.4}$$

Then by Theorem 4.1.1 *we have the following conclusions.*

(I) *If L is a rational function, then $A \equiv C$, L is constant and*

$$\begin{aligned} 0 &= B_1 - LD_1 \\ &= A\lambda + B - \lambda' - L(C\mu + D - \mu'). \end{aligned}$$

(II) *If L is a transcendental function, then one of the following cases holds:*

(i) $B_1 \equiv A\lambda + B - \lambda' \equiv C\mu + D - \mu' \equiv D_1 \equiv 0$ *and v_1, w_1 have no zeros.*

(ii) $A = -C$ *and B_1/A, D_1/C are non-zero constants, and*

$$v_1 = c_1 + c_2 e^{A_1},$$

$$w_1 = c_3 + c_4 e^{-A_1},$$

where $c_j \in \mathbb{C} \quad (j = 1, 2, 3, 4)$, $A_1' = A$ and $L =$ (constant) e^{A_1}.

If, in addition, L has finite order in case (ii)*, then A, B_1, C, D_1 are polynomials and so is A_1.*

Now, we give some examples to show that the cases in Corollary 4.3.1 can happen. Here we can take $\lambda = 1$, $\mu = -1$ for example.

Example 4.3.1. We choose v_1 to satisfy

$$v_1' = Av_1 + B_1,$$

with A, B_1 entire of order less than 1, and with

$$T(r, A) + T(r, B_1) = S(r, v_1).$$

(For example, $v_1 = e^z + 1$, $A = 1$, $B_1 = -1$) and we write

$$v_1 = Lw_1$$

with L a non-zero constant, so that

$$w_1' = Cw_1 + D_1, \qquad C = A, \qquad LD_1 = B_1.$$

Then we set

$$v = v_1 + \lambda, \qquad w = w_1 + \mu,$$

and $v - \lambda$, $w - \mu$ have the same zeros.

Example 4.3.2. We give an example to show that case (i) in (II) can happen. We choose v_1, w_1 as follows

$$v_1 = e^z, \qquad w_1 = e^{-z}. \tag{4.3.5}$$

Then, we have

$$v_1' = e^z, \qquad w_1' = -e^{-z}. \tag{4.3.6}$$

So, by using (4.3.2), (4.3.3), (4.3.5) and (4.3.6), we can write

$$Av_1 + B_1 = v_1' = e^z = v_1 \quad \Rightarrow \quad A = 1, B_1 \equiv 0,$$

and

$$Cw_1 + D_1 = w_1' = -e^{-z} = -w_1 \quad \Rightarrow \quad C = -1, D_1 \equiv 0.$$

This shows that this case is possible.

Example 4.3.3. We finally give an example to show that case (ii) in (II) is possible. Let

$$v_1 = c_1 + c_2 e^{A_1}, \qquad w_1 = c_3 + c_4 e^{-A_1}, \tag{4.3.7}$$

where $c_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, 2, 3, 4)$, are chosen so that v_1, w_1 have the same zeros, $A_1' = A$ and $L = \frac{v_1}{w_1} =$ (constant) e^{A_1}. Then, we have

$$v_1' = c_2 A e^{A_1}, \qquad w_1' = -c_4 A e^{-A_1}. \tag{4.3.8}$$

Now, we need

$$\begin{aligned} Av_1 + B_1 &= v_1' && \text{by (4.3.2)} \\ &= c_2 A e^{A_1} && \text{by (4.3.8)} \\ &= A(v_1 - c_1) && \text{by (4.3.7)} \\ &= Av_1 - c_1 A \end{aligned}$$

Therefore, we set $B_1 = -c_1 A$ and so B_1/A is a non-zero constant. Also, we need

$$\begin{aligned} Cw_1 + D_1 &= w_1' && \text{by (4.3.3)} \\ &= -c_4 A e^{-A_1} && \text{by (4.3.8)} \\ &= -A(w_1 - c_3) && \text{by (4.3.7)} \\ &= -Aw_1 + c_3 A \end{aligned}$$

Therefore, $A = -C$, $D_1 = c_3 A$ and so D_1/C is a non-zero constant. This shows that this case also can happen.

CHAPTER 5

The case of one homogeneous and one non-homogeneous differential equation

This chapter aims to study the same problem as in the previous chapters but when we have a combination of homogeneous and non-homogeneous differential equations.

