The Dimensions of Spaces of Holomorphic Second–Order Cusp Forms with Characters

Thomas Blann, Mmath.

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“Some mathematician has said that enjoyment lies in the search for truth, not in the finding of it”

Anna Karenina by Leo Tolstoy

This thesis is dedicated to Janneke Dobben

Abstract

To each pair of characters (χ, ψ) on a Fuchsian group of the first kind we associate a space of functions generalizing the space of second–order cusp forms. We determine the dimensions of these spaces and construct explicit bases. We separate two cases according to the weight. The first case deals with weight higher than 2 whilst the second deals with the more complicated case of weight 2.

An application of these results to Percolation Theory is provided in the last section.

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1 Introduction

Automorphic forms are an important area of modern number theory, particularly in the setting of modular forms which describe some of the most interesting and deep problems of the day. One such case is Andrew Wiles' proof of Fermat's Last Theorem, a question posed so simply that a ten–year old can comprehend it, and yet over 350 years later, the answer was solved in no small part by the use of modular forms.

Automorphic forms are a natural generalization of periodic functions and were first studied and developed by Henri Poincaré in his doctoral thesis during the early 1880's (calling them Fuchsian functions at the time, after Lazarus Fuchs). Felix Klein developed the theory of automorphic functions (he gave them the name in 1890) in his 1884 book on the icosahedron, which connected the fields of algebra and geometry.

The characteristic property of automorphic functions is that they satisfy a transformation law given by

$$f(\gamma z) = f(z)$$

for some transformations γ acting on the upper half plane. More generally, we can define modular forms of weight k and character χ: If k is an even integer, Γ a Fuchsian group of the first kind and χ a character on Γ, then we call a holomorphic function f on the upper half plane $\mathfrak{H}$ a modular form of weight k if it satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)\chi(\gamma)$$

and if it is "of moderate growth at the cusps" (a condition which will be explained in more detail in the next section).

The theory of second–order automorphic forms has emerged in various guises over the last decade especially. Some of the first contexts in which it appeared were problems in the theory of classical modular forms and in Percolation Theory, a subject in physics which studies the movement of an object inside porous materials (both natural and manmade) and in mathematics which models the action by describing the behaviour of connected clusters in a random graph. The original question of second–order forms of even weight was posed by Don Zagier, regarding their exact dimensions.

A large part of the research on second–order automorphic forms has been foundational. For instance, N. Diamantis and C. O'Sullivan ([DO]) proved formulae about the dimensions of spaces of second–order modular and cusp forms of even weight (denoted $M_k^2(\Gamma)$ and $S_k^2(\Gamma)$ respectively) and that was later extended to all orders ([DS]). In conjunction with this, bases have been found and an analogue of the classical Eichler–Shimura isomorphism (also helping to potentially provide a natural geometric interpretation). The method for achieving the results has been by considering certain Poincaré series $P_{\mathfrak{a}m}(z, L)_k$.

However, the dimension results of [DO] and [DS] cannot be applied directly in the context of Percolation Theory because the second–order automorphic forms appearing in [KZ] use characters. This motivates the introduction and study of an extended space of second–order automorphic forms. We show that in the case with non–trivial characters, the dimensions attain their *a priori* maximum. This contrasts with the case without characters where the dimension of second–order forms of weight > 2 is equal to the natural upper bound as given in [CDO] whilst the weight 2 case differs from this bound by 1.

In Chapter 2, we begin by giving the basic definitions and ideas that will be needed. We recall the definition of a modular and a cusp form of weight k and extend the definitions to include a character. We set out the foundations of parabolic cohomology as applied to our setting and proceed to define the modular symbol which is crucial to the study of our work. Finally, we define and establish the basic properties of the twisted Poincaré series.

The material in Chapters 3 and 4 is original work and draws from the joint paper ([BD]) with N. Diamantis. Chapter 3 centres on finding the dimensions of $S_k^2(\Gamma; \chi, \psi)$ and $M_k^2(\Gamma; \chi, \psi)$ when $k \geqslant 4$ is an even integer. We desire to create explicit bases for these spaces and do so by considering certain Poincaré series $P_{\mathfrak{a}m}(z, L, \chi)$.

In Chapter 4, we wish to attain the same outcomes for $k = 2$, namely bases and dimensions of $S_2^2(\Gamma; \chi, \psi)$ and $M_2^2(\Gamma; \chi, \psi)$. We cannot employ the same technique in this case because the function $P_{\mathfrak{a}m}(z, L, \chi)_2$ are no longer absolutely convergent, so instead we consider the series $Z_m(z, s; f, \chi)$ which is a linear combination of two separate non–holomorphic functions $U_m(z, s, k; \chi)$ and $G_m(z, s; \overline{f}, \chi)$. We have to perform some analysis in order to analytically continue it to our desired region of absolute convergence and once we have proved these theorems which bound our functions, we construct the basis functions from $Z_m(z, s; f, \chi)$. We are then able to specify a set of linear combinations of these $Z_m(z, s; f, \chi)$ which is a basis for $S_2^2(\Gamma; \chi, \psi)/(S_2(\Gamma; \chi) + S_2(\Gamma; \psi))$ and hence we can determine the dimension.

Chapter 5 outlines a potential application to Percolation Theory in the form of crossing probabilities, in which we consider a rectangle and model the probability that one cluster connects the left and right vertical edges of the rectangle. This probability is shown to be a second–order modular form with character, as is the probability that a cluster connects the left and right edges whilst the top and bottom edges remain unconnected.

2 Background definitions

In this chapter, we give some background definitions and material which we will subsequently build up and study in the following chapters.

First, we recall the Fuchsian group Γ and the subgroup $\Gamma_{\mathfrak{a}}$ for a cusp point $\mathfrak{a}$, with a quick thought on the generators. We then introduce the character χ belonging to the group Γ which will be used throughout the study. Next, we define the slash operator $|_{k,\chi}$ which acts on a function f.

Secondly, we set out the formal definitions of both the modular and cusp forms of weight k. These are then developed into second–order modular and cusp forms of weight k and type (χ, ψ), whose spaces we are interested in finding the dimensions of (we denote them $M_k^2(\Gamma; \chi, \psi)$ and $S_k^2(\Gamma; \chi, \psi)$ respectively). We then give a definition on the Poincaré series.

Finally, we shall discuss the cohomology of the spaces $M_k^2(\Gamma; \chi, \psi)$ and $S_k^2(\Gamma; \chi, \psi)$ which leads on to a proposition that shall simplify matters when we calculate a basis for the weight 2 second–order cusp space at the culmination of the study. It will be necessary when evaluating the functions which construct our basis functions to consider certain integrals, and we define those integrals here along with some useful properties. The chapter ends with a bound on the dimensions of the spaces $M_k^2(\Gamma; \chi, \psi)$ and $S_k^2(\Gamma; \chi, \psi)$.

2.1 The group Γ and subgroup $\Gamma_{\mathfrak{a}}$

We start with a few definitions and some notation associated with the primary Poincaré series which we consider.

Let $\Gamma \subset PSL_2(\mathbb{R})$ be a Fuchsian group of the first kind acting on the upper half plane $\mathfrak{H} = \{z = x + iy |\ y > 0\}$ in the usual way, with the non–compact quotient $\Gamma\backslash\mathfrak{H}$. V is the volume of $\Gamma\backslash\mathfrak{H}$. Let $\mathfrak{a}, \mathfrak{b}$ be representatives of inequivalent cusp points of a fundamental domain $\mathfrak{F}$ and let $\sigma_{\mathfrak{a}}$, $\sigma_{\mathfrak{b}} \in SL_2(\mathbb{R})$ be their associated scaling matrices.

$\Gamma_{\mathfrak{a}}$ is the subgroup of all elements of Γ fixing the cusp $\mathfrak{a}$, and

$$\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & mh \\ 0 & 1 \end{pmatrix} \middle| m \in \mathbb{Z} \right\}$$

for some $h \in \mathbb{R}$ and specifically with $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$.

We let $\gamma_{\mathfrak{a}}$ denote a generator of $\Gamma_{\mathfrak{a}}$ and define the matrix T to be $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We will want to know the generators of the group Γ; if we suppose that $\Gamma\backslash\mathfrak{H}$ has genus g with r elliptic fixed points and p cusp points, then Γ is generated by:

- $2g$ hyperbolic elements γ_i, $1 \leqslant i \leqslant 2g$;
- r elliptic elements ε_i, $1 \leqslant i \leqslant r$; and
- p parabolic elements π_i, $1 \leqslant i \leqslant p$

satisfying the $r + 1$ relations

$$[\gamma_1, \gamma_{g+1}] \dots [\gamma_g, \gamma_{2g}]\varepsilon_1 \dots \varepsilon_r \pi_1 \dots \pi_p = 1, \qquad \varepsilon_j^{e_j} = 1 \tag{1}$$

for $1 \leqslant j \leqslant r$ and $e_j \geqslant 2$ integers. $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$ of a,b, as described by [HK], (10).

It is useful here to define formally a character of Γ and a few associated properties.

Definition 2.1. *Let χ be a character of Γ. Suppose that $\chi(\gamma_{\mathfrak{a}}) = e^{2\pi i y_{\mathfrak{a}}}$ for some $0 \leqslant y_{\mathfrak{a}} < 1$. Then $\mathfrak{a}$ is singular if $y_{\mathfrak{a}} = 0$ and non–singular otherwise. Similarly, we say that the character χ is singular if $\chi(\gamma_{\mathfrak{a}}) = 1$; otherwise it is non–singular.*

Definition 2.2. *Let χ be a character of Γ, $\overline{\chi}$ its conjugate and k be an even integer. We define the slash operator $|_{k,\chi}$ on a function $f : \mathfrak{H} \to \mathbb{C}$ by*

$$(f|_{k,\chi}\,\gamma)(z) = f(\gamma z)(cz+d)^{-k}\overline{\chi(\gamma)}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$.

We extend the action to $\mathbb{C}[PSL_2(\mathbb{R})]$ by linearity.
For convenience, set $j(\gamma, z) = cz + d$ so that we can re–write

$$(f|_{k,\chi}\,\gamma)(z) = f(\gamma z)j(\gamma, z)^{-k}\overline{\chi(\gamma)}.$$

When χ is the trivial character $\mathbf{1}$, we set $|_k$ for $|_{k,\mathbf{1}}$. We call k the weight.

Similarly, we can extend the definition of the slash operator to include two characters, so that if we let χ, ψ be two characters of Γ, and let $|_{k,\psi}$ act on f followed by $|_{k,\chi}$ acting on the result, we get:

$$\begin{aligned}((f|_{k,\chi}\,\gamma)|_{k,\psi}\,\delta)(z) &= (f|_{k,\chi}\,\gamma)(\delta z)j(\delta, z)^{-k}\overline{\psi(\delta)} \\ &= f(\gamma\delta z)j(\gamma, \delta z)^{-k}\overline{\chi(\gamma)}\;\overline{\psi(\delta)}\end{aligned}$$

where $\gamma, \delta \in PSL_2(\mathbb{R})$.

Remark: We set for use later the notation

$$\varepsilon(\gamma, z) = \frac{j(\gamma, z)}{|j(\gamma, z)|}. \tag{2}$$

It is useful here to note for further reference an identity of $|j(\gamma, z)|^2$, (see [Iw1], (1.11)): For $\gamma \in \Gamma$, we have

$$|j(\gamma, z)|^2 = \frac{\operatorname{Im}(z)}{\operatorname{Im}(\gamma z)}. \tag{3}$$

Note that since $z = x + iy$, $\operatorname{Im}(z) = \operatorname{Im}(x + iy) = y$ so this simplifies to $|j(\gamma, z)|^2 = y/\operatorname{Im}(\gamma z)$.

We will use the Poincaré series (see [Iw2]) in the analysis of the weight $k = 2$ case and modified versions of it in the case $k \geqslant 4$, so we give a formal definition now:

Definition 2.3. *Let k be an integer, m a non–negative integer and $\mathfrak{a}$ a cusp. Then we define the Poincaré series $P_{\mathfrak{a}m}(z)$ to be*

$$P_{\mathfrak{a}m}(z) = P_{\mathfrak{a}m}(z)_k = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\, e^{2\pi i m \sigma_{\mathfrak{a}}^{-1}\gamma z}.$$

2.2 Modular forms of orders one and two

Ultimately, we would like to find a basis for $M_k^2(\Gamma;\chi)$, the space of second–order modular forms of weight k with character χ for Γ, and $S_k^2(\Gamma;\chi)$, the space of holomorphic second–order cusp forms, and then to determine their dimensions. We set out a formal definition of the spaces of orders one and two before proceeding.

Let $z = x + iy$. We say that "f is holomorphic at the cusps" if, for each cusp $\mathfrak{a}$, $(f|_k\, \sigma_{\mathfrak{a}})(z) \ll y^c$ as $y \to \infty$ uniformly in x for some constant c. We will say "f vanishes at the cusps" if, for each cusp $\mathfrak{a}$, $(f|_k\, \sigma_{\mathfrak{a}})(z) \ll y^c$ as $y \to \infty$ uniformly in x for every constant c.

Definition 2.4. *Let k be an even integer, $\chi \in \Gamma$ be a character and let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic function. f is called a modular (resp. cusp) form of weight k with character χ if:*

i) $f|_{k,\chi}(\gamma - 1) = 0 \qquad \forall\, \gamma \in \Gamma$*; and*

ii) f is holomorphic (resp. vanishes) at the cusps.

Their space is denoted by $M_k(\Gamma;\chi)$ (resp. $S_k(\Gamma;\chi)$).

Definition 2.5. *Let k be an even integer, $\chi,\psi \in \Gamma$ be two characters and let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic function. f is called a second–order modular form of weight k and type (χ,ψ) if:*

i) $f|_{k,\chi}(\gamma - 1) \in M_k(\Gamma;\psi) \qquad \forall\, \gamma \in \Gamma$*; and*

ii) There is an $f_0 \in M_k(\Gamma;\psi)$ such that for all parabolic $\pi \in \Gamma$, $f|_{k,\chi}(\pi - 1) = ((\psi\overline{\chi})(\pi) - 1)\, a_\pi f_0$ for some $a_\pi \in \mathbb{C}$; and

iii) f is holomorphic at the cusps.

Their space is denoted by $M_k^2(\Gamma;\chi,\psi)$.

Condition ii) means that $f|_{k,\chi}(\pi - 1) = 0$ whenever $\chi(\pi) = \psi(\pi)$ and, otherwise, $f|_{k,\chi}(\pi - 1) = c_\pi f_0$ for some f_0 independent of π and a $c_\pi \in \mathbb{C}\backslash\{0\}$.

Definition 2.6. *Let k be an even integer, $\chi,\psi \in \Gamma$ be two characters and let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic function. f is called a second–order cusp form of weight k and type (χ,ψ) if:*

i) $f|_{k,\chi}(\gamma - 1) \in S_k(\Gamma;\psi) \qquad \forall\, \gamma \in \Gamma$*; and*

ii) *There is an* $f_0 \in M_k(\Gamma;\psi)$ *such that for all parabolic* $\pi \in \Gamma$, $f|_{k,\chi}(\pi - 1) = ((\psi\overline{\chi})(\pi) - 1)a_\pi f_0$ *for some* $a_\pi \in \mathbb{C}$*; and*

iii) f *vanishes at the cusps.*

Their space is denoted by $S_k^2(\Gamma;\chi,\psi)$*.*

Remark: The percolation crossing formulas $\pi_b, \pi_{\bar{b}}$ and n studied in [DK] are almost in the space $M_0^2(\Gamma(2);\mathbf{1},\psi)$, where ψ is the character of $\eta(z)^4$ and η is the Dedekind eta function. The reason that they are not in the space is because they fail to be holomorphic at all the cusps. This justifies the comment made in the Introduction about the need to extend the study of second–order forms to the case of poles at the cusps.

A basis of $S_k(\Gamma;\chi)$ is formed by $\{P_{m_1}, ..., P_{m_s}\}$ for unique, different integers $m_1, ..., m_s$. A reference for this fact for trivial χ is [Iw1] and for general characters is [Ra].

2.3 Cohomology associated to $S_k^2(\Gamma;\chi,\psi)$ and $M_k^2(\Gamma;\chi,\psi)$

Now, we shall state the definition of parabolic cohomology as applied to our case (see [Sh1], P. 223). We use a notation that makes the dependence on the character and other invariants explicit.

Definition 2.7. *Let* χ *be a character of* Γ*. For* $\gamma \in \Gamma$*, consider a representation* ρ_χ *of* Γ *such that*

$$\rho_\chi(\gamma)(v) = \chi(\gamma)\, v \quad \forall\, v \in \mathbb{C}.$$

Then we set

$$\begin{aligned} Z^1_{par}(\Gamma,\rho_\chi) &:= \{f : \Gamma \to \mathbb{C} | f(\gamma_1\gamma_2) = \rho_\chi(\gamma_1)(f(\gamma_2)) + f(\gamma_1), \forall\, \gamma_1, \gamma_2 \in \Gamma, \\ &\qquad f(\pi_i) = (\rho_\chi(\pi_i) - 1)(a_i) \;\; (i = 1, \dots, p) \text{ for some } a_i \in \mathbb{C}\} \\ B^1_{par}(\Gamma,\rho_\chi) &:= B^1(\Gamma,\rho_\chi) := \{f : \Gamma \to \mathbb{C} | \exists\, a \in \mathbb{C} | \forall\, \gamma \in \Gamma, f(\gamma) = (\rho_\chi(\gamma) - 1)a\}. \end{aligned}$$

Then

$$H^1_{par}(\Gamma,\rho_\chi) := Z^1_{par}(\Gamma,\rho_\chi)/B^1_{par}(\Gamma,\rho_\chi).$$

In order to simplify the notation, we write $H^1_{par}(\Gamma,\chi)$ instead of $H^1_{par}(\Gamma,\rho_\chi)$.

Now, for characters χ,ψ of Γ, we fix a basis of $M_k(\Gamma;\psi)$ to be $\{f_i\}_{i=1}^d$ where $d:=\dim(M_k(\Gamma;\psi))$. Let $f\in M_k^2(\Gamma;\chi,\psi)$. Then

$$f|_{k,\chi}(\gamma-1)=\sum_{i=1}^{d}c_i(\gamma^{-1})f_i \tag{4}$$

for some $c_i(\gamma^{-1})\in\mathbb{C}$. We take γ^{-1} because we want the induced cocycle to be in terms of a left action. $f\in M_k^2(\Gamma;\chi,\psi)$ implies that

$$\begin{aligned} f|_{k,\chi}(\gamma-1) &= f|_{k,\chi}(\gamma-1)|_{k,\psi}\delta = f|_{k,\chi}((\gamma-1)\delta)\,\overline{\psi(\delta)}\,\chi(\delta)\\ &= (f|_{k,\chi}(\gamma\delta-1)-f|_{k,\chi}(\delta-1))\,\overline{\psi(\delta)}\,\chi(\delta). \end{aligned}$$

So, for $i=1,\ldots,d$, we have $c_i(\gamma^{-1})=(c_i(\delta^{-1}\gamma^{-1})-c_i(\delta^{-1}))\,\overline{\psi(\delta)}\,\chi(\delta)$.

