USE OF THE FINITE ELEMENT METHOD FOR THE VIBRATION ANALYSIS OF ROTATING MACHINERY

- By -

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ABSTRACT

The finite element method has been used to predict the effect of rotation on the vibrational characteristics of structures.

Kinematics of rotary motion were studied and numerical adaptation for the calculation of the acceleration matrices has been accomplished. Successful computation of the eigenvalue equation was achieved and a solution algorithm based on a modified QL method was then used.

Existing three dimensional isoparametric finite elements were modified to add the extra acceleration matrices. Comparison with existing methods shows a very good agreement, typical discrepancies being 1-2%.

It was shown that the centrifugal loading creates an initial stress field the effect of which can incorporated by the introduction of a geometric stiffness matrix. While the initial stresses are the largest contributors to the changes in the natural frequencies, they had only a very small effect on the mode shapes of any of the structures that were examined. The centripetal accelerations were found to have a smaller influence on the natural frequencies. It was found that the natural frequencies changed with the angular speed according to the Southwell equation.

The effect of the Coriolis acceleration on the natural frequencies is negligibly small. The mode shapes of rotating structures are affected by the Coriolis acceleration component and for some structures this effect is significant.

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LIST OF SYMBOLS

A	Area
$[A]$	Matrix relating constants to nodal values.
$[B]$	Matrix relating strains to nodal displacements.
$c_1, c_2, c_3, c_4, \ldots$ etc	Different multiplier functions
C	Torsional rigidity.
$[C]$	Coriolis Matrix.
$[D]$	Elasticity matrix.
e	Symbol for 'element'
f	Frequency in Hz = $(2\pi\omega)$
F	Applied load.
$\{F\}$	Load vector.
g	Acceleration due to gravity.
g	Symbol for 'global'.
G	Eigenvalue matrix or equation
G	Shear modulus.
$[G]$	Gyroscopic matrix.
h	Thickness.
i	$\sqrt{-1}$
I	Moment of inertia.
$[I]$	Unit matrix.
J	Torsional Constant.
$[J]$	Jacobian matrix.
k	Shear correction factor.
K	Spring stiffness
KE	Kinetic energy.

LIST OF SYMBOLS (continued)

l	Length.
$[L]$	Lower triangular matrix.
M	Mass.
M	Bending Moment.
$[M]$	Mass matrix.
n	Integer.
n	Symbol for a 'nodal' quantity.
$\lfloor N \rfloor$	Row polynomial of shape functions.
P	Applied pressure or loading intensity.
$\lfloor P \rfloor$	Row matrix of polynomial terms.
PE	Potential energy.
q	Mass per unit length.
$[Q]$	Matrix of polynomial functions.
$\dot{q}$	Vector of velocities.
r	Radius, radial.
R	Radius.
$[S]$	Stiffness Matrix.
SE	Strain energy.
t	time.
$[T]$	Transformation Matrix.
u_x, u_y, u_z	Cartesian displacement.
U	Total energy ($U = KE + PE$, $\frac{dU}{dt} = 0$)
$[U]$	Upper triangular matrix.
V	volume.
w	weight per unit length.
W.D.	Work done.

LIST OF SYMBOLS (continued)

α	Coefficient, conductivity.
$\{\alpha\}$	List of arbitrary constants.
β	Angle.
δ or ς	Displacement.
ε	Strain.
ζ, η, ξ	Curvilinear coordinates.
ϑ	Angle.
$\lambda = \omega^2$	Eigenvalue.
ν	Poisson's Ratio.
ρ	Density.
σ	Stress.
$[\sigma]$	Initial stress matrix.
Φ_x, Φ_y, Φ_z	Angles of rotation.
ω	Radian frequency.
$(\dot{})$	A dot above a variable indicates differenciation with respect to time.

MATRIX NOTATION

$\{\ \}$	Column vector.
$\|[\ ]\|$	Determinant of a matrix.
$[\ ]^T$	Rectangular matrix.
$[\ ]^{-1}$	Transposition of a matrix.
$[\ ]$	Inversion of a matrix ($[A]^{-1}[A] = [A][A]^{-1} = [I]$

CHAPTER ONE

INTRODUCTION

Often, designers are subject to constraints such as economy on materials, light weight, high strength, reliability and low factors of safety. Satisfying all these requirements means that the design procedure should take account of all the factors involved during the service period of the product. In practice this is very difficult to carry out simultaneously for many reasons which may depend on the path followed in design:

a- Theoretical analysis may involve large equations which in many cases are difficult or impossible to solve.

b- If the design is based upon experimental investigation it may be very expensive and limited to prototypes that are closely related to the models that were investigated.

The most obvious way out of these problems is to use simplifications which would lead to a possible solution that is acceptable within the limits of safe operation. One of the suggested simplifications is to cancel a number of the problem variables that have no major effect on the probability of failure of the object concerned. If the problem continues to be large to handle then the next step would be to study these variables individually and later on superimpose their total effect.

1.1. Object of This Work

This study is directed towards investigating the effects of centrifugal forces on the vibration characteristics of rotating machinery. Although the material explained could be applied

generally to any rotating system, concern will be given to rotating cantilever blades for the purpose of comparison with some existing studies and since failure of these components can cause tragic consequences, for instance a helicopter can end up falling if any of its rotor blades fail for any reason since it has no wings at all. A less disastrous example would be the fracture of a blade from a turbine which is built to a very close tolerance and it is very easy to imagine what would happen if the fragment stands in the way of other blades.

Centrifugal effects on rotating systems have not been a subject of systematic study although individual lateral studies in different engineering problems have been undertaken. Among these problems one can mention three of the more frequently apparent ones; those are, vibration problems (natural frequencies and mode shapes), stress analysis, and large displacement problems. The contribution of centrifugal effects to those problems tends to vary from one situation to another, in some cases it helps to reduce large displacements and structure failure, and in other situations it becomes the major factor in failure probability. It is certain however that for optimum design conditions, these effects should be taken into account. The need arises for certain means by which calculation of these effects can be made and at the same time be universal for a wide range of engineering problems. Obviously steps 'a' and 'b' are not universal for all kinds of problems. This is why a change to numerical methods was made. The finite element method was chosen since it proved to be versatile for analysing different problems. Many finite element schemes are in existence[119] and are continually modified to increase their versatility and

applicability to a wide range of engineering problems.

This study is intended to fill the gap in these finite element schemes by presenting a modification to standard element routines which allows them to take care of the effects of centrifugal forces and accelerations. For the sake of comparison the element that is chosen from the various element families is a general three dimensional isoparametric element which is based on displacement assumptions. Particular attention will be made to investigate the effect of the coriolis acceleration on the vibration characteristics since earlier investigations did not concentrate on these effects.

1.2. Layout of this Text :

On the next chapter which concentrates on the background for this subject it will be found that apart from the experimental work that has been carried out theoretical investigations are in existence. Most of the centrifugal problems are found to be dealing with the vibrations of turbine and fan blading and it is due to the fact that the use of digital computers did not start commercially until the late fifties, the analytical solutions could not be extended to include more than the centripital force effects. Although mention was made out for the Coriolis acceleration effects as well as the shear deformation, the majority of investigations tried to avoid including these terms in their formulation of equations of motion or include them in the early stages of formulation to go on with simplifications that would lead to exclude them in the end. Through the survey of literature it is found that two investigations[62,14] tried to prove that shear deformation and rotary inertia produce negligible effects and hence could be dropped out of the analysis. Other investigations[1,5] went to experimental

methods to provide this proof. However, the author thinks that the accuracy of experimental measurements is usually low and it may well be that these terms disappear within the limits of experimental error. Logically it may appear that centrifugal forces produce radial stresses in long rotating members and hence tend to reduce the shear deformation to a negligible amount by stiffening them.

Chapter three in this text gives a brief explanation of the finite element method in connection with the elements used in this investigation. Chapter four gives the theoretical formulation necessary to put the centrifugal effects into action with a finite element program, while chapter five gives the programming details. Chapters six and seven are concerned with the results and conclusion. A set of appendicies is put at the end of the text to give a complimentary discription for some of the items which are indirectly related to this study. Appendix E contains a side work which the author is suggesting for large library programs and appendix F contributes a case study which shows the merits of the finite element method.

CHAPTER TWO

BACKGROUND

2.1. Introduction

Certainly failures of turbine and fan blades are the largest and most dramatic types of failure that occur in rotating machinery and it seems logical that most studies concerned with these machines tend to concentrate about these blades. It is known that for any theoretical study, simplifying assumptions are used to start the analysis, and for the present problem it is thought necessary to start the analysis with the vibrations of non-rotating cantilever blades. Two main contributions are given to the vibration of a single degree of freedom system. The first is the energy method[109, 108]. It can be stated that for any instant of free vibration, the energy of a conservative system (a system that has no energy dissipation) is partly potential and partly kinetic. The potential energy is stored in the form of work done against a force field such as gravity or work done in elastic deformation. The kinetic energy is stored in the mass by virtue of its velocity. It becomes obvious that the total energy is a constant and its rate of change is zero. This statement is given in mathematical form as:

$$KE + PE = U \tag{2.1}$$

$$\frac{dU}{dt} = 0 \tag{2.2}$$

The natural frequency of a system can be determined if the vibrational motion was assumed to be harmonic.

The second method is the Rayleigh method [109]. In this

principle, it is shown that the fundamental frequency of vibration for systems with distributed masses can be determined with good accuracy by assuming any resonable deflection curve. Moreover the fundamental frequency will be the correct frequency if the true deflection of the vibrating system was assumed.

The chief advantage of the Rayleigh method is the relative simplicity of the final equations. Unfortunately only the fundamental frequency can be determined with accuracy using this method. However in certain cases, an acceptable estimate of higher natural frequencies is also possible.

2.2. Vibration of Non-Rotating Cantilever Blades:

2.2.1. Historical Review:

Probably Carnegie has the longest series of investigation in the field of vibrations of cantilever blading, and following his investigations in their order of appearance would give a complete historical review about the advances in the study of turbine blades.

In his investigations Carnegie used both the energy method and Rayleigh method to approach the problem of vibration of pre-wisted cantilever blading. In many instances he made use of the Rayleigh method extensively to obtain expressions for the natural frequency of gradually developed forms of pretwisted cantilever blades of which the final form that was studied was an aerofoil cross section form which represents a prototype of the actual turbomachinery blade. In more recent papers the investigator changed to the more general dynamic problem of the vibrations of rotaing blades.

A summary the various solution methods for the problem of vibration of cantilever blading in their order of appearance is given below:

SUMMARY - DIFFERENT SOLUTION METHODS FOR VIBRATION OF CANTILEVER BLADES* (the basic approach to centrifugal effects)

1- CONTINUUM MODEL APPROACH:

a) Rayleigh - Ritz.

1952 Lo, 1959 Carnegie.

b) Galerkin.

1965 Rao.

c) Transformation.

1966 Montoya.

d) Collocation.

1965, 1972 Rao.

2- DISCRETE MODEL APPROACH:

a) Holzer - Myklestad.

1973 Rao et al., 1974-5 Ansari, 1977 Dawson and Davies.

b) Matrix.

1951 Plunket, 1976 Murty.

c) Finite Difference.

1960 Carnegie et al., 1967-1972 Carnegie and Thomas.

d) Polynomial Frequency.

1977 Roa and Banarjee.

e) Finite Element.

1970 Ahmed et al., 1972 Dokainish and Rawtany, 1973 Bossak and Zienkiewicz, 1973 Gupta, 1973 Trompete et al., 1974 Filstrup, 1974 Thomas et al., 1975 Tovey, 1975 Allen et al., 1975 Vissar et al., 1976 Thomas et al., 1976 Barlaw, 1977 Dokumaci.

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*See bibliography of Authors at the end of this text.

This section will not be dealt in with great detail as it is very elementary and most of its materials could be found in text books on vibrations and applied mechanics such as for instance Anderson [3]. However it should be mentioned that the method given by Carnegie is found to be satisfactory that is to relate all the equations for the more sophisticated system to the very simplest one by the use of multiplier functions that depend in most of the cases on the geometry and boundary conditions.

2.2.2. Displacement Analysis for Static Bending of Cantilever Blades:

In reference [19] the static bending of pretwisted cantilever blading was developed using the calculus of variations on the equation of the total potential energy to obtain the equations of stability and upon solving these stability equations expressions for the static deflection were found. The angle of pretwist was not to exceed $\frac{\pi}{2}$ radians. Both the case of concentrated loading and distributed loading were considered as well as different cross sections of the cantilever blades such as rectangular aerofoil and square. The total potential energy was found as the sum of the strain energy and the gravitational potential energy. The strain energy is the energy stored in elastic deformation in the blade due to three present moments (refer to figure 2-1):

a- Bending in the x direction.

b- Bedding in the y direction.

c- Torsion.

And hence the widly used term bending - bending - torsion for these

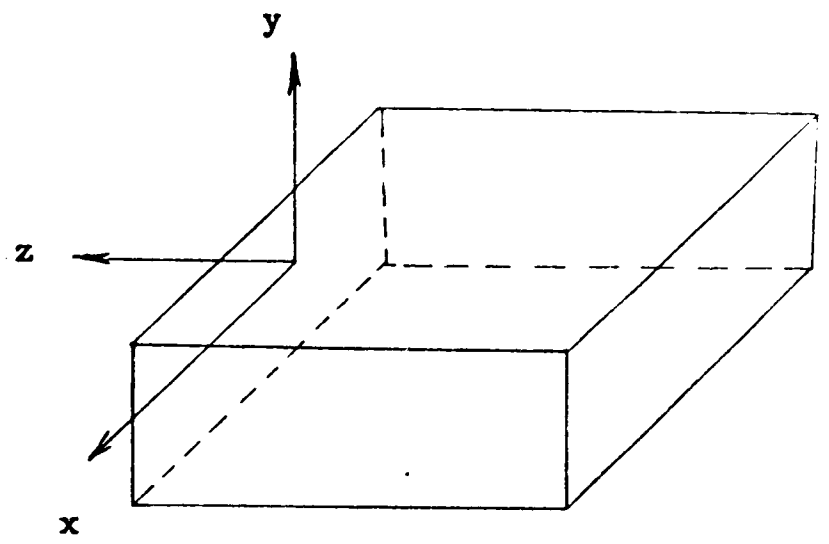

Figure (2.1) Choice of the coordinate axes.

types of problems. The total strain energy obtained by superposition thus becomes,

$$SE = \int_0^L \left\{ \frac{EI_{yy}}{2}\left(\frac{d^2x}{dz^2}\right)^2 + EI_{xy}\left(\frac{d^2x}{dz^2}\cdot\frac{d^2y}{dz^2}\right) + \frac{EI_{xx}}{2}\left(\frac{d^2y}{dz^2}\right)^2 + \frac{C}{2}\left(\frac{d\vartheta}{dz}\right) \right\} dz \qquad \text{.....(2.3)}$$

where,

ϑ is the static angle of elastic twist or torsional deflection and C is the torsional stiffness given by $T = C\,\frac{d\vartheta}{dz}$ and T is the torque.

The gravitational potential energy is given by the expression

$$U_g = \int_0^L w\,(\,y + r_x\,\vartheta\,)\,dz \qquad (2.4)$$

where,

w is the weight per unit length.

y is the elastic deflection (in the direction of gravitational force, assumed).

r_x is the distance between the centre-of-flexure and the centroid in the X direction.

Finally the total potential energy becomes,

$$PE = SE + U_g \qquad (2.5)$$

Upon substituting equations (2.3) and 2.4) into equation (2.5) and applying Euler characteristic equations for stationary conditions (see Appendix IV of reference 19) for a concentrated load, the resulting simultaneous equations will become,

$$\left.\begin{aligned} \frac{d^2}{dz^2}\left\{EI_{xx}\left(\frac{d^2y}{dz^2}\right) + EI_{xy}\left(\frac{d^2x}{dz^2}\right)\right\} &= 0 \\ \frac{d^2}{dz^2}\left\{EI_{xx}\left(\frac{d^2x}{dz^2}\right) + EI_{xy}\left(\frac{d^2y}{dz^2}\right)\right\} &= 0 \end{aligned}\right\} \qquad (2.6)$$

Integrating twice and substituting for boundary conditions a new

set of simultaneous equations would be obtained which, upon solving, yield the expressions for the deflections in the x and y directions.

$$y_L = \frac{-wL^3}{3EI_{xx}} \cdot C_1 \qquad (2.7)$$

$$x_L = \frac{wL^3}{3EI_{yy}} \cdot C_2 \qquad (2.8)$$

where,

$$C_1 = \left[\frac{1}{2}\left(1 + \frac{I_{xx}}{I_{yy}}\right) - \frac{3}{8}\left(1 - \frac{I_{xx}}{I_{yy}}\right)\left(\frac{\sin 2\alpha_L - 2\alpha_L}{\alpha_L^3}\right)\right] \qquad (2.9)$$

$$C_2 = \left[\frac{3}{8}\left(1 - \frac{I_{xx}}{I_{yy}}\right)\left(\frac{\cos 2\alpha_L + 2\alpha_L^2 - 1}{\alpha_L^3}\right)\right] \qquad (2.10)$$

and α_L is the angle of pretwist at the free end.

For a blade of rectangular cross section and thickness ratio $\frac{n}{m}$,

$$I_{xx} = \frac{mn^3}{12}, \quad I_{yy} = \frac{nm^3}{12}, \quad \frac{I_{xx}}{I_{yy}} = \frac{n^2}{m^2}$$

it becomes evident from the above relations that C_1 and C_2 are functions of the geometry.

Recalling that a new set of stability equations can be obtained for the case of distributed loading and hence a new set of multipliers.

$$y_L = C_3 \cdot \frac{-wL^4}{8EI_{xx}} = \left[\frac{1}{2}\left(1 + \frac{I_{xx}}{I_{yy}}\right) - \frac{3}{4}\left(1 - \frac{I_{xx}}{I_{yy}}\right)\left(\frac{\cos 2\alpha_L + 2\alpha_L - 1}{\alpha_L^4}\right)\right] \cdot \frac{-wL^4}{8EI_{xx}} \qquad (2.11)$$

$$x_L = C_4 \cdot \frac{wL^4}{8EI_{yy}} = \left[\frac{3}{4}\left(1 - \frac{I_{xx}}{I_{yy}}\right)\left(\frac{\sin 2\alpha_L - 2\alpha_L + \frac{4}{3}\alpha_L^3}{\alpha_L^4}\right)\right] \cdot \frac{wL^4}{8EI_{yy}} \qquad (2.12)$$

Again $\frac{wL^4}{8EI}$ is the standard result for a blade without pretwist.

In comparing tne deflections resulting from equations (2.7) to (2.12) with the deflections resulting from experimental tests Carnegie concluded that the given equations can be used to calculate the

deflections with high accuracy since both the calculated and experimental deflections agree with each other.

2.2.3. Vibration Analysis of Static Cantilever Blades:

In the previous section formulation of the equations for the static bending of cantilever blading was successfully developed. Since the above mentioned equations are accurate judging by the closeness with experimental results it is found equally suitable for further use in the vibration analysis. However it should be mentioned at this point that there is more than one approach to the vibration problem. The first of these involves solving the differential equations of motion of the cantilever blades , a method which is very exact and easy for cases of simple geometry. In reference [20] a derivation of the differential equations of motion was being made. The other method is the Rayleigh method (Appendix A) which can be employed with very reasonable accuracy for the fundamental frequency of vibration. The exactness of the Rayleigh method could be checked for any particular problem (such as the lateral vibration of straight cantilever blades of rectangular cross section) by chosing a suitable deflection curve to represent the shape of the fundamental mode of motion, and using equation (4.7) of Appendix A. The resulting natural frequency would only be higher than the calculated frequency by a small amount provided that the functions used satisfy the end boundary conditions. Table (2.1) shows some numerical values.

From a geometrical point of view a straight blade of rectangular cross section has three basic modes of vibration , two of which are due to bending in a principal plane and one due to torsion. This implies that there should be three corresponding deflection curves for these motions. When a blade is pretwisted

TABLE 2.1 FREQUENCIES OF LATERAL VIBRATIONS FOR A RECTANGULAR CANTILEVER BLADE: COMPARISON BETWEEN EXACT AND RAYLEIGH METHODS

ASSUMED END DEFLECTION (Units of Length)	REFERENCE	EXACT FREQUENCY	FREQUENCY BY RAYLEIGH METHOD	% ERROR
$y = \frac{q}{2EI_{xx}}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right)$	Carnegie[20]	$\omega_n = \frac{3{\cdot}515}{l^2}\sqrt{\frac{EI_{xx}}{q}}$	$\omega_n = \frac{3.53}{l^2}\sqrt{\frac{EI_{xx}}{q}}$	0.25%
$y = \frac{q}{6EI_{xx}}\left\{3\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)^3\right\}$	Thomson[108]		$\omega_n = \frac{3{\cdot}56}{l^2}\sqrt{\frac{EI_{xx}}{q}}$	1.26%

q = mass per unit length.

it would allow bending to occur simultaneously in both the x and y planes, and a corresponding frequency of coupled bending-bending vibration will result. It becomes evident in this case that for the Rayleigh method to be applied two deflection forms would become necessary to obtain the value of the coupled bending frequency.

As the geometry of the blade becomes more complicated greater possiblity of frequency coupling occurs, for instance a blade of aerofoil section will have bending in each of the principal planes accompanied by torsion to allow for the centroid and the centre of flexure to coincide. Hence two corresponding fundamental frequencies of coupled bending-torsion. Furthermore, if this blade was pretwisted a coupled bending-bending-torsion frequency is more likely to occur. Again this coupled trio would need three simultaneous deflection curves of which two are bending in the x and y planes and the third is in torsion.

Carnegie[20] went on to find expressions for the natural frequency of vibration for all the abovementioned cases, and as mentioned before he used to relate the formuli for the more complicated geometrical shapes to the simpler ones by some coefficients and correction factors that depend upon the geometry itself in most of the cases. Furthermore, this investigator concluded that the static deflection relationships are the most obvious forms to be considered for the shapes of elastic curves of the blades in their fundamental mode of dynamic motion. Hence the relationships of the previous sections (2.2.2. and 2.2.3.) were employed to obtain further useful data, most of which is tabulated in table 2.2.

Correction factors were necessary for the case of torsional vibration as the results of experimental tests had shown higher values for the corresponding frequencies of vibration. Carnegie suggests that this difference could be attributed to the increase

TABLE 2.2 VIBRATION OF STATIC CANTILEVER BLADING : DIFFERENT GEOMETRICAL SHAPES

CASE NO.	VIBRATION MODE	PRETWIST	CROSS SECTION	THEORETICAL, NOMINAL OR EXACT FREQUENCY	FREQUENCY GIVEN BY CARNEGIE
1	lateral	none	rectangular	$\omega_n = \frac{3.52}{l^2}\sqrt{\frac{E I_{xx}}{q}}$	$\omega_n = \frac{3.53}{l^2}\sqrt{\frac{E I_{xx}}{q}}$
2	torsional	none	rectangular	$\omega_n = \frac{n\pi}{2l}\sqrt{\frac{C}{I_{cg}}}$	$\omega_n = \frac{n\pi}{2l}\sqrt{\frac{C}{I_{cg}}}\sqrt{1 + \frac{n^2 \pi^2 E t^3 m^3}{576 C l^2}}$
3	lateral	yes	rectangular	$\omega_n = \frac{3.53 K}{l^2}\sqrt{\frac{E I_{xx}}{q}}$	*
4	torsional	yes	rectangular	same as case 2	same as case 2
5	coupled b-t	none	aerofoil	$\omega_n = \frac{n\pi K}{2l}\sqrt{\frac{C}{I_{cg}}}$	$\omega_n^2 = \frac{1}{l}\left\{\frac{\frac{l^2}{20EI_{xx}} + \frac{r_x^2}{3C}}{\frac{2.311 w l^4}{576 E^2 I_{xx}^2} + \frac{2 I_{cg} r_x^2}{15 C^2} + \frac{37 w r_x^2 l^2}{840 E I_{xx} C}}\right\}$
6	torsional	none	aerofoil	$\omega_n = \frac{n\pi K}{2l}\sqrt{\frac{C}{I_{cg}}}$	$K = \sqrt{1 + \frac{E n^2}{48 l^2 C}\int_A b^2 t^3 \, db}$
7	coupled b-b-t	yes	aerofoil	$\omega_n = \frac{3.53 K}{l^2}\sqrt{\frac{E I_{xx}}{q}}$	$K^* = \frac{2\left(1 + \frac{I_{xx}}{I_{yy}}\right) + 15\left(1 - \frac{I_{xx}}{I_{yy}}\right)A}{\left(1 + \frac{I_{xx}}{I_{yy}}\right)^2 + \left(1 - \frac{I_{xx}}{I_{yy}}\right)^2 B + \left(1 - \frac{I_{xx}}{I_{yy}}\right)^2 C}$
9	torsional	yes	aerofoil	same as case 2	same as case 2

* see Carnegie[20] for the values of K,A,B, and C.

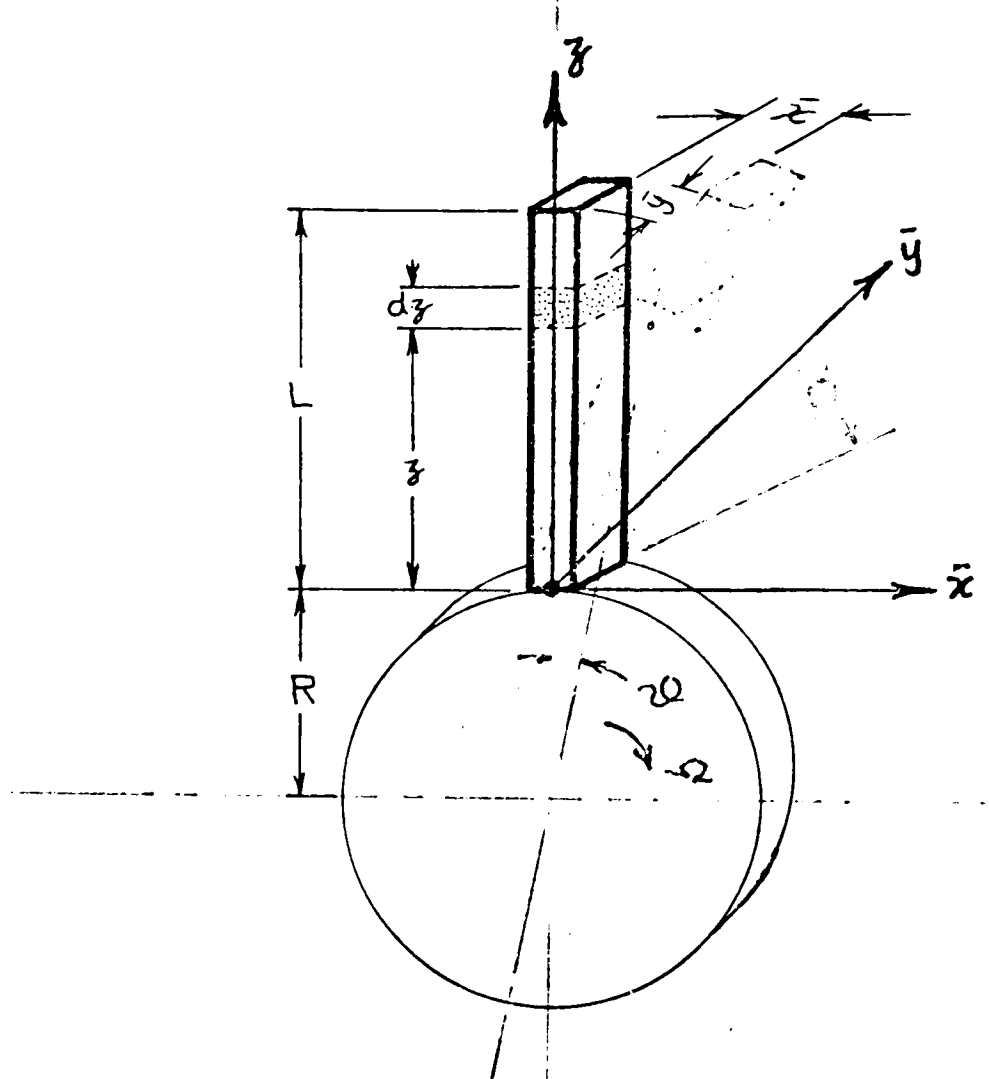

FIGURE 2.1. A : BLADE DISC ASSEMBLY.

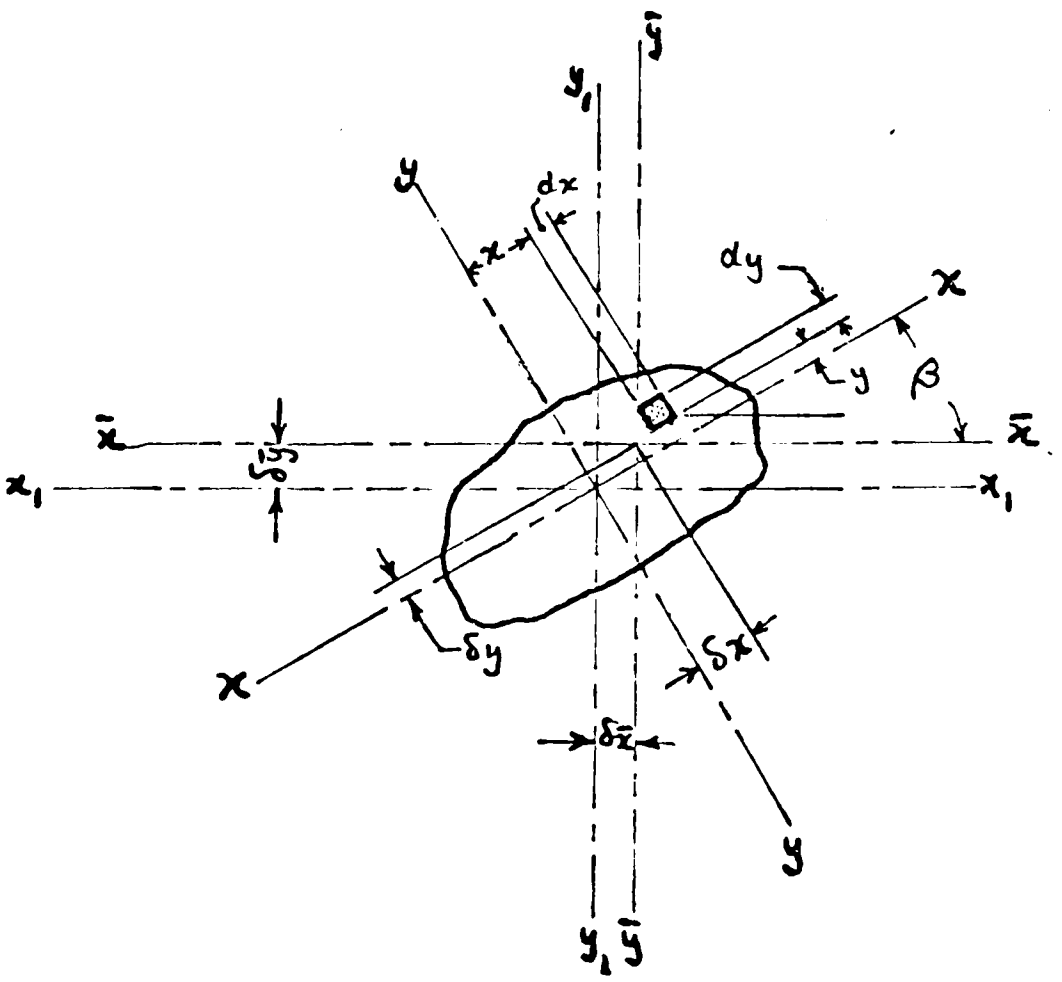

FIGURE 2.1. B : RELATIONSHIP BETWEEN COORDINATE AXES.

of the torsional stiffness of a vibrating blade under pure torsional oscillation due to the fact that its longitudinal fibers are subject to bending. The reader is referred to the original paper for further detail.

In conclusion, it was found that the relations tabulated in table 2.2 agree with measured values taken from vibration tests conducted on sets of blades of both uniform rectangular and uniform aerofoil cross section pertwisted between 0 and 90 degrees, hence justifying the use of the Rayleigh's method as a simple but reliable tool for the calculation of natural frequencies of vibration.

Further material about this subject could be found in text books about vibration such as those listed in references[3,77,108,109].

2.3. Vibrations of Rotating Cantilever Blades

2.3.1. The Energy Approach

The expression for the potential energy resulting from centrifugal forces can be obtained by considering a general blade of length L fixed radially at an angle φ on a rotating disc of radius R . The speed of rotation being Ω rev/min. By considering a rotating coordinate system the disc could be assumed fixed with a radial centrifugal force imposed on it. In such a case it can be assumed to have two components of deflection, one being in the plane of rotation and one out of the plane of rotation. When the blade deflects out of the plane of rotation the centrifugal force on an elemental section dz (Figure 2-2A) will take the same direction as the zz plane and hence its component in the $\bar{y}$ direction* will be zero,

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*here $\bar{y}$ and later $\bar{x}$ are auxilliary coordinate system inclined at an angle φ to the normal y and x axes respectively[21].

and it follows that the potential energy stored by the element is also zero, mathematically

$$PE_{\underline{y}} = 0 \tag{2.13}$$

The centrifugal force acting in the plane of rotation is resolved into two components; radial and circumferential. The radial component is given by dF_z ,

$$dF_z = dF \cos\beta \tag{2.14}$$

where,

$$dF = q\, dz\ \Omega^2\ (R + z) \tag{2.15}$$

and q is the mass per unit length.

For small angles β , $\cos\beta \cong 1$ and the force component in the z direction is the same as dF given by (2.15) and the z component of the potential energy becomes,

$$PE_z = dF \cdot \Delta$$

where Δ is the small displacement of the centre of the infinitesimal element. By simple geometry[21] it can be shown that

$$d\Delta = \tfrac{1}{2}\left\{\left(\frac{d\bar{x}}{dz}\right)^2 + \left(\frac{d\bar{y}}{dz}\right)^2\right\} dz \tag{2.17}$$

and the total displacement Δ of the infinitesimal element at z is given by

$$\Delta = \tfrac{1}{2}\int_0^z \left\{\left(\frac{d\bar{x}}{dz}\right)^2 + \left(\frac{d\bar{y}}{dz}\right)^2\right\} dz \tag{2.18}$$

considering small displacements and using equations (2.15) and (2.18)

the potential energy in the z direction for the entire blade becomes

$$PE_z = \frac{\Omega^2}{2} \int_0^L \left[q\,(R + z) \int_0^z \left\{ \frac{d\bar{x}}{dz}^2 + \frac{d\bar{y}}{dz} \right\}^2 dz \right] dz \qquad (2.19)$$

Now the circumferential components of the centrifugal force can be written as:

$$dF_{\bar{x}} = dF \sin\beta \qquad (2.20)$$

where,

$$\sin\beta = \frac{\bar{x}}{R + z} \qquad (2.21)$$

Substituting equations (2.21) and (2.15) into equation (2.20) yields

$$dF_{\bar{x}} = q\ \Omega^2\ \bar{x}\ dz \qquad (2.22)$$

knowing that $dF_{\bar{x}}$ increases from zero to $dF_{\bar{x}}$ then an average value is taken as $\frac{dF_{\bar{x}}}{2}$. The corresponding potential energy for the entire blade will be,

$$PE_{\bar{x}} = -\int_0^L \left[\frac{q}{2}\ \Omega^2\ \bar{x}^2 \right] dz \qquad (2.23)$$

The total potential energy resulting from centrifugal forces will be the sum of its components in the $\bar{x}$, $\bar{y}$, and z directions,

$$PE_\Omega = PE_{\bar{x}} + PE_{\bar{y}} + PE_z \qquad (2.24)$$

substituting equations (2.23),(2.13) and (2.19) into equation (2.24) to give

$$PE_\Omega = \frac{\Omega^2}{2} \int_0^L \left[q\,(R + z) \int_0^z \left\{ \frac{d\bar{x}}{dz}^2 + \frac{d\bar{y}}{dz} \right\}^2 \right] dz - w\,\bar{x}^2 \Big] dz \qquad .....(2.25)$$

Now the relations for coordinate transfer are given by:

$$\left.\begin{aligned} \bar{x} &= x_1 \sin\varphi + y_1 \cos\varphi \\ \bar{y} &= x_1 \cos\varphi + y_1 \sin\varphi \end{aligned}\right\} \qquad (2.26)$$

where x_1 and y_1 are the shifts of the centroid in the x and y directions respectively resulting from an aerofoil cross section over a straight rectangular cross section:

$$\left.\begin{aligned} x_1 &= x + \zeta_y \vartheta \\ y_1 &= y + \zeta_x \vartheta \end{aligned}\right\} \qquad (2.27)$$

and ϑ is the angle of elastic twist.

Substituting equations (2.26) into equation (2.25) gives:

$$PE_\Omega = \frac{\Omega^2}{2} \int_0^L \left[q\,(R + z) \int_0^z \left\{ \left(\frac{dx_1}{dz}\right)^2 + \left(\frac{dy_1}{dz}\right)^2 \right\} dz \right.$$

$$\left. -q\left(x_1^2 \sin^2\varphi - 2\, x_1\, y_1 \sin\varphi \cos\varphi + y_1^2 \cos^2\varphi \right) \right] dz$$

.....(2.28)

The strain energy of a blade vibrating in combined bending-bending-torsion was obtained earlier in this text and was given by equation (2.3). When gravitational effects are ignored the total potential energy for the blade will be the sum of both equations (2.3) and (2.28),

$$PE_{total} = PE_\Omega + SE \qquad (2.29)$$

The expressions for the kinetic energy of a blade vibrating in combined bending-bending-torsion is considered to be the sum of two parts, the first of which is the kinetic energy of the mass, concentrated at the centroid, PE_m . The second part is the kinetic energy due to rotation about the centroid, PE_ϑ . Hence the instantaneous kinetic energy of the mass at the centroid of the infinitesimal element is given by:

$$d\,KE_m = \tfrac{1}{2}\, q \left[(\dot{x}_1)^2 + (\dot{y}_1)^2 \right] dz \qquad (2.30)$$

using equations (2.27) into (2.30) leads to,

$$d\,KE_m = \tfrac{1}{2}\,q\left[(\dot{x} + \varsigma y\,\dot{\vartheta})^2 + (\dot{y} + \varsigma x\,\dot{\vartheta})^2\right] dz \qquad (2.31)$$

and the total instantaneous kinetic energy will be,

$$KE_m = \tfrac{1}{2}\,q \int_0^L \left[(\dot{x} + \varsigma y\,\dot{\vartheta})^2 + (\dot{y} + \varsigma x\,\dot{\vartheta})^2\right] dz \qquad (2.32)$$

The instantaneous kinetic energy due to rotation about the centroid is given by,

$$d\,KE = \frac{I_{cg}}{2}\,dz\,(\dot{\vartheta})^2 \qquad (2.33)$$

and the total instantaneous kinetic energy will be,

$$KE = \tfrac{1}{2}\int_0^L I_{cg}\,(\dot{\vartheta})^2\,dz \qquad (2.34)$$

the total kinetic energy for the blade will be the sum of equations (2.32) and (2.34), thus

$$KE_{total} = KE_m + KE \qquad (2.35)$$

or

$$KE_{total} = \tfrac{1}{2}\,q \int_0^L \left[(\dot{x} + \varsigma y\,\dot{\vartheta})^2 + (\dot{y} + \varsigma x\,\dot{\vartheta})^2 + \frac{I_{cg}}{q}\,(\dot{\vartheta})^2\right] dz \qquad \ldots\ldots\ldots(2.36)$$

Finally the natural frequency of the fundamental mode can be obtained by using the Rayleigh principle and equating equations (2.36) for the kinetic energy with (2.29) for the potential energy will result in the required frequency.

Carnegie[21] was the first to calculate the kinetic and potential energies in this way and he gave an example problem to test the method. The motion was assumed to be harmonic and a straight rotating cantilever blade of rectangular cross section was

chosen. The following equation was used to represent the static deflection curve:

$$y_o = \frac{q}{2 E I_{xx}} \left[\frac{L^2 z^2}{2} - \frac{L z^3}{3} + \frac{z^4}{12} \right] \qquad (2.37)$$

and the resulting fundamental circular frequency was:

$$\omega_n = \frac{3.53\ K}{L^2} \sqrt{\frac{E I_{xx}}{w}} \qquad (2.38)$$

where,

$$K = \left\{ 1 + \frac{w \Omega^2}{E I_{xx}} \left[\frac{R L^3}{8} + \frac{L^4}{10.6} - \frac{L^4 \cos^2}{12.45} \right] \right\}^{\frac{1}{2}} \qquad (2.39)$$

By comparing ω_n from equation (2.38) with the corresponding ω_n shown on table 2.2 (case number 1) it will be found that both are the same except for the factor K of equation (2.39) which is clearly a function of the speed of rotation and blade geometry. It is evident from equation (2.39) that as the rotational speed increases, the value of K and hence the fundamental frequency will also be increased. The rise of the frequency with the speed of rotation is also explained by the fact that the centrifugal forces on the blades tend to reduce the lateral bending by increasing the in-plane force component, this will ensure higher stiffness to mass ratio and hence a higher natural frequency.

2.3.2. Including Effects due to Torsion, Shear Deflection, and Rotary Inertia

So far, total energy expressions for a blade vibrating in combined bending-bending-torsion have been made. It is now wished to make an allowance for bending of longitudinal fibers occuring due to torsion. Using Figure (2.3) it can be shown that for small angles

of elastic twist, the deflection of an element of the blade at a distance n from the centre of torsion will be:

$$y = b\,\vartheta \tag{2.40}$$

From mechanics of materials the shearing force dV is given by

$$dV = - E I_{xx} \left(\frac{d^3\vartheta}{dz^3} \right) \tag{2.41}$$

where the negative sign indicated a negative shearing force for a positive deflection y . Substituting equation (2.40) into equation (2.41) will result in,

$$dV = - E I_{xx}\, b \left(\frac{d^3\vartheta}{dz^3} \right) \tag{2.42}$$

and the torque due to this shearing force is,

$$dV \cdot b = dT = - E I_{xx}\, b^2 \left(\frac{d^3\vartheta}{dz^3} \right) \tag{2.43}$$

for the elemental fibre of figure 2.3, $I_{xx} = d\,b\,t^3/12$ and hence,

$$dT = - \frac{E}{12} \left(\frac{d^3\vartheta}{dz^3} \right) b^2\, t^3\, db \tag{2.44}$$

and the total torque becomes

$$T = - \frac{E}{12} \left(\frac{d^3\vartheta}{dz^3} \right) \int_A b^2\, t^3\, db \tag{2.45}$$

the strain energy due to this elastic torsion will be $(\frac{1}{2}\, T\, d\vartheta)$,

$$\text{or} \quad SE_T = \int_0^L \frac{E}{24} \frac{d\vartheta}{dz} \frac{d^3\vartheta}{dz^3} \int b^2\, t^3\, db \tag{2.46}$$

Adding equation (2.46) to equation (2.3), will give the total potential energy for a blade vibrating in combined bending-bending-torsion and allowing for bending of longitudinal fibres. The total kinetic energy expression for the blade will remain the same as equation (2.36)[22].

Two additional terms are required for the total kinetic energy if the effects of rotary inertia are to be included, they result from

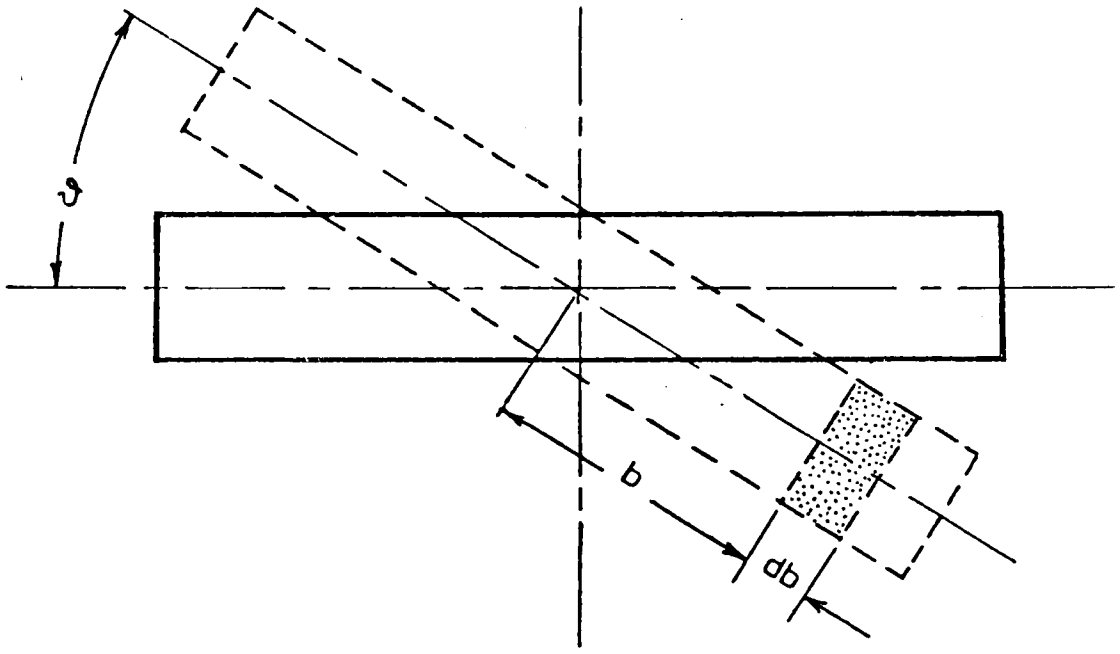

FIGURE 2-3 Blade cross section twisted by an angle ϑ

rotations φ_{1r} and φ_{2r} in the $x_1 z$ and $y_1 z$ planes respectively, and the resulting extra kinetic energy will be[23]

$$KE_r = \int_0^L \left\{ \frac{I_{yy}\ \rho\ (\dot{\varphi}_{1r})^2}{2} + \frac{I_{xx}\ \rho\ (\dot{\varphi}_{2r})^2}{2} \right\} dz \qquad \ldots\ldots(2.47)$$

Letting $\varphi_{1r} = [\ \varphi_{1b} + (\varsigma_y \vartheta)'\ ]$ and $\varphi_{2r} = [\ \varphi_{2b} + (\varsigma_x \vartheta)'\ ]$ (where φ_{1b} and φ_{2b} are slopes due to bending and the prime sign means a differentiation with respect to z), and summing with equation (2.36) the final expression for the kinetic energy allowing for rotary inertia becomes,

$$KE = \tfrac{1}{2} \int_0^L \Big\{ q \left[(\dot{x} + \varsigma_y\ \dot{\vartheta})^2 + (\dot{y} + \varsigma_x\ \dot{\vartheta})^2 \right] + I_{cg} (\dot{\vartheta})^2 + \rho \left[I_{yy} (\varphi_{1b} + \varsigma_y \dot{\vartheta} + \varsigma_y' \dot{\vartheta})^2 + I_{xx} (\varphi_{2b} + \varsigma_x \dot{\vartheta} + \varsigma_x' \dot{\vartheta})^2 \right] \Big\} dz \qquad \ldots\ldots(2.48)$$

If the effect of shear deflection is to be included, the expression for total potential energy must be modified by adding the strain energy expressions resulting from this deflection, thus

$$d\ SE = \tfrac{1}{2} \left\{ F_x\ \varsigma x_s + F_y\ \varsigma y_s + Mx\ \varsigma\varphi_{1b} + My\ \varsigma\varphi_{2b} \right\} . \qquad (2.49)$$

where x_s and y_s are the corresponding deflections due to shear.

Substituting for F , and M into equation (2.49) by appropriate parameters obtained in references[19,23] and integrating over the length of the blade, the extra strain energy resulting from shear deflection will be

$$SE_s = \int_0^L \left\{ \frac{A\ G}{2\ K} \left[(x' - \varphi_{1b})^2 + (y' - \varphi_{2h})^2 \right] \right\} dz \qquad (2.50)$$

where A is the cross sectional area of the blade and k is a torsional constant. Summing equation (2.50) with equation (2.3) will give the final expression for the total potential energy for a blade vibrating in combined bending-bending-torsion and allowing

for shear deflection.

2.3.3. Procedure to Obtain the Equation of Motion:

In the previous sections, expressions for the lowest natural frequency of rotating and non-rotating cantilever blading were obtained, based upon energy expressions that were developed for different boundary conditions. It is found very helpful for every boundary condition using the calculus of variation (see Appendix G). The procedure for obtaining these equations is standard and can be employed for all cases, using Hamilton's Principle which states that the minimum of the integral

$$I = \int_{t_1}^{t_2} L \, dt \qquad (2.51)$$

for fixed t_1 and t_2 with L (the Lagrangian)

$$L = KE - PE \qquad (2.52)$$

give the Euler-Lagrange equations which are nothing but the equations of motion of the system.

In the rest of this section an attempt will be made to find one (since the others follow the same line of steps) set of equation of motion for a pretwisted blade of uniform asymmetric cross-section. Using equations (2.3.) and (2.36) to write the Lagrangian,

$$L = \int_0^L \left[\tfrac{1}{2} q \left\{ (\dot{x} + S_y \dot{\vartheta})^2 + (\dot{y} + S_x \dot{\vartheta})^2 + \frac{I_{cg}}{q} (\dot{\vartheta})^2 \right\} - \left\{ \frac{E I_{xx}}{2} \left(\frac{\partial^2 y}{\partial z^2} \right)^2 + E I_{xy} \left(\frac{\partial^2 y}{\partial z^2} \right) \left(\frac{\partial^2 x}{\partial z^2} \right) + \frac{E I_{yy}}{2} \left(\frac{\partial^2 x}{\partial z^2} \right)^2 + \frac{C}{2} \left(\frac{\partial \vartheta}{\partial z} \right)^2 \right\} \right] dz \qquad \text{.....(2.53)}$$

and application of Hamilton's principle results in

$$\int_{t_1}^{t_2} L\,dt = \int_{t_0}^{t_1}\int_0^l \left[\tfrac{1}{2} q\left\{ (\dot{x} + S_y \dot{\vartheta})^2 + (\dot{y} + S_x \dot{\vartheta})^2 + \frac{I_{cg}}{q} (\dot{\vartheta})^2 \right\} \right.$$

$$\left. - \frac{E}{2}\left\{ I_{xx}\left(\frac{\partial^2 y}{\partial z^2}\right)^2 + 2 I_{xy}\left(\frac{\partial^2 y}{\partial z^2}\right)\left(\frac{\partial^2 x}{\partial z^2}\right) + I_{yy}\left(\frac{\partial^2 x}{\partial z^2}\right) + C\left(\frac{\partial \vartheta}{\partial z}\right)^2 \right\} \right] dz\,dt$$

.....(2.54)

Stationary values of this integral which is of the form of equation (G.28), are obtained by operating equations (G.29), (G.30) and (G.31) to it. The resulting differential equations of motion will be

$$\frac{\partial^2}{\partial z^2}\left\{ E I_{xx}\left(\frac{\partial^2 y}{\partial z^2}\right) + E I_{xy}\left(\frac{\partial^2 x}{\partial z^2}\right)\right\} = - q\,(\ddot{y} - S_x \ddot{\vartheta}) \qquad (2.55)$$

$$\frac{\partial^2}{\partial z^2}\left\{ E I_{yy}\frac{\partial^2 x}{\partial z^2} + E I_{xy}\left(\frac{\partial^2 y}{\partial z^2}\right)\right\} = - q\,(\ddot{x} - S_y \ddot{\vartheta}) \qquad (2.56)$$

$$\frac{\partial}{\partial z}\left\{ C\left(\frac{\partial^2 \vartheta}{\partial z^2}\right)\right\} = q\left\{ S_y \ddot{x} + S_x \ddot{y} + S_y^2 \ddot{\vartheta} + S_x^2 \ddot{\vartheta} + \frac{I_{cg}\ddot{\vartheta}}{q}\right\} \qquad (2.57)$$

As can be seen, these equations are fairly complicated and are only soluble for a small number of cases where simplifying boundary conditions are used.

2.4. Experimental Work:

Normally, the results from an analytical procedure are backed up by experimental results to provide the necessary proof that the analytical procedure can be safely used in future as a substitute for the experimental set up that may not be convenient to proceed later on for various reasons. One of the causes behind this experimental assesment being the difference between the assumed analytical boundary conditions and the actual environments and a valuable example can be drawn from the Author's experience during the course of this investigation. It was a good opportunity that access was

possible to a readily designed and built turbomachinery blade on which experimental vibration tests were made (explained later in this chapter) by clamping the non-rotating blade at the root between two wooden blocks thus providing a plastic support for the blade. Due to the difficulties involved in the attempt to exactly model the blade root by 3-Dimensional finite elements, a very primitive method was used instead by assuming an infinite stiffness at the bottom of the blade shank, this provided a hypothetical rigid support and the difference in results of the two methods was in order of magnitude of 3 for the fundamental frequency (see chapter 6). At the first instant, it was thought that a mistake is the most evident factor, however after modelling the blade by various other methods and like other investigations[86,112] springs were assumed to exist at the root which means that a finite stiffness is applied at the shank bottom nodes, it became certain that the choice of boundary conditions has a very significant effect on the results.

Setting an experimental procedure requires information about three definite parts,

1- Definition of the problem variables, blade geometry and method of clamping:

Probably an ideal experimental setting would be made using the actual parts or machinery to be tested, however due to economic reasons, and or technical difficulties it becomes common practice to choose models of smaller or actual size being produced in a cheaper method and that help a better control on the variables to be studied. Normally the problem variables will define the methods

and techniques used to study these variables, and hence it is at this stage where the two other factors (vibration excitation and monitoring) are considered carefully.

2- Methods of Vibration Excitation :

Early experimental investigations were based upon static cantilever blading rather than rotating blades due to the difficulties involved in monitoring vibrations of the latter kind. Rao[94] mentioned that the experimental investigation of blade vibrations started with marine propellers that were producing excessive noise and hence ncessitating a study of the natural frequencies and vibration modes. For this particular problem tests were performed on flat blades of uniform thickness held firmly at the root and excited with a violin bow. Other methods included the use of magneto-striction elements and electro-magnetic excitation where the blade is vibrated by a variable frequency electromagnet, often with a DC bias to improve the magnetic circuit, the difficulties encountered is that it is only restricted to magnetic materials and to low vibratory energy of longer blades. Mechanical flexing was used for higher excitation energies but the problem of limited low frequency range is still standing same as with the case of mechanical excitation Truman et al.[113] criticised the jet nozzle method and suggested an alternative pulsed air method.

At present, easier methods are used with the introduction of standard electro-magnetic or hydraulic shakers which are used for higher power requirements, while for low

power requirements hybrid analogue systems are used with impact hammers. The latter being found useful in low frequency situations with lightly damped structures.

3- Vibration Monitoring:

From a technological point of view, monitoring vibrations is much more involved than exciting these vibrations. It is believed that attaching strain gauges on the moving parts is the best method of recording the deformations and their frequency, but unfortunately there is a great difficulty involved in transfer of the signals from the rotating strain gauges to the stationary recording device. The suggestion of using sliprings has been enhanced by Carnegie [24] and Andrews [4], both were satisfied with it. An alternative way is possible through the use of mercury cells*. Additional problems are encountered with high temperatures and high centrifugal action. of course, these problems will require proper heat resisting properties for the strain gauge cement. The reader is referred to Kemp et al.[66] for such types of gauges.

Wallace [116] in his paper gives some useful information about the application of transducers in different practical problems.

Modern monitoring equipment normally consist of three basic parts:

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*literature available from the manufacturers

a- The transducer which can be either of two groups, the seismic transducer (acceleration or velocity transducer) which measures the absolute motion of moving parts at the point of attachment, the second group consist of various probes which measure the relative motion between moving parts.

b- The signal conditioning unit that makes decisions with regard to significant parts of the wide frequency band of the response spectrum of the transducer.

c- The analysing system which helps interpreting the data provided and relating it to the problem environment.

Rao [93] gives a historical review of the development of these methods and text books [55] give more detail about the subject.

2.4.1. Some Particular Experimental Investigations:

In this section detailed explanation of some experimental investigation will be given. The author find them necessary since the first reflects the importance monitoring vibrations on the work site [32], while the others show the effect of rotation on the vibration characteristics of stationary members. A third group of investigations will be concerned with the change of some of the problem variables.

2.4.1.1. Davies et al. [32]

In this work a team of investigators combined the application of theoretical and standard experimental techniques to the diagnosis of some operational problems on modern power generation plants. Attention will be given to one problem only that lies within the scope of this text. This problem is concerned with a large fan that had

critical speeds within its range of normal operational speeds.

Lateral vibrations were measured in terms of radial displacement of rotating parts relative to each other.

The site instrumentation included a mobile trolly containing signal conditioning equipment which enables vibration signals to be observed and recorded on magnetic tape for further off-site processing.

The tape is then fed through an analogue digital converter to a digital computer programmed to analyse the vibration of the required components by using a fast fourier transform software pack.

By plotting the displacement against the speed of rotation a complete view of the situation was obtained. The vibrations were reduced by increasing the vertical stiffness of the supporting bearing and reducing the overhung distance. By changing these two factors the gyroscopic effects are reduced to minimum as well.

2.4.1.2. R.F. French[52]

In this work an attempt is made to simulate the actual running conditions of a gas turbine to study the vibration and fatigue of the blades in their actual centrifugal field.

The investigation concentrated on the argument that the old methods of statically testing the blades would not give a very clear idea about the vibration characteristics, and although an imrovment of adding a spring luad at the tip of the blade[122] is made but still it is not satisfactory as it produces a constant load on the test part, while the centrifugal loading would generate a load gradient along the blade.

The test rig consists of a 500 HP electric motor which is coupled to a speed increaser through an eddy current clutch coupling,

thus giving an infinitely variable speed range of 500-10 000 rev/min. Vibrations were excited by equally spaced air jets directed at or near the tip of the blade. A provision is made for high temperature testing by adding a combustion chamber to the air circuit.

With some difficulty, electrical resistance strain gauges were used for the strains and thermocouples for the temperature measurements. The strain gauge and thermocouple leads were brought and down the face of the disc then out through the centre of the shaft to a slip ring. A frequency counter was used for fatigue testing while a magnetic tape was used for stresses or vibrations. Information from the magnetic tape were fed to an x-y recorder to show the vibration response.

2.4.1.3. Bigret[13]

In his paper Bigret[13] gives a brief explanation of three experimental procedures to study the vibrations of steam turbine blades. These three cases are:

a- determination of the resonant frequencies for stationary blades.

b- Determination of the resonant frequencies during rotation in a special rig.

c- Study of vibration behaviour in service.

It is found that only part -b- is of importance for this text and it will be explained in more detail. The main point of interest in here will be the method of signal transmission.

The power driver is a DC motor which supplies variable speed to the test rotor.

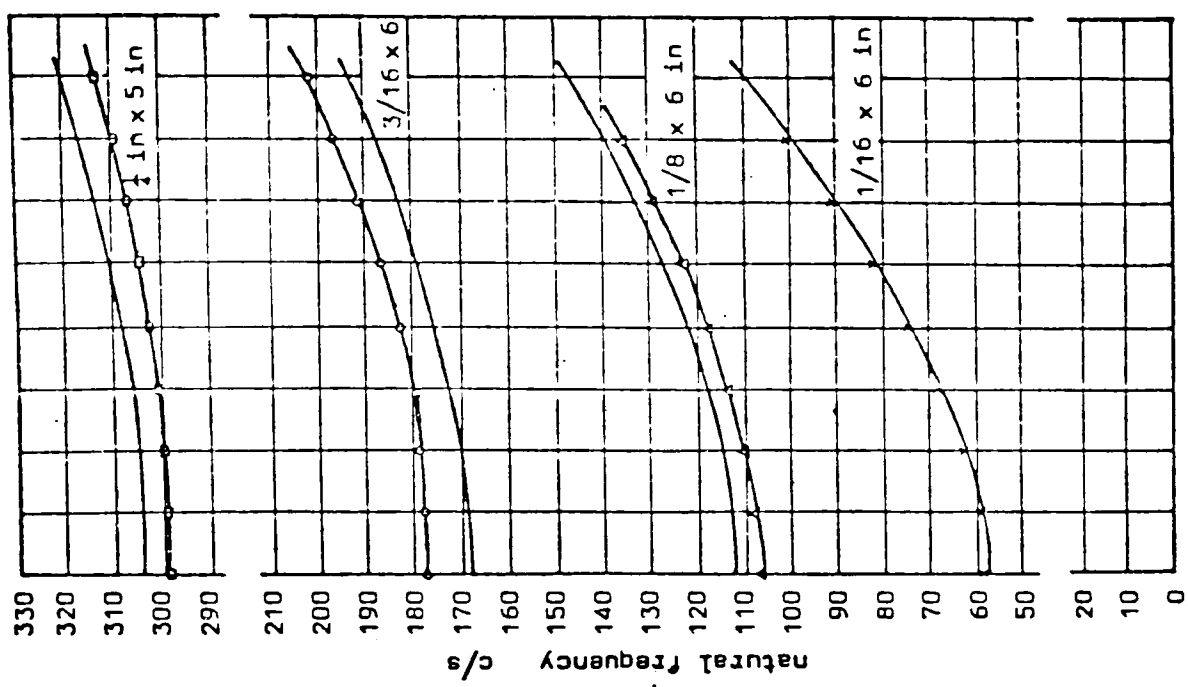
Vibration is excited by two opposite electro-magnets having their poles 5 mm away from the blade tips. Another part of the excitation comes from the centrifugal effects.

