# Super-inflation and perturbations in LQC, and scaling solutions in curved FRW universes

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*To Mum and Dad*

and

*all the stars shining down on me*

# Abstract

We investigate phenomenologies arising from two distinct sets of modifications introduced in Loop Quantum Cosmology (LQC), namely, the inverse volume and the holonomy corrections. We find scaling solutions in each setting and show they give rise to a period of super-inflation soon after the universe starts expanding. This type of inflation is explicitly shown to resolve the horizon problem with far fewer number of e-foldings compared to the standard inflationary model. Scalar field perturbations are obtained and we demonstrate their near scale invariance in agreement with the latest observations of the Cosmic Microwave Background (CMB). Consideration of tensor perturbations of the metric results in a large blue tilt for these fluctuations, which implies their amplitude will be suppressed by many orders of magnitude on the CMB compared to the predictions of the standard inflation. This LQC result is shared by the ekpyrotic model and the model of a universe sourced by a phantom field.

Exploring a correspondence map at the cosmological background level between braneworld cosmologies and the inverse volume corrected LQC, we discover this map not to hold at the level of linear perturbations. This is found to be due to the different behaviour of the rate of the Hubble parameter in the two classes of models.

A complete dynamical analysis of Friedmann-Robertson-Walker spacetimes we carry out results in the most general forms of late time attractor scaling solutions. Our examination includes expanding and contracting universes when a scalar field evolves along a positive or a negative potential. Known results in the literature are demonstrated to correspond to certain limits of our solutions.

# List of publications

- E. J. Copeland, D. J. Mulryne, N. J. Nunes and M. Shaeri, *"Super-inflation in Loop Quantum Cosmology"*, Phys. Rev. D **77** (2008) 023510 [arXiv:0708.1261 [gr-qc]].

- E. J. Copeland, D. J. Mulryne, N. J. Nunes and M. Shaeri, *"The gravitational wave background from super-inflation in Loop Quantum Cosmology"*, Phys. Rev. D **79** (2009) 023508 [arXiv:0810.0104 [astro-ph]].

- E. J. Copeland, S. Mizuno and M. Shaeri, *"Dynamics of a scalar field in Robertson-Walker spacetimes"*, Phys. Rev. D **79** (2009) 103515 [arXiv:0904.0877 [astro-ph.CO]].

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# Contents

# Chapter 1

# Introduction

In physics and philosophy there are unanswered questions regarding the origin of the universe we live in. We do not yet have the answers, but by asking the questions we aim to gain better understanding of the clues that nature has given us.

In recent years we have entered an era of precision cosmology on the experimental front. This has made it possible to probe the physical world we live in and test theories that have been around for decades. Einstein's General Relativity (GR), that was first proposed about a century ago [1], is only recently being tested on very small scales to shed light on the validity limit of this theory. On the other hand, very large scales are being probed with a variety of experimental tools both on the Earth and out in space. A comprehensive up-to-date progress in experimental tests of GR is presented in [2]. These are exciting times for theoretical and experimental physicists as we push existing mathematical tools and cutting edge technologies available to their limits in their search for the underlying physical rules of nature.

In theoretical physics much effort has been made to explore the world on the smallest scales. One of the fundamental tasks is to define what one means by the smallest scales. We have learnt from GR that space and time can be treated equivalently on large scales, but by evolving these equations back in time, the universe shrinks and one hits the Big Bang singularity. Existence of infinities or singularities in a theory is a sign of its inability to successfully describe those regimes. So, Einstein's equations need to be modified on small scales.

Experimentalists are taking the top-down approach to see at what scale they may begin to see deviations from Einstein's exact equations, whilst theorists are working towards bottom-up methods motivated by unifying gravity with other known forces in nature. It is widely thought that a quantum theory of gravity should not only successfully combine two of the best theories in physics (i.e. Quantum Field Theory and GR), but it should also be able to predict the corrections that need to be made to these theories outside of their regions of validity.

The power of a theory lies in making predictions and describing the governing rules of nature as they are. Since a quantum theory of gravity is concerned with extremely small scales, it is believed to be most significant at the high energies of the very early universe. There are currently many approaches being developed towards possible theories of quantum gravity. Two of the most popular are string theory (or M-theory) [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and Loop Quantum Gravity (LQG) [14]. The former relies on the existence of higher dimensions and generally a background metric is assumed as a starting point, whereas the latter is an attempt to quantise GR in four dimensions in a background independent way and it predicts a discrete nature for spacetime. The two theories are very different in their approach to the question of the nature of gravity on small scales. String theory related advances on the theoretical side have been quite impressive and attempts are constantly being made to make predictions that could one day be verified. LQG, by comparison, is a relatively new theory, which is rapidly gaining momentum in its development.

Because these theories deal with the nature of spacetime at the quantum level, it is very difficult to link their dynamical description of nature to what we could test in a laboratory in the foreseeable future. What we have learnt from the extremely precise Cosmic Microwave Background (CMB) observations, first discovered in 1965 [15], is that there may be signatures left from the very early universe that we can look for. If a theory of quantum gravity is to modify the known classical equations during the evolution of the very early universe, perhaps it has left imprints which we can observe by exploring cosmological models they would describe. The most feasible method to test the predictions of any such theory, therefore, is through its power of phenomenological predictions in the field of cosmology.

In this thesis our main focus will be on investigating cosmologies motivated by LQG. Loop Quantum Cosmologies (LQC) have been developed by quantising the Friedmann-Robertson-Walker (FRW) universe using quantisation techniques developed in the full theory [16]. One particularly useful approach to studying the early universe in the context of LQC is that of deriving and studying *effective* equations of motion. While such equations cannot probe fully quantum regimes, they nevertheless incorporate quantum modifications into classical evolution equations whilst avoiding the interpretational difficulties inherent in fully quantum equations. In isotropic settings two sets of modifications have been predominately considered. The first originates from the spectra of quantum operators related to the inverse triad, while the second arises from the use of holonomies as a basic variable in the quantisation scheme. This terminology will be introduced in the following chapter. It is still unclear how these modifications are related. We will, thus, treat both sets separately throughout this thesis, but we recognise that to make full predictions for LQC these corrections (possibly together with any others which have not yet been identified) have to be combined appropriately to provide an accurate description of the theory.

This thesis is organised such that in chapter 2 we present a short summary of the setup of the underlying LQG and LQC models. Some of the terminologies used in this context are introduced and much of the mathematical framework is omitted, but basic results and predictions relevant to our study are quoted from the full theory. We also introduce some simple braneworld scenarios inspired by string theory, which we will refer to in some of the following chapters. The phase space analysis and stability properties of a dynamical system are also outlined here, as this procedure is adopted in the proceeding investigation. The aim of chapter 3 is to concentrate on cosmological models inspired by the ideas behind LQC and to examine possible predictions arising from these models. In particular, we describe the ways in which inflation can be realised in LQC when scaling cosmological background solutions are considered, and we see that these inflationary scenarios are of a different nature to what we know of the standard inflation. We will explicitly show that this type of inflation resolves the horizon problem with a far fewer number of e-foldings than is required in the standard inflation. The dynamical behaviour of our scaling solutions is also analysed and we comment on the stability properties.

Our next step is to carry out a thorough examination of the scalar field perturbations we derive for the two types of corrections in LQC, and we conclude that it is indeed possible to obtain a near scale invariant spectral index for these fluctuations. This is consistent with our current knowledge of the most recent observations [17]. The majority of the work explained in this chapter has been published in the Physical Review journal [18].

In chapter 4 we explore the properties of tensor perturbations in LQC and reach the conclusion that their spectral tilt is very large and blue, which means their amplitudes on the CMB would be suppressed by many orders of magnitude compared to the predictions of the standard inflation. Similarities between this LQC result and similar calculations in the context of an ekpyrotic universe and also a universe sourced by a phantom field are also highlighted. For the case of the holonomy corrections, we also comment on a maximum scale predicted by the governing equations for which the calculation of gravitational wave fluctuations is physically meaningful. The bulk of this set of results is published in the Physical Review journal [19]. An attempt is made in chapter 5 to examine an existing correspondence map between the inverse volume corrected LQC and the modified braneworld cosmologies at the background level [20]. We then explore to see whether such a map would also hold at the level of linear perturbations around our scaling solutions. We will see that a map constructed as in [20] does not automatically relate these models to one another at the level of fluctuations, mainly due to the different behaviour of the rate of the Hubble parameter in these two models.

There has been much new interest in collapsing universes, especially that these have been predicted by the fundamental theories which have been of interest to our study, namely LQC and the ekpyrotic model (in the context of braneworld cosmologies). Another common feature is that in both theories the inflaton can evolve along a negative potential. Combining these possibilities in regions where spatial curvature is more likely to be significant, we carry out a thorough dynamical analysis of FRW universes in chapter 6 for an expanding/contracting volume sourced by an inflaton on a positive/negative potential. This investigation results in obtaining the most general forms of background scaling solutions, and we demonstrate that in certain limits they reduce to the previously known solutions in the literature. We have published these results in the Physical Review journal [21]. We will end this thesis by presenting some conclusions and a short discussion on possible future extensions to our work in chapter 7.

# Chapter 2

# Background Information

## 2.1 Introduction

An intriguing and attractively neat theory, which has gathered much support amongst theoretical physicists and cosmological experimentalists, is the theory of inflation, first proposed by Guth [22] amongst others [23, 24]. An inflationary epoch is currently the most promising model for the origin of large-scale structure in the universe [22, 25, 26]. The predictions of inflation are fully compatible with the most recent observations suggesting that structure originated from a near scale-invariant, Gaussian and adiabatic primordial density fluctuations [27]. Despite these successes, however, a number of important questions remain. In particular, in what fundamental theory will inflation be seen to arise? Having asked the question, it is worth noting that there have been successful implementations of inflationary models motivated by fundamental theories such as string theory [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and Loop Quantum Gravity (LQG) (see [14] for reviews), which are currently the two main candidates in this field.

The field of LQG-inspired cosmologies is still in its early days, requiring much more development before its full predictive potential can be probed. So far, the general cosmological framework investigated in this context is for the case where spatial isotropy and homogeneity are assumed and the mathematical approach of LQG is applied to this model [16]. This is known as Loop Quantum Cosmology (LQC). In this chapter we review the basic ideas behind these theories, focusing mainly on the phenomenological aspects rather than the complicated mathematical background. After introducing the terminology used in LQG and a very brief outline of its setup, we describe the much simpler setup of LQC in more detail in section 2.2. We emphasise that describing the mathematical foundations of these theories is beyond the scope of this work, but certain results from the underlying structure have been quoted in order to relate the cosmological implications to the mathematical framework.

Much effort has so far been made in making cosmological predictions based on models inspired by string theory [29, 30, 31]. This is commonly done through the idea of braneworld cosmologies, which we will describe briefly in this chapter. Simple braneworld scenarios, which are of interest to our work and we refer to in later chapters, together with some of the basic ideas behind the physical picture are presented in section 2.3.

Given the importance of inflation, due to its simple structure and the successes it has had in cosmology, in the course of providing most of the background information used in this thesis, we will see that the standard picture of slow-roll inflation is not the only method by which inflation can arise. The assumption of slow-roll inflation limits the classes of cosmologies one can consider in order to explain the cosmological observations. The concept of 'fast-roll' inflation and its relation to the standard slow-roll inflation is introduced in section 2.4. This broadens the options available to theorists in order to design a realistic picture of the physical processes at the high energies near the Big Bang.

Another aspect to consider in the dynamics of the very early universe is its stability nature and the asymptotic behaviour of the solutions. We do not observe chaotic characteristics in the evolution of the universe today, so it is reasonable to think of late time stability as a desirable physical feature when formulating a theory or investigating a theoretical model. We will outline the analysis procedure for determining the stability of a dynamical system and its asymptotic behaviour in section 2.5 and present results, which we will return to a number of times in the following chapters.

## 2.2 Loop Quantum Cosmology (LQC)

LQC is the application of quantisation techniques developed in the context of Loop Quantum Gravity (LQG), to a symmetric (isotropic and homogeneous) model of the universe [32, 16] (see also [33, 34] for an overview and recent advances). LQG is a background independent and non-perturbative canonical quantisation of Einstein's General Relativity (GR) [14], which means the formulation does not rely on the pre-existence of a classical background metric or any small fields. This idea is motivated by asking a full theory of quantum gravity to have GR as its low energy limit, and so it must also be independent of the background at the quantum level.

### 2.2.1 Setup of Loop Quantum Gravity (LQG)

The quantisation technique starts by rewriting the spatial part of the metric in GR, $q_{ab}$, in terms of a system of triads, $e^a_i$, where the spatial indices $a, b, \cdots = 1, 2, 3$; and $i, j, \cdots = 1, 2, 3$ for the internal indices. The inverse metric can then be obtained from $q^{ab} = e^a_i e^b_i$, where we sum over the index $i$. This method clearly does not change the physics, but introduces further gauge freedom in the analysis, since the metric is invariant under rotations of the triad under an SO(3) gauge group, and also remains invariant under reflections. Ashtekar variables [35, 36, 37] are then introduced in the form of *densitised triads* $E^a_i = |det e^b_j|^{-1} e^a_i$, and the *Ashtekar connections*, $A^i_a = \Gamma^i_a + \gamma K^i_a$, where $\gamma$ is the Barbero-Immirzi parameter taking positive real values [38, 39], $K^i_a \equiv K_{ab} e^b_i$ are the extrinsic curvature coefficients, and $\Gamma^i_a$ is the spin connection, which can be given in terms of the triad. In this scheme, Ashtekar connections are analogous to the configuration space, and the triads to the momentum space of quantum mechanics. These new parameters are canonically conjugate to one another, and their introduction allows gravity to appear as a canonical gauge theory (with the gauge group being that of triad rotations) for which background independent quantisation techniques have been developed [40, 41].

Usually, in quantum field theories, quantisation relies on evaluating integrals of the configuration and momentum space over three-dimensional regions for which an integration measure is required. This does not pose any problems in ordinary quantum field theories, as the formulism is based on a background. In LQG, however, since background independence is desirable, one is faced with having to adopt a different quantisation technique. One such method, not relying on the metric, is developed using the path integral approach to quantisation. In this formulism, *holonomies* of the Ashtekar connections are defined in the context of LQG by exponentiating the connections along a closed one-dimensional loop [1], $e$, embedded in a 3-dimensional spatial manifold, in a path-ordered fashion [32]:

$$h_e(A) = \mathcal{P} \exp \int_e A^i_a \tau_i \dot{e}^a d\sigma \,, \tag{2.1}$$

where $\tau_i = -\frac{i}{2}\sigma_i$ are the SU(2) generators related to the Pauli matrices $\sigma_i$, $\dot{e}^a$ is the tangent vector to the curve $e$, and $\sigma$ is the parameter that varies along the curve, $e$. The path ordering is required in a non-commutative algebra, as is the case here, since the rotations in three dimensions is a non-abelian gauge group.

Similarly, *fluxes* are formed from integrating the triad over a two-dimensional surface [32]

$$F_S(E) = \int_S \tau^i E^a_i n_a d^2 y \,, \tag{2.2}$$

where $n_a$ is the co-normal vector to the surface, $S$.

Starting from a 'ground state', which does not depend on the connections, multiplying by the holonomies creates states that depend on the connections along the loops used in the process. In this sense, holonomies can be considered as 'creation operators'. By this operation one constructs a Hilbert space where states are functionals of the connections [42, 43, 44]. In the space of holonomies and fluxes, the action of fluxes on a state is non-zero only when there are

[1] Hence, the origin and the name of this quantisation mechanism of gravity.

intersections between the loops used in generating the state and the surface through which the flux is defined [42]. Analogies have been made between the angular momentum operator in quantum mechanics and these intersection points, but no concrete mathematical links have yet been established. By analogy, it has been argued [45, 46] that the flux operator has eigenvalues which do not change continuously, and therefore, has a discrete spectrum containing zero. Since the triad contains all the information about the geometry of space, and the flux is defined in terms of the triad, this demands a discrete spatial geometry. The kinematics is set up in this fashion, and it has been quite challenging to understand the nature of time or indeed what one means by dynamics in this formulism. The only mathematically acceptable meaning of dynamics in this framework is obtained through a Hamiltonian formulation [47].

### 2.2.2 Setup of LQC

Obtaining cosmology from the mathematical framework of the full quantum gravity theory has proven to be a challenging task especially since the notion of time is not well understood deep in the quantum regime. Therefore, describing the *evolution* of the universe out of the quantum gravity phase becomes a vague notion. Thus, the community has made considerable efforts searching for models, which could provide us with insight into possible phenomenologies arising from LQG [32, 48, 49, 50]. In doing so, the simplest case to consider is the isotropic case, where the number of unknowns in the theory is reduced considerably, allowing one to analyse a simplified model. This is motivated by the most up-to-date observational data [17], as the universe on the very large scales seems to be approximated rather well by an isotropic system. This is not to say that we would expect the universe to be isotropic at the quantum regime, but that by assuming an isotropic universe and quantising it, let us see what quantum corrections are fed into the classical equations in order to describe the deviation from the classical description.

Considering an isotropic universe, in the context of LQC, the isotropic connections and triads are given by single parameters $c$ and $p$, respectively. These parameters are canonically conjugate: $\{c, p\} = \frac{8\pi G\gamma}{3}$, and they are related to the scale factor, $a$, by [48]

$$p = a^2, \qquad c = \frac{1}{2}(k + \gamma\dot{a}) , \tag{2.3}$$

where a dot denotes differentiation with respect to proper time, $t$, $G$ is the gravitational constant, and $k$ is the intrinsic curvature taking values of $+1, 0, -1$ for a closed, flat, or open universe, respectively. The proper time, $t$, refers to the time coordinate of a Friedmann-Robertson-Walker (FRW) universe whose line element is given by

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\xi^2)\right) , \tag{2.4}$$

where $r$, $\theta$, and $\xi$ refer to the spherical coordinates. We note that by setting the metric to be that of an FRW universe, one fixes the gauge and is no longer working with a background independent theory, but rather, with a much smaller class of models. This is the current framework in which LQC is being investigated. We also note that similarly to the case of the full theory, the parameter, $p$, encoding the information about spatial geometry has a discrete spectrum once quantised.

The classical Friedmann equation is given by

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} a^{-3}\mathcal{H}_{matter}(a) - \frac{k}{a^2} , \tag{2.5}$$

in the absence of a cosmological constant, where $H$ is the Hubble parameter, and $\mathcal{H}_{matter}$ is the matter Hamiltonian. For a single scalar field, $\phi$, on a potential $V(\phi)$, and having conjugate momentum $p_\phi$, the classical matter Hamiltonian is given by

$$H_\phi(a, \phi, p_\phi) = \frac{1}{2}|p|^{-3/2}p_\phi^2 + |p|^{3/2}V(\phi) , \tag{2.6}$$

where the gradient term has been ignored due to the homogeneity assumption. By evolving the classical equations of motion back in time, one approaches the Big Bang singularity as $a \to 0$, and this can be understood in terms of the matter Hamiltonian diverging in this limit.

In the LQC framework, this singularity (as well as the one predicted by GR at the centre of a black hole) is removed, and instead, there is a precise mathematical description of the nature of geometry close to these classical singularities. The details of this formulism can be found in [32, 44, 51, 50, 53, 54] and are beyond the scope of this work. In order to investigate the cosmological phenomenology, we accept this result and work in the 'semi-classical' regime of LQC, which we now explain.

The parameter space of the scale factor can be considered to consist of three distinct regions. First, $a \approx l_p$, where the scale factor is of the order of the Planck length and spacetime geometry is discrete; and although considerable effort has been made to describe the kinematics of geometries at this level [52], it is still rather difficult to draw precise conclusions on the dynamics [14]. The second region is the so-called 'semi-classical' region: $l_p \ll a < a_*$, where quantum corrections are thought to modify the classical equations of motion. Therefore, allowing one to make the assumption that the continuous differential equations are good approximations to the difference equations of the deep quantum regime. The scale $a_*$ is related to the Planck length, but it also depends on the quantisation ambiguities of LQC, which is the reason why there is lack of understanding within the community on how far away from the Planck scale it should be. Finally, for $a \gg a_*$, one recovers the classical dynamics of the universe. The *effective* equations arising from the semi-classical regime are believed to bridge the gap between the description of quantum space and the classical behaviour of the universe. As the universe evolves, and once the quantum corrections become negligible, these equations should asymptote to the classical form. Two main distinct types of quantum alterations of the classical picture are introduced in the context of LQC, which are discussed below.

**a) Inverse Volume Corrections**

In order to quantise the matter Hamiltonian (2.6), we immediately face a problem since the triad operator, $\hat{p}$, has a discrete spectrum containing zero; and therefore, its inverse is not well-defined, leading to the Hamiltonian (2.6) diverging. It turns out that although the inverse of $\hat{p}$ is ill-defined (due to its discrete spectrum containing zero), the classical parameter $|p|^{-3/2}$, which is the inverse volume factor appearing in (2.6), can be directly quantised resulting in a well-defined operator, $\widehat{|p|^{-3/2}}$ [55]. This method was first carried out in the full theory [56], and has also been applied to the very symmetric case of LQC [57, 58, 59].

Classical expressions can often be written in many equivalent ways, but each will have a separate quantisation differing from others in details, but sharing the main properties [58]. When quantising the classical inverse volume, the different quantisations can be parameterised, outside of the deep quantum regime, by the function $d(p)_{j,l}$ as the *effective* inverse volume [60, 61, 48, 49]:

$$H_\phi(a, \phi, p_\phi) = \frac{1}{2} d(p)_{j,l} p_\phi^2 + |p|^{3/2} V(\phi)\,, \tag{2.7}$$

where $d(p)_{j,l} \equiv |p|^{-3/2} D_l(3|p|/\gamma j l_p^2)$, and $D_l$ is a complicated function well approximated by

$$\begin{aligned} D_l(q) \;=\; & \Big[\frac{3}{2l}\, q^{1-l}\Big(\frac{1}{l+2}\big((q+1)^{l+2} - |q-1|^{l+2}\big) \\ & - \frac{1}{l+1} q\big((q+1)^{l+1} - sgn(q-1)|q-1|^{l+1}\big)\Big)\Big]^{3/(2-2l)}\,, \end{aligned} \tag{2.8}$$

where $l$ and $j$ are the LQC quantisation ambiguities, and their regions of validity are given by $j \in \frac{1}{2}\mathbb{N}$, and $0 < l < 1$. To simplify the notation, we will drop the subscript of $D_l$ in the rest of this text, keeping in mind its dependence on the quantisation ambiguity.

In this case, the critical scale separating the semi-classical behaviour from the classical one is $a_* \equiv \sqrt{\frac{\gamma j}{3}} l_p$. In order to observe any deviation from the classical dynamics in the effective equations of motion, one needs to use a large value of $j$ so that $a_*$ is suitably away from the deep quantum regime. For $a \gg a_*$, $D(a)$ tends to unity, allowing one to recover the classical Hamiltonian; but in the semi-classical region, $D(a)$ introduces modifications to Eq.(2.6) and the phenomenology is altered accordingly. In particular, for small values of the scale factor, Eq.(2.8) can be approximated by

$$D(a) = D_* a^n \,, \tag{2.9}$$

where $D_*$ is a constant, and $n = 3(2-l)/(1-l)$ takes values in the range $6 < n < \infty$. It is worth noting that in the semi-classical regime, the effective inverse volume is an *increasing* function of the scale factor in an expanding universe, which clearly differs from the classical case. Classically, the inverse volume (and hence, the Hamiltonian) diverges as $a \to 0$, whereas taking these quantum corrections into account, the effective inverse volume tends to zero in this limit and remains well-defined. In this case, it is clear from (2.7) that the energy density at very small values of the scale factor is dominated by the scalar potential, as the kinetic term tends to zero. We also note from Eq.(2.7) that these quantum corrections modify the kinetic energy term of the scalar field in the semi-classical region, but leave the potential term unchanged.

In the Hamiltonian formulism, the matter equations of motion are given by the Poisson bracket of the canonically conjugate parameters and the matter Hamiltonian as

$$\begin{aligned} \dot{\phi} &= \{\phi, H_{matter}\} = d(p)_{j,l} p_\phi \\ \dot{p_\phi} &= \{p_\phi, H_{matter}\} = -|p|^{3/2} V_{,\phi}(\phi) \,. \end{aligned} \tag{2.10}$$

These can be combined to give the corresponding Klein-Gordon equation of the field

$$\ddot{\phi} - H \frac{d \ln d(p)_{j,l}}{d \ln a} \dot{\phi} + |p|^{3/2} d(p)_{j,l} V_{,\phi}(\phi) = 0 \,. \tag{2.11}$$

Upon using the definition of the effective inverse volume, this then reduces to

$$\ddot{\phi} + 3H\left(1 - \frac{1}{3}\frac{d\ln D(a)}{d\ln a}\right)\dot{\phi} + D(a)V_{,\phi}(\phi) = 0\,, \tag{2.12}$$

where a subscript $\phi$ denotes differentiation with respect to the scalar field. The qualitative behaviour of matter is, therefore, changed in the small scale factor regime. In particular, the sign of the friction term is different compared to the classical field dynamics described by $D \to 1$. This means that although classically (i.e. when $\frac{d\ln d(p)_{j,l}}{d\ln a} \to -3 < 0$), in a contracting universe ($H < 0$), a scalar field experiences antifriction and speeds up as it evolves along its potential, when the quantum gravity effects are important (i.e. $\frac{d\ln d(p)_{j,l}}{d\ln a} \to -3 + n > 0$), such a field would experience friction and would slow down as it evolves towards $a \to 0$. At this point, the field becomes frozen as it slows down so much that $\dot{\phi} \approx 0$, and the dynamics is determined purely by the potential. By this time, the scale factor has reached close to its minimum size and the bounce occurs. After the bounce, the universe starts expanding ($\dot{a} > 0$) in the semi-classical regime experiencing antifriction and $\phi$ evolves down the potential. This behaviour continues until the classical limit is reached, where the field feels the friction term of Eq. (2.11) and slowly moves towards its minimum. In this sense, the nature of gravity is repulsive in the semiclassical regime, as it enhances the growth of the universe in its expanding phase. This phenomenon has been interpreted as *inflation* arising from LQC before the classical behaviour is reproduced [62, 48, 63, 64], perhaps followed by slow-roll inflation.

