A MARTINGALE APPROACH TO OPTIMAL PORTFOLIOS WITH JUMP-DIFFUSIONS AND BENCHMARKS

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Abstract

We consider various portfolio optimization problems when the stock prices follow jump-diffusion processes. In the first part the classical optimal consumption-investment problem is considered. The investor's goal is to maximize utility from consumption and terminal wealth over a finite investment horizon. We present results that modify and extend the duality approach that can be found in Kramkov and Schachermayer (1999). The central result is that the optimal trading strategy and optimal equivalent martingale measure can be determined as a solution to a system of non-linear equations.

In another problem a benchmark process is introduced, which the investor tries to outperform. The benchmark can either be a generic jump-diffusion process or, as a special case, a wealth process of a trading strategy. Similar techniques as in the first part of the thesis can be applied to reach a solution. In the special case that the benchmark is a wealth process, the solution can be deduced from the first part's consumption-investment problem via a transform of the parameters. The benchmark problem presented here gives a different approach to benchmarks as in, for instance, Browne (1999b) or Pra et al. (2004). It is also, as far as the author is aware, the first time that martingale methods are employed for this kind of problem. As a side effect of our analysis some interesting relationships to Platen's benchmark approach (cf. Platen (2006)) and change of numeraire techniques (cf. German et al. (1995)) can be observed.

In the final part of the thesis the set of trading strategies in the previous two problems are restricted to constraints. These constraints are, for example, a prohibition of short-selling or the restriction on the number of assets. Conditions are provided under which a solution to the two problems can still be found. This extends the work of Cvitanic and Karatzas (1993) to jump-diffusions where the initial market set-up is incomplete.

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Chapter 1

Introduction

This thesis studies various portfolio selection problems when stock prices follow jump-diffusion processes. In recent years jump-diffusion models as well as Lévy process models have become increasingly popular in the academic and financial literature. This is mainly due to the short-comings of the classical Black-Scholes model developed in Black and Scholes (1973).

Empirical studied of stock market returns as have been carried out by Cont (2001), Campbell et al. (1996), Pagan (1996), and others show that the distribution of stock market returns is leptokurtic, i.e. returns have higher peaks and heavier tails. Often jumps occur in the prices of stocks that cannot be explained by a Brownian motion driven model. These jumps also have a gain/loss asymmetry, meaning that one has large down movements in stock prices but not equally large up movements. Another feature often observed in the stock price distributions is that of volatility clustering: Large changes in prices are often followed by large changes and small changes tend to be followed by small changes.

To accommodate these kind of empirical observations Lévy and jump-diffusion models have been used which can capture many of the empirical features of stock price returns. The difference between Lévy and jump-diffusion models is thereby broadly the following. Lévy processes are Markov process that can have potentially infinite active jumps. When modelling a financial time series with a Lévy process model a Brownian motion component is not necessary needed as the process can essentially move by jumps. Lévy models can accommodate many empirical facts observed about stock price distributions and are very popular. However, because of the independent increment property (i.e. the Markov property) of Lévy processes they cannot model the effect of volatility clustering.

Jump-diffusion models on the other hand are generally assumed to have finite jumps during a finite time interval which represent rare events. The distribution of the jumps is usually assumed to be known so that they are easy to simulate. Because the model parameters can be time dependent and possibly random it is possible to model the effect of volatility clustering with jump-diffusions. For the case of constant model parameters a jump-diffusion becomes a Lévy process. Standard literature on Lévy process and jump-diffusions and their application to finance include Barndorff-Nielson et al. (2001), Cont and Tankov (2003), and Hanson (2009).

In the academic literature there is a wide range of portfolio selection problems. Most common are problems formulated either in the mean-variance framework pioneered by Markowitz (1952) or problems of expected utility maximization type as first considered by Merton (1969) and (1971) for a diffusion type model.

This thesis treats portfolio optimization problems of the later type which are formed in an expected utility maximization setting. Of these problems two classes will be considered. The first one is the classical problem of expected utility maximization of the terminal wealth of an investor and his/her consumption during the investment horizon. The second one is the problem of outperforming a benchmark process using again an expected utility maximization approach.

The first problem has been fully solved in the complete market case by Karatzas et al. (1987) and Cox and Huang (1989) among others. They use a martingale approach to solve the optimal investment-consumption problem considered by Merton (1969, 1971). This approach extends Merton's results as more general models can be considered that do not need to be of Markovian type as in Merton's work. In this thesis the results of Karatzas et al. (1987) and Cox and Huang (1989) and others for the geometric Brownian motion model will be extended to a jump-diffusion model for the stock prices. The jump-diffusion models lead to incompleteness in the market where the equivalent martingale measure (EMM) can no longer be determined uniquely. The incomplete market case is more delicate but results have been obtained in, for example, Cvitanic and Karatzas (1992) when the incompleteness is produced by additional Brownian motions in the model. Particularly Kallsen (2000) and Kramkov and Schachermayer (1999) developed results for Lévy processes and general semimartingale models respectively. Important relationships, similarities, and differences to their work will be pointed out whenever relevant.

The main result of this part of the thesis is the extension and combination of the ideas of the mentioned papers to obtain a solution to the consumption-investment problem when stock prices are driven by jump-diffusion processes. In particular the duality approach by Kramkov and Schachermayer (1999) is modified and extended in such a way that the optimal trading strategy and optimal EMM can be found as a solution to a system of non-linear equations.

In the next part of the thesis we consider the problem that an investor wants to outperform a benchmark. This benchmark can be a stock index, a stock portfolio, an exchange rate, or a similar benchmark. The benchmark is assumed to follow a jump-diffusion process and the investor's wealth process is given as in the previous parts of the thesis. Again, it is assumed that the stocks are driven by jump-diffusions. The investor's way to try to outperform the benchmark is to maximize expected utility from the ratio between investor's wealth and the benchmark. This ratio will be called relative wealth and is popular in the literature. However, as far as the author is aware, it hasn't been defined in a jump-diffusion context.

A martingale approach will be developed to present a general approach to active portfolio selection problem which has many parallels to the first part of the thesis. Benchmark related problems have been treated mostly from a statistical point of view in the literature assuming certain distributions for the stock returns rather than using a model for the stock prices. A paper that probably comes closest to the approach presented here is Browne (1999b). He uses a geometric Brownian motion model for the stock prices and the benchmark process to solve the problem of reaching a performance goal before a deadline. The solution is found using stochastic control methods. Other papers like Popova et al. (2007) consider the problem of maximizing the probability of beating the benchmark. However, a martingale approach hasn't been presented in the literature for these kind of problems.

As a byproduct of the analysis some relationships to Platen's benchmark approach (cf. Platen (2005)) can be found. Interpreting the model set-up as a change of numeraire using change of numeraire techniques as developed by Geman et al. (1995) clarifies the relationship between the discounted wealth process of an investor and his/her relative wealth process.

In a final chapter the two previous problems are constrained in the sense that the underlying trading strategy has to be in a closed convex set. This allows for example for short selling constraints or prohibition of borrowing. It extends the approach developed in Cvitanic and Karatzas (1992) and (1993) to the jump-diffusion framework, first for the consumption-investment problem and later for the benchmark problem.

The thesis is structured as follows. Chapter 2 revises all necessary tools from stochastic calculus and stochastic differential equation for jump-diffusion processes. The stock market model will be introduced and analysed in Chapter 3, with the final section of Chapter 3 introducing the concept of utility functions, which will accompany us throughout the thesis. These two chapters will provide an extensive review of the martingale methods used for the optimal portfolio problems. Chapter 4 presents the consumption-investment problem in the jump-diffusion framework. Martingale methods will be developed to solve the problem and will later relate to partial differential equations. The problem of outperforming a benchmark process is solved in Chapter 5, and in addition, relationships to change of numeraire techniques will be established in this chapter. Links to Platen's benchmark approach (cf. Platen (2005)) will be highlighted and some extensions of his approach towards jump-diffusion will be made. The constrained version of the problem of Chapter 4 and 5 will be solved in Chapter 6, which concludes this thesis.

Chapter 2

Results on Stochastic Calculus and Stochastic Differential Equations

A stochastic process is a family of random variables $(X(t))_{t\in[0,T]}$ indexed by time. The time parameter can be either discrete or continuous, but we will only consider the continuous case. In this thesis we consider only processes that are of càdlàg type. That is processes that have right continuous sample paths with left limits. The processes that are introduced in this chapter either have càdlàg path or have a modification that has càdlàg paths. In what follows we always consider the càdlàg version of the process.

Throughout the thesis the space $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space. On this probability space there exists an increasing family of σ-algebras $(\mathcal{F}_t)_{t\in[0,T]}$ that forms an information flow or filtration. The filtration together with the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ are called a filtered probability space. A process $X(t)$ is called adapted or non-anticipating with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ if each $X(t)$ is revealed at time t, i.e. each $X(t)$ is $\mathcal{F}_t$-measurable. A process $X(t)$ is always adapted to its history or natural filtration which is given by

$$\mathcal{F}_t^X = \sigma(X(t), C | s \in [0,t], C \in \mathcal{N})$$

where $\mathcal{N}$ is the set of all the null sets of the state space of the process. $\mathcal{F}_t^X$ contains all the information of X_t up to time t, that is, it contains all the information about the realized sample path of $X(t)$. If the filtration that a process is adapted to is not specified, then the natural filtration is assumed.

2.1 Brownian Motion and the Poisson Process

Two fundamental examples of stochastic processes are the Brownian motion $B(t)$ and the Poisson process $P(t)$. They form the toolbox to create jump-diffusion processes. Both processes are Markov processes which means that they are without memory of all but the prior state. If we want to make predictions about the future state then all that is needed is the current state of the process, rather than the complete history of the process. Both Brownian motion and Poisson process have càdlàg sample paths (right-continuous with left limits) and are Lévy processes. The paths of a Brownian motion are continuous whereas the paths of a Poisson process are piecewise constant.

Definition 2.1. *An adapted process $(\boldsymbol{B}(t))_{t\geq 0}$ taking values in $\mathbb{R}^n$ is called an n-dimensional Brownian motion if*

1. *the process $(\boldsymbol{B}(t))_{t\geq 0}$ has independent and stationary increments;*
2. *the increments $\boldsymbol{B}(t) - \boldsymbol{B}(s)$ are normally distributed with mean zero and variance matrix $(t-s)\boldsymbol{\sigma}$, for a given, non random matrix $\boldsymbol{\sigma}$;*
3. *the sample paths of $(\boldsymbol{B}(t))_{t\geq 0}$ are a.s. continuous.*

The last item (iii) is not essential if B is also separable. If instead of (iii) B is also a separable stochastic process it it can be shown that every Brownian motion has a modification with continuous sample paths. It is then assumed that we always consider the version with the continuous sample paths.

A Brownian motion is called standard if its variance matrix $\boldsymbol{\sigma}$ is equal to the unit matrix I. For a one-dimensional Brownian motion $B(t)$ it can be shown (cf. Hanson (2002)) that its auto-covariance is given by

$$\mathrm{Cov}\big[B(t), B(s)\big] = \min(t, s).$$

The Poisson process is a discontinuous process that counts the number of random occurrences of some events which happen in a certain time interval. The inter arrival time between two events occurring is exponentially distributed.

Definition 2.2. *Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter $\lambda > 0$ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $(P(t))_{t\geq 0}$ defined by*

$$P(t) = \sum_{n\geq 1} \mathbf{1}_{\{t\geq T_n\}}$$

is called a Poisson process with intensity λ. The function $\mathbf{1}_{\{.\}}$ is thereby the indicator function.

A Poisson process has piecewise constant sample paths and it increases by jumps of size 1. Its increments $P(t) - P(s)$ are independent and stationary and have a Poisson distribution with intensity $(t-s)\lambda$ for $t > s \geq 0$, i.e.

$$\mathbb{P}\big[P(t) - P(s) = n\big] = e^{-\lambda(t-s)} \frac{\big(\lambda(t-s)\big)^n}{n!} \quad \forall n \in \mathbb{N} \cup \{0\}.$$

From the definition the time between jumps of a Poisson process is exponentially distributed so that

$$\mathbb{P}\big[\tau_i \leq t\big] = 1 - e^{-\lambda t},$$

where τ_i is the inter jump-time between the $(i-1)th$ and ith jump. The auto-covariance of a Poisson process $P(t)$ is given by

$$\text{Cov}\big[P(t), P(s)\big] = \lambda \min(t, s).$$

See Hanson (2002) for a proof and more details on Poisson processes.

2.2 Poisson Random Measures and Jump Measures

We will introduce the one dimensional Poisson random measure in this section. Most of the material is borrowed from Cont and Tankov (2003) and additional information can be found therein.

Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a measure space $(E, \mathcal{E}, \mu)$ with $E = [0, T] \times \mathbb{R} \backslash \{0\}$ and $\mathcal{E}$ being the Borel σ-algebra of E, a Poisson random measure N is a integer valued random measure

$$N : \Omega \times \mathcal{E} \to \mathbb{N} \cup \{0\}$$

such that

1. $N(\omega, \cdot)$ is a Random measure on E, that is $N(A) = N(\cdot, A)$ is an integer valued random variable for bounded measurable $A \subset E$;

2. $N(\cdot, A)$ is a Poisson random variable with parameter $\mu(A)$ for each measurable $A \subset \mathcal{E}$;

3. for disjoint measurable sets $A_1, \ldots, A_n \in \mathcal{E}$, the random variables $N(A_1), \ldots, N(A_n)$ are independent.

Given a certain sequence of non-anticipating random times $(T_n)_{n \geq 1}$ and a sequence of random variables $(Y_n)_{n \geq 1}$ that are revealed at the random times T_n (i.e. Y_n is $\mathcal{F}_{T_n}$ measurable) each Poisson random measure N can be represented as a counting measure.

It counts the number of events that have occurred at times T_n if at these times the corresponding Y_n has hit a certain set. That is, if δ denotes the dirac delta function then the Poisson random measure can be represented as

$$N = \sum_{n \geq 1}^{n} \delta_{(T_n, Y_n)}. \tag{2.1}$$

The random times T_n can be seen as random jump times and Y_n corresponds to the jump-sizes at time T_n. If we take $[s,t) \in [0,T]$ and $C \in \mathcal{B}(\mathbb{R})$ the Poisson measure counts the number of jumps between time s and t whose size lie in the set C:

$$N\big(\omega, [s,t) \times C\big) = \sum_{n \geq 1}^{n} \delta_{(T_n(\omega), Y_n(\omega))}\big([s,t) \times C\big).$$

It is important that each Poisson random measure corresponds to a sequence of processes $(T_n, Y_n)_{n \geq 1}$ such that (2.1) holds. However, the converse is in general not true. That means that for a sequence $(T_n, Y_n)_{n \geq 1}$, a measure formed by (2.1) is in general not a Poisson random measure. This is because for $A \subset \mathcal{E}$, $N(\cdot, A)$ is generally not a Poisson random variable any more. Also often the independence property doesn't hold. Yet, processes that are formed in such a way can be interesting and are called marked point processes.

A way to construct a marked point process is by using a càdlàg stochastic process $X(t)$. Let us denote by $\Delta X(t) = X(t) - X(t-)$ the jump size (possibly zero) of $X(t)$ at time t. Then every such càdlàg process X has at most countable number of jumps, that is the set

$$\{t \in [0,T] | \Delta X(t) \neq 0\} \tag{2.2}$$

is countable (cf. Cont and Tankov (2003)). The elements of (2.2) can be arranged in a sequence $(T_n)_{n \geq 1}$ which are the random jump times of X. At the jump times X has discontinuity of size

$$Y_n := X(T_n) - X(T_n-) \in \mathbb{R}^d \setminus \{0\}.$$

The jump times together with the jump sizes form a marked point process $(T_n, Y_n)_{n \geq 1}$ on $[0,T] \times \mathbb{R}^d \setminus \{0\}$ which contains all the information about the jumps of the process X. Following the steps above the associated random jump measure of X is

$$J_X = \sum_{n \geq 1} \delta_{(T_n(\omega), Y_n(\omega))} = \sum_{t \in [0,T]}^{\Delta X(t) \neq 0} \delta_{(t, \Delta X(t))}. \tag{2.3}$$

As mentioned before this jump measure is in generally not a Poisson random measure. However, a case when the jump measure is in fact a Poisson measure is when the process X is a Lévy process. We will come back to this when we discuss the Lévy-Itô representation of Lévy processes in Section 2.3. We have discussed how to construct jump measures and Poisson random measure from marked point processes and càdlàg process. On the other hand it is possible to construct jump processes from Poisson random measures.

Let N be a Poisson random measure with intensity measure μ and let $f : E \to \mathbb{R}$ be a measurable function that satisfies

$$\int_0^t \int_{\mathbb{R}\backslash\{0\}} |f(s,y)|\mu(ds \times dy) < \infty.$$

Then a jump processes $X(t)$ can be constructed as the stochastic integral with respect to the Poisson measure N:

$$X(t) = \int_0^t \int_{\mathbb{R}\backslash\{0\}} f(s,y)N(ds,dy) = \sum_{\{n\geq 1, T_n \in [0,t]\}} f(T_n, Y_n).$$

Probably the simplest example of a jump process constructed through a Poisson random measure is a Poisson process. The Poisson process counts the numbers of jumps of size 1 with intensity λ. The corresponding Poisson random measure is given by

$$N_P(\omega, [s,t) \times C) = \begin{cases} \#\{i \geq 1 | T_i(\omega) \in [s,t)\} & \text{if } 1 \in C \\ 0 & \text{if } 1 \notin C \end{cases}$$

for $t > s \geq 0$ and $C \in \mathcal{B}(\mathbb{R})$ and where $(T_i)_{i\geq 1}$ are the random jump times of the process. The intensity measure μ_P of N_P is then given by

$$\mu_P([s,t), C) = \lambda(t-s)$$

if C contains the jump size 1, and is equal to zero otherwise. The Poisson process is thus,

$$P(t) = \int_0^t N_P(ds, \{1\}) = \int_0^t N_P(ds)$$

The intensity measure μ of a jump process can be time dependent μ_t or time homogeneous μ. We will only consider the later case and moreover the intensity measure is assumed to have the form $\mu(dt, dy) = \nu(dy)dt$.

2.3 Lévy Processes

A class of stochastic processes for which a lot of research has been carried out are Lévy processes. Important examples of Lévy processes are Brownian motions and Poisson processes.

Definition 2.3. *A càdlàg stochastic process $(X(t))_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$ such that $X(0) = 0$ is called a Lévy process if it possesses the following properties:*

1. *independent increments: for* $t_0 < t_1 < \ldots < t_n$ *the random variables* $X(t_0), X(t_1) - X(t_0), \ldots, X(t_n) - X(t_{n-1})$ *are independent;*

2. *stationary increments: the law of* $X(t+h) - X(t)$ *does not depend on* t*;*

3. *stochastic continuity:* $\forall \epsilon > 0 \lim_{h \to 0} P\big(|X(t+h) - X(t)| \geq \epsilon\big) = 0.$

The last property doesn't imply the continuity of sample paths but ensures that Lévy processes don't have jumps at fixed (non-random) times.

Each Lévy process has a Lévy measure ν that characterizes the jumps of the process. It is defined by

$$\nu(C) := \mathbb{E}\big[\#\{t \in [0,1] | \Delta X(t) \neq 0, \Delta X(t) \in C\}\big], \quad C \in \mathcal{B}(\mathbb{R}). \tag{2.4}$$

It counts the expected number of jumps between the time interval from 0 to 1 whose size belongs to C. Notice that it doesn't depend on a time variable.

A Lévy process can be decomposed into a non-random drift part, a continuous random part, represented by a Brownian motion, and a jump part, given by two integrals with respect to a Poisson random measure. One of the Poisson integrals represent large jumps, the other one integrates over small jumps. The big jumps are often restricted to be finite but the small jumps can occur infinitely often. If that is the case the Lévy process is called infinitely active.

Theorem 2.4 (Lévy-Itô decomposition)**.** *Let* $X(t)$ *be a Lévy process on* $\mathbb{R}$ *and* ν *it's Lévy measure satisfying*

$$\int_{|x|\leq 1} |x|^2 \nu(dx) < \infty, \quad \int_{|x|\geq 1} \nu(dx) < \infty \tag{2.5}$$

and let J_X *be its associated jump measure from* (2.3)*, which is in fact a Poisson random measure. There exists a scalar* $\alpha \in \mathbb{R}$ *and a Brownian motion* B *with variance* σ *such that*

$$\begin{aligned} X(t) &= \alpha t + B(t) + X^l(t) + \lim_{\epsilon \downarrow 0} X^\epsilon(t), \qquad \textit{where} \\ X^l(t) &= \int_0^t \int_{|x|\geq 1} x J_X(ds \times dx), \quad \textit{and} \\ X^\epsilon(t) &= \int_0^t \int_{\epsilon \geq |x| > 1} x\big(J_X(ds \times dx) - \nu(dx)dt\big). \end{aligned}$$

The triplet (σ, ν, α) completely characterizes the Lévy process and is therefore often called the characteristic triplet. The condition (2.5) ensures that the processes doesn't blow up because of infinite number of jumps around zero. Also the second condition ensures that the number of jumps of absolute size above 1 is finite.

An example of a Lévy process of pure jump type is the compound Poisson process.

Definition 2.5. *A compound Poisson process with intensity $\lambda > 0$ and jump size distribution f is a stochastic process $X(t)$ defined by*

$$X(t) = \sum_{i=1}^{P(t)} Y_i$$

where jump sizes Y_i are i.i.d. with distribution f and $P(t)$ is a Poisson process with intensity λ independent from $(Y)_{i\geq 1}$.

The sample paths of a compound Poisson process $X(t)$ are càdlàg piecewise constant functions and the jump times $(T_i)_{i\geq 1}$ can be expresses as partial sums of independent exponential random variables with parameter λ. Every compound Poisson process $X(t)$ can be associated to a random measure on $[0,\infty)\times\mathbb{R}$ that describe the jumps of X:

$$J_X(B) = \#\{\big(t, X(t) - X(t-)\big) \in A\},$$

for measurable $A \subset [0,\infty)\times\mathbb{R}$. The jump measure J_X is in fact a Poisson random measure with intensity measure $\mu(dx \times dt) = \nu(dx)dt = \lambda f(dx)dt$. The measure ν is thereby the Lévy measure as in (2.4). The compound Poisson process can be thus expresses as

$$X(t) = \sum_{s\in[0,t]} \Delta X(s) = \int_0^t \int_{\mathbb{R}^d} x J_X(ds \times dx).$$

2.4 Jump-Diffusion Processes

In the following let $\boldsymbol{B}(t) = \big(B_1(t), \dots, B_n(t)\big)$ be a n-dimensional Brownian motion, whose components are mutually independent standard Brownian motions. Further there are m mutually independent jump process given by the Poisson random measures $N_h(dt, dy)$ with corresponding jump size functions γ_h, and intensity measures $\nu_h(dy)dt$ which do not depend on the time variable for $h = 1, \dots, m$. The Brownian motions and jump processes are assumed to be mutually independent too.

A jump-diffusion process is a stochastic process $X(t)$ that follows a stochastic differential equation of the form

$$dX(t) = \alpha(t, X(t))dt + \boldsymbol{\beta}(t, X(t))^{\intercal} d\boldsymbol{B}(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_h\big(s, y, X(t-)\big) N_h(ds, dy), \quad (2.6)$$

for some progressively measurable processes $\alpha, \boldsymbol{\beta}$, and γ_h, $h = 1, \dots, m$. Note that the name jump-diffusion has nothing to do with a diffusion even if we neglect the jumps. The parameter functions are time dependent and random which does not hold for a diffusion.

Nevertheless, the name jump-diffusion is very common in the literature for processes like this.

In the case that the parameter functions are time homogeneous and non-random, i.e.

$$dX(t) = \alpha(X(t))dt + \boldsymbol{\beta}(X(t))^{\intercal}d\boldsymbol{B}(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_h\big(y, X(t-)\big)N_h(ds, dy). \qquad (2.7)$$

we call a solution to (2.7) a *Lévy diffusion*. In that case $X(t)$ is a Lévy process. Additionally, if there are also no jumps, then the process is a diffusion:

$$dX(t) = \alpha(X(t))dt + \boldsymbol{\beta}(X(t))^{\intercal}d\boldsymbol{B}(t). \qquad (2.8)$$

The following theorem, borrowed from Øksendal and Sulem (2007), gives a result on the existence and uniqueness of solutions to jump-diffusion stochastic differential equations. If the parameter functions satisfy at most linear growth and Lipschitz continuity then a solution exists and is unique.

Theorem 2.6. *Consider the stochastic differential equation* (2.6) *with* $X(0) \in x_0 \in \mathbb{R}$ *where* $\alpha : [0,T] \times \mathbb{R} \to \mathbb{R}$, $\boldsymbol{\beta} : [0,T] \times \mathbb{R} \to \mathbb{R}^n$, *and* $\gamma_h : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *for* $h = 1, \dots, m$ *satisfy the following conditions*

1. *There exists a constant* $C_1 < \infty$ *such that*

$$||\boldsymbol{\beta}(t,x)||^2 + |\alpha(t,x)|^2 + \int_{\mathbb{R}\setminus\{0\}} \sum_{h=1}^{m} |\gamma_h(t,x,z)|^2 \nu_k(dz) \le C_1(1 + |x|^2) \qquad (2.9)$$

for all $x \in \mathbb{R}$.

2. *There exists a constant* $C_2 < \infty$ *such that*

$$\begin{aligned} &||\boldsymbol{\beta}(t,x) - \boldsymbol{\beta}(t,y)||^2 + |\alpha(t,x) - \alpha(t,y)|^2 \\ &+ \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} |\gamma_h(t,x,z) - \gamma_h(t,y,z)|^2 \nu_h(dz) \le C_2|x-y|^2 \end{aligned} \qquad (2.10)$$

for all $x, y \in \mathbb{R}$.

Then there exists a unique càdlàg adapted solution $X(t)$ *such that*

$$\mathbb{E}[|X(t)|^2] < \infty \quad \textit{for all } t \in [0,T].$$

We will often consider stochastic differential equations that have a geometric form which means that parameter functions have the form $\alpha(t,x) = x\widetilde{\alpha}(t)$, $\boldsymbol{\beta}(t,x) = x\widetilde{\boldsymbol{\beta}}(t)$, and $\gamma_h(t,y,x) = x\widetilde{\gamma}_h(t,y)$, $h = 1, \dots, m$. The stochastic differential equation (2.6) becomes then

$$dX(t) = X(t)\alpha(t)dt + X(t)\boldsymbol{\beta}(t)^{\intercal}d\boldsymbol{B}(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} X(t-)\gamma_h\big(s,y\big)\big)N_h(ds,dy). \quad (2.11)$$

In this case the conditions for the existence and uniqueness of a solution to (2.11) given in (2.9) and (2.10) simplify to

$$||\boldsymbol{\beta}(t)||^2 + |\alpha(t)|^2 + \int_{\mathbb{R}\setminus\{0\}} \sum_{h=1}^{m} |\gamma_h(t,z)|^2 \nu_k(dz) < \infty \tag{2.12}$$

for all $t \in [0,T]$.

2.5 Stochastic Calculus with Jump-Diffusions

As before we are going to work with an n-dimensional Brownian motion $\boldsymbol{B}(t)$, whose components are mutually independent standard Brownian motions, and m mutually independent jump processes given by the Poisson random measures $N_h(dt,dy)$ with corresponding jump size functions γ_h, and intensity measures $\nu_h(dy)$. In a general set up the intensity measures can be time dependent, however this will not be the case in this thesis. The Brownian motions and jump processes are assumed to be mutually independent.

For our purposes it is enough to consider one dimensional processes that are driven by several sources of randomness:

$$dX(t) = \alpha(t)dt + \boldsymbol{\beta}(t)^{\intercal} d\boldsymbol{B}(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_h(t,y) N_h(dt,dy), \quad t \in [0,T]. \tag{2.13}$$

For convenience reasons we assume that the number of jumps that can occur during any time interval is finite. This allows us to neglect certain convergence problems when dealing with infinite active jumps (see Cont and Tankov (2003) for details).

For a jump-diffusion process X denote by X^c its continuous parts. Then Itô's formula for jump-diffusions is basically the same as Itô's formula for continuous processes (Brownian motion) but with adding the jump increments.

Theorem 2.7 (Itô's Formula for Jump-Diffusions). *Let $X(t)$ be a jump-diffusion process as in* (2.13). *If $f \in C^{1,2}([0,T],\mathbb{R})$ then*

$$\begin{aligned} df\big(t,X(t)\big) = {} & \frac{\partial f}{\partial t}\big(t,X(t-)\big)dt + \frac{\partial f}{\partial x}\big(t,X(t-)\big)dX^c(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\big(t,X(t-)\big)\boldsymbol{\beta}(t)\boldsymbol{\beta}(t)^{\intercal}dt \\ & + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \Big\{ f\big(t,X(t-)+\gamma_h(t,y)\big) - f\big(t,X(t-)\big) \Big\} N_h(dt,dy) \end{aligned} \tag{2.14}$$

for $t \in [0,T]$. In other form, we have

$$df\big(t,X(t)\big) = \frac{\partial f}{\partial t}\big(t,X(t-)\big)dt + \frac{\partial f}{\partial x}\big(t,X(t-)\big)\alpha(t)dt$$
$$+ \frac{\partial f}{\partial x}\big(t,X(t-)\big)\boldsymbol{\beta}(t)^{\intercal}d\boldsymbol{B}(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\big(t,X(t-)\big)\boldsymbol{\beta}(t)\boldsymbol{\beta}(t)^{\intercal}dt$$
$$+ \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\Big\{f\big(t,X(t-)+\gamma_h(s,y)\big) - f\big(t,X(t-)\big)\Big\}N_h(dt,dy).$$

Denote by $\Delta X(t)$ the size of the jump (possibly zero) at time t. Then Itô's formula can also be expressed as

$$f\big(t,X(t)\big) = f\big(0,X(0)\big) + \int_0^t \frac{\partial f}{\partial t}\big(s,X(s-)\big)ds$$
$$+ \int_0^t \frac{\partial f}{\partial x}\big(s,X(s-)\big)dX(s) + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}\big(s,X(s-)\big)\boldsymbol{\beta}(s)\boldsymbol{\beta}(s)^{\intercal}ds$$
$$+ \sum_{0\le s\le t}^{\Delta X(s)}\Big[f\big(s,X(s-)+\Delta X(s)\big) - f\big(s,X(s-)\big) - \Delta X(s)\frac{\partial f}{\partial x}\big(s,X(s-)\big)\Big].$$

Note, however, that in the case of an infinitely active jump process the sum involving the jump terms can be infinite. In that case, the sum

$$\sum_{0\le s\le t}^{\Delta X(s)}\Big[f\big(s,X(s-)+\Delta X(s)\big) - f\big(s,X(s-)\big) - \Delta X(s)\frac{\partial f}{\partial x}\big(s,X(s-)\big)\Big] \tag{2.15}$$

doesn't necessary converge. We therefore often assume that f and its two derivatives are bounded by a constant C. Then

$$|f\big(s,X(s-)+\Delta X(s)\big) - f\big(s,X(s-)\big) - \Delta X(s)\frac{\partial f}{\partial x}\big(s,X(s-)\big)| \le C\Delta X(s)^2$$

and the sum in (2.15) is indeed finite. However, as the number of jumps are assumed to be finite in a finite time interval there is no need to worry about these convergence issues.

Itô's formula takes a similar form in the case that the underlying process is a geometric jump diffusion $Y(t)$:

$$dY(t) = Y(t-)\Big[\alpha(t)dt + \boldsymbol{\beta}(t)^{\intercal}d\boldsymbol{B}(t) + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\gamma_h(t,y)N_h(dt,dy)\Big]. \tag{2.16}$$

Theorem 2.8 (Itô's Formula for geometric Jump-Diffusions)**.** *Let $Y(t)$ be a geometric jump-diffusion process as in (2.16). If $f \in C^{1,2}([0,T],\mathbb{R})$ then*

$$df\big(t,Y(t)\big) = \frac{\partial f}{\partial t}\big(t,Y(t-)\big)dt + Y(t)\frac{\partial f}{\partial x}\big(t,Y(t-)\big)\alpha(t)dt$$
$$+ Y(t)\frac{\partial f}{\partial x}\big(t,Y(t-)\big)\boldsymbol{\beta}(t)^{\intercal}d\boldsymbol{B}(t) + \frac{1}{2}Y(t)^2\frac{\partial^2 f}{\partial x^2}\big(t,Y(t-)\big)\boldsymbol{\beta}(t)\boldsymbol{\beta}(t)^{\intercal}dt$$
$$+ \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\Big\{f\big(t,Y(t-)[1+\gamma_h(t,z)]\big) - f\big(t,Y(t-)\big)\Big\}N_h(dt,dz). \tag{2.17}$$

for $t \in [0, T]$.

A proof can be found in Øksendal and Sulem (2007).

Strongly related to Itô's formula is Itô's product rule that determines the dynamics of the product of two stochastic processes. It is an immediate consequence of the multivariate Itô's formula. As before we denote for a jump-diffusion process $X(t)$ its change in the continuous part by $dX^c(t)$ and its change in the discontinuous part by $\Delta X(t)$, so that $dX(t) = dX^c(t) + \Delta X(t)$.

Lemma 2.9. *Let* $X(t)$ *and* $Y(t)$ *be two jump-diffusion processes. Itô's product rule determines the change of the product of the two processes by*

$$\begin{aligned} d\big(X(t)Y(t)\big) =& X(t-)dY(t) + Y(t-)dX(t) + dX(t)dY(t) \\ =& X(t-)dY^c(t) + Y(t-)dX^c(t) + dX^c(t)dY^c(t) \\ &+ X(t-)\Delta Y(t) + Y(t-)\Delta X(t) + \Delta X(t)\Delta Y(t). \end{aligned} \tag{2.18}$$

Remark.

1. Products that may arise in equation (2.18) are, in Itô's mean square sense, as the following

$$dtdt = 0, \quad dB_i(t)dt = 0, \quad dt \int_{\mathbb{R}\setminus\{0\}} \gamma_h(t, y)N_h(dt, dy) = 0,$$

   and

$$dB_i(t) \int_{\mathbb{R}\setminus\{0\}} \gamma_h(t, y)N_h(dt, dy) = 0,$$

   for $i = 1, \ldots, n$, and $h = 1, \ldots, m$.

2. Products of Brownian motions and Poisson measure integrals are, respectively,

$$dB_i(t)dB_j(t) = \begin{cases} 0, & \text{if } i \neq j \\ dt, & \text{if } i = j \end{cases}$$

   for $i, j = 1, \ldots, n$, and

$$\begin{aligned} &\int_{\mathbb{R}\setminus\{0\}} \gamma_h(t, y)N_h(dt, dy) \int_{\mathbb{R}\setminus\{0\}} \widetilde{\gamma}_l(t, y)N_l(dt, dy) \\ &= \begin{cases} 0, & \text{if } h \neq l \\ \int_{\mathbb{R}\setminus\{0\}} \gamma_h(t, y)\widetilde{\gamma}_h(t, y)N_h(dt, dy), & \text{if } h = l \end{cases} \end{aligned}$$

   for $h, l = 1, \ldots, m$.

3. The last term of the first equation in (2.18), $dX(t)dY(t)$ is the change of the quadratic variation process. The quadratic variation process for two processes (at least semimartingales) is defined through

$$[X,Y]_t := X(t)Y(t) - \int_0^t X(s-)dY(s) - \int_0^t Y(s-)dX(s), \quad t \in [0,T].$$

It is different from the $\langle X, Y\rangle$ process, which consists only of the continuous parts of $[X,Y]$, i.e. $\langle X, Y\rangle = [X,Y]^c$.

As an example, if it is assumed that the two processes X and Y are geometric jump-diffusions and have dynamics

$$dX(t) = X(t-)\Big[\alpha(t)dt + \boldsymbol{\beta}(t)^\intercal d\boldsymbol{B}(t) + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \gamma_h(t,y)N_h(dt,dy)\Big],$$

and

$$dY(t) = Y(t-)\Big[\widetilde{\alpha}(t)dt + \widetilde{\boldsymbol{\beta}}(t)^\intercal d\boldsymbol{B}(t) + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \widetilde{\gamma}_h(t,y)N_h(ds,dy)\Big],$$

respectively, then their product is following the stochastic differential equation

$$\begin{aligned} d\big(X(t)Y(t)\big) = X(t-)Y(t-)\Big[&\Big(\widetilde{\alpha}(t) + \alpha(t) + \widetilde{\boldsymbol{\beta}}(t)^\intercal\boldsymbol{\beta}(t)\Big)\,dt + \Big(\widetilde{\boldsymbol{\beta}}(t) + \boldsymbol{\beta}(t)\Big)^\intercal\,d\boldsymbol{B}(t) \\ &+ \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\Big(\widetilde{\gamma}_h(t,y) + \gamma_h(t,y) + \widetilde{\gamma}_h(t,y)\gamma_h(t,y)\Big)N_h(dt,dy)\Big]. \end{aligned}$$

The section is concluded discussing a special form of a jump process which is a generalization of the compound Poisson process introduced in Definition 2.5. Consider a pure jump process X that is the solution to the stochastic differential equation

$$dX(t) = \int_{\mathbb{R}\setminus\{0\}} \gamma(t,y)N(dt,dy)$$

for some Poisson random measure $N(t,\cdot)$. Assume further that the dynamics of the process X can be written in the form $dX(t) = \Gamma(t)dP(t)$ for a Poisson process $P(t)$, where $\Gamma(t)$ represents the size of the jump at time t if a jump actually occurs. If $t_1, t_2, \ldots$ denote the (random) jump times of the Poisson process, then the process X has the solution

$$X(t) = X(0) + \int_0^t\int_{\mathbb{R}\setminus\{0\}} \gamma(s,y)N(ds,dy) = X(0)\int_0^t \Gamma(s)dP(s) = X(0) + \sum_{\ell=1}^{P(t)}\Gamma(t_\ell)$$

The process is a generalization of the compound Poisson process since the jump-sizes are now time dependent $\Gamma(t)$. If we are carrying out a similar analysis for a geometric jump-process of the form

$$dY(t) = Y(t-)\int_{\mathbb{R}\setminus\{0\}} \gamma(s,z)N(ds,dz) = Y(t-)\int_{\mathbb{R}\setminus\{0\}} \Gamma(s)dP(s), \tag{2.19}$$

it is necessary to use Itô's formula for jump-diffusions to get to a solution. Applying Itô's formula Theorem 2.17 to $\log(x)$ and $Y(t)$ shows since $\log(y + y\gamma) - \log(y) = \log(1 + \gamma)$ that

$$d\log(Y(t)) = \int_{\mathbb{R}\setminus\{0\}} \log\big(1+\gamma(t,z)\big)N(dt,dz) = \log\left(1+\Gamma(t)\right)dP(t) = \sum_{\ell=1}^{dP(t)} \log\left(1+\Gamma(t_\ell)\right)$$

The solution to (2.19) can then be deduced to be

$$Y(t) = Y(0)\prod_{\ell=1}^{P(t)}\left(1+\Gamma(t_\ell)\right).$$

We will come back to this particular case when we discuss the stock market model in Chapter 3.

2.6 Martingales, Compensated Jump Processes and Change of Measure

Throughout this thesis the concept of a martingale will play a central role. A martingale is a stochastic càdlàg process $X(t)$ that is adapted to a filtration $\mathcal{F}_t$, $\mathbb{E}[|X(t)|]$ is finite for any $t \in [0,T]$, and

$$\mathbb{E}[X(t)|\mathcal{F}_s] = X(s), \quad 0 \leq s \leq t \leq T.$$

Typical examples for martingales are the Brownian motion $B(t)$ or the geometric Brownian motion $\exp(-\frac{1}{2}a^2t + aB(t))$ for some $a \in \mathbb{R}$. A Poisson process is in general not a martingale. However a compensated version of a Poisson process can be formed that is a martingale. For a Poisson process $N(t)$ with intensity λ the compensated Poisson process is

$$\widetilde{N}(t) = N(t) - \lambda t.$$

It represents a centered version of the Poisson process. The martingale property follows from

$$\begin{aligned}\mathbb{E}[\widetilde{N}(t)|\mathcal{F}_s] &= \mathbb{E}[N(t)-N(s)|\mathcal{F}_s] + N(s) - \lambda t\\ &= \mathbb{E}[N(t)-N(s)] + N(s) - \lambda t = N(s) + \lambda(t-s) - \lambda t = \widetilde{N}(s).\end{aligned}$$

Similarly, a jump-process $\int_{\mathbb{R}\setminus\{0\}}\gamma(t,y)N(dt,dy)$ constructed through a Poisson random measure $N(\cdot,\cdot)$ is in general not a martingale. However, integrating with respect to the compensated Poisson random measure $\widetilde{N}(dt,dy) := N(dt,dy) - \nu(dy)dt$, where ν is the intensity measure of the Poisson random measure, makes the process a martingale (given that γ satisfies the integrability condition). That is

$$\int_{\mathbb{R}\setminus\{0\}}\gamma(t,y)\widetilde{N}(dt,dy) = \int_{\mathbb{R}\setminus\{0\}}\gamma(t,y)N(dt,dy) - \int_{\mathbb{R}\setminus\{0\}}\gamma(t,y)\nu(dy)dt$$

is a martingale as long as $\int_{\mathbb{R}\setminus\{0\}} |\gamma(t,y)|\nu(dy) < \infty$ for all $t \in [0,T]$.

Often a particular stochastic process is not a martingale but can be transformed into a martingale by changing the underlying probability measure. This procedure is called change of probability measure and its central result is the Girsanov theorem. The probabilities of the possible paths of a stochastic process are thereby reweighed, which alters the probability distribution of the random outcomes of the process. One thereby changes the probability measure from the original measure P to the new measure Q. To do this it is necessary that the new measure Q is equivalent to P. This means that all null sets of P are also null sets of Q and vice versa. In other words that means that events that can not happen in P can also not happen in Q and vice versa. For example, if we change the probability law of a Brownian motion then the sample paths still have to be continuous under Q. On the other hand, if we change the law of a jump process with fixed jump size c then under the new measure the jump size also has to be of size c. However, the intensity of a measure changed process may vary because it doesn't effect almost surely properties of the sample paths.

Theorem 2.10 (Girsanov's theorem). *Assume that there exist predictable processes $\boldsymbol{\theta}^D(t) = (\theta_1^D(t), \dots, \theta_n^D(t)) \in \mathbb{R}^n$ and $\boldsymbol{\theta}^J(t,y) = (\theta_1^J(t,y), \dots, \theta_m^J(t,y)) \in \mathbb{R}^m$, with $\theta_k^J > 0$, for $k = 1, \dots, m$, such that the process*

$$\begin{aligned} Z_\theta(t) := \exp\Big(&-\frac{1}{2}\int_0^t ||\boldsymbol{\theta}^D(s)||^2 ds + \int_0^t \boldsymbol{\theta}^D(s)^\intercal d\boldsymbol{B}(s) \\ &+ \sum_{h=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} \log\theta_h^J(s,y) N_h(ds,dy) + \sum_{h=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} \big(1 - \theta_h^J(s,y)\big)\nu_h(dy)ds\Big) \end{aligned} \tag{2.20}$$

exists for $0 \le t \le T$ and satisfies

$$\mathbb{E}\big[Z_\theta(T)\big] = 1.$$

Define the probability measure Q on $\mathcal{F}_T$ by

$$dQ = Z_\theta(T)dP.$$

Define the process $\boldsymbol{B}^Q(t) \in \mathbb{R}^n$ and the random measures $\widetilde{N}_h^Q(dt,dy) \in \mathbb{N} \cup \{0\}$ by

$$d\boldsymbol{B}^Q(t) = d\boldsymbol{B}(t) - \boldsymbol{\theta}^D(t)dt, \tag{2.21}$$

and

$$\widetilde{N}_h^Q(dt,dy) = N_h(dt,dy) - \theta_h^J(t,y)\nu_h(dy)dt, \quad h = 1, \dots, m. \tag{2.22}$$

Then $\boldsymbol{B}^Q$ is a n-dimensional standard Brownian motion with respect to Q and $\widetilde{N}_h^Q$, $h = 1, \dots, m$, are compensated Poisson random measures under Q, in the sense that the *processes*

$$M_h(t) := \int_0^t \int_{\mathbb{R}\backslash\{0\}} \rho_h(s,y)\widetilde{N}_h^Q(ds,dy), \quad t \in [0,T], \quad h = 1,\dots,m,$$

are local Q-martingales, for all predictable process $\rho_h(s,y)$ that satisfy

$$\int_0^T \int_{\mathbb{R}\backslash\{0\}} \rho_h(s,y)^2\theta_h^J(s,y)^2\nu_h(dy)ds < \infty, \quad a.s.$$

for all $h = 1,\dots,m$.

For a proof see Øksendal and Sulem (2007).

Remarks.

1. In case that the processes $\boldsymbol{\theta}^D$ and $\boldsymbol{\theta}^J$ satisfy $\theta_j^D(t) = 0$ for $j = 1,\dots,n$, and $\theta_h^J(t) = 1$ for $h = 1,\dots,m$ and $t \in [0,T]$ a.s. the original probability measure can be recovered. That is $Z_\theta(t) = 1$, $t \in [0,T]$, and $Q = P$ a.s.

2. The measure transformation in Grisanov's theorem is often stated without given explicit formulas for the martingale $Z_\theta(t)$, $t \in [0,T]$, and the new Brownian motions and Poisson random measures. That is, given a $(\mathcal{F}_t)$-martingale $Z(t)$ with $Z(0) = 1$, we can always change the probability measure from P to Q via $\frac{dQ}{dP}|_{\mathcal{F}_T} = Z(T)$.

Girsanov's change of measure theorem can be used to transform the jump-diffusion process given in (2.13) into a local Q martingale. For processes $\boldsymbol{\theta}^D$ and $\boldsymbol{\theta}^J$ as given in the above theorem, the Brownian motion and Poisson random measure can be changed according to (2.21) and (2.22). The dynamics of the jump-diffusion $X(t)$ as defined in (2.6) are then given in terms of the Q measure by

$$\begin{aligned} dX(t) &= \Big(\alpha(t) + \boldsymbol{\beta}(t)^\intercal\boldsymbol{\theta}^D(t) + \sum_{h=1}^m \int_{\mathbb{R}\backslash\{0\}} \gamma_h(t,y)\theta_h^J(t,y)\nu_h(dy)\Big)dt \\ &\quad + \boldsymbol{\beta}(t)^\intercal d\boldsymbol{B}^Q(t) + \sum_{h=1}^m \int_{\mathbb{R}\backslash\{0\}} \gamma_h(t,y)N_h^Q(dt,dy), \quad t \in [0,T]. \end{aligned}$$

If the process $\boldsymbol{\theta} = (\boldsymbol{\theta}^D, \boldsymbol{\theta}^J)$ actually satisfies for $t \in [0,T]$ the equation

$$\alpha(t) + \boldsymbol{\beta}(t)^\intercal\boldsymbol{\theta}^D(t) + \sum_{h=1}^m \int_{\mathbb{R}\backslash\{0\}} \gamma_h(t,y)\theta_h^J(t,y)\nu_h(dy) = 0,$$

then the process $X(t)$ becomes a local Q-martingale:

$$dX(t) = \boldsymbol{\beta}(t)^\intercal d\boldsymbol{B}^Q(t) + \sum_{h=1}^m \int_{\mathbb{R}\backslash\{0\}} \gamma_h(t,y)N_h^Q(dt,dy), \quad t \in [0,T].$$

The martingale Z_θ is essential in the change of probability measure as described in Girsanov's theorem above. It will play a crucial role whenever we utilize martingale methods in the following chapters. It will be important to understand the dynamics of this process Z_θ.

Lemma 2.11. *The process $Z_\theta(t)$ defined in* (2.20) *is a (local) martingale and it's dynamics are given by the SDE*

$$dZ_\theta(t) = Z_\theta(t-)\sum_{j=1}^{n}\theta_j^D(t)dB_j(t) - Z_\theta(t-)\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}(1-\theta_k^J(t,y))\widetilde{N}_h(dt,dy). \quad (2.23)$$

Proof. From (2.20) follows that

$$\begin{aligned} d\log Z_\theta(t) \quad = \quad & -\frac{1}{2}\sum_{j=1}^{n}|\theta_j^D(t)|^2dt + \sum_{j=1}^{n}\theta_j^D(t)dB_j(t) \\ & +\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\log\theta_h^J(t,y)N_h(dt,dy) + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\big(1-\theta_h^J(t,y)\big)\nu_h(dy)dt. \end{aligned}$$

Applying Itô's formula to $f(x) = e^x$ with $x = \log Z_\theta(t)$ leads to the result. □

Further, Itô's formula can be used to calculate the dynamics of the reciprocal of Z_θ. Taking the function $f(x) = 1/x$ and applying Itô's formula Theorem 2.14 on Z_θ and f, one can see since $f'(x) = -1/x^2$, $f''(x) = 2/x^3$, and $f(x+\Delta x) - f(x) = f(x\theta)) - f(x) = 1/x\,(1-\theta)/\theta$ that

$$d\frac{1}{Z_\theta(t)} = -\frac{1}{Z_\theta(t-)}\sum_{j=1}^{n}\theta_j^D(t)dB_j^Q(t) + \frac{1}{Z_\theta(t-)}\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\frac{1-\theta_h^J(t,y)}{\theta_h^J(t,y)}\widetilde{N}_h^Q(dt,dy). \quad (2.24)$$

Independent from how the martingale Z_θ in Girsanov's theorem looks like, as long as $Z(0) = 1$ a.s. it can be used to change the measure. When calculating with conditional expectations the so called generalized Bayes theorem can be used to change between Q and P measure.

Lemma 2.12. *Let Z be a $\mathcal{F}_t$-martingale with $Z(0) = 1$ for which we change the probability measure from P to Q through $dQ = Z(T)dP$. For fixed $t \in [0,T]$ let Y be a $\mathcal{F}_t$-measurable random variable satisfying*

$$\mathbb{E}^Q[|Y|] < \infty.$$

Then Bayes' rule states that for $s \le t$

$$\mathbb{E}^Q[Y|\mathcal{F}_s] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}_s], \quad P-a.s., \quad Q-a.s.$$

Proof. Let $s \leq t \leq T$ and $A \in \mathcal{F}_s$ then

$$\begin{aligned}\mathbb{E}^Q\Big[1_A \frac{1}{Z(s)}\mathbb{E}(YZ(t)|\mathcal{F}_s)\Big] &= \mathbb{E}\Big[1_A \mathbb{E}(YZ(t)|\mathcal{F}_s)\Big] \\ &= \mathbb{E}\Big[1_A \mathbb{E}(YZ(t))\Big] = \mathbb{E}^Q[1_A Y].\end{aligned}$$

□

The section is concluded with the martingale representation theorem. The theorem has a crucial role in the creation of replication strategies. It states that every $(\mathcal{F}_t)$-martingale can be represented in terms of a a sum of integrals with respect to the Brownian motions and compensated Poisson ranom measures. For a Poisson measure $N(dt, dy)$ with Lévy measure $\nu(dy)$ denote as before the compensated Poisson random measure by $\widetilde{N}(dt, dy) := N(dt, dy) - \nu(dy)dt$.

Theorem 2.13 (Martingale Representation Theorem). *Any $(P, \mathcal{F}_t)$-martingale $M(t)$ has the representation*

$$M(t) = M(0) + \int_0^t \sum_{j=1}^n a_j^D(s)dB_j(s) + \int_0^t \sum_{k=1}^m \int_E a_k^J(s,y)\widetilde{N}_k(ds, dy)$$

where a_j^D, $j = 1, \ldots, n$, are predictable and square integrable, and a_k^J are predictable, marked processes, that are integrable with respect to $\nu_k(dy)$, $k = 1, \ldots, m$.

For a proof (of the one dimensional case) see Runggaldier (2003). For a more generalized treatment, including more information on Itô's formula and change of measure the reader is referred to Jacod and Shiryaev (2003).

Chapter 3

Stock Prices and Portfolio Models under Jump-Diffusions

Models where the stock price follows a geometric Brownian motion have been very successful in the financial literature. The first to use such a model was Samuelson (1969). Later it was formulated within the framework of Itô's stochastic calculus by Merton (1969, 1971) and the relationship to martingale methods have been developed by Harrison and Kreps (1971) and Harrison and Pliska (1981). First geometric Brownian motion models, as in Black and Scholes (1971) and Merton (1973), had the simple form

$$dS(t) = \alpha S(t)dt + \sigma S(t)dB(t).$$

They were then extended to allow parameters that are time dependent or random, and Brownian motions that are multidimensional. Although a geometric Brownian motion model is not necessary a very precise model for a stock price it remains a popular model because of its simplicity. Measure change techniques as developed in Pliska (1986), Cox and Huang (1989), and Karatzas, Lehoczky, and Shreve (1987) can be applied to transform the discounted stock prices into martingales under a unique equivalent martingale measure (EMM) Q.

Despite the great success of the Black-Scholes model it has several shortcomings. Several empirical research have been shown that stock returns are not normal distributed. Rather, empirical stock return distributions, as investigated in Anderson et al. (2002), are skewed negatively and are leptokurtic, with higher peaks and heavier tails. The volatility smile of option prices shows that implied volatility is not constant as in the Black-Scholes model (cf. Bates (1996). Further, the Black-Scholes model can not incorporate sudden price movements as can be observed in crashes and rallies.