Using the transformations $\gamma^{-1}\longmapsto\gamma$ and $\delta^{-1}\longmapsto\delta$, we can write this as

$$c_i(\delta\gamma)=\overline{\psi(\delta)}\,\chi(\delta)\,c_i(\gamma)+c_i(\delta).$$

By condition ii) of Definition 2.5, $c_i(\pi_i)\in(\rho_{\overline{\psi}\cdot\chi}(\pi_i)-1)(\mathbb{C})$ where $i=1,\ldots,p$, and so c_i induces an element $[c_i]$ of $H^1_{par}(\Gamma;\overline{\psi}\cdot\chi)$. So, we have a linear map

$$\begin{aligned} \phi &: M_k^2(\Gamma;\chi,\psi)\to H^1_{par}(\Gamma;\overline{\psi}\cdot\chi)\otimes M_k(\Gamma;\psi)\\ \phi &: f\longmapsto\sum_{i=1}^{d}[c_i]\otimes f_i. \end{aligned}$$

Analogously, we have the map

$$\phi' \;:\; S_k^2(\Gamma;\chi,\psi)\to H^1_{par}(\Gamma;\overline{\psi}\cdot\chi)\otimes S_k(\Gamma;\psi).$$

Proposition 2.8. *The kernel of the map ϕ (resp. ϕ') is isomorphic to the image of $M_k(\Gamma;\chi)+M_k(\Gamma;\psi)$ (resp. $S_k(\Gamma;\chi)+S_k(\Gamma;\psi)$) under the natural injection into $M_k^2(\Gamma;\chi,\psi)$ (resp. $S_k^2(\Gamma;\chi,\psi)$).*

Proof. We can easily see that $M_k(\Gamma;\chi)+M_k(\Gamma;\psi)\subset\ker(\phi)$. In the opposite direction, suppose that $f\in\ker(\phi)$. Then we have $c_i\in B^1(\Gamma;\overline{\psi}\cdot\chi)$ or $c_i=a_i(\overline{\psi(\gamma)}\cdot\chi(\gamma)-1)$ for some constants $a_i\in\mathbb{C}$. Then (4) implies

$$f|_{k,\chi}(\gamma-1)=\left(\sum_{i=1}^{d}a_if_i\right)(\psi(\gamma)\cdot\overline{\chi(\gamma)}-1).$$

Since $F:=\sum_{i=1}^{d}a_if_i\in M_k(\Gamma;\psi)$, the righthand side is equal to $\overline{\chi(\gamma)}F|_k\gamma-F=F|_{k,\chi}(\gamma-1)$. Therefore, $f-F\in M_k(\Gamma;\chi)$ which implies the assertion. The proof of the statement of the cuspidal case is performed similarly. $\square$

In preparation for our next definition and in order to estimate the dimension of $H^1_{par}(\Gamma, \chi)$, we define the following map:

Definition 2.9. *Let χ be a character of Γ. We associate to each $F = (f, \overline{g}) \in S_2(\Gamma;\chi) \bigoplus \overline{S_2(\Gamma;\overline{\chi})}$ and $a \in \mathfrak{H} \cup \mathrm{Cusps}(\Gamma)$ a map $L_F(a, \cdot) : \Gamma \to \mathbb{C}$ given by*

$$L_F(a,\gamma) = \int_a^{\gamma a} f(w)dw + \overline{\int_a^{\gamma a} g(w)dw}. \tag{5}$$

Definition 2.10. *For $f \in S_2(\Gamma)$, an arbitrary $z \in \mathfrak{H} \cup \mathrm{Cusps}(\Gamma)$ and $\mathfrak{a}$ a cusp, we define*

$$\Lambda_f(\mathfrak{a}, z) := \int_{\mathfrak{a}}^{z} f(w)dw \quad \textit{where } f \in S_2(\Gamma;\chi). \tag{6}$$

We now set out two lemmas which we will need in the next chapter in order to evaluate our main results. Firstly, we consider the integral between ∞ and $\gamma\infty$ of a function $f \in S_2(\Gamma)$ and relate it to $\Lambda_f(\mathfrak{a}, z)$:

Lemma 2.11. *For $f \in S_2(\Gamma)$, $z \in \mathfrak{H} \cup \mathrm{Cusps}$ and a cusp $\mathfrak{a}$, we have*

$$\int_\infty^{\gamma\infty} f(w)dw = \Lambda_f(\mathfrak{a}, \gamma z) - \Lambda_f(\mathfrak{a}, z) \tag{7}$$

Proof.

$$\begin{aligned}
\int_\infty^{\gamma\infty} f(w)dw &= \int_\infty^{z} f(w)dw + \int_z^{\gamma z} f(w)dw + \int_{\gamma z}^{\gamma\infty} f(w)dw \\
&= \int_\infty^{z} f(w)dw + \int_z^{\gamma z} f(w)dw + \int_z^{\infty} f(\gamma w)d(\gamma w) \\
&\quad \text{[using a change of variables]} \\
&= \int_\infty^{z} f(w)dw + \int_z^{\gamma z} f(w)dw + \int_z^{\infty} f(w)dw \\
&\quad \text{[since } f \text{ is a weight 2 form]} \\
&= \int_z^{\gamma z} f(w)dw \\
&= \int_z^{\mathfrak{a}} f(w)dw + \int_{\mathfrak{a}}^{\gamma z} f(w)dw \\
&= -\int_{\mathfrak{a}}^{z} f(w)dw + \int_{\mathfrak{a}}^{\gamma z} f(w)dw \\
&= -\Lambda_f(\mathfrak{a}, z) + \Lambda_f(\mathfrak{a}, \gamma z).
\end{aligned}$$

□

We replicate the previous theorem with $f \in S_2(\Gamma;\chi)$:

Lemma 2.12. *For $f \in S_2(\Gamma;\chi)$, $z \in \mathfrak{H} \cup \mathrm{Cusps}$ and a cusp $\mathfrak{a}$, we have*

$$\int_{\infty}^{\gamma\infty} f(w)dw = (1-\chi(\gamma))\, \Lambda_f(\infty, z) + \Lambda_f(\mathfrak{a}, \gamma z) - \Lambda_f(\mathfrak{a}, z). \tag{8}$$

Proof.

$$\begin{aligned}
\int_{z}^{\gamma z} f(w)dw &= \int_{z}^{\infty} f(w)dw + \int_{\infty}^{\gamma\infty} f(w)dw + \int_{\gamma\infty}^{\gamma z} f(w)dw \\
&= \int_{z}^{\infty} f(w)dw + \int_{\infty}^{\gamma\infty} f(w)dw + \int_{\infty}^{z} f(\gamma w)d(\gamma w) \\
&= \int_{z}^{\infty} f(w)dw + \int_{\infty}^{\gamma\infty} f(w)dw - \chi(\gamma)\int_{z}^{\infty} f(w)dw \\
&= (\chi(\gamma)-1)\int_{\infty}^{z} f(w)dw + \int_{\infty}^{\gamma\infty} f(w)dw \\
\Rightarrow \int_{\infty}^{\gamma\infty} f(w)dw &= (1-\chi(\gamma))\int_{\infty}^{z} f(w)dw + \int_{z}^{\gamma z} f(w)dw.
\end{aligned}$$

But by the previous Lemma 2.11, $\int_z^{\gamma z} f(w)dw = \Lambda_f(\mathfrak{a}, \gamma z) - \Lambda_f(\mathfrak{a}, z)$ and since ∞ is a cusp, $\int_\infty^z f(w)dw = \Lambda_f(\infty, z)$ by definition and so we have the desired result

$$\int_{\infty}^{\gamma\infty} f(w)dw = (1-\chi(\gamma))\, \Lambda_f(\infty, z) + \Lambda_f(\mathfrak{a}, \gamma z) - \Lambda_f(\mathfrak{a}, z). \qquad \square$$

Using Lemma 2.12, we have

$$\int_{z_1}^{\gamma z_1} f(w)dw = \int_{z_2}^{\gamma z_2} f(w)dw + (1-\chi(\gamma))\int_{z_1}^{z_2} f(w)dw \quad \forall\, z_1, z_2 \in \mathfrak{H} \cup \mathrm{Cusps}(\Gamma) \tag{9}$$

which shows that $L_F(a,\cdot) \in Z^1_{par}(\Gamma,\chi)$ and that it depends on a only up to coboundaries. Using a special case of the Eichler–Shimura isomorphism (cf. [CDO], Ch 8), the map

$$S_2(\Gamma;\chi) \bigoplus \overline{S_2(\Gamma;\overline{\chi})} \to H^1_{par}(\Gamma,\chi)$$

which sends F to the cohomology class $[L_F]$ of $L_F(a,\cdot)$ is an isomorphism. Consequently, along with Proposition 2.8, we deduce that

$$\dim\ M_k^2(\Gamma;\chi,\psi) \leqslant d_0\ \dim\ M_k(\Gamma;\psi) + \dim\left(M_k(\Gamma;\chi) + M_k(\Gamma;\psi)\right) \tag{10}$$

where $d_0 := \dim\ S_2(\Gamma;\overline{\psi}\cdot\chi) + \dim\ S_2(\Gamma;\overline{\chi}\cdot\psi)$. In particular, $M_k^2(\Gamma;\chi,\psi)$ is finite dimensional.

Similarly,

$$\dim\ S_k^2(\Gamma;\chi,\psi) \leqslant d_0\ \dim\ S_k(\Gamma;\psi) + \dim\,(S_k(\Gamma;\chi)+S_k(\Gamma;\psi)). \tag{11}$$

We wish to fix a basis of $H^1_{par}(\Gamma;\chi)$. Suppose that f_i with $i=1,\ldots,\dim\ S_2(\Gamma;\chi)$ is a basis of $S_2(\Gamma;\chi)$ and that $f_{j+\dim\ S_2(\Gamma;\chi)}$ with $j=1,\ldots,\dim\ S_2(\Gamma;\overline{\chi})$ is a basis of $S_2(\Gamma;\overline{\chi})$. Consider the basis of the space $S_2(\Gamma;\chi)\bigoplus\overline{S_2(\Gamma;\overline{\chi})}$ formed by $F_i := (f_i,0)$ $(i=1,\ldots,\dim\ S_2(\Gamma;\chi))$ and $F_{j+\dim\ S_2(\Gamma;\chi)} := (0,\overline{f}_{j+\dim\ S_2(\Gamma;\chi)})$ $(j=1,\ldots,\dim\ S_2(\Gamma;\overline{\chi}))$. Then the set

$$\{[L_i]; i=1,\ldots,\dim\ S_2(\Gamma;\chi)+\dim\ S_2(\Gamma;\overline{\chi})\}$$

with

$$L_i := L_{F_i}(a_i,\cdot) \tag{12}$$

for a choice of $a_i\in\mathfrak{H}\cup\mathrm{Cusps}(\Gamma)$ is a basis of $H^1_{par}(\Gamma,\chi)$.

Lemma 2.13. *Let $F=(f,\bar{g})\in S_2(\Gamma;\chi)\bigoplus\overline{S_2(\Gamma;\overline{\chi})}$ and $L_F(a,\gamma)=\int_a^{\gamma a}f(w)dw+\overline{\int_a^{\gamma a}g(w)dw}$ with $a\in\mathfrak{H}\cup\mathrm{Cusps}(\Gamma)$ be as above. Then for each $z\in\mathfrak{H}\cup\mathrm{Cusps}(\Gamma)$ and $\gamma\in\Gamma$ we have*

$$L_F(a,\gamma)=\Lambda_f(a,\gamma z)+\overline{\Lambda_g(a,\gamma z)}-\chi(\gamma)\left(\Lambda_f(a,z)+\overline{\Lambda_g(a,z)}\right).$$

Proof. Let $z\in\mathfrak{H}\cup\mathrm{Cusps}(\Gamma)$. Then for $f\in S_2(\Gamma;\chi)$ we have

$$\int_a^{\gamma a}f(w)dw=\int_a^{\gamma z}f(w)dw+\int_{\gamma z}^{\gamma a}f(w)dw$$

and similarly for $g\in S_2(\Gamma;\overline{\chi})$. Upon a change of variables, the second integral equals

$$\int_z^a f(\gamma w)d(\gamma w)=-\chi(\gamma)\int_a^z f(w)dw$$

and hence we deduce the result. □

3 Bases of $S_k^2(\Gamma;\chi,\psi)$ and $M_k^2(\Gamma;\chi,\psi)$ for $k \geqslant 4$

We shall concentrate initially on the case when the even integer $k \geqslant 4$. We begin by defining the Poincaré series with character $P_{\mathfrak{a}m}(z,\chi)_k$ which spans the space $S_k(\Gamma;\chi)$. If there are p^* inequivalent cusps, then the space $M_k(\Gamma;\chi)$ is spanned by $P_{\mathfrak{a}m}(z,\chi)_k$ (where m is a positive integer) and the p^* linearly independent series $P_{\mathfrak{a}0}(z,\chi)_k$. We shall extend the definition of the Poincaré series to include a homomorphism L (we define the series as $P_{\mathfrak{a}m}(z,L,\chi)_k$). In order to prove that this is absolutely convergent and holomorphic, we bound L. Finally, we give a proposition to prove which spaces $P_{\mathfrak{a}m}(z,L,\chi)_k$ and $P_{\mathfrak{a}0}(z,L,\chi)_k$ live in and the chapter culminates in a proof of the dimensions of $S_k^2(\Gamma;\chi,\psi)$ and $M_k^2(\Gamma;\chi,\psi)$.

3.1 The Poincaré series with character – $P_{\mathfrak{a}m}(z, L, \chi)_k$

Let $k \geqslant 4$ be an even integer, $p > 0$, $\mathfrak{a}$ a cusp of Γ and χ a character in Γ. Let p^* denote the number of inequivalent cusps which are singular in χ.

Definition 3.1. *Let k be an integer, m a positive integer and χ a character of Γ. Then we define the following Poincaré series for a cusp $\mathfrak{a}$:*

$$P_{\mathfrak{a}m}(z,\chi)_k = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} e^{2\pi i(m+y_{\mathfrak{a}})\gamma z}\, j(\gamma,z)^{-k}\,\overline{\chi(\gamma)} \tag{13}$$

where $\chi(\gamma_{\mathfrak{a}}) = e^{2\pi i y_{\mathfrak{a}}}$ for some $y_{\mathfrak{a}} \in [0,1)$.

For each fixed cusp $\mathfrak{a}$, the space $S_k(\Gamma;\chi)$ is spanned by the Poincaré series $P_{\mathfrak{a}m}(z,\chi)_k$ as m ranges over the positive integers (see [Ra], Theorem 5.2.4 or [HK], Section 2). A basis for the space $M_k(\Gamma;\chi)$ when $k \geqslant 4$ is comprised of the above Poincaré series $P_{\mathfrak{a}m}(z,\chi)_k$ together with the p^* linearly independent $P_{\mathfrak{a}0}(z,\chi)_k$ as $\mathfrak{a}$ varies over p^* inequivalent singular cusps.

When $m = 0$ and $\mathfrak{a}$ is non–singular in χ, the series (13) are called holomorphic Eisenstein series. If we let $E_k(\Gamma;\chi)$ denote the space spanned by these Eisenstein series of weight k, then we have the direct sum

$$M_k(\Gamma;\chi) = E_k(\Gamma;\chi) \bigoplus S_k(\Gamma;\chi). \tag{14}$$

To prove that the dimensions of $S_k^2(\Gamma;\chi,\psi)$ and $M_k^2(\Gamma;\chi,\psi)$ attain the upper bounds (10) and (11) we consider the Poincaré series with homomorphism as defined thus:

Definition 3.2. *Let k be an integer, m a non–negative integer, $\mathfrak{a}$ a cusp and χ a character of Γ. We define*

$$P_{\mathfrak{a}m}(z,L,\chi)_k = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} L(\mathfrak{a},\gamma)\, e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma z}\, j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-k}\,\overline{\chi(\gamma)} \tag{15}$$

where $L \in Z^1_{par}(\Gamma;\chi\cdot\overline{\psi})$ is a cocycle associated with the cusp $\mathfrak{a}$ and $\chi(\gamma_{\mathfrak{a}}) = e^{2\pi i x_{\mathfrak{a}}}$.

To show that these series are absolutely convergent and holomorphic for $k \geqslant 4$ we need to bound L. To this end we prove:

Lemma 3.3. *Let χ be a character of Γ. For any $f \in S_2(\Gamma;\chi)$, $z_0 \in \mathfrak{H} \cup \mathrm{Cusps}(\Gamma)$, all $z \in \mathfrak{H}$ and any cusp $\mathfrak{a}$, we have*

$$\int_{z_0}^{z} f(w)dw \ll \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}z)^{\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}z)^{-\varepsilon} + 1$$

uniformly in x, with the implied constant depending on $f, \mathfrak{F}, \mathfrak{a}$ and ε but independent of z.

Proof. By a change of variables, we have

$$\int_{\infty}^{\sigma_{\mathfrak{a}} z} f(w)dw = \int_{\sigma_{\mathfrak{a}}^{-1}\infty}^{z} (f|_2\sigma_{\mathfrak{a}})(w)dw.$$

However, $f|_2\sigma_{\mathfrak{a}} \in S_2(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}};\chi')$ for some character χ' (by [Ra], Theorem 4.9.3). Further, for every Fuchsian group of the first kind G, a character χ on G, $f \in S_2(G;\chi)$ and $z \in \mathfrak{H}$, we have $|yf(z)| \ll 1$. Indeed, this holds, by compactness, in the closure of a fundamental domain of $G\backslash\mathfrak{H}$. On the other hand, $|\operatorname{Im}(\gamma z)f(\gamma z)| = |yf(z)| \ \ \forall\, \gamma \in G$ and thus, the bound holds on the entire $\mathfrak{H}$. Therefore, $(f|_2\sigma_{\mathfrak{a}})(w) \ll \operatorname{Im}(w)^{-1} \ \ \forall\, w \in \mathfrak{H}$. This implies that

$$\begin{aligned}\int_{\sigma_{\mathfrak{a}}^{-1}\infty}^{z} (f|_2\sigma_{\mathfrak{a}})(w)dw &= \int_{\sigma_{\mathfrak{a}}^{-1}\infty}^{\infty} (f|_2\sigma_{\mathfrak{a}})(w)dw + \int_{\infty}^{z} (f|_2\sigma_{\mathfrak{a}})(w)dw \\ &= \int_{\sigma_{\mathfrak{a}}^{-1}\infty}^{\infty} (f|_2\sigma_{\mathfrak{a}})(w)dw + \int_{\infty}^{n+x+iy} (f|_2\sigma_{\mathfrak{a}})(w)dw\end{aligned}$$

where we write $z = n + x + iy$ with $0 \leqslant x < 1$ and $n \in \mathbb{Z}$. The last integral is equal to

$$\int_{\infty}^{x+iy} (f|_2\sigma_{\mathfrak{a}}T^n)(w)dw = e^{2\pi i n y_{\mathfrak{a}}} \int_{\infty}^{x+iy} (f|_2\sigma_{\mathfrak{a}})(w)dw$$

for some $y_{\mathfrak{a}} \in \mathbb{R}$ since $f|_2\sigma_{\mathfrak{a}} \in S_2(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}};\chi')$. This implies that

$$\begin{aligned}\int_{\sigma_{\mathfrak{a}}^{-1}\infty}^{z} (f|_2\sigma_{\mathfrak{a}}T^n)(w)dw &= \int_{\sigma_{\mathfrak{a}}^{-1}\infty}^{\infty} (f|_2\sigma_{\mathfrak{a}}T^n)(w)dw \\ &\quad + e^{2\pi i n x_{\mathfrak{a}}}\left(\int_{\infty}^{1} (f|_2\sigma_{\mathfrak{a}})(x+it)dt + \int_{1}^{y} (f|_2\sigma_{\mathfrak{a}})(x+it)dt\right) \\ &\ll 1 + \int_{1}^{y} \Big|(f|_2\sigma_{\mathfrak{a}})(x+it)\Big|dt \\ &\ll 1 + \int_{1}^{y} \frac{1}{t}dt = 1 + \log y\end{aligned}$$

uniformly in x, with the implied constant depending on $\mathfrak{a}, f$ and $\mathfrak{F}$. Since, for all ε, $\log(y^{\varepsilon}) < y^{\varepsilon} + y^{-\varepsilon} \ \ \forall\, y > 0$, we deduce that

$$\int_{\infty}^{\sigma_{\mathfrak{a}} z} f(w)dw \ll 1 + y^{\varepsilon} + y^{-\varepsilon}$$

with the implied constant further depending on ε. Upon the transformation $z \longmapsto \sigma_{\mathfrak{a}}^{-1}z$, the result follows immediately. □

We now determine which holomorphic space the series $P_{\mathfrak{a}m}(z, L_i(\mathfrak{a}, \cdot), \chi)_k$ lives in.