5.1 Introduction

In Chapter 2 and Chapter 3, we studied when solutions of two homogeneous differential equations have (mostly) the same zeros. In Chapter 4, we looked at the same problem but with a pair of first order non-homogeneous differential equations.

In this chapter, we study the same problem but with one homogeneous and one non-homogeneous differential equation. In particular, we consider the first equation to be homogeneous of the second order with a polynomial coefficient and the second equation to be non-homogeneous of the first order with entire coefficients.

5.2 The result

We now state our result in this chapter.

Theorem 5.2.1. *Suppose that $P \not\equiv 0$ is a polynomial of degree n, and w solves*

$$w'' + Pw = 0, \tag{5.2.1}$$

and $w \not\equiv 0$ has infinitely many zeros. Suppose that $v \not\equiv 0$ solves

$$v' = Av + B, \tag{5.2.2}$$

where A, B are entire and $AB \not\equiv 0$, and

$$T(r, A) + T(r, B) = o\left(r^{(n+2)/2}\right). \tag{5.2.3}$$

Suppose that $L = \frac{w}{v}$ has finitely many zeros and poles (i.e. w and v have the same zeros with finitely many exceptions).

Then A is a polynomial and there exists a polynomial Q such that

$$w = c_1 e^Q \int e^{-2Q} dz \qquad \textit{and} \qquad v = c_2 e^{A_1} \int e^{-2Q} dz,$$

and

$$P = -\left(Q'' + Q'^2\right) \qquad \textit{and} \qquad B = c_2 e^{A_1 - 2Q},$$

where $A_1' = A$ and c_1, c_2 are constants.

Here $\int e^{-2Q} dz$ means

$$d + \int_0^z e^{-2Q(t)} dt$$

for some constant d.

Proof. We have

$$w = Lv. \tag{5.2.4}$$

So,

$$w'' = L''v + 2L'v' + Lv''. \tag{5.2.5}$$

but

$$
\begin{aligned}
v'' &= A'v + Av' + B' \\
&= A'v + A(Av + B) + B' \\
&= v(A^2 + A') + AB + B'. \qquad (5.2.6)
\end{aligned}
$$

We also have, using (5.2.1), (5.2.2), (5.2.4), (5.2.5) and (5.2.6),

$$
\begin{aligned}
0 &= w'' + Pw \\
&= L''v + 2L'v' + Lv'' + PLv \\
&= L''v + 2L'(Av + B) + L[v(A^2 + A') + AB + B'] + PLv \\
&= \left[L'' + 2L'A + L(A^2 + A') + PL\right] v + 2L'B + L(AB + B'). \qquad (5.2.7)
\end{aligned}
$$

Let

$$M = \frac{L'}{L}. \qquad (5.2.8)$$

Then

$$\frac{L''}{L} = M' + M^2. \qquad (5.2.9)$$

We divide (5.2.7) by L, and by using (5.2.8) and (5.2.9), we get

$$
\begin{aligned}
0 &= \left[\frac{L''}{L} + 2\frac{L'}{L}A + A^2 + A' + P\right] v + 2\frac{L'}{L}B + AB + B' \\
&= \left[M' + M^2 + 2MA + A^2 + A' + P\right] v + 2MB + AB + B'. \qquad (5.2.10)
\end{aligned}
$$

The next step is to estimate the growth of M.

We know that $\rho(w) = \frac{n+2}{2}$ from [6]. Therefore,

$$m(r, \frac{w'}{w}) = O\left(\log r\right) = o\left(r^{(n+2)/2}\right). \qquad (5.2.11)$$

Claim 5.1. We claim that $T(r, M) = o\left(r^{(n+2)/2}\right)$.

To show this we know that $N(r, M) = O(\log r)$ since M has finitely many poles.

Write (5.2.2) as

$$\left(ve^{-A_1}\right)' = e^{-A_1}B,$$

where $A_1' = A$.