The effective Friedmann equation for the inverse volume quantum corrections is obtained from combining Eq.(2.5), (2.7), and (2.10):

$$H^2 = \frac{8\pi G}{3}\left(\frac{\dot{\phi}^2}{2D(a)} + V(\phi)\right) - \frac{k}{a^2}\,. \tag{2.13}$$

Another important aspect of these LQC corrections can be illustrated by differentiating this equation for a spatially flat universe, and using Eq.(2.12):

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{D}\left(1 - \frac{1}{6}\frac{d\ln D}{d\ln a}\right) , \tag{2.14}$$

and since, $\frac{\ddot{a}}{a} = \dot{H} + H^2$, one can then see that an expanding universe will be accelerating (i.e. $\ddot{a} > 0$) in the semi-classical regime, regardless of the form of the potential. This implies that inflation can be obtained in this region by including the inverse volume quantum corrections, and is an important result in the context of LQC.

Numerical analysis of an expanding (contracting) universe has shown its smooth turning point and transition to the contracting (expanding) universe whilst avoiding the classical singularity [65, 66, 44, 34, 54]. The requirement for the bounce to occur is $\dot{a} = 0$ and $\ddot{a} > 0$ at the same instant. This describes a bouncing scenario, and is an important characteristic feature of LQC. It is not yet clear if (or how) this property would manifest itself in the full theory of LQG [67].

By examining the gravitational sector, it has been shown [68, 69, 70] that the gravitational Hamiltonian also picks up quantum corrections due to the appearence of the inverse volume term. This modification is distinct from $D(a)$, since it only affects the gravitational sector, and is denoted by the function $S(a)$, which appears in the Friedmann equation as

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}S(a)\left(\frac{\dot{\phi}^2}{2D(a)} + V(\phi)\right) , \tag{2.15}$$

where $S(a) = S_* a^r$ in the semi-classical regime, with $S_* = \frac{3}{2}a_*^{-r}$, where $r$ is a constant. We will treat $r$ as an arbitrary variable in the range $3 < r < \infty$ within the semi-classical region, and we also note that classical behaviour corresponds to $r \to 0$. In its form, $S(a)$ varies with the scale factor in a similar way to $D(a)$. In our work, we consider Eq.(2.15) to be the most general effective Friedmann equation arising from this type of quantum corrections in LQC. It is worth noting that the function $S(a)$ does not alter the Klein-Gordon equation (2.12) for the scalar field.

**b) Holonomy Corrections**

This type of modification arises from the use of holonomies as a basic variable in the quantisation scheme, and a discrete spatial geometry is described in this case, too [51, 71, 50]. This particular type of correction is a consequence of the loops on which holonomies are computed having a non-zero minimum area, which is given by the eigenvalues of the area operator in the parent LQG theory [55]. The origin of these quantum corrections is not as intuitive as in the previous case because interpretation of including higher orders of the isotropic connection, $c$, can not be easily made in terms of physical quantities (e.g. volume of space, in the case of the effective densities).

A simple model incorporating such corrections in an isotropic and spatially flat universe sourced by a single scalar field has been proposed [51, 71, 50, 34, 53, 54] which illustrates the LQC phenomenology corresponding to this class of quantum corrections. The effective Friedmann equation is given by the simple expression

$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{2\sigma}\right) , \tag{2.16}$$

where $\rho = \frac{\dot{\phi}^2}{2} + V(\phi)$ is the energy density due to the scalar field, and $2\sigma$ is the critical value of the energy density corresponding to the maximum value of $\rho$ at the time of the bounce. The effect is, therefore, the simple addition of a negative $\rho^2$ term. It is clear that the right hand side of Eq.(2.16) is not permitted to be negative, so in a collapsing universe, as the energy density increases towards $2\sigma$, the universe undergoes a bounce, since it can not increase beyond this critical value. In its most general form, this critical density is a function of time [51, 71, 50], but in a perfectly isotropic universe, it is a constant given in terms of the Barbero-Immirzi parameter, $\sigma = 3/(2\gamma^2)$ [69]. These quantum corrections do not modify the classical Klein-Gordon equation of motion governing the dynamics of the scalar field. By differentiating (2.16) for a universe with its energy density dominated by a scalar field, one obtains

$$\dot{H} = -\frac{\dot{\phi}^2}{2}\left(1 - \frac{\rho}{\sigma}\right) , \tag{2.17}$$

in units where $8\pi G = 1$. One can then show that for the values of the energy density in the range $\sigma < \rho < 2\sigma$, $\dot{H}$ is positive and, similarly to the inverse volume scenario, the universe undergoes a super-inflationary expansion. Since $\ddot{a}/a = \dot{H} + H^2$, the univese is accelerating and this condition holds also very close to the bounce (i.e. $\rho \to 2\sigma$). Furthermore, it is clear from (2.16) that the simultaneous condition for the bounce (i.e. $\dot{a} = 0$) is also achieved at this limit. Even though this modified Friedmann equation is an *effective* equation with its core assumptions only strictly valid in the semi-classical regime, numerical simulations have shown that this correction is remarkably accurate near the bounce [51, 71, 33, 34, 53, 54, 72].

As we will see in the next subsection, this simple form of the Friedmann equation above can also be achieved in the bouncing braneworld scenarios, which are motivated in the context of String theory. It remains to be seen whether there is anything deep in this relationship.

## 2.3 Braneworlds avoiding the Big Bang

In braneworld scenarios, the standard model particles are confined to *branes* embedded in a higher dimensional *bulk*, and only gravity is permitted to propagate in the bulk (see [73, 74] for a review and background information). In the simplest of these cases, there is only one large extra higher dimension after compactification. In the Randall-Sundrum (R-S) model [75], for example, we live on a 3-brane (consisting of three spatial and one time dimensions) embedded in an extra spatial dimension, where the four dimensional metric is multiplied by an exponential warp factor, which is a function of the extra dimension. The effective Friedmann equation on the brane is modified in the R-S model at high densities [76, 77]:

$$H^2 = \frac{1}{3}\rho\left(1 + \frac{\rho}{2|\sigma|}\right) , \tag{2.18}$$

where $\sigma > 0$ is the brane tension, and the four dimensional cosmological constant on the brane has been ignored. This model does not solve the Big Bang singularity issue, but is one of the most popular and well studied models. It has been proposed, for example in [78, 79, 80, 81, 82], that what we understand as the Big Bang singularity could be interpreted as the collision of branes in a bulk. One effective scenario obtained in this context is the cyclic model [83, 84] containing the ekpyrotic phase.

The cyclic model is based on assuming that there is an attractive force between two boundary branes, which causes them to approach each other. Branes drawing nearer each other is seen as the universe (on the brane) collapsing, which is often referred to as the ekpyrotic phase. They eventually collide and separate. At the time of collision, quantum fluctuations on the branes result in different parts of each brane making contact with the opposite brane at slightly different times, leading to different parts of the universe starting to expand and cool at slightly different times.

Once the branes have separated and are far apart, they slow down and the attraction force between them will bring them closer together once again, resulting in another ekpyrotic phase and another brane collision. Repeating this process brings about the cyclic behaviour.

Even though these models are motivated in the context of higher dimensions, in this work we concentrate on the four-dimensional effective description. In particular, let us focus on the collapsing ekpyrotic phase, which is a more general evolutionary era, and although it is neatly incorporated within the particular model of a cyclic universe, it may also arise in alternative physical scenarios.

### 2.3.1 The Ekpyrotic model

The ekpyrotic model of cosmology claims to provide an alternative means of explaining the Big Bang cosmology puzzles before the universe starts expanding (see [85] for a recent review). It also claims to produce a scale-invariant spectrum of scalar perturbations in line with current observations [87]. Furthermore, it predicts a large blue tilt and very small amplitudes for the tensor perturbations [88]. It is important to note that near scale invariant curvature perturbations in such models have been strongly opposed in the literature once metric perturbations [89] and matching conditions between modes in a collapsing phase and those in an expanding era are considered [90]. Such perturbations have also been investigated in the context of multiple fields evolving during an ekpyrotic collapse [91, 92]. Here, a brief description of the basic ideas of this model is presented. We limit this introduction to considering only the ekpyrotic phase (not embedded into a larger picture such as the cyclic model); and when talking about perturbations, we consider only the scalar field fluctuations of a single scalar field in this setup.

Starting from the standard Friedmann equations in the Friedmann-Robertson-Walker (FRW) universe

$$H^2 = \frac{1}{3}\rho - \frac{k}{a^2}\,, \tag{2.19}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3P)\,, \tag{2.20}$$

where $\rho$ and $P$ are the energy density and the pressure of the fluid, respectively; and the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0\,, \tag{2.21}$$

for a constant equation of state $w \equiv \frac{P}{\rho}$, one can show that

$$\rho \propto a^{-3(1+w)}\,. \tag{2.22}$$

In a universe containing radiation, matter, and a scalar field, the Friedmann equation becomes

$$H^2 = \frac{1}{3}\rho - \frac{k}{a^2} = \frac{1}{3}\left(\frac{\rho_r}{a^4} + \frac{\rho_m}{a^3} - \frac{3k}{a^2} + \frac{\rho_\phi}{a^{3(1+w_\phi)}}\right), \tag{2.23}$$

where the subscripts $r$, $m$, and $\phi$ refer to radiation, matter and the scalar field, respectively. The Friedmann equation (2.19) can be rewritten as

$$\Omega - 1 = \frac{k}{a^2 H^2}, \tag{2.24}$$

where $\Omega \equiv \frac{\rho}{\rho_{crit}}$, where $\rho$ is the total energy density, and $\rho_{crit} = 3H^2$. Our current observations suggest an almost flat universe [27]: $|\Omega - 1| \lesssim 10^{-2}$, and extrapolating (2.23) back in time, suggests a much smaller value for the curvature at early times. This poses a fine-tuning problem in cosmology, which a successful candidate for the theory of the very early universe should address. Inflationary scenarios have been incredibly successful in providing an explanation for the flatness problem [22, 25]. They simultaneously resolve the horizon problem, which is related to why different regions in the sky, that would not have been in causal contact in the past, are seen to have very similar temperatures.

Inflation is defined as an era where $\ddot{a} > 0$. From (2.20), it is clear that, assuming positive energy density, the pressure of the fluid needs to be negative. The simplest way to achieve this is by using a scalar field, where the energy density would be given by $\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi)$, and the pressure by $P = \frac{\dot{\phi}^2}{2} - V(\phi)$. It is clear that when the energy density is dominated by the scalar field and the condition $V(\phi) \gg \frac{\dot{\phi}^2}{2}$ is satisfied (i.e. slowly moving scalar field), inflation takes place with $H \approx$ constant.

It is shown, for the first time in [22], that if the universe undergoes a rapid exponential expansion at very early times such that about 60 e-foldings of growth is achieved, the flatness and the horizon problems are resolved. i.e. when

$$\ln\left(\frac{a_e}{a_i}\right) \approx 60\,, \tag{2.25}$$

where the subscripts $e$ and $i$ denote the end and the start of the inflationary era, respectively. A more physically motivated definition for a general inflationary scenario to solve the horizon problem can be given in terms of the size of the comoving Hubble length. This is because the general inflationary condition $\ddot{a} > 0$, is equivalent to the decrease in the rate of change of the Hubble radius, i.e. $\frac{d(\frac{1}{aH})}{dt} < 0$. And, also, as originally argued in [22] and pointed out by others more recently [86], it is the sharp reduction of $\frac{1}{aH}$ during inflation that resolves the horizon problem. A more general and more physical definition for the condition to solve this cosmological problem can, therefore, be written in terms of the comoving Hubble length:

$$\ln\left(\frac{a_e H_e}{a_i H_i}\right) = \ln\left(\frac{\dot{a}_e}{\dot{a}_i}\right) \approx 60\,. \tag{2.26}$$

It is then clear that for the particular case of the standard inflationary scenario, since the Hubble parameter is almost a constant, Eq. (2.26) reduces to the more familiar form of Eq. (2.25). We will keep this in mind and return to this point in the next chapter, where we will carry out an explicit investigation of how LQC is capable of resolving the horizon problem.

By far, the most powerful result of inflation is its ability to account for the density perturbation patterns observed on the Cosmic Microwave Background (CMB) [17] and the accuracy with which this is done, together with the explanation it offers for the origin of the large scale structure from quantum perturbations.

During inflation, when the slow-roll condition is satisfied, from (2.23), it is clear that in an expanding universe, the scalar field dominates the evolution of the universe as the scale factor increases. It is also clear that during this rapid expansion the contribution from the curvature is exponentially suppressed by the scalar field resulting, effectively, in an exponentially flat universe.

The ekpyrotic model offers an alternative explanation to inflation with many similar outcomes. Instead of a very flat positive potential for the slow-roll process, imagine a very steep negative one, and instead of the universe expanding, consider a contracting scenario. The equation of state then is given by

$$w_\phi = \frac{\frac{\dot{\phi}^2}{2} - V(\phi)}{\frac{\dot{\phi}^2}{2} + V(\phi)}\,, \tag{2.27}$$

where $V(\phi)$ is negative and steep in the ekpyrotic model, and since the field is falling down fast due to the slope of the potential, $w_\phi \gg 1$. From (2.23), it is clear that in a collapsing universe, once again, the scalar field becomes dominant at late times (as $a \to 0$), and in the process of contraction, for large values of $w_\phi$, the curvature contribution is soon overtaken by the scalar field, resulting in an effectively flat universe.

In order to realise such a physical scenario in the context of ekpyrotic collapse, one first needs to obtain the background cosmology. For a universe filled with a scalar field, imposing the solution to the Friedmann equation and the continuity equation

$$H^2 = \frac{1}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right)\,, \qquad \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0\,, \tag{2.28}$$

to be a late time attractor, one finds the scaling background [101]

$$a = (-t)^q\,, \qquad \phi = \sqrt{2q}\ln(-t)\,, \qquad V(\phi) = -V_0 e^{-\sigma\phi}\,, \tag{2.29}$$

where $t$ is the cosmic time increasing from $-\infty$ to $0_-$, $V_0 = q(1-3q)$, and $\sigma = \sqrt{\frac{2}{q}}$. To find this cosmology, one may find it convenient to employ the useful relation $\dot{H} = -\frac{\dot{\phi}^2}{2}$. For a contracting universe, $q > 0$, and a steep potential implies $q \to 0_+$. The equation of state in this limit is given by

$$w = \frac{2}{3q} - 1 \gg 1\,. \tag{2.30}$$

Note that the contraction process is very slow, and the scalar field on this steep potential can be thought of as a stiff fluid. Studying the quantum fluctuations on this background can produce a near scale-invariant spectrum of density perturbations in accordance with current observations [87]. The ekpyrotic model also predicts a large blue tilt for the gravitational waves produced during this era, in great contrast to the small red tilt of tensor perturbations obtained during inflation [88]. Consequently, the amplitude of these fluctuations is suppressed by many orders of magnitude at large scales, compared to the levels predicted by inflation. It has been claimed that this is the decisive test, and if gravitational waves are observed on the CMB at levels of the order predicted by inflation, the ekpyrotic model would be ruled out [88, 85].

### 2.3.2 The Shtanov-Sahni model

Another four dimensional effective cosmology model in the context of braneworlds, which avoids the Big Bang singularity and is of interest in our phenomenological studies in this work, is the simple Shtanov-Sahni model [78, 79, 94]. The setup in this case is very similar to that of the R-S model with the exception of the extra dimension being timelike. In this model, the brane tension is found to be negative, and the Friedmann equation is modified such that

$$H^2 = \frac{1}{3}\rho\left(1 - \frac{\rho}{2|\sigma|}\right) , \tag{2.31}$$

which at first instance has a very similar form to (2.18), but on closer inspection, quite different phenomenology is described here, and in fact, it is more similar to the Holonomy corrections in LQC given by Eq. (2.16). Most importantly, an upper limit is implied on the possible level of the energy density, close to which a bounce can occur, as discussed in subsection (2.2.2). Notice that the additional $\rho^2$ term in (2.31) modifies $\dot{H}$ from its form in standard inflation

$$\dot{H} = -\frac{\dot{\phi}^2}{2}\left(1 - \frac{\rho}{\sigma}\right) . \tag{2.32}$$

Here, too, soon after the universe starts expanding (i.e. $\sigma < \rho \leq 2\sigma$), an inflationary phase begins. Later in the expansion process, as the energy density decreases (i.e. $0 < \rho \leq \sigma$), the condition for inflation becomes $H^2 > -\dot{H}$.

## 2.4 Fast-roll Inflation

In cosmologies where the energy density is dominated by a single scalar field, both inflationary solutions and scaling solutions have been demonstrated to be stable late time attractors (see [95, 96, 97] for early work in this field). Scaling solutions are those in which the kinetic energy of the scalar field scales with its potential energy when the energy density of the universe is dominated by the scalar field. Standard slow-roll inflation has been studied extensively, and its properties and predictions have been calculated to high precisions [98, 26, 99]. However, it is generally difficult to motivate such incredibly flat potentials from fundamental theories. Steep potentials arise more commonly from underlying physics (e.g. [80, 100, 18]). In order to see if such forms of scalar potentials can be supported in the context of scaling solutions, as possible alternatives to slow-roll inflation, one must first demonstrate that currently observed phenomena could be incorporated within this framework. One of the most important achievements of inflation has been its ability to explain the near scale-invariance of density perturbations on the CMB. We now briefly review the conditions under which this effect can be generated through a steep potential, first discussed in [101].

Let us consider the simple case of a single scalar field, $\phi$, on a potential, $V(\phi)$, where the energy density $\rho = \frac{\dot{\phi}^2}{2} + V$, and the pressure $P = \frac{\dot{\phi}^2}{2} - V$ , lead to the equation of state $w = P/\rho$ being a constant for scaling solutions. In perturbation theory, the scalar perturbations of the metric, for a spatially flat background in the Newtonian gauge, can be encoded by a single gauge invariant variable, $\Phi$, the Newtonian potential, such that [102]

$$ds^2 = a^2(\tau)\{-(1 + 2\Phi\,(\tau, \bar{x}))\, d\tau^2 + (1 - 2\Phi(\tau, \bar{x}))\, d\bar{x}^2\}\,, \tag{2.33}$$

where $\tau$ is the conformal time related to proper time, $t$, by $dt = a\, d\tau$.

By perturbing the scalar field $\phi \to \phi + \delta\phi$ obeying the equation of motion

$$\phi'' + 2\mathcal{H}\phi' - \nabla^2\phi + a^2 V_{,\phi} = 0\,, \tag{2.34}$$

the differential equation for the perturbations in Fourier space becomes

$$(\delta\phi)'' + 2\mathcal{H}(\delta\phi)' + (k^2 + a^2 V_{,\phi\phi})(\delta\phi) = 0\,, \tag{2.35}$$

where a prime denotes differentiation with respect to $\tau$, and $\mathcal{H} \equiv \frac{a'}{a}$.

Assuming a cosmological background, where the scale factor behaves as $a = (-\tau)^{\frac{1}{\tilde{\epsilon}-1}}$, for a scaling solution to the standard Friedmann equations, the equation of state is related to the slope of the potential by

$$\tilde{\epsilon} \equiv \frac{3}{2}(1+w) = \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2\,. \tag{2.36}$$

We note here, that $\tilde{\epsilon}$ has the usual form of the slow-roll parameter. Upon solving for $\delta\phi$ in (2.35), once normalisation has been done on small scales, where quantum fluctuations are most important, in such a way that no particle production takes place as modes evolve, the condition for near scale-invariance translates to [101]

$$\frac{\tilde{\epsilon}}{(\tilde{\epsilon}-1)^2} \ll 1\,. \tag{2.37}$$

Notice that this condition is satisfied in two different limits of $\tilde{\epsilon}$. First, when $\tilde{\epsilon} \to 0$, which corresponds to $w \approx -1$ (i.e. standard inflationary scenario), and second, when $\tilde{\epsilon} \to \infty$, corresponding to $w \gg 1$, which implies steep potentials. We will refer to inflation arising from these potentials as 'fast-roll' inflation in the remainder of this work.

Explicit calculations in [101] have shown that steep potentials in ekpyrotic models generate a near scale-invariant spectrum of scalar perturbations. The difficulty arises in quantifying any small deviation from exact scale-invariance. In such models, since the field is evolving rapidly down the potential, it is not appropriate to use the slow-roll parameters in an expansion, as $\tilde{\epsilon}$ will clearly have a very large value. It is convenient, therefore, to introduce a 'fast-roll' parameter, instead, defined as

$$\epsilon \equiv \frac{1}{2\tilde{\epsilon}} = \left(\frac{V}{V_{,\phi}}\right)^2 , \tag{2.38}$$

which will take small values in fast-roll inflation. By assuming a nearly constant $\tilde{\epsilon}$, one obtains

$$\frac{d\ln\tilde{\epsilon}}{dN} = -2\left(\frac{V_{,\phi}}{V}\right)\left(\frac{\phi'}{aH}\right)\eta , \tag{2.39}$$

where $N = \ln a$, and $\eta$ is small and is defined by

$$\eta \equiv 1 - \frac{VV_{,\phi\phi}}{V_{,\phi}^2} . \tag{2.40}$$

If one takes this parameter to be a constant (i.e. $\frac{d\ln\eta}{dN} = 0$), one could then use the parameters $\epsilon$ and $\eta$ to express the amount of deviation from exact scale-invariance in scalar field fluctuations. We will use this method in the context of LQC in the next chapter and the details of the calculation, which have been omitted here, will be clearly explained.

## 2.5 Stability Analysis

Throughout this work stability analysis is carried out on many dynamical systems. A brief note is included here to cover the basic approach taken and to outline the methods of phase plane analysis. (This is a well-studied concept in the context of dynamical systems and chaos theory. See [103] for a summary and various applications of this method). Here, we demonstrate this by taking a two-dimensional autonomous dynamical system of equations, where the ordinary differential equations (ODEs) do not depend on the independent variable, which is usually understood to be a dynamical parameter such as time. It is then easy to generalise this approach to higher dimensional and more complicated scenarios.

Let us consider the following equations

$$\dot{x} \equiv \frac{dx}{dt} = F(x(t), y(t)) \tag{2.41}$$
$$\dot{y} \equiv \frac{dy}{dt} = G(x(t), y(t)), \tag{2.42}$$

where $F$ and $G$ are continuously differentiable functions. We note that the gradient of the solution curve through each point $(x, y)$ on the phase plane is given by $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{G}{F}$. In order to find the critical points $(x_0, y_0)$ of the system, one must solve for $G = F = 0$, and then by considering the local behaviour about these points, their stability nature can be determined. Upon perturbing our variables around the fixed points such that $x = x_0 + \delta$, and $y = y_0 + \sigma$, where $\delta$ and $\sigma$ are taken to be small, after substituting back into (2.41) -(2.42) and expanding, one obtains

$$\frac{d\delta}{dt} = F|_{(x_0,y_0)} + \delta\frac{\partial F}{\partial x}|_{(x_0,y_0)} + \sigma\frac{\partial F}{\partial y}|_{(x_0,y_0)} + \dots \tag{2.43}$$
$$\frac{d\sigma}{dt} = G|_{(x_0,y_0)} + \delta\frac{\partial G}{\partial x}|_{(x_0,y_0)} + \sigma\frac{\partial G}{\partial y}|_{(x_0,y_0)} + \dots \tag{2.44}$$

The linear system of ODEs for the small linear changes near the fixed points can be written as

$$\begin{pmatrix} \dot{\delta} \\ \dot{\sigma} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta \\ \sigma \end{pmatrix}, \tag{2.45}$$

where the $\mathcal{M}$-matrix is

$$\mathcal{M} = \begin{pmatrix} \frac{\partial F}{\partial x}|_{(x_0,y_0)} & \frac{\partial F}{\partial y}|_{(x_0,y_0)} \\ \frac{\partial G}{\partial x}|_{(x_0,y_0)} & \frac{\partial G}{\partial y}|_{(x_0,y_0)} \end{pmatrix} \tag{2.46}$$

In our work, we consider cases where the $\mathcal{M}$-matrix is non-singular. Ignoring higher orders, Eq. (2.45) can be solved by finding the eigenvalues of the $\mathcal{M}$-matrix, and the solution will have the form

$$\begin{pmatrix} \delta \\ \sigma \end{pmatrix} = c_1 \begin{pmatrix} \bar{\delta}_1 \\ \bar{\sigma}_1 \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} \bar{\delta}_2 \\ \bar{\sigma}_2 \end{pmatrix} e^{\lambda_2 t} , \tag{2.47}$$

where $\lambda_i$ are the eigenvalues of $\mathcal{M}$, $\begin{pmatrix} \bar{\delta}_i \\ \bar{\sigma}_i \end{pmatrix}$ are the eigenvectors, and $c_i$ are constants set by the initial conditions, and where $i = 1, 2$ in this case.

The stability nature of the critical points is then classified in terms of the relation between the eigenvalues. There are five different classes: 1) $\lambda_1$ and $\lambda_2$ are both real and unequal, with the same sign. If they are both positive (negative), this corresponds to an unstable (asymptotically stable) node. 2) $\lambda_1$ and $\lambda_2$ are both real with either equal or unequal magnitudes, but have opposite signs, which corresponds to an unstable saddle point. 3) $\lambda_1 = \lambda_2$, so they are necessarily real. If they are both positive (negative), the critical point will be unstable (asymptotically stable). 4) $\lambda_i$ are complex conjugate pairs in which case the stability is determined by the sign of the real part. For a positive (negative) real part, the solution is an unstable spiral source (asymptotically stable spiral sink). 5) $\lambda_i$ are pure imaginary conjugate pairs in which case the solutions are always stable centres.

By continuity, one can extend the local behaviour around the fixed points to obtain a global picture of the phase plane. In doing so, one extends the linearity condition to the global picture. This is an assumption and needs to be checked or justified in cases under study. The most critical scenario sensitive to small changes in the values taken by the eigenvalues, which could dramatically change the dynamical behaviour described by the linear equations, is when they are both imaginary numbers. A small positive real change from this turns a stable centre to an unstable spiral. One needs to keep such assumptions and caveats in mind when carrying out stability analysis for a dynamical system.

# Chapter 3

# Super-inflation in LQC

## 3.1 Introduction

Given the importance of inflation and the need to explore all possibilities of accommodating it in alternative theories of quantum gravity, in this chapter we turn our attention to inflation in the context of Loop Quantum Gravity (LQG) [14].