A signal transmission system was used. The strains in the blades were obtained by means of piezoxide transducer and semiconductor gauges. Lead wires from the transducers are connected to a transmitting antenna which consists of a free wire stretched radially due to the centrifugal forces. The transmitter circuitry consist of high frequency generators which are fixed on the rotor with their associated batteries. The receiver antenna is a steel ring placed around the rotor.

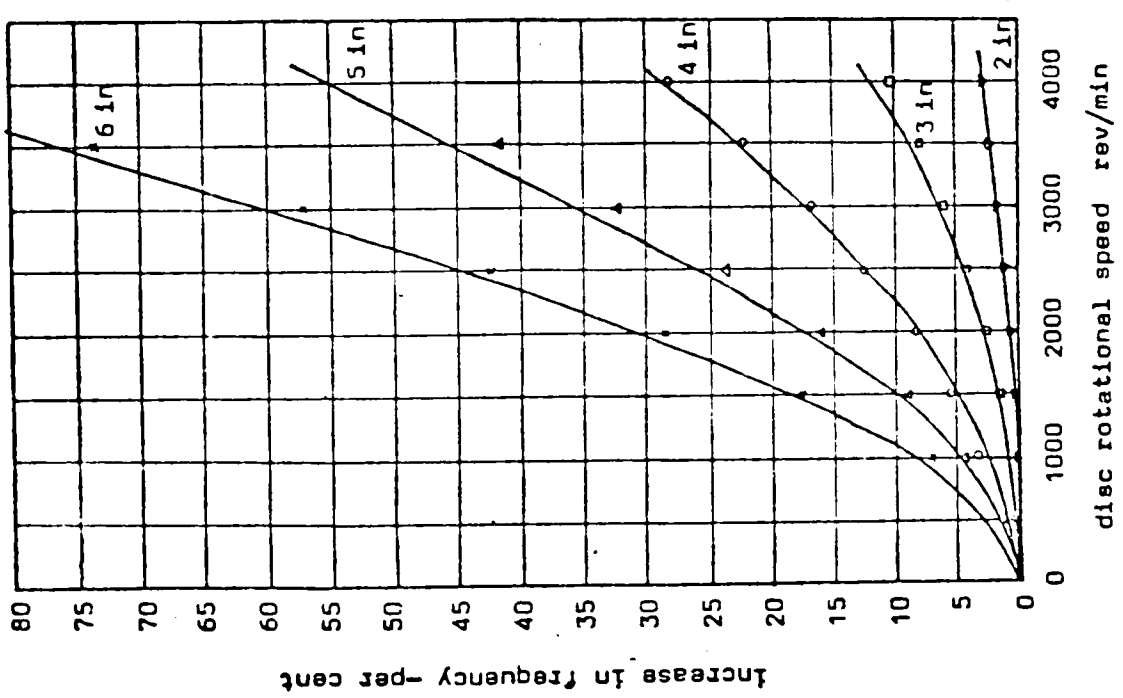
Bigret concludes that it is a simple matter to predict the resonant frequencies and modes of vibration of stationary blades. Further he suggested that an algorithm used on a digital computer can calculate the effect of centrifugal acceleration due to rotation but he doubts the possibility of obtaining the effects of coupling between the blades themselves only. With the aid of the setting just explained above he was able to obtain an idea about the coupling effects but not the modes of vibration. The interesting point that the author of this text had noticed is that Ewins[40,41] who presented his paper[40] at the same conference mentioned both experimental and theoretical studies concerning the effect of coupling.

2.4.1.4. Carnegie et al.[24]

It was thought necessary to finish the part about experimental work by giving results obtained from an actual experimental installation. The work of this paper was divided into two parts, the first entails the construction of a test rig capable of operating up to a speed of 20 000 rev/min. The second part included experimental results of the work carried out in a lower speed range of 4 000 rev/min. Also given, a complimentary theoretical investigation using the finite difference method based on equations of motion given by Carnegie[24].



b- Effect of changing blade thickness.



observed values, x o
theoretical values

a- Effect of changing blade length.

FIGURE 2-3 EFFECT OF CHANGING BLADE DIMENSIONS - FREQUENCY-SPEED CHARACTERISTIC CURVES. (TAKEN FROM REFERENCE 24)

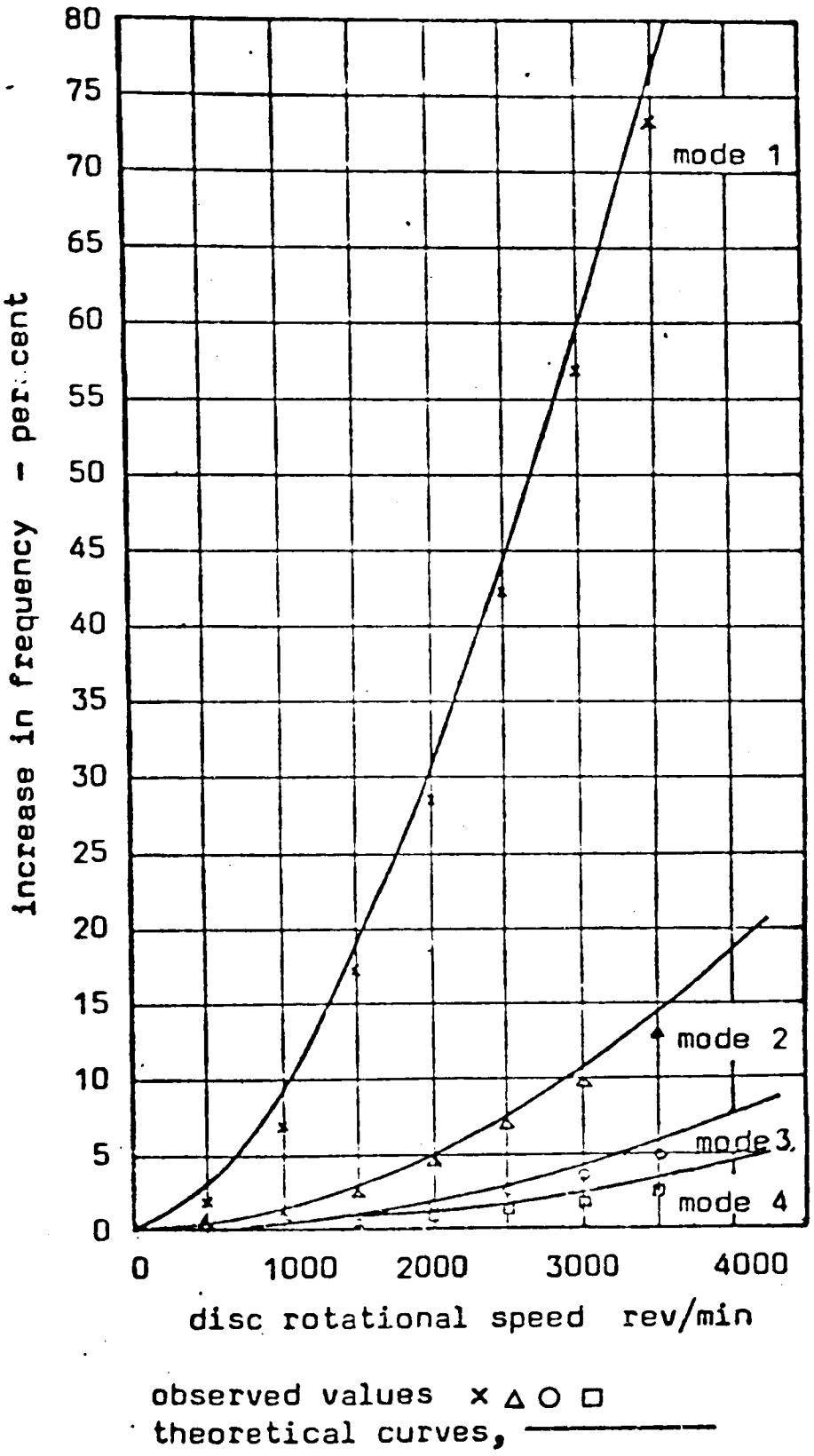
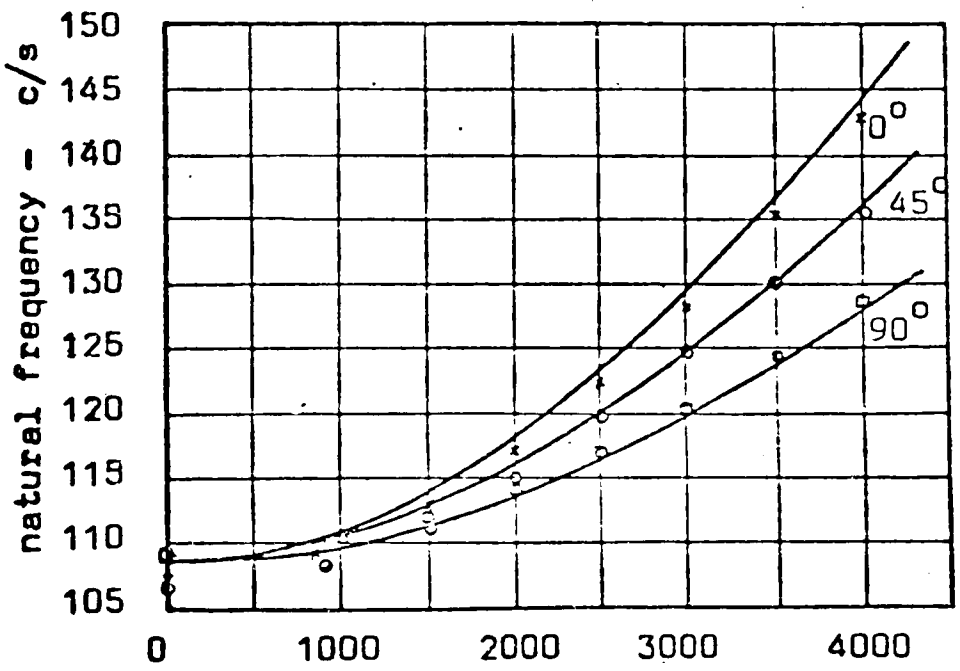


FIGURE 2-4 CHANGE OF THE VIBRATIONAL MODES WITH SPEED

(taken from reference 24)



observed values x, O, □;
stagger angles 0,45,90 degrees respectively
theoretical curves, ————

FIGURE 2-5 AEROFOIL STUDY - EFFECT OF CHANGING STAGGER ANGLES

(taken from reference 24)

The main point of interest was the lateral vibration in the weak plane of bending of slender blades of uniform rectangular and uniform aerofoil cross section. The effect of centrifugal forces on the natural frequency of the first four modes of such vibration was obtained, together with the influence of the stagger angle parameter. Rectangular blades of different length and thickness were used in addition to one blade of a uniform aerofoil cross section. The stagger angle was varied in steps of 15 degrees over the range of 0 to 90 degrees. The results are presented in the same way suggested by DiTorronto[35], that is, showing the effect of rotational speed on the natural frequency of blades as a percentage increase in the frequency above the non-rotating frequency.

The results are shown in Figures 2-3 to 2-5 showing the effects of changing the various parameters.

2.5. The Numerical Approach :

The problem of vibration of rotating cantilever blades received much attention with the availability of digital computers and the evolution of numerical methods in the early 1950's. It continuously developed from easy equations and simple boundary conditions to the inclusion of all the variables involved. The survey given in section 2.2.1. is a good guide to the history of the numerical methods as applied to this problem. Concentration will only be given to studies employing the finite element method, however explanation of one paper in each of the other methods will be given.

2.5.1. Lumped Parameter Systems :

The work of Ansari[5,6] is well distinguished in this respect and a brief description of his work will be given below.

Ansari gave an analysis for the problem of non-linear modes

of vibration of a complicated (uniform pretwisted blade of unsymmetrical cross section) cantilever beam, mounted on the periphery of a rotating disc. The analysis was extended to include the effect of shear deformation, rotary inertia, and coriolis accelerations. The analysis was based on a discrete mathematical model that follows the Ritz energy method and it was possible to obtain the fundamental frequency of vibration of the blade for such a linear problem. The author points out that the non-linear terms in the equation of motion of the system arises from the Coriolis acceleration present. In his introduction, Ansari points out that previous investigations attempted to include such effects as shear deformation and rotary inertia and came out with complicated equations that are only soluble for a limited number of cases, others just ignored these terms and ended up with linear equations.

To produce the discrete mathematical model, the blade was divided into a series of rigid bodies*, the mass of each body was assumed to be concentrated at a certain mass centre in the body so that the field between two successive mass centres is massless but it possesses bending, shear, and torsional flexibilities.

The blade is identified by an arbitrary reference axis of which the mass centres are separated by different distances. Four assumptions were made to aid the analysis,

1- The bodies are long enough in the radial direction that they could be modelled as beams.

2- The displacements of the mass centres towards the reference axis (i.e. inwardly) are small that squares and higher

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*analogous to elements in finite element terminology.

powers of displacements could be neglected.

3- The mass centre axis is inextentional.

4- Amplitudes are moderately large and therefore any deviation from the vibratory modes of the corresponding linear system would be small.

By defining a set of axes as given by Myklestad* the positions and velocity vector for the i-th mass centre could be written as,

$$r_i = x_i + y_j + z_k$$

$$\dot{r}_i = u_i\, i + v_i\, j + w_i\, k \qquad (2.58)$$

where x, y, z, and u, v, w, are defined in terms of geometry.

Expressions for the kinetic energy of translation and the kinetic energy of rotation are given by,

$$KE_{i\ tr} = \tfrac{1}{2}\, m_i\, (u_i^2 + v_i^2 + w_i^2) \qquad (2.59)$$

where the subscript '$_i$' denotes the i-th section, also,

$$KE_{i\ rot} = \tfrac{1}{2}\, (\omega_1^{2'''} A_i + \omega_2^{2'''} B_i + \omega_3^{2'''} C_i) \qquad (2.60)$$

where the ω's are expressed later in terms of geometry, and A_i, B_i, and C_i are moments of inertia.

In the expression of the energy of rotation, the products of inertia were neglected since they are numerically small and they introduce unwanted complication.

The total energy will be given by the sum of equations (2.59) and (2.60)

$$KE_{total\ i} = KE_{i\ tr} + KE_{i\ rot} \qquad (2.61)$$

and for the whole blade,

$$KE_{total} = \sum_{i=1}^{n} KE_i$$

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*"NATURAL MODES OF HELICOPTER BLADES", report written for Bell and Helicopter Company, Fortworth Texas, Unpublished.

After substitutions, a final but long expression for the total kinetic energy was given.

By omitting all the terms that lead to non-linearities and excluding effects of rotation, the expression for the kinetic energy could be written in matrix form as:

$$KE = \frac{1}{2} \left\{ \dot{q} \right\}^T \left[M \right] \left\{ \dot{q} \right\} \qquad (2.62)$$

where $\left\{ \dot{q} \right\}$ is the vector of velocities.

and $\left[M \right]$ is a symmetric inertia matrix which is composed of partial matricies $\left[\bar{m}_i \right]$

$$\left[\bar{m}_i \right] = \begin{bmatrix} m_i & 0 & m_i r_{z_i} & 0 & 0 \\ & m_i & m_i r_{y_i} & 0 & 0 \\ & & (m_i(r_{z_i}+r_{y_i}^2)+A_i) & 0 & 0 \\ \text{SYMMETRIC} & & & (B_i \sin^2\vartheta_i + C_i \cos^2\vartheta_i) & (B_i - C_i)\cos\vartheta_i \sin\vartheta_i \\ & & & & (B_i \cos^2\vartheta_i + C_i \sin^2\vartheta_i) \end{bmatrix} \qquad \ldots\ldots (2.63)$$

A generalized mass matrix was given by:

$$\left[M \right] = \left[U^T \right] \left[m \right] \left[U \right] \qquad (2.64)$$

where $\left[U \right]$ is the system modal matrix (as was called by Ansari)

Pointing out to a previous reference of Ansari[5], this investigator gave an expression for the potential energy that includes the contribution to the centrifugal force field as follows:-

$$PE = \frac{1}{2} \sum_{j=1}^{2} \omega_j^2 M_j \dot{q}_j \qquad (2.65)$$

where ω_j is the j-th natural frequency of the linear rotating blade, corresponding to the j-th normal mode.

Applications using the Ritz method to different problems were given. Two types of problems were discussed;

a) Free vibrations.

b) Steady state vibrations.

Both cases are based upon energy minimization and expressions for the harmonic amplitudes were given.

In conclusion, Ansari suggests that the effect of Coriolis forces is similar to that of a hard spring which gradually becomes stiffer with increasing amplitudes. Furthermore, the effect of coupling due to Coriolis forces is significant (range of frequency ratio $\frac{\text{Excitation frequency}}{\text{Natural frequency}}$ is 1 to 10) at high rotational speeds encountered in blade problems and cannot therefore be ignored.

2.5.2. The Finite Difference Method:

At present, tha finite difference method is well known for the solution of differential equations and can be applied to a large number of problems provided that correct boundary conditions are used. A number of text books exist on the subject [65,48]. Further it has been considered as a standard numerical method in much of the literature [29]. It is out of the scope of this text to attempt a detailed discussion of the finite difference method. It will be dealt with to some extent in Appendix F through an attempt to compare it with the finite element method and an experimental method in a particular case study.

The author had found only two investigations[24,25] concerned with the vibration of rotating machinery that employ the finite difference method. Carnegie[24] used a longer version of equation (2.56) taken from (reference 21). The equation refers to free vibration of coupled bending-bending-torsion of a pretwisted blade

rotating with angular velocity Ω . Equations (2.55) and (2.56) were ignored as the assumption is made that vibration occurs only in the weak plane of bending. Equation (2.56) was further modified by simplifying assumptions and the finite difference forms were introduced for treatment by matrix iteration techniques. Two meshes were generated; a ten section mesh was used in the early stages and a twenty section mesh was used later on. Boundary conditions for the blade were $x = \acute{x} = 0$ at the root and $\acute{\acute{x}} = \acute{\acute{\acute{x}}} = 0$ at the free end. The results were shown earlier in figures 2-3 to 2-5.

The second investigation[25] dealt with the vibration characteristics of non-rotating tapered blading. The work of the paper was divided into three parts,

a- The vibration of a uniform beam of rectangular cross section was considered in order to develop the method of analysis and to obtain an estimate of errors involved by comparing the results obtained by other investigators.

b- For beams of square cross section at the root, subjected to various width tapers in the range of -0.75 and 1.0 and various pretwist angles between 0 and 90 degrees, the frequency parameter ratios and mode shapes are presented for selected values of taper and pretwist.

c- The effects of both width and depth of taper on the vibration of pretwisted beams of square cross section at the root were investigated.

Based on an earlier paper by the authors, modified equations of motion were obtained and they represent an eigenvalue problem,

$$[M]\{y\} = \lambda\{y\} \qquad (2.66)$$

The eigen values and eigen vectors were obtained by an iterative procedure and backsubstitution. Due to the large core requirements such a problem would take, only a small mesh was possible and mesh sizes of 10,15, and 20 were attempted.

The conclusions drawn were that pretwisted blades execute coupled bending--bending vibration since for square root cross sections the centroid coincides with the centre of flexure. Further, the variation in the frequency of vibration of a uniform beam due to pretwist and the ratio of width to depth of the cross section. The principal flexural rigidities of a tapered beam of otherwise uniform cross section will be unequal except at the root. When such a beam is pretwisted coupling occurs between the bending motions in the two mutually perpendicular planes containing the xx and yy axes of the root cross sections. The degree of coupling is a function of taper and angle of pretwist.

2.5.3. The Finite Element Method :

Probably the earliest finite element study that mentioned turbine blades was that of Ahmed et al.[1] in 1970. They introduced a new thick shell element for vibration problems that can be used for highly distorted shapes such as turbine blades. Unfortunately the element was intended for the free vibration of stationary bodies only. Never the less the investigators used it for a test problem on the vibration analysis of a turbine blade. Discripancies of 20% on lower modes and up to 30% on higher modes from experimental results were obtained. However it was stated that the element suggested was not intended for extremely thin sections and hence this big difference with experimental results.

The earliest finite element work concerned with the problem

of vibration of rotating blades was that of Dokainish and Rawtani[36]. The two investigators idealised the problem by considering the blades as cantilevers, and since they have a low aspect ratio then they were treated as cantilever plates rather than beams, their main interest was to obtain the natural frequencies and mode shapes.

The dynamical condition was converted to an equivalent statical condition by the use of D'Alembert's principle and hence the stresses due to in-plane centrifugal forces were included.

Triangular bending elements were employed for a quarter of the plate which was chosen according to symmetry conditions. Three degreesof bending freedom were assumed at each node of the triangular elements (ζ , $\frac{\partial \zeta}{\partial y'}$, $-\frac{\partial \zeta}{\partial x'}$) where 'ζ' is the deflection of an arbitrary point on the mid plane of the plate and x , and y are the local set of axes for the elements.

The bending stiffness matrix was obtained following a standard finite element procedure as given by Zienkiewicz[124] for the case where in-plane stresses in the middle surface of the plate are absent. However due to the presence of in-plane stresses, the strain energy stored per unit volume of the element becomes higher than the usual bending strain energy. The difference was attributed to "centrifugal strain energy", and hence the authors arrived at the conclusion that the total stiffness matrix is equal to the sum of the bending stiffness matrix and the centrifugal stiffness matrix. The author of this text suggests that the order of magnitude of the bending stiffness matrix is very much larger than the order of magnitude of the centrifugal stiffness matrix that in most of the cases the latter term can be lost when it is put on a computer.

Equating the work done by the external distributed force with the work done by the nodal forces made it possible to obtain an

expression for the nodal forces that depend upon the bending mass matrix, the nodal displacements, and the nodal accelerations. Finally the two investigators modified the equation of motion of the system to the form of an eigen value problem, the eigen values of which give the values of $\frac{1}{\omega^2 + \Omega^2 \sin^2 \vartheta}$ and of course the eigen vectors will give the mode shapes.

The results were presented in the form of curves showing the variation of the natural frequencies (given as a dimensionless factor $\beta = w\, l^2 (\rho t/D)^{\frac{1}{2}}$, $D = E\, t^3/12(1 - \mu^2)$) with the speed of rotation (wich was again given as the ratio of: Ω / fundamental frequency of the non rotating plate) for different values of the setting angle and aspect ratio.

Two basic conclusions were drawn; firstly that all the natural frequencies were observed to increase with the speed of rotation. This is due to the increase in stiffness resulting from the in-plane centrifugal forces. Secondly, it was observed that the rotating speed does not produce any significant coupling between the bending and the torsional modes of vibration. The author of this text finds a contradiction between this statement and what Carnegie[21] is suggesting; there exist coupling between the torsional and bending modes of vibration and the author of this text takes the second view as the one that is more likely to occur, and some of the results in chapter 6 indicates clearly this issue.

For quick calculation of the natural frequencies of the rotating blades, the two investigators suggested the use of Southwell coefficients as given by the equation,

$$\omega^2 = \omega_o^2 + S\, \Omega^2 \qquad (2.67)$$

where, ω is the frequency of the rotating blade.

ω_o is the frequency of the non-rotating blade.

Ω is the speed of rotation.

S is the Southwell coefficient.

A list of the above mentioned coefficient was given in a table in the paper for different values for the setting angle β (the angle between the plane of the rotation and the plane of the plate) and the rotational speed. It should be mentioned at this stage that Carnegie[24] suggested that the square of the frequency varies linearly with the square of the speed of rotation and he was able to prove it experimentally, this of course confirmed equation (2.67).

In a more recent work[17] the case of rotating solids was discussed through the development of an isoparametric solid element for the free vibration problems. Bossak and Zienkiewicz suggested that the effect of initial stress due to centrifugal forces or due to the second order strain components and, when the displacements are small, it manifests itself through a "stress stiffness matrix", which can be handled by an iteration process[124]. The problem of rotation was idealized by assuming a rotating coordinate axes system, and thus the rest of the formulation follows the same lines used for problems with stationary axes set. The formulation included the effects of centripital and Coriolis components of acceleration but these effects were excluded at the solution stage due to the argument that coriolis acceleration components introduce complex terms. Omission of these terms was justified by comparison with experimental tests within the practical operational speeds (further details are given in Chapter 4).

The effect of Gauss integration on the accuracy of results was investigated and it was concluded that for stiffness calculations

a set of $2 \times 2 \times 2$ Gauss points is more accurate than a set of $3 \times 3 \times 3$ Gauss points. During the course of this study the effect of Gauss integration was also investigated by the author of this text, see chapter six.

The two investigators studied two different types of problems and were able to compare their solutions with experimental work that had been carried out on these problems. The problems were,

a- Problems involving stationary cantilever plates and blades.

b- Problems involving rotating cantilever plates and blades.

Both types of problems have been experimented on and the results of the present investigation were compared with a number of experimental studies that were described earlier in this chapter. It was found that most of the results agreed with each other to within 1%.

Gupta[43] was more concerned that such problems as those related to rotating machinery of free spinning structures would take very large core sizes and he suggested a solution that follows the sturm sequence method. This is an interactive method that finds only one eigen value at a time.

Trompette and Lalanne[112] formulated expressions for the kinetic and potential energy for a three dimensional rotating solid element including large strains.

A turbine blade was modelled from isoparametric solid finite elements (24 nodes per element). Provision was made for the root since idealization by isoparametric finite elements for the exact shape of the root is fairly elaborate and increases the size of the problem drastically. Instead spring elements[59] were used at the root. Temperature calculations were allowed for to be included using the Rayleigh quotient method. Also, the Rayleigh

quotient method was used to calculate the effect of speed on the natural frequency. Stationary blade vibration results compared very well with experimental results for the first three modes, which is expected from a finite element analysis as indicated by earlier by the work of Ahmed et al.[1].

The equations of motion obtained in this investigation was very much the same as that obtained by Bossak and Zienkiewicz[17]. However, no further attempt was made to include the Coriolis acceleration component, and in fact it was being neglected at the solution stage. The simultaneous iterative method was used for the solution without reducing the number of degrees of freedom of the system.

Filstrup[44] divided the state of stress within an operating turbine or compressor blade into steady-state and vibratory components. He used flat shell elements of 5 degrees of freedom per node on the NASTRAN finite element program to calculate these stresses. It was suggested that the steady state stresses (alternatively called initial stresses) are due to centrifugal forces, pressure acting on the blade surface, and thermal gradients, but only the centrifugal forces were considered in loading the blade and neither pressure nor temperature loading were considered. Two types of meshes were used, a fine mesh of 400 triangular flat shell elements and a coarse mesh of 240 triangular flat shell elements.

Also made, the vibration analysis of non-rotating blades as well as for blades rotating at 3600 rev/min.

Both the stationary vibration analysis and the steady-state analysis follow the same method employed by the PAFEC finite element program, but the author of this text regrets that no contribution

was made in the work of Filstrup to the method of calculating the natural frequencies and mode shapes of the rotating blade.

Filstrup suggested that equation (2.67) can be used to relate the frequencies of the rotating blade to those of the stationary blades. The value of Southwell coefficient in equation (2.67) depends upon the geometry of the blade and its position on the disc, and also it depends on the initial stresses present.

A further practical application was made by this investigator, its plotting the Campbell diagrams (alternatively called spoke diagrams- see reference 111).

The blade natural frequencies and multiples of the turbine shaft speed are plotted against speed so that it then becomes possible to obtain a complete view of the operational limits. If a certain speed was chosen then the possibility of resonance occurs if one of the natural frequencies intersects with one of the harmonics.

In conclusion, Filstrup suggests that the use of flat shells is very suitable for blade idealizations for both low and high aspect ratios for the blades. Further more it was deduced that the natural frequencies converge more rapidly with finer meshes than do the displacements which in turn converge more rapidly than the stresses. This means that a coarse mesh is just sufficient for frequency calculations while finer meshes are required for stress calculations.

Probably the earliest intended design of finite elements for blade vibration applications was in 1973 when Thomas and Dokumaci[104] described the formulation of improved finite elements for tapered beams and later[105] they introduced simple finite elements for pretwisted blade vibration.

In their first paper the two investigators introduced two

tapered elements for the vibration analysis using a Hermitian polynomial displacement function of the sixth order (see section 3.2). The idea behind the improvement can be explained as follows; it is known from the finite element theory (see chapter 3) that the problem working variables can be expressed as a linear combination of a number of parameters (unknown as yet) which will be approximated by nodal values. The nodal values must provide continuity over the whole system. If the number of unknown parameters equals the number of conditions provided, then, the transformation to nodal values is unique.However,if in an attempt to refine this approximation the order of the approximation is increased, then the transformation is no longer unique, since there will be many ways of adding more defined values. Thus by raising the order of the polynomial function the two investigators were able to obtain more accurate results with much smaller matrix sizes. However, it was concluded that the increase in accuracy due to a more correct representation of the field distribution within elements can not make up for the loss of accuracy resulting from a coarser division of the field.

In their second paper, Thomas and Dokumaci established a simple formulation for two twisted finite elements that employ fifth order polynomials and used them in the calculation of natural frequencies of pretwisted beams for several boundary conditions. The results were compared with an exact method and it was noticed that they converged rapidly. One important note was given through the discussions about the presence of negative eigen values which they attributed to the rounding of errors during the process of preparing the eigen equation. From the experience gained throughout the course of the present study, negative eigen values normally indicate a rigid

body mode or failure of a particular mode to exist. It is realized however that in practical terms there is nothing like a negative frequency of oscillation. While working with dimensionless terms a negative frequency ratio should indicate vibrations which are out of phase.

Unfortunately the two investigators did not include any effect of rotation in both of their papers.

Like the work of Filstrup which was mentioned earlier in this section, Allen and Erickson[2] carried out an analysis of a free standing turbine blade using the NASTRAN finite element program. The stiffening effect of the centrifugal force field was considered by using NASTRAN's differential stiffness option*. This option allows for a first order approximation for large deformation effects, such as those considered in beam-column action. The differential stiffness procedure is done by solving the problem first assuming no effect of deflection on loading. Then, using the resulting displacements normal to the plate elements and the membrane forces in each element, a linearised first approximation of the stiffening influence of the membrane forces is obtained in the form of a differential stiffness matrix. It is assumed that the applied loads from which the differential stiffness is derived remain fixed in magnitude and direction during motion of the structure, and that the points of application move with the structure. The elements of the differential stiffness matrix are added to the corresponding elements in the basic stiffness matrix and the problem is solved again. It was mentioned that an iteration process of solving and

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*Refer to MacNeal,R.H., "THE NASTRAN THEORETICAL MANUAL",NASA SP-221(01), (1972), and MacCormick,C.W., "THE NASTRAN USER'S MANUAL",NASA SP-222(01) (1973). It is regrettable that it was not possible to get hold of NASTRAN Newsletter as it was not public. Author.

obtaining more refined differential stiffness matrices is necessary if a higher accuracy is required. The results obtained by the two investigators were well conforming with the results of actual experiments carried out on the same blades.

Visser et al.[114] and later Schaller at al.[98] carried out finite element studies to obtain the stress and stress distributions for a ceramic rotating turbine blade. The study was made with the aid of a family of three dimensional isoparametric elements of various displacement polynomials. Mixed or graded elements (see also reference 15) were used to interface between two different types of element .

The investigators gave a brief description of the finite element procedure based upon the principle of virtual work. It was understood from the explanation that the blade was assumed stationary and that the load acting on it consists of a centrifugal force field caused by a rotational speed of 3 600 rev/min. The author of this text finds that the procedure used is the same as the one employed by the PAFEC finite element program as given in Appendix C.3.

Tovey[111] in his thesis had included the effect of centrifugal forces by calculating the radial forces (using D'Alembert principle) and was able to calculate the stresses by dividing the forces by the areas at any point in the blade. Then, integration along the whole blade would give the total stresses. He assumed that no other stresses are present and he entered the 'radial stresses' into the total stiffness matrix of the element. Thus at the end Tovey collected the total stiffness matrix, made from two components; the ordinary or basic stiffness matrix without the centrifugal effects plus the 'stability matrix' which contains the centrifugal effects.

A more complete analysis based on the super parametric shell elements was given by Thomas and Mota Soares[106]. The two investigators developed a model for the dynamic analysis of rotating structures of different shape and thickness. Being shell type elements, they were found vey suitable for rotating blade analysis. The analysis accounted for the 'centrifugal mass and stiffness' in addition to the centrifugal loading vector. The reduced integration technique was used to evaluate the strain energy of the structure. In a very similar way to the NASTRAN's approach to the treatment of the centrifugal steady-state stress explained earlier in this section, the two investigators made their analysis starting by the statement that the centrifugal force acting on the structure will produce a steady-state displacement and oscillatory motion will occur about an equilibrium position, and hence a non-linear system. They suggested a first approximation as follows; the rotating structure was considered to have large steady-state elastic displacements with small deformations. Thus the stress and strain tensor can be referred to the original system of undeformed geometry. On this basis a Lagrangian energy expression was made. The gyroscopic effect was also included in the final equation of motion and due to its presence the system no longer stays linear. A good suggestion was made to separate the linear and non linear terms into two equations, thus

$$\left\{ [S] - [G] + [\sigma] \right\} \{ \varsigma_o \} = \{ F_c \} \tag{2.68}$$

and $$[M] \{ \ddot{\varsigma}_1 \} + [G] \{ \dot{\varsigma}_1 \} + \left\{ [S] - [G] + [\sigma] \right\} \{ \varsigma_1 \} = \{ F_1 \} \tag{2.69}$$

where $[G]$ is the gyroscopic matrix,

$[\sigma]$ is the initial stress matrix,

the subscripts ($_o$ and $_1$) denote steady-state and

oscillatory motions, $\{F_c\}$ is the centrifugal load vector, and $\{F_1\}$ is the forcing function.

Thomas and Mota Soares suggested that at low velocities equation (2.67) reduces to,

$$[S] \{S_o\} = \{F_c\} \qquad (2.70)$$

since the stiffness matrix is very big compared to the other two matrices . However at higher velocities an iterative method was suggested*.

The gyroscopic matrix was neglected according to the argument that it is only proportional to the angular velocity while the other terms were either proportional to the square of the velocity or independent. Accordingly, equation (2.69) was reduced to normal eigen value equation (see also Appendix D),

$$[M]\{\ddot{S}\} + [S]\{S\} = \{F\} \qquad (2.71)$$

It should be mentioned that equation (2.71) is only applicable at low velocities. More over, it is the same form of free vibration used on the PAFEC finite element program.

Barlow[8] studied the accelerations and velocities present in a structure subjected to compound motions and using finite element principles he related them to nodal parameters and thus was able to calculate equivalent nodal loads. He suggested that structures subjected to compound motions such as aero-engines would normally have gyroscopic inertias acting on the engine rotating parts due to coupling of the rotational velocities of the engine and

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*it was pointed out that the method follows reference number (8) but unfortunately no mention of reference number (8) was made in the reference material list.

airframe. Such gyroscopic forces would cause oscillating stresses when applied to high speed rotating assemblies and hence increase the possibility of fatigue failure. Barlow demonstrated the validity of his method by giving an analysis of the stresses due to gyroscopic and centrifugal forces in an aircraft engine blade model.

2.5.4. Others :

In the previous section, the state of the art of the blade vibrations was discussed, based upon the finite element method which included three types of studies;

a- Vibrations of stationary blades.

b- Vibrations of rotating blades.

c- Stress analysis due to the centrifugal loading of blades.

However it was felt necessary before closing this chapter to give a brief review of the other available material regarding the problem of rotating machinery in general which will concentrate mainly on two points;

a- Extra work using the finite element method which include that of,

Putter and Manner[89], who studied the natural frequency of rotating beams using fifth order polynomial beam finite elements.

Gupta and Rao[43], derived the system matrices of twisted beam finite elements with linearly varying breadth and depth.

Dżygadlo[39] discussed the dynamic model of a rotating gas turbine disc using finite elements.

Newman and Filstrup[84] presented a number of three dimensional isoparametric finite elements.

Henry and Lalanne[56] modelled a compressor blade by triangular

elements and formulated energy expressions for the rotating blades.

Carnegie et al.[27] suggested an improved method for the quick convergence of eigen value problems by introducing more degrees of freedom using the matrix displacement method.

Kirkhope and Wilson[68] studied the vibrations of thin rotating discs and they formulated angular finite elements which have only four degrees of freedom and later on[69] they investigated the vibration modes of bladed discs.

Ulrich Holzlohner[62] discussed transients and time dependent problems.

Murty and Murthy[81] presented a finite element formulation for the natural frequencies of tapered and pretwisted rotors.

Filstrup[45] applied the finite element method to a rotating group of lashed turbine blades.

Kuo[70] demonstrated blade dynamics using the NASTRAN finite element program.

Kennedy and Gorman[67] studied the centrifugal and thermal stresses induced in rotating discs using annular finite elements.

Thomas and Belek[107] investigated the effect of the blade coupling using finite elements.

Rieger et al.[97] presented the dynamic stiffness concept as a general method for the vibration characteristics of rotor bearing systems.

Finally a more complete bibliography of finite elements is recommended by Whiteman[119].

b- Various other studies that include,

Downham[38] who concentrated on the vibration monitoring as a criteria for practical vibration studies.

Carnegie and Dawson[26] suggested transforming the differential equation of motion of asymmetrical aerofoil blades into a set of simultaneous first order equations and solving them by a step-by-step finite difference procedure.

Rao and Carnegie[91,92] wrote a couple of papers regarding the non-linear vibration of rotating blades using various Ritz processes, and later on[94] they explained a numerical procedure based on an extended Holzer method.

Rao[93] gave a review of the methods used for the vibration analysis of turbine blades.

Rao and Banarjee[95] used the method of polynomial frequency equation to study the coupled vibrations of rotating cantilever blades.

Swaminathan and Rao[103] formulated expressions for the kinetic energy and potential energy and minimized the Lagrangian obtained according to the Ritz process and they solved the equations for the first three natural frequencies.

Murthy and Pierce[80] used the transmission matrix method to predict the effect of phase angle on multibladed rotor flutter.

Dawson and Davies[33] presented an extension to Myklestad's method to allow the natural frequencies of lumped mass system to be obtained.

Murty and Prabhakaran[82] used a lumped inertia force method to investigate the vibration characteristics of a linearly tapered cantilever beam.

Fu[53] derived the basic equations for a computer analysis for an equivalent lumped parameter system which simulates a pretwisted rotating or non-rotating Timoshenko beam in coupled bending-bending-torsion vibrations.

Murthy[79] used the transmission matrix method to determine the natural frequencies and mode shapes of rotor blades.

Hsu Lo[63] suggested that there exists non-linear terms in the problem of bending vibrations of rotating cantilever beams. The presence of the Coriolis acceleration is the cause of this non-linearity. He idealised the problem by assuming the beam is rigid in bending throughout all its length except at the root where a spiral spring was used to connect the beam to the rotating shaft.

Gotham and Smailes[54] studied the particular case of pin fixed compressor blades.

Montoya[78] used the Runge-Kutta numerical integration method for the coupled bending and torsional vibrations of rotating pretwisted blades.

Ewins[41] studied the vibration modes of mistuned bladed discs, while Whitehead[117] investigated the effect of mistuning on forced vibrations.

Schaller et al.[98] gave a three dimensional study of ceramic rotating blades and they studied the effect of friction, contact area, and root fixing geometry on the root stresses.

De Silva et al.[34] used the finite difference method for the optimum design of rotating machinery.

Shapiro[100] gave a review of the avilable computer programs for the vibrations of rotating machinery.

Finally the proceedings of a conference on the vibration in rotating machinery arranged by the Institute of Mechanical Engineers and held in Cambridge in 1976 will be pointed out to the reader.

CHAPTER 3

THE FINITE ELEMENT METHOD

3.1. Introduction

No doubt, the finite element method has developed in the past decade to become the most favoured method for engineering design. The vast amount of literature available makes it undesireable to elaborate on the fundamentals of the method. It will be attempted to limit the contents of this chapter to the particular points relevant to this investigation, trying as much as possible to include the personal experience from the application point of view. A number of text books are available and the author recommends three of them according to their field of study;

- K.H. Heubner[64] for a theoretical approach and as a standard book on the subject.
- L.J. Segerlind[99] for applications.
- K.J. Bathe and E.L. Wilson[10] for computer implimentation.

Whiteman's[119] Bibliography will be found very useful for any person working on the subject.

3.2. Field Discretization

The field is defined as any medium exhibiting a physical change due to external effects acting on it. A steel structure that is displaced by an external force could be referred to as a field while its displacement is considered as the problem variable. The finite element procedure involves dividing this field into smaller parts or elements interconnected with each other by nodes. Corresponding nodes on interconnected elements should have the same value for the problem variable, so that if the problem variable is displacement

then two elements sharing a node will have the same displacement at that node. Exceptions from the above rule can exist, taking for example a stress element having an average stress which is redistributed equally on all the nodes of the element, thus it would appear that the same node have a different value for an adjacent element. The problem variable is conveyed by the node by allowing it to have a number of degrees of freedom. Thus for a problem involving displacements, the maximum number of degrees of freedom a node can take is six; three of which are translational in the three coordinate directions, and three corresponding rotational degrees of freedom.

Choosing the elements will mainly be dependent upon the type of problem, for example in a problem for calculating the deflection of a concrete pile due to a concentrated load at the free end, the pile can be idealized by beam (one dimensional) elements, while a platform on a ship should be idealized by plate elements which of course are two dimensional. There are certain situations where some elements are preferred to others even though they are of the same dimension group, for example, in problems involving in-plane stresses the field can be modelled more acurately by using in-plane elements which have two degrees of freedom per node, while a problem for plate vibrations must be modelled so that the elements have some freedom for flexing out of the plane of the elements. Three dimensional elements are more expensive than other elements since they involve more matrix operations and larger core sizes. Otherwise, three dimensional elements are very accurate and can model almost any field. Special types of elements are used for various applications but they are only useful for the application that they are

made for.

Discretization of the field into a number of different elements is normally dependent upon engineering judgement, but choosing a simple mesh of elements is desireable for a two fold reason, first for economy reasons and second to avoid mistakes. The following rules are found useful;

a- Very small angles can cause problems in calculations and it is advised that they should be avoided as much as possible.

b- When using three dimensional elements attention should be maintained to avoid situations where one side of the element is 10 times longer than the other side since this will stiffen the element more than necessary and hence less accurate results.

c- Avoid using mismatched elements such as using an element with one or more midside nodes adjacent to another which has no midside nodes, mixed or graded elements[15] can solve this problem.

d- Most available literature explain the triangular element, however it is advised that the quadrilateral isoparametric series of elements are not much more expensive to used and they are more accurate. Probably triangular elemements are more easy to understand.

The sequence of ordering the elements can affect the size of the problem (computer core) and the time for solution. For programs that have a wave method of carrying on the solution it becomes necessary to order the elements in such a way that the wave front is optimized to be as small as possible. The wave front is defined

as the width of one row of elements attached to each other. Figure 3-1 shows an idealization of a gyro-rotor. The rotor is modelled by using 8-noded isoparametric axisymmetric elements. A fine discretization is made, but as can be seen, it is not homogeneous, the crowded area is where higher stresses are expected and hence the larger number of elements will provide better accuracy. The element ordering is shown to meander by arrow heads and the element numbers are written with bold figures. The widest front is indicated by the elements 18 to 23 which are pointed out by the larger empty triangles. Afterwards the front will be constant, meandering on groups of four elements.

Most of the sophisticated finite element programs allow their user to define his nodes randomly but uniquely. Later on he can define his elements by reordering the nodes in a topology module. Gaps between node numbers are preferably avoided to reduce mistakes and to minimise problem sizes for automatic schemes. It should be mentioned at this point that a facility for automatic discretization with limitations does exist on most advanced finite element programs. With such a facility the user can automatically order the node numbers, coordinates, wave front, and the way the field is discretized. So far, the author of this text knows of no facility for automatically ordering the element sequence for a comlicated field.

3.3. Interpolation and Transformations :

In the finite element approximation method a change of a variable within a certain field is assumed to aquire piecewise continuity over the subdomain (elements) via the nodes. Further more the change of the variable inside the elements is done by assuming that the change follows the so called shape functions, which are functions that should satisfy certain criteria[124] ,

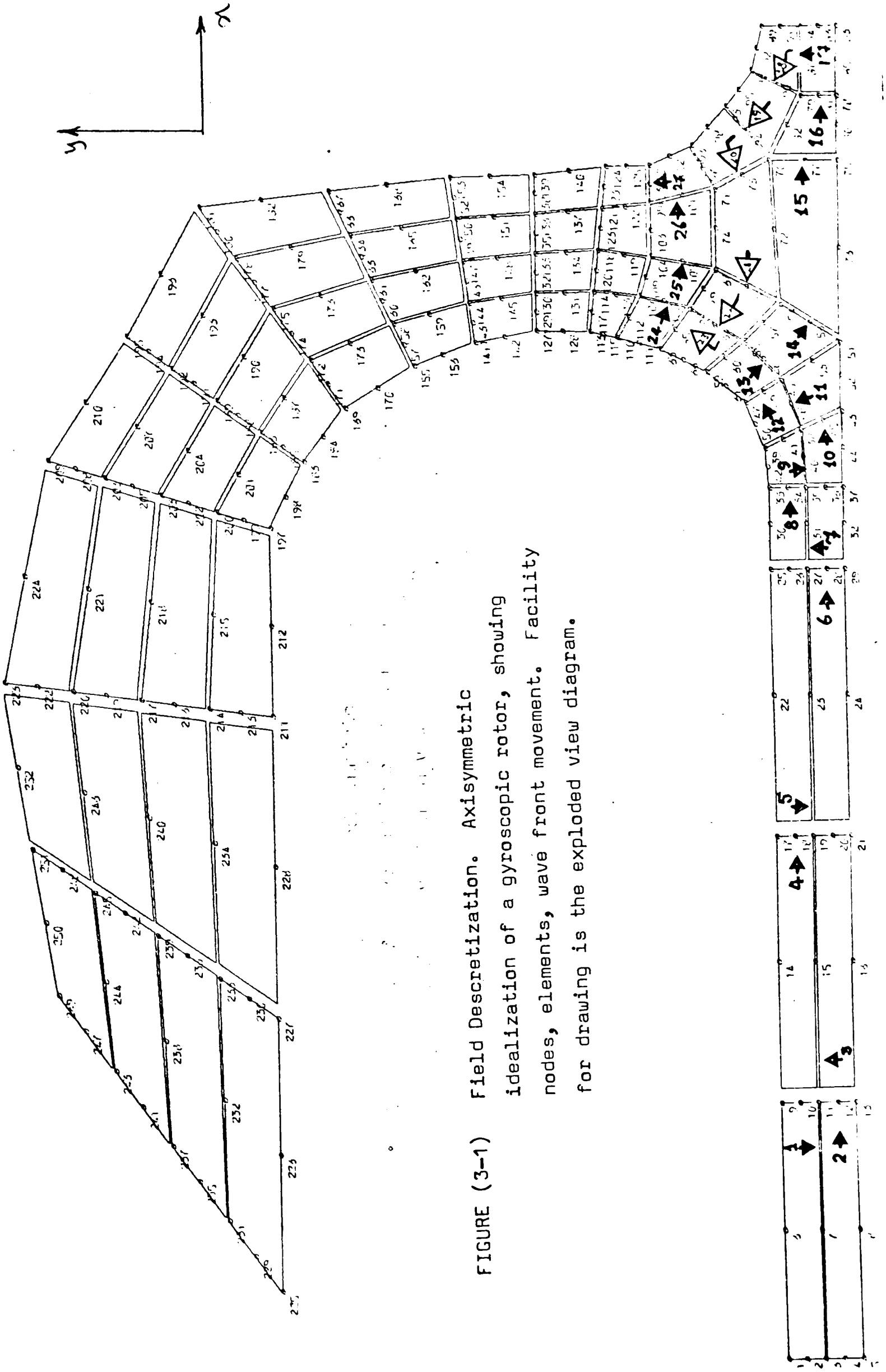


FIGURE (3-1) Field Descretization. Axisymmetric idealization of a gyroscopic rotor, showing nodes, elements, wave front movement. Facility for drawing is the exploded view diagram.

repeated here for convenience;

a- The shape functions must provide continuity of the field variable only between the elements (i.e. slope continuity is not required, or in other words, derivatives of the change not necessarily be constant).

b- For the displacement approach, the shape function is to allow any arbitrary form to be taken so that the constant strain criterion can be observed.

Normally polynomials are chosen for the shape functions for the simplicity in representing them on the computer and the ease in numerically integrating and differentiating them. Thus for the displacement approach which will be adopted throughout all this work, the displacement vector in any three dimensional body can be written as:

$$\{U\} = \sum_i [N_i] \{u_i\} \qquad (3.1)$$

where

$\{U\}$ is any displacement in the body and can be taken from any continuous equation relating displacements to other external effects such as loads,

$[N_i]$ is a generalized shape function,

while $\{u_i\}$ are the displacements measured at the nodes.

In the above equation (3.1) the summation will be taken over all the elements involved.

The displacements in the element are given by

$$\{u_e\} = [u_x, u_y, u_z]^T = [p]\{\alpha\} \qquad (3.2)$$

where,

$[p]$ is a polynomial function whose order and number of

variables involved depends on the element geometry and accuracy required,

$\{\alpha\}$ is a list of constant coefficients whose number is the same as the number of nodes on the element.

The polynomial terms can be decided by obtaining the binomial terms for three variables (corresponding to the three dimensional coodinates), $\{(x+y)+z\}^{n-2}$ where 'n' is the number of nodes on the element. The first of these elements taken to be of order zero (i.e. unity), and the rest of the terms will be chosen from the binomial terms taking into account two priorities,

a- Simpler orders come first then higher orders.

b- Mixed variables are preferred to pure (single) variables of the same order. (i.e. $x^3 y$ is preferred to x^4 or y^4).

The polynomial terms can be arranged to build a tetrahedron of which (see figure 3-2);

a- One edge contains the polynomial terms for one dimensional problems (one variable only being involved).

b- One side face contains the polynomial terms for 2-D problems.

Various elements with their polynomial representation are shown on figure (3-3).

It is noticed that in figure (3-3) the coordinate set $\xi\ \eta\ \zeta$ was used instead of the normal coordinate system XYZ, the reason being due to the use of isoparametric elements. The isoparametric element family was introduced for the analysis of complicated shape structures where the sides and faces of the elements can no longer be

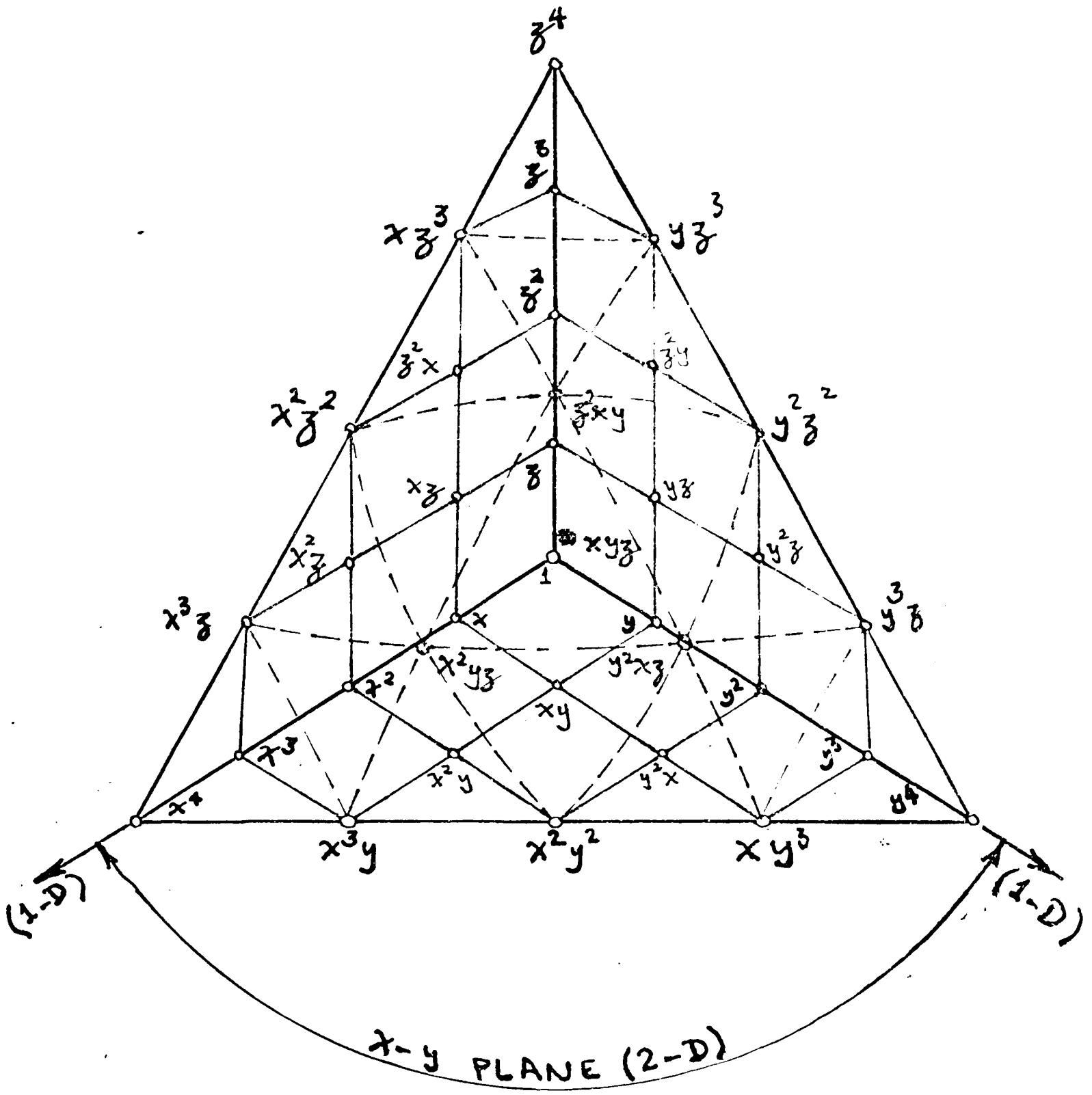
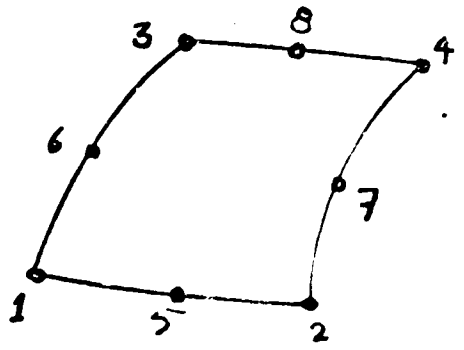


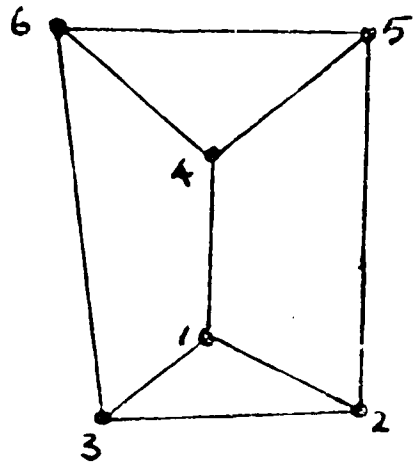
FIGURE (3-2) The polynomial tetrahedron, a guide for choosing polynomial terms for the shape functions. (dotted lines represent the X-Y-Z plane, drawn a little curved to separate them from the external surfaces. One internal point XYZ does exist and its link with the other points and surfaces is omitted to avoid crowding of the lines.)



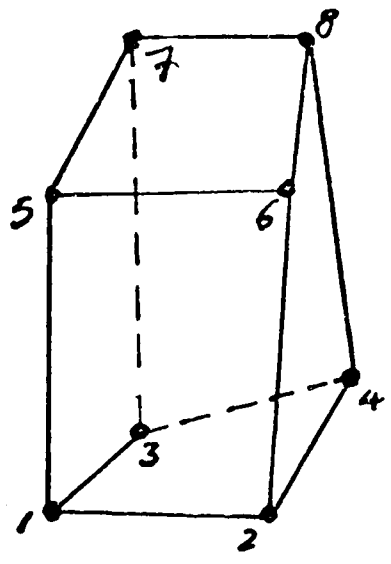
$$1 + \xi + \eta + \xi\eta + \xi^2 + \eta^2 + \xi^2\eta + \xi\eta^2$$

a) 8-noded Quadrilateral.

$$1 + \xi + \eta + \zeta + \xi\eta + \eta\zeta$$



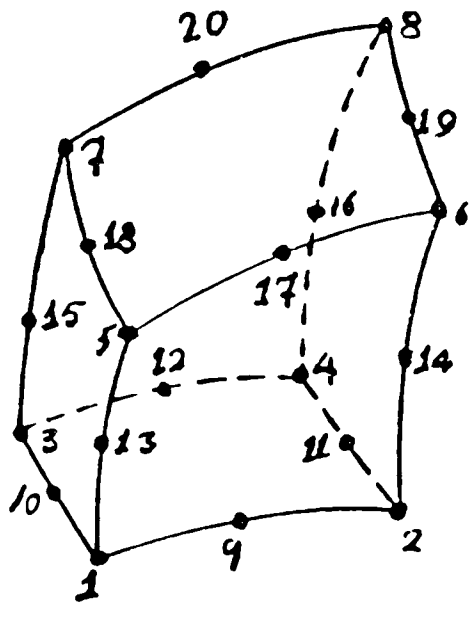
b) 6-noded Pentahedron.



$$1 + \xi + \eta + \zeta + \xi\eta + \eta\zeta + \zeta\xi + \xi\eta\zeta$$

c) 8-noded Hexahedron.

$$1 + \xi + \eta + \zeta + \xi\eta + \eta\zeta + \zeta\xi + \xi^2 + \eta^2 + \zeta^2$$
$$+ \xi^2\eta + \eta^2\xi + \zeta^2\xi + \xi^2\zeta + \eta^2\zeta + \zeta^2\xi + \zeta^2\eta + \xi\eta\zeta$$
$$+ \xi^2\eta\zeta + \xi\eta^2\zeta + \xi\eta\zeta^2$$



d) 20-noded Hexahedron.

FIGURE (3-3) Some elements used in this investigation with their polynomial terms, obtained with the aid of figure (3-2).

straight. It is due to the fact that numerical integration becomes very difficult for elements of complicated shapes that these elements are transformed to a curvilinear set of coordinates where the new shape of the elements will appear with straight sides and extends at an equal distance of one unit on both sides of every coordinate axis and hence the name isoparametric. It will be realized that the integration in this case will be taken between the limits (-1 , and 1). It just happens that transforming to the new set of coordinates follows exactly the same lines and rules layed out for representing the displacement of any particular point on the element and hence the polynomial terms would be just the same that it was thought unnecessary to repeat them. Figure (3-4) shows the isoparametric transformation of the 20 - noded hexahedron of figure (3-3) into the new curvilinear coordinate system leaving the rest of the transformations of figure (3-3) to the readers imagination.

Using the polynomial transformation, an analogous equation to equation (3.2) can be written for the coordinate transfers;

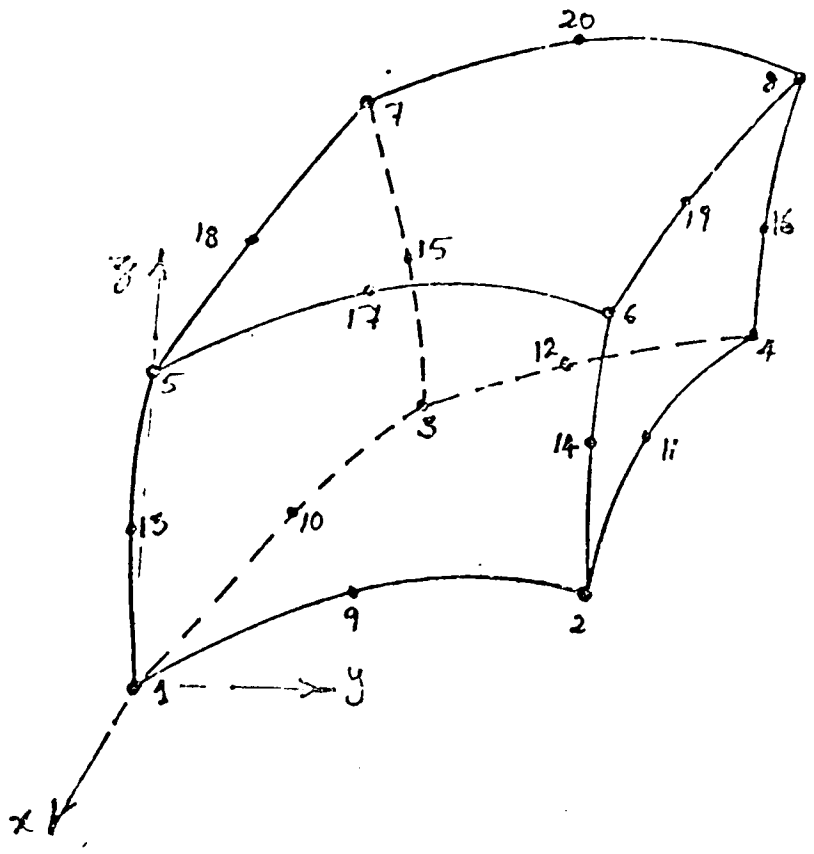
$$x = \lfloor p \rfloor \{\alpha\} \qquad (3.3)$$

where, x is any cartesian coordinate value on the 'x' axis,

$\lfloor p \rfloor$ is a raw matrix of polynomial terms (see figures 3-2 and 3-3),

and $\{\alpha\}$ is a column vector of coefficients whose number is equal to the number of polnomial terms which again equals the number of nodes on the element.