Then, there exists a constant c such that

$$v = e^{A_1}\left(c + \int_0^z e^{-A_1(t)}B(t)dt\right).$$

Also, using (5.2.3), we can write

$$\log M(r, A) \leq 3T(2r, A) = o\left(r^{(n+2)/2}\right),$$

and so

$$M(r, A) \leq \exp\left(o\left(r^{(n+2)/2}\right)\right).$$

Now,

$$A_1(z) = A_1(0) + \int_0^z A(t)dt.$$

So,

$$\begin{aligned} M(r, A_1) &\leq |A_1(0)| + rM(r, A) \\ &\leq O(1) + r\exp\left(o\left(r^{(n+2)/2}\right)\right) \\ &\leq \exp\left(o\left(r^{(n+2)/2}\right)\right). \end{aligned}$$

Therefore, we get

$$M(r, v) \leq \exp\exp\left(o\left(r^{(n+2)/2}\right)\right),$$

and so

$$T(r, v) \leq \log M(r, v) \leq \exp\left(o\left(r^{(n+2)/2}\right)\right).$$

We use Lemma 2.3 in [13, p.36] with $R = 2r$ to get

$$\begin{aligned} m(r, \frac{v'}{v}) &= O\left(\log r\right) + O\left(\log^+ T(2r, v)\right) \\ &\leq o\left(r^{(n+2)/2}\right). \end{aligned}$$

Now, we have $M = \frac{w'}{w} - \frac{v'}{v}$. So

$$m(r, M) = o\left(r^{(n+2)/2}\right).$$

We also have $N(r, M) = O(\log r)$.

Hence,

$$\begin{aligned} T(r, M) &= o\left(r^{(n+2)/2}\right) + O\left(\log r\right) \\ &= o\left(r^{(n+2)/2}\right). \end{aligned}$$

This completes the proof of Claim 5.1.

Using Claim 5.1 and (5.2.10), we get

$$T(r, M' + M^2 + 2MA + A^2 + A' + P) = o\left(r^{(n+2)/2}\right)$$

and

$$T(r, 2MB + AB + B') = o\left(r^{(n+2)/2}\right).$$

Also, using Theorem 1.4.4, we have

$$\begin{aligned} T(r, v) &\geq N\left(r, \frac{1}{v}\right) - O(1) \\ &\geq \text{(constant)} \cdot r^{(n+2)/2}. \end{aligned}$$

Therefore, we must have

$$M_1 = M' + M^2 + 2MA + A^2 + A' + P \equiv 0 \tag{5.2.12}$$

and

$$M_2 = 2MB + AB + B' \equiv 0 \tag{5.2.13}$$

because otherwise we can write $v = -M_2/M_1$ to get a contradiction.

We now divide (5.2.13) by B to get

$$2M + A + \frac{B'}{B} \equiv 0.$$

So, B'/B has finitely many poles and so B has finitely many zeros. Then we can write B in the form

$$B = P_1 e^{P_2},$$

where P_1, P_2 are polynomials.

But then we can write

$$\frac{B'}{B} = R_1,$$

where R_1 is rational.

Thus, we also can write

$$M = -\frac{A}{2} + R_2, \tag{5.2.14}$$

where $R_2 = -\frac{1}{2}R_1$ is rational.

Substitute (5.2.14) in (5.2.12); we obtain

$$\begin{aligned} 0 &\equiv \left(-\frac{A'}{2} + R_2'\right) + \left(\frac{A^2}{4} - AR_2 + R_2^2\right) + \left(-A^2 + 2AR_2\right) + A^2 + A' + P \\ &\equiv \left(\frac{A'}{2} + R_2'\right) + \left(\frac{A^2}{4} + AR_2 + R_2^2\right) + P \\ &\equiv \left(\frac{A}{2} + R_2\right)' + \left(\frac{A}{2} + R_2\right)^2 + P. \end{aligned} \tag{5.2.15}$$

Now, let

$$N = \frac{A}{2} + R_2 = \frac{A}{2} - \frac{B'}{2B}.$$

Also, let

$$H = e^{\frac{A_1}{2}} B^{-\frac{1}{2}}. \tag{5.2.16}$$

So, we get

$$\frac{H'}{H} = \frac{A}{2} - \frac{B'}{2B} = N. \tag{5.2.17}$$

Substituting (5.2.17) in (5.2.15), we obtain

$$0 = N' + N^2 + P = \frac{H''}{H} + P,$$

and so

$$H'' + PH = 0.$$

Thus, $\rho(H) < \infty$, H is entire and B has no zeros.

Then,

$$m\left(r, \frac{H'}{H}\right) = O\left(\log r\right)$$

Therefore, A is a polynomial, by (5.2.17).

Since B has no zeros, from (5.2.16) we can write

$$H = e^{Q}, \tag{5.2.18}$$

where Q is a polynomial.