Several alternative models have been suggested to deal with these shortcomings. Stochastic and local volatility models have been developed in Cox and Ross (1976), Hull and White (1987), Stein and Stein (1991) and Heston (1993) among others. They allow the volatility change over time, either by using a random factor as in stochastic volatility models or in a deterministic sense when using local volatility models. This allows the volatility to vary over time, and allows periods of high volatility as well as periods with relatively low volatility. Another type of model that has become popular is that of Lévy process models. Lévy models have been investigated for example in Barndorff-Nielson and Shephard (2001) and Eberlein (2002).

Similar to Lévy processes are jump-diffusions which also allow the volatility and the distribution of jump-amplitudes to vary in time. Among the first to use jump-diffusion models was Merton (1976) who suggested a model with log-normal distributed jumps. A double exponential jump model is analysed in Kou (2002), and log-uniform distributed jumps are considered in Yan and Hanson (2006). Their complexity ranges from compound Poisson processes to processes that involve Poisson random measures. This chapter will introduce the jump-diffusion model framework.

3.1 The Stock Price Model

In this section the stock price model that follows a jump-diffusion process is introduced. It is assumed that there are $k(\leqslant m+n)$ stocks in the market that are driven by n mutually independent Brownian motions $B_1(t), \cdots, B_n(t)$ and m mutually independent jump processes that are determined by homogeneous Poisson random measures $N_1(dt, dy_1), \cdots, N_m(dt, dy_m)$ with intensities $\nu_1(dy), \cdots, \nu_m(dy)$ (cf. Section 2.4), where the m jump processes are also independent of the n Brownian motions. All of these processes are assumed to be defined on a common probability space. The stock prices are assumed to follow geometric jump-diffusion processes in the sense that they follow the stochastic differential equations

$$\frac{dS_i(t)}{S_i(t-)} = \alpha_i(t)dt + \sum_{j=1}^{n} \xi_{ij}(t)dB_j(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,y)N_h(dt,dy), \tag{3.1}$$

for $1 \leqslant i \leqslant k$ and $t \in [0,T]$. Thereby represents $T(>0)$ the investment horizon which will be assumed to be finite. $\alpha_i(t)$ represents the drift rate of the i^{th} asset, $\xi_{ij}(t)$ the volatility of the i^{th} asset with respect to the j^{th} Brownian motion, and $\gamma_{i\ell}(t,y)$ is such that $\int_{\mathbb{R}\setminus\{0\}} \gamma_{ij}(t,y)N_j(dt,dy)$ describes the effect of the j^{th} jump process on the i^{th} stock price at time t. The model parameters α_i, ξ_{ij}, and $\gamma_{i\ell}$ are assumed to be progressively measurable processes such that all the involved integrals exist, and $\gamma_{i\ell}$ are assumed further to have the property that $1 + \int_{\mathbb{R}\setminus\{0\}} \gamma_{i\ell}\, N_\ell(dt,dy) \geq 0$ so that all the S_i remain non-negative in $[0, T]$ (cf. Runggaldier (2003)). In particular, that means that the model parameters have to satisfy

$$\int_0^T |\alpha_i(t)|^2 dt < \infty, \quad \int_0^T |\xi_{ij}(t)|^2 dt < \infty, \quad \text{and} \int_0^T \int_{\mathbb{R}\setminus\{0\}} |\gamma_{ih}(t, y)|^2 \nu_h(dt, dy) < \infty \tag{3.2}$$

for $i = 1, \ldots, k$, $j = 1, \ldots, n$, and $h = 1, \ldots, m$, to guarantee the existence and uniqueness of a solution to (3.1). Note also that we have assumed that the Poisson random measures are not infinitely active to make the analysis simpler. If this assumption is dropped then we could replace N_h by its compensated version $\widetilde{N}_h$. This would also require some corresponding modifications later.

As the underlying filtration $(\mathcal{F}_t)$, we take the canonical filtration constructed by the Brownian motions B_i, $1 \leqslant i \leqslant n$, the Poisson random measures N_j, $1 \leqslant j \leqslant m$, as well as all null sets of the underlying probability measure P:

$$\begin{aligned}\mathcal{F}_t := \sigma\Big\{\big(B_i(s), N_j((0, s], A), C\big) \quad \Big| \quad & A \in \mathcal{B}(\mathbb{R} \setminus \{0\}), C \in \mathcal{N}, s \in [0, t], \\ & 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\},\end{aligned}$$

with $\mathcal{N}$ being the collection of all P-null sets and $\mathcal{B}(\mathbb{R} \setminus \{0\})$ being the Borel-σ-algebra on $\mathbb{R} \setminus \{0\}$.

In the following let us discuss the effects that the jump processes have on the stock prices. As mentioned before it is required that $\int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t, y) N_h(dt, dy) \geq -1$, a.s., to ensure that the stock prices are non-negative. In a model driven by Brownian motions (without jumps) the stock price is always positive. However, in the presents of jumps this is no longer the case. Due to the nature of the model, the m underlying jump processes won't jump at the same time almost surely. In fact, any two jump processes will not jump at the same time almost surely. If, however, one of the m jumps is so big that the stock price jumps to zero, then the stock price will not recover and will stay at zero until the end of the investment horizon $[0, T]$. This is the case when $\int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t, y) N_h(dt, dy) = -1$ occurs for any of the $h = 1, \ldots, m$. Then the firm goes bankrupt and its stock is worthless. This conditions on jump-size distributions is often stated in the literature (e.g. Runggaldier (2003)), however a similar non-bankruptcy condition on the trading strategy is often not clearly pointed out and will be discussed later.

A more restrictive condition on the jump process that guarantees positivity of the stock process is that $\int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t, y) N_h(dt, dy) > -1$ for all $h = 1, \ldots, m$ ν_h-a.s. Then it is guaranteed that the i^{th} stock price is always positive and no bankruptcy can occur. Let this be the case for a moment so that the stock price SDE (3.1) can be solved using a log transform. Using Itô's formula Theorem 2.17 applied to (3.1) for $f(x) = \log(x)$ leads to

$$\begin{aligned} d\log S_i(t) = &\Big(\alpha_i(t) - \frac{1}{2}\sum_{j=1}^{n}\xi_{ij}^2(t)\Big)dt + \sum_{j=1}^{n}\xi_{ij}(t)dB_j(t) \\ &+ \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \log(1+\gamma_{ih}(t,y))N_h(dt,dy), \end{aligned} \tag{3.3}$$

for $i = 1, \ldots, k$. Define the log drift rate by

$$\mu_i(t) := \alpha_i(t) - \frac{1}{2}\sum_{j=1}^{n}\xi_{ij}^2(t).$$

Then the SDE (3.3) can be solved to

$$\begin{aligned} S_i(t) =&S_i(0)\exp\left(\int_0^t \mu_i(s)ds + \sum_{j=1}^{n}\int_0^t \xi_{ij}(s)dB_j(s)\right. \\ &\left.+\sum_{h=1}^{m}\int_0^t\int_{\mathbb{R}\setminus\{0\}} \log(1+\gamma_{ih}(s,y))N_h(ds,dy)\right), \end{aligned} \tag{3.4}$$

for $i = 1, \ldots, k$. Following the discussion at the end of Section 2.5 assume that the jump processes can be written in the form

$$\int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,y)N_h(dt,dy) = \Gamma_{ih}(t)dP_h(t) \tag{3.5}$$

for some Poisson processes P_h and jump sizes Γ_{ih}. Also, let $t_{h1}, t_{h2}, \ldots$ denote the jump times of $N_h(t)$. Then the stock prices can be written in the form

$$S_i(t) = S_i(0)\exp\Big(\int_0^t \mu_i(s)ds + \sum_{j=1}^{n}\int_0^t \xi_{ij}(s)dB_j(s)\Big)\prod_{h=1}^{m}\prod_{\ell=1}^{P_h(t)}\big(1+\Gamma_{ih}(t_{h\ell})\big),$$

for $i = 1, \ldots, k$, and with the convention that $\prod_{l=1}^{0} = 1$.

Now consider the case when stock prices can actually jump to zero, i.e. $1 + \Gamma_{ih}(t) = 0$ is a possible outcome when a jump occurs. The time a company goes bankrupt and its stock price jumps to zero is then the smallest $t \geq 0$ such that

$$\prod_{h=1}^{m}\prod_{\ell=1}^{P_h(t)}\big(1+\Gamma_{ih}(t_{h\ell})\big) = 0.$$

Denote the time of bankruptcy of the i^{th} stock by τ_i, so that

$$\tau_i = \inf\Big\{t \geq 0 \Big| \prod_{h=1}^{m}\prod_{\ell=1}^{P_h(t)}\big(1+\Gamma_{ih}(t_{h\ell})\big) = 0\Big\}.$$

Clearly, τ_i is a stopping time and since a stock that has jumped to zero won't recover, the stock prices process $S_i(t)$ is actually a stopped stochastic process with $S_i(t) > 0$ for $t < \tau_i$ and $S_i(t) = 0$ for $t \geq \tau_i$.

The stock price model we consider is a fairly general one. If one makes some simplification one can obtain simpler models. For example, the stock prices become Lévy processes if the model coefficient are constant and γ_{ih} are functions of y only, i.e.

$$\frac{dS_i(t)}{S_i(t-)} = \alpha_i dt + \sum_{j=1}^{n} \xi_{ij} dB_j(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(y) N_h(dt, dy).$$

If on the other hand, there are no jumps in (3.1),

$$\frac{dS_i(t)}{S_i(t-)} = \alpha_i(t)dt + \sum_{j=1}^{n} \xi_{ij}(t) dB_j(t),$$

then one recovers the general Brownian motion driven model as used, for example, in Karatzas et al. (1987).

3.2 The Risk-less Asset and State Price Densities

Apart from the k stocks it is assumed that there is a risk-less asset in the market. This is often either assumed to be a bank account or a zero coupon bond with no default risk. The risk-free asset is described by the ordinary differential equation

$$dS_0(t) = S_0(t)r(t)dt, \quad t \geq 0; \quad S_0(0) = 1, \tag{3.6}$$

where $r(t)$ is the risk-free rate process. The risk-free rate process $r(t)$ is a progressively measurable process w.r.t. $(\mathcal{F}_t)$ that satisfies $\int_0^T |r(t)|dt < \infty$ a.s. The risk-less asset can be written as

$$S_0(t) = e^{\int_0^t r_s ds}.$$

Remark. Since $\int_0^T |r(t)|dt$ is a.s. finite the process $S_0(t)$ as well as the process $1/S_0(t)$, $t \in [0, T]$, is a.s. finite. This can be seen by observing for $t \in [0, T]$ that $\int_0^t |r_s|ds \leq \int_0^T |r_s|ds < \infty$. But this shows that $\int_0^t r_s ds \leq \int_0^t |r_s|ds < \infty$ as well as $-\int_0^t r_s ds \leq \int_0^t |r_s|ds < \infty$ for $t \in [0, T]$.

The discounted stock prices are the stock prices divided by the risk-less asset price process, i.e. $S_i(t)/S_0(t)$ for $i = 1, \ldots, k$. They are denoted by an over line, so that $\overline{S_i}(t) := S_i(t)/S_0(t)$. It can be easily verified that

$$\frac{d\overline{S}_i(t)}{\overline{S}_i(t-)} = \overline{\alpha}_i(t)dt + \sum_{j=1}^{n} \xi_{ij}(t) dB_j(t) + \sum_{k=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ik}(t, y) N_h(dt, dy) \tag{3.7}$$

with

$$\overline{\alpha}_i(t) = \alpha_i(t) - r(t),$$

for all $i = 1, \dots, k$. In the financial literature it is common to change the measure from P to Q using Girsanov's Theorem 2.10 such that the discounted stock prices become (local) martingales. This measure change requires that the non-martingale drift in the expression (3.7) becomes zero. However, unlike models that are driven by Brownian motions only the measure change is not unique so that there are several equivalent martingale measures (EMM) Q under which the discounted stock prices become martingales. These measure changes can be parametrized (as already discussed in Section 2.6) by a pair of process $\boldsymbol{\theta} = \left(\boldsymbol{\theta}^D, \boldsymbol{\theta}^J\right)$ and the Radon-Nikodym density of the measure change is then given by $\frac{dQ}{dP} \mid_{\mathcal{F}_T} = Z_\theta(T)$ for Z_θ as defined in (2.20). The next definition makes this idea more rigorous.

Definition 3.1. *A jump-diffusion Girsanov kernel is a pair of predictable vector processes $\boldsymbol{\theta} = \left(\boldsymbol{\theta}^D, \boldsymbol{\theta}^J\right)$ such that the following properties are satisfied:*

(i) $\boldsymbol{\theta}^D(t) = (\theta_1^D(t), \dots, \theta_n^D(t)) \in \mathbb{R}^n$ *and* $\boldsymbol{\theta}^J(t,y) = (\theta_1^J(t,y), \dots, \theta_m^J(t,y)) \in \mathbb{R}^m$, *with* $\theta_h^J > 0$, *for* $h = 1, \dots, m$, $y \in \mathbb{R} \setminus \{0\}$, *and* $t \in [0,T]$,

(ii) Z_θ *defined as in* (2.20) *is actually a martingale with* $\mathbb{E}[Z_\theta(T)] = 1$.

(iii)

$$\overline{\alpha}_i(t) + \sum_{j=1}^{n} \xi_{ij}(t)\theta_j^D(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ik}(t,y)\theta_k^J(t,y)\nu_k(dy) = 0, \tag{3.8}$$

a.s. for $i = 1, \dots, k$ *and* $t \in [0,T]$.

Further, Θ *is defined as the set containing all jump-diffusion Girsanov kernels.*

Conditions (i) and (ii) in the above definition are the same conditions on the process $\boldsymbol{\theta}$ as in Girsanov's theorem 2.10. This means in particular that for each $\boldsymbol{\theta} \in \Theta$ a probability measure Q can be associated through $dQ = Z_\theta(T)dP$. Condition (iii) then guarantees that the non-martingale drift in (3.7) vanishes. In the case that there are no jumps the condition on $\boldsymbol{\theta}$ in (3.8) becomes

$$\overline{\alpha}_i(t) + \sum_{j=1}^{n} \xi_{ij}(t)\theta_j^D(t) = 0, \quad \text{a.s.} \tag{3.9}$$

Given that $\boldsymbol{\xi} = (\xi_{ij})$ has full rank the Girsanov kernel is uniquely determined and the set of all Girsanov kernels Θ is a singleton. One then calls the market model complete. In contrast, with jumps, the market is incomplete because the Girsanov kernel is not uniquely determined. Financial products can not perfectly hedged any more.

The probability measure Q that has been constructed in the way above is called the martingale measure associated to $\boldsymbol{\theta} \in \Theta$. To parametrize martingale measures in terms of a $\boldsymbol{\theta}$ has been very popular in the financial literature and $\boldsymbol{\theta}$ is often called the market price of risk. The use of market prices of risk to change the drift and jump-intensity of one dimensional jump-diffusions has been demonstrated in Runggaldier (2003). The following result extends it to the multi-dimensional case.

Proposition 3.2. *Let $\boldsymbol{\theta} \in \Theta$. Then the discounted stock prices $\overline{S_i}(t)$ are local martingales under the new probability measure Q:*

$$\frac{d\overline{S_i}(t)}{\overline{S_i}(t-)} = \sum_{j=1}^{n} \xi_{ij}(t) dB_j^Q(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,y) \widetilde{N}_h^Q(dt,dy)$$

for $t \in [0,T]$.

Proof. The process $\boldsymbol{\theta} \in \Theta$ corresponds to a measure Q under which, following Girsanov's theorem,

$$\begin{aligned} dB_j^Q(t) &= dB_j(t) - \theta_j^D(t)dt, \quad \text{and} \\ \widetilde{N}_h^Q(dt,dy) &= N_h(dt,dy) - \theta_h^J(t,y)\nu_h(dy)dt \end{aligned} \tag{3.10}$$

are Brownian motions and Poisson random measures, respectively, for $j = 1,..,n$, $h = 1,\ldots,m$, and $t \in [0,T]$. Substituting (3.10) into (3.7) and observing (3.8), the proposition follows. ☐

It is possible to use measure change techniques but still work under the original probability measure P. This is done using a process often called the state price density.

Definition 3.3. *For $\boldsymbol{\theta} \in \Theta$ the process defined by $H_\theta(t) := Z_\theta(t)/S_0(t)$ is called the state price density of the jump-diffusion Girsanov kernel $\boldsymbol{\theta} = (\boldsymbol{\theta}^D, \boldsymbol{\theta}^J)$.*

The state price density with Girsanov kernel $\boldsymbol{\theta}$ is the discounted version of the process Z_θ. It follows the stochastic differential equation

$$\begin{aligned} dH_\theta(t) = &- H_\theta(t-)r(t)dt + H_\theta(t-)\boldsymbol{\theta}^D(t)d\boldsymbol{B}(t) \\ &+ H_\theta(t-)\sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} (\theta_k^J(t,y) - 1)\widetilde{N}_k(dt,dy), \quad t \in [0,T]. \end{aligned} \tag{3.11}$$

and starts almost surely at one, $H_\theta(0) = 1$. The state price density $H_\theta(t)$ can be used to transfer Q-martingale measure results into the real world measure P. Since discounted stock prices $\overline{S}_i$ are local Q-martingales the product $H_\theta S_i$ is a local P-martingale for $\boldsymbol{\theta} \in \Theta$.

3.3 Portfolios and the Wealth Process

For a financial agent who wants to invest into the market a trading strategy can be described in various ways. A portfolio can be described by the amount of money invested into the $k+1$ assets (stocks and bond) or it can be expressed as the proportion of money invested into each asset. A third way would be to specify the number of assets hold for each asset class.

Considering the third option first and denote by $u_i(t)$, $i = 1, \dots k$ the number of stocks of type i hold at time t, and by $u_0(t)$ the number of risk-free zero bonds. If the initial endowment of an investor is $x > 0$, and the value of a portfolio, which we call the wealth process of the investor, is denoted by V_u^x, then the wealth process should satisfy $V_u^x(0) = x$ and

$$V_u^x(t) = \sum_{i=0}^{k} u_i(t) S_i(t).$$

Further, if one assumes that an investor does not withdraw or deposit further money into the investment portfolio (such a portfolio is called self-financing), then the change in portfolio should be given by

$$dV_u^x(t) = \sum_{i=0}^{k} u_i(t) dS_i(t).$$

Substituting (3.1) then gives the infinitesimal change in wealth as

$$\begin{aligned} dV_u^x(t) &= u_0(t) S_0(t) r(t) dt \\ &\quad + \sum_{i=1}^{k} u_i(t) S_i(t-) \left(\alpha_i(t) dt + \sum_{j=1}^{n} \xi_{ij}(t) dB_j(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,y) N_h(dt,dy) \right). \end{aligned}$$

The term just after the sum in the above equation, $u_i(t)S_i(t-)$, is however, just the amount of money that an investor holds in asset i at time t. If this amount of money is denoted by ϕ_i such that $\phi_i(t) := u_i(t)S_i(t-)$ then the change in the wealth equation can be rewritten as

$$\begin{aligned} dV_\phi^x(t) &= \phi_0(t) r(t) dt \\ &\quad + \sum_{i=1}^{k} \phi_i(t) \left(\alpha_i(t) dt + \sum_{j=1}^{n} \xi_{ij}(t) dB_j(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,y) N_h(dt,dy) \right). \end{aligned} \tag{3.12}$$

On the other hand, the total of the amount invested into each asset should be equal to the wealth process at each time t, i.e. $\sum_{i=0}^{k} \phi_i(t) = V_\phi^x(t)$, and if one considers the proportion of wealth invested into the i^{th} stock, denoted by $\pi_i(t)$ then clearly $\pi_i(t) = \phi_i(t)/V_\phi^x(t-)$.

Consequently, $\pi_i(t)V_\phi^x(t-) = \phi_i(t)$ and the proportional amount of wealth invested into the risk-less asset is $\left(V_\phi^x(t-) - \sum_{i=1}^k \phi_i(t)\right)/V_\phi^x(t-)$. Thus, if the trading strategy is represented by the proportion invested in each asset then the dynamics of (3.12) can be written as

$$\begin{aligned}\frac{dV_\pi^x(t)}{V_\pi^x(t-)} &= r(t)dt + \sum_{i=1}^k \pi_i(t)\Bigg(\big(\alpha_i(t) - r(t)\big)dt \\ &\quad + \sum_{j=1}^n \xi_{ij}(t)dB_j(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,y)N_h(dt,dy)\Bigg).\end{aligned}$$

This will be the usually way how trading strategies will be described in this work. The strategy vector $\boldsymbol{\pi}$ determines the proportions of wealth invested into the k stocks. Once a strategy is set for all the stocks, the proportional amount of money invested into the risk-less bond is uniquely determined by $1 - \sum_{i=1}^n \pi_i(t)$. A positive value for $\pi_i(t)$ represents a long position whereas a negative $\pi_i(t)$ stands for a short position in the i^{th} asset. This is also true for the risk-less asset in the sense that whether $1 - \sum_{i=1}^n \pi_i(t)$ is positive or negative determines whether the investment has deposited or borrowed money from the bank account. If any of the $\pi_i(t)$ exceeds one, then the agent invests more money into the i^{th} stock than he/she posses and it becomes necessary to short a position in at least one other stock or bond.

Formally, a proportional portfolio strategy $\boldsymbol{\pi}(t) = (\pi_1(t), \ldots, \pi_n(t))$ is a progressively measurable process that is a.s. bounded on $[0,T]$. To write the wealth process of a trading strategy $\boldsymbol{\pi}$ into a more convenient vector form the following processes are introduced. The discounted drift rate vector $\overline{\boldsymbol{\alpha}}(t)$ is defined by $\overline{\boldsymbol{\alpha}}(t) := \big(\alpha_1(t) - r(t), \ldots, \alpha_n(t) - r(t)\big)^\intercal$, and the Brownian motion volatility matrix $\boldsymbol{\xi}$ is the matrix process given by $\boldsymbol{\xi}(t) = (\xi_{ij})(t)$. Further, if the vector process $\boldsymbol{\gamma}_h(t,y) := \big(\gamma_{1h}(t,y), \ldots, \gamma_{kh}(t,y)\big)^\intercal$ is introduced for $h = 1, \ldots, m$, then

$$\frac{dV_\pi^x(t)}{V_\pi^x(t-)} = r(t)dt + \boldsymbol{\pi}(t)^\intercal \left[\overline{\boldsymbol{\alpha}}(t)dt + \boldsymbol{\xi}(t)d\boldsymbol{B}(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)N_h(dt,dy)\right], \quad (3.13)$$

for $t \in [0,T]$. For a proportional portfolio process $\boldsymbol{\pi}(t)$ the wealth process V_π^x satisfies the stochastic differential equation

$$\begin{aligned}\frac{dV_\pi^x(t)}{V_\pi^x(t-)} &= \Big(1 - \sum_{i=1}^k \pi_i(t)\Big)\frac{dS_0(t)}{S_0(t-)} + \sum_{i=1}^k \pi_i(t)\frac{dS_i(t)}{S_i(t-)}, \quad t \in [0,T]; \\ V_\pi^x(0) &= x.\end{aligned} \quad (3.14)$$

In the previous section discounted stock prices where discussed which are the stock price processes divided by the risk-less asset S_0. Similarly, the discounted wealth process of a trading strategy $\boldsymbol{\pi}$ is the process $\overline{V}_\pi^x = V_\pi^x / S_0$. Since $d\overline{V}_\pi^x(t) = dV_\pi^x(t)/S_0(t) - r(t)V_\pi^x(t)/S_0(t)dt$ the dynamics of the discounted wealth process $\overline{V}_\pi^x$ are given by

$$\frac{d\overline{V}_\pi^x(t)}{\overline{V}_\pi^x(t-)} = \boldsymbol{\pi}(t)^\intercal \left[\overline{\boldsymbol{\alpha}}(t)dt + \boldsymbol{\xi}(t)d\boldsymbol{B}(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)N_h(dt,dy) \right], \tag{3.15}$$

for $t \in [0,T]$. Its relationship to the discounted stock prices $\overline{S}_i$ is considering (3.7)

$$\frac{d\overline{V}_\pi^x(t)}{\overline{V}_\pi^x(t-)} = \sum_{i=1}^k \pi_i(t) \frac{d\overline{S}_i(t)}{\overline{S}_i(t-)}, \quad t \in [0,T].$$

Defining a portfolio in terms of proportions makes only sense when the underlying wealth process is bigger than zero. In the absence of jumps this condition is always satisfied as long as the initial wealth x is positive. However, in the presence of jumps an additional condition is needed to satisfy the positivity or non-negativity of wealth. This requirement will be that

$$\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y)N_h(dt,dy) \geq -1, \tag{3.16}$$

for all $h = 1,\dots,m$. The set of strategies $\boldsymbol{\pi}$ that satisfy the above requirement (3.16) will be denoted by Π. If further, the condition (3.16) on $\boldsymbol{\pi}$ is stronger so that

$$\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y)N_h(dt,dy) > -1, \tag{3.17}$$

then the wealth process is almost surely positive. The set of strategies that satisfy this condition shall be denoted by Π_+. Similar, how the stock price SDEs have been solved, the wealth process SDE (3.13) can be solved for a trading strategy $\boldsymbol{\pi} \in \Pi_+$. Taking $f(x) = \log(x)$ and using the Itô's formula Theorem 2.17 one can derive

$$\begin{aligned} d\log(V_\pi^x(t)) &= \mu_\pi(t)dt + \boldsymbol{\pi}(t)^\intercal \boldsymbol{\xi}(t)d\boldsymbol{B}(t) \\ &\quad + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \log(1 + \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y))N_h(dt,dy), \end{aligned}$$

where μ_π is defined by

$$\mu_\pi(t) := r(t) + \overline{\alpha}_\pi(t) - \frac{1}{2}\sigma_{\pi\pi}(t), \quad t \in [0,T], \tag{3.18}$$

and $\sigma_{\pi\pi}$ by

$$\sigma_{\pi\pi}(t) := \boldsymbol{\pi}(t)^\intercal \boldsymbol{\sigma}(t)\boldsymbol{\pi}(t), \quad t \in [0,T].$$

Then the wealth SDE can be solved to

$$\begin{aligned} V_\pi^x(t) = x\exp\Bigg(&\int_0^t \mu_\pi(s)ds + \int_0^t \boldsymbol{\pi}(s)^\intercal \boldsymbol{\xi}(s)d\boldsymbol{B}(s) \\ &+ \sum_{h=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} \log(1 + \boldsymbol{\pi}(s)^\intercal \boldsymbol{\gamma}_h(s,y))N_h(ds,dy) \Bigg) \end{aligned} \tag{3.19}$$

If the jumps of the stock prices can be expressed in a form (3.5) and the vector of jump sizes is defined by $\boldsymbol{\Gamma}_h(t) := (\Gamma_{1h}(t), \ldots, \Gamma_{kh}(t))^\intercal$, then the jumps of the wealth process can be expressed as

$$\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y) N_h(dt,dy) = \boldsymbol{\pi}(t)^\intercal \boldsymbol{\Gamma}_h(t) dP_h(t),$$

for $t \in [0,T]$. Further, the above solution of the wealth process SDE (3.19) can then be written as

$$V_\pi^x(t) = x \exp\left(\int_0^t \mu_\pi(s) ds + \int_0^t \boldsymbol{\pi}(s)^\intercal \boldsymbol{\xi}(s) d\boldsymbol{B}(s) \right) \prod_{h=1}^{m} \prod_{\ell=1}^{P_h(t)} \left(1 + \boldsymbol{\pi}(t_{h\ell})^\intercal \boldsymbol{\Gamma}_h(t_{h\ell})\right),$$

where $t_{h1}, t_{h2}, \ldots$ denote the jump times of the h^{th} jump process.

So far, it was assumed that an investor does not withdraw money from his/her investment portfolio during the investment horizon. This will be allowed in what follows and the concept of consumption will be introduced in the next definition.

Definition 3.4. *A consumption process $c(t)$ is a non-negative progressively measurable process such that*

$$\int_0^T c(t) dt < \infty, \quad a.s.$$

For initial wealth $x > 0$ and a portfolio process $\boldsymbol{\pi}(t)$ the wealth process with consumption $c(t)$ is assumed to satisfy the stochastic differential equation

$$\begin{aligned} dV_{\pi,c}^x(t) = &\left(V_{\pi,c}^x(t-) r(t) - c(t)\right) dt + V_{\pi,c}^x(t-) \boldsymbol{\pi}(t)^\intercal \left[\overline{\boldsymbol{\alpha}}(t) dt + \boldsymbol{\xi}(t) d\boldsymbol{B}(t)\right] \\ &+ V_{\pi,c}^x(t-) \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y) N_h(dt,dy), \quad t \in [0,T]. \end{aligned} \tag{3.20}$$

If the trading strategy is expressed in absolute terms $\boldsymbol{\phi}$, with ϕ_i representing the amount of money invested into the i^{th} stock, then the wealth process SDE changes to

$$\begin{aligned} dV_{\phi,c}^x(t) = &\left(V_{\phi,c}^x(t-) r(t) - c(t)\right) dt + \boldsymbol{\phi}(t)^\intercal \overline{\boldsymbol{\alpha}}(t) dt \\ &+ \boldsymbol{\phi}(t)^\intercal \boldsymbol{\xi}(t) d\boldsymbol{B}(t) + \sum_{i=1}^{k} \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \phi_i(t) \gamma_{ik}(t,y) N_h(dt,dy), \end{aligned} \tag{3.21}$$

for all $t \in [0,T]$. Because of the consumption a wealth process might become zero or even negative. It should not be allowed to obtain negative wealth, but consumption processes that lead to zero wealth will be allowed. A discussion of the consequences for the trading strategy $\boldsymbol{\phi}$ or $\boldsymbol{\pi}$ will be given in Section 3.4.

Like the discounted stock prices the discounted wealth process is given as the wealth process divided by the risk-less asset $\overline{V}^x_{\pi,c}(t) := V^x_{\pi,c}(t)/S_0(t)$, so that

$$\begin{aligned}\overline{V}^x_{\pi,c}(t) = x &- \int_0^t \frac{1}{S_0(s)} c(s)ds + \int_0^t \overline{V}^x_{\pi,c}(s-)\boldsymbol{\pi}(s)^\intercal \overline{\boldsymbol{\alpha}}(s)ds \\ &+ \int_0^t \overline{V}^x_{\pi,c}(s-)\boldsymbol{\pi}(s)^\intercal \boldsymbol{\xi}(s)d\boldsymbol{B}(s) \\ &+ \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \overline{V}^x_{\pi,c}(t-)\boldsymbol{\pi}_i(s)^\intercal \boldsymbol{\gamma}_h(s,y)N_h(ds,dy).\end{aligned} \tag{3.22}$$

From (3.22) and (3.7) it can be verify that the discounted wealth process $\overline{V}^x_{\pi,c}(t)$ can be related to the discounted stock prices $\overline{S}_i(t)$ through

$$d\overline{V}^x_{\pi,c}(t) = \sum_{i=1}^k \overline{V}^x_{\pi,c}(t-)\pi_i(t)\frac{d\overline{S}_i(t)}{\overline{S}_i(t-)} - \frac{1}{S_0(t)}c(t)dt.$$

In the previous section it has been shown that the discounted stock prices are local Q-martingales for a martingale measure Q defined by a process $\boldsymbol{\theta} \in \Theta$. Thus, ignoring consumption the wealth process should be a local Q martingale too.

Proposition 3.5. *Let $x \geq 0$ be some initial wealth and $(\boldsymbol{\pi}, c)$ a consumption-investment strategy. Then for $\boldsymbol{\theta} \in \Theta$ the process $M^Q(t)$, $t \in [0,T]$, defined by*

$$M^Q(t) := \overline{V}^x_{\pi,c}(t) + \int_0^t \frac{1}{S_0(u)} c(u)du, \tag{3.23}$$

is a local Q-martingale, where Q is the martingale measure corresponding to the Girsanov kernel $\boldsymbol{\theta}$ defined in Definition 3.1.

Proof. Substituting

$$\begin{aligned} dB_j(t) &= dB^Q_j(t) + \theta^D_j(t)dt, \quad \text{and} \\ N_h(dt,dy) &= \widetilde{N}^Q_k(dt,dy) + \theta^J_k(t,y)\nu_k(dy)dt, \end{aligned}$$

into (3.22) and observing that

$$\alpha_i(t) + \sum_{j=1}^n \xi_{ij}(t)\theta^D_j(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \gamma_{ik}(t,y)\theta^J_k(t,y)\nu_k(dy) = 0,$$

a.s. for $t \in [0,T]$ and $i = 1,\ldots,k$ together with $\boldsymbol{\theta} \in \Theta$ reveals that

$$M^Q(t) = x + \int_0^t \sum_{i=1}^k \overline{V}^x_{\pi,c}(t-)\pi_i(u)\Big[\sum_{j=1}^n \xi_{ij}(u)dB^Q_j(u) + \sum_{k=1}^m \int_{\mathbb{R}\setminus\{0\}} \gamma_{ik}(u,y)N^Q_h(du,dy)\Big].$$

Clearly, $M^Q(t)$ is a local Q-martingale. □

The state price density can be used to derive a similar result for the state price modified wealth and consumption process. If $\boldsymbol{\theta} \in \Theta$ is a jump-diffusion Girsanov kernel and $(\boldsymbol{\pi}, c)$ is a consumption-investment strategy then the process

$$M(t) := H_\theta(t) V^x_{\pi,c}(t) + \int_0^t H_\theta(s) c(s) ds \tag{3.24}$$

is a (local) P-martingale. These two results extend the results for a generalized geometric Brownian motion model in Karatzas et al. (1987) to a jump-diffusion model. Clearly, if there is no consumption then the discounted wealth process is a (local) martingale under the measure Q induced by a Girsanov kernel $\boldsymbol{\theta} \in \Theta$:

$$d\overline{V}^x_\pi(t) = \overline{V}^x_\pi(t-) \boldsymbol{\pi}(t)^\intercal \left[\boldsymbol{\xi}(t) d\boldsymbol{B}^Q(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y) \widetilde{N}^Q_h(dt,dy) \right]. \tag{3.25}$$

3.4 Admissible Strategies and Budget Constraints

When allowing stock prices to jump it is necessary to take some special care. There are two interesting things that can happen when stock prices jump. First, a stock price can jump to zero leading to bankruptcy of the company. This is the case when the event

$$\int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,y) N_h(dt,dy) = -1$$

happens for at least one of the $h = 1, \ldots, m$. In such a case the i^{th} stock company is going bankrupt and the stock price will not recover from zero. Interestingly, when the model has jumps one can construct trading strategies $\boldsymbol{\pi}$ that lead to possible zero or even negative wealth even though the stock prices have only done 'usual' jumps. This phenomenon has briefly been discussed in the last section. Consider, for clarity, the case without consumption. The wealth process of a trading strategy $\boldsymbol{\pi}$ could jump to zero if (cf. (3.16))

$$\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y) N_h(dt,dy) = -1,$$

is a possible event. If $\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y) N_h(dt,dy) < -1$ then the wealth process could become negative and, clearly, this should not be allowed.

We explain in the following what will happen to the trading strategy $\boldsymbol{\pi}$ in the case that either a stock or even the whole wealth process becomes zero. If a stock price jumps to zero, then the investor immediately 'sells' or 'buys' all all remaining stocks and sets its trading strategy to zero. That means if, for some $i = 1, \ldots, k$, $S_i(t)$ jumps to zero at (random) time $\tau_i \in [0,T)$, then $\phi_i(t) = \pi_i(t) = 0$ for all $t \in [\tau_i, T]$. The investor will only have k assets left, instead of the $k+1$ assets that he had before (assuming of cause that no other stock has gone bust yet). In the worst case, if all assets jump to zero during the investment period, the investor will have to put all his funds into the risk-less asset. However, this is a rather theoretical observation as it will be hardly happening in practice.

Consider what happens if the wealth process becomes zero. We mentioned earlier that we won't allow negative wealth, however we allow the possibility of zero wealth. The wealth process can become zero because of two reasons. First, it can become zero because of jumping stock prices, that is the investor follows a trading strategy $\boldsymbol{\pi}(t)$ for which $\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y)N_h(dt,dy) = -1$, is a possible outcome and is actually happening during the investment period $[0,T]$. The second reason is that consumption is bringing the portfolio value down to zero. The time of bankruptcy will in either case be denoted by

$$\tau_V := \inf\left\{t \in [0,T] : V^x_{\pi,c}(t) = 0\right\}.$$

In the case of bankruptcy the investors wealth is zero $V^x_{\pi,c}(\tau_V) = \sum_{i=1}^k \phi_i(\tau_V) = 0$. In that case the investor immediately sells all long positions and buys all short positions so that $\phi_i(t) := 0$ for all $i = 1,\dots,k$ and $t \in (\tau_V, T]$. The same is true for the proportional trading process, so that $\pi_i(t) := 0$ for all $i = 1,\dots,k$ and $t \in (\tau_V, T]$. Notice, however, that since $\pi_i(t)$ is a proportion process it could be set equal to any fixed number in the case of zero wealth. It will, however, turn out to be more convenient to set it equal to zero too.

Summarizing the above, if the i^{th} stock goes bankrupt, the i^{th} trading strategy is set to zero,

$$\phi_i(t) = \pi_i(t) = 0, \quad t \in (\min\{\tau_i, \tau_V\}, T],$$

where τ_i is the time of bankruptcy of the i^{th} stock as described above. If the portfolio goes bankrupt, all trading strategies are set to zero.

We are only interested in trading strategies and consumption plans for which the wealth process is non-negative. These strategies are called admissible.

Definition 3.6. *The investment-consumption pair $(\boldsymbol{\pi}, c)$ is called admissible if*

$$V^x_{\pi,c}(t) \geq 0, \quad t \in [0,T], a.s. \tag{3.26}$$

The set of all admissible investment-consumption pairs $(\boldsymbol{\pi}, c)$ will be denoted by $\mathcal{A}(x)$. For the case that there is no consumption the set of admissible trading strategies $\boldsymbol{\pi}$ will be denoted by Π and is not dependent on the initial endowment $x > 0$. In that case a condition for $\boldsymbol{\pi}$ to be in Π is (3.16)*, which is*

$$\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y)N_h(dt,dy) \geq -1,$$

for all $h = 1, \dots, m$. Strategies $\boldsymbol{\pi}$ without consumption that lead to a.s. positive wealth are collected in the set Π_+. These are the strategies that satisfy

$$\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y) N_h(dt,dy) > -1,$$

for all $h = 1, \dots, m$.

For positive initial endowment $x > 0$, $\mathcal{A}(x)$ is clearly non-empty since an investor could always invest all his funds into the risk-less asset, i.e. $\boldsymbol{\pi}_i(t) \equiv 0$, and not consume at all, i.e. $c(t) = 0$, for all $t \in [0,T]$. In this case the portfolio would evolve like the risk-less asset.

Lemma 3.7. *The set of all admissible investment-consumption pairs $\mathcal{A}(x)$ is convex, i.e. for $(\boldsymbol{\pi}, c) \in \mathcal{A}(x)$ and $(\widetilde{\boldsymbol{\pi}}, \widetilde{c}) \in \mathcal{A}(x)$ we have $\lambda(\boldsymbol{\pi}, c) + (1-\lambda)(\widetilde{\boldsymbol{\pi}}, \widetilde{c}) \in \mathcal{A}(x)$ for $\lambda \in [0,1]$.*

Proof. We need to show that for any $(\boldsymbol{\pi}, c), (\widetilde{\boldsymbol{\pi}}, \widetilde{c}) \in \mathcal{A}(x)$ the strategy $(\lambda\boldsymbol{\pi} + (1-\lambda)\widetilde{\boldsymbol{\pi}}, \lambda c + (1-\lambda)\widetilde{c})$ is also in $\mathcal{A}(x)$ for all $\lambda \in [0,1]$. But clearly, for $x > 0$,

$$V^x_{\lambda\pi+(1-\lambda)\widetilde{\pi},\lambda c+(1-\lambda)\widetilde{c}}(t) = \lambda V^x_{\pi,c}(t) + (1-\lambda) V^x_{\widetilde{\pi},\widetilde{c}}(t) \geq 0, \quad t \in [0,T], \quad \lambda \in [0,1].$$

□

The set of admissible strategies constrains the set of all possible trading strategy - consumption pairs. Yet, together with the measure change techniques from the previous sections it is possible to give some constraint on the wealth process. These constraints are often called budget constraints. To prove them the following lemma is needed.

Lemma 3.8. *Let $x > 0$ be an initial endowment and let Q be a measure constructed from the process $\boldsymbol{\theta} \in \Theta$. The process M^Q defined in (3.23) for $(\boldsymbol{\pi}, c) \in \mathcal{A}(x)$ is a Q supermartingale. Similar, the process M defined in (3.24) for $(\boldsymbol{\pi}, c) \in \mathcal{A}(x)$ is a P supermartingale.*

Proof. Notice that M^Q and M are local martingales that are bounded from below. By Fatou's lemma they must be supermatinales. □

The supermartingale properties proven above are crucial to derive the following budget constraints.

Proposition 3.9 (Budget Constraint)**.** *Let $x > 0$ and let $\boldsymbol{\theta} \in \Theta$ be a jump-diffusion Girsanov kernel with corresponding probability measure Q. Then every admissible pair*

$(\boldsymbol{\pi}, c) \in \mathcal{A}(x)$ *satisfies the budget constraint*

$$\mathbb{E}^Q\left[\int_0^T \frac{c(s)}{S_0(s)}ds + \frac{V^x_{\pi,c}(T)}{S_0(T)}\right] \leq x, \tag{3.27}$$

or, equivalently,

$$\mathbb{E}\left[\int_0^T H_\theta(s)c(s)ds + H_\theta(T)V^x_{\pi,c}(T)\right] \leq x. \tag{3.28}$$

Proof. Lemma 3.8 shows that $M^Q(t) = \overline{V}^{x,\pi,c}(t) + \int_0^t \frac{c(s)}{S_0(s)}ds$ is a supermartingale. Thus,

$$\mathbb{E}^Q\left[\overline{V}^{x,\pi,c}(T) + \int_0^T \frac{c(s)}{S_0(s)}ds\right] = \mathbb{E}^Q\left[M^Q(T)\middle|\mathcal{F}_0\right] \leq M^Q(0) = x.$$

$\square$

The above budget constraint states that under the Q measure the discounted terminal wealth together with the discounted total consumption over the investment period $[0, T]$ should not exceed the initial endowment $x > 0$. It gives therefore an indicator what to expect from an optimal trading strategy.

Since consumption and the wealth processes of admissible strategies is almost surely non-negative, the above budget constraint should also hold in the case of zero terminal wealth and in the case of no consumption.

Corollary 3.10. *Assume the same conditions as in Proposition 3.9. Then the budget constraints for consumption and terminal wealth, respectively, given by*

$$\begin{aligned} \mathbb{E}\left[\int_0^T H_\theta(s)c(s)ds\right] &\leq x, \text{ and} \\ \mathbb{E}\left[H_\theta(T)V^x_{\pi,c}(T)\right] &\leq x, \end{aligned}$$

respectively, are satisfied.

Proof. The corollary follows immediately from Proposition 3.9 since the consumption process and the terminal wealth of an admissible strategy is almost surely non-negative.

$\square$

3.5 Utility Functions

Utility functions will play a central role in this thesis. They are a measure of relative satisfaction from some good. In the framework of this thesis this good will be money. This kind of formulation of preferences goes back to Bernoulli (1738), and an axiomatic theory was initiated by von Neumann and Morgenstern (1947). This is why utility functions are also often called von Neumann-Morgenstern utility functions.

Definition 3.11. *A utility function $U : (0,\infty) \to \mathbb{R}$ is a strictly increasing and strictly concave C^1 function that satisfies the Inada conditions*

$$U'(0+) = \lim_{x \downarrow 0} U'(x) = \infty, \quad \textit{and} \quad U'(\infty) = \lim_{x \to \infty} U'(x) = 0.$$

A utility function U is strictly increasing since an investor prefers higher to lower levels of consumption or terminal wealth. It is concave since investors are assumed to be risk-averse. Some examples of utility functions are the so called power and log utility functions

$$U^{(p)}(x) := \frac{x^p}{p} \quad \textit{and} \quad U^{(0)}(x) := \log x,$$

for $p \in (-\infty, 1) \backslash \{0\}$.

Definition 3.12. *Let U be a utility function. The convex dual of U is defined as the function*

$$U^*(y) := \sup_{x>0} \{U(x) - xy\}, \quad y > 0. \tag{3.29}$$

The convex dual of a utility function U is the Legendre-Fenchel transformation of the function $-U(-x)$. It will become important in Section 4.4 when dealing with the dual problem.

Let the inverse of U' be denoted by I so that

$$x = I(U'(x)) = U'(I(x)), \quad \text{for } x > 0.$$

Since U is strictly increasing and strictly concave, U' must be strictly decreasing, and therefore I is also strictly decreasing.

Lemma 3.13. *Let U be a utility function and let U^* be the convex dual of U. Then $U^* : (0,\infty) \to \mathbb{R}$ is convex, nonincreasing, lower semicontinous, and satisfies*

(i)

$$U^*(y) = U(I(y)) - yI(y), \quad y > 0; \tag{3.30}$$

(ii)

$$U(x) = \inf_{y>0} \{U^*(y) + xy\}, \quad x > 0;$$

(iii)

$$U(x) = U^*(U'(x)) + xU'(x), \quad x > 0;$$

(iv)

$$U(x) \le U^*(y) + xy, \quad x, y > 0 \tag{3.31}$$

Moreover, equality holds if $x = I(y)$.

For a proof and more details see Karatzas and Shreve (1998) and Rockafellar (1970). Notice also that, combining (3.30) and (3.31) also shows that

$$U(I(y)) \geq U(x) + y\big(I(y) - x\big) \quad x, y > 0. \tag{3.32}$$

Later we will also make use of the relationship that for $b > a > 0$

$$bI(b) - aI(a) - \int_a^b I(y)dy = U(I(b)) - U(I(a)). \tag{3.33}$$

This follows immediately from the integration by parts formula

$$U(I(b)) - U(I(a)) = \int_a^b U'(I(y))dI(y) = \int_a^b ydI(y) = yI(y)\big|_a^b - \int_a^b I(y)dy.$$

Chapter 4

Optimization of the Expected Utility of Consumption and Investment

4.1 Introduction

One of the pioneering works in the field maximizing the expected utility of consumption and terminal wealth was carried out by Merton (1969, 1971). He considers the problem of the form

$$\max \quad \mathbb{E}\left[\int_0^T U_1(t, c(t))dt + U_2(V^x_{\pi,c}(T))\right], \tag{4.1}$$

where c is a consumption rate and $V^x_{\pi,c}(T)$ is the terminal wealth of a strategy $(c, \boldsymbol{\pi})$ with initial endowment $x > 0$. The stock price model that he is considering is of Markovian type and has the form of a geometric Brownanian motion

$$\frac{dP_i(t)}{P_i(t)} = \alpha_i(t, P(t))dt + \sigma_i(t, P(t))dZ_i(t)$$

where Z_i are Brownian motions. The problem is solved using a stochastic control approach, which works well with the Markovian structure of the model.

The introduction of martingale methods and change of measure techniques for stock prices by Harrison and Kreps (1979) and Harrison and Pliska (1981) made way for a different approach to tackle Merton's asset allocation problem (4.1). Instead of using stochastic control theory, the problem was solved by using an equivalent martingale measure together with convex analysis tools. The portfolio allocation problem can then be solved in two steps in a complete market. First one determines the optimal terminal wealth and optimal consumption, then one derives the optimal trading strategy using the martingale representation theorem. These methods for portfolio optimization were developed by Pliska (1985), Karatzas et al. (1987), Cox and Huang (1989), and others. They, like Merton, use Itô processes to describe stock price movements and the market they construct is complete. They consider the same stock market model as has been introduced in Chapter 3 in (3.1) but without jumps. That is the stock prices are assumed to follow the stochastic differential equations

$$\frac{dS_i(t)}{S_i(t)} = \alpha_i(t)dt + \sum_{j=1}^{n} \xi_{ij}(t)dB_j(t),$$

for $i = 1, \ldots, k$ and $t \in [0, T]$. In a complete market, the martingale approach to the problem proceeds the following way. First, the measure is changed so that the discounted stock prices and therefore the discounted wealth process (without consumption) become martingales. One can then define a budget constraint that every admissible consumption-investment strategy has to satisfy. It is shown that any random variable that satisfies the budget constraint can be replicated. Thus, one finds the optimal consumption and optimal terminal wealth and replicates them.

For an incomplete market, where the martingale measure is not unique, the solution to the problem is more delicate. Karatzas et al. (1991), He and Pearson (1991) and Cvitanić and Karatzas (1992) consider the problem of constraint consumption and terminal wealth where the incomplete market is a special case of this class of problems. A stochastic control approach to the optimal consumption problem with jump-diffusions has been carried out by Benth et al. (1999) and by Framstad et al. (1998). They use a Lévy type jump diffusion model of the form

$$dP(t) = P(t-) \left[\mu dt + \sigma dB(t) + \int_{-1}^{\infty} z\widetilde{N}(dt, dz) \right],$$

to solve an infinite time horizon maximum consumption problem

$$\max \quad \mathbb{E}\left[\int_0^{\infty} e^{-\delta t} \frac{c^{\gamma}(t)}{\gamma} dt \right]$$

for $\gamma \in (0, 1)$ and $\delta > 0$. As mentioned above, dynamic programming is used to find a solution to the problem.

The case for either the consumption, or the terminal wealth, problem when the stock is driven by a Lévy type model is treated by Kallsen (2000) using a martingale approach. The optimization problem is stated in the form

$$\max \quad \mathbb{E}\left[\int_0^{T} U(\kappa(t))dK(t) \right],$$

where $\int_0^t \kappa(s)dK(s)$ is the discounted consumption rate and $K(t)$ is the consumption clock, which determines the time of consumption. For instance, $K(t) = t$ is denoting consumption uniform in time, or $K(t) = 1_{[T,\infty)}$ represents terminal consumption. If $K(t) = t$, then the problem considered is that in (4.1) but without terminal wealth. If on the other hand $K(t) = 1_{[T,\infty)}$, then the problem is that of maximizing terminal wealth which is the one in (4.1) without consumption term. The model is a geometric Lévy process of the form

$$\overline{S}_i(t) = \overline{S}_i(0)e^{L_i(t)},$$

where $\overline{S}_i$ are the discounted stock prices and L_i are Lévy processes. Explicit solutions for the cases of logarithmic, power and exponential utilities are derived.

An often cited paper that develops a general duality approach for a general incomplete semimartingale model is that of Kramkov and Schachermayer (1999). The problem is that of optimal terminal wealth when the risk-less interest rate is assumed to be zero. The problem is then that of

$$\max \quad \mathbb{E}\left[U\left(V_\pi^x(T)\right)\right],$$

which is similar to Merton's problem (4.1) but without consumption. In their results, they consider in particular the asymptotic elasticity of utility functions as condition for several key assertions, like, for example, the existence and uniqueness of a solution to the problem.

Callegaro and Vargiolu (2009) obtain results for HARA utility functions under multi-dimensional models of pure jump type. Another paper involving jump-diffusion models is that by Bardhan and Chao (1995) who use a Brownian motion model with Poisson jumps for a problem of the form (4.1). However, the market they consider is actually complete. They proceed similarly to Karatzas et al. (1990), but have a different way of changing the measure.

In this chapter we study optimal trading strategies for the problem of maximizing expected utility of consumption and terminal wealth under a multidimensional jump-diffusion model. This is the classical Merton problem (4.1) in a jump-diffusion framework. For a pure diffusion model in a complete market, the unique auxiliary process determined by the dual approach is the optimal consumption process and it, together with the unique random variable determined similarly, gives the optimal trading strategy. For our incomplete market, there does not always exist a corresponding trading strategy for the pair determined in such a way. We modify the Karatzas and Shreve (1998) approach for the problem under a pure diffusion process in a complete market by showing that the optimal consumption process and the optimal trading strategy are determined by the martingale measure whose parameter is a solution to a system of (non-linear) equations. This is in contrast with the standard duality approach, used in Kramkov and Schachermayer (1999) for the terminal wealth and in Karatzas and Shreve (1998), where the optimal EMM is first obtained by solving the dual problem that uses the convex conjugate of the utility function. Then the original problem is solved as a constraint optimization problem. The constraint is thereby the budget constraint. Our modification of the duality approach has the advantage that the optimal martingale measure, as well as the optimal consumption process and trading strategy, can be directly obtained by solving a system of non-linear equations. As a special case, we also derive the concrete result for HARA utilities when the parameters in the model are deterministic functions. It can be varified that our approach also holds for either the consumption, or the terminal wealth, problem. Since both the Lévy model and the pure jump model with constant parameters are special cases of a general jump-diffusion model, the results presented here extend the results of both Kallsen (2000) and Callegaro and Vargiolu (2009).

The chapter is structered as follows. Section 4.2 introduces the optimization problem. A set of auxiliary process will then be introduced in Section 4.3. These auxiliary process have the special properties when compared with admissible consumption and terminal wealth: expected utility from the auxiliary processes never underperforms expected utility from admissible consumption and terminal wealth. The actual investment-consumption problem will be solved in Section 4.4 and applications follow in terms of power utility in Section 4.5. The problem of either maximizing consumption or terminal wealth is discussed in Section 4.7. The chapter is completed with a transition to partial differential equation problems, making heavily use of Komogorov's equation in Section 4.8

4.2 The Investment-Consumption Problems

We consider a utility maximization problem where an investor obtains utility from consumption and from terminal wealth of his/her investment. Utility functions are thereby functions as introduced in Section 3.5.