Proposition 3.4. *Let $k \geqslant 4$ be an even integer and characters $\chi, \psi \in \Gamma$. For each $\mathfrak{a} \in \mathrm{Cusps}(\Gamma)$ and $L_i(\mathfrak{a}, \cdot) \in Z^1_{par}(\Gamma; \chi \cdot \overline{\psi})$ as in (12), with $i = 1, \ldots, d_0$ and $d_0 = \dim(S_2(\Gamma; \chi \cdot \overline{\psi})) + \dim(S_2(\Gamma; \psi \cdot \overline{\chi}))$, we have:*

$$\begin{aligned} P_{\mathfrak{a}0}(z, L_i(\mathfrak{a}, \cdot), \chi)_k &\in M^2_k(\Gamma; \chi, \psi) \quad \textit{if } \mathfrak{a} \textit{ is singular in } \psi; \\ P_{\mathfrak{a}m}(z, L_i(\mathfrak{a}, \cdot), \chi)_k &\in S^2_k(\Gamma; \chi, \psi) \quad \textit{if } m > 0. \end{aligned}$$

Proof. We first show that each term of the series is independent of the representative in $\Gamma_{\mathfrak{a}}\backslash\Gamma$. The cocycle condition of $L_i(\mathfrak{a}, \cdot)$ implies that $L_i(\mathfrak{a}, \gamma_{\mathfrak{a}}\gamma) = \chi(\gamma_{\mathfrak{a}})\overline{\psi(\gamma_{\mathfrak{a}})}L_i(\mathfrak{a}, \gamma)$ because clearly $L_i(\mathfrak{a}, \gamma_{\mathfrak{a}}) = 0$. Using the identity $\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}} = T\sigma_{\mathfrak{a}}^{-1}$, where we recall that T is the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we deduce that

$$\begin{aligned} &L_i(\mathfrak{a}, \gamma_{\mathfrak{a}}\gamma)\, j(\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\gamma, z)^{-k}\, e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\gamma z}\, \overline{\chi(\gamma_{\mathfrak{a}}\gamma)} \\ &\quad = L_i(\mathfrak{a}, \gamma)\, \chi(\gamma_{\mathfrak{a}})\, \overline{\psi(\gamma_{\mathfrak{a}})}\, j(T\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\, e^{2\pi i(m+x_{\mathfrak{a}})T\sigma_{\mathfrak{a}}^{-1}\gamma z}\, \overline{\chi(\gamma_{\mathfrak{a}}\gamma)} \\ &\quad = L_i(\mathfrak{a}, \gamma)\, j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k}\, e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma z}\, \overline{\chi(\gamma)}. \end{aligned}$$

To prove the convergence, we first note that, by Lemma 3.3 and Lemma 2.13,

$$L_i(\mathfrak{a}, \gamma) \ll \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{-\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} z)^{\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} z)^{-\varepsilon} + 1$$

for $i = 1, \ldots, d_0$. Therefore

$$\begin{aligned} &P_{\mathfrak{a}m}(z, L_i(\mathfrak{a}, \cdot), \chi)_k \\ &\quad \ll \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} (\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{-\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} z)^{\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} z)^{-\varepsilon} + 1) \\ &\qquad \times |j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)|^{-k} \\ &\quad = y^{-k/2} \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} (\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{k/2+\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{k/2-\varepsilon}) \\ &\qquad + y^{-k/2}(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} z)^{\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} z)^{-\varepsilon} + 1) \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{k/2} \end{aligned} \tag{16}$$

for any $\varepsilon > 0$ and the implied constant depending on ε. Since the non–holomorphic Eisenstein series

$$E_{\mathfrak{a}}(z, s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s} \tag{17}$$

is absolutely convergent for s with $\mathrm{Re}(s) > 1$, the equation (16) implies the absolute and uniform (for z in compact sets in $\mathfrak{H}$ – see (3.11) of [Iw1]) convergence of $P_{\mathfrak{a}m}(z, L_i, \chi)_k$ for $k/2 - \varepsilon > 1$ and hence for $k \geqslant 4$.

Next, we determine the growth at the cusps. We recall from [Iw1] the functions

$$\phi_{\mathfrak{ab}}(s) = \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}\sum_{c} c^{-2s} S_{\mathfrak{ab}}(0,0;c)$$

and the Kloosterman sum (see section 2.5 of [Iw1] for specific details)

$$S_{\mathfrak{ab}}(0,0;c) = \#\left\{ d(mod\ c) : \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\}$$

and $W_s(z)$ the usual Whittaker function defined as

$$W_s(z) = 2y^{1/2}K_{s-1/2}(2\pi y)e^{x},$$

with $K_\nu(z)$ the Bessel K-function

$$K_\nu(z) = \frac{\pi}{2}\frac{1}{sin(\pi\nu)}\left(\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+1-\nu)}\left(\frac{z}{2}\right)^{-\nu+2k} - \sum_{k=0}^{\infty}\frac{1}{k!\Gamma(k+1+\nu)}\left(\frac{z}{2}\right)^{\nu+2k}\right).$$

Then we recall that $E_{\mathfrak{a}}(z,s)$ has the Fourier expansion at the cusp $\mathfrak{b}$

$$\begin{aligned} E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z, s) &= \delta_{\mathfrak{ab}}y^s + \phi_{\mathfrak{ab}}(s)y^{1-s} + \sum_{m\neq 0}\phi_{\mathfrak{ab}}(m,s)W_s(mz) \\ &= \delta_{\mathfrak{ab}}y^s + \phi_{\mathfrak{ab}}(s)y^{1-s} + O(e^{-2\pi y}) \end{aligned} \tag{18}$$

as $y \to \infty$ with an implied constant depending only on s and Γ (see (6.20) of [Iw1]). This is valid for all $s \in \mathbb{C}$. This and the fact that $L_i(\mathfrak{a}, I) = 0$ where I is the identity element of Γ, yields

$$\begin{aligned} & j(\sigma_{\mathfrak{b}}, z)^{-k}P_{\mathfrak{a}m}(\sigma_{\mathfrak{b}}, L_i(\mathfrak{a},\cdot),\chi)_k \\ &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} L_i(\mathfrak{a},\gamma) j(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}, z)^{-k} e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z} \\ &\ll y^{-k/2}\sum_{\substack{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma \\ \gamma\neq\Gamma_{\mathfrak{a}}}} |L_i(\mathfrak{a},\gamma)|\,\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)^{k/2} \\ &\ll y^{-k/2}\Big(|E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z, k/2-\varepsilon) - \delta_{\mathfrak{ab}}y^{k/2-\varepsilon}| \\ &\quad +(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\sigma_{\mathfrak{b}}z)^{\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\sigma_{\mathfrak{b}}z)^{-\varepsilon} + 1)|E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z, k/2) - \delta_{\mathfrak{ab}}y^{k/2}|\Big). \end{aligned}$$

Since $\mathrm{Im}(gz) \asymp y^{-1}$ for $g \in SL_2(\mathbb{R})\backslash\{I\}$, this is $\ll y^{-k/2} \times y^{1-(k/2-\varepsilon)} = y^{1-k+\varepsilon}$ as $y \to \infty$ uniformly in x. Therefore, $P_{\mathfrak{a}m}(z, L_i(\mathfrak{a}, \cdot), \chi)_k$ vanishes at the cusps for $m > 0$ as well as $m = 0$.

To verify the transformation law, we re–write $P_{\mathfrak{a}m}(z, L_i(\mathfrak{a}, \cdot), \chi)_k$ in the form

$$P_{\mathfrak{a}m}(z, L_i(\mathfrak{a}, \cdot), \chi)_k = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} L_i(\mathfrak{a}, \gamma) e^{2\pi i (m+x_{\mathfrak{a}})\cdot}|_k \sigma_{\mathfrak{a}}^{-1}\gamma \overline{\chi(\gamma)}$$

and thus we have

$$\begin{aligned} P_{\mathfrak{a}m}(z, L_i(\mathfrak{a}, \cdot), \chi)_k|_{k,\chi}\delta &= \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} L_i(\mathfrak{a}, \gamma) e^{2\pi i (m+x_{\mathfrak{a}})\cdot}|_k \sigma_{\mathfrak{a}}^{-1}\gamma\delta \overline{\chi(\gamma\delta)} \\ &= \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} L_i(\mathfrak{a}, \gamma\delta^{-1}) e^{2\pi i (m+x_{\mathfrak{a}})\cdot}|_k \sigma_{\mathfrak{a}}^{-1}\gamma \overline{\chi(\gamma)}. \end{aligned}$$

This and the cocycle condition of $L_i(\mathfrak{a}, \cdot)$ imply that

$$\begin{aligned} P_{\mathfrak{a}m}(\cdot, L_i(\mathfrak{a}, \cdot), \chi)_k|_{k,\chi}(\delta - 1) &= \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} L_i(\mathfrak{a}, \delta^{-1}) e^{2\pi i (m+x_{\mathfrak{a}})\cdot}|_k \sigma_{\mathfrak{a}}^{-1}\gamma \chi(\gamma)\overline{\psi(\gamma)}\ \overline{\chi(\gamma\delta)} \\ &= L_i(\mathfrak{a}, \delta^{-1}) P_{\mathfrak{a}m}(\cdot, \psi). \end{aligned} \tag{19}$$

Therefore, condition i) of the definition of $S_k^2(\Gamma; \chi, \psi)$, (2.6) (resp. $M_k^2(\Gamma; \chi, \psi)$, (2.5)) holds for the series $P_{\mathfrak{a}m}(z, L_i(\mathfrak{a}, \cdot), \chi)$, if $m > 0$ (resp. $P_{\mathfrak{a}0}(z, L_i(\mathfrak{a}, \cdot), \chi)$, if $\mathfrak{a}$ is singular in ψ).

Equation (19) also shows condition ii) of the definition of second–order forms: By (9) applied with $\gamma = \pi$ parabolic, $z_1 = \mathfrak{a}$ and $z_2 =$ a fixed point of π, we deduce that $L_i(\mathfrak{a}, \pi) = (\chi(\pi)\overline{\psi(\pi)} - 1)a_\pi$ for some constant $a_\pi \in \mathbb{C}$. Since the cocycle condition of $L_i(\mathfrak{a}, \cdot)$ implies that $L_i(\mathfrak{a}, \pi^1) = -\psi(\pi)\overline{\chi(\pi)}L_i(\mathfrak{a}, \pi)$, we deduce that $P_{\mathfrak{a}m}(\cdot, L_i(\mathfrak{a}, \cdot), \chi)_k|_{k,\chi}(\pi - 1)$ has the form stipulated by condition ii) of the definition. □

3.2 The dimensions of $S_k^2(\Gamma; \chi, \psi)$ and $M_k^2(\Gamma; \chi, \psi)$ for $k \geqslant 4$ an even integer

We are now in a position to finish the chapter with our main aim, to determine the dimensions of the two spaces $S_k^2(\Gamma; \chi, \psi)$ and $M_k^2(\Gamma; \chi, \psi)$ (for $k \geqslant 4$ an even integer).

Theorem 3.5. *For $k \geqslant 4$ an even integer and $d_0 := \dim\left(S_2(\Gamma; \chi \cdot \overline{\psi}) + S_2(\Gamma; \psi \cdot \overline{\chi})\right)$, we have*

$$\begin{aligned} \dim S_k^2(\Gamma; \chi, \psi) &= d_0 \dim S_k(\Gamma; \psi) + \dim\left(S_2(\Gamma; \chi) + S_2(\Gamma; \psi)\right) && (20) \\ \dim M_k^2(\Gamma; \chi, \psi) &= d_0 \dim M_k(\Gamma; \psi) + \dim\left(M_2(\Gamma; \chi) + M_2(\Gamma; \psi)\right). && (21) \end{aligned}$$

Proof. To obtain a basis for $S_k^2(\Gamma; \chi, \psi)$, we first consider the set A of series $P_{\infty j}(z, L_i(\infty, \cdot), \chi)$, as j runs over integers yielding a basis $P_{\infty j}(z, \psi)$ for $S_k(\Gamma; \chi)$ and as i runs over integers in $\{1, \dots, d_0\}$ inducing a basis $[L_i]$ of $H^1_{par}(\Gamma; \chi \cdot \overline{\psi})$. With (19), these series are all linearly independent because the linear independence of L_i implies the linear independence of $L_i(\infty, \cdot)$. We further consider a basis B of $S_k(\Gamma; \chi) + S_k(\Gamma; \psi)$. As such a basis, we may choose the union of bases of $S_k(\Gamma; \chi)$ and $S_k(\Gamma; \psi)$, if $\psi \not\equiv \chi$, or, otherwise, a basis of $S_k(\Gamma; \chi)$. Because of (19) and the fact that the L_i's (resp. j's) are chosen to induce a basis of $H^1_{par}(\Gamma; \chi \cdot \overline{\psi})$ (resp. $S_k(\Gamma; \chi)$), the set $A \cup B$ is linearly independent and, in particular, the set $A \cap B = \emptyset$. The cardinality of $A \cup B$ equals the upper bound in (10), so $A \cup B$ is a basis of $S_k^2(\Gamma; \chi, \psi)$. This proves (20).

A similar argument, using the fact that $P_{\mathfrak{a}0}(z, \psi)_k$ with $\mathfrak{a}$ running over the inequivalent cusps of $\Gamma \backslash \mathfrak{H}$ which are singular in terms of ψ form a basis of $E_k(\Gamma; \psi)$, yields (21). $\square$

Remark: The dimensions appearing in Theorem 3.5 can be evaluated explicitly using the formulae for the dimensions of modular forms for $k > 0$ as presented, for instance, in [HK]: If χ is a character in Γ, then, with the notation used in (1), set $q = p + \sum_{j=1}^{r}(1 - 1/e_f)$, $\chi(\pi_i) = e(x_i)$ and $\chi(\varepsilon_i) = ((k + a_j)/(2e_j))$ for some $x_i \in [0, 1), e_j \in [0, e_j - 1]$. Then we have

$$\dim M_k(\Gamma; \chi) = k(g - 1 + q/2) - \sum_{i=1}^{p} x_i - \sum_{j=1}^{r} a_j/e_j - g + 1$$

and

$$\dim S_k(\Gamma; \chi) = k(g - 1 + q/2) - \sum_{i=1}^{p} x_i - \sum_{j=1}^{r} a_j/e_j - g + 1 - p^* + \delta$$

where $\delta = 0$ unless $k = 2$ and $\chi \equiv 1$.

4 The basis of $S_2^2(\Gamma; \chi, \psi)$

As mentioned in the introduction, the case $k = 2$ requires a different treatment. Before we formulate and prove the main theorem, we introduce some important functions which we will need, and we will prove some auxiliary propositions which are of independent interest.

We begin with definitions of three functions: $U_{\mathfrak{a}m}(z, s, k; \chi)$, $Q_{\mathfrak{a}m}(z, s, n; \overline{f}, \chi)$ and $G_{\mathfrak{a}m}(z, s; \overline{f}, \chi)$. These sets of functions will provide the analysis required to provide the basis elements at the end of the chapter. We provide the necessary bounds on the functions.

Next, the requisite Spectral Theory is laid out for later use and we introduce the Maass raising and lowering operators. Some useful identities and formulae are also calculated.

The last of the "big" analytical functions $Z_m(z, s; f, \chi)$ is presented before we are ready to tackle our main objective: The basis elements and dimension of $S_2^2(\Gamma; \chi, \psi)$. This is achieved in three stages. First, we adjust the function $Z_m(z, s; f, \chi)$ until it is in our required form. Then, we use these amended functions to create our basis elements. Finally, we can determine the dimension of the space $S_2^2(\Gamma; \chi, \psi)$.

4.1 Preliminary results and definitions

As in the previous chapter, we consider a Fuchsian group of the first kind Γ with non–compact quotient. In addition, we assume that ∞ is a cusp for Γ. We fix characters χ, ψ on Γ such that $\chi(T) = e^{2\pi i m y_\infty}$ and $\psi(T) = e^{2\pi i m x_\infty}$ where $0 \leqslant y_\infty, x_\infty < 1$ and m is an integer. We begin by defining the Poincaré series with character:

Definition 4.1. *For m a non–negative integer, $z \in \mathfrak{H}$, $s \in \mathbb{Z}$ with $\mathrm{Re}(s) > 1$, $\mathfrak{a}$ a cusp, $\gamma \in \Gamma$, and χ a character in Γ, we define the non–holomorphic, weight k Poincaré series with character χ*

$$U_{\mathfrak{a}m}(z,s,k;\chi) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \, e^{2\pi i (m+y_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma z} \, \varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k} \overline{\chi(\gamma)}.$$

For simplicity, we use the following shorthand functions:

- $U_{\mathfrak{a}m}(z,s;\chi) := U_{\mathfrak{a}m}(z,s,0;\chi)$;
- $U_m(z,s,k;\chi) := U_{\infty m}(z,s,k;\chi)$; and
- $U_m(z,s,k) := U_{\infty m}(z,s,k;\mathbf{1})$ where $\mathbf{1}$ represents the trivial character.

For this next function, we fix $f(z) \in S_2(\Gamma; \overline{\chi}\cdot\psi)$ and set

$$I_{\mathfrak{a}n}(z_n) = \begin{cases} f_{\mathfrak{a}}^{(-n)}(z_n) & \text{if } n < 0 \\ \int_{i\infty}^{z_n} \dots \int_{i\infty}^{z_2} \int_{i\infty}^{z_1} f_{\mathfrak{a}}(z_0) dz_0 dz_1 \dots dz_{n-1} & \text{if } n \geqslant 0 \end{cases}$$

for $f_{\mathfrak{a}}(z) = f(\sigma_{\mathfrak{a}} z)/j(\sigma_{\mathfrak{a}}, z)^2$. Thus, we have $\frac{d}{dz} I_{\mathfrak{a}n}(z) = I_{\mathfrak{a}(n-1)}(z)$ and we can use this to extend the definition of $I_{\mathfrak{a}n}$ to all integers n. Thus:

$$\begin{aligned} I_{\mathfrak{a}1}(\sigma_{\mathfrak{a}}^{-1} z) &= \Lambda_f(\mathfrak{a}, z) = \int_{\mathfrak{a}}^{z} f(z) dw; \text{ and} \\ I_{\mathfrak{a}0}(\sigma_{\mathfrak{a}}^{-1} z) &= f(z)\, j(\sigma_{\mathfrak{a}}^{-1}, z)^{-2}. \end{aligned} \tag{22}$$

We now define a function $Q_{\mathfrak{a}m}(z,s,n;\overline{f},\chi)$:

Definition 4.2. *For $f \in S_2(\Gamma;\chi)$, $z \in \mathfrak{H}$, $\mathfrak{a}$ a cusp, n an integer, $\gamma \in \Gamma$, and χ a character in Γ, we have*

$$Q_{\mathfrak{a}m}(z,s,n;\overline{f},\chi) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{I_{\mathfrak{a}n}(\sigma_{\mathfrak{a}}^{-1}\gamma z)} \, \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \, e^{2\pi i (m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma z} \, \overline{\chi(\gamma)}.$$

We can show that, for all $l \in \mathbb{Z}$, $\overline{I_{\infty n}(z+l)} = e^{2\pi i l(y_\infty - x_\infty)}\overline{I_n(z)}$. Therefore, our function $Q_{\mathfrak{a}m}(z,s,n;\overline{f},\chi)$ is well–defined (for s in a right halfplane which is to be determined in Theorem 4.4).