Since w and H solves the same equation and are linearly independent (because w has zeros but H does not), we can write

$$\left(\frac{w}{H}\right)' = \frac{\text{(constant)}}{H^2}.$$

Therefore,

$$w = c_1 e^{Q} \int e^{-2Q} dz, \tag{5.2.19}$$

where c_1 is a constant and Q is a polynomial.

Now, we have

$$\frac{L'}{L} = M = -\frac{A}{2} + R_2 = N - A = \frac{H'}{H} - A.$$

So, we can write

$$L = \text{(constant)} \cdot He^{-A_1}.$$

Therefore, we have

$$v = \frac{w}{L} = \frac{\text{(constant)} \cdot H \int H^{-2} dz}{He^{-A_1}}.$$

Hence, using (5.2.18), we obtain

$$v = c_2 e^{A_1} \int e^{-2Q} dz, \tag{5.2.20}$$

where c_2 is a constant and $A_1' = A$.

Now, from (5.2.19) and (5.2.20) we notice that w and v have the same zeros.

Also, differentiating (5.2.20), using (5.2.18), we have

$$\begin{aligned} v' &= c_2 A e^{A_1} \int H^{-2} dz + c_2 e^{A_1} H^{-2} \\ &= Av + c_2 e^{A_1} H^{-2} \\ &= Av + c_2 e^{A_1 - 2Q}. \end{aligned}$$

Comparing this with (5.2.2), we get

$$B = c_2 e^{A_1 - 2Q}.$$

Moreover, $H = e^Q$ solves $H'' + PH = 0$ and so

$$-P = \frac{H''}{H} = Q'' + Q'^2.$$

Hence,

$$P = -\left(Q'' + Q'^2\right).$$

This completes the proof of Theorem 5.2.1. □

Example 5.2.1. Take Q to be a polynomial. Let

$$w = e^Q \int_0^z e^{-2Q} dz.$$

Then,

$$w' = Q'w + e^{-Q},$$

and

$$\begin{aligned} w'' &= Q''w + Q'w' - Q'e^{-Q} \\ &= Q''w + Q'\left(Q'w + e^{-Q}\right) - Q'e^{-Q} \\ &= \left(Q'' + Q'^2\right) w. \end{aligned}$$

So, we have $P = -\left(Q'' + Q'^2\right)$.

Now, let A_1 be another polynomial and let

$$v = e^{A_1} \int_0^z e^{-2Q} dz.$$

Note that v has same zeros as w. Now, we have

$$v' = Av + e^{A_1 - 2Q},$$

where $A = A_1'$.

We choose A_1 so that

$$\deg(A_1 - 2Q) < \frac{\deg(P) + 2}{2}.$$

For example, let $A_1 = 2Q$.

CHAPTER 6

Solutions taking different values at the same points

In this chapter, we consider two linear second order differential equations and we use Nevanlinna theory to determine when the solutions of these differential equations can take the value 0 and a non-zero value at (mostly) the same points. The content of this chapter and Chapter 5 was submitted to *Journal of Inequalities and Applications*.

6.1 Introduction

The previous chapters studied common zeros of the solutions of two homogeneous or non-homogeneous differential equations.

In this chapter we study the case where the solutions of two second order homogeneous differential equations take the value 0 and a non-zero value at (mostly) the same points. We have the following result.

Theorem 6.1.1. *Suppose that $P \not\equiv 0$ is a polynomial of degree n, and A is an entire function, suppose that w and v solve*

$$w'' + Pw = 0, \quad (6.1.1)$$

$$v'' + Av = 0, \quad (6.1.2)$$

and $vw \not\equiv 0$*. Let* $v-1$ *and* w *have, with finitely many exceptions, the same zeros and the same multiplicities. Then one of the following holds.*

(A) w *has finitely many zeros and* v *is a polynomial and* $A = 0$*.*

(B) w *has infinitely many zeros and* P, A *are non-zero constants and* $\frac{v-1}{w}$ *is non-constant and*

$$\begin{aligned} w &= \lambda_1 e^{\sigma z} + \lambda_2 e^{-\sigma z}, \\ v &= \lambda_3 e^{2\sigma z}, \end{aligned} \tag{6.1.3}$$

where $\sigma, \lambda_1, \lambda_2, \lambda_3$ *are non-zero constants.*

Example 6.1.1. If $w = e^z - e^{-z}$ and $v = e^{2z}$, then

$$v - 1 = e^{2z} - 1 = e^z(e^z - e^{-z}) = e^z w.$$

Hence, $v-1$ has the same zeros as w. Here $P = -1$ and $A = -4$.