The constant coefficients $\{\alpha\}$ can be obtained by writing a number of linear equations which are exactly the same as equation (3.3) but each has a different value for the nodal coordinates for



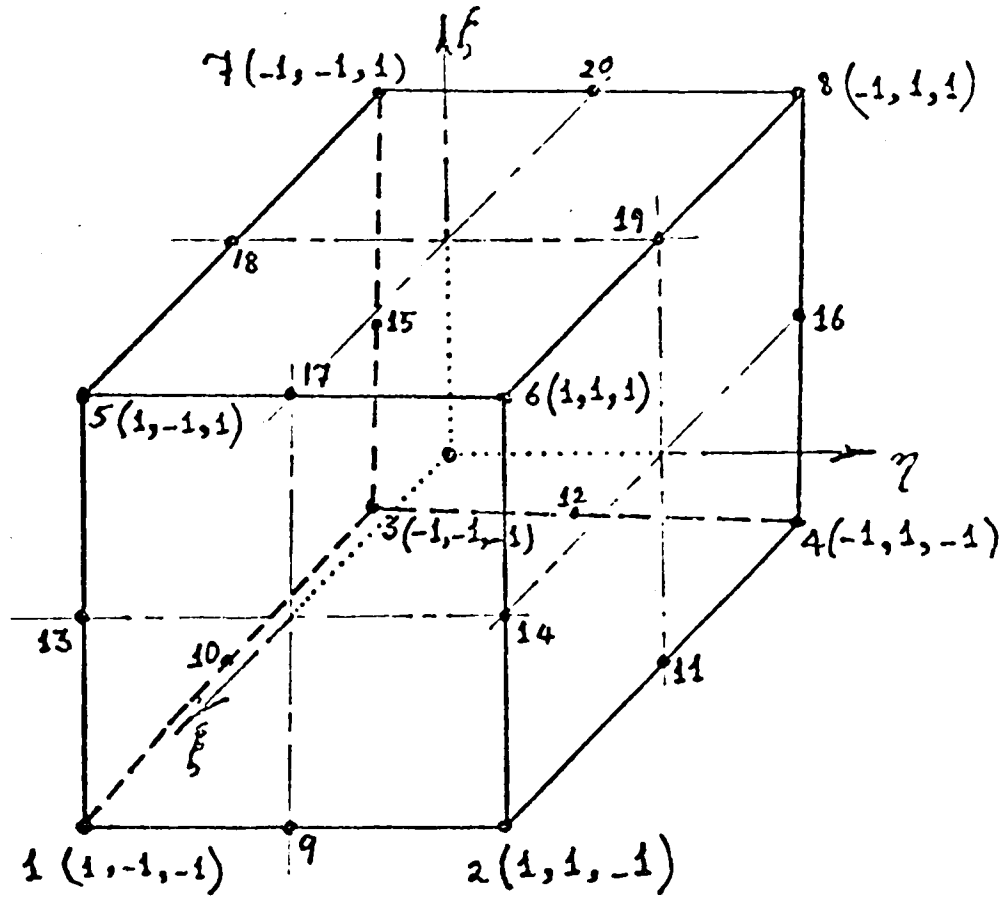


FIGURE (3-4) Transforming the element into the curvilinear coordinate system.

both coordinate sets, thus,

$$\{x_n\} = [A]\ \{\alpha_x\} \tag{3.4}$$

where, the subscript '$_n$' refers to the nodes,

the matrix $[A]$ is a square matrix of polynomial terms.

Solving for $[\alpha]$ in equation (3.4) above, we get,

$$\{\alpha_x\} = [A^{-1}]\ \{x_n\} \tag{3.5}$$

Similar expressions can be written to obtain the corresponding $\{\alpha_y\}$ and $\{\alpha_z\}$ for the y and z coordinates respectively.

Equation (3.5) above can be substituted in equation (3.3) to obtain the 'x' coordinate value for any point on the element,

$$x_e = [P]\ [A_x^{-1}]\ \{x_n\} \tag{3.6}$$

and in a similar way follows y_e and z_e thus,

$$y_e = [P]\ [A_y^{-1}]\ \{y_n\} \tag{3.7}$$

$$\text{and} \quad z_e = [P]\ [A_z^{-1}]\ \{z_n\} \tag{3.8}$$

3.4. Displacement Elasticity Relationships :

It is established from the theory of elasticity[28] that the state of strain at a point can be given by the tensor*

$$\begin{pmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{pmatrix} \tag{3.9}$$

---

*This is the only form of tensor notation that will appear in this text. Tensors for cartesian coordinate systems are a shorter method of compiling a number of equations. Even though the second order tensor shown above looks very much like a matrix but every element of the tensor represent a set of three equations that have three terms on one of the sides. For further detail see reference 28.

Specifying three displacements u, v, w in the x, y, z directions respectively it is possible to write the strains of equations (3.9) or derivatives of the displacements as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad , \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \quad , \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{3.10}$$

and

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3.11}$$

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{3.12}$$

$$\gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{3.13}$$

In the above equations ε is the direct strain and γ is the shearing strain.

Using matrix notation it becomes possible to write the above equations as,

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \\ \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix} \tag{3.14}$$

and for finite element approximations we have,

$$\{\varepsilon\} = [B]\ \{u\} \tag{3.15}$$

where,

$[B]$ is the matrix relating the strains to displacements, and it consists of differential operators (see reference 57)

$\{u\}$ is the list of nodal displacements.

Assuming linear elastic behaviour that is, linear stress-strain

relations, and studying the stress tensor given by

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yx} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \tag{3.16}$$

Where, 'σ' represents the direct stress and 'τ' the shearing stress. It becomes possible to link the stresses to the strains via an elasticity matrix $[D]$ thus,

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = [D]\{\varepsilon\} = [D] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{3.17}$$

For an elastic material there are 21 material constants, the number which can be figured from equation (3.17) by counting the number of elements for the upper half of the matrix $[D]$ which is a 6x6 matrix, thus assuming $[D]$ as symmetrical we have the number of material constants equal to,

$$\text{No.} = \frac{6}{2} \times (6 + 1) = 21 \tag{3.18}$$

However it is known that most materials have the same properties in different directions (i.e. they are **iso**tropic) and hence the number of **non-zero** elements is reduced to 9, thus the final form of the $[D]$ matrix will be

$$D = \frac{E}{2(1+\nu)(1-2\nu)} \begin{bmatrix} 2(1-\nu) & 2\nu & 2\nu & 0 & 0 & 0 \\ & 2(1-\nu) & 2\nu & 0 & 0 & 0 \\ & & 2(1-\nu) & 0 & 0 & 0 \\ & & & 1-2\nu & 0 & 0 \\ & \text{SYMMETRICAL} & & & 1-2\nu & 0 \\ & & & & & 1-2\nu \end{bmatrix} \quad \ldots\ldots\ldots (3.19)$$

Where, 'E' is the constant of elasticity.

and 'ν' is Poisson's ratio.

3.5. Element Matrices (three dimensional)

In this section concern will be given to the derivation of the stiffness and mass matrices for 3D elements.

In the displacement approach for a discrete system, the system forces are related to the diplacements by the equation,

$$\{F\} = [S]\{u_i\} \qquad (3.20)$$

where 'F' represents the forces and $[S]$ is a square matrix called the stiffness matrix, which is merely the system transfer function. It should be pointed out that equation (3.20) which is the virtual work equation in discretized form is analogous to the equation of a spring.

The stiffness matrix can be obtained from the expression for the strain energy which is evaluated over the whole volume of the element by the integral,

$$SE = \frac{1}{2}\int_V \{\varepsilon\}^T \{\sigma\}\, dv \qquad (3.21)$$

Using relations (3.17) and (3.15), it is possible to write the strain energy as,

$$SE = \tfrac{1}{2}\int_V \{u\}^T [B]^T [D] [B] \{u\} \, dv \qquad (3.22)$$

Numerical integration of the above equation is almost impossible for complicated structures (see also section (3.3.)) without the introduction of curvilinear coordinates and the change to the isoparametric family of elements. For this family, the elements are transformed into very simple shapes in the $\xi\,\eta\,\zeta$ domain with details of transformation given earlier in this chapter, in this case the Kernel $[B^T][D][B]$ is a function of ξ, η, ζ and it is more convenient to write

$$dv = dx\ dy\ dz = |[J]|\ d\xi\, d\eta\, d\zeta \qquad (3.23)$$

where $|[J]|$ is the determinant of the Jacobian Matrix,

$$[J] = \begin{bmatrix} \partial x/\partial\xi & \partial y/\partial\xi & \partial z/\partial\xi \\ \partial x/\partial\eta & \partial y/\partial\eta & \partial z/\partial\eta \\ \partial x/\partial\zeta & \partial y/\partial\zeta & \partial z/\partial\zeta \end{bmatrix} \qquad (3.24)$$

Hence the final form of the strain energy expression will be given by,

$$SE = \tfrac{1}{2} \{u_e\}^T [S_e] \{u_e\} \qquad (3.25)$$

where the subscript '$_e$' is introduced to denote the element, and the stiffness matrix $[S_e]$ will be given by

$$[S_e] = [A^*]^T \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} [B]^T [D] [B] |[J]| \, d\xi\, d\eta\, d\zeta \, [A^*] \qquad .....(3.26)$$

The matrix $[A^*]$ is composed of smaller matrices $[A^{-1}]$ arranged on the diagonal of $[A^*]$. $[A^{-1}][A] = [1]$, and the matrix $[A]$ is defined earlier in this chapter.

It is found at this stage that proving equation (3.20) above becomes fairly simple, since by the theorem of Castigliano, the

differentiation of the strain energy with respect to the displacement gives the set of forces existing in the direction of the displacement, and hence differentiation equation (3.25) leads to,

$$\frac{d\,SE}{d\,u} = \left\{ F \right\} = \left[S_e \right] \left\{ u_e \right\} \qquad (3.27)$$

since $\left[S_e \right]$ is independent of the displacement, it was assumed a constant during differentiation. Comparing equations (3.27) and (3.20) shows that they are identical and that equation (3.26) is a true representation of the stiffness matrix. The alternative way of proving equation (3.27) is to minimize the expression for the total potential energy which is normally the strain energy plus the work done by the extended loads.

The mass matrix will be obtained from the expression for the kinetic energy,

$$KE = \tfrac{1}{2} \int_V \rho \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) d\,v \qquad (3.28)$$

where ρ is the density, (u, v, w) were defined earlier as the displacements in the three cartesian coordinate directions, and the dot (•)indicates differentiation with respect to time. Taking

$$\left\{ u_e \right\} = \left[u \;,\; v \;,\; w \right]^T \qquad (3.29)$$

and using equations (3.6), (3.7), and (3.8) it becomes possible to write equation (3.28) as

$$KE = \tfrac{1}{2}\, \omega^2 \left\{ u_e^T \right\} \left[M_e \right] \left\{ u_e \right\} \qquad (3.30)$$

where $\left[M_e \right]$ is the element mass matrix given by,

$$\left[M_e \right] = \int_V \left[N^T \right] \rho \left[N \right] d\,v \qquad (3.31)$$

or in the more detailed way of appendix C,

$$\left[M_e \right] = \left[A^{*T} \right] \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} \begin{bmatrix} [Q] & 0 & 0 \\ 0 & [Q] & 0 \\ 0 & 0 & [Q] \end{bmatrix} \left| [J] \right| \, d\xi \, d\eta \, d\zeta \, \left[A^* \right]$$

.....(3.32)

where $\left[Q \right] = \left[N^{*T} \right] \left[N^{*} \right]$ and $\left[N^{*} \right]$ is a polynomial function, written in this way to distinguish it from the generalized shape function $\left[N \right]$.

3.6. Merging of Element Matrices :

The process of collecting the element matrices in one system matrix is called 'Merging'. Once the element matrices are prepared as explained in the previous section merging can place based on the following rules:

a- The element matrices are assumed to be blocks of data which consist of a number of terms the position of which relative to each other is not to be disturbed.

b- The element matrices are to be entered on the diagonal of a larger system matrix one after another according to the element ordering suggested by the user (see section 3.2.). However before transfering the element matrices to the system matrix they must be changed (transformed) from local coordinates to global coordinates using direction cosines.

c- The element matrices overlap by a numbers of diagonal terms which is equal to the number of degrees of freedom being shared between these elements (see figure 3-5). Thus for two (3D) elements sharing one face of 4 nodes each of which has three degrees of freedom, the element will be overlapped by 4x3 = 12 diagonal terms while the stiffness matrix of every element will have an order of n x 3 where 'n' is the number of nodes on the element.

d- Whenever there are terms overlapping, the system stiffness of that term will be the algebraic sum of the component

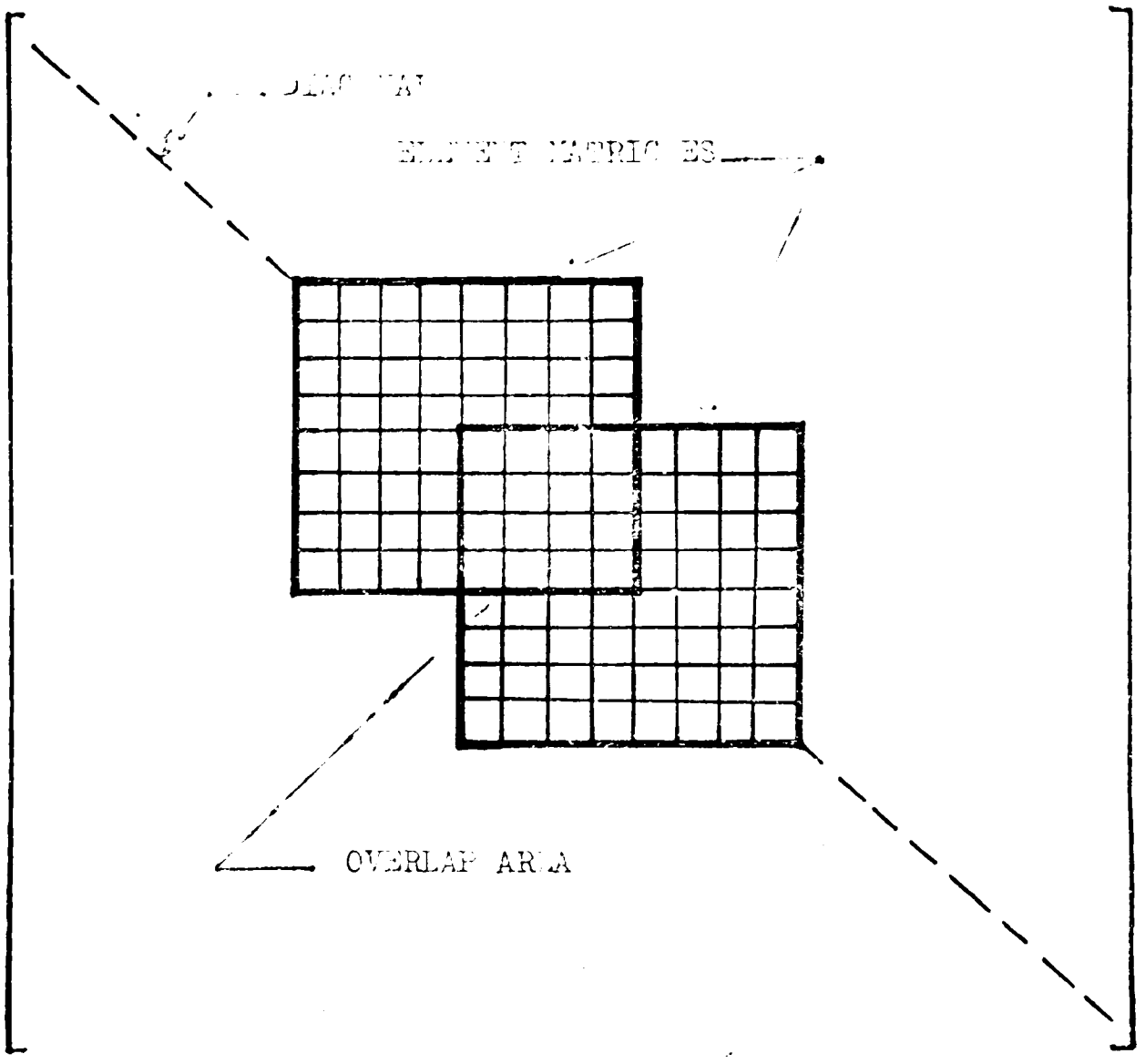


FIGURE (3-5) Merging of element matrices in the larger system matrix. Two 8-noded element matrices are shown to share 4 nodes. Note that for space limitations each small square is assumed to be further devided to nine extra smaller squares. The area outside the element matrices will have zero terms.

stiffnesses of the elements for that particular point.

e- The displacements and forces will be merged in the same manner as that used for the stiffness matrix bearing in mind that they are column vectors rather than two dimensional matrices .

f- A knowledge of the system boundary conditions can reduce the size of the system matrices by a certain amount since a structure that is fixed to earth by a number of points can have zero displacements at these particular points and hence a corresponding infinite stiffness which substituted by zeroes at the solution stage.

g- The size of the system matrix based upon the above merging method will seem to be drastically large since it will have an order which equals

$(NE \times NDOF/_{element}) - NSF - NCF$

where NE is the number of elements, $NDOF/_{element}$ is the number of degrees of freedom per element, NSF is the number of shared freedoms, and NCF is the number of constrained freedoms. However, there are methods which eliminate the huge number of zero terms that appear below and above the diagonal of the system matrix and compress the system size. Further explanation on handling large problems will be given later on in this chapter.

3.7. Numerical Integration:

It was seen that both equations (3.26) and (3.32) had integration signs. In most of the cases such integrations can only be evaluated numerically. Numerical integration has been known for

a long time, it involves dividing the field to be integrated into equal parts of known geometical shapes of which the area is easy to be approximately obtained, thus the sum of all the individual parts will lead to the value of the integral of the specified region. Thus in mathematical terms for a rectangular approximation we have,

$$\int_a^b y(x)\, dx \simeq \sum_{i=1}^{n} A_i\, y(x_i) \qquad (3.33)$$

where A_i is the width of a rectangle and $y(x_i)$ is the value of the function $y(x)$ at any particular interval and it represents the hight of the rectangle. However using equation (3.33) on a digital computer will consume a relatively long time if good accuracy is required, hence a faster converging approximation will be preferable. The choice is settled on the Gauss integration method which is suitable for power functions such as polynomial.

The main idea behind the Gaussian integration is that in the selection of an integration formula such as that of equation (3.33), it may not be wise to specify that the arguments (x_i) be equally spaced. The question that is to be asked in such cases is what choice of the x_i and A_i together will bring maximum accuracy. In fact[10],

$$A_i = \int_a^b L_i(x)\, dx, \quad i = 1,2,3,.....n \qquad (3.34)$$

where L_i is the lagrange multiplier function given by

$$L_i(x) = \frac{(x-x_0)\,(x-x_1)\,.....(x-x_{i-1})\,(x-x_{i+1})...(x-x_n)}{(x_i-x_0)\,(x_i-x_1)\,.....(x_i-x_{i-1})\,(x_i-x_{i+1})...(x_i-x_n)} \qquad .....(3.35)$$

having the properties $L_i(x_k) = 0$ for $k \neq i$, $L_i(x_i) = 1$

the arguments X_1 ..... X_n are zeros of the nth degree polynomial

ORDER	X_i	A_i
2	+0.5773502919	1.0000000000
	-0.5773502919	1.0000000000
3	+0.7745966692	0.5555555556
	-0.7745966692	0.5555555556
	0.0000000000	0.8888888889
4	+0.8611363116	0.3478548451
	-0.8611363116	0.3478548451
	+0.3399810436	0.6521451549
	-0.3399810436	0.6521451549

TABLE 3-1

POINTS AND WEIGHTS FOR THE GAUSS INTERGATION

satisfying the condition,

$$\int_a^b p(x)\, x^k\, dx = 0 \qquad , k = 0, 1, 2, \ldots\ldots\ldots n-1 \tag{3.36}$$

thus taking $a = -1$ and $b = 1$ a table of points X_i and weights A_i can be produced. Such a table for the Gauss orders 2 to 4 is shown on table (3-1).

Finally to evaluate the triple integral

$$\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta$$

it must first be transformed to the triple summation

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} A_i\, A_j\, A_k \quad f(\xi_i, \eta_j, \zeta_k)$$

However due to the symmetrical nature of the isoparametric elements, the limits l,m,n will all be the same.

It was shown[1] that a Gauss order of 2 is sufficient for turbine blade applications and gives better results than the corresponding order 3. Not withstanding this, Gauss orders of both 2 and 3 have been investigated.

3.8. Vibration Problems:

The formulation of vibration problems for finite elements goes in very much the same way as the analytical method. The analogy can be made by representing the vibrating (not rotating) element by a simple damped spring-mass system[3] whose equation of motion is of the second order given by:

$$M\ddot{u} + C\dot{u} + Su = f(t) \tag{3.39}$$

where M is the mass.

u is the displacement.

C is the damping coefficient.

S is the spring stiffness.

f(t) is the time base exciting function.

The analogous descretized quation would be

$$\left[M_e\right]\left\{\ddot{u}_e\right\} + \left[C_e\right]\left\{\dot{u}\right\} + \left[S_e\right]\left\{u_e\right\} = \left\{f_e\right\} \qquad (3.38)$$

and for the whole system, all the matrices have to be merged together as explained in section 3.6. to form the system equation replacing the subscript 'e' in equation (3.38) by 's' for system.

Simpler forms of equation (3.38) are possible through various boundary conditions such as in the case of undamped vibration, the term $\left[C\right]$ in equation (3.38) is omitted and further still by putting $\left\{f_e\right\} = 0$ for natural vibration of the form,

$$\left[M\right]\left\{\ddot{u}\right\} + \left[S\right]\left\{u\right\} = \left\{0\right\} \qquad (3.39)$$

assuming a periodic solution for equation (3.39) given by,

$$\left\{u\right\} = \left\{u_o\right\} \cos \omega t \qquad (3.40)$$

substituting in equation (3.39) yield

$$\left[S\right]\left\{u_o\right\} \cos\omega t - \left[M\right]\left\{u_o\right\} \omega^2 \cos \omega t = 0$$

$$= \left(\left[S\right] - \omega^2 \left[M\right]\right)\left\{u_o\right\} = 0 \qquad (3.41)$$

Equation (3.41) represent a typical eigenvalue problem which has the necessary condition for a non trivial solution of the form shown in equation (3.30) to appear as,

$$\left|\left[S\right] - \omega^2 \left[M\right]\right| = 0 \qquad (3.42)$$

The eigen values of equation (3.41) gives the natural frequencies of the system while each eigenvalue determines a vector which will give the relative movement of the vibrating structure which is called the mode shape of the structure at that particular frequency. A number of methods can be used to solve and evaluate the eigenvalues and eigenvectors of equation (3.41) (see references 120, 83) both as analytical and numerical and at present a lot of computer programs exist for this purpose. The solution can be

either to obtain the whole eigen system or to obtain a specific eigenvalue. In either case further modification of equation (3.41) has to be performed before submitting it to the solution stage. Preparation can be made by first modifying equation (3.41) as follows:

$$\left[S \right]\left\{ u_o \right\} = \omega^2 \left[M \right] \left\{ u_o \right\} \qquad (3.43)$$

then by inverting $\left[S \right]$ to the other side and letting $\lambda = \frac{1}{\omega^2}$ we get

$$\left[S \right]^{-1} \left[M \right] \left\{ u_o \right\} = \lambda \left\{ u_o \right\} \qquad (3.44)$$

using the choleski decomposition (see Appendix D) the matrix $\left[S^{-1} \right]$ can be decomposed into

$$\left[S \right]^{-1} = \left[L^T \right]^{-1} \left[L \right]^{-1} \qquad (3.45)$$

substituting in equation (3.44) results

$$\left[L^T \right]^{-1} \left[L \right]^{-1} \left[M \right] \left\{ u_o \right\} = \lambda \left\{ u_o \right\}$$

or

$$\left[L \right]^{-1} \left[M \right] \left\{ u_o \right\} = \left[L \right]^T \left\{ u_o \right\} \qquad (3.46)$$

defining

$$X = \left[L \right]^T \left\{ u_o \right\} \qquad (3.47)$$

and upon substituting into equation (3.46) the final form of equation (3.41) will be

$$\left[A \right] \left\{ X \right\} = \lambda \left\{ X \right\} \qquad (3.48)$$

which is exactly the same as equation (D.25) in Appendix D to which the reader will be referred for the method of solution. It should be mentioned at this stage that general solution algorithms are in existance (see for example reference 121 or 83).

3.9. Arrangements for Large Problems - Economization of the Degrees of Freedom.

Requiring more accuracy from finite elements means an extra investment on increasing one or more of the following factors:

a- Total mumber of elements in the descretized field.

b- Number of nodes per element.

c- Number of degrees of freedom per node.

Increasing any of the above factors will lead to the same result which is larger program sizes and higher running expenses. The latter (expenses) being a function of the program size. Further more, the type of problem plays a major role in deciding the program size, for instance vibration problems need more than double the size of the static problems since vibration problems require extra space for the mass matrix which the statics problems would not need at all. In review of the methods used to deal with large problems, the Author finds three distinct procedures with regard to the way the problem is tackled,

a- Organizational: It will be seen after reading section 3.6. that the merged element matrices will form a conventionally diagonal matrix which consists of a large number of zero elements off its conventional diagonal. A smart program writer can make use of this fact and store the diagonal matrices in a much smaller form which is normally called the banded form, finally at the solution stage the matrix is used row by row while the necessary zero terms being added in their appropriate places making use of the knowledge about the way these terms have been removed in the first place. Another method would be to use backing store (disc or magnetic tape) to store data which can be retrieved bit by bit when required. The third method would be by creating a data front which is a special way of ordering the elements so that the width

of the system matrix is maintained to a minimum (see section 3.2. and references 58,59, and 87).

b- Numerical: Eigen value problems are a good example of large problems which have a very expensive exact solution. For this reason approximate solutions are used. Mathematicians and mumerical analysts have devised a number of these solutions and the best is to refer to their expertise in this field (see section D.3. and references 9 and 10) It should be mentioned that some of these methods deal with finding only a certain eigen value at a time while others solve for the whole eigen system.

c- Analytical: A method was divised to reduce the size of the system matrices based upon personal judgement[124] or automatically by deciding the degree of freedom with high ratio of mass to stiffness since these degrees of freedom have higher effect on the natural frequency using the simple relation as a guide,

$$\omega_n \alpha \sqrt{\frac{S}{M}} \qquad (3.49)$$

where S is the stiffness and M is the mass, and since our interest is to obtain the fundamental frequency in most of the cases it is found that the lowest fundamental is obtained with the highest ratio of mass to stiffness this approximation is based on the justification that the engineer always find his way through short cuts to the required result making his steps simpler through ignoring complicated terms which are within the engineering tolerence criterion. Such degrees of freedom are normally called 'Masters' or more scientifically independent degrees

of freedom. Other degrees of freedom will be linearly related to the independent degrees of freedom and they are called slaves or again using the scientific terminology the dependent degrees of freedom. This method is explained in references 57,58, and 61 and will not be elaborated in here. However further reference will be made in a later chapter.

3.10. Centrifugal Loading:

It was mentioned in chapter 2 that most of the existing finite element programs deal with the centrifugal loading as being part of the body loads and implementing it is done by modifying the force vector in equation (3.20), normally the body loads are represented by equivalent nodal loads by assuming it as a distributed load and equating the work done by this load on the element with the work done by the nodes results in calculating the nodal loads. Further details will be found in reference 57 and 58.

CHAPTER 4

FUNDAMENTAL ANALYSIS - VIBRATIONS FOR ROTATING MACHINERY

Unlike the previous chapter which was a general chapter for the finite element method, this chapter will be particularly devoted to including the centrifugal effects on the calculation of the modes and frequencies of vibration for different elements. The process which constitutes the major part of this work consists of;

a- Formulation of equations for a 3D isoparametric element.

b- Preparation of the system matrices prior to the solution stage.

c- Solution of equations.

4.1. Kinematic Analysis of a Point on a Rotating Body :

Prior to the analysis it will be found useful to look into a simple situation, that of a block sliding along a rotating rod OA. OA is rotating at a speed $\Omega = d\vartheta / dt$, which is not necessarily a constant. If the block 'B' of Figure 4-1 is situated at an instantaneous radius 'r' from the origin 'O' (polar representation), then its cartesian coordinates will be:

$$x = r \cos \vartheta \qquad (4.1)$$

and

$$y = r \sin \vartheta \qquad (4.2)$$

The cartesian velocities and accelerations of B are obtained by differentiating equations (4.1) and (4.2) twice with respect to time, thus,

$$\frac{dx}{dt} = \dot{x} = \dot{r} \cos \vartheta - r \sin \vartheta \frac{d\vartheta}{dt} \qquad (4.3)$$

and

$$\frac{d^2x}{dt^2} = \ddot{x} = \ddot{r}\cos\vartheta - 2\dot{r}\sin\vartheta\,\frac{d\vartheta}{dt} - r\cos\vartheta\left(\frac{d\vartheta}{dt}\right)^2 - r\sin\vartheta\,\frac{d^2\vartheta}{dt^2}$$

or

$$\ddot{x} = \ddot{r}\cos\vartheta - 2\dot{r}\Omega\sin\vartheta - r\,\Omega^2\cos\vartheta - r\alpha\sin\vartheta \qquad (4.4)$$

and similarly,

$$\ddot{y} = \ddot{r}\sin\vartheta + 2\dot{r}\Omega\cos\vartheta - r\,\Omega^2\sin\vartheta + r\alpha\cos\vartheta \qquad (4.5)$$

where,

$$\alpha = \frac{d\Omega}{dt} = \frac{d\dot{\vartheta}}{dt} = \frac{d^2\vartheta}{dt^2} = \dot{\Omega} \qquad (4.6)$$

Now by choosing the 'x' coordinate in a way that $\vartheta = 0$, then we get,

$$\cos\vartheta = 1 \quad \text{and} \quad \sin\vartheta = 0 \qquad (4.7)$$

substituting this into equations (4.4) and (4.5) will give,

$$\ddot{x} = \ddot{r} - r\,\Omega^2 \qquad (4.8)$$

and

$$\ddot{y} = 2\dot{r}\,\Omega + r\alpha \qquad (4.9)$$

It is apparent from equations (4.8) and (4.9) that there is a total of four acceleration components of which the first two are radial or parallel to the rod (equation 4.8) and the other two which are tangential components are perpendicular to the rod (equation 4.9).

Following the same lines of analysis, we now move to the next problem of the rotating body OA of figure (4-2). If it is assumed that a point 'P' on OA is being distorted from its original position P_1 by an angle $\Psi = \int \Omega\,dt$ due to rotation plus two

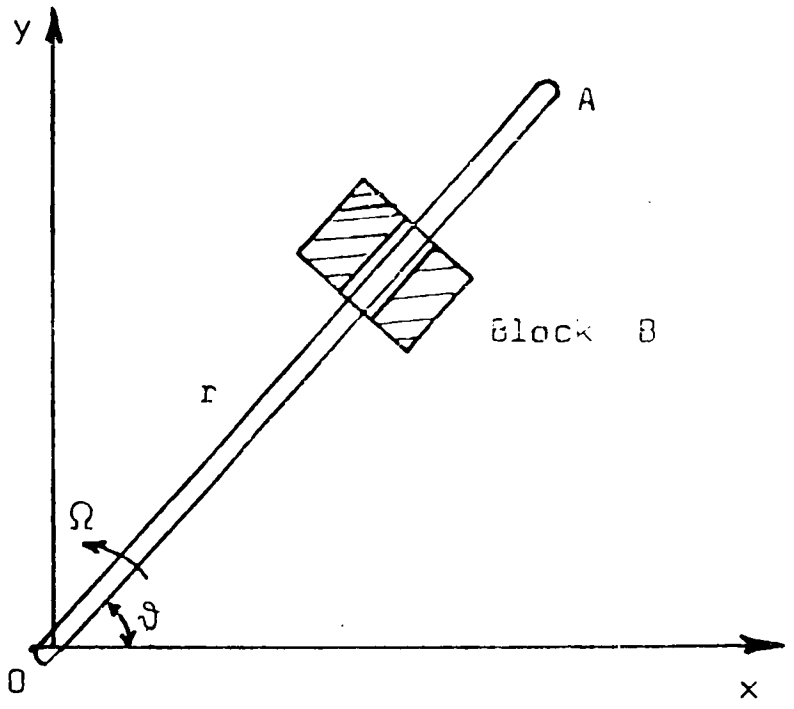


FIGURE 4-1 Block sliding on a rotating rod

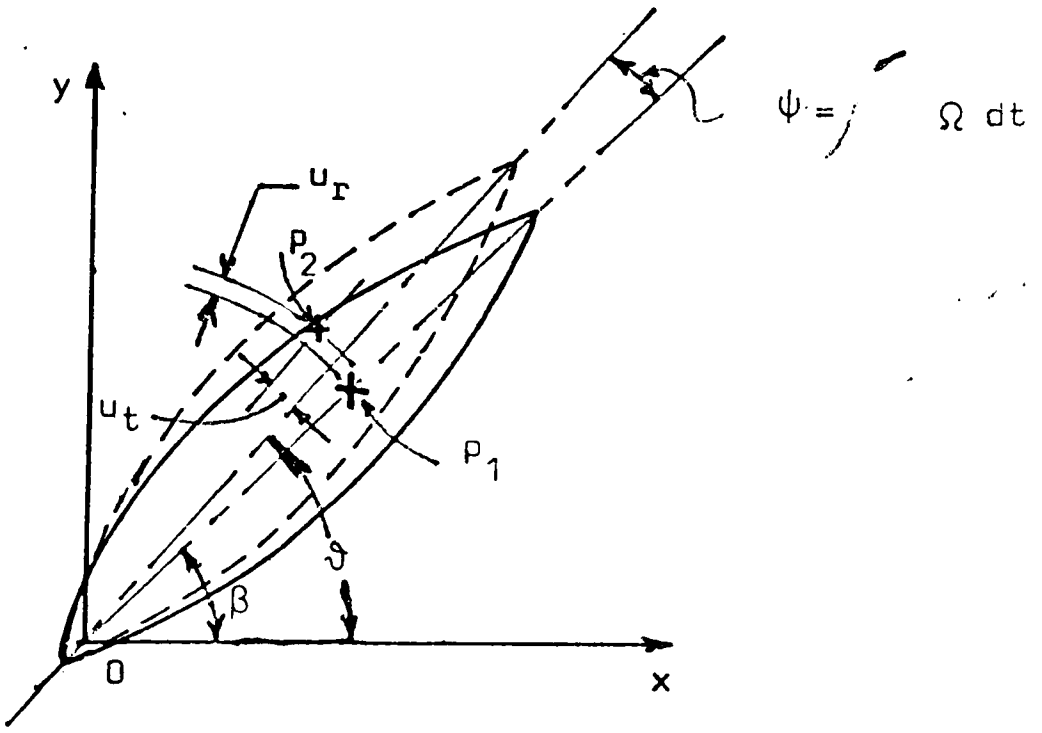


FIGURE 4-2 Kinematics of a point on a rotating body

extra deflections u_r and u_t in the radial and tangential directions respectively due to the centrifugal action, then it becomes easy to imagine that point 'P' is in an analogous situation to that of block 'B' of figure (4-1). There exists a minor detail however, that is, in the first case of figure (4-1) the block was assumed to be completely independent of OA in the radial direction while it is forced to move with OA in the tangential direction, and hence its a one degree of freedom system, while in the second case, even if point 'P' is a part of OA it will be assumed that it is linked to OA by radial and tangential springs that it will possess two degrees of freedom. The accelerations of point 'P' can be obtained by differentiating its coordinates of position twice with respect to time. Referring to figure (4-2), the new coordinates of position of point P_2 are given by,

$$x = (R + u_r) \cos \vartheta - u_t \sin \vartheta \qquad (4.10)$$

and

$$y = (R + u_r) \sin \vartheta + u_t \cos \vartheta \qquad (4.11)$$

$$\vartheta = \beta + \psi \quad \text{and} \quad \psi = \int \Omega \, dt \qquad (4.12)$$

Differentiation with respect to time gives,

$$\dot{x} = -(R + u_r) \sin \vartheta \frac{d\vartheta}{dt} + \dot{u}_r \cos \vartheta - \left[u_t \cos \vartheta \frac{d\vartheta}{dt} + \dot{u}_t \sin \vartheta \right]$$

$$= -(R + u_r) \sin \vartheta \cdot \Omega + \dot{u}_r \cos \vartheta - \left[u_t \cos \vartheta \cdot \Omega + \dot{u}_t \sin \vartheta \right]$$

.....(4.13)

and

$$\dot{y} = (R + u_r) \cos \vartheta \cdot \Omega + \dot{u}_r \sin \vartheta - u_t \sin \vartheta \cdot \Omega + \dot{u}_t \cos \vartheta$$

.....(4.14)

where,

$$\frac{d\vartheta}{dt} = \frac{d(\beta + \psi)}{dt} = \frac{d\psi}{dt} = \Omega \qquad (4.15)$$

The second differentiation with respect to time of x can be greatly simplified if Ω is assumed to be a constant and $\frac{d\vartheta}{dt} = 0$, then

$$\ddot{x} = -\Omega\left\{(R + u_r)\cos\vartheta\,\frac{d\vartheta}{dt} + \dot{u}_r\sin\vartheta\right\} - \dot{u}_r\sin\vartheta\,\frac{d\vartheta}{dt} + \ddot{u}_r\cos\vartheta$$
$$-\left[\Omega\left\{-u_t\sin\vartheta\,\frac{d\vartheta}{dt} + \dot{u}_t\cos\vartheta\right\} + \dot{u}_r\cos\vartheta\,\frac{d\vartheta}{dt} + \ddot{u}_t\sin\vartheta\right] \qquad \ldots\ldots(4.16)$$

and

$$\ddot{y} = \Omega\left\{-(R + u_r)\sin\vartheta\,\frac{d\vartheta}{dt} + \dot{u}_r\cos\vartheta\right\} + \dot{u}_r\cos\vartheta\,\frac{d\vartheta}{dt} + \ddot{u}_r\sin\vartheta$$
$$-\Omega\left\{u_t\cos\vartheta\,\frac{d\vartheta}{dt} + \dot{u}_t\sin\vartheta\right\} - \dot{u}_t\sin\vartheta\,\frac{d\vartheta}{dt} + \ddot{u}_t\cos\vartheta \qquad \ldots\ldots(4.17)$$

Again arranging for the coordinate system that the relations of equations (4.7) are true then equations (4.16) and (4.17) reduce to

$$\ddot{x} = -\Omega^2(R + u_r) + \ddot{u}_r - 2\Omega\dot{u}_t \qquad (4.18)$$

and

$$\ddot{y} = 2\Omega\dot{u}_r + \ddot{u}_t - \Omega^2 u_t \qquad (4.19)$$

If it is assumed that point 'P' is displaced in the 'z' direction as well then there will be a z component of acceleration which clearly consists of u_z only since the displacement in the 'z' direction is only u_z thus the third equation corresponding to equations (4.18) and (4.19) is,

$$\ddot{z} = \ddot{u}_z \qquad (4.20)$$

In sorting out the accelerations of equations (4.18) to (4.20) it is found that they consist of three parts:

1- Centripetal acceleration proportional to the displacements.

2- Coriolis accelerations which are proportional to the velocities.

3- Relative accelerations which form the rest of the terms.

Using the shape function notation of (section 3.3.) to enable applying the finite element method, the acceleration vector can be written in terms of the nodal accelerations as follows.

$$\{a\} = \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [N]\{\ddot{u}\} + 2\Omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} [N]\{\dot{u}\}$$

$$- \Omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} [N]\{u\} - \Omega^2 R \qquad (4.21)$$

4.1.1. Provision for Rotation About More Than One Axis

It is now wished to obtain a more general form of equation (4.21) to include the effect of rotation about more than one axis. However this time vector notation will be used since the number of terms involved is much less than the previous method.

Consider the point 'P' to be moving on a rotating body of speed

$$\underline{\Omega} = \Omega_x \underline{i} + \Omega_y \underline{j} + \Omega_z \underline{k} \qquad (4.22)$$

where $\underline{\Omega}$ is the total vector and Ω_x, Ω_y, and Ω_z are its components in the three unit directions $\underline{i}$, $\underline{j}$, and $\underline{k}$ respectively.

The instantaneous position of point 'P' is given by the vector

$$\underline{r} = (R + u_r)\underline{i} + u_t \underline{j} + u_z \underline{k} \qquad (4.23)$$

where R is the original radius of P_1 figure (4-2) and u_r, u_t, u_z are the distortions after a time Δt.

The velocity of the vector $\underline{r}$ is obtained differentiating respect

$$\dot{\underline{r}} = \frac{\partial \underline{r}}{\partial t} + \underline{\Omega} \times \underline{r} = \dot{u}_r \underline{i} + \dot{u}_t \underline{j} + \dot{u}_z \underline{k} + \underline{\Omega} \times \underline{r} \qquad (4.24)$$

Making use of the unit vector products,

$$\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} \tag{4.25}$$

$$\underline{i} \times \underline{j} = - \underline{j} \times \underline{i} = \underline{k} \tag{4.26}$$

$$\underline{j} \times \underline{k} = - \underline{k} \times \underline{j} = \underline{i} \tag{4.27}$$

$$\underline{k} \times \underline{i} = - \underline{i} \times \underline{k} = \underline{j} \quad \text{or} \quad \underline{i} \times \underline{k} = - \underline{j} \tag{4.28}$$

It becomes easy to form the product of two general vectors,

$$\underline{A} = x_1 \underline{i} + y_1 \underline{j} + z_1 \underline{k}$$

and

$$\underline{B} = x_2 \underline{i} + y_2 \underline{j} + z_2 \underline{k}$$

then

$$\underline{A} \times \underline{B} = (y_1 z_2 - z_1 y_2) \underline{i} - (x_1 z_2 - z_1 x_2) \underline{j} + (x_1 y_2 - y_1 x_2) \underline{k}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \tag{4.29}$$

and similarly,

$$\underline{\Omega} \times \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \Omega_x & \Omega_y & \Omega_z \\ R+u_r & u_t & u_z \end{vmatrix}$$

$$= (\Omega_y u_z - \Omega_z u_t) \underline{i} - \left\{ \Omega_x u_z - \Omega_z (R+u_r) \right\} \underline{j}$$

$$+ \left\{ \Omega_x u_t - \Omega_y (R+u_r) \right\} \underline{k} \tag{4.30}$$

Substituting equation (4.30) into equation (4.24) leads to

$$\dot{\underline{r}} = (\dot{u}_r + \Omega_y u_z - \Omega_z u_t) \underline{i} - \left\{ -\dot{u}_t + \Omega_x u_z - \Omega_z (R+u_r) \right\} \underline{j}$$

$$+ \left\{ \dot{u}_z + \Omega_x u_t - \Omega_y (R+u_r) \right\} \underline{k} \tag{4.31}$$

The accelerations can be found by a second differentiation with respect to time,

$$\ddot{\underline{r}} = \frac{d\dot{\underline{r}}}{dt} + \underline{\Omega} \times \dot{\underline{r}}$$

$$= (\dot{u}_r + \Omega_y \dot{u}_z - \Omega_z \dot{u}_t)\,\underline{i} + (\dot{u}_t - \Omega_x \dot{u}_z + \Omega_z \dot{u}_r)\,\underline{j}$$

$$+(\dot{u}_z + \Omega_x \dot{u}_t - \Omega_y \dot{u}_r)\,\underline{k} + \underline{\Omega} \times \underline{r} \qquad (4.32)$$

Again making use of the product of equation (4.29)

$$\underline{\Omega} \times \dot{\underline{r}} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \Omega_x & \Omega_y & \Omega_z \\ \dot{u}_r + y u_z - z u_t & \dot{u}_t - x u_z + z(R+u_r) & \dot{u}_z + x u_t - y(R+u_r) \end{vmatrix}$$

.....(4.33)

Expanding and rearranging gives,

$$\underline{\Omega} \times \dot{\underline{r}} = \left[\Omega_y \dot{u}_z - \Omega_z \dot{u}_t + \Omega_x(\Omega_y u_t + \Omega_z u_z) - (R + u_r)(\Omega_y^2 + \Omega_z^2)\right]\,\underline{i}$$

$$+\left[\Omega_z \dot{u}_r - \Omega_x \dot{u}_z + \Omega_y\left\{\Omega_z u_z + \Omega_x(R + u_r)\right\} - u_t(\Omega_z^2 + \Omega_x^2)\right]\,\underline{j}$$

$$+\left[\Omega_x \dot{u}_t - \Omega_y \dot{u}_r + \Omega_z\left\{\Omega_x(R + u_r) + \Omega_y u_t\right\} - u_z(\Omega_x^2 + \Omega_y^2)\right]\,\underline{k}$$

.....(4.34)

Finally substitution in equation (4.32) will give the acceleration vector,

$$\ddot{\underline{r}} = \left[\ddot{u}_r + 2(\Omega_y \dot{u}_z + \Omega_z \dot{u}_t) + \Omega_x(\Omega_y u_t + \Omega_z u_z) - (R + u_r)(\Omega_y^2 + \Omega_z^2)\right]\,\underline{i}$$

$$+\left[\ddot{u}_t + 2(\Omega_z \dot{u}_r - \Omega_x u_z) + \Omega_y\left\{\Omega_z u_z + \Omega_x(R + u_r)\right\} - u_t(\Omega_z^2 + \Omega_x^2)\right]\,\underline{j}$$

$$+\left[\ddot{u}_z + 2(\Omega_x u_t - \Omega_y u_r) + \Omega_z\left\{\Omega_x(R + u_r) + \Omega_y u_t\right\} - u_z(\Omega_x^2 + \Omega_y^2)\right]\,\underline{k}$$

.....(4.35)

Initial examination of equation (4.35) shows that it consists of three different accelerations, relative accelerations proportional to $\ddot{u}$, Coriolis accelerations proportional to $2\Omega\dot{u}$ and the third is the centripital accelerations which form the third terms $\Omega^2 u$. A fourth group of acceleration terms does exist and its terms are proportional to $\Omega^2 R$ which the author of this text calls it acceleration of position. However these last accelerations will not appear on the left hand side of the differential equation of motion and hence will not affect the results. Using matrix notation , a corresponding but more general equation than equation (4.21) can be obtained, hence,

$$\{a\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [N]\{\ddot{u}\} + 2\begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} [N]\{\dot{u}\}$$

$$+ \begin{bmatrix} -(\Omega_y^2 + \Omega_z^2) & \Omega_x\Omega_y & \Omega_x\Omega_z \\ \Omega_y\Omega_x & -(\Omega_z^2 + \Omega_x^2) & \Omega_y\Omega_z \\ \Omega_z\Omega_x & \Omega_z\Omega_y & -(\Omega_x^2 + \Omega_y^2) \end{bmatrix} [N]\{u\} + \begin{Bmatrix} \Omega_y^2 + \Omega_z^2 \\ \Omega_y\ \Omega_x \\ \Omega_z\ \Omega_x \end{Bmatrix} R$$

.....(4.36)

Thus by putting $\Omega_x = \Omega_y = 0$, $\Omega_z = \Omega$ into equation (4.36) the equation reduces to equation (4.21)

4.1.2. A Further Step into Generality

In the analysis of the previous section it was assumed that there are radial and tangential components of displacement and that the axes set was so chosen that the initial position of point

P has a radius R which is always coinsiding with the radial component of displacement this will complicate the shape functions and will limit the application to certain geometries only and a more complete analysis can be made by considering R as a position vector of constant length and that the displacements are composed of components in the three cartesian directions thus,

$$\underline{R} = r_x \underline{i} + r_y \underline{j} + r_z \underline{k} \qquad (4.37)$$

and

$$\underline{u} = u_x \underline{i} + u_y \underline{j} + u_z \underline{k} \qquad (4.38)$$

Following the same procedure used in the previous section, a similar equation to (4.36) can be obtained with a more complete matrix for the accelerations of position, thus

$$\{a\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} [N]\{\ddot{u}\} + 2 \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} [N]\{\dot{u}\}$$

$$+ \begin{bmatrix} -(\Omega_y^2 + \Omega_z^2) & \Omega_x \Omega_y & \Omega_x \Omega_z \\ \Omega_y \Omega_x & -(\Omega_z^2 + \Omega_x^2) & \Omega_y \Omega_z \\ \Omega_z \Omega_x & \Omega_z \Omega_y & -(\Omega_x^2 + \Omega_y^2) \end{bmatrix} \{[N]\{u\} - R\}$$

$$\text{or } \{a\} = [I][N]\{\ddot{u}\} + 2[\Omega_1][N]\{\dot{u}\} + [\Omega_2]\{[N]\{u\} - R\} \qquad (4.39)$$

As was mentioned earlier in this chapter, the accelerations of position are not included in the left hand side of the equation of motion and hence they would not affect the results. Equation (4.39) has standard shape functions which are used for a general 3-D brick finite element while the shape functions of equation (4.36) have to be modified to care for the non-standard displacements.

4.2. Formulation of the Main Equation of Motion :

In the previous chapter general comments on vibration problems were given and methods of solution with limitations and boundary conditions were discussed. Also the discretized form of a general vibration equation was given by equation (3.38) which is repeated below for convenience,

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [S]\{u\} = \{F\} \qquad (4.40)$$

where $[M]$ is the mass matrix.

$[S]$ is a modified stiffness matrix

$[C]$ is a Coriolis matrix resulting from the Coriolis effect.

$\{F\}$ is a force vector containing the internal and external loads.

Equation (4.40) looks very much like a general forced damped vibration equation with $[C]$ corresponds to the damping matrix.

The derivation of equation (4.40) above is started with the virtual work process[17], thus

$$[S]\{u\} - \{F\} = \{0\} \qquad (4.41)$$

which is merely the same as equation (3.20) of the previous chapter. In equation (4.41) $[S]$ is a generalised stiffness matrix given by the form,

$$[S] = \int_V [B^T][D][B] \qquad (4.42)$$

and $\{F\}$ is the load matrix given by:

$$\{F\} = \int_v [N^T]\{X\}\, dv \qquad (4.43)$$

The dynamic forces are calculated with the aid of D'Alembert's principle and body forces could be written as:

$$\{X\} = \rho \{a\} \qquad (4.44)$$

where ρ is the density and $\{a\}$ the present accelerations as given by equation (4.21) or equation (4.39).

Substituting equation (4.44) into equation (4.43) results in,

$$\{F\} = \int_v [N^T] \rho \{a\}\, dv \qquad (4.45)$$

substituting equation (4.21) or equation (4.39) into equation (4.45) then into equation (4.41) gives the final form of the equation of motion,

$$[M]\{\ddot{u}\} + 2\,[C]\{\dot{u}\} + (\,[S] - [S^*]\,)\,\{u\} = \{\Omega_2\}R \qquad .....(4.46)$$

where the matrix $[C]$ is the Coriolis acceleration matrix given by,

$$[C] = \int_v [N^T]\, \rho\, [\Omega_1]\{N\}\, dv \qquad (4.47)$$

and S^* is the centripital acceleration matrix,

$$[S^*] = \int_v [N^T]\, \rho\, [\Omega_2][N]\, dv \qquad (4.48)$$

The two matrices $[\Omega_1]$ and $[\Omega_2]$ are matrices of speed components as defined by equation (4.39), $[\Omega_2] = [\Omega_1]^2$.

Knowing from a glance on the matrix $[\Omega_1]$ that it is skew symmetric, it becomes easy to understand that the Coriolis matrix is also skew symmetric.

The right hand side of equation (4.46) contains the term $[\Omega_2]R$ which is not a function of the displacement but it represents a prescribed body force, which can be treated statically together with any other prescribed forces present. The first step of the calculation must therefore be static analysis that is based on the original equation (4.41) from which an initial stress matrix $[\sigma]$ is obtained as

$$[\sigma] = \begin{bmatrix} \sigma_x I & \tau_{xy} I & \tau_{xz} I \\ & \sigma_y I & \tau_{yz} I \\ \text{(symmetric)} & & \sigma_z I \end{bmatrix} \qquad (4.49)$$

where I is a 3*3 identity matrix.

The second step would be to calculate the geometric stiffness matrix $[S_g]$ given by

$$[S_g] = \int_v [G^T][\sigma][G]\, dv \qquad (4.50)$$

where the matrix $[G]$ is a matrix defining detrivatives, it is related to the matrix $[B]$ of equation (3.14) simply through their definitions. The reader is referred to Zienkiewics[124] for more detail about the geometric stiffness matrix. The writer finds references (17) and (125) rather confusing and are inconsistant in defining the matrix $[G]$ or the representation of the geometric stiffness matrix.

Having obtained the geometric stiffness matrix, the right hand term of equation (4.45) could be transferred to the left and the equation can now be written in a more general form:

$$[M]\{\ddot{u}\} + 2[C]\{\dot{u}\} + [S_T]\{u\} = \{0\} \qquad (4.51)$$

where

$$[S_T] = [S] - [S^*] + [S_g] \qquad (4.52)$$

4.3. Method of Solution:

Equation (4.51) represent a Hermitian eigen value problem to which a number of solution methods are available (see references 120, 10), among the solution methods are

a- Sturm sequence method (42)

b- The method of Newmark (30)

c- Recurrence formulas (124, 64)

d- Modified QL Algorithm (121)

It should be mentioned at this stage that most of the investigations dealing with centrifugal effects had omitted the terms proportional to the velocity such as the damping terms or specifically the Coriolis term inorder to reduce the problem to an ordinary eigen value problem such as that of equation (3.41). The PAFEC finite element program solves equation (3.41) by using the QR Algorithm.

The work of this text will follow a modified QL Algorithm. By shifting the eigen equation to an upper Hessenberg form, the QL method is used to solve this equation and finally back substitution is used to obtain the eigen system of the original equation.

4.3.1. Formulation of the eigen equation:

In this section an attempt will be made to obtain the eigen equation in order to be able to solve it following the procedure suggested above.

Starting by the suggestion given by reference (42) to rewrite equation (4.40) as follows:

$$\begin{bmatrix} 0 & -[M] \\ [M] & [C] \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \dot{u} \end{Bmatrix} + \begin{bmatrix} [M] & 0 \\ 0 & [S] \end{bmatrix} \begin{Bmatrix} \dot{u} \\ u \end{Bmatrix} = \{0\} \qquad (4.53)$$

This is possible by partitioning (see Appendix D). Equation (4.53) can be written as;

$$A \dot{y} + B y = 0 \qquad (4.54)$$

where 'A' is skew symmetric and 'B' is symmetric and positive definite.

A possible solution of equation (4.54) is,

$$y = e^{pt} \qquad (4.55)$$

substituting into equation (4.54) to obtain;

$$(B + pA)\, y = 0 \qquad (4.56)$$

equation (4.56) can be written in the form

$$(G + \gamma I)u = 0 \qquad (4.57)$$

where $G = L^{-1} A L^{-T}$, $\gamma = 1/p$, $\omega = \sqrt{-p}$ (4.58)

and 'I' being the unit matrix and 'L' is a lower triangular matrix found by the Choleski decomposition of the matrix 'B' as explained in Appendix D,

$$B = LL^T \qquad (4.59)$$

Equation (4.57) represent the eigen value equation with the real matrix 'G' to which the above mentioned solution procedure can now be applied.

4.3.2. Reduction to Upper Hessenberg Form:

Prior to the explanation of the reduction procedure it will be necessary to lay down some definitions.

Consider a system of linear equations such as the one mentioned in Appendix D (viz equation D.1 or the short form of equation D.2) which will be repeated here for convenience

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{Bmatrix} \qquad (4.60)$$

Probably the reader is familiar with the solution of such a system by direct algebraic elimination in which the column vector Y is regarded as fixed and then it becomes required to determine all the column vectors for which $AX = Y$. In these terms the system represented m linear equations and n unknowns. It will be found useful to regard the vector Y as a variable, in which case we get a system of m linear homogeneous equations in m + n variables:

$$-Y + AX = 0 \qquad (4.61)$$

Furthermore, it is more convenient to remove the minus sign by letting $V = -Y$, to obtain in block form

$$[I]\{V\} + [A]\{X\} = \left\{\begin{matrix} I \\ \hline A \end{matrix}\right\} \left\{\begin{matrix} V \\ \hline X \end{matrix}\right\} = 0 \qquad (4.62)$$

or in expanded form,

$$\begin{aligned} v_1 \cdot \cdot \cdot \cdot + a_{11} x_1 + \cdot \cdot + a_{1n} x_n &= 0 \\ v_2 \cdot \cdot \cdot + a_{21} x_1 + \cdot \cdot + a_{2n} x_n &= 0 \\ \cdot \cdot \cdot + \cdot \cdot + \cdot \cdot + \cdot \cdot \cdot & \\ \cdot \cdot + \cdot \cdot + \cdot \cdot + \cdot \cdot \cdot & \\ v_m + a_{m1} x_1 + \cdot \cdot + a_{mn} x_n &= 0 \end{aligned} \qquad (4.63)$$

It seems very trivial to attempt to solve for the v's in terms of the x's in the system (4.63) since the number of unknowns is m+n while the system has only m linear equations. However our

objective is to solve for as many as possible of the x_j in terms of the v's and the rest of the x's. To solve the system (4.63), it will be exchanged for another system which has exactly the same solution but which is preferable in some sence, until the following system is achieved:

$$\begin{array}{ccccccccccc}
x_{1'} & & + b_{11} & x_{(r+1)'} & + \cdots + & b_{1s} & x_{n'} & + c_{11} & v_1 & + \cdots + & c_{1m} & v_m = 0 \\
\ddots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & x_{r'} & + b_{r1} & x_{(r+1)'} & + \cdots + & b_{rs} & x_{n'} & + c_{r1} & v_1 & + \cdots + & c_{rm} & v_m = 0 \\
 & & 0 & x_{(r+1)'} & + \cdots + & 0 & x_{n'} & + c_{t1} & v_1 & + \cdots + & c_{tm} & v_m = 0 \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & 0 & x_{(r+1)'} & + \cdots + & 0 & x_{n'} & + c_{tm} & v_1 & + \cdots + & c_{mm} & v_m = 0
\end{array}$$

.....(4.64)

where $\{1',\ldots,n'\}$ is a permutation of $\{1,\ldots,n\}$, $r \leqslant m$, $s = n-r$, and $t = r+1$. r is called the rank of the matrix 'A' of coefficients of the system (4.63), and any solution of (4.64) is obtained by assigning the v's any set of values which satisfy the last m-r equations and assigning arbitrary values to $x_{(r+1)'},\ldots,x_{n'}$. In case the values of the v's are specified, (4.64) has a non-trivial solution if and only if the last m-r equations reduce to $0 = 0$.

From the above it becomes possible to conclude that: two systems of linear equations are said to be equivalent if and only if the solution of either system is a solution of the other. In manipulating equations it is wished to be certain that any operations performed will produce a system that is equivalent to the original system. There are some operations that are permissible under this requirement. First, it is apparent that the solution is not affected by the order in which the equations are written.

Hence any permutation of the arrangement of equations will produce an equivalent system. Second, any equation may be related by a non zero scalar multiple of itself. Since such a scalar has a reciprocal, the process can be reversed and the systems are equivalent. Third, an equation can be replaced by the sum of itself and any other equation in the system. Thus we can summarize three kinds of what is called Elementary Operations for a linear system of homogeneous equations:

a- Permutation of any two equations.

b- Multiplication of any equation by a non zero scalar.

c- Addition of one equation to another.

It should be mentioned that the three elementary operations correspond to row operations which are useful in evaluating determinants by replacing a given determinant by an equal determinants which is simpler. In this context, "Simpler" means that the new determinant contains more zeros or at least a more useful arrangement of zeros.

The next step after defining elementary operations is to define elementary matrices . An Elementary Matrix is any matrix which can be obtained by performing a single elementary row operation on the Identity matrix.

There are three types of elementary matrices , one for each type of elementary row operation. Recalling the definition of the identity or unit matrix as given by equation (D.11) of Appendix D we can describe these matrices ,

a- Permutative matrix: Let P_{ij} be the matrix obtained from I by permutating the ith and jth rows. Then

$$P_{ij} = I - a_{ii} + a_{ji} - a_{jj} + a_{ij} \qquad (4.65)$$

b- Multiplication by a constant: Let $M_{i(c)}$ denote the matrix obtained from I by multiplying the ith row by $c \neq 0$. $M_{i(c)}$ is obtained by adding $c-1$ to the element in the (i,i) position, so,

$$M_{i(c)} = I + (c-1)\, a_{ii} \tag{4.66}$$

c- Addition: Let A^{*}_{ij} denote the matrix obtained from I by adding row i to row j where $i \neq j$. Clearly,

$$A^{*}_{ij} = I + a_{ij} \tag{4.67}$$

The merits of the three elementary matrices P_{ij} , $M_{i(c)}$, A^{*}_{ij} stems from the fact that an elementary row operation on arbitrary rectangular matrix A may be performed by permutating A (i.e., multiplying A on the left) by the corresponding elementary matrix. Now since there is an identity matrix of any dimension, it follows that there exists three elementary matrices of each dimension. If A is m x n, then any premultiplying matrix B must have m columns for B•A to be defined.