We also set the definition of $G_{\mathfrak{a}m}(z,s;\overline{f},\chi)$:

Definition 4.3. *For $f \in S_2(\Gamma;\chi)$, $\mathfrak{a}$ a cusp, $\gamma \in \Gamma$, and χ a character in Γ, we have*

$$G_{\mathfrak{a}m}(z,s;\overline{f},\chi) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \frac{\overline{\Lambda_f(\mathfrak{a},\gamma z)}}{j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^2}\, \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s\, e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma z}\,\overline{\chi(\gamma)}.$$

The next theorem describes the original regions of convergence of the above series and their bounds. To state it, we recall the notation of [Iw1], (2.42) for the invariant height:

$$y_\Gamma(z) = \max_{\mathfrak{a}}(\max_{\gamma\in\Gamma}(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))).$$

If ψ (or $|\psi|$) is smooth with weight 0, then

$$\psi(z) \ll y_\Gamma(z)^A$$

if and only if $\psi(\sigma_{\mathfrak{a}} z) \ll y^A$ for each cusp $\mathfrak{a}$ as $y \to \infty$. We also use the notation

$$y_{\mathfrak{F}}(z) = \max_{\mathfrak{a}}(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} z))$$

for $z \in \mathfrak{F}$.

4.2 Bounding $U_{\mathfrak{a}m}(z,s,k;\chi), Q_{\mathfrak{a}m}(z,s,1;\overline{f},\chi)$ and $G_{\mathfrak{a}m}(z,s-1;\overline{f},\chi)$

We are now ready to describe the behaviour of the series $U_{\mathfrak{a}m}(z,s,k;\chi)$, $Q_{\mathfrak{a}m}(z,s,1;\overline{f},\chi)$, the derivative of $Q_{\mathfrak{a}m}(z,s,1;\overline{f},\chi)$, and $G_{\mathfrak{a}m}(z,s-1;\overline{f},\chi)$ in the region of absolute convergence.

Theorem 4.4. *Let k be an even integer, $\sigma = \mathrm{Re}(s) > 1$, and χ, ψ characters on Γ with $\chi(\gamma_{\mathfrak{a}}) = e^{2\pi i y_{\mathfrak{a}}}$ and $\psi(\gamma_{\mathfrak{a}}) = e^{2\pi i x_{\mathfrak{a}}}$. For $f \in S_2(\Gamma;\overline{\chi}\cdot\psi)$, the series $U_{\mathfrak{a}m}(z,s,k;\chi)$, $Q_{\mathfrak{a}m}(z,s,1;\overline{f},\chi)$, $Q'_{\mathfrak{a}m}(z,s,1;\overline{f},\chi)$ and $G_{\mathfrak{a}m}(z,s-1;\overline{f},\chi)$ converge absolutely and uniformly on compact sets to analytic functions of s. For these s we have:*

i) $U_{\mathfrak{a}0}(z,s,k;\chi) \ll y_\Gamma(z)^\sigma$

ii) $U_{\mathfrak{a}m}(z,s,k;\chi) \ll 1, \quad m > 0$

iii) $Q_{\mathfrak{a}m}(z,s,1;\overline{f},\chi) \ll y_\Gamma(z)^{1/2-\sigma/2}, \quad m \geqslant 0$

iv) $yQ'_{\mathfrak{a}m}(z,s,1;\overline{f},\chi) \ll (|m|+1)y_\Gamma(z)^{1/2-\sigma/2}, \quad m \geqslant 0$

v) $yG_{\mathfrak{a}m}(z,s-1;\overline{f},\chi) \ll y_\Gamma(z)^{1/2-\sigma/2}, \quad m \geqslant 0$

where the implied constants depend on s,k,f,χ,ψ *and* Γ *but not on* m.

Proof. We will prove the five statements separately and in order.

i) Begin with the Eisenstein series (17). For each cusp $\mathfrak{a}$ and taking $m=0$, we have

$$U_{\mathfrak{a}0}(z,s,k;\chi) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s\,\overline{\chi(\gamma)} \ll E_{\mathfrak{a}}(z,s)\overline{\chi}$$

by definition of $U_{\mathfrak{a}m}$. Since by (18), $E_{\mathfrak{a}}(z,s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$ and satisfies

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s) = \delta_{\mathfrak{a}\mathfrak{b}}y^s + \phi_{\mathfrak{a}\mathfrak{b}}(s)y^{1-s} + O(e^{-2\pi y})$$

as $y\to\infty$ (see [Iw1]), we deduce that

$$U_{\mathfrak{a}0}(z,s,k;\chi) \ll E_{\mathfrak{a}}(z,\sigma)\overline{\chi} \ll y_\Gamma(z)^\sigma$$

as $y\to\infty$.

ii) We first note that, for $m>0$, $|e^{2\pi i(m+y_{\mathfrak{a}})\sigma_{\mathfrak{a}}^{-1}\gamma z}| \leqslant 1$. This implies that

$$\begin{aligned} U_{\mathfrak{a}m}(\sigma_{\mathfrak{a}}z,s,k;\chi) &\ll y^\sigma e^{-2\pi(m+y_{\mathfrak{a}})y} + \sum_{\substack{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma\\ \gamma\notin\Gamma_{\mathfrak{a}}}} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{a}}z)^\sigma \\ &\ll y^\sigma e^{-2\pi(m+y_{\mathfrak{a}})y} + |E_{\mathfrak{a}}(\sigma_{\mathfrak{a}}z,\sigma) - y^\sigma| \ll 1. \end{aligned}$$

Consider now another cusp $\mathfrak{b}\neq\mathfrak{a}$ – then

$$U_{\mathfrak{a}m}(\sigma_{\mathfrak{b}}z,s,k;\chi) \ll E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,\sigma) \ll \phi_{\mathfrak{a}\mathfrak{b}}(s)y^{1-\sigma} \ll 1. \tag{23}$$

Therefore, for all cusps $\mathfrak{b}$, $U_{\mathfrak{a}m}(\sigma_{\mathfrak{b}}z,s,k;\overline{\chi}) \ll 1$ as $y\to\infty$.

iii) We separate again into two cases:
$\mathfrak{b}\neq\mathfrak{a}$: Lemma 3.3 implies that for any $\varepsilon>0$, we have

$$\Lambda_f(\mathfrak{a},\gamma\sigma_{\mathfrak{b}}z) = \Lambda_f(\mathfrak{a},\mathfrak{a}\mathfrak{a}^{-1}\gamma\sigma_{\mathfrak{b}}z) \ll \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)^\varepsilon + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)^{-\varepsilon} + 1 \tag{24}$$

for any cusp $\mathfrak{b}$ and any $z \in \mathfrak{H}$. The implied constant is only dependent on ε, f and Γ. Therefore, for $\sigma > 1+\varepsilon$

$$Q_{\mathfrak{a}m}(\sigma_{\mathfrak{a}} z, s, 1; \overline{f}, \chi) \ll E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, \sigma+\varepsilon) + E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, \sigma-\varepsilon) + E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, \sigma) \ll y^{1-\sigma+\varepsilon} \tag{25}$$

as $y \to \infty$ by (18).

$\mathfrak{a} = \mathfrak{b}$: We first observe that (with Lemma 3.3)

$$\begin{aligned}
&Q_{\mathfrak{a}m}(\sigma_{\mathfrak{a}} z, s, 1; \overline{f}, \chi) \\
&\quad = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{F_{\mathfrak{a}}(\sigma_{\mathfrak{a}} z)} \, \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma \sigma_{\mathfrak{a}} z)^s \, e^{2\pi i (m+\alpha) \sigma_{\mathfrak{a}}^{-1} \gamma \sigma_{\mathfrak{a}} z} \, \overline{\chi(\gamma)} \\
&\quad \ll \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} |\Lambda_f(\mathfrak{a}, \gamma \sigma_{\mathfrak{a}} z)| \, \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma \sigma_{\mathfrak{a}} z)^{\sigma} \\
&\quad \ll |\Lambda_f(\mathfrak{a}, \sigma_{\mathfrak{a}} z)| \\
&\qquad + \sum_{\substack{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma \\ \gamma \notin \Gamma_{\mathfrak{a}}}} \left(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma \sigma_{\mathfrak{a}} z)^{\sigma+\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma \sigma_{\mathfrak{a}} z)^{\sigma-\varepsilon} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma \sigma_{\mathfrak{a}} z)^{\sigma} \right)
\end{aligned} \tag{26}$$

as $y \to \infty$. Since, with a change of variables $w \longmapsto \sigma_{\mathfrak{a}} w$,

$$\Lambda_f(\mathfrak{a}, \sigma_{\mathfrak{a}} z) = \int_{\infty}^{z} f(w) dw$$

we deduce that

$$\Lambda_f(\mathfrak{a}, \sigma_{\mathfrak{a}} z) \ll e^{-2\pi y} \tag{27}$$

as $y \to \infty$. Therefore, with (26), $Q_{\mathfrak{a}m}(\sigma_{\mathfrak{a}} z, s, 1; \overline{f}, \chi) \ll y^{1-\sigma+\varepsilon}$ for $\sigma > 1+\varepsilon$ as $y \to \infty$.

Setting (in both cases) $\varepsilon = (\sigma - 1)/2$, we deduce iii).

iv) Differentiating for $\gamma \in PSL_2(\mathbb{R})$, we have

$$2iy\frac{d}{dz}\left(\overline{\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{a}}\gamma z)}\ \mathrm{Im}(\gamma z)^s\, e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}\gamma z}\,\overline{\chi(\gamma)}\right)$$

$$\begin{aligned}
&= 2iy\,\overline{\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{a}}\gamma z)}\,\frac{d}{dz}\left(\mathrm{Im}(\gamma z)^s\right)\ e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}\gamma z}\,\overline{\chi(\gamma)}\\
&\quad +2iy\,\overline{\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{a}}\gamma z)}\ \mathrm{Im}(\gamma z)^s\,\frac{d}{dz}\left(e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}\gamma z}\right)\ \overline{\chi(\gamma)}\\
&= 2iy\,\overline{\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{a}}\gamma z)}\,\frac{s}{2i}\,\frac{\mathrm{Im}(\gamma z)^{s-1}}{j(\gamma,z)^2}\ e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}\gamma z}\,\overline{\chi(\gamma)}\\
&\quad +2iy\,\overline{\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{a}}\gamma z)}\ \mathrm{Im}(\gamma z)\frac{2\pi i(m+x_\infty)}{j(\gamma,z)^2}\ e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}\gamma z}\,\overline{\chi(\gamma)}\\
&= sy\,\overline{\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{a}}\gamma z)}\ \mathrm{Im}(\gamma z)^s\,\frac{|j(\gamma,z)|^2}{yj(\gamma,z)^2}\ e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}\gamma z}\,\overline{\chi(\gamma)}\\
&\quad -4\pi(m+x_\infty)y\,\overline{\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{a}}\gamma z)}\ \mathrm{Im}(\gamma z)^{s+1}\,\frac{|j(\gamma,z)|^2}{yj(\gamma,z)^2}\ e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}\gamma z}\,\overline{\chi(\gamma)}\\
&= s\,\overline{\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{a}}\gamma z)}\ \mathrm{Im}(\gamma z)^s\, e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}\gamma z}\,\varepsilon(\gamma,z)^{-2}\,\overline{\chi(\gamma)}\\
&\quad -4\pi(m+x_\infty)\,\overline{\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{a}}\gamma z)}\ \mathrm{Im}(\gamma z)^{s+1}\, e^{2\pi i(m+x_{\mathfrak{a}})\sigma_{\mathfrak{a}}\gamma z}\,\varepsilon(\gamma,z)^{-2}\,\overline{\chi(\gamma)}
\end{aligned}$$

Then

$$\begin{aligned}
yQ'_{\mathfrak{a}m}(z,s,1;\overline{f},\chi) &\ll |s|\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}|\Lambda_f(\mathfrak{a},\gamma z)|\,\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{\sigma}\\
&\quad +|m+x_\infty|\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}|\Lambda_f(\mathfrak{a},\gamma z)|\,\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{\sigma+1}\\
&\ll (|m|+1)\,y_\Gamma(z)^{1-\sigma+\varepsilon}.
\end{aligned}$$

Take $\varepsilon = \frac{\sigma-1}{2}$ to get the desired result.

v) Since

$$yG_{\mathfrak{a}m}(z,s;\overline{f},\chi) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\Lambda_f(\mathfrak{a},\gamma z)}\,\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s+1}\, e^{2\pi i(m+x_\infty)\sigma_{\mathfrak{a}}^{-1}\gamma z}\,\varepsilon(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-2}\,\overline{\chi(\gamma)}$$

working as in the proof of iii), we deduce v).

$\square$

4.3 Analytic continuations

Now that we have established the original domain of convergence and bounds for the series which we will need for the construction, we can proceed with their meromorphic continuation to the region we will require. We start be recalling some necessary known results from Spectral Theory.

Definition 4.5. *For $f, g \in S_k(\Gamma)$, the Petersson Inner Product is defined by*

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathfrak{H}} f(z)\overline{g(z)}\, y^k \, d\mu z.$$

The weight k of the inner product used throughout the chapter is weight 2.

Let

$$\Delta = -4y^2 \frac{d}{dz}\frac{d}{d\bar{z}}$$

be the hyperbolic Laplacian and, for χ a character of Γ, let

$$L^2(\Gamma\backslash\mathfrak{H}, \chi) = \{f : \mathfrak{H} \to \mathbb{C} \text{ is smooth, square integrable and} f(\gamma z) = \chi(\gamma) f(z) \;\; \forall\gamma \in \Gamma\}.$$

Any $f \in L^2(\Gamma\backslash\mathfrak{H})$ may be decomposed into constituent parts from the discrete and continuous spectrum of Δ. The Roelcke–Selberg decomposition amounts to the identity

$$\xi(z) = \sum_{j=0}^{\infty}\langle \xi, \eta_j(z)\rangle + \frac{1}{4}\sum_{\mathfrak{b}}{}' \int_{-\infty}^{\infty} \langle \xi, E_{\mathfrak{a}}(\cdot, 1/2 + ir; \chi)\rangle\, E_{\mathfrak{b}}(z, 1/2 + ir; \chi) dr \qquad (28)$$

where η_j denotes a complete orthonormal basis of Maass forms, with corresponding eigenvalues $\lambda_j = s_j(1 - s_j)$ and where the sum $\sum'$ ranges over all inequivalent cusps in terms of which χ is singular (see [He], Sections 1 and 2 of Chapter 7). We write $s_j = \sigma_j + it_j$, with $\sigma_j \geqslant 1/2$ and $t_j \geqslant 0$. The eigenvalues, counted with multiplicity, are ordered as $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$.

Recall that the Weyl–Selberg formula (Equation 7.8 of [Ve]) implies that

$$\#\{j |\, |\lambda_j| \leqslant T\} \ll T. \qquad (29)$$

The decomposition (28) is absolutely convergent for each fixed z and uniform on compact subsets of $\mathfrak{H}$, provided that ξ and $\Delta\xi$ are smooth and bounded.

For each j, the Fourier expansion of η_j is

$$\eta_j(\sigma_{\mathfrak{a}} z) = \rho_{\mathfrak{a}j}(0,\chi) y^{1-s_j} + \sum_{m\neq 0} \rho_{\mathfrak{a}j}(m,\chi)\, W_{s_j}((m+y_\infty)z). \tag{30}$$

For all but finitely many of the j, we have $\lambda_j \geqslant 1/4$; that is, $\sigma_j = 1/2$ and $\rho_{\mathfrak{a}j}(0,\chi) = 0$. For those j with $\lambda_j < 1/4$, we have $\rho_{\mathfrak{a}j}(0,\chi) = 0$ if χ is non–singular at $\mathfrak{a}$ ([Pr], section 3). The constant δ_Γ which we use throughout is chosen so that $1-\delta_\Gamma > \sigma_1 \geqslant 1/2$.

We will first prove some bounds for η_j and the coefficients. We use the bound (8.11) from [JO]

$$W_s(nz) \ll \frac{|s|^{2k}+1}{(|n|y)^{2k-1+\sigma}} |\Gamma(s)| \tag{31}$$

which is valid for $k \geqslant 0$ and $\sigma = \mathrm{Re}(s) > 1/2 - k$, and the implied constant is dependent on σ and k. We will also make use of James Stirling's classical formula for the Gamma function

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2} e^{-\pi|t|/2} \quad \text{as} \quad |t| \to \infty. \tag{32}$$

Since there are at most finitely many j with $\rho_{\mathfrak{a}j}(0,\chi) \neq 0$, we have

$$\rho_{\mathfrak{a}j}(0,\chi) \ll 1. \tag{33}$$

Further, with the Bruggeman–Kuznetsov formula (see [Pr], for instance),

$$\rho_{\mathfrak{a}j}(m,\chi) \ll \frac{|t_j|}{\sqrt{|m+y_{\mathfrak{a}}|}} e^{\pi|t_j|/2}, \quad m \neq 0. \tag{34}$$

In both of the inequalities, the implied constant depends only on Γ and χ.

We can now further bound $\eta_j(z)$: If we substitute (31), (32), (33) and (34) into (30), we deduce that

$$\eta_j(z) \ll y_\Gamma(z)^{1/2} + (|t_j|^{7/2}+1) y_\Gamma(z)^{-3/2} \tag{35}$$

where the implied constant depends on Γ and χ. Now we finally recall similar estimates for the Fourier coefficients of $E_{\mathfrak{a}}(z,s,\chi)$ following from [Pr]. They will be used in the continuous element of the spectral decomposition.

$$|\phi_{\mathfrak{a}\mathfrak{b}}(1/2+ir;\chi)| \ll 1. \tag{36}$$

$$\int_T^{T+1} |\phi_{\mathfrak{a}\mathfrak{b}}(m,1/2+ir;\chi)|^2 dr \ll \frac{T^2}{|m+y_{\mathfrak{a}}|} e^{\pi T}. \tag{37}$$

$$\phi_{\mathfrak{a}\mathfrak{b}}(m,1/2+ir;\chi) \ll |m+y_\infty|^2 \quad \text{for } r \in [T,T+1]. \tag{38}$$

4.4 Maass raising and lowering operators

We now give the required theory and applications of Maass raising and lowering operators, as found in [DO].

Let $C^{\infty}(\Gamma\backslash\mathfrak{H}, k; \chi)$ denote the space of smooth functions ϕ on $\mathfrak{H}$ which transform as

$$\phi(\gamma z) = \varepsilon(\gamma, z)^k \phi(z)\chi(\gamma)$$

for $\gamma \in \Gamma$. Notice that the weight defined by this formula differs from the one previously defined.

Define the *Maass raising and lowering operators* thus:

- $R_k = 2iy\frac{d}{dz} + \frac{k}{2}$
- $L_k = -2iy\frac{d}{d\bar{z}} - \frac{k}{2}$.

They satisfy

$$(R_k f)|_{k+2}\, \gamma = R_k(f|_k\, \gamma) \quad \text{and} \quad (L_k f)|_{k-2}\, \gamma = L_k(f|_k\, \gamma)$$

and so we have the pair of maps

- $R_k : C_{\infty}(\Gamma\backslash\mathfrak{H}, k; \chi) \longrightarrow C_{\infty}(\Gamma\backslash\mathfrak{H}, k+2; \chi)$
- $L_k : C_{\infty}(\Gamma\backslash\mathfrak{H}, k) \longrightarrow C_{\infty}(\Gamma\backslash\mathfrak{H}, k+2)$.

For $n > 0$, we can write

- $R^n := R_{k+2n-2} \dots R_{k+2} R_k$ and
- $L^n := L_{k-2n+2} \dots R_{k-2} R_k$.

We also set R^0 and L^0 for the identity operator.