It is assumed that there are k stocks in the market as introduced in Chapter 3. Each of these stocks follow a stochastic jump-diffusion differential equation as in (3.1), which was given there as

$$\frac{dS_i(t)}{S_i(t-)} = \alpha_i(t)dt + \sum_{j=1}^{n} \xi_{ij}(t)dB_j(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,y)N_h(dt,dy),$$

for $i = 1, \ldots, k$. The initial wealth of an investor is given by $x > 0$ so that following a consumption-investment strategy $(\boldsymbol{\pi}, c) \in \mathcal{A}(x)$ his/her wealth evolves according to the stochastic differential equation

$$\begin{aligned} dV^x_{\pi,c}(t) = &\ \big(V^x_{\pi,c}(t-)r(t) - c(t)\big)\,dt + \boldsymbol{\pi}(t)V^x_{\pi,c}(t-)^\intercal\left[\overline{\boldsymbol{\alpha}}(t)dt + \boldsymbol{\xi}(t)d\boldsymbol{B}(t)\right] \\ &+ V^x_{\pi,c}(t-)\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal\boldsymbol{\gamma}_h(t,y)N_h(dt,dy), \quad t\in[0,T], \end{aligned}$$

which has been introduced in (3.20). Admissible strategy $\mathcal{A}(x)$ are thereby the strategy that lead almost surely to non-negative wealth and have been introduced in Definition 3.6.

Because of the jump processes the market is incomplete. There are more sources of randomness than stocks in the market and so an equivalent martingale measure is not unique. We have, however, parametrized each martingale measures by a pair $\boldsymbol{\theta} = (\boldsymbol{\theta}^D, \boldsymbol{\theta}^J)$ as in Definition 3.1 and denoted the set of all eligible parametrizations by Θ. An equivalent martingale measure can be constructed by setting $dQ = Z_\theta(T)dP$, where $Z_\theta(T)$ as in (2.20). The discounted asset prices become martingales under this measure and the state price density H_θ defined in Definition 3.3 for a $\boldsymbol{\theta} \in \Theta$ can be used to take advantage of these martingale properties while still working under the original probability measure P. A typical application of H_θ is Proposition 3.9, where it has been shown that all admissible strategies $(\boldsymbol{\pi}, c) \in \mathcal{A}(x)$ satisfy the budget constraint

$$\mathrm{E}\left[\int_0^T H_\theta(s-)\,c(s)\,ds + H_\theta(T)V^x_{\pi,c}(T)\right] \le x,$$

for some initial wealth $x > 0$. This budget constraint will work as an optimization constraint when the investor is considering the optimal consumption-investment problem. It is assumed that the investment horizon is finite, i.e. $T < \infty$. We will denote the utility from consumption by U_1 which will be a time dependent utility function, and the utility from terminal wealth will be given by the utility function U_2. The admissible strategies need to be constraint to be adequate for the optimization problem. Define the set

$$\widetilde{\mathcal{A}}(x) := \left\{(\boldsymbol{\pi}, c) \in \mathcal{A}(x)\Big|\quad \mathbb{E}\left[\int_0^T U_1\big(t, c(t)\big)^- dt + U_2(V^x_{\pi,c}(T))^-\right] > -\infty\right\}, \tag{4.2}$$

where $x^- := \min\{0, X\}$, and further define for initial wealth $x > 0$ and a strategy $(\boldsymbol{\pi}, c) \in \widetilde{\mathcal{A}}(x)$ the objective function by

$$J(x; \boldsymbol{\pi}, c) := \mathbb{E}\left[\int_0^T U_1\big(t, c(t)\big)dt + U_2(V^x_{\pi,c}(T))\right]. \tag{4.3}$$

The problem of maximizing consumption and terminal wealth, denoted by $\Phi(x)$, is that of finding an optimal pair $(\widehat{\boldsymbol{\pi}}, \widehat{c}) \in \widetilde{\mathcal{A}}(x)$ such that

$$\Phi(x) := \sup_{(\boldsymbol{\pi},c)\in\widetilde{\mathcal{A}}(x)} J(x; \boldsymbol{\pi}, c) = J(x; \widehat{\boldsymbol{\pi}}, \widehat{c}). \tag{4.4}$$

With a slight abuse of notation we will also refer to the optimal performance function $\Phi(x)$ as the optimization problem. This optimization problem consist of the problem of finding $\Phi(x)$ as well as of finding an optimal strategy pair $(\widehat{\boldsymbol{\pi}}, \widehat{c})$.

The optimization problem is solved in two steps. First, for each martingale measure represented by $\boldsymbol{\theta}$ an 'optimal' auxiliary terminal wealth and consumption process is constructed. This pair outperforms, in the expected utility sense, every admissible consumption - terminal wealth pair. In the second step it is checked under what conditions the auxiliary terminal wealth and consumption performs exactly the same as an admissible consumption - terminal wealth pair. This is the case when the EMM parameter $\boldsymbol{\theta}$ and the optimal trading strategy $\boldsymbol{\pi}$ satisfy a set of non-linear equations.

4.3 Auxiliary Processes

In the following we introduce for each martingale measure parametrized by $\boldsymbol{\theta}$ a pair of consumption and terminal wealth (c_θ, Y_θ). It turns out that the expected utility of these pairs always performs at least as good as any admissible consumption - terminal wealth pair. That means the expected utility of (c_θ, Y_θ) is never smaller than the objective function $J(x; \boldsymbol{\pi}, c)$ of any admissible pair $(\boldsymbol{\pi}, c) \in \widetilde{\mathcal{A}}(x)$.

We write $I_1(t, \cdot)$ and I_2 for the inverse of $U_1'(t, \cdot)$ and U_2' respectively, where $U_1'(t, \cdot)$ denotes the partial derivative of U_1 with respect to its (second) variable representing the consumption. For $\boldsymbol{\theta} = (\theta^D, \theta^J) \in \Theta$ recall that the state price density H_θ is defined by (3.3) as $H_\theta(t) = Z_\theta(t)/S_0(t)$, where Z_θ is the Radon-Nikodym density of the measure change as in (2.20). We can then define the function

$$\mathcal{X}_\theta(y) := \mathbb{E}\left[\int_0^T H_\theta(t-) I_1(t, yH_\theta(t-))dt + H_\theta(T) I_2(yH_\theta(T))\right], \quad y > 0, \tag{4.5}$$

and the set

$$\widetilde{\Theta} := \big\{\boldsymbol{\theta} \in \Theta \mid \mathcal{X}_\theta(y) < \infty, \text{ for } y > 0\big\}.$$

Note that the subset $\widetilde{\Theta}$ of Θ depends on U_1 as well as on U_2 and that, for $\boldsymbol{\theta} \in \widetilde{\Theta}$, $\mathcal{X}_\theta$ maps $(0, \infty)$ into itself continuously and, by the properties of utility functions, is strictly decreasing with $\mathcal{X}_\theta(0+) = \infty$ and $\mathcal{X}_\theta(\infty) = 0$. For fixed $x > 0$ and $\boldsymbol{\theta} \in \widetilde{\Theta}$, we now define respectively the process $c_\theta(t)$, for $t \in [0, T]$, and the random variable Y_θ by

$$\begin{aligned} c_\theta(t) &:= I_1\big(t, \mathcal{X}_\theta^{-1}(x) H_\theta(t-)\big), \\ Y_\theta &:= I_2\big(\mathcal{X}_\theta^{-1}(x) H_\theta(T)\big). \end{aligned} \tag{4.6}$$

Clearly, both c_θ and Y_θ are non-negative. Further, in the complete pure diffusion model of Karatzas and Shreve (1998), the EMM parameter $\boldsymbol{\theta}$ is unique, and the corresponding unique $c_\theta(t)$ and Y_θ are respectively the optimal consumption process and terminal wealth random variable. However, in an incomplete market setting, $\boldsymbol{\theta}$ is not unique and c_θ as well as Y_θ depend on $\boldsymbol{\theta}$. The auxiliary consumption c_θ and auxiliary terminal wealth Y_θ have the property that they outperform any admissible consumption-terminal wealth pair - as can be seen in the next lemma.

Lemma 4.1. *For any $\boldsymbol{\theta} \in \widetilde{\Theta}$, $c_\theta(t)$, $t \in [0,T]$, and Y_θ defined by* (4.6) *satisfy*

(*i*) $\mathrm{E}\left[\int_0^T H_\theta(t-)c_\theta(t)dt + H_\theta(T)Y_\theta\right] = x;$

(*ii*) $\mathrm{E}\left[\int_0^T U_1(t,c_\theta(t))^- dt + U_2(Y_\theta)^-\right] > -\infty;$

(*iii*) $J(x;\boldsymbol{\pi},c) \leq \mathrm{E}\left[\int_0^T U_1(t,c_\theta(t))dt + U_2(Y_\theta)\right]$ *for all* $(\boldsymbol{\pi},c) \in \widetilde{\mathcal{A}}(x)$,

where J is defined by (4.3).

Proof. (*i*) This follows from the definition (4.5) of $\mathcal{X}_\theta$ and the construction of c_θ and Y_θ.
(*ii*) From (3.32), it follows that

$$\begin{aligned} U_1(t,c_\theta(t)) &\geq U_1(t,1) + \mathcal{X}_\theta^{-1}(x)\, H_\theta(t)\{c_\theta(t)-1\} \geq U_1(t,1) - \mathcal{X}_\theta^{-1}(x)\, H_\theta(t) \\ U_2(Y_\theta) &\geq U_2(1) + \mathcal{X}_\theta^{-1}(x)\, H_\theta(T)\{Y_\theta - 1\} \geq U_2(1) - \mathcal{X}_\theta^{-1}(x)\, H_\theta(T). \end{aligned}$$

The second inequality in both cases follows from $c_\theta(t)$ and Y_θ, respectively, being non-negative. Moreover, observe that $U_1(t,c_\theta(t))^- \geq \big(U_1(t,1) - \mathcal{X}_\theta^{-1}(x)H_\theta(t)\big)^- \geq -|U_1(t,1)| - \mathcal{X}_\theta^{-1}(x)H_\theta(t)$ for $t \in [0,T]$ and $U_2(Y_\theta)^- \geq -|U_2(1)| - \mathcal{X}_\theta^{-1}(x)H_\theta(T)$ and so

$$\begin{aligned} & \mathbb{E}\left[\int_0^T U_1(t,c_\theta(t))^- dt + U_2(Y_\theta)^-\right] \\ \geq\ & -\int_0^T |U_1(t,1)|\, dt - \mathcal{X}_\theta^{-1}(x)\,\mathbb{E}\left[\int_0^T H_\theta(t)\, dt\right] \\ & -|U_2(1)| - \mathcal{X}_\theta^{-1}(x)\,\mathbb{E}\left[H_\theta(T)\right] \\ =\ & -\int_0^T |U_1(t,1)|\, dt - \mathcal{X}_\theta^{-1}(x)\,\mathbb{E}^Q\left[\int_0^T \frac{1}{S_0(t)} dt\right] \\ & -|U_2(1)| - \mathcal{X}_\theta^{-1}(x)\,\mathbb{E}^Q\left[\frac{1}{S_0(T)}\right] \\ >\ & -\infty, \end{aligned}$$

where the last inequality follows from S_0 being bounded away from zero.
(*iii*) Again from (3.32), it follows that for any $(\pi,c) \in \widetilde{\mathcal{A}}(x)$

$$\begin{aligned} U_1(t,c_\theta(t)) &\geq U_1(t,c(t)) + \mathcal{X}_\theta^{-1}(x)\, H_\theta(t)\{c_\theta(t) - c(t)\}, \quad t \in [0,T], \\ U_2(Y_\theta) &\geq U_2(V_{\pi,c}^x(T)) + \mathcal{X}_\theta^{-1}(x)\, H_\theta(T)\{Y_\theta - V_{\pi,c}^x(T)\}. \end{aligned}$$

Then, the result (i) and Proposition 3.9 together give

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T U_1(t, c_\theta(t))\,dt + U_2(Y_\theta)\right] \geq J(x;\pi,c) \\
&\qquad + \mathcal{X}_\theta^{-1}(x)\,\mathbb{E}\left[\int_0^T H_\theta(t)\{c_\theta(t) - c(t)\}\,dt + H_\theta(T)\{Y_\theta - V_{\pi,c}^x(T)\}\right] \\
&= J(x;\pi,c) + \mathcal{X}_\theta^{-1}(x)\left\{x - \mathbb{E}\left[\int_0^T H_\theta(t)\,c(t)\,dt + H_\theta(T)\,V_{\pi,c}^x(T)\right]\right\} \\
&\geq J(x;\pi,c).
\end{aligned}
$$

□

The above lemma has some similarities to a lemma appearing in Kallsen (2000). In Kallsen's lemma, a consumption process is assumed to be given (he considers a generalised optimal consumption problem). If a Girsanov martingale Z then satisfies certain conditions similar to, for example Lemma 4.1*(i)*, then it has been shown that a similar result as in Lemma 4.1 *(iii)* is satisfied. However, in our approach the argument is slightly different. Instead, of fixing a consumption process and determining an eligible martingale Z, a measure change marginal has already been fixed by $\boldsymbol{\theta}$. The corresponding c_θ and Y_θ may or may not be an eligible consumption process and an eligible terminal wealth respectively. Conditions under which these terms are actually admissible consumption process and admissible terminal wealth will be given later. They represent the key result of this chapter.

An important consequence of Lemma 4.1(iii) is that

$$
\begin{aligned}
&\sup_{(\widetilde{\pi},\widetilde{c})\in\widetilde{\mathcal{A}}(x)} \mathbb{E}\left[\int_0^T U_1(t,\widetilde{c}(t))\,dt + U_2(V_{\widetilde{\pi},\widetilde{c}}^x(T))\right] \\
&\qquad \leq \inf_{\widetilde{\theta}\in\widetilde{\Theta}} \mathbb{E}\left[\int_0^T U_1(t, c_{\widetilde{\theta}}(t))\,dt + U_2(Y_{\widetilde{\theta}})\right],
\end{aligned}
\tag{4.7}
$$

so that the expected utility corresponding to these auxiliary processes outperforms or is at least equal to the expected utility of any admissible investment-consumption pair.

4.4 The Solution to the Optimal Investment-Consumption Problem

We have used the set of jump diffusion Girsanov kernels Θ as in Definition 3.1 to specify the equivalent martingale measures. This set had to be constraint to $\widetilde{\Theta}$ to fit technical requirements of our optimization problem $\Phi(x)$ as defined in (4.4). We are now interested in a particular Girsanov kernel (or EMM) $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}$ for which the infimum in (4.7) is attained and in fact equality holds. That is, $\widehat{\boldsymbol{\theta}}$ should satisfy

$$\Phi(x) = \inf_{\theta\in\widetilde{\Theta}} \mathbb{E}\left[\int_0^T U_1(t, c_\theta(t))dt + U_2(Y_\theta)\right] = \mathbb{E}\left[\int_0^T U_1(t, c_{\widehat{\theta}}(t))dt + U_2(Y_{\widehat{\theta}})\right]$$

We will call this measure the optimal martingale measure for the optimization problem $\Phi(x)$. Recall thereby that with an abuse of notation we refer by $\Phi(x)$ not only to the optimal performance function but also to the problem of finding the optimal strategy $(\widehat{\boldsymbol{\pi}}, \widehat{c})$. The optimal consumption process for the problem $\Phi(x)$ then turns out to be the consumption process $c_{\widehat{\theta}}$ as defined in (4.6) for the optimal measure represented by $\widehat{\boldsymbol{\theta}}$. Furthermore, we will be able to derive a trading strategy $\pi_{\widehat{\theta}}$ that is optimal for the problem $\Phi(x)$ in the sense that the supremum of $\mathbb{E}\left[\int_0^T U_1(t, c(t))dt + U_2(V^x_{\pi,c}(T))\right]$ over all $(\boldsymbol{\pi}, c) \in \widetilde{\mathcal{A}}$ is attained by $(\boldsymbol{\pi}_{\widehat{\theta}}, c_{\widehat{\theta}})$. In fact, the trading strategy $\boldsymbol{\pi}_{\widehat{\theta}}$ together with the optimal measure given by $\widehat{\theta}$ are obtained by solving a system of (non-linear) equations.

Definition 4.2. *A martingale measure Q obtained by $\frac{dQ}{dP}|_{\mathcal{F}_T} = Z_\theta(T)$ in terms of a $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}$ is called optimal for the optimization problem* (4.4) *if the infimum in* (4.7) *is attained, i.e.*

$$\mathbb{E}\left[\int_0^T U_1(t, c_{\widehat{\theta}}(t))dt + U_2(Y_{\widehat{\theta}})\right] = \inf_{\theta\in\widetilde{\Theta}} \mathbb{E}\left[\int_0^T U_1(t, c_\theta(t))dt + U_2(Y_\theta)\right],$$

where c_θ and Y_θ are defined by (4.6) *respectively.*

In the following such optimal martingale measures Q are linked to optimal investment-consumption pairs which solve (4.4). This will allow us to derive a trading strategy $\boldsymbol{\pi}_{\widehat{\theta}}$ such that (4.4) is solved at $(\boldsymbol{\pi}_{\widehat{\theta}}, c_{\widehat{\theta}})$. For this, we consider, for any $\boldsymbol{\theta} \in \widetilde{\Theta}$, the martingale M_θ defined by

$$M_\theta(t) := \mathbb{E}\left[\int_0^T H_\theta(s-)c_\theta(s)ds + H_\theta(T)Y_\theta \;\middle|\; \mathcal{F}_t\right] \tag{4.8}$$

and the process J_θ defined by

$$J_\theta(t) := \int_0^t H_\theta(s-)c_\theta(s)ds, \qquad t \in [0, T], \tag{4.9}$$

where c_θ and Y_θ are defined by (4.6) and H_θ is defined in Definition 3.3 by

$$\begin{aligned} H_\theta(t) = \exp\Big(&- \int_0^t r(s)ds - \frac{1}{2}\int_0^t ||\boldsymbol{\theta}^D(s)||^2 ds + \int_0^t \boldsymbol{\theta}^D(s)^\intercal d\boldsymbol{B}(s) \\ &+ \sum_{h=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} \log \theta_h^J(s,y) N_k(ds, dy) + \sum_{k=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} \big(1 - \theta_h^J(s,y)\big)\nu_h(dy)ds\Big). \end{aligned}$$

The martingale M_θ is non-negative and, by Lemma 4.1(i), $M_\theta(0) = x$ holds a.s. for every $\boldsymbol{\theta} \in \widetilde{\Theta}$. Next, define the process V_θ for $\boldsymbol{\theta} \in \widetilde{\Theta}$ by

$$V_\theta(t) := \frac{1}{H_\theta(t)}\big\{M_\theta(t) - J_\theta(t)\big\}, \quad t \in [0, T]. \tag{4.10}$$

Then $V_\theta(t)$ satisfies $V_\theta(0) = x$, $V_\theta(T) = Y_\theta$ as well as $V_\theta(t) \geq 0$ for all $t \in [0, T]$. Further, as M_θ defined in (4.8) is a martingale, the Martingale Representation Theorem 2.13 has shown that every martingale can be written in form of two integrals, one with respect to a Brownian motion and the other one with respect to a compensated Poisson random measure. To get such a martingale representation let $\boldsymbol{a}^D(t)$, and $a_i^J(t, y)$, $i = 1, ..., m$, be the essentially unique martingale representation coefficients of $M_\theta(t)$ so that the dynamics of M_θ can be written in the form

$$dM_\theta(t) = \boldsymbol{a}^D(t)^\intercal d\boldsymbol{B}(t) + \sum_{i=1}^{m} \int_{\mathbb{R}\setminus\{0\}} a_i^J(t, y)\, \widetilde{N}_i(dt, dy). \tag{4.11}$$

Although not explicitly stated, these martingale coefficients depend also on $\boldsymbol{\theta} \in \widetilde{\Theta}$. We can now formulate the central result of this chapter. The following result characterizes the optimal strategies and shows that, under certain conditions, such strategies relate to $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}$ with the corresponding Q being optimal. In particular, for such a $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}$, the corresponding $c_{\widehat{\theta}}$ defined by (4.6) is the optimal consumption strategy and the corresponding $V_{\widehat{\theta}}$ defined by (4.10) is the optimal portfolio wealth process.

Recall that in Definition 3.6 the set Π of all trading strategies $\boldsymbol{\pi}$ for which

$$1 + \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t, y) \geq 0$$

holds for $t \in [0, T]$, $h = 1, ..., m$, and ν_h-almost all $y \in \mathbb{R} \setminus \{0\}$ has been introduced. These are the trading strategies that guarantee that a jump in the stock prices does not lead to negative wealth.

Theorem 4.3. *Suppose that there exist a $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}$ and a trading strategy $\boldsymbol{\pi}_{\widehat{\theta}} \in \Pi$ that satisfy*

$$\begin{aligned} V_{\widehat{\theta}}(t-)\boldsymbol{\pi}_{\widehat{\theta}}(t)^\intercal \boldsymbol{\xi}(t) &= \frac{1}{H_{\widehat{\theta}}(t-)}\boldsymbol{a}^D(t) - V_{\widehat{\theta}}(t-)\,\widehat{\boldsymbol{\theta}}^D(t), \\ V_{\widehat{\theta}}(t-)\boldsymbol{\pi}_{\widehat{\theta}}(t)^\intercal \boldsymbol{\gamma}_h(t, y) &= \frac{1}{H_{\widehat{\theta}}(t-)}\frac{a_h^J(t, y)}{\widehat{\theta}_h^J(t, y)} - V_{\widehat{\theta}}(t-)\frac{\widehat{\theta}_h^J(t, y) - 1}{\widehat{\theta}_h^J(t, y)}, \end{aligned} \tag{4.12}$$

for $h = 1, ..., m$, and ν_ℓ-almost all $y \in \mathbb{R} \setminus \{0\}$. Assume further that (3.20) has a solution for $(\boldsymbol{\pi}, c) = (\boldsymbol{\pi}_{\widehat{\theta}}, c_{\widehat{\theta}})$, where $c_{\widehat{\theta}}$ is defined by (4.6). Then $(\boldsymbol{\pi}_{\widehat{\theta}}, c_{\widehat{\theta}})$ is a solution to the problem (4.4) of maximizing expected utility of consumption and terminal wealth under multi-dimensional jump-diffusion models and the corresponding wealth process is given by

$$V^x_{\pi_{\widehat{\theta}}, c_{\widehat{\theta}}}(t) = V_{\widehat{\theta}}(t), \qquad a.s.\ , t \in [0, T],$$

where V_θ is defined by (4.10).

Proof. To prove that $V^x_{\pi_{\widehat{\theta}},c_{\widehat{\theta}}} = V_{\widehat{\theta}}$ a.s. we show that both process follow the same stochastic differential equation. Let $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}$ and $\boldsymbol{\pi}_{\widehat{\theta}} \in \Pi$ satisfy the system of equations (4.12). It follows from the definition (4.10) of $V_{\widehat{\theta}}$ that

$$dV_{\widehat{\theta}}(t) = d\frac{M_{\widehat{\theta}}(t)}{H_{\widehat{\theta}}(t)} - d\frac{J_{\widehat{\theta}}(t)}{H_{\widehat{\theta}}(t)}, \quad t \in [0,T]. \tag{4.13}$$

The dynamics of M_θ in terms of its martingale representation can be found in (4.11). The dynamics of the reciprocal of H_θ can be determined by Itô's formula for geometric jump-diffusions (2.17) when applied to the function $f(x) = 1/x$ and H_θ. From (3.11) the dynamics of the state price density are

$$\begin{aligned} dH_\theta(t) = &- H_\theta(t-)r(t)dt + H_\theta(t-)\boldsymbol{\theta}^D(t)d\boldsymbol{B}(t) \\ &+ H_\theta(t-)\sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} (\theta^J_k(t,y)-1)\widetilde{N}_k(dt,dy), \quad t \in [0,T]. \end{aligned}$$

Clearly, $f'(x) = -1/x^2$, $f''(x) = 2/x^3$, and $f(x+(\theta-1)x) - f(x) = \frac{1}{x}\frac{1-\theta}{\theta}$ so that $df(H_\theta)$ can be calculated as

$$\begin{aligned} d\frac{1}{H_\theta(t)} = &\frac{1}{H_\theta(t-)}\left(r(t) + ||\boldsymbol{\theta}^D(t)||^2 + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \left(\theta^J_k(t,y)-1\right)\nu_k(dy)\right)dt \\ &- \frac{1}{H_\theta(t-)}\boldsymbol{\theta}^D(t)^\intercal d\boldsymbol{B}(t) - \frac{1}{H_\theta(t-)}\sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \frac{\theta^J_k(t,y)-1}{\theta^J_k(t,y)} N_k(dt,dy). \end{aligned}$$

The dynamics of the products $M_{\widehat{\theta}}(t)\frac{1}{H_{\widehat{\theta}}(t)}$ and $J_{\widehat{\theta}}(t)\frac{1}{H_{\widehat{\theta}}(t)}$ in (4.13) can then be calculated using Itô's product rule (2.18)

$$\begin{aligned} d\big(X(t)Y(t)\big) = &X(t-)dY^c(t) + Y(t-)dX^c(t) + dX^c(t)dY^c(t) \\ &+ X(t-)\Delta Y(t) + Y(t-)\Delta X(t) + \Delta X(t)\Delta Y(t). \end{aligned}$$

Thus, the products are

$$\begin{aligned} d\frac{M_{\widehat{\theta}}(t)}{H_{\widehat{\theta}}(t)} = &\frac{M_{\widehat{\theta}}(t-)}{H_{\widehat{\theta}}(t-)}\Bigg(\Bigg[r(t) + ||\widehat{\boldsymbol{\theta}}^D(t)||^2 - \frac{1}{M_{\widehat{\theta}}(t)}\boldsymbol{a}^D(t)^\intercal\boldsymbol{\theta}^D(t) \\ &+ \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \left(\widetilde{\theta}^J_k(t,y) - 1 - \frac{1}{M_{\widehat{\theta}}(t-)}a^J_h(t,y)\right)\nu_k(dy)\Bigg]dt \\ &+ \left[\frac{1}{M_{\widehat{\theta}}(t-)}\boldsymbol{a}^D(t) - \widehat{\boldsymbol{\theta}}^D(t)\right]^\intercal d\boldsymbol{B}(t) \\ &+ \left[\sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \frac{1}{M_{\widehat{\theta}}(t-)}\frac{a^J_h(t,y)}{\widehat{\theta}^J_h(t,y)} - \frac{\widehat{\theta}^J_h(t,y)-1}{\widehat{\theta}^J_h(t,y)}\right]N_h(dt,dy)\Bigg), \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} d\frac{J_{\widehat{\theta}}(t)}{H_{\widehat{\theta}}(t)} = c_{\widehat{\theta}}(t) + &\frac{J_{\widehat{\theta}}(t-)}{H_{\widehat{\theta}}(t-)}\Bigg(\Bigg[r(t) + ||\widehat{\boldsymbol{\theta}}^D(t)||^2 + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \left(\widehat{\theta}^J_k(t,y)-1\right)\nu_k(dy)\Bigg]dt \\ &-\widehat{\boldsymbol{\theta}}^D(t)^\intercal d\boldsymbol{B}(t) - \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \frac{\widehat{\theta}^J_k(t,y)-1}{\widehat{\theta}^J_k(t,y)} N_k(dt,dy)\Bigg). \end{aligned} \tag{4.15}$$

We can now substitute (4.14) and (4.15) into (4.13) to obtain the dynamics of $dV_{\widehat{\theta}}$. Notice also that because of (4.10) $V_\theta(t) = \frac{1}{H_\theta(t)}\{M_\theta(t) - J_\theta(t)\}$. Thus,

$$\begin{aligned}
dV_{\widehat{\theta}}(t) = & -c_{\widehat{\theta}}(t)dt + V_{\widehat{\theta}}(t-)\left(r(t) + ||\boldsymbol{\theta}^D(t)||^2 + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \left(\theta_h^J(t,y) - 1\right)\nu_h(dy)\right)dt \\
& - \frac{1}{H_{\widehat{\theta}}(t-)}\left(\boldsymbol{a}^D(t)^\intercal \boldsymbol{\theta}^D + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} a_h^J(t,y)\nu_h(dy)\right)dt \\
& + \left(\frac{1}{H_{\widehat{\theta}}(t-)}\boldsymbol{a}^D - V_{\widehat{\theta}}(t-)\widehat{\boldsymbol{\theta}}^D(t)\right)^\intercal d\boldsymbol{B}(t) \\
& + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\left(\frac{1}{H_{\widehat{\theta}}(t-)}\frac{a_h^J(t,y)}{\widehat{\theta}_h^J(t,y)} - V_{\widehat{\theta}}(t-)\frac{\widehat{\theta}_h^J(t,y) - 1}{\widehat{\theta}_h^J(t,y)}\right)N_h(dt,dy)
\end{aligned} \tag{4.16}$$

for $t \in [0,T]$. Substituting (4.12) into (4.16) and also multiplying each equation by $\widehat{\boldsymbol{\theta}}^D$ and $\widehat{\theta}_h^J$ respectively, and substituting into (4.16) shows that $V_{\widehat{\theta}}$ is given by

$$\begin{aligned}
dV_{\widehat{\theta}}(t) \quad = \quad & \left[V_{\widehat{\theta}}(t-)r(t) - c_{\widehat{\theta}}(t) + V_{\widehat{\theta}}(t-)\,\boldsymbol{\pi}_{\widehat{\theta}}(t)^\intercal \overline{\alpha}(t)\right]dt + V_{\widehat{\theta}}(t-)\,\boldsymbol{\pi}_{\widehat{\theta}}(t)^\intercal \boldsymbol{\xi}(t)\,d\boldsymbol{B}(t) \\
& + \sum_{i=1}^{k}\sum_{j=1}^{m}\int_{\mathbb{R}\setminus\{0\}} V_{\widehat{\theta}}(t-)\,\pi_{\widehat{\theta},i}(t)\,\gamma_{ij}(t,y)\,N_j(dt,dy), \quad t \in [0,T].
\end{aligned}$$

Thereby, we also used that $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}$ satisfies (3.8). Then this SDE coincides with the stochastic differential equation (3.20) satisfied by the portfolio wealth process $V^x_{\pi_{\widehat{\theta}},c_{\widehat{\theta}}}$ that follows the strategy $(\boldsymbol{\pi}_{\widehat{\theta}}, c_{\widehat{\theta}})$. Observing further that $V_{\widehat{\theta}}(0) = V^x_{\pi_{\widehat{\theta}},c_{\widehat{\theta}}}(0) = x$, we conclude that $V_{\widehat{\theta}}(t) = V^x_{\pi_{\widehat{\theta}},c_{\widehat{\theta}}}(t)$ a.s. for all $t \in [0,T]$.

To show that $(\boldsymbol{\pi}_{\widehat{\theta}}, c_{\widehat{\theta}})$ is a solution to (4.4), by Lemma 4.1 and (4.7), we only need to show that

$$\mathbb{E}\left[\int_0^T U_1(t,c(t))dt + U_2(V^x_{\pi,c}(T))\right] = \mathbb{E}\left[\int_0^T U_1(t,c_{\widehat{\theta}}(t))dt + U_2(Y_{\widehat{\theta}})\right]$$

for $(\boldsymbol{\pi}, c) = (\boldsymbol{\pi}_{\widehat{\theta}}, c_{\widehat{\theta}})$. However, this is true if $V^x_{\pi_{\widehat{\theta}},c_{\widehat{\theta}}}(T) = Y_{\widehat{\theta}}$ a.s. The latter follows from the facts that $V^x_{\pi_{\widehat{\theta}},c_{\widehat{\theta}}}(t) = V_{\widehat{\theta}}(t)$ a.s., $t \in [0,T]$, and that, by the construction of $V_{\widehat{\theta}}$, $V_{\widehat{\theta}}(T) = Y_{\widehat{\theta}}$ a.s. □

Theorem 4.3 shows that it is possible to obtain an optimal measure for the optimization problem (4.4) as well as an optimal trading strategy $\boldsymbol{\pi}_{\widehat{\theta}}$ by solving the system of equations given by (4.12). Recall however that the martingale representation processes $a_i^D(t)$ and a_h^J in (4.12) depend also indirectly on the unknown $\widehat{\boldsymbol{\theta}}$. Therefore the system of equation can be difficult to solve. In the next section, we will however give some examples for cases where such a solution can indeed be found. Nevertheless, Theorem 4.3 does not address the uniqueness issue as Kallsen (2000) did for the optimal consumption problem and Kramkov and Schachermayer (1999) did for the optimal terminal wealth problem, both in the framework of a general incomplete semimartingale model.

We now turn to solutions for the dual problem of the 'primal' optimization problem (4.4). In (3.29) the convex dual U^* of a utility function U has been defined as the Legendre-Fenchel transform of the function $-U(-x)$, that is

$$U^*(y) := \sup_{x>0} \big\{ U(x) - x\, y \big\}, \quad y > 0.$$

The convex dual satisfies for $x, y > 0$, the Fenchel inequality (3.31) which was given by

$$U(x) \leq U^*(y) + x\, y$$

Equality in the Fenchel inequality holds for $x = I(y)$.

Theorem 4.4. *For $y > 0$, assume that $\boldsymbol{\theta}_y \in \widetilde{\Theta}$ minimizes*

$$J^*(y, \boldsymbol{\theta}) := \mathrm{E}\left[\int_0^T U_1^*(yH_\theta(t-))dt + U_2^*(yH_\theta(T)) \right]$$

such that

$$\Phi^*(y) := \inf_{\boldsymbol{\theta} \in \widetilde{\Theta}} J^*(y, \boldsymbol{\theta}) = J^*(y, \boldsymbol{\theta}_y) < \infty. \tag{4.17}$$

Assume further that, for such a $\boldsymbol{\theta}_y$, there exists a $\boldsymbol{\pi}_{\theta_y} \in \Pi_+$ such that (4.12) *is satisfied. Then*

$$\Phi^*(y) = \sup_{x>0} \big\{ \Phi(y) - xy \big\}, \quad y > 0,$$

where Φ is defined by (4.4).

Proof. Note first that, by Theorem 4.3, $(\boldsymbol{\pi}_{\theta_y}, c_{\theta_y})$ solves (4.4).

Fix $y > 0$. Then, for any $x > 0$, $(\boldsymbol{\pi}, c) \in \widetilde{\mathcal{A}}(x)$ and $\boldsymbol{\theta} \in \widetilde{\Theta}$, from (3.31) and Proposition 3.9 it follows that

$$\begin{aligned}
J(x; \boldsymbol{\pi}, c) &= \mathbb{E}\left[\int_0^T U_1(t, c(t))\, dt + U_2(V_{\pi,c}^x(T)) \right] \\
&\leq \mathbb{E}\left[\int_0^T U_1^*(t, yH_\theta(t-))\, dt + U_2^*(yH_\theta(T)) \right] \\
&\quad + y\, \mathbb{E}\left[\int_0^T H_\theta(t-)\, c(t)\, dt + H_\theta(T)\, V_{\pi,c}^x(T) \right] \\
&= J^*(y; \boldsymbol{\theta}) + y\, \mathbb{E}\left[\int_0^T H_\theta(t-)\, c(t)\, dt + H_\theta(T)\, V_{\pi,c}^x(T) \right] \\
&\leq J^*(y; \boldsymbol{\theta}) + yx.
\end{aligned} \tag{4.18}$$

The arbitrariness of $x > 0$, $(\boldsymbol{\pi}, c) \in \widetilde{\mathcal{A}}(x)$ and $\boldsymbol{\theta} \in \widetilde{\Theta}$ implies that

$$\Phi(x) \leq \Phi^*(y) + xy, \quad x, y > 0,$$

so that

$$\sup_{x>0} \{\Phi(x) - xy\} \leq \Phi^*(y), \quad y > 0. \tag{4.19}$$

On the other hand, by (3.31), for given $y > 0$ equality holds in (4.18) if there exists $(\boldsymbol{\pi}, c) \in \widetilde{\mathcal{A}}(x)$ such that

$$\begin{aligned} c(t) &= I_1(t, yH_\theta(t-)), \quad t \in [0,T], \\ V^x_{\pi,c}(T) &= I_2(yH_\theta(T)), \\ \mathbb{E}\Big[\int_0^T H_\theta(t-)\, c(t)\, dt &+ H_\theta(T)\, V^x_{\pi,c}(T)\Big] = x. \end{aligned}$$

Then, $\mathcal{X}_\theta(y) = x$, $c = c_\theta$ and $V^x_{\pi,c}(T) = Y_\theta$. In particular, equality holds in (4.18) for $(\boldsymbol{\pi}, c) = (\boldsymbol{\pi}_{\theta_y}, c_{\theta_y})$ and $\boldsymbol{\theta} = \boldsymbol{\theta}_y$:

$$J(x; \boldsymbol{\pi}_{\theta_y}, c_{\theta_y}) = J^*(y; \boldsymbol{\theta}_y) + xy, \quad x, y > 0.$$

Now, since $(\boldsymbol{\pi}_{\theta_y}, c_{\theta_y})$ solves the optimization problem (4.4) and $\boldsymbol{\theta}_y$ is optimal for the dual problem of (4.4), we have

$$\Phi^*(y) = J^*(y, \boldsymbol{\theta}_y) = J(x; \boldsymbol{\pi}_{\theta_y}, c_{\theta_y}) - xy = \Phi(x) - xy \leq \sup_{z>0} \{v(z) - zy\},$$

for $y > 0$. This, together with (4.19), leads to the required result. □

If the assumptions in Theorem 4.4 are satisfied then θ_y represents the optimal martingale measure of problem (4.4).

The dual problem (4.17) is very similar to the dual problem introduced in Kramkov and Schachermayer (1999) but is slightly different. The difference is that the Radon-Nikodym density of the measure change is not parametrized in terms of $\boldsymbol{\theta}$ in Kramkov and Schachermayer's work as it is done here. Instead the measure change density is left in a general form dQ/dP. They assume zero interest rates such that, in our case, $H_\theta = Z_\theta$. Further, they only consider the pure terminal wealth optimization problem without consumption. In Kramkov and Schachermayer's work the dual problem is then that of finding an martingale measure Q out of a set of eligible measures $\mathcal{M}$ such that

$$\Phi^*(y) = \inf_{Q \in \mathcal{M}} \mathbb{E}\left[U^*\left(y\frac{dQ}{dP}\right)\right], \quad y > 0.$$

Then again the relationships hold that

$$\Phi^*(y) = \sup_{x>0} \{\Phi(x) - xy\}, \quad y > 0, \text{ and}$$

$$\Phi(x) = \inf_{y>0} \{\Phi^*(y) - xy\}, \quad x > 0.$$

If we relate it to our work, the results coincide (neglecting consumption) when we substitute $dQ/dP = Z_\theta$. Yet, a major drawback of not parametrizing the Radon-Nikodym density is that in the general case it can not guaranteed to find the density of the optimal measure Q. In contrast, in our approach the density is always given by Z_θ.

4.5 Utility Functions of Power Type

The results of the previous section are now applied to the power and logarithmic utility functions. The optimal trading strategy and martingale measure is obtained, and in certain cases a closed form for the auxiliary consumption c_θ and terminal wealth Y_θ can be stated. We introduce the convention that $\frac{1}{\beta}x^\beta = \log(x)$ for $\beta = 0$ and $x > 0$ to allow us to write the logarithmic utility function in terms of power utility.

For $\beta_i < 1$, $i = 1, 2$, let the utility functions for consumption and for terminal wealth be given by

$$U_1(t,x) = \frac{x^{\beta_1}}{\beta_1}, \quad \text{and} \quad U_2(x) = \frac{x^{\beta_2}}{\beta_2}, \tag{4.20}$$

respectively. The inverse of U_1' is $I_1(t,y) = y^{1/(\beta_1-1)}$ and that of U_2' is $I_2(y) = y^{1/(\beta_2-1)}$. The optimization problem (4.4) then takes the form

$$\Phi(x) = \sup_{(\pi,c)\in\widetilde{\mathcal{A}}(x)} \mathbb{E}\left[\int_0^T \frac{c(t)^{\beta_1}}{\beta_1} dt + \frac{V^{x,c,\pi}(T)^{\beta_2}}{\beta_2}\right],$$

for $x > 0$. As far as the author is aware, such a problem has never been treated in the literature in the context of jump-diffusion processes or similar models.

To derive a condition for the optimal trading strategy $\widehat{\boldsymbol{\pi}}$ and the optimal EMM $\widehat{\boldsymbol{\theta}}$ the martingale representation coefficients in M_θ in (4.8) needs to be calculated. To do this, we decompose the state price density H_θ corresponding to $\boldsymbol{\theta} \in \widetilde{\Theta}$ as a product of a deterministic process and a martingale

$$H_\theta^{\beta_i/(\beta_i-1)}(t) = h_\theta^{(i)}(t)\, \widetilde{H}_\theta^{(i)}(t), \quad t \in [0,T], \quad i = 1,2, \tag{4.21}$$

where $h_\theta^{(i)}$ is the deterministic process given by

$$\begin{aligned} h_\theta^{(i)}(t) := \exp\Bigg\{ &-\frac{\beta_i}{\beta_i - 1}\int_0^t r(s)\,ds + \frac{1}{2}\frac{\beta_i}{(\beta_i-1)^2}\int_0^t ||\boldsymbol{\theta}^D(s)||^2 ds \\ &+ \sum_{h=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left[\theta_h^J(s,y)^{\beta_i/(\beta_i-1)} - 1 + \frac{\beta_i}{(\beta_i-1)}\big(1-\theta_h^J(s,y)\big)\right] \nu_h(dy)\,ds \Bigg\}, \end{aligned} \tag{4.22}$$

and $\widetilde{H}_\theta^{(i)}$ is the martingale given by

$$\begin{aligned} \widetilde{H}_\theta^{(i)}(t) \;\; := \;\; \exp\Bigg\{ &\frac{\beta_i}{\beta_i - 1}\int_0^t \boldsymbol{\theta}^D(s)^\intercal d\boldsymbol{B}(s) - \frac{1}{2}\frac{\beta_i^2}{(\beta_i-1)^2}\int_0^t ||\boldsymbol{\theta}^D(s)||^2 ds \\ &+ \sum_{h=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} \log\left(\theta_h^J(s,y)^{\beta_i/(\beta_i-1)}\right) N_h(dy,ds) \\ &- \sum_{h=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} \left[\theta_h^J(s,y)^{\beta_i/(\beta_i-1)} - 1\right] \nu_h(dy)\,ds \Bigg\}. \end{aligned}$$

To determine the dynamics of $\widetilde{H}_\theta^{(i)}$ we can apply Itô formula (2.14) to the function $f(x) = e^x$ and to the exponent of $\widetilde{H}_\theta^{(i)}$. Clearly, $f'(x) = f''(x) = e^x$ and $f(x+\log\theta^{\frac{\beta}{\beta-1}}) - f(x) = e^x \left[\theta^{\frac{\beta}{\beta-1}} - 1\right]$. Thus, for $t \in [0,T]$,

$$d\widetilde{H}_\theta^{(i)}(t) = \widetilde{H}_\theta^{(i)}(t-)\left\{\frac{\beta_i}{\beta_i - 1}\boldsymbol{\theta}^D(t)^\intercal d\boldsymbol{B}(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \left[\theta_h^J(t,y)^{\beta_i/(\beta_i-1)} - 1\right]\widetilde{N}_h(dt,dy)\right\}. \tag{4.23}$$

To derive a optimal condition on the Girsanov kernel and the trading strategy as in (4.12) it is necessary to calculate $\mathcal{X}_\theta$ and the martingale M_θ in its martingale representation form. From the definition in (4.5) the function $\mathcal{X}_\theta$ is defined by

$$\mathcal{X}_\theta(y) = \mathbb{E}\left[\int_0^T H_\theta(t-)I_1(t, yH_\theta(t-))dt + H_\theta(T)I_2(yH_\theta(T))\right], \quad y > 0,$$

Substituting $I_1(t,y) = y^{1/(\beta_1-1)}$ and $I_2(y) = y^{1/(\beta_2-1)}$ and using the decomposition (4.21) leads to

$$\mathcal{X}_\theta(y) = \mathbb{E}\left[\int_0^T y^{\frac{1}{\beta_1-1}} h_\theta^{(1)}(t)\widetilde{H}_\theta^{(1)}(t-)dt + y^{\frac{1}{\beta_2-1}} h_\theta^{(2)}(T)\widetilde{H}_\theta^{(2)}(T)\right].$$

Since $\widetilde{H}_\theta^{(i)}$ are martingales the function $\mathcal{X}_\theta$ has the form

$$\mathcal{X}_\theta(y) = y^{\frac{1}{\beta_1-1}}\mathbb{E}\left[\int_0^T h_\theta^{(1)}(t)dt\right] + y^{\frac{1}{\beta_2-1}}\mathbb{E}\left[h_\theta^{(2)}(T)\right].$$

$\mathcal{X}_\theta$ has a unique inverse and we denote by $\widehat{y}_\theta$ the unique point such that $\mathcal{X}_\theta(\widehat{y}_\theta) = x$ for given x. Next, the martingale M_θ is defined in (4.8) by

$$M_\theta(t) = \mathbb{E}\left[\int_0^T H_\theta(s-)c_\theta(s)ds + H_\theta(T)Y_\theta \;\middle|\; \mathcal{F}_t\right]$$

To calculate M_θ the optimal auxiliary consumption c_θ and terminal wealth Y_θ needs to be calculated. They are given in (4.6) by

$$c_\theta(t) = I_1\left(t, \mathcal{X}_\theta^{-1}(x)H_\theta(t-)\right),$$
$$Y_\theta = I_2\left(\mathcal{X}_\theta^{-1}(x)H_\theta(T)\right).$$

Let $\widehat{y}$ denote the unique value for which $\mathcal{X}_\theta(y) = x$. Then substituting $I_1(t,y) = y^{1/(\beta_1-1)}$ and $I_2(y) = y^{1/(\beta_2-1)}$ into the above equations shows that

$$c_\theta(t) = \widehat{y}_\theta^{\frac{1}{\beta_1-1}} H_\theta(t-)^{\frac{1}{\beta_1-1}},$$
$$Y_\theta = \widehat{y}_\theta^{\frac{1}{\beta_2-1}} H_\theta(T)^{\frac{1}{\beta_2-1}}.$$

Define the two functions

$$K_{1,\theta}(t) := \mathbb{E}\left[\int_t^T h_\theta(s)ds\middle|\mathcal{F}_t\right], \quad t \in [0,T], \tag{4.24}$$

and

$$K_{2,\theta}(t) := \mathbb{E}\left[h_\theta(T)\middle|\mathcal{F}_t\right], \quad t \in [0,T]. \tag{4.25}$$

Substituting c_θ and Y_θ into the definition of M_θ as well as using the decomposition (4.21) brings M_θ into the form

$$\begin{aligned} M_\theta(t) = \widehat{y}_\theta^{\frac{1}{\beta_1-1}} \left(\int_0^t H_\theta(s-)^{\frac{\beta_1}{\beta_1-1}} ds\right) + \widehat{y}_\theta^{\frac{1}{\beta_1-1}} \widetilde{H}_\theta^{(1)}(t) K_{1,\theta}(t) \\ + \widehat{y}_\theta^{\frac{1}{\beta_2-1}} \widetilde{H}_\theta^{(2)}(t) K_{2,\theta}(t), \end{aligned}$$

for $\theta \in \widetilde{\Theta}$. The dynamics of M_θ are given by

$$dM_\theta(t) = \widehat{y}_\theta^{\frac{1}{\beta_1-1}} K_{1,\theta}(t) d\widetilde{H}_\theta^{(1)}(t) + \widehat{y}_\theta^{\frac{1}{\beta_2-1}} K_{2,\theta}(t) d\widetilde{H}_\theta^{(2)}(t),$$

which again can be written in the following form when considering (4.23):

$$\begin{aligned} dM_\theta(t) = & \left[\widehat{y}_\theta^{\frac{1}{\beta_1-1}} \widetilde{H}_\theta^{(1)}(t) K_{1,\theta}(t) \frac{\beta_1}{\beta_1 - 1}\right. \\ & \left. + \widehat{y}_\theta^{\frac{1}{\beta_2-1}} \widetilde{H}_\theta^{(2)}(t) K_{2,\theta}(t) \frac{\beta_2}{\beta_2 - 1}\right] \boldsymbol{\theta}^D(t) d\boldsymbol{B}(t) \\ & + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \left[\widehat{y}_\theta^{\frac{1}{\beta_1-1}} \widetilde{H}_\theta^{(1)}(t-) K_{1,\theta}(t) \left(\theta_h^J(t,y)^{\frac{\beta_1}{\beta_1-1}} - 1\right)\right. \\ & \left. + \widehat{y}_\theta^{\frac{1}{\beta_2-1}} \widetilde{H}_\theta^{(2)}(t) K_{2,\theta}(t) \left(\theta_h^J(t,y)^{\frac{\beta_2}{\beta_2-1}} - 1\right)\right] \widetilde{N}_h(dt,dy). \end{aligned}$$

To simplify the above expression let us introduce the following two processes

$$\begin{aligned} C_\theta(t) &:= \mathbb{E}\left[\int_t^T H_\theta(s-) c_\theta(s) \middle| \mathcal{F}_t\right] = \widehat{y}_\theta^{\frac{1}{\beta_1-1}} \widetilde{H}_\theta^{(1)}(t) K_{1,\theta}(t), \quad \text{and} \\ W_\theta(t) &:= \mathbb{E}\left[H_\theta(T) Y_\theta | \mathcal{F}_t\right] = \widehat{y}_\theta^{\frac{1}{\beta_2-1}} \widetilde{H}_\theta^{(2)}(t) K_{2,\theta}(t). \end{aligned}$$

Then the martingale representation coefficients of M_θ are given by

$$\begin{aligned} \boldsymbol{a}^D(t) &= \left[C_\theta(t) \frac{\beta_1}{\beta_1 - 1} + W_\theta(t) \frac{\beta_2}{\beta_2 - 1}\right] \boldsymbol{\theta}^D(t), \quad \text{and} \\ a_h^J(t,y) &= C_\theta(t) \left(\theta_h^J(t,y)^{\frac{\beta_1}{\beta_1-1}} - 1\right) + W_\theta(t) \left(\theta_h^J(t,y)^{\frac{\beta_2}{\beta_2-1}} - 1\right), \end{aligned} \tag{4.26}$$

for $h = 1, \ldots, m$. They will be used in the next proposition.

Proposition 4.5. *Under the assumptions set at the beginning of this section, the optimal pair $(\boldsymbol{\pi}_{\widehat{\theta}}, \widehat{\boldsymbol{\theta}})$ that solves the problem (4.4) of maximizing expected utility of consumption and terminal wealth has to satisfy*

$$\begin{aligned} \boldsymbol{\xi}(t)\boldsymbol{\pi}_{\widehat{\theta}}(t) &= \frac{1}{C_{\widehat{\theta}}(t-) + W_{\widehat{\theta}}(t-)} \left\{\frac{1}{\beta_1 - 1} C_{\widehat{\theta}}(t-) + \frac{1}{\beta_2 - 1} W_{\widehat{\theta}}(t-)\right\} \widehat{\boldsymbol{\theta}}^D(t) \\ \boldsymbol{\gamma}_h(t,y)\boldsymbol{\pi}_{\widehat{\theta}}(t) &= \frac{1}{C_{\widehat{\theta}}(t-) + W_{\widehat{\theta}}(t-)} \left\{\left(\widehat{\theta}_h^J(t,y)^{\frac{1}{\beta_1-1}} - 1\right) C_{\widehat{\theta}}(t-)\right. \\ & \qquad \left. + \left(\widehat{\theta}_h^J(t,y)^{\frac{1}{\beta_2-1}} - 1\right) W_{\widehat{\theta}}(t-)\right\}, \end{aligned} \tag{4.27}$$

for $h = 1, \dots, m$. Further, if such an optimal pair exits then the optimal wealth process is given by $V_{\widehat{\theta}}(t) = \frac{1}{H_{\widehat{\theta}}(t)} \left(C_{\widehat{\theta}}(t) + W_{\widehat{\theta}}(t) \right)$.

Proof. The conditions on the optimal trading strategy and optimal martingale measure $(\pi_{\widehat{\theta}}, \widehat{\theta})$ in (4.12) simplify in the one dimensional case to

$$\begin{aligned} \boldsymbol{\xi}(t)\boldsymbol{\pi}(t) &= \frac{1}{H_\theta(t-)V_\theta(t-)}\boldsymbol{a}^D(t) - \boldsymbol{\theta}^D(t), \quad \text{and} \\ \boldsymbol{\gamma}_h(t,y)\boldsymbol{\pi}(t) &= \frac{1}{\theta^J(t,y)}\left[\frac{a_h^J(t,y)}{H_\theta(t-)V_\theta(t-)} + 1 - \theta_h^J(t,y)\right]. \end{aligned} \tag{4.28}$$

From (4.10) and the definition of M_θ in (4.8) and J_θ in (4.9) follows that the optimal wealth process satisfies $H_{\widehat{\theta}}(t)V_{\widehat{\theta}}(t) = C_{\widehat{\theta}}(t) + W_{\widehat{\theta}}(t)$. Substituting this together with the martingale representation coefficients (4.26) into (4.28) shows that the optimal strategy and Girsanov kernel have to satisfy (4.27). □

In the following, some special choices of the model parameters and the risk aversion parameters β_1 and β_2 are discussed. Depending on the parameter choices, these results relate to different work by different authors. Consider first the case when $\beta_1 = \beta_2 = \beta$. In that case a closed form solution can be given for the optimal consumption, the optimal terminal wealth, and the optimal wealth process. The next corollary gives a closed form for the auxiliary processes defined in (4.6) and (4.10).