The subject of elementary matrices can be expanded to various types of application and the reader is referred to any text book on the subject such as reference (51). One application that will be given here with concern is the reduction of a matrix A to upper-Hessenberg form, or in other words, to a matrix H such that

$$H_{ij} = 0 \quad (i \geq j+1) \tag{4.68}$$

The reduction may be achieved in a stable manner by the use of either stabilized elementary matrices or elementary unitary matricies.

The reduction by stabilized elementary matrices takes place in $n-2$ major steps; immediately before the rth step the original matrix A_1 has been reduced to A_r which is of upper Hessenberg form in its first r-1 columns. For real matrices the rth step is then

as follows:

a) Determine the maximum of the quantities $\left|a_{ir}^{(r)}\right|$ $(i=r+1,..,n)$. If the maximum is zero the r-th step is complete. Otherwise this maximum element is denoted by $a_{(r+1),r}^{(r)}$ and the procedure continues as follows.

b) Rows $(r+1)'$ and $(r+1)$ are interchanged then the same is done with columns $(r+1)'$ and $(r+1)$.

c) The following steps are repeated for each value of i from $r+2$ to n;

Computation of $n_{i,r+1} = a_{ir}^{(r)} / a_{r+1,r}^{(r)}$, subtraction of $n_{i,r+1}$ times row r+1 from row i and addition of $n_{i,r+1}$ times column i to column r+1 .

The relationship between A_r and A_{r+1} may be expressed in the form

$$A_{r+1} = N_{r+1}^{-1} I_{r+1,(r+1)} A_r I_{r+1,(r+1)} N_{r+1} \qquad (4.69)$$

where $I_{r+1,(r+1)}$ is an elementary permutation matrix and N_{r+1} is an elementary matrix with

$$(N_{r+1})_{i,r+1} = n_{i,r+1} \quad (i = r+2,\ldots,n) \text{ and } (N_{r+1})_{ij} = \delta_{ij} \qquad \ldots\ldots(4.70)$$

Otherwise* an alternative procedure can be explained as follows:

The row and column interchanges used in this reduction effectively determine a permutation matrix P such that when $A_1^{\ddagger} = P_1 A_1 P^T$ is reduced no row interchanges are required. It further determines

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* δ_{ij} is a substitute for the elements of the identity matrix,

$\delta_{ij} = 0 \quad (i \neq j)$, $\delta_{ij} = 1 \quad (i = j)$

a unit lower triangular matrix N with r_1 as its first column such that,

$$A_1^{\ddagger} N = N H \qquad (4.71)$$

where H is of upper-Hessenberg form. From equation (4.71) N and H can be determined directly column by column by equating in succession columns 1,2,...,n of the two sides and interchanges can be determined at the same time.

Reference 76 gives algorithms for the reduction procedures explained together with the necessary backsubstitution procedures. The algorithms are written in ALGOL and since the PAFEC finite element program is written in FORTRAN it was necessary to translate these programs into FORTRAN in order to be able to use them on the PAFEC program. The method used to check the correctness of the translated versions was by comparing results from the FORTRAN program with the results of sample problems given in the original paper. The second check which is used more than often in this work was to reverse the process and see whether the original data is obtained. In the present case, the backsubstitution routine was used to give the required test.

4.3.3. Solution by the QL Algorithm :

Eigen value problems have their share among the contributions to the digital solution of problems. There are many dedicated algorithms in this respect[120,121,9,10,83] and in many cases it is left to the user to decide which algorithm he is going to use according to his requirments. It is known that the L R transformation of Rutishauser is a powerful method for finding the eigenvalues of symmetric banded matric es. Little attention however had been given to its application to the more difficult problem of finding

the eigen system of a general unsymmetric matrix. If the matrix is large then two main difficulties are likely to occur. Firstly, the triangular decomposition, which forms the basis of this method is not always numerically stable, and secondly the amount of computation will be very great. The Rutishauser method consists of forming a sequence of matrices $A^{(r)}$ where $A^{(r)} = A$ the original matrix, and $A^{(r+1)}$ is derived from $A^{(r)}$ by decomposing it into lower and upper triangular matrices , $L^{(r)}$ and $U^{(r)}$ and forming the product of these in the reversed order, hence giving the relations:

$$\left.\begin{aligned} A^{(r)} &= L^{(r)}\ U^{(r)} \\ A^{(r+1)} &= U^{(r)}\ L^{(r)} \end{aligned}\right\} \qquad (4.72)$$

It can be readily seen that the process consists of a series of similarity transformations on the original matrix:

$$\begin{aligned} A^{(r+1)} &= L^{(r)^{-1}}\ A^{(r)}\ L^{(r)} \\ &= L^{(r)^{-1}}\ L^{(r-1)^{-1}} \ldots\ L^{(1)^{-1}}\ A\ L^{(1)}\ L^{(2)} \ldots\ L^{(r)} \end{aligned} \qquad (4.73)$$

Thus with some conditions to be fulfilled and as $r \rightarrow \infty$, $A^{(r)}$ tends to an upper triangular matrix of which the diagonal elements are the eigenvalues arranged in order of modulus, the first being the largest.

Knowing from section (4.3.1) that the eigen equation G is skew symmetric , it was thought that the easiest way to solve it is by considering it completely unsymmetric and solving it by the QL algorithm of Francis. Francis[49,50] wrote a similar algorithm to the L R algorithm (arbitrarily calling it the Q L algorithm) except that the transformations involved are all unitary and can thus

be expected to be numerically stable. Francis suggested that there are various advantages in first converting the matrix to almost-triangular form, the main advantage being the reduction of the amount of computation involved. It will be realized that this idea was used in this study as it was explained earlier in the previous section.

The Q R algorithm with shifts of the origin is described by the relations:

$$\left.\begin{aligned} & Q^{(r)}\,(A^{(r)} - K^{(r)}\,I) = U^{(r)} \\ \text{and} \quad & A^{(r+1)} = U^{(r)}\,Q^{(r)T} + K^{(r)}\,I \end{aligned}\right\} \qquad (4.74)$$

thus giving

$$A^{(r+1)} = Q^{(r)}\,A^{(r)}\,Q^{(r)T} \qquad (4.75)$$

where $Q^{(r)}$ is orthogonal, $U^{(r)}$ is an upper triangular matrix, and $K^{(r)}$ is the shift of origin. When the initial matrix $A^{(1)}$ is of upper Hessenberg form then it is easy to show that this is true of all $A^{(r)}$.

By comparing equation (4.75) to equation (4.73) it will be realized that Q in the first equation is analogous to L in the second equation except that Q is orthogonal.

If all the $K^{(r)}$ are taken to be zero then in general $A^{(r)}$ tends to a form in which $a^{(r)}_{i+1,i}$, $a^{(r)}_{i+2,i+1} = 0$ $(i=1,2,\ldots,n-2)$ and hence all the eigen values are either isolated on the diagonal or they are the eigenvalues of a 2×2 diagonal submatrix. Origin shifts are normally used to obtain convergence and for rapid results each shift should ultimately be close to an eigen value of $A^{(1)}$.

ALGOL programs have been written to solve problems using the QR method and the reader is referred to reference (121) and reference (72) for further information. The author of this text has translated

these programs into FORTRAN in order to make them compatible with the PAFEC finite element program which is written in FORTRAN.

As mentioned in the previous section, the method of testing the translated programs is by operating them on the test data of the original papers as well as comparing them with the NAG(83) set of subroutine libraries.

4.3.4. Backsubstitution :

Arrays and pointers left from the reduction to upper Hessenberg form of the original matrix are normally kept and passed over from one routine to the other until the last stage when the eigen system of the transformed matrix is obtained. The backsubstitution process is used to obtain the eigen system of the original matrix with the aid of these pointers and vectors.

CHAPTER FIVE

PROGRAMMING TECHNIQUES AND DETAILS

The program coding that was written for this investigation was prepared in the form of independent scientific subroutine segments which would allow any user to select any routine for the particular application that he may need. A large proportion of the coding was concerned with the solution of a Hermitian eigenvalue problem. Manipulation routines are written to allow a general vibration problem to be solved, that is, the solution routine accepts three matrices ,

a- A mass or inertia matrix which is proportional to the acceleration.

b- A damping or Coriolis matrix which is proportional to the velocity.

c- A stiffness plus/or an initial stress matrix which is proportional to the displacement.

and solves for the eigen system of the problem.

To make a more versatile program, it was chosen to write compatible segments which allow computation of the above matrices on the PAFEC finite element program (which is explained in appendix C) and automatically call the manipulation routines and solution routines then join the results with the PAFEC program for further processing such as plotting the mode shapes.

5.1. Necessary Conditions for Compatibility:

It is important to make certain that all the program segments being written to run with another program such as the PAFEC finite element program must be compatible to the best possible way in order

to make the resulting program completely efficient. During the preparation of this study, concern was given to a number of rules,

a- Whenever possible the written program was made to use the data base (arrangement of the data base is explained in appendix C) and to avoid using independent arrays to save core.

b- The notation used was the same as the PAFEC notation except for some variables to which evocative names were given.

c- Whenever possible the PAFEC suite of subroutines was used to prevent duplication of the effort and increasing the number of segments.

d- Although the PAFEC 75 system control module (see Appendix C) is very flexible and allows a wide range of applications, a big effort was paid to avoid mixing between the phases and minor changes were put to only three phases individually. These phases are: phase 4 (the automatic data generation phase), phase 6 where the calculation of element matrices is made, and phase 7 which is the solution phase.

e- Only one user file was created for the necessary alterations in every phase since there is a separate subroutine library for every phase.

f- To be compatible with the automatic nature of the PAFEC program all the necessary changes to existing subroutines were made in a way that these changes would not be followed if the centrifugal module is not in existence.

g- Excessive usage of core was avoided where possible.

5.2. Modifications Necessary Before the Matrix Generation:

Unfortunately, the PAFEC 75 finite element program does not have the facility for calculating the combined effect of more than one loading case, such as combined modes and frequencies with prescribed stresses or displacements. Knowing (see appendix C) that the PAFEC policy is to reduce the human effort in designing a problem as much as possible, it becomes necessary to modify some of the routines even at earlier stages of handling a problem. Actually the changes start as early as the automatic data generation phase in which the user's supplied data is interpretted following the rules layed out in reference 59.

Alterations for the above mentioned phase (phase 4) took two different directions. The first of these was based upon the philosophy that these changes should be performed in a way that allows the program to work for more than one solution type at the same time. This means that there are three areas that must be surveyed in order to fulfill these requirements. These areas are:

a- The automatic choice of the solution type.

b- The prefront data generation routines.

c- The instantaneous size of the stiffness matrix.

However, due to the large number of changes that were required, the first path was unregrettably ignored.

Instead, it was decided to let the system assume one kind of solution for the early stages and make the necessary modifications to the later routines. It was decided to fool the system that there is only the modes and frequencies to be calculated since we are mainly concerned with effect of centrifugal action on the

vibration charactaristics. The presence of the centrifugal module will be ignored in phase 4 and no body loads will be assumed.

5.3. Modifications to Obtain the Coriolis and Other Acceleration Matrices :

In chapter four the necessary formulation for the vibration of rotating bodies was made theoretically and in this section the practical implementation of these equations on the PAFEC 75 finite element program will be explained.

Two alternative methods are suggested for the computation of the Coriolis and other matrices , each method leads to the same results but has advantages and disadvantages over the other method.

a- The first method : It will seen by comparing equations (4.47) and (4.48) for the acceleration matrices (repeated here for convenience) with equation (3.31) for the mass matrix that

$$[C] = \int_v [N^T]\, \rho\, [\Omega_1][N]\, dv \qquad (4.47)$$

$$[S^*] = \int_v [N^T]\, \rho\, [\Omega_2][N]\, dv \qquad (4.48)$$

$$[M_e] = \int_v [N^T]\, \rho\, [N]\, dv \qquad (3.31)$$

they are identical except for the small matrices $[\Omega_1]$ and $[\Omega_2]$ and it becomes easy to obtain the matrices $[\Omega_1]$ and $[\Omega_2]$ if a coding for the mass matrix exists. Unfortunately this is not the case with the PAFEC finite element program for which the method of obtaining the mass and stiffness matrices is explained in section (C.3.) of the Appendix. Figure (C-1) of Appendix C shows a flow chart of an element routine from which it is possible to find out a starting point of modification for the acceleration

matrices , it is the subroutine R37008 which performs the post- and premultiplication by the matrix $\left[A^{-1}\right]$. It will be seen from figure (C-2) which is a flow chart for the subroutine R37008 that carrying out the multiplication is done only once for one degree of freedom on every node on the element followed by a triple 'DO' loop which is set up to redistribute these values in their appropriate place on the mass matrix for all the three degrees of freedom of the node. It is exactly at this point the difficulty arises. A simple insertion of a matrix $\left[\Omega_1\right]$ or $\left[\Omega_2\right]$ cannot be made with the existine coding. A new idea is suggested, it is to let the mass matrix be prepared first in the normal way, then splitting it into two identical halves or matrices by taking its square root. This is possible and the reader is referred to Appendix D for a more detailed explanation of a square root procedure. A conventionally diagonal matrix $\left[\Omega^{\ddagger}\right]$ of size (3x number of nodes on the element)^2 can be formed, it is supposed to have either the matrix $\left[\Omega_1\right]$ or $\left[\Omega_2\right]$ arranged on the diagonal as many times as the number of nodes on the element while the rest of the matrix will be left null. An extra post- and pre-multipication can then be carried out to obtain any of the acceleration matrices .

So far, it does not seem to be a difficult matter to carry out the above operations. However, the actual situation is not as easy as it appears to be since as it is the case with any matrix handling routine, the square root subroutine makes a cubic increase of time directly with the increase of the size of matrices it is handling, thus from initial test runs it was found to converge on three iterations only in a short time for

a 2x2 matrix, while for a 20-noded element mass matrix of size 60x60 it was found that subroutines make an average of 50 iterations and aproximately 10 times as much of the necessary time to calculate the mass matrix itself. However, remembering that the mass matrix is initially prepared by permutation of one third of the total number of degrees of freedom on the element (i.e. for the normal calculation cycle it would take 3 times more than the short cut) the conclusion is that going through the long winded route of splitting the mass matrix is still favourable since it is independent of the method of calculating the mass matrix and hence it will be universal for all kinds of elements. The question that could be posed now is how to compromise between core and time requirements ? An answer cannot be given without pointing out to specific computer configuration and accounting methods, but in general since the square root method is unavoidably consuming a great amount of time then it would be advisable to restrict the work in phase 6 (calculation of element matrices to finding out only the square roots and allowing for the rest to be resumed after a restart in phase 7 (solution).

It should be stated at this stage that putting the necessary changes inside the subroutine R37008 will make the method limited to only the 3D group of element which is the only group that employs this subroutine, instead it will be suggested that these changes take place at a later stage in the subroutine R14000 (see table 5-1 for a list of these routines) which is called by all the element families at the end of every element generation in order to store the element matrices on a sequential file.

In the second method it was chosen to write the necessary

coding and allocate more computer space for the computation of the Coriolis matrix and the change to the stiffness matrix inside the routine R37008 (see table 5-1).

5.3.1. Economization of the Degrees of Freedom and Merging of Matrices :

In section (4.4.1) of the previous chapter it was shown that a problem will be encountered in handling large core requirements, hence condensation of degrees of freedom should be made. For the PAFEC program this has been done and an automatic choice of dependent and independent degrees of freedom is also possible (see references 57 and 61). Unfortunately these facilities are only available for the simpler problem of equation (3.39).

$$\left[M \right]\left\{ \ddot{u} \right\} + \left[S \right]\left\{ u \right\} = \left\{ 0 \right\} \qquad (3.39)$$

Hence it becomes necessary to prove that the extra terms of equation (4.65) can also follow the economization procedure.

$$\left[M \right]\left\{ \ddot{u} \right\} + \left[C \right]\left\{ \dot{u} \right\} + \left[S \right]\left\{ u \right\} = \left\{ 0 \right\} \qquad (4.65)$$

Referring again to the dynamic equation (4.65) and following the same lines given by Zienkiewicz[124] , the expression for the potential energy would be written as

$$PE = \left(\left[M \right]\left\{ \ddot{u} \right\} \right)^T \left\{ u \right\} + \left(\left[C \right]\left\{ \dot{u} \right\} \right)^T \left\{ u \right\} + \left(\left[S \right]\left\{ u \right\} \right)^T \left\{ u \right\} \qquad (5.1)$$

using D'Alembert principle for dynamic loads. After some transformation it becomes possible to write the expression for the minimum potential energy

$$\frac{\partial PE}{\partial \left\{ u \right\}} = \left[M^* \right]\left\{ \bar{u} \right\} + \left[C^* \right]\left\{ \bar{u} \right\} + \left[S^* \right]\left\{ \bar{u} \right\} \qquad (5.2)$$

where $\left\{ \bar{u} \right\}$ is the independent degrees of freedom array and

$$\left\{ u \right\} = \left\{ \begin{matrix} \bar{u} \\ \hline \bar{\bar{u}} \end{matrix} \right\} \tag{5.3}$$

in which $\left\{ \bar{\bar{u}} \right\}$ is the dependent degrees of freedom array so that

$$\left\{ \bar{\bar{u}} \right\} = \left[L \right] \left\{ \bar{u} \right\} \tag{5.4}$$

in equation (5.4) $\left[L \right]$ is a matrix specifying the dependence of $\left\{ \bar{\bar{u}} \right\}$ upon $\left\{ \bar{u} \right\}$. Finally ignoring some of the inertia terms[61] we can write:

$$\left\{ u \right\} = \begin{bmatrix} I \\ L \end{bmatrix} \left\{ \bar{u} \right\} \tag{5.5}$$

following the same terminology it becomes easy to note that

$$\begin{aligned} \left[M^* \right] &= \begin{bmatrix} I \\ L \end{bmatrix}^T \left[M \right] \begin{bmatrix} I \\ L \end{bmatrix} \\ \left[C^* \right] &= \begin{bmatrix} I \\ L \end{bmatrix}^T \left[C \right] \begin{bmatrix} I \\ L \end{bmatrix} \end{aligned} \tag{5.6}$$

$$\left[S^* \right] = \begin{bmatrix} I \\ L \end{bmatrix}^T \left[S \right] \begin{bmatrix} I \\ L \end{bmatrix}$$

Hence the same procedure could be followed to economize the degrees of freedom as before.

The reduction procedure is explained and justified in references 58 and 61 both for the mass and stiffness matrices. Following the same lines of analysis justification of the procedure for reducing the Coriolis matrix can also be established and need not be further elaborated here.

In writing the reduction routine for the Coriolis matrix some time saving was obtained by making use of the fact that the Coriolis matrix is skew symmetric. Any multiplication by the diagonal terms will result in a zero value and hence wherever these multiplcations occured they were ignored.

As far as other core requirements are considered, an automatic workspace generation routine was written by the author to allow temporary workspace modules to be created on the data base. A large amount of core was saved by the introduction of a small function subprogram which simulates the matrix $[C^{\dagger}]$ of the previous section instead of creating the whole matrix.

Merging the mass and stiffness matrices did not change since modifications on these matrices are carried out before the call to the merging routine, but never the less the routine had to be modified to merge the third array, the Coriolis array. The modification was simple and straight forward. It involved similar instructions to those used in merging the mass matrix, being effective only in the case of the presence of the centrifugal module.

Helping function subprograms were written to aid handling information between the data base which is a long unidimensional array and the other bits of the program. A list of all the sub-routines involved or written by the author is given at the end of this chapter, and lists of the FORTRAN coding could be obtained by request.

5.4. Preparation of the Eigen Equation :

Although the procedure used to prepare the eigen equation follows the general pattern explained in section (4.4.1) of the previous chapter, its preparation is more involved with the problem of core economy and its found equally important to give a brief idea about it. The preparation is made by the subroutine H52102 which is written by the author. A flow chart of the routine is shown below while a more detailed explanation is given in appendix H.

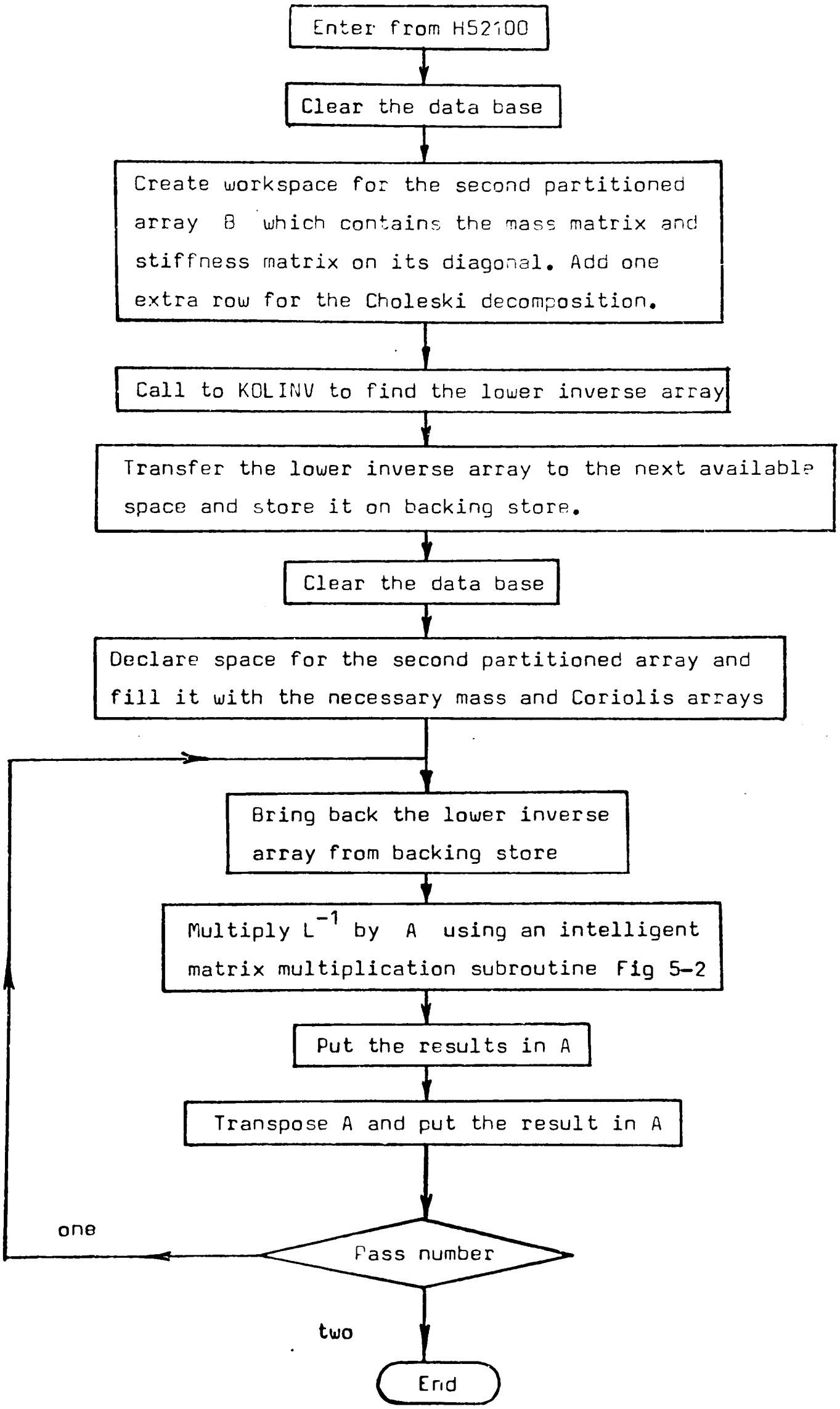


Figure 5-1 Flow chart for the preparation of the eigen equation

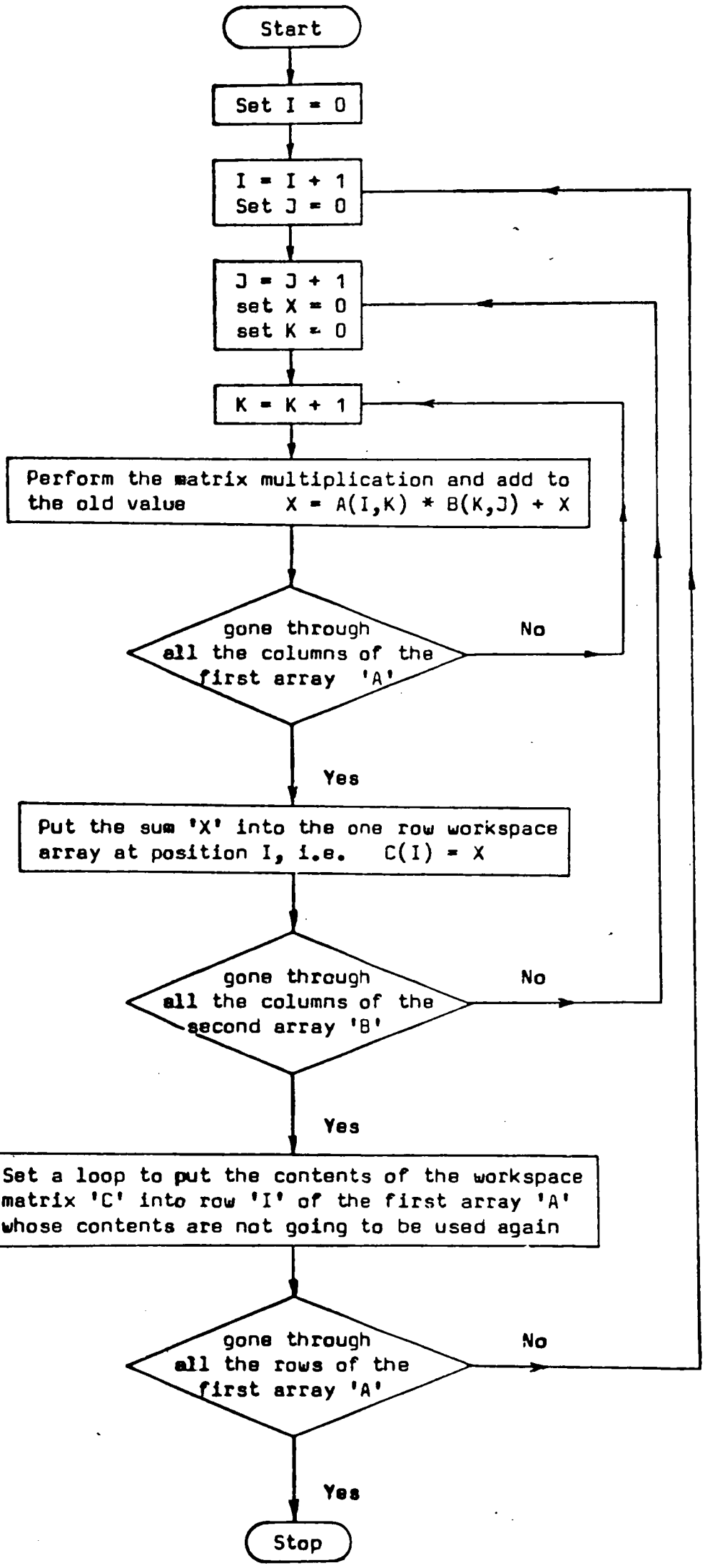


Figure 5-2 Flow chart for a core saver multiplication routine

5.5. Scaling the Matrices Prior to Manipulation :

In the forgoing chapters, formulation of various kinds of the vibration and eigenvalue equations was made, no matter how different they were all of these equations shared two entities, the mass and the stiffness. Theoretically, solution for these problems should give correct answers irrespective of the values of the mass and stiffness. However, this is not true in practice since the order of magnitude of the stiffness over the mass is quite often larger than the number of significant digits of most digital computers. This reduces the reliability of the numerical solution procedure unless extra precautions are taken. Recalling that in section (4.4.2) a discussion was made regarding elementary processes in which it was stated that for a set of linear equations a modification of one row could be made by multiplying it by a constant without affecting the results. Based on this principle a subroutine can be written to scale a square matrix so that the majority of its elements will vary around an order of unity, this is done by dividing every term on the array by a scaling factor which is given by the following product equation:

$$\text{Scaling factor} = \left(\prod a_{ii} \right)^{1/n} \quad , \; i = 1,2,3, \ldots .n \qquad (5.7)$$

where $\prod$ indicates product of all the diagonal terms a_{ii} of the matrix $\left[A \right]$ which is of order n.

This procedure can be applied to the partitioned arrays of equation (4.74) which is repeated below for convenience,

$$\left[\begin{array}{c|c} 0 & -[M] \\ \hline [M] & [C] \end{array}\right] \left\{\begin{array}{c} \ddot{u} \\ \hline \dot{u} \end{array}\right\} + \left[\begin{array}{c|c} [M] & 0 \\ \hline 0 & [S] \end{array}\right] \left\{\begin{array}{c} \dot{u} \\ \hline u \end{array}\right\} = 0 \qquad (4.47)$$

The second partitioned array can be multiplied by a scaling

factor which will balance it, however it can be easily seen that due to the difference between the mass and stiffness matrices the scaling procedure will take an average which will not improve the partitioned matrix at all. Instead only the stiffness matrix will be scaled and the changes to equation (4.47) will appear as

$$\left\{\begin{bmatrix} I \\ \hline D^{-1} \end{bmatrix}\right\} \left[\begin{array}{c|c} 0 & -[M] \\ \hline [M] & [C] \end{array}\right] \left\{\begin{matrix} u \\ \hline \dot{u} \end{matrix}\right\} + \left\{\begin{bmatrix} I \\ \hline D \end{bmatrix}\right\} \left[\begin{array}{c|c} [M] & 0 \\ \hline 0 & [S] \end{array}\right] \left\{\begin{matrix} \dot{u} \\ \hline u \end{matrix}\right\} = [0] \qquad \text{.....(5.8)}$$

where the stiffness matrix $[S]$ is modified by a diagonal matrix $[D]$ which must be inverted before multiplying it by the first partitioned array.

Equation (5.8) can be written as ,

$$\left\{\begin{bmatrix} I \\ \hline D^{-1} \end{bmatrix}\right\} \left[\begin{array}{c|c} 0 & -[M] \\ \hline [M] & [C] \end{array}\right] \left\{\begin{matrix} u \\ \hline \dot{u} \end{matrix}\right\} + \left[\begin{array}{c|c} [M] & 0 \\ \hline 0 & [S^*] \end{array}\right] \left\{\begin{matrix} \dot{u} \\ \hline u \end{matrix}\right\} = [0] \qquad \text{.....(5.9)}$$

where

$$[S^*] = [D][S] \qquad (5.10)$$

and

$$[D][D^{-1}] = [I] \qquad (5.11)$$

Thus as the second partitioned array becomes properly scaled Choleski decomposition can be affected without loss of accuracy. The alternative approach to the problem of loss of accuracy is the use of double precision facilities which is normally available on most digital computers, and the author's reply to the argument that these facilities increase the core requirements is that

intelligent routines can be written to make use of double precision facilities for rounding of errors only and thus optimum use of core is achieved.

One final comment on the scaling of equation (5.9) is that multiplying the first partitioned array by $\left[D^{-1}\right]$ will reduce the accuracy of obtaining the effect of the Coriolis acceleration component on the vibration characteristics.

5.6. Solution :

As explained earlier the solution of the vibration equation goes in three stages for which three different subroutines were written. A general short subroutine was written to allocate the necessary workspace and make calls to all the other routines involved including the preparation routine for the eigen equation. The frequencies of vibration will come out on the imaginary part of the eigen values after backsubstitution in pairs of equal value and opposite sign, while the mode shapes will be the eigenvector corresponding to the positive imaginary eigenvalues.

Finally to match the results in the PAFEC data base clearing the data base of the unnecessary modules and putting the results in their appropriate modules is carried out. The easiest way of avoiding troubles of loosing modules or shifting them by mistake is to store the address part of every module information in IBASE separately from the entry point to the particular solution routines and then restore them just before returning to the calling segment.

TABLE 5-1 LIST OF SUBPROGRAMS USED BY THE AUTHOR

NO	TYPE*	NAME	PHASE USING IT	FUNCTION	MODIFICATION FOR EXISTING SUBPROGRAMS
1	S	R02530	4	Detarmine the solution type (IM) and DOF nod.	Change IM.
2	S	R25000	4	Controls the perfront data generation.	Change the size of arrays.
3	S	R25400	4	Determine the instantaneous size of system matrix.	Change the size of arrays.
4	S	R37110	6	20- noded 3D element routine.	Change the Guass integration.
5	S	R37100	6	8- noded 3D element routine.	Change the Gauss integration.
6	S	R37002	6	Polynomial coefficients and local element exes.	Change DCA.
7	S	R37008	6	Post and premultiplication by $\left[A^{-1} \right]$ outside the integration loop.	Change mass and stiffness matricies.
8	S	R14000	6	Write to sequential file.	Store the Coriolis Matrix.
9	F	ISINGL	6+7	Double - single symetrical array allocation transformation.	
10	F	I10	6+7	Double - single general array allocation transformation.	
11	S	R09500	6	Initialize the data base.	Change to request for modules

TABLE 5-1 LIST OF SUBPROGRAMS USED BY THE AUTHOR (continued)

NO	TYPE*	NAME	PHASE USING IT	FUNCTION	MODIFICATION FOR EXISTING SUBPROGRAMS
12	S	R09800	6+7	Module location trensfer routine.	Add more features.
13	S	MODIC	6+7	Module Header directory.	
14	S	H52103	7	Reduction of an unsymmetrical matrix to upper Hessenberg.	
15	S	H52110	7	Finds the eigen system of a real matrix that was reduced to upper Hessenberg.	
16	S	H52111	7	Back substitution to obtain the eigen system of the original matrix.	
17	S	CDIV	7	COMPLEX Division.	
18	S	R52000	7	Main natural frequency solution routine.	Create more modules.
19	S	R52201	7	Preparation and solution of the eigen value equation.	Introduces basic changes
20	S	R14001	7	Read matrices from sequential file.	Bring Coriolis matrix.
21	S	R14002	7	Main merging routine.	Merge Coriolis matrix.
22	S	R14802	7	Guyan reduction for frontal solution.	Reduce Coriolis terms.
23	F	BMAT	7	Ficticious matrix simulation function.	

TABLE 5-1 LIST OF SUBPROGRAMS USED BY THE AUTHOR (continued)

NO	TYPE*	NAME	PHASE USING IT	FUNCTION	MODIFICATION FOR EXISTING SUBPROGRAMS
24	F	AMAT	7	Similar to BMAT.	
25	S	H52102	7	Prepare the Hermitian eigen value equation.	
26	S	PAFKOL	7	Choleski transformation routine.	
27	S	MATINV	7	Matrix inversion routine.	Extra debugging features.
28	S	MATDIV	7	Matrix division	Extra debugging features.
29	S	H52100	7	Main Hernitian eigen value solution routine.	
30	S	WSPACE	6+7	Finds an unused workspace module.	
31	S	WRCALL	6+7	Writes a line of 16 characters for debugging purposes.	
32	S	H52115	7	Special purpose matrix handling routine.	
33	F	H52116	7	Special purpose matrix handling function.	
34	S	KOLSKI	7	Performs choleski transformation with core saving.	
35	S	KOLINV	7	Prepares the lower inverse array.	
36	S	R15515	7	Empties base from irrelevant routines.	More modules added.
37	S	H15515	7	Same as R15515 but effects different routine.	

TABLE 5-1 LIST OF SUBPROGRAMS USED BY THE AUTHOR (continued)

NO	TYPE*	NAME	PHASE USING IT	FUNCTION	MODIFICATION FOR EXISTING SUBPROGRAMS
38	S	H11100	7	Matrix partition tranfer routine.	
39	F	CORCOF	7	Array simulation function.	
40	S	H37028	7	Finds the coriolis matrix from the square root of the mass matrix and modifies matrices	
41	S	SQROOT	6	Finds the square root of a square matrix.	
42	S	KINV	NONE	General routine for matrix inversion, uses KOLINV with maximum core saving.	

* S : Subroutine subprogram.

F : Function subprogram.

CHAPTER SIX

NUMERICAL APPLICATIONS AND RESULTS

6.1. Introduction:

Calculation of the natural frequencies of stationary structures is one of the main facilities available on the PAFEC finite element program. Calculation and plotting of the eigenvectors is also possible and it gives the vibrational modes for the structures being studied. This chapter displays the effect of rotation on the vibrational characteristics. All the element families on the PAFEC system can be used for vibration studies, but for the present investigation concentration was given to problems involving three dimensional elements since the theory given earlier in chapter four was tailored for these elements.

6.2. Structures Used in this Chapter:

A number of different structures will be used in the next section for the purposed of testing the program. These are:

a- A simple two degree of freedom structure which is composed of two freely vibrating masses and springs, figure (6-1). The masses are sandwiched between frictionless walls to restrict their motion to one direction only.

b- A straight cantilever beam of square cross section as shown in figure (6-2). The beam has the dimensions of 20 x 1 x 1 cm and it is fixed radially on a rotating rigid disc of radius 20 cm. The structure is idealised by five 20-noded elements of the three dimensional isoperametric family. The material used is steel having the following properties

$M_1 = 4$ kg

$M_2 = 10$ kg

$K_1 = 625\ N\ m^{-1}$

$K_2 = 1000\ N\ m^{-1}$

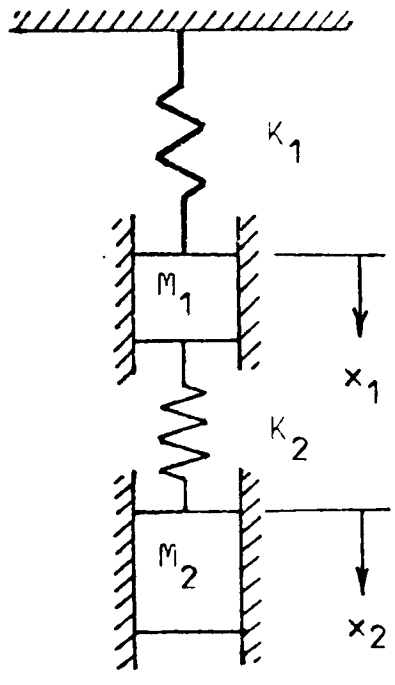
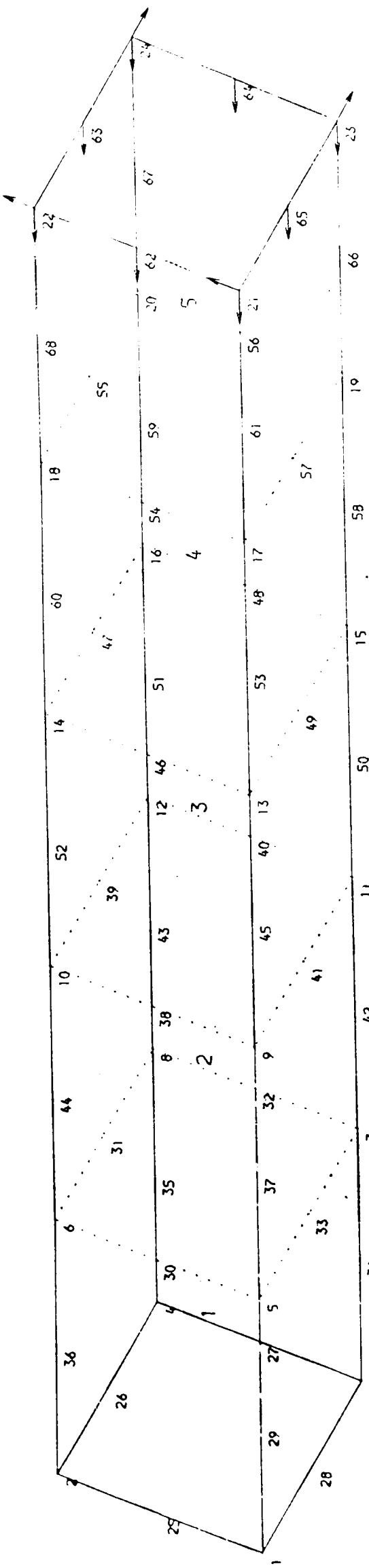


Figure (6-1) A two degree of freedom system.



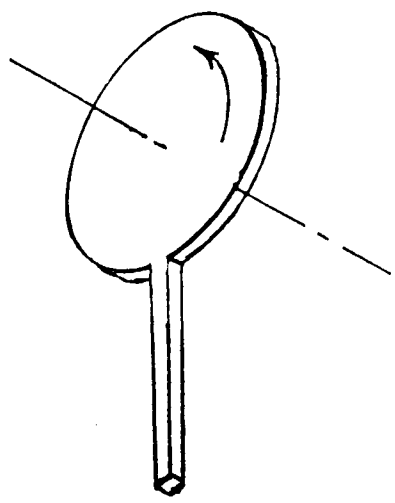


Figure (6-2) Structure 'b': A straight cantilever beam of dimensions 20 x 1 x 1 cm, fixed radially on a rotating disc of radius 20 cm. The beam is idealised by five 20-noded isoparametric brick elements with the base partially constrained.

Modulus of Elasticity = 209×10^9 N/m^2

Density = 7800 kg/m^3

Poisson's Ratio = 0.3

The constraints used on the cantilever root are only partial clamping in order to simulate the simple beam condition for the sake of comparison.

c- A straight cantilever beam of dimensions 20 x 5 x 5 cm fixed radially on a rotating disc of radius 20 cm. The beam is idealised by five 8-noded elements of the three dimensional isoparametric family. The material is steel with the same properties as in 'b' above. Constraints are only partial as shown in figure (6-3).

d- Same as structure 'c' above with all its dimensions and detail except that full constraints are applied at the root.

e- A rectangular cantilever beam of dimensions 15.24 x 2.54 x 0.476 cm (6 x 1 x 3/16 in) figure (6-4). The beam is idealised by twenty four 20-noded elements of the three dimensional isoparametric family. The root is partially clamped.

f- The arrangement shown in figure (6-5) consists of a cantilever clamped on the periphery of a rotating disc. The longitudinal axis of the cantilever is parallel to the axis of rotation. The disc radius is 20 cm and the beam has the dimensions of 20 x 5 x 5 cm. The root is fully clamped.

g- A pretwisted aerofoil cross section compressor blade, figure (6-6). The blade is idealised by thirty six 16-noded elements of the three dimensional isoparametric family. Constraints are

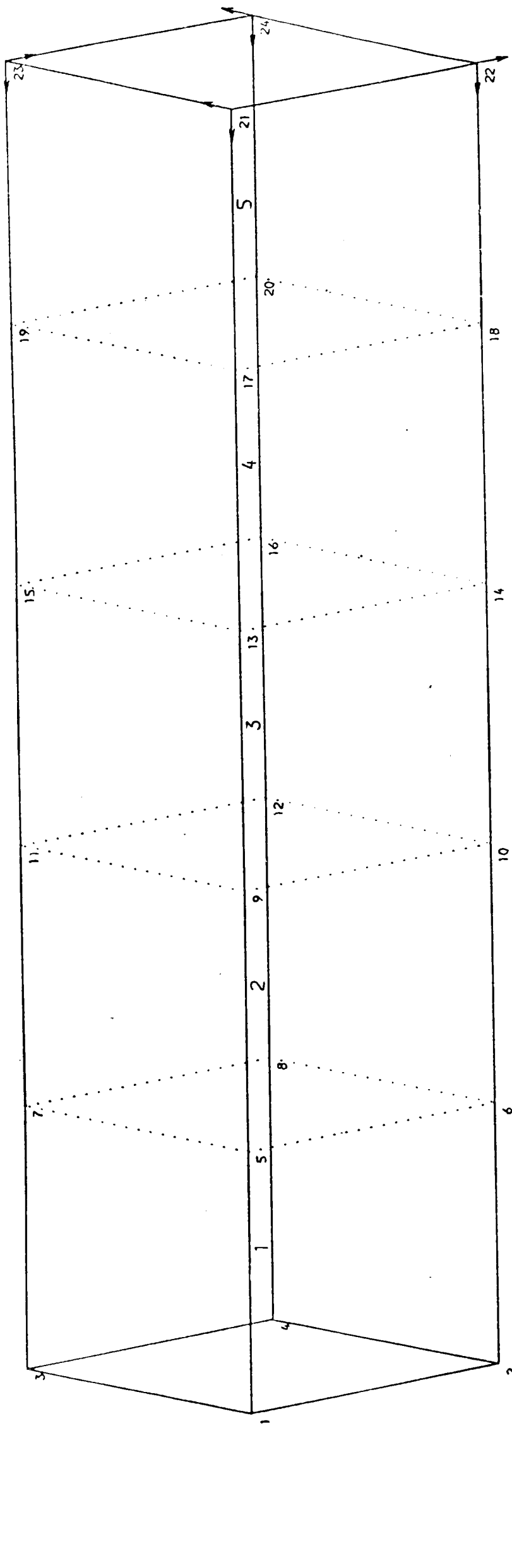
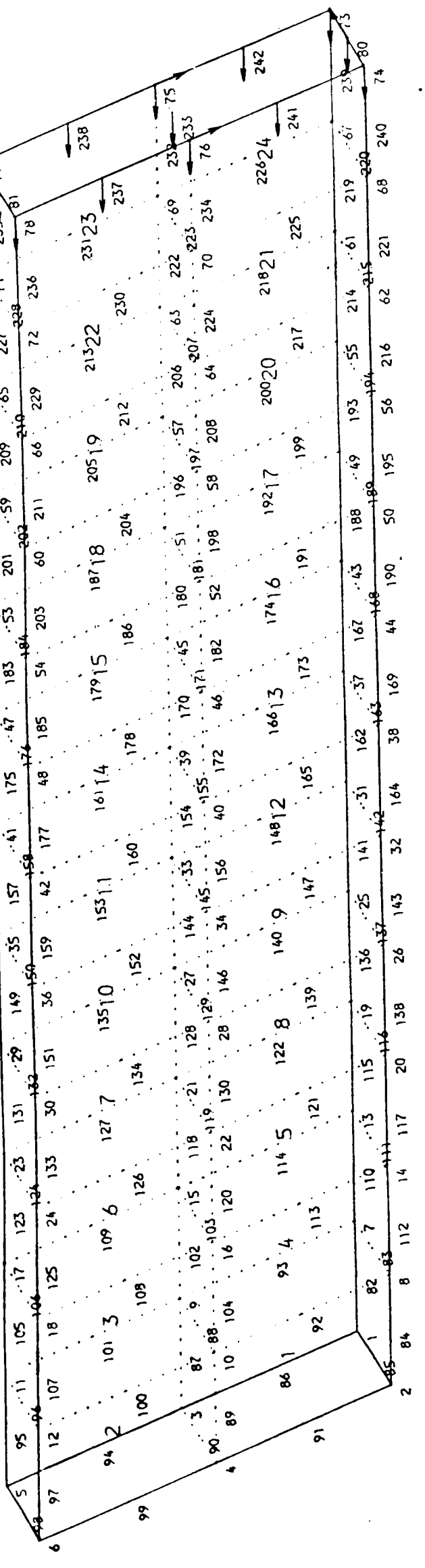


Figure (6-3) Structure 'c': A straight cantilever beam of dimensions 20 x 5 x 5 cm, fixed radially on a rotating disc of radius 20 cm. The beam is idealised by five 8-noded isoparametric brick elements with the base partially constrained.



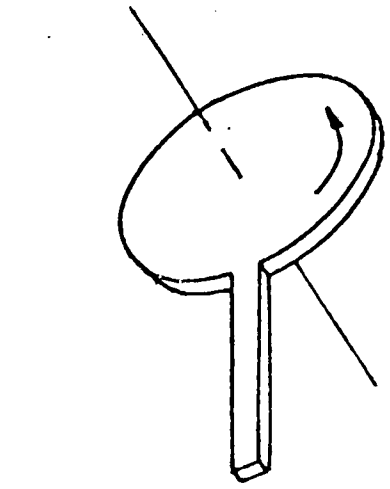
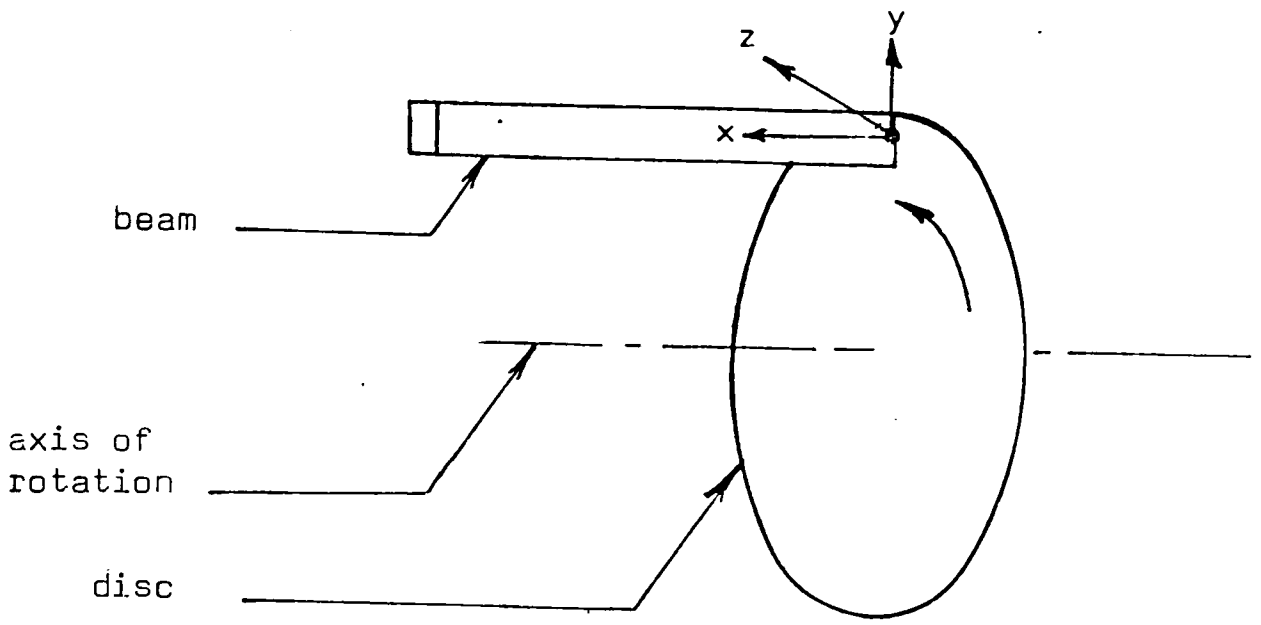


Figure (6-4) Structure 'e': A straight cantilever plate of dimensions 15.24 x 2.54 x .476 cm, fixed on a rotating disc of radius 7.62 cm. The cantilever is idealised by twenty four 16-noded isoparametric brick elements with the root partially constrained.



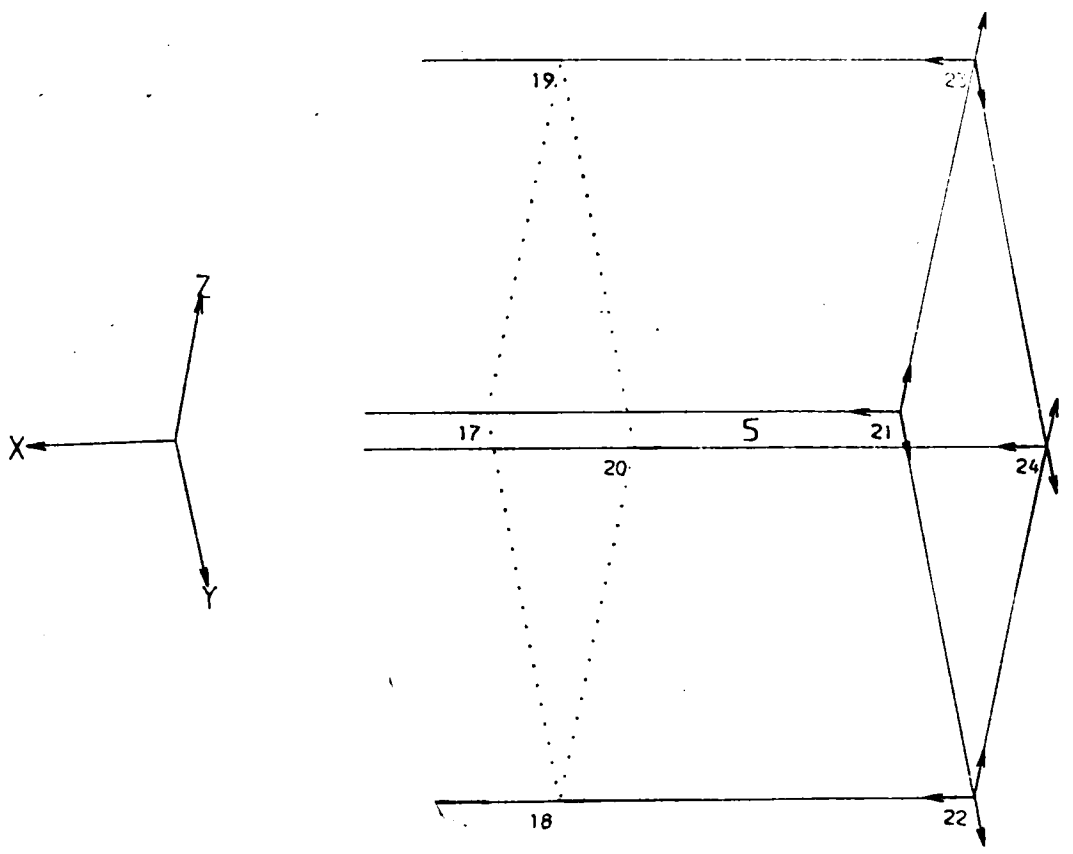


Figure (6-5) Structure 'f': A straight cantilever beam of dimensions 20 x 5 x 5 cm, fixed on a rotating disc so that the longitudinal axis is parallel to the axis of rotation (top). The beam is idealised by five 8-noded elements with the base fully clamped (bottom).

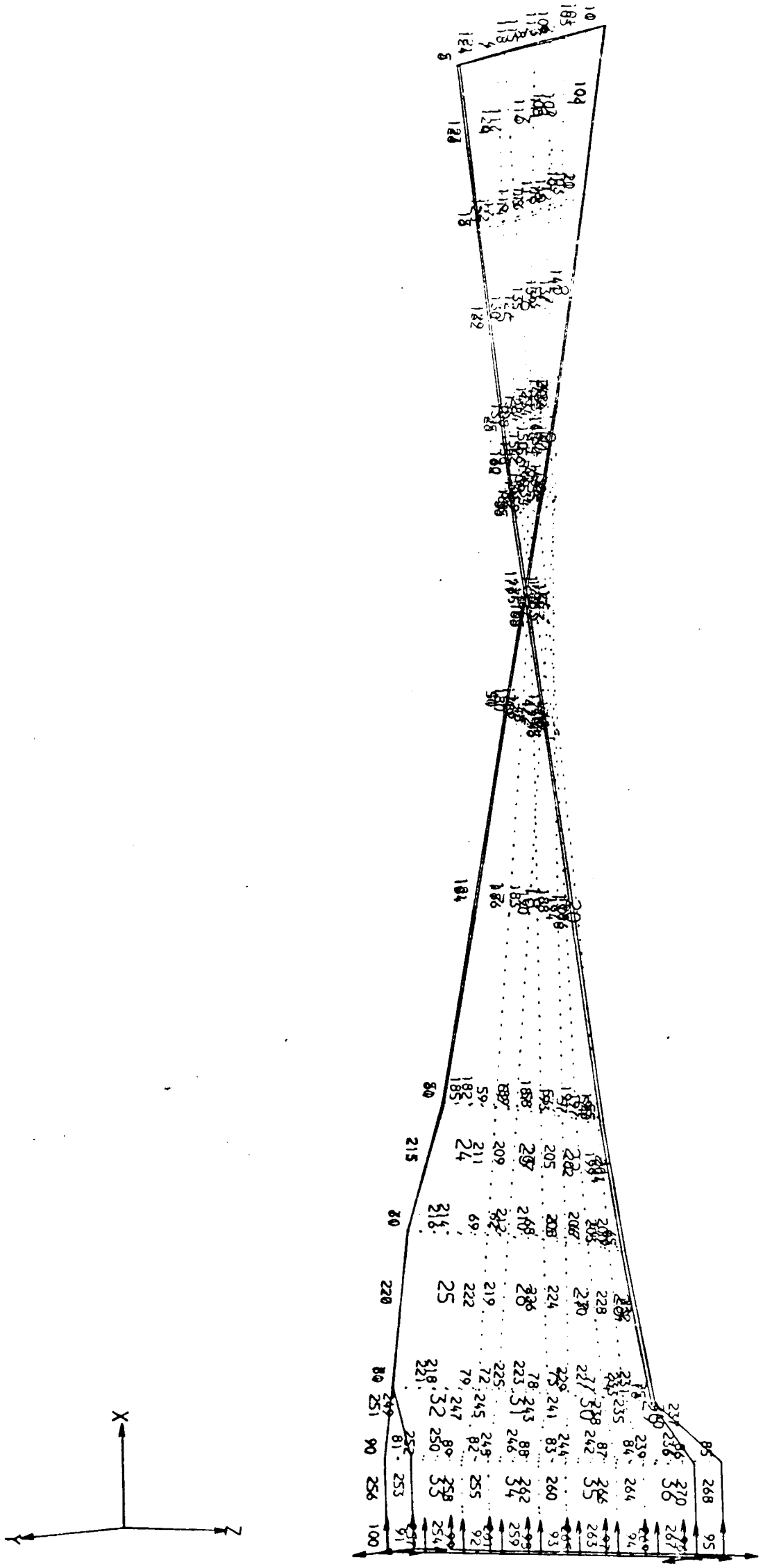


Figure 6-6 Idealization of a pretwisted compressor blade

partially applied at the root. The material is titanium with the following properties,

Modulus of Elasticity = 115.7×10^9 N/m^2

Density = 4500 kg/m^3

Poisson's Ratio = 0.321

h- A low aspect ratio, rotating tapered cantilever beam of length 5 cm, width 2.5 cm. height at the root 0.6 cm, and height at the free end 0.3 cm, The beam is rotated about its root at a stagger of 45^o. Ultra high speed of 52000 rev/min was used for the sake of comparison. The beam is idealised by eight 20-noded three dimensional isoparametric elements, figure 6-7.

6.3. Program Testing:

6.3.1. Testing the Solution Routine:

A number of methods were used to assess and test the performance of the solution procedure described in chapter five of this text:

a- By comparison with examples given by Wilkinson[121] who wrote some of the routines in ALGOL thus establishing the correctness of the FORTRAN translation by the author of this text. The results agreed up to nine significant digits.

b- Testing the program against a known eigenvalue problem. The problem chosen is structure 'a' of section 6.2. which consists of two freely vibrating masses and springs figure (6-1).

Assuming the two masses M_1 and M_2 to be displaced by the values x_1 and x_2 respectively, the governing differential equations would be,

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (6.1)$$

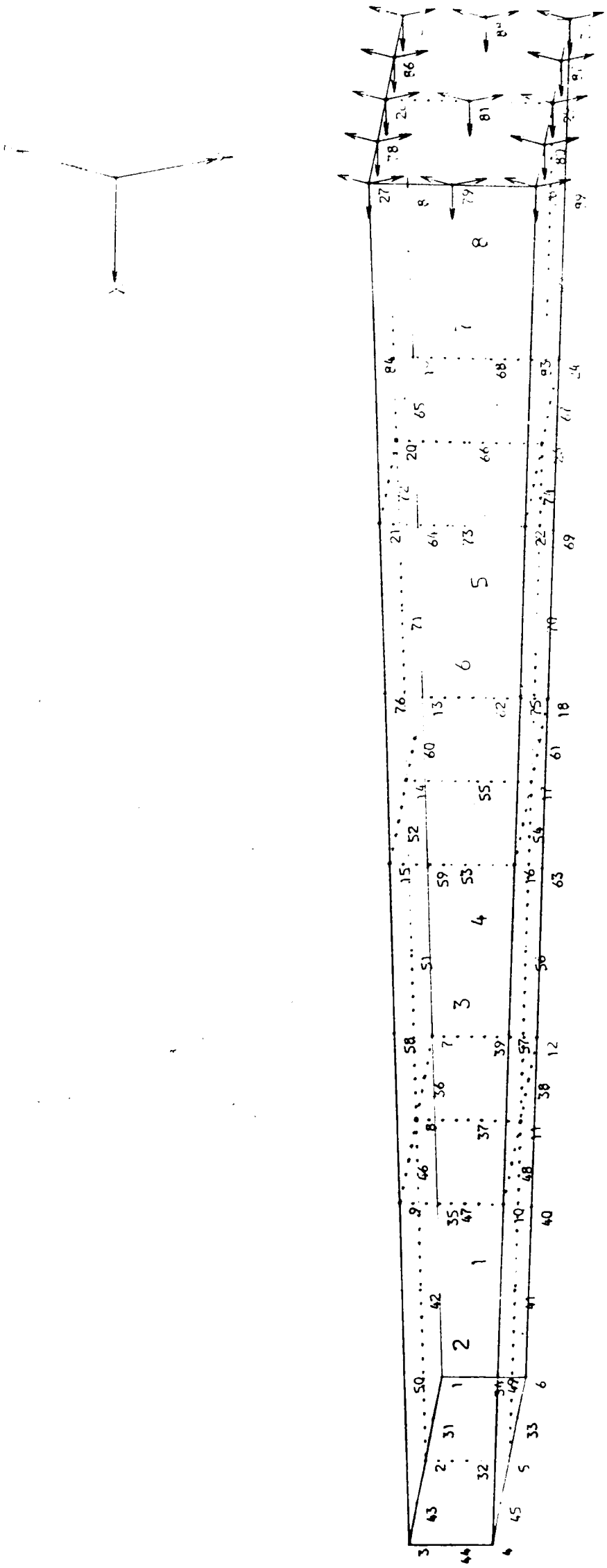


Figure (6-7) Structure 'h': A low aspect ratio tapered cantilever beam of length 5 cm, width 2.5 cm, height at the root 0.6 cm, and height at the free end 0.3 cm. The beam is idealised by eight 20-noded isoparametric brick elements with the root fully constrained. The beam is rotated about the root at a stagger of 45^{0}. The beam is rotated about the 'z' axis at 52 000 rev/min.