Next, for each $\tau \in PSL_2(\mathbb{R})$, we define the operator $\theta_{\tau,k} : C^{\infty}(\mathfrak{H}) \longrightarrow C^{\infty}(\mathfrak{H})$ given by

$$\theta_{\tau,k}\phi(z) = \frac{\phi(\tau z)}{\varepsilon(\tau, z)^k}. \tag{39}$$

Using the elementary identities

$$\theta_{\tau,k+2} R_k = R_k \theta_{\tau,k} \quad \text{and} \tag{40}$$

$$\theta_{\tau,k-2} L_k = L_k \theta_{\tau,k} \tag{41}$$

we can derive the relation for $f_{\mathfrak{a}}(z) = f(\sigma_{\mathfrak{a}} z)/j(\sigma_{\mathfrak{a}}, z)^2$

$$L^r\left(y\overline{f_{\mathfrak{a}}(z)}\right) = L^r\theta_{\sigma_{\mathfrak{a}},-2}\left(y\overline{f(z)}\right) \tag{42}$$
$$= \theta_{\sigma_{\mathfrak{a}},-2r-2}L^r\left(y\overline{f(z)}\right) \tag{43}$$
$$= L^r\left(y\overline{f(z)}\right)\Big|_{\sigma_{\mathfrak{a}}z}\varepsilon(\sigma_{\mathfrak{a}}, z)^{2r+2}. \tag{44}$$

Lemma 9.2 of [JO] implies that

$$\begin{aligned} R_kU_{\mathfrak{a}m}(z,s,k;\chi) &= (s+k/2)U_{\mathfrak{a}m}(z,s,k+2;\chi) \qquad (45)\\ &\quad -4\pi(m+y_\infty)U_{\mathfrak{a}m}(z,s+1,k+2;\chi)\\ L_kU_{\mathfrak{a}m}(z,s,k;\chi) &= (s-k/2)U_{\mathfrak{a}m}(z,s,k-2;\chi). \qquad (46)\end{aligned}$$

In [DO] ((8.18) and Lemma 4.3) it is essentially proved that:

Lemma 4.6. *i) For any s with $1/2 \leqslant \mathrm{Re}(s) \leqslant 1-\delta$ with $\delta > 0$ and $l \geqslant 0$, we have*

$$\begin{aligned} R^n\left(W_s((m+y_\infty)z)\right) &\ll (|m+y_\infty|y)^{n-2l-3/2}\left(|s|^{2l+2+n}+1\right)|\Gamma(s)|\\ L^n\left(W_s((m+y_\infty)z)\right) &\ll (|m+y_\infty|y)^{n-2l-3/2}\left(|s|^{2l+2+n}+1\right)|\Gamma(s)|\end{aligned}$$

as $y\to\infty$. The implied constant depends on n, l and δ.

ii) For $n \geqslant 0$, we have

$$R^n\left(\eta_j(z)\right), L^n\left(\eta_j(z)\right) \ll (|t_j|^n+1)y_\Gamma^{1/2} + (|t_j|^{2n+5}+1)y_\Gamma^{-3/2} \tag{47}$$

with the implied constant depending on n and Γ alone.

iii) For $n \geqslant 0$, we have

$$\int_T^{T+1}|R^nE_{\mathfrak{a}}(z,1/2+ir)|^2dr, \int_T^{T+1}|L^nE_{\mathfrak{a}}(z,1/2+ir)|^2dr \ll T^{4n+12}y_\Gamma(z). \tag{48}$$

4.5 Bounding the Poincaré series $U_{\mathfrak{a}m}(z,s,k;\chi)$ for $m>0$

Theorem 4.7. *Let k be an even integer, $m>0$ and χ be a character in terms of which ∞ is singular. Then the Poincaré series $U_{\mathfrak{a}m}(z,s,k;\chi)$ has an analytic continuation for all s with $\mathrm{Re}(s) > 1-\delta_\Gamma$ and*

$$U_{\mathfrak{a}m}(z,s,k;\chi) \ll y_\Gamma(z)^{1/2}$$

for those s, where the implied constant depends on s, m, k and Γ only.

Proof. First, we prove the result for $k=0$ and then extend it based on weight raising and lowering operators.

Equation (44) (as seen in [Pr]) gives explicitly the spectral decomposition of $U_{\mathfrak{a}m}(z,s,0)$:

$$U_m(z,s,0)2^{2s-1}\pi^{s-1}(m+y_\infty)^{s-1/2}\Gamma(s)=\sum_{j=1}^{\infty}\Gamma(s-s_j)\,\Gamma(s-1+s_j)\,\overline{\rho_{\infty j}}(m)\eta_j(z)$$

$$+\frac{1}{4\pi}\sum_{\mathfrak{b}}{}' \int_{-\infty}^{\infty}\Gamma(s-1/2-ir)\,\Gamma(s-1/2+ir)\,\overline{\phi_{\infty\mathfrak{b}}}(m,1/2+ir;\chi)E_{\mathfrak{b}}(z,1/2+ir;\chi)dr$$

where $\rho_{\infty j}(m)$ are the Fourier coefficients of Maass forms η_j and

$$E_{\mathfrak{b}}(z,s,\chi)=\delta_{\mathfrak{a}\infty}y^s+\delta_{0y_\infty}\phi_{\infty\mathfrak{b}}(s)y^{1-s}+\sum_{l\neq 0}\phi_{\infty\mathfrak{b}}(l,s,\chi)\,W_s((l+y_\infty)z).$$

(Recall that the eigenvalues of η_j satisfy $\lambda_i=s_j(1-s_j)$).

Definition 4.8. *In the spectral decomposition of $U_m(z,s,0)$, we define the discrete spectral component to be*

$$U_m(z,s,0)_D=\sum_{j=1}^{\infty}\Gamma(s-s_j)\,\Gamma(s-1+s_j)\,\overline{\rho_{\infty j}}(m)\,\eta_j(z)$$

and the continuous spectral component to be

$$U_m(z,s,0)_C=$$
$$\frac{1}{4\pi}\sum_{\mathfrak{b}}{}' \int_{-\infty}^{\infty}\Gamma(s-1/2-ir)\,\Gamma(s-1/2+ir)\,\overline{\phi_{\infty\mathfrak{b}}}(m,1/2+ir;\chi)\,E_{\mathfrak{b}}(z,1/2+ir;\chi)dr.$$

We first observe that the discrete part has the required properties and that these are preserved by the Maass operators. Now, note that, with (34) and (32), we have

$$\Gamma(s-s_j)\Gamma(s-1+s_j)\overline{\rho_{\infty j}}(m)\ll\frac{|t_j|^{2\sigma-1/2}}{\sqrt{m+y_\infty}}e^{-\pi|t_j|/2}. \tag{49}$$

This inequality, together with Lemma 4.6 and (29), imply that, for fixed s, the series

$$\sum_{j=1}^{\infty}\Gamma(s-s_j)\Gamma(s-1+s_j)\overline{\rho_{\infty j}}(m)R^n(\eta_j(z))$$

converges uniformly for z in any compact set and is bounded by $|m+y_\infty|^{-1/2}y_\Gamma(z)^{1/2}$. It further converges uniformly for s in compact sets with $\mathrm{Re}(s)>1-\delta_\Gamma$ giving an analytic function of s.

Since, upon interchanging summation and differentiation, we get

$$R^n \left(\sum_{j=1}^{\infty} \Gamma(s-s_j)\,\Gamma(s-1+s_j)\,\overline{\rho_{\infty j}}(m)\,\eta_j(z) \right)$$
$$= \sum_{j=1}^{\infty} \Gamma(s-s_j)\,\Gamma(s-1+s_j)\,\overline{\rho_{\infty j}}(m)\,R^n\left(\eta_j(z)\right)$$

we deduce that

$$R^n \left(\sum_{j=1}^{\infty} \Gamma(s-s_j)\,\Gamma(s-1+s_j)\,\overline{\rho_{\infty j}}(m)\,\eta_j(z) \right)$$

is analytic for $\mathrm{Re}(s) > 1-\delta_\Gamma$. Furthermore, for these s, this function satisfies

$$R^n \left(\sum_{j=1}^{\infty} \Gamma(s-s_j)\,\Gamma(s-1+s_j)\,\overline{\rho_{\infty j}}(m)\,\eta_j(z) \right) \ll \sqrt{\frac{y_\Gamma}{|m+y_\infty|}}$$

where the implied constant depends on s, n and Γ.

The same statement holds for similar calculations for L^n.

For the analysis of the continuous spectrum component, we need an auxiliary lemma which slightly generalizes Lemma 4.11 of [DO]:

Lemma 4.9. *For $\psi(r)$ smooth on $[T, T+1]$ and $\mathfrak{a}$ singular in terms of χ, we have*

$$\frac{d}{dz}\int_T^{T+1} \psi(r)\,E_{\mathfrak{a}}(z,1/2+ir;\chi)dr = \int_T^{T+1} \psi(r)\left(\frac{d}{dz}E_{\mathfrak{a}}(z,1/2+ir;\chi)\right)dr.$$

Proof. With the Fourier expansion of (18), we have

$$E_{\mathfrak{a}}(z,1/2+ir;\chi) = \delta_{\mathfrak{a}\infty}\,y^{1/2+ir} + \delta_{y_\infty 0}\,\phi_{\mathfrak{a}\infty}(1/2+ir;\chi)\,y^{1/2-ir} \qquad (50)$$
$$+ \sum_{m\neq 0} \phi_{\mathfrak{a}\infty}(m,1/2+ir;\chi)\,W_{1/2+ir}((m+y_\infty)z).$$

Combine (31), (36) and (38) to see that

$$\int_T^{T+1} \Big(\delta_{\mathfrak{a}\infty}\left|y^{1/2+ir}\right| + \delta_{y_\infty 0}\left|\phi_{\mathfrak{a}\infty}(1/2+ir;\chi)\,y^{1/2-ir}\right|$$
$$+ \sum_{m\neq 0}\left|\phi_{\mathfrak{a}\infty}(m,1/2+ir;\chi)\,W_{1/2+ir}((m+y_\infty)z)\right| \Big) dr < \infty$$

and hence, by Corollary 8.6 of [DO]

$$\int_T^{T+1} \psi(r)\,E_{\mathfrak{a}}(z,1/2+ir;\chi)dr$$
$$= \int_T^{T+1} \psi(r) \Big(\delta_{\mathfrak{a}\infty}\left|y^{1/2+ir}\right| + \delta_{y_\infty 0}\left|\phi_{\mathfrak{a}\infty}(1/2+ir;\chi)\,y^{1/2-ir}\right|$$
$$+ \sum_{m\neq 0}\left|\phi_{\mathfrak{a}\infty}(m,1/2+ir;\chi)\,W_{1/2+ir}((m+y_\infty)z)\right| \Big) dr$$

and hence we can interchange the summation and integration to get

$$\begin{aligned}
&\int_T^{T+1} \psi(r)\, E_{\mathfrak{a}}(z, 1/2+ir;\chi) dr \qquad (51)\\
&\quad = \int_T^{T+1} \psi(r) \left(\delta_{\mathfrak{a}\infty}\, y^{1/2+ir} + \delta_{y_\infty 0}\, \phi_{\mathfrak{a}\infty}(1/2+ir;\chi)\, y^{1/2-ir}\right) dr\\
&\quad\quad + \sum_{m\neq 0} \int_T^{T+1} \psi(r)\, \phi_{\mathfrak{a}\infty}(m, 1/2+ir;\chi)\, W_{1/2+ir}((m+y_\infty)z).
\end{aligned}$$

Using Lemma 4.6, we can deduce a similar expression for the derivative of $E_{\mathfrak{a}}(z, 1/2+ir;\chi)$:

$$\begin{aligned}
&\int_T^{T+1} \psi(r) \left(\frac{d}{dz} E_{\mathfrak{a}}(z, 1/2+ir;\chi)\right) dr = \qquad (52)\\
&\int_T^{T+1} \psi(r) \left(\delta_{\mathfrak{a}\infty} \frac{d(y^{1/2+ir})}{dz} + \delta_{y_\infty 0}\, \phi_{\mathfrak{a}\infty}(1/2+ir;\chi) \frac{d(y^{1/2+ir})}{dz}\right) dr\\
&\quad + \sum_{m\neq 0} \int_T^{T+1} \psi(r) \left(\phi_{\mathfrak{a}\infty}(m, 1/2+ir;\chi)\, W_{1/2+ir}((m+y_\infty)z\right) dr
\end{aligned}$$

Since the derivative of each term in the righthand side of (51) equals the corresponding term of (52) this completes the proof. □

The arguments of the proof of this lemma can be iterated to give

$$\int_T^{T+1} \psi(r)\, E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, 1/2+ir;\chi) dr = \int_T^{T+1} \psi(r)\, (R^n E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, 1/2+ir;\chi)) dr \qquad (53)$$

$$\int_T^{T+1} \psi(r)\, E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, 1/2+ir;\chi) dr = \int_T^{T+1} \psi(r)\, (L^n E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, 1/2+ir;\chi)) dr. \qquad (54)$$

Based on these identities, we can now analytically continue and bound $R^n U_m(z, s, 0;\chi)$:

$$\sum_{\mathfrak{b}}{}' \int_{-\infty}^{\infty} \Gamma(s-1/2-ir)\, \Gamma(s-1/2+ir)\, \overline{\phi_{\infty\mathfrak{b}}}(m, 1/2+ir;\chi)\, E_{\mathfrak{b}}(z, 1/2+ir;\chi) dr. \qquad (55)$$

Specifically,

$$\begin{aligned}
&\sum_{\mathfrak{b}}{}' R^n \int_T^{T+1} \Gamma(s-1/2-ir)\, \Gamma(s-1/2+ir)\, \overline{\phi_{\infty\mathfrak{b}}}(m, 1/2+ir;\chi)\, E_{\mathfrak{b}}(z, 1/2+ir) dr\\
&\ll \sum_{\mathfrak{b}}{}' \int_T^{T+1} \Gamma(s-1/2-ir)\, \Gamma(s-1/2+ir)\, \overline{\phi_{\infty\mathfrak{b}}}(m, 1/2+ir;\chi)\, R^n E_{\mathfrak{b}}(z, 1/2+ir) dr.
\end{aligned}$$

With the Cauchy–Schwarz inequality and Stirling's estimate this is

$$\ll |T|e^{-\pi|T|}\sqrt{\int_T^{T+1} |\phi_{\infty\mathfrak{b}}(m, 1/2+ir;\chi)|^2 dr}\sqrt{\int_T^{T+1} |R^n E_{\mathfrak{b}}(m, 1/2+ir;\chi)|^2 dr}.$$

Finally, with Lemma 4.6 iii) and (37), we deduce that this is

$$\ll |T|^{2n+8}\, e^{-\pi|T|/2}\, |m+y_\infty|^{-1/2}\, y_\Gamma(z)^{1/2}. \tag{56}$$

Adding over all $T \in \mathbb{Z}$, we deduce that

$$R^n(U_{\mathfrak{a}m}(z,s,0)_C), L^n(U_{\mathfrak{a}m}(z,s,0)_C) \ll |m|^{-1/2}\, y_\Gamma(z)^{1/2}$$

is analytic in $\mathrm{Re}(s) > 1-\delta_\Gamma$ and is bounded by $|m+y_\infty|^{-1/2}\, y_\Gamma(z)^{1/2}$.

The same statement and bound holds for the function of s obtained after application of L^n.

We may now finish the proof of Theorem 4.7. With (45) we see that

$$\begin{aligned} R_0 U_{\mathfrak{a}m}(z,s,0) &= s\, U_{\mathfrak{a}m}(z,s,2) - 4\pi m\, U_{\mathfrak{a}m}(z,s+1,2) \text{ and} \\ R_2 R_0 U_{\mathfrak{a}m}(z,s,0) &= s(s+1)\, U_{\mathfrak{a}m}(z,s,4) - 4\pi m(2s+2)\, U_{\mathfrak{a}m}(z,s+1,4) \\ &\quad + (4\pi m)^2\, U_{\mathfrak{a}m}(z,s+2,4). \end{aligned}$$

In general, for $k \geqslant 0$

$$\begin{aligned} U_{\mathfrak{a}m}(z,s,2k) \;=\; & \frac{1}{s(s+1)\dots(s+k-1)}\Big(R^k U_{\mathfrak{a}m}(z,s,0) \\ & + p_1(m,s)\, U_{\mathfrak{a}m}(z,s+1,2k) + \ldots + p_k(m,s)\, U_{\mathfrak{a}m}(z,s+k,2k)\Big) \end{aligned} \tag{57}$$

where the polynomials $p_i, i = 1,\ldots,k$ are in variables m and s. Therefore, using Lemmas 4.6, 4.9 and Theorem 4.4 ii), the righthand side of (57) is analytic for $\mathrm{Re}(s) > 1-\delta_\Gamma$ and bounded by $y_\Gamma(z)^{1/2}$. Similarly for $k < 0$ and we deduce the theorem. □

Before we continue, we choose, once and for all, a constant δ_Γ depending on Γ with $0 < \delta_\Gamma < 1/2$. It is chosen so that poles appearing from the discrete spectrum have real part less that $1-\delta_\Gamma$.

4.6 The twisted Poincaré series $Q_{\mathfrak{a}m}(z,s,1;\overline{f},\chi)$

The purpose of this subsection is to establish the meromorphic continuation of the functions $Q_{\mathfrak{a}m}(z,s,1;\overline{f},\chi)$ and its differential, transforming by χ. Again, it has both independent interest and it is necessary for the construction of our basis functions. To prove the continuation, we will first need an auxiliary theorem:

Theorem 4.10. *For $m > 0$, $-n \leqslant 0$, characters χ,ψ of Γ and $f \in S_2(\Gamma; \overline{\chi} \cdot \psi)$, the series $Q_m(z, s+n+1, -n; \overline{f}, \chi)$ is analytic for s for $\mathrm{Re}(s) > 1 - \delta_\Gamma$. For these s*

- $Q_m(z, s+n+1, -n; \overline{f}, \chi) \ll e^{-\pi y_\Gamma(z)}$
- $R_0 Q_m(z, s+n+1, -n; \overline{f}, \chi) \ll e^{-\pi y_\Gamma(z)}$

with the implied constant depending on m, n, f, s, χ and Γ only.

Proof. We first prove the formula:

$$\overline{f^{(n)}(\gamma z)} = (-2i)^{-n} \, \mathrm{Im}(\gamma z)^{-(n+1)} \sum_{r=0}^{n} (-1)^{n-r} \, \varepsilon(\gamma, z)^{-2(r+1)} \binom{n}{r} \frac{(n+1)!}{(r+1)!} \, L^r (y\overline{f(z)}) \, \overline{\chi(\gamma)}$$

for $f \in S_2(\Gamma)$ and $\gamma \in \Gamma$. We proceed with induction on n:

Now, when $n = 0$, our formula gives

$$\overline{f^{(0)}(\gamma z)} = \overline{f(\gamma z)} = \mathrm{Im}(\gamma z)^{-1} \, \varepsilon(\gamma, z)^{-2} \, y\overline{f(z)} \; \overline{\chi(\gamma)}.$$

The righthand side equals:

$$\begin{aligned}
& \frac{y^{-1}}{|j(\gamma, z)|^{-2}} \, \frac{j(\gamma, \overline{z})^{-2}}{|j(\gamma, \overline{z})|^{-2}} \, y\overline{f(z)} \; \overline{\chi(\gamma)} \\
& \text{[Using (2) and (3)]} \\
= \; & \frac{j(\gamma, \overline{z})^{-2}}{|j(\gamma, z)|^{-4}} \overline{f(z)} \; \overline{\chi(\gamma)} \\
& \text{[Since } |j(\gamma, \overline{z})| = |j(\gamma, z)|\text{]} \\
= \; & \frac{j(\gamma, \overline{z})^{-2}}{j(\gamma, \overline{z})^{-2} \, j(\gamma, \overline{z})^{-2}} \overline{f(z)} \; \overline{\chi(\gamma)} \\
= \; & j(\gamma, \overline{z})^{2} \overline{f(z)} \; \overline{\chi(\gamma)} \\
= \; & \overline{j(\gamma, z)}^{2} \; \overline{f(z)} \; \overline{\chi(\gamma)}.
\end{aligned}$$

This equals the lefthand side by Definition 2.2 as required. So it certainly works for $n = 0$.