Corollary 4.6. *Let $x > 0$ and let U_1 and U_2 be given by (4.20) for $\beta_1 = \beta_2 = \beta < 1$. Further, define for $h_\theta^{(i)} = h_\theta$ as in (4.22) the deterministic function*

$$K_\theta(t) := \mathbb{E}\left[\int_t^T h_\theta(s)ds + h_\theta(T)\big|\mathcal{F}_t\right], \quad t \in [0,T].$$

Then, the auxiliary processes (4.6) and (4.10) corresponding to $\boldsymbol{\theta} \in \widetilde{\Theta}$ for the investment-consumption problem (4.4) are respectively

$$\begin{aligned} c_\theta(t) &= x \cdot \frac{H_\theta(t)^{1/(\beta-1)}}{K_\theta(0)}, \\ Y_\theta &= x \cdot \frac{H_\theta(T)^{1/(\beta-1)}}{K_\theta(0)}, \quad \textit{and} \\ V_\theta(t) &= x\frac{\widetilde{H}_\theta(t)}{H_\theta(t)}\frac{K_\theta(t)}{K_\theta(0)}, \end{aligned} \tag{4.29}$$

for $t \in [0,T]$.

Proof. Since $I_1(t,y) = y^{1/(\beta-1)}$ and $I_2(y) = y^{1/(\beta-1)}$, for $y > 0$ and $t \in [0,T]$, and since the martingale part $\widetilde{H}_\theta$ of $H_\theta(t)^{\beta/(\beta-1)}$ satisfies $\widetilde{H}_\theta(0) = 1$, for $y > 0$, it follows from

(4.5) that

$$\begin{aligned}\mathcal{X}_\theta(y) &= y^{1/(\beta-1)}\mathbb{E}\left[\int_0^T H_\theta(t)^{\beta/(\beta-1)}dt + H_\theta(T)^{\beta/(\beta-1)}\right]\\ &= y^{1/(\beta-1)}K_\theta(0).\end{aligned}$$

Thus, the inverse of $\mathcal{X}_\theta$ is given by $\mathcal{X}_\theta^{-1}(x) = x^{\beta-1}K_\theta(0)^{1-\beta}$, for $x > 0$, so that by (4.6)

$$c_\theta(t) = x\,\frac{H(t)^{1/(\beta-1)}}{K_\theta(0)}, \qquad Y_\theta = x\,\frac{H(T)^{1/(\beta-1)}}{K_\theta(0)}.$$

Finally we have, by (4.10), that

$$\begin{aligned}V_\theta(t) &= \frac{1}{H_\theta(t)}\mathbb{E}\left[\int_t^T H_\theta(s)c_\theta(s) + H_\theta(T)Y_\theta \mid \mathcal{F}_t\right]\\ &= x\frac{1}{H_\theta(t)K_\theta(0)}\mathbb{E}\left[\int_t^T H_\theta(s)^{\beta/(\beta-1)}ds + H_\theta(T)^{\beta/(\beta-1)} \mid \mathcal{F}_t\right]\\ &= x\frac{\widetilde{H}_\theta(t)K_\theta(t)}{H_\theta(t)K_\theta(0)}, \qquad \text{for } t \in [0,T].\end{aligned}$$

□

Clearly, to get a representation for the optimal consumption, terminal wealth, and optimal wealth process one only has to substitute the optimal Girsanov kernel $\widehat{\boldsymbol{\theta}}$ into the definitions in (4.29).

Let us consider the condition on the optimal trading strategy and optimal Girsanov kernel in (4.27). For the case $\beta_1 = \beta_2 = \beta$ these conditions simplifies to

$$\boldsymbol{\xi}(t)\,\boldsymbol{\pi}_{\widehat{\theta}}(t) = \frac{1}{\beta-1}\widehat{\boldsymbol{\theta}}^D(t), \quad \text{and} \quad \boldsymbol{\gamma}_h(t,y)\,\boldsymbol{\pi}_{\widehat{\boldsymbol{\theta}}}(t) = \widehat{\theta}_h^J(t,y)^{1/(\beta-1)} - 1. \tag{4.30}$$

Further, taking condition (3.8) on $\widehat{\boldsymbol{\theta}}$ into account, it can be seen that the optimal trading strategy $\widehat{\boldsymbol{\pi}}$ has to satisfy a non-linear equation that will be stated in the next corollary.

Corollary 4.7. *Assuming that $\beta_1 = \beta_2 = \beta$, in addition to the hypothesis of Proposition 4.5, the optimal trading strategy $\widehat{\boldsymbol{\pi}}$ has to satisfy the non-linear equation*

$$\overline{\boldsymbol{\alpha}}(t) - (1-\beta)\boldsymbol{\sigma}(t)\widehat{\boldsymbol{\pi}}(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)\{1 + \widehat{\boldsymbol{\pi}}(t)^\intercal \boldsymbol{\gamma}_h(t,y)\}^{\beta-1}\nu_h(dy) = 0. \tag{4.31}$$

If, on the other hand, there are no jumps in the market, i.e. $\boldsymbol{\gamma} \equiv 0$, then the above condition simplifies, and the optimal trading strategy $\widehat{\boldsymbol{\pi}}$ is given by

$$\widehat{\boldsymbol{\pi}} = \frac{1}{(1-\beta)}\boldsymbol{\sigma}(t)^{-1}\overline{\boldsymbol{\alpha}}(t), \quad t \in [0,T]. \tag{4.32}$$

Further, the Girsanov kernel $\boldsymbol{\theta}$ under which the discounted asset prices are martingales is uniquely determined by (3.8) and is given by $\boldsymbol{\theta}^D(t) = -\boldsymbol{\xi}(t)^{-1}\overline{\boldsymbol{\alpha}}(t)$. The optimal trading strategy has then also the form

$$\widehat{\boldsymbol{\pi}}(t) = \frac{1}{(\beta-1)}\boldsymbol{\xi}(t)^{-1}\boldsymbol{\theta}^D(t), \quad t \in [0,T],$$

which confirms the condition (4.30). This particular example, without a jump component, has been thoroughly investigated in Karatzas et al. (1987) and numerous other papers.

For the remainder of this section let us consider only the one-dimensional case, with one dimensional parameters. This is mainly for clarity purposes and the results can be extended to the multi-dimensional case with obvious changes. For the case that the model parameters are constants, the model becomes a Lévy process. The Girsanov kernels can then usually also assumed to be constant such that they need to satisfy a simplified version of (3.8) given by

$$\overline{\alpha} + \xi\theta^D + \int_{\mathbb{R}\backslash\{0\}} \gamma(y)\theta^J(y)\nu(dy) = 0.$$

Actually, since γ is just a function of the jump size y and the jump size distribution is determined by the intensity measure ν, the model can be reformulated so that γ is not needed. If one adjust the intensity measure ν (and with that the Poisson random measure N) the stock prices actually follow the stochastic differential equation

$$dS(t) = S(t-)\left[\alpha dt + \xi dB(t) + \int_{\mathbb{R}\backslash\{0\}} yN(dt,dy)\right].$$

The condition on the Girsanov kernels becomes then

$$\overline{\alpha} + \xi\theta^D + \int_{\mathbb{R}\backslash\{0\}} y\theta^J(y)\nu(dy) = 0.$$

The optimal trading strategy $\widehat{\pi}$ is then also constant and has to satisfy the optimality condition (4.31), which simplifies to

$$\alpha - r - (1-\beta)\xi^2\widehat{\pi} + \int_{\mathbb{R}\backslash\{0\}} \frac{y}{(1+\widehat{\pi}y)^{1-\beta}}\nu(dy) = 0. \tag{4.33}$$

This compares with the results of Kallsen (1999), who analysed the case of power utility in a Lévy model. In his version the utility function is given by $U(x) = \frac{x^{1-p}}{1-p}$ for $p \in \mathbb{R}_+\backslash\{0,1\}$, and the optimal trading strategy has to satisfy

$$\alpha - r - p\,\xi^2\widehat{\pi} + \int_{\mathbb{R}\backslash\{0\}} \left[\frac{y}{(1+\widehat{\pi}y)^p} - h(y)\right]\nu(dy) = 0, \tag{4.34}$$

where h is some truncation function. The truncation function is needed as Kallsen also allows infinitely many small jumps that are not considered in our model framework. If these are taking away, h disappears, and the two conditions (4.33) and (4.34) coincide since $\beta = 1 - p$. Notice that, although Kallsen considers the problem of maximizing either consumption or terminal wealth, one can still make a comparison for the power utility case. This is because it will turn out in the next section that for the power utility case the optimal trading strategy is the same for all three problems.

The result can also be related to the paper from Goll and Kallsen (2003). In the paper the authors present an explicit solution to the optimal portfolio problem under a semi-martingale model with logarithmic utility. As in Kallsen (1999) the authors formulate the problem in terms of an generalized optimal consumption problem. If the results are stated in the above discussed Lévy model format, then Goll and Kallsen's optimality condition become equivalent to (4.33) for the log case (i.e. $\beta = 0$). Expressed in the terms of this thesis, Goll and Kallsen's condition becomes that the optimal trading strategy $\widehat{\pi}$ should satisfy

$$\sup\{\Lambda(\pi - \widehat{\pi}) : \psi \in \Pi\} = 0,$$

where

$$\Lambda(\pi) = (\alpha - r)\pi - \pi^2\xi^2 + \int_{\mathbb{R}\setminus\{0\}} \left[\frac{\pi y}{1 + \widehat{\pi} y} - \pi h(y)\right] \nu(dy). \tag{4.35}$$

h is thereby again some truncation function, which disappears in the framework of the thesis. The supremum in (4.35) is in particular obtained if π satisfies (4.33) with $\beta = 0$.

For the case that the stock prices are governed by a pure jump model with drift but without Brownian motion part, the condition on the optimal trading strategy $\widehat{\pi}$ changes from (4.31) to

$$\alpha(t) - r(t) + \int_{\mathbb{R}\setminus\{0\}} \gamma(t, y)\{1 + \widehat{\pi}(t)\gamma(t, y)\}^{\beta - 1}\nu(dy) = 0, \quad \text{a.s.}$$

It is worth noting, however, that in that case the set of Girsanov kernels Θ is in general not a singleton. The condition on the Girsanov kernels (3.8) becomes

$$\alpha(t) - r(t) + \int_{\mathbb{R}\setminus\{0\}} \gamma(t, y)\theta_J(t, y)\nu(dy) = 0, \quad \text{a.s.}, \tag{4.36}$$

and has generally not a unique solution for all $t \in [0, T]$. Yet, a case when there exists a unique martingale measure, is when there is only one jump size at each time $t \in [0, T]$. Denoting this single jump size by $\gamma(t)$, the condition on θ^J changes to

$$\alpha(t) - r(t) + \gamma(t)\theta^J(t)\nu = 0, \quad \text{a.s.},$$

where $\nu \geq 0$ is the constant intensity. The unique Girsanov kernel is then given by

$$\theta^J(t) = -\frac{\alpha(t) - r(t)}{\gamma(t)\nu}, \quad t \in [0, T].$$

In the case of a single jump size $\gamma(t)$ at each time $t \in [0,T]$, the optimal trading strategy takes the form, from (4.36), as

$$\widehat{\pi}(t) = \frac{1}{\gamma(t)} \left[\left(-\frac{\alpha(t) - r(t)}{\gamma(t)\nu} \right)^{1-\beta} - 1 \right], \quad t \in [0,T].$$

Expressed in terms of the Girsanov kernel θ^J, the optimal trading strategy is $\widehat{\pi}(t) = \frac{1}{\gamma(t)} \left[\theta^J(t)^{\beta-1} - 1 \right]$, which again confirms condition (4.30) for the jump part.

The case where the utility for consumption differs from the utility for terminal wealth can be significantly more complex. If there are no jumps in the market and the stock prices are driven by a geometric Brownian motion and the condition on the trading strategy is as in (4.27)

$$\xi(t)\widehat{\pi}(t) = \frac{1}{C_\theta(t-) + W_\theta(t-)} \left\{ \frac{1}{\beta_1 - 1} C_\theta(t-) + \frac{1}{\beta_2 - 1} W_\theta(t-) \right\} \theta^D(t) \tag{4.37}$$

However, now the Girsanov kernel is known (cf. (3.9)) to be

$$\theta^D(t) = -\frac{\alpha(t) - r(t)}{\xi(t)}$$

and is therefore unique. This means that the expression on the right hand side of (4.37) can be calculated easily as it depends on a unique θ^D.

On the other hand, if there is no Brownian motion in the model, the condition (4.27) is

$$\begin{aligned} \gamma(t,y)\widehat{\pi}_{\widehat{\theta}}(t) = \frac{1}{C_{\widehat{\theta}}(t-) + W_{\widehat{\theta}}(t-)} & \left\{ \left(\widehat{\theta}^J(t,y)^{\frac{1}{\beta_1 - 1}} - 1 \right) C_{\widehat{\theta}}(t-) \right. \\ & \left. + \left(\widehat{\theta}^J(t,y)^{\frac{1}{\beta_2 - 1}} - 1 \right) W_{\widehat{\theta}}(t-) \right\}. \end{aligned} \tag{4.38}$$

In contrast to (4.37) the right hand side of (4.38) is not easily calculated analytically. This is because, if there are several jump sizes in the model, the optimal Girsanov kernel $\widehat{\theta}^J$ is not easily determined since the set of Girsanov kernels is not a singleton. Instead a jump Girsanov kernel θ^J has to satisfy

$$\overline{\alpha}(t) + \int_{\mathbb{R}\setminus\{0\}} \gamma(t,y)\theta^J(t,y)\nu(dy) = 0, \quad \text{a.s.,} \quad t \in [0,T], \tag{4.39}$$

which has no unique solution and the market is incomplete. Thus, $\widehat{\theta}$ has to be found otherwise. If, however, there is only one jump size, then θ^J is the unique solution of (4.39) and the market is complete. Then the expression on the right hand side of (4.38) can be calculated analytically.

The section is concluded by giving some graphical illustrations of the expected consumption and wealth processes for a concrete model. Also it is shown how the expected consumption and wealth evolve over time. The simplest model is assumed, namely that of constant parameters, with N being a Poisson process with parameter ν and single jump size, and we take the parameters $\alpha = .1$, $r = .05$, $\xi = .3$, $\nu = .2$ and $\gamma = -.1$. This means that the yearly drift is 10%, the riskless rate is 5%, the standard deviation with respect to the Brownian motion is given by 30%, jumps are expected to happen every 5 years ($1/\nu$), and if a jump happens the stock price drops by 10%. It is further assumed that investor's initial wealth is £100 and the investment horizon is 10 years, i.e. $T = 10$. As utility function the logarithmic utility is chosen so that $\beta = 0$. Figure 4.1 shows the corresponding optimal expected portfolio value and expected consumption for the optimization problem (4.4) with both utilities being logarithmic and with the given constant parameters.

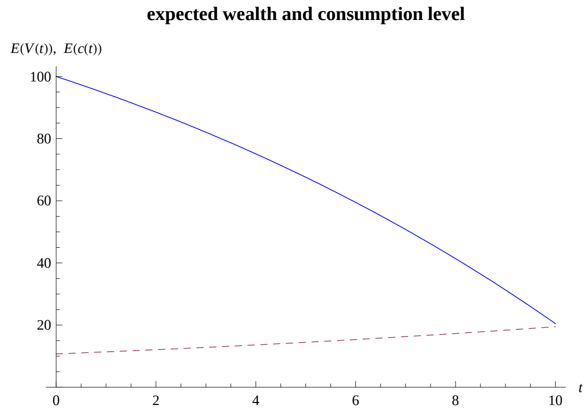

Figure 4.1: Expected consumption (dashed line) and wealth over time when maximizing consumption and terminal wealth with parameters $\alpha = .1$, $r = .05$, $\xi = .3$, $\nu = .2$, $\gamma = -.1$, $\beta = 0$, $T = 10$

To see the impact of jumps on the optimal trading strategy we compare two optimal trading strategies: one is for a jump-diffusion model and the other is in the absence of jumps. For the jump diffusion case we use the same parameters as above. For the diffusion model we drop the jump parameters with the remaining parameters unchanged, so that $\alpha_D = .1$, $r_D = .05$, $\xi_D = .3$, $\nu_D = 0$, and $\gamma_D = 0$. It turns out that the optimal strategy for the jump-diffusion model is $\widehat{\pi}_{JD} = .33$ and the optimal strategy for the diffusion model is $\widehat{\pi}_D = .56$.

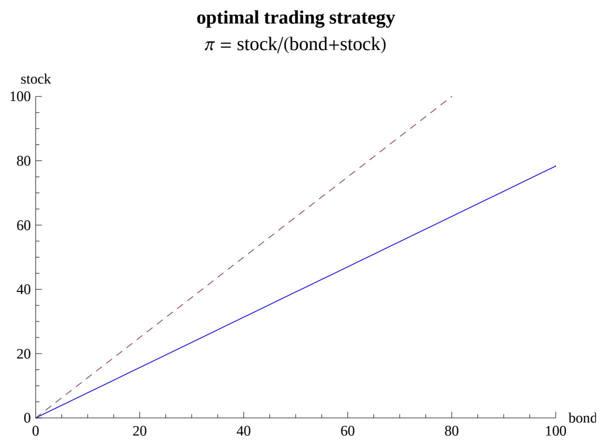

Figure 4.2: Optimal trading strategy in a jump diffusion and diffusion (dashed line) case with parameters $\beta = 0$, $\alpha = .1$, $r = .05$, $\xi = .3$, $\nu_{JD} = .2$, $\gamma_{JD} = -.1$, and $\nu_D = 0$, $\gamma_D = 0$ respectively.

Figure 4.2 shows how much money an investor should invest in the stock for every unit of money invested in the bond. Recall that the money invested in the stock is $\pi/(1-\pi)$ times the amount invested into the bond. The continuous line represents the optimal strategy in the jump diffusion case $\widehat{\pi}_{JD} = .33$, and the dashed line represents the optimal strategy for the diffusion model $\widehat{\pi}_D = .56$.

4.6 Several Examples of Jump Diffusion Models under Power Utility

In this section we consider the existence and uniqueness of an optimal trading strategy in the power utility case with constant parameters in a one dimensional framework. Thus the stock price is assume to follow the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \alpha dt + \xi dW(t) + \int_{\mathbb{R}\setminus\{0\}} \gamma(y)\nu(dy), \quad t \in [0,T].$$

The stock price is assume to be non-negative so that the jump size $\gamma(y)$ has to satisfy $\gamma(y) \geq -1$ for all $y \in \mathbb{R} \setminus \{0\}$ (cf. Section 3.1). Let A denote the subset $A \subset [-1, \infty)$ in which for the given model all jumps $\gamma(\cdot)$ of the stock price lie. Further, denote by a and b the infimum and, respectively, the supremum of the set A. For the purpose of the following considerations introduce for $a = 0$ and $b = \infty$ the convention that $1/0 = \infty$ and $-1/\infty = 0$, respectively.

For a trading strategy π to be admissible it needs to satisfy $1 + \pi\gamma(y) \geq 0$ for all $y \in \mathbb{R} \setminus \{0\}$. This leads to three possible forms of the set of admissible strategies:

(a) If $-1 \leq a < 0 < b \leq \infty$ then an admissible strategy satisfies $\pi \in [-1/b, -1/a]$.

(b) If $-1 \leq a \leq b \leq 0$ then an admissible strategy satisfies $\pi \in (-\infty, -1/a]$.

(c) If $0 \leq a \leq b \leq \infty$ then an admissible strategy satisfies $\pi \in [-1/b, \infty)$.

For the remainder of this section assume that the stock jumps sizes are of the form $-1 \leq a < 0 < b \leq \infty$ as in (a) above. For the following, the other two cases can be treated in an analogue way to (a).

Proposition 4.8. *Let $a = \inf_{y \in \mathbb{R}\setminus\{0\}} \gamma(y)$ and $b = \sup_{y \in \mathbb{R}\setminus\{0\}} \gamma(y)$, and let $-1 \leq a < 0 < b \leq \infty$. Further assume that the integrals*

$$\int_{\mathbb{R}\setminus\{0\}} \gamma(y) \left(1 - \frac{\gamma(y)}{a}\right)^{\beta-1} \nu(dy) \text{ and } \int_{\mathbb{R}\setminus\{0\}} \gamma(y) \left(1 - \frac{\gamma(y)}{b}\right)^{\beta-1} \nu(dy) \tag{4.40}$$

are defined (possibly being $\pm\infty$). If the model parameters satisfy the conditions

$$\begin{aligned} \overline{\alpha} - \frac{1}{a}(\beta-1)\sigma + \int_{\mathbb{R}\setminus\{0\}} \gamma(y) \left(1 - \frac{\gamma(y)}{a}\right)^{\beta-1} \nu(dy) \leq 0 \quad &\text{and} \\ \overline{\alpha} - \frac{1}{b}(\beta-1)\sigma + \int_{\mathbb{R}\setminus\{0\}} \gamma(y) \left(1 - \frac{\gamma(y)}{b}\right)^{\beta-1} \nu(dy) \geq 0, & \end{aligned} \tag{4.41}$$

then there exists a unique optimal solution $\widehat{\pi} \in [-1/b, -1/a]$ to the optimal investment-consumption problem under power utility with constant parameters with one stock in the market. The unique solution can be found by solving

$$\overline{\alpha} + (\beta-1)\sigma\widehat{\pi} + \int_{\mathbb{R}\setminus\{0\}} \gamma(y) \left(1 + \widehat{\pi}\gamma(y)\right)^{\beta-1} \nu(dy) = 0. \tag{4.42}$$

Proof. In Proposition 4.5 condition (4.31) has been stated for an admissible trading strategy to be optimal. This translates in the current model set up to (4.42). Using this optimality condition, define the function

$$g(\pi) := \overline{\alpha} + (\beta-1)\sigma\pi + \int_{\mathbb{R}\setminus\{0\}} \gamma(y) \left(1 + \pi\gamma(y)\right)^{\beta-1} \nu(dy), \quad \pi \in [-1/b, -1/a]. \tag{4.43}$$

Further consider its first derivative

$$\frac{\partial g}{\partial \pi}(\pi) = (\beta-1)\sigma + (\beta-1)\int_{\mathbb{R}\setminus\{0\}} \gamma(y)^2 \left(1 + \pi\gamma(y)\right)^{\beta-2} f(y)dy.$$

Since $(\beta - 1)$ is the only negative term, the first derivative of g is negative for all $\pi \in [-1/b, -1/a]$. That means in particular that g is downward sloping and a unique solution $\widehat{\pi}$ to (4.42) must exists if $g(-1/a) \leq 0 \leq g(-1/b)$. But this is exactly the condition stated in the proposition. $\square$

Remark 4.9. *(i) The argument in the above proof is that the function* $g(\cdot)$ *in* (4.43) *has a solution. A sufficient condition is condition* (4.41)*. Thereby guarantees* (4.40) *that the integrals in* (4.41) *actually exist. In the following examples we will give, if necessary, explicit additional conditions on the model parameters to ensure that the integrals in* (4.40) *are defined.*

(ii) For the case that $\int_{\mathbb{R}\setminus\{0\}} \gamma(y)\left(1-\frac{\gamma(y)}{a}\right)^{\beta-1}\nu(dy) = -\infty$*, the first condition in* (4.41) *is still assumed satisfied. The condition* (4.41) *is violated if*
$\int_{\mathbb{R}\setminus\{0\}} \gamma(y)\left(1-\frac{\gamma(y)}{a}\right)^{\beta-1}\nu(dy) = +\infty$.

(iii) On the other hand, if $\int_{\mathbb{R}\setminus\{0\}} \gamma(y)\left(1-\frac{\gamma(y)}{b}\right)^{\beta-1}\nu(dy) = +\infty$*, the second condition in* (4.41) *is still treated as fulfilled. If however* $\int_{\mathbb{R}\setminus\{0\}} \gamma(y)\left(1-\frac{\gamma(y)}{b}\right)^{\beta-1}\nu(dy) = -\infty$*, condition* (4.41) *is violated.*

For the remainder of this section we check how the existence and uniqueness condition in (4.41) manifest for various jump-diffusion models. Consider first the Kou model.

In the model suggested by Kou (2002), the logarithm of the asset price is assumed to follow a sum of Brownian motion and a compound Poisson process where the jump sizes are double exponentially distributed. Compound Poisson processes have been discussed in Definition 2.5. This allows to capture two empirical phenomena in stock prices. One is the asymmetric leptokurtic feature of stock price returns, and the other one is the feature to capture volatility smiles. In Kou's model the stock price follows the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \alpha dt + \xi dW(t) + d\left(\sum_{l=1}^{N(t)}(V_l - 1)\right), \quad t \in [0,T]. \tag{4.44}$$

Thereby is N a Poisson process with rate $\lambda > 0$, (V_l) is a sequence of i.i.d. non-negative random variables such that $Y = \log(V)$ has an asymmetric double exponential distribution with the density

$$f(y) = p\eta_1 e^{-\eta_1 y}1_{\{y\geq 0\}} + (1-p)\eta_2 e^{\eta_2 y}1_{\{y<0\}}, \tag{4.45}$$

where $\eta_1 > 1$, $\eta_2 > 0$, and $0 \leq p \leq 1$ represents the probability of an upward jump.

In other words, the random variable Y could be written as

$$Y = \log(V) = \begin{cases} \zeta^+, & \text{with probability } p \\ -\zeta^-, & \text{with probability } 1-p \end{cases} \tag{4.46}$$

where ζ^+ and ζ^- are exponential random variables with parameters η_1 and η_2 respectively. The parameters α and σ in (4.44) are assumed to be positive constants. The parameters η_1 and η_2 describe the severity of the up- and downward jump respectively. The bigger η_1 the more likely it is that upward jumps are big, and the bigger η_2 the more likely it is that downward jumps are big.

The Kou model is an example of the generalised model set up of (3.1). It is a one dimensional stock model for a market with a single stock. In the set up of (3.1), the drift and diffusion parameters become constants and the jump part takes the form of a compound jump process such that the $\int_{\mathbb{R}\setminus\{0\}} \gamma(t,y)N(dt,dy)$ translates to $(V-1)\,dN(t)$. Alternatively, the jump part of the changing stock price can be written as $\left(e^Y-1\right)dN(t)$ where Y is double exponential distributed with parameters η_1, η_2 and p, as described in (4.46).

For the set up used in this thesis, this translates to $\gamma(t,y) = e^y - 1$ and the jump intensity measure takes the form $\nu(dy) = \lambda f(dy)$ where f is given by (4.45). Since the jumps don't allow the stock price to become negative or zero (cf. Section 3.1), the stochastic differential equation (4.44) can be solved to

$$S(t) = S(0)\exp\left(\left(\alpha - \frac{\sigma}{2}\right)t + \xi W(t)\right)\prod_{l=1}^{N(t)} V_l, \quad t \in [0,T].$$

This is using the same argument as has been used when (3.4) has been derived.

Example 4.10. *Assume that the stock price follows the Kou model as given in* (4.44)*. If the model parameters satisfy* $0 < \beta + \eta_2 < 1$ *then the existence and uniqueness condition given in* (4.41) *becomes*

$$\lambda\left(\frac{p}{1-\eta_1} + \frac{1-p}{1+\eta_2}\right) \leq \overline{\alpha}. \tag{4.47}$$

If the model parameters satisfy $\beta + \eta_2 > 1$*, in addition to* (4.47)*, the the additional condition*

$$\overline{\alpha} + \lambda\left(\frac{p\eta_1}{(\beta-\eta_1)(\beta-1-\eta_1)} - \frac{(1-p)\eta_2}{(\beta+\eta_2)(\beta-1+\eta_2)}\right) \leq 0 \tag{4.48}$$

needs to be satisfied to guarantee the existence and uniqueness of an optimal trading strategy. The unique optimal solution $\widehat{\pi} \in [0,1]$ *to the optimal investment-consumption problem under power utility in the Kou model can then be found by solving*

$$\overline{\alpha} + (\beta-1)\sigma\widehat{\pi} + \lambda\int_{\mathbb{R}\setminus\{0\}} (e^y-1)\left(1+\widehat{\pi}\,(e^y-1)\right)^{\beta-1} f(y)dy = 0. \tag{4.49}$$

Proof. The corollary follows from Proposition 4.8. Since in the Kou model the size of jumps reaches from $a = -1$ to $b = \infty$, only trading strategies in the set $[0,1]$ are admissible. The condition for existence and uniqueness derived in the proof Proposition 4.8 becomes therefore $g(0) \geq 0 \geq g(1)$, with g as in (4.43). The first integral can be evaluated as

$$
\begin{aligned}
g(0) &= \overline{\alpha} + \lambda \int_{\mathbb{R}\setminus\{0\}} (e^y - 1)\, f(dy) \\
&= \overline{\alpha} + \lambda \int_{\mathbb{R}\setminus\{0\}} (e^y - 1) \left(p\eta_1 e^{-\eta_1 y} 1_{\{y\geq 0\}} + (1-p)\eta_2 e^{\eta_2 y} 1_{y<0} \right) dy \\
&= \overline{\alpha} + \lambda p \eta_1 \int_0^\infty e^{(1-\eta_1)y} dy - \lambda p \eta_1 \int_0^\infty e^{-\eta_1 y} dy \\
&\quad + \lambda(1-p)\eta_2 \int_{-\infty}^0 e^{(1+\eta_2)y} dy - \lambda(1-p)\eta_2 \int_{-\infty}^0 e^{\eta_2 y} dy \\
&= \overline{\alpha} - \lambda p \frac{\eta_1}{1-\eta_1} - \lambda p + \lambda(1-p)\frac{\eta_2}{1+\eta_2} - \lambda(1-p) \\
&= \overline{\alpha} - \lambda \left(\frac{p}{1-\eta_1} + \frac{1-p}{1+\eta_2} \right).
\end{aligned}
\tag{4.50}
$$

The second integral, $g(1)$, is given by

$$
\begin{aligned}
g(1) &= \overline{\alpha} + (\beta-1)\sigma + \lambda \int_{\mathbb{R}\setminus\{0\}} (e^y - 1)\, e^{(\beta-1)y} f(dy) \\
&= \overline{\alpha} + \lambda \int_{\mathbb{R}\setminus\{0\}} (e^y - 1)\, e^{(\beta-1)y} \left(p\eta_1 e^{-\eta_1 y} 1_{\{y\geq 0\}} + (1-p)\eta_2 e^{\eta_2 y} 1_{y<0} \right) dy \\
&= \overline{\alpha} + \lambda p \eta_1 \int_0^\infty e^{(\beta-\eta_1)y} dy - \lambda p \eta_1 \int_0^\infty e^{(\beta-1-\eta_1)y} dy \\
&\quad + \lambda(1-p)\eta_2 \int_{-\infty}^0 e^{(\beta+\eta_2)y} dy - \lambda(1-p)\eta_2 \int_{-\infty}^0 e^{(\beta-1+\eta_2)y} dy.
\end{aligned}
$$

If $0 < \beta+\eta_2 < 1$ then the last integral in the above equation is infinite so that $g(1) = -\infty$. Thus, $g(1) < 0$ as required in Proposition 4.8. If $\beta + \eta_2 > 1$, then $g(1)$ is finite and can be evaluated as

$$
\begin{aligned}
g(1) &= \overline{\alpha} - \lambda p \frac{\eta_1}{\beta-\eta_1} + \lambda p \frac{\eta_1}{\beta-1-\eta_1} + \lambda(1-p)\frac{\eta_2}{\beta+\eta_2} - \lambda(1-p)\frac{\eta_2}{\beta-1+\eta_2} \\
&= \overline{\alpha} + \lambda \left(\frac{p\eta_1}{(\beta-\eta_1)(\beta-1-\eta_1)} - \frac{(1-p)\eta_2}{(\beta+\eta_2)(\beta-1+\eta_2)} \right).
\end{aligned}
\tag{4.51}
$$

Thus, to guarantee that $g(0) \geq 0 \geq g(1)$ the conditions as stated in (4.47) and (4.48) need to be satisfied. If they are satisfied an optimal solution $\widehat{\pi}$ exists and is unique. □

Remark 4.11. *The conditions* $0 < \beta + \eta_2 < 1$ *and* $\beta + \eta_2 > 1$ *in the above example ensure that the integrals in* (4.40) *are actually defined. That means it is necessary and sufficient that the model parameters satisfy either* $0 < \beta + \eta_2 < 1$ *or* $\beta + \eta_2 > 1$ *to ensure that the integrals in* (4.40) *are defined.*

If the stock price follows a similar model as before but has now jumps following a Gamma distribution, similar conditions can be given. Assume a stock model as in (4.44) but let V_l be Gamma distributed such that for $k, \theta > 0$ the probability density function is given by

$$
f(y) = y^{k-1} \frac{e^{-y/\theta}}{\theta^k \Gamma(k)}, \quad y \geq 0, \tag{4.52}
$$

whereby Γ is the Gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, $x \in \mathbb{R}$.

Example 4.12. *If the stock price follows the model as given in* (4.44)*, but where jumps follow a Gamma distribution as in* (4.52) *with intensity* $\lambda > 0$*, the existence and uniqueness condition given in* (4.41) *becomes*

$$\begin{aligned} \overline{\alpha} + \lambda(k\theta - 1) \geq 0, \quad \textit{and} \\ \overline{\alpha} + (\beta - 1)\sigma + \frac{\lambda}{\Gamma(k)} \Big(\theta^\beta \Gamma(k+\beta) - \theta^{\beta-1}\Gamma(k+\beta-1) \Big) \leq 0. \end{aligned} \tag{4.53}$$

If condition (4.53) *is satisfied, the unique optimal solution* $\widehat{\pi} \in [0,1]$ *to the optimal investment-consumption problem under power utility with Gamma distributed jumps can be found by solving*

$$\overline{\alpha} + (\beta - 1)\sigma\widehat{\pi} + \lambda \int_{\mathbb{R}\setminus\{0\}} (y-1) \Big(1 + \widehat{\pi}(y-1)\Big)^{\beta-1} f(y) dy = 0. \tag{4.54}$$

Proof. This corollary is derived in the same way as Corollary (4.10). The jump parameter becomes $\gamma(y) = y - 1$ and the jumps have the density (4.52). Conditions (4.53) will be shown to be equivalent to $g(0) \geq 0 \geq g(1)$, with g as in (4.43). First,

$$g(0) = \overline{\alpha} + \lambda \int_0^\infty (y-1) f(y) dy = \overline{\alpha} + \lambda(k\theta - 1).$$

Since it is needed that $g(0) \geq 0$ a necessary condition is that $\overline{\alpha} + \lambda(k\theta - 1) \geq 0$. Calculating the second condition gives

$$\begin{aligned} g(1) &= \overline{\alpha} + (\beta-1)\sigma + \lambda \int_0^\infty (y-1) y^{\beta-1} f(y) dy \\ &= \overline{\alpha} + (\beta-1)\sigma + \frac{\lambda}{\theta^k \Gamma(k)} \left(\int_0^\infty y^{k-1+\beta} e^{y/\theta} dy - \int_0^\infty y^{k-1+\beta-1} e^{y/\theta} dy \right) \\ &= \overline{\alpha} + (\beta-1)\sigma + \frac{\lambda}{\theta^k \Gamma(k)} \Big(\theta^{k+\beta}\Gamma(k+\beta) - \theta^{k+\beta-1}\Gamma(k+\beta-1) \Big) \\ &= \overline{\alpha} + (\beta-1)\sigma + \frac{\lambda}{\Gamma(k)} \Big(\theta^\beta \Gamma(k+\beta) - \theta^{\beta-1}\Gamma(k+\beta-1) \Big) \end{aligned}$$

As it is needed that $g(1) \leq 0$, condition (4.53) needs to be satisfies for a unique optimal strategy to exist. □

As a final example log normal distributed jumps are considered. The stock model is again assumed to follow a stochastic differential equation as in (4.44) but this time V is log normal distributed with parameters (m, s) and density

$$f(x) = \frac{1}{xs\sqrt{2\pi}} e^{-\frac{(\ln(x)-m)^2}{2s^2}}, \quad x \in \mathbb{R}_+. \tag{4.55}$$

Example 4.13. *If the stock price follows the model as given in* (4.44)*, but where jumps follow a log normal distribution with density* (4.55) *with intensity* $\lambda > 0$*, the existence and uniqueness condition given in* (4.41) *becomes*

$$\begin{aligned} \overline{\alpha} + \lambda\left(e^{m+\frac{s^2}{2}} - 1\right) \geq 0, \quad \textit{and} \\ \overline{\alpha} + (\beta-1)\sigma + \lambda\left(e^{\frac{s^2\beta^2}{2}+\beta m} - e^{\frac{s^2(\beta-1)^2}{2}+(\beta-1)m}\right) \leq 0. \end{aligned} \tag{4.56}$$

If condition (4.56) *is satisfied, the unique optimal solution* $\widehat{\pi} \in [0,1]$ *to the optimal investment-consumption problem under power utility with log normal distributed jumps can be found by solving* (4.54) *with* f *being the log normal density as in* (4.55).

Proof. As in the previous proofs. The values $g(0)$ and $g(1)$ needs to be calculated:

$$g(0) = \overline{\alpha} + \lambda \int_{\mathbb{R}} (y-1) f(y) dy = \overline{\alpha} + \lambda\left(\mathbb{E}[V] - 1\right) = \overline{\alpha} + \lambda\left(e^{m+\frac{s^2}{2}} - 1\right).$$

This derives the first part of (4.56) since it is required that $g(0) \geq 0$. To calculate $g(1)$ consider first only the integral part

$$\begin{aligned} \int_{\mathbb{R}} (y-1) y^{\beta-1} f(y) dy &= \int_{\mathbb{R}} y^{\beta} f(y) dy - \int_{\mathbb{R}} y^{\beta-1} f(y) dy \\ &= \frac{1}{s\sqrt{2\pi}}\left(\int_{\mathbb{R}_+} y^{\beta-1} e^{-\frac{(\ln y - m)^2}{2s^2}} dy - \int_{\mathbb{R}_+} y^{\beta-2} e^{-\frac{(\ln y-m)^2}{2s^2}} dy\right). \end{aligned}$$

To calculate the integrals apply the substitution $x = \ln y$ such that $dy = e^x dx$. After simplification, the integral takes then the form

$$\begin{aligned} \int_{\mathbb{R}} (y-1) y^{\beta-1} f(y) dy = \frac{1}{s\sqrt{2\pi}} &\left(\int_{\mathbb{R}} e^{-\frac{1}{2s^2}\left(\left[x-(s^2\beta+m)\right]^2-(s^2\beta)^2-2s^2\beta m\right)} dy\right. \\ &\left. - \int_{\mathbb{R}} e^{-\frac{1}{2s^2}\left(\left[x-(s^2(\beta-1)+m)\right]^2-(s^2(\beta-1))^2-2s^2(\beta-1)m\right)} dy\right) \\ = e^{\frac{s^2\beta^2}{2}+\beta m} - e^{\frac{s^2(\beta-1)^2}{2}+(\beta-1)m}. \end{aligned}$$

Thus, $g(1)$ can be calculated as

$$g(1) = \overline{\alpha} + (\beta-1)\sigma + \lambda\left(e^{\frac{s^2\beta^2}{2}+\beta m} - e^{\frac{s^2(\beta-1)^2}{2}+(\beta-1)m}\right).$$

□

4.7 Optimizing the Expected Utilities of Consumption or Terminal Wealth

The previous sections were focusing on the combined consumption-investment problem. This is, the investor objective is to maximize his/her terminal wealth as well as the consumption during the investment period. This combined problem can be naturally split into two problems where a financial agent is either interested in maximizing expected utility from terminal wealth or expected utility from consumption only. The two problems can be solved in an analogue way as the combined investment-consumption problem has been solved. The objective functions for the problems of consumption or terminal wealth are given, respectively, by

$$J_1(x;\boldsymbol{\pi},c) = \mathbb{E}\left[\int_0^T U_1(t,c_t)dt\right] \quad \text{and} \quad J_2(x;\boldsymbol{\pi}) = \mathbb{E}\left[U_2(V^x_{\pi,0}(T))\right],$$

and the admissible strategies for both optimization problems are changing to

$$\begin{aligned}\widetilde{\mathcal{A}}_1(x) &= \left\{(\boldsymbol{\pi},c)\in\mathcal{A}(x)\Big|\ \ \mathbb{E}\left[\int_0^T U_1(t,c_t)^- dt\right] > -\infty\right\}, \text{ and}\\ \widetilde{\mathcal{A}}_2(x) &= \left\{(\boldsymbol{\pi},c)\in\mathcal{A}(x)\Big|\ \ \mathbb{E}\left[U_2(V^x_{\pi,c}(T))^-\right] > -\infty\right\}.\end{aligned}$$

The problem of maximizing consumption is then that of finding an optimal pair $(\widehat{\boldsymbol{\pi}}_1,\widehat{c}_1)\in\widetilde{\mathcal{A}}_1(x)$ such that

$$\Phi_1(x) := \sup_{(\pi,c)\in\widetilde{\mathcal{A}}_1(x)} J_1(x;\boldsymbol{\pi},c) = J_1(x;\widehat{\boldsymbol{\pi}}_1,\widehat{c}_1). \tag{4.57}$$

The problem of maximizing terminal wealth, on the other hand, is that of finding an optimal strategy $\widehat{\boldsymbol{\pi}}_2$ such that $(\widehat{\boldsymbol{\pi}}_2,0)\in\widetilde{\mathcal{A}}_2(x)$ and

$$\Phi_2(x) := \sup_{(\pi,0)\in\widetilde{\mathcal{A}}_2(x)} J_2(x;\boldsymbol{\pi}) = J_2(x;\widehat{\boldsymbol{\pi}}_2) \tag{4.58}$$

Again with a slight abuse of notation, we refer to the performance functions Φ_1 and Φ_2 respectively as the optimization problems. These problems consist of finding the optimal performance functions, the optimal trading strategy, and the optimal consumption plan for the consumption only problem.

The two problems relate naturally to the combined investment-consumption problem. If one considers maximizing consumption only then terminal wealth should be zero, otherwise the utility of consumption could be increased by consuming the positive terminal wealth. However, maximizing $J(x;\boldsymbol{\pi},c)$ under the constraint that $V^x_{\pi,c}(T)=0$ is equivalent to maximizing $J_1(x;\boldsymbol{\pi},c)$. On the other hand, if one is interested in maximizing terminal wealth only then no consumption should take place since consumption only reduces wealth. However, if consumption is zero, then maximizing $J(x;\boldsymbol{\pi},0)$ is equivalent to maximizing $J_2(x;\boldsymbol{\pi})$. Thus, the problems of maximizing consumption or terminal wealth can be related to (4.4) by introducing the constraints $V^x_{\pi,c}(T)=0$, a.s., and $c\equiv 0$, a.s., respectively.

To obtain analogue results as in Section 4.3 and 4.4 for the consumption only problem it is necessary to make some modifications. For $y>0$ define the function

$$\mathcal{X}_{1,\theta}(y) := \mathbb{E}\left[\int_0^T H_\theta(t) I_1(t, yH_\theta(t))dt\right], \tag{4.59}$$

which is the consumption only equivalent to (4.5). The set of jump-diffusion Girsanov kernels is constraint to the set

$$\widetilde{\Theta}_1 := \big\{\theta \in \Theta | \quad \mathcal{X}_{1,\theta}(y) < \infty, y > 0\big\}.$$

Defining for $\boldsymbol{\theta} \in \widetilde{\Theta}$ the auxiliary consumption and terminal wealth pair

$$\begin{aligned} c_{1,\theta}(t) &:= I_1(t, \mathcal{X}_{1,\theta}^{-1}(x) H_\theta(t)), \quad \text{and} \\ Y_{1,\theta} &:= 0, \end{aligned} \tag{4.60}$$

then the corresponding version of Lemma 4.1 is given as

(*i*) $\mathbb{E}\left[\int_0^T H_\theta(t) c_{1,\theta}(t) dt\right] = x$,

(*ii*) $\mathbb{E}\left[\int_0^T U_1(t, c_{1,\theta}(t))^- dt\right] > -\infty$,

(*iii*) $J_1(x; \boldsymbol{\pi}, c) \leq \mathbb{E}\left[\int_0^T U_1(t, c_{1,\theta}(t)) dt\right]$ for all $(\boldsymbol{\pi}, c) \in \widetilde{\mathcal{A}}_1(x)$.

It follows that

$$\begin{aligned} \mathbb{E}\left[\int_0^T U_1(t, c(t)) dt\right] &\leq \sup_{(\widehat{\pi}, \widehat{c}) \in \widetilde{\mathcal{A}}_1(x)} \mathbb{E}\left[\int_0^T U_1(t, \widehat{c}(t)) dt)\right] \leq \inf_{\widehat{\theta} \in \widetilde{\Theta}_1} \mathbb{E}\left[\int_0^T U_1(t, c_{\widehat{\theta}}(t)) dt\right] \\ &\leq \mathbb{E}\left[\int_0^T U_1(t, c_{1,\theta(t)}) dt\right] \end{aligned}$$

and, thus, in particular

$$\mathbb{E}\left[\int_0^T U_1(t, c(t)) dt\right] \leqslant \mathbb{E}\left[\int_0^T U_1(t, c_{1,\theta(t)}) dt\right]$$

for any $(\boldsymbol{\pi}, c) \in \widetilde{\mathcal{A}}_1(x)$ and $\theta \in \widetilde{\Theta}_1$. Equivalent to Definition 4.2 the optimal martingale measure can be also defined for the consumption only problem.

Definition 4.14. *A martingale measure Q obtained by $\frac{dQ}{dP}|_{\mathcal{F}_T} = Z_\theta(T)$ in terms of $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}_1$ is called optimal for the optimization problem $\Phi_1(x)$ in (4.57) if it satisfies*

$$\mathbb{E}\left[\int_0^T U_1(t, c_{1,\widehat{\theta}}(t)) dt\right] = \inf_{\theta \in \widetilde{\Theta}_1} \mathbb{E}\left[\int_0^T U_1(t, c_{1,\theta}(t)) dt\right]$$

where $c_{1,\theta}$ as defined in (4.60).

For the terminal wealth problem the modifications are similar. The function $\mathcal{X}_\theta$ needs to be changed to

$$\mathcal{X}_{2,\theta}(y) := \mathbb{E}\left[H_\theta(T) I_2(y H_\theta(T))\right], \quad y > 0, \tag{4.61}$$

and the set of martingale measures is constraint to

$$\widetilde{\Theta}_2 := \big\{\theta \in \Theta | \quad \mathcal{X}_{2,\theta}(y) < \infty, y > 0\big\}. \tag{4.62}$$

The auxiliary consumption has to be zero as already mentioned in the remarks above. Further, the auxiliary terminal wealth is defined by

$$
\begin{aligned}
c_{2,\theta}(t) &:= 0, \quad t \in [0,T], \text{ and} \\
Y_{2,\theta} &:= I_2\left(\mathcal{X}_{2,\theta}^{-1}(x)\right).
\end{aligned} \tag{4.63}
$$

The equivalent results from Lemma 4.1 are then given for $\boldsymbol{\theta} \in \Theta_2$ by

(i) $\mathbb{E}\left[H_\theta(T)Y_{2,\theta}\right] = x$,

(ii) $\mathbb{E}\left[U_2(Y_{2,\theta})^-\right] > -\infty$,

(iii) $J_2(x;\boldsymbol{\pi},c) \leq \mathbb{E}\left[U_2(Y_{2,\theta})\right]$ for all $(\boldsymbol{\pi},c) \in \widetilde{A}_2(x)$.

Similarly, for the terminal wealth problem it follows that

$$
\mathbb{E}\left[U_2(V_{\pi,0}^x(T))\right] \quad \leq \sup_{(\widehat{\pi},\widehat{c})\in\widetilde{\mathcal{A}}_2(x)} \mathbb{E}\left[U_2(V_{\widehat{\pi},0}^x(T))\right] \leq \inf_{\widehat{\theta}\in\widetilde{\Theta}_2} \mathbb{E}\left[U_2(Y_{\widehat{\theta}})\right] \leq \mathbb{E}\left[U_2(Y_{2,\theta})\right],
$$

thus, in particular

$$
\mathbb{E}\left[U_2(V_{\pi,0}^x(T))\right] \leqslant \mathbb{E}\left[U_2(Y_{2,\theta})\right]
$$

for any $(\boldsymbol{\pi},c) \in \widetilde{\mathcal{A}}_2(x)$ and $\boldsymbol{\theta} \in \widetilde{\Theta}_2$. An optimal EMM can be defined for this problem in the following way.

Definition 4.15. *A martingale measure Q obtained by $\frac{dQ}{dP}|_{\mathcal{F}_T} = Z_\theta(T)$ in terms of $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}_2$ is called optimal for the optimization problem $\Phi_2(x)$ in* (4.58) *if it satisfies*

$$
\mathbb{E}\left[U_2(Y_{2,\widehat{\theta}})\right] = \inf_{\theta\in\widetilde{\Theta}_1} \mathbb{E}\left[U_2(Y_{2,\theta})\right]
$$

where $Y_{2,\theta}$ as defined in (4.63).

To apply the results from Theorem 4.3 to either the consumption or terminal wealth problem. The processes M_θ, J_θ, and V_θ defined in (4.8), (4.9), and (4.10) respectively need adjustment. For $i = 1,2$ define the following processes

$$
\begin{aligned}
M_{i,\theta}(t) &:= \mathbb{E}\left[\int_0^T H_\theta(s)c_{i,\theta}(s)ds + H_\theta(T)Y_{i,\theta}\middle|\mathcal{F}_t\right], \\
J_{i,\theta}(t) &:= \int_0^t H_\theta(s)c_{i,\theta}(s)ds, \quad \text{and} \\
V_{i,\theta}(t) &:= \frac{1}{H_\theta(t)}\big(M_{i,\theta}(t) - J_{i,\theta}(t)\big),
\end{aligned} \tag{4.64}
$$

respectively. $i = 1$ represents thereby the consumption problem and $i = 2$ the terminal wealth problem.

Proposition 4.16. *For the consumption problem $i = 1$ and for the terminal wealth problem $i = 2$, the optimality condition in* (4.12) *on the optimal Girsanov kernel $\widehat{\boldsymbol{\theta}}$ and the optimal trading strategy $\boldsymbol{\pi}_{\widehat{\theta}}$ takes respectively the form*

$$\begin{aligned}
\boldsymbol{\pi}_{\widehat{\theta}}(t)^{\intercal}\boldsymbol{\xi}(t) &= \frac{1}{H_{\widehat{\theta}}(t-)V_{i,\widehat{\theta}}(t-)}\boldsymbol{a}^{(i),D}(t) - \widehat{\boldsymbol{\theta}}^{D}(t), \\
\boldsymbol{\pi}_{\widehat{\theta}}(t)^{\intercal}\boldsymbol{\gamma}_h(t,y) &= \frac{1}{H_{\widehat{\theta}}(t-)V_{i,\widehat{\theta}}(t-)}\frac{a_h^{(i),J}(t,y)}{\widehat{\theta}_h^J(t,y)} - \frac{\widehat{\theta}_h^J(t,y)-1}{\widehat{\theta}_h^J(t,y)},
\end{aligned} \tag{4.65}$$

for $h = 1, ..., m$, and ν_h-almost all $y \in \mathbb{R}\setminus\{0\}$. Thereby is $V_{i,\theta}$ defined as in (4.64)*, and $\boldsymbol{a}^{(i),D}$, and $\boldsymbol{a}^{(i),J}$ are the martingale representation coefficients of the martingale $M_{i,\theta}$ defined in* (4.64)*. If the conditions are satisfied then the wealth process for the corresponding problem is given by $V_{i,\widehat{\theta}}(t)$.*

As an application of the above proposition let us continue the power utility example from the last section. Interestingly, it turns out that the optimal Girsanov kernel and trading strategy $\left(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\pi}}\right)$ have to satisfy the same conditions (4.31) in the individual terminal wealth and consumption problems as they do in the combined terminal wealth and consumption problem. There is no need to differentiate the risk aversion parameter β for terminal wealth and consumption since they are now separate problems. So let $\beta < 1$ the power utility risk aversion of the investor, so that $U(t,x) = U(x) = \frac{x^\beta}{\beta}$ and $I(y) = y^{\frac{1}{\beta-1}}$.