While the consequences of the quantum evolution in the semi-classical regime of LQC are fascinating, it is difficult to connect it to existing theories of the early universe which tend to be based on classical background dynamics, and in particular to inflation. While the two sets of modifications discussed in the previous chapter have rather different origins, it appears that the qualitative effects of both the inverse volume effects and the $\rho^2$ term are rather similar. In particular, they both give rise to a period of super-inflation during which the Hubble factor rapidly increases, rather than remaining nearly constant as is the case during standard slow-roll inflation. Given that such a super-accelerating phase appears to be a robust prediction of LQC, it is important to study both the background dynamics, and particularly the cosmological perturbations, which such a phase gives rise to. Considering a universe sourced by a scalar field, a number of important results have already been obtained. A scaling solution for the effective equations which arise from the inverse scale factor modifications has been derived [63], and a number of attempts made at studying perturbations in the super-inflationary regime [109, 100]. Similarly the scaling solution for the $\rho^2$ effective equation has also been derived [110, 111].

It is also interesting to note the close connections between the super-inflationary phases in LQC and the evolution of a universe sourced by a phantom field, and with the ekpyrotic evolution of a collapsing universe [80, 81, 87, 83]. In all these cases the magnitude of the Hubble rate grows with time. Moreover, the scale factor duality discussed in [112] maps the ekpyrotic collapse onto the super-inflationary scaling solution for the inverse volume modified equations. On the other hand, another duality maps the ekpyrotic collapse phase onto the dynamics of a universe sourced by a phantom field [112]. These three regimes are therefore all related to one another at the background cosmology level. Furthermore, given that the ekpyrotic collapse is thought to offer a method for the generation of scale-invariant perturbations [101] (as is the dual super-inflationary phase sourced by a phantom field [113]), it is reasonable to expect that a similar mechanism may operate in the super-inflationary phases of LQC. Indeed such a mechanism has been discussed previously [100], though its relation to ekpyrotic and phantom models was not emphasised.

In this study we aim to explore further the phenomenology of super-inflation in LQC. One complication, however, is that the relative status of the two sets of modifications discussed is at present unclear. We therefore take a pragmatic approach and study the dynamics when each of the modifications is considered in turn, but not including both sets of modifications simultaneously. In this chapter, we focus our attention on perturbations in the scalar field as a first approximation. This approach allows us to establish a framework for dealing with metric perturbations in LQC. We will complete this study by turning our attention to metric perturbations in the next chapter.

This chapter is organised as follows. In section 3.2, we introduce the cosmological evolution equations which arise in LQC including the inverse volume corrections. Solutions are obtained including those showing scaling behaviour, and the primordial spectrum of scalar perturbations is calculated for each solution in terms of 'fast-roll' parameters. The stability of these solutions is then discussed. In section 3.3, we analyse the dynamics when the modification is induced by a $\rho^2$ correction to the Friedmann equation. Concentrating on the evolution just after the bounce, we demonstrate the existence of the super-inflationary solution, obtain the scaling dynamics of the system, the primordial spectrum of scalar perturbations as well as the stability of the background solutions. Section 3.4 discusses the way in which super-inflation in LQC can solve the horizon problem with a small number of e-foldings.

## 3.2 Effective field equations with LQC inverse volume corrections

The first set of modified equations which we consider are those which incorporate the two functions, $D_{j,l}(a)$ and $S_j(a)$, defined in section 2.2.2, into the dynamics. These functions arise because of the presence of powers of the inverse scale factor in the Hamiltonian constraint for an isotropic and homogeneous universe.

The modified Friedmann equation is given by

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3} S \left(\frac{\dot{\phi}^2}{2D} + V(\phi)\right) \quad , \tag{3.1}$$

where $\kappa^2 = 8\pi G$. In what follows we choose units in which $\kappa = 1$. Comparing this equation with (2.15), we note that we have omitted the intrinsic curvature contribution as we assume this to be a reasonable assumption. The equation of motion for the scalar field takes the form [100]

$$\ddot{\phi} + 3H \left(1 - \frac{1}{3}\frac{d\ln D}{d\ln a}\right)\dot{\phi} - D\nabla^2\phi + DV_{,\phi} = 0\,, \tag{3.2}$$

where a subscript $\phi$ means differentiation with respect to the field. Comparing this equation to Eq. (2.12), we see that the gradient term has been included in order for us to be able to do a perturbation analysis. Strictly speaking, this term violates the homogeneity assumtion, but we assume its effect on the background cosmology can be safely neglected. These equations can also be combined to give the Raychaudhuri equation

$$\dot{H} = -\frac{S\,\dot{\phi}^2}{2D}\left[1 - \frac{1}{6}\frac{d\ln D}{d\ln a} + \frac{1}{6}\frac{d\ln S}{d\ln a}\right] + \frac{S\,V}{6}\frac{d\ln S}{d\ln a}\ . \tag{3.3}$$

### 3.2.1 Scaling dynamics

We will be interested in the regime $a \ll a_\star$, where the function $D_{j,l}(a)$ may be approximated by a power law of the form $D(a) = D_\star a^n$, where $D_\star$ and $n$ are constants defined in the previous chapter with $n$ taking values in the range $6 < n < \infty$. Likewise, the function $S(a)$ may be similarly approximated by $S(a) = S_\star a^r$, where $S_\star$ and $r$ are also constants introduced in the last chapter, with $r$ in the range $3 < r < \infty$. For $a \gg a_\star$ (i.e. in the classical limit), $S_\star \approx D_\star \approx 1$ and $r = n = 0$. Inserting this form for the functions $S$ and $D$ into Eq. (3.3), we can clearly see that for an expanding universe, and with $n > 6 + r$, which occurs for all $l$, $\dot{H}$ is necessarily positive (assuming that the potential is either positive or the term involving $SV$ can be neglected). Hence super-inflation is occurring. We will confine ourselves to these situations in what follows.

To study this regime further, it proves convenient to introduce the variables

$$x \equiv \frac{\dot{\phi}}{\sqrt{2D\rho}}, \qquad y \equiv \frac{\sqrt{|V|}}{\sqrt{\rho}}\ . \tag{3.4}$$

where $\rho \equiv \frac{\dot{\phi}^2}{2D} + V(\phi)$. Using these definitions, the Friedmann equation (3.1) and the equation of motion for the scalar field (3.2) can be written, for an expanding universe, in terms of a system of first order differential equations as

$$
\begin{aligned}
x_{,N} &= -3\alpha x \pm \sqrt{\frac{3}{2}}\lambda y^2 + 3\alpha x^3\,, &(3.5)\\
y_{,N} &= -\sqrt{\frac{3}{2}}\lambda x y + 3\alpha x^2 y\,, &(3.6)\\
\lambda_{,N} &= -\sqrt{6}\lambda^2(\Gamma-1)x + \frac{1}{2}(n-r)\lambda\,, &(3.7)
\end{aligned}
$$

where

$$\lambda \equiv -\sqrt{\frac{D}{S}}\frac{V_{,\phi}}{V}\,, \qquad \Gamma \equiv \frac{V\,V_{,\phi\phi}}{V_{,\phi}^2}\,, \tag{3.8}$$

with $\alpha = 1 - n/6 < 0$ and $N = \ln a$. These variables are subject to the constraint equation

$$x^2 \pm y^2 = 1. \tag{3.9}$$

The plus and minus signs correspond to positive and negative potentials, respectively. Using the constraint equation to substitute for $y$ in Eq. (3.5) renders Eq. (3.6) redundant.

The resulting system defined by Eq. (3.5) together with the constraint equation and (3.7) has three fixed points for $\lambda \neq 0$. Two of them represent kinetic energy dominated solutions, valid for all values of $\lambda$:

$$
\begin{aligned}
x &= -1\,, \quad y = 0\,, \quad \Gamma = 1 - \frac{\sqrt{6}}{12\lambda}(n-r)\,, &(3.10)\\
x &= +1\,, \quad y = 0\,, \quad \Gamma = 1 + \frac{\sqrt{6}}{12\lambda}(n-r)\,, &(3.11)
\end{aligned}
$$

and the third point is a scaling solution for which the kinetic and potential energies evolve in a constant ratio to one another:

$$
\begin{aligned}
x &= \frac{\lambda}{\sqrt{6}\,\alpha}, \qquad y = \sqrt{\pm\left(1 - \frac{\lambda^2}{6\alpha^2}\right)}\,, &(3.12)\\
\Gamma &= 1 + \frac{\alpha}{2\lambda^2}(n-r)\,. &(3.13)
\end{aligned}
$$

The scaling solution is therefore well defined for $\lambda^2 < 6\alpha^2$ for positive potentials and for $\lambda^2 > 6\alpha^2$ for negative potentials. In the remainder of this analysis, we will focus mainly on the scaling solution for negative potentials as this is the case that, as we shall see, leads to a scale invariant power spectrum of the perturbed field.

Considering the fixed point for the scaling solution (3.12), one can write

$$\frac{\dot{\phi}}{\sqrt{2D\rho}} = \sqrt{\frac{S}{D}}\frac{\phi_{,N}}{\sqrt{6}} = \frac{\lambda}{\sqrt{6}\alpha}\,, \tag{3.14}$$

which upon integration gives

$$\phi = \frac{2\lambda}{(n-r)\alpha}\sqrt{\frac{D}{S}}\,, \tag{3.15}$$

where we have set the integration constant to zero without loss of generality.

Then inserting this relation into the definition of $\lambda$ in (3.8) gives

$$V = V_0\,\phi^{\beta}\,, \tag{3.16}$$

where $\beta = \frac{-2\lambda^2}{(n-r)\alpha} > 0$, and $V_0$ is a constant.

Considering now the fixed point for $y$ we have

$$\frac{V}{\rho} = \frac{V\,S}{3H^2} = 1 - \frac{\lambda^2}{6\alpha^2}\,. \tag{3.17}$$

Obtaining an expression for $\phi$ in terms of $a$, from (3.15) in the semi-classical regime, and substituting this into the form of the potential (3.16), and using (3.17), the Hubble parameter can be found from which, by integration, the behaviour of the scale factor can be determined as a function of time. This is calculated to be a power law function, and in terms of the conformal time, $\tau$, where $dt = ad\tau$, we find

$$a(\tau) = (-\tau)^{p}\,, \tag{3.18}$$

where for an expanding universe $\tau$ is negative and increasing towards zero, and

$$p = \frac{2\alpha}{2\bar{\epsilon} - (2+r)\alpha}\,, \tag{3.19}$$

where, for direct comparison with the previous chapter and Eq. (2.36), we have introduced the slow-roll parameter $\bar{\epsilon} \equiv \lambda^2/2$, and from (3.12), $\lambda = -\sqrt{2\bar{\epsilon}}$ for $\dot{\phi} > 0$. Using this form of $a$ we find that $H = \frac{p}{a\tau}$, and it is straightforward to show that

$$\phi'(\tau) = -\frac{2\sqrt{2\bar{\epsilon}}}{2\bar{\epsilon} - (2+r)\alpha}\sqrt{\frac{D}{S}}\,\frac{1}{\tau}\,, \tag{3.20}$$

$$V(\tau) = \frac{4(3\alpha^2 - \bar{\epsilon})}{(2\bar{\epsilon} - (2+r)\alpha)^2}\,\frac{1}{S(a\tau)^2}\,, \tag{3.21}$$

where a prime means differentiation with respect to conformal time, $\tau$. Equations (3.18) - (3.21) form the basis of our analysis.

### 3.2.2 Power spectrum of the perturbed field

For a universe which evolves according to the scaling solution, the primordial spectrum of scalar perturbations produced by this super-inflationary phase was previously calculated in Ref. [100]. It was found that the spectrum tends to exact scale invariance for $\beta \gg 1$ (i.e. $\bar{\epsilon} \gg 1$), without any fine tuning of the quantisation parameters of LQC. The purpose of this section is to review how scale invariance arises for the scaling solution with $\beta \gg 1$, and to generalise the analysis of [100] in order to allow for potentials which do not give rise to exact scaling solutions.

In order to calculate the spectrum of perturbations, we now perturb the scalar field equation (3.2) to linear order by using $\phi \to \phi + \delta\phi$, and $V_{,(\phi+\delta\phi)} = V_{,\phi} + \delta\phi V_{,\phi\phi} + O(\delta\phi^2)$. The perturbation in the field $\delta\phi$ then satisfies the equation

$$\delta\phi'' = \left[-2\frac{a'}{a} + \frac{D_{,\tau}}{D}\right]\delta\phi' + D\left[\nabla^2 - a^2 V_{,\phi\phi}\right]\delta\phi\,, \tag{3.22}$$

which can be written in the form [100]

$$u'' + \left(-D\nabla^2 + m_{\rm eff}^2\right)\, u = 0\,, \tag{3.23}$$

where $u$ is defined as $u \equiv aD^{-1/2}\delta\phi$ and the effective mass of the field $u$ is given by

$$m_{\text{eff}}^2 = -\frac{(aD^{-1/2})''}{aD^{-1/2}} + a^2 D V_{,\phi\phi} \ . \tag{3.24}$$

This change of variable clearly does not change the physical outcome, but it allows (3.22) to be written in such a way that the first order term is eliminated. It is then convenient to compare (3.23) with a harmonic oscillator equation. We pause here to note that the first term inside the brackets of Eq. (3.23) is reminiscent of the form obtained in the context of dynamics of a fluid with a varying speed of sound (see [114, 115, 116, 117] for examples). Here, the function $D(a)$ plays the role of the square root of the effective speed of sound, $C_s$. As it has recently been shown [118], for the case of $C_s \approx t^{\tilde{p}}$, where $\tilde{p}$ is a constant, there are two conditions under which scale invariance of the field fluctuations may be obtained. Namely, $\tilde{p} \to 0$ and $\tilde{p} \to -2$. In the case of the modified LQC equation (3.23), $\tilde{p} = \frac{np}{2(1+p)}$. The first condition corresponds to either the classical case of $n \to 0$, or $p \to 0$, which as we shall see later, is the scenario we will be interested in. The second condition corresponds to $n \to -4\frac{1+p}{p}$, which in the limit of $p \to 0$ results in very large values for $n$. Although this has not been ruled out in the theory, the largest value quoted in numerical simulations of LQC is of the order of 12. Therefore, the assumption that allows us to proceed from this point to calculating the power spectrum of the field fluctuations is that $D(a)$ is a very slowly varying function of time. We will see later in the chapter that this assumption is justified in our arguments since the scale factor will be demonstrated to be a slowly varying function of time during the super-inflationary phase under study.

Eq. (3.23) can also be obtained by varying the perturbed part of the action in inverse volume corrected LQC [100]

$$\delta S = \frac{1}{2} \int d\tau d^3\mathbf{x} (u'^2 - D\delta^{ij}\partial_i u \partial_j u - m_{eff}^2 u^2) \, . \tag{3.25}$$

In order to consider quantum fields, we start by canonical quantisation of this action to find the conjugate momentum, $\pi_u$:

$$\pi_u = \frac{\partial \mathcal{L}}{\partial u'} = u' \,. \tag{3.26}$$

In quantum theory, the variables $u$ and $\pi_u$ become operators $\hat{u}$ and $\hat{\pi}_u$, respectively, obeying the usual commutation relations

$$[\hat{u}(\tau,\mathbf{x}),\hat{u}(\tau,\mathbf{y})] = [\hat{\pi}_u(\tau,\mathbf{x}),\hat{\pi}_u(\tau,\mathbf{y})] = 0\,, \qquad [\hat{u}(\tau,\mathbf{x}),\hat{\pi}_u(\tau,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y})\,, \tag{3.27}$$

and the field equation for $\hat{u}$ is given by Eq. (3.23) once the following transformation is made:

$$u \to \hat{u}\,, \qquad \text{and} \qquad u'' \to \hat{u}''\,. \tag{3.28}$$

Decomposing $\hat{u}$ in Fourier modes $u_k$, we have

$$\hat{u}(\tau,\mathbf{x}) = \int \frac{\mathbf{d^3k}}{\mathbf{(2\pi)^3}}\,[\mathbf{u_k}(\tau)\mathbf{\hat{a}_k}\,\mathbf{e^{ik.x}} + \mathbf{u^*_k}(\tau)\,\mathbf{\hat{a}^\dagger_k}\,\mathbf{e^{-ik.x}}]\,, \tag{3.29}$$

where $u_k$ satisfies the equation of motion

$$u''_k + \left(Dk^2 + m^2_{\rm eff}\right)\, u_k = 0\,, \tag{3.30}$$

in Fourier space. The creation and annihilation operators $\hat{a}_k$ and $\hat{a}^\dagger_k$ obey the usual commutation relations

$$[\hat{a}_{k_1},\hat{a}_{k_2}] = [\hat{a}^\dagger_{k_1},\hat{a}^\dagger_{k_2}] = 0\,, \qquad [\hat{a}_{k_1},\hat{a}^\dagger_{k_2}] = \delta_{k_1 k_2}\,. \tag{3.31}$$

It is then straightforward, but tedious, to verify that the commutation relations in (3.27) and (3.31) are equivalent as long as the Wronskian condition below is satisfied

$$u^*_k\,\frac{du_k}{d\tau} - u_k\,\frac{du^*_k}{d\tau} = -i\,. \tag{3.32}$$

For this system a unique vacuum state $|0\rangle$ can be defined [102] such that

$$\hat{a}_k|0\rangle = 0\,,\ \forall k\,, \tag{3.33}$$

where the annihilation operators, $\hat{a}_k$, are the coefficients of the positive frequency modes in the expansion of $\hat{u}$ given by Eq. (3.29). Since we are interested in investigating quantum fluctuations arising from the vacuum, we will concentrate on the forward moving (i.e. positive frequency) waves once solutions to (3.30) are found, and then set the negative frequency components to zero. This will also eliminate the possibility of particle production in the vacuum.

The power spectrum for the quantum field, $\hat{u}(\tau, \mathbf{x})$, which can be expanded in Fourier space as in (3.29), is defined by [102]

$$\mathcal{P}_u \equiv \frac{k^3}{2\pi^2}|u_k|^2 \,. \tag{3.34}$$

The general solution to Eq. (3.30) is found to be given in terms of the Bessel funtions as

$$u_k(\tau) = c_1\sqrt{-\tau}\, J_{|\nu|}(x) + c_2\sqrt{-\tau}\, Y_{|\nu|}(x)\,, \tag{3.35}$$

where

$$\nu = -\frac{\sqrt{1-4\,m_{\rm eff}^2\,\tau^2}}{2+np}\,, \qquad\qquad x = \frac{2}{2+np}\frac{k\sqrt{D}}{aH}\,, \tag{3.36}$$

and where $c_1$, $c_2$, and $m_{\rm eff}\tau$ are constants. This solution can also be rewritten using the Hankel functions so that

$$u_k(\tau) = \frac{c_1 - ic_2}{2}\sqrt{-\tau}\, H^{(1)}_{|\nu|}(x) + \frac{c_1 + ic_2}{2}\sqrt{-\tau}\, H^{(2)}_{|\nu|}(x)\,, \tag{3.37}$$

where $H^{(1)}_{|\nu|}(x) = J_{|\nu|}(x) + iY_{|\nu|}(x) \propto e^{ix}$ and $H^{(2)}_{|\nu|}(x) = J_{|\nu|}(x) - iY_{|\nu|}(x) \propto e^{-ix}$ are Hankel functions of the first and second kind, respectively. By rewriting the solution in terms of these functions, one is able to conveniently decompose it in terms of two plane waves propagating in two opposite directions. As mentioned above, we follow the usual convention of standard inflationary scenario and select only the advanced part by setting $c_1 = ic_2$, and ignoring the retarded wave. Eq. (3.37) is then reduced to

$$u_k(\tau) = c_1\sqrt{-\tau}\, H^{(2)}_{|\nu|}(x)\,. \tag{3.38}$$

In order to find the constant $c_1$, one imposes the Wronskian condition (3.32). This mode normalisation should be done at very high frequencies (i.e. $\frac{k\sqrt{D}}{aH} \to \infty$), where quantum fluctuations in the vacuum are most important. By imposing this condition the solution (3.38) becomes

$$u_k(\tau) = \sqrt{\frac{\pi}{2(2+np)}}\sqrt{-\tau}\, H^{(2)}_{|\nu|}(x)\,. \tag{3.39}$$

On large scales (i.e. $x \propto \frac{k\sqrt{D}}{aH} \to 0$), by using the appropriate limits of the Bessel functions

$$J_{|\nu|}(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}\,, \qquad Y_{|\nu|}(x) \to -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^{\nu}\,, \tag{3.40}$$

the power spectrum defined in Eq. (3.34), yields

$$\mathcal{P}_u \propto k^{3-2|\nu|}\,(-\tau)^{1-|\nu|(np+2)}\,. \tag{3.41}$$

Scale invariance of the power spectrum is then attained when the spectral tilt $\Delta n_u \equiv 3 - 2|\nu|$ is zero.

For a universe evolving according to the scaling solution Eq. (3.18), we have

$$m^2_{\rm eff}\tau^2 = -2 + (3-2n)p + \frac{1}{2}(6+2n-n^2)p^2\,, \tag{3.42}$$

where we have used Eqs. (3.20)-(3.21), and (3.24). We can see from Eqs. (3.36), (3.41), and (3.42) that scale invariance occurs whenever $p \to 0$, which, as we referred to, does indeed imply that $\bar{\epsilon} \gg 1$ and consequently $V < 0$ from Eq. (3.21). There is one other value of $p$ for which scale invariance is attained, $p = -4/(n+4)$; however, we will not consider it any further, as there will be an explicit dependence on the value of $n$ in this case, and this parameter is not well bounded within the context of LQC.

We would now like to generalise the form of the potential we are dealing with so that it no longer has to be of exactly the form which gives rise to the scaling solution. For standard slow-roll inflation, where the kinetic energy is small compared with the potential energy, it is possible to account for potentials of a form more general than a scaling potential by introducing slow-roll parameters. Such parameters parametrise the steepness of the potential, and how this steepness evolves as the field moves along the potential. For a given field potential they also allow the dynamics which follow from a more general potential to be expanded locally about the dynamics which follow from a scaling potential with the same local steepness. The power spectrum which follows from the general potential can then also be written in terms of the slow-roll parameters.

For the case at hand we would like to develop a similar expansion scheme. However, for the regime which we are considering in which $\bar{\epsilon} \gg 1$, it is clear that the kinetic energy is of approximately the same magnitude as the potential energy, and therefore the slow-roll approximation is inadequate. Indeed in this case the field is evolving rapidly along a steep negative potential, and we refer to the evolution as the 'fast-roll' regime, which we briefly touched on in the previous chapter. Our strategy will therefore be to determine other suitable small parameters which characterise the steepness and curvature of the potential, and which we will refer to as 'fast-roll' parameters. The derived parameters we will arrive at are similar to those described in the last chapter, where fast-roll parameters were required to parametrise general potentials in the ekpyrotic scenario. This similarity is natural since, as we have already mentioned in the introduction, the evolution of the super-inflationary scaling solution in LQC is dual to the ekpyrotic collapse.

The first step in accommodating a more general class of potentials is to allow $\bar{\epsilon}$ to become time dependent. From its definition it then follows that

$$\bar{\epsilon}' = -(2\bar{\epsilon})^{3/2}\,\eta\sqrt{\frac{S}{D}}\phi'\,, \tag{3.43}$$

where we have defined

$$\eta \equiv 1 - \frac{V_{,\phi\phi}V}{V_{,\phi}^2} - \frac{1}{2}\frac{V}{V_{,\phi}}\left(\frac{D_{,\phi}}{D} - \frac{S_{,\phi}}{S}\right), \tag{3.44}$$

which can also be written in terms of the background quantities as

$$\eta \equiv 1 - \frac{V_{,\phi\phi}V}{V_{,\phi}^2} - \frac{1}{2}(n-r)\frac{V}{V_{,\phi}}\frac{a'}{a\,\phi'}. \tag{3.45}$$

Likewise, we can calculate $\eta'$ in terms of a third parameter $\xi^2$,

$$\eta' = -\sqrt{2\bar{\epsilon}}\,\xi^2\sqrt{\frac{S}{D}}\phi', \tag{3.46}$$

where

$$\begin{aligned}
\xi^2 \quad \equiv \quad & \left[1 + \frac{V_{,\phi\phi\phi}V}{V_{,\phi\phi}V_{,\phi}} - 2\frac{V_{,\phi\phi}V}{V_{,\phi}^2}\right]\frac{V_{,\phi\phi}V}{V_{,\phi}^2} + \\
& -\frac{1}{2}\left[1 + \frac{D_{,\phi\phi}V}{D_{,\phi}V_{,\phi}} - \frac{D_{,\phi}V}{DV_{,\phi}} - \frac{V_{,\phi\phi}V}{V_{,\phi}^2}\right]\frac{D_{,\phi}V}{DV_{,\phi}} + \\
& +\frac{1}{2}\left[1 + \frac{S_{,\phi\phi}V}{S_{,\phi}V_{,\phi}} - \frac{S_{,\phi}V}{SV_{,\phi}} - \frac{V_{,\phi\phi}V}{V_{,\phi}^2}\right]\frac{S_{,\phi}V}{SV_{,\phi}}.
\end{aligned} \tag{3.47}$$

In particular, for the scaling solution where $\bar{\epsilon}$ is constant, one can verify that $\eta = \xi^2 = 0$ exactly. Since we are considering potentials which are close to the form of a scaling potential, we expect that there will be solutions to the equations of motion of a form very similar to that given by Eqs. (3.18), (3.19), (3.20) and (3.21), when $\bar{\epsilon}$ is slowly varying. Assuming that Eqs. (3.18), (3.19) and (3.20) are indeed good approximations, from (3.43) and (3.46), we have that

$$\frac{d\ln\bar{\epsilon}}{d\ln a} \approx 4\frac{\bar{\epsilon}\,\eta}{\alpha}, \qquad \frac{d\ln\eta}{d\ln a} \approx 2\frac{\bar{\epsilon}\,\xi^2}{\eta\,\alpha}. \tag{3.48}$$

Imposing that $\bar{\epsilon}$ and $\eta$ are slowly varying, and since $\bar{\epsilon}$ is large in the regime which gives rise to scale invariance, requires $\eta$ and $\xi^2$ to be small, i.e. the potential must be nearly power law in form which is in agreement with our assumption. We refer to $\eta$ and $\xi$ as the second and third fast-roll parameters, and for convenience introduce the first fast-roll parameter as $\epsilon = 1/2\bar{\epsilon}$. In terms of relative magnitude we find, from the form of the potential (3.16), $\beta = \frac{-4\bar{\epsilon}}{(n-r)\alpha}$ and $\bar{\epsilon} \gg 1$, that $\eta \sim \epsilon$ and $\xi^2 \sim \mathcal{O}(\epsilon^2)$. Using these relations it is possible to verify, by substituting Eqs. (3.18) – (3.21) into the Friedmann equation and the equation of motion, that these solutions are indeed valid up to second order in fast-roll parameters for a general negative potential confirming the consistency of our analysis.