Suppose now that the identity holds for n – we prove that it holds for $n + 1$:

First, we use a change of variables $z \longmapsto \gamma^{-1}z$. Then

$$\begin{aligned}
&\overline{f^{(n+1)}(z)} \\
= &\ \frac{d}{d\bar{z}}\Bigg((-2i)^{-n}\operatorname{Im}(z)^{-(n+1)}\sum_{r=0}^{n}(-1)^{n-r}\varepsilon(\gamma^{-1},z)^{2(r+1)} \\
&\times\binom{n}{r}\frac{(n+1)!}{(r+1)!}L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)}\Bigg) \\
= &\ (-2i)^{-n}\frac{-(n+1)}{-2i}\operatorname{Im}(z)^{-(n+1)-1}\times \\
&\sum_{r=0}^{n}(-1)^{n-r}\varepsilon(\gamma^{-1},z)^{2(r+1)}\binom{n}{r}\frac{(n+1)!}{(r+1)!}L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)} \\
&+(-2i)^{-n}\operatorname{Im}(z)^{-(n+1)}\sum_{r=0}^{n}(-1)^{n-r}\frac{d}{d\bar{z}}\left(\theta_{\gamma^{-1},-2(r+1)}(L^r(y\overline{f(\cdot)}))|_{\gamma^{-1}z}\right)\,\overline{\chi(\gamma)} \\
= &\ (-2i)^{-(n+1)}\operatorname{Im}(z)^{-((n+1)+1)}(n+1)\times \\
&\sum_{r=0}^{n}(-1)^{(n+1)-r}\varepsilon(\gamma^{-1},z)^{2(r+1)}\binom{n}{r}\frac{(n+1)!}{(r+1)!}L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)} \\
&+(-2i)^{-(n+1)}\operatorname{Im}(z)^{-((n+1)+1)}\sum_{r=0}^{n}\Bigg((-1)^{n-r}\varepsilon(\gamma^{-1},z)^{2(r+1)}(L^r(y\overline{f(\cdot)}))|_{\gamma^{-1}z} \\
&\times((L_{-2(r+1)}(y\overline{f(\cdot)}))|_{\gamma^{-1}z}+(r+1))\Bigg)\,\overline{\chi(\gamma)}
\end{aligned}$$

$$\begin{aligned}
&\Bigg[\text{Since } \frac{d}{d\bar{z}}\left(\theta_{\gamma^{-1},-2(r+1)}(L^r(y\overline{f(\cdot)}))|_{\gamma^{-1}z}\right) \\
&=-\frac{1}{2iy}\theta_{\gamma^{-1},-2(r+1)}(L^r(y\overline{f(\cdot)}))|_{\gamma^{-1}z}((L_{-2(r+1)}(y\overline{f(\cdot)}))|_{\gamma^{-1}z}+(r+1))\Bigg]
\end{aligned}$$

$$\begin{aligned}
= &\ (-2i)^{-(n+1)}\operatorname{Im}(z)^{-((n+1)+1)}\times \\
&\Bigg\{\sum_{r=0}^{n}(-1)^{(n+1)-r}\varepsilon(\gamma^{-1},z)^{2(r+1)}\binom{n}{r}\frac{(n+1)!}{(r+1)!}(n+1)L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)} \\
&+\sum_{r=0}^{n}(-1)^{(n+1)-r}\varepsilon(\gamma^{-1},z)^{2(r+1)}\binom{n}{r}\frac{(n+1)!}{(r+1)!}(r+1)L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)} \\
&+\sum_{r=0}^{n}(-1)^{n-r}\varepsilon(\gamma^{-1},z)^{2((r+1)+1)}\binom{n}{r}\frac{(n+1)!}{(r+1)!}L^{r+1}(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)}\Bigg\}
\end{aligned}$$

[Using (41), ie. $\theta_{\tau,k-2}L_k = L_k\theta_{\tau,k}$]

Now, use a change of variable on the third summation $r+1 \longmapsto r$:

$$
\begin{aligned}
= \;& (-2i)^{-(n+1)} \operatorname{Im}(z)^{-((n+1)+1)} \times \\
& \Bigg\{ \sum_{r=0}^{n} (-1)^{(n+1)-r} \varepsilon(\gamma^{-1}, z)^{2(r+1)} \binom{n}{r} \frac{(n+1)!}{(r+1)!} (n+1) L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\, \overline{\chi(\gamma)} \\
& + \sum_{r=0}^{n} (-1)^{(n+1)-r} \varepsilon(\gamma^{-1}, z)^{2(r+1)} \binom{n}{r} \frac{(n+1)!}{(r+1)!} (r+1) L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\, \overline{\chi(\gamma)} \\
& + \sum_{r=1}^{n+1} (-1)^{(n+1)-r} \varepsilon(\gamma^{-1}, z)^{2(r+1)} \binom{n}{r} \frac{(n+1)!}{(r+1)!} L^{r+1}(y\overline{f(\cdot)})|_{\gamma^{-1}z}\, \overline{\chi(\gamma)} \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \;& (-2i)^{-(n+1)} \operatorname{Im}(z)^{-((n+1)+1)} \times \\
& \Bigg\{ (-1)^{(n+1)} \varepsilon(\gamma^{-1}, z)^{2} L^0(y\overline{f(\cdot)})|_{\gamma^{-1}z} \binom{n}{0} (n+1)!(n+1)\overline{\chi(\gamma)} \\
& + \varepsilon(\gamma^{-1}, z)^{2((n+1)+1)} L^{n+1}(y\overline{f(\cdot)})|_{\gamma^{-1}z}\, \overline{\chi(\gamma)} \\
& + (-1)^{(n+1)} \varepsilon(\gamma^{-1}, z)^{2} L^0(y\overline{f(\cdot)})|_{\gamma^{-1}z} \binom{n}{0} (n+1)!\overline{\chi(\gamma)} \\
& + \sum_{r=1}^{n} (-1)^{(n+1)-r} \varepsilon(\gamma^{-1}, z)^{2(r+1)} L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\, \overline{\chi(\gamma)} \times \\
& \Bigg\{ \binom{n}{r} \frac{(n+1)!}{(r+1)!}(n+1) + \binom{n}{r-1} \frac{(n+1)!}{r!} + \binom{n}{r} \frac{(n+1)!}{(r+1)!}(r+1) \Bigg\} \Bigg\}
\end{aligned}
$$

Dealing separately with the binomial coefficients for the time being, we have

$$
\begin{aligned}
& \binom{n}{r} \frac{(n+1)!}{(r+1)!}(n+1) = \frac{n!}{r!(n-r)!} \frac{(n+1)!}{(r+1)!}(n+1) \\
& = \frac{(n+1)!}{r!(n-r)!} \frac{(n+1)!}{(r+1)!} = \frac{(n+1)!}{r!((n+1)-r)!} \frac{(n+1)!}{r)!} \frac{((n+1)-r)}{(r+1)} \\
& = \binom{n+1}{r} \frac{(n+1)!}{r)!} \frac{((n+1)-r)}{(r+1)}
\end{aligned}
$$

and also using Pascal's Triangle, $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$, then

$$
\binom{n}{r} \frac{(n+1)!}{(r+1)!}(n+1) + \binom{n}{r-1} \frac{(n+1)!}{r!} + \binom{n}{r} \frac{(n+1)!}{(r+1)!}(r+1)
$$

$$
= \binom{n+1}{r} \frac{(n+1)!}{r!} \frac{((n+1)-r)}{(r+1)} + \binom{n}{r-1} \frac{(n+1)!}{r!} + \binom{n}{r} \frac{(n+1)!}{(r+1)!}(r+1)
$$

$$
= \binom{n+1}{r} \frac{(n+1)!}{r!} \frac{((n+1)-r)}{(r+1)} + \binom{n+1}{r} \frac{(n+1)!}{r!}
$$

$$= \binom{n+1}{r}\frac{(n+1)!}{r!}\left(\frac{((n+1)-r)}{(r+1)}+1\right)$$

$$= \binom{n+1}{r}\frac{(n+1)!}{r!}\left(\frac{((n+1)-r)+r+1}{(r+1)}\right)$$

$$= \binom{n+1}{r}\frac{(n+1)!}{r!}\frac{((n+1)+1)}{(r+1)} = \binom{n+1}{r}\frac{((n+1)+1)!}{(r+1)!}$$

Returning to our equation with this, we have:

$$\begin{aligned}
&\overline{f^{(n+1)}(z)} \\
= \;& (-2i)^{-(n+1)}\,\mathrm{Im}(z)^{-((n+1)+1)} \times \\
&\Big\{(-1)^{(n+1)}\varepsilon(\gamma^{-1},z)^2 L^0(y\overline{f(\cdot)})|_{\gamma^{-1}z}\binom{n}{0}(n+1)!((n+1)+1)\overline{\chi(\gamma)} \\
&+\varepsilon(\gamma^{-1},z)^{2((n+1)+1)}L^{n+1}(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)} \\
&+\sum_{r=1}^{n}(-1)^{(n+1)-r}\varepsilon(\gamma^{-1},z)^{2(r+1)}\binom{n+1}{r}\frac{((n+1)+1)!}{(r+1)!}L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)}\Big\}
\end{aligned}$$

$$\begin{aligned}
= \;& (-2i)^{-(n+1)}\,\mathrm{Im}(z)^{-((n+1)+1)} \times \\
&\Big\{\sum_{r=0}^{0}(-1)^{(n+1)-r}\varepsilon(\gamma^{-1},z)^{2(r+1)}L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\binom{n}{r}\frac{((n+1)+1)!}{(r+1)!}\overline{\chi(\gamma)} \\
&+\sum_{r=n+1}^{n+1}(-1)^{(n+1)-r}\varepsilon(\gamma^{-1},z)^{2(r+1)}L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)} \\
&+\sum_{r=1}^{n}(-1)^{(n+1)-r}\varepsilon(\gamma^{-1},z)^{2(r+1)}\binom{n+1}{r}\frac{((n+1)+1)!}{(r+1)!}L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)}\Big\}
\end{aligned}$$

$$\begin{aligned}
= \;& (-2i)^{-(n+1)}\,\mathrm{Im}(z)^{-((n+1)+1)} \times \\
&\sum_{r=1}^{n}(-1)^{(n+1)-r}\varepsilon(\gamma^{-1},z)^{2(r+1)}\binom{n+1}{r}\frac{((n+1)+1)!}{(r+1)!}L^r(y\overline{f(\cdot)})|_{\gamma^{-1}z}\,\overline{\chi(\gamma)}
\end{aligned}$$

Take $z \longmapsto \gamma z$ and we have the formula as required.

Now, for $n \geqslant 0$ and taking $\mathfrak{a}$ to equal ∞, we have

$$Q_m(z,s,-n;\overline{f},\chi) = \sum_{\gamma'\in\Gamma_\infty\backslash\Gamma}\overline{f_\infty^{(n)}(\sigma_\infty^{-1}\gamma z)}\,\mathrm{Im}(\sigma_\infty^{-1}\gamma z)^s e^{2\pi i(m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z}\overline{\chi(\gamma)}. \qquad (58)$$

So, if we name $\Gamma' = \sigma_\infty^{-1}\Gamma\gamma_\infty$ and note that $\sigma_\infty^{-1}\Gamma_\infty\gamma_\infty = \Gamma_\infty$, we find

$$\begin{aligned} Q_m(\sigma_\infty z, s, -n; \overline{f}, \chi) &= \sum_{\gamma' \in \Gamma_\infty \backslash \Gamma'} \overline{f_\infty^{(n)}(\gamma' z)}\, \mathrm{Im}(\gamma' z) e^{2\pi i(m+\alpha)\gamma' z}\overline{\chi(\gamma)} \\ = & \ (-2i)^{-n} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{(n+1)!}{(r+1)!} L^r(y\overline{f_{\mathfrak{a}}(z)}) \\ & \times \sum_{\gamma' \in \Gamma_\infty \backslash \Gamma'} \mathrm{Im}(\gamma' z)^{s-(n+1)} e^{2\pi i(m+\alpha)\gamma' z} \varepsilon(\gamma', z)^{-2(r+1)} \overline{\chi(\gamma)} \\ = & \ (-2i)^{-n} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{(n+1)!}{(r+1)!} L^r(y\overline{f_\infty(z)}) \varepsilon(\sigma_\infty, z)^{-2(r+1)} \\ & \times U_m(\sigma_\infty z, s-(n+1), 2(r+1); \chi). \end{aligned}$$

Using the commutativity relations (40) and (41) implies that

$$L^r(y\overline{f_\infty(z)}) = L^r(y\overline{f(z)})|_{\sigma_\infty z}\, \varepsilon(\sigma_\infty, z)^{2(r+1)} \tag{59}$$

and then

$$\begin{aligned} & Q_m(z, s, -n; \overline{f}, \chi) \\ & \quad = (-2i)^{-n} \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \frac{(n+1)!}{(r+1)!} L^r(y\overline{f(z)}) U_m(z, s-(n+1), 2(r+1); \chi). \end{aligned} \tag{60}$$

Together with Theorem 4.7, this gives an analytic continuation of $Q_m(z, s, -n; \overline{f}, \chi)$ to $\mathrm{Re}(s) > 2 + n - \delta_\Gamma$.

To complete the proof, it remains to prove the bound. First, with (41), we observe that $L^r\left(y\overline{f(z)}\right)$ has exponential decay at every cusp $\mathfrak{b}$ because

$$\begin{aligned} \theta_{\sigma_{\mathfrak{b}}, -2(r+1)} L^r\left(y\overline{f(z)}\right) &= L^r\left(\theta_{\sigma_{\mathfrak{b}}, -2} y\overline{f(z)}\right) \\ &= L^r\left(y\overline{j(\gamma, \sigma_{\mathfrak{b}} z)^{-2}}\ \overline{f(\sigma_{\mathfrak{b}} z)}\right) \\ &= L^r\left(y \sum_{n=1}^{\infty} \overline{a_{\mathfrak{b}}(n)}\ \overline{e^{2\pi i n z}}\right). \end{aligned}$$

Therefore

$$L^r\left(y\overline{f(z)}\right) \ll y_\Gamma(z)^{r+1} e^{-2\pi y_\Gamma(z)}$$

for an implied constant depending on r, f and Γ. An application of this and Theorem 4.7 to (60) gives

$$Q_m(z, s+n+1, -n; \overline{f}, \chi) \ll e^{\pi y_\Gamma(z)} \tag{61}$$

for $\mathrm{Re}(s) > 1 - \delta_\Gamma$.

By repeating the proof *mutatis mutandis*, it can be shown the same result for $R_0 Q_m(z, s+n+1, -n; \overline{f}, \overline{\chi})$. However, it is quicker to just apply R_0 to both sides of (60) and then use the easily verifiable (with (45)) identity

$$\begin{aligned}
&R_0 \left(L^r \left(y\overline{f(z)} \right) U_m(z, s-n-1, 2(r+1); \chi) \right) \\
&= \left(R_{-2(r+1)} \, L^r \left(y\overline{f(z)} \right) \right) U_m(z, s-n-1, 2(r+1); \chi) \\
&\quad + L^r \, y\overline{f(z)} \, R_{2(r+1)} \, U_m(z, s-n-1, 2(r+1); \chi) \\
&= R_{-2(r+1)} \, L^r \left(y\overline{f(z)} \right) U_m(z, s-n-1, 2(r+1); \chi) \\
&\quad + L^r \left(y\overline{f(z)} \right) \Big((s-n+1) \, U_m(z, s-n-1, 2(r+2); \chi) \\
&\quad - 4\pi m \, U_m(z, s-n, 2(r+2); \chi) \Big).
\end{aligned}$$

□

Using this theorem, we can now prove:

Theorem 4.11. *For $m > 0$, the cusp $\mathfrak{a}$ taken to be ∞, χ, ψ characters on Γ and $f \in S_2(\Gamma; \overline{\chi} \cdot \psi)$, both of the series*

- $(s-1) \, Q_{\mathfrak{a}m}(z, s, 1; \overline{f}, \chi)$
- $Q'_{\mathfrak{a}m}(z, s, 1; \overline{f}, \chi)$

have continuations to meromorphic functions of s when $\mathrm{Re}(s) > 1 - \delta_\Gamma$, and when this is satisfied

- $(s-1) \, Q_{\mathfrak{a}m}(z, s, 1; \overline{f}, \chi) \ll y_\Gamma(z)^{1/2}$
- $Q'_{\mathfrak{a}m}(z, s, 1; \overline{f}, \chi) \ll y_\Gamma(z)^{1/2}$

with the implied constants depending on s, m, f, Γ and χ. If $\chi \not\equiv 1$, then $Q_{\mathfrak{a}m}(z, s, 1; \overline{f}, \chi)$ is analytic in this region; otherwise, it has a simple pole at $s = 1$ with residue $2i\overline{\langle f, P_{\mathfrak{a}m}(\cdot)_2 \rangle}$ where $\langle \cdot, \cdot \rangle$ denotes the Petersson Inner Product.

Proof. The function $Q_{\mathfrak{a}m}(z, s, 1; \overline{f}, \chi)$ satisfies

$$\langle Q_{\mathfrak{a}m}(\cdot, s, 1; \overline{f}, \chi), Q_{\mathfrak{a}m}(\cdot, s, 1; \overline{f}, \chi) \rangle < \infty \tag{62}$$

for $\mathrm{Re}(s) > 1$ because of Theorem 4.4 iii). The spectral decomposition then implies

$$\begin{aligned}
Q_{\mathfrak{a}m}(z, s, 1; \overline{f}, \chi) &= \sum_{j=0}^{\infty}{}' \langle Q_{\mathfrak{a}m}(\cdot, s, 1; \overline{f}, \overline{\chi}), \eta_j \rangle \, \eta_j(z) \\
&+ \frac{1}{4\pi} \sum_{\mathfrak{b}} \int_{-\infty}^{\infty} \langle Q_{\mathfrak{a}m}(\cdot, s, 1; \overline{f}, \chi), E_{\mathfrak{b}}(\cdot, 1/2 + ir, \overline{\chi}) \rangle \, E_{\mathfrak{b}}(\cdot, 1/2 + ir, \chi) dr
\end{aligned} \tag{63}$$

The computation of the inner products appearing in this formula is more subtle than that of the corresponding products for U_m and we will need the following lemma proved in Section 9 of [JO]. Although the proof there referred to trivial characters, the identical argument applies to the general case because the character appears only in the Petersson scalar product.