Following any text book on vibrations leads to the frequency equation the solution of which gives the two natural frequencies for the system as

$\omega_1 = 0.914519$ Hz and $\omega_2 = 3.46224$ Hz

Substitution of the above frequencies gives the following eigenvectors,

$$V_1 = \begin{Bmatrix} 0.669824 \\ 1 \end{Bmatrix} \quad \text{and} \quad V_2 = \begin{Bmatrix} 1 \\ -0.267929 \end{Bmatrix}$$

The mass and stiffness matrices of equation (6.1) were fed to the solution procedure to obtain the following

$\omega_1 = 0.914519$ Hz $\omega_2 = 3.46224$ Hz

$$V_1 = \begin{Bmatrix} 0.669824 \\ 1.0 \end{Bmatrix} \quad \text{and} \quad V_2 = \begin{Bmatrix} 1.0 \\ -0.267932 \end{Bmatrix}$$

by using an extra segment to calculate the eigenvalue equation as suggested in chapter 5.

c- Partial comparison of larger problems showing the effect of rotation without the Coriolis component can be made with other existing solution packages such as the NAG library of reference (83) or the existing facilities on the PAFEC program for free undamped vibration. Starting by the partitioned equation (4.53) which is repeated below,

$$\left[\begin{array}{c|c} 0 & [-M] \\ \hline [M] & [C] \end{array}\right] \begin{Bmatrix} \ddot{u} \\ \dot{u} \end{Bmatrix} + \left[\begin{array}{c|c} [M] & 0 \\ \hline 0 & [S] \end{array}\right] \begin{Bmatrix} \dot{u} \\ u \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4.53)$$

the Coriolis matrix can be nulled so that the equation turns into that of a free undamped vibration, thus

$$\left[\begin{array}{c|c} 0 & [-M] \\ \hline [M] & 0 \end{array}\right] \begin{Bmatrix} \ddot{u} \\ \dot{u} \end{Bmatrix} + \left[\begin{array}{c|c} [M] & 0 \\ \hline 0 & [S] \end{array}\right] \begin{Bmatrix} \dot{u} \\ u \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (6.2)$$

and after departitioning

$$[-M]\{\dot{u}\} + [M]\{\dot{u}\} = \{0\} \qquad (6.3)$$

$$[M]\{\ddot{u}\} + [S]\{u\} = \{0\} \qquad (6.4)$$

Obviously equation (6.3) gives no information at all since the two entities on the left hand side cancel each other, leaving equation (6.4) which is exactly the same as equation (3.39) or the equation used by the PAFEC system for free undamped vibration. The affect of equation (6.3) will be to double the size of the problem without any appreciable change on it. Structure 'b' of section 6.2. was used to make the necessary comparison.

The present solution procedure gives complex frequencies and vectors. However, the imaginary part of the frequencies and vectors is found to be at least twelve orders of magnitude smaller than the real part as should be expected if the Coriolis component of acceleration is omitted.

By comparing the real part of frequencies and vectors with their counterparts obtained by using the existing PAFEC solution packages it was found that they agreed up to the seventh significant digit.

Thus from the above tests it is concluded that the new solution method is accurate and can be used for further studies of vibration problems.

6.3.2. Testing the Effects of Rotation (Without Coriolis):

Structure 'b' of section 6.2. was used to study the correctness of the implimentation of the effect of rotation (i.e. initial stress field and centrifugal acceleration terms) without the Coriolis terms.

The fundamental natural frequency of the stationary structure can be calculated using the formula

$$\omega_{stationary} = \frac{3.5156}{L^2}\sqrt{\frac{EI}{\rho A}} \qquad (6.5)$$

to be 209.28 Hz. Using equation (2.39) which is repeated below

$$K = \frac{\omega_{rotating}}{\omega_{stationary}} = \sqrt{1+\frac{\Omega^2 R^4 \rho A}{EI}\left[\frac{1}{8}\left(\frac{L}{R}\right)^3 + \frac{1}{10.6}\left(\frac{L}{R}\right)^4 - \frac{\cos^2\varphi}{12.45}\left(\frac{L}{R}\right)^4\right]}$$

.....(2.39)

to obtain a multiplier which converts the stationary frequency into that of a rotating structure gives the following values taking $\Omega = 50$ rad/s,

$K_1 = 1.0243$ for stagger angle 0^o.

$K_2 = 1.0381$ for stagger angle 90^o.

hence the corresponding rotational frequencies would be

$\omega_1 = 214.36$ Hz and $\omega_2 = 217.24$ Hz.

Results from the finite element program give

		% error
$\omega_{stationary}$ = 209.70 Hz		0.20
$\omega_{rotating,stagger\ 0^o}$ = 219.89 Hz		2.36
$\omega_{rotating,stagger\ 90^o}$ = 225.50 Hz		3.51

An alternative method of comparison would be by comparing results taken from structure 'e' with the graph of figure 2-3b which is taken from a paper by Carnegie[24]. The following figures were obtained

	Carnegie's values	Finite element values	Percentage difference
Stationary	168 Hz	170 Hz	1.19
Rotating at 3000 rev/min	183 Hz	184 Hz	0.546

Although there is a small error involved in interpretting values off the Carnegie[24] graph, it would be expected that structure 'e' would give better accuracy than structure 'b' since it has a larger number of elements in its idealization.

Thus the coding for calculating the frequencies of vibration of rotating structures is accurate and reliable.

6.3.3. Testing the Implementation of the Coriolis Effect

Three different methods were used to evaluate various aspects of the implimentation of the Coriolis effects

a- It is possible to make a qualitative but not quantitative evaluation of the Coriolis effects. This took the form of checking whether the Coriolis effect produces a positive or negative change on the frequencies of vibration, and whether it would produce larger effects on some structures than others. As an example structure 'c' of the previous section has a Coriolis force acting to soften or destiffen the structure in a tangential direction in the plane of rotation, as a consequence the frequencies should drop in value. This effect can be seen in the results from the program.

$\Omega = 0$	$\Omega = 3000$ rev/min without Coriolis	$\Omega = 3000$ rev/min with Coriolis
$\omega = 1170.045$ Hz	$\omega = 1172.575$ Hz	$\omega = 1172.461$ Hz

b- It was pointed out that including the Coriolis component would increase the array sizes by a factor of 4. This means that problems with more than 60 degrees of freedom can never be solved on the Nottingham 1906 A computer without reverting to a method for the reduction of the degrees of freedom. Testing the reduction

procedure for the Coriolis matrix is an easy task and it involves comparison between the structures that are solved without reduction with corresponding ones that are solved by removing some of the degrees of freedom. Tables (6-1) shows a comparison of the results taken using structure 'f'. The small percentage error indicate the validity of the reduction procedure numerically. Discussion of the reduction procedure was made in chapter five.

c- Direct comparison is possible due to the very recent publication of reference 129. Low aspect ratio cantilevers such as structure 'h' were used and rotated at ultra high speeds to be able to see the Coriolis effect. Table (6-2) gives a comparison between the results discussed in reference 129 and the present results. It is noticed that there is a persistant difference of about 3-5% between the two sets. This is due to the better accuracy of the 20-noded three dimensional elements over the triangular plate elements used in reference 129. The Coriolis effect on the first bending frequency is quoted to be 3.45% in reference 129 while the present investigation calculates it to be 3.48% which is a very good agreement. However in comparing the Coriolis effect on the first torsional mode it is found that reference 129 mentions a very high value of 2.53% corresponding to a present value of 0.53%. The author of this text believes that the present value is more accurate due to the fact that at any section perpendicular to the axis of the beam torsional vibrations would give velocities in opposite directions, and hence the Coriolis forces cancel each other. The small discrepancy in the present results may well be due to rounding of errors or due to the automatic choice of master degrees of freedom. Furthermore, reference 129 suggests that the Coriolis effect is proportional to the speed of rotation while the other rotation effects are proportional to the square of the speed of rotation and hence the Coriolis effect never becomes predominant.

Table 6-1 Comparison of results obtained with and without reducing the number of degrees of freedom

Solution	Mode Number	Frequency Hz		% Error	Remarks
		All DOF*	12 DOF*		
Free vibration	1	1196.100	1198.491	0.199	structure 'f': straight
	2	1196.101	1199.049	0.246	cantilever of dimensions
Rotation without Coriolis calculation	1	1194.661	1197.488	0.236	20 x 5 x 5 cm, fixed on a
	2	1194.662	1198.033	0.282	rotating disc of radius 20 cm.
Rotation with Coriolis Calculation	1	1194.660	1196.893	0.186	Speed of rotation 3000 rev/min.
	2	1194.661	1198.627	0.331	

*DOF = Degress of freedom

Table (6-2) Coriolis effects, comparison between reference 129 and the present investigation. Structure 'h', tapered rotating cantilever of length 50mm, width 25mm, height at root 6mm, height at free end 3mm, stagger angle 45^{o}, and speed of rotation 52 000 rev/min.

Case No	Description	Mode	Reference 129	Present Investigation	% Difference	Coriolis Effect	
						Ref 129	Present
1	Stationary	Bending Torsion	2166.82 Hz * 7922.43 Hz	2250.36 Hz 7475.40 Hz	3.85 -5.46	Bending 3.45%	Bending 3.48%
2	Rotating, Coriolis Excluded	Bending Torsion	* 2309.56 Hz * 7962.20 Hz	2377.04 Hz 7531.88 Hz	2.92 -5.40	Torsion 2.53%	Torsion 0.53%
3	Rotating, Coriolis Included	Bending Torsion	* 2223.57 Hz * 7764.44 Hz	2297.11 Hz 7492.21 Hz	3.31 -3.51		

Notes: 1- Frequency values preceeded by an asterisk is interpreted from the graphs of reference 129 giving rise to a small error.

2- The percentage difference is given by $\frac{(\text{frequency from ref 129} - \text{present frquency})}{\text{frequency from ref 129}} * 100$

3- The Coriolis effect is given by $\frac{(\text{frequency from case 3} - \text{frequency from case 2})}{\text{frequency from case 3}} * 100$

This acceptable logic contradicts the results displayed in reference 129 on two different levels. First on a proportionality level, graphs given in the paper indicates that as the speed is increased the Coriolis effect becomes larger than other rotational effects wich does not agree with what was previously mentioned 'the Coriolis effect never becomes predominant'. Second, references (17 and 63) suggest that on a direct level, at any particular speed the Coriolis effect is never larger than other rotational effects which again contradicts the results of reference 129 suggesting that the latter may be wrong. The present results agree qualitatively with other investigations.

6.4. Particular Effects:

6.4.1. Effects of Rotation Without Including Coriolis Calculations

The initial stress field created by centrifugal rotation is the largest contributor to the changes in the frequencies of vibration. Beams fixed radially on rotating discs would normally increase their frequency as the speed is increased due to the stiffening effect resulting from the centrifugal force field. The centripetal accelerations have an opposite effect and they tend to soften the structure and hence reduce the frequencies by an amount which is normally smaller than the increase brought by the initial stresses. Table (6-3) shows a comparative study for the changes on the frequencies of structure 'd' due to the effects of initial stresses alone, and initial stresses plus centripetal accelerations. A plot of the square of the frequency versus the square of the speed of rotation is shown in figure (6-8).

The effect of speed does not always act to increase the frequency of vibration. In some structures the opposite is

Table 6-3 Effect of rotation on the frequencies of vibration, first mode only.

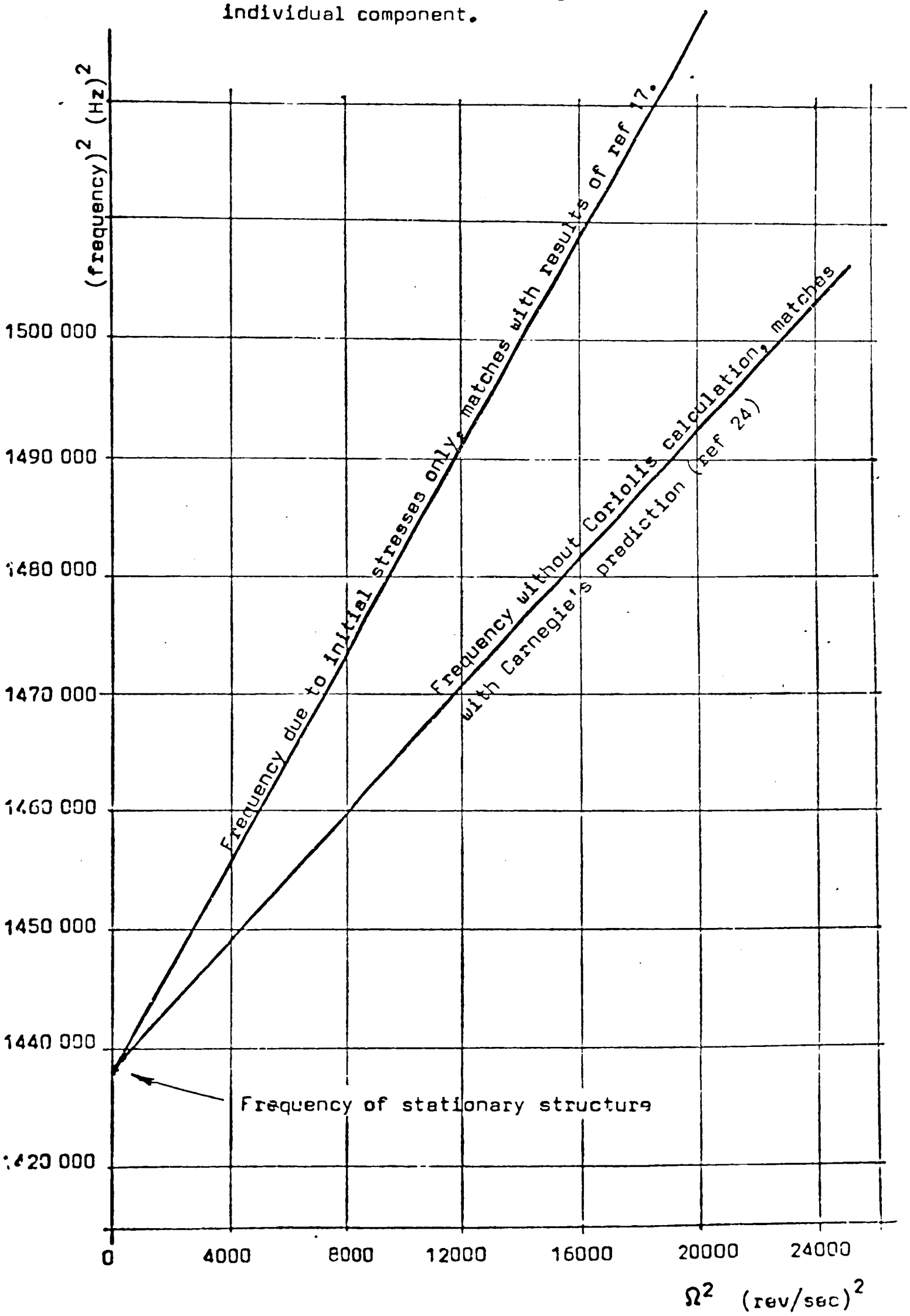
Speed rev/sec	Frequency Hz		% Change from stationary	
	1st effect	2nd effect	1st effect	2nd effect
0	1199.152	1199.152	0.00	0.00
50	1203.721	1201.961	0.38	0.23
100	1217.324	1210.348	1.51	0.93
150	1239.663	1224.201	3.37	2.08
200	1270.279	1243.335	5.93	3.68

1st effect is due to the presence of initial stresses only.

2nd effect is due to the presence of initial stresses plus the centripetal accelerations.

Structure 'd': Cantilever beam of dimensions 20 x 5 x 5 cm, fixed on a rotating disc of radius 20 cm.

Figure 6-8 Frequency-speed plot showing the effect of each individual component.



expected. Taking structure 'f' for example, the centrifugal action will help the beam to flex more and hence the frequencies of vibration are reduced. The fundamental frequency of vibration was seen to reduce from 1198.491 Hz when stationary to 1197.488 Hz when rotating at 3000 rev/min.

Neither the initial stresses nor the centripetal accelerations affect the mode shapes of simple beams such as structure 'd'. Slight changes in the mode shapes may occur due to the choice of automatic masters which is affected as changes are introduced on the stiffness matrix (see section 3.9.).

Changes taking place on the mode shapes of structures with complicated geometry cannot be easily predicted and every individual case may be studied on its own. Taking for example the pretwisted compressor blade shown on figure (6-6). The blade is idealised by 16-noded elements which are more adaptable than the 20-noded elements for plate applications. The frequency of vibration of the first mode was seen to increase from 31.673 Hz to 42.965 Hz as the speed of rotation was increased from 0 to 3000 rev/min. The frequencies of higher modes are shown in table (6-4). Rotation affects the mode shapes as can be seen on figures (6-9) and (6-10) which show a comparison for the two lowest modes.

6.4.2. Effect of the Coriolis Component:

By studying table 6-5 it can be clearly seen that the Coriolis acceleration component has a very small effect on the fundamental frequency of vibration of structure 'd'. The plot shown on figure (6-11) shows that the Coriolis effect is small at lower speeds of rotation but gets bigger as the speed is increased. This non-linear behaviour is expected since the

Table 6-4 Effect of rotation on the frequencies of vibration of a pretwisted compressor blade. (Speed 3000 rev/min)

Mode No.	Frequency (Hz) and percentage change from stationary				
	Stationary	Rotating excluding Coriolis effects	%Change	Rotating including Coriolis effects	%Change
1	31.67	42.96	35.64	42.08	32.88
2	91.11	140.12	53.79	140.66	54.38
3	219.97	305.31	38.85	307.04	39.69
4	240.58	317.60	32.01	316.57	31.58
5	435.67	491.22	12.72	491.27	12.73
6	589.85	640.63	8.60	636.94	7.98
7	654.06	717.81	9.74	717.75	9.73
8	871.18	908.54	4.28	905.54	3.94

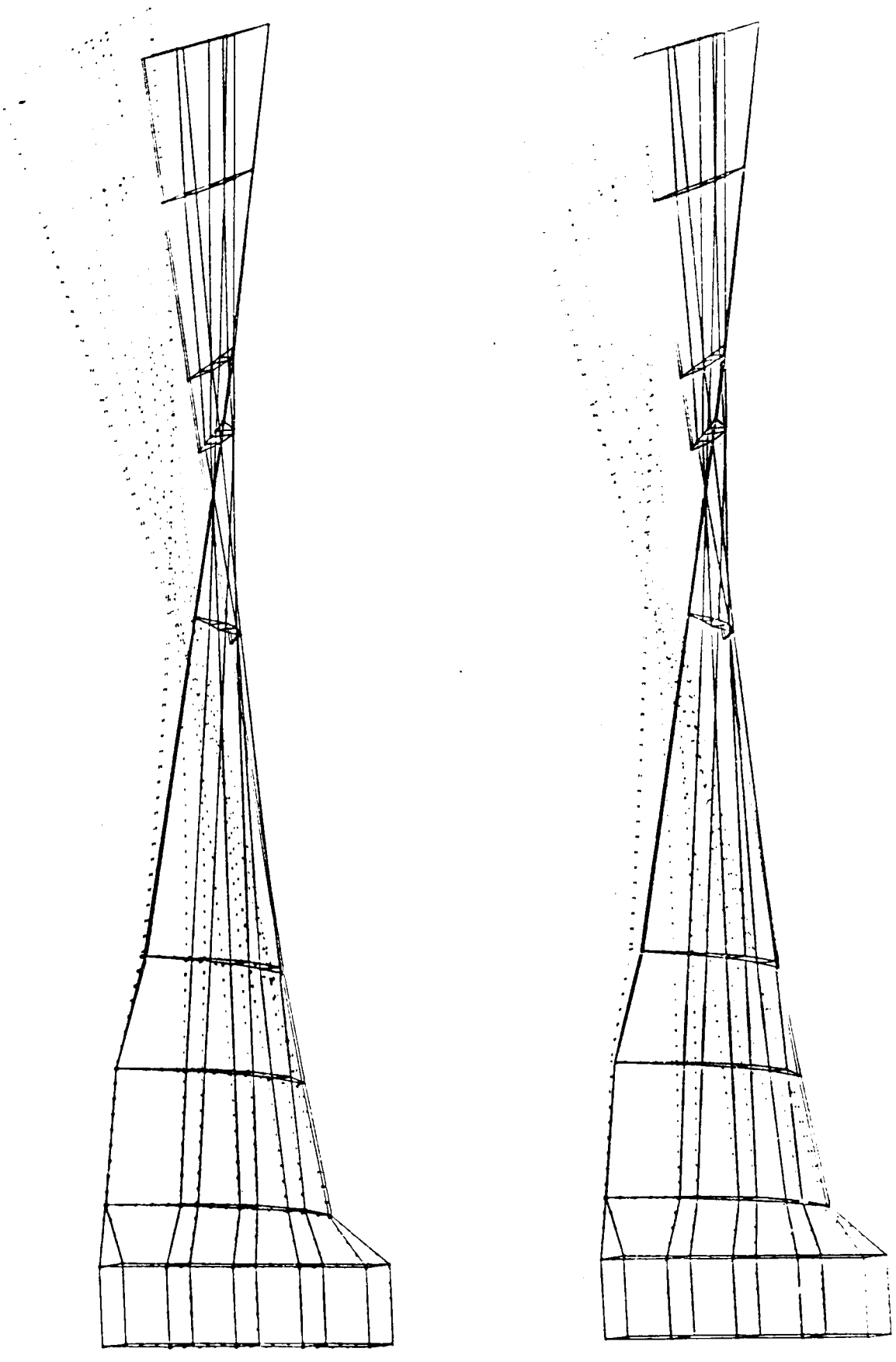


Figure (6-9) Effect of rotation (without Coriolis) on the first mode shape of a pretwisted compressor blade when rotated at 3000 rev/min.

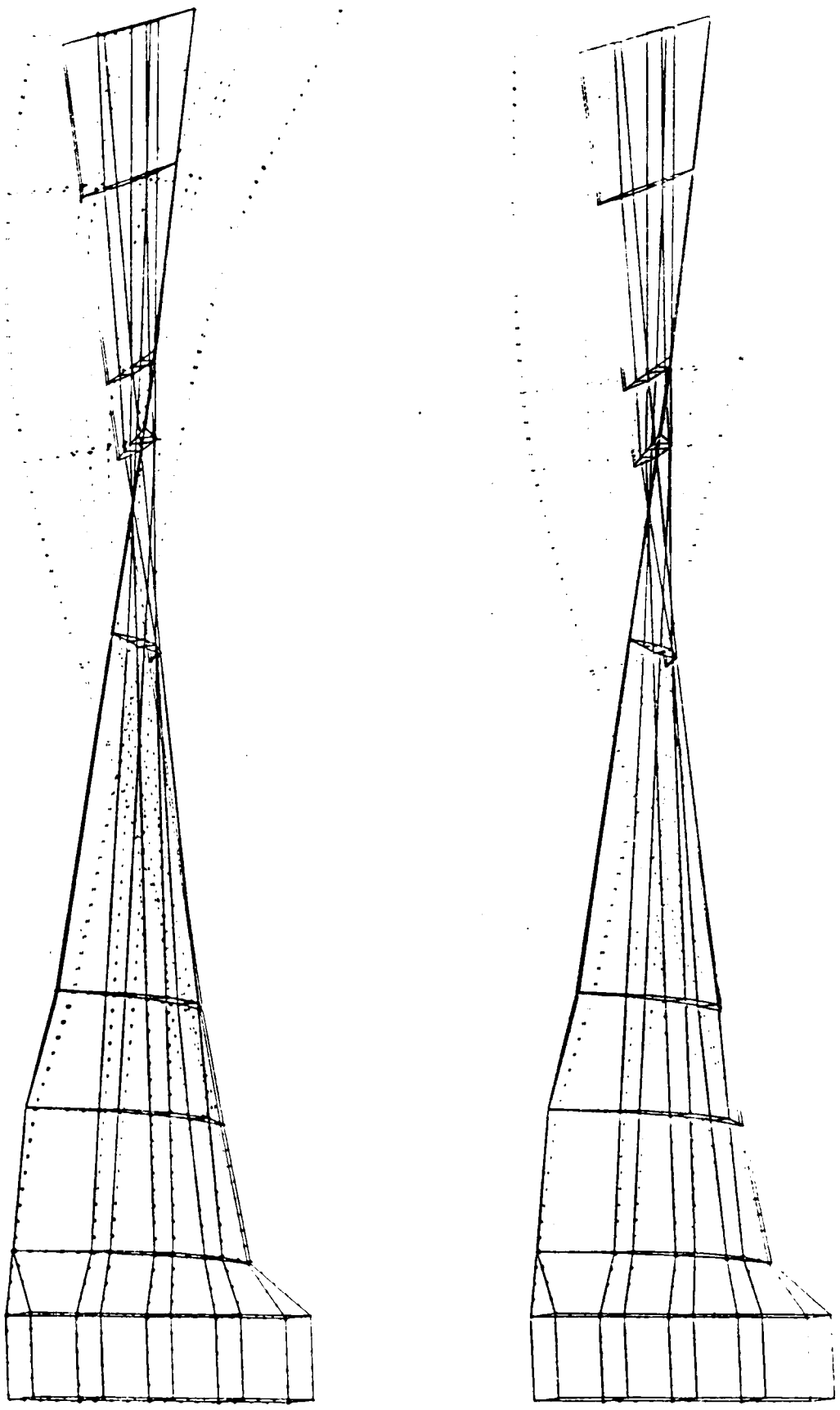


Figure (6-10) Effect of rotation (without Coriolis) on the second mode shape of a pretwisted compressor blade when rotated at 3000 rev/min.

Table 6-5 Effect of rotation on the frequencies of vibration, first mode only.

Speed rev/sec	Frequency Hz			percentage change from stationary		
	1st effect	2nd effect	3rd effect	1st effect	2nd effect	3rd effect
0	1199.152	1199.152	1199.152	0.0	0.0	0.0
50	1203.721	1201.961	1201.957	0.38	0.23	0.23
100	1217.324	1210.348	1209.129	1.51	0.93	0.83
150	1239.663	1224.201	1218.006	3.37	2.08	1.57
200	1270.279	1243.335	1230.302	5.93	3.68	2.59

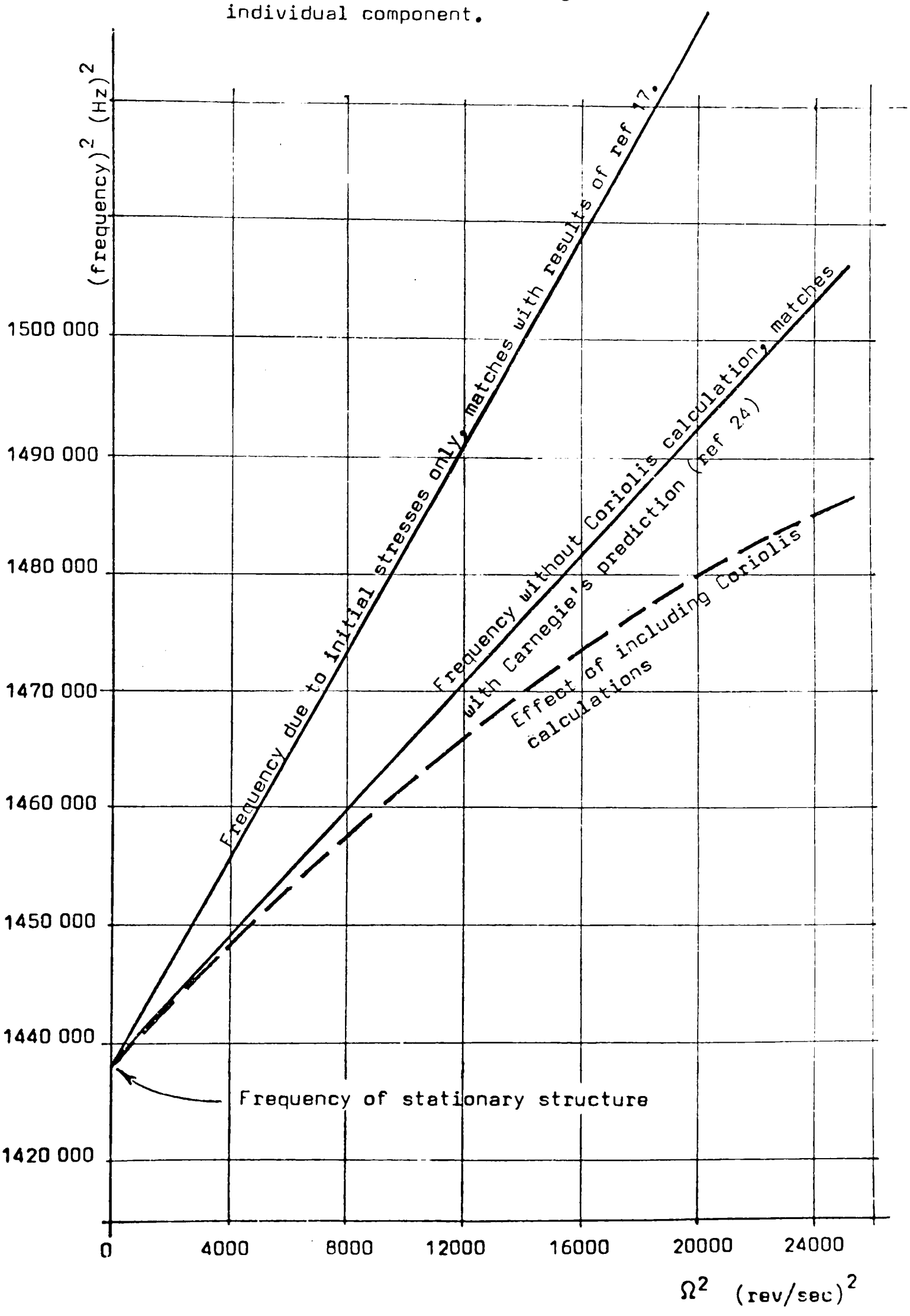
1st effect is due to the presence of initial stresses only.

2nd effect is due to the presence of initial stresses plus the centripetal accelerations.

3rd effect is due to the presence of Coriolis accelerations in addition to the previous effects.

Structure 'd': Cantilever beam of dimensions 20 x 5 x 5 cm, fixed on a rotating disc of radius 20 cm.

Figure 6-11 Frequency-speed plot showing the effect of each individual component.



Coriolis acceleration component is proportional to the angular velocity while the initial stresses and the centripetal accelerations are functions of the square of the angular velocity.

It was pointed out earlier that the Coriolis acceleration component produces a complex eigenvector which includes the time and space domains of the degrees of freedom on the structure. The motion of each degree of freedom is described by three complex coordinate axes corresponding to x,y, and z. When expressed by their equivalent trigonometric relations, each complex coordinate gives information about the amplitude and phase angle of the motion in that particular direction, thus for the x coordinate we have;

$$a_x + \underline{i}\, b_x = a_x \cos \omega_r t + b_x \sin \omega_r t \qquad (6.6)$$

$$= (a_x^2 + b_x^2)^{\frac{1}{2}} \cos(\omega_r t + \psi_x) \qquad (6.7)$$

where a_x and b_x are the real and imaginary parts of the x coordinate.

ω_r is the real component of the complex frequency.

ψ_x is the phase angle = $\tan^{-1} \frac{b_x}{a_x}$

Thus by plotting all the displacements of one cycle at different positions in time the vibrational mode shape can be obtained.

Figure (6-12) shows the first bending mode of an edge line on structure 'f' both including and excluding the Coriolis effects. Other edge lines give the same pattern. Due to large phase angles between the coordinate axes, the first bending mode is seen to change with Coriolis effects so that instead of vibrating in one plane as in the stationary case, it is vibrating in an elliptical motion. This behaviour is expected since while rotating and

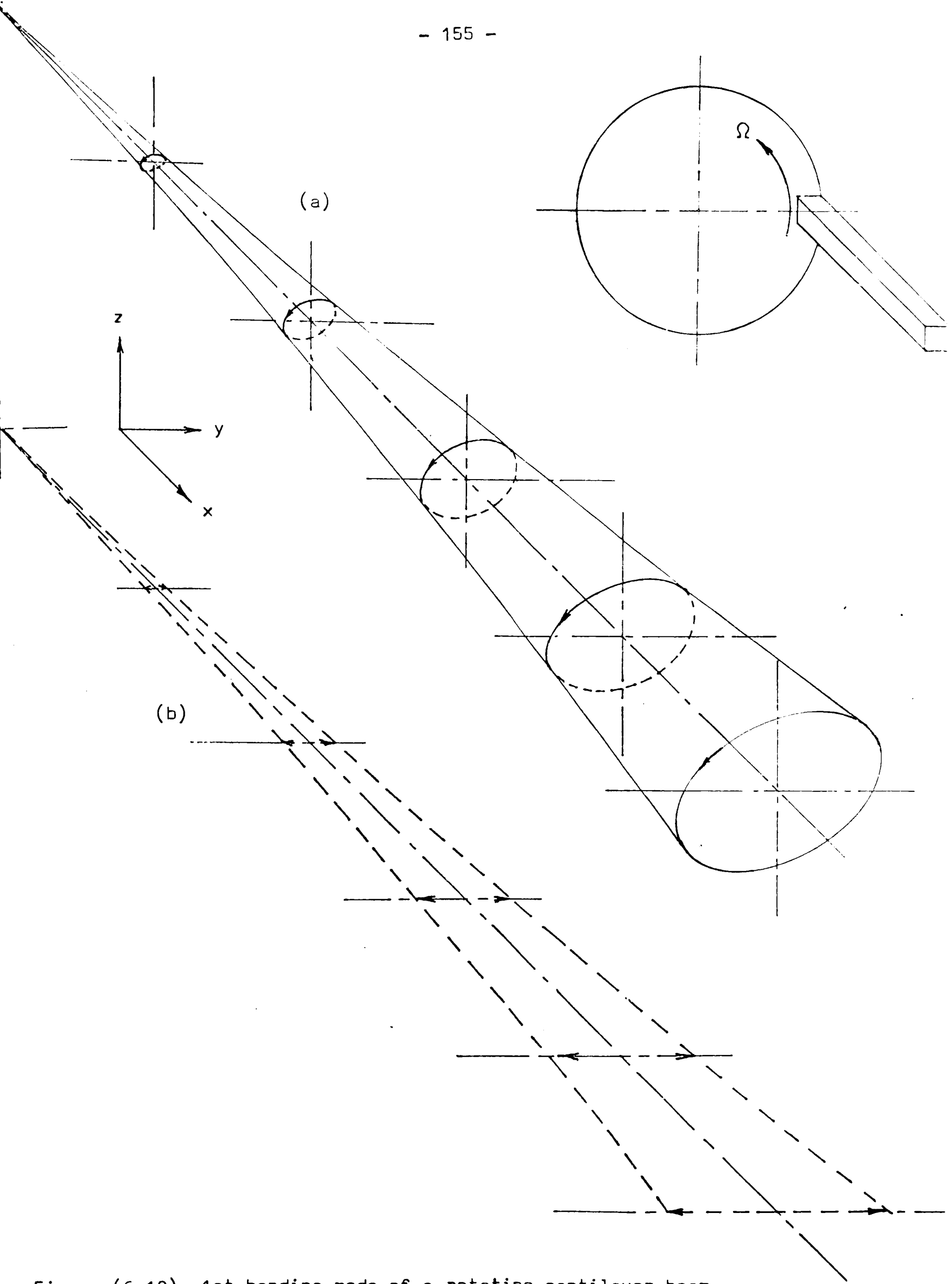


Figure (6-12) 1st bending mode of a rotating cantilever beam structure 'f'. a- including Coriolis effects, b- excluding Coriolis effects. Axial displacements are not plotted.

vibrating in one plane, the Coriolis forces will cause motion to occur in a perpendicular plane and hence the complex mode shape of figure (6-12).

The second bending mode (figure 6-13) is following the same pattern in that the Coriolis terms are bringing lateral movements. However, it is important to notice that the velocity terms ($\dot{r}$) have opposite directions at the exaggerated sections of figure (6-13). Accordingly the Coriolis forces would have opposite directions and hence the total Coriolis effect on the mode shape is not as pronounced as its effect on the first bending mode. This is generally true for higher mode shapes. Hence ignoring the effect of Coriolis acceleration on the higher mode shapes is a reasonable engineering short cut for which earlier investigators chose to select. It is felt however that ignoring the Coriolis effect on the first mode is not justified due to the detectable change introduced despite the low ratio of speed to natural frequency that was used for this particular problem. The ratio of speed to natural frequency for the second mode is much lower than the first mode yet there is a very slight change on the mode shape indicating that at higher speeds the Coriolis effect on higher modes must be taken into account.

Comparing plots of mode shapes resulting from the use of more accurate elements with corresponding plots obtained by using less accurate elements does not show a discernable difference. This important issue is discussed in the next chapter.

It becomes evident from the above that the effect of the Coriolis terms on the mode shapes is more pronounced than their effects on the natural frequencies.

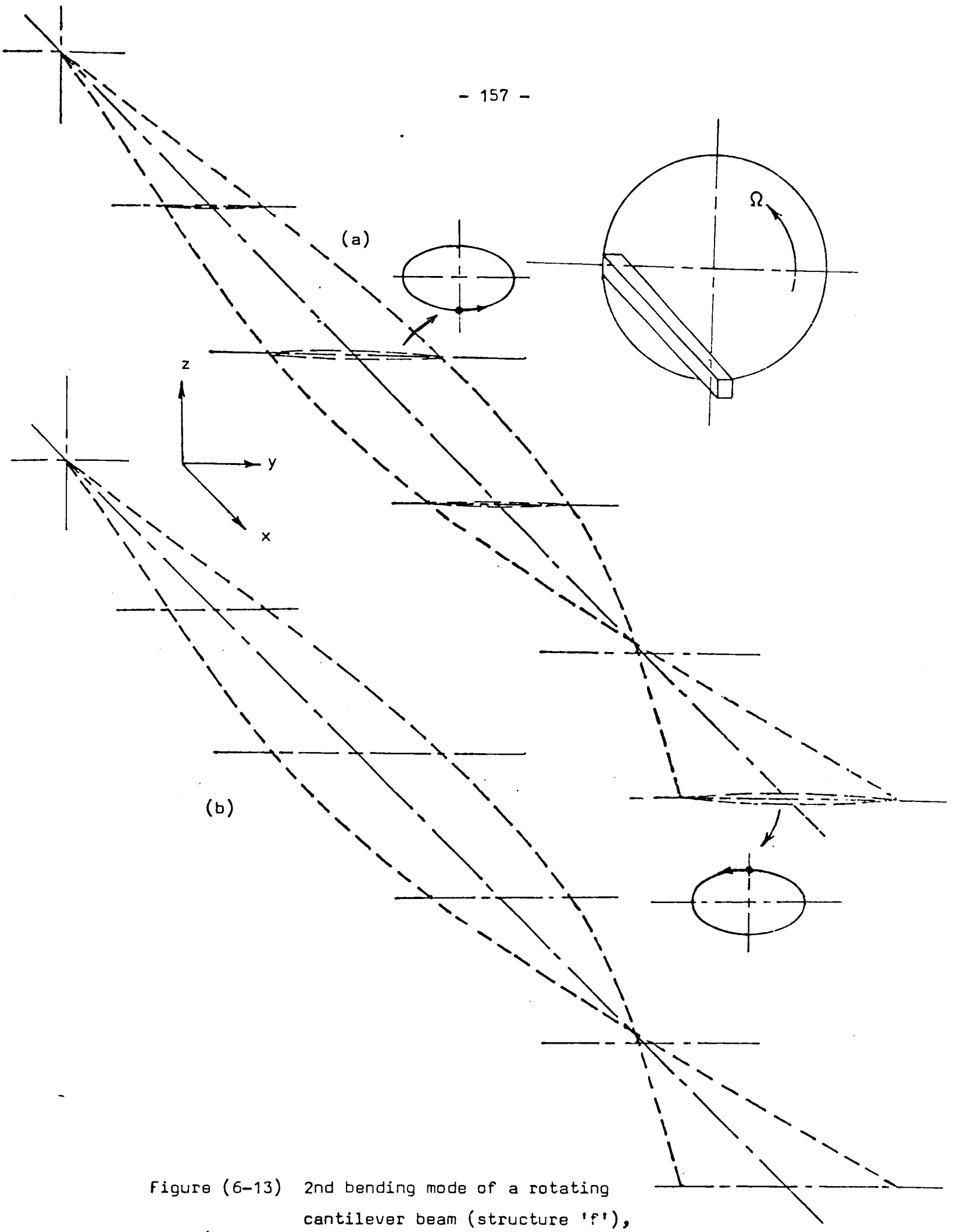


Figure (6-13) 2nd bending mode of a rotating cantilever beam (structure 'f'), axial displacements not plotted. a- Including Coriolis effects, notice the dirctions of motion on exaggerated sections. b- Excluding Coriolis effects.

CHAPTER SEVEN

DISCUSSION

7.1. Introduction:

Choosing the finite element method for this study was based upon a comparative study given in appendix 'F'. The finite element method was compared with other methods such as the finite difference method and an experimental method. It was found that the finite element method was more accurate and easier to apply. The same is true for a large number of engineering problems where complex geometry is involved.

The main reason behind choosing the PAFEC system was its availability. A number of problems were studied in chapter six using the PAFEC system. The author's impression is that the system is very versatile and has a number of useful facilities in addition to obtaining satisfactory results once a suitable idealization of the problem is made. The author regrets that although documentation for data input is well explained, the need exists for a Users' Experience Document. Also information for program developers is not adequate.

Choosing the family of three dimensional isoparametric elements for the study of vibrational problems was made after studying the theory and programming methods behind this and other families of elements.

7.2. Vibration of Non-Rotating Structures:

The elements that were used in all the problem idealizations in this study were the 8 - noded, 16 - noded, and the 20 - noded three dimensional isoparametric elements. The 8 - noded element was chosen

because it offered a larger computer time and core saving compared with the 20-noded element. It was found that the penalties of using this element were its poorer accuracy and a limited utility for modelling thin blades.

Experience with various problems indicates that the idealization of the structure plays a major role in the accuracy of the results. However it was found that with the best possible combination , the frequencies of the 20-noded elements were far more accurate than the frequencies of the 8-noded elements.(which are based on a simpler polynomial shape function than the 20-noded elements). The 8-noded elements are normally stiffer and idealizing any structure with one such element in any one direction results in a much higher stiffness. This produces smaller displacements for static problems and higher natural frequencies for dynamic problems. The author finds that if one 20-noded element is substituted for eight 8-noded elements (to give the same problem size) improved frequencies are obtained. This means that the 8-noded elements are not desirable for blade vibration problem idealization.

The use of reduced integration techniques will not improve the efficiency of the 8-noded element since they are modelled by a first order polynomial for which the order of the Gauss integration does not affect the accuracy of the integration. As for the 20-noded elements, their response to changing the order of Gauss integration and constraints is more pronounced than the alternative 8-noded elements and improvements of 25% were noticed on some of the results. The difficulty encountered with the use of the 20-noded elements is that they occupy a large area of computer storage which enforces a limit on the total number of elements that can be used. Comparisons of the results obtained by these elements with other

methods have been carried out by other investigators (references 87 and 58). The results of the present investigation do not differ from that of reference 58 since both use the same program.

The 8-noded elements can be used for the prediction of the mode shapes of vibration since these were not found to be greatly affected by the accuracy of the natural frequency results.

7.3. Vibration of Rotating Structures :

In order to study the effect of rotation on the vibration characteristics of structures, two cases have bean used;

a- Straight, rectangular cross section cantilever beams.

b- Pretwisted, aerofoil cross section compressor fan blade.

The beams had different dimensions and constraints and were studied with a view to obtaining better information about the behaviour of the modified elements with changes in geometry. At the early stages in the preparation of the various acceleration matrices it was found very useful to use a structure of two or even one element to make certain of the correctness of the modifications introduced. All the matrices that were prepared were checked by alternative programs at least once (the geometric stiffness matrix was checked by three different programs). Although a bit tedious, this method of testing was found to be very convenient.

Exhaustive testing was carried on the pretwisted blade to

make its results compare favourably with experimental results. However a difference of (30%) was still observed and the author of this text attributes the difference to the idealization of the blade root in the finite element model. A recent investigation by Nagamatsu et al[127] was faced with the same problem. The investigators suggested that the difference between the experimental and numerical frequency values was due to the fact that the actual roots of the blade did not satisfy complete clamping conditions that were used in the numerical study .

The method used to assess the validity of the results described in the previous chapter was to split the assessment into two main parts, the first was intended to establish the correctness of the computer programs by feeding known data to give expected results. In the second part, real structures were idealized and fed to the PAFEC program and the results were compared with other published material. In both part excellent conformity was obtained.

In comparing the results of the present work with the results of other investigations it was found that when the Coriolis acceleration component is not included the change of the frequency of vibration with the angular speed follows the Southwell equation (2.78) which agrees completely with other studies[21,24,36]. This qualitative result is backed up by a quantitative comparison with the results obtained by Carnegie[24] which showed a very small percentage error of 0.546 . Rotation may change the sequence of the mode shapes since it increases some of the frequencies and reduces others but it does not change the first three mode shapes of straight cantilever beams.

When the Coriolis effects are included, the frequencies

of vibration were not found to change appreciably at moderate angular speeds. The results were found in agreement with the work of references(63 and 129). Coriolis acceleration components were found to introduce a complex eigenvector with differing phase angles for different coordinate directions and hence their effect cannot be ignored.

It was mentioned that implimenting the Coriolis calculations requires large computer time and storage. This fact may discourage carrying on these calculations due to the small changes brought by the Coriolis terms. The author suggests the following economical procedure to overcome this difficulty. Since the frequencies are not affected much by the Coriolis components,then their calculation can be made accurately by using 20-noded elements without introducing the coriolis terms. In a second run, the mode shapes can be obtained using a less accurate element such as the 8-noded element which consumes much less computer time and core storage even when the Coriolis terms are included.

7.4. Program Considerations:

Solution procedures have been investigated, such as the Sturm sequence method[42], the method of Newmark[30], and a modified QL method[121]. The modified QL solution procedure has been found to give satisfactory results in comparison to other existing procedures [88]. Although the present work had criticised the method of solution that calculates one eigenvalue at a time, the sturm sequence program suggested by Gupta[42] was attempted but unfortunately no useful output was obtained. It is believed that there are some errors in the program such as the repeated statements in the main segment at the lines denoted by SPINO193 and SPINO194. Nagamatsu et

al[127] pointed out that the above program could be used but unfortunately they did not try to use it, instead they used the surface iteration method.

One of the largest time consuming program segments used in this investigation was the square root method for calculating the Coriolis matrix. The calculation of the square root of an array is an iterative process, the number of iterations of which is seen to increase with the size of the matrix. Thus for the 60 x 60 arrays associated with the 20-noded 3 D elements, the number of iterations averaged 35. In comparison to the normal calculation of the Coriolis matrix (see table 7-1), it is found the the square root method takes as much as 4 times more time for the 8-noded elements, and about 9 times as much for the 20-noded elements. Never the less the author finds that this method has its own advantage that the programmer can prepare the extra matrices independently of the method used to prepare the mass and other matrices.

Feeding the initial stresses causes some delay since a reading peripheral is allocated to read them from magnetic tape in addition to the calculation of the necessary changes on the stiffness matrix.

Other clocking times show that the time difference rises by a factor of 50 between the application of the solution routines suggested in this text and the normal PAFEC routines. However one should not forget two facts;

a- The PAFEC routines are incapable of solving the full skew symmetric eigenvalue equation in which the Coriolis term occurs.

Table 7-1 Clocking times for various runs on a cantilever of five 3-D elements.

Phase No	No of nodes	CPU Time* (Milliseconds)			
		1	2	3	4
6	8	0.03	0.54	0.14	----
6	20	0.32	13.33	1.47	2.04
7	8	0.46	26.23	26.06	26.33
7	20	2.55	27.38	27.49	29.16

*The numbers 1, 2, 3, and 4 have the following meanings

a- In phase 6:

1- Free vibration of non-rotating structure, no calculation of Coriolis or other matricies.

2- Vibration of rotating structures, calculation of Coriolis matrix by square root method, initial stresses not considered.

3- Same as 2 but Coriolis matrix calculated by normal method.

4- Same as 3 in addition to calculating the initial stress matrix.

b- In phase 7:

1- Vibration of non-rotating structure, using existing PAFEC routines.

2, 3, and 4- Vibration of rotating structure, using own solution routines.

b- The present routines run into the trouble of more processing to obtain more precise results by rounding of errors in addition to the use of double precision variables which makes the time used much longer than the time consumed by the corresponding PAFEC subroutines. Furthermore, preparation of the eigen equation is longer and more involved on the present solution routines.

7.5. Suggestions for Further Work:

It is unfortunate that most of the time of this project was spent on ensuring a trouble free program that would suit the existing PAFEC 75 program. The result was that only a few applications could be studied. It would be hoped that the method could be applied to more cases in the future. Such investigation would add to the knowledge gained in the course of this study concerning the feasibility of the method.

During the process of writing the computer coding, great care was taken to avoid unnecessry usage of computer time and core, however in debugging the program, less attention is paid to these principles and nence deviations from them may occur. For example, a large amount of core saving could be achieved in the preparation of the eigenvalue equation by the use of partiticning techniques.

The QZ algorithm[128] is a computer program to solve a general eigenvalue equation and could be used for the present investigation. By the time the author became aware of the presence of such an algorithm he had already chosen his solution programs. The shortage of time did not allow investigating the advantages and disadvantages of using it. Further work can be made to test this algorithm.

CHAPTER EIGHT

CONCLUSION

This study was directed towards obtaining the effects of rotation on the vibration characteristics of structures. Numerical prediction was made by modifying three dimensional isoparametric finite elements. Comparison with existing work has been made. The following conclusions have been drawn.

1- Literature search: The theoretical analyses made by Carnegie were found to be the most comprehensive to date. Numerical adaptations by Mota Soares were found very suitable for finite element application.

   Experimentally, a modal analysis procedure can be used for vibration studies. It was used in this investigation for the vibration analysis of the compressor fan blade to provide a comparison with the results of the numerical procedure.

2- The finite element method: Three dimensional isoparametric elements that are based on displacement assumptions were chosen and described for the purpose of modifying them for the vibration studies.

3- Fundamental analysis: Kinematics of rotating structures have been developed and the equation of motion was derived. A study of the solution methods was made and a modified QL algorithm was suggested.

4- Results: It was found that the natural frequencies changed with the speed of rotation according to the Southwell equation (2.78). Comparison with literature shows that in

the absence of the Coriolis acceleration the results conform with equation (2.39) with a minor difference of 1% only. The initial stresses were found to be the largest contributor to the changes on the natural frequencies while the centripetal accelerations were contributing a smaller fraction. When Coriolis effects are excluded, the lower mode shapes of rotating straight cantilever beams are not changed.

It was possible to calculate the effects of the Coriolis acceleration component on the natural frequencies. It was found that its effect was to decrease the frequency by typically 2% at high speeds of rotation. The effect increases with speed but never becomes predominant over the changes by the initial stresses. This effect is considered to be sufficiently small for the Coriolis component to be neglected in the calculation of natural frequencies. Comparison with the very recent work of reference (129) gives very close agreement. The effect of Coriolis components on the modes shapes can not be ignored and in some cases it becomes dominant, especially at higher speeds.

An economic procedure for incorporating the Coriolis effect was suggested. It consists of calculating the natural frequencies accurately using 20-noded elements without the including Coriolis effects, then the mode shapes can be obtained with good accuracy using 8-noded elements.

5- General: Carrying out this study has led to other important side work on three different problems;

a- A time consuming but useful method for obtaining the

Coriolis and other acceleration matrices from the stiffness and mass matrices independantly from the procedure used for the preparation of these matrices.

b- The introduction of an offset plate element which proved to be accurate and reliable for various design applications.

c- The development of a standardisation program for large FORTRAN library programs, which can be used as a programming aid.

APPENDIX A

A.1. RAYLEIGH'S METHOD

A.1.1. Simple Beam Theory

It was stated earlier in this text that the fundamental frequency of vibration can be obtained with good accuracy by assuming a suitable elastic deflection curve. The proof of this statement is given in (Ref 77) and will not be repeated in here, however an attempt will be made to obtain the frequency of vibration for beams.

Taking a small portion of a deflected beam as the one shown in figure A - 1.

From the geometry of the figure:

$$d\vartheta = \frac{dx}{dR} \tag{A.1}$$

where $d\vartheta$ is a very small angle, and ϑ is the slope of the elastic curve.

Assuming harmonic motion, the kinetic energy of the vibrating beam becomes,

$$KE_{max} = \tfrac{1}{2}\int y^2\, dm = \tfrac{1}{2}\int y^2\, dm \tag{A.2}$$

The work done on the beam in the form of elastic deformation represents the potential energy of the beam. Thus if ϑ is the slope of the beam after deformation:

$$\vartheta = \frac{dy}{dx} \tag{A.3}$$

the work done is

$$WD \equiv PE = \tfrac{1}{2}\int M d\vartheta \tag{A.4}$$

where M is the bending moment.

Recalling that the radius R is related to the bending moment by the following equation

$$\frac{1}{R} = \frac{M}{EI} \tag{A.5}$$

which is called the flexure equation for beams.

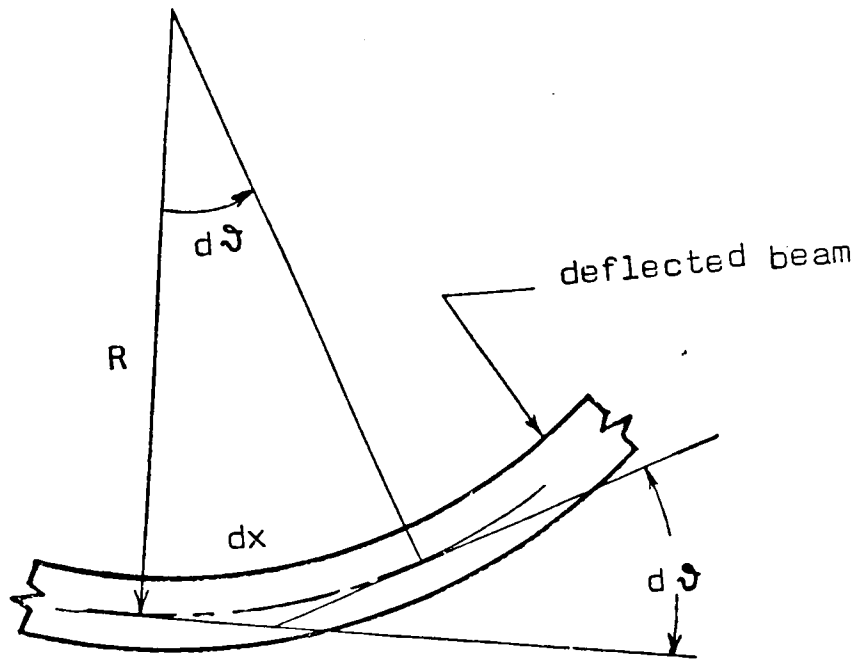


FIGURE A-1 Geometrical relations for a deflected beam

Upon substituting equations (A.1), (A.3), and (A.5) into equation (A.4) the expression for the work done becomes

$$WD = \tfrac{1}{2}\int \frac{M^2}{EI}\, dx = \tfrac{1}{2}\int EI\left(\frac{d^2y}{dx^2}\right) dx \qquad (A.6)$$

Equating the kinetic and the potential energies result the fudamental frequency of the beam

$$\omega^2 = \frac{\int EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int y^2\, dm} \qquad (A.7)$$

A.2. Extensions for Lumped Mass Systems :

The above mentioned method can be applied for lumped - mass systems with some extensions as follows:

For a system with n - degrees of freedom that is moving in its fundamental frequency mode, the shape of the fundamental mode could be estimated (say) through experience, and the motion is approximated by

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_n \end{Bmatrix} X \qquad (A.8)$$

in which $\{\gamma\}$ represents the estimated shape and X is a generalized coordinate.

By definition[3], the kinetic energy of a lumped mass system is a positive definate quadratic form in the generalised velocities $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$. It becomes possible to write an expression for the kinetic energy as:

$$KE = \tfrac{1}{2} \left\{\dot{x}_i\right\}^T \left[m_{ij}\right] \left\{\dot{x}_i\right\} \quad , i,j = 1, n \qquad (A.9)$$

The elastic potential energy or strain energy stored in a mechanical lumped mass system is equal to the work done by the

applied forces in deforming the system from the position of static equilibrium hence

$$PE \equiv SE = \tfrac{1}{2} \left\{ x_i \right\}^T \left\{ F_i \right\} , \quad i = 1,n \tag{A.10}$$

or

$$PE = \tfrac{1}{2} \left\{ F_i \right\}^T \left\{ x_i \right\} , \quad i = 1,n$$

Assuming a linear force - displacement relationship we can write

$$\left\{ F_i \right\} = \left[S_{ij} \right] \left\{ x_i \right\}, \quad i,j = 1,n \tag{A.11}$$

where, $\left[S_{ij} \right]$ is a stiffness matrix.

If we rename $\left[S_{ij} \right]^{-1}$ by the flexibility matrix $\left[K_{ij} \right]$ we can write equation (A.11) as;

$$\left\{ x_i \right\} = \left[k_{ij} \right] \left\{ F_i \right\} \qquad i,j = 1,n \tag{A.12}$$

Combining either equations (A.11) or (A.12) with equation (A.10) will result in a quadratic form in the displacements or forces respectively

$$PE = \tfrac{1}{2} \left\{ x_i \right\}^T \left[S_{ij} \right] \left\{ x_i \right\}, \quad i,j = 1,n \tag{A.13}$$

$$PE = \tfrac{1}{2} \left\{ F_i \right\}^T \left[k_{ij} \right] \left\{ F_i \right\}, \quad i,j = 1,n$$

Substituting the transformation equation (A.8) into the expressions for the kinetic energy and the potential energy to give:

$$\begin{aligned} KE &= \tfrac{1}{2} \left\{ \gamma \right\}^T \left[m \right] \left\{ \gamma \right\} \dot{X}^2 \\ PE &= \tfrac{1}{2} \left\{ \gamma \right\}^T \left[S \right] \left\{ \gamma \right\} X^2 \end{aligned} \tag{A.14}$$

from which we can define a general mass and stiffness associated with X as

$$\begin{aligned} \left[M \right] &= \left\{ \gamma \right\}^T \left[m \right] \left\{ \gamma \right\} \\ \left[S \right] &= \left\{ \gamma \right\}^T \left[S \right] \left\{ \gamma \right\} \end{aligned} \tag{A.15}$$

correspondingly, the general inertia and elastic forces become

$$F_i = - \frac{d}{dt} \frac{\partial T}{\partial \dot{X}} = - \left[M \right] \ddot{X} \tag{A.16}$$

$$F_e = - \frac{\partial PE}{\partial X} = - \left[k \right] X \tag{A.17}$$

In a similar way a generalized expression for the external force

acting on the system can be obtained from the work done by the external forces acting in a virtual displacement x .

$$\delta W D = \left\{ \delta x_i \right\}^T \left\{ F_{ex} \right\} \qquad (A.18)$$

or

$$\delta W D = \delta X \left\{ \gamma_i \right\}^T \left\{ F_{ex} \right\} \quad , i = 1,n$$

than the generalized force will be

$$F_{ex} = \left\{ \gamma \right\}^T \left\{ F_{ex} \right\} \qquad (A.19)$$

Finally the equation of motion could be written as

$$\Sigma F = - \left[M \right] \ddot{X} - \left[S \right] X + F_{ex} = 0 \qquad (A.20)$$

from which an approximation for the fundamental eigen value in free vibration could be obtained

$$\omega^2 = \frac{\left[S \right]}{\left[M \right]} = \frac{\left\{ \gamma_i^T \right\} \left[S_{ij} \right] \left\{ \gamma_i \right\}}{\left\{ \gamma_i^T \right\} \left[M_{ij} \right] \left\{ \gamma_i \right\}} \quad , i \, , j = 1,\ldots,n \qquad (A.21)$$

As was stated earlier in this text (sec 2.1), a close estimate of the eigen value will be obtained with a fairly good guess of the mode shape, furthermore the approximate value of the fundamental eigen value will always be larger than the exact value. (109) This last statement could be assesed by the fact that normally a beam system consists of an infinite number of degrees of freedom and reducing it to a system with a finite number of degrees of freedom will imply more constraints on the system which will only make it stiffer and hence increase the value of its fundamental eigen value.

APPENDIX B

DEFFINITION OF THE CORIOLIS ACCELERATION COMPONENT

Consider a rod AB, rotating counter clockwise about the point 'A' with an angular velocity Ω. Assume a block 'C', sliding radially outwards along AB at a constant speed 'V' relative to the rod (see figure B-1).