Substituting Eq. (3.19) into (3.42) and then evaluating $\nu$ in (3.36) to first order in the parameters $\epsilon$ and $\eta$, we obtain

$$\Delta n_u \approx 4\epsilon \left[1 - \frac{n}{12}\left(1 + \frac{n}{6} - r\right) - \frac{r}{2}\right] - 4\eta\,, \tag{3.49}$$

where we have made use of $\Delta n_u \equiv 3 - 2|\nu|$ in Eq. (3.41). We note that although we have used the solution to Eq. (3.30) which is valid only when $m_{\rm eff}\tau$ is constant, the solution will remain sufficiently accurate provided that $\epsilon$ does not vary significantly as a given $k$ mode crosses the horizon from the small wavelength to the long wavelength regime. This is simply the condition that $\eta$ is small, which we have assumed already. The spectral tilt can therefore be calculated at any given scale by inserting into Eq. (3.49) the values that $\epsilon$ and $\eta$ take as this scale crosses the horizon. In particular, to compare with observations we need to consider the mode which corresponds to the largest scales on the Cosmic Microwave Background (CMB).

Finally we also note that we could include time derivatives of the powers $n$ and $r$. The generalisation to do so is straightforward and, following the calculation process that led to Eq. (3.49), yields a more complicated expression for the spectral index

$$\begin{aligned}\Delta n_u \;\approx\;& 4\epsilon \left[1 - \frac{n}{12}\left(1 + \frac{n}{6} - r\right) - \frac{r}{2}\right] - 4\eta \\ &+ \frac{\epsilon}{9}\tau\, n' \left[8 - 3(n-6) - 4(r+n)\right] \\ &+ \frac{\epsilon}{9}\tau\, r' \left[-8 - 4(n-6)\right]\,.\end{aligned} \tag{3.50}$$

An explicit time dependance has now entered the expression for the spectral tilt, which complicates the interpretation of this result. It does, however, imply that for $n \propto r \propto \ln \tau$, a small constant value for the tilt will be recovered. For simplicity we ignore such time derivatives and assume them to be small in our region of interest.

### 3.2.3 Stability of the fixed points

The analysis we have performed so far is intriguing. It appears that in the $a \ll a_*$ regime, the scaling solution which follows from a steep negative potential can give rise to a scale-invariant power spectrum of scalar field perturbations, and moreover we can generalise the analysis to potentials which deviate from the scaling potential.

However, there is another element which is involved in building a convincing theory for the origin of scale-invariant perturbations. That is, it would be highly desirable if the scaling solution was an attractor, so that initial conditions not exactly on the solution would evolve towards it, and the solution's stability against small local perturbations would be assured.

To determine the stability of the scaling solution, we study the nature of the fixed points of Eqs. (3.5) and (3.7). We do this by linearising the equations about the fixed points and determining the corresponding eigenvalues ($\omega$) in each case, using the method outlined in the previous chapter. For the kinetic energy dominated solutions, valid for an arbitrary $\lambda$, in Eqs. (3.10) and (3.11), we find their respective eigenvalues of the $\mathcal{M}$-matrix

$$\mathcal{M} = \begin{pmatrix} -3\alpha - \sqrt{6}x\lambda + 9\alpha x^2 & -\sqrt{\frac{3}{2}}(x^2-1) \\ -\sqrt{6}(\Gamma-1)\lambda^2 & -2\sqrt{6}x(\Gamma-1)\lambda + \frac{1}{2}(n-r) \end{pmatrix} \tag{3.51}$$

to be

$$\omega_+ = 6\alpha + \sqrt{6}\lambda\,, \qquad \omega_- = -\frac{1}{2}(n-r)\,, \tag{3.52}$$

$$\omega_+ = 6\alpha - \sqrt{6}\lambda\,, \qquad \omega_- = -\frac{1}{2}(n-r)\,. \tag{3.53}$$

Since $n > r$ and $\alpha < 0$, the first fixed point is stable for $\lambda < -\sqrt{6}\alpha$ and the second for $\lambda > \sqrt{6}\alpha$. Turning to the scaling solution, Eq. (3.12), we find

$$\omega_{\pm} = -\frac{1}{4\alpha}\left(\theta \pm \sqrt{\theta^2 + 8\,\alpha\,(n-r)(\lambda^2 - 6\alpha^2)}\right)\,, \tag{3.54}$$

where $\theta = \alpha\,(6-r) - \lambda^2 < 0$. The scaling solution is therefore stable whenever $\lambda^2 > 6\alpha^2$, which coincides with the region of existence of this solution. In Fig. 3.1 we show the evolution of the ratio $-\dot{\phi}^2/2DV$ obtained by numerically solving the equations of motion. We can see that it approaches the value given by $x^2/y^2$ where $x$ and $y$ are given by Eqs. (3.12). Tyically the evolution only reaches the attractor when $a > a_\star$ which is far outside the region where the appoximation $D \approx a^n$ is valid. We conclude that this solution must be extremely fine tuned in order to deliver the dynamics and power spectrum as described in the previous subsections.

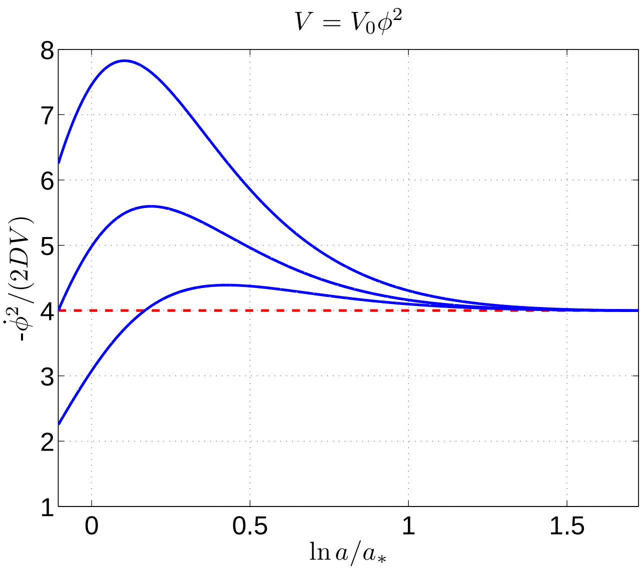


Figure 3.1: The evolution of the ratio $-\dot{\phi}^2/2DV$ obtained by numerically solving the equations of motion for three different initial conditions (solid line). They approach the scaling solution given by $x^2/y^2$ where $x$ and $y$ are given by Eqs. (3.12) (dashed line). We used as parameters $V_0 = -10^{-20}$, $\phi_{\rm init} = 1$, in Planck units and $n = 15$, $r = 3$ and $a_{\rm init} = 0.9a_\star$, in a flat universe.

In the second part of this chapter we will be dealing with a second possibility of obtaining a super-inflationary regime that also leads to a scale invariant power spectrum with the advantage that the stability of the scaling solution is no longer a dangerous issue.

## 3.3 Effective dynamics with quadratic corrections

The second modification to classical dynamics which we adopt follows from considering that holonomies are the basic variables for quantisation in LQC. This modification gives rise to a Friedmann equation of the following form [51, 71, 50, 34, 53, 54]:

$$H^2 = \frac{1}{3}\rho\left(1 - \frac{\rho}{2\sigma}\right), \tag{3.55}$$

which is identical to Eq. (2.16) that we first introduced in section 2.2.2. Once again we are assuming either a flat universe or that the curvature contribution can be safely neglected. It is interesting that this form of the Friedmann equation is identical to the form which arises in braneworld scenarios with a single time-like extra dimension spacetime [79]. In the braneworld case $\sigma$ represents the brane tension while for the LQC case $2\sigma$ represents the critical energy density arising from quantum geometry effects which leads to the scale factor undergoing a bounce as $\rho$ approaches it from below.

We are interested in high density regimes where $\rho$ approaches the bounding value of $2\sigma$. In this case, the term within brackets tends to zero, and the behaviour of the equations alters significantly compared with the classical behaviour. Indeed in this regime we have $\dot{H} > 0$ and for an expanding universe super-inflation takes place. We will again consider a scalar field dominated universe, hence, $\rho = \dot{\phi}^2/2 + V(\phi)$. We stress that we are studying inverse volume and quadratic corrections separately, hence we do not include the $D$ and $S$ functions in the Friedmann equation and in the definition of energy density. The scalar field equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0\,, \tag{3.56}$$

is unchanged from the classical form.

### 3.3.1 Scaling dynamics

It was shown in Ref. [110] (also see [111]) that the effective equations (3.55)-(3.56) also allow a scaling solution in which the kinetic and potential energy vary in proportion to one another. In this case the potential must be of the form $V = V_0\cosh(\phi)$. However, the scaling solution which was found implies an evolution for the scale factor, which is not of a power law kind, and it has not been possible to obtain analytical solutions to the equations of motion of the scalar fluctuations, or for the spectrum of perturbations. It is, thus, quite difficult to make general comments on the scale-dependence of such a spectrum. Instead of focusing on the scaling solution we, therefore, take a different approach and ask whether there is a form of the scalar field potential which does result in a power law evolution of the scale factor. We are interested in the regime in which $\rho \approx 2\sigma$, when super-inflation occurs, and where $H \approx 0$. Inserting the power-law ansatz,

$$a(t) = (-t)^m\,, \tag{3.57}$$

where $m$ is a constant, into the time derivative of the Hubble rate,

$$\dot{H} = -\frac{\dot{\phi}^2}{2}\left(1-\frac{\rho}{\sigma}\right)\,, \tag{3.58}$$

we see that for $\rho \approx 2\sigma$ the kinetic energy of the field is $\dot{\phi}^2/2 \approx -m/t^2$ which upon integration gives

$$\phi \approx \pm\sqrt{-2m}\,\ln t\,, \tag{3.59}$$

from which we clearly need $m < 0$. Using Eq. (3.55) and expanding in $6H^2/\sigma \ll 1$, solving for $\rho$, we have that

$$\rho \approx 2\sigma - 3H^2\,, \tag{3.60}$$

and it follows from the definition of the energy density of the field and Eq. (3.59) that

$$V \approx 2\sigma - U_0 e^{-\lambda\phi}\,, \tag{3.61}$$

where $U_0 = 3m^2 - m > 0$ and $\lambda = 1/\sqrt{-2m}$. It is evident that, in this regime, scaling exists between $V - 2\sigma$ and the kinetic energy $\dot{\phi}^2/2$. We now look for a more precise description of the dynamics.

The form of the potential (3.61) motivates us to define the new variables:

$$x \equiv \frac{\dot{\phi}}{\sqrt{4\sigma - 2\rho}}, \qquad y \equiv \frac{\sqrt{U}}{\sqrt{2\sigma - \rho}}\,, \tag{3.62}$$

such that $\rho \lesssim 2\sigma$ and $V(\phi) = 2\sigma - U(\phi)$. In terms of a system of first order differential equations, the equation of motion of the scalar field now reads,

$$x_{,N} = -3x - \sqrt{\frac{3}{2}}\,\lambda\, y^2 - 3x^3 \tag{3.63}$$

$$y_{,N} = -\sqrt{\frac{3}{2}}\,\lambda x y - 3x^2 y \tag{3.64}$$

$$\lambda_{,N} = -\sqrt{6}\,\lambda^2(\Gamma - 1)x + 3x^2\left(\frac{2\sigma}{\rho} - 1\right)\sqrt{\frac{2\sigma}{\rho}}\,, \tag{3.65}$$

where $\lambda$ and $\Gamma$ are defined as

$$\lambda \equiv -\frac{U_{,\phi}}{U}\sqrt{\frac{2\sigma}{\rho}}\,, \qquad \Gamma \equiv \frac{U\,U_{,\phi\phi}}{U_{,\phi}^2}\,. \tag{3.66}$$

The variables $x$, and $y$ are also related by the constraint condition

$$x^2 - y^2 = -1. \tag{3.67}$$

Considering the regime discussed above where $2\sigma/\rho \approx 1$, we can see $\lambda$ is a constant (remember $\lambda = 1/\sqrt{-2m}$), and by integrating $\lambda$ in (3.66), we see that the $U$ part of the scalar potential is given by

$$U = U_0\, e^{-\lambda\phi}\,, \tag{3.68}$$

as we were expecting from Eq. (3.61).

We consider the section of the phase space in which $x < 0$, $y > 0$, and $\lambda > 0$. Taking $\lambda$ to be constant, and substituting the constraint equation (3.67) into Eq. (3.63), results in an autonomous system with three fixed points. Two of them are non-physical solutions with

$$x = \pm i\,, \qquad y = 0\,, \tag{3.69}$$

and the third is a scaling solution with

$$x = -\frac{\lambda}{\sqrt{6}}\,, \qquad y = \sqrt{1 + \frac{\lambda^2}{6}}\,. \tag{3.70}$$

The scaling solution is valid for all real values of $\lambda$.

As in the previous case, it is straightforward to show that as the universe evolves according to this solution,

$$\frac{\dot{\phi}}{\sqrt{4\sigma - 2\rho}} = \frac{\phi_{,N} H}{\sqrt{4\sigma - 2\rho}} = -\frac{\lambda}{6}\,, \tag{3.71}$$

which upon integration, in the region $\rho \approx 2\sigma$, yields

$$\phi \approx -\lambda \ln a\,. \tag{3.72}$$

The potential in Eq. (3.68) then is given in terms of the scale factor by

$$U \approx U_0 a^{\lambda^2}\,, \tag{3.73}$$

and since this is a scaling solution (i.e. $\dot{\phi}^2 \propto U$), the Hubble parameter is found to be

$$H^2 \propto a^{\lambda^2}\,, \tag{3.74}$$

and it is then easy to show that the scale factor undergoes a power law evolution

$$a(\tau) = (-\tau)^p\,, \tag{3.75}$$

where $\tau$ is the confrmal time, and where

$$p = -\frac{1}{\bar{\epsilon}+1}\,, \tag{3.76}$$

and $\bar{\epsilon}$ is here defined as $\bar{\epsilon} = (U_{,\phi}/U)^2/2 \approx \lambda^2/2$. This is of course what we expect since we began by searching for such a solution using the ansatz Eq. (3.57). The time derivative of the field and the potential yields,

$$\phi' = \frac{\sqrt{2\bar{\epsilon}}}{\bar{\epsilon}+1}\frac{1}{\tau}\,, \tag{3.77}$$

$$V = 2\sigma - \frac{3+\bar{\epsilon}}{(1+\bar{\epsilon})^2}\frac{1}{(a\tau)^2}\,. \tag{3.78}$$

We are now ready to compute the spectrum of the scalar field perturbations produced by this power-law solution.

### 3.3.2 Power spectrum of the perturbed field

In this section we follow the same approach we took in the previous analysis of the scalar field perturbations. Similarly to our analysis presented in the previous section, in terms of conformal time, the perturbation equation for the scalar field, $\phi$, can be written as

$$\delta\phi'' = -2\frac{a'}{a}\delta\phi' + \left(\nabla^2 - a^2 V_{,\phi\phi}\right)\delta\phi\,, \tag{3.79}$$

which in turn can be written in terms of $u \equiv a\delta\phi$ as

$$u'' + \left(-\nabla^2 + m_{\rm eff}^2\right) u = 0\ \ , \tag{3.80}$$

and the effective mass of the field $u$ is

$$m_{\rm eff}^2 = \left(-\frac{a''}{a} + a^2 V_{,\phi\phi}\right)\ , \tag{3.81}$$

and by substituting Eqs. (3.75) - (3.78) into Eq. (3.81), we find

$$m_{\rm eff}^2\tau^2 = -2 + 3p + 3p^2. \tag{3.82}$$

The change of variable from $\delta\phi$ to $u$ is not essential, but helps intuitive understanding of the physics of the system by eliminating the first order term in (3.79) and reducing it to what resembles a harmonic osscilator. Eq. (3.80) can also be obtained by varying the perturbed part of the action in holonomy corrected LQC

$$\delta S = \frac{1}{2}\int d\tau d^3\mathbf{x}(u'^2 - \delta^{ij}\partial_i u\partial_j u - m_{eff}^2 u^2)\,. \tag{3.83}$$

Following the steps taken in the previous section to produce Eq.s (3.26)-(3.29), yields Fourier modes $u_k$, that satisfy

$$u_k'' + \left(k^2 + m_{\rm eff}^2\right)\, u_k = 0\,, \tag{3.84}$$

and the power spectrum is then defined by

$$\mathcal{P}_u \equiv \frac{k^3}{2\pi^2}|u_k|^2\,, \tag{3.85}$$

and it is clear that the Wronskian condition given by (3.32) remains unchanged.

The general solution to Eq. (3.84) is found to be given in terms of the Bessel funtions as

$$u_k(\tau) = c_1\sqrt{-\tau}\,J_{|\nu|}(x) + c_2\sqrt{-\tau}\,Y_{|\nu|}(x)\,, \tag{3.86}$$

which can be rewritten using the Hankel functions so that

$$u_k(\tau) = \frac{c_1 - ic_2}{2}\sqrt{-\tau}\,H_{|\nu|}^{(1)}(x) + \frac{c_1 + ic_2}{2}\sqrt{-\tau}\,H_{|\nu|}^{(2)}(x)\,, \tag{3.87}$$

whenever $m_{\rm eff}\tau$ is constant. The constant, $\nu$, and the variable, $x$, are given by

$$\nu = -\frac{\sqrt{1-4\,m_{\rm eff}^2\,\tau^2}}{2}\,, \qquad\qquad x = \frac{k}{aH}\,, \tag{3.88}$$

where $c_1$ and $c_2$ are constants. $H^{(1)}_{|\nu|}(x)$ and $H^{(2)}_{|\nu|}(x)$ are Hankel functions of the first and second kind, respectively. Using these functions we can once again conveniently decompose the solution in terms of two plane waves propagating in two opposite directions. We follow our previous line of argument, and that of the usual convention of standard inflationary scenario, and select only the advanced part by setting $c_1 = ic_2$. This sets the retarded wave equal to zero, which, as argued in the previous section, has the consequence of no particles being produced in the vacuum. Eq. (3.87) is then reduced to

$$u_k(\tau) = c_1\sqrt{-\tau}\,H^{(2)}_{|\nu|}(x)\,. \tag{3.89}$$

In order to find the constant $c_1$, following our previous method, we impose the Wronskian condition (3.32), and find

$$u_k(\tau) = \sqrt{\frac{\pi}{4}}\sqrt{-\tau}\,H^{(2)}_{|\nu|}(x)\,. \tag{3.90}$$

On large scales (i.e. $\frac{k}{aH} \to 0$), by using the appropriate limits of the Bessel functions, as before, the power spectrum can be approximated by

$$\mathcal{P}_u \propto k^{3-2|\nu|}\,(-\tau)^{1-2|\nu|}\,. \tag{3.91}$$

Consequently, the spectral tilt is defined, as before, by $\Delta n_u \equiv 3-2|\nu|$. By substituting (3.82) into (3.88), we see that we expect to have scale invariance of field perturbations for $p \to 0$. Once again, therefore, scale invariance occurs when the field $\phi$ is rolling down a steep potential and the kinetic energy is not negligible, but comparable to $V - 2\sigma$. Hence the evolution should again be understood as a fast-roll regime.

Clearly, scale invariance is also obtained for $p \to -1$ or $\bar{\epsilon} \ll 1$ which corresponds to the standard slow-roll regime that we are not concerned with for the purposes of this work.

In order to extend this analysis to general potentials, as we did for the previous system of modified equations we studied, we allow $\bar{\epsilon}$ to depend on conformal time such that

$$\bar{\epsilon}' = -(2\bar{\epsilon})^{3/2}\eta\,\phi'\,, \tag{3.92}$$

which defines the fast-roll parameter $\eta$,

$$\eta \equiv 1 - \frac{U_{,\phi\phi}U}{U_{,\phi}^2}\,. \tag{3.93}$$

Similarly, $\eta'$ can be written in terms of the next order fast-roll parameter $\xi^2$ as

$$\eta' = -\sqrt{2\bar{\epsilon}}\,\xi^2\,\phi'\,, \tag{3.94}$$

with

$$\xi^2 \equiv \left(1 + \frac{U_{,\phi\phi\phi}U}{U_{,\phi\phi}U_{,\phi}} - 2\frac{U_{,\phi\phi}U}{U_{,\phi}^2}\right)\frac{U_{,\phi\phi}U}{U_{,\phi}^2}\,. \tag{3.95}$$

For the exact scaling solution, it can be verified that both $\eta$ and $\xi^2$ vanish. As in the previous case, we can use Eqs. (3.75), (3.76) and (3.77) as approximate solutions, and hence we have

$$\frac{d\ln\bar{\epsilon}}{d\ln a} \approx 4\bar{\epsilon}\eta\,, \qquad \frac{d\ln\eta}{d\ln a} \approx 2\frac{\bar{\epsilon}\xi^2}{\eta}\,, \tag{3.96}$$

where we have again defined a further fast-roll parameter by $\epsilon = 1/2\bar{\epsilon}$. This means that for a large and slowly varying $\bar{\epsilon}$ the parameter $\eta$ must be small, and for a slowly varying $\eta$ the parameter $\xi^2$ must also be small. Hence, the $U$ part of the scalar potential must be close to exponential.

Using Eqs. (3.75) - (3.77) in the expression for the effective mass of $u$, Eq. (3.81), where $\bar{\epsilon}$ is now time dependent, and then using Eq. (3.88), we find that $\Delta n_u \equiv 3 - 2|\nu|$ is to first order

$$\Delta n_u = -4(\epsilon - \eta)\,. \tag{3.97}$$

Again the assumption that $m_{\rm eff}\tau$ is nearly constant as a given mode evolves outside the cosmological horizon is valid. Hence the spectral tilt for a given $k$ mode can be calculated by inserting the values the fast-roll parameters took as the mode crossed the horizon in Eq. (3.97). It is clear that the system we have investigated here results in a scale-invariant spectrum of scalar field perturbations for $\epsilon \ll 1$ and $\eta \ll 1$.

### 3.3.3 Stability of the fixed points

Following our previous stability analysis, using the constraint Eq. (3.67) to substitute for $y$ in (3.63) and (3.65), and linearising the system, we find the $\mathcal{M}$-matrix

$$\mathcal{M} = \begin{pmatrix} -3-\sqrt{6}x\lambda - 9x^2 & -\sqrt{\frac{3}{2}}(x^2+1) \\ -\sqrt{6}(\Gamma-1)\lambda^2 + 6x\mu & -2\sqrt{6}x(\Gamma-1)\lambda + 3x^2\mu \end{pmatrix}, \tag{3.98}$$

where $\mu = \left(\frac{2\sigma}{\rho} - 1\right)\sqrt{\frac{2\sigma}{\rho}}$. The eigenvalues, $\omega$, are then given: for the unphysical kinetic energy dominated solution, Eq. (3.69),

$$\omega = 6 \pm i\sqrt{6}\lambda\,, \tag{3.99}$$

and, hence, this solution is unstable. While for the scaling solution, Eq. (3.70)

$$\omega = -\frac{1}{2}(6+\lambda^2)\,, \tag{3.100}$$

and, hence, this point is a *stable* attractor for *all* values of $\lambda$. A numerical analysis of this system shown in Fig. 3.2 supports our analytical results presented here. The figure shows the evolution of the ratio $\dot{\phi}^2/(2\sigma - V)$ obtained by numerically solving the equations of motion for three different initial conditions. They approach the scaling solution given by $2x^2/y^2$ where $x$ and $y$ are given by Eqs. (3.70). When $a \gg a_\star$ the quadratic corrections become negligable and the numerical evolution diverges from this attractor.

In our calculation we have neglected metric perturbations, and instead, we have only considered scalar perturbations and been able to find scale-invariance through the scalar field perturbations.

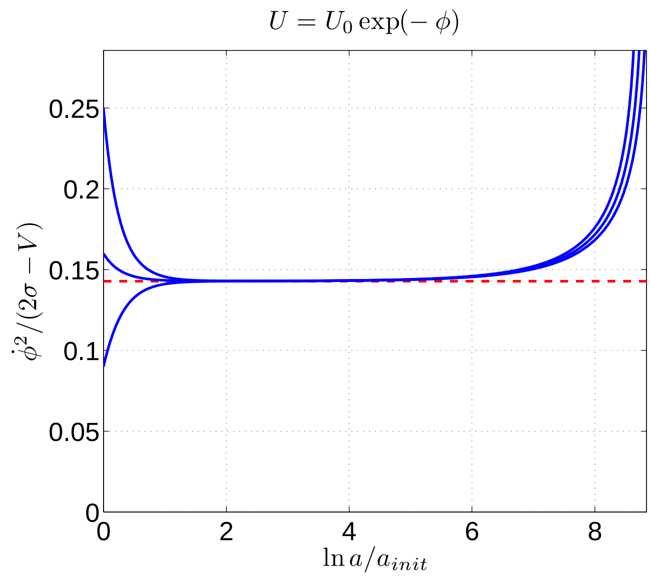


Figure 3.2: The evolution of the ratio $\dot{\phi}^2/(2\sigma - V)$ obtained by numerically solving the equations of motion for three different initial conditions (solid lines). They approach the scaling solution given by $2x^2/y^2$ where $x$ and $y$ are given by Eqs. (3.70) (dashed line). We used as parameters $U_0 = 10^{-2}$, $\phi_{\rm init} = 1$ and $\sigma = 0.41$ in Planck units, in a flat universe.