For the study of this decomposition, we employ the following lemma:

Lemma 4.12. *Let ξ_1, ξ_2 and ψ be any smooth functions such that $\xi_i(\gamma z) = \chi(\gamma)\xi_i(z)$ and $\psi(\gamma z) = \chi(\gamma)\psi(z)$. If $(\Delta - \lambda)\xi_1 = \xi_2, (\Delta - \lambda')\psi = 0$ and*

$$\begin{aligned} \xi_1, R_0\xi_1, \Delta\xi_1 &\ll y_\Gamma(z)^A, \\ \psi, R_0\psi &\ll y_\Gamma(z)^B, \end{aligned}$$

for $A + B < 0$ and $R_0 = 2iy\frac{d}{dz}$ the raising operator, then

$$\langle \xi_1, \psi \rangle = \frac{1}{\lambda' - \lambda}\langle \xi_2, \psi \rangle.$$

Proof. See [JO], Section 9. Although the proof there referred to the trivial character, the identical argument applies generally because the character only appears in the definition of the Petersson scalar product. □

Now, for all $n \in \mathbb{Z}$, we have

$$\begin{aligned} (\Delta - s(1-s))\, Q_m(z, s, n; \overline{f}, \chi) = &- 8\pi i(m + x_{\mathfrak{a}})\, Q_m(z, s+2, n-1; \overline{f}, \chi) \\ &+ 4\pi s(m + x_{\mathfrak{a}})\, Q_m(z, s+1, n; \overline{f}, \chi) \\ &+ 2is\, Q_m(z, s+1, n-1; \overline{f}, \chi). \end{aligned}$$

We apply Lemma 4.12 to $\xi_1 = Q_m(z, s, n; \overline{f}, \chi)$ and $\psi = \eta_j$, and recall that $(\Delta - s_j(1-s_j)\eta_j) = 0$. To check the growth conditions, we will need the following result: Now, we have $\eta_j(z), R_0\eta_j(z) \ll y_\Gamma(z)^{1/2}$ by Lemma 4.6 and

$$\begin{aligned} Q_m(z, s, n; \overline{f}, \chi) &\ll y_\Gamma(z)^{1/2-\sigma/2} \\ R_0 Q_m(z, s, n; \overline{f}, \chi) &\ll y_\Gamma(z)^{1/2-\sigma/2} \\ \Delta Q_m(z, s, n; \overline{f}, \chi) &\ll y_\Gamma(z)^{1/2-\sigma/2} \end{aligned}$$

for $\sigma = \mathrm{Re}(s) > 1$ by Theorem 4.4 iii) and iv) and Theorem 4.10. So we may use Lemma 4.12 to get, for $\mathrm{Re}(s) > 2$,

$$\begin{aligned} \langle Q_m(\cdot, s, 1; \overline{f}, \chi), \eta_j \rangle = &\frac{1}{(s_j - s)(1 - s_j - s)}\Big(- 8\pi i m \langle Q_m(\cdot, s+2, 0; \overline{f}, \chi), \eta_j \rangle \\ &+ 4\pi m s \langle Q_m(\cdot, s+1, 1; \overline{f}, \chi), \eta_j \rangle + 2is \langle Q_m(\cdot, s+1, 0; \overline{f}, \chi), \eta_j \rangle \Big). \end{aligned}$$

Iterating M times, we deduce that for $\mathrm{Re}(s) > 2$,

$$\langle Q_m(\cdot, s, 1; \overline{f}, \chi), \eta_j \rangle = \sum_l \frac{P_l(m,s)}{R_l(m,s)} \langle Q_m(\cdot, s+M+c_l, 1-d_l; \overline{f}, \chi), \eta_j \rangle \tag{64}$$

with integers c_l, d_l satisfying $0 \leqslant c_l, d_l \leqslant M$, $d_l \leqslant M + c_l$ and $P_l(m,s)$ is a polynomial in m and s alone of degree M in m and of degree M in s. Further,

$$R_l(s_j, s) = \prod_b (s_j - b - s)(1 - s_j - b - s) \tag{65}$$

where, for each l, the product is over some subset of integers b in $\{0, 1, \ldots, 2M\}$ of cardinality M.

Now, for s with $\mathrm{Re}(s) > 1 - \delta_\Gamma$, we have:
Case I: $d_l = 0$. Then, with Theorem 4.4 iii), $Q_m(\cdot, s+M+c_l, 1; \overline{f}, \chi)$ is analytic and, for $M \geqslant 1$, it satisfies

$$Q_m(z, s+M+c_l, 1; \overline{f}, \chi) \ll y_\Gamma(z)^{1/4-M/2}. \tag{66}$$

Therefore, with Cauchy–Schwarz,

$$\langle Q_m(\cdot, s+M+c_l, 1; \overline{f}, \chi), \eta_j \rangle \ll \sqrt{||y_\Gamma(z)^{-1/4}|| \cdot ||\eta_j||} = \sqrt{||y_\Gamma(z)^{-1/4}||} \ll 1.$$

Case II: $0 < d_l \leqslant M$. Then, by Theorem 4.10, we have

$$Q_m(\cdot, s+M+c_l, 1-d_l; \overline{f}, \chi) \ll e^{-\pi y_\Gamma(z)}. \tag{67}$$

Hence, with Cauchy–Schwarz,

$$\langle Q_m(z, s+M+c_l, 1-d_l; \overline{f}, \chi), \eta_j \rangle \ll 1.$$

Therefore, and taking into account (65), in both cases, the righthand side of (64) is analytic for s with $\mathrm{Re}(s) > 1 - \delta_\Gamma$. Furthermore,

$$\langle Q_m(\cdot, s, 1; \overline{f}, \chi), \eta_j \rangle \ll |s_j|^{-2M} \ll |\lambda_j|^{-M}$$

with implied constants depending on s, m, M, f, χ and Γ only. If, in addition, we use (29) and (35), we deduce, for each $j > 0$, that

$$\sum_{T \leqslant |\lambda_j| < T+1} \langle Q_m(\cdot, s, 1; \overline{f}, \chi), \eta_j \rangle \eta_j(z) \ll T^{1-M} y_\Gamma(z)^{1/2} + T^{11/4-M} y_\Gamma(z)^{-3/2}.$$

Fix an $M \geqslant 4$. Then, adding over all $T \in \mathbb{Z}_{\geqslant 0}$, we get

$$\sum_{j=1} \langle Q_{\mathfrak{a}m}(\cdot, s, 1; \overline{f}, \chi), \eta_j \rangle \eta_j(z) \ll y_\Gamma(z)^{1/2}$$

for all s with $\mathrm{Re}(s) > 1 - \delta_\Gamma$ and an implied constant which depends solely on s, m, f and Γ.

The index $j = 0$ plays a role only for the trivial character χ because otherwise $\eta_0 = 0$. Then, we have the constant eigenfunction $\eta_0 = V^{-1/2}$. Unfolding gives:

$$\begin{aligned}\langle Q_m(\cdot, s, 1; \overline{f}, \chi), \eta_0 \rangle \eta_0 =& \frac{-\overline{a_{\mathfrak{a}}(m)}\Gamma(s-1)}{2\pi i m (4\pi m)^{s-1}} \\ =& \frac{-\overline{a_{\mathfrak{a}}(m)}\Gamma(s-1)}{2\pi i m} \left(\frac{1}{s-1} + O(1) \right) \\ =& 2i\overline{\langle f, P_m(\cdot)_2 \rangle} \left(\frac{1}{s-1} + O(1) \right)\end{aligned}$$

as $s \to 1$ since $\langle f, P_m(\cdot)_2 \rangle = \frac{a(m)}{4\pi m}$. Here, $a(m)$ is the m-th Fourier coefficient of $f(z)$.

We next consider the contribution of the continuous part. We first note that the Fourier expansion of $E_{\mathfrak{a}}(z, s; \chi)$ at any cusp together with (31), (36) and (38) implies that

$$E_{\mathfrak{a}}(z, 1/2 + ir; \chi) \ll y_\Gamma(z)^{1/2} \tag{68}$$

for $r \in [T, T+1]$. This allows us to use Lemma 4.12 to get, for $\mathrm{Re}(s) \geqslant 2$,

$$\begin{aligned}&\langle Q_m(\cdot, s, 1; \overline{f}, \chi), E_{\mathfrak{b}}(\cdot, 1/2 + ir; \chi) \rangle = \\ &\sum_l \frac{P_l(m,s)}{R_l(1/2+ir, s)} \langle Q_m(\cdot, s + M + c_l, 1 - d_l; \overline{f}, \chi), E_{\mathfrak{b}}(\cdot, 1/2 + ir; \chi) \rangle.\end{aligned} \tag{69}$$

The quantities P_l, R_l, c_l, d_l and the summation range are the same as in (64).

Applying (66), (67) and (68) to (69), we deduce that for $M > 1$, the righthand side of (69) converges. This gives the analytic continuation of the lefthand side for $\mathrm{Re}(s) > 1 - \delta_\Gamma$. Furthermore, (69) implies that, for $z_0 \in \mathfrak{H}$, we have

$$\int_T^{T+1} \langle Q_m(\cdot, s, 1; \overline{f}, \chi), E_{\mathfrak{b}}(\cdot, 1/2 + ir) \rangle E_{\mathfrak{b}}(\cdot, 1/2 + ir) dr \tag{70}$$

$$\begin{aligned}&= \sum_l P_l(m,s) \int_T^{T+1} \frac{\langle Q_m(\cdot, s + M + c_l, 1 - d_l; \overline{f}, \chi), E_{\mathfrak{b}}(\cdot, 1/2 + ir) \rangle}{R_l(1/2+ir, s)} \\ &\times E_{\mathfrak{b}}(z_0, 1/2 + ir) dr \\ &= \sum_l P_l(m,s) \int_T^{T+1} \int_{\mathfrak{F}} \frac{Q_m(z, s + M + c_l, 1 - d_l; \overline{f}, \chi)}{R_l(1/2+ir, s)} \\ &\times \overline{E_{\mathfrak{b}}(z, 1/2 + ir)} E_{\mathfrak{b}}(z_0, 1/2 + ir) d\mu z \, dr.\end{aligned} \tag{71}$$

By (65), (66), (67) and (68), the integrand satisfies

$$\begin{aligned}&\frac{Q_m(z,s+M+c_l,1-d_l;\overline{f},\chi)}{R_l(1/2+ir,s)}\overline{E_{\mathfrak{b}}(z,1/2+ir)}E_{\mathfrak{b}}(z_0,1/2+ir)\\&\ll |r|^{-2M}y_\Gamma(z)^{1/4-M/2}y_\Gamma(z)^{1/2}y_\Gamma(z_0)^{1/2}\end{aligned}$$

and therefore the last integral in (71) is absolutely and uniformly convergent. We can thus interchange the limits of integration to obtain

$$\begin{aligned}&\sum_l P_l(m,s)\int_{\mathfrak{F}} Q_m(z,s+M+c_l,1-d_l;\overline{f},\chi)\\&\times\int_T^{T+1}\frac{\overline{E_{\mathfrak{b}}(z,1/2+ir;\overline{\chi})}}{R_l(1/2+ir,s)}E_{\mathfrak{b}}(z_0,1/2+ir;\overline{\chi})dr\;d\mu z.\end{aligned}\tag{72}$$

Since, by Cauchy–Schwarz,

$$\begin{aligned}&\int_T^{T+1}\frac{\overline{E_{\mathfrak{b}}(z,1/2+ir;\overline{\chi})}}{R_l(1/2+ir,s)}E_{\mathfrak{b}}(z_0,1/2+ir;\overline{\chi})dr\\&\ll T^{-2M}\sqrt{\int_T^{T+1}|E_{\mathfrak{b}}(z,1/2+ir;\overline{\chi})|^2dr\cdot\int_T^{T+1}|E_{\mathfrak{b}}(z_0,1/2+ir\overline{\chi})|^2dr},\end{aligned}$$

a special case of Lemma 4.6 iii) implies that the sum in (72) is

$$\begin{aligned}&\ll\sum_l|P_l(m,s)|\int_{\mathfrak{F}}|Q_m(z,s+M+c_l,1-d_l;\overline{f},\chi)|\\&\times T^{-2M}\,y_\Gamma(z)^{1/2}\,T^6\,y_\Gamma(z_0)^{1/2}\,T^6d\mu z\;dr\\&\ll\sum_l y_\Gamma(z_0)^{1/2}|P_l(m,s)|\,T^{12-2M}\int_{\mathfrak{F}}y_\Gamma(z)^{3/4-M/2}d\mu z.\end{aligned}$$

Hence, for M chosen large enough,

$$\int_{-\infty}^{\infty}\langle Q_m(\cdot,s,1;\overline{f},\chi),E_{\mathfrak{b}}(\cdot,1/2+ir;\overline{\chi})\rangle\,E_{\mathfrak{b}}(z,1/2+ir\chi)dr\ll y_\Gamma(z)^{1/2}.\tag{73}$$

This completes the proof of the meromorphic continuation for $\mathrm{Re}(s)>1-\delta_\Gamma$ and the bound of $Q_m(z,s,1;\overline{f},\chi)$. The analytic continuation and bound for $R_0Q_m(z,s,1;\overline{f},\chi)$ are proved in an entirely analogous manner, once we apply R_0 to both sides of (63). In fact, the only essential difference is that, when χ is the trivial character, there is no pole because R_0 eliminates it. □

4.7 $Z_m(z,s;f,\chi)$ and an identity on $G_m(z,s;\overline{f},\chi)$

We are now ready to construct the functions which will give the basis elements we are seeking.

Definition 4.13. *For $f \in S_2(\Gamma; \overline{\chi} \cdot \psi)$ and* $\mathrm{Re}(s) > 0$ *set*

$$Z_m(z,s;f,\chi) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{L_f(\infty,\gamma)} \frac{\mathrm{Im}(\gamma z)^s}{j(\gamma,z)^2} e^{2\pi i (m+x_\infty)} \overline{\chi(\gamma)}. \tag{74}$$

By Lemma 2.12, for all $z \in \mathfrak{H}$,

$$\overline{L_f(\infty,\gamma)} = \overline{\Lambda_f(\infty,\gamma z)} - \chi(\gamma)\overline{\psi(\gamma)}\;\overline{\Lambda_f(\infty,\gamma)}$$

and thus

$$Z_m(z,s;f,\chi) = G_m(z,s;\overline{f},\chi) - \overline{\Lambda_f(\infty,z)} y^{-1} U_{\mathfrak{a}m}(z,s+1,2;\chi). \tag{75}$$

This expression together with Theorem 4.4 implies that $Z_m(z,s;f,\chi)$ converges absolutely for $\mathrm{Re}(s) > 0$.

Before proceeding on to the main theorem of the section, we first prove the following proposition:

Proposition 4.14. *Let m be a non–negative integer, $f \in S_2(\Gamma;\chi,\psi)$ such that either χ is singular or else χ is non–singular but ψ is singular, and $\mathfrak{a}$ a cusp. $Z_{\mathfrak{a}m}(z,s;f,\chi)$ has an analytic continuation to* $\mathrm{Re}(s) > -\delta_\Gamma$ *and*

$$\frac{d}{dz} Q_m(z,s,1;\overline{f},\chi) = 2\pi i (m+x_\infty)\, G_m(z,s;\overline{f},\chi) - \frac{is}{2}\, G_m(z,s-1;\overline{f},\chi) \tag{76}$$

or, equivalently

$$G_m(z,s;\overline{f},\chi) = \frac{4\pi(m+x_\infty)}{s+1}\, G_m(z,s-1;\overline{f},\chi) + \frac{2i}{s+1}\, Q'_m(z,s,1;\overline{f},\chi). \tag{77}$$

Proof.

$$\begin{aligned}
\frac{d}{dz} Q_{\mathfrak{a}m}(z,s,1;\overline{f},\chi) &= \frac{d}{dz}\left(\sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\int_{\mathfrak{a}}^{\gamma z} f(w)dw}\; \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s\, e^{2\pi i (m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z}\, \overline{\chi(\gamma)} \right) \\
&= \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\int_{\mathfrak{a}}^{\gamma z} f(w)dw}\; \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \frac{d}{dz}\left(e^{2\pi i (m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z} \overline{\chi(\gamma)} \right) \\
&\quad + \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\int_{\mathfrak{a}}^{\gamma z} f(w)dw} \frac{d}{dz}\left(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \right)\; e^{2\pi i (m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z}\, \overline{\chi(\gamma)} \\
&\quad + \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \left(\frac{d}{dz} \overline{\int_{\mathfrak{a}}^{z} f(w)dw} \right) \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s\, e^{2\pi i (m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z}\, \overline{\chi(\gamma)}
\end{aligned}$$

We use the results:

- $\frac{d}{dz}(e^{2\pi i(m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z}) = 2\pi i(m+\alpha)\,\frac{d}{dz}(\sigma_{\mathfrak{a}}^{-1}\gamma z)\,e^{2\pi i(m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z}$;
- $\frac{d}{dz}(\sigma_{\mathfrak{a}}^{-1}\gamma z) = j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-2}$;
- $\frac{d}{dz}(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s) = s\,\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1}\,\frac{d}{dz}(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))$;
- $\frac{d}{dz}(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)) = \frac{d}{dz}\left(\frac{(\sigma_{\mathfrak{a}}^{-1}\gamma z)-(\overline{\sigma_{\mathfrak{a}}^{-1}\gamma z})}{2i}\right) = \frac{1}{2i}\frac{d}{dz}\left(\sigma_{\mathfrak{a}}^{-1}\gamma z - \overline{\sigma_{\mathfrak{a}}^{-1}\gamma z}\right)$
  $= \frac{1}{2i}(\frac{d}{dz}(\sigma_{\mathfrak{a}}^{-1}\gamma z) - \frac{d}{dz}(\overline{\sigma_{\mathfrak{a}}^{-1}\gamma z})) = \frac{1}{2i}j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-2}$; and
- $\frac{d}{dz}\overline{\int_{\mathfrak{a}}^{z} f(w)dw} = 0$.

Putting these together, we have

$$\begin{aligned}
&\frac{d}{dz}Q_{\mathfrak{a}m}(z,s,1;\overline{f},\chi)\\
=\;& \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\int_{\mathfrak{a}}^{\gamma z} f(w)dw}\;\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s\left(2\pi i(m+\alpha)\,j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-2}\,e^{2\pi i(m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z}\,\overline{\chi(\gamma)}\right)\\
&+\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\int_{\mathfrak{a}}^{\gamma z} f(w)dw}\left(s\,\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1}\cdot\frac{1}{2i}\right)e^{2\pi i(m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z}\,\overline{\chi(\gamma)}+0\\
=\;& 2\pi i(m+\alpha)\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\int_{\mathfrak{a}}^{\gamma z} f(w)dw}\,\frac{\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s}{j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^2}e^{2\pi i(m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z}\,\overline{\chi(\gamma)}\\
&-\frac{si}{2}\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}\overline{\int_{\mathfrak{a}}^{\gamma z} f(w)dw}\;\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s-1}\,e^{2\pi i(m+\alpha)\sigma_{\mathfrak{a}}^{-1}\gamma z}\,\overline{\chi(\gamma)}\\
=\;& 2\pi i(m+\alpha)\,G_{\mathfrak{a}m}(z,s;\overline{f},\chi)-\frac{si}{2}\,G_{\mathfrak{a}m}(z,s-1;\overline{f},\chi).
\end{aligned}$$

Rearrange and use the substitution $s \longmapsto s+1$ to get

$$G_{\mathfrak{a}m}(z,s;\overline{f},\chi) = \frac{4\pi(m+\alpha)}{s+1}\,G_{\mathfrak{a}m}(z,s+1;\overline{f},\chi)+\frac{2i}{s+1}\,Q'_{\mathfrak{a}m}(z,s+1,1;\overline{f},\chi).$$

Theorem 4.11 gives $G_{\mathfrak{a}m}(z,s;\overline{f},\chi)$ an analytic continuation to $\mathrm{Re}(s) > -\delta_\Gamma$ and combining with Theorem 4.7, both terms on the righthand side of (75) have analytic continuations to $\mathrm{Re}(s) > -\delta_\Gamma$, proving that $Z_{\mathfrak{a}m}(z,s;f,\chi)$ has an analytic continuation. □

4.8 The basis element functions

Theorem 4.15. *Let m be a non–negative integer, χ,ψ characters in Γ and $f \in S_2(\Gamma;\overline{\chi}\cdot\psi)$. Then $Z_m(z,s;f,\chi)$ admits a meromorphic continuation to* $\mathrm{Re}(s) > -\delta_\Gamma$.

It satisfies:

$$\text{i)}\quad Z_m(\gamma z,0;f,\chi)\,j(\gamma,z)^{-2}\overline{\chi(\gamma)} = Z_m(z,0;f,\chi) + \overline{L_f(\infty,\gamma^{-1})}\,\chi(\gamma)\,\overline{\psi(\gamma)}\,y^{-1}\,U_m(z,1,2;\psi);$$

$$\text{ii)}\quad yZ_m(\gamma z,0;f) \ll y_{\mathfrak{F}}(z)^{1/2};\ \text{ and}$$

$$\text{iii)}\quad \frac{d}{d\bar{z}}Z_m(\gamma z,0;f) = -\delta(\psi)y^{-2}\,\overline{\langle f,P_m(\cdot)_2\rangle}.$$

where the implied constant in ii) is independent of z and $\delta(\psi) := 1$ if $\psi \equiv 1$ and 0 otherwise.