Then the function $\mathcal{X}_{1,\theta}$ as defined in (4.59) takes the form

$$\mathcal{X}_{1,\theta}(y) = y^{\frac{1}{\beta-1}}\mathbb{E}\left[\int_0^T H_\theta(t)^{\frac{\beta}{\beta-1}}dt\right] = y^{\frac{1}{\beta-1}}K_{1,\theta}(0),$$

where $K_{1,\theta}$ is defined as in (4.24). Then the inverse of $\mathcal{X}_{1,\theta}$ is given by $\mathcal{X}_{1,\theta}^{-1}(x) = \left(\frac{x}{K_{1,\theta}(0)}\right)^{\beta-1}$ so that the martingale $M_{1,\theta}$ defined in (4.64) can be calculated as

$$\begin{aligned}
M_{1,\theta}(t) &= \left(\frac{x}{K_{1,\theta}(0)}\right)^{\beta-1}\mathbb{E}\left[\int_0^T H_\theta(s)^{\frac{\beta}{\beta-1}}ds\big|\mathcal{F}_t\right] \\
&= \left(\frac{x}{K_{1,\theta}(0)}\right)^{\beta-1}\left[\int_0^t H_\theta(s)^{\frac{\beta}{\beta-1}}ds + \widetilde{H}_\theta(t)K_{1,\theta}(t)\right]
\end{aligned}$$

Clearly, the dynamics of $M_{1,\theta}$ can be calculated as

$$\begin{aligned}
dM_{1,\theta}(t) &= \left(\frac{x}{K_{1,\theta}(0)}\right)^{\beta-1}K_{1,\theta}(t)d\widetilde{H}_\theta(t) \\
&= \left(\frac{x}{K_{1,\theta}(0)}\right)^{\beta-1}K_{1,\theta}(t)\widetilde{H}_\theta(t-)\left[\frac{\beta}{\beta-1}\theta^D(t)dB(t)\right. \\
&\qquad \left. + \int_{\mathbb{R}\setminus\{0\}}\left(\theta^J(t,y)^{\frac{\beta}{\beta-1}} - 1\right)\widetilde{N}(dt,dy)\right]
\end{aligned}$$

On the other hand from (4.64) we have

$$H_\theta(t)V_{1,\theta}(t) = \left(\frac{x}{K_{1,\theta}(0)}\right)^{\beta-1} \mathbb{E}\left[\int_t^T H_\theta(s)^{\frac{\beta}{\beta-1}} ds \big| \mathcal{F}_t\right] = \left(\frac{x}{K_{1,\theta}(0)}\right)^{\beta-1} K_{1,\theta}(t)\widetilde{H}_\theta(t),$$

so that the dynamics of $M_{1,\theta}$ can be rewritten as

$$dM_{1,\theta}(t) = H_\theta(t)V_{1,\theta}(t)\left[\frac{\beta}{\beta-1}\theta^D(t)dB(t) + \int_{\mathbb{R}\setminus\{0\}} \left(\theta^J(t,y)^{\frac{\beta}{\beta-1}} - 1\right)\widetilde{N}(dt,dy)\right] \tag{4.66}$$

Now, on the other hand consider the martingale for the terminal wealth problem. We follow the same procedure. Then the function $\mathcal{X}_{2,\theta}$ as defined in (4.61) takes the form

$$\mathcal{X}_{2,\theta}(y) = y^{\frac{1}{\beta-1}}\mathbb{E}\left[H_\theta(T)^{\frac{\beta}{\beta-1}}\right] = y^{\frac{1}{\beta-1}}K_{2,\theta}(0),$$

where $K_{2,\theta}$ is defined as in (4.25).

The inverse of $\mathcal{X}_{2,\theta}$ is given by $\mathcal{X}_{2,\theta}^{-1}(x) = \left(\frac{x}{K_{2,\theta}(0)}\right)^{\beta-1}$ so that the martingale $M_{2,\theta}$ defined in (4.64) can be calculated as

$$M_{2,\theta}(t) = \left(\frac{x}{K_{2,\theta}(0)}\right)^{\beta-1} \mathbb{E}\left[H_\theta(T)^{\frac{\beta}{\beta-1}} \big| \mathcal{F}_t\right] = \left(\frac{x}{K_{2,\theta}(0)}\right)^{\beta-1} \widetilde{H}_\theta(t)K_{2,\theta}(t).$$

Considering that from (4.64)

$$H_\theta(t)V_{2,\theta}(t) = \left(\frac{x}{K_{2,\theta}(0)}\right)^{\beta-1} \mathbb{E}\left[H_\theta(T)^{\frac{\beta}{\beta-1}} \big| \mathcal{F}_t\right] = \left(\frac{x}{K_{2,\theta}(0)}\right)^{\beta-1} K_{2,\theta}(t)\widetilde{H}_\theta(t),$$

so that, as before, the dynamics of $M_{2,\theta}$ can be calculated as

$$\begin{aligned} dM_{2,\theta}(t) &= \left(\frac{x}{K_{1,\theta}(0)}\right)^{\beta-1} K_{1,\theta}(t)d\widetilde{H}_\theta(t) \\ &= H_\theta(t)V_{2,\theta}(t)\left[\frac{\beta}{\beta-1}\theta^D(t)dB(t) + \int_{\mathbb{R}\setminus\{0\}} \left(\theta^J(t,y)^{\frac{\beta}{\beta-1}} - 1\right)\widetilde{N}(dt,dy)\right] \end{aligned} \tag{4.67}$$

Comparing the martingale coefficients from (4.66) and (4.67) one clearly sees that the martingale coefficients that are given for $i = 1, 2$ by

$$\begin{aligned} a^{(i),D}(t) &= H_\theta(t)V_{2,\theta}(t)\frac{\beta}{\beta-1}\theta^D(t), \text{ and} \\ a^{(i),J}(t,y) &= H_\theta(t)V_{2,\theta}(t)\int_{\mathbb{R}\setminus\{0\}} \left(\theta^J(t,y)^{\frac{\beta}{\beta-1}} - 1\right), \end{aligned}$$

and give the same condition on $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\pi}})$ when substituted into (4.65). This condition is the same as in (4.30), which is

$$\xi(t)\,\pi_{\widehat{\theta}}(t) = \frac{1}{\beta-1}\widehat{\theta}^D(t), \quad \text{and} \quad \gamma(t,y)\,\pi_{\widehat{\theta}}(t) = \widehat{\theta}^J(t,y)^{1/(\beta-1)} - 1.$$

Thus, the optimal trading strategy for all three problems and the case $\beta_1 = \beta_2 = \beta$ has to satisfy (4.31)

$$\alpha(t) - r(t) - (1-\beta)\xi(t)^2\widehat{\pi}(t) + \int_{\mathbb{R}\setminus\{0\}} \gamma(t,y)\{1+\widehat{\pi}(t)\gamma(t,y)\}^{\beta-1}\nu(dy) = 0.$$

Analogue to Corollary 4.6 it is possible to derive some results on the auxiliary consumption, terminal wealth, and wealth process for $\theta \in \widetilde{\Theta}_i$, $i = 1, 2$. For the consumption problem the auxiliary processes are given as

$$\begin{aligned} c_{1,\theta}(t) &= x\frac{H_\theta(t)^{1/(\beta-1)}}{K_{1,\theta}(0)}, \quad t \in [0,T], \\ Y_{1,\theta} &= 0, \end{aligned}$$

where $K_{1,\theta}$ as defined in (4.24) with h_θ defined by (4.22) and the auxiliary wealth process $V_{1,\theta}(t)$ is given by

$$V_{1,\theta}(t) = x \cdot \frac{\widetilde{H}_\theta(t)}{H_\theta(t)} \frac{K_{1,\theta}(t)}{K_{1,\theta}(0)}, \quad t \in [0,T].$$

The corresponding auxiliary processes for the terminal wealth problem are given by

$$\begin{aligned} c_{2,\theta}(t) &\equiv 0, \\ Y_{2,\theta} &= x\frac{H_\theta(T)^{1/(\beta-1)}}{h_\theta(T)} \end{aligned}$$

and the auxiliary wealth process $V_{2,\theta}(t)$ as defined in (4.64) is given by

$$V_{2,\theta}(t) = x\frac{\widetilde{H}_\theta(t)}{H_\theta(t)}, \quad t \in [0,T]. \tag{4.68}$$

These results can be proven analogously to the proof of Corollary 4.6. If the optimal EMM can be obtained in form of $\widehat{\boldsymbol{\theta}}$, then the optimal consumption, terminal wealth, and wealth process can be obtained by substituting $\boldsymbol{\theta}$ by the optimal $\widehat{\boldsymbol{\theta}}$ in the above equations.

When maximizing utility for consumption only the expected consumption and wealth processes behave similarly to those in the investment-consumption problem. This is shown in Figure 4.3, where all parameters are as given in the previous section.

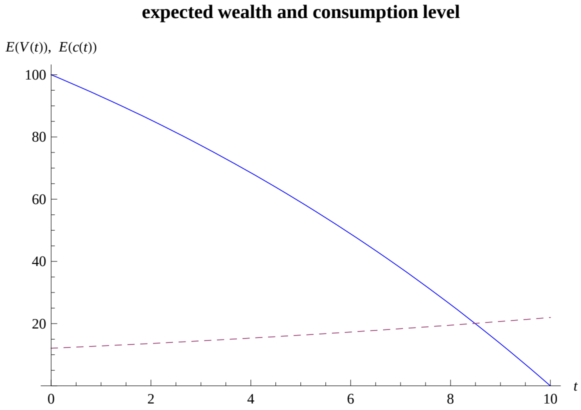

Figure 4.3: Expected wealth and consumption (dashed) over time when maximizing consumption with parameters $\alpha = .1$, $r = .05$, $\xi = .3$, $\nu = .2$, $\gamma = -.1$, $\beta = 0$, $T = 10$

However, for the terminal wealth maximization problem the situation is very different from that in the investment-consumtion problem, since there is no consumption. It can be seen in Figure 4.4 that the expected wealth increases exponentially in time, where again all parameters are as given in the previous section.

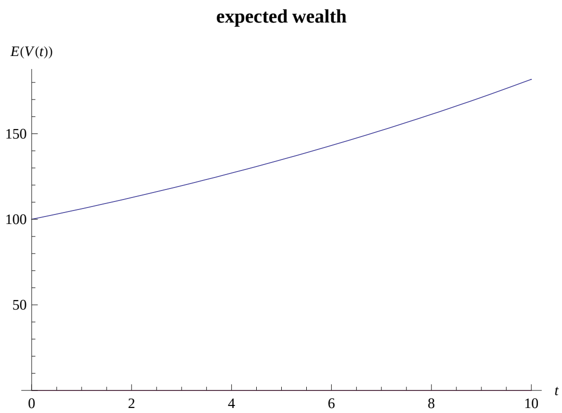

Figure 4.4: Expected wealth over time when maximizing terminal wealth with parameters $\alpha = .1$, $r = .05$, $\xi = .3$, $\nu = .2$, $\gamma = -.1$, $\beta = 0$, $T = 10$

4.8 Relationship to Partial Differential Equations

In this section we relate the martingale results on optimal portfolios to partial differential equations. The key tool to make the transition from stochastic processes to partial differential equations is the Kolmogorov equation. The results of the Kolmogorov equation can particularly be applied to the optimal terminal wealth problem on which we are going to restrict ourself in this section.

The model parameters $\boldsymbol{\alpha}, \boldsymbol{\xi}, \boldsymbol{\gamma}$ and r are assumed to be deterministic, continuously differentiable with bounded spacial gradients. In particular, it is assumed that all jump-diffusion Girsanov kernels are deterministic too. That is the set of processes Θ shall only contain deterministic processes. This means in particular that the stock prices S_i, $i = 1, \ldots, n$, and the risk-less asset S_0 become Markov processes. Further, the Radon-Nikodym densities Z_θ as well as the state price density H_θ are then also Markov processes. However, the wealth process V_π^x of a trading strategy $\boldsymbol{\pi}$ is Markovian if and only if the trading strategy $\boldsymbol{\pi}$ is a Markov process. If this is the case $\boldsymbol{\pi}$ is called a Markov or feedback strategy. Its value then depends on the current time t and the current level of wealth.

To relate the martingale approach results to PDEs and finally to the Hamilton-Jacobi-Bellman equation of stochastic control it is necessary to modify the terminal wealth problem of Section 4.7. For $(s, v) \in [0, T] \times (0, \infty)$ introduce the optimization problem

$$\Phi(s, v) = \sup_{\pi \in \mathcal{A}(s,v)} \mathbb{E}\left[U\big(V_\pi^{s,v}(T)\big)\right], \quad (s, v) \in [0, T] \times (0, \infty), \tag{4.69}$$

where $V_\pi^{s,v}$ is the wealth process of the strategy $\boldsymbol{\pi}$ that starts at v at time s, instead of x at time 0. This wealth process was given as the solution to the SDE (3.13) by

$$\frac{dV_\pi^{s,v}(t)}{V_\pi^{s,v}(t-)} = r(t)dt + \boldsymbol{\pi}(t)^\intercal \left[\overline{\boldsymbol{\alpha}}(t)dt + \boldsymbol{\xi}(t)d\boldsymbol{B}(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t, y) N_h(dt, dy)\right],$$

with new initial condition $V_\pi^{s,v}(s) = v$. The set of admissible trading strategies is thereby

$$\mathcal{A}(s, v) := \{\boldsymbol{\pi} \mid V_\pi^{s,v} \geq 0, \text{ and } \mathbb{E}[\min\{0, U\left(V_\pi^{s,v}(T)\right)\}] > -\infty\}.$$

This extends naturally the problem $\Phi_2(x)$ in (4.58) in the sense that the wealth problem starts now at time s instead of time 0. In the case that $s = 0$ and $v = x$ the optimization problem $\Phi_2(x)$ from (4.58) is recovered.

For $\boldsymbol{\theta} \in \Theta$ we allow the state price density H_θ from Definition 3.3 to start at h at time s and denote it in this case by $H_\theta^{s,h}$. It then satisfies the properties

$$H_\theta^{s,h}(t) = h H_\theta^{s,1}(t), \quad \text{and } H_\theta^{0,1}(t) = H_\theta^{0,1}(s) H_\theta^{s,1}(t),$$

for $0 \leq s \leq t \leq T$. Since for $\boldsymbol{\theta} \in \Theta$ the density $Z_\theta^{s,z}$ that starts at point z at time s is still a martingale since $Z_\theta^{0,1}$ is a martingale, the state price density $H_\theta^{s,h}$ still preserves its usual properties and behaves like H_θ but with a different starting point. In particular, for $\boldsymbol{\theta} \in \Theta$, what holds true for Z_θ and H_θ still applies to $Z_\theta^{s,z}$ and $H_\theta^{s,h}$ respectively.

Analogue to the definition of $\mathcal{X}_{2,\theta}$ in (4.61) let the function $\mathcal{X}_\theta(s,y)$ be defined for $(s,y) \in [0,T] \times (0,\infty)$ by

$$\mathcal{X}_\theta(s,y) := \mathbb{E}\left[H_\theta^{s,1}(T) I\big(y H_\theta^{s,1}(T)\big)\right]. \tag{4.70}$$

To guarantee that $\mathcal{X}_\theta$ has an inverse function $\mathcal{X}_\theta^{-1}$ the jump-diffusion Girsanov kernels are restricted to the set

$$\widetilde{\Theta}_T := \{\boldsymbol{\theta} \in \Theta \,|\, \mathcal{X}_\theta(s,y) < \infty, (s,y) \in [0,T] \times (0,\infty)\}.$$

The above set $\widetilde{\Theta}_T$ relates to the set $\widetilde{\Theta}_2$ introduced in (4.62) but is now time extended, hence the T subscript. It is assumed that $\widetilde{\Theta}_T$ is not empty - which is the case if $\widetilde{\Theta}_2$ is non-empty.

Similar to the definition of $Y_{1,\theta}$ in (4.63), define for $\boldsymbol{\theta} \in \widetilde{\Theta}_T$ and $(s,v) \in [0,T] \times (0,\infty)$ the random variable

$$Y_\theta^{s,v} = I\big(\mathcal{X}_\theta^{-1}(s,v) H_\theta^{s,1}(T)\big), \tag{4.71}$$

where I is, as usual, the inverse of the first derivative of the utility function U. From its construction it clearly has to satisfy a budget constraint of the form

$$\mathbb{E}\left[H_\theta^{s,1}(T) Y_\theta^{s,v}\right] = v. \tag{4.72}$$

This again relates to the budget constraint $\mathbb{E}[H_\theta(T) Y_{2,\theta}] = x$ of the terminal wealth only problem $\Phi_2(x)$. Since $H_\theta(T) = H_\theta^{s,1}(T) H_\theta(S)$ the above budget constraint can also be written in the form $\mathbb{E}\left[H_\theta(T) Y_\theta^{s,v} | \mathcal{F}_s\right] = H_\theta(s) v$.

Remark.
The idea for the definition of the random variable $Y_\theta^{s,v}$ in (4.71) comes from considering the constraint optimization problem

$$\max_{Y_\theta^{s,v}} \mathbb{E}\left[U(Y_\theta^{s,v})\right] \quad \text{s.t.} \quad \mathbb{E}\left[H_\theta^{s,1}(T) Y_\theta^{s,v}\right] = v.$$

A Lagrange multiplier technique $L(Y_\theta^{s,v}, y) := \mathbb{E}\left[U(Y_\theta^{s,v}) + y\big(v - H_\theta^{s,1}(T) Y_\theta^{s,v}\big)\right]$ can be applied that leads to the optimal candidate

$$Y_\theta^{s,v} = I\big(y H_\theta^{s,1}(T)\big).$$

If $Y_\theta^{s,v}$ should satisfy the budget constraint (4.72) it has to be of the form (4.71).

Theorem 4.17 (Kolmogorov equation)**.** *Let $X(t)$ be the solution of*

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB(t) + \int_{\mathbb{R}\setminus\{0\}} c(t, X(t), y) N(dt, dy)$$

for smooth (continuously differentiable) a, b, and c with bounded spacial gradients. Define

$$u(s,x) := \mathbb{E}^{s,x}[f\big(X(T)\big)] \tag{4.73}$$

for a C^2 function f with compact support. Then u satisfies the Kolmogorov equation

$$\begin{aligned}\partial_s u(s,x) + a(s,x)\partial_x u(s,x) + \frac{1}{2}b^2(s,x)\partial_{xx}u(s,x)\\ + \int_{\mathbb{R}\setminus\{0\}} \big[u(s,x+c(s,x,y)) - u(s,x)\big]\nu(y)dy = 0\end{aligned} \tag{4.74}$$

with terminal condition $u(T,x) = f(x)$.

A proof of the theorem can be found in Hanson (2009). A version where the stochastic process follows a diffusion process can be found in Schuss (1980). The theorem shows how stochastic processes link to the solution of certain partial differential equations. The Kolmogorov equation can be modified to allow a discounted version of the function $f(X(T))$.

Corollary 4.18. *Let $X(t)$ be the solution of*

$$dX(t) = a(t,X(t))dt + b(t,X(t))dB(t) + \int_{\mathbb{R}\setminus\{0\}} c(t,X(t),y)N(dt,dy)$$

for smooth a, b, and c that are such that a unique solution exists. Let r be a continuous integrable discount rate. Define

$$v(s,x) := \mathbb{E}^{s,x}[e^{-\int_s^T r(u)du} f\big(X(T)\big)] \tag{4.75}$$

for a C^2 function f with compact support. Then v satisfies the modified Kolmogorov equation

$$\begin{aligned}\partial_s v(s,x) + a(s,x)\partial_x v(s,x) + \frac{1}{2}b^2(s,x)\partial_{xx}v(s,x)\\ + \int_{\mathbb{R}\setminus\{0\}} \big[v(s,x+c(s,x,y)) - v(s,x)\big]\nu(y)dy = r(s)v(s,x)\end{aligned} \tag{4.76}$$

with terminal condition $v(T,x) = f(x)$.

Proof. Define the function $u(s,x) := e^{\int_s^T r(w)dw}v(s,x)$. Then it is known from Theorem 4.17 that u satisfies the equation (4.74). Differentiating u one obtains the derivatives

$$\begin{aligned}\partial_s u(s,x) &= -r(s)e^{\int_s^T r(w)dw}v(s,x) + e^{\int_s^T r(w)dw}\partial_s v(s,x),\\ \partial_x u(s,x) &= e^{\int_s^T r(w)dw}\partial_x v(s,x), \text{ and}\\ \partial_{xx}u(s,x) &= e^{\int_s^T r(w)dw}\partial_{xx}v(s,x).\end{aligned}$$

Substituting this into equation (4.74) and multiplying by $e^{-\int_s^T r(w)dw}$ leads to equation (4.76). ☐

Before the first application of the Kolmogorov equation is presented, it is necessary to determine an expression for $\mathcal{X}_\theta(s,y)$ of (4.70) under Q expectation.

Lemma 4.19. *For $\boldsymbol{\theta} \in \Theta$ the function $\mathcal{X}_\theta$ defined in* (4.70) *is given under Q-expectation as*

$$\mathcal{X}_\theta(s,y) = \mathbb{E}^Q\left[e^{-\int_s^T r(u)du} I\big(yH_\theta^{s,1}(T)\big)\right]. \tag{4.77}$$

Proof. In the following the property $Z_\theta^{0,1}(T) = Z_\theta^{0,1}(s)Z_\theta^{s,1}(T)$ is used. Consider the right hand side of (4.77). Then

$$\begin{aligned}
\mathbb{E}^Q\left[e^{-\int_s^T r(u)du} I\big(yH_\theta^{s,1}(T)\big)\right] &= \mathbb{E}\left[Z_\theta^{0,1}(T)e^{-\int_s^T r(u)du} I\big(yH_\theta^{s,1}(T)\big)\right] \\
&= \mathbb{E}\left[Z_\theta^{0,1}(s)\mathbb{E}\left[H_\theta^{s,1}(T)I\big(yH_\theta^{s,1}(T)\big)\big|\mathcal{F}_s\right]\right] \\
&= \mathbb{E}\left[Z_\theta^{0,1}(s)\right]\mathbb{E}\left[H_\theta^{s,1}(T)I\big(yH_\theta^{s,1}(T)\big)\right] \\
&= \mathbb{E}\left[H_\theta^{s,1}(T)I\big(yH_\theta^{s,1}(T)\big)\right] \\
&= \mathcal{X}_\theta(s,y) \quad \text{as in (4.70).}
\end{aligned}$$

□

The modified Kolmogorov equation introduced in Corollary 4.18 can be applied to the state price density $H_\theta^{s,1}$ under Q-dynamics and the function $\mathcal{X}_\theta(s,y)$ under Q-expectation as in (4.77)

Corollary 4.20. *Let $\boldsymbol{\theta} \in \widetilde{\Theta}_T$. Then the function $\mathcal{X}_\theta(s,y)$ defined in* (4.70) *solves the stochastic Dirichlet boundary value problem*

$$\begin{aligned}
\partial_s\mathcal{X}_\theta(s,y) + y\left(-r(s) + ||\boldsymbol{\theta}_D(s)||^2 - \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \big(\theta_h^J(ds,dz) - 1\big)\nu_h(dz)ds\right)\partial_y\mathcal{X}_\theta(s,y) \\
+\frac{1}{2}y^2||\boldsymbol{\theta}_D(s)||^2\partial_{yy}\mathcal{X}_\theta(s,y) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \left[\mathcal{X}_\theta\big(s, y\theta_h^J(s,z)\big) - \mathcal{X}_\theta(s,y)\right]\theta_h^J(s,z)\nu_h(dz) \\
-r(s)\mathcal{X}_\theta(s,y) = 0,
\end{aligned} \tag{4.78}$$

for $(t,y) \in [0,T) \times (0,\infty)$, and

$$\mathcal{X}_\theta(T,y) = I(y), \qquad \text{for } Q \text{ almost all } y \in (0,\infty).$$

Proof. Consider the Q-dynamics of the state price density $H_\theta^{s,1}$ that starts almost surely at 1 at time s. Since $\widetilde{N}^Q(dt,dz) = N(dt,dz) - \theta_J(dt,dz)\nu(dz)dt$ we have $\widetilde{N}(dt,dz) = \widetilde{N}^Q(dt,dz) + \big(\theta_J(dt,dz) - 1\big)\nu(dz)dt$. Thus, the dynamics of the state price density $H_\theta^{s,1}$ under Q can be derived from 3.11 as

$$\begin{aligned}
dH_\theta^{s,1}(t) = H_\theta^{s,1}(t-)\Bigg[\left(-r(t) + ||\boldsymbol{\theta}_D(t)||^2 + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \big(\theta_h^J(dt,dz) - 1\big)^2\nu_h(dz)\right)dt \\
+\boldsymbol{\theta}_D(t)^\intercal d\boldsymbol{B}^Q(t) + \int_{\mathbb{R}\setminus\{0\}} \big(\theta_h^J(dt,dz) - 1\big)\widetilde{N}_h^Q(dt,dz)\Bigg].
\end{aligned}$$

The compensated Poisson measure $\widetilde{N}_h^Q(dt,dz)$ has the intensity measure $\theta_h^J(dt,dy)\nu_h(dy)$. That means that the jump related drift term in $dH_\theta^{s,1}$ is given by

$$\int_{\mathbb{R}\setminus\{0\}} \left[\left(\theta_h^J(dt,dz)-1\right)^2 - \left(\theta_h^J(dt,dz)-1\right)\theta_h^J(dt,dz)\right]\nu_h(dz)$$
$$= -\int_{\mathbb{R}\setminus\{0\}} \left(\theta_h^J(dt,dz)-1\right)\nu_h(dz)$$

for $h=1,\dots,m$. From (4.77), the function $\mathcal{X}_\theta(s,y)$ has under Q-expectation the form

$$\mathcal{X}_\theta(s,y) = \mathbb{E}^Q\left[e^{-\int_s^T r(u)du} I\left(yH_\theta^{s,1}(T)\right)\right].$$

Then the Kolmogorov backward equation for jump-diffusion processes Corollary 4.18 can be applied to $\mathcal{X}_\theta$ and $H_\theta^{s,1}$ to derive equation (4.78). □

The partial differential equation (4.78) will help to derive a feedback form of the optimal trading strategy for the problem $\Phi(s,v)$ in (4.69). In particular, it is shown that the optimal trading strategy $\widehat{\boldsymbol{\pi}}$ to $\Phi(s,v)$ is a Markov strategy and depends only on the current time and the current level of wealth. The notion of the optimal martingale measure will be used and is taken over from Section 4.7. As in Definition 4.15 the following is the time extended analogue.

Definition 4.21. *A martingale measure Q obtained by $\frac{dQ}{dP}|_{\mathcal{F}_T} = Z_\theta(T)$ in terms of $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}_T$ is called optimal for the optimization problem $\Phi(s,v)$ in (4.69) if it satisfies*

$$\mathbb{E}\left[U_2(Y_{\widehat{\theta}}^{s,v})\right] = \inf_{\theta\in\widetilde{\Theta}_T} \mathbb{E}\left[U_2(Y_\theta^{s,v})\right],$$

where $Y_\theta^{s,v}$ is defined as in (4.71).

Lemma 4.22. *Let $\widehat{\boldsymbol{\theta}}$ represent the optimal equivalent martingale measure for the problem $\Phi(0,x)$ in (4.69) for $x>0$. Then the optimal trading strategy $\widehat{\boldsymbol{\pi}}$ can be written in feedback form $\widehat{\boldsymbol{\pi}}(s,v)$ and satisfies for $(s,v)\in[0,T]\times(0,\infty)$*

$$\begin{aligned}
\widehat{\boldsymbol{\pi}}(s,v)^{\intercal}\boldsymbol{\xi}(s) &= \frac{1}{v}\frac{\mathcal{X}_{\widehat{\theta}}^{-1}(s,v)}{\partial_x\mathcal{X}_{\widehat{\theta}}^{-1}(s,v)}\widehat{\boldsymbol{\theta}}_D(s),\\
\sum_{i=1}^k \widehat{\pi}_i(s,v)\gamma_{ih}(s,z) &= \frac{1}{v}\mathcal{X}_{\widehat{\theta}}\left(s,\widehat{\theta}_h^J(s,z)\mathcal{X}_{\widehat{\theta}}^{-1}(s,v)\right)-1,
\end{aligned}\tag{4.79}$$

for all $z\in\mathbb{R}\setminus\{0\}$ and $h=1,\dots,m$.

Proof. Let $\widehat{\boldsymbol{\theta}}$ be the optimal EMM. From (4.71) the optimal terminal wealth is then given by

$$Y_{\widehat{\theta}}^{s,v} = I\left(\mathcal{X}_{\widehat{\theta}}^{-1}(s,v)H_{\widehat{\theta}}^{s,1}(T)\right).$$

Since the discounted wealth process is a martingale under the martingale measure Q it follows that, substituting $Y^{s,v}_{\widehat{\theta}}$,

$$\begin{aligned} V^x_{\widehat{\pi}}(t) &= e^{\int_0^t r(u)du}\mathbb{E}^Q\left[e^{-\int_0^T r(u)du}Y^{0,x}_{\widehat{\theta}}\big|\mathcal{F}_t\right] \\ &= \mathbb{E}^Q\left[e^{-\int_s^T r(u)du}I\big(\mathcal{X}^{-1}_{\widehat{\theta}}(0,x)H^{0,1}_{\widehat{\theta}}(T)\big)\big|\mathcal{F}_t\right] \end{aligned} \tag{4.80}$$

As has been seen in (4.77), the function $\mathcal{X}_\theta$ defined in (4.70) can be written under Q-expectation as

$$\mathcal{X}_\theta(s,y) := \mathbb{E}^Q\left[e^{-\int_s^T r(u)du}I\big(yH^{s,1}_\theta(T)\big)\right].$$

Comparing this with the last expression and using that $H^{0,1}_\theta(T) = H^{0,1}(t)H^{t,1}(T)$, the optimal wealth process, given in (4.80), has the form

$$V^x_{\widehat{\pi}}(t) = \mathcal{X}_{\widehat{\theta}}\left(t, \mathcal{X}^{-1}_{\widehat{\theta}}(0,x)H^{0,1}_{\widehat{\theta}}(t)\right) \tag{4.81}$$

For fixed $x > 0$ define $\widehat{y} = \mathcal{X}^{-1}_{\widehat{\theta}}(0,x)$. It is known that the discounted wealth process is a martingale and has dynamics given by (3.22). The same should be true when discounting the process in (4.81). Itô's formula 2.17 is applied to $\frac{\mathcal{X}_{\widehat{\theta}}\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t)\right)}{S_0(t)}$ (under Q dynamics) so that

$$\begin{aligned} d\overline{V}^x_{\widehat{\pi}}(t) =& d\frac{\mathcal{X}_{\widehat{\theta}}\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t)\right)}{S_0(t)} \\ =& \frac{1}{S_0(t)}\Big\{\partial_s\mathcal{X}_{\widehat{\theta}}\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\right) + \widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\Big[-r(s) + ||\widehat{\boldsymbol{\theta}}_D(s)||^2 \\ & - \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} (\widehat{\theta}^J_h(ds,dz) - 1)\nu_h(dz)ds\Big]\,\partial_y\mathcal{X}_\theta\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\right) \\ & + \frac{1}{2}\widehat{y}^2H^{0,1}_{\widehat{\theta}}(t-)^2||\widehat{\boldsymbol{\theta}}_D(s)||^2\partial_{yy}\mathcal{X}_{\widehat{\theta}}\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\right) \\ & + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \Big[\mathcal{X}_\theta\big(s,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\widehat{\theta}^J_h(s,z)\big) - \mathcal{X}_\theta\big(s,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\big)\Big]\,\theta^J_h(s,z)\nu_h(dz)ds \\ & - r(s)\mathcal{X}_\theta\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\right)\Big\} \\ & + \frac{1}{S_0(t)}\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\partial_y\mathcal{X}_{\widehat{\theta}}\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\right)\widehat{\boldsymbol{\theta}}_D(t)^\intercal d\boldsymbol{B}^Q \\ & + \frac{1}{S_0(t)}\sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \Big[\mathcal{X}_{\widehat{\theta}}\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\theta^J_h(t,z)\right) - \mathcal{X}_{\widehat{\theta}}\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\right)\Big]\,\widetilde{N}^Q_h(dt,dz). \end{aligned}$$

The term in the curly brackets is according to (4.78) zero, so that the above term simplifies to

$$\begin{aligned} d\overline{V}^x_{\widehat{\pi}}(t) =& \frac{1}{S_0(t)}\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\partial_y\mathcal{X}_{\widehat{\theta}}\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\right)\widehat{\boldsymbol{\theta}}_D(t)^\intercal d\boldsymbol{B}^Q \\ & + \frac{1}{S_0(t)}\sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \Big[\mathcal{X}_{\widehat{\theta}}\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\theta^J_h(t,z)\right) - \mathcal{X}_{\widehat{\theta}}\left(t,\widehat{y}H^{0,1}_{\widehat{\theta}}(t-)\right)\Big]\,\widetilde{N}^Q_h(dt,dz). \end{aligned} \tag{4.82}$$

Observing the term $\partial_y \mathcal{X}_{\widehat{\theta}}\left(t, \widehat{y} H_{\widehat{\theta}}^{0,1}(t-)\right)$ in the Brownian motion part of the above equation, it can be seen that since $V_{\widehat{\pi}}^x(t-) = \mathcal{X}_{\widehat{\theta}}\left(t, \widehat{y} H_{\widehat{\theta}}^{0,1}(t-)\right)$ it has to hold that

$$\partial_y \mathcal{X}_{\widehat{\theta}}\left(t, \widehat{y} H_{\widehat{\theta}}^{0,1}(t-)\right) = \partial_y \mathcal{X}_{\widehat{\theta}}\left(t, \mathcal{X}_{\widehat{\theta}}^{-1}\left(t, V_{\widehat{\pi}}^x(t-)\right)\right) = \frac{1}{\partial_x \mathcal{X}_{\widehat{\theta}}^{-1}\left(t, V_{\widehat{\pi}}^x(t-)\right)}$$

where the inverse function theorem has been applied in the last step. Thus, using that $\widehat{y} H_{\widehat{\theta}}^{0,1}(t-) = \mathcal{X}_{\widehat{\theta}}^{-1}\left(t, V_{\widehat{\pi}}^x(t-)\right)$ the equation (4.82) can be written as

$$\begin{aligned} d\overline{V}_{\widehat{\pi}}^x(t) =& \frac{1}{S_0(t)} \frac{\mathcal{X}_{\widehat{\theta}}^{-1}\left(t, V_{\widehat{\pi}}^x(t-)\right)}{\partial_x \mathcal{X}_{\widehat{\theta}}^{-1}\left(t, V_{\widehat{\pi}}^x(t-)\right)} \widehat{\boldsymbol{\theta}}_D(t)^\intercal d\boldsymbol{B}^Q \\ &+ \frac{1}{S_0(t)} \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \left[\mathcal{X}_{\widehat{\theta}}\left(t, \mathcal{X}_{\widehat{\theta}}^{-1}\left(t, V_{\widehat{\pi}}^x(t-)\right) \theta_h^J(t,z)\right) - V_{\widehat{\pi}}^x(t-)\right] \widetilde{N}_h^Q(dt, dz). \end{aligned}$$

The above equation can be compared with the dynamics of a discounted wealth process of a trading strategy $\boldsymbol{\pi}$, which are found in (3.22) as

$$d\overline{V}_\pi^x(t) = \overline{V}_\pi^x(t) \boldsymbol{\pi}(t)^\intercal \boldsymbol{\xi}(t) d\boldsymbol{B}^Q + \overline{V}_\pi^x(t) \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t) \boldsymbol{\gamma}_h(t,z) \widetilde{N}_h^Q(dt, dz).$$

For these two processes to be the same, the optimal trading strategy has be a Markov process that satisfies (4.79). $\square$

It becomes clear from the condition on the optimal trading strategy (4.79) that $\widehat{\boldsymbol{\pi}}$ has to be Markovian. The next corollary will be important, when the Hamilton Jacobi Bellman equation is proven in Theorem 4.24.

Corollary 4.23. *For $\boldsymbol{\theta} \in \widetilde{\Theta}$, the function $G_\theta(s,y)$ defined by*

$$G_\theta(s,y) := \mathbb{E}\left[U\left(I\left(y H_\theta^{s,1}(T)\right)\right)\right] \tag{4.83}$$

satisfies the stochastic Dirichlet boundary value problem

$$\begin{aligned} &\partial_s G_\theta(s,y) - y\left[\sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \left(\theta_h^J(s,z) - 1\right) \nu_h(dz) + r(s)\right] \partial_y G_\theta(s,y) \\ &\quad + \frac{1}{2} y^2 ||\boldsymbol{\theta}_D(s)||^2 \partial_{yy} G_\theta(s,y) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \left[G_\theta\big(s, y\theta_h^J(s,z)\big) - G_\theta(s,y)\right] \nu_h(dz) = 0, \end{aligned} \tag{4.84}$$

for $(s,y) \in [0,T) \times (0,\infty)$, and

$$G_\theta(T,y) = U\big(I(y)\big), \quad \textit{for } P \textit{ almost all } y \in (0,\infty).$$

Proof. The dynamics of the state price density $H_\theta^{s,1}$ starting at s are given in (3.11) by

$$dH_\theta^{s,1}(t) = H_\theta^{s,1}(t-)\left[-r(t)dt + \boldsymbol{\theta}_D(t)d\boldsymbol{B}(t) + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\left(\theta_h^J(t,z)-1\right)\widetilde{N}_h(dt,dz)\right].$$

Applying Kolmogorov backward equation Theorem 4.17 to the state price density $H_\theta^{s,1}$ and the function $G_\theta(s,y)$ as in (4.83) shows that G_θ has to satisfy the stochastic Dirichlet boundary value problem. □

The function $G_\theta(s,y)$ defined in the above corollary gives an alternative way to express the optimal performance function $\Phi(s,v)$. Since the optimal terminal wealth is given by

$$Y_{\widehat{\theta}}^{s,v} = I\big(\mathcal{X}_{\widehat{\theta}}^{-1}(s,v)H_{\widehat{\theta}}^{s,1}(T)\big),$$

for the optimal Girsanov kernel $\widehat{\theta}$ and $\mathcal{X}_\theta(s,y)$ as defined in (4.77), the optimal performance function can be expressed as

$$\Phi(s,v) = G_{\widehat{\theta}}\left(s,\mathcal{X}_{\widehat{\theta}}^{-1}(s,v)\right).$$

This will become useful in the proof of the next theorem.

Theorem 4.24. *Let $\Phi(s,v)$ be the optimal performance function of the optimal terminal wealth problem specified in (4.69). Then Φ satisfies the Hamilton-Jacobi-Bellman equations of stochastic programming*

$$\begin{aligned}\partial_s\Phi(s,v) + \max_{\boldsymbol{\pi}\in\mathbb{R}^k}\Bigg\{ & v\big(r(s)+\boldsymbol{\pi}(s)^\intercal\overline{\boldsymbol{\alpha}}(s)\big)\partial_v\Phi(s,v) + \frac{1}{2}||\boldsymbol{\pi}(s)^\intercal\boldsymbol{\xi}(s)||^2v^2\partial_{vv}\Phi(s,v) \\ & + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\left[\Phi\left(s,v\left(1+\gamma_h^\pi(s,z)\right)\right)-\Phi(s,v)\right]\nu_h(dz)\Bigg\} = 0,\end{aligned} \tag{4.85}$$

on the set $[0,T)\times(0,\infty)$ and satisfies the boundary condition

$$\Phi(T,v) = U(v), \quad v>0.$$

Proof. The equation (4.85) is proven by calculating the derivatives of Φ and confirming that they indeed satisfy the above PDE. In the following let $\widehat{\boldsymbol{\theta}}\in\widetilde{\Theta}$ represent the optimal martingale measure of the problem Φ. Then the optimal terminal wealth is given by (4.71)

$$Y_{\widehat{\theta}}^{s,v} = I\left(\mathcal{X}_{\widehat{\theta}}^{-1}(s,v)H_{\widehat{\theta}}^{s,1}(T)\right).$$

Since $\Phi(s,v)=\mathbb{E}\left[U\left(Y_{\widehat{\theta}}^{s,v}\right)\right]$ it can be also written in terms of $G_{\widehat{\theta}}$ from (4.83) as

$$\Phi(s,v) = G_{\widehat{\theta}}\left(s,\mathcal{X}_{\widehat{\theta}}^{-1}(s,v)\right)$$

To obtain the derivatives of Φ it is necessary to calculate the derives of $G_{\widehat{\theta}}$. From (3.33) and the definition of $\mathcal{X}_\theta$ and G_θ in (4.70) and (4.83) respectively, it can be deduced that for $y > a > 0$ and $s \in [0, T]$

$$y\mathcal{X}_\theta(s, y) + a\mathcal{X}_\theta(s, a) - \int_a^y \mathcal{X}_\theta(s, \lambda)d\lambda = G_\theta(s, y) + G(s, a).$$

Differentiating the above equation, the derivatives of G_θ are

$$\partial_y G_\theta(s, y) = y\partial_y \mathcal{X}_\theta(s, y), \quad \text{and} \quad \partial_{yy} G_\theta(s, y) = \partial_y \mathcal{X}_\theta(s, y) + y\partial_{yy}\mathcal{X}_\theta(s, y). \tag{4.86}$$

The first derivative of Φ with respect to v can then be calculated as

$$\begin{aligned}
\partial_v \Phi(s, v) &= \partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s, v)\partial_y G\left(s, \mathcal{X}_{\widehat{\theta}}^{-1}(s, v)\right) \\
&= \partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s, v)\mathcal{X}_{\widehat{\theta}}^{-1}(s, v)\partial_y \mathcal{X}_{\widehat{\theta}}\left(s, \mathcal{X}_{\widehat{\theta}}^{-1}(s, v)\right) \\
&= \mathcal{X}_{\widehat{\theta}}^{-1}(s, v),
\end{aligned}$$

where we have used the inverse function theorem in the last equation. The second derivative with respect to v is therefore $\partial_{vv}\Phi(s, v) = \partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s, v)$. To obtain the maximum in the curly brackets in (4.85) the first derivative with respect to $\boldsymbol{\pi}$ is set equal to zero. Define the vector $\boldsymbol{\gamma}_h(t, y) := (\gamma_{1h}(t, y), \ldots, \gamma_{kh}(t, y))$, for $h = 1, \ldots, m$, then the optimal trading strategy has to satisfy the equations

$$\begin{aligned}
&\overline{\boldsymbol{\alpha}}(s)\partial_v \Phi(s, v) + v\boldsymbol{\pi}(s)^\intercal \boldsymbol{\xi}(s)\boldsymbol{\xi}(s)^\intercal \partial_{vv}\Phi(s, v) \\
+&\sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(s, z)\partial_v \Phi\left(s, v\big(1 + \boldsymbol{\pi}(s)^\intercal \boldsymbol{\gamma}_h(s)\big)\right)\nu_h(dz) = \mathbf{0}.
\end{aligned}$$

Substitution the derivatives of Φ this becomes

$$\begin{aligned}
&\overline{\boldsymbol{\alpha}}(s)\mathcal{X}_{\widehat{\theta}}^{-1}(s, v) + v\boldsymbol{\pi}(s)^\intercal \boldsymbol{\xi}(s)\boldsymbol{\xi}(s)^\intercal \partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s, v) \\
+&\sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(s, z)\mathcal{X}_{\widehat{\theta}}^{-1}\left(s, v\big(1 + \boldsymbol{\pi}(s)^\intercal \boldsymbol{\gamma}_h(s)\big)\right)\nu_h(dz) = \mathbf{0}.
\end{aligned}$$

The above expression is equal to zero if the optimal strategy satisfies the conditions in (4.79) which confirm the previous results.

Further, the derivative of Φ with respect to time can be calculated using again (4.86)

$$\begin{aligned}
\partial_s \Phi(s, v) &= \partial_s G_{\widehat{\theta}}\left(s, \mathcal{X}_{\widehat{\theta}}^{-1}(s, v)\right) + \partial_y G_{\widehat{\theta}}\left(s, \mathcal{X}_{\widehat{\theta}}^{-1}(s, v)\right)\partial_s \mathcal{X}_{\widehat{\theta}}^{-1}(s, v) \\
&= \partial_s G_{\widehat{\theta}}\left(s, \mathcal{X}_{\widehat{\theta}}^{-1}(s, v)\right) + \frac{\mathcal{X}_{\widehat{\theta}}^{-1}(s, v)}{\partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s, v)}\partial_s \mathcal{X}_{\widehat{\theta}}^{-1}(s, v).
\end{aligned}$$

Let us define for notational convenience $\zeta := \mathcal{X}_{\widehat{\theta}}^{-1}(s, v)$ for a fixed pair (s, v). If the partial derivatives of Φ are substituted into the HJB equation one obtains

$$\partial_s G_{\widehat{\theta}}(s,\zeta) + \frac{\zeta}{\partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)} \partial_s \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)$$

$$\max_{\boldsymbol{\pi}\in\mathbb{R}^k} \Bigg\{ v\big(r(s) + \boldsymbol{\pi}(s)^\intercal \overline{\boldsymbol{\alpha}}(s)\big)\zeta + \frac{1}{2}||\boldsymbol{\pi}(t)^\intercal \boldsymbol{\xi}(t)||^2 v^2 \partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)$$

$$\sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \left[G_{\widehat{\theta}}\left(s, \mathcal{X}_{\widehat{\theta}}^{-1}\left(s, v\big(1 + \boldsymbol{\pi}(s)^\intercal \boldsymbol{\gamma}_h(s,z)\big)\right)\right) - G_{\widehat{\theta}}(s,\zeta) \right] \nu_h(dz) \Bigg\} = 0,$$

Observing that because of (3.8), $\overline{\boldsymbol{\alpha}}(s) = -\boldsymbol{\xi}(t)^\intercal \widehat{\boldsymbol{\theta}}_D(s) - \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}(s,y)\widehat{\theta}_J(s,z)\nu(dz)$ the optimal trading strategy can be substituted into the above equation using condition (4.79):

$$\partial_s G_{\widehat{\theta}}(s,\zeta) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \left[G_{\widehat{\theta}}\left(s, \theta_h^J(s,z)\zeta\right) - G_{\widehat{\theta}}(s,\zeta) \right] \nu_h(dz) + \frac{\zeta}{\partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)} \partial_s \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)$$
$$+ v r(s)\zeta - \frac{1}{2} \frac{\zeta^2}{\partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)} ||\boldsymbol{\theta}_D(s)||^2 - \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \zeta \left[\mathcal{X}_{\widehat{\theta}}\left(s, \theta_h^J(s,z)\zeta\right) - v \right] \theta_h^J(s,z)\nu_h(dz) = 0.$$

$G_{\widehat{\theta}}$ has to satisfy (4.84) which changes the above equation to

$$\zeta \left[\sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \big(\theta_h^J(s,z) - 1\big)\nu(dz) + r(s) \right] \partial_y G_{\widehat{\theta}}(s,\zeta)$$
$$-\frac{1}{2}\zeta^2 ||\boldsymbol{\theta}_D(s)||^2 \partial_{yy} G_{\widehat{\theta}}(s,\zeta) + \frac{\zeta}{\partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)} \partial_s \mathcal{X}_{\widehat{\theta}}^{-1}(s,v) + v r(s)\zeta$$
$$-\frac{1}{2} \frac{\zeta^2}{\partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)} ||\boldsymbol{\theta}_D(s)||^2 - \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \zeta \left[\mathcal{X}_{\widehat{\theta}}\left(s, \theta_h^J(s,z)\zeta\right) - v \right] \theta_h^J(s,z)\nu_h(dz) = 0.$$

From (4.86), we have that $\partial_y G_{\widehat{\theta}}(s,\zeta) = \zeta \partial_y \mathcal{X}_{\widehat{\theta}}(s,\zeta)$ and $\partial_{yy} G_{\widehat{\theta}}(s,\zeta) = \partial_y \mathcal{X}_{\widehat{\theta}}(s,\zeta) + \zeta \partial_{yy} \mathcal{X}_{\widehat{\theta}}(s,\zeta)$. Further, $v = \mathcal{X}_{\widehat{\theta}}\left(\mathcal{X}_{\widehat{\theta}}^{-1}(s,v)\right) = \mathcal{X}_{\widehat{\theta}}(s,\zeta)$. Substituting all this into the above equation and dividing by ζ gives then

$$\left[\sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \big(\theta_h^J(s,z) - 1\big)\nu(dz) + r(s) \right] \zeta \partial_y \mathcal{X}_{\widehat{\theta}}(s,\zeta)$$
$$-\frac{1}{2}\zeta ||\boldsymbol{\theta}_D(s)||^2 \Big(\partial_y \mathcal{X}_{\widehat{\theta}}(s,\zeta) + \zeta \partial_{yy} \mathcal{X}_{\widehat{\theta}}(s,\zeta) \Big) + \frac{1}{\partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)} \partial_s \mathcal{X}_{\widehat{\theta}}^{-1}(s,v) + \mathcal{X}_{\widehat{\theta}}(s,\zeta) r(s)$$
$$-\frac{1}{2} \frac{\zeta}{\partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)} ||\boldsymbol{\theta}_D(s)||^2 - \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \left[\mathcal{X}_{\widehat{\theta}}\left(s, \theta_h^J(s,z)\zeta\right) - \mathcal{X}_{\widehat{\theta}}(s,\zeta) \right] \theta_h^J(s,z)\nu_h(dz) = 0.$$

Differentiating $v = \mathcal{X}_{\widehat{\theta}}\left(\mathcal{X}_{\widehat{\theta}}^{-1}(s,v)\right)$ shows that

$$\partial_s \mathcal{X}_{\widehat{\theta}}(s,\zeta) + \partial_y \mathcal{X}_{\widehat{\theta}}(s,\zeta) \partial_s \mathcal{X}_{\widehat{\theta}}^{-1}(s,v) = 0.$$

and therefore $\partial_y \mathcal{X}_{\widehat{\theta}}(s,\zeta) \partial_s \mathcal{X}_{\widehat{\theta}}^{-1}(s,v) = -\partial_s \mathcal{X}_{\widehat{\theta}}(s,\zeta)$. Further, since $\frac{1}{\partial_v \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)} = \partial_y \mathcal{X}_{\widehat{\theta}}(s,\zeta)$

the above equation becomes

$$\left[\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\left(\theta_h^J(s,z)-1\right)\nu_h(dz)-||\boldsymbol{\theta}_D(s)||^2+r(s)\right]\zeta\partial_y\mathcal{X}_{\widehat{\theta}}(s,\zeta)$$
$$-\frac{1}{2}\zeta^2||\boldsymbol{\theta}_D(s)||^2\partial_{yy}\mathcal{X}_{\widehat{\theta}}(s,\zeta)-\partial_s\mathcal{X}_{\widehat{\theta}}(s,\zeta)+\mathcal{X}_{\widehat{\theta}}(s,\zeta)r(s)$$
$$-\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}}\left[\mathcal{X}_{\widehat{\theta}}\left(s,\theta_h^J(s,z)\zeta\right)-\mathcal{X}_{\widehat{\theta}}(s,\zeta)\right]\theta_h^J(s,z)\nu_h(dz)=0.$$

but this equations is simply (4.78) mulitplied by -1. Hence, it has been proven to hold true. □

The HJB equation gives us an alternative tool to derive the optimal performance function $\Phi(s,v)$ as a solution to a partial differential equation with boundary condition. This is convenient when it is difficult to derive $\Phi(s,v)$ through the martingale approach. Using a PDE method bears also the advantage of being able to use numerical methods to solve the HJB equation.

4.9 Conclusion

We have treated the problem of maximizing expected utility from terminal wealth and consumption as defined in (4.4) in a general jump-diffusion framework. This modified and extended the approach of Karatzas and Shreve (1998) for the consumption-investment problem. The problem has been tackled by defining a set of auxiliary variables c_θ and Y_θ in (4.6), which have the property that expected utility from them is at least as high as from any admissible consumption-terminal wealth pair (cf. Lemma 4.1). We have then minimized over the set of Girsanov kernels Θ in order to find a trading strategy that replicates a auxiliary terminal wealth Y_θ and for which $c = c_\theta$. A condition under which this is possible has been given in Theorem 4.3, requiring that $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ solve a system of non-linear equations (4.12). Relationships to the duality approach, as for example studied in Kramkov and Schachermayer (1999) have been established in Theorem 4.4.

The results have been applied to the power utility case where we also considered various model variations, like for example pure Brownian motion driven models, or pure jump models with drift. Relationships have been established to similar work at that stage.

In similar manners the problem has been solved for the maximization of either consumption or terminal wealth in Proposition 4.16, which is the equivalent of Theorem 4.3 for the individual problems. In the final section the martingale methods have been related to PDE methods and a feedback form of the optimal trading strategy has been developed in Lemma 4.22. The final section has been concluded by establishing the Hamilton-Jacobi-Bellman equation of dynamic programming in Theorem 4.24.

Chapter 5

Active Portfolio Management: Outperforming a Benchmark Portfolio

5.1 Introduction

In the previous chapter an investor aim was to optimize his/her portfolio in the expected utility sense. However, in reality an investor usually evaluates his/her performance by comparing their performance to that of their peers or to a benchmark like for example the FTSE 100. Active portfolio managers thus attempt to outperform mutual funds or benchmarks. This is in contrast to passive portfolio managers who try to track the performance of a benchmark. Often active portfolio managers claim that their skills allow them to perform better than the market represented by a mutual fund or benchmark. This claim allows them to charge a much higher fee than their passively managing peers.

There is a great controversy whether or not active portfolio management brings any benefit to the investor. Shukla and Trzcinka (1992) argue that actively managed portfolios do not perform better than passively managed portfolios. Other researchers argue that there are funds that outperform the benchmarks and, even after transaction costs, perform better than passively managed funds. Papers supporting this idea are Keim (1999), Chen et al. (2000), and Wermers (2000).

The effectiveness of actively managed funds will not be discussed in this chapter. Instead, the aim is to provide tools for an investor to make decisions on how to optimize his/her portfolio with respect to a benchmark portfolio. The problem will be formulated as an expected utility maximization problem of maximizing expected utility from terminal relative wealth of the investor. Relative wealth is thereby defined as the ratio between the investor's wealth process and the benchmark portfolio wealth process. In the literature there are various papers on benchmark related problems.