After a time interval Δt it will be found that the block had moved from C to $\acute{C}$ and point B had moved to $\acute{B}$ thus the rod makes an angle $\Delta\vartheta$ with its original position. As can be seen from the figure, the absolute velocity of the block at its original position is composed of two components, V along the rod, and r perpendicular to the rod, while at the new position the velocity components will be,

$$V_R = V \cos(\Delta\vartheta) - (r + \Delta r) \cdot \Omega \sin(\Delta\vartheta) \qquad (B.1)$$

radially or along the rod, and

$$V_T = V \sin(\Delta\vartheta) + (r + \Delta r) \cdot \Omega \cos(\Delta\vartheta) \qquad (B.2)$$

perpendicular to the new position of the rod.

For small angles $\Delta\vartheta$, it is possible to write,

$$\sin(\Delta\vartheta) = \Delta\vartheta = \Omega \Delta t,$$
$$\cos(\Delta\vartheta) = 1 \qquad (B.3)$$
$$\Delta r = V \Delta t$$

Substituting relations (B.3) into equations (B.1) and (B.2), and neglecting higher orders lead to,

$$V_R = V - r \cdot \Omega^2 \cdot \Delta t \qquad (B.4)$$

and

$$V_T = r \cdot \Omega + 2.V.\Omega.\Delta t \qquad (B.5)$$

Differntiating equations (B.4) and (B.5) with respect to time results in the acceleration components,

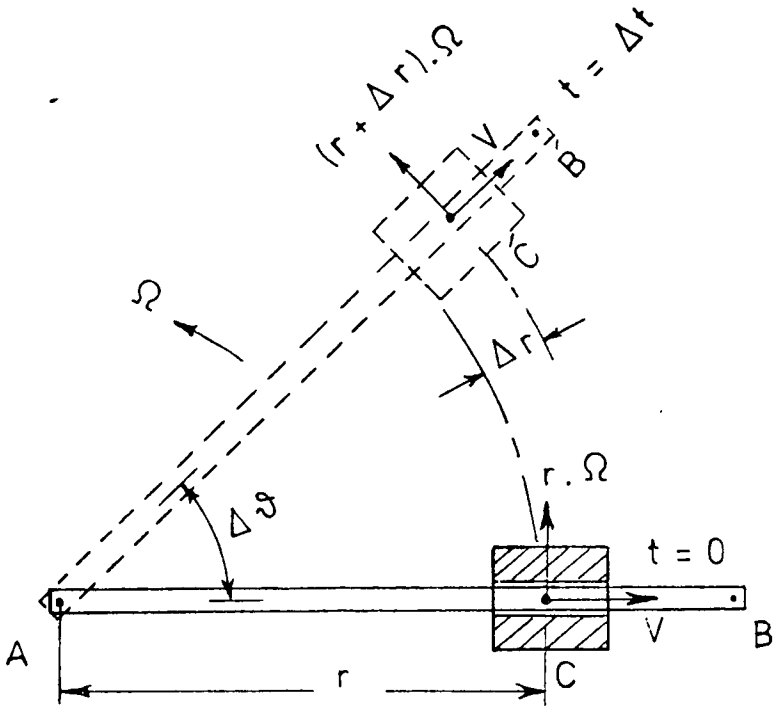


FIGURE B-1 Crank slider mechanism

$$A_R = r \cdot \Omega^2 \qquad (B.6)$$

and $$A_T = 2 \cdot V \cdot \Omega \qquad (B.7)$$

where A_R and A_T are radial (along the rod) and tangental (perpendicular to the rod) acceleration components respectively. Equation (B.6) represent the acceleration of point C on the rod with which the centre of the block is coincident, while the right hand term of equation (B.7) which is called the Coriolis acceleration component is the acceleration of the block relative to the rod, which can be seen to result from two effects, first, the changing magnitude of the tangential component $r.\Omega$ of the velocity of the block, and second, the changing direction of the radial component V.

APPENDIX C

THE PAFEC SCHEME

PAFEC (Program for Automatic Finite Element Calculations) was started in the University of Nottingham in the late sixties. Initially, it was intended to be a research tool to ease finite element studies. As the finite element method rapidly developed to become a chosen method for engineering design and application, PAFEC was developed in accordance, and commercial users started to exist. At this point a separate administration was allocated inside the Department of Mechanical Engineering to maintain the program and support advice for the increasing number of users. As the number of industrial users increased, the program was further developed to reduce input information to a minimum with increasing attention to keep the user as far as possible from involving in the internal details of the program or machine instructions. At present, the program is very flexible, can do alot of automatic data check and assume defaults when necessary, in addition to the possiblity of linking users own coding and restart from various points on the program. In the last quarter of 1976, PAFEC was completely commerciallized when the PAFEC group separated from the University of Nottingham and a patent was affected to all the codes that were in existance.

The use of the scheme is well documented in users manuals, both in short and detailed forms[58 to 62] . But despite the 'systems manual' which was only recently released, the Author of this text still finds a gap in documentation for the more advanced user who might wish to add or develop some parts of the scheme.

C.1. PAFEC 70 + System and Associated Preprocessor

This system was the first PAFEC program to appear in package form, probably it was written for the second generation of computers. It had a good simple preprocessor to automatically run the program. The program used to run in two stages or phases, the first stage is to read, interpret and store input information, and in the second stage, element matrices are created, merged and solved.

A prepocessor is a separately running computer program which can be employed routinely to ease and improve the use of the finite element processor programs at an overall cost saving. PAFEC 70 + is supplied with a FORTRAN preprocessor "PAFECAA" which reads the input data and writes a FORTRAN program together with the necessary machine instructions to run it as a normal program. PAFECAA is being controlled by another machine interactive 'Macro' (PAFECA) which is a lower level language program written in machine instruction language (at present GEORGE 3 and 4 for the ICL 1906 computer).

The 70+ system is composed of a general subroutine library which is used in both stages of operation and an element subroutine library which is used in the second stage only. Other specific libraries are in existence, such as mesh generation and drawing subroutine libraries etc and they are used at different stages depending upon what is guessed from the users input information.

When running a program, 70+ will store the input data in three main arrays;

a- CPDOC :- Array for coordinates of position, Displacement numbers and Direction cosines.

b- PLO :- Array for applied loads, later on for calculated displacements.

c- DLIST :- Data List, an array which is used as a data base for all the rest of the finite element date. Pointers are used to address the start and end of various information on DLIST and are normally put in the beginning of this array.

The reader is referred to the users manual [58] for other supported facilities and information.

C.2. PAFEC 75 And Associated Processors:

The need for this system came from a number of factors of which one can mention,

a- The development of mesh generation facility made it possible have very large problem sizes which gave rise to the need for means of reallocating the core requirements at various stages of running time.

b- The need of users to allocate workspaces and access various data during operation

c- Economization on unnecessary repetitions of complete runs in the event of errors being encountered or the use of iteration cycles.

d- Increase in coding to the extent that flooding the core became a common event. This of course led to the requirement for further classification of coding and more libraries of subroutines.

e- Increase in the number of commercial users made the program developers realize that the program should elaborate more on reducing the manual work to the degree that even basic knowledge of programming is not required from the users.

This raised the need for a new processor.

f- In addition to point 'a' above, there was an increasing demand to exchange data between core and backing store to allow more space for larger problems.

g- The development of the third generation of computers made it possible to use more facilities that the syste had to develop in accordance.

h- The need for standardization.

Thus, the new system came with the following features,

a- Modular; Various modules are to hold either initial or calculated data. The modules are created internally, externally, with fixed or variable lengths. Morover, the modules can be exchanged between core and backing store easily and can be erased from either or both of them.

b- Compact; Only two main common arrays are used, BASE for carrying the modules, and IBASE to hold information about the module lengths, position, and existance on core, in addition to holding various error flags and control integers. A lot of the internal subroutines have no arguments at all but they collect information from the common arrays BASE and IBASE.

c- Partitioned; The program operates in ten distinguished stages or phases. These phases are as follows:

1- Reading initial data and control module and setting up the data base .

2- Mesh generation if requested by user.

3- Drawing elements and nodes.

4- Generation of freedoms.

5- Drawing of freedoms and constraints.

6- Generation of element matricies.

7- Solution of equations.

8- Drawing of displacements

9- Calculation of stresses.

10- Drawing of stresses.

d- Advanced processor; The processor, often called the controller, is a supervisor program that analyses the instructions given in the control module by the user to decide how the program is going to be run (the reader is referred to the published literature about the control module options). The processor writes machine instructions and various FORTRAN codes to run the program at different phases in addition to monitoring the program events in anticipation for future runs. Also, the processor checks a large number ofsyntax in the users supplied data.

e- Flexible; Due to the big number of subroutines made available to the user and programmer to enable them to manipulate data and modules very easily without causing any loss of other information.

The disadvantages of the 75 system are listed below:

a- Needs file storage on a large scale that will add extra expense for file store investment.

b- Takes more time in transfers between core and backing store.

c- Loses efficiency due to repetition of data retrieval and extra steps in processing units of information rather than blocks of information at a time due to the need of space

on core.

d- Involvement of the processor in machine interaction that makes linking the program to other machines very difficult if not impossible in addition to the difficulty in implimenting the program on different machines (the latter is a problem that concerns the program developers and is out of the scope of this text).

C.3. Preparation of the Stiffness and Mass Matricies:

Derivation of the element matrices in the PAFEC scheme is explained in references (58 and 59) and need not be elaborated here. Only the equations for the mass and stiffness matrices will be given in here, thus the stiffness matrix of a three dimensional isoparametric element using the displacement approach is given by,

$$\left[S_e\right] = \left[A\sharp^T\right]\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}\left[B^T\right]\left[D\right]\left[B\right]\left\|\left[J\right]\right\| d\xi\ d\eta\ d\zeta \left[A\sharp\right] \qquad .....(C.1)$$

where, $\left[A\sharp\right]$ is a matrix that is composed of smaller matrices $\left[A^{-1}\right]$ arranged on the diagonal of $\left[A\sharp\right]$, $\left[A^{-1}\right]\left[A\right] = I$ and the matrix $\left[A\right]$ consists of row substitutions of the nodal values (ξ, η, ζ) into the shape functions, $\left[J\right]$ is the Jacobian matrix and $\left\|\left[J\right]\right\|$ is its determinant, $\left[B\right]$ and $\left[D\right]$ are the matrices that relate displacements to strains and stresses to strains respectively.

The mass matrix for these elements, obtained from expressions for the kinetic energy of a vibrating but not translating or rotating solid elements is given by

$$\left[M_e\right] = \left[A\ddagger^T\right] \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} \begin{bmatrix} [Q] & 0 & 0 \\ 0 & [Q] & 0 \\ 0 & 0 & [Q] \end{bmatrix} \left|\left[J\right]\right| \, d\xi \, d\eta \, d\zeta \left[A\ddagger\right] \quad \ldots\ldots \text{(C.2)}$$

where, $\left[Q\right] = \left[N^{*T}\right]\left[N^{*}\right]$ and $\left[N^{*}\right]$ is a polynomial function and the asterisk is used to defer it from a general shape function (see section 3.2.). It is seen that both integrations of equations (C.1) and (C.2) are made with reference to an isoparametric representation of the element which must be transferred to the proper axes using the pre- and post- multiplication by $\left[A\ddagger\right]$.

The main interest of this section is to show how the mass and stiffness matrices are prepared once their equations are known. Going back to (reference 61) to look for a general 3-D element routine, such as the one reproduced in figure (C-1). It will be found that this routine consists of three main parts,

a- Preparation of constants and matrices before the integration loop, this includes finding the Gauss integration points, setting the $\left[A^{-1}\right]$ matrix and various polynomial coefficients in addition to setting the elasticity matrix $\left[D\right]$.

b- Building a triple integration loop to create the Jacobian and perform the product $\left[B^T\right]\left[D\right]\left[B\right]$.

c- Perform the post- and pre- multiplication by $\left[A^{-1}\right]$ and merging the element matrices with the system matrix.

The first part of 'c' above is done using the subroutine R37008 for which a flow chart is given in figure (C-2). It is clearly seen from the flow chart that carrying out the multiplication is done only once for one degree of freedom on every node on the

element, then a triple DO loop is set up to redistribute these values in their appropriate place on the mass or stiffness matrix for all the three degrees of freedom on the node. Since the element is assumed to be isotropic in all the three coordinate directions, it is hoped by this procedure that some time can be saved in making one calculation out of three similar ones.

C.4. Scope and Validity:

At present PAFEC is a well developed computer program that can be adopted to numerous kinds of engineering problems and a proper judgement about its validity can be made by comparing it with other programs of its kind. This task has been carried out by a number of contributors in the text, "STRUCTURAL MECHANICS COMPUTER PROGRAMS" which is edited by Pilkey and others, published by the University Press of Virginia, Charlottesville, 1974.

Figure (C-1) General 3D element routine, taken from reference (60).

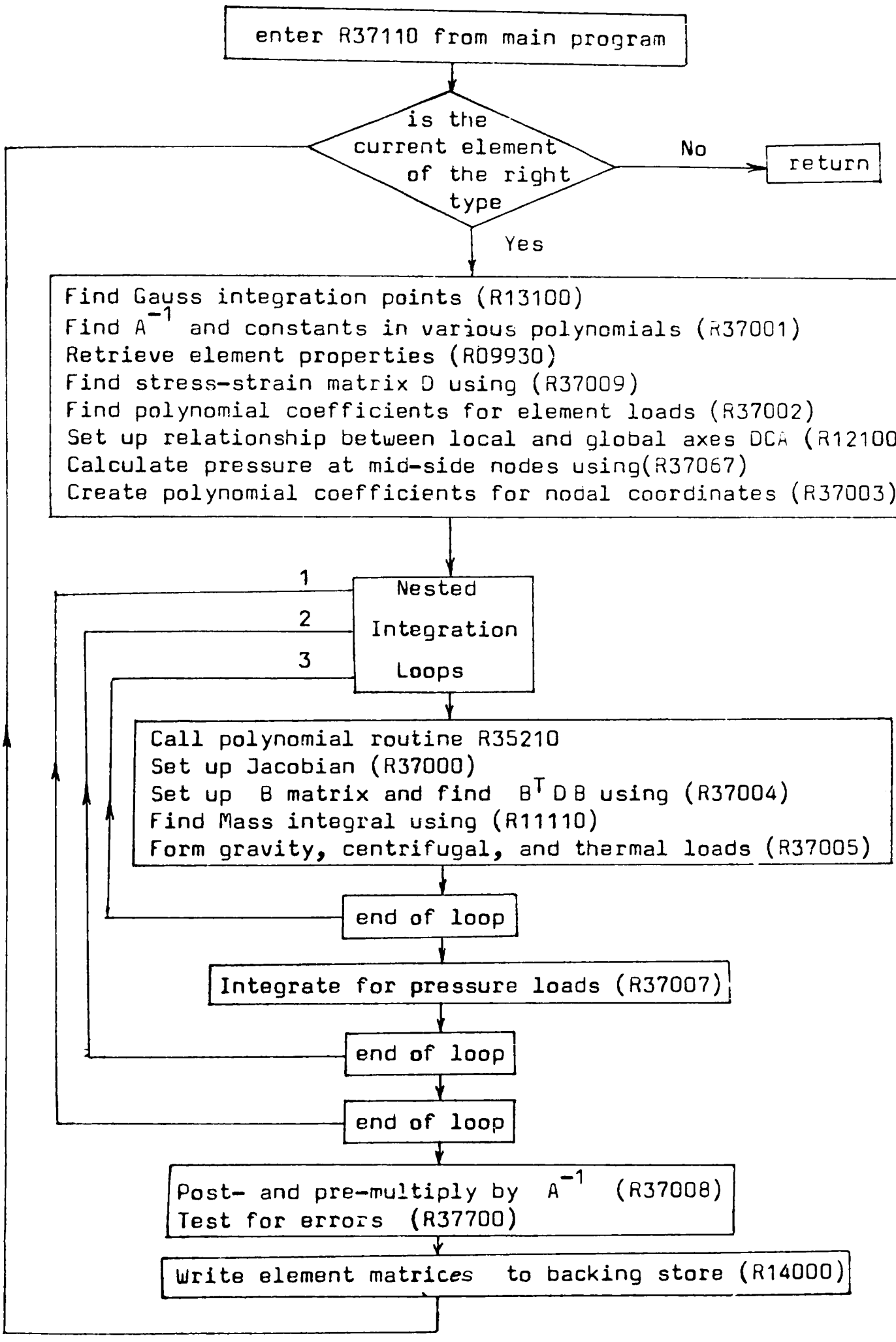
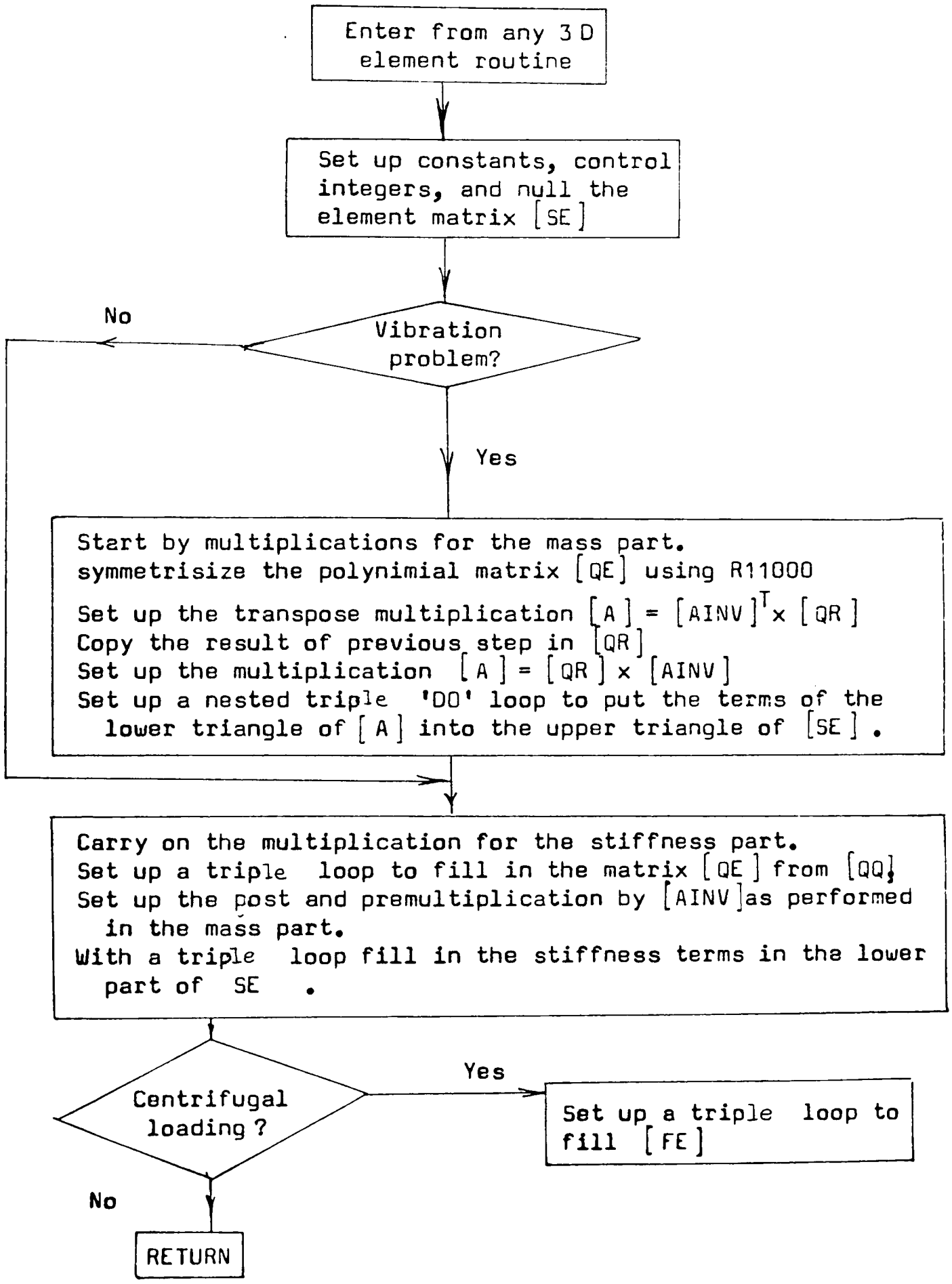


Figure C-2 Flowchart for the routine R37008 which performs the post- and premultiplication by $[A]^{-1}$ outside the integration loop.



APPENDIX D

SOME PROPERTIES OF MATRICES

D.1. Definitions and Basic Operations:

Matrices were introduced to make it possible to write a large number of data and equations in a shorter form. For example a set of simultaneous linear equations which is written in algebraic form

$$\begin{aligned}
a_{11}\,x_1 + a_{12}\,x_2 + \cdots\cdots + a_{1m}\,x_m &= y_1 \\
a_{21}\,x_1 + a_{22}\,x_2 + \cdots\cdots + a_{2m}\,x_m &= y_2 \\
\cdots\quad & \\
a_{m1}\,x_1 + a_{m2}\,x_2 + \cdots\cdots + a_{mm}\,x_m &= y_m
\end{aligned} \qquad (D.1)$$

can be written in a neater form using matrix notation as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{Bmatrix} \qquad (D.2)$$

It is seen from equation (D.2) that the coefficients a_{ij} are separated from the variables x_i. The matrix of coefficients a_{ij} is called a square matrix while the matrix of entities x_i or y_i is a column matrix. A row matrix is a matrix having one row of elements, while a general double dimensional matrix would be a rectangular matrix of order mxn, where m and n correspond to the number of rows and number of columns respectively. The multiplication of matrices is the procedure which makes equation (D.1) and

(D.2) equivalent. The rule for matrix multiplication can be shown on two second order matrices such as

$$\left[A\right] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \left[B\right] = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad (D.3)$$

and to multiply the matrix $\left[A\right]$ by the matrix $\left[B\right]$ a third matrix $\left[C\right]$ can be formed row by row by multiplying each row in turn of $\left[A\right]$ by the corresponding elements of each and every column of $\left[B\right]$ giving

$$\left[C\right] = \left[A\right].\left[B\right] = \begin{bmatrix} a_{11}\,b_{11} + a_{12}\,b_{21} & a_{11}\,b_{12} + a_{12}\,b_{22} \\ a_{21}\,b_{11} + a_{22}\,b_{21} & a_{21}\,b_{12} + a_{22}\,b_{22} \end{bmatrix} \qquad \text{.....(D.4)}$$

The product $\left[C\right]$ of the two matrices $\left[A\right]$ and $\left[B\right]$ above is defined only when they are conformable; that is when the number of columns of $\left[A\right]$ equal the number of rows of $\left[B\right]$.

Two matrices will be equal if and only if they are of the same size and corresponding pairs of elements are equal. In much the same way, matrix addition or subtraction involves simply adding or subtracting the corresponding elements. Thus we have,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 7 & -6 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 11 & -1 \end{bmatrix} \qquad (D.5)$$

or in subscript element notation,

$$\left[C_{ij}\right] = \left[a_{ij}\right] + \left[b_{ij}\right]$$

In general matrix multiplication is not commutative, but associative and distributive, so it is possible to write,

$$\left[A\right]\left[B\right] \neq \left[B\right]\left[A\right]$$

$$\left[A\right]\left(\left[B\right]\left[C\right]\right) = \left(\left[A\right]\left[B\right]\right)\left[C\right] \qquad (D.7)$$

$$\left[A\right]\left(\left[B\right] + \left[C\right]\right) = \left[A\right]\left[B\right] + \left[A\right]\left[C\right]$$

The transpose of a matrix is the exchange of its rows by its columns thus

$$\left[a_{ij}^{T}\right] = \left[a_{ji}\right] \qquad (D.8)$$

A symmetric matrix is a matrix that has its upper triangle equals its lower triangle i.e.,

$$\left[a_{ij}\right] = \left[a_{ji}\right] = \left[a_{ij}^{T}\right] \qquad (D.8A)$$

of course equation (D.8A) can be considered as a corollary of equation (D.8).

A skew symmetric matrix is a matrix that has its upper triangle equals to -1 multiplied by the elements of the lower triangle,

$$\left[a_{ij}\right] = -\left[a_{ji}\right], \quad \left[a_{ii}\right] = \left[a_{jj}\right] = 0 \qquad (D.9)$$

The elements $\left[a_{ij}\right]$ of a matrix can be zero, real, quadrature or complex numbers.

The diagonal matrix is a matrix that has non-zero values on the diagonal while it has a zero every where else, thus,

$$a_{ij} = 0 \quad \text{for all} \quad i \neq j \qquad (D.10)$$

A special case of the diagonal matrix is the 'unit' matrix which has all the diagonal terms equal to unity, so we can write,

$$\left[a_{ij}\right] = 0 \quad \text{for all } i \neq j$$
$$\left[a_{ii}\right] = \left[a_{jj}\right] = 1 \qquad (D.11)$$

Inverting a matrix $\left[A\right]$ is the process of dividing the unit matrix by the matrix $\left[A\right]$,

$$\left[A^{-1}\right] = \frac{[I]}{[A]} \qquad \text{or} \quad [A]\,[A^{-1}] = [I] \tag{D.12}$$

where $[I]$ is the unit or unitary matrix.

Partitioning a matrix is the process of parting elements of the matrix by horizontal or vertical lines, or both,

$$\left[\begin{array}{c|ccc|c} 3 & 2 & 1 & 4 \\ 1 & 5 & -1 & 3 \\ \hline -4 & -2 & 0 & 6 \end{array}\right] = \left[\begin{array}{c|c|c} [a] & [b] & [c] \\ \hline [d] & [e] & [f] \end{array}\right] \tag{D.13}$$

where,

$$[a] = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad [b] = \begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix}, \quad [c] = \begin{bmatrix} 4 \\ 3 \end{bmatrix},$$

$$[d] = [-4], \quad [e] = [-2 \quad 0], \text{ and } [f] = [6]$$

The so called submatrices $[a]$ to $[f]$ are normaly treated as matrix elements with the necessary caution that during multiplication the partitioned matrices should follow the conformability requirement.

By partitioning, it is aimed to reduce some labour when dealing with matrices of order higher than three. However, whilst the amount of calculation to be carried out in the final stage of the solution is not shortened by partitioning, it is generally found less tiring to deal with products and matrix operations in submatrix groups than to work on the whole of the original matrices without splitting them up.

The trace of a matrix $[A]$ is the sum of all its diagonal element, thus,

$$\text{tr}\,[A] = \sum_{i=1}^{n} a_{ii} \tag{D.14}$$

D.2. Orthogonal Transformations:

Transforming sets of axes is a common practice in finite element

studies, and it can be done with the use of matrix notation. A particularly simple matrix that transforms displacements when expressed in terms of orthogonal axis directions is a matrix of nine direction cosines given by $\left[\alpha_{ij}\right]$,

$$\left[\alpha\right] = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{bmatrix} \qquad (D.15)$$

which relates the three directions of an axis system with the three directions of a rotated axis system. For each axis system that consists of three mutually perpendicular directions, the direction cosines are interrelated by a set of six equations formed by correspondence with the dot product of the three unit vectors

$$\begin{aligned} \underline{i} \cdot \underline{i} &= \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1 \\ \underline{i} \cdot \underline{j} &= \underline{j} \cdot \underline{k} = \underline{k} \cdot \underline{i} = 0 \end{aligned} \qquad (D.16)$$

and typical equations can be formed such as,

$$\left.\begin{aligned} \alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 &= 1 \\ \alpha_{11} \cdot \alpha_{21} + \alpha_{12} \cdot \alpha_{22} + \alpha_{13} \cdot \alpha_{33} &= 0 \\ \alpha_{21}^2 + \alpha_{22}^2 + \alpha_{23}^2 &= 1 \\ \vdots \quad\quad \vdots \quad\quad \vdots & \\ \text{e t c} & \end{aligned}\right\} \qquad (D.17)$$

Thus the complete set of equations could be represented by matrix notation as the product,

$$\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{bmatrix} \begin{bmatrix} 11 & 21 & 31 \\ 12 & 22 & 32 \\ 13 & 23 & 33 \end{bmatrix} = \left[\alpha\right]\left[\alpha^T\right] = \left[I\right] \qquad .....(D.18)$$

Equation (D.18) is called an orthogonal relation and the matrix is called an orthogonal matrix.

Comparing equation (D.18) with the earlier definition of equation (D.12) shows that for the orthogonal matrix $[\alpha]$, the inverse and transpose are identical,

$$[\alpha^T] = [\alpha^{-1}] \qquad (D.19)$$

Since forming the transpose is a very easy operation, here its equivalence to the inverse makes handling many manipulations involving direction cosines very simple.

Going back to the problem of transformation, suppose that the matrix $[A]$ of equation (D.2) relates the vectors $\{x\}$ and $\{y\}$. Assume that a new base was introduced and its new coordinates are $\{\bar{x}\}$ and $\{\bar{y}\}$, it is required to find the matrix $[A]$ which relates $\{x\}$ to $\{y\}$ interms of the original matrix $[A]$. To do this we assume a transformation matrix $[\alpha]$ so that

$$\{x\} = [\alpha]\{\bar{x}\} \text{ and } \{y\} = [\alpha]\{\bar{y}\} \qquad (D.20)$$

substituting the relations (D.20) into (D.2) we get,

$$[\alpha]\{\bar{y}\} = [A][\alpha]\{\bar{x}\} \qquad (D.21)$$

solving for $\{\bar{y}\}$ leads to

$$[\bar{y}] = [\alpha^{-1}][A][\alpha]\{\bar{x}\} \qquad (D.22)$$

and thus we can define $[A]$ as

$$[A] = [\alpha^{-1}][A][\alpha] \qquad (D.23)$$

Using equation (D.19) into equation (D.23) leads to the very simple conclusion that if an orthogonal matrix is used in the change of basis then we get the simple relation

$$[A] = [\alpha^T][A][\alpha] \qquad (D.24)$$

D.3. Application to Eigen Values and Vectors

If $\{x\}$ is a column matrix, and $[A]$ is a square matrix of

the same order of $\{x\}$, then the product $[A]\{x\}$ is also a column. Our intention is to find those columns $\{x\}$ so that $[A]\{x\}$ is proportional to $\{x\}$. In other words, we are looking for the solution of the homogeneous equations

$$[A]\{x\} = \lambda\{x\} \tag{D.25}$$

In order to find a non trivial solution, the determinant* of the matrix of coefficients must be set to zero, thus

$$\det\left([A] - \lambda I\right) = 0 \tag{D.26}$$

Assume for simplicity that the order of the matrices is only three, then equation (D.26) becomes

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = 0 \tag{D.27}$$

expanding to obtain

$$\lambda^3 - \lambda^2(a_{11}+a_{22}+a_{33}) + \lambda\left\{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right\} - \det[A] = 0 \tag{D.28}$$

Equation (D.28) represent a cubic polynomial possessing three roots for λ called the characteristic roots or eigen values of the matrix $[A]$. For a general matrix $[A]$ of order 'n' there will be 'n' characteristic roots. Equation (D.28) is know as the characteristic equation of $[A]$. The vector $\{x\}$, chosen that

$$\left([A] - \lambda I\right)\{x\} = 0 \tag{D.29}$$

is known as the characteristic vector or the eignvector of $[A]$.

---

* It is assumed that the reader is familiar with determinants, otherwise reference (7) is suggested.

It should be noticed that a change of basis can be made in the following manner; suppose that the matrix $[A]$ of equation (D.2) represents a linear transformation then the change of basis is done using

$$\{x\} = [\alpha]\{\bar{x}\} \tag{D.30}$$

Substituting for $\{\bar{x}\}$ in equation (D.25) yields

$$[\bar{A}]\{\bar{x}\} = \lambda\{\bar{x}\} \tag{D.31}$$

where $[\bar{A}]$ is given by equation (D.24).

The transformation of equation (D.31) is referred to as similarity transformation. If $[\alpha]$ is orthogonal then the transformation is called orthogonal similarity transformation. Such transformations are very helpful in obtaining eigen values or then change the problem nature from matrix inversion to matrix transpose.

D.4. Reduction to Diagonal Form

For a given symmetric matrix $[A]$, it is possible to find an orthogonal matrix $[\Pi]$ such that the product $[\Pi]^T [A][\Pi]$ is diagonal.

Let the normalized characteristic vectors $\{x_i\}$ correspond to the roots λ_i the matrix

$$[\Pi] = (x_1\ x_2\ x_3\ \ldots\ldots\ x_n) \tag{D.32}$$

is orthogonal. This condition still holds even with identical roots and further, signs can be adjusted that the determinant $|[\Pi]|$ is + 1. Thus the product $[A][\Pi]$ becomes,

$$[A][\Pi] = [A](x_1\ x_2\ \ldots\ x_n)$$

$$= (\lambda_1 x_1\ \lambda_2 x_2\ \ldots\ldots\ \lambda_n x_n) \tag{D.33}$$

$$= (x_1\ x_2\ \ldots\ x_n)\begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \lambda_n \end{bmatrix} = [\Pi][D]$$

and hence $\left[\Pi\right]^T \left[A\right]\left[\Pi\right] = \left[D\right]$ (D.34)

The matrix $\left[D\right]$ in equation (D.34) above is a diagonal matrix consisting of the eigen values arranged on the diagonal in order.

In a similar way, general non-symmetric matrices can be reduced to an upper Hessenburg form[121 and 76].

D.5 Powers of Square Matrices

If a square matrix $\left[A\right]$ is given, it is required to find a form for $\left[A\right]^n$ where 'n' is a positive integer. Let $\left[A\right]$ be diagonalized by the matrix $\left[T\right]$ so that

$$\left[T\right]^{-1}\left[A\right]\left[T\right] = \left[D\right] \qquad (D.35)$$

then

$$\left[A\right]^n = \left[T\right]\left[D\right]\left[T\right]^{-1}\cdot\left[T\right]\left[D\right]\left[T\right]^{-1}\cdots\left[T\right]\left[D\right]\left[T\right]^{-1}$$

$$= \left[T\right]\begin{bmatrix} \lambda_1^n & \cdots & \cdots \\ 0 & \lambda_2^n & \cdots \\ 0 & 0 & \ddots \\ \vdots & \vdots & \ddots \lambda_m^n \end{bmatrix}\left[T\right]^{-1} \qquad (D.36)$$

From the above analysis it was seen that only integer powers of a matrix can be taken, but in fact square roots or any real power of a matrix can be made and the reader is referred to[12] for proof. For the case of a square root of a matrix equation (D.36) becomes,

$$\left[A\right]^{\frac{1}{2}} = \left[T\right]\begin{bmatrix} \lambda_1^{\frac{1}{2}} & 0 & 0 \cdots 0 \\ 0 & \lambda_2^{\frac{1}{2}} & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ & & \lambda_n^{\frac{1}{2}} \end{bmatrix}\left[T\right]^{-1} \qquad (D.37)$$

D.6. The Choleski Method of Matrix Decomposition:

Theoretically, it is possible to decompose a matrix into two triangular matrices, one of them is a lower triangular matrix and the other is an upper triangular matrix. Further more, symmetric

matrices can be decomposed to give the upper half as the transpose of the lower half, thus

$$[A] = [L][U] \quad \text{for unsymmetric Matrices} \qquad (D.38)$$

$$[A] = [L][L^T] \quad \text{for symmetric Matrices} \qquad (D.39)$$

To illustrate the method, an example of a set of equations of order three will be given. It is required to find the coefficients l_{ij} such that,

$$\begin{bmatrix} a_{11} & \text{symmetric} & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \qquad \text{..... (D.40)}$$

Multiplying the two matrices of coefficients on the right hand side of the above equation will result in

$$\left.\begin{aligned} a_{11} &= l_{11}^2 \\ a_{21} &= l_{11}\, l_{21} \\ a_{31} &= l_{11}\, l_{31} \\ a_{22} &= l_{21}^2 + l_{22}^2 \\ a_{32} &= l_{31}\, l_{21} + l_{32}\, l_{22} \\ a_{33} &= l_{31}^2 + l_{32}^2 + l_{33}^2 \end{aligned}\right\} \qquad (D.41)$$

Using the above set of equations a generalization for a matrix of order n can be made, thus for a matrix of n rows, the i th row is given by,

$$\sum_{k=1}^{j} l_{ik}\, l_{jk} = a_{ij} \quad \text{giving } l_{ij} = \left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk} \right) / l_{jj} \;, \; (j=1,2,3,..,i-1)$$

.....(D.42)

and for the diagonal elements

$$\sum_{k=1}^{i} l_{ik}^2 = a_{ii} \text{ giving } l_{ii} = \left(a_{ii} - \sum_{k=1}^{J-1} l_{ik}^2 \right)^{\frac{1}{2}} \quad (D.43)$$

The choleski decomposition or factorization is very useful in solving a set of simultaneous equations such as equation (D.2). The procedure can be performed in two stages. Each involve the solution of a set of equations with a triangular matrix, such a solution is simplified by a forward- or back substitution. Referring again to equation (D.2) it is found that the equation reduces to

$$[L][U]\{x\} = \{y\} \quad (D.44)$$

and a vector $\{B\}$ is found such that

$$[L] \cdot \{B\} = \{y\} \quad (D.45)$$

after which solution can proceed for the equation

$$[U] \cdot \{X\} = \{B\} \quad (D.46)$$

It should be mentioned that in general, finite element matrices are symmetric and our procedure can be further reduced by assuming that

$$[U] = [L^T] \quad (D.47)$$

APPENDIX E

AN AUTOMATIC FORTRAN-FORTRAN REARRANGING PROGRAM FOR LARGE LIBRARY PROGRAMS

E.1 Introduction

At present, standardization is an accepted fact which is favoured for any type of subject released to public or exchanged between a number of people, whether it was a manufactured article, published material, or even a way of eating or dressing clothes which to some extent follow a certain standard of fashion and morality. The question that lays itself in here is how far should one go along with standardization. The answer can be derived from the fact that classified information can be much easier to learn and memorize than other scattered ones, a good example can be taken from peoples ability to memorize hundreds of musical notes and rhythms whether they know to play a musical instrument or not, but the reason of course being that music always follow certain rules. Thus the answer would be that standardization should be carried out in any possible way that makes things more classified and then easier to understand and memorize.

It is established that FORTRAN is a standard international problem oriented scientific higher level language for digital computers, and no doubt that today there exist hundreds of sophisticated programs written in FORTRAN and presented in the form of complete packages to do various types of operations and calculations amongst which one can mention Matrix Handling, Problem Solving, Structural Analysis, Statistical Studies, Graphics, Computer Aided Deign, etc. Although the majority of users see these packages as a collection of black box routines, a number of specific users will be in direct contact with the contents of these packages, and most probably they will need to understand

parts of it, modify or append to other parts of it, being able to do so since they know the language (FORTRAN) which in itself is a standard that follows certain rules of syntax. However due to the difference between the ways of individual thinking, it is still found that these codes are a little out of standard. To clarify this point an example would be given with regard to statement label sequence. Assume that a programmer writes a program with a number of statement label that start by 10 and increases by a factor of 10 every time a new label is required, after finishing the program he found that he needs to add extra statements and labels between some labelled statements, then it is sensible enough to give the new statement labels values which lie between the labels which the statements are to be inserted in between, or add ten to the last (normally highest) label achieved and continue in the normal way. Thus for a long program this process will become more and more difficult as the writer will tend to loose track of numbers, and again this process will be much more difficult for an external writer who would want to further modify the program. Thus the need arises for a method by which these programs can be rewritten or rearranged so that they are faster to be followed and understood, and at the same time they become easier to be modified. These are the reasons for introducing the present rearranging program which are mainly based on the argument that "although FORTRAN is a standard language on its own, its output listings must be further standardized if they are to be further processed".

E.2. Features of the Suggested program:

a- Reorders the statement labels so that they appear in an ascending order on two different levels:

1- FORMAT statement labels will be increased from 1 to 99 in steps of 1.

2- Other statement labels will start from 100 to 5190 increasing in steps of 10.

This feature was introduced to ease following and modifying the package program coding.

b- Modifies all the statements that refer to other labelled statements with accordance to the new numbering scheme explained in 'a' above.

c- Modifies all the DO statements so that their indexes appear in the form (Ln), where 'L' (abbreviation for loop) is an implicit integer variable, and 'n' is an integer constant which points out the loop nesting, the opposite is unfortunately not true since the possiblity of occurance of a loop index outside the do loop makes it necessary to store information about every variable used as an index for a DO loop, and hence 'n' is actually increasing with the number of different variables used for DO loop indexing. With the aid of this feature the user is automatically notified of the presence of DO loops and extra loops can be inserted depending upon feature 'e' below.

d- With reference to point 'c' above, all references to the loop index within a certain loop will be modified and replaced by the new loop index.

e- All the statements inside a DO loop will be indented by a number of spaces which equals to (3* the nesting level). The indenting stops as the loop terminator is encountered. This will allow the user to easily recognise loop nesting, and the existance of loops in general.

f- Brings all the FORMAT statements to the end of the segment just before the end statement in order to let the user track the FORMAT statments easily since there may be more than one reference to the same FORMAT inside one program segment.

g- COMMENT statements will be inscribed by two lines above and below the comment regardless of the comment length in lines. This facility will distinguish comments from other executable statements.

h- For programs with alot of different subroutine calls, an extra facility exists that allows automatically inserting a Comment statement which describes the called subroutine immediately after the CALL statement.

i- Up to 64 characters will be allowed in a statement and the rest will be transfered to the next line as a continuation statement. This feature was introduced to allow the program to be photocopied on A4 size paper without the need to reduce its size.

j- For a text file with a number of various segments, the order of statement labels will be reset for every new segment to start as in point 'a' above.

k- Other minor facilities such as including 50 lines of coding in one page and page numbering plus statement numbering (in columns 73 - 80) etc.... will not be given in much detail due to the shortage of space.

E.3 ORGANIZATIONAL DETAILS:

The program is mainly written in FORTRAN, normally started as a semicompiled version. It is operated on the ICL 1960 machine by

a small GEORGE3 Macro* which assigns the I/O devices, loads the program on core, and starts the execution.

The text file consits of a MASTER segment and subprogram segments as follows

MASTER ARNG4 :-This is the main segment, it allocates all the main arrays, sets up the character codes and other initial information, reads the DATA textfile and controls the calls to other segments.

SUBROUTINE MON :- This is used for debugging the present program (unnecessary).

FUNCTION IQ :- Characters comparison function, checks two characters for similarity.

SUBROUTINE SLASHO :- This is called after printing an output line on the line printer to reprint a slash (/) over the letter 'o' to differ it from the zero.

SUBROUTINE RITA :- This is the main output printing routine, it does other functions before or after printing a statement.

SUBROUTINE NTST1 :- This subroutine classifies the FORTRAN STATEMENTS and it looks for statement labels.

FUNCTION JSUM :- This function interprets the statement label various digits and delivers the number as an integer.

---

* GEORGE3 is a machine operation language and a Macro is a small program that uses such a language.

SUBROUTINE STANAL :- Changes DO loop indexes and replace variable names.

SUBROUTINE ARNGE :- Reorders statement labels.

SUBROUTINE SEARCH :- Relevant to feature 'n' of the previous section, searches for subroutine description to output it after a CALL statement.

E.4 CLASSIFICATION OF STATEMENTS:

In the previous section, it was mentioned that the Subroutine NTST1 classifies the FORTRAN statements, this classification is necessary inorder to be able to carry on the various other tasks. The procedure would be a little simpler if these statements were the only statements existing in the program, but unfortunately the existance of a wide choice of variables makes it very likely that variables will be mixed with some FORTRAN statements. However as a start it will be assumed that there is no such mixing between variables and statements. The classification goes in very much the same way to that of ref. (Sale, see bibliography at the end of this Appendix). Scanning normally starts from column 7 on the records (1 record is equivalent to the contents of 80 column punched card), and only two characters are chosen at a time. If these two characters match with the first two characters of a FORTRAN statement then the rest of the characters will be compared for correctness with some other special tests that are particular to that statement. Further grouping of statements are made for function - similar statements such as read and write statements since they follow the same syntax. Table E.1 shows the various groups of statements together with the method of checking them.

E.5. PRINCIPLE OF OPERATION

The text file containing the source to be rearranged is brought and processed in core segment - by - segment using the master ARNG4 which looks for the segment terminator as it stores it record-by-record on the character array 'IA' in core. Once the terminator is located, processing this segment will start by scanning it twice as follows:

a- In the first scan four main checks are made,

1- Continuation records are indicated and decision is made as to whether they are a continuation of a FORMAT statement or continuation of an ordinary statement, for the first case the record is moved to the end of the segment, while in the second case it is left above.

2- The presence of a statement label is indicated by looking into the contents of the first five columns of a record.

3- One point (2) above shows the presence of a statement label then the whole record is studied for the presence of a FORMAT, if it is not present then ARNGE is called to interpret and replace the label according to the new numbering scheme (see point a.2. of section E.2.) keeping at the same time a record of the old value of the label so that in the second scan all the references to this label can be traced and changed in accordance. If instead a FORMAT was indicated, then ARNGE is called to reorder the label according to the FORMAT labelling scheme.

4- DO loops are checked for their indexes, and notes are taken for the index name preparing for the next scan.

b- In the second scan, the Master segment ARNG4 calls the statement analyser routine NTST1 for every statement to find references to the statement labels and correct them according to the first scan, while variable names are also studied and may be changed if necessary. NTST1 calls both ARNGE and STANAL to renumber the labels and rename the variables respectively.

Before the Master ARNG4 moves the control to process the next statement it calls the output routine RITA to write the statement just being processed to the line printer. RITA tests comments to insert a line before the comment and another one after it, also it can call SLASHO to slash the letter 'o' as well as performing a counting process to write page numbers, line or statement numbers, and finally before returning to the main segment, it calls the subroutine ROUT to look for subroutine calls and put a comment to describe the called routine as an extra feature. After a segment is rearranged all the arrays are cleared and a new segment is processed. This procedure continues until the end of the text file.

The flow chart shown on figure (E-1) further explains the principle of operation. Figure (E-2) is a test sample while the rest of the appendix is left for the listing of the whole program (being arranged by itself).

E.6 SCOPE AND LIMITATIONS:

Due to the nature of the program, it will be assumed that the text file to be arranged is syntax free, that is, it has been compiled by a FORTRAN compiler and the errors were debugged before this program can be used.

Normally large library programs are presented in short segments of not more than 500 records each. Knowing this fact makes it possible to reduce the amount of core used for these programs. The present listing however assumes a maximum length of segment of 850 records.

On the ICL 1906 machine the program takes 29k words and it consumes about 78 seconds to arrange a text file of length 1200 lines.

Only two machine dependent routines were used to allow faster character comparison which can be easily replaced by equivalent routines for other machines.

E.7. References:

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"PHRASE STRUCTURES IN FORTRAN".
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"FLOWCODER"
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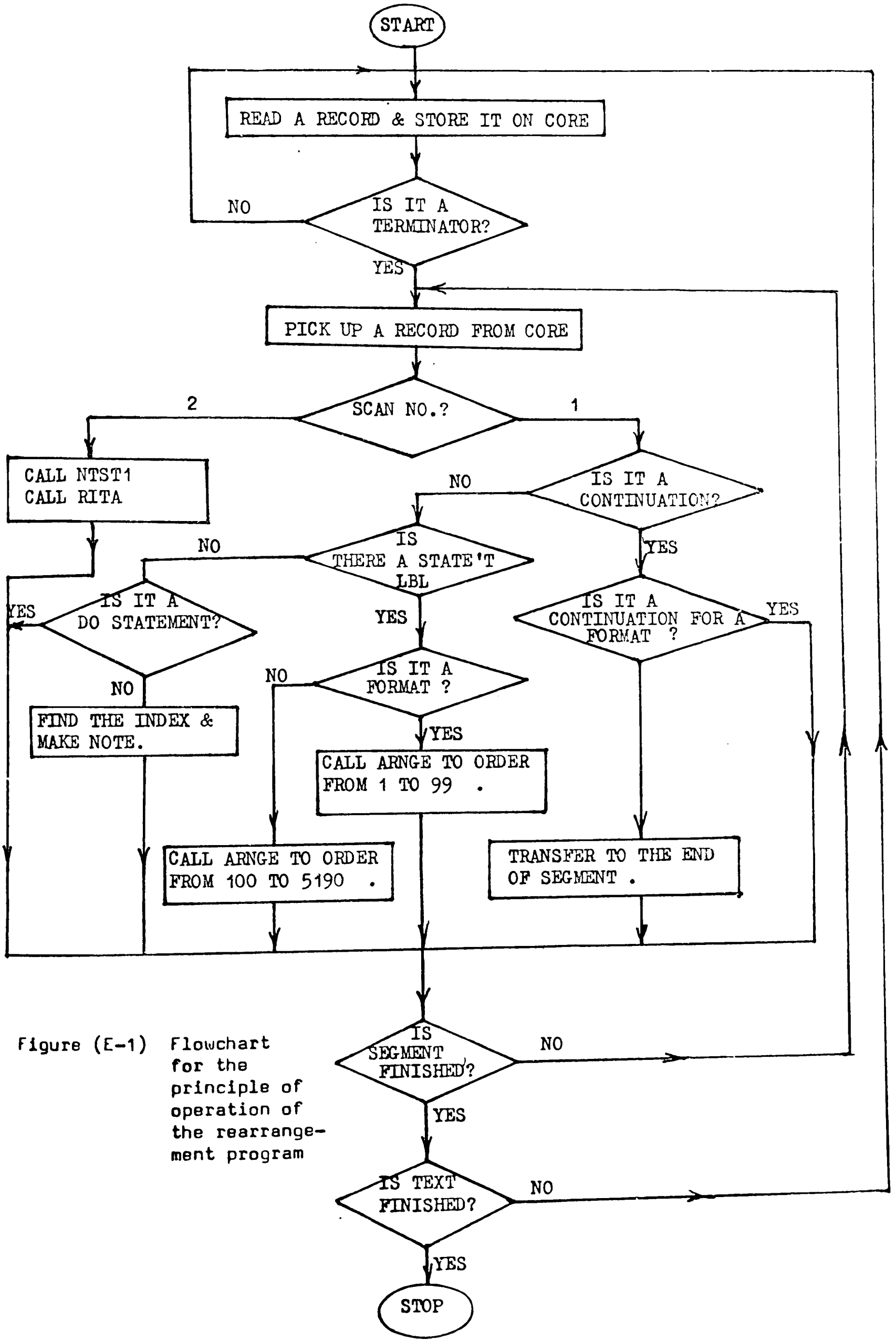


Figure (E-1) Flowchart for the principle of operation of the rearrangement program