Example 6.1.2. We give an example to show that the zeros of $v-1$ and w must necessarily have the same multiplicities. To show this let

$$w = \sin\frac{z}{2}, \qquad v = \cos z.$$

Then $w = 0 \Leftrightarrow \frac{z}{2} = k\pi$ where $k \in \mathbb{Z}$.

Also $v = 1 \Leftrightarrow z = k2\pi$ where $k \in \mathbb{Z}$.

Therefore, w and $v-1$ have the same zeros but the zeros are simple for w, double for $v-1$. Here $P = \frac{1}{4}$ and $A = 1$.

6.2 A lemma

In order to prove Theorem 6.1.1, we must state and prove the following lemma.

Lemma 6.2.1. *Let $P_1, \dots, P_n \in \mathbb{C}$ be distinct, and let $A_1, \dots, A_n$ be rational functions, such that*

$$1 \equiv A_1(z)e^{P_1 z} + \cdots + A_n(z)e^{P_n z}. \tag{6.2.1}$$

Then there exists $k \in \{1, \dots, n\}$ such that $P_k = 0$ and $A_k = 1$, and $A_j = 0$ for $j \neq k$.

Proof. The proof is by induction. It is obvious that the lemma is true when $n = 1$. Assume that the lemma is true for $m \leq n - 1$. Differentiating (6.2.1), we get

$$0 \equiv B_1(z)e^{P_1 z} + \cdots + B_n(z)e^{P_n z}, \qquad B_j = A_j' + P_j A_j.$$

Now we have two cases to consider.

Case (1): Suppose there exists k such that $B_k \not\equiv 0$. Without loss of generality let $k = 1$, then we can write

$$0 = 1 + \frac{B_2}{B_1}e^{(P_2 - P_1)z} + \cdots + \frac{B_n}{B_1}e^{(P_n - P_1)z}.$$

Since we assumed the lemma is true for $m \leq n - 1$, there exists $j \in \{2, \dots, n\}$ such that $P_j - P_1 = 0$. But this contradicts our assumption that $P_1, \dots, P_n$ are distinct.

Case (2): Suppose that $B_j = 0$ for each j, i.e.

$$A_j' + P_j A_j \equiv 0.$$

If $P_j \neq 0$, then $A_j \equiv 0$ because otherwise we have

$$\frac{A_j'}{A_j} + P_j \equiv 0, \qquad A_j(z)e^{P_j z} = c \in \mathbb{C} \setminus \{0\}.$$

But this contradicts the fact that $P_j \neq 0$.

So we have $A_j \equiv 0$ for $P_j \neq 0$. Thus (6.2.1) becomes (for some k)

$$1 = A_k(z)e^{0z} = A_k(z),$$

and $P_k = 0$ and $A_k = 1$. □

6.3 Proof of Theorem 6.1.1

If $A \equiv 0$ then v is a polynomial and w has finitely many zeros. Assume henceforth that $A \not\equiv 0$.

We next note that, outside a set of finite measure, by Theorem 1.4.2,

$$T(r, v) \sim N\left(r, \frac{1}{v-1}\right) \leq N\left(r, \frac{1}{w}\right) + O(\log r) \leq O\left(r^{(n+2)/2}\right) + O(\log r). \quad (6.3.1)$$

In particular, if w has finitely many zeros, then v is a polynomial, which gives $A = 0$. This completes the proof of part (A) in the conclusion.

Assume henceforth that w has infinitely many zeros. Then (6.3.1) implies that $\rho(v) \leq (n+2)/2$, and so A is a polynomial of degree at most n by Wiman-Valiron theory [14]. Also $A \not\equiv 0$, since $v - 1$ has infinitely many zeros.

Now, two cases have to be considered.

Case (I) Assume that P is a non-zero constant; then $n = 0$ and A is constant. Therefore, we can write

$$w = c_1 e^{\alpha z} + c_2 e^{-\alpha z}, \qquad v = d_1 e^{\beta z} + d_2 e^{-\beta z}, \quad (6.3.2)$$

where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $c_j, d_j \in \mathbb{C}$ and $c_j \neq 0 \quad (j = 1, 2)$.