## 3.4 Number of $e$-folds

Before concluding, it is important to address the question of whether a sufficient amount of super-inflation can occur in LQC to account for the largest scale perturbations observed on the CMB. This in turn is equivalent to asking whether the super-inflationary phases can solve the cosmological horizon problem.

It is clear that only a small number of $e$-folds of super-inflation can be considered generic for either of the modified sets of evolution equations we have studied [62]. This might be considered disappointing, since experience from standard inflation suggests that approximately 60 $e$-folds of inflation are required for consistency with observations. However, the inflationary periods we have been studying are considerably different to standard inflation and this has a dramatic effect.

Solving the horizon problem is essentially the requirement that $aH$ grows sufficiently during an early stage of the universe's evolution. While in standard inflation this is accomplished by $a$ changing rapidly as $H$ remains nearly constant, in our case the converse appears to be possible, that $H$ increases sufficiently as $a$ remains nearly constant. The number of $e$-folds usually only refers to the change in $a$, and so we only expect a small number of e-folds to be necessary in a super-inflationary phase, provided that $H$ changes sufficiently. To confirm the expectation that our super-inflationary phases can indeed solve the horizon problem, let us now quantify our qualitative arguments.

During super-inflation, perturbation modes exit the cosmological horizon, and once super-inflation ends, modes start to re-enter. Let us consider a perturbation mode with wavenumber $k$, such that $k$ exited the cosmological horizon $N(k)$ $e$-folds before the end of the super-inflationary phase, and re-entered sometime later. The mode re-entering the horizon today, $k_*$, must satisfy $k_* = a_0 H_0$, where subscript 0 indicates quantities at the present epoch. Comparing this with the generic $k$ mode we have:

$$\frac{k}{a_0 H_0} = \frac{a_k H_k}{a_0 H_0} = \frac{a_k H_k}{a_{\rm end} H_{\rm end}} \frac{a_{\rm end}}{a_{\rm reh}} \frac{a_{\rm reh}}{a_{\rm eq}} \frac{a_{\rm eq}}{a_0} \frac{H_{\rm end}}{H_0} \ , \tag{3.101}$$

where subscript 'end' labels quantities at the end of inflation, 'reh' at reheating, and 'eq' at matter radiation equality. For simplicity, assuming that the universe behaves as if it is matter dominated between the end of inflation and reheating, we have

$$\frac{k}{a_0 H_0} = \frac{a_k H_k}{a_{\rm end} H_{\rm end}} \left(\frac{\rho_{\rm end}}{\rho_{\rm reh}}\right)^{-1/3} \left(\frac{\rho_{\rm reh}}{\rho_{\rm eq}}\right)^{-1/4} \left(\frac{\rho_{\rm eq}}{\rho_0}\right)^{-1/3} \frac{H_{\rm end}}{H_0} \ , \tag{3.102}$$

which can also be written as

$$\frac{k}{a_0 H_0} = \frac{a_k H_k}{a_{\rm end} H_{\rm end}} \left(\frac{\rho_{\rm end}}{\rho_{\rm reh}}\right)^{-1/12} \left(\frac{\rho_{\rm eq}}{\rho_0}\right)^{-1/12} \left(\frac{\rho_{\rm end}}{\rho_0}\right)^{-1/4} \frac{H_{\rm end}}{H_0} \ , \tag{3.103}$$

which upon using $H^2 = \frac{1}{3}\rho$, becomes

$$\frac{k}{a_0 H_0} = \frac{a_k H_k}{a_{\rm end} H_{\rm end}} \left(\frac{\rho_{\rm end}}{\rho_{\rm reh}}\right)^{-1/12} \left(\frac{a_{\rm eq}}{a_0}\right)^{-1/4} \left(\frac{H_{\rm end}}{M_{\rm Pl}}\right)^{1/2} \left(\frac{M_{\rm Pl}}{H_0}\right)^{1/2} . \quad (3.104)$$

Then employing the known evolution of the universe from equality to the present day (i.e $\frac{a_0}{a_{\rm eq}} = 1 + z_{\rm eq} \approx 1000$), together with the measured value of the Hubble rate at the present epoch, $H_0$, we find:

$$\ln\left[\frac{k}{a_0 H_0}\right] \approx 68 + \ln\left[\frac{a_k H_k}{a_{\rm end} H_{\rm end}}\right] - \frac{1}{2}\ln\left[\frac{M_{\rm Pl}}{H_{\rm end}}\right] - \frac{1}{3}\ln\left[\frac{\rho_{\rm end}}{\rho_{\rm reh}}\right]^{1/4} . \quad (3.105)$$

The energy scale at the end of inflation must be determined by requiring that the magnitude of the curvature perturbation accounts for the temperature anisotropies in the CMB. Since we have only worked with the scalar field perturbation this is so far undetermined in our model and we therefore take $H_{\rm end}$ to be the highest possible scale, i.e. $H_{\rm end} = M_{\rm Pl}$. A lower scale would lead to fewer required $e$-folds. Further considering $k = k_*$ and assuming instant reheating, we find that

$$\ln\left(\frac{a_{\rm end} H_{\rm end}}{a_{k_*} H_{k_*}}\right) \approx 68 \, . \quad (3.106)$$

In order to determine the number of $e$-folds of super-inflation required, we now consider the cases in which the scale factor is undergoing pure power-law behaviour, $a \propto (-\tau)^p$. Using this together with Eq. (3.106), we find

$$\ln\left(\frac{\tau_{\rm end}}{\tau_{k_*}}\right)^{-1} \approx 68 \, , \quad (3.107)$$

and in turn

$$N(k_*) = \ln\left(\frac{a_{\rm end}}{a_{k_*}}\right) = -68\, p \, . \quad (3.108)$$

Recalling that $p$ must be small and negative for scale invariance, we see that only a small number of $e$-folds are required. Although considering behaviour which deviates from pure power law behaviour will alter Eq. (3.108), it is clear that the conclusion of only a small number of e-folds being necessary will remain valid. We will proceed in the next chapter to investigate the role of tensor perturbations in LQC.

# Chapter 4

# The gravitational wave background from super-inflation in LQC

## 4.1 Introduction

A key result of the inflationary scenario of the very early universe [22, 25, 26] is that it gives rise to a stochastic background of gravitational wave radiation (tensor perturbations) [122]. Such a background is in principle observable and could be used to distinguish between different models of inflation [123]. Alternative proposals such as the ekpyrotic/cyclic scenario [80, 81, 82, 83, 87, 84, 88, 85] or phantom super-inflation [113, 124] lead to very different predictions for the spectral tilt of tensor perturbations when compared with the simplest inflationary models [88], implying the gravitational wave background could also be a powerful discriminator between competing theories. In this chapter we will be concerned with the gravitational wave background produced by super-inflationary scenarios within Loop Quantum Cosmology (LQC).

Following the analysis we carried out in the previous chapter, one point to note here is the assumption of a purely isotropic universe. In the regime of very small scale factor, the discrete structure of space is so strong that any inhomogeneous configuration would be far from being isotropic. Therefore, despite being able to write the effective perturbation equations for a purely isotropic scenario in this regime, one should keep in mind that in reality care needs to be taken regarding the cosmological interpretation of perturbations and their evolution [48, 125].

In the previous chapter we did not consider the spectrum of tensor perturbations produced in the cosmologies we discussed. This is the question to which we now turn. This question is particularly timely since the effect of the two kinds of modifications on the evolution of tensor perturbations in LQC has only recently been derived [69].

Similar calculations have already been carried out and reported in the literature for different types of cosmologies. In particular, gravitational wave perturbations for ekpyrotic models of a collapsing universe [88], and for phantom superinflaion scenarios [124] have been computed. Despite being based on different physical concepts and behaviours, both cosmologies predict a strongly blue tilted spectrum of tensor perturbations, and since these have not yet been observed, it suggests that in both scenarios, the amplitude of these fluctuations are suppressed on large scales by many orders of magnitude compared to those predicted by standard inflation. This is not unexpected, as it has been pointed out in Ref. [112] that a duality exists between the ekpyrotic collapse and the dynamics of a universe sourced by a phantom field. Moreover, a scale factor duality maps the ekpyrotic collapse onto the superinflationary scaling solutions in LQC [63].

Given the connections found between LQC and the ekpyrotic and phantom scenarios, one might expect that LQC also predicts a large blue tilted spectrum for the tensor perturbations. However, as the connections are at the level of the background equations and since LQC corrections also arise in the evolution equations of tensor perturbations themselves, this expectation may not be realised and a careful analysis is necessary to confirm it.

The structure of this chapter is as follows. In Section 4.2, we discuss the inverse triad modifications. First we review the background dynamics which give rise to a scale invariant spectrum of scalar field perturbations, and then we consider the evolution of tensor perturbations in this setting calculating their spectrum. We then repeat the exercise for holonomy corrections in section 4.3. Finally we conclude in section 4.4.

## 4.2 Tensor dynamics with inverse triad corrections

We first consider the cosmological equations which follow from including modifications associated with the inverse triad in the semi-classical regime of LQC. Whilst recognising the careful attention this regime deserves, and acknowledging more work is required to clarify the validity of the dynamical equations we use together with the assumption of isotropy, we hope to pave the way by demonstrating a method of calculation which can easily be employed once new light is shed on currently uncertain sections of the theory.

The unperturbed isotropic modified Friedmann equation is given by Eq. (2.15), which in terms of the conformal time becomes

$$\mathcal{H}^2 \equiv \left(\frac{a'}{a}\right)^2 = \frac{\kappa^2}{3} S\left(\frac{\phi'^2}{2D} + a^2 V(\phi)\right) \quad , \tag{4.1}$$

where a dash represents differentiation with respect to conformal time, $\tau$, and we have assumed that any curvature contribution has become subdominant. The scalar field equation of motion is similarly given by Eq. (2.12), which in terms of $\tau$ becomes

$$\phi'' + 2\mathcal{H}\left(1 - \frac{1}{2}\frac{d\ln D}{d\ln a}\right)\phi' + a^2 D V_{,\phi} = 0\,. \tag{4.2}$$

In this chapter our interest is in the evolution of tensor perturbations about this isotropic background. Tensor perturbations are defined as the transverse and trace free part of the perturbed spatial metric, and represent gravitational wave perturbations propagating on the unperturbed background spacetime. They can be further decomposed into two polarisation modes represented by $\times$ and $+$, and in LQC, with inverse triad modifications included, the equation of motion for these modes has recently been derived to be [69]

$$h''_{\times,+} + 2\mathcal{H}\left[1 - \frac{1}{2}\frac{d\ln S}{d\ln a}\right] h'_{\times,+} - S^2\nabla^2 h_{\times,+} = 0 \ . \tag{4.3}$$

$h_{\times,+}$ is a tensorial quantity, but from here on to avoid clutter we will drop the $\times$ and $+$ subscripts, and take $h$ to represent the magnitude of one of the polarisation modes, but always keep in mind that both modes are present.

### 4.2.1 The background power law solution and scale invariant scalar field dynamics

As we demonstrated previously, in section 3.2, in the regime $a \ll a_\star$, there exists a power law solution to the equations of motion (4.1)–(4.2) which is a stable attractor to the dynamics. This solution exists for negative power law potentials of the form $V = V_0\,\phi^\beta$ (with $V_0 < 0$) and gives rise to the dynamics

$$a(\tau) = A(-\tau)^p\,, \tag{4.4}$$

$$\phi'(\tau) = -\sqrt{p\,\frac{2+(r+2)p}{\alpha}}\,\sqrt{\frac{D}{S}}\,\frac{1}{\tau}\,, \tag{4.5}$$

$$V(\tau) = \left(3 - \frac{2+(r+2)p}{2\alpha}\right)\frac{1}{S(a\tau)^2}\,, \tag{4.6}$$

where for an expanding universe $\tau$ is negative and increasing from $-\infty$ towards zero. $A$ is an arbitrary normalisation constant, and $\alpha = 1 - n/6$ with $\beta$ and $p$ being related through

$$\beta = -\frac{2}{p}\,\frac{2+(r+2)p}{n-r}\,. \tag{4.7}$$

Notice the equivalence between these equations and (3.20) - (3.21) derived in the previous chapter, where we expanded about this solution in terms of fast-roll parameters, in order to generalise the potentials which can be considered. Here, for simplicity we will consider only this exact solution.

For $-1 < p < 0$ the solution given above represents a universe undergoing super-inflationary evolution during which the Hubble rate increases. A particularly interesting case then occurs as $p$ tends to zero from below. This represents a universe in which the scale factor is almost constant, but $H$ increases rapidly. Considering scalar field perturbations about the background field, we showed in section 3.2 that the spectrum of scalar field perturbations attains scale-invariance in this limit. Moreover, because $H$ increases so rapidly, the horizon problem is solved during this phase with only a small number of $e$-folds required. This raises the intriguing possibility that these perturbations could be responsible for the observed CMB anisotropies and hence for structure in the universe. If this were the case, no period of standard inflation in which $H$ remains nearly constant for roughly 60 or more $e$-folds of expansion would be required (where number of $e$-folds is defined as $N = \ln(a/a_{\rm i})$). A natural, and indeed important question is: 'what is the spectrum of primordial tensor perturbations which accompanies this scale invariant spectrum of scalar field perturbations?'. This is the question to which we now turn.

### 4.2.2 The primordial spectrum of tensor perturbations

To calculate the spectrum of gravitational waves produced during super-inflation we follow the standard procedure. Given that $-a^2h/2$ is canonically conjugate to $h'/(2S)$ such that [69]

$$\left\{\frac{1}{2S}h'(t,x), -\frac{a^2}{2}\,h(t,y)\right\} = \delta^3(x,y)\,, \tag{4.8}$$

the system is quantised by promoting $h$ and $h'$ to operators and the Poisson brackets to commutators. We have

$$\left[\hat{h}', \hat{h}\right] = -4i\frac{S}{a^2}\delta^3(x,y)\,. \tag{4.9}$$

$\hat{h}$ is then decomposed into Fourier modes

$$\hat{h} = \int \frac{d^3k}{(2\pi)^{3/2}}\left[h_k(\tau)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + h_k^*(\tau)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right]\,, \tag{4.10}$$

where, considering Eq. (4.3), we see that each mode $h_k$ obeys the evolution equation

$$h_k'' + 2\mathcal{H}\left(1 - \frac{1}{2}\frac{d\ln S}{d\ln a}\right)h_k' + S^2k^2h_k = 0 \ , \tag{4.11}$$

in Fourier space. The power spectrum for one polarisation state of tensorial fluctuations is given by the standard expression

$$\mathcal{P}_h = \frac{k^3}{2\pi^2}\,|h_k|^2 \quad . \tag{4.12}$$

In order to evaluate Eq. (4.12) we must solve Eq. (4.11). Considering the power law solution for the regime $a \ll a_\star$ (given by Eq. (4.4)), and using the form of $S$ in this regime ($S = S_* a^r$), we find that Eq. (4.11) becomes

$$h_k'' + 2\,\frac{p}{\tau}\left(1 - \frac{r}{2}\right)h_k' + k^2S_\star^2A^{2r}(-\tau)^{2rp}h_k = 0\,, \tag{4.13}$$

with the general solution

$$h_k(\tau) = c_1\frac{S^{1/2}}{\mathcal{H}^{1/2}a}J_\nu(x) + c_2\frac{S^{1/2}}{\mathcal{H}^{1/2}a}Y_\nu(x)\,, \tag{4.14}$$

where $J_\nu(x)$ and $Y_\nu(x)$ are Bessel functions of the first and the second kind, respectively. $c_1$ and $c_2$ are constants, and the constant $\nu$ and the variable $x$ are given by

$$\nu = \frac{1+p(r-2)}{2(1+pr)}\,, \qquad\qquad x = \frac{-pSk}{(1+pr)\mathcal{H}}\,. \tag{4.15}$$

Eq. (4.14) can also be rewritten in terms of the Hankel functions, so that

$$h_k(\tau) = \frac{c_1 - ic_2}{2}\frac{S^{1/2}}{\mathcal{H}^{1/2}a}H_\nu^{(1)}(x) + \frac{c_1 + ic_2}{2}\frac{S^{1/2}}{\mathcal{H}^{1/2}a}H_\nu^{(2)}(x)\,, \tag{4.16}$$

where $H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$ and $H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$ are the Hankel functions (also known as the Bessel functions of the third kind). As mentioned in the previous chapter, this decomposition of our solution allows it to be written in terms of two plane waves moving in opposite directions. Once again, we adopt the general convention of standard inflationary scenario and select the forward moving wave. Neglecting the retarded wave, by setting $c_1 = c_2$, reduces Eq. (4.16) to

$$h_k(\tau) = c_1 \frac{S^{1/2}}{\mathcal{H}^{1/2} a} H_\nu^{(2)}(x)\,. \tag{4.17}$$

Before going any further from this point, it is worth pausing to give a brief outline of the analysis which should be followed to find the constant, $c_1$. We start from the perturbed densitised triad and the extrinsic curvature perturbations in the classical regime, given by [69]

$$\delta E = -\frac{1}{2}\bar{p}h\,, \tag{4.18}$$

$$\delta K = \frac{1}{2}[h' + \mathcal{H}h]\,, \tag{4.19}$$

where $\bar{p} = a^2$ (not to be confused with $p$ in Eq. (4.4)), and where, for simplicity, we have dropped all indices. Using the knowledge that these quantities are canonically conjugate to one another

$$\{\delta K_x\,,\,\delta E_y\} = \delta^{(3)}(x-y)\,, \tag{4.20}$$

one obtains

$$\{\frac{1}{2}[h' + \mathcal{H}h]_x\,,\,-\frac{1}{2}a^2 h_y\} = \{\frac{1}{2a}u'_x\,,\,-\frac{1}{2}a u_y\}\,, \tag{4.21}$$

where $u \equiv ah$. Once these parameters have been promoted to operators and the Poisson brackets to commutation relations, we get

$$[-\frac{1}{4}\hat{u}'_x\,,\,\hat{u}_y] = i\delta^{(3)}(x-y)\,, \tag{4.22}$$

and it is then clear that $-\frac{1}{4}\hat{u}'$ is the conjugate momentum of $\hat{u}$.

In Fourier space, by expanding $\hat{u}(\tau,\mathbf{x})$

$$\hat{u}(\tau,\mathbf{x}) = \int \frac{\mathbf{d^3k}}{(\mathbf{2\pi})^{\mathbf{3/2}}}[\mathbf{w_k}(\tau)\mathbf{\hat{a}_k e^{ik.x}} + \mathbf{w_k^*}(\tau)\mathbf{\hat{a}_k^\dagger e^{-ikx}}]\,, \tag{4.23}$$

and calculating the commutation relation of (4.22), upon using the usual commutation relations for the creation and the annihilation operators, $\hat{a}^\dagger$ and $\hat{a}$, we find

$$[\hat{u}'_x\,,\,\hat{u}_y] = (u^*_k u_{k,\tau} - u_k u^*_{k,\tau})\delta^{(3)}(x-y)\,. \tag{4.24}$$

Comparing this result to Eq. (4.22), we arrive at the Wronskian condition

$$u^*_k u_{k,\tau} - u_k u^*_{k,\tau} = -4i\,. \tag{4.25}$$

Notice that although this is the usual form in which this condition is generally written, in our analysis, there is no need (other than for convenience) to introduce the parameter $u$ in the formulism. However, it is not always possible to introduce an analogous parameter which simplifies the formulism, so one may need to go back a step and expand the tensor perturbations operator, $\hat{h}$, directly (as in Eq. (4.10)). The effect of this on the Wronskian condition is easily seen to be

$$h^*_k h_{k,\tau} - h_k h^*_{k,\tau} = -\frac{4}{a^2}i\,, \tag{4.26}$$

and by following the method above, it can be shown that the conjugate momentum of $\hat{h}$ is given by $-\frac{a^2}{4}\hat{h}'$.

In the case of the inverse triad corrections, Eq. (4.19) is found to be modified such that [69]

$$\delta K = \frac{1}{2}[\frac{h'}{S} + \mathcal{H}h]\,. \tag{4.27}$$

Once again, by employing Eq.s (4.18) and (4.20), the conjugate momentum of $\hat{h}$ is found to be $-\frac{a^2}{4S}\hat{h}'$, and the Wronskian condition becomes

$$h^*_k h_{k,\tau} - h_k h^*_{k,\tau} = -\frac{4S}{a^2}i\,. \tag{4.28}$$

This condition is equivalent to requiring no particle production in the vacuum for it to be well defined, and so must be satisfied at very early times when the quantum fluctuations are most likely to create particles inside the vacuum. This corresponds to the limit of very high frequencies (low wavelengths), i.e.

$Sk/\mathcal{H} \to \infty$. In this regime, Eq. (4.17) can be approximated by

$$h_k(\tau) \approx c_1 \frac{S^{1/2}}{\mathcal{H}^{1/2} a} \sqrt{\frac{-2(1+pr)\mathcal{H}}{p\pi Sk}}\; e^{-i\frac{pSk}{(1+pr)\mathcal{H}}} \,. \tag{4.29}$$

By substituting this result back into Eq. (4.28), and finding the constant $c_1$, the exact solution for the tensor perturbations is found to be

$$h_k(\tau) = \frac{S^{1/2}}{\mathcal{H}^{1/2} a} \sqrt{\frac{-p\,\pi}{1+rp}}\; H_\nu^{(2)}(x) \,. \tag{4.30}$$

This solution has the expected behaviour that each $k$ mode begins in an oscillatory state where normalisation occurs, and evolves into a non-oscillatory state once each given $k$ mode crosses a suitably defined horizon. From the form of (4.30) and (4.15), it is clear that for tensor modes horizon crossing occurs when $k \approx -aH/pS$. A given mode can only be considered to become a classical perturbation once it crosses this horizon.

Taking the solution (4.30) and employing (4.12) we find that

$$\mathcal{P}_h = \frac{2^{2(\nu-1)}}{\pi^2} \frac{\Gamma(\nu)^2}{\Gamma(3/2)^2} \left(\frac{k_0}{k_{\rm e}}\right)^{3-2\nu} \left(\frac{k}{k_0}\right)^{3-2\nu} H_{\rm e}^2 \,, \tag{4.31}$$

where we have evaluated (4.30) in the limit where the modes are outside the defined horizon (i.e. $x \to 0$), by using the appropriate limits of the Bessel functions given by Eq. (3.40), and it can be seen that we have also accounted for both polarisation states. The mode $k_0$ corresponds to the largest scales on the CMB and $k_{\rm e}$ is defined as the last mode to cross the horizon at the end of super-inflation ($k_{\rm e} = \mathcal{H}_{\rm e}/S_{\rm e} \approx \mathcal{H}_{\rm e}$).

A couple of observations are in order. The first is that in the limit of interest, $p \to 0$, the spectrum is blue tilted with a tensor spectral index given by

$$n_t = 2 + \frac{2p}{1+rp} \approx 2 \,, \tag{4.32}$$

where $n_t$ is defined, as usual, by $P_h \propto k^{n_t}$. This implies that on large scales the spectrum is hugely suppressed. The second important point is that the magnitude of the spectrum is fixed by the Hubble rate at the end of inflation. This also fixes the magnitude of scalar perturbations and ultimately has to be normalised such that the scalar perturbations have the correct magnitude to account for the CMB anisotropies. In the scenario at hand such a normalisation is difficult to determine since the scalar field perturbations (which are close to scale invariant) must be related to curvature perturbations (as discussed in the previous chapter and in [18]), and this step cannot be performed at present. Nevertheless in the following section we will make the reasonable assumption that $H_{\rm e}$ corresponds to the GUT scale ($10^{14} GeV$), in order to calculate the present-day spectrum of gravitational waves produced by this super-inflationary scenario. By assuming a radiation dominated universe between now and the end of inflation, today's temperature of $T_0 \approx 3K$, Boltzmann's constant of $k_B \approx 10^{-13} GeV\, K^{-1}$, and assuming the temperature drops with the scale factor such that $\frac{T_0}{T_e} \propto \frac{a_e}{a_0}$, the GUT level of energy corresponds to $H_{\rm e} \approx 10^{-6}$. Furthermore, we will see that the conclusion - that the spectrum is unobservably small - is insensitive to the choice of normalisation within reasonable bounds.

### 4.2.3 The present-day spectrum

At the end of super-inflation all classical perturbation modes are outside the horizon. We will assume that reheating occurs instantaneously at the energy scale $H_{\rm e}$, and hence that the universe becomes radiation dominated at this point. Moreover we will assume the universe is classical after reheating, and any quantum corrections (similar to $D$ or $S$) are absent from the dynamics. From this point onwards modes will begin to re-enter the cosmological horizon. We schematically illustrate this dynamics in Fig. 4.1.

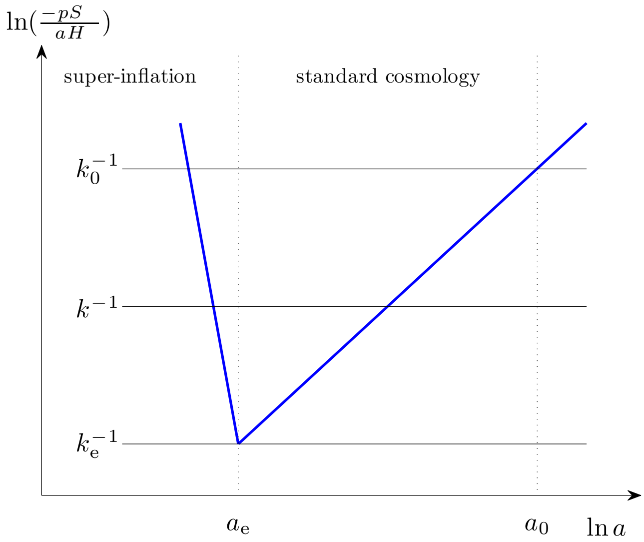


Figure 4.1: Schematic illustration of the evolution of modes $k$ which exit the horizon (thick solid line) during super-inflation and re-enter during the standard radiation or matter era. $k_0^{-1}$ is the scale corresponding to the size of the observable universe.

To convert the primordial spectrum, Eq. (4.31), to the spectrum which would be observed today we employ the numerically obtained transfer function [128, 129]

$$T(k) = \left(\frac{k_0}{k}\right)^2 \left[1 + \frac{4}{3}\frac{k}{k_{\rm eq}} + \frac{5}{2}\left(\frac{k}{k_{\rm eq}}\right)^2\right]^{1/2}, \tag{4.33}$$

where $k_0 = H_0 a_0$, is again the $k$ mode corresponding to the largest scales today, and $k_{\rm eq} = a_{\rm eq} H_{\rm eq}$, where eq stands for quantities at radiation matter equality.