TABLE (E-1) CLASSIFICATION OF FORTRAN STATEMENTS

STATEMENT	DESCRIPTION	LOGIC	CHANGES
COMMENT	not executable	letter 'C' in the first column	none
CONTINUATION	follows what comes before it	not COMMENT, character in column 6, rest follows beginning of statement	follow type of statement
READ (n_1,n_2) list	group I/O	1st & 2nd 2 chars, bracket, comma, 1 pair brackets only.	n_2, list if the variable is a DO index
WRITE(n_1,n_2) list		5 chars and same as for READ	---DO--------
GO TO n	transfer of control	1st & 2nd 2 chars followed by integer only.	n
GO TO $(n_1,n_2...)$,I	conditional transfer	same as GO TO, brackets, commas, integer variable or constant.	$n_1,n_2...$,I
IF (condition)n_1,n_2,n_3	Arithmetic IF	1st 2chars, outer most brackets, 2 commas & 3 integer constants	n_1,n_2,n_3
IF (logic) statement	Logical IF	1st 2chars, decimal points not followed by integer constants	treat statement after the IF as a new statement
DO n I = n_1,n_2,n_3	cycle	1st 2chars, followed by integer const followed by integer variable, (=)	n, I, may be n_1 n_2, if they are variables.
END	segment terminator	1st 2chars, 2nd 2chars, rest of record empty	none
**** or FINISH	file terminator (not FORTRAN)	same as for END	none
CALL	linke to other segment	1st [illegible] followed by variable name	none
Arithmetic, Continue RETURN...etc	others	they are t[illegible] as rec[illegible] [illegible] th[illegible]	v[illegible] ames
FORMAT ( )		[illegible]	[illegible]

```
      MASTER ARUGN
C-----------------------------------------------------------------------
C     DIMENSION IDIR(6600)
C  THIS FACILITY IS FOR PROGRAMS WITH LARGE NO OF S/R,IGNORED
C-----------------------------------------------------------------------
      DIMENSION ISTF(100),IVTAB(1,100),ITTL(5)
      DIMENSION ITERM(3),IPOS(2),NOL(25),NFORM(100),KDD(8)
      DIMENSION IB( 80),N(500),IA(10,650),NUM(2
     +4),IFORT(6,5)
C-----------------------------------------------------------------------
C     MAX SEGMENT LENGTH 650 RECORDS,CHANGE THE SECOND DIMENSION
C     OF 'IA' IF MORE IS REQUIRED
C-----------------------------------------------------------------------
      DATA ITL/' MAIN   '/
      DATA NUM,IFORT /66576,328720,590864,853008,11
     +15152,[illegible],[illegible]
     +,[illegible],[illegible],2425872,4260880,7406608,6358032,6620
     +176,-[illegible],
     +426[illegible],[illegible],-7535600,[illegible],7[illegible]30896,71444
     +64,[illegible],[illegible],
     +[illegible],-7011312,-3603440,-2292720,-6437024,-6
     +[illegible],-5962736,
     +-7[illegible],-6749168,-4652016,-7011312,-3603440,-4389872
     +,-[illegible],
     +-6[illegible],-4389872,-4389872,-7273456,-8059888,-5962736
     +,-[illegible],
     +-[illegible],6358032,4260880,-3603440,4260880,-7273456,-3
     +[illegible],
     +-[illegible],-4389872,4260880,4260880,-4914160,4260880,42
     +[illegible],-[illegible]
     +2,[illegible],6358032,4260880,4260880,-6059[illegible]/
      DATA ITERM /'END     ','****    ','FINISH  '/
      DATA ISC/1H./
      DATA NUM(17)/'*'/
      DATA ITTL(1)/'M'/
      DATA ITTL(2)/'A'/
      DATA ITTL(3)/'I'/
      DATA ITTL(4)/'N'/
      DATA ITTL(5)/'-'/
C-----------------------------------------------------------------------
C     READ(4,4)(IDIR(I),I=1,4541)
C     THIS IS TEXT FILE WHICH CONTAINS S/R DESCRIPTIONS
C     READ(5,[illegible]) NUM,IFORT
C     WRITE (6,[illegible]) NUM,IFORT
C
C-----------------------------------------------------------------------
      IPAG = 1
C-----------------------------------------------------------------------
C     PAGING & NUMBERING FACILITY IGNORED IN S/R RITA
C     READ(5,5) ISIGN
C   25 FORMAT (T8,A1)
C-----------------------------------------------------------------------
  100 IC = 0
      IFORM = 0
      IN = 1
      KCOUNT = 1
      IF = 1
      IVTAB(1,1) = NUM(11)
      IPAGE = 0
      IFAG =
      IPOS(1) = 1
      IPOS(2) = 1
      WRITE(6, 7 )
      KSTOP = [illegible]
      KCOUNT = 0
C-----------------------------------------------------------------------
C .. .. .. ..  READ CARDS
C-----------------------------------------------------------------------
  110 DO 200 L1 = 1,650
      J = L1
      READ (5, 6) (IB( L3 ) , L3 =1,80)
C-----------------------------------------------------------------------
C .. .. .. .. .. CHECK ON TERMINATOR
C-----------------------------------------------------------------------
         DO 120 L2 =1,80
  120    CALL COPY (1,IA(1, L1 ), L2 ,IB( L2 ),1)
C-----------------------------------------------------------------------
C.........  BLANK THE END FIELD .............
C-----------------------------------------------------------------------
         DO 130 L2 =1,8
  130    KDD( L2 )=NUM (11)
      K = 0
         DO 160 L2 =1,72
         IF (IQ(IB( L2 ),NUM(11))) 140, 160, 140
  140    K = K + 1
         IF (K - 9) 150, 200, 200
  150    KDD(K) = IB( L2 )
  160    CONTINUE
         DO 170 L2 =1,8
  170    CALL COPY (1,IAF, L2 ,KDD( L2 ),1)
         DO 180 L2 =1,3
         IZ = 0
         CALL COMPA (IAF,ITERM( L2 ),IZ)
         IF(IZ-2) 190, 180, 190
  180    CONTINUE
      L2 =4
  190 GO TO ( 210, 220, 220, 200),L2
  200 CONTINUE
     WRITE(6, 8)
     GO TO 210
  210 KSTOP  = 1
     JKELP = J - 1
     GO TO 230
  220 KSTOP = 2
     JKELP = J - 1
  230   DO 730  L3 =1,JKELP
  240   IF (JKELP.EQ. L3 -1)GOTO 740
         DO 250 L2 =1,80
  250    CALL COPY(1,IB( L2 ),1,IA(1, L3 ), L2 )
C-----------------------------------------------------------------------
C     WRITE (6,16) IB
C .. .. .. ..  CHECK CONTINUATION AND COMMENT CARDS
C-----------------------------------------------------------------------
  260   IF (IQ(IB(1),NUM (13))) 270, 330, 270
  270   IF(IQ(IB(6  ) , NUM(11))) 260, 330, 280
  280   IF(IFORM) 290, 730, 290
  290   IFC = IFC + 1
         DO 300  L4 = 1,80
  300    CALL COPY (1,ISTF( L4 ),1,IA(1, L3 ), L4 )
      JKELP = JKELP - 1
      JM1 = J - 2
         DO 310  L5 = L3 ,JM1
           DO 310  L4 = 1,80
  310    CALL COPY (1,IA(1, L5 ), L4 ,IA(1, L5 +1), L4 )
         DO 320  L4 = 1,80
  320    CALL COPY (1,IA(1,J-1), L4 ,ISTF( L4 ),1)
      GO TO 240
  330 L = 0
      IFORM = 0
```

```
C .. .. .. ..    CHECK WHETHER THERE IS A STATEMENT NUMBER
C-----------------------------------------------------------------------
            DO 370  L5 = 1,5
            IF(ID(IB( L5 )  , NUM(11))) 340, 370, 340
  340          DO 350 L4 = 1,10
               IF(ID(IB( L5 )  , NUM( L4 ))) 350, 360, 350
  350          CONTINUE
            GO TO 730
  360       L = L + 1
  370       CONTINUE
            IF (L) 520, 520, 380
  380    LCOUNT = 0
C-----------------------------------------------------------------------
C .. .. .. .. ... CHECK ON FORMATS
C-----------------------------------------------------------------------
            IF (ID(IB(1),IAG)) 400, 390, 400
  390       IP(1) = NUM(11)
            CALL COPY (1,IA(1, L3 ),1,ID(1),1)
            GO TO 750
  400          DO 430  L6 = 2,72
               IF(ID (IB(  L6 ) , NUM(11))) 410, 430, 410
  410          LCOUNT = LCOUNT + 1
               IF (LCOUNT - 5) 420, 420, 440
  420          IF(ID(IFORT (6,LCONT) , IB( L6 ))) 440, 430, 440
  430          CONTINUE
            IF (LCOUNT - 5) 440, 460, 460
  440       CONTINUE
            CALL ARRAYF( 1, 5,NUM,IB,6,NFORM,1,IPOS,IFOR
     *      M,ITAG,NUL,IFAIL)
            IPOS(IFORM + 1) = IPOS(IFORM + 1)  + 1
C-----------------------------------------------------------------------
C     ARRANGE THE REVERSE STORING
C-----------------------------------------------------------------------
            DO 450  L2 =1,5
  450       CALL COPY(1,IA(1, L3 ), L2 ,IB( L2 ),1)
            GO TO 520
  460       IFORM = 1
  470       CONTINUE
            CALL ARRAYF( 1, 5,NUM,IB,6,NFORM,1,IPOS,IFOR
     *      M,ITAG,NUL,IFAIL)
               DO 490  L2 =1,5
  480          CALL COPY(1,IA(1, L3 ), L2 ,IB( L2 ),1)
            IPOS(IFORM + 1) = IPOS(IFORM + 1)  + 1
               DO 490  L4 = 1,80
  490          CALL COPY (1,ISTF( L4 ),1,IA(1, L3 ), L4 )
            JM1 = J - 2
               DO 500  L5 = L3 ,JM1
                  DO 500  L4 = 1,80
  500          CALL COPY (1,IA(1, L5 ), L4 ,IA(1, L5 +1), L4 )
               DO 510  L4 = 1,80
  510          CALL COPY (1,IA(1,J-1), L4 ,ISTF( L4 ),1)
            JFEED = JFEED - 1
            GO TO 240
  520       KK = 1
C-----------------------------------------------------------------------
C**** CHECK ON DO LOOPS
C-----------------------------------------------------------------------
               DO 550  L4 = 2,72
               IF (ID(IB( L4 ),NUM(11))) 530, 550, 530
  530          IF (IABS(IB( L4 ))-IABS(IFORT(7,K+))) 730, 540, 730
  540          K, = KK + 1
               IF (KK.GT.2) GO TO 560
  550          CONTINUE
            GO TO 730
  560       L4 =  L4 + 1
```

```
C **** FIND THE START & END ADDRESSES OF THE DO LOOP INFO
C-----------------------------------------------------------------------
            DO 620  L7 = L4 ,72
            IF (IST) 610, 570, 610
  570       IF(ID(IB( L7 ),NUM(11))) 580, 620, 580
  580          DO 590  L5 = 1,10
               IF(ID(IB( L7 ),NUM( L5 ))) 590, 610, 590
  590          CONTINUE
            IF ( L7 .EQ. L4 )GO10 730
            IF(ID(IB( L7 ),NUM(11))) 600, 610, 600
  600       IST = L7
            GO TO 620
  610       IF(ID(IB( L7 ),NUM(17))) 620, 630, 620
  620       CONTINUE
         WRITE (6, 10 ) IB
            GO TO 730
  630       IF =  L7 - 1
C-----------------------------------------------------------------------
C       WRITE (6,6646)(IP(IL),IL=IST,IE)
C-----------------------------------------------------------------------
            KK = 1
  640          DO 660  L7 = IST,IE
               IF (ID(IB( L7 ),NUM(11))) 650, 660, 650
  650          KWD(KK) = IB ( L7 )
               KK = KK + 1
  660          CONTINUE
C-----------------------------------------------------------------------
C **** BLANK TH REST OF THE VARIABLE NAME
C-----------------------------------------------------------------------
               DO 670  L7 = KK,8
  670          KWD( L7 )= NUM(11)
               DO 680  L2 =1,8
  680          CALL COPY(1,IAF, L2 ,KWD( L2 ),1)
            NV1 = NV -1
            IF(NV1) 710, 710, 690
  690          DO 700  L5 =1,NV1
               IZ=8
               CALL COMPS(IVTAB(1, L8 ),IAF,IZ)
               IF (IZ-2 ) 720, 700, 720
  700          CONTINUE
  710       CALL COPY (8,IVTAB(1,NV),1,IAF,1)
            NV = NV+1
  720       CONTINUE
  730       CONTINUE
  740 NV = NV - 1
C-----------------------------------------------------------------------
C       WRITE (6,6666)(IVTAB(1,I),I=1,NV)
C-----------------------------------------------------------------------
  750       DO 760  L4 = 1,J
               DO 760  L5 = 1,80
  760          CALL COPY (1,IP( L5 ),1,IA(1, L4 ), L5 )
            IMAGE = 0
            IF ( L4 .LT.JFEED)GO TO 770
            IMAGE = 1
            GO TO 780
  770       CALL NTST1(IB,6,NFORM,NUM,IFORT,IPOS,ITAG,N
     *      OP,ITEST,IVTAB,
     *      NV,J,IA, L4 ,JFEED)
  780       CALL PITA(IB,IPAG,NUM,ITAG,ITEST,IMAGE,IFORT
     *      ,ICOUNT,ITTL
     *      ,IPIC,IA,J)
               DO 790  L5 = 1,80
  790       CALL COPY(1,IA(1, L4 ), L5 ,IB( L5 ),1)
  800 IF (KSTOP - 1) 100, 100, 810
```

```
    2 FORMAT (120A1)
    3 FORMAT(1X,10A1)
    4 FORMAT (10I10)
    5 FORMAT(80A1)
    6 FORMAT(,0A1)
    7 FORMAT (1H7)
    8 FORMAT (' NUMBER OF RECORDS MORE THAN FILE SIZE')
    9 FORMAT(1X,80A1)
   10 FORMAT (' FOLLOWING LINE WAS SUSPECTED AS A DO LOOP'/1X,80A1)
   11 FORMAT (' DO LOOP INDICATED, INDEX ',8A1)
   12 FORMAT(10(3X,A8))
   13 FORMAT(1H1)
      END



      SUBROUTINE HON(I)
C-----------------------------------------------------------------------
C     USED FOR DEBUGGING
C-----------------------------------------------------------------------
      WRITE (6, 1)I
      RETURN
    1 FORMAT(' POSITION ' , I6)
      END



      FUNCTION IQ(IA,IB)
C-----------------------------------------------------------------------
C     COMPARES TWO CHARACTERS FOR EQUALITY
C-----------------------------------------------------------------------
      IQ = 0
      IZZ = 1
      CALL COMP(IZZ,IA,1 ,IB,1)
C-----------------------------------------------------------------------
C     ICL S/R FOR CHARACTER COMPARISON
C-----------------------------------------------------------------------
      IF (IZZ) 110, 100, 110
  100 IQ = 1
  110 RETURN
      END


      SUBROUTINE SLASHO    (IA,NUM)
C-----------------------------------------------------------------------
C     SLASHES THE LETTER 'O' TO DIFFER IT FROM THE ZERO
C-----------------------------------------------------------------------
      DIMENSION IA(80) ,NUM(24)
         DO 120 L1 =1,80
         IF (IQ(IA( L1 ),NUM(15))) 100, 110, 100
  100    IA ( L1 )=NUM(11)
         GO TO 120
  110    IA ( L1 )=NUM(19)
  120    CONTINUE
      WRITE (6, 1) IA
      RETURN
    1 FORMAT ('0',80A1)
      END
```

```
      SUBROUTINE RITA(IA,IPAG,NUM,ITAG,ITEST
     *,IFORM,IFORT,NCOMT,ITTL
     *,IDIR,IATOT,LPOINT)
C-----------------------------------------------------------------------
C     MAIN LINE OUTPUT S/R
C-----------------------------------------------------------------------
      DIMENSION IA(80),ITTL(5),NUM(24),IB(80),IDIR(6600)
      DIMENSION IFORT(8,5), IATOT(10,350)
      DATA JTL/'RITA    '/
C-----------------------------------------------------------------------
C     WRITE (6,37) IA
C-----------------------------------------------------------------------
      IP1 = ITAG + ITEST
      INDEX = 0
C-----------------------------------------------------------------------
C ****     CHECK PRESENCE OF COMMENTS
C-----------------------------------------------------------------------
      IF (IQ(IA(1),NUM(18))) 160, 100, 160
  100 IPOS = 24
  110 IF (IQ(IA(24),NUM(11))) 130, 150, 130
  120 IPOS = IPOS+1
      IF(IPOS.GT.10)GO TO 150
      GO TO 110
  130 IAA= IA(24)
      IS = IPOS+1
         DO 140  L1 = IS,80
         IF (IQ(IA( L1 ),IAA)) 150, 140, 150
  140    CONTINUE
      IPAG = 0
      GO TO 460
  150 IF(IPAG) 460, 460, 450
  160 IF (IPAG) 170,.180, 190
  170 WRITE (6, 3)
  180 IPAG = 1
C-----------------------------------------------------------------------
C **** CHECK CONTINUATION CARDS
C-----------------------------------------------------------------------
  190 IF(IP1) 200, 200, 210
  200 JC = 0
      GO TO 260
  210 IF (IQ(IA(6),NUM(11))) 220, 230, 220
  220 INDEX = 1
C-----------------------------------------------------------------------
C ****    LOOK AFTER FORMATS
C-----------------------------------------------------------------------
  230 IF (IFORM) 240, 240, 470
  240 ITEST = 0
C-----------------------------------------------------------------------
C     CALL ROUT(IA,NUM,IDIR)
C ****     INDEXING CYCLE STARTS ******
C-----------------------------------------------------------------------
  250 JC = IP1  * 3
  260 JSTOP = JC
      NBL = 0
      N10 = 0
         DO 280  L1 = 7,72
C-----------------------------------------------------------------------
C **** CHECK BLANKS IN THE BEGINNING OF THE STATEMENT
C-----------------------------------------------------------------------
         IF (IQ(IA( L1 ),NUM(11)) ) 290, 270, 290
  270    JC = JC - 1
         NBL = NBL+1
         IF (JC.LE.0) N10 = 10
```

```
      IF (NBL.EQ.JSTOR) GO TO 480
      IF(NTO) 300, 300, 310
  300 IF (IC) 460, 480, 330
  310 IPOS = 7 + JSTOR
          DO 320  L2 = L1 ,72
          IB(IPOS) = IA( L2 )
          IPOS = IPOS + 1
  320     IA( L2 )= NUM(11)
       GO TO 430
  330 JS = 72 - JC + 1
C-------------------------------------------------------------------------
C ****  JS IS THE THE STARTING POINT FOR INDENTATIONS
C-------------------------------------------------------------------------
      JSS = L1
      ITOT = 72 +JS
      L = IC
         DO 340 L3 =JS,72
         JP = ITOT - L3
C-------------------------------------------------------------------------
C ****  IF SPACE IS AVAILABLE , THEN JUST INDENT
C-------------------------------------------------------------------------
         IF(IO (IA(JR) ,NUM(11))) 380, 340, 380
  340    L = L - 1
         DO 350  L3 =7,72
         JP = 79 - L3
         IF (IO(IA(JR),NUM(11))) 360, 350, 360
  350    CONTINUE
      GO TO 430
  360 J7 = JR + 7
         DO 370  L3 =7, JR
         IP = J7 - L3
         JPC = IR +JC
         IA (JPC) = IA(IR)
  370    IA(IR) = NUM (11)
      GO TO 430
C-------------------------------------------------------------------------
C ****   SPACE PROBLEM, INDENT WHAT COULD BE INDENTED
C        AND PUT THE REST ON A CONTINUATION LINE
C-------------------------------------------------------------------------
  380 JR = JR + 1
      L4 = 7 + L        + JC
         DO 390 L3 =1,78
  390    IB( L3 )=NUM(11)
      IB (6) = NUM(21)
         DO 400 L3 =1,L
         IP = JR - L3
         IIR = L4 - L3
         IB(IIR) = IA(IR)
  400    IA(IP) = NUM (11)
      JS = JSS
      IPOS = 72
      JR = JR - L
      I7 = JS +JR - 1
        DO 410  L3 =JS,JR
        IP = I7 - L3
        IB(IPOS) = IA(IR)
  410   IPOS = IPOS - 1
        DO 420  L3 =1,JSTOR
        I7 = L3 +6
  420   IB(I7) = NUM(11)
      WRITE (6, 5) (IA(MI),MI=1,72)
C-------------------------------------------------------------------------
C        CALL SLASHO (IA,NUM)
C-------------------------------------------------------------------------
```

```
      IF (INDEX) ...
C-------------------------------------------------------------------------
C   41 WRITE(6,46)ITTL,MCONT
C-------------------------------------------------------------------------
  430 CONTINUE
      MCONT = MCONT + 1
  440 WRITE (6, 4) (IB(MI),MI=1,72)
C-------------------------------------------------------------------------
C        CALL SLASHO (IB,NUM)
C-------------------------------------------------------------------------
      GO TO 510
  450 WRITE (6, 3)
  460 WRITE (6, 5) IA
C-------------------------------------------------------------------------
C        CALL SLASHO (IA,NUM)
C-------------------------------------------------------------------------
      IF (IPAG.EQ.0) RETURN
      IPAG = - 1
      RETURN
  470 IFORM = 0
  480 IF (INDEX) 500, 490, 500
C-------------------------------------------------------------------------
C   51 WRITE(6,46)ITTL,MCONT
C-------------------------------------------------------------------------
  490 CONTINUE
      MCONT = MCONT + 1
  500 WRITE (6, 4)(IA(MI),MI = 1,72)
C-------------------------------------------------------------------------
C 105 CALL SLASHO (IA,NUM)
C69     CALL ROUT (IA,NUM,IDIR,IATOT,LPOINT)
C-------------------------------------------------------------------------
  510 CONTINUE
      RETURN
    1 FORMAT(' C& ',80A1)
    2 FORMAT(' ',I74,5A1,I3)
    3 FORMAT(' C---------------------------------------------------------
     *---------------')
    4 FORMAT (' ',80A1)
    5 FORMAT (1X,80A1)
      END


      SUBROUTINE NTST1 (IA,N,NFORM,NUM,IFO
     *RT,IPOS,ITAG,NDL,ITEST
     *,IVTAB,NV,ISTOP,IPROG,ISTART,JKEEP)
      DIMENSION IVTAB(1,100),IPOS(2),IBB(80),IPROG(10,350)
C-------------------------------------------------------------------------
C        ANALIZES FORTRAN STATEMENTS
C-------------------------------------------------------------------------
      DIMENSION KV(5),NT(10),NDL(25),KDN(5),IDDD(80)
      DIMENSION IA(80),N(500),NUM(24), IFORT(8,
     *5),NFORM(100),IO(80)
      DATA NT/'(',')',',','+','-',1H/,'*','=','.','L'/
      DATA JTL/'NTST1   '/,LENGTH/80/
      IF1 = 0
      IFLAG = 0
C-------------------------------------------------------------------------
C .. ..    IF ITS A COMMENT THEN JUST JUMP IT
C-------------------------------------------------------------------------
      IF(IO (IA(1) , NUM(18))) 100, 1020, 100
C-------------------------------------------------------------------------
C .. .. .. ..    CHECK FOR NESTED DO LOOPS
C  CHECK IF THE STATEMENT NUMBER COINCIDES WITH A DO LOOP TERMINATOR
C-------------------------------------------------------------------------
  100 KK =0
```

```
C-------- LOOK FOR STATEMENT NUMBER
C-----------------------------------------------------------------------
      DO 160  L1 =1,5
      IF(IO (IA( L1 ),NUM(11))) 110, 140, 110
  110    DO 120   L2 = 1,10
         IF(IO  ( IA( L1 ),KUM( L2 ))) 120, 130, 120
  120    CONTINUE
      GO TO 200
  130 KK = KK + 1
  140 CONTINUE
      IF (KK) 200, 200, 150
  150 IF (ITAG) 160, 200, 170
  160 IF (ITAG+26) 200, 200, 170
  170 ISUB = JSUB (1,5,NUM,IA,KUM,KDN,JTL,LENGTH,IFAIL)
      IATAG = IABS(ITAG)
  180 IF(KPL(I TAG) - ISUB) 200, 190, 200
  190 ITAG = ITAG - (ISIGN(1,ITAG))
      ITEST = 1
      IF(ITAG) 180, 200, 180
  200 KK = 0
C-----------------------------------------------------------------------
C .. .. .. .. ..  FETCH THE FIRST 5 CHARACTERS IN THE STATEMENT
C-----------------------------------------------------------------------
      DO 220 L1 =7,72
      IF(IO (IA( L1 ),NUM(11))) 210, 220, 210
  210 KK = KK + 1
      KK(KK) = IA( L1 )
      IF (KK - 5) 220, 230, 230
  220 CONTINUE
      IF (KK) 1020, 1020, 230
  230 JT = 0
      LEVLT = 0
C-----------------------------------------------------------------------
C  .. ..  CHECK IF IT CONFORMS WITH ONE OF THE SPECIAL FORTRAN STATEMENTS
C-----------------------------------------------------------------------
      DO 260 L3 =2,7
      LL = 0
         DO 250 L4 =1,2
         IF(IO (KV( L4 ),IFORT( L3 , L4 ))) 260, 240, 260
  240    JT = L3
         LL = LL + 1
         IF(LL - 2) 250, 440, 440
  250    CONTINUE
  260 CONTINUE
C-----------------------------------------------------------------------
C     IF THE CONTROL REACHED THIS POINT THEN THE GIVEN STATEMENT
C     IS NOT A SPECIAL FORTRAN STATEMENT
C-----------------------------------------------------------------------
  270 IF (KK.LT.5) GO TO 410
C-----------------------------------------------------------------------
C     IN WHAT FOLLOWS, RECORDS WITH CHARACTERS IN COLUMNS 60 TO 72
C     WILL BE DUMPED ON A CONTINUATION CARD.
C
C     START TO LOOK FOR CHARACTERS IN COLUMNS 60 TO 72
C-----------------------------------------------------------------------
  280    DO 290  L3 =60 ,72
         IF (IO(IA( L3 ),NUM(11))) 300, 290, 300
  290    CONTINUE
C-----------------------------------------------------------------------
C      NOT FOUND ANY
C-----------------------------------------------------------------------
      GO TO 410
C-----------------------------------------------------------------------
C     CHARACTER FOUND. DECIDE A PROPER CUTTING POINT THAT STARTS
C     WITH SOMETHING THAT COMES AFTER AN OPERATOR
C-----------------------------------------------------------------------
      IP = 67 - L3
C-----------------------------------------------------------------------
C     LOOK FOR ANY OPERATOR
C-----------------------------------------------------------------------
         DO 310  L4 =1,8
         IF (IO(IA( L3 ),NT( L4 ))) 310, 350, 310
  310    CONTINUE
  320 CONTINUE
      IF (IF1.NE.1) GO TO 410
  330 CALL COPY(1,IAF,1,IPROG(1,ISTART+1),6)
      IF (IABS(IAF).EQ.IABS(NUM(11)) )GO TO 1050
         DO 340   L3 =1,80
  340    CALL COPY(1,IA( L3 ),1,IPROG(1,ISTART+1), L3 )
      ISTART = ISTART + 1
      IF1 = 72
      CALL STANAL(7,IF1,IVTAB,NV,IA,NUM)
      IS = 7
      GO TO 740
  350 KS = 7
C-----------------------------------------------------------------------
C      BLANK IBB
C-----------------------------------------------------------------------
      DO 360  L3 =1,60
  360 IBB( L3 )=NUM(11)
      IBB(6) = NT(4)
C-----------------------------------------------------------------------
C      STORE IN IBB
C-----------------------------------------------------------------------
      DO 370  L3 =IP,72
      IBB (KS) = IA( L3 )
      IA( L3 )=NUM(11)
  370 KS = KS + 1
      IST1 = ISTART +1
      ISTST = ISTOP+ISTART+1
C-----------------------------------------------------------------------
C     MOVE THE RECORDS TO INTRODUCE A SPACE TO INSERT A CONTINUATION
C      LINE
C-----------------------------------------------------------------------
      DO 380  L3 =IST1,ISTOP
      IR = ISTST - L3
         DO 380  L4 =1,10
  380 CALL COPY8 (IPROG( L4 ,IR+1),IPROG( L4 ,IR))
C-----------------------------------------------------------------------
C     NOW INSERT
C-----------------------------------------------------------------------
      DO 390  L3 =1,80
  390 CALL COPY (1,IPROG(1,IST1), L3 ,IBB( L3 ),1)
C-----------------------------------------------------------------------
C     RESET THE LIMITS
C-----------------------------------------------------------------------
      JPEEP=JPEEP+1
      ISTOP = ISTOP + 1
      IF (IF1 - 1) 410, 400, 410
  400 IF1 = 0
      GO TO 330
  410 IS = 7
      IE = 72
      IF (IFLAG.EQ.0) GO TO 430
         DO 420   L3 =7,IFLAG
  420    IA( L3 )=ID( L3 )
      GO TO 1050
  430 CALL STANAL(IS,IE,IVTAB,NV,IA,NUM)
      GO TO 1050
  440 GO TO ( 1020, 450, 450, 530, 530, 700, 920),JT
```

```
C-----------------------------------------------------------------------
C .. .. .. ..    TEST INPUT OUTPUT STATEMENTS
C-----------------------------------------------------------------------
  450 JP2 = JT + 2
      DO 460 L3 =3,JP2
      IF (IQ(KV( L3 ),IFORT(JT, L3 ))) 270, 460, 270
  460 CONTINUE
      DO 470 L3 =7,72
      IF (IQ(IA( L3 ),NUM(14))) 470, 480, 470
  470 CONTINUE
      GO TO 270
  480 IE = L3 - 1
      DO 490  L5 =11,IE
      IF(IQ(IA( L5 ),NUM(12))) 490, 500, 490
  490 CONTINUE
      GO TO 270
  500 IB1 = L5 + 1
      CALL ARGGF(IB1 ,IE,NUM,IA,N,NFORM,2,IPOS,1,
     *ITAG,NDL,IFAIL)
      IF1 = L5 -1
      DO 510 L3 =1,IE
      IF (IABS(IA( L3 ))-IABS(NUM(13))) 510, 520, 510
  510 CONTINUE
      GO TO 270
  520 IS = L3 + 1
      CALL STANAL(IS,IF1,IVTAB,NV,IA,NUM)
      IF1 =72
      CALL STANAL(IF,IF1,IVTAB,NV,IA,NUM)
      GO TO 1020
C-----------------------------------------------------------------------
C .. .. .. ..    CHECK THE GO TO AND THE COMPUTED GO TO STATEMENTS
C-----------------------------------------------------------------------
  530 DO 540 L3 =3,4
      IF(IQ (KV( L3 ),IFORT(JT, L3 ))) 1020, 540, 1020
  540 CONTINUE
      IF(IQ (KV(5) , NUM(13))) 550, 600, 550
  550 DO 560 L6 = 11 ,72
      IR = 83 - L6
      IF(IQ (IA(IR) , NUM(11))) 570, 560, 570
  560 CONTINUE
  570 IE = IR
      DO 580  L6 = 11,IE
      IR =  IE-  L6 + 11
      IF(IQ (IA(IR) , NUM(15))) 580, 590, 580
  580 CONTINUE
  590 IS = IR + 1
      CALL ARGGF(IS,IE,NUM,IA,N,NFORM,2,IPOS,0,IT
     *AG,NDL,IFAIL)
      GO TO 1020
  600 DO 610  L6 = 11,72
      IF(IQ (IA( L6 ),NUM(13))) 610, 620, 610
  610 CONTINUE
  620 IS =  L6 + 1
  630 DO 650  L6 = IS ,72
      IF(IQ (IA( L6 ), NUM(12))) 640, 670, 640
  640 IF(IQ (IA( L6 ), NUM(14))) 650, 660, 650
  650 CONTINUE
      CALL STANAL(7,72,IVTAB,NV,IA,NUM)
      GO TO 1020
  660 IE1 = 7
      CALL STANAL(IF,IE1,IVTAB,NV,IA,NUM)
      IF =  L6 - 1
      CALL ARGGF(IS,IE,NUM,IA,N,NFORM,2,IPOS,0,IT
     *AG,NDL,IFAIL)
      GO TO 1020
  670 IE =  L6 - 1
      CALL ARGGF(IS,IE,NUM,IA,N,NFORM,2,IPOS,0,IT
     *AG,NDL,IFAIL)
      DO 680  L6 = IS , 72
      IF(IQ (IA( L6 ), NUM(12))) 680, 690, 680
  680 CONTINUE
  690 IS =  L6 + 1
      GO TO 630
C-----------------------------------------------------------------------
C .. .. .. ..    CHECK THE IF STATEMENT
C-----------------------------------------------------------------------
  700 IF (IQ(KV(3),NUM(13))) 270, 710, 270
  710 DO 720  L6 = 9,72
      IF (IQ(IA( L6 ), NUM(13))) 720, 730, 720
  720 CONTINUE
      GO TO 270
  730 IS = L6 +1
      ISJS = IS
      IE = 72
      CALL STANAL(IS,IE,IVTAB,NV,IA,NUM)
      ICONT = 1
  740 DO 790  L6 = ISJS ,72
      IF(IQ (IA( L6 ) , NUM(14))) 770, 750, 770
  750 ICONT = ICONT - 1
      IF (ICONT) 790, 760, 790
  760 IF =  L6 -1
      GO TO 850
  770 IF(IQ (IA( L6 ), NUM(13))) 790, 780, 790
  780 ICONT = ICONT + 1
  790 CONTINUE
      IF1 = 1
C-----------------------------------------------------------------------
C....... VERY POSSIBLE THAT THE 'IF' CONTINUES ON THE NEXT
C....... CARD
C-----------------------------------------------------------------------
      GO TO 250
  800 CONTINUE
      IS = IF + 2
      ICC=0
  810 DO 820  L6 = IS,72
      IF(IQ (IA( L6 ), NUM(12))) 820, 830, 820
  820 CONTINUE
      GO TO 800
  830 CONTINUE
      IE1 =  L6 - 1
      DO 840  L5 =1,80
  840 IBBB( L5 )=IA( L5 )
      CALL ARGGF(IS,IE1,NUM,IA,N,NFORM,2,IPOS,0,I
     *TAG,NDL,IFAIL)
      IF (IFAIL.NE.1) GO TO 860
      DO 850  L5 =1,80
  850 IA( L5 )=IBBB( L5 )
      GO TO 800
  860 DO 870  L6 = IS ,72
      IF(IQ (IA( L6 ), NUM(12))) 870, 880, 870
  870 CONTINUE
  880 IS =  L6 + 1
      ICC = ICC + 1
      IF (ICC - 2) 810, 890, 890
  890 CALL ARGGF(IS,IS+5,NUM,IA,N,NFORM,2,IPOS,0,
     *ITAG,NDL,IFAIL)
      GO TO 1020
  900 IFLAG = IF      + 1
      DO 910  L6 = 7,IFLAG
      IB ( L6 )= IA( L6 )
  910 IA ( L6 )= NUM(11)
      GO TO 200
```

```
C-----------------------------------------------------------------------
C .. .. .. ..   CHECK THE 'DO' STATEMENT
C-----------------------------------------------------------------------
  920    DO 930  L6 = 8,72
C-----------------------------------------------------------------------
C **** CHECK THE PRESENSE OF THE LETTER "O"
C-----------------------------------------------------------------------
         IF(IO(IA( L6 ), NUM(15))) 930, 940, 930
  930    CONTINUE
      GO TO 1050
  940 IS =  L6 + 1
      KEN = 0
C-----------------------------------------------------------------------
C **** CYCLE TO FIND THE FIRST ALPHABETIC CHARACTER
C-----------------------------------------------------------------------
         DO 980  L6 = IS , 72
         IF(IO (IA( L6 ), NUM (11))) 950, 980, 950
  950       DO 970  L7 = 1,10
C-----------------------------------------------------------------------
C **** LOOK FOR NUMERIC CHARACTERS
C-----------------------------------------------------------------------
            IF(IO (IA( L6 ), NUM( L7 ))) 970, 960, 970
  960       KEN = KEN + 1
C-----------------------------------------------------------------------
C **** A NUMBER OF MORE THAN FIVE DIGITS IS IN ERROR
C-----------------------------------------------------------------------
            IF (KEN - 5 ) 980, 990, 1020
  970       CONTINUE
         IF(KEN.LE.0) RETURN
         GO TO 990
  980    CONTINUE
  990 IE =  L6 - 1
         DO 1000 L7 = 8,72
         IF(IO(IA( L7 ),NUM(17))) 1000, 1010, 1000
 1000    CONTINUE
      RETURN
 1010 CALL STANAL( L6 , L7 ,IVTAB,NV,IA,NUM)
      ITAG = ITAG + 1
      CALL ARAGE(IS,IL,NUM,IA,N,MFORM,2,IPOS,0,IT
     *AG,OPL,IFAIL)
      IFAIL = 0
      NDL (ITAG) =  JSUM(IS,IE,NUM,IA,KMN,KV,JTL,LENGTH
     *,IFAIL)
      IFF = 72
      CALL STANAL( L7 +1,IFF,IVTAB,NV,IA,NUM)
      RETURN
 1020 IF (IFLAG) 1030, 1050, 1030
 1030    DO 1040 L6 = 7,IFLAG
 1040    IA( L6 )= ID ( L6 )
 1050 RETURN
      END
```

```
      FUNCTION JSUM (IS,IE,NUM,IA,KMN,KD,JTL,LENGTH,IFAIL)
C-----------------------------------------------------------------------
C     COLLECTS A NUMBER FROM SEPARATE DIGITS AND DELIVERS IT
C      AS ONE INTEGER
C-----------------------------------------------------------------------
      DIMENSION NUM(24) ,IA(LENGTH),KD(5)
  100 KK = 5
         DO 140 L1 =IS,IE
         IF(IO(IA( L1 ),NUM(11))) 110, 140, 110
  110       DO 130 L2 =1,10
            IF(IO (NUM( L2 ),IA( L1 ))) 130, 120, 130
  120       KD(KK) = L2 - 1
            KK = KK - 1
            IF (KK) 150, 150, 140
  130       CONTINUE
         GO TO 170
  140    CONTINUE
      IF(KK-5) 150, 170, 170
  150 KMN = KK + 1
      L1 =1
      ISUM = 0
         DO 160 L3 = KMN,5
         ISUM = ISUM + KD ( L3 )* L1
  160    L1 = L1 *10
      JSUM = ISUM
      RETURN
  170 IF (IFAIL .NE.100) GO TO 180
      IFAIL = 1
      RETURN
  180 WRITE (6, 1)JTL,(IA(I),I=IS,IE)
      RETURN
    1 FORMAT(' ERROR- EITHER NO STATEMENT LABLE OR LABLE CONTAINS NON NU
     ,MERIC CHARACTER'/1X,A6,10X,80A1)
      END
```

```
      SUBROUTINE STANAL (IS,IE,IVTAB,NV,IA,NUM)
C-----------------------------------------------------------------------
C     CHANGES VARIABLE NAMES
C-----------------------------------------------------------------------
      DIMENSION IVTAB(1,100),IA(80),NUM(24),NT(10),KDD(5)
      DIMENSION KD(5)
      DATA JTL/'STANAL  '/,LENGTH/8/,IACO/'0'/
      DATA NT/'(',')',',','+','-',1H/,'*','=','.','L'/
C-----------------------------------------------------------------------
C ****STORE STARTING ADDRESS       ISSTO= IS
C-----------------------------------------------------------------------
      IPPT = 0
      IFLAG = 0
      ISSTO = IS
  100 IC = 1
  110    DO 570 L1 =IS,IE
         IF( L1 .EQ.IS)IPSTOR= L1
         IF (IO(IS(73),IACO)) 130, 120, 130
  120    L1 =IS+ IC
         GO TO 210
C-----------------------------------------------------------------------
C **** GET RID OF THE BLANKS
C-----------------------------------------------------------------------
  130    IF (IABS(IA( L1 ))-IABS(NUM(11))) 140, 570, 14
```

```
C ****  IF A CHARACTER IS PRESENT CHECK IT IS NOT A SPECIAL CHARACTER
C-----------------------------------------------------------------------
  140       DO 150 L2 =1,9
            IF (IABS(IA( L1 ))-IABS(NT( L2 ))) 150, 170, 150
  150       CONTINUE
  160   IF (IC.LE.0) GO TO 570
        KD(IC) = IA( L1 )
        IF(IC.EQ.1)ILSTOR = L1
        IC = IC + 1
C-----------------------------------------------------------------------
C ****  UP TO 6 CHARACTERS PER VARIABLE NAME ARE ALLOWED
C-----------------------------------------------------------------------
        IF (IC.GT.7)  IC = -1
        GO TO 570
  170   IF (IC-1) 180, 180, 190
  180   IS = L1 +1
        GO TO 190
  190   IF ( L1 -IS) 570, 570, 200
  200   IE1 = L1 -1
  210      DO 220 L2 =IC,8
  220      KD( L2 )=NUM(11)
        JJ1 = IC -1
C-----------------------------------------------------------------------
C ****  CHECK THE PRESENSE OF A NUMERIC CHARACTER
C-----------------------------------------------------------------------
            DO 230  L3 = 1,10
            IF(IABS(KD(1))-IABS(NUM( L3 ))) 230, 550, 230
  230       CONTINUE
        IF (IMSTOP) 250, 250, 240
C-----------------------------------------------------------------------
C ****  IS THE VARIABLE PRECEDED BY A DECIMAL POINT
C-----------------------------------------------------------------------
  240   IF(IABS(IA(ILSTOP -1)) - IABS(NT(9))) 250, 260, 250
  250   IF (IC - 1) 550, 550, 280
  260   IF (IDPT - 1) 540, 270, 270
  270   IDPT =0
  280   IC = 1
            DO 290  L2 =1,8
  290       CALL COPY (1,IAF, L2 ,KD( L2 ),1)
  300       DO 430 L4 = 1, NV
            IF(IFLAG) 310, 310, 340
C-----------------------------------------------------------------------
C ****  COMPARE THE NAME WITH OTHER VARIABLE NAMES
C-----------------------------------------------------------------------
  310       CALL COMPA (IVTAB(1, L4 ),IAF,IZ)
            IF (IZ-1) 430, 320, 460
  320       CONTINUE
            KD(1) = NT(10)
               DO 330 L5 =2,8
  330          KD( L5 )=NUM(11)
C-----------------------------------------------------------------------
C ****  SET VARIABLE TO ITS POSITION IN IVTAB
C  ****  AND CHANGE NUMERICAL VALUE OF POS. INTO CHARACTERS
C-----------------------------------------------------------------------
            ISUM = L4
  340          DO 350 L5 =IS,IE1
  350          IA( L5 )=NUM(11)
            L5 =10
               DO 360 L6 = 1,4
               IF ( L5 -ISUM/ L5 * L5 ) 360, 360, 370
  360          L5 = L5 *10
            L6 = 5
  370          DO 380 L5 =1, L6
               JP =  L6 + 1 - L5
               JI = 10**(JP - 1)
               JF = ISUM/JI
               KD( L5 +1)=NUM(JF+1)
  380          ISUM = ISUM - JI*JF
            L5 = L6 + 1
            JJ1 = L5 +1
            ISPAS = IE1 - IS
C-----------------------------------------------------------------------
C ****  MOVING SPACES
C-----------------------------------------------------------------------
            ILO = ISPAS - JJ1
            ISTOR = ILO
            IF(ILO) 390, 460, 460
  390       IS1 = L1 +1
            IS2 = IS1
            IBO = IABS(IBO)
               DO 450  L7 =IS2 ,72
               IF (IABS(IA( L7 ))-IABS(NUM(11))) 450, 410, 450
  410          IIS = L7 + IS1
                  DO 420  L6 = IS1 , L7
                  IR = IIS - L6
                  IA(IR) = IA(IR - 1)
  420             IA(IR-1) = NUM(11)
               IF ( L7 -IE) 440, 440, 430
  430          IE = IE + 1
  440          IBO = IBO - 1
               IS1 = IS1 + 1
               L1 = L1 +1
               IE1 = IE1 + 1
               IS2 = L7
               IF (IBO) 460, 460, 400
  450          CONTINUE
            IS = ISSTO
            RETURN
C-----------------------------------------------------------------------
C ****  PUT THE NEW VARIABLE IN HERE
C-----------------------------------------------------------------------
  460       IPOS = L1 -JJ1
            IA(IPOS -1) = NUM(11)
               DO 470  L6 = 1,JJ1
               IA (IPOS) = KD( L6 )
  470          IPOS = IPOS + 1
            IFLAG = 0
            GO TO 550
  480       CONTINUE
C-----------------------------------------------------------------------
C ****  DOES THE VARIABLE START WITH AN 'L'
C-----------------------------------------------------------------------
          IF(IABS(KD(1)) - IABS(NT(10))) 550, 490, 550
  490       DO 510  L3 = 2,JJ1
C-----------------------------------------------------------------------
C ****  CHECK PRESENSE OF A NUMERAL
C-----------------------------------------------------------------------
               DO 500  L2 =1,10
               IF (IABS(KD( L3 ))- IABS(NUM( L2 ))) 500, 510, 500
  500          CONTINUE
            GO TO 550
  510       CONTINUE
          ISUM = JNUM (2,JJ1,NUM,KD,KDG,KDD,JTI,LENGTH,IVALE)
C-----------------------------------------------------------------------
C ****  CHANGE CHARACTERS INTO A NUMERICAL VALUE
C-----------------------------------------------------------------------
  520     NV = NV + 1
          ISUM = NV
C-----------------------------------------------------------------------
C ****  COPY THE VARIABLE INTO VARIABLES TABLE
C-----------------------------------------------------------------------
            DO 530  L7 =1,8
  530       CALL COPY(1,IVTAB(1,NV), L7 ,KD( L7 ),1)
C-----------------------------------------------------------------------
C       WRITE(6,321)NV,IVTAB(1,NV)
C-----------------------------------------------------------------------
```

```
      IFLAG = 1
      GO TO 330
C-----------------------------------------------------------------
C **** CHECK WHETER THE WHOLE STATEMENT IS FINISHED
C-----------------------------------------------------------------
 540     IPPT = 1
 550     IF ( L1 -IE) 560, 580, 560
 560     IS = IE1+2
         IF (IS.GT.IE) GO TO 600
         IF (IO(IA(73)),IACO)) 100, 600, 100
 570     CONTINUE
      IF (IC-1) 610, 610, 580
 580  IF (IO(IA(73)),IACO)) 590, 600, 590
 590  IASTO = IA(73)
      IE1 = IS + IC - 1
      IA(73) = IACO
      IF (IS.GT.IE) GO TO 600
      GO TO 110
 600  IS = IASTO
      IA(73) = IASTO
 610  RETURN
   1  FORMAT(I3,3X,A8)
      END


      SUBROUTINE ARNSL (IS,IE,NUM,IA,N,NFO
     *RM,IP,IPOS,IFORM,ITAG,NDL,
     *IFAIL)
C-----------------------------------------------------------------
C       CHANGES STATEMENT LABLES
C-----------------------------------------------------------------
      DIMENSION NFORM(100),IPOS(2),NDL(25)
      DIMENSION NUM(24),IA(80),N(500),KD(5)
      DATA JTL,LENGTH/'APAGE    ',6/
C-----------------------------------------------------------------
C****    EXPLAIN THE NUMBER
C-----------------------------------------------------------------
      IFAIL = 100
      ISUM =     JNUM(IS,IE,NUM,IA,KUM,KD,JTL,LENGTH,IFAIL)
      IF (IFAIL.EQ.1) RETURN
C-----------------------------------------------------------------
C**** BLANK THE FIELD
C-----------------------------------------------------------------
 100     DO 110 L1 = IS,IE
 110     IA ( L1 )= NUM (11)
         DO 120 L2 = KPO , 5
 120     KD( L2 )= NUM (11)
      IHO = IE - IS
      GO TO( 360, 130),IP
 130  IF (IFORM) 140, 160, 140
 140  IPOS2 = IPOS(2)
         DO 150 L3 =1,IPOS2
         IF (ISUM - NFORM( L3 )) 150, 180, 150
 150     CONTINUE
      GO TO 1.0
 160  IPOS1 = IPOS(1)
         DO 170 L3 =1,IPOS1
         IF (ISUM - N( L3 )) 170, 180, 170
 180  ISUM = L3
      IF (IFORM) 200, 190, 200
 190  ISUM = 10*ISUM + 90
 200  L5 =10
      ITAG = ITAG
 210  KT = 0
         DO 220 L4 =1,4
         IF( L5 -ISUM/ L5 * L5 ) 220, 220, 230
 220     L5 = L5 *10
 230     DO 240 L5 =1, L4
         L6 = L4 +1- L5
         JP = 10**( L6 -1)
         JF = ISUM / JP
         KD( L5 )=NUM(JF +1)
 240     ISUM = ISUM - JP * JF
      IF(IHO - L4 ) 250, 300, 300
 250  IHJ = L4 -IHO
      IE1 = IE + 1
 260     DO 270 L6 =IE1 , 72
         IF (IO(IA( L6 ),NUM(11))) 270, 280, 270
 270     CONTINUE
 280  IA( L6 )=IA( L6 -1)
      L6 = L6 -1
      IF( L6 -IE1) 290, 290, 280
 290  IHJ = IHJ - 1
      IF (IHJ) 300, 300, 260
 300  ISPJ = IS + L4 -1
      KTK = 0
         DO 310  L6 =IS,ISPJ
         KTK = KTK + 1
 310     IA( L6 )=KD(KTK)
      IF (IP.GT.1)  GO TO 390
      KOUNT = 5
         DO 330  L7 = 1,5
         IPV = 6- L7
         IF(IO(IA(IPV),NUM(11))) 340, 320, 340
 320     KOUNT = KOUNT - 1
 330     CONTINUE
 340  IF (KOUNT.EQ.5) RETURN
         DO 350  L7 = 1 ,KOUNT
         L5 =6- L7
         IPV= KOUNT + 1 - L7
         IA(L5)= IA(IPV)
 350     IA(IPV ) = NUM(11)
      RETURN
 360  IF (IFORM) 370, 380, 370
 370  IPOS2 = IPOS(2)
      NFORM(IPOS2) = ISUM
      L3 =IPOS 2
      GO TO 1.0
 380  IPOS1 = IPOS(1)
      N(IPOS1) = ISUM
      L3 =IPOS(1)
      GO TO 1.0
 390  KTOT =  ISPJ + IS
         DO 400  L8 =IS,ISPJ
         IR = KTOT - L8
 400     IA(IR+1) = IA(IR)
      IA (IR)= NUM(11)
      IE = ISPJ + 1
      RETURN
      END
```

```
      SUBROUTINE ROUT (IA,LOP,IDIR,IATOT,LPOINT)
C-----------------------------------------------------------------------
C     LOOKS FOR S/P DESCRIPTION TO COLLECT THE CALLS
C-----------------------------------------------------------------------
      DIMENSION HOP(24),IATOT(10,350),IA(30),K
     *D(12),I IR(6600)
      DATA ICALL/'CALL'/
C-----------------------------------------------------------------------
C.......... FIND THE FIRST 11 CHARACTERS ..............
C-----------------------------------------------------------------------
      K = 0
         DO 110 L1 =7,72
         IF (IOCIA( L1 ),HOP(11))) 100, 110, 100
  100    K = K +1
         KD(K) = IA( L1 )
         IF (K .GE.11) GO TO 120
  110    CONTINUE
C-----------------------------------------------------------------------
C......... COMPARE WITH CALL  .......
C-----------------------------------------------------------------------
      IF (K.LT.4)RETURN
  120    DO 130 L1 =1,4
         IF = 1
         CALL COMP (I2,ICALL, L1 ,KD( L1 ),1)
         IF (IZ) 130, 170, 130
  130    CONTINUE
      K = 1
         DO 140 L1 =5,11
         IF (IOCKD( L1 ),HOP(13))) 140, 150, 140
  140    CONTINUE
      IF (IOCIA(11),HOP(11))) 170, 160, 170
  150 IF ( L1 -5) 170, 17), 160
  160 IF = L1 - 1
      CALL SEARCH (IE,KD,IDIR,IATOT,LPOINT)
  170 RETURN
      END

      SUBROUTINE SEARCH(IE,KD,IDIR,IATOT,LPOINT)
      DIMENSION IDII (6600),IATOT(10,350), KD(12)
      DATA IBLANK/'     '/
      K = 1
         DO 110 L1 =5,IE
         CALL COPY(1,IC,K,KD( L1 ),1)
  100    K = K + 1
      IS = K
         DO 110 L1 =IS,8
         CALL COPY (1,IC,K,IBLANK,1)
  110    K = K+1
      CALL COPY(1,ID,1,KD(1),1)
         DO 120 L1 =2,8
  120    CALL COPY(1,ID, L1 ,IC, L1 -1)
      CALL CO PA(ID,IATOT(1,LPOINT+1),IZ)
      IF (IZ-2) 160, 130, 160
  130    DO 140 L1 =1,396
         IP = 6
         CALL COMPA (IDIR( L1 ),IC,IZ)
         IF (IZ - 2) 150, 140, 150
  140    CONTINUE
      WRITE (6, 1)IC
      RETURN
  150 L1 =( L1 -1)*10 + 6)1
      L2= L1 + 9
      WRITE (6, 2)(IDIR(I),I = L1 , L2 )
  160 RETURN
    1 FORMAT (' C',A8,' IS NOT FOUND IN DIRECTORY')
    2 FORMAT (' C',10A8)
      END
```

```
C  THIS IS A TEST SEGMENT
      WRITE (6,151) WI
151   FORMAT (6F10.5)
      DO 10 I = 1,100
      L1 = I + 10
      DO 20 J = I , L1
      L1 = J
      L2 = I
      G = 3.5
      DO 15 M = L1,L2
      L4 = 5
15    CONTINUE
      A = 3.5 + L1 + L2 + L3 + L4 + I + J + M + K
20    CONTINUE
      GO TO (1,2,3,4,10,10,10),I
   10 CONTINUE
1        Z = 10.5
      GO TO 40
2     A = B
      GO TO 120
   3  A = L1
    4 B = J
   40 CONTINUE
120   CONTINUE
         IF ( A.EQ.B) GO TO 1
      IF ( A ) 1,2,3
      READ (5,50) A
50    FORMAT (A8)
      CALL EXIT
      END
      FINISH
```

(A)

```
C-----------------------------------------------------------------------
C  THIS IS A TEST SEGMENT
C-----------------------------------------------------------------------
      WRITE (6, 1 ) WI
         DO 120 L1 =1,100
         L4 = L1 +10
            DO 110 L2 = L1 , L4
            L4 = L2
            L5 = L1
            G = 3.5
               DO 100 L3 = L4 , L5
               L6 =5
100            CONTINUE
            A = 3.5 + L4 + L5 + L7 + L6 + L1 + L2 + L3 +K
110         CONTINUE
         GO TO ( 130, 140, 150, 160, 120, 120, 120),L1
120      CONTINUE
130   Z = 10.5
      GO TO 170
140   A = B
      GO TO 180
150   A = L4
160   B = L2
170   CONTINUE
180   CONTINUE
      IF ( A.EQ.B) GO TO 130
      IF ( A ) 130, 140, 150
      READ (5, 2) A
      CALL EXIT
    1 FORMAT '(6F10.5)
    2 FORMAT (A8)
      END

      FINISH
```

(B)

FIGURE (E-2) TEST SAMPLE OF PROGRAM OUT PUT;A)NOT ARRANGED.
B)ARRANGED).

APPENDIX F

CASE STUDY: ANALYSIS OF TWO-WAY SINGLE PANEL SLABS LOADED BY A CONCENTRATED LOAD AT THE CENTRE

F,1. Introduction:

Two way slabs are structural systems that consist of a flat plate simply supported or built in with beams which span between columns. Simple or multiple pannels may extend over the beams in one or two directions. They find application in reinforced conceete buildings as floor systems in addition to their use in mechanical parts.

The aim of this study is to obtain an idea 'quantitativewise' about the effectiveness, applicabitity and fensibibty of the Finite Element method as used to analyse engineering structures. The study will be based upon comparison between three methods of solution, the experimental, the finite difference, and the finite element method. In using the latter method, different results were obtained according to different idealizations of the problem. The nearest idealization to the exprimental model required programming a new element into the present PAFEC 70+ scheme. This element is called an Offset Plate Bending and Stretching Element.

F.2. The Experimental Method:

Experimental tests were carried out using perspex models for the plate and the supporting beams, while steel columns fixed rigidly to a steel frame are used to support the whole assembly, so that it could be assumed (according to the big difference in mechanical properties of perspex and steel and knowing that the load used is only 10 Newtons) that the columns are of infinite stiffness. Strain gauges were used all over the plate and stiffeners to

measure the strains and hence moment coefficients while a mechanical extensometer was put to indicate the deflection under the point of application of the load.

Strain gauges are known for reliable strain measurements and the results obtained from this method could be reasonabley accurate. Thus the results of the experimental method will be taken as the judging factor that will give an idea of the other two methods used.

F.3. The Finite Difference Method:

The finite difference method was long known to be a versatile method that could be used to approximate solutions to many types of slab problems. Any combination of beam and slab configuration subject to any arbitrary loading system could be handled by this method. Just like the finite element method, the finite difference method satisfies the requirements of equilibrium and compatibiliy at only a finite number of points* rather than satisfying these conditions at every point on the slab and beams. However the process of limitting the effect of all the points to a finite number of points will be carried out on the expence of accuracy. The process of satisfying equilibrium and compatibiliy of each of 'n' points in the slab results in a set of 'n' linear algebraic equations which are written in terms of the deflection of each of the points considered. The set of equations is then solved simultaneously to give the deflection at each point.

---

*By this assumption is meant that both the finite element and the finite difference methods satisfy compatibility and equilibrium equations. However the difference between the two is that the finite difference method satisfies the above mentioned equations at a finite number of points while the finite element method satisfies them at a finite number if interconnected elements rather than points. (see also reference 125).

The operators required for the slab beam system were formed from the conditions of equilibrium of the model rather than from the governing differential equations. Modifications were made to cater for internal discontinueties and other conditions.

To carry on the complete solutions, the following assumptions were made in addition to those required for plates and shells as given for instance in reference (110). These assumptions are:

a- The behaviour of beam in both bending and torsion is linearly elastic.

b- The neutral axis of the beams coinsides with the middle plane of the slab, (not offset beams).

c- The forces and moments acting on a beam are assumed to be distributed along a line and not over a finite width.

d- Torsion in the beam is assumed to be uniform between the nodal points and the effects of warping are neglected.

e- The action of the column is assumed to be concentrated at a point, located at the interaction of the column centre-line and the midde plane of the slab.

Due to the enormous core requirements of such program, only one quarter of the slab-beam system was analysed and modification for the conditions of symmetry were made.

In comparing the results of the finite difference method with the experimental results, reasonable agreement was noticed between them. This agreement was good on two different levels; first, the was exellent agreement in the bending moment trends as well as the deflections as the edge beam stiffness was changed, and second, there existed relatively good agreement in the numerical values of

the deflections as well as bending momeuts.

One last comment to the above mentioned method is that as the problem geometries or boundary conditions become complicated, it becomes harder to apply the finite difference to the problem. In the first case (complicated geometry) finer and finer meshes must be used and in the second case (boundary conditions) more difference operators have to be involved.

F.4. The Finite Element Method:

As it was stated earlier in this Appendix, the finite element method reduces a problem with an infinite number to one with a finite number of unknowns by partitioning the problem region into smaller regions or elements and by carefully expressing the working variables of the problem in terms of approximating functions* that are assumed to be existing within each element. Elements are distinguished by nodal points on their boundaries (or more rarely at the interior). The approximating functions (usually polynomials, since they are easy to integrate and differentiate) are expressed in terms of values of the problem working variables at these nodal points.

As the above mentioned relations are set up, the nodal values of the working variable become the new unknowns. As soon as these unknowns are found, the approximating functions become known and the distribution of the working variable in the whole problem could then be obtained by merging the finite elements together to obtain the full system matrix.

5.4.1. Problem Idealization:

In using such a versatile finite element program, many approaches

---

*The approximating fuctions should satisfy certain criteria. See reference (124).

to the problem can be made. However the best idealization should be one which is most near to reality or at least to the experimental method. Two main idealizations were taken into consideration, one that is near to the experimental model, and one which is near to the finite difference method model. The dimensions and properties of all the models were assumed to be the same in all the cases being considered.

A diagramatic explanation of the models made in all the three methods is shown on figure F-1.

Probably model number (1) of the finite element models is the nearest to the experimental model as the offset plate elements offer a certain width upon which they are acting and unlike the finite difference model in which the beams are limitted to one line of action. Model (5) of the finite element models is the nearest to the finite difference model since the beam elements are defined by two nodes only and it could be implimented that they act on the line joining the two nodes. It should be pointed out at this stage that the beam element data on the PAFEC finite element program contains a measure of the beam width which again is a merit in comparison to the finite difference method.

Model number (2) offers an axis of symmetry that coinsides with the axis of the slab but it is not near to the experimental model and on the other hand it is not near to the finite difference method since the plate elements offer a finite width of action.

Model number (3) offer a line of action for the stiffness but is not suitable in the since that it offers no rotational stiffness between the stiffeners and the plate.

Model number (4) approaches the experimental model in the manner

that the beams are offset and their axis do not coinside with the axis of the plate. While on the other hand, it is still near to the finite difference model since the beams have one line of action.

Three types of meshes were used for the finite element models. the first is a fine mesh which is just similar to the mesh used in the finite difference models. The second is a coarse mesh of four elements on the slab only. The third mesh is a non-uniform mesh which provides accuracy at the points of importance only.

Constraints were put as shown in the diagrams. Obviously all the constraints which appear on the internal edges are due to symmetry since only one quarter of the slab was taken.

F.5. Modification of a Combined Bending and Stretching Element into an Offset Plate Element:

It is required to transfer the displacements and forces to an element which is offset from another depending on the forces and displacements that exist on the second element. Thus in figure (F-2), an element is offest by an amount ε_z from an existing node on another parallel element.

It is known that bending-stretching elements have five degrees of freedom per node[57], of which two degrees of freedom are for in-plane interaction and another three are for out of plane interaction*. These degrees of freedom could be multiplied by a transformation matrix $[\,T\,]$ to give the displacements on the element.

---

* in the PAFEC 70+ scheme this isset by putting a DATA statement in the element routina. For instance in R44210 which is a bending stretching routine we have DATA IDG/1,2,3,4,5,0/

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \\ \Phi_x \\ \Phi_y \end{Bmatrix}_{\text{element}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \varepsilon \\ 0 & 1 & 0 & -\varepsilon & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_x \\ u_Y \\ u_z \\ \Phi_x \\ \Phi_y \end{Bmatrix}_{\text{node}} \qquad (F.1)$$

or

$$\{ U_e \} = [T] \{ U_n \} \qquad (F.2)$$

where e,n refer to element and node respectively.

Also a transformation of forces could be effected and it will appear as,

$$\{ F_e \} = [T] \{ F_n \} \qquad (F.3)$$

The relation between the forces and displacements is given by:

$$\{ F_e \} = [S_e] \{ U_e \} \qquad (F.4)$$

However, inorder to merge the elements together it is required to obtain the above relation at the nodal points (where the elements are assumed to be linked together) rather than at the element ends. This could be fulfilled by simply substituting equations (F.2) and (F.3) in equation (F.4) above to obtain:

$$[T^{-T}] \{ F_n \} = [S] [T] \{ U_n \} \qquad (F.5)$$

or

$$\{ F_n \} = [T^T] [S] [T] \{ U_n \} \qquad (F.6)$$

From equation (F.6) it appears that a transformation is possible by modifying the stiffness matrix and the last equation (F.6) is re-written in the following form,

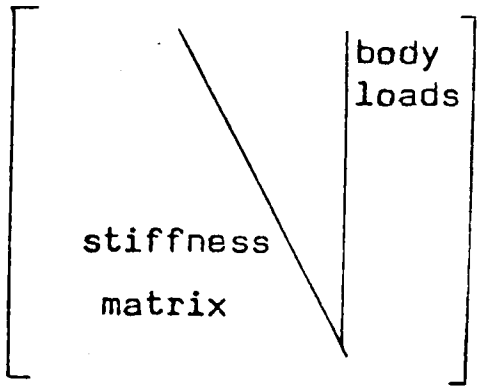
$$\{ F_n \} = [S_{new}] \{ U_n \} \qquad (F.7)$$

where $[S_{new}]$ is the new stiffness matrix.

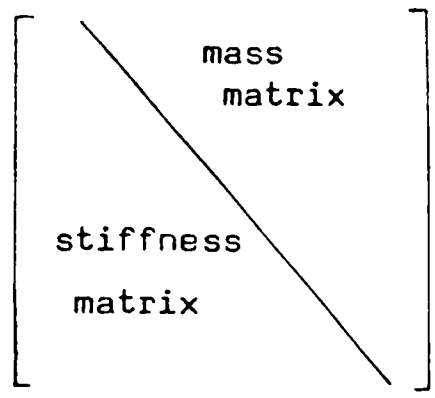
Thus the routine R44710 (bending and stretching plate element routine on the PAFEC finite element program) is modified to become an offset element routine by just changing the parts related to the stiffness matrix in the appropiate manner. It should be remenbered though that the above procedure was made for only one node, but it could be arranged to suit the whole either partition products or whole matrix products as follows:

$$\left\{ F_n \right\} = \begin{bmatrix} T^T & & & & \\ & T^T & & & \\ & & T^T & & \\ & & & T^T & \\ & & & & \ddots \end{bmatrix} \left[S \right] \begin{bmatrix} T & & & & \\ & T & & & \\ & & T & & \\ & & & T & \\ & & & & \ddots \end{bmatrix} \left\{ U_n \right\} \qquad (F.9)$$

The structure of the system matrix as set up by the PAFEC scheme is shown below,



for static applications



for dynamic applications

Actually one half of the stiffness matrix is stored in the lower triangle since it is known to be symmetrical for such elements.

The other half of the system array would be occupied by either a column vector of the body loads as it is the case with problems of static nature or it may be filled with a mass matrix as is the case with dynamic problems. In both cases (static and dynamic) one have to symmetrize the stiffness matrix before transforming it. Further, another transformation would be required for the body load or the

mass matrix whichever is relevant.

F.5.1. Modification of the Mass Matrix

Dynamic problems need extra force terms to be added over the static problemrs, those are the forces required to overcome the inertia. The equation linking forces and displacements could be given by:

$$\left\{ F_{tot} \right\} = \left(\left[s_e \right] - \omega^2 \left[M_e \right] \right) \left\{ u_c \right\} \qquad (F.9)$$

where, $\left\{ F_{tot} \right\}$ is the array of total forces.

Equation (F.9) above contains two parts; the first of which is the previous static term and the second is the dynamic forces term, like the static forces, the dynamic forces are given as a product of a matrix $\left[M \right]$ by the displacements matrix. The displacements are assumed to vary hormonically at a frequency ' ω ' rad/sec.

Substituting equations (F.2) and (F.3) into equation (F.9) and rearranging results in

$$\left\{ F_{tot} \right\}_{node} = \left[T^T \right] \left(\left[s_e \right] - \omega^2 \left[M_e \right] \right) \left[T \right] \left\{ u_n \right\} \qquad \text{.....(F.10)}$$

It is clear from equation (F.10) above that the mass matrix M_e is modified in the same manner the stiffness matrix was modified.

F.6. Results:

All the experimental results as well as the results of the finite difference method are taken from reference[123] and since they are so many it was thought unnecessary to repeat them in here. Only the results of various finite element model idealizations as given in figure (F-1) will be shown on figures (F-3 to F 13). Each of these figures take one of four titles:

a- drawing of the mesh and choice of model.

b- drawing of the constraints.

c- Drawing of the displaced shape diagram.

d- A comparison drawing that shows the change of one variable.

As mentioned earlier in this appendix, the main interest is to discover the merits of using the finite element method over the finite difference method as an analysis procedure, hence it was found unnecessary to follow the aspects of the finite difference program (ZMA) with detail since it is explained elsewhere[123] including the calculation of the deflection coefficients and the moment coefficients rather than just comparing the actual deflections due to the concentrated load at the centre.

F.7. Discussion of Results:

From the different possible problem idealizations, it is found that the finite element method is a flexible and easy to apply method for better geometrical similarity with the actual model in comparison to the finite difference method.

Correctness of the solution could be check by comparing the effect of reducing the edge deflection as the edge beam stiffness is increased, as well as the reduction of the deflections in general.

As a matter of quick reference, all the deflections under the point of application of the loads were put to be equal on all the displaced shape diagrams rather than putting them as the absolute displacements. All the other displacements are scaled in accordance. By this way it becomes possible to note the differences quickly.

All the results taken from models 2,3, and 4 of the finite elements idealization models were better than those obtained by the finite difference method. Their values vary differently between the experimental values and the finite difference values. In comparison with experimental results, the difference was found to

be in the range of 6% to 14% .,

The values obtained from model (5) which is the nearest idealization to the finite difference model were very close to those obtained by the finite difference model, but their numerical values is higher which means that they are still nearer to the experimental model.

Results obtained from model number (1) are found to be completely conforming with experimental results (0.5% difference only), while with the finite difference method the results showed a deviation of up to 25% from the experimental results.

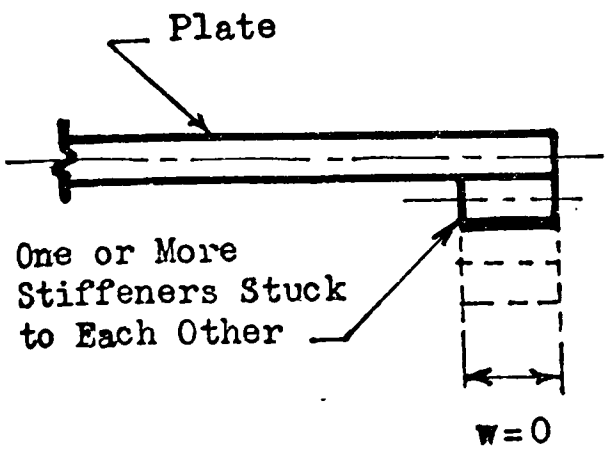
Results obtained with the crude meshes showed good agreement with the experimental results, and when comparing them with results obtained by the finite difference method the appeared to be fairly reasonable.

F.8. Conclusion:

The finite element method seems to give better results than the finite difference method in general. Results could be obtained at early stages with fairly crude meshes and would still be valid to a great extent. As far as versatility is concerned it is certain that a slightly sophisticated program could be adopted to solve a large varicty of problems, while for the finite difference method a new program has to be written every time since operators must be specified differently for every problem.