Browne (1999a) considers the problem of reaching an investment goal before a finite deadline. In another paper, Browne (1999b), extends his previous work by considering the problem of reaching a certain goal before falling below it to a predefined shortfall. Other objectives include minimizing the expected time to reach a performance goal, maximizing the expected reward obtained upon reaching the goal, as well as minimizing the expected penalty paid upon falling to the shortfall level. The model that he is using for the stock prices is that of Merton (1971), which is

$$dS_i(t) = S_i(t)\left[\alpha_i(t)dt + \sum_{j=1}^{k}\xi_{ij}(t)dW_j(t)\right], \tag{5.1}$$

for $i = 1, \ldots, k$. This is similar to the model (3.1) used in this thesis but with deterministic coefficients and without jumps. The wealth stochastic differential equation is then given by

$$dV_\pi^x(t) = V_\pi^x(t)\left[\left(r + \boldsymbol{\pi}(t)^\intercal\overline{\boldsymbol{\alpha}}\right)dt + \boldsymbol{\pi}(t)^\intercal\boldsymbol{\xi}d\boldsymbol{W}(t)\right] \tag{5.2}$$

which is the same as in (3.13) but without jumps and with deterministic coefficients. The benchmark, which can be a stock index, an inflation rate, an exchange rate, or any other kind of benchmark, is described by the stochastic differential equation

$$dY(t) = Y(t)\left[adt + \boldsymbol{b}^\intercal d\boldsymbol{W}(t) + \beta d\widetilde{W}(t)\right], \tag{5.3}$$

where $\widetilde{W}$ is a Brownian motion independent from the multidimensional Brownian motion $\boldsymbol{W}$. Browne then defines a ratio process by $Z(t) = V_\pi^x(t)/Y(t)$, which is normalized to start at 1 almost surely. This ratio process is then used in conjunction with stochastic control to find answers to the problems mentioned above. Typical for Browne's work is to work with an investment goal, than an investor tries to reach, and a shortfall level, which an investor tries to avoid. In a different paper Popova et al. (2007) also investigate the problem of maximizing the probability of beating a benchmark within some investment horizon. The authors also consider other problems related to minimizing the expected shortfall relative to a benchmark. However, in contrast to the papers by Browne the paper of Popova et al. (2007) does not specify a specific model for the stock prices but instead assumes that the returns are normal distributed.

The same benchmark model framework as in Browne (1999b) is used in a paper by Pra et al. (2004). They consider a benchmark tracking problem, where the benchmark process is given as in (5.3). The same ratio process $Z_\pi(t) = V_\pi^x(t)/Y(t)$ is used to minimize a cost function that punishes variation of the investor's wealth from the benchmark. The cost function is given in the form $C(z) = (z-1)^2$ so that the optimization problem, that they consider, is given as the problem of finding a trading strategy $\boldsymbol{\pi}$ that minimizes

$$\mathbb{E}\left[\int_0^T C\big(Z_\pi(t)\big)dt\right]$$

Similar to Browne (1999b) the way the problem is solved is by applying stochastic control techniques. The choice of constant parameters makes stochastic control a natural tool of choice. A different approach is chosen by Teplá (2001). In his paper the objective is to maximize expected utility of the terminal wealth of the investor under the constraint that the investor's wealth process always exceeds the benchmark process. All involved processes are modelled again as geometric Brownian motions as in Merton (1971). The problem is different from all the others in the sense that it is a constraint optimization problem.

A completely different starting point is taken by Fernholz (1999). Rather than focusing on optimization problems he investigates the relationship between a financial agent's wealth process and a benchmark process which is again a portfolio's wealth process. The wealth process of a financial agent is given thereby by

$$dV_\pi^x(t) = V_\pi^x(t)\left[\left(\boldsymbol{\pi}(t)^\intercal \boldsymbol{\alpha}(t)\right)dt + \boldsymbol{\pi}(t)^\intercal \boldsymbol{\xi}(t) d\boldsymbol{W}(t)\right].$$

Notice that this is slightly different than (5.2) since Fernholz only considers pure stock portfolios. Because there is no risk-less asset the drift is given by $\boldsymbol{\alpha}$ rather than the discounted version $\overline{\boldsymbol{\alpha}}$. However, apart from the lack of a risk-less asset this is a generalization of the models used in Browne (1999b) and Pra et al.(2004) since now the model parameters are time dependent and potentially non-deterministic. A benchmark strategy, denoted by $\boldsymbol{\eta}$, is a trading strategy like $\boldsymbol{\pi}$ and the benchmark process is then the portfolio wealth process of the strategy $\boldsymbol{\eta}$:

$$dV_\eta^x(t) = V_\eta^x(t)\left[\left(\boldsymbol{\eta}(t)^\intercal \boldsymbol{\alpha}(t)\right)dt + \boldsymbol{\eta}(t)^\intercal \boldsymbol{\xi}(t) d\boldsymbol{W}(t)\right].$$

Of central importance in Fernholz work is the concept of relative return. The relative return of the i^{th} stock price, driven by the stochastic differential equation (5.1) is defined for a benchmark portfolio $\boldsymbol{\eta}$ by $\log\left(S_i/V_\eta^x\right)$. Similarly, the relative return of two portfolio's $\boldsymbol{\pi}$ and $\boldsymbol{\eta}$ is defined by $\log\left(V_\pi^x/V_\eta^x\right)$, which is obviously independent of the choice of $x > 0$. In contrast to other authors who consider various optimization problems, Fernholz analyses general market behaviour. This includes the long term behaviour of the market, market diversity, and the relationship between stocks with small and high market capitalization (cf. Fernholz (1998), Fernholz (1999), Fernholz (2001)). When investigating market diversity he makes use of the entropy which favours the log framework. Also the analysis of the long term behaviour of the stock market and portfolios favours a log model since the long term behaviour of a portfolio's wealth process is just the drift of the log wealth. We shall use a similar form of relative return to that of Fernholz but will use the simple ratio process V_π^x/V_η^x instead of $\log\left(V_\pi^x/V_\eta^x\right)$, which is also more popular in the literature. Later we will call V_π^x/V_η^x the relative wealth process between $\boldsymbol{\pi}$ and $\boldsymbol{\eta}$. This layout is more appropriate when considering optimization problems in this chapter.

A different approach to benchmarks, pioneered by Platen (2006), is his so called benchmark approach. In his approach the wealth process of the growth optimal portfolio (GOP) is taken as a numeraire in the financial market. Financial derivatives can then be priced under the real world probability measure P, hence, the pricing concept is called real world asset pricing. For a portfolio $\boldsymbol{\pi}$ the wealth process V_π^x is driven by (5.2) which is the same as in (3.13) but without jumps. The growth rate of the wealth process is then the drift of the logarithm of V_π^x which can be read from the SDE

$$d\log(V_\pi^x)(t) = \left(r(t) + \boldsymbol{\pi}(t)^\intercal \overline{\boldsymbol{\alpha}}(t) - \boldsymbol{\pi}(t)^\intercal \boldsymbol{\sigma}(t)\boldsymbol{\pi}(t)\right)dt + \boldsymbol{\pi}(t)^\intercal \boldsymbol{\xi}(t)dW(t)$$

as $\mu_\pi(t) = r(t) + \boldsymbol{\pi}(t)^\intercal \overline{\boldsymbol{\alpha}}(t) - \boldsymbol{\pi}(t)^\intercal \boldsymbol{\sigma}(t)\boldsymbol{\pi}(t)$ and was defined in (3.18). If one chooses $\boldsymbol{\pi}$ such that the growth rate μ_π is maximized, then one obtains, by setting the first derivative of μ_π with respect to $\boldsymbol{\pi}$ equal to zero, the growth optimal portfolio by

$$\boldsymbol{\pi}^*(t) = \boldsymbol{\sigma}(t)^{-1}\overline{\boldsymbol{\alpha}}(t) = \boldsymbol{\xi}(t)^{-1}\boldsymbol{\lambda}(t), \tag{5.4}$$

where $\boldsymbol{\lambda}$ are the market prices of risk given by $\boldsymbol{\lambda}(t) := \boldsymbol{\xi}(t)^{-1}\overline{\boldsymbol{\alpha}}(t)$. The portfolio $\boldsymbol{\pi}^*$ as defined in (5.4) is called the growth optimal portfolio process and maximizes the long term performance of a wealth process (cf. Goll and Kallsen (2003), Long (1990), Artzner (1997)). Its wealth process $X = V_{\pi^*}^x$ satisfies the stochastic differential equation

$$dX(t) = X(t)\left[\left(r(t) + ||\boldsymbol{\lambda}(t)||^2\right)dt + \boldsymbol{\lambda}(t)d\boldsymbol{W}(t)\right].$$

This can be easily confirmed by substituting $\boldsymbol{\pi}^*$ into the wealth equation (5.2). Platen describes then benchmarking as the procedure of discounting asset prices using X as the numeraire. In this way a financial derivative $C(t)$ can be priced using the pricing rule

$$C(t) = X(t)\mathbb{E}\left[\frac{C(T)}{X(T)}\Big|\mathcal{F}_t\right], \quad t \in [0,T].$$

The important property of this pricing approach is that the equivalent martingale measure is the original real world measure P. Although Platen's approach is mainly used for pricing derivatives it has some interesting features that relate to the analysis carried out in this chapter. It will be referred back to when it becomes relevant.

In this chapter we are going to consider an optimization problem where an investor's aim is to outperform a financial benchmark. This benchmark is given by a jump diffusion process of the form

$$\frac{dV_B(t)}{V_B(t-)} = a(t)dt + \boldsymbol{b}(t)^\intercal d\boldsymbol{W}(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} c_h(t,y)N_h(dt,dy). \tag{5.5}$$

In contrast to Browne (1999b) we are not interested in reaching a performance goal, neither are we interested in solving a tracking problem as in Pro et al. (2004). Instead, the financial agent's aim is to maximize the expected utility from terminal relative wealth $V_\pi(T)/V_B(T)$.

To do so martingale methods are applied to transform the relative wealth process into a martingale. As will be seen in Section 5.3.1 and 5.3.2, this will relate to change of numeraire techniques and the benchmark approach pioneered by Platen (2005) respectively. The tools developed in Chapter 4 can then be applied to find a solution to the expected utility optimization problem, which is done in Section 5.4. Section 5.5 discusses the case of power utility functions. The chapter is concluded by linking the PDE results of Section 4.8 to the special case when the benchmark is the wealth process of some strategy $\boldsymbol{\eta}$. This is carried out in Section 5.6.

5.2 Benchmarks and the Relative Wealth Process

The stock price processes are assumed to follow a jump-diffusion model as introduced in Section 3.1. This means that the k stock prices follow the SDEs (3.1) and a risk-less asset is given in the market that is the solution to the ordinary differential equation (3.6). The wealth process of a financial agent that starts with initial endowment $x > 0$ and that follows a trading strategy $\boldsymbol{\pi}$ is then following the stochastic differential equation (3.13) which is given by

$$\frac{dV_\pi^x(t)}{V_\pi^x(t-)} = r(t)dt + \boldsymbol{\pi}(t)^\intercal \left[\overline{\boldsymbol{\alpha}}(t)dt + \boldsymbol{\xi}(t)d\boldsymbol{W}(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)N_h(dt,dy) \right],$$

for $t \in [0,T]$. The trading strategy $\boldsymbol{\pi}$ is assumed to be admissible in the sense of Definition 3.6. This means that $\boldsymbol{\pi} \in \Pi$ and, in particular, $\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y)N_h(dt,dy) \geq -1$ a.s. for all $h = 1,\dots,m$ and $t \in [0,T]$. There is no consumption in the model.

An investor's performance is compared to the performance of a benchmark process V_B, which is again a jump-diffusion process that satisfies the stochastic differential equation (5.5) and with initial condition $V_B(0) = x$. We made a slight change in notations. The n-dimensional Brownian motion in the wealth process has been denoted by $\boldsymbol{W}$ instead of $\boldsymbol{B}$. This is to avoid confusion since in this chapter B will be reserved for benchmark related notations.

To guarantee the existence and uniqueness of a solution to the benchmark process SDE (5.5) some integrability conditions are required (cf. (2.12)). For the process V_B these integrability conditions translate to

$$\int_0^T |a(t)|^2 dt < \infty, \quad \int_0^T ||\boldsymbol{b}(t)||^2 dt < \infty, \quad \text{and} \quad \int_0^T \int_{\mathbb{R}\setminus\{0\}} |c_h(t,y)|^2 \nu_h(dt,dy) < \infty.$$

for $h = 1,\dots,m$. As has been discussed in Section 3.3 and 3.4 some extra care has to be taken when working with jump-diffusion process if one wants to guarantee positivity of the process. From the discussions in the above two sections the jump parameters $\boldsymbol{c} = (c_1,\dots,c_m)^\intercal$ need to satisfy

$$\int_{\mathbb{R}\setminus\{0\}} c_h(t,y) N_h(dt,dy) > -1, \quad a.s., \quad t \in [0,T], \tag{5.6}$$

for all $h = 1,\dots,m$ to guarantee that the benchmark process V_B is almost surely positive.

In many cases the benchmark process V_B takes the form of a wealth process of a trading strategy $\boldsymbol{\eta}$. The model parameters a, $\boldsymbol{b}$, and $\boldsymbol{c}$ take then the form

$$\begin{aligned} a(t) &= r(t) + \boldsymbol{\eta}(t)^\intercal \overline{\boldsymbol{\alpha}}(t), \\ b(t) &= \boldsymbol{\xi}(t)^\intercal \boldsymbol{\eta}(t), \text{ and} \\ c_h(t,y) &= \boldsymbol{\eta}(t)^\intercal \boldsymbol{\gamma}_h(t,y), \quad y \in \mathbb{R}\setminus\{0\}. \end{aligned} \tag{5.7}$$

To ensure the positivity of the benchmark wealth (5.6) the benchmark portfolio strategy $\boldsymbol{\eta}$ needs to be in Π_+ as defined in Definition 3.6. In particular, the trading strategy $\boldsymbol{\eta}$ satisfies then $\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\eta}(t)^\intercal \boldsymbol{\gamma}_h(t,y) N_h(dt,dy) > -1$ a.s. for all $h = 1,\dots,m$ and $t \in [0,T]$.

Definition 5.1. *Let $\boldsymbol{\pi} \in \Pi$ be a trading strategy with corresponding wealth determined by the stochastic differential equation* (3.13) *and let V_B be a benchmark process as in* (5.5).

1. *The relative wealth process of a strategy $\boldsymbol{\pi} \in \Pi$ relative to the benchmark process V_B is defined as the process $V_{\pi/B}(t) := V_\pi^x(t)/V_B(t)$ for $t \in [0,T]$.*
2. *In case the benchmark V_B is represented by a portfolio process $\boldsymbol{\eta} \in \Pi_+$, the relative wealth of $\boldsymbol{\pi}$ relative to $\boldsymbol{\eta}$ is defined by $V_{\pi/\eta}(t) := V_\pi^x(t)/V_\eta^x(t)$ for $t \in [0,T]$ and for arbitrary $x > 0$.*

The relative wealth process is thus just the ratio process between the investor's wealth process and the benchmark process. It is always normalized in the sense that it starts almost surely at 1. To determine the dynamics of the relative wealth process $V_{\pi/B}$ it is necessary to determine the dynamics of the reciprocal process $1/V_B$. This is done by applying Itô's formula Theorem 2.14 on V_B and the function $f(x) = 1/x$. Then $f'(x) = -1/x^2$, $f''(x) = 2/x^3$, and $f(x(1+c)) - f(x) = -\frac{c}{x(1+c)}$, and therefore

$$
\begin{aligned}
d\frac{1}{V_B(t)} &= \frac{1}{V_B(t-)}\big(-a(t) + \boldsymbol{b}(t)^\intercal \boldsymbol{b}(t)\big)dt - \frac{1}{V_B(t-)}\boldsymbol{b}(t)^\intercal d\boldsymbol{W}(t) \\
&\quad - \frac{1}{V_B(t-)}\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \frac{c_h(t,y)}{1+c_h(t,y)} N_h(dt,dy).
\end{aligned}
$$

By Itô's product rule Lemma 2.18, the dynamics of the product $V_{\pi/B} = V_\pi^1 \cdot 1/V_B$ are

$$
\begin{aligned}
\frac{dV_{\pi/B}(t)}{V_{\pi/B}(t-)} &= \Big(r(t) - a(t) + \boldsymbol{b}(t)^\intercal \boldsymbol{b}(t) + \boldsymbol{\pi}(t)^\intercal\big(\overline{\boldsymbol{\alpha}}(t) - \boldsymbol{\xi}(t)\boldsymbol{b}(t)\big)\Big)dt \\
&\quad + \big(\boldsymbol{\pi}(t)^\intercal \boldsymbol{\xi}(t) - \boldsymbol{b}(t)^\intercal\big)d\boldsymbol{W}(t) + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y) - c_h(t,y)}{1+c_h(t,y)} N_h(dt,dy).
\end{aligned}
\tag{5.8}
$$

The term in the jump-part of the above expression might need some explanation. From the product rule Lemma 2.9 the jump terms are given by

$$
\Delta\big(X(t)Y(t)\big) = X(t-)\Delta Y(t) + Y(t-)\Delta X(t) + \Delta X(t)\Delta Y(t).
$$

Thus, for each $h = 1, \dots, m$ the jump sizes must satisfy

$$
\boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y) - \frac{c_h(t,y)}{1+c_h(t,y)} - \frac{c_h(t,y)\boldsymbol{\pi}(t)\gamma_h(t,y)}{1+c_h(t,y)} = \frac{\boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y) - c_h(t,y)}{1+c_h(t,y)}.
$$

Comparing the relative wealth equation (5.8) with the wealth equation (3.13) one can see that we again obtain a stochastic differential equation that is driven by Brownian motions and Poisson random measure jumps. In the relative wealth equation there are now parts of the dynamics that can not be controlled by the trading strategy $\boldsymbol{\pi}$. However, it will be shown in (5.14) that if the benchmark is actually a wealth process of a trading strategy $\boldsymbol{\eta}$, then one can transform the relative wealth process into a wealth process stochastic differential equation.

Following the previous considerations about the almost surely positivity of the benchmark and the non-negativity of the investor's wealth process, the following lemma can be formed.

Lemma 5.2. *The relative wealth process $V_{\pi/B}$ of a strategy $\boldsymbol{\pi} \in \Pi$ relative to a benchmark process V_B is non-negative. In particular, for all $t \in [0,T]$*

$$
\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y) - c_h(t,y)}{1+c_h(t,y)} N_h(dt,dy) \geq -1, \quad a.s. \tag{5.9}
$$

Proof. Since the relative wealth process is defined as the ratio of an non-negative process and a positive process respectively it cannot be negative almost surely. If (5.8) is non-negative then (5.9) must almost surely hold. □

Remark.

The above result appears at first rather counter-intuitive. To get an intuitive understanding why it holds consider the informal argument. Since the benchmark jumps are bigger -1 it must be that $c_h(t,y) > -1$. But then $1 + c_h(t,y) > 0$ and so $-c_h(t,y)/(1 + c_h(t,y) > -1$ since $-c_h(t,y) > -1 - c_h(t,y)$. That $\boldsymbol{\pi}(t)^\intercal \gamma_h(t,y) \geq -1$ is clear since $\boldsymbol{\pi} \in \Pi$. Thus, it also holds that $\boldsymbol{\pi}(t)^\intercal \gamma_h(t,y)/(1 + c_h(t,y) \geq -1$ since $1 + c_h(t,y) > 0$.

For the case that the benchmark is the wealth process of a portfolio $\boldsymbol{\eta}$, i.e. the model parameters satisfy (5.7), the relative wealth $V_{\pi/\eta}$ follows the stochastic differential equation

$$\begin{aligned}\frac{dV_{\pi/\eta}(t)}{V_{\pi/\eta}(t-)} &= \big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\big)^\intercal \Big[\big(\overline{\boldsymbol{\alpha}}(t) - \boldsymbol{\sigma}(t)\boldsymbol{\eta}(t)\big)dt + \boldsymbol{\xi}(t)d\boldsymbol{W}(t) \\ &\quad + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{\gamma}_h(t,y)}{1 + \boldsymbol{\eta}(t)^\intercal \boldsymbol{\gamma}_h(t,y)} N_h(dt,dy)\Big].\end{aligned} \tag{5.10}$$

This can be immediately verified by substituting (5.7) into (5.8). Following Platen's (2002) terminology it is now possible to perform benchmarking. Benchmarking is the procedure of taking V_B as the numeraire and expressing all asset prices in its terms. For each $i = 1, \ldots, k$, the i^{th} stock price relative to the benchmark V_B is defined by $V_{i/B} := S_i/V_B$. The stock prices are determined by the dynamics (3.1). Itô's product rule Lemma 2.18 can then be applied to dS_i and $d1/V_B$ to derive the dynamics of the relative stock prices by

$$\begin{aligned}dV_{i/B}(t) = V_{i/B}(t-) &\Bigg[\Big(r(t) - a(t) + \overline{\alpha}_i(t) + ||\boldsymbol{b}(t)||^2 - \sum_{j=1}^{n} \xi_{ij}(t) b_j(t)\Big)dt \\ &+ \sum_{j=1}^{n} \Big(\xi_{ij}(t) - b_j(t)\Big)dB_j(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{\gamma_{ih}(t,y) - c_h(t,y)}{1 + c_h(t,y)} N_h(dt,dy)\Bigg]\end{aligned} \tag{5.11}$$

Further, the risk-less asset is given by

$$dS_0(t) = S_0(t) r(t) dt, \quad t \in [0,T].$$

Thus, the risk-less asset relative to the benchmark V_B is defined by $V_{0/B} = S_0/V_B$ and follows the dynamics

$$dV_{0/B}(t) = V_{0/B}(t-)\left[\big(r(t) - a(t) + ||\boldsymbol{b}(t)||^2\big)dt - \boldsymbol{b}(t)d\boldsymbol{B}(t)\right.$$
$$\left. - \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{c_h(t,y)}{1+c_h(t,y)} N_h(dt,dy)\right], \quad t \in [0,T] \tag{5.12}$$

The relative wealth process $V_{\pi/B}$ can be related to the relative stock price process via the equation

$$\frac{dV_{\pi/B}(t)}{V_{\pi/B}(t-)} = \Big(1 - \sum_{i=1}^{k} \pi_i(t)\Big)\frac{dV_{0/B}(t)}{V_{0/B}(t)} + \sum_{i=1}^{k} \pi_i(t)\frac{dV_{i/B}(t)}{V_{i/B}(t)}, \tag{5.13}$$

for any $x > 0$. Thus, its return is the weighted averaged (with possibly negative weights) of the benchmarked asset returns. This, is a similar expression as has been obtained in (3.14). However, now the part that was taken by the deterministic risk-less asset dynamics dS_0/S_0 in (3.14) is now given by the benchmarked risk-less asset $dV_{0/B}/V_{0/B}$ in (5.13), which is driven by a stochastic differential equation which means in particular that it is non-deterministic.

If the benchmark process is given by the wealth process of a trading strategy $\boldsymbol{\eta} \in \Pi_+$, some interesting relationships can be established to the model discussed in Chapter 3. First, if the benchmark portfolio only invests into the risk-less asset, i.e. $\boldsymbol{\eta} \equiv 0$, then the benchmark wealth process is equal to the discounted wealth process as given in (3.15). The risk-less asset S_0 is then the benchmark. Second, since the model parameters and $\boldsymbol{\eta}$ are assumed to be exogenous given processes, the relative wealth process $V_{\pi/\eta}$ as in (5.10) can be related to the wealth process SDE (3.13). The two processes coincide if one chooses the following parameters in the wealth process equation:

$$\widetilde{r} = 0, \quad \widetilde{\overline{\boldsymbol{\alpha}}} = \overline{\boldsymbol{\alpha}} - \boldsymbol{\sigma}\boldsymbol{\eta}, \quad \widetilde{\boldsymbol{\xi}} = \boldsymbol{\xi}, \quad \text{and } \widetilde{\gamma}_{ih} = \frac{\gamma_{ih}(t,y)}{1 + \boldsymbol{\eta}(t)^\intercal \boldsymbol{\gamma}_h(t,y)}, \tag{5.14}$$

$x = 1$ and the trading strategy is $\widetilde{\boldsymbol{\pi}} = \boldsymbol{\pi} - \boldsymbol{\eta}$.

5.3 Relative Wealth under Martingale Measures

The change of measure methods that have been applied to discounted asset prices in Section 3.2 can also be applied to the benchmark framework. The aim is to find a set of EMMs Q represented by Girsanov kernels $\boldsymbol{\theta}$ under which the relative asset prices are (local) martingales. As in Section 3.2 this is done by introducing Q-Brownian motions $d\boldsymbol{W}^Q$ and Q-compensated Poisson measures $\widetilde{N}_h^Q(dt,dy)$ as has been done in (2.21) and (2.22) respectively. One can then eliminate the non-martingale drift term when substituting $d\boldsymbol{W}^Q$ and $\widetilde{N}_h^Q(dt,dy)$ into the relative asset price SDEs (5.11) and (5.12).

Consider first the benchmarked risk-less asset price process $V_{0/B}$ that has the dynamics as given in (5.12). The non-martingale drift should be zero, thus, substituting $d\boldsymbol{W}(t) = d\boldsymbol{W}^Q(t) + \boldsymbol{\theta}^D(t)dt$ and $N_h(dt, dy) = \widetilde{N}_h^Q(dt, dy) + \theta_h^J(t, y)$ for $h = 1, \ldots, m$ into (5.12) it is required that

$$r(t) - a(t) + ||\boldsymbol{b}(t)||^2 - \boldsymbol{b}(t)^\intercal \boldsymbol{\theta}^D(t) - \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{c_h(t,y)}{1 + c_h(t,y)} \theta_h^J(t,y)\nu_h(dt, dy) = 0, \quad (5.15)$$

for $V_{0/B}$ to become a Q-martingale. If this condition is satisfied then the Q-dynamics of $V_{0/B}$ are given by

$$dV_{0/B}(t) = V_{0/B}(t-) \left[-\boldsymbol{b}(t) d\boldsymbol{B}^Q(t) - \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{c_h(t,y)}{1 + c_h(t,y)} \widetilde{N}_h^Q(dt, dy) \right],$$

for $t \in [0, T]$. Taking the same considerations for $V_{i/B}$, a condition on the Girsanov kernels of a measure change under which the benchmarked stock processes $V_{i/B}$ as given in (5.11) are martingales, is again when the non-martingale drift is zero. Thus,

$$\begin{aligned} r(t) - a(t) + \overline{\alpha}_i(t) + ||\boldsymbol{b}(t)||^2 - \sum_{j=1}^{n} \xi_{ij}(t) b_j(t) + \sum_{j=1}^{n} \left(\xi_{ij}(t) - b_j(t)\right) \theta_j^D(t) \\ + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{\gamma_{ih}(t,y) - c_h(t,y)}{1 + c_h(t,y)} \theta_h^J(t,y) \nu_h(dt, dy) = 0. \end{aligned}$$

However, taking also into account that the Girsanov kernels $\boldsymbol{\theta}$ should satisfy (5.15), because of the risk-less asset, the above condition simplifies to

$$\overline{\alpha}_i(t) + \sum_{j=1}^{n} \xi_{ij}(t) \left(\theta_j^D(t) - b_j(t)\right) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{\gamma_{ih}(t,y)}{1 + c_h(t,y)} \theta_h^J(t,y)\nu_h(dt, dy) = 0, \quad (5.16)$$

for all $i = 1, \ldots, k$ and $t \in [0, T]$. If the two conditions (5.15) and (5.16) are satisfied then the relative stock prices $V_{i/B}$ become martingales and have dynamics

$$\begin{aligned} dV_{i/B}(t) = V_{i/B}(t-) \Bigg[& \sum_{j=1}^{n} \Big(\xi_{ij}(t) - b_j(t)\Big) dB_j^Q(t) \\ & + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{\gamma_{ih}(t,y) - c_h(t,y)}{1 + c_h(t,y)} \widetilde{N}_h^Q(dt, dy) \Bigg], \quad t \in [0, T], \end{aligned}$$

for $i = 1, \ldots, k$. As in Definition 3.1 let us define the B-Girsanov kernels as follows.

Definition 5.3. *A B-Girsanov kernel is a pair of predictable vector processes $\boldsymbol{\theta} = \left(\boldsymbol{\theta}^D, \boldsymbol{\theta}^J\right)$ that satisfies (i) and (ii) in Definition 3.1, and for which conditions* (5.15) *and* (5.16) *are satisfied:*

$$\begin{aligned} a(t) - r(t) - ||\boldsymbol{b}(t)||^2 + \boldsymbol{b}(t)^\intercal \boldsymbol{\theta}^D(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{c_h(t,y)}{1 + c_h(t,y)} \theta_h^J(t,y)\nu_h(dt, dy) = 0, \\ \overline{\alpha}_i(t) + \sum_{j=1}^{n} \xi_{ij}(t) \left(\theta_j^D(t) - b_j(t)\right) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{\gamma_{ih}(t,y)}{1 + c_h(t,y)} \theta_h^J(t,y)\nu_h(dt, dy) = 0, \end{aligned} \quad (5.17)$$

a.s. for all $i = 1, \dots, k$ and $t \in [0, T]$. The set of all B-Girsanov kernels is denoted by Θ_B.

Thus, the B-Girsanov kernel is the benchmark equivalent to the Girsanov kernel from Definition 3.1. The main difference is that condition (3.8) has been changed into (5.17). Clearly, for any such B-Girsanov kernel the relative wealth process $V_{\pi/B}$ of a portfolio $\boldsymbol{\pi} \in \Pi$ is also a martingale under the EMM Q. It's dynamics are given by

$$\begin{aligned} dV_{\pi/B}(t) &= V_{\pi/B}(t-)\big(\boldsymbol{\pi}(t)^\intercal \boldsymbol{\xi}(t) - \boldsymbol{b}(t)^\intercal\big) d\boldsymbol{W}^Q(t) \\ &\quad + V_{\pi/B}(t-) \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{\{\boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y)\} - c_h(t,y)}{1 + c_h(t,y)} \widetilde{N}_h^Q(dt, dy), \end{aligned} \tag{5.18}$$

for $t \in [0, T]$. This can be easily verified by observing (5.13). The above SDE (5.18) is the benchmark equivalent to the discounted wealth process without consumption that is a martingale under appropriate measure change (cf. (3.25)). For the case that the benchmark is the risk-less asset, i.e. $a = r$, $\boldsymbol{b} \equiv 0$, $c_h \equiv 0$, the two equations (5.18) and (3.25) coincide.

For the special case that the benchmark is a wealth process of a portfolio $\boldsymbol{\eta} \in \Pi_+$ the relative wealth process is clearly also a Q-martingale as in (5.18). Additionally, condition (5.17) simplifies.

Corollary 5.4. *Let $\boldsymbol{\theta} \in \Theta_B$ be a B-Girsanov kernel with corresponding EMM Q. If the benchmark is the wealth process of a benchmark strategy $\boldsymbol{\eta} \in \Pi_+$, then the relative wealth dynamics are given by*

$$\begin{aligned} dV_{\pi/\eta}(t) &= V_{\pi/\eta}(t-)\big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\big)^\intercal \boldsymbol{\xi}(t) d\boldsymbol{W}^Q(t) \\ &\quad + V_{\pi/\eta}(t-) \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{\big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\big)^\intercal \boldsymbol{\gamma}_h(t,y)}{1 + \boldsymbol{\eta}(t)^\intercal \boldsymbol{\gamma}_h(t,y)} \widetilde{N}_h^Q(dt, dy). \end{aligned} \tag{5.19}$$

Further, the conditions (5.17) *on $\boldsymbol{\theta} \in \Theta_B$ simplify to the single condition*

$$\overline{\boldsymbol{\alpha}}(t) - \boldsymbol{\sigma}(t)\boldsymbol{\eta}(t) + \boldsymbol{\xi}(t)\boldsymbol{\theta}^D(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{\gamma}_h(t,y)\theta_h^J(t,y)}{1 + \boldsymbol{\eta}(t)^\intercal \boldsymbol{\gamma}_h(t,y)} \nu_h(dt, dy) = 0, \tag{5.20}$$

a.s. for $i = 1, ..., k$ and $t \in [0, T]$.

Proof. The equation (5.19) can be derived by substituting (5.7) into (5.18). Similarly, (5.20) is derived by substituting (5.7) into (5.16). Using the same substitution (5.15) becomes

$$\begin{aligned} &\boldsymbol{\eta}(t)^\intercal \left(\overline{\boldsymbol{\alpha}}(t) - \boldsymbol{\sigma}(t)\boldsymbol{\eta}(t) + \boldsymbol{\xi}(t)\boldsymbol{\theta}^D(t)\right) \\ &+ \sum_{i=1}^{k} \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{\eta_i(t)\gamma_{ih}(t,y)\theta_h^J(t,y)}{1 + \sum_{j=1}^{n} \eta_j(t)\gamma_{ih}(t,y)} \nu_h(dt, dy) = 0. \end{aligned}$$

But this is satisfied if (5.20) is satisfied. □

This section is concluded by providing a budget constraint similar to the one in Corollary 3.10.

Corollary 5.5. *Let $\boldsymbol{\theta} \in \Theta_B$ be a B-Girsanov kernel with corresponding martingale measure Q. If $\boldsymbol{\pi} \in \Pi$ then*

$$\mathbb{E}^Q\big[V_{\pi/B}(T)\big] \leq 1. \tag{5.21}$$

Proof. Because of Lemma 5.2 $V_{\pi/B}$ is a.s. bounded below by zero under P. Since P and Q are equivalent the same is true under Q. According to (5.18) is also a local Q-martingale. Thus, $V_{\pi/B}$ is a supermartingale and (5.21) follows from the supermartingale property. □

This budget constraint is a similar one as found in Corollary 3.10. However, this constraint is for benchmarks and has naturally the constraint 1 as all benchmark processes are assumed to start at 1 a.s. Also there is no discounting needed since the benchmark process acts as the numeraire.

5.3.1 Relationship to Change of Numeraire Techniques

In this section we analyse the relationship between the measure transformation of Chapter 3 and 4 to the measure change carried out in this chapter. The transition of the EMM in Chapter 3 to the EMM of this chapter is often called a change of numeraire in the literature. In the previous chapters the numeraire was given by the risk-less asset S_0, whereas in this chapter the numeraire is a benchmark process V_B. The set of martingale measure has been parametrized by the so called jump-diffusion Girsanov kernels $\boldsymbol{\theta}$ from Definition 3.1 in Chapter 3. The Girsanov kernels thereby had to satisfy the crucial condition (3.8). In contrast, in this chapter the measure has been changed such that the benchmarked asset prices are martingales. The set of Girsanov kernels $\boldsymbol{\theta}$ that parametrize such a measure change has been called B-Girsanov kernels and have been defined in Definition 5.3. The main difference between the two Girsanov kernels is that the B-Girsanov kernels have to satisfy the two conditions in (5.17), instead of (3.8).

For the purpose of the following considerations let Q_0 denote EMMs obtained from a Girsanov kernels under which the discounted asset prices are martingales, and let Q_B denote EMMs obtained from B-Girsanov kernels under which the benchmarked asset prices are martingales. What has been done in Chapter 3 and 4 is that the original probability measure P has been changed to Q_0, and in this chapter the measure has been changed from P to Q_B. However, what really happened is that the numeraire has changed from S_0 to V_B. Instead of changing measure from P to eiter Q_0 or Q_B it is also possible to change the measure directly from Q_0 to Q_B. This is a technique, developed by Geman et al. (1995), called the change of numeraire and provides a way to make the transition from the measure Q_0 to the measure Q_B. We briefly explain the technique for our model set up.

From change of numeraire methods as introduced in Geman et al. (1995), it is known that the martingale $Z_{0,B}$ that transforms the probability measure from Q_0 to Q_B via

$$\frac{dQ_B}{dQ_0}\Big|_{\mathcal{F}_T} = Z_{0,B}(T)$$

is given by

$$Z_{0,B}(t) := \frac{S_0(0)}{V_B(0)}\frac{V_B(t)}{S_0(t)} = \frac{1}{x}\frac{V_B(t)}{S_0(t)}, \quad t \in [0,T],$$

for some $x > 0$. $Z_{0,B}$ is the discounted benchmark process, normalized to start at 1. Its differential form can be deduces from (3.22) as being

$$\begin{aligned} dZ_{0,B}(t) &= Z_{0,B}(t-)\boldsymbol{b}(t)^{\intercal} d\boldsymbol{W}^{Q_0} + Z_{0,B}(t-)\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} c_h(t,y)\widetilde{N}_h^{Q_0}(dt,dy); \\ Z_{0,B}(0) &= 1. \end{aligned} \tag{5.22}$$

These are the dynamics of the density that makes the transition from the probability measure Q_0 under which the discounted stock prices are martingales to the probability measure Q_B under which the relative stock prices are martingales. Rewriting (5.22) in terms of its Girsanov kernel $\boldsymbol{\theta}_{0,B} = \left(\boldsymbol{\theta}_{0,B}^D, \boldsymbol{\theta}_{0,B}^J\right)$ (cf. (2.23)), the above stochastic differential equation becomes

$$dZ_{0,B}(t) = Z_{0,B}(t-)\boldsymbol{\theta}_{0,B}^D(t)^{\intercal} d\boldsymbol{W}^{Q_0} - Z_{0,B}(t-)\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \left(1-\theta_{(0,B),h}^J(t,y)\right)\widetilde{N}_k^{Q_0}(dt,dy)$$

But comparing this with (5.22) shows that the Girsanov kernel for the measure transformation from Q_0 to Q_B must be given by

$$\boldsymbol{\theta}_{0,B}^D(t) := \boldsymbol{b}(t), \quad \text{and} \quad \theta_{(0,B),h}^J(t,y) = 1 + c_h(t,y) \tag{5.23}$$

for $t \in [0,T]$, $y \in \mathbb{R}\setminus\{0\}$, and $h = 1,\ldots,m$. Thus, given a Q_0 standard Brownian motion $\boldsymbol{W}^{Q_0}$, a Q_B standard Brownian motion is constructed by

$$\boldsymbol{W}^{Q_B}(t) := \boldsymbol{W}^{Q_0}(t) - \int_0^t \boldsymbol{\theta}_{0,B}(s)ds, \quad t \in [0,T], \tag{5.24}$$

and the Q_B compensated Poisson random measures are given by

$$\widetilde{N}_h^{Q_B}(dt,dy) := N_h^{Q_0}(dt,dy) - \theta_{(0,B),h}^J(t,y)\nu_h^{Q_0}(dy)dt, \quad h = 1,\ldots,m. \tag{5.25}$$

Thus, the new Brownian motion $\boldsymbol{W}^{Q_B}(t)$ of (5.24) and the discounted Poison random measures of (5.25) enables us to change the equivalent measure from Q_0 to Q_B. Let us recall what different measure changes we have made. Under the original probability measure P the relative wealth process $V_{\pi/B}$ follows the dynamics as given in (5.8). If the relative wealth process were expressed in Q_0-measure dynamics it would be

$$\begin{aligned}\frac{dV_{\pi/B}(t)}{V_{\pi/B}(t-)} = &\Big[r(t) - a(t) + ||\boldsymbol{b}(t)||^2 - \big(\boldsymbol{\xi}(t)\boldsymbol{\pi}(t) + \boldsymbol{\theta}_0^D(t)\big)^\intercal \boldsymbol{b}(t)\\ &-\sum_{h=1}^{k}\int_{\mathbb{R}\setminus\{0\}} \frac{c_h(t,y)}{1+c_h(t,y)}\left(1+\boldsymbol{\pi}(t)^\intercal\boldsymbol{\gamma}_h(t,y)\right)\theta_{0,h}^J(t,y)\nu_h(dy)\Big]\,dt\\ &+\big(\boldsymbol{\pi}(t)^\intercal\boldsymbol{\xi}(t) - \boldsymbol{b}(t)^\intercal\big)d\boldsymbol{W}^{Q_0}(t)\\ &+\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{\pi}(t)^\intercal\boldsymbol{\gamma}_h(t,y) - c_h(t,y)}{1+c_h(t,y)}\widetilde{N}_h^{Q_0}(dt,dy).\end{aligned}$$

If, however, the substitution (5.24) and (5.25) are carried out to change the numeraire from S_0 to V_B (thus changing the measure from Q_0 to Q_B), then the relative wealth process dynamics transform from the above SDE into the familiar form (5.18).

5.3.2 Relationship to Platen's Benchmark Approach

In the following we are going to link our work to the benchmark approach pioneered by Platen (2005). In Platen's work the growth optimal portfolio plays a central role. This is the portfolio that maximizes the growth rate of a portfolio, which is the expected log wealth. It has been shown in the literature that such a portfolio has many interesting properties. One of them is, for example, that it maximizes the long term performance of a portfolio (cf. Goll and Kallsen (2003)). Platen considers the stock market model driven by Brownian motions but without jump processes. He is working with the portfolio that maximizes growth, which is, in a no-jump model given by $\boldsymbol{\eta}^*(t) = \boldsymbol{\sigma}(t)^{-1}\overline{\boldsymbol{\alpha}}(t) = \big(\boldsymbol{\xi}(t)^\intercal\big)^{-1}\boldsymbol{\lambda}(t)$, were $\boldsymbol{\lambda}(t) := \boldsymbol{\xi}(t)^{-1}\overline{\boldsymbol{\alpha}}(t)$ is called the market price of risk. He then takes the wealth process of the GOP $\boldsymbol{\eta}^*$ and takes it as the numeraire when pricing financial assets. He calls the change of numeraire from S_0 to $V_{\pi^*}^x$ benchmarking. A financial contract C can then be priced using the general pricing rule

$$\frac{C(t)}{V_{\eta^*}^1(t)} = \mathbb{E}\left[\frac{C(T)}{V_{\eta^*}^1(T)} \mid \mathcal{F}_t\right], \quad t\in[0,T]. \tag{5.26}$$

The very interesting property of this method of asset pricing is that the EMM is actually the real world measure P, which is why he is calling it real world pricing.

As mentioned before, Platen's work is carried out assuming that stock prices are driven by Brownian motions and the market is complete. Our aim here is to extend these results to the incomplete market case with jump-diffusions. The satisfactory answer is that this is possible.

Recall from equation (4.31) that the optimal strategy for optimizing expected power wealth is

$$\overline{\boldsymbol{\alpha}}(t) - (1-\beta)\boldsymbol{\sigma}(t)\widehat{\boldsymbol{\pi}}(t) + \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)\{1+\widehat{\boldsymbol{\pi}}(t)^{\intercal}\boldsymbol{\gamma}_h(t,y)\}^{\beta-1}\nu_h(dy) = 0.$$

a.s. for $h = 1, \ldots, m$. Thus, the portfolio $\boldsymbol{\eta}^*$ that maximises log terminal wealth ($\beta = 0$) has to satisfy

$$\overline{\boldsymbol{\alpha}}(t) - \boldsymbol{\sigma}(t)\boldsymbol{\eta}^*(t) + \int_{\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{\gamma}_h(t,y)}{1+\boldsymbol{\eta}^*(t)^{\intercal}\boldsymbol{\gamma}_h(t,y)}\nu_h(dy) = 0, \quad \text{a.s.} \tag{5.27}$$

In Platen's results (cf. Platen (2005)), which does not include jumps, the asset processes and wealth process, that are benchmarked by the wealth process of the GOP, are martingales under the real world measure P. Let us verify that the same is true in a jump-diffusion market. It follows from (4.30) that the optimal measure for the expected log wealth maximization problem is given by the Girsanov kernels

$$\boldsymbol{\theta}^{*,D}(t) = -\xi(t)\eta^*(t), \quad \text{and} \quad \boldsymbol{\theta}_h^{*,J}(t,y) = \frac{1}{1+\boldsymbol{\eta}^*(t)^{\intercal}\boldsymbol{\gamma}_h(t,y)} \tag{5.28}$$

From (4.68) we can derive the optimal wealth process of the power utility terminal wealth problem if the optimal EMM is known. Substituting (5.28) into (4.68) and observing that $\widetilde{H}_\theta \equiv 1$ for $\beta = 0$ shows that

$$V_{\eta^*}^x(t) = x\frac{1}{H_{\theta^*}(t)}, \quad t \in [0,T].$$

From (3.3) and (5.28) the wealth process of the GOP $\boldsymbol{\eta}^*$ is therefore given by

$$\begin{aligned} V_{\eta^*}^x(t) = x\exp\Bigg(&\int_0^t r(s)ds + \frac{1}{2}\int_0^t ||\boldsymbol{\eta}^*(t)^{\intercal}\boldsymbol{\xi}(t)||^2 ds + \int_0^t \boldsymbol{\eta}^*(t)^{\intercal}\boldsymbol{\xi}(t)d\boldsymbol{B}(t) \\ &+\sum_{h=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} \log\left(1+\boldsymbol{\eta}^*(s)^{\intercal}\boldsymbol{\gamma}_h(s)\right) N_h(ds,dy) \\ &-\sum_{h=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{\eta}^*(s)^{\intercal}\boldsymbol{\gamma}_h(s,y)}{1+\boldsymbol{\eta}^*(s)^{\intercal}\boldsymbol{\gamma}_h(s,y)}\nu_h(dy)\Bigg), \quad t \in [0,T]. \end{aligned}$$

It can also be expresses in differential form, where it looks almost similar to the wealth process of a trading strategy

$$dV_{\eta^*}^x(t) = V_{\eta^*}^x(t-)\left[r(t)dt + \boldsymbol{\eta}^*(t)^{\intercal}\boldsymbol{\xi}(t)d\boldsymbol{B}(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\eta}^*(t)^{\intercal}\gamma_h(t,y)\widetilde{N}_h(dt,dy)\right].$$

As a matter of fact, this is the evolution of a wealth process under risk-neutrality when the risk-less asset is chosen as the numeraire. We summarize our analysis in the next proposition.

Proposition 5.6. *If the benchmark process is given by the wealth process of the GOP* $\boldsymbol{\eta}^*$ *as implicitly defined in* (5.27)*, then all benchmarked (relative) asset prices are martingales under the real world measure* P*:*

$$V_{0/\boldsymbol{\eta}^*}(t) = \frac{1}{x} H_{\theta^*}(t) S_0(t) = \frac{1}{x} Z_{\theta^*}(t),$$
$$V_{i/\boldsymbol{\eta}^*}(t) = \frac{1}{x} H_{\theta^*}(t) S_i(t) = \frac{1}{x} Z_{\theta^*}(t) \overline{S}_i(t),$$

for $i = 1, \ldots, k$ *and* $t \in [0, T]$*. Further, for a given portfolio* $\boldsymbol{\pi} \in \Pi(x)$ *the relative wealth process is a martingale under* P*:*

$$V_{\pi/\eta^*}(t) = \frac{1}{x} H_{\theta^*}(t) V_\pi^x(t) = \frac{1}{x} Z_{\theta^*}(t) \overline{V}_\pi^x(t).$$

Using Platen's terminology one can now perform real world pricing in the jump-diffusion market. That is if there is a financial claim C, we can use the growth optimal portfolio as the numeraire and derive the pricing rule (5.26). We refer to Platen (2005) for more details on the case where the stocks are driven by Brownian motions only.

5.4 Martingale Approach to Active Portfolio Selection

An investor is interested in finding a trading strategy that maximizes expected utility from terminal generalized relative wealth. That is to find

$$\Phi_B = \sup_{\pi \in \widetilde{\mathcal{A}}_B} \mathbb{E}\big[U\big(V_{\pi/B}(T)\big)\big], \tag{5.29}$$

whereby

$$\widetilde{\mathcal{A}}_B = \big\{\boldsymbol{\pi} \in \Pi \,|\, \mathbb{E}\big[\min\big\{U\big(V_{\pi/B}(T)\big), 0\big\}\big] > -\infty\big\} \tag{5.30}$$

represents the set of admissible trading strategies for the above problem. As usual, the optimization problem in (5.29) will be referred to by Φ_B.

The way Φ_B is solved is similar to the procedure in Chapter 4. First, an auxiliary random variable $Y_B^{\boldsymbol{\theta}}$ is introduced for each B-Girsanov kernel $\boldsymbol{\theta}$. Each of these random variables performs at least as good as any admissible trading strategy when comparing expected utility from the auxiliary random variable and terminal relative wealth:

$$\mathbb{E}[U(Y_B^{\boldsymbol{\theta}})] \geq \mathbb{E}[U\big(V_{\pi/B}(T)\big)], \quad \boldsymbol{\pi} \in \widetilde{\mathcal{A}}_B.$$

Then under some conditions a replication strategy $\boldsymbol{\pi} \in \widetilde{\mathcal{A}}_B$ can be found that replicated $Y_B^{\boldsymbol{\theta}}$ for some $\boldsymbol{\theta} \in \Theta_B$. Since non of the $Y_B^{\boldsymbol{\theta}}$ underperforms any admissible terminal relative wealth the replication strategy must create terminal relative wealth that is not outperformed by any other admissible relative wealth, thus is optimal.

Let Q denote the EMM associated to the B-Girsanov kernel $\boldsymbol{\theta} \in \Theta_B$ as defined in Definition 5.3. The analogue function to $\mathcal{X}_\theta$ as defined in (4.5) is for this problem defined by

$$\mathcal{X}_B^\theta(y) := \mathbb{E}^Q\big[I\big(yZ_B^\theta(T)\big)\big], \quad y > 0, \tag{5.31}$$

for a B-kernel Girsanov $\boldsymbol{\theta} \in \Theta_B$. The set of $\boldsymbol{\theta} \in \Theta_B$ for which the above function is finite is denoted by $\widetilde{\Theta}_B$ and is assumed to be non-empty:

$$\widetilde{\Theta}_B := \big\{\boldsymbol{\theta} \in \Theta_B | \mathcal{X}_B^\theta(y) < \infty, \text{ for } y > 0\big\}.$$

Then for each $\boldsymbol{\theta} \in \widetilde{\Theta}_B$ the function $\mathcal{X}_B^\theta$ is continuous and maps $(0, \infty)$ into itself. Further, $\mathcal{X}_B^\theta$ is strictly decreasing and satisfies $\mathcal{X}_B^\theta(0+) = \infty$ and $\mathcal{X}_B^\theta(\infty) = 0+$. This means in particular that there exists a unique $\widehat{y} > 0$ for which $\mathcal{X}_B^\theta(\widehat{y}) = 1$.

This function $\mathcal{X}_B^\theta$ allows us to define a random variable Y_B^θ with some desirable properties. Among others these are that Y_B^θ never underperforms relative wealth (in the expected utility sense) of any admissible trading strategy. The following lemma can be proven in an analogue way as Lemma 4.1.

Lemma 5.7. *Let $\boldsymbol{\theta} \in \widetilde{\Theta}_B$ with associated measure Q. Define a random variable $Y_B^\theta := I\big(\widehat{y}Z_B^\theta(T)\big)$ for $\widehat{y}$ as described above, then*

(i) $\mathbb{E}^Q[Y_B^\theta] = 1$;

(ii) $\mathbb{E}\big[U(Y_B^\theta)^-\big] > -\infty$;

(iii) $\mathbb{E}[U\big(V_{\pi/B}(T)\big)] \leq \mathbb{E}\big[U(Y_B^\theta)\big], \quad \forall \boldsymbol{\pi} \in \widetilde{\mathcal{A}}_B$.

Remarks.

1. Lemma 5.7 (i) act as a budget constraint similar to be found in (5.21), giving an indication that every replication strategy should be admissible.

2. Item (ii) guarantees that any replication strategy for Y_B^θ is actually in the admissible set $\widetilde{\mathcal{A}}_B$ from equation (5.30).

3. An inequality that immediately follows (iii) is that for all $\boldsymbol{\pi} \in \widetilde{\mathcal{A}}_B$

$$\mathbb{E}[U\big(V_{\pi/B}(T)\big)] \leq \sup_{\widetilde{\pi} \in \widetilde{\mathcal{A}}_B} \mathbb{E}[U\big(V_{\widetilde{\pi}/B}(T)\big)] = \Phi_B \leq \mathbb{E}\big[U(Y_B^\theta)\big]. \tag{5.32}$$

   Thus, Y_B^θ never underperforms the terminal relative wealth of the optimal trading strategy.

The above Lemma is the version of Lemma 4.1 tailored to the benchmark problem $\mathbf{\Phi}_B$ in (5.29). We have constructed a well behaved random variable Y_B^{θ} that performs at least as well as any admissible terminal relative wealth for the problem Φ_B. If it is possible to find a trading strategy whose relative wealth replicates for some $\boldsymbol{\theta} \in \widetilde{\Theta}_B$ the random variable Y_B^{θ} then this trading strategy should be optimal for the problem Φ_B.

The objective is to find a trading strategy $\boldsymbol{\pi}$ and a B-Girsanov kernel $\boldsymbol{\theta}$ for which equality holds in equation (5.32). Such a B-Girsanov kernel should attain the infimum on the right hand side of (5.32). If it exists it will be called the optimal B-Girsanov kernel for problem Φ_B. Similarly to Definition 3.1 we make for the benchmark framework the following definition.