Using this transfer function we can calculate the present day spectrum of tensor perturbations once we fix $H_{\rm e}$ and $k_{\rm e}/k_0$. A sensible estimate for $H_{\rm e}$ is the GUT scale, while an absolute upper limit is given by the Planck scale. If we also make the assumption that as many modes exit the tensor horizon as scalar modes exit the scalar horizon, $k_0/k_{\rm e}$ can be fixed by the requirement that the horizon problem be solved. In the previous chapter, as well as in [18], we found that this required $k_0/k_{\rm e} \approx e^{-60}$.

A useful physical quantity that can be used to express the present day spectrum of gravitational waves is $\Omega_{\rm gw}$, the gravitational wave energy per unit logarithmic wave number in units of the critical density, $\rho_{\rm crit} = 3H_0^2$, [88, 128]:

$$\Omega_{\rm gw}(k) \equiv \frac{k}{\rho_{\rm crit}} \frac{d\rho_{\rm gw}}{dk} = \frac{1}{6} \left(\frac{k}{k_0}\right)^2 T^2(k)\,\mathcal{P}_h\,, \tag{4.34}$$

where the combination $T^2(k)\,\mathcal{P}_h$ represents the present-day gravitational wave power spectrum; and we note that for its fixed values, the energy density varies with $k^2$. In general, if the k-dependence of the transfer function (4.33) is given by $k^\gamma$, the scale dependence of $\Omega_{\rm gw}$ will be $k^{2+2\gamma+n_t}$.

Figure 4.2 shows a plot of the present tensor abundance for a number of choices of the parameter $p$, together with the strongest observational constraints from current and future experiments searching for gravitational waves [88, 130]. Note that the gradient of the curves on large scales (i.e. $\gamma \to -2$), is $-2+n_t$; and on small scales (i.e. $\gamma \to -1$), it will be $n_t$. This is clearly seen in Fig. 4.2 for the case of the standard inflation, where $p \to -1$ from below, where the curve appears almost horizontal for all frequencies $f > f_{\rm eq}$, which corresponds to the almost scale-invariance of gravitational modes in this case. From the value of $(a_{\rm eq}H_{\rm eq})^{-1} = 14h^{-2}Mpc$, where $h \approx 0.7$, one obtains the corresponding frequency of $f_{\rm eq} = 10^{-15}$ Hz. $f_0$ for today is similarly found to be $f_0 = 10^{-18}$Hz. Since we have fixed $H_e = 10^{-6}$, and we also find for a radiation dominated universe $\frac{a_{\rm eq}H_{\rm eq}}{a_e H_e} \propto \frac{a_e}{a_{\rm eq}} \propto 10^{-23}$ (where we have used $1+z_{\rm eq} = \frac{a_0}{a_{\rm eq}} \approx 10^4$), we obtain $a_e H_e = 10^8$, corresponding to the frequencies in the region of $f_e = 10^8$ Hz.

It can be seen from this figure that by relaxing the assumtion of $H_e = 10^{-6}$, corresponding to the GUT scale, one would effectively shift the intersection point of the three lines to the left. Although this has the effect of bringing the $\Omega_{\rm gw}$ of today (for $p > -1$) up on the vertical axis, the amplitude of tensor fluctuations remains suppressed by quite a few orders of magnitude on the CMB, compared to the standard inflationary case. This is a direct result of the large blue tilt of the spectrum.

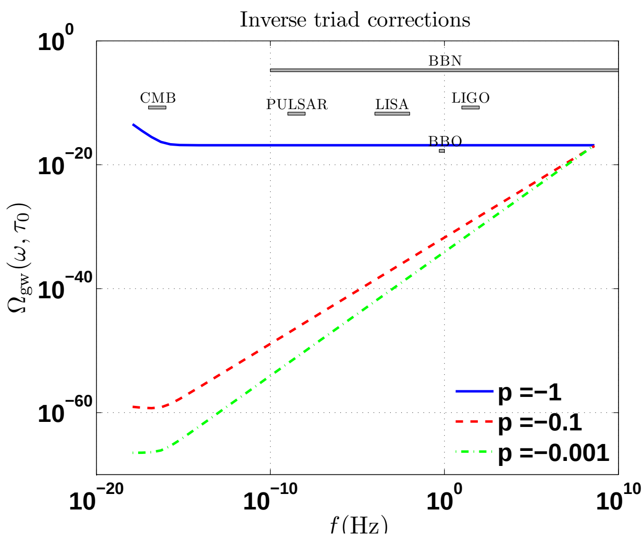


Figure 4.2: Predicted present abundance of gravitational waves produced during the super-inflationary phase of LQC with inverse volume corrections, assuming that $H_{\rm e} = 10^{-6}$ in Planck units. The solid line corresponds to the standard inflation abundance ($n = r = 0$). Also indicated are the present bounds and future sensitivities.

## 4.3 Tensor dynamics with holonomy corrections

We now turn our attention to the second set of effective equations, those which arise from considering that holonomies are the basic operators to be quantised in LQC.

The isotropic unperturbed dynamics is described, by the Friedmann equation [44, 51, 71, 50, 34, 53, 54]:

$$\mathcal{H}^2 = \frac{1}{3}a^2\rho\left(1 - \frac{\rho}{2\sigma}\right) , \tag{4.35}$$

which is modified from the classical equation by the inclusion of a $-\rho^2$ term, where $\sigma = 3/(2\gamma^2)$ is a constant in an exactly isotropic model, and where $\gamma$ is the Barbero-Immirzi parameter [36, 37]. This is identical to Eq. (2.16) first introduced in chapter 2, but here is written in terms of conformal time, $\tau$. We are assuming either a flat universe or that the curvature term can be safely neglected. Our interest here is in a scalar field dominated universe, and hence

$\rho = \phi'^2/2a^2 + V(\phi)$. The scalar field equation of motion remains unchanged from its classical form given by Eq. (3.56), which in terms of conformal time can be written as

$$\phi'' + 2\mathcal{H}\phi' + a^2 V_{,\phi} = 0\,. \tag{4.36}$$

We stress that as we are studying inverse volume and quadratic corrections separately, we do not include the $D$ and $S$ functions in the equations of motion.

The form of the evolution equation for tensor perturbations when holonomy corrections are included has recently been derived to be [69]

$$h''_{\times,+} + 2\mathcal{H}h'_{\times,+} - \nabla^2 h_{\times,+} + T_Q h_{\times,+} = 2\Pi_Q \ , \tag{4.37}$$

where

$$T_Q = \frac{a^2}{3}\frac{\rho^2}{2\sigma}\,, \tag{4.38}$$

$$\Pi_Q = \frac{1}{2}\frac{\rho}{2\sigma}\left(a^2\rho - \phi'^2\right) h_{\times,+}\,, \tag{4.39}$$

are quantum corrections to the classical dynamics that become unimportant when $\rho \ll 2\sigma$. From here on we again drop the $\times$ and $+$ subscripts as we did in the inverse triad case.

### 4.3.1 Power-law solution and scale invariant scalar field perturbations

We are interested in high density regimes where $\rho$ approaches the bounding value of $2\sigma$. In this case, the term within brackets of Eq. (4.35) tends to zero and the behaviour of the equations alters significantly compared with the classical behaviour. Indeed in this regime we have $\dot{H} > 0$ (seen from Eq. (2.17)) and for an expanding universe super-inflation takes place. In section 3.3 (and also in [18]), we showed that in this regime there exists an approximate power-law solution for a potential of the form $V = 2\sigma - U_0 e^{-\lambda\phi}$ (i.e. as given by Eq. (3.61)). Moreover, this solution is an attractor as we demonstrated both analytically and numerically in section 3.3 (also in [18]). The full background cosmology solution is given by

$$a(\tau) = A(-\tau)^p \,, \tag{4.40}$$

$$\phi'(\tau) = \sqrt{-2p(p+1)}\,\frac{1}{\tau} \,, \tag{4.41}$$

$$V(\tau) = 2\sigma - \frac{p(2p-1)}{(a\tau)^2} \tag{4.42}$$

where $A$ is an arbitrary normalisation constant. $\lambda$ and $p$ are related through

$$\lambda^2 = -2\,\frac{p+1}{p} \,. \tag{4.43}$$

Once again $-1 < p < 0$ corresponds to a universe undergoing super-inflation (i.e. $\dot{H} > 0$), and moreover the limit $p \to 0$ from below leads to a scale-invariant spectrum of scalar field perturbations, as seen from Eq. (3.91) discussed in section 3.3. Furthermore, as was the case for the super-inflationary solution we studied in the presence of inverse volume corrections, only a small number of $e$-folds are required to solve the horizon problem. In the previous chapter we generalised the form of the potential that could be considered by expanding about this solution, but for simplicity in this work we will consider only this exact form.

We now turn our attention to the question of what spectrum of tensor perturbations accompanies the scale-invariant spectrum of scalar field perturbation in this version of super-inflation.

### 4.3.2 The primordial spectrum of tensor perturbations

We again follow the standard procedure for calculating the spectrum of tensor perturbations. In our perfectly isotropic scenario, Eq.s (4.18) and (4.19) remain unaltered. Imposing Eq. (4.20) then implies that $h'$ is canonically conjugate to $2a^2h$ such that

$$\left\{\frac{1}{2}h'(t,x), -\frac{a^2}{2}\,h(t,y)\right\} = \delta^3(x,y) \,. \tag{4.44}$$

To quantise the system $h$ and $h'$ are promoted to operators and Fourier decomposed according to Eq. (4.10). Each $h_k$ mode satisfies the evolution equation given by substituting Eq.s (4.38)-(4.39) into Eq. (4.37) during the sacling solution:

$$h_k'' + 2\frac{a'}{a}\,h_k' + \left[k^2 - \frac{4\sigma}{3}(-\tau)^{2p} - \frac{2p(p+1)}{\tau^2}\right] h_k = 0\,, \tag{4.45}$$

where we have used $\rho \approx 2\sigma$, and Eq. (4.41) has been substituted into Eq. (4.39). This equation does not have an exact analytical solution [1]. We study the dynamical behaviour by analysing this equation in some particular and physically interesting limits. At early times (i.e. $\tau \to -\infty$), we note that the second term inside the brackets is more dominant than the third term. As the system evolves in time, at late times (i.e. $\tau \to 0_-$), the situation is reversed and the third term dominates over the second. This is true for all modes. From our discussion in section 3.4 we know that during the super-inflationary phase, the term proportional to $\tau^{-2}$ increases by about $e^{140}$ (from Eq. (3.107)), and the term proportional to the square of the scale factor increases by about $e^{140p}$ (from Eq. (3.108)). In the limit that $p \to 0$, which is the case we are interested in, this means that during the super-inflationary evolution the second term inside the brackets of Eq. (4.45) can be treated as a constant relative to the third term. We, therefore, consider an *effective* wave number $k_{eff}^2 = k^2 - \frac{4\sigma}{3}(-\tau)^{2p}$, which introduces a shift in the wave number $k$, and we rewrite the mode equation in terms of this new parameter

$$h_k'' + 2\frac{a'}{a}\,h_k' + \left[k_{eff}^2 - \frac{2p(p+1)}{\tau^2}\right] h_k = 0\,. \tag{4.46}$$

For very small scales (i.e. $k \to \infty$), we can use the approximation $k_{eff}^2 \approx k^2$, and following a similar method to what we did in the previous section, the

[1] This equation is different to what we quoted in [19], where the second term inside the bracket was mistakenly absent. In that paper we gave a full analytic solution for the equation we had derived. Much of the analysis in this section is similar to the paper, but we also make comments on how the presence of a term proportional to the scale factor in the mode equation could be treated and interpreted.

solution to Eq. (4.46) is found to be

$$h_k(\tau) = \frac{1}{\mathcal{H}^{1/2}a}\sqrt{-p\,\pi}\,H_\nu^{(2)}(-k\tau)\,, \tag{4.47}$$

where

$$\nu = \frac{1}{2}\sqrt{1+4p+12p^2}\,, \tag{4.48}$$

where we have normalised the solution by the requirement that $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ satisfy the usual raising and lowering operator algebra while $a^2\hat{h}$ and $\hat{h}'$ satisfy their commutation algebra, with only the forward moving solution being selected in the asymptotic past (selecting the adiabatic vacuum). Note that this normalisation has been done on the very small scales, where quantum fluctuations are thought to be most important. We do not expect the difference between $k_{eff}$ and $k$ to be significant in this limit.

Finally, utilising Eq. (4.12) and evaluating this expression in the limit that the modes are outside the horizon ($k < -\mathcal{H}/p$ in this case), leads us to the same expression for the primordial power spectrum (4.31), with the difference that now $\nu$ is given by Eq. (4.48). The spectral index can clearly be seen to be given by

$$n_t \approx 2 - 2p \approx 2 \tag{4.49}$$

in the limit $p \to 0$, and the amplitude fixed by $H_{\rm e}$. We note here that in cases where the dynamics of tensor perturbations could be represented by the standard equation of motion and the evolution of the scale factor is given by Eq. (4.40), having $p \to 0$ will inevitably result in $n_t = 2$. This has indeed been shown to be true explicitly for the collapsing ekpyrotic scenario [88] and for the evolution of a universe sourced by a phantom field [124]. In the full Eq. (4.46), for the physically meaningful case of $k^2_{eff}$ being a positive quantity, there is a lower limit on the wave number $k$, which we denote as $k_{min}$ and is set by the largest value $\frac{4\sigma}{3}(-\tau)^{2p}$ can take. Since $\sigma = \frac{3}{2\gamma^2}$ depends on the fundamental loop parameter $\gamma$, which is of order unity [38, 39], the limiting quantity is the size of the square of the scale factor. $(-\tau)^{2p}$ is largest at the end of the super-inflationary era, and for all modes much greater than this value the argument and the analysis above remains valid. One thing to notice is that for modes of the order of the scale factor, in calculating the spectral tilt, we have from Eq. (4.12)

$$\mathcal{P}_h \propto k^3 \, k_{eff}^{-2\nu} \, . \tag{4.50}$$

For the scaling solution we have been considering, in the limit of $p \to 0$, from Eq. (4.48) we have $\nu \to 1/2$. It is clear that the value of the power spectrum of (4.31) is only approximately valid when $k$ is close to the value by which it is shifted, and in fact the power spectrum (4.50) will pick up modifications in lower powers of $k$. The spectral tilt of approximately 2 will still be the leading order result for this scenario. It is not physically meaningful to consider frequencies lower than $k_{min}$. We, therefore, make the assumption that this sets the maximum scale of the universe for which Eq. (4.46) holds. Whilst the combination $k^2 - \frac{4\sigma}{3}(-\tau)^{2p}$ is much less than $\tau^{-2}$ at late times on this scale, the spectral tilt calculated above is correct. This does not mean that the universe can not grow beyond this scale, but rather that our results here suggest we can not make any predictions on primordial gravitational waves corresponding to scales greater than $k_{min}$. We, thus, assume that for the purpose of our study we are only dealing with modes that are physically meaningful if Eq. (4.46) is to hold.

At this point, making some reasonable assumptions, we can proceed, as we did for the solution in the inverse triad case, to calculate the present day spectrum of gravitational wave perturbations using the transfer function Eq. (4.33). We show the abundance of tensors in Fig. 4.3 and note that the result is very similar to the inverse volume case.

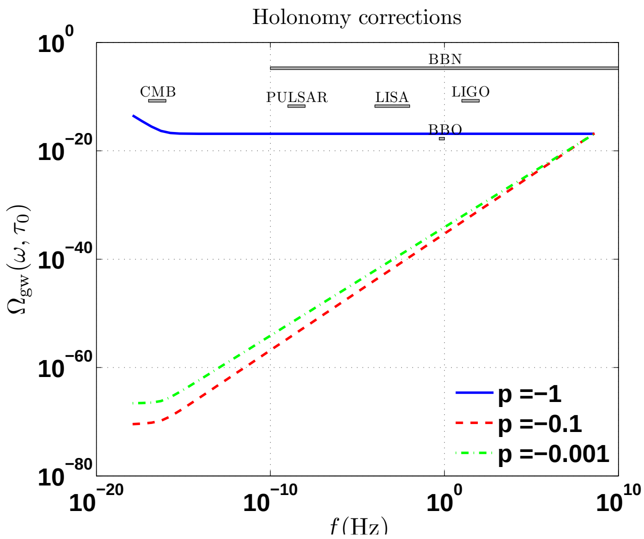


Figure 4.3: Predicted present abundance of gravitational waves produced during the super-inflationary phase of LQC with holonomy corrections. The solid line corresponds to the standard inflation abundance. Also indicated are the present bounds and future sensitivities.

## 4.4 Discussion

We conclude that super-inflation in both versions of the corrections in LQC predicts a strong blue spectrum of gravitational waves, hence, their abundance is strongly suppressed on the large scales and it is many orders of magnitude smaller than the value predicted in the standard inflationary scenario. In the case of the holonomy corrections, we have the extra feature of a maximum scale being introduced for which the calculation of the spectral tilt of the gravitational waves can be performed. In our analysis we have made the assumption that the scales we can probe today are smaller than this value. This is a new result and has not been mentioned elsewhere in the literature.

It is now important to discuss our results in the light of other investigations in the literature. In particular, it is claimed in Ref. [131] that the abundance of gravitational waves generated during super-inflation under inverse volume corrections is above the current bounds. We note however that the authors did not consider the evolution of the background in a scaling solution and the $S(a)$

correction was not used. Moreover, the full expression for the tensor perturbations was obtained more recently [69] and therefore it was not used in that work. In a more recent analysis [126] focusing on the holonomy corrections, it is also found, like in our work, that the spectrum of gravitational waves must be blue. However, a scaling solution was not used and the expansion of the universe was assumed to be close to de Sitter, hence, a direct comparison with our work is in fact not possible.

We have highlighted the similarities between our predictions of tensor perturbation spectral index in the context of LQC and those obtained in the ekpyrotic collapse models and also when the universe is sourced by a phantom field.

One needs to be cautious about making general conclusions in the context of LQC, as the theory is currently far from complete. As mentioned previously, the introduction of inhomogeneities in the small $a$ regime is likely to break the assumption of an isotropic universe. Attempts are being made to gain better understanding of this region [125]. Moreover, there is the possibility that higher order perturbative correction terms could play a role, or that quantum backreaction might significantly modify the background dynamics [132].

Despite the ambiguities and uncertainties in the theory, if we were to make observational claims for the scaling solutions we have considered in the current state of LQC, it would be that if gravitational waves are observed, they would rule out this scenario of superinflation in LQC as it stands.

# Chapter 5

# Field perturbations: LQC mapped to Braneworlds

## 5.1 Introduction

A successful and consistent picture of the evolution of the universe is provided by the standard model of cosmology based on General Relativity. It is widely believed that by approaching the validity limit of GR, this picture will be modified. One expects this to happen at the high energy levels of the very early universe, and modifications to arise in the Friedmann equations when the curvature of the universe is quite high. A theory of Quantum Gravity is expected to shed light on the nature of these corrections by probing the quantum properties of spacetime near the Big Bang. There have been a number of attempts at formulating such a fundamental theory, most popular of which are either motivated by higher dimensions and are related to string/M-theory (e.g. [73, 74]), or are inspired by Loop Quantum Gravity (LQG) based on four spacetime dimensions [14]. We will consider braneworld cosmologies in the former class of theories and we limit ourselves to the LQC model in the latter class.

We discussed in Chapter 2 the setup of simple braneworld cosmologies and the types of corrections they can introduce to the Friedmann equations. In the course of the previous chapters we have also reviewed in detail the nature of modifications that arise in the context of LQC. Despite the different underlying physical pictures in these scenarios, since they both ultimately aim to describe what one observes in the universe, and assuming both approaches are correct in the modifications they introduce beyond the standard description of GR, we are interested in finding out how one could relate their predictions in a quantitative fashion. The work of [20] has considered a possible map between the inverse volume corrections of LQC and typical braneworld cosmologies. This correspondence maps the background cosmologies of the two sets of models, and since the number of free parameters on both sides are unequal, a particular LQC model can be mapped to a class of braneworld models and visa versa. The map is produced by demanding that both cosmologies result in the same effective Hubble parameter. Since this is a typical parameter that could be measured, any physically viable theory is expected to make the correct prediction for it. By making the assumption that both theories under investigation are physical descriptions of our universe, to relate their predictions at the cosmological background level, one obtains a map between the two in this way [20]. Notice that such a map does not prove if/how these theories may be equivalent.

The aim of this chapter is to investigate whether this map would also link other physically measurable quantities predicted through the corrections these theories introduce to our understanding of cosmology. We begin by reviewing the outline of the calculations of [20] for the more general LQC model described in Chapter 2. Considering the general modifications made to the Friedmann equations in braneworld-inspired cosmologies, we derive the correspondence map at the cosmological background level in section 5.2. We will proceed in section 5.3 to map the LQC model of the inverse triad corrections to a class of braneworld cosmologies at the background level using phase space analysis. Since stability is an important physical property in cosmology, we check and see if this property is mapped directly across from LQC to braneworlds. The scale-dependence of the scalar field perturbations in the braneworld cosmologies is derived in section 5.4, followed by a comparison made in section

5.5 between these perturbations and the near scale-invariant perturbations we have presented in the previous two chapters in the case of the LQC model. We do this to investigate whether the two cosmologies also correspond to one another at the perturbation level. We will conclude in section 5.6.

## 5.2 Relating LQC and Braneworld cosmologies at the background level

A constraint equation relating braneworld cosmologies and LQC-inspired cosmologies at the background level was derived in [20]. For our purposes, we derive a more general form for this equation, when considering the inverse volume corrections to the Friedmann equation and the equation of motion in LQC are given by Eqs. (2.15) and (2.12), also reproduced below for easy reading:

$$H_l^2 = \frac{\kappa^2}{3} S \left( \frac{\dot{\phi}^2}{2D} + V(\phi) \right) \ , \tag{5.1}$$

$$\ddot{\phi} + 3H_l \left( 1 - \frac{1}{3} \frac{d \ln D}{d \ln a} \right) \dot{\phi} + DV_{,\phi} = 0 \ . \tag{5.2}$$

In these equations, $\phi$ and $V$ are the LQC scalar field and scalar potential, respectively, as before.

The braneworld modifications to the classical Friedmann equation and the equation of motion for the homogeneous scalar field, $\chi$, are such that

$$H_b^2 = \frac{\kappa^2}{3} \rho_\chi L^2(\rho_\chi) \ , \tag{5.3}$$

$$\ddot{\chi} + 3H_b \dot{\chi} + W_{,\chi} = 0 \ , \tag{5.4}$$

where $L^2(\rho_\chi)$ represents the quantum gravitational modifications in braneworld models. We have already mentioned the Randall-Sundrum model in section 2.3, where $L(\rho_\chi) = \sqrt{1 + \frac{\rho_\chi}{2\sigma}}$ (from Eq. (2.18)), and the Shtanov-Sahni model, where $L(\rho_\chi) = \sqrt{1 - \frac{\rho_\chi}{2\sigma}}$ (from Eq. (2.31)). Other examples and different forms of the function $L(\rho_\chi)$ can also be found in [133]. $\chi$ and $W$ represent the scalar field and its potential in the braneworld cosmology, respectively; and $\rho_\chi = \frac{\dot{\chi}^2}{2} + W(\chi)$ is the energy density associated with the homogeneous scalar field, $\chi$. For both cosmologies we have made the assumption that the curvature term is subdominant. These equations are valid in the absence of any fluid, where the energy density is dominated by the scalar fields.

By allowing the evolution of both systems to be governed by the same Hubble parameter (i.e. $H_l = H_b$), motivated by the reasons mentioned above, we obtain the constraint equation

$$\frac{2\dot{\phi}^2}{\dot{\phi}^2 + 2DV}\left(1 - \frac{1}{6}\frac{d\ln D}{d\ln a}\right) - \frac{r}{3} = -\frac{1}{3}\frac{d\ln\left(\rho_\chi L^2(\rho_\chi)\right)}{d\ln a}\,. \tag{5.5}$$

It is easy to check that the left hand side of Eq. (5.5) can be written as $\frac{2\left(1-\frac{n}{6}\right)}{1+2c} - \frac{r}{3}$, where $c = \frac{DV}{\dot{\phi}^2}$ is a constant when considering scaling solutions. After substituting these into the constraint equation and integrating, we find that, up to a constant

$$\rho_\chi L^2(\rho_\chi) = a^{\frac{n-6}{1+2c}+r}. \tag{5.6}$$

Using Eq. (5.3), the scale factor of the braneworld cosmology is found as a function of conformal time, $\tau$, to be

$$a = A_0(-\tau)^\alpha \ , \tag{5.7}$$

where $\alpha$ is defined as

$$\alpha \equiv \frac{-2(1+2c)}{(1+2c)(2+r)+n-6}\,, \tag{5.8}$$

and $A_0 = \left(\frac{-1}{\sqrt{3}\alpha}\right)^\alpha$ is a positive constant. Notice that this is consistent with an expanding universe, where one requires $\alpha < 0$. As an ansatz, let us assume we can write the braneworld scalar field and its potential as power-law functions:

$$\chi = \chi_0(-\tau)^\Lambda, \qquad W = W_0(-\tau)^\sigma. \tag{5.9}$$

where $\chi_0$ and $W_0$ are constants. By substituting Eq. (5.9) into Eq. (5.4), and requiring every term to evolve with the conformal time in the same way, we obtain the following relations:

$$\sigma = 2(\Lambda - 1 - \alpha), \qquad 2W_0 A_0^2 = -\Lambda^2 \chi_0^2 \frac{\Lambda - 1 + 2\alpha}{\Lambda - 1 - \alpha}. \tag{5.10}$$

We note that the first of these relations is equivalent to a scaling requirement on the braneworld cosmology, meaning the same equation can be found for $\sigma$ if we demanded $\dot{\chi}^2/W$ to be a constant. This is a desirable outcome and we will discuss the stability properties of these solutions later in this chapter. By substituting our ansatz (5.9) into Eq. (5.3), we find that the general form of the braneworld modifications made to the standard cosmology satisfying the constraint equation (5.5), is

$$L^2(\rho_\chi) \propto \rho_\chi^{\frac{-\Lambda}{\Lambda - 1 - \alpha}} \quad . \tag{5.11}$$

This is an important result and it suggests that the Hubble parameter evolves as a power-law function of the scalar field energy density. Since $L^2(\rho_\chi)$ represents the high energy corrections made to the standard cosmology, we expect it to become more significant as we go back in time towards the Big Bang, where the energy density of the universe would have been much greater than its value later on in the evolution of the universe. Let us consider the case where $L^2(\rho_\chi) = \rho_\chi^m$, where $m > 0$. In this case using Eq.s (5.3), (5.4),(5.10), and (5.11), we obtain

$$\Lambda = \frac{m(1+\alpha)}{1+m}, \tag{5.12}$$

$$\chi_0^2 = \frac{2(3\alpha^2)^{-\frac{\alpha m - 1}{1+m}}(1+m)}{3\alpha m^2(1+\alpha)}, \tag{5.13}$$

$$W_0 = \frac{(3\alpha^2)^{\frac{1+\alpha}{1+m}}(2\alpha - 1 + 3\alpha m)}{3\alpha(1+m)}. \tag{5.14}$$

Notice that since $\alpha < 0$, $W_0$ is positive definite, and the form of the potential as a function of the scalar field is derived, using Eq. (5.9), as $W \propto \chi^{-2/m}$. i.e. $W(\chi)$ is a positive, inverse power-law potential.