Proof. Beginning with equation (77), Theorems 4.4 v) and 4.11 then imply that $G_m(z,s;\overline{f},\chi)$ is absolutely convergent for $\sigma = \mathrm{Re}(s) > 0$. In combination with Proposition 4.7, this gives the assertation of the theorem because of (75).

i) For $\mathrm{Re}(s)$ large, we have

$$Z_m(\gamma z,s;f,\chi)\,j(\gamma,z)^{-2}\,\overline{\chi(\gamma)} = Z_m(z,s;f,\chi) - \overline{L_f(\infty,\gamma)}\ \overline{\chi(\gamma)}\,\psi(\gamma)\,y^{-1}\,U_m(z,s+1,2;\psi)$$

or

$$Z_m(\gamma z,s;f,\chi)j(\gamma,z)^{-2}\overline{\chi(\gamma)} = Z_m(z,s;f,\chi) + \overline{L_f(\infty,\gamma^{-1})}\ \overline{\psi(\gamma)}\chi(\gamma)y^{-1}U_m(z,s+1,2;\psi).$$

i) follows on from the analytic continuation we have just proved.

ii) Recall that Theorem 4.4 v) and Theorem 4.11 imply:

$$yG_m(z,1;\overline{f},\chi) \ll y_\Gamma(z)^{1/2};\quad \text{and}$$
$$yQ'_m(z,1,1;\overline{f},\chi) \ll y_\Gamma(z)^{1/2}.$$

Furthermore, for $m > 0$, we have $U_m(z,1,2;\psi) \ll y_\Gamma(z)^{1/2}$ by Theorem 4.7. Since, with (27), for all cusps $\mathfrak{a},\mathfrak{b}$

$$\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{a}}z) \ll e^{-2\pi y}$$

$$\Lambda_f(\mathfrak{a},\sigma_{\mathfrak{b}}z) = \int_\infty^{\mathfrak{b}} f(z)dz + \Lambda_f(\mathfrak{b},\sigma_{\mathfrak{b}}z) \ll 1,\quad \mathfrak{a} \neq \mathfrak{b}$$

as $y \to \infty$, we deduce that $\overline{\Lambda_f(\infty,z)}\,U_m(z,1,2;\psi) \ll y_\Gamma(z)^{1/2}$.

On the other hand, equations (75) and (77) give

$$\begin{aligned} yZ_m(z,0;f,\chi) &= 4\pi(m+x_\infty)y\,G_m(z,1;\overline{f},\chi) \\ &\quad + 2iy\,Q'_m(z,1,1;\overline{f},\chi) \\ &\quad - \overline{\Lambda_f(\infty,z)}y^{-1}\,U_m(z,1,2;\psi). \end{aligned}$$

An application of the bounds proved above to this identity gives the desired result.

iii) Finally, for $\mathrm{Re}(s)$ large, we have

$$\frac{d}{d\overline{z}} Z_m(z,s;f,\chi) = \frac{is}{2y^2}\left(Q_m(z,s+1,1;\overline{f},\chi) - \overline{\Lambda_f(\infty,z)}\, U_m(z,s+1;\psi)\right).$$

Combining this with Theorems 4.7 and 4.11, we have iii). $\square$

4.9 The basis elements

We are now in a position by using Theorem 4.15 to identify a basis of $S_2^2(\Gamma;\chi,\psi)$.

First, let (λ,μ) be a pair in $S_2(\Gamma;\overline{\chi}\cdot\psi)\times S_2(\Gamma;\psi)$. Since the set $\{y^{-1}U_m(z,1,2;\psi)|\ m>0\}$ spans $S_2(\Gamma;\psi)$ (see [Ra]), there is a linear combination of $y^{-1}U_m(z,1,2;\psi)$ which equals μ. If we apply the same linear combination in Theorem 4.15, we obtain a linear combination $Z_{\lambda,\mu}$ of $Z_m(z,0;\lambda,\chi)$ (with $m>0$) such that, for all $\gamma\in\Gamma$, we have:

$$Z_{\lambda,\mu}|_{2,\chi}(\gamma-1) = \overline{\langle\gamma,\lambda\rangle}\mu; \tag{78}$$

$$yZ_{\lambda,\mu} \ll y_{\mathfrak{F}}(z)^{1/2}; \quad \text{and} \tag{79}$$

$$y^2\frac{d}{d\overline{z}}Z_{\lambda,\mu} = \delta(\psi)\,\overline{\langle f, P_m(\cdot)_2\rangle} \quad \text{i.e. } Z_{\lambda,\mu} \text{ is holomorphic unless } \psi\not\equiv 1. \tag{80}$$

Theorem 4.16. *Let $\{f_1,\dots,f_{d_1}\}$ be an orthonormal basis for $S_2(\Gamma;\psi)$, let $\{g_1,\dots,g_{d_2}\}$ be an orthonormal basis of $S_2(\Gamma;\overline{\chi}\cdot\psi)$ and let $\{h_1,\dots,h_{d_3}\}$ be an orthonormal basis of $S_2(\Gamma;\overline{\psi}\cdot\chi)$. If $\psi\not\equiv 1$, then the set*

$$\mathcal{S} = \{Z_{g_j,f_i}\}_{1\leqslant i\leqslant d_1, 1\leqslant j\leqslant d_2} \cup \{f_i\Lambda_{h_j}(\infty,\cdot)\}_{1\leqslant i\leqslant d_1, 1\leqslant j\leqslant d_3} \tag{81}$$

is a basis of

$$S_2^2(\Gamma;\chi,\psi)/(S_2(\Gamma;\chi)+S_2(\Gamma;\psi)). \tag{82}$$

If $\psi\equiv 1$ then a basis is

$$\mathcal{T} = \{Z_{f_i,f_j}\}_{1\leqslant i\neq j\leqslant d_1} \cup \{Z_{f_i,f_i}-Z_{f_1,f_1}\}_{1< i\leqslant d_1} \cup \{f_i\Lambda_{h_j}(\infty,\cdot)\}_{1\leqslant i\leqslant d_1, 1\leqslant j\leqslant d_2}. \tag{83}$$

Proof. Using Theorem 4.15, all of the functions (in both cases) belong to $S_2^2(\Gamma;\chi,\psi)$. Now, using Proposition 2.8 and equation (11), the proof in the case of $\chi\not\equiv 1$ reduces to proving that the (projections of the) elements of the set $\mathcal{S}$ are linearly independent. Suppose that for some constants $k_{ij}, l_{ij}, m_i, n_i\in\mathbb{C}$, we have

$$\sum_{i,j} k_{ij}Z_{g_i,f_j} + \sum_{i,j} l_{ij}f_i\Lambda_{h_j}(\infty,\cdot) \in S_2(\Gamma;\chi)+S_2(\Gamma;\psi).$$

With $\gamma \in \Gamma$, let $\gamma - 1$ act on both sides of the equation via $|_{2,\chi}$. Then

$$\sum_{i,j} k_{ij} \overline{L_{f_i}(\infty, \gamma)} f_j + \sum_{i,j} l_{ij} L_{f_i}(\infty, \gamma) f_j \in \left(1 - \overline{\chi(\gamma)}\psi(\gamma)\right) S_2(\Gamma; \psi).$$

Therefore, for each j, the sum

$$\sum_i \left(k_{ij} \overline{L_{f_i}(\infty, \gamma)} + l_{ij} L_{f_i}(\infty, \gamma) \right)$$

is an Eichler coboundary and therefore, by injectivity of the Eichler–Shimura isomorphism, this implies that

$$\sum_i k_{ij} f_i = \sum_i l_{ij} f_i = 0.$$

The linear independence of f_i implies that all k_{ij} and l_{ij} vanish.
The modifications required in the case $\chi \equiv 1$ are now clear: Proposition 5.2 of [DO] implies that, if $g \in S_2(\Gamma; \chi)$ is non–zero, then there is no cuspidal f such that

$$f|_2(\gamma - 1) = \overline{L_g(\infty, \gamma)} g.$$

Therefore, the projections of the elements of $\mathcal{T}$ suffice to generate $S_2^2(\Gamma; \chi, \psi)/(S_2(\Gamma; \chi) + S_2(\Gamma; \psi))$. Since, as before, they are also linearly independent, that implies the theorem. $\square$

Corollary 4.17. *Let Γ be a Fuchsian group of the first kind with non–compact quotient and such that $\dim(S_2(\Gamma; \psi)) \neq 0$. Then:*

i) if $\chi \not\equiv 1$, the dimension of $S_2^2(\Gamma; \chi, \psi)/(S_2(\Gamma; \chi) + S_2(\Gamma; \psi))$ is

$$\dim(S_2(\Gamma; \psi)) \left(\dim(S_2(\Gamma; \psi \cdot \overline{\chi})) + \dim(S_2(\Gamma; \chi \cdot \overline{\psi})) \right);$$

ii) if $\chi \equiv 1$, the dimension of $S_2^2(\Gamma; \chi, \psi)/(S_2(\Gamma; \chi) + S_2(\Gamma; \psi))$ is

$$\dim(S_2(\Gamma; \psi)) \left(\dim(S_2(\Gamma; \psi)) + \dim(S_2(\Gamma; \overline{\psi})) \right) - 1.$$

5 Applications in Percolation Theory

One possible application of the study into holomorphic second–order automorphic forms with characters is in the field of Percolation Theory. This has been particularly studied in the collaborative paper between N. Diamantis and P. Kleban ([DK]). The motivation for this is detailed in [KZ] – for instance, if we take a rectangle with horizontal to vertical ratio r, we can consider the "horizontal" crossing probability $\Pi_h(r)$, i.e. the probability that we can find a "path" connecting the left and right vertical edges of the rectangle. We can explicitly calculate $\Pi_h(r)$ ([Ca]), and similarly the probability $\Pi_{h\bar{v}}(r)$ that we cross "horizontally but not vertically" (namely that the left and right edges of the rectangle are connected whilst the top and bottom edges remain unconnected) ([Wa], [Du]).

To describe the connection between second–order cusp forms with characters and Percolation Theory, we begin with these crossing probabilities which are characterized as second–order forms but with exponential growth at some cusps. This exponential growth is what distinguishes this type of function from the forms already studied. In fact, in [DK] it is shown that, in some sense, these crossing probabilities are determined by the "principle parts" of their Fourier expansions. It is therefore natural to ask how much freedom we are allowed once we fix those principle parts.

If we consider the transformations $S : z \longmapsto -1/z$ (taking $z = ir$) and $T : z \longmapsto z + 1$, we can associate to them behavioural changes – the transformation S arises due to physical symmetries of the crossing problem, and the transformation T can be associated to the structure of the probability formulae themselves, namely a "conformal block" (see [KZ] for further details). We should not expect modular behaviour on a rectangle because it does not possess the appropriate symmetry and yet the differential on the horizontal crossing probability, $\Pi_h'(r)$, is a modular form. Furthermore, the differential $\Pi_{h\bar{v}}'(r)$ has unusual modular behaviour which gives rise to the definition of the n-th – order modular form, which were defined independently in another context by Chinta, Diamantis and O'Sullivan ([CDO]). Second–order modular forms arise in Percolation Theory via the difference in sign under S of $\prod_h'$ and $\prod_{hv}'$, where $\Pi_{hv}'(r)$ is the probability of finding a path which crosses both horizontally and vertically (mathematically, $\Pi_{hv}(r) = \Pi_h(r) - \Pi_{h\bar{v}}(r)$).

I aim to reproduce the appropriate information from N. Diamantis and P. Kleban's paper [DK] and to show how my research into second–order cusp forms may be of some relevance.

5.1 Crossing Probabilities

First, we set the matrices $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and let $\Gamma(2)$ be the group of matrices in $SL_2(\mathbb{Z})$ congruent mod 2 to I_2. The generators of $\Gamma(2)$ are $g_1 := T^2$ and $g_2 := ST^2S^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. We have the standard Dedekind eta function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

where $q := e^{2\pi i z}$ and let the classical modular function for $\Gamma(2)$ be

$$\lambda(z) := 16 \frac{\eta(z/2)^8 \eta(2z)^{16}}{\eta(z)^{24}} = 1 - \frac{\eta(z/2)^{16} \eta(2z)^8}{\eta(z)^{24}}.$$

We set χ for the character of the function η^4.

For $z = ir$, $\lambda(z)$ is the cross–ratio of the four points to which the corners of the rectangle are mapped. We write

$$C := \frac{2^{1/3\pi^2}}{3\Gamma(1/3)^3}$$

and then $\Pi_h(\lambda(ir))$ satisfies

$$\frac{d}{dr} \left(\Pi_h(\lambda(ir)) \right) = -4\sqrt{3} C \eta(ir)^4.$$

Similarly, $\Pi_{h\bar{v}}(\lambda(ir))$ satisfies

$$\frac{d}{dr} \left(\Pi_{h\bar{v}}(\lambda(ir)) \right) = -8\sqrt{3} f_2(ir)$$

where

$$f_2(z) := \frac{2\pi i}{3} \eta(z)^4 \int_{\infty}^{z} \frac{\eta(w/2)^8 \eta(2w)^8}{\eta(w)^{12}} dw.$$

$\eta(z)$ is a weight $\frac{1}{2}$ cusp form on $SL_2(\mathbb{Z})$ with character, and we will denote the character of $\eta(z)^4$ by χ. $f_2(z)$ is a second–order modular form.

Let

$$\phi(z) := \frac{f_2(z)}{\eta(z)^4} = \frac{C}{2} \frac{\Pi'_{h\bar{v}}(z)}{\Pi'_h(z)}.$$

It can be shown ([DK]) that

$$\phi(z) = \frac{1}{2^{8/3}}\lambda(z)^{2/3}\ {}_2F_1\left(\frac{1}{3},\frac{2}{3};\frac{5}{3};\lambda(z)\right)$$

where ${}_2F_1\left(a,b;c;\lambda(z)\right)$ is the Gauss hypergeometric function and the three crossing probabilities we are interested in are given by, for $0<\alpha<1$ and $1<\beta$:

$$\begin{aligned}
\Pi_h^b(\alpha,\beta) &= \frac{(\beta+\alpha)\ {}_2F_1(1,\frac{4}{3};\frac{5}{3};1-\frac{\alpha}{\beta})-2\beta}{4\sqrt{3}\pi\beta^2(\beta-\alpha)};\\
\Pi_h^{\bar b}(\alpha,\beta) &= \frac{(\beta+\alpha)\ {}_2F_1(1,\frac{4}{3};\frac{5}{3};\frac{\alpha}{\beta})+2\beta}{4\sqrt{3}\pi\beta^2(\beta-\alpha)};\\
\nu_h(\alpha,\beta) &= \frac{(\beta^2+2\alpha\beta-(\beta^2-\alpha^2))\ {}_2F_1(1,\frac{4}{3};\frac{5}{3};\frac{\alpha}{\beta})}{4\sqrt{3}\pi\beta^2(\beta-\alpha)^2}.
\end{aligned}$$

Replacing α with $\lambda(z)$ and β with 1 we follow the notation:

$$\begin{aligned}
p_{\bar b}(z) &:= \Pi_h^{\bar b}(\lambda(z),1);\\
p_b(z) &:= \Pi_h^{b}(\lambda(z),1);\\
n(z) &:= \nu_h(\lambda(z),1).
\end{aligned}$$

We are now in a position to consider our second–order automorphic form $f\in S_k^2(\Gamma;\chi,\psi)$. First we formulate a definition on f:

Definition 5.1. *For a set of characters $\{\chi_1,\ldots,\chi_n\}$ on Γ, a weakly holomorphic n-th –order modular form on Γ of weight k and type $(\chi_1,\ldots,\chi_n)$ is a holomorphic function f on the upper half plane $\mathfrak{H}$, meromorphic at the cusps and such that at each element $\gamma_i\in\Gamma$, f satisfies*

$$f|_{k,\chi_1}(\gamma_1-1)|_{k,\chi_2}(\gamma_2-1)\ldots|_{k,\chi_n}(\gamma_n-1)=0.$$

Then clearly, by definition, $f\in S_2^2(\Gamma;\chi,\psi)$ is a second–order modular form of weight 2 and type (χ,ψ). If we take $\chi(\gamma)=1$ for all $\gamma\in\Gamma(2)$, then the function $f_2(z)$ is a second–order, weight 2 form of type $(1,\psi)$ and we can use this to prove that the three crossing probabilities $p_{\bar b}(z)$, $p_b(z)$ and $n(z)$ are (weakly holomorphic) second–order modular forms on $\Gamma(2)$ of weight 0, type $(1,\psi)$. Before commencing, note that $\Gamma(2)$ has three inequivalent cusps at ∞, 0 and at 1, and their three corresponding scaling matrices are I, $U:=\begin{pmatrix}0&-1\\1&1\end{pmatrix}=ST$ and $U^2=\begin{pmatrix}-1&-1\\1&0\end{pmatrix}$.

The classification theorem of [DK] then asserts:

Theorem 5.2. *As functions of z, $p_{\bar{\mathfrak{b}}}(z)$, $p_{\mathfrak{b}}(z)$ and $n(z)$ are weakly holomorphic second–order modular forms on $\Gamma(2)$ of weight 0, type $(1,\psi)$, i.e. holomorphic functions f such that, for each cusp $\mathfrak{a}$,*

$$f|_0(\sigma_{\mathfrak{a}}(x+iy)) = O(e^{ay}) \quad (a \in \mathbb{R}), \qquad as\ \ y \to \infty,\ \ uniformly\ \ in\ \ x$$

and, for each $\gamma \in \Gamma(2)$, $f|_0(\gamma - 1)$ transforms as a weight 0 form with character χ. All these functions have Fourier expansions at the cusps. The first power of q appearing in the expansion of $p_{\bar{\mathfrak{b}}}$ at ∞ (resp. 0, -1) is 1 (resp. $q^{-5/6}$, $q^{2/3}$). The corresponding first powers for $p_{\mathfrak{b}}$ and n are $q^{-1/3}, 1, q^{2/3}$ and $q^{1/2}, q^{-1}, q^{2/3}$.

Proof. See [DK], Theorem 3.1. □

As in [KZ], the derivatives of such probabilities are often more interesting than the actual probabilities themselves. With the use of the identity

$$(f|_0\gamma)'(z) = (f(\gamma z))' = (f|_2\gamma)(z)$$

we deduce that the derivatives of $p_{\bar{\mathfrak{b}}}$, $p_{\mathfrak{b}}$ and n are weight *two* 'weakly holomorphic forms of type $(\mathbf{1},\chi)$'. A natural question then is how many weight two 'weakly holomorphic second–order forms of type $(\mathbf{1},\chi)$' of given principal parts there are. For our purposes, 'principal part' at a cusp $\mathfrak{a}$ will be the part of the Fourier expansion at $\mathfrak{a}$ consisting of non–negative powers of q.

The results of the last section provide a way to answer this question. Suppose that the dimension of $S_2^2(\Gamma(2);\mathbf{1},\chi)$ is d. Then, if f_1, f_2 are weakly holomorphic forms of weight 2 and type $(\mathbf{1},\chi)$ with the same principal part at each cusp, their difference $f_1 - f_2$ belongs to $S_2^2(\Gamma(2);\mathbf{1},\chi)$. Condition i) of the definition of $S_2^2(\Gamma(2);\mathbf{1},\chi)$ follows from the definition of weakly holomorphic forms and condition iii) from the fact that f_1 and f_2 have the same principal parts at the cusp. Condition ii) follows from equation (3.3) of [DK]:

$$p_{\bar{\mathfrak{b}}}|_0(\gamma - 1) = d_\gamma G$$

for some $d_\gamma \in \mathbb{C}$ and a G transforming as a weight 0 form with character χ, as well as the analogous equations for $p_{\mathfrak{b}}$ and n.

Therefore, two 'weakly holomorphic second–order forms of type $(\mathbf{1},\chi)$' with the same principal part at each cusp can only differ by a function ranging in vector space with dimension

$$\dim(S_2(\Gamma(2),\chi))\,(\dim(S_2(\Gamma(2),\chi)) + \dim(S_2(\Gamma(2),\overline{\chi}))) - 1.$$

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