Definition 5.8. *An equivalent martingale measure Q represented by $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}_B$ is called optimal for the problem Φ_B in* (5.29) *if it satisfies*

$$\mathbb{E}\big[U(Y_B^{\widehat{\theta}})\big] = \inf_{\theta \in \widetilde{\Theta}_B} \mathbb{E}\big[U(Y_B^{\theta})\big].$$

Our aim is to find such an optimal martingale measure and to find a replication strategy that leads to terminal relative wealth Y_B^{θ} for some optimal $\widehat{\boldsymbol{\theta}}$. Thus, we need to define a martingale M_B^{θ} for each $\boldsymbol{\theta} \in \widetilde{\Theta}_B$ which is the benchmark equivalent to M_θ in (4.8). Its martingale representation coefficients can then be used to construct a condition on the optimal trading strategy and optimal B-Girsanov kernel. To do this define for $\boldsymbol{\theta} \in \widetilde{\Theta}_B$ the martingale M_B^{θ} by

$$M_B^{\theta}(t) := \mathbb{E}\big[Z_B^{\theta}(T) Y_B^{\theta} | \mathcal{F}_t\big], \quad t \in [0, T]. \tag{5.33}$$

Following the martingale representation Theorem 2.13 there must exist processes $\boldsymbol{a}^D$ and $\boldsymbol{a}^J$ that satisfy the integrability condition as in Theorem 2.13 and for which

$$M_B^{\theta}(t) = 1 + \int_0^t \boldsymbol{a}^D(s)^\intercal d\boldsymbol{W}(s) + \sum_{h=1}^m \int_0^t \int_{\mathbb{R}\setminus\{0\}} a_h^J(s, y) \widetilde{N}_h(ds, dy). \tag{5.34}$$

These martingale representation coefficients can be used to give conditions for a replication strategy and an optimal EMM. The next theorem is the benchmark equivalent to Theorem 4.3. The proof is done in the same manner as in Theorem 4.3 but will be carried out to clarify where the following optimality conditions on the optimal trading strategy and optimal B-Girsanov kernel are coming from.

Theorem 5.9. *If $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}_B$ and $\widehat{\boldsymbol{\pi}} \in \widetilde{\mathcal{A}}_B$ satisfy for all $h = 1, \ldots, m$ and $t \in [0, T]$*

$$\begin{aligned} \boldsymbol{\xi}(t)^\intercal \widehat{\boldsymbol{\pi}}(t) - \boldsymbol{b}(t) &= \frac{1}{M_B^{\widehat{\theta}}(t-)} \boldsymbol{a}^D(t) - \widehat{\boldsymbol{\theta}}^D(t), \\ \frac{\widehat{\boldsymbol{\pi}}(t)^\intercal \boldsymbol{\gamma}_h(t, y) - c_h(t, y)}{1 + c_h(t, y)} &= \frac{1}{\widehat{\theta}_h^J(t, y)} \left(\frac{a_h^J(t, y)}{M_B^{\widehat{\theta}}(t-)} + 1 - \widehat{\theta}_h^J(t, y) \right), \end{aligned} \tag{5.35}$$

for ν_h almost surely all $y \in \mathbb{R} \setminus \{0\}$, $h = 1, \ldots, m$, then $\widehat{\boldsymbol{\pi}}$ is an optimal strategy for Φ_B and

$$V_{\widehat{\pi}/B}(T) = Y_B^{\widehat{\theta}}. \tag{5.36}$$

Thereby is M_B^{θ} the martingale defined in (5.33) for $\boldsymbol{\theta} \in \widetilde{\Theta}_B$ and $\boldsymbol{a}^D$ and $\boldsymbol{a}^J$ are its martingale representation coefficients as in (5.34). Further, $\widehat{\boldsymbol{\theta}}$ represents the optimal EMM for the problem Φ_B. The optimal relative wealth process is given by

$$V_{\widehat{\pi}/B}(t) = V_B^{\widehat{\theta}}(t) = \mathbb{E}^Q\left[Y_B^{\widehat{\theta}} \mid \mathcal{F}_t\right], \quad t \in [0,T], \tag{5.37}$$

where $V_B^{\widehat{\theta}}$ is defined in (5.38).

Proof. We show that if $\boldsymbol{\pi}$ and $\boldsymbol{\theta}$ are satisfying (5.35), then $\boldsymbol{\pi}$ is indeed replicating Y_B^{θ}. If this is the case, then because of (5.32), $\boldsymbol{\pi}$ must be optimal (given that it is admissible for the problem) and equality holds in (5.32). Define for $\boldsymbol{\theta} \in \widetilde{\Theta}_B$ the process

$$V_B^{\theta}(t) := \frac{1}{Z_B^{\theta}(t)} M_B^{\theta}(t), \quad t \in [0,T]. \tag{5.38}$$

Then from Itô's product rule, equation (5.34), and (2.24), the process V_B^{θ} follows the stochastic differential equation

$$\begin{aligned}\frac{dV_B^{\theta}(t)}{V_B^{\theta}(t-)} = &\left\{||\boldsymbol{\theta}^D(t)||^2 - \frac{1}{M_B^{\theta}(t-)}\boldsymbol{a}^D(t)^{\intercal}\boldsymbol{\theta}^D(t) - \sum_{h=1}^{m}\left(\frac{a_h^J(t,y)}{M_B^{\theta}(t-)} + 1 - \theta_h^J(t)\right)\nu_h(dy)\right\}dt \\ &+ \left(\frac{1}{M_B^{\theta}(t-)}\boldsymbol{a}^D(t) - \boldsymbol{\theta}^D(t)\right)^{\intercal} d\boldsymbol{W}(t) \\ &+ \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \frac{1}{\theta_h^J(t)}\left(\frac{a_h^J(t,y)}{M_B^{\theta}(t-)} + 1 - \theta_h^J(t)\right) N_h(dt,dy).\end{aligned}$$

Under the to $\boldsymbol{\theta}$ associated Q measure the process V_B^{θ} becomes a martingale so that

$$\begin{aligned}\frac{dV_B^{\theta}(t)}{V_B^{\theta}(t-)} = &\left(\frac{1}{M_B^{\theta}(t-)}\boldsymbol{a}^D(t) - \boldsymbol{\theta}^D(t)\right)^{\intercal} d\boldsymbol{W}^Q(t) \\ &+ \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \frac{1}{\theta_h^J(t)}\left(\frac{a_h^J(t,y)}{M_B^{\theta}(t-)} + 1 - \theta_h^J(t)\right) N_h^Q(dt,dy).\end{aligned} \tag{5.39}$$

Notice further that V_B^{θ} has the properties that $V_B^{\theta}(0) = 1$ a.s. and $V_B^{\theta}(T) = Y_B^{\theta}$ a.s. Since all involved stochastic differential equations were assumed to have a unique solution, a relative wealth process $V_{\pi/B}$ associated to the strategy $\boldsymbol{\pi}$ is equal to V_B^{θ} (and is therefore building a replication portfolio), if the two stochastic differential equations (5.18) and (5.39) are equal. This, however, is exactly the case if (5.35) holds. Finally, the representation (5.37) follows from (5.36) since $V_{\widehat{\pi}/B}$ is a martingale under the optimal EMM Q. Also observe that $V_{\widehat{\pi}/B}(T) = Y_B^{\widehat{\theta}}$ a.s. □

The above theorem gives a tool to determine the optimal trading strategy for the investment problem Φ_B. It is the benchmark equivalent of Theorem 4.3. However, the main difference between the above theorem and Theorem 4.3 is that the benchmark Girsanov kernels have to satisfy two conditions (cf. (5.17)) as opposed to only one condition given by (3.8). This makes it more difficult to find cases of utility functions for which more explicit conditions on the optimal strategy and optimal Girsanov kernel can be given. Fortunately, one class of utility functions for which this is the case are the already familiar power utility functions. They will be discussed in more detail in Section 5.5

Corollary 5.10. *Let the same assumptions as in Theorem 5.9 be satisfied. For the case that the benchmark is the wealth pocess of the benchmark strategy $\boldsymbol{\eta} \in \Pi_+$ the optimality conditions on $(\widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\pi}})$ in (5.35) change to*

$$\boldsymbol{\xi}(t)^\intercal\big(\widehat{\boldsymbol{\pi}}(t) - \boldsymbol{\eta}(t)\big) = \frac{1}{M_B^{\widehat{\theta}}(t-)}\boldsymbol{a}^D(t) - \widehat{\boldsymbol{\theta}}^D(t),$$

$$\frac{\big(\widehat{\boldsymbol{\pi}}(t) - \boldsymbol{\eta}(t)\big)^\intercal\boldsymbol{\gamma}_h(t,y)}{1 + \boldsymbol{\eta}(t)^\intercal\boldsymbol{\gamma}_h(t,y)} = \frac{1}{\widehat{\theta}_h^J(t,y)}\left(\frac{a_h^J(t,y)}{M_B^{\widehat{\theta}}(t-)} + 1 - \widehat{\theta}_h^J(t,y)\right),$$

for ν_h almost surely all $y \in \mathbb{R}\setminus\{0\}$, $h = 1, \dots, m$.

5.5 The Case of Power and Logarithmic Utility

Consider a power utility function $U(x) = \frac{x^\beta}{\beta}$ for some $\beta < 1$, with the usual convention that that $\frac{x^\beta}{\beta} = \log(x)$ for $\beta = 0$. If I denotes the inverse of the first derivative of U, then it is given as $I(y) = y^{\frac{1}{\beta-1}}$ for $y > 0$.

Proposition 5.11. *Let $\beta < 1$. If $\widehat{\boldsymbol{\pi}} \in \widetilde{\mathcal{A}}_B$ satisfies*

$$\begin{aligned}\Big(\overline{\boldsymbol{\alpha}}(t) + (\beta-1)\boldsymbol{\sigma}(t)\widehat{\boldsymbol{\pi}} - \beta\boldsymbol{\xi}(t)\boldsymbol{b}(t)\Big)_i + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,z)\frac{(1+\boldsymbol{\pi}(t)^\intercal\boldsymbol{\gamma}_h(t,z))^{\beta-1}}{(1+c_h(t,z))^\beta}\nu_h(dz) &= 0,\\ a(t) - r(t) + (\beta-1)\boldsymbol{b}(t)^\intercal\boldsymbol{\xi}(t)\widehat{\boldsymbol{\pi}}(t) - \beta\boldsymbol{b}(t)^\intercal\boldsymbol{b}(t) \qquad\qquad & \\ + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} c_h(t,z)\frac{(1+\widehat{\boldsymbol{\pi}}(t)^\intercal\boldsymbol{\gamma}_h(t,z))^{\beta-1}}{(1+c_h(t,z))^\beta}\nu_h(dz) &= 0,\end{aligned} \tag{5.40}$$

a.s. for all $i = 1, \dots k$ and $t \in [0,T]$, then it is optimal for the problem Φ_B with power or logarithmic utility $U(x) = \frac{x^\beta}{\beta}$ (and $U(x) = \log(x)$ if $\beta = 0$).

Proof. To derive an optimality condition for the strategy in problem Φ_B it is necessary to compute $\mathcal{X}_B^\theta$, Y_B^θ, and M_B^θ as defined in (5.31), in Lemma 5.7, and in (5.33) respectively. For given $\boldsymbol{\theta} \in \widetilde{\Theta}_B$ the function $\mathcal{X}_B^\theta$ is given for the power utility case by

$$\mathcal{X}_B^\theta(y) = \mathbb{E}^Q\big[I\big(yZ_B^\theta(T)\big)\big] = y^{\frac{1}{\beta-1}}\mathbb{E}\big[Z_B^\theta(T)^{\frac{\beta}{\beta-1}}\big], \quad y > 0.$$

The process $Z_B^\theta(T)^{\frac{\beta}{\beta-1}}$ can be factorized into a martingale part $\zeta_M(t)$ and a non-martingale part $\zeta_N(t)$ so that $Z_B^\theta(T)^{\frac{\beta}{\beta-1}} = \zeta_M(t)\zeta_N(t)$. The function $\mathcal{X}_B^\theta$ then simplifies to $\mathcal{X}_B^\theta(y) = y^{\frac{1}{\beta-1}}\mathbb{E}[\zeta_N(T)]$, so that the $\widehat{y}$ that solves the equation $\mathcal{X}_B^\theta(\widehat{y}) = 1$ is then given by

$$\widehat{y} := \frac{1}{\mathbb{E}[\zeta_N(T)]^{\beta-1}}.$$

The number $\widehat{y}$ can be used to derive an expression for the optimal terminal relative wealth of the auxiliary problem, Y_B^θ, as defined in Lemma 5.7. Substituting I and $\widehat{y}$ into the definition of Y_B^θ determines it as

$$Y_B^\theta = I\big(\widehat{y}Z_B^\theta(T)\big) = \widehat{y}^{\frac{1}{\beta-1}}Z_B^\theta(T)^{\frac{1}{\beta-1}}.$$

What is left is to determine the martingale M_B^θ from (5.33):

$$M_B^\theta(t) = \mathbb{E}\big[Z_B^\theta(T)Y_B^\theta|\mathcal{F}_t\big] = \widehat{y}^{\frac{1}{\beta-1}}\mathbb{E}\big[Z_B^\theta(T)^{\frac{\beta}{\beta-1}}|\mathcal{F}_t\big] = \widehat{y}^{\frac{1}{\beta-1}}\mathbb{E}[\zeta_N(T)]\zeta_M(t) = \zeta_M(t)$$

for $t \in [0,T]$. As $\zeta_M(t)$ is the martingale part of $Z_B^\theta(t)^{\frac{\beta}{\beta-1}}$ it can be easily seen that it follows the stochastic differential equation

$$d\zeta_M(t) = \zeta_M(t-)\frac{\beta}{\beta-1}\boldsymbol{\theta}^D(t)^\intercal d\boldsymbol{W}(t) + \zeta_M(t-)\sum_{h=1}^m\int_{\mathbb{R}\setminus\{0\}}\Big(\theta_h^J(t,z)^{\frac{\beta}{\beta-1}} - 1\Big)\widetilde{N}_h(dt,dz).$$

Thus, the martingale representation coefficients of M_B^θ are given by

$$\boldsymbol{a}^D(t) = M_B^\theta(t-)\frac{\beta}{\beta-1}\boldsymbol{\theta}^D(t), \quad \text{and} \quad a_h^J(t,z) = M_B^\theta(t-)\Big(\theta_h^J(t,z)^{\frac{\beta}{\beta-1}} - 1\Big)$$

for $t \in [0,T]$, $z \in \mathbb{R}\setminus\{0\}$, and $h = 1,\dots m$. Substituting the above martingale coefficients into (5.35) shows that the optimal B-Girsanov kernel $\widehat{\boldsymbol{\theta}}$ and the optimal trading strategy $\widehat{\boldsymbol{\pi}}$ should satisfy

$$\begin{aligned}\boldsymbol{\xi}(t)^\intercal\widehat{\boldsymbol{\pi}}(t) - \boldsymbol{b}(t) &= \frac{1}{\beta-1}\widehat{\boldsymbol{\theta}}^D(t)\\ \frac{\widehat{\boldsymbol{\pi}}(t)^\intercal\boldsymbol{\gamma}_h(t,y) - c_h(t,y)}{1+c_h(t,y)} &= \Big(\widehat{\theta}_h^J(t,y)^{\frac{1}{\beta-1}} - 1\Big),\end{aligned}$$

for $h = 1,\dots,m$. Solving the above equations with respect to $\boldsymbol{\theta}^D$ and $\boldsymbol{\theta}^J$ one obtains

$$\begin{aligned}\widehat{\boldsymbol{\theta}}^D(t) &= (\beta-1)\left(\boldsymbol{\xi}(t)^\intercal\widehat{\boldsymbol{\pi}}(t) - \boldsymbol{b}(t)\right), \quad \text{and}\\ \widehat{\theta}_h^J(t,y) &= \left(\frac{1+\widehat{\boldsymbol{\pi}}(t)\boldsymbol{\gamma}_h(t,y)}{1+c_h(t,y)}\right)^{\beta-1},\end{aligned} \tag{5.41}$$

for $h = 1,\dots,m$ and ν_h-almost surely all $y \in \mathbb{R}\setminus\{0\}$. However, a B-Girsanov kernel has also to satisfy the additional constrains (5.17). Substituting the above optimal $\widehat{\boldsymbol{\theta}}^D$ and $\widehat{\theta}_h^J$, $h = 1,\dots,m$ into these conditions leads to the optimality condition (5.40). $\square$

It is not always guaranteed to find an optimal trading strategy $\boldsymbol{\pi}$ that satisfies the optimality condition in (5.40). In the following some conditions are considered under which it can be guaranteed that an optimal trading strategy exists for the power utility case. Consider first the case where there are no jumps in the wealth process of the investor and in the benchmark process. Then the optimality conditions in (5.40) become

$$\begin{aligned} \overline{\boldsymbol{\alpha}}(t) + (\beta - 1)\boldsymbol{\sigma}(t)\widehat{\boldsymbol{\pi}} - \beta\boldsymbol{\xi}(t)\boldsymbol{b}(t) = 0, \quad & \text{and} \\ a(t) - r(t) + (\beta - 1)\boldsymbol{b}(t)^{\intercal}\boldsymbol{\xi}(t)\widehat{\boldsymbol{\pi}}(t) - \beta\boldsymbol{b}(t)^{\intercal}\boldsymbol{b}(t) = 0, \quad & \text{a.s.,} \quad t \in [0, T]. \end{aligned} \tag{5.42}$$

Lemma 5.12. *Let there be no jumps in the market and the benchmark. If the drift of the benchmark satisfies*

$$a(t) = r(t) + \boldsymbol{b}(t)^{\intercal}\boldsymbol{\xi}^{-1}(t)\overline{\boldsymbol{\alpha}}(t), \tag{5.43}$$

then the optimal trading strategy for the relative wealth problem Φ_B *in* (5.29) *with power utility* $U(x) = x^{\beta}/\beta$, $\beta < 1$, *is given by*

$$\widehat{\boldsymbol{\pi}}(t) = \frac{1}{1-\beta}\boldsymbol{\sigma}^{-1}(t)\left(\overline{\boldsymbol{\alpha}}(t) - \beta\boldsymbol{\xi}(t)\boldsymbol{b}(t)\right), \quad t \in [0, T]. \tag{5.44}$$

Independent from (5.43), *for the case that the benchmark follows a trading strategy* $\boldsymbol{\eta} \in \mathcal{A}_{+}$, *the optimal trading strategy is*

$$\widehat{\boldsymbol{\pi}}(t) = \frac{1}{1-\beta}\left(\boldsymbol{\sigma}^{-1}(t)\overline{\boldsymbol{\alpha}}(t) - \beta\boldsymbol{\eta}(t)\right), \quad t \in [0, T]. \tag{5.45}$$

Proof. Solving the first equation in (5.42) with respect to $\widehat{\boldsymbol{\pi}}$ gives (5.44). Substiting the solution of the first equation, i.e. (5.44), into the second equation in (5.42) leads to the condition (5.43). For the case that the benchmark is a wealth process of a strategy $\boldsymbol{\eta}$ the condition is (5.43) is automatically satisfied and $\boldsymbol{b}(t) = \boldsymbol{\xi}(t)^{\intercal}\boldsymbol{\eta}(t)$ as in (5.7). □

The form of the optimal trading strategy $\widehat{\boldsymbol{\pi}}$ in (5.44) for the power utility case without jumps takes a very similar form as the solution in the non-benchmark framework, which was given in (4.32) by

$$\widehat{\boldsymbol{\pi}} = \frac{1}{(1-\beta)}\boldsymbol{\sigma}(t)^{-1}\overline{\boldsymbol{\alpha}}(t).$$

Consider now the case that the benchmark is given as the wealth process of the GOP, thus $\boldsymbol{\eta}(t) = \boldsymbol{\sigma}(t)^{-1}\overline{\boldsymbol{\alpha}}(t)$ as of (5.4). Then from (5.45) the optimal trading strategy is also the GOP $\widehat{\boldsymbol{\pi}}(t) = \boldsymbol{\sigma}(t)^{-1}\overline{\boldsymbol{\alpha}}(t)$. Thus, if an investor has preferences in form of power utility and the benchmark is given by the GOP, then he/she should also choose the GOP as optimal trading strategy. This is quite interesting as it supports arguments as in Goll and Kallsen (2000), Kelly (1956), and other, that argue the GOP is the best portfolio of a rational investor.

If there are instead of the Brownian motion parts jump processes in the model the stock market model becomes a jump-model with drift. In that case the optimality condition in (5.40) is given by

$$\begin{aligned}
\overline{\alpha}_i(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,z) \frac{(1+\boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,z))^{\beta-1}}{(1+c_h(t,z))^\beta} \nu_h(dz) &= 0,\\
a(t) - r(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} c_h(t,z) \frac{(1+\widehat{\boldsymbol{\pi}}(t)^\intercal \boldsymbol{\gamma}_h(t,z))^{\beta-1}}{(1+c_h(t,z))^\beta} \nu_h(dz) &= 0,
\end{aligned}$$

a.s. for $i = 1, \dots, k$. For this to become a simpler condition in form of for example only one equation, the c_h should somehow relate to the k jump sizes γ_{ih}. This is the case if c_h is a linear combination of the γ_{ih} such that $c_h(t,y) = \sum_{i=1}^k \mu_i(t)\gamma_{ih}(t,y)$ for some $\mu_1, \dots, \mu_k$. But then it is also necessary that $a(t) - r(t) = \sum_{i=1}^k \mu_i(t)\overline{\alpha}_i(t)$ such that $\boldsymbol{\mu}$ is actually a benchmark strategy (cf. (5.7)). If this is the case then the condition on the optimal strategy simplifies to

$$a(t) - r(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} c_h(t,z) \frac{(1+\widehat{\boldsymbol{\pi}}(t)^\intercal \boldsymbol{\gamma}_h(t,z))^{\beta-1}}{(1+c_h(t,z))^\beta} \nu_h(dz) = 0.$$

For the very special case when there is only one risky asset in the market and with only one jump process with fixed jump size γ and intensity ν the optimal trading strategy can be written in explicit form as

$$\widehat{\pi}(t) = \frac{1}{\gamma(t)} \left(\left(\frac{r(t) - a(t)}{c(t)\nu} \right)^{1-\beta} (1 + c(t)) - 1 \right), \quad t \in [0,T].$$

Thus, we have seen that in the presents of jumps a simplified condition on the optimal trading strategy can be given when the benchmark is a portfolio wealth process. In this case the drift has to satisfy $a(t) - r(t) = \boldsymbol{\eta}(t)^\intercal \overline{\boldsymbol{\alpha}}(t)$. From the consideration on the no-jump case the drift also has to satisfy the condition $a(t) - r(t) = \boldsymbol{b}(t)^\intercal \boldsymbol{\xi}^{-1}(t)\overline{\boldsymbol{\alpha}}(t)$ as in (5.43). However, if both of these condition should be satisfied then the diffusion term must become $\boldsymbol{b}(t) = \boldsymbol{\xi}(t)^\intercal \boldsymbol{\eta}(t)$.

Corollary 5.13. *Let $\beta < 1$. If the benchmark is given by the wealth process of the trading strategy $\boldsymbol{\eta} \in \Pi_+$ and $\widehat{\boldsymbol{\pi}} \in \widetilde{\mathcal{A}}_B$ satisfies*

$$\begin{aligned}
&\overline{\boldsymbol{\alpha}}(t) - \beta\boldsymbol{\sigma}(t)\boldsymbol{\eta}(t) - (1-\beta)\boldsymbol{\sigma}(t)\widehat{\boldsymbol{\pi}}(t)\\
&\qquad + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \frac{(1+\widehat{\boldsymbol{\pi}}(t)^\intercal \boldsymbol{\gamma}_h(t,y))^{\beta-1}}{(1+\boldsymbol{\eta}(t)^\intercal \boldsymbol{\gamma}_h(t,y))^\beta} \boldsymbol{\gamma}_h(t,y)\nu_h(dy) = 0,
\end{aligned} \tag{5.46}$$

then $\widehat{\boldsymbol{\pi}}$ is optimal for the problem $\Phi_B(x)$ for the power utility case $U(x) = x^\beta/\beta$.

5.6 Stochastic Control and Benchmark Portfolios

Let us consider the case when a benchmark process is given by a wealth process of a portfolio $\boldsymbol{\eta} \in \Pi_+$, so that the relative wealth process $V_{\pi/\eta}$ has dynamics as given in (5.10). This can be extended to a relative wealth process $V^{s,v}_{\pi/\eta}(t)$ that starts at v at time $s \in [0,T]$. Then $V^{s,v}_{\pi/\eta}$ satisfies also the stochastic differential equation (5.10) for $t \in [s,T]$ with initial condition that $V^{s,v}_{\pi/\eta}(s) = 1$.

From the discussion at the end of Section 5.2 the relative wealth can be transformed into the wealth process stochastic differential equation if the substitutions (5.14) are carried out. In such a case and if the trading strategy is $\widetilde{\boldsymbol{\pi}} = \boldsymbol{\pi} - \boldsymbol{\eta}$, then the relative wealth process is equivalent to

$$\frac{dV^{s,v}_{\widetilde{\pi}}(t)}{V^{s,v}_{\widetilde{\pi}}(t-)} = \widetilde{r}(t)dt + \widetilde{\boldsymbol{\pi}}(t)^{\intercal}\left[\widetilde{\widetilde{\boldsymbol{\alpha}}}(t)dt + \widetilde{\boldsymbol{\xi}}(t)d\boldsymbol{B}(t) + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \widetilde{\boldsymbol{\gamma}}_h(t,y)N_h(dt,dy)\right],$$

for $t \in [s,T]$.

Using this parallel between wealth process and relative wealth process it is possible to transfer the results on partial differential equations from Section 4.8 to an extended relative wealth problem where the parameters are Markov processes and the starting point of the relative wealth is given by v at time $s \in [0,T]$. In this section it is assumed that all model parameters $\boldsymbol{\alpha}$, r, $\boldsymbol{\xi}$, $\boldsymbol{\gamma}$, as well as the benchmark strategy $\boldsymbol{\eta}$ are Markov processes depending on time and current relative wealth v. Further, the set of B-Girsanov kernels is also assumed to contain only Markov kernels, i.e. kernels that are Markov processes. Since we only going to work with benchmark strategies $\boldsymbol{\eta}$ in this section, let us call the B-Girsanov kernels $\boldsymbol{\eta}$-Girsanov kernels to emphasis the dependency on $\boldsymbol{\eta}$.

Define for $(s,v) \in [0,T] \times (0,\infty)$ the optimization problem of maximizing terminal relative wealth when starting in (s,v) by

$$\Phi_\eta(s,v) := \sup_{\boldsymbol{\pi} \in \mathcal{A}_\eta(s,v)} \mathbb{E}\left[U\left(V^{s,v}_{\pi/\eta}(T)\right)\right], \tag{5.47}$$

where the set of admissible trading strategies is defined for the problem by

$$\mathcal{A}_\eta(s,v) := \left\{\boldsymbol{\pi} \in \Pi \mid \quad \mathbb{E}\left[\min\{0, U(V^{s,v}_{\pi/\eta}(T)\} > -\infty\right]\right\}.$$

As in Section 4.8 the function $\mathcal{X}^{\theta}_{B}$ in (5.31) is extended to accommodate time dependency. Thus for $(s,y) \in [0,T] \times (0,\infty)$ define

$$\mathcal{X}^{\theta}_{\eta}(s,y) := \mathbb{E}^{Q}\left[I\left(Z^{s,y}_{\eta,\theta}(T)\right)\right], \quad y > 0.$$

Thereby is $Z^{s,y}_{\eta,\theta}$ the Radon-Nikodym density of the change in probability measure from P to Q of the $\boldsymbol{\eta}$-Girsanov kernel $\boldsymbol{\theta} \in \Theta_\eta$ that starts almost surely at y at time s. The

η-Girsanov kernel is thereby chosen out from the set for which the function $\mathcal{X}_\eta^\theta(s,y)$ has an inverse which is

$$\widetilde{\Theta}_{T,\eta} := \left\{ \boldsymbol{\theta} \in \Theta_\eta \mid \mathcal{X}_\eta^\theta(s,y) < \infty, \quad (s,y) \in [0,T] \times (0,\infty) \right\}.$$

The state space inverse of the function $\mathcal{X}_\eta^\theta(s,y)$ will be denoted by $\mathcal{Y}_\eta^\theta(s,v)$ for $(s,v) \in [0,T] \times (0,\infty)$. In the $\boldsymbol{\theta} \in \widetilde{\Theta}_{T,\eta}$ auxiliary market let

$$Y_{\eta,\theta}^{s,v} = I\left(Z_\theta^{s,y}(T)\right) \tag{5.48}$$

be the auxiliary terminal relative wealth. This construction of $Y_{\eta,\theta}^{s,v}$ is directly taken over from (4.71). The optimal equivalent martingale measure for the problem $\Phi_\eta(s,v)$ in (5.47) can then be defined in the following way, similar to Definition 4.21.

Definition 5.14. *A martingale measure Q obtained by $\frac{dQ}{dP}|_{\mathcal{F}_T} = Z_\theta(T)$ in terms of $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}_{T,\eta}$ is called optimal for the optimization problem $\Phi_\eta(s,v)$ in (5.47) if it satisfies*

$$\mathbb{E}\left[U\left(Y_{\eta,\widehat{\theta}}^{s,v}\right)\right] = \inf_{\theta \in \widetilde{\Theta}_{T,\eta}} \mathbb{E}\left[U\left(Y_{\eta,\theta}^{s,v}\right)\right],$$

where $Y_{\eta,\theta}^{s,v}$ is defined as in (5.48).

The results on partial differential equations from Section 4.8 can now be applied to the problem $\Phi_\eta(s,v)$ of (5.47) using the transform (5.14). For the optimal trading strategy and the optimal measure of the problem $\Phi_\eta(s,v)$ the equivalent condition as in Lemma 4.22 can be derived.

Lemma 5.15. *Let $\widehat{\boldsymbol{\theta}}$ represent the optimal equivalent martingale measure for the problem $\Phi_\eta(0,x)$ in (5.47) for $x > 0$. Then the optimal trading strategy $\widehat{\boldsymbol{\pi}}$ can be written in feedback form $\widehat{\boldsymbol{\pi}}(s,v)$ and satisfies for $(s,v) \in [0,T] \times (0,\infty)$ the equations*

$$\begin{aligned} (\widehat{\boldsymbol{\pi}}(s,v) - \boldsymbol{\eta}(s))^\intercal \boldsymbol{\xi}(s) &= \frac{1}{v} \frac{\mathcal{Y}_\eta^{\widehat{\theta}}(s,v)}{\partial_x \mathcal{Y}_\eta^{\widehat{\theta}}(s,v)} \widehat{\boldsymbol{\theta}}^D(s), \\ \frac{(\widehat{\boldsymbol{\pi}}(s,v) - \boldsymbol{\eta}(s))^\intercal \boldsymbol{\gamma}_h(s,z)}{1 + \boldsymbol{\eta}(s)^\intercal \boldsymbol{\gamma}_h(s,z)} &= \frac{1}{v} \mathcal{X}_{\widehat{\theta}}\left(s, \widehat{\theta}_h^J(s,z) \mathcal{X}_{\widehat{\theta}}^{-1}(s,v)\right) - 1, \end{aligned} \tag{5.49}$$

for all $z \in \mathbb{R} \setminus \{0\}$ and $h = 1, \ldots, m$.

Thus, the optimal trading strategy is again Markovian and so the optimal relative wealth process is also a Markov process. Using the same transformations (5.14), the main result from Section 4.8, the Hamilton-Jacobi-Bellman equation, can also be transferred to the benchmark problem.

Theorem 5.16. *Let $\Phi_\eta(s,v)$ be the optimal performance function of the optimal terminal wealth problem specified in* (5.47). *Then Φ_η satisfies the Hamilton-Jacobi-Bellman equations of stochastic programming*

$$\begin{aligned}\partial_s\Phi(s,v) + \max_{\boldsymbol{\pi}\in\mathbb{R}^k}\Big\{ v\big(\,(\boldsymbol{\pi}(s)-\boldsymbol{\eta}(s))^\intercal\,(\overline{\boldsymbol{\alpha}}(s)-\boldsymbol{\sigma}(s)\boldsymbol{\eta}(s))\,\big)\partial_v\Phi(s,v)\\ +\frac{1}{2}||\,(\boldsymbol{\pi}(s)-\boldsymbol{\eta}(s))^\intercal\,\boldsymbol{\xi}(s)||^2v^2\partial_{vv}\Phi(s,v)\\ +\sum_{h=1}^m\int_{\mathbb{R}\setminus\{0\}}\left[\Phi\left(s,v\frac{1+\boldsymbol{\pi}(s)^\intercal\boldsymbol{\gamma}_h(s,z)}{1+\boldsymbol{\eta}(s)^\intercal\boldsymbol{\gamma}_h(s,z)}\right)-\Phi(s,v)\right]\nu_h(dz)\Big\}=0,\end{aligned}\tag{5.50}$$

on the set $[0,T)\times(0,\infty)$ and satisfies the boundary condition

$$\Phi_\eta(T,v)=U(v),\quad v>0.$$

Proof. The Hamilton-Jacobi-Bellman equation (5.50) follows from Theorem 4.24 and the substitution (5.14). The term in the jump-part of the HJB equation (5.50) follows from

$$1+\widetilde{\gamma}_h^{\widehat{\pi}}(s,z)=1+\widetilde{\boldsymbol{\pi}}(s)^\intercal\widetilde{\gamma}_h(s,z)=1+\frac{(\boldsymbol{\pi}(s)-\boldsymbol{\eta}(s))^\intercal\,\boldsymbol{\gamma}_h(s,z)}{1+\boldsymbol{\eta}(s)^\intercal\boldsymbol{\gamma}_h(s,z)}=\frac{1+\boldsymbol{\pi}(s)^\intercal\boldsymbol{\gamma}_h(s,z)}{1+\boldsymbol{\eta}(s)^\intercal\boldsymbol{\gamma}_h(s,z)}.$$

□

5.7 Conclusion

We have carried out an expected utility maximization problem where the objective was to maximize the terminal relative wealth. To do this we have changed the probability measure into an equivalent martingale measure Q such that the relative wealth became a Q-martingale. The crucial condition on the B-Girsanov kernels, which are responsible for the measure change, was thereby equation (5.17). Using the measure change, relationships to change of numeraire techniques, as developed by Geman et al (1995), has been discussed in Section 5.3.1. The numeraire has been changed from the risk-less asset S_0 as used in Chapter 3 and 4 to the benchmark process V_B of this chapter. Further relationships to Platen's work on the benchmark approach have been established and extended to the incomplete jump-diffusion framework in Section 5.4.

The martingale methods for portfolio optimization developed in Chapter 4 have been applied to the benchmark problem, which has let to a condition on the optimal trading strategy and the optimal martingale measure, which has been shown in Theorem 5.9. The case of power utility has been analysed and we have investigated under which conditions more closed form solutions to the problem exist. Conditions for that have been given in Lemma 5.12 and Corollary 5.13.

The chapter has been concluded by linking the PDE results from the previous chapter to the benchmark problem so that the Hamilton-Jacobi-Bellman equation for the benchmark problem could be derived in Theorem 5.16.

Chapter 6

Expected Utility Maximization under Constraints

6.1 Introduction

In this chapter we consider the problems of the previous two chapters when the investor's trading strategy is constrained to a non-empty closed convex set $K \subset \mathbb{R}^k$ that contains the origin. The set K represents the constraint on the trading strategy which could be for example the constraint of no short selling, a restricting on the number of assets to trade with, or some constraints in form of boundaries on the proportion strategies π_i.

In the literature constraints of these kind have been treated by various authors. The paper that is probably most relevant to the approach in this chapter is that by Cvitanic and Karatzas (1992). They study the problem of optimal consumption and investment when the portfolio is constrained to take values in a given convex set K. They model the stock prices as continuous Itô process which is essentially the model that has been used throughout this thesis but without jumps. The approach to solve the constrained optimization problem is to embed it into a suitable family of unconstrained problems. One then tries to single out a member of this family for which the optimal portfolio actually obeys the constraint and thereby solves the original problem as well.

In a later paper Cvitanic and Karatzas (1993) transfer the results from the constrained consumption-investment problem to contingent claim pricing and hedging. In a different paper, Bardhan (1993) extends the results from Cvitanic and Karatzas (1992) by considering additionally some minimum requirements on the consumption rate and the wealth process of the investor over the investment horizon. The arguments for these additional constraints are cash flow and regulatory requirements.

In this chapter, and in particular in the first part, the results of Cvitanic and Karatzas (1992) are extended to the jump-diffusion model. The procedure is similar as in their paper but now the additional requirement is to handle the market incompleteness that arises from the jump-diffusion model. This is done by adding an addition layer of embedding which deals with the jump-diffusion incompleteness. As in Cvitanic and Karatzas (1992) the constrained problem will be transferred to a family of auxiliary problems that are not constrained. Each of these auxiliary problems are solved by again relating them to a family of auxiliary problems, each parametrized by a kernel of an equivalent martingale measure. This part will be using heavily the methods developed in Chapter 4. One gets therefore two layers of auxiliary problems, one that deals with the constraints and the other one that deals with the market incompleteness. The original problem is then solved by finding a strategy for which the solution of all layers coincide and that obeys also the constraint K.

In the second part of the chapter constraints are applied to the benchmark optimization problem of Chapter 5. It is assumed that the benchmark is the wealth process of a benchmark strategy $\boldsymbol{\eta}$. Again a financial agent's aim is to outperform the benchmark portfolio. Now it is the difference of the two proportional trading strategies that is not allowed to leave the convex set K. This allows to model various constraints. For example, the trading strategy $\boldsymbol{\pi}$ could not be allowed to be too different from $\boldsymbol{\eta}$ in absolute terms say. Another constraint could be that an investor has to invest at least as much money into a particular stock as strategy $\boldsymbol{\eta}$ does. The problem is solved by combining the results from Chapter 4, Chapter 5, and the results from the first part of this chapter. As far as the author is aware such kind of problem has not been treated in the literature.

Portfolio optimization in the context of benchmarks and constraints has been approached rather differently in the academic literature. Often it is assumed that an investor's portfolio should satisfy some tracking error constraints when trying to outperform a benchmark. Examples for this kind of treatment are Bajeux-Bsnainou et al. (2007), Jorion (2003), and El-Hassan and Kofman (2003). However, these problems are often solved in a mean-variance framework rather than the expected utility maximization approach used here. Also the authors make assumptions on the distributions of the returns of the financial assets rather than specifying a continuous time model for the stock price dynamics as it is done in this thesis with jump-diffusion processes.

Papers that consider drawdown constraints and value-at-risk constraints are Alexander and Baptista (2006) and Alexander and Baptista (2008), respectively. They also consider a mean-variance approach and work with the expected return and the variance of asset returns. A paper that analyse the problem of index tracking with constrained portfolios is written by Maringer and Oyewumi (2007).

The chapter is outlines as follows. Section 6.2 introduces the constrained version of the consumption-investment problem of Chapter 4. A family of $\boldsymbol{\upsilon}$-problems will be introduced which will be solved in Section 6.3. The central result of the chapter, the solution to the constrained consumption-investment problem, will also be presented in this section. Section 6.4 deals with the problem of constrained relative wealth maximization which is the constrained version of the problem treated in Chapter 5. In a final section, Section 6.5, various constraints, namely upper and lower boundary constraints, are applied to the problems in a power utility setting.

6.2 The Constrained Investment-Consumption Problem and the Family of $\boldsymbol{\upsilon}$-Problems

In the following we solve the optimal consumption-investment problem of Chapter 4 which was given by $\Phi(x)$ in (4.4) under constraints. Our main reference for this work will be the paper by Cvitanic and Karatzas (1992) who solve the problem without jumps. Their results will be extended to the general jump-diffusion case. The stock market model is as usual the model given in (3.1) such that the wealth process of an investor is the solution of the stochastic differential equation (3.20):

$$\begin{aligned} dV^x_{\pi,c}(t) = &\ \big(V^x_{\pi,c}(t-)r(t) - c(t)\big)\, dt + \boldsymbol{\pi}(t) V^x_{\pi,c}(t-)^\intercal \left[\overline{\boldsymbol{\alpha}}(t)dt + \boldsymbol{\xi}(t)d\boldsymbol{B}(t)\right] \\ &+ V^x_{\pi,c}(t-) \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\pi}(t)^\intercal \boldsymbol{\gamma}_h(t,y) N_h(dt,dy), \quad t \in [0,T]. \end{aligned}$$

The set $\mathcal{A}(x)$ is as usual the set of admissible strategies $(\boldsymbol{\pi}, c)$ for which the wealth process is almost surely non-negative. It has been defined in Definition 3.6 and is always non-empty since the strategy $(\boldsymbol{\pi}, c) = (\mathbf{0}, 0)$ is always in it. In the following we are interested in restricting the portfolio strategies to certain convex sets that represent constraints.

Definition 6.1. *A closed convex set $K \in \mathbb{R}^k$ that contains the origin is called a constraint.*

Examples of constraints are short-selling constraints $K = [0,\infty)^k$, prohibition of borrowing $K = \{\boldsymbol{p} \in \mathbb{R}^k \mid \sum_{i=1}^k p_i \le 1\}$, or restriction on the number of assets $K = \{\boldsymbol{p} \in \mathbb{R}^k \mid p_{M=1} = \ldots = p_k = 0\}$ for a $M \in \{1,\ldots,k-1\}$. We introduce the convention that $\boldsymbol{\pi} \in K$ if and only if $\boldsymbol{\pi}(t) \in K$ for almost surely all $t \in [0,T]$.

The problem of maximizing consumption and terminal wealth under the constraint K is then that of finding an optimal strategy pair $(\widehat{\boldsymbol{\pi}}, \widehat{c})$ and an optimal performance function

$\Phi_K(x)$ defined by

$$\begin{aligned}\Phi_K(x) &:= \sup_{(\boldsymbol{\pi},c)\in\mathcal{A}_K} \mathbb{E}\left[\int_0^T U_1\big(t,c(t)\big)dt + U_2(V^x_{\pi,c}(T))\right] \\ &= \mathbb{E}\left[\int_0^T U_1\big(t,\widehat{c}(t)\big)dt + U_2(V^x_{\widehat{\pi},\widehat{c}}(T))\right],\end{aligned} \tag{6.1}$$

where the set of admissible strategies for the problem is given by

$$\mathcal{A}_K(x) := \left\{(\boldsymbol{\pi},c)\in\mathcal{A}(x)\middle|\quad \mathbb{E}\left[\int_0^T U_1\big(t,c(t)\big)^- dt + U_2(V^x_{\pi,c}(T))^-\right] > -\infty, \quad \boldsymbol{\pi}\in K\right\}.$$

As usual, the constrained optimization problem of finding an optimal strategy $(\widehat{\boldsymbol{\pi}},\widehat{c})$ as well as the optimal performance function given in (6.1) will be referred to, with a slight abuse of notation, by $\Phi_K(x)$.

To handle the constraint we need to heavily borrow tools from the theory of convex analysis. A standard reference on convex analysis is Rockafellar (1970). Bismut (1973) offers an application of convex analysis to stochastic control. A key concept of convex analysis is that of the support function of a closed convex set.

Definition 6.2. *Let K be a closed convex set. The function $\zeta:\mathbb{R}^k\to[0,\infty]$ defined by*

$$\zeta(\boldsymbol{v}) := \sup_{\boldsymbol{p}\in K}\{-\boldsymbol{p}^\intercal\boldsymbol{v}\}, \quad \boldsymbol{v}\in\mathbb{R}^k$$

is called the support function of the convex set $-K$. The effective domain, on which ζ is finite, will be denoted by $\widetilde{K}$.

From the definition follows immediately that $\boldsymbol{p}\in K$ if

$$\zeta(\boldsymbol{v}) + \boldsymbol{p}^\intercal\boldsymbol{v} \geq 0, \quad \forall\boldsymbol{v}\in\widetilde{K}.$$

Some examples of support functions for various constrains are (cf. Cvitanic and Karatzas (1992)) no short selling $K=[0,\infty)^k$, $\zeta\equiv 0$ on $\widetilde{K}=K$; no borrowing $K=\{\boldsymbol{p}\in\mathbb{R}^k \mid \sum_{i=1}^k p_i\leq 1\}$, $\zeta(\boldsymbol{v})=-v_1$ on $\widetilde{K}=\{\boldsymbol{v}\in\mathbb{R}^k\mid v_1=\ldots=v_k\leq 0\}$; or restriction on the number of assets $K=\{\boldsymbol{p}\in\mathbb{R}^k\mid p_{M=1}=\ldots=p_k=0\}$, $\zeta\equiv 0$ on $\widetilde{K}=\{\boldsymbol{v}\in\mathbb{R}^k\mid v_1=\ldots=v_M=0\}$.

To apply the concept of support functions to stochastic process we need to make a further introduction of a set of processes for which the support function is well behaved.

Definition 6.3. *Define by $\mathcal{D}_K$ the set containing all $(\mathcal{F}_t)$-measurable processes $\boldsymbol{v}$: $[0,T]\times\Omega\to\widetilde{K}$ that satisfy*

$$\mathbb{E}\left[\int_0^T \zeta(\boldsymbol{v}(t))\right] < \infty.$$

Thus a trading strategy $\boldsymbol{\pi}$ satisfies the constraint K, i.e. $\boldsymbol{\pi} \in K$, if

$$\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal \boldsymbol{\upsilon}(t) \geq 0, \quad t \in [0,T], \quad \forall \boldsymbol{\upsilon} \in \mathcal{D}_K. \tag{6.2}$$

The equation (6.2) is thus the key tool to relate a trading strategy $\boldsymbol{\pi} \in K$ to the support function ζ.

The idea how the constrained consumption-investment problem is solved is the following. For each of the $\boldsymbol{\upsilon} \in \mathcal{D}_K$ a so called $\boldsymbol{\upsilon}$-market is constructed. In such an $\boldsymbol{\upsilon}$-market an $\boldsymbol{\upsilon}$-wealth process $V_{\pi,c}^{x,(\upsilon)}$ is constructed that has the property that as long as its trading strategy $\boldsymbol{\pi}$ satisfies the constraint K it is always as least as big as the original wealth process: $V_{\pi,c}^{x,(\upsilon)} \geq V_{\pi,c}^{x}$ if $\boldsymbol{\pi} \in K$. Within the $\boldsymbol{\upsilon}$-market another optimization problem is formulated which will be called the $\boldsymbol{\upsilon}$-problem. This $\boldsymbol{\upsilon}$-problem is an unconstrained problem and can be solved using methods from Chapter 4. One then checks for what $\boldsymbol{\upsilon}$ the solution of the $\boldsymbol{\upsilon}$ coincides with the solution with the constrained problem with the trading strategy also satisfying $\boldsymbol{\pi} \in K$. It will be seen that the inequality (6.2) will play a central role.

To introduce the $\boldsymbol{\upsilon}$-market we follow the approach in Cvitanic and Karatzas (1992) by changing the risk-less asset rate r and the stock drift rates $\boldsymbol{\alpha}$. This will lead to the desired properties of the $\boldsymbol{\upsilon}$-wealth process $V_{\pi,c}^{x,(\upsilon)}$ as will be seen in Lemma 6.4. Thus, for a $\boldsymbol{\upsilon} \in \mathcal{D}_K$ define the processes

$$\begin{aligned} r_\upsilon(t) &:= r(t) + \zeta(\boldsymbol{\upsilon}(t)) \\ \boldsymbol{\alpha}_\upsilon(t) &:= \boldsymbol{\alpha}(t) + \boldsymbol{\upsilon}(t) + \zeta(\boldsymbol{\upsilon}(t))\,\mathbf{1}, \end{aligned} \tag{6.3}$$

where $\mathbf{1} = (1,\ldots,1)^\intercal$ is the k-dimensional unit vector. The $\boldsymbol{\upsilon}$-market is then the market under which the asset prices have the parameters as defined in (6.3). That is the $\boldsymbol{\upsilon}$-stock price processes $S_i^{(\upsilon)}$ are defined as the solution to the stochastic differential equations

$$\frac{dS_i^{(\upsilon)}(t)}{S_i^{(\upsilon)}(t-)} = \alpha_{\upsilon,i}(t)dt + \sum_{j=1}^{n} \xi_{ij}(t)dB_j(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,y) N_h(dt,dy),$$

for $i = 1,\ldots,k$. Equivalently, the $\boldsymbol{\upsilon}$-risk-less asset $S_0^{(\upsilon)}$ is defined as the solution to the ordinary differential equation

$$dS_0^{(\upsilon)}(t) = S_0^{(\upsilon)}(t) r_\upsilon(t) dt, \quad t \geq 0; \quad S_0(0) = 1.$$

The original asset price process as in (3.1) and (3.6) can be recovered for $\boldsymbol{\upsilon} \equiv 0$.

The $\boldsymbol{\upsilon}$-wealth process $V_{\pi,c}^{x,(\upsilon)}$ of a strategy $(\boldsymbol{\pi}, c)$ is then simply the wealth process of the strategy $(\boldsymbol{\pi}, c)$ when the asset prices are following the stochastic differential equations as above. Thus, $V_{\pi,c}^{x,(\upsilon)}$ is solving the stochastic differential equation

$$\begin{aligned} dV_{\pi,c}^{x,(\upsilon)}(t) = & \left(V_{\pi,c}^{x,(\upsilon)}(t-)r_\upsilon(t) - c(t)\right) dt \\ & + \boldsymbol{\pi}(t)^\intercal V_{\pi,c}^{x,(\upsilon)}(t-) \left[\overline{\boldsymbol{\alpha}}_\upsilon(t)dt + \boldsymbol{\xi}(t)d\boldsymbol{B}(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)N_h(dt,dy)\right], \end{aligned} \tag{6.4}$$

for $t \in [0,T]$. This relates to the original wealth process SDE (3.20) but with r replaced by r_υ and $\overline{\boldsymbol{\alpha}}$ replaced by $\overline{\boldsymbol{\alpha}}_\upsilon := \boldsymbol{\alpha}_\upsilon - r_\upsilon = \overline{\boldsymbol{\alpha}} + \boldsymbol{\upsilon}$. Obviously, $\boldsymbol{\upsilon}$-wealth and original wealth coincided if $\boldsymbol{\upsilon} \equiv 0$. For $\boldsymbol{\upsilon} \in \mathcal{D}_K$ the set of strategies $(\boldsymbol{\pi}, c)$ that lead to almost surely non-negative $\boldsymbol{\upsilon}$-wealth is denoted by $\mathcal{A}_\upsilon(x)$ for initial wealth $x > 0$. Strategies in $\mathcal{A}_\upsilon(x)$ are called $\boldsymbol{\upsilon}$-admissible.

Notice that the difference of (6.4) to (3.20) becomes more clear when we substitute (6.3) into (6.4). Then

$$\begin{aligned} dV_{\pi,c}^{x,(\upsilon)}(t) = & \left(V_{\pi,c}^{x,(\upsilon)}(t-)r(t) - c(t)\right) dt + V_{\pi,c}^{x,(\upsilon)}(t-)\left(\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal \boldsymbol{\upsilon}(t)\right) dt \\ & + \boldsymbol{\pi}(t)^\intercal V_{\pi,c}^{x,(\upsilon)}(t-) \left[\overline{\boldsymbol{\alpha}}(t)dt + \boldsymbol{\xi}(t)d\boldsymbol{B}(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)N_h(dt,dy)\right]. \end{aligned}$$

Neglecting the $V_{\pi,c}^{x,(\upsilon)}(t-)\left(\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal \boldsymbol{\upsilon}(t)\right) dt$ term this is exactly the normal wealth equation (3.20). It should now be intuitively clear why $V_{\pi,c}^{x,(\upsilon)} \geq V_{\pi,c}^{x}$. The following lemma proofs it.