Notice further that by substituting Eq. (5.12) into the expression for the rate of change of the Hubble parameter

$$\dot{H} = \frac{1}{2}\dot{\chi}^2 L^2(\rho_\chi)\frac{1+\alpha}{\Lambda - 1 - \alpha}, \tag{5.15}$$

we find that $\dot{H} < 0$, and using the behaviour of our braneworld scale factor in Eq. (5.7), this implies $\alpha < -1$.

## 5.3 Stability analysis of the Braneworld cosmology

There is a class of cosmological solutions, referred to as the 'scaling solutions', which are usually considered to have the important desirable property of describing asymptotic late time behaviour of cosmological backgrounds, and are often found to be stable attractors [95, 96, 97, 153, 110]. Our focus here is based on such solutions in the context of the modified braneworld cosmologies we are considering. We saw in Chapter 3 the form of the background cosmology which yields scaling solutions in the semi-classical regime of LQC, and a stability analysis was also carried out on these solutions in section 3.2.3. This section is dedicated to using a similar procedure to determine the late time behaviour and stability criteria for the corresponding braneworld cosmology. By employing the phase space analysis method developed in [149, 150] for such scenarios, we proceed by defining the following two parameters:

$$x \equiv \frac{\dot{\chi}}{\sqrt{2\rho_\chi}}, \qquad y \equiv \frac{\sqrt{W}}{\sqrt{\rho_\chi}}. \tag{5.16}$$

It is then easy to show that Eq.s (5.3)-(5.4) can be written in terms of these parameters in the following form:

$$x_{,N} = -3x(1-x^2) + \sqrt{\frac{3}{2}}\lambda y^2 \,, \tag{5.17}$$

$$y_{,N} = \sqrt{\frac{3}{2}}\lambda xy + 3x^2 y \,, \tag{5.18}$$

$$\lambda_{,N} = -\sqrt{6}(\Gamma - 1)\lambda^2 x + 6\lambda \frac{d\ln L(\rho_\chi)}{d\ln \rho_\chi} x^2 \,, \tag{5.19}$$

where $\lambda$ and $\Gamma$ are defined as

$$\lambda \equiv -\frac{1}{L(\rho_\chi)}\frac{W_{,\chi}}{W}, \qquad \Gamma \equiv \frac{W_{,\chi\chi}W}{W_{,\chi}^2}. \tag{5.20}$$

In the model under consideration,

$$\lambda_{,N} = -\sqrt{6}(\Gamma - 1)\lambda^2 x + 3m\lambda x^2 \,. \tag{5.21}$$

The $x$ and $y$ parameters defined here, in the absence of any fluid, are subject to the constraint equation

$$x^2 + y^2 = 1. \tag{5.22}$$

Taking $\lambda$ to be constant, and substituting this constraint equation into Eq. (5.17), makes Eq. (5.18) redundant, and together with Eq. (5.21) results in an autonomous system with three fixed points:

$$x_{c1} = 1\,, \qquad y_{c1} = 0\,, \qquad \Gamma_{c2} = 1 + \frac{\sqrt{3/2}}{\lambda}m\,, \tag{5.23}$$

$$x_{c2} = -1\,, \qquad y_{c2} = 0\,, \qquad \Gamma_{c2} = 1 - \frac{\sqrt{3/2}}{\lambda}m\,, \tag{5.24}$$

$$x_{c3} = \frac{\lambda}{\sqrt{6}}\,, \qquad y_{c3} = \sqrt{1 - \frac{\lambda^2}{6}}\,, \qquad \Gamma_{c3} = 1 + \frac{m}{2}\,, \tag{5.25}$$

The first two points correspond to kinetic energy dominated solutions, and the third point is a scaling solution, where the kinetic and the potential energies of the scalar field scale together throughout the evolution of the universe. In order to analyse the late time behaviour and stability conditions for our system, we study the nature of these fixed points. We do this by linearising Eq.s (5.17) and (5.21) about the fixed points and determining the corresponding eigenvalues in each case, using the methods outlined in Chapter 2. The $\mathcal{M}$-matrix for these equations is found to be

$$\mathcal{M} = \begin{pmatrix} -3-\sqrt{6}x\lambda+9x^2 & -\sqrt{\frac{3}{2}}(x^2-1) \\ -\sqrt{6}(\Gamma-1)\lambda^2+6m\lambda x & -2\sqrt{6}x(\Gamma-1)\lambda+3mx^2 \end{pmatrix} \tag{5.26}$$

admitting the following eigenvalues for the kinetic energy dominated fixed points

$$\omega_{c1+} = 6-\sqrt{6}\lambda\,, \qquad \omega_{c1-} = -3\,, \tag{5.27}$$

$$\omega_{c2+} = 6+\sqrt{6}\lambda\,, \qquad \omega_{c2-} = -3\,. \tag{5.28}$$

The first of these is stable for $\lambda \geq \sqrt{6}$, and the second is stable when $\lambda \leq -\sqrt{6}$. The eigenvalues of the scaling solution are

$$\omega_{c3+} = 0\,, \qquad \omega_{c3-} = -3\,, \tag{5.29}$$

which means this solution is unconditionally stable. By considering the scaling solution, we observe that the valid range of $\lambda$ for the existence of these solutions is given by $\lambda^2 \leq 6$. In fact, as the scalar field rolls down its positive inverse power-law potential, initially, the kinetic energy dominated solution is the stable fixed point, where the potential is steep. Thereafter, the potential drops and flattens, where the scaling solution then becomes the late-time attractor.

We have demonstrated that the scaling solutions of the corresponding braneworld scenario to the LQC model considered in this chapter are also late time attractors. This is a desirable result, since stability can be viewed as a physical property of our universe and the fact that both cosmologies under study, related via the constraint Eq. (5.5), are asymptotically stable is encouraging.

## 5.4 Power Spectrum of the perturbed field

In this section we proceed to calculate the vacuum fluctuations of the scalar field, $\chi$. We will then make a comparison between our results here and the spectrum of the scalar field perturbations calculated in Chapter 3. We start by splitting the field into homogeneous and perturbation parts: $\chi \to \chi + \delta\chi$. The perturbation in the field, $\delta\chi$, satisfies

$$(\ddot{\delta\chi}) + 3H(\dot{\delta\chi}) + (W_{,\chi\chi} - \nabla^2)(\delta\chi) = 0\,, \tag{5.30}$$

which upon introducing $u = a\delta\chi$, can be written as

$$u'' + \left(-\nabla^2 + m_{eff}^2\right) u = 0\,, \tag{5.31}$$

where

$$m_{eff}^2 = -\frac{a''}{a} + W_{,\chi\chi} a^2\,. \tag{5.32}$$

Following the general methods employed in this type of analysis, first described in chapter 3, we quantise $u$ by promoting it to an operator $\hat{u}$. An appropriate conjugate momentum $\hat{\pi}_u$ can also be defined such that a similar relation to Eq. (3.27) holds in this case. Decomposing $\hat{u}$ in Fourier modes, $u_k$, gives Eq. (3.29), written here for easy reading:

$$\hat{u}(\tau, \mathbf{x}) = \int \frac{\mathbf{d^3k}}{(\mathbf{2\pi})^\mathbf{3}} \left[\mathbf{u_k}(\tau)\hat{\mathbf{a}}_\mathbf{k}\, \mathbf{e}^{\mathbf{ik.x}} + \mathbf{u}^*_\mathbf{k}(\tau)\, \hat{\mathbf{a}}^\dagger_\mathbf{k}\, \mathbf{e}^{-\mathbf{ik.x}}\right], \tag{5.33}$$

where the modes satisfy the mode equation

$$u_k'' + \left(k^2 + m_{eff}^2\right) u_k = 0\,. \tag{5.34}$$

The power spectrum of these modes can then be calculated from the definition

$$\mathcal{P}_u \equiv \frac{k^3}{2\pi^2}|u_k|^2\,. \tag{5.35}$$

The general solution to Eq. (5.34) is given by

$$u_k(\tau) = \sqrt{\frac{\pi}{2|2+np|}}\left(d_1\sqrt{-\tau}\,H^{(1)}_{|\nu|}(x) + d_2\sqrt{-\tau}\,H^{(2)}_{|\nu|}(x)\right) \tag{5.36}$$

whenever $m_{\rm eff}\tau$ is constant, and where

$$\nu = -\frac{\sqrt{1-4\,m_{\rm eff}^2\,\tau^2}}{2}\,. \tag{5.37}$$

$H^{(1)}_{|\nu|}(x)$ and $H^{(2)}_{|\nu|}(x)$ are Hankel functions of the first and second kind, respectively. Following the procedure outlined in the previous two chapters, the constants $d_1$ and $d_2$ can be chosen such that only the forward moving wave is chosen. In the long wavelength limit, the power spectrum yields

$$\mathcal{P}_u \propto k^{3-2|\nu|}\,(-\tau)^{1-2|\nu|}\,. \tag{5.38}$$

We expect to have scale-invariance when this power spectrum is independent of $k$. In order to quantify the scale dependence properties of the field perturbations, we define a set of slow-roll parameters; and produce an expansion in these parameters describing the deviation of our field perturbations from exact scale-invariance.

We define our first slow-roll parameter as

$$\bar{\epsilon} \equiv \frac{1}{2}\lambda^2 = \frac{1}{2}\frac{1}{L^2(\rho_\chi)}\left(\frac{W_{,\chi}}{W}\right)^2\,, \tag{5.39}$$

and we will soon explain that this is an appropriate parameter to choose for a slow-roll inflationary scenario in these models. For the model we have been considering, this parameter can be written as

$$\bar{\epsilon} = \frac{1+\alpha}{\alpha(1+m)}\,, \tag{5.40}$$

or alternatively, we have that $\alpha = \frac{1}{(1+m)\bar{\epsilon}-1}$. The background cosmology can then be written in terms of these parameters:

$$
\begin{aligned}
a &= A_0(-\tau)^{\frac{1}{(1+m)\bar{\epsilon}-1}}\,, & (5.41)\\
\chi' &= -\frac{\sqrt{2\bar{\epsilon}}}{(1+m)\bar{\epsilon}-1}\left[\frac{3}{((1+m)\bar{\epsilon}-1)^2}\right]^{\frac{-m}{2((1+m)\bar{\epsilon}-1)}}(-\tau)^{\frac{m\bar{\epsilon}}{(1+m)\bar{\epsilon}-1}-1}, & (5.42)\\
W &= \frac{3-\bar{\epsilon}}{((1+m)\bar{\epsilon}-1)^2}\left[\frac{3}{((1+m)\bar{\epsilon}-1)^2}\right]^{-\frac{m}{1+m}}\frac{1}{(a\tau)^{\frac{2}{1+m}}}\,. & (5.43)
\end{aligned}
$$

By substituting Eq. (5.41) - (5.43), into Eq. (5.32), we find

$$
m_{eff}^2\tau^2 = \frac{3\alpha(\alpha+1)(1+m)^2-(m\alpha-1)(m(\alpha-1)-2)}{(1+m)^2}\,. \qquad (5.44)
$$

It is then evident from Eq. (5.37)-(5.38) that scale-invariance is obtained in the braneworld scenario for $\alpha \to -1$, and $\alpha \to -\frac{m(2m+3)}{2m^2+6m+3}$. The latter is outside the valid range of $\alpha$ (for our assumption of $m > 0$), but the former falls within the acceptable range, and corresponds to larger values of the scalar field, where the potential is flat and the field is slowly rolling down. As we evolve towards a stable scaling solution, we expect this behaviour of the field to lead to slow-roll inflation. The limit of $\alpha \to -1$ also implies that $\bar{\epsilon} \ll 1$, and it can consequently be treated as a conventional slow-roll parameter.

We would like to be able to generalise our argument for the potentials which are not of the exact form that give rise to the scaling solutions we have discussed, but approach these solutions asymptotically. We can account for small deviations from exact solutions by allowing the slow-roll parameters to be time dependent, but we also impose the condition that they are only slowly varying with time. It then follows from differentiating Eq. (5.39) that

$$
\bar{\epsilon}' = -(2\bar{\epsilon})^{3/2}\,L(\rho_\chi)\eta(\tau)\chi'\,, \qquad (5.45)
$$

where the second slow-roll parameter, $\eta$, is defined as

$$
\eta(\tau) \equiv 1+\frac{W}{W_{,\chi}}\frac{L(\rho_\chi)_{,\chi}}{L(\rho_\chi)}-\frac{W_{,\chi\chi}W}{W_{,\chi}^2}\,. \qquad (5.46)
$$

It is also easy to show that $\eta' = -\sqrt{2\bar{\epsilon}}\, L(\rho_\chi)\xi^2\chi'$, where

$$
\begin{aligned}
\xi^2 \;\equiv\;& \frac{W_{,\chi\chi}W}{W_{,\chi}^2}\left(1+\frac{W_{,\chi\chi\chi}W}{W_{,\chi\chi}W_{,\chi}}-2\frac{W_{,\chi\chi}W}{W_{,\chi}^2}\right) \\
-\;& \frac{WL_{,\chi}}{W_{,\chi}L}\left(1+\frac{L_{,\chi\chi}W}{L_{,\chi}W_{,\chi}}-\frac{L_{,\chi}W}{LW_{,\chi}}-\frac{W_{,\chi\chi}W}{W_{,\chi}^2}\right)\,. \qquad (5.47)
\end{aligned}
$$

By imposing the variations of $\bar{\epsilon}$ and $\eta$ with time (i.e. $\frac{d\ln\bar{\epsilon}}{d\ln a} \approx 4\bar{\epsilon}\eta$, and $\frac{d\ln\eta}{d\ln a} \approx 2\frac{\bar{\epsilon}\,\xi^2}{\eta}$) to be negligible, one observes that $\eta$ and $\xi^2$ should be very small. We will drop $\xi^2$ and higher orders in the perturbation expansion in what follows.

In order to quantify the spectral tilt of the scalar field perturbations, we substitute the appropriate slow-roll parameters into the expression for the effective mass Eq. (5.32). The deviation from exact scale-invariance is given, from Eq. (5.38), by $\Delta n_s = 3 - \sqrt{1-4m_{eff}^2\tau^2}$. After expanding to first order we find

$$\Delta n_s \approx 2\bar{\epsilon}\,. \qquad (5.48)$$

Clearly, when $\bar{\epsilon} \ll 1$, near scale-invariance is achieved. This is in line with the limit of $\alpha \to -1$ for the case of our late time scaling solution.

## 5.5 Relating LQC and Braneworld cosmologies at the perturbation level

In this section we make a comparison between the spectral tilt of perturbations of the scalar field, $\phi$, in LQC given by Eq. (3.49) and the braneworld scalar field perturbations. In order to establish if the two cosmologies are related at the perturbation level in the same way they correspond to each other at the classical level, we ask the following question. When the scalar field power spectrum of the braneworld cosmology is rewritten in terms of the LQC parameters, would the scale-invariance condition of our LQC model also result in scale-invariance in this rewritten expression? We begin by writing Eq. (5.48) in terms of the LQC parameters. In doing so, we first substitute the result of the correspondence relation between the braneworlds and the LQC model derived earlier in the form of Eq. (5.8) into Eq. (5.40). Eq. (5.48) then becomes

$$\Delta n_s \approx -\frac{r(1+2c)+n-6}{(1+2c)(1+m)}\,, \tag{5.49}$$

where $c = DV/\dot{\phi}^2$ is a constant for the scaling solution considered in our previous chapters. We now refer to the model we considered in the context of LQC in chapter 3. By assuming we operate in the semi-classical regime (i.e. $D = D_* a^n$), and setting $D_* = 1$ without loss of generality, we can evaluate $c$ from Eqs.(3.20)-(3.21), and find

$$c = \frac{\frac{1}{12}(n-6)^2 - \bar{\epsilon}_{Loops}}{2\bar{\epsilon}_{Loops}}\,, \tag{5.50}$$

where the subscript *Loops* refers explicitly to the LQC model. Rewriting Eq. (3.19), we obtain an expression for $\bar{\epsilon}_{Loops}$:

$$\bar{\epsilon}_{Loops} = -\frac{(n-6)(2+p(2+r))}{12p}\,, \tag{5.51}$$

where $p$ describes how the LQC scale factor behaves through Eq. (3.18). By substituting this into Eq. (5.50), the expression for the *braneworld* scalar field perturbations derived above in Eq. (5.49) becomes

$$\Delta n_s \approx \frac{2(1+p)}{p(1+m)}\,, \tag{5.52}$$

which includes the LQC parameter $p$. We note here that in the limit $p \to 0$, the scalar field power spectrum in the braneworld cosmology blows up; but, as shown in Chapter 3, this is indeed the limit of $p$ for which scale-invariance is obtained in LQC. This discrepancy needs to be examined more closely to gain better understanding.

We have seen that by perturbing the scalar field $\chi$, (i.e.$\chi \to \chi + \delta\chi$), the perturbations satisfy Eq. (5.30). From Eq. (5.20), $W_{,\chi\chi}$ can be written as

$$W_{,\chi\chi} = \Gamma\lambda^2 W L^2(\rho_\chi)\,, \tag{5.53}$$

which upon using the definition of $y$ in Eq. (5.16) becomes

$$W_{,\chi\chi} = \Gamma\lambda^2 y^2 \rho_\chi L^2(\rho_\chi)\,. \tag{5.54}$$

For the scaling solution described by (5.25), we have

$$W_{,\chi\chi} = 3\lambda^2(1+\frac{m}{2})(1-\frac{\lambda^2}{6})H^2\,, \tag{5.55}$$

and once we employ the definition of $\bar{\epsilon}_{Branes}$ in (5.39), it gives

$$W_{,\chi\chi} = 6H^2(1+\frac{m}{2})\bar{\epsilon}_{Branes}(1-\frac{\bar{\epsilon}_{Branes}}{3})\,, \tag{5.56}$$

where the subscript *Branes* has been introduced to distinguish the slow-roll parameters of each model explicitly. Similarly, for the LQC model the perturbation equation of the scalar field, $\phi$, is given by Eq. (3.22), which for easy reading and flow of our line of argument is also given below in terms of cosmic time, $t$.

$$\ddot{(\delta\phi)} + 3H(1-\frac{n}{3})\dot{(\delta\phi)} + D(V_{,\phi\phi} - \nabla^2)(\delta\phi) = 0\,, \tag{5.57}$$

where we have assumed we are working in the semi-classical regime (i.e. $D \propto a^n$). $V_{,\phi\phi}$ for our scaling cosmology is calculated through a similar method outlined above, to be

$$V_{,\phi\phi} = \frac{6H^2}{D}\bar{\epsilon}_{Loops}\left[1 - \frac{\bar{\epsilon}_{Loops}}{3} + \frac{(n-r)\left(1-\frac{n-r}{6}\right)}{\bar{\epsilon}_{Loops}} + \frac{n}{12\left(1-\frac{n}{6}\right)}\right]\,. \tag{5.58}$$

Let us start our investigation into the discrepancy by writing the LQC scalar field, $\phi$, in terms of the braneworld scalar field, $\chi$. Since the corespondence relation derived in this chapter is based on demanding the Hubble parameters of both models to be equal, the behaviour of the scale factors would also be identical, up to a constant. Therefore, from Eq.s (3.18) and (5.7) we have

$$p = \alpha\,. \tag{5.59}$$

From the dependence of the scalar field, $\phi$, on conformal time for the scaling solution in LQC [100, 18]

$$\phi = \phi_0(-\tau)^{((n-r)\alpha/2)}\,, \tag{5.60}$$

where $\phi_0$ is a constant, and using Eq. (5.9), we have

$$\phi = C\,\chi^{\gamma}, \tag{5.61}$$

where $\gamma = \frac{(n-r)\alpha(1+m)}{2m(1+\alpha)}$, and $C = \frac{\phi_0}{\chi_0^{\gamma}}$. By considering small variations in the scalar fields, we will then have

$$\begin{aligned}
(\delta\phi) &= C\gamma\chi^{\gamma-1}(\delta\chi) && (5.62)\\
(\dot{\delta\phi}) &= C\gamma\left[(\gamma-1)\chi^{\gamma-2}\dot{\chi}(\delta\chi)+\chi^{\gamma-1}(\dot{\delta\chi})\right] && (5.63)\\
(\ddot{\delta\phi}) &= C\gamma(\gamma-1)\left[(\gamma-2)\chi^{\gamma-3}\dot{\chi}^2(\delta\chi)+\chi^{\gamma-2}\ddot{\chi}(\delta\chi)+\chi^{\gamma-2}\dot{\chi}(\dot{\delta\chi})\right] && (5.64)\\
&+ C\gamma\left[(\gamma-1)\chi^{\gamma-2}\dot{\chi}(\dot{\delta\chi})+\chi^{\gamma-1}(\ddot{\delta\chi})\right]\,. && (5.65)
\end{aligned}$$

Substituting these into Eq. (5.57) and upon using the expression for $V_{,\phi\phi}$ given by (5.58), we find the corresponding braneworld perturbation equation for the case where Eq. (5.61) is satisfied, as

$$(\ddot{\delta\chi}) + 3H\left(1-\frac{n-r}{3}+\frac{2\Lambda(\gamma-1)}{3\alpha}\right)(\dot{\delta\chi}) + \left[6H^2\left(q_L+\mu\right) - D\nabla^2\right](\delta\chi) = 0\,, \tag{5.66}$$

where $\mu = \frac{\Lambda(\gamma-1)}{6\alpha^2}\left(\Lambda(\gamma-1)-1-\alpha-3\alpha(1-\frac{n}{3})\right)$, and $q_L$ is given by

$$q_L = \bar{\epsilon}_{Loops}\left[1 - \frac{\bar{\epsilon}_{Loops}}{3} + \frac{(n-r)\left(1-\frac{n}{6}\right)}{\bar{\epsilon}_{Loops}} + \frac{n-r}{12\left(1-\frac{n}{6}\right)}\right]. \tag{5.67}$$

Let us remind ourselves that for the LQC and the braneworld models considered here to correspond to one another at the background level, the constraint Eq. (5.5) must hold. If they are also to correspond to one another at the perturbation level, we expect Eq. (5.66) to be equivalent to Eq. (5.30), which was derived independently for the braneworld scenario. This implies the following two criteria must be met simultaneously:

$$\frac{2\Lambda(\gamma-1)}{3\alpha} - \frac{n-r}{3} = 0\,, \qquad q_B = q_L + \mu\,, \tag{5.68}$$

where from Eq.s (5.30) and (5.57), $q_B = (1+\frac{m}{2})\bar{\epsilon}_{Branes}(1-\frac{\bar{\epsilon}_{Branes}}{3})$. Using Eq. (5.12) and the form of $\gamma$ given above, the first of these criteria can be shown to be satisfied either for $m = 0$, which corresponds to the classical cosmology case without the introduction of any quantum gravitational corrections; or when $\alpha \to -1$, which is the case for our braneworld scaling solution considered here. The latter is the case we are interested in.

The second of the criteria above, in the limit $\alpha \to -1$, implies that

$$q_B \approx q_L + \frac{n}{12}(3 - \frac{n}{2})\,. \tag{5.69}$$

Note that despite the forms of $q_B$ and $q_L$ being similar, they differ from one another by orders of magnitude. In LQC, $q_L$ is very large due to the scalar field potential, $V$, being very steep and consequently, the 'slow-roll' parameter, $\bar{\epsilon}_{Loops}$, has a huge value. In the braneworld cosmology, however, $q_B$ is of the order of $\bar{\epsilon}_{Branes}$, which is the slow-roll parameter and is very small due to the nearly flat scalar field potential, $W$, in the region where the scaling solution is a stable attractor. Thus, Eq. (5.69) is not satisfied unless $n$ is very large, and severely fine-tuned. This is not practical, as although there is no upper bound specified for the quantisation parameter, $n$, in LQC, the highest feasible value in the literature is close to 12 [48].

## 5.6 Discussion

It turns out that by imposing the Hubble parameter to be identical for the inverse volume corrected LQC and the braneworld cosmology through applying the constraint equation (5.5), it is not possible to obtain the same behaviour of the scalar field perturbations in the two cosmologies when we consider scaling solutions. The fundamental reason for this is the fast-roll nature of inflation in the semi-classical regime of LQC which is obtained in the limit of the scale factor growth parameter, $p \to 0$. By asking the Hubble parameters in both cosmologies to be identical, we are implicitly carrying this parameter, without any modification, across to the braneworld cosmology. This parameter has been denoted by $\alpha$ in the braneworld cosmology, and in fact, it is easy to show that $\alpha = p$ for this correspondence to hold.

We showed in Chapter 3 that in the semi-classical region of LQC we have a super-inflationary era irrespective of the form of the potential. It is easily shown that the parameter space of $p$ for which super-inflation is obtained is $-1 < p < 0$, and it is in this range of $p$ that one expects scale-invariance. However, we have also shown here that this is outside the allowed range of $\alpha$ in the braneworld cosmology we have investigated. Therefore, the inflationary regimes of the two cosmologies do not map onto one another. It is for this fundamental underlying reason that we do not see a correspondence between the scale invariance criteria of the two models.