Since w and $v - 1$ have the same zeros with finitely many exceptions, we can write

$$v - 1 = R_1 e^{P_1} w, \quad (6.3.3)$$

where R_1 is a rational function and P_1 is a polynomial. We know that $\deg(P_1) \leq 1$ because $\rho(w), \rho(v) \leq 1$. We can now write

$$d_1 e^{\beta z} + d_2 e^{-\beta z} - 1 = R_1 e^{\gamma z}\left(c_1 e^{\alpha z} + c_2 e^{-\alpha z}\right),$$

where $\gamma \in \mathbb{C}$, and so

$$1 = d_1 e^{\beta z} + d_2 e^{-\beta z} - c_1 R_1 e^{(\gamma+\alpha)z} - c_2 R_1 e^{(\gamma-\alpha)z}.$$

Now by using Lemma 6.2.1, R_1 is constant and so we can write (6.3.3) as

$$\frac{v-1}{w} = e^{\gamma z+\delta}, \tag{6.3.4}$$

where δ is constant.

Therefore,

$$\begin{aligned} d_1 e^{\beta z} + d_2 e^{-\beta z} - 1 &= \left(e^{\gamma z+\delta}\right)\left(c_1 e^{\alpha z} + c_2 e^{-\alpha z}\right) \\ &= c_1 e^{(\alpha+\gamma)z+\delta} + c_2 e^{(-\alpha+\gamma)z+\delta}. \end{aligned} \tag{6.3.5}$$

Now, by using Lemma 6.2.1 we get

$$\alpha+\gamma = \beta, -\beta \text{ or } 0, \qquad -\alpha+\gamma = \beta, -\beta \text{ or } 0,$$

and $\alpha+\gamma$, $-\alpha+\gamma$, β, $-\beta$, 0 cannot all be different.

We must now try six cases:

I(a): If $\alpha+\gamma=\beta$ and $-\alpha+\gamma=-\beta$, then $\gamma=0$ and $\alpha=\beta$. But this contradicts (6.3.5). Thus, this case cannot happen.

I(b): If $\alpha+\gamma=\beta$ and $-\alpha+\gamma=0$, then $\beta=2\gamma$ and $\alpha=\gamma$. Substituting these in (6.3.5) gives

$$d_1 e^{2\gamma z} + d_2 e^{-2\gamma z} - 1 = h_1 e^{2\gamma z} + h_2,$$

where h_1, h_2 are constants, which yields $d_2 = 0$. Putting this in (6.3.2) gives (6.1.3) with $\sigma=\gamma$.

I(c): If $\alpha+\gamma=-\beta$ and $-\alpha+\gamma=\beta$, then $\gamma=0$ and $\alpha=-\beta$. But this contradicts (6.3.5). Thus, this case cannot happen.

I(d): If $\alpha+\gamma=-\beta$ and $-\alpha+\gamma=0$, then $\beta=-2\gamma$ and $\alpha=\gamma$. Substituting these in (6.3.5) gives

$$d_1 e^{-2\gamma z} + d_2 e^{2\gamma z} - 1 = h_1 e^{2\gamma z} + h_2,$$

where h_1, h_2 are constants, which yields $d_1 = 0$. Putting this in (6.3.2) gives (6.1.3) with $\sigma=\gamma$.

I(e): If $\alpha+\gamma = 0$ and $-\alpha+\gamma = \beta$, then $\beta = 2\gamma$ and $\alpha = -\gamma$. Substituting these in (6.3.5) gives

$$d_1 e^{2\gamma z} + d_2 e^{-2\gamma z} - 1 = h_1 + h_2 e^{2\gamma z},$$

where h_1, h_2 are constants, which yields $d_2 = 0$. Putting this in (6.3.2) gives (6.1.3) with $\sigma = \gamma$.

I(f): If $\alpha + \gamma = 0$ and $-\alpha + \gamma = -\beta$, then $\beta = -2\gamma$ and $\alpha = -\gamma$. Substituting these in (6.3.5) gives

$$d_1 e^{-2\gamma z} + d_2 e^{2\gamma z} - 1 = h_1 + h_2 e^{2\gamma z},$$

where h_1, h_2 are constants, which yields $d_1 = 0$. Putting this in (6.3.2) gives (6.1.3) with $\sigma = \gamma$.