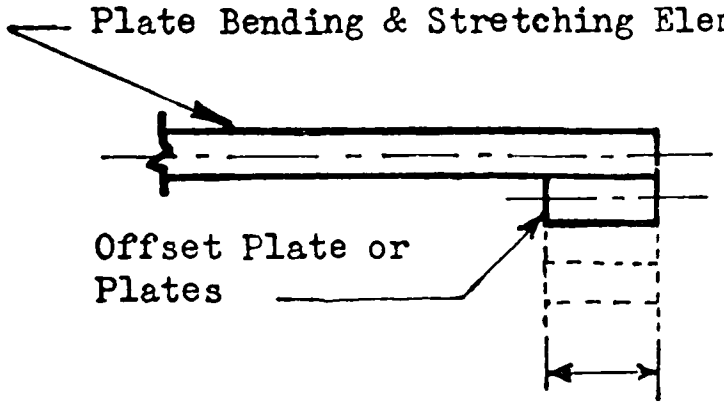
FIGURE F-1 Beam - Slab Idealization for Different Models

EXPERIMENTAL MODEL

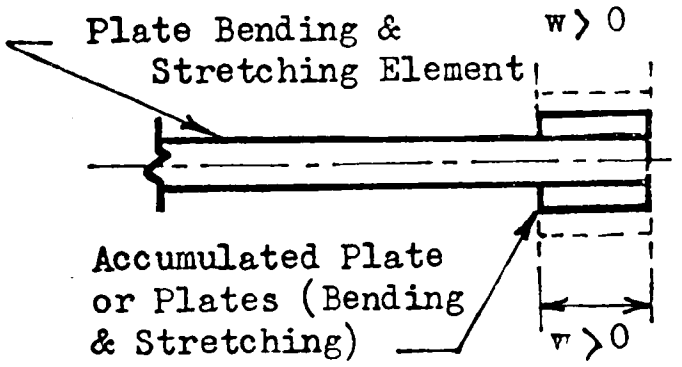


FINITE ELEMENT MODEL

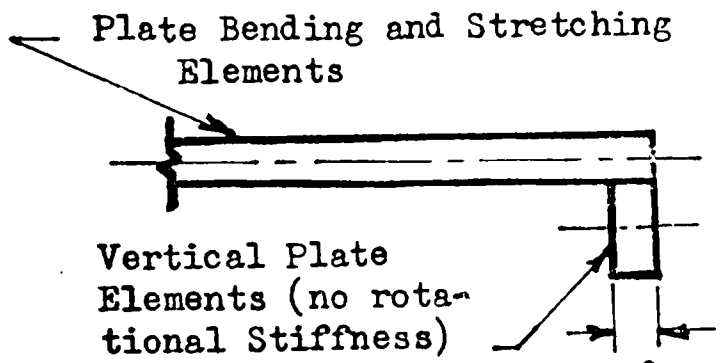
1)



2)

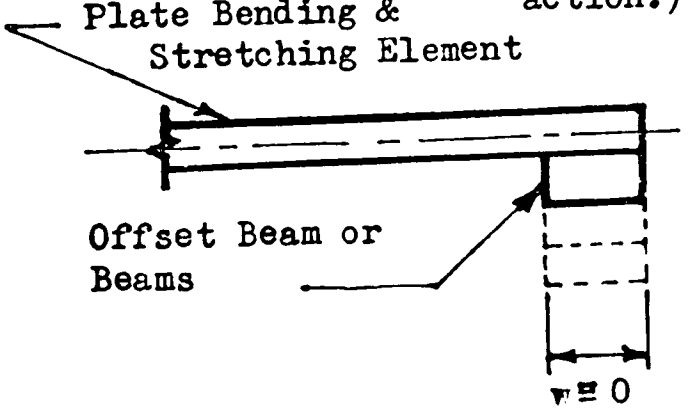


3)



(beam acting on one line of action.)

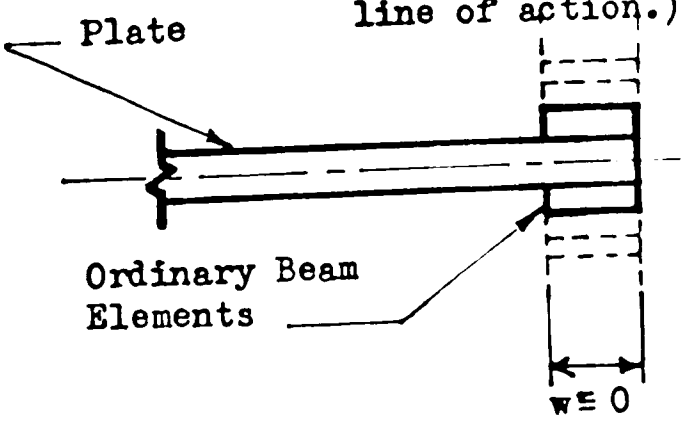
4)

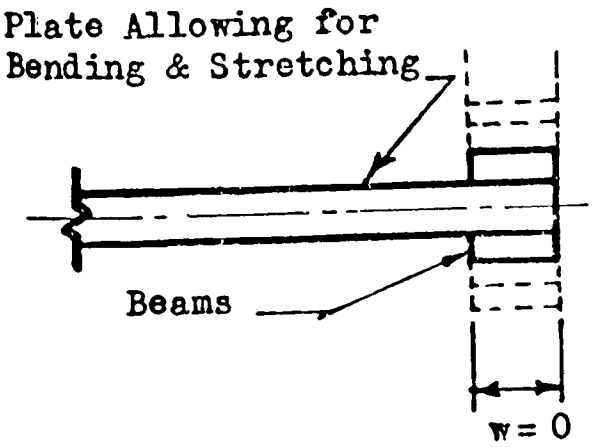


(beam acting on one line of action.)

FINITE DIFFERENCE MODEL

5)





(beams act on one line of action)

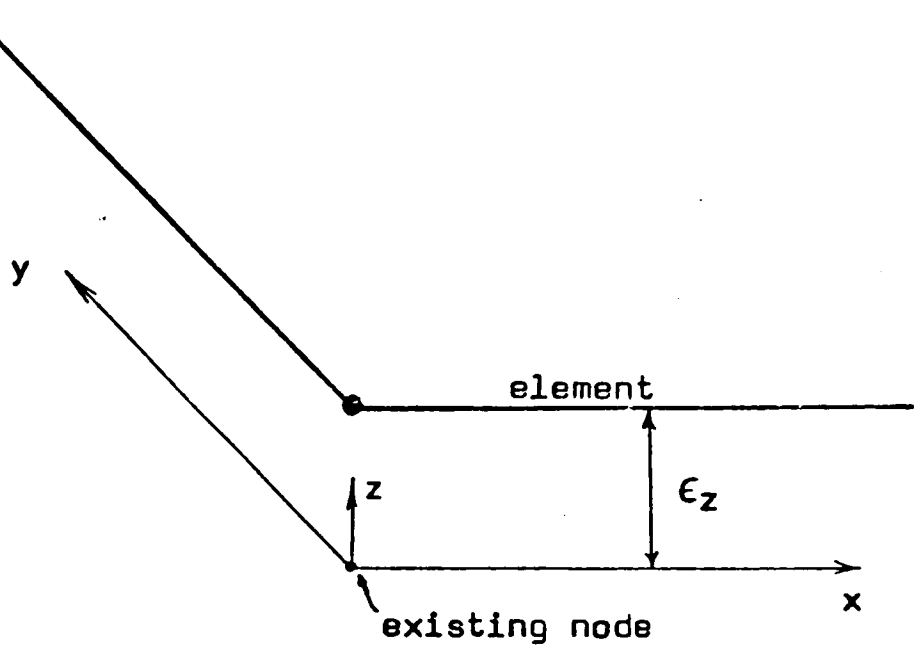


Figure (F-2) Two plate elements offset by ϵ_z

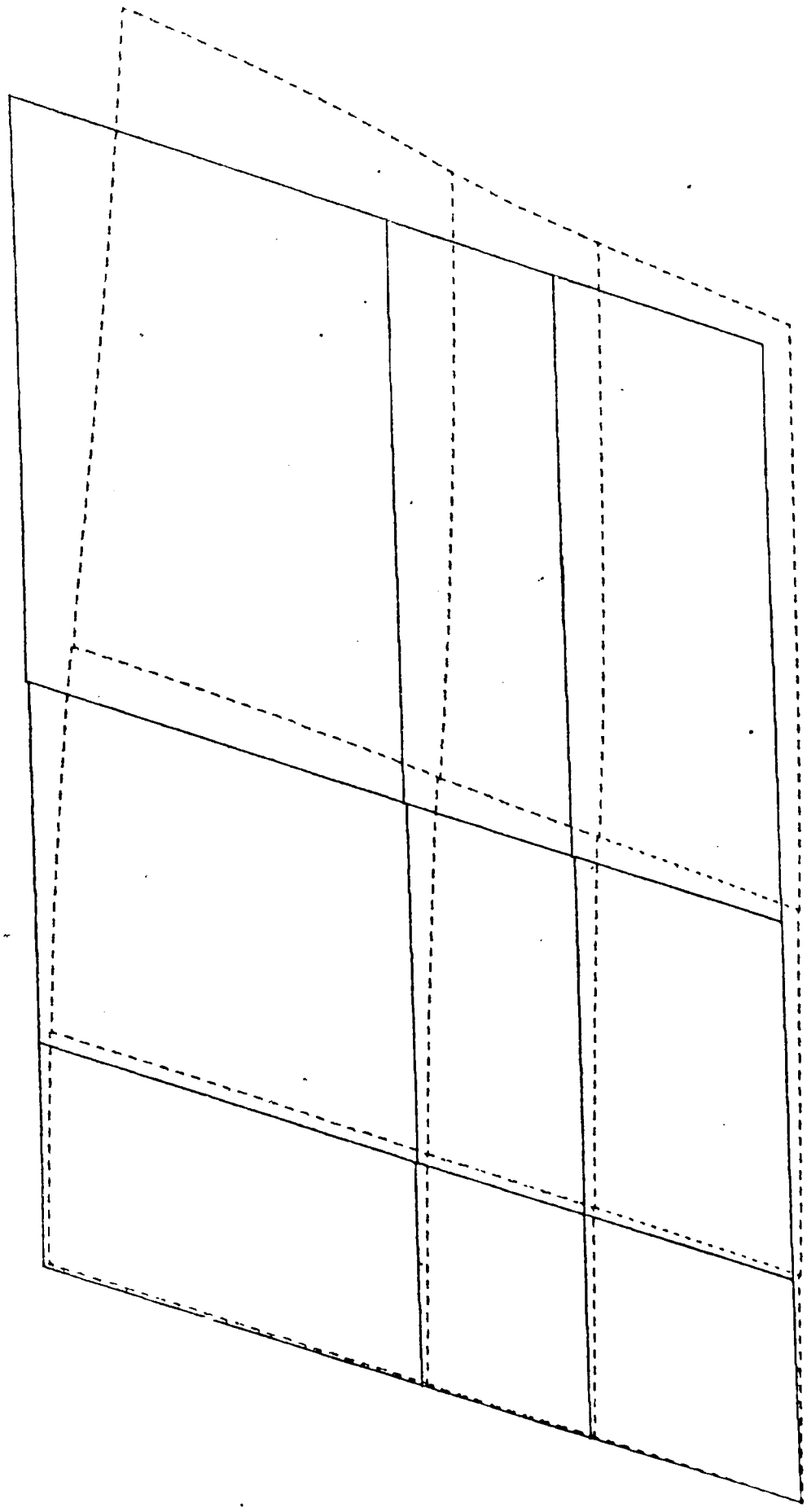


FIGURE F-3 Displaced Shape Diagram for Finite Element Model Number 1. Number of Stiffeners 1.

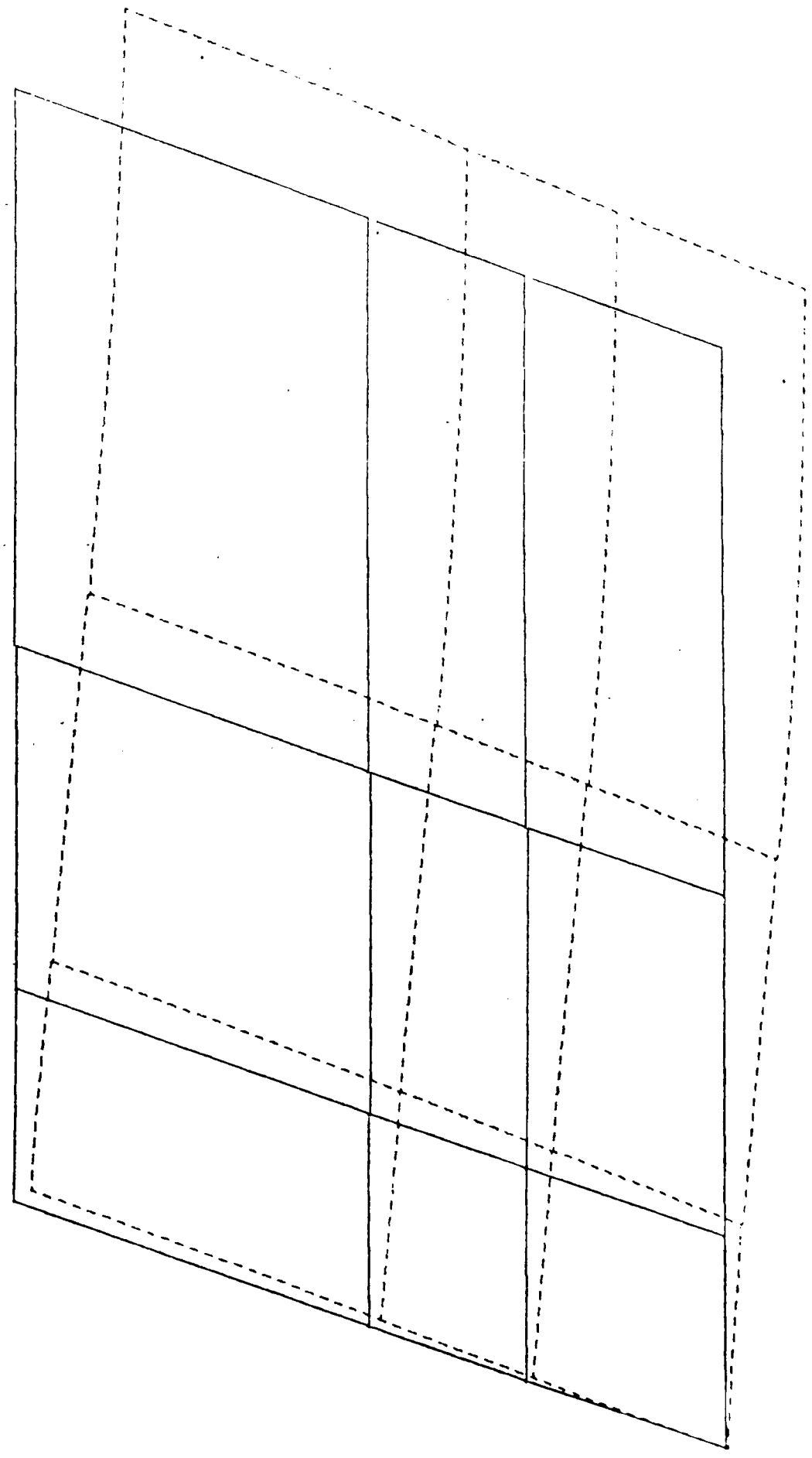


FIGURE F-4 Displaced Shape Diagram for Finite Element Model Number 2.
Number of Stiffiners 1

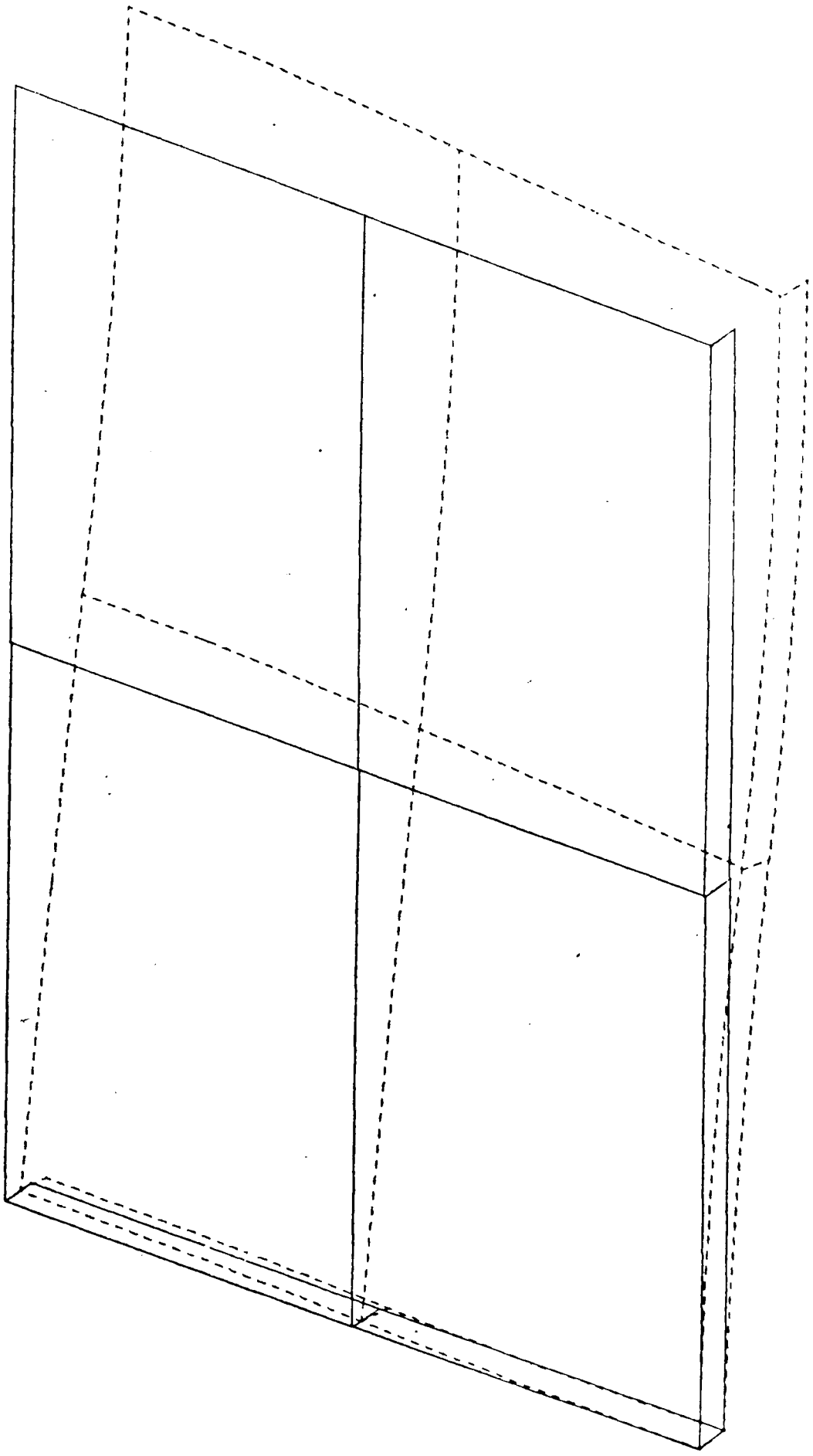


FIGURE F-5 Displaced Shape Diagram for Finite Element Model Number 3.
Number of Stiffeners 1
Crude Mesh

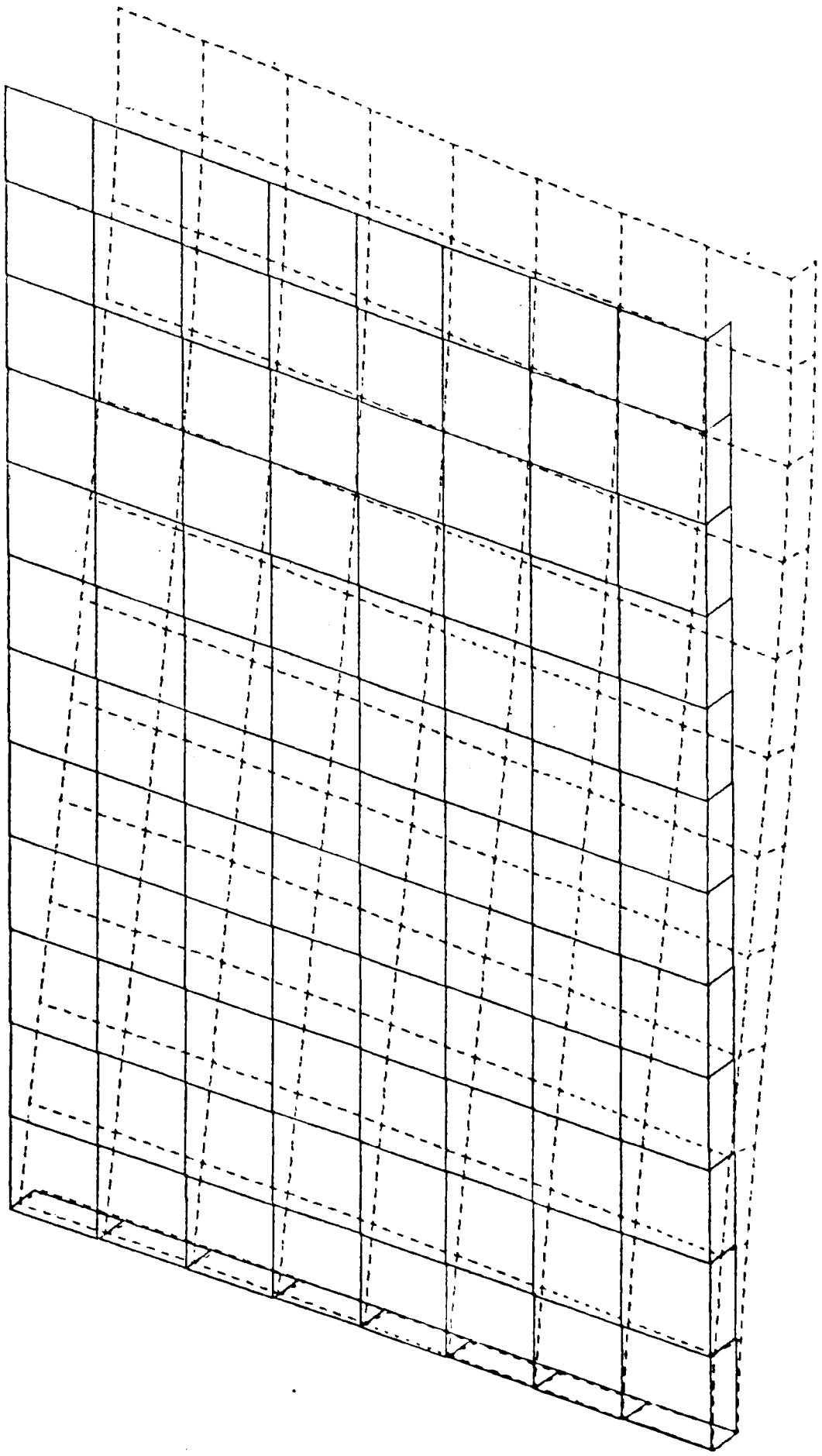


Figure F-6 Displaced Shape Diagram for Finite Element Model Number 3.
Number of stiffeners 1
Fine Mesh

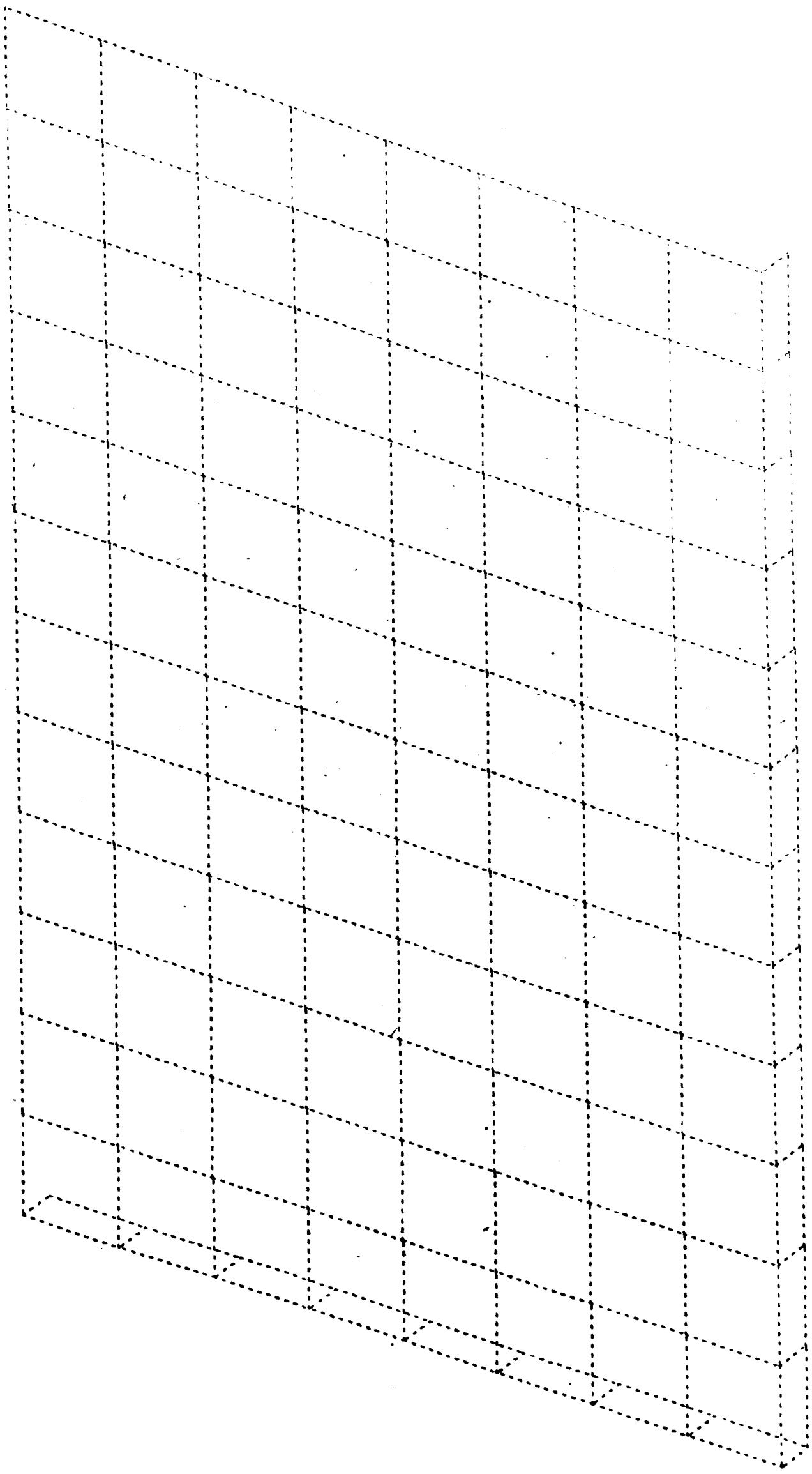


FIGURE F-7 Mesh Generation for Model Number 3.
Number of Stiffeners 1
Fine Mesh

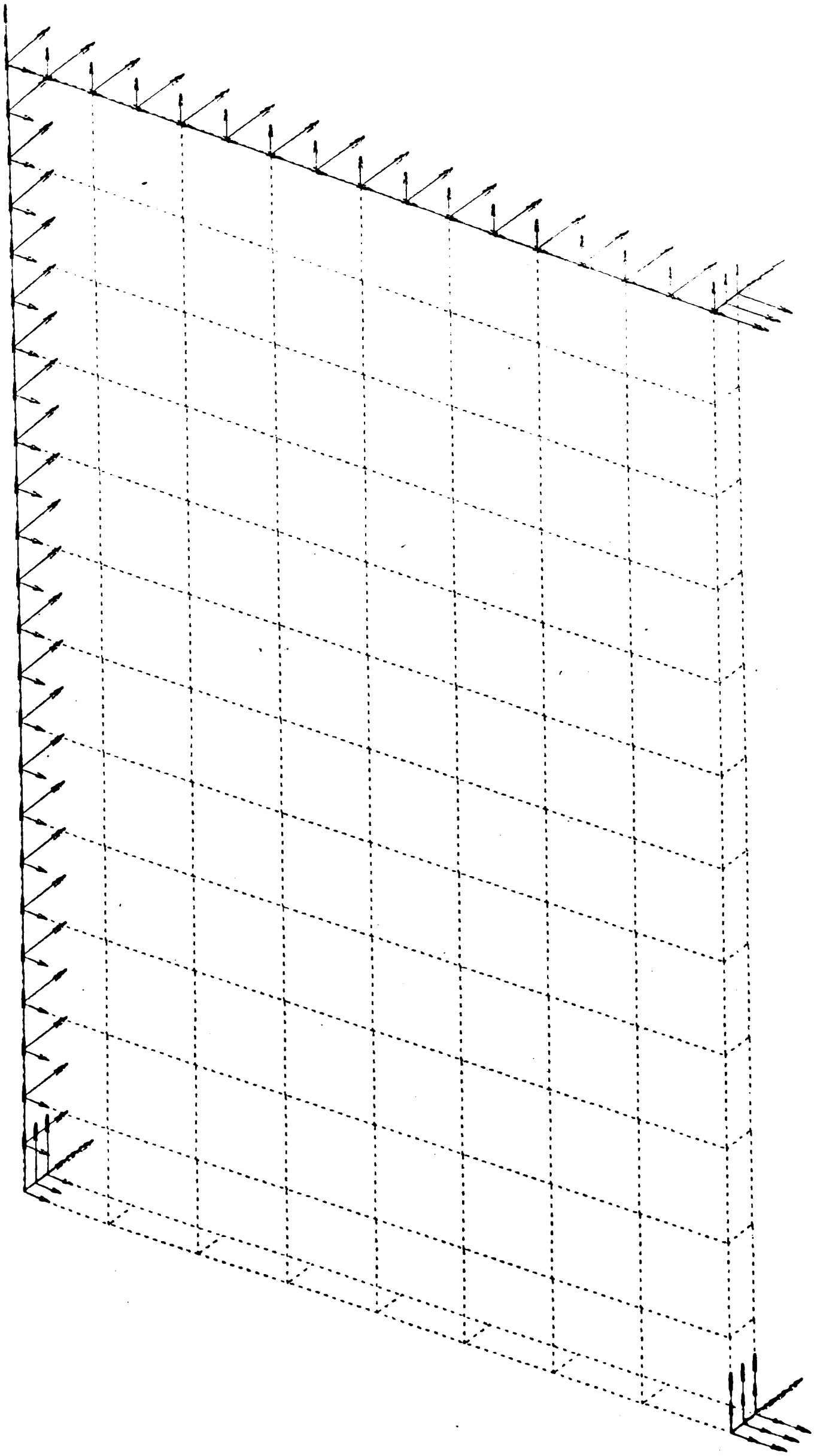


Figure F-8 Constraints for Finite Element Model Number 3.
Number of Stiffeners 1
Fine Mesh

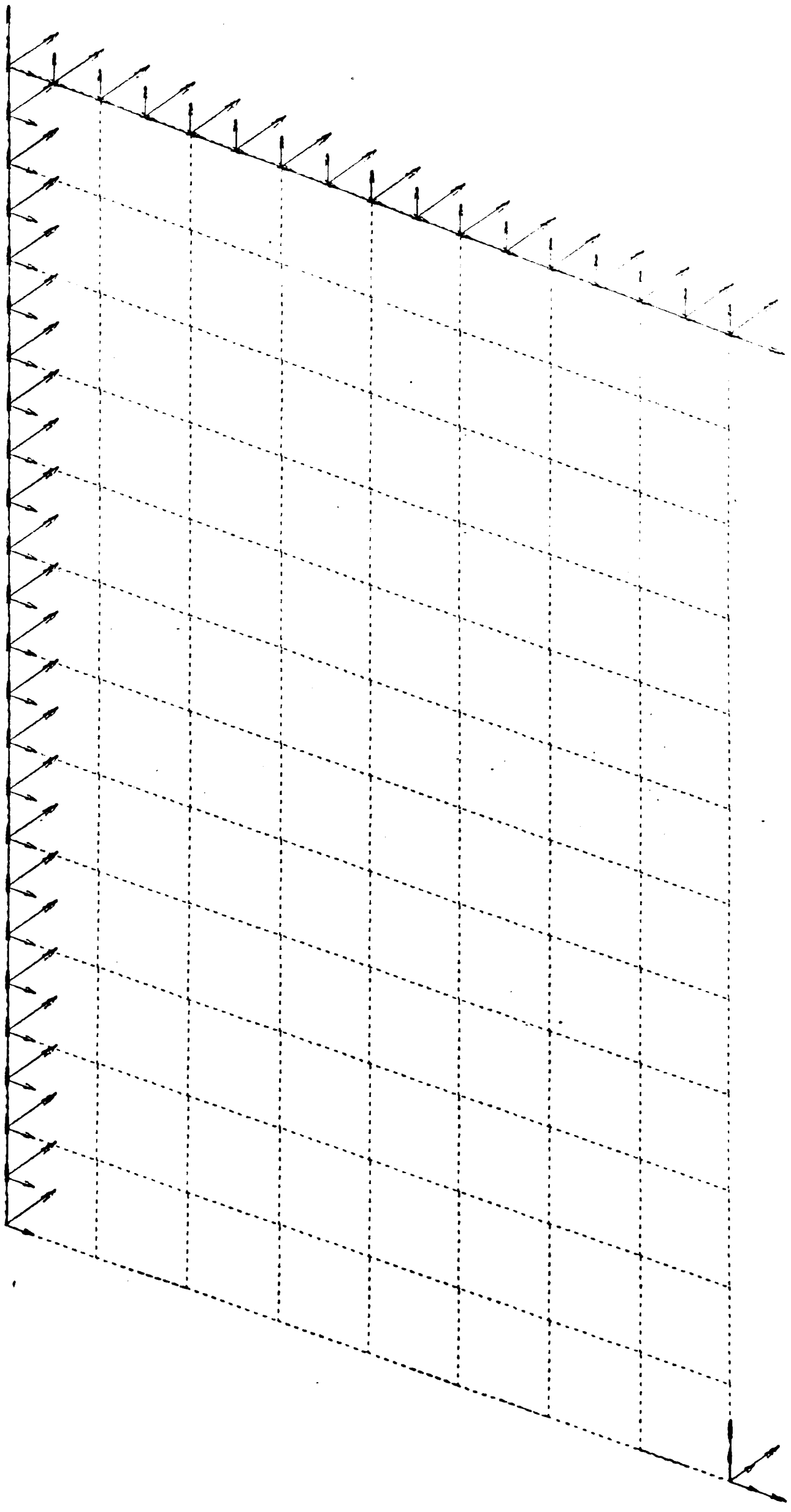


FIGURE F-9 Constraints for Finite Element Model Numbers 1,2,4, and 5.
Number of Stiffeners 1
Fine Mesh

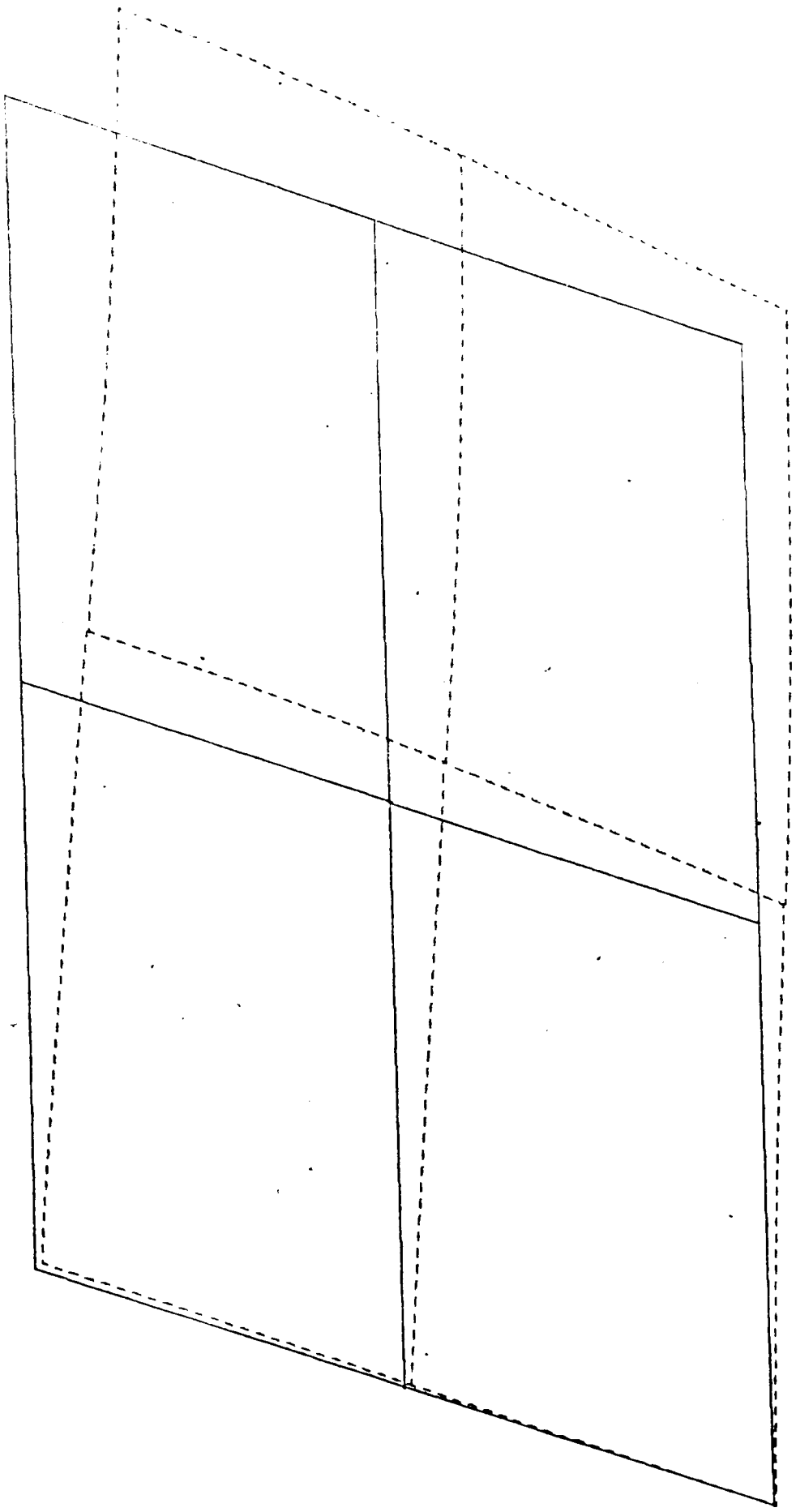


FIGURE F-10 Displaced Shape Diagram for Finite Element Model Number 4.
Number of Stiffeners 1
Crude Mesh

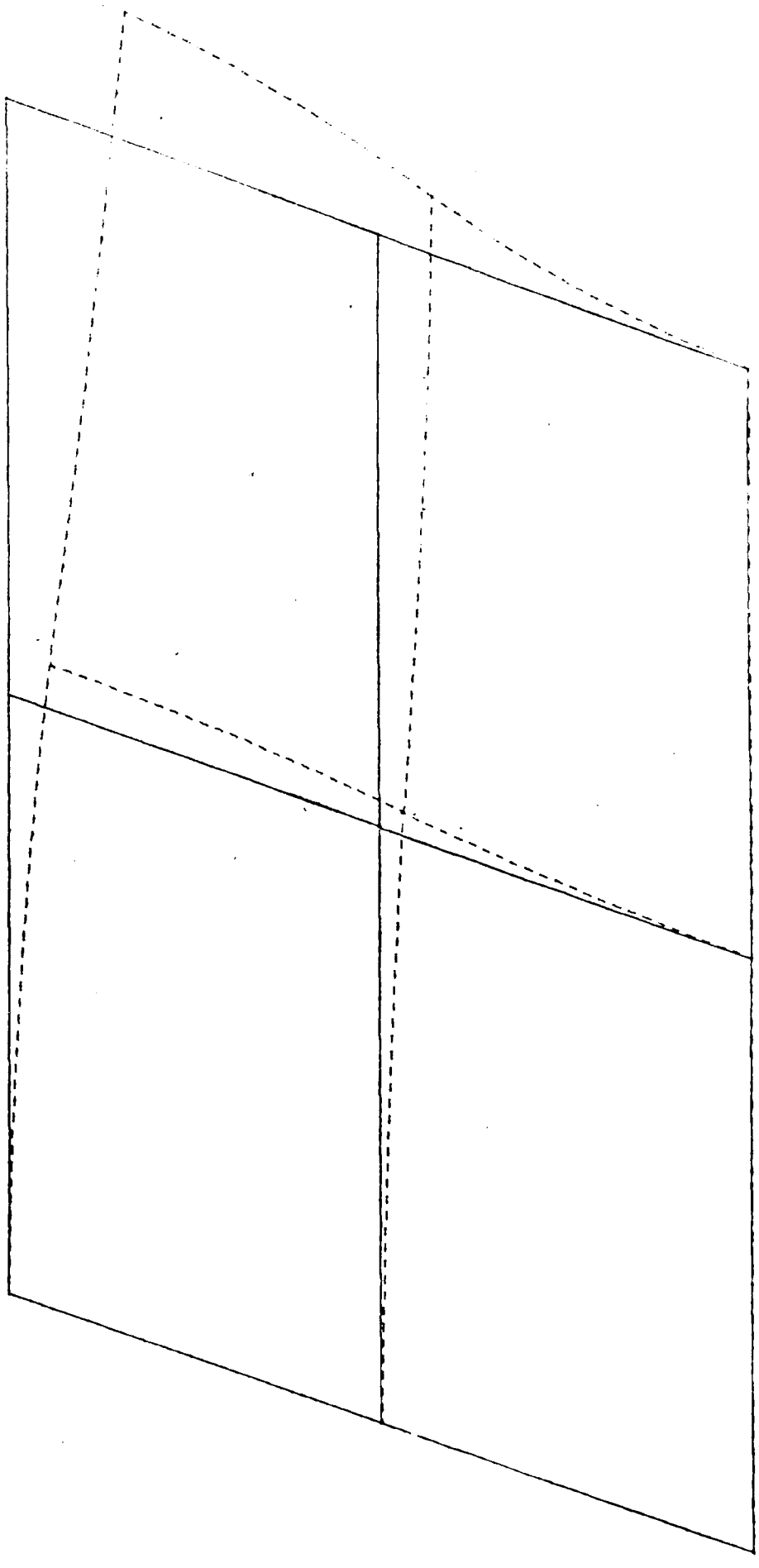


FIGURE F-11 Displaced Shape Diagram for Finite Element Model Number 4.
Number of Stiffeners 10
Crude Mesh

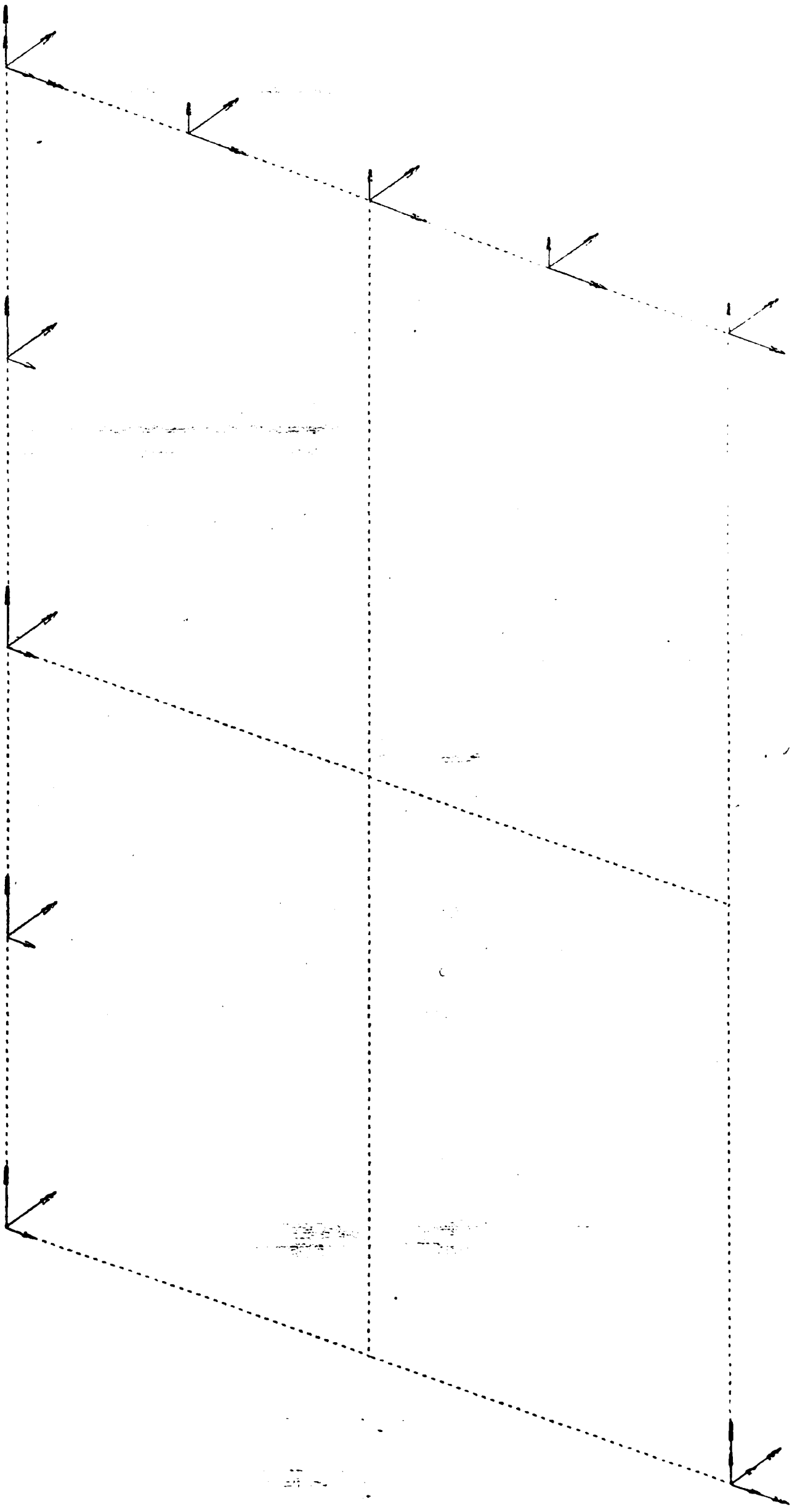


FIGURE F-12 Constraints for Model Numbers 4 and 5

Number of Stiffeners 1

Crude Mesh

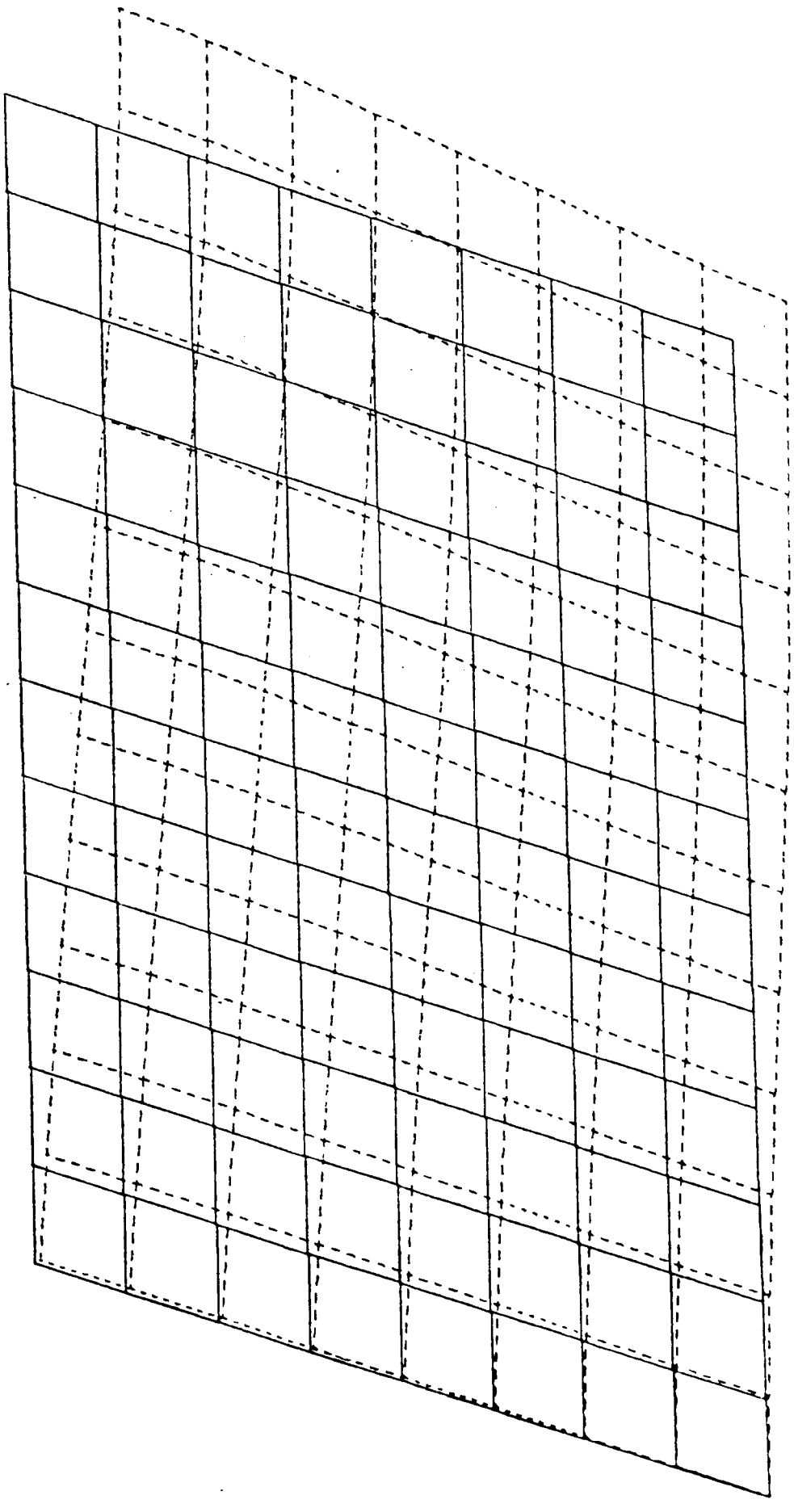


FIGURE F-13 Displaced Shape Diagram for Model Number 4.
Number of Stiffeners 1
Fine Mesh

APPENDIX G

VARIATIONAL PRINCIPLES

The calculus of variations plays a good role in the subject of dynamics, particularly for conservative systems. It is of particular use to the engineer in two classes of dynamic equations and is directly analogous to similar principles in other fields such as the principle of minimum strain energy in elasticity. At present, the variational procedures are widely used due to the development of numerical techniques. The second class of dynamics that is making use of the variational techniques is the true optimization of dynamic performance of moving objects and systems. A typical example would be the programming of magnitude and direction of rocket engine thrust to achieve optimum performance.

G.1. Original Principles from Ordinary Calculus:

A familiar problem of elementary calculus is to find the maximum or minimum values of algebraic functions such as:

$$w = f(y) \tag{G.1}$$

A mathematical development was made to ease finding points of zero slope of w plotted against y, hence the conditions for extreme values,

$$\frac{dw}{dy} = 0 \tag{G.2}$$

If we attempt to extend 'w' to be a function of two variables

$$w = f(y,z) \tag{G.3}$$

then we can express the change in 'w' as

$$dw = \frac{\partial w}{\partial y}\, dy + \frac{\partial w}{\partial z}\, dz \tag{G.4}$$

Now for dw to be zero and only zero for an arbitrary dy and dz the following condition must be satisfied;

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial z} = 0 \tag{G.5}$$

The points for which the above equation holds are called stationary values of 'w' and it includes maxima, minima and other flat areas*. The nature of the stationary point could be governed by the relation;

$$\left(\frac{\partial^2 w}{\partial y \partial z}\right)^2 < \frac{\partial^2 w}{\partial y^2} \cdot \frac{\partial^2 w}{\partial z^2} \qquad (G.6)$$

So if $\frac{\partial^2 w}{\partial y^2}$ and $\frac{\partial^2 w}{\partial z^2}$ are positive then the stationary point is a minimum; if they are negative then we have a maximum.

Generalizing equation (G.3) means that we have a function of 'n' variables or coordinates,

$$w = f(u_i) \quad , \quad i = 1, 2, 3, \ldots n \qquad (G.7)$$

and we can think of extending equztions (G.4) and (G.5) so that we have general equations,

$$dw = 0 = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \, du_i \qquad (G.8)$$

$$\frac{\partial f}{\partial u_i} = 0 \quad , \; i = 1, 2, 3, \ldots\ldots n \qquad (G.9)$$

and the nature of the stationary points will be defined by the second derivatives $\frac{\partial^2 f}{\partial u_i \partial u_j}$, $i,j = 1, 2, 3, \ldots\ldots, n$.

It is well known that for problems with more than one variable, variables can be eliminated by having more equations, each equation makes possible eliminating one variable. This is made possible to some of the problems by adding boundary conditions or constrains.

---

*It is suggested that they are areas since the plot of equation (G.3) is like a geographical map.

Thus we may find the stationary value of $w = f(x, y, z)$ subject to the constraint $\phi(x,y,z) = 0$. Using the elimination principle just mentioned above, it is possible to eliminate z from $w = f(x,y,z)$. Here again we can write equation (G.8) in expanded form for the differential of w , thus

$$dw = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0 \qquad (G.10)$$

subject to the constraint among the differentials dx, dy, dz that,

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \qquad (G.11)$$

or using the summation notation,

$$d\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial u_i} = 0 \qquad (G.12)$$

Solving (G.11) for dz and substituting in (G.10) to get

$$dw = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy - \frac{\frac{\partial f}{\partial z}}{\frac{\partial \phi}{\partial z}} \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \right) = 0$$

.....(G.13)

then it becomes easier now to consider only x and y as independent and the necessary conditions for stationary values become

$$\frac{\partial f}{\partial x} - \frac{\frac{\partial f}{\partial z}}{\frac{\partial \phi}{\partial z}} \cdot \frac{\partial \phi}{\partial x} = 0 \quad , \quad \frac{\partial f}{\partial y} - \frac{\frac{\partial f}{\partial z}}{\frac{\partial \phi}{\partial z}} \cdot \frac{\partial \phi}{\partial y} = 0$$

.....(G.14)

and $\Phi(x,y,z) = 0$

Solving the above three equations will lead to the coordinates (x,y,z) of the stationary value of w.

G.2. Application to Integrals:

A widely used application of the calculus of variation is finding an entire function $y(x)$ which gives an extreme, or stationary value of a quantity I, which could be defined for a given form of 'f' as

$$I = \int_{x_1}^{x_2} f\left[x,\ y(x),\ \dot{y}(x)\right] dx \qquad (G.15)$$

where the prime on the third term indicates differentiation of $y(x)$ with respect to x.

It is evident that the curve $y(x)$ can assume any path between the limits (x_1, y_1), and there can be an open choice to $y(x)$ so that I becomes a minimum or a maximum, however in this particular problem it is wished to find the function $y(x)$ which will give the integral of equation (G.15) a minimum value between its limits, based on the assumption that the form of f in terms of x, y, z is already known.

With reference to figure (G.1) it is possible to choose a curve,

$$Y(\alpha,x) = y(x) + \alpha\eta(x) \qquad (G.16)$$

which is neighbouring to $y(x)$ and extends between the same limts of $y(x)$. In equation (G.16) is not a function of x, and $\eta(x)$ is a smooth curve chosen that

$$\eta(x_1) = \eta(x_2) = 0 \qquad (G.17)$$

Thus when 'α' is small, the neighbouring curve Y approaches

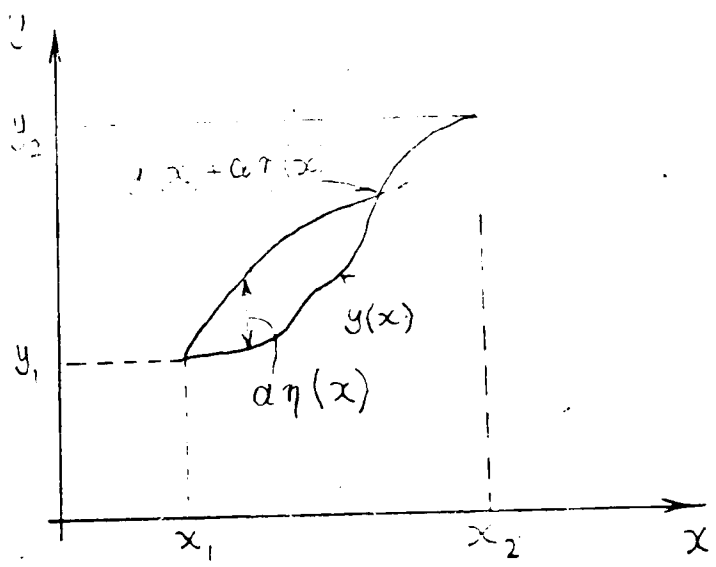


Figure G.1 Showing the function $y(x)$ and its neighbouring curve $Y(\alpha,x)$

y(x) and the integral I will have almost its minimum value.

$$I(\alpha) = \int_{x_1}^{x_2} f(x, y + \alpha\eta \ , \ \acute{y} + \alpha\acute{\eta} \ )dx \qquad (G.18)$$

The limit as $\alpha \to 0$ will give the curve y(x). This condition occurs when $\frac{dI(\alpha)}{d\alpha} = 0$, i.e. the minimum of I, this

$$\frac{dI(\alpha)}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial}{\partial\alpha} f(x, y + \alpha\eta \ , \ \acute{y} + \alpha\acute{\eta} \ ) \, dx$$

$$= \int_{x_1}^{x_2} \frac{\partial}{\partial\alpha} f(x, Y \ , \ \acute{Y} \ ) \, dx \qquad (G.19)$$

But since 'x' is not a function of 'α' then $\frac{\partial f(x)}{\partial\alpha} = 0$.

using the chain rule on equation (G.19) we get,

$$\frac{dI}{d\alpha} = 0 = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial\alpha} + \frac{\partial f}{\partial \acute{Y}} \cdot \frac{\partial \acute{Y}}{\partial\alpha} \right) dx \qquad (G.20)$$

Since it was assumed,

$$Y = y + \alpha\eta \qquad \text{and} \qquad \acute{Y} = \acute{y} + \alpha\acute{\eta}$$

then it follows that,

$$\frac{dY}{d\alpha} = \frac{\partial Y}{\partial\alpha} = \eta \qquad \text{and} \qquad \frac{d\acute{Y}}{d\alpha} = \frac{\partial \acute{Y}}{\partial\alpha} = \acute{\eta}$$

So

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial Y} \eta(x) + \frac{\partial f}{\partial \acute{Y}} \acute{\eta}(x) \right] dx \qquad (G.21)$$

It is required to have $\frac{dI}{d\alpha} = 0$ for $\alpha = 0$, then equation (G.21) reduces to

$$\left.\frac{dI}{d\alpha}\right|_{\alpha = 0} = 0 = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial \acute{y}} \acute{\eta}(x) \right] dx$$

.....(G.22)

note that Y was replaced by 'y' since they are the same at $\alpha = 0$.

Integrating the second term of equation (G.22) by parts lead to

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial \acute{y}} \acute{\eta}(x)\, dx = \frac{\partial f}{\partial \acute{y}} \eta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial \acute{y}}\right) dx$$

.....(G.23)

Substituting into equation (G.22) of equation (G.23) and rearranging terms gives,

$$\left.\frac{dI}{d\alpha}\right|_{\alpha = 0} = 0 = \frac{\partial f}{\partial y} \eta(x) \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \acute{y}}\right)\right] \eta(x)\, dx$$

.....(G.24)

making use of equation (G.17) leads to,

$$0 = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \acute{y}}\right)\right] \eta(x)\, dx \qquad (G.24)$$

but since $\eta(x)$ was chosen arbitrarily then the integral of equation (G.24) can only be zero if,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \acute{y}}\right) = 0 \qquad (G.25)$$

Equation (G.25) is known as an Euler-Lagrange equation and is a necessary (although not sufficient) condition for a minimum or maximum of I . In effect, its a transfer function to obtain $y(x)$.

Quite often the integral of equation (G.15) contains more than one derivative of y such as $\acute{y}$ or $\acute{\acute{y}}$, several different functions of x or even the extreme case of having several independent variables such as x, ϑ, t ... etc, and a more general form of (G.15) would be

$$I = \int_{t_o}^{t_1} \int_{x_1}^{x_2} f\Big[x, y(x), \acute{y}(x), \acute{\acute{y}}(x)..., z(x), \acute{z}(x),...,t, y(t), \acute{y}(t),..... z(t)....\Big] dx\, dt \qquad (G.26)$$

following the same lines of analysis used above, a more general equation or set of equations corresponding to equation (G.25) can be obtained, thus,

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \acute{y}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial \acute{\dot{y}}}\right) + \cdots - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial \dot{y}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial f}{\partial \ddot{y}}\right) + \cdots = 0$$

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \acute{z}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial f}{\partial \acute{\dot{z}}}\right) + \cdots - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial \dot{z}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial f}{\partial \ddot{z}}\right) + \cdots = 0 \qquad \text{(G.27)}$$

. . . . . .

where the dots indicate differentiation with respect to time (e.g. $\dot{y} = dy/dt$). Clearly from equation (G.27) that there is a separate equation for every dependent variable.

A particular case of equations (G.26) and (G.27) is the one used in section 2.3.3. in this text, where we have the dependent variables (x, y, ϑ) as functions of (z and t), thus,

$$I = \int_{t_o}^{t_1}\int_{z_1}^{z_2} f\,(y,\acute{y},\dot{y},\acute{\dot{y}},\ddot{y},x,\acute{x},\dot{x},\acute{\dot{x}},\ddot{x},\vartheta,\acute{\vartheta},\dot{\vartheta},\acute{\dot{\vartheta}},\ddot{\vartheta},z,t)\,dz\,dt \qquad \text{.....(G.28)}$$

and the corresponding Euler-Lagrange equations would be,

$$\frac{\partial f}{\partial y} - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \acute{y}}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial f}{\partial \acute{\dot{y}}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial \dot{y}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial f}{\partial \ddot{y}}\right) = 0 \qquad \text{(G.29)}$$

$$\frac{\partial f}{\partial x} - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \acute{x}}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial f}{\partial \acute{\dot{x}}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial f}{\partial \ddot{x}}\right) = 0 \qquad \text{(G.30)}$$

$$\frac{\partial f}{\partial \vartheta} - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \acute{\vartheta}}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial f}{\partial \acute{\dot{\vartheta}}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial \dot{\vartheta}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial f}{\partial \ddot{\vartheta}}\right) = 0 \qquad \text{(G.31)}$$

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