Lemma 6.4. *Let K be a constraint and let $\boldsymbol{\upsilon} \in \mathcal{D}_K$. If $(\boldsymbol{\pi}, c) \in \mathcal{A}_K(x)$ then*

$$V_{\pi,c}^{x,(\upsilon)}(t) \geq V_{\pi,c}^{x}(t), \quad t \in [0,T]. \tag{6.5}$$

Proof. Let $(\boldsymbol{\pi}, c) \in \mathcal{A}_K(x)$. Consider the dynamics of an $\boldsymbol{\upsilon}$-wealth process discounted by the (original) risk-less asset S_0 under an EMM Q_0 with corresponding Girsanov kernel $\boldsymbol{\theta} \in \Theta$:

$$\begin{aligned} d\frac{V_{\pi,c}^{x,(\upsilon)}(t)}{S_0(t)} = & -\frac{c(t)}{S_0(t)}dt + \left(\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal \boldsymbol{\upsilon}(t)\right)\frac{V_{\pi,c}^{x,(\upsilon)}(t-)}{S_0(t)}dt \\ & + \frac{V_{\pi,c}^{x,(\upsilon)}(t-)}{S_0(t)}\boldsymbol{\pi}(t)^\intercal \left[\boldsymbol{\xi}(t)d\boldsymbol{B}^{Q_0}(t) + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)\widetilde{N}_h^{Q_0}\right] \end{aligned}$$

Define the process

$$\Delta(t) := \frac{V_{\pi,c}^{x,(\upsilon)}(t) - V_{\pi,c}^{x}(t)}{S_0(t)}, \quad t \in [0,T].$$

If we can show that $\Delta(t) \geq 0$ almost surely then the lemma is proven. Consider the dynamics of Δ which are

$$
\begin{aligned}
d\Delta(t) &= \Delta(t-)\boldsymbol{\pi}(t)^\intercal \left[\boldsymbol{\xi}(t)d\boldsymbol{B}^{Q_0}(t) + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)\widetilde{N}_h^{Q_0}(dt,dy)\right] \\
&\quad + \frac{V_{\pi,c}^{x,(v)}(t-)}{S_0(t)}\left(\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal\boldsymbol{\upsilon}(t)\right)dt \\
&= \Delta(t-)\Big[\left(\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal\boldsymbol{\upsilon}(t)\right)dt \qquad\qquad (6.6) \\
&\quad +\boldsymbol{\pi}(t)^\intercal\left(\boldsymbol{\xi}(t)d\boldsymbol{B}^{Q_0}(t) + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)\widetilde{N}_h^{Q_0}(dt,dy)\right)\Big] \\
&\quad + \frac{V_{\pi,c}^{x}(t-)}{S_0(t)}\left(\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal\boldsymbol{\upsilon}(t)\right)dt.
\end{aligned}
$$

Further define the jump-diffusion process $J(t)$ as the solution to the stochastic differential equation

$$
\begin{aligned}
\frac{dJ(t)}{J(t-)} = &\Bigg[\sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \frac{||\boldsymbol{\pi}(t)^\intercal\boldsymbol{\gamma}_h(t,y)||^2}{1+\boldsymbol{\pi}(t)^\intercal\boldsymbol{\gamma}_h(t,y)}\boldsymbol{\theta}_h^J(t,y)\nu_h(dy) \\
&- \boldsymbol{\pi}(t)^\intercal\boldsymbol{\sigma}(t)\boldsymbol{\pi}(t) - \left(\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal\boldsymbol{\upsilon}(t)\right)\Bigg]dt \qquad\qquad (6.7) \\
&- \boldsymbol{\pi}(t)^\intercal\boldsymbol{\xi}(t)d\boldsymbol{B}^{Q_0}(t) - \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{\pi}(t)^\intercal\boldsymbol{\gamma}_h(t,y)}{1+\boldsymbol{\pi}(t)^\intercal\gamma_h(t,y)}\widetilde{N}_h^{Q_0}(dt,dy).
\end{aligned}
$$

Notice that the process $J(t)$ is non-negative since $1 - \frac{\boldsymbol{\pi}^\intercal\boldsymbol{\gamma}_h}{1+\boldsymbol{\pi}^\intercal\gamma_h} = \frac{1}{1+\boldsymbol{\pi}^\intercal\gamma_h} > 0$. Also the dynamics of the process ΔJ can be calculated through Itô's product rule Lemma 2.18 applied on (6.6) and (6.7) as

$$
d\left(\Delta(t)J(t)\right) = \frac{J(t-)V_{\pi,c}^{x}(t-)}{S_0(t)}\left(\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal\boldsymbol{\upsilon}(t)\right)dt,
$$

so that most terms cancel out. Since $\Delta(0)$ we have that

$$
\Delta(t) = \frac{1}{J(t)}\int_0^t \frac{J(s-)V_{\pi,c}^{x}(s-)}{S_0(s)}\left(\zeta(\boldsymbol{\upsilon}(s)) + \boldsymbol{\pi}(s)^\intercal\boldsymbol{\upsilon}(s)\right)ds, \quad t \in [0,T].
$$

The above expression is non-negative if $\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal\boldsymbol{\upsilon}(t) \geq 0$. But comparing with (6.2) this is the case since $\boldsymbol{\pi} \in K$. □

The same result to (6.5) above can be found in Cvitanic and Karatzas (1992) for the non-jump case. Hence, the above lemma naturally extends it to the jump-diffusion framework.

In the following we want to apply the usual measure change techniques to the $\boldsymbol{\upsilon}$-markets. Recalling the conditions under which the discounted original wealth process $\overline{V}_{\pi,c}^{x}$ becomes a martingale if consumption is neglected, namely that a jump-diffusion Girsanov kernel $\boldsymbol{\theta} \in \Theta$ has to satisfy condition (3.8). Then it is natural that the $\boldsymbol{\upsilon}$-wealth process

$V_{\pi,c}^{x,(v)}$ becomes a Q-martingale if, again neglecting consumption, it is discounted by the $\boldsymbol{v}$-risk-less asset and the Girsanov kernel $\boldsymbol{\theta}$ satisfies

$$\overline{\alpha}_{i,v}(t) + \sum_{j=1}^{n} \xi_{ij}(t)\theta_{j,v}^{D}(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ik}(t,y)\theta_{k,v}^{J}(t,y)\upsilon_k(dy) = 0, \tag{6.8}$$

a.s. for all $i = 1, \dots, k$ and $t \in [0,T]$. For a given constraint K and $\boldsymbol{v} \in \mathcal{D}_K$ as in Definition 6.3 a process pair $\boldsymbol{\theta}_v = (\boldsymbol{\theta}_v^D, \boldsymbol{\theta}_v^J)$ is called a $\boldsymbol{v}$-Girsanov kernel if it satisfies conditions (i) and (ii) of Definition 3.1 and additionally (6.8). Thus, an $\boldsymbol{v}$-Girsanov kernel is exactly the same as the Girsanov kernel of Definition 3.1 but tailored to the $\boldsymbol{v}$-market. The set of all $\boldsymbol{v}$-Girsanov kernel shall be denoted by Θ_v for $\boldsymbol{v} \in \mathcal{D}_K$.

Using the usual introduction of measure changed Brownian motion and Poisson random measure

$$\begin{aligned} dB_j^Q(t) &= dB_j(t) - \theta_{j,v}^D(t)dt, \quad \text{and} \\ \widetilde{N}_h^Q(dt,dy) &= N_h(dt,dy) - \theta_{h,v}^J(t,y)\nu_h(dy)dt \end{aligned}$$

for $h = 1, ..., m$. The discounted $\boldsymbol{v}$-asset prices $\overline{S}_i^{(v)} = S_i^{(v)}/S_0^{(v)}$ and the discounted $\boldsymbol{v}$-wealth without consumption are Q martingales. Notice that discounting in an $\boldsymbol{v}$-market is thereby using the $\boldsymbol{v}$-risk-less asset instead of the original risk-less asset.

Proposition 6.5. *Let $\boldsymbol{v} \in \mathcal{D}_K$ for a constraint K. Further let Q be an EMM constructed through a $\boldsymbol{v}$-Girsanov kernel $\boldsymbol{\theta} \in \Theta_v$. The discounted $\boldsymbol{v}$-stock prices (discounted by the $\boldsymbol{v}$ risk-less asset) are martingales under Q:*

$$\frac{d\overline{S}_i^{(v)}(t)}{\overline{S}_i^{(v)}(t-)} = \sum_{j=1}^{n} \xi_{ij}(t)dB_j^Q(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \gamma_{ih}(t,y)\widetilde{N}_h^Q(dt,dy), \quad t \in [0,T].$$

Further, for a given strategy $(\boldsymbol{\pi}, c) \in \mathcal{A}_v(x)$ the process M_θ^v defined by

$$M_\theta^v(t) := \overline{V}_{\pi,c}^{x,(v)}(t) + \int_0^t \frac{1}{S_0^{(v)}(s)} c(s)ds, \quad t \in [0,T], \tag{6.9}$$

is a Q martingale.

This section is concluded by introducing a set of auxiliary problems, henceforth called $\boldsymbol{v}$-problems. Each $\boldsymbol{v} \in \mathcal{D}_K$ has lead to an $\boldsymbol{v}$-stock market and in each of these stock markets an consumption-investment problem is considered. For fixed $\boldsymbol{v} \in \mathcal{D}_K$ the problem of maximizing consumption and terminal wealth in an $\boldsymbol{v}$-market is that of finding an optimal strategy pair $(\widehat{\boldsymbol{\pi}}, \widehat{c})$ that maximizes the right hand side of the optimal performance function

$$\begin{aligned} \Phi_v(x) &:= \sup_{(\boldsymbol{\pi},c)\in\widetilde{\mathcal{A}}_v} \mathbb{E}\left[\int_0^T U_1\big(t,c(t)\big)dt + U_2(V_{\pi,c}^{x,(v)}(T))\right] \\ &= \mathbb{E}\left[\int_0^T U_1\big(t,\widehat{c}(t)\big)dt + U_2(V_{\widehat{\pi},\widehat{c}}^{x,(v)}(T))\right], \end{aligned} \tag{6.10}$$

where the set of admissible strategies for the problem is given by

$$\widetilde{\mathcal{A}}_{\upsilon}(x) := \Big\{(\boldsymbol{\pi}, c) \in \mathcal{A}_{\upsilon}(x) \Big| \quad \mathbb{E}\left[\int_0^T U_1\big(t, c(t)\big)^- dt + U_2(V_{\pi,c}^{x,(\upsilon)}(T))^-\right] > -\infty\Big\}.$$

The above optimization problem $\Phi_{\upsilon}(x)$ is the $\boldsymbol{\upsilon}$-market equivalent of the problem $\Phi(x)$ in (4.4). It will help us to solve the constraint optimization problem $\Phi_K(x)$ in (6.1). Both problems will be solved in the next section.

6.3 Solving the $\boldsymbol{\upsilon}$-Problems and the Constrained Problem

In this section martingale methods are applied to solve the $\boldsymbol{\upsilon}$-problems defined in the previous section. The consumption-investment problem in the $\boldsymbol{\upsilon}$-markets can be solved in an analogue way to the problem $\Phi(x)$ of (4.4) that has been solved in Chapter 4. However, there are some points that need clarification. Unlike problem $\Phi(x)$ there is now a family of problems $(\Phi_{\upsilon}(x))_{\boldsymbol{\upsilon}\in\mathcal{D}_K}$ for each constraint K. For each of these problems, i.e. for each $\boldsymbol{\upsilon} \in \mathcal{D}_K$, there exists a set of EMMs parametrized by a $\boldsymbol{\upsilon}$-Girsanov kernel $\boldsymbol{\theta}_{\upsilon}$. This set of EMMs is given by the set of all $\boldsymbol{\upsilon}$-kernels Θ_{υ}. In problem $\Phi(x)$ some $\boldsymbol{\upsilon}$-processes have been defined for each jump-diffusion Girsanov kernel $\boldsymbol{\theta} \in \Theta$. Here again some $\boldsymbol{\upsilon}$-processes representing auxiliary consumption and terminal wealth will be defined again for each $\boldsymbol{\upsilon}$-Girsanov kernel $\boldsymbol{\theta}_{\upsilon} \in \Theta_{\upsilon}$. However, the reader is reminded that these sets of $\boldsymbol{\upsilon}$-processes will exist for each $\boldsymbol{\upsilon} \in \mathcal{D}_K$. That is unlike in problem $\Phi(x)$ where there is not just one sub-problem, whereas here there are two sub-problems. The first layer, given for each $\boldsymbol{\upsilon}$, will handle the constraint given by K. The second layer, given by a $\boldsymbol{\theta}_{\upsilon}$ for each $\boldsymbol{\upsilon}$ will handle the issues arising because of the incompleteness of the market due to the jump-diffusion model.

For $\boldsymbol{\theta}_{\upsilon} \in \Theta_{\upsilon}$, the concept of the state price density H_{θ} as defined in (3.3) for Girsanov kernels is naturally extended to the $\boldsymbol{\upsilon}$-market by defining $H_{\theta}^{(\upsilon)}(t) := Z_{\theta}^{(\upsilon)}(t)/S_0^{(\upsilon)}(t)$. It follows the dynamics given by

$$\frac{dH_{\theta}^{(\upsilon)}(t)}{H_{\theta}^{(\upsilon)}(t-)} = -r_{\upsilon}(t) + \boldsymbol{\theta}_{\upsilon}^D(t)^{\intercal} d\boldsymbol{B}(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \left(\theta_{k,\upsilon}^J(t, y) - 1\right) \widetilde{N}_h(dt, dy) \tag{6.11}$$

for $t \in [0, T]$. Equivalent to Proposition 3.9 in an $\boldsymbol{\upsilon}$-market the following budget constraint is satisfied.

Corollary 6.6. *Let K be a constraint and let $\boldsymbol{\upsilon} \in \mathcal{D}_K$. Given an $\boldsymbol{\upsilon}$-Girsanov kernel $\boldsymbol{\theta}_{\upsilon} \in \Theta_{\upsilon}$ with corresponding EMM Q, every admissible pair $(\boldsymbol{\pi}, c) \in \mathcal{A}_{\upsilon}(x)$, $x > 0$, satisfies the budget constraint*

$$\mathbb{E}^Q\left[\int_0^T \frac{c(s)}{S_0^{(\upsilon)}(s)} ds + \overline{V}_{\pi,c}^{x,(\upsilon)}(T)\right] = \mathbb{E}\left[\int_0^T H_{\theta}^{(\upsilon)}(s) c(s) ds + H_{\theta}^{(\upsilon)}(T) V_{\pi,c}^{x,(\upsilon)}(T)\right] \leq x, \tag{6.12}$$

Proof. Equation (6.12) follows immediately from (6.9) and the non-negativity of $\overline{V}_{\pi,c}^{x,(\upsilon)}$, i.e. (6.9) is a supermartingale. $\square$

For fixed $\boldsymbol{\upsilon} \in \mathcal{D}_K$ the problem of maximizing consumption and terminal wealth introduced in (6.10) can be solved in an analogue way as the original consumption-investment problem has been solved in Chapter 4. The procedure is analogue to Section 4.3 and Section 4.4. Most proofs will be omitted since they are one to one analogue as in the two sections. We briefly state the $\boldsymbol{\upsilon}$-market analogue of the function $\mathcal{X}_\theta$ as defined in (4.5) and the $\boldsymbol{\upsilon}$-analogue of the variables c_θ and Y_θ from definition (4.6). As in (4.5), for $\boldsymbol{\upsilon} \in \mathcal{D}_K$ the function $\mathcal{X}_\theta^{(\upsilon)}$ is defined by

$$\mathcal{X}_\theta^{(\upsilon)}(y) := \mathbb{E}\left[\int_0^T H_\theta^{(\upsilon)}(t-) I_1(t, yH_\theta^{(\upsilon)}(t-))dt + H_\theta^{(\upsilon)}(T) I_2(yH_\theta^{(\upsilon)}(T))\right], \quad y > 0,$$

The set of $\boldsymbol{\upsilon}$-Girsanov kernels are reduced to a set where $\mathcal{X}_\theta^{(\upsilon)}$ is well behaved, i.e.

$$\widetilde{\Theta}_\upsilon := \left\{\boldsymbol{\theta}_\upsilon \in \Theta_\upsilon \mid \mathcal{X}_\theta^{(\upsilon)}(y) < \infty, \text{ for } y > 0\right\}.$$

For $\boldsymbol{\theta}_\upsilon \in \widetilde{\Theta}_\upsilon$ the function $\mathcal{X}_\theta^{(\upsilon)}$ has an inverse, which will be denoted by $\mathcal{Y}_\theta^{(\upsilon)}$. In an $\boldsymbol{\upsilon}$-market an auxiliary consumption process and auxiliary terminal wealth is defined for initial wealth $x > 0$ and $\boldsymbol{\theta}_\upsilon \in \widetilde{\Theta}_\upsilon$ by

$$\begin{aligned} c_\theta^{(\upsilon)}(t) &:= I_1\left(t, \mathcal{Y}_\theta^{(\upsilon)}(x) H_\theta^{(\upsilon)}(t-)\right), \\ Y_\theta^{(\upsilon)} &:= I_2\left(\mathcal{Y}_\theta^{(\upsilon)}(x) H_\theta^{(\upsilon)}(T)\right). \end{aligned} \tag{6.13}$$

These definitions are the $\boldsymbol{\upsilon}$-equivalent to (4.6). It is obvious that for a constraint K and $\boldsymbol{\upsilon} \in \mathcal{D}_K$, the variables $c_\theta^{(\upsilon)}$ and $Y_\theta^{(\upsilon)}$ satisfy items (i) to (iii) in Lemma 4.1 in the $\boldsymbol{\upsilon}$-market setting, that is with respect to the $\boldsymbol{\upsilon}$-state price density $H_\theta^{(\upsilon)}$, and trading strategies are in $(\boldsymbol{\pi}, c) \in \widetilde{\mathcal{A}}_\upsilon(x)$. There is also a further property illustrating the relationship between the three optimization problems (the original and the two auxiliary) which will be shown in the next lemma.

Lemma 6.7. *Let $\boldsymbol{\upsilon} \in \mathcal{D}_K$ for some constraint K and let $\boldsymbol{\theta}_\upsilon \in \widetilde{\Theta}_\upsilon$. Then $c_\theta^{(\upsilon)}$ and $Y_\theta^{(\upsilon)}$ as defined in* (6.13) *satisfy*

$$\begin{aligned} \mathbb{E}\left[\int_0^T U_1(t, c(t))dt + U_2\left(V_{\pi,c}^x(T)\right)\right] &\le \mathbb{E}\left[\int_0^T U_1(t, c(t))dt + U_2\left(V_{\pi,c}^{x;(\upsilon)}(T)\right)\right] \\ &\le \mathbb{E}\left[\int_0^T U_1(t, c_\theta^{(\upsilon)}(t))dt + U_2(Y_\theta^{(\upsilon)})\right], \end{aligned} \tag{6.14}$$

for all $(\boldsymbol{\pi}, c) \in \mathcal{A}_K(x)$.

Proof. Only the first inequality needs to be proven in (6.14), since the second is analogue to (iii) in Lemma 4.1. However, the first inequality follows from U_2 being almost surely an increasing function and because $V^x_{\pi,c}(T) \geq V^{x,(\upsilon)}_{\pi,c}(T)$ since $(\boldsymbol{\pi}, c) \in \mathcal{A}_K(x)$, as shown in Lemma 6.4. □

An optimal EMM for the problem $\Phi_\upsilon(x)$ in (6.10) is the $\boldsymbol{\upsilon}$-equivalent of Definition 4.2. That is, Q is optimal for $\Phi_\upsilon(x)$ if its $\boldsymbol{\upsilon}$-Girsanov kernel representation $\widehat{\boldsymbol{\theta}}_\upsilon \in \widehat{\Theta}_\upsilon$ satisfies

$$\mathbb{E}\left[\int_0^T U_1(t, c^{(\upsilon)}_{\widehat{\theta}}(t))dt + U_2(Y^{(\upsilon)}_{\widehat{\theta}})\right] = \inf_{\theta \in \widetilde{\Theta}_\upsilon} \mathbb{E}\left[\int_0^T U_1(t, c^{(\upsilon)}_{\theta}(t))dt + U_2(Y^{(\upsilon)}_{\theta})\right].$$

Having introduced all processes and variables needed, we can transfer the results on the optimal consumption-investment problem from Theorem 4.3 to the $\boldsymbol{\upsilon}$-market problem $\Phi_\upsilon(x)$.

Corollary 6.8. *Let $\boldsymbol{\upsilon} \in \mathcal{D}_K$ for a constraint K. For $\boldsymbol{\theta}_\upsilon \in \widetilde{\Theta}_\upsilon$ define the processes*

$$\begin{aligned} M^{(\upsilon)}_\theta(t) &:= \mathbb{E}\left[\int_0^T H^{(\upsilon)}_\theta(s-)c^{(\upsilon)}_\theta(s)ds + H^{(\upsilon)}_\theta(T)Y^{(\upsilon)}_\theta \;\Big|\; \mathcal{F}_t\right] \\ J^{(\upsilon)}_\theta(t) &:= \int_0^t H^{(\upsilon)}_\theta(s-)c^{(\upsilon)}_\theta(s)ds \\ V^{(\upsilon)}_\theta(t) &:= \frac{1}{H^{(\upsilon)}_\theta(t)}\{M^{(\upsilon)}_\theta(t) - J^{(\upsilon)}_\theta(t)\}, \end{aligned} \tag{6.15}$$

where $c^{(\upsilon)}_\theta$ and $Y^{(\upsilon)}_\theta$ are defined as in (6.13). If there exist a $\boldsymbol{\upsilon}$-kernel $\widehat{\boldsymbol{\theta}}_\upsilon \in \widetilde{\Theta}_\upsilon$ and a trading strategy $\boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}} \in \Pi$ that satisfy the set of non-linear equations

$$\begin{aligned} \boldsymbol{\xi}(t)\boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}}(t) &= \frac{1}{H^{(\upsilon)}_{\widehat{\theta}}(t-)V^{(\upsilon)}_{\widehat{\theta}}(t-)}\boldsymbol{a}^D_\upsilon(t) - \widehat{\boldsymbol{\theta}}^D_\upsilon(t), \\ \boldsymbol{\gamma}_h(t,y)\boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}}(t) &= \frac{1}{H^{(\upsilon)}_{\widehat{\theta}}(t-)V^{(\upsilon)}_{\widehat{\theta}}(t-)}\frac{a^J_{h,\upsilon}(t,y)}{\widehat{\theta}^J_{\upsilon,h}(t,y)} - \frac{\widehat{\theta}^J_{\upsilon,h}(t,y) - 1}{\widehat{\theta}^J_{\upsilon,h}(t,y)}, \end{aligned} \tag{6.16}$$

for $h = 1, \dots, m$, and ν_h-almost all $y \in \mathbb{R} \setminus \{0\}$, then $\left(\boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}}, c^{(\upsilon)}_{\widehat{\theta}}\right)$ is a solution to the $\boldsymbol{\upsilon}$-problem $\Phi_\upsilon(x)$ in (6.10). Thereby are, as usual, $\boldsymbol{a}^D_\upsilon$ and $a^J_{h,\upsilon}$ the martingale representation coefficients of $M^{(\upsilon)}_\theta$ defined in (6.15). Further, the optimal $\boldsymbol{\upsilon}$-wealth process of $\left(\boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}}, c^{(\upsilon)}_{\widehat{\theta}}\right)$ is given by

$$V^x_{\boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}}, c^{(\upsilon)}_{\widehat{\theta}}}(t) = V^{(\upsilon)}_{\widehat{\theta}}(t), \qquad a.s., \ t \in [0, T].$$

We have formed a family of $\boldsymbol{\upsilon}$-problems wich parameterize the constraint K using the processes $\boldsymbol{\upsilon} \in \mathcal{D}_K$. In each $\boldsymbol{\upsilon}$-problem a consumption-investment problem has been solved in a jump-diffusion setting using the results from Chapter 4, and in particular Theorem 4.3.

It is now possible to link the family of $\boldsymbol{\upsilon}$-problems to the original constraint investment-consumption problem $\Phi_K(x)$ in (6.1). Consider thereby the following. It is know from the proof of Lemma 6.4 that $V^{x,(\upsilon)}_{\pi,c}$ and $V^x_{\pi,c}$ coincide if

$$\zeta(\boldsymbol{\upsilon}(t)) + \boldsymbol{\pi}(t)^\intercal \boldsymbol{\upsilon}(t) = 0, \quad t \in [0,T].$$

If a solution to a $\boldsymbol{\upsilon}$-problem satisfies this and in addition to that the $\boldsymbol{\upsilon}$-optimal trading strategy $\widehat{\boldsymbol{\pi}}^{(\upsilon)}$ satisfies the constraint K, i.e. $\widehat{\boldsymbol{\pi}}^{(\upsilon)} \in K$, then from (6.14) this must be also an optimal trading strategy for the constrained problem $\Phi_K(x)$. Hence, we have just proven the following theorem.

Theorem 6.9. *Let $\boldsymbol{\upsilon} \in \mathcal{D}_K$ for a constraint K. Assume further that $\boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}} \in \Pi$ satisfies* (6.16) *for some optimal kernel $\widehat{\boldsymbol{\theta}}_\upsilon \in \widetilde{\Theta}_\upsilon$. If $\boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}}$ satisfies for $t \in [0,T]$*

$$\zeta\left(\boldsymbol{\upsilon}(t)\right) + \boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}}(t)^\intercal \boldsymbol{\upsilon}(t) = 0, \quad \textit{as well as} \quad \boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}} \in K,$$

then $\left(\boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}}, c^{(\upsilon)}_{\widehat{\theta}}\right)$ is an optimal trading strategy for the problem (6.1) *given by*

$$\Phi_K(x) := \sup_{(\boldsymbol{\pi},c)\in\mathcal{A}_K} \mathbb{E}\left[\int_0^T U_1\big(t, c(t)\big)dt + U_2(V^x_{\pi,c}(T))\right].$$

Further, if $\boldsymbol{\pi}^{(\upsilon)}_{\widehat{\theta}}$ and $\widehat{\boldsymbol{\theta}}_\upsilon$ satisfy all the above conditions then the optimal consumption process and the optimal terminal wealth are given as in (6.13) *for $\widehat{\boldsymbol{\theta}}_\upsilon$, and the EMM Q associated to $\widehat{\boldsymbol{\theta}}_\upsilon$ is optimal for the problem $\Phi_K(x)$.*

6.4 Benchmarks and Constraints

It has been seen that when a benchmark process is given as the wealth process of a trading strategy $\boldsymbol{\eta}$ the results in Chapter 4 and Chapter 5 are strongly interlinked. In this section the aim is to carry these relationships over to the case of constrained portfolios. In the previous section a financial agent's aim was to maximize terminal wealth and consumption when his/her trading strategy $\boldsymbol{\pi}$ is constrained to a convex set K. Similar considerations can be made for the benchmark problem when one assumes that the difference between the investor's trading strategy $\boldsymbol{\pi}$ and the benchmark strategy $\boldsymbol{\eta}$ has to lie within a constraint. This can be interpreted as that an investor still wants to outperform the benchmark strategy, but has now the constraint that the strategy $\boldsymbol{\pi}$ can not be too different from the benchmark's. Thus, the problem has become more similar to a more passive problem but where one still tries to outperform a benchmark. Let us formally define the problem that we are going to solve in this section. Let again K be a given constraint as of Definition 6.1, and let $\boldsymbol{\eta} \in \Pi_+$ be a given benchmark strategy such that the relative wealth process is given by $V_{\pi/\eta}$ as in (5.10). The optimization problem of maximizing expected utility from terminal relative wealth under the constraint K is then the problem of finding an optimal trading strategy $\widehat{\boldsymbol{\pi}}$ that attains the maximum in the optimal performance function

$$\Phi_K^\eta = \sup_{\boldsymbol{\pi}\in\mathcal{A}_K^\eta} \mathbb{E}\left[U\left(V_{\pi/\eta}(T)\right)\right], \tag{6.17}$$

where the set of admissible trading strategies for the problem is given by

$$\mathcal{A}_K^\eta := \left\{\boldsymbol{\pi}\in\Pi \mid \mathbb{E}\left[U\left(V_{\pi/\eta}(T)^-\right)\right] > -\infty; \quad \boldsymbol{\pi}-\boldsymbol{\eta}\in K\right\}.$$

The problem can be solved in the same style as problem $\Phi_K(x)$ in the previous section. Because of (5.7) most results can be immediately transferred to the problem Φ_K^η of this section if consumption is equal to zero $c\equiv 0$. The support function ζ of the set $-K$ as defined in Definition 6.2 can be used without changes since clearly $\boldsymbol{\pi}-\boldsymbol{\eta}\in K$ if and only if

$$\zeta(\boldsymbol{\upsilon}(t)) + (\boldsymbol{\pi}(t)-\boldsymbol{\eta}(t))^\intercal\,\boldsymbol{\upsilon}(t) \geq 0, \quad t\in[0,T], \tag{6.18}$$

for all $\boldsymbol{\upsilon}\in\mathcal{D}_K$. Analogue to the $\boldsymbol{\upsilon}$-market introduced in Section 6.2 we will now introduce a $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-market tailored to the constrained benchmark problem. To relate the wealth process V_π^x in (3.13) to the relative wealth process $V_{\pi/\eta}$ in (5.10) we have carried out the substitution (5.14). In particular, for the risk-less rate and for the discounted $\boldsymbol{\alpha}$-rates this substitution is

$$\begin{aligned}\widetilde{r}(t) &= 0, \quad \text{and}\\ \widetilde{\overline{\boldsymbol{\alpha}}}(t) &= \overline{\boldsymbol{\alpha}}(t)-\boldsymbol{\sigma}(t)\boldsymbol{\eta}(t).\end{aligned}$$

To now transfer the $\boldsymbol{\upsilon}$-wealth process $V_\pi^{x,(\upsilon)}$ into an $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-relative wealth process $V_{\pi/\eta}^{(\upsilon)}$ one has to carry out the substitutions as in (6.3), which become

$$\begin{aligned}\widetilde{r}_\upsilon(t) &:= \widetilde{r}(t)+\zeta(\boldsymbol{\upsilon}(t)) = \zeta(\boldsymbol{\upsilon}(t)),\\ \widetilde{\overline{\boldsymbol{\alpha}}}_\upsilon(t) &:= \widetilde{\overline{\boldsymbol{\alpha}}}(t)+\boldsymbol{\upsilon}(t) = \overline{\boldsymbol{\alpha}}(t)-\boldsymbol{\sigma}(t)\boldsymbol{\eta}(t)+\boldsymbol{\upsilon}(t),\\ \widetilde{\boldsymbol{\xi}}(t) &:= \boldsymbol{\xi}(t),\\ \widetilde{\gamma}_{ih} &:= \frac{\gamma_{ih}(t,y)}{1+\boldsymbol{\eta}(t)^\intercal\boldsymbol{\gamma}_h(t,y)},\end{aligned} \tag{6.19}$$

$x=1$ and the trading strategy is set to $\widetilde{\boldsymbol{\pi}} := \boldsymbol{\pi}-\boldsymbol{\eta}$ in the $\boldsymbol{\upsilon}$-wealth process. The $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-relative wealth process $V_{\pi/\eta}^{(\upsilon)}$ satisfies then the stochastic differential equation

$$\begin{aligned}\frac{dV_{\pi/\eta}^{(\upsilon)}(t)}{V_{\pi/\eta}^{(\upsilon)}(t-)} = &\left(\zeta(\boldsymbol{\upsilon}(t)) + \left(\boldsymbol{\pi}(t)-\boldsymbol{\eta}(t)\right)^\intercal\boldsymbol{\upsilon}(t)\right)dt\\ &+\left(\boldsymbol{\pi}(t)-\boldsymbol{\eta}(t)\right)^\intercal\Big[\left(\overline{\boldsymbol{\alpha}}(t)-\boldsymbol{\sigma}(t)\boldsymbol{\eta}(t)\right)dt+\boldsymbol{\xi}(t)d\boldsymbol{W}(t)\\ &+\sum_{h=1}^m\int_{\mathbb{R}\setminus\{0\}}\frac{\gamma_h(t,y)}{1+\boldsymbol{\eta}(t)^\intercal\boldsymbol{\gamma}_h(t,y)}N_h(dt,dy)\Big], \quad t\in[0,T].\end{aligned}$$

Comparing the $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$ relative wealth with the original relative wealth in (5.10) one can see that

$$\frac{dV^{(\upsilon)}_{\pi/\eta}(t)}{V^{(\upsilon)}_{\pi/\eta}(t-)} = \frac{dV_{\pi/\eta}(t)}{V_{\pi/\eta}(t-)} + \big(\zeta(\boldsymbol{\upsilon}(t)) + \big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\big)^{\intercal}\boldsymbol{\upsilon}(t)\big)\, dt.$$

In particular, it can be deduces from (6.18) that if $\boldsymbol{\pi} - \boldsymbol{\eta} \in K$ then, clearly, $V^{(\upsilon)}_{\pi/\eta}(t) \geq V_{\pi/\eta}(t)$.

As usual in the $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-market we can perform a measure change such that the discounted(!) relative $\boldsymbol{\upsilon}$-wealth process becomes a martingale. This is now different from the procedure in Chapter 5, since we derive our results from a transformation of the $\boldsymbol{\upsilon}$-wealth process of the previous section. Before, when transforming a wealth process into a relative wealth process the risk-less asset has become the constant process $S_0 \equiv 1$ since we have set $r \equiv 0$, however, now in the transformation (6.19) from the $\boldsymbol{\upsilon}$-market to the $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-market the $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-risk-less asset is given as the solution of the ordinary differential equation

$$dS_0^{(\upsilon),\eta}(t) = \zeta(\boldsymbol{\upsilon}(t))S_0^{(\upsilon),\eta}(t)dt, \quad t \in [0,T],$$

and $S_0^{(\upsilon),\eta}(0) = 1$. Thus, we have performed two substitutions. First, the wealth process has been transformed into a relative wealth process through (5.14), and, second the substitution in (6.3) transforms the relative wealth into an $\boldsymbol{\upsilon}$-type market as in the previous sections. These two substitutions have been summarized in (6.19). We call the newly constructed market a $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-market to emphasis the dependency on η.

The $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-Girsanov kernels are then a pair of processes $\boldsymbol{\theta}_{\upsilon,\eta} = (\boldsymbol{\theta}^J_{\upsilon,\eta}, \boldsymbol{\theta}^D_{\upsilon,\eta})$ defined like the $\boldsymbol{\upsilon}$-Girsanov kernels in the previous section but with the condition

$$\overline{\boldsymbol{\alpha}}(t) + \boldsymbol{\upsilon}(t) - \boldsymbol{\sigma}(t)\boldsymbol{\eta}(t) + \boldsymbol{\xi}(t)\boldsymbol{\theta}^D(t) + \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{\gamma}_h(t,y)\theta^J_h(t,y)}{1+\boldsymbol{\eta}(t)^{\intercal}\boldsymbol{\gamma}_h(t,y)}\nu_h(dt,dy) = 0,$$

instead of condition (6.8). This condition is basically the $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-equivalent of (5.20) but has now an additional $\boldsymbol{\upsilon}$-term. For a given constraint K and a given $\boldsymbol{\upsilon} \in \mathcal{D}_K$ the set of all $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-Girsanov kernels will be denoted by Θ^η_υ. Clearly, if Q denotes the probability measure associated to the $\boldsymbol{\upsilon}$-$\boldsymbol{\eta}$-Girsanov kernel $\boldsymbol{\theta}_{\upsilon,\eta} \in \Theta^\eta_\upsilon$, and if $\boldsymbol{W}^Q$ and $\widetilde{\boldsymbol{N}}^Q$ as defined as in (2.21) and (2.22) for $\boldsymbol{\theta}_{\upsilon,\eta}$ respectively then the discounted relative wealth process is a Q-martingale:

$$\begin{aligned} d\frac{V^{(\upsilon)}_{\pi/\eta}(t)}{S_0^{\upsilon,\eta}(t)} = \frac{V^{(\upsilon)}_{\pi/\eta}(t-)}{S_0^{\upsilon,\eta}(t-)}\big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\big)^{\intercal}\Big[&\boldsymbol{\xi}(t)d\boldsymbol{W}^Q(t) \\ &+ \sum_{h=1}^{m}\int_{\mathbb{R}\setminus\{0\}} \frac{\boldsymbol{\gamma}_h(t,y)}{1+\boldsymbol{\eta}(t)^{\intercal}\boldsymbol{\gamma}_h(t,y)}\widetilde{N}^Q_h(dt,dy)\Big] \end{aligned}$$

It is possible to transfer the results from the previous section to the constrained relative wealth problem Φ^η_K in (6.17)

Proposition 6.10. *Let K be a constraint and let $\boldsymbol{\upsilon} \in \mathcal{D}_K$. Define the function*

$$\mathcal{X}_\theta^{(\upsilon),\eta}(y) = \mathbb{E}\left[H_\theta^{(\upsilon),\eta}(T) I\left(y H_\theta^{(\upsilon),\eta}(T)\right)\right], \quad y > 0,$$

and the set for which it is finite for all $y > 0$ by

$$\widetilde{\Theta}_\upsilon^\eta = \left\{\boldsymbol{\theta} \in \Theta_{\upsilon,\eta} \mid \mathcal{X}_\theta^{(\upsilon),\eta}(y) < \infty, \quad y > 0\right\}.$$

For $\boldsymbol{\theta} \in \widetilde{\Theta}_{\upsilon,\eta}$ denote by $\mathcal{Y}_\theta^{(\upsilon),\eta}$ the inverse of $\mathcal{X}_\theta^{(\upsilon),\eta}$ and define

$$\begin{aligned} M_\theta^{(\upsilon),\eta}(t) &:= \mathbb{E}\left[H_\theta^{(\upsilon),\eta}(T) I\left(\mathcal{Y}_\theta^{(\upsilon),\eta}(x) H_\theta^{(\upsilon),\eta}(T)\right) \mid \mathcal{F}_t\right] \\ V_\theta^{(\upsilon),\eta}(t) &:= \frac{M_\theta^{(\upsilon),\eta}(t)}{H_\theta^{(\upsilon),\eta}(t)}, \quad t \in [0,T]. \end{aligned} \tag{6.20}$$

If $\widehat{\boldsymbol{\pi}} \in \Pi$ and $\widehat{\boldsymbol{\theta}} \in \widetilde{\Theta}_\upsilon^\eta$ satisfy the conditions

$$\begin{aligned} (\widehat{\boldsymbol{\pi}}(t) - \boldsymbol{\eta}(t))^\intercal \boldsymbol{\xi}(t) &= \frac{1}{H_\theta^{(\upsilon),\eta}(t) V_\theta^{(\upsilon),\eta}(t)} a^D(t) - \widehat{\boldsymbol{\theta}}^D(t) \\ (\widehat{\boldsymbol{\pi}}(t) - \boldsymbol{\eta}(t))^\intercal \boldsymbol{\gamma}_h(t,y) &= \frac{1}{H_\theta^{(\upsilon),\eta}(t) V_\theta^{(\upsilon),\eta}(t)} \frac{a_h^J(t,y)}{\widehat{\theta}_h^J(t,y)} - \frac{\widehat{\theta}_h^J(t,y) - 1}{\widehat{\theta}_h^J(t,y)} \end{aligned}$$

and in addition to that satisfy

$$\begin{aligned} \zeta(\boldsymbol{\upsilon}(t)) + (\widehat{\boldsymbol{\pi}}(t) - \boldsymbol{\eta}(t))^\intercal \boldsymbol{\upsilon}(t) &= 0 \\ \widehat{\boldsymbol{\pi}}(t) - \boldsymbol{\eta}(t) &\in K, \end{aligned} \tag{6.21}$$

then $\widehat{\boldsymbol{\pi}}$ is optimal for the problem Φ_K^η in (6.17). The optimal relative wealth process is then given by $V_{\widehat{\pi}/\eta}(t) = V_{\widehat{\theta}}^{(\upsilon),\eta}(t)$ defined in (6.20). Hence, the optimal terminal relative wealth is given by

$$V_{\widehat{\pi}/\eta}(T) = I\left(\mathcal{Y}_{\widehat{\theta}}^{(\upsilon),\eta}(x) H_{\widehat{\theta}}^{(\upsilon),\eta}(T)\right).$$

6.5 Applications to various Constraints

In this section the results of the previous sections on constrained consumption-investment problem and the constrained benchmark problem is tried on concrete constraints. As usual, to get some closed form results we will be working with the power utility. Consider the constrained consumption-investment problem from (6.1) first.

Let K be, as before, the convex set that contains the origin and represents the constraint. $\mathcal{D}_K$ is defined as in (6.3) so that $\boldsymbol{\upsilon} \in \mathcal{D}_K$ specifies a $\boldsymbol{\upsilon}$-market. The same utility functions as in Section 4.5 are chosen so that the utility from consumption is $U_1(t,x) = \frac{x^{\beta_1}}{\beta_1}$ for $\beta_1 < 1$ and the utility from terminal wealth is $U_2(x) = \frac{x^{\beta_2}}{\beta_2}$ for $\beta_2 < 1$.

The optimality condition (6.16) for an optimal trading strategy-Girsanov kernel pair $(\boldsymbol{\pi}^{(v)}_{\widehat{\theta}}, \widehat{\boldsymbol{\theta}}_v)$ in the $\boldsymbol{v}$-market has the same form as the optimality condition (4.12) in the normal market. Thus, for the power utility case the optimality condition for an optimal trading strategy and an optimal $\boldsymbol{v}$-Girsanov kernel for the $\boldsymbol{v}$-problem takes the same form as condition (4.27), which is then in the $\boldsymbol{v}$-market

$$
\begin{aligned}
\boldsymbol{\xi}(t)\boldsymbol{\pi}^{(v)}_{\widehat{\theta}}(t) &= \frac{1}{C^{(v)}_{\widehat{\theta}}(t-) + W^{(v)}_{\widehat{\theta}}(t-)} \left\{ \frac{1}{\beta_1 - 1} C^{(v)}_{\widehat{\theta}}(t-) + \frac{1}{\beta_2 - 1} W^{(v)}_{\widehat{\theta}}(t-) \right\} \widehat{\boldsymbol{\theta}}^D_v(t) \\
\boldsymbol{\gamma}_h(t,y)\boldsymbol{\pi}^{(v)}_{\widehat{\theta}}(t) &= \frac{1}{C^{(v)}_{\widehat{\theta}}(t-) + W^{(v)}_{\widehat{\theta}}(t-)} \left\{ \left(\widehat{\theta}^J_{v,h}(t,y)^{\frac{1}{\beta_1 - 1}} - 1 \right) C^{(v)}_{\widehat{\theta}}(t-) \right. \\
&\qquad \left. + \left(\widehat{\theta}^J_{v,h}(t,y)^{\frac{1}{\beta_2 - 1}} - 1 \right) W^{(v)}_{\widehat{\theta}}(t-) \right\},
\end{aligned}
\tag{6.22}
$$

for $h = 1, \ldots, m$. Thereby are $C^{(v)}_{\theta}$ and $W^{(v)}_{\theta}$ defined by

$$
\begin{aligned}
C^{(v)}_{\theta}(t) &:= \mathbb{E}\left[\int_t^T H^{(v)}_{\theta}(s-) c^{(v)}_{\theta}(s) \middle| \mathcal{F}_t \right], \quad \text{and} \\
W^{(v)}_{\theta}(t) &:= \mathbb{E}\left[H^{(v)}_{\theta}(T) Y^{(v)}_{\theta} \middle| \mathcal{F}_t \right],
\end{aligned}
$$

respectively, with $H^{(v)}_{\theta}$ as in (6.11), and $c^{(v)}_{\theta}$ and $Y^{(v)}_{\theta}$ as defined in (6.13) for $\boldsymbol{\theta}_v \in \widetilde{\Theta}_v$.

Since we are going to work under constraints which in turn add complexity to the problem it appears appropriate to restrict our analysis to the simpler case where the utility from terminal wealth is the same as the utility drawn from consumption, such that $\beta_1 = \beta_2 = \beta$. For this case the above condition (6.22) becomes for the optimal strategy $\widehat{\boldsymbol{\pi}}_v$

$$
\overline{\boldsymbol{\alpha}}(t) + \boldsymbol{v}(t) - (1-\beta)\boldsymbol{\sigma}(t)\widehat{\boldsymbol{\pi}}_v(t) + \sum_{h=1}^{m} \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y)\{1 + \widehat{\boldsymbol{\pi}}_v(t)^{\intercal}\boldsymbol{\gamma}_h(t,y)\}^{\beta-1} \nu_h(dy) = 0.
\tag{6.23}
$$

This is the $\boldsymbol{v}$-equivalent to (4.31), which is not surprising since $\boldsymbol{v}$-Girsanov kernels need to satisfy the condition (6.8) which has exactly the same form as the condition on the Girsanov kernels in (3.8)

Proposition 6.11. *Let K be a convex set containing the origin and let $\beta < 1$. If for $\boldsymbol{v} \in \mathcal{D}_K$ the condition* (6.23) *are satisfied as well as*

$$
\zeta(\boldsymbol{v}(t)) + \widehat{\boldsymbol{\pi}}_v(t)^{\intercal}\boldsymbol{v}(t) = 0, \quad t \in [0,T],
\tag{6.24}
$$

and $\widehat{\boldsymbol{\pi}}_v \in K$. Then $\widehat{\boldsymbol{\pi}}_v$ is the optimal trading strategy for the problem $\Phi_K(x)$ in (6.1) *with power utility $U_1(t,x) = U_2(x) = x^{\beta}/\beta$.*

Let us for the purpose of the following discussion write $\boldsymbol{v} \in \widetilde{K}$ as a synonym for $\boldsymbol{v} \in \mathcal{D}_K$, which is in the same sense as we write $\boldsymbol{\pi} \in K$. The procedure of solving the consumption-investment problem under constraints for the power utility case is then the following. First, one tries to find the solutions of the two equations (6.23) and (6.24) with respect to $(\boldsymbol{\pi}, \boldsymbol{v})$, then one checks for each solution if $(\boldsymbol{\pi}, \boldsymbol{v}) \in K \times \widetilde{K}$. If this is the case for at least one pair $(\widehat{\boldsymbol{\pi}}, \widehat{\boldsymbol{v}})$ one has found a solution to the constrained optimization problem and the optimal trading strategy is given by $\widehat{\boldsymbol{\pi}}$. The optimal terminal wealth and consumption process can be derived from $\widehat{\boldsymbol{v}}$.

Finding all solutions to (6.23) and (6.24) can quickly become a complex problem in the multidimensional setting which should be solved using numerical methods. For the one-dimensional case some further analytical results can be obtained. In a one-dimensional model with one stock, one Brownian motion, and one Poisson measure, let us consider the constraint of a lower bound, an upper bound, and a combination of both. A lower bound constraint is defined by the set $K_l := \{p \in \mathbb{R} \mid l \leq p\}$ for a $l \in \mathbb{R}$. Then $\zeta_l(v) = -l\, v$ for all $v \in \widetilde{K}_l = \mathbb{R}_+ := [0, \infty)$. Denote by π_β the solution for the unconstrained power utility consumption-investment problem as stated for the multidimensional case in (4.31). Then π_β solves, in the one-dimensional case, the equation

$$\overline{\alpha}(t) - (1-\beta)\sigma(t)\pi_\beta(t) + \int_{\mathbb{R}\setminus\{0\}} \gamma(t,y)\{1 + \pi_\beta(t)\gamma(t,y)\}^{\beta-1}\nu(dy) = 0.$$

For the problem with the lower boundary the possible solutions to the equations (6.23) and (6.24), that are also in $K_l \times \mathbb{R}_+$, are then

1. $(\pi_\beta, 0)$ if $\pi_\beta \geq l$, and
2. (l, v_l) if $v_l(t) := -\overline{\alpha}(t) + (1-\beta)\sigma(t)\, l - \int_{\mathbb{R}\setminus\{0\}} \gamma(t,y)\,(1 + l\,\gamma(t,y))^{\beta-1}\,\nu(dy) > 0.$

Thus, the optimal trading strategy for the lower boundary constraint consumption-investment problem is given by

$$\widehat{\pi}(t) = \begin{cases} \pi_\beta(t), & \text{if } \pi_\beta(t) \geq l \\ l, & \text{if } v_l(t) > 0, \end{cases}$$

and can not be provided for all other cases. The optimal consumption and wealth processes can then be calculated using the appropriate $\widehat{v}$ of either $\widehat{v}(t) = 0$ if $\widehat{\pi}(t) = \pi_\beta$, or $\widehat{v}(t) = v_l(t)$ if $\widehat{\pi}(t) = l$ at time $t \in [0, T]$.

In a similar manner an upper bound constraint of the form $K_u := \{p \in \mathbb{R} \mid p \leq u\}$ can be considered for $u \in \mathbb{R}$. Then $\zeta_u(v(t)) = -u\, v(t)$ for $v \in \widetilde{K}_u = \mathbb{R}_-$, and solution of (6.23) and (6.24), that also lie in $K_u \times \mathbb{R}_-$, are

1. $(\pi_\beta, 0)$ if $\pi_\beta \leq u$, and

2. (u, υ_u) if $\upsilon_u(t) := -\overline{\alpha}(t) + (1-\beta)\sigma(t)\,u - \int_{\mathbb{R}\setminus\{0\}} \gamma(t,y)\,(1+u\,\gamma(t,y))^{\beta-1}\,\nu(dy) < 0$.

Thus, the optimal trading strategy for the upper boundary constrained consumption-investment problem is given by

$$\widehat{\pi}(t) = \begin{cases} \pi_\beta(t), & \text{if } \pi_\beta(t) \le u \\ u, & \text{if } \upsilon_u(t) < 0, \end{cases}$$

and does not exist otherwise. The optimal consumption and wealth processes can then be calculated using the appropriate $\widehat{\upsilon}$ of either $\widehat{\upsilon}(t) = 0$ if $\widehat{\pi}(t) = \pi_\beta$, or $\widehat{\upsilon}(t) = \upsilon_u(t)$ if $\widehat{\pi}(t) = u$ at time $t \in [0, T]$.

Notice that the constraint of no short-selling is just a special case of lower boundary with $l = 0$. Equally, the constraint of no borrowing is the special case that there is an upper boundary constraint with $u = 1$. Equal analysis as the above can be carried out when one considers an upper and lower boundary of the form $K_{lu} = \{p \in \mathbb{R} \mid l \le p \le u\}$. Then

$$\zeta_{lu}(\upsilon(t)) = \begin{cases} -l\,\upsilon(t), & \text{if } \upsilon(t) > 0 \\ -u\,\upsilon(t), & \text{otherwise,} \end{cases}$$

and $\widetilde{K}_{lu} = \mathbb{R}$. Solutions of (6.23) and (6.24) that lie in $K_{lu} \times \mathbb{R}$ are

1. $(\pi_\beta, 0)$ if $l \le \pi_\beta \le u$,
2. (u, υ_u) where $\upsilon_u(t) = -\overline{\alpha}(t) + (1-\beta)\sigma(t)\,u - \int_{\mathbb{R}\setminus\{0\}} \gamma(t,y)\,(1+u\,\gamma(t,y))^{\beta-1}\,\nu(dy)$ as above, and
3. (l, υ_l) with $\upsilon_l(t) = -\overline{\alpha}(t) + (1-\beta)\sigma(t)\,l - \int_{\mathbb{R}\setminus\{0\}} \gamma(t,y)\,(1+l\,\gamma(t,y))^{\beta-1}\,\nu(dy)$ as above.

Thus, as long as it is possible to find a solution to the unconstrained problem it is always possible to find a solution to the constraint problem and the optimal trading strategy is given by

$$\widehat{\pi}(t) = \begin{cases} l, & \text{if } \pi_\beta(t) < l \\ \pi_\beta(t), & \text{if } l \le \pi_\beta(t) \le u \\ u, & \text{if } u < \pi_\beta(t). \end{cases}$$

The optimal υ's are respectively, υ_l, 0, and υ_u.

In a very analogue way results can be obtained for the constrained benchmark problem with power utility.

Corollary 6.12. *Let K be a constraint, $\beta < 1$, and let $\boldsymbol{\eta} \in \Pi_+$ be a benchmark strategy. If for a $\boldsymbol{\upsilon} \in \mathcal{D}_K$ the two conditions*

$$\begin{aligned}
&\overline{\boldsymbol{\alpha}}(t) + \boldsymbol{\upsilon}(t) - \boldsymbol{\sigma}(t)\boldsymbol{\eta}(t) - (1-\beta)\boldsymbol{\sigma}(t)\left(\widehat{\boldsymbol{\pi}}_\upsilon^\eta(t) - \boldsymbol{\eta}(t)\right) \\
&\quad + \sum_{h=1}^m \int_{\mathbb{R}\setminus\{0\}} \boldsymbol{\gamma}_h(t,y) \frac{(1+\widehat{\boldsymbol{\pi}}_\upsilon^\eta(t)^\intercal \boldsymbol{\gamma}_h(t,y))^{\beta-1}}{(1+\boldsymbol{\eta}(t)^\intercal \boldsymbol{\gamma}_h(t,y))^\beta} \nu_h(dy) = 0, \text{ and} \\
&\qquad \zeta(\boldsymbol{\upsilon}(t)) + \left(\widehat{\boldsymbol{\pi}}_\upsilon^\eta(t) - \boldsymbol{\eta}(t)\right)^\intercal \boldsymbol{\upsilon}(t) = 0,
\end{aligned}$$

a.s., are satisfied and further $\widehat{\boldsymbol{\pi}}_\upsilon^\eta - \boldsymbol{\eta} \in K$, then $\widehat{\boldsymbol{\pi}}_\upsilon^\eta$ is an optimal trading strategy for the problem Φ_K^υ in (6.17) for the power utility case $U(x) = x^\beta/\beta$.

Proof. The corollary follows from Proposition 6.11. In (6.23) the substitutions (5.14) has to be carried out and condition (6.24) has to be replaced by (6.21). □

It is now possible to apply the same constraints as before. Thus, for example with upper and lower constraints $K_{lu} = \{p \in \mathbb{R} \mid l \le p \le u\}$ the optimal trading strategy is given by

$$\widehat{\pi}(t) = \begin{cases} l, & \text{if } \pi_\beta^\eta(t) - \eta(t) < l, \\ \pi_\beta^\eta(t), & \text{if } l \le \pi_\beta^\eta(t) - \eta(t) \le u, \\ u, & \text{if } u < \pi_\beta^\eta(t) - \eta(t), \end{cases}$$

where π_β^η denotes the solution to the unconstrained optimal relative wealth problem Φ_η in (5.29) under power utility. It has to satisfy, in the one dimensional case (cf. (5.46)),

$$\overline{\alpha}(t) - \beta\sigma(t)\eta(t) - (1-\beta)\sigma(t)\pi_\beta^\eta(t) + \int_{\mathbb{R}\setminus\{0\}} \frac{\left(1+\pi_\beta^\eta(t)\gamma(t,y)\right)^{\beta-1}}{(1+\eta(t)\gamma(t,y))^\beta} \gamma(t,y)\nu(dy) = 0.$$

6.6 Conclusion

We have considered the optimal portfolio selection problem of the previous chapters under constraints. To solve constrained investment-consumption problem it has been embedded into a set of $\boldsymbol{\upsilon}$-problems which again has been associated to a family of $\boldsymbol{\theta}$-problems that dealt with the market incompleteness arising from the jump-diffusion model. The main result has been stated in Theorem 6.9. It gives a condition under which a trading strategy is optimal. The results have been then transferred to the benchmark problem and its main result is summarized in Proposition 6.10. Constraints have been demonstrated in form of upper and lower boundaries on the trading strategy for the power utility case.

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