From these cases we find that $\gamma \neq 0$ and so $\frac{v-1}{w} = e^{\gamma z+\delta}$ is non-constant. Also, we have (6.1.3), and case (B) of the conclusion.

Case (II) Suppose that P is non-constant. We will show that this leads to a contradiction. Let

$$\frac{v-1}{w} = M = Le^{Q}, \tag{6.3.6}$$

where L is a rational function and Q is an entire function.

From (6.3.1) we have $\rho(v) < \infty$ and so Q is a polynomial.

Also from (6.3.6), we have

$$v = Mw + 1, \qquad v' = M'w + Mw',$$

$$v'' = M''w + 2M'w' + Mw'' = 2M'w' + w(M'' - PM) = -Av = -A(Mw+1).$$

So,

$$2M'w' + (M'' + AM - PM)w = -A. \tag{6.3.7}$$

Now we have two cases to consider.

Case (i): If M is constant, then either $A = P$ and $A = 0$, so that $P = 0$, which is a contradiction, or

$$w = \frac{-A}{AM - PM}$$

is a rational function, which is a contradiction since w has infinitely many zeros.

Case (ii): If M is non-constant, then $M' \not\equiv 0$. Therefore,

$$w' + \left(\frac{M''}{2M'} + \frac{(A-P)M}{2M'}\right) w = \frac{-A}{2M'}, \tag{6.3.8}$$

where $\frac{M''}{2M'} + \frac{(A-P)M}{2M'}$ is rational because $\frac{M'}{M} = \frac{L'}{L} + Q'$ is rational and $\frac{M''}{M}$ is rational and so $\frac{M''}{M'} = \frac{M''}{M} / \frac{M'}{M}$ is rational.

Also,

$$\frac{-A}{2M'} = \frac{-A}{2(L' + Q'L)e^{Q}}.$$

Then we can write (6.3.8) as

$$w' = Rw + Se^{-Q}, \tag{6.3.9}$$

where

$$R = \frac{PM - AM - M''}{2M'}, \qquad S = \frac{-A}{2(L' + Q'L)}$$

are rational functions and Q is a polynomial.

Let $U = Se^{-Q}$, then we can write (6.3.9) as

$$w' = Rw + U. \tag{6.3.10}$$

Now we have two cases to consider:

Case ii(a): If $R \equiv 0$ in (6.3.10), then (6.3.9) gives

$$w' = Se^{-Q},$$

and so

$$w = \frac{-w''}{P} = \frac{-(S' - Q'S)}{P} e^{-Q},$$

which is a contradiction since w has infinitely many zeros.

Case ii(b): Assume that $R \not\equiv 0$ in (6.3.10); then (6.3.10) gives

$$w'' = Rw' + R'w + U'.$$

Now (6.1.1) and (6.3.10) give

$$-Pw = R(Rw + U) + R'w + U',$$

and so

$$\left(R' + R^2 + P\right) w + RU + U' = 0.$$

Therefore,

$$R' + R^2 + P \equiv 0, \tag{6.3.11}$$

because if not, w has finitely many zeros, a contradiction. Also,

$$RU + U' \equiv 0. \tag{6.3.12}$$

Put

$$G = \frac{1}{U} = Te^Q, \tag{6.3.13}$$

where $T = 1/S$ is a rational function.

Then,

$$R = \frac{-U'}{U} = \frac{G'}{G}.$$

From (6.3.11) we get

$$0 = R' + R^2 + P = \frac{G''}{G} + P,$$

and so

$$G'' + PG = 0. \tag{6.3.14}$$

So, G solves (6.1.1) and since $P \not\equiv 0$ and is a polynomial of degree n, we see that G is a transcendental entire function with finitely many zeros and has order $(n + 2)/2$.

Since w and G solve the same equation but w has infinitely many zeros and G has finitely many zeros, w and G are linearly independent and we can write

$$w'G - wG' = c,$$

where c is a non-zero constant. So

$$\left(\frac{w}{G}\right)' = \frac{c}{G^2}.$$

By integrating we get

$$w = G\int^z \frac{c}{G^2}d\zeta. \tag{6.3.15}$$

Also, using (6.3.6), (6.3.13) and (6.3.15),

$$\begin{aligned} v = 1 + Mw &= 1 + Le^Q Te^Q \int^z \frac{c}{G^2}d\zeta \\ &= 1 + HG^2\int^z \frac{c}{G^2}d\zeta, \end{aligned} \tag{6.3.16}$$

where $H = L/T$ is a rational function.

Now we can assume that $c = 1$ because if $c \neq 1$, we can multiply w by $1/c$.

We differentiate (6.3.16) to get

$$\begin{aligned} v' &= H'G^2\int^z \frac{1}{G^2}d\zeta + 2HGG'\int^z \frac{1}{G^2}d\zeta + H \\ &= H + K(v-1), \end{aligned} \tag{6.3.17}$$

where

$$K = \frac{H'}{H} + 2\frac{G'}{G} \tag{6.3.18}$$

is a rational function.

So, from (6.1.2) and (6.3.17), we get

$$-Av = v'' = H' + K'(v-1) + K[H + K(v-1)],$$

and so

$$\begin{aligned} 0 &= v(K' + K^2 + A) + (H' - K' + KH - K^2) \\ &= vU_1 + U_2 \end{aligned}$$

where $U_1 = K' + K^2 + A$ and $U_2 = H' - K' + KH - K^2$.

Since v is transcendental and U_1, U_2 are rational functions, we must have

$$U_1 = K' + K^2 + A = 0 \tag{6.3.19}$$

and

$$U_2 = H' - K' + KH - K^2 = 0. \tag{6.3.20}$$

Claim 6.1. We claim that $H \equiv K$.

To show this, let $H \not\equiv K$.

From (6.3.20) we have

$$\frac{H' - K'}{H - K} + K = 0.$$

From (6.3.18) we get

$$\frac{H' - K'}{H - K} = -K = \frac{-H'}{H} - 2\frac{G'}{G}.$$

We integrate to get

$$H - K = \frac{a}{HG^2},$$

where a is a constant, and so

$$G^2 = \frac{a}{H(H - K)},$$

but this contradicts the fact that H and K are rational functions and G is a transcendental function. This completes the proof of Claim 6.1.

Once we have Claim 6.1, (6.3.17) gives

$$v' = H + H(v-1) = Hv,$$

and so

$$H = \frac{v'}{v}. \tag{6.3.21}$$

By (6.3.21), v has finitely many zeros, so we can write

$$v = P_1 e^{Q_1},$$

where P_1, Q_1 are polynomials, $P_1 \not\equiv 0$, and Q_1 is non-constant because v is transcendental.

Therefore,

$$w = \frac{v-1}{Le^Q} = \frac{P_1 e^{Q_1} - 1}{Le^Q}.$$

Now we can write this as

$$w = R_1 e^{S_1} + R_2 e^{S_2}, \tag{6.3.22}$$

where $R_1 = P_1/L \not\equiv 0$, $R_2 = -1/L \not\equiv 0$ are rational functions and $S_1 = Q_1 - Q$, $S_2 = -Q$ are polynomials.

Here, $R_1 e^{S_1}$ and $R_2 e^{S_2}$ are linearly independent because Q_1 is non-constant. Now we get

$$0 = w'' + Pw = J_1 e^{S_1} + J_2 e^{S_2} = J_1 e^{Q_1 - Q} + J_2 e^{-Q},$$

where J_1, J_2 are rational and satisfy

$$J_k = R_k'' + 2R_k' S_k' + R_k(S_k'' + S_k'^2 + P).$$

Therefore, $J_1 = J_2 = 0$ because otherwise $e^{Q_1} = -J_2/J_1$. Thus $R_1 e^{S_1}$, $R_2 e^{S_2}$ both solve $y'' + Py = 0$ and have finitely many zeros, and they are linearly independent.

Hence, P is constant by [6], which contradicts our assumption in Case (II) that P is non-constant. $\square$

Appendix: Problems for future work

In this thesis, we have studied some problems related to the behaviour of the zeros of solutions of differential equations. These involved polynomial and transcendental coefficients, homogeneous and non-homogeneous equations. Moreover, we have studied in Chapter 4 the case where the solutions of two second order homogeneous differential equations can take the value 0 and a non-zero value at (nearly) the same points.

Further cases can be considered in future. For example, we can look at when the solutions of two differential equations can have the same zeros in the unit disc or in a half plane; also, when the solutions of two differential equations take different values at (mostly) the same points in the unit disc or in a half plane.

Similar problems could perhaps be investigated for difference equations instead of differential equations.

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