We realise that in order to be able to compare theory predictions with observations, one would ideally like to be considering the curvature perturbations. However, this has not been the primary aim of this chapter. Our main purpose has been to compare the LQC and the braneworld cosmologies at the perturbation level; and as discussed in Chapter 3, our current knowledge about the metric perturbations in LQC is incomplete. This is why we concentrated on calculating the scalar field fluctuations in LQC. We then proceeded to focus on the power spectrum of the braneworld scalar field perturbations in order to make a fair comparison between the two models.

Other attempts have also been made in the literature to identify possible correspondences between LQC and braneworld cosmologies. Despite the apparent similarity between the form of LQC modified cosmologies due to holonomy corrections given by Eq. (2.16) and the Shtanov-Sahni model, Eq. (2.31), no formal link has been proposed to relate these models. However, in [111] a thorough investigation has led to obtaining a dual relationship between the holonomy corrected model of LQC and the Randall and Sundrum (R-S) model at the level of background scaling solutions. This is a result that theoretically links the brane tension in the R-S model to the fundamental loop parameter, $\gamma$. This can be done by assuming the energy levels below which the Hubble parameter is not well defined are of similar magnitudes. In [135] an independent line of study has led to a similar result relating the holonomy corrections in LQC to the R-S model of cosmology by considering a deformed Heisenberg algebra.

These maps have been considered for a perfectly isotropic and homogeneous universe, but there is no fundamental reason why this property should be maintained at extremely high energy densities of the very early universe, where these corrections would be significant. It would be interesting to see how introduction of inhomogeneities and anisotropy alters such a result. Investigating the behaviour of perturbations to see if the same map links both models at the level of fluctuations would also shed light on the strength of such a relationship. This task is beyond the scope of our investigation in this chapter for the models we have discussed in the context of holonomy corrected LQC scenario.

# Chapter 6

# Dynamics of a scalar field in Robertson-Walker spacetimes

## 6.1 Introduction

Scalar fields have played a very important role in modern cosmology. Today's observed acceleration of the universe, for example, may be explained by the dynamics of a scalar field (for a review, see [136]). The scenarios proposed to solve the initial conditions of the standard Big Bang theory such as inflation [22, 25, 26], pre-big bang [137, 138, 139], and the ekpyrotic/cyclic [80, 83] scenarios also usually require a scalar field. One of the aspects which makes these scenarios interesting is the existence of attractor solutions which implies that the dynamical system becomes insensitive to the initial conditions. A class of such attractor solutions is referred to as scaling solutions, where the energy density can be divided into contributions which scale with one another throughout the evolution of the universe. Investigating the nature of scaling solutions allows one to understand the asymptotic behaviour of a particular cosmology and helps determine whether such behaviour is stable or just a transient feature.

In a spatially flat Friedmann-Robertson-Walker (FRW) universe filled with a perfect fluid and a canonical scalar field, such scaling solutions are obtained through a simple exponential potential [96]. The attractor behaviour in this system has been analyzed extensively [140, 141, 142, 143, 144]. More recently they have been obtained for a wide class of modified cosmologies proposed by fundamental theories. The cosmologies include those with a non-minimally coupled scalar field [145, 146, 147, 148], braneworld cosmologies [149, 150, 151, 152, 153, 154], loop quantum cosmology, [63, 20, 18], and phantom cosmology [155].

Fundamental theories may introduce modifications to the Friedmann equation in various models that may or may not be of a similar form. Since the stability of the background cosmology is understood to be a desirable property, the pragmatic approach is to accommodate such modifications in the phase space analysis of the standard dynamical system. For the case of a flat FRW universe, the analytical method of obtaining the scaling solutions has been extended to a general set-up, where the Hubble parameter is given either by an arbitrary power [158, 159] or an arbitrary function [110] of the total energy density.

According to the latest results of WMAP5 [17], although limits being placed on the flatness of our observable universe are squeezing the possible outcomes, $-0.063 < \Omega_k < 0.017$, there is still the possibility that we live in a spatially curved FRW universe. It is appropriate, therefore, to investigate the consequences of curvature on the stability of the background cosmology.

In this chapter, using the method developed in [110] in the context of braneworld corrections made to the standard cosmology, we will obtain the scaling solutions in spatially curved FRW universes and classify the asymptotic behaviour of the systems. We aim to present a general approach to analysing such systems by deriving the form of the scalar potential leading to late time attractors. In doing so, we will demonstrate that our results reduce to the known results in the literature once applied to the corresponding models [160, 161]. As was done in [144] for the case of a flat FRW universe, we consider positive and negative potentials of the scalar field, since negative potentials can provide interesting cosmological scenarios both in an expanding [162, 163, 18] and a collapsing universe [80, 83].

The rest of the chapter is arranged as follows. In section (6.2) we present the equations of motion and introduce the variables which allow the scaling solutions to be determined. After analyzing the stability of these solutions in section (6.3), we obtain the general relations which hold when the scaling solutions are obtained in section (6.4). Then, we apply these results to the open and the closed FRW universe in section (6.5) and section (6.6), respectively. Finally, we summarise in section (6.7).

## 6.2 Equations of motion

We consider Friedmann-Robertson-Walker (FRW) cosmologies such that the dynamics is determined by an effective Friedmann equation of the form

$$H^2 = \frac{1}{3}\rho L^2(\rho(a))\,, \tag{6.1}$$

Notice the similarity between this equation and (5.3) presented in the previous chapter in the context of braneworld cosmologies. We introduce the notation $H = \epsilon\sqrt{H^2}$, where $\epsilon = \pm 1$ corresponds to an expanding or a contracting universe, respectively. Modifications to standard relativistic cosmology are parameterized by the function $L(\rho(a))$ and this is assumed to be positive–definite without loss of generality.

We will investigate models where the universe is sourced by a self–interacting scalar field $\phi$ with potential $V(\phi)$ together with a barotropic fluid with equation of state $p_\gamma = (\gamma - 1)\rho_\gamma$, where $\gamma$ is the adiabatic index. The energy density and pressure of the homogeneous scalar field are given by $\rho_\phi = \dot{\phi}^2/2 + V$ and $p_\phi = \dot{\phi}^2/2 - V$, respectively. We note here that the effective adiabatic index of the scalar field is given by $\gamma_\phi = (\rho_\phi + p_\phi)/\rho_\phi$. As in conventional cosmologies, we assume that the energy–momenta of these matter fields are covariantly conserved and this implies that

$$\begin{aligned} \dot{\rho}_\gamma &= -3\gamma H \rho_\gamma, & (6.2)\\ \ddot{\phi} &= -3H\dot{\phi} - V_{,\phi}\,, & (6.3) \end{aligned}$$

Eqs. (6.1)–(6.3) close the system that determines the cosmic dynamics. As in chapter 5, we introduce the variables [149, 150]

$$X \equiv \frac{\dot{\phi}}{\sqrt{2\rho}}\,, \qquad Y \equiv \frac{\sqrt{|V|}}{\sqrt{\rho}}\,, \tag{6.4}$$

where $\rho$ is the total energy density of the universe, and we adopt the notation $V = \alpha|V|$ for $\alpha = \pm 1$ for positive and negative potentials, respectively. Eqs. (6.1)-(6.3) can be rewritten in the form

$$X_{,N} = -3X + \epsilon\alpha\lambda\sqrt{\frac{3}{2}}Y^2 + \frac{3}{2}X[2X^2 + \gamma(1 - X^2 - \alpha Y^2)]\,, \tag{6.5}$$

$$Y_{,N} = -\epsilon\lambda\sqrt{\frac{3}{2}}XY + \frac{3}{2}Y[2X^2 + \gamma(1 - X^2 - \alpha Y^2)]\,, \tag{6.6}$$

$$\lambda_{,N} = -\epsilon\sqrt{6}\lambda^2(\Gamma - 1)X + 3\lambda[2X^2 + \gamma(1 - X^2 - \alpha Y^2)]\,\frac{d\ln L}{d\ln\rho}\,, \tag{6.7}$$

where

$$\lambda \equiv -\frac{1}{L}\frac{V_{,\phi}}{V} \qquad \Gamma \equiv V\frac{V_{,\phi\phi}}{(V_{,\phi})^2}\,. \tag{6.8}$$

For this new set of variables, the definition of the total energy density implies the constraint equation

$$X^2 + \alpha Y^2 + \frac{\rho_\gamma}{\rho} = 1\,. \tag{6.9}$$

From Eq. (6.2) the scale factor, $a$, is expressed in terms of the fluid energy density, $\rho_\gamma$, as

$$a = a_{(i)}\rho_{\gamma\,(i)}^{\frac{1}{3\gamma}}\rho_\gamma^{-\frac{1}{3\gamma}}\,, \tag{6.10}$$

where the quantities with subscript $(i)$ are evaluated at some initial time.

Differentiating $\lambda$, given by Eq. (6.8), with respect to the scalar field, $\phi$, for the special case of $\lambda$ being a constant, one can show that

$$\Gamma = 1 + \frac{d \ln L}{d \ln |V|} \,. \tag{6.11}$$

This is the case we consider for the rest of the chapter.

## 6.3 Stability

In order to carry out the stability analysis for this case, it is sufficient to solve Eqs. (6.5)-(6.6) for variables $X$ and $Y$, since Eq. (6.7) reduces to the constraint equation (6.11). For this system of equations we find the following set of physical fixed points

$$\begin{aligned}
X_c &= \pm 1 & Y_c &= 0\,, & (6.12)\\
X_c &= 0 & Y_c &= 0\,, & (6.13)\\
X_c &= \sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}\,\epsilon & Y_c &= \sqrt{\frac{3(2-\gamma)\gamma}{2\lambda^2}}\,\alpha\,, & (6.14)\\
X_c &= \frac{\lambda\epsilon}{\sqrt{6}} & Y_c &= \sqrt{\left(1-\frac{\lambda^2}{6}\right)}\,\alpha\,. & (6.15)
\end{aligned}$$

In what follows we describe the classification of these solutions in terms of their region of existence and stability for an expanding universe containing a scalar field with a positive potential (i.e. $\epsilon = 1$, and $\alpha = 1$). Later, we will generalise our analysis for an expanding or a contracting universe in which the scalar field could have either a positive or a negative potential. These are summarised in Table 6.1.

The first two points in (6.12) correspond to the scalar field kinetic energy dominated solutions. The third point (6.13) corresponds to the fluid dominated solution. Upon imposing the constraint Eq. (6.9), the fourth point (6.14) is a solution that exists for $\lambda \neq 0$, $0 < \gamma < 2$, and $\lambda^2 > 3\gamma$; and it describes a scenario where, for a given fluid, the contribution of the scalar field density to the total energy density scales with that of the fluid to the total density. i.e. $X_c^2 + \alpha Y_c^2 = \frac{3\gamma}{\lambda^2}$. For convenience, we refer to this solution as the fluid-scalar field scaling solution throughout this chapter. The final point (6.15) arises if $\lambda^2 < 6$, and the constraint Eq. (6.9) implies that, in this case, the energy density of the universe is dominated by the scalar field, having an effective adiabatic index $\gamma_\phi = \frac{\lambda^2}{3}$. This describes a scaling solution, where as the universe evolves, the kinetic energy and the potential energy of the scalar field scale together. We refer to this solution as the scalar field dominated scaling solution throughout this chapter.

Having obtained the scaling solutions, we need to investigate their stability to small fluctuations. Considering perturbations of the form

$$X = X_c + \delta X\,, \qquad Y = Y_c + \delta Y\,, \tag{6.16}$$

where $\delta X \propto e^{wN}$, and $\delta Y \propto e^{wN}$, and expanding Eqs. (6.5)-(6.6), for the kinetic energy dominated solutions, following the method outlined in Chapter 2, the eigenvalues of the $\mathcal{M}$-matrix

$$\mathcal{M} = \begin{pmatrix} -3 - \sqrt{6}\epsilon x\lambda + 9\alpha x^2 + \frac{3}{2}\gamma\frac{\rho_\gamma}{\rho} & \sqrt{\frac{3}{2}}(1 - x^2 - \frac{\rho_\gamma}{\rho}) \\ -\sqrt{6}\epsilon(\Gamma - 1)\lambda^2 + 12\lambda X\frac{d\ln L}{d\ln\rho} & -2\sqrt{6}\epsilon x(\Gamma - 1)\lambda + 3\left(2X^2 + \gamma\frac{\rho_\gamma}{\rho}\right)\frac{d\ln L}{d\ln\rho} \end{pmatrix} \tag{6.17}$$

will be

$$w_+ = 3(2-\gamma) \qquad w_- = 3 \mp \sqrt{\frac{3}{2}}\lambda\epsilon\,, \tag{6.18}$$

respectively. In this case, stability is only achieved for fluids with adiabatic index of $\gamma > 2$, which is not satisfied for known realistic fluids. These points, are therefore considered to be unstable. For the fluid dominated scenario, the eigenvalues are given by

$$w_+ = \frac{3\gamma}{2} \qquad w_- = -3 + \frac{3\gamma}{2}\,, \tag{6.19}$$

which is clearly only stable for negative values of the adiabatic index, $\gamma$. We will therefore consider this solution to be unstable for all realistic types of fluid.

The fluid-scalar field scaling solution yields the eigenvalues (recall $0 < \gamma < 2$, and $\lambda^2 > 3\gamma$)

$$w_{\pm} = \frac{3}{4}(\gamma - 2)\left(1 \pm \sqrt{1 - \frac{8\gamma(3\gamma - \lambda^2)}{\lambda^2(\gamma - 2)}}\right), \tag{6.20}$$

which implies unconditional stability when these solutions exist. The same analysis yields the following eigenvalues for the scalar field dominated scaling solutions

$$w_{+} = \frac{1}{2}(\lambda^2 - 6) \qquad w_{-} = \lambda^2 - 3\gamma\,, \tag{6.21}$$

which indicates that when these solutions exist, as long as $\lambda^2 < 3\gamma$, stability is guaranteed. We also note that in a contracting universe ($\epsilon = -1$), an attractor solution correpsonds to one with *positive* real eigenvalues. This is because the parameter with respect to which the dynamical system of (6.5)-(6.6) is described, $N$, becomes a decreasing function of time. Keeping this in mind, the general consideration of various combinations of an expanding/contracting universe containing a scalar field with a positive/negative potential are captured in Table 6.1 below. For the case of $L = 1$ (i.e. a flat FRW universe), a similar classification was done for simple exponential potentials in [144].

| | | $X = 1$, $Y = 0$ | | $X = -1$, $Y = 0$ | | $X = 0$, $Y = 0$ | | $X = \sqrt{\frac{3}{2}}\frac{\gamma}{\lambda}\epsilon$, $Y = \sqrt{\frac{3(2-\gamma)\gamma}{2\lambda^2}\alpha}$ | | $X = \frac{\lambda\epsilon}{\sqrt{6}}$, $Y = \sqrt{\left(1 - \frac{\lambda^2}{6}\right)\alpha}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\epsilon$ | $\alpha$ | exists | stable | exists | stable | exists | stable | exists | stable | exists | stable |
| 1 | 1 | $\forall\lambda,\gamma$ | No | $\forall\lambda,\gamma$ | No | $\forall\lambda,\gamma$ | No | $\lambda \neq 0,\ \lambda^2 > 3\gamma$ | when exists | $\lambda^2 < 6$ | $\lambda^2 < 3\gamma$ |
| 1 | -1 | $\forall\lambda,\gamma$ | No | $\forall\lambda,\gamma$ | No | $\forall\lambda,\gamma$ | No | No | - | $\lambda^2 > 6$ | No |
| -1 | 1 | $\forall\lambda,\gamma$ | $0 > \lambda > -\sqrt{6}$ | $\forall\lambda,\gamma$ | $0 < \lambda < \sqrt{6}$ | $\forall\lambda,\gamma$ | No | $\lambda \neq 0,\ \lambda^2 > 3\gamma$ | No | $\lambda^2 < 6$ | No |
| -1 | -1 | $\forall\lambda,\gamma$ | $0 > \lambda > -\sqrt{6}$ | $\forall\lambda,\gamma$ | $0 < \lambda < \sqrt{6}$ | $\forall\lambda,\gamma$ | No | No | - | $\lambda^2 > 6$ | when exists |

Table 6.1: This table summarises the existence and stability conditions for an expanding ($\epsilon = 1$) or a contracting ($\epsilon = -1$) universe containing a fluid with the adiabatic index $\gamma$ and a scalar field with either a positive ($\alpha = 1$) or a negative ($\alpha = -1$) potential.

## 6.4 General relations for Scaling Solutions

In the remainder of this chapter we concentrate on the solutions given by Eqs. (6.14)-(6.15). We aim to derive the scalar field potentials which correspond to these late time attractors for particular forms of $L(\rho)$ based on spatially curved cosmologies. Working at the fixed points, from $Y_c = \sqrt{\frac{|V|}{\rho}}$, and the fact that $\rho_\phi/\rho = X_c^2 + \alpha Y_c^2 =$ constant for these solutions, $L(\rho)$ can be described as $L(V)$. Moreover, from the definition of $\lambda$, one can show that

$$\int \frac{dV}{VL} = -\lambda\phi\,. \tag{6.22}$$

Given a specific form of $L$, therefore, it is possible to derive the scalar potential resulting in the scaling solution by integrating Eq. (6.22). We note that, as demonstrated in [110], Eq. (6.22) is equivalent to imposing the constraint Eq. (6.11) throughout the evolution of the field.

Furthermore, since at the fixed points, $X = X_c$, $\phi$ is a monotonically varying function of cosmic time, $t$, and can be considered as a suitable dynamical variable for the system. We note here that this assumption is only valid when $X_c \neq 0$. In general, the scalar field Eq. (6.3) can be expressed in the form

$$\dot{\rho}_\phi = -3H\dot{\phi}^2\,, \tag{6.23}$$

and for the scaling solutions, this equation can be expressed as

$$\dot{\phi} = -\alpha\epsilon\frac{1}{Y_c\sqrt{3}}\left(\frac{\rho_\phi}{\rho}\right)\frac{1}{L(V)}\frac{V_{,\phi}}{\sqrt{|V|}}\,, \tag{6.24}$$

but since $V$ and $\phi$ are related through Eq. (6.22), this can be integrated to find $t$ as a function of the scalar field

$$t = -\alpha\epsilon Y_c\sqrt{3}\left(\frac{\rho}{\rho_\phi}\right)\int^\phi d\phi L(V)\frac{\sqrt{|V|}}{V_{,\phi}}\,. \tag{6.25}$$

## 6.5 Open FRW universe

Here, we consider what form of the scalar field potential provides the fixed point solution characterised by $(X, Y) = (X_c, Y_c)$ in the open FRW universe. In this case, $L(\rho)$ is given by

$$L(\rho) = \sqrt{1 + \frac{3}{\rho a^2}}\,. \tag{6.26}$$

In our analysis, we assume that $\lambda \neq 0$, but we return to this point and consider the special case of $\lambda = 0$ at the end of this section. We note at this point that an expanding (contracting) open universe obeying the Friedmann equation (6.1) could stop its expansion (contraction) process and begin to contract (expand) if the scalar field has a negative (positive) potential. It is, therefore, interesting to study the physics obtained from different combinations of $\epsilon$ and $\alpha$ in an open universe. We begin our discussion by concentrating first on an expanding ($\epsilon = 1$) universe sourced by a positive potential ($\alpha = 1$) scalar field.

### 6.5.1 Case A: Fluid-scalar field scaling solutions

When the universe is expanding ($\epsilon = 1$), the solution given by (6.14) exists and is stable for $\lambda^2 > 3\gamma$, and the scale factor $a$ can be expressed in this case as

$$a = A\rho_\gamma^{-\frac{1}{3\gamma}}\,, \quad \text{with} \;\; A = a_{(i)}\rho_{\gamma(i)}^{\frac{1}{3\gamma}}\left(\frac{\lambda^2}{\lambda^2 - 3\gamma}\right)^{\frac{1}{3\gamma}}\,. \tag{6.27}$$

The correction function $L$ given by Eq. (6.26) can then be rewritten as

$$L(\rho) = \sqrt{1 + \frac{3}{A^2}\rho^{\frac{2-3\gamma}{3\gamma}}}\,. \tag{6.28}$$

By considering the fixed point, substituting this into Eq. (6.22) and integrating, yields the scaling solution potential

$$|V(\phi)|^{\mu} = Y_c^{2\mu}\left(\frac{A^2}{3}\right)\mathrm{cosech}^2\left(\frac{\lambda}{2}\mu\phi\right) , \tag{6.29}$$

where $\mu = (2-3\gamma)/3\gamma$. Notice that the valid range of $\mu$, for the region of existence of these solutions, is $-2/3 < \mu < 0$ and $0 < \mu < \infty$. The special case of $\mu = 0$, where Eq. (6.29) is no longer valid, is discussed later.

Using the definition of $Y$ at the fixed point, and substituting Eq. (6.29) into Eq. (6.28), the correction function can be written in terms of the scalar field as $L(\phi)$. Upon substituting this form back in Eq. (6.25) the time dependence of the scalar field can be evaluated as

$$t = \epsilon\frac{\sqrt{3}\lambda}{3\gamma}\left(\frac{3}{A^2}\right)^{\frac{1}{2\mu}}\int^{\phi} d\phi\,\sinh^{1/\mu}\left(\frac{\lambda\mu}{2}\phi\right) . \tag{6.30}$$

Classifying the behaviour of the evolution equations above in terms of the sign of the parameter $\mu$, and recognising the sign of the argument inside the brackets remains invariant under the transformation $\lambda \to -\lambda$ and $\phi \to -\phi$, we choose to work in the first quadrant without loss of generality.

For $\mu > 0$, we find that at early times, as $\phi \to 0$, the asymptotic form of the potential relating to the curvature dominated universe is a power law function $V \propto \phi^{-\frac{2}{\mu}}$. However, at late times, as $\phi \to \infty$, the potential of a fluid-scalar field dominated universe is of an exponential form $V \sim \exp[-\lambda\phi]$.

On the other hand, for $\mu < 0$, at early times, as $\phi \to \infty$, where the curvature is negligible, the asymptotic form of the potential is obtained to be an exponential one $V \sim \exp[-\lambda\phi]$. Once the universe becomes dominated by the curvature at late times, as $\phi \to 0$, the potential can be approximated by a power law function $V \propto \phi^{-\frac{2}{\mu}}$, recovering the standard result $a \sim t$.

For $\mu = 0$, since the correction function $L$ is a constant and can be thought of as modification to Newton's gravitational constant in Eq. (6.1), the expansion law becomes that of the flat Friedmann cosmology, and the potential yielding this late time attractor will then be the exponential potential as found in [96]. We further note that this scenario refers to the special case where the contribution of the scalar field, the fluid, and the curvature to the Friedmann equation scale together.

We note, according to the results illustrated in Table 6.1, that the fixed points (6.14) do not exist for negative potentials. This is true for an expanding or a contracting universe. Although in an expanding (contracting ) universe, a negative (positive) scalar potential could slow down the growth of the scale factor and cause the universe to collapse (expand), we do not expect the late time attractors to be given by these scaling solutions. We can also see from Table 6.1 that this set of solutions does not exist in a contracting universe sourced by a negative scalar field potential.

### 6.5.2 Case B: Scalar field dominated scaling solution

When the universe is expanding ($\epsilon = 1$) the solution given by (6.15), is an attractor for $\lambda^2 < 3\gamma$, and the scalar field dominates over the fluid, resulting in the effective adiabatic index $\gamma_\phi = \frac{\lambda^2}{3}$. The scale factor is then given by

$$a = B\rho^{-\frac{1}{\lambda^2}}\,, \quad \text{with} \;\; B = a_{(i)}\rho_{(i)}^{\frac{1}{\lambda^2}}\,. \tag{6.31}$$

The form of the correction function $L$ given by Eq. (6.26) can then be rewritten as

$$L(\rho) = \sqrt{1 + \frac{3}{B^2}\rho^{\frac{2-\lambda^2}{\lambda^2}}}\,. \tag{6.32}$$

By considering the attractor solution, substituting this form of the correction function into Eq. (6.22), and integrating, yeilds the scaling solution potential

$$|V(\phi)|^\nu = Y_c^{2\nu}\left(\frac{B^2}{3}\right)\text{cosech}^2\left(\frac{\lambda}{2}\nu\phi\right)\,, \tag{6.33}$$

where $\nu = (2-\lambda^2)/\lambda^2$, and the form of the potential is valid for $\lambda^2 \neq 2$ .

By using the definition of $Y$ at the fixed point, and substituting Eq. (6.33) into Eq. (6.32) the correction function can be written in terms of the scalar field as $L(\phi)$. From this form the time dependence of the scalar field can be evaluated using Eq.(6.25) to be

$$t = \epsilon \frac{\sqrt{3}}{\lambda} \left( \frac{3}{B^2} \right)^{\frac{1}{2\nu}} \int^{\phi} d\phi \sinh^{1/\nu} \left( \frac{\lambda \nu}{2} \phi \right) , \tag{6.34}$$

By analogy, the asymptotic behaviour of the universe in this scenario can be obtained using the same method as employed in *Case A*. These results are summarised in Table 6.2.

| | *Case A* | | | *Case B* | | |
|---|---|---|---|---|---|---|
| | $\mu < 0$ | $\mu = 0$ | $\mu > 0$ | $\nu < 0$ | $\nu = 0$ | $\nu > 0$ |
| Early times | $V \sim \exp[-\lambda\phi]$ negligible curvature | $V \sim \exp[-2\phi]$ scaling curvature | $V \sim \phi^{-\frac{2}{\mu}}$ curvature dominated | $V \sim \exp[-\lambda\phi]$ negligible curvature | $V \sim \exp[-2\phi]$ scaling curvature | $V \sim \phi^{-\frac{2}{\nu}}$ curvature dominated |
| Late times | $V \sim \phi^{-\frac{2}{\mu}}$ curvature dominated | $V \sim \exp[-2\phi]$ scaling curvature | $V \sim \exp[-\lambda\phi]$ negligible curvature | $V \sim \phi^{-\frac{2}{\nu}}$ curvature dominated | $V \sim \exp[-2\phi]$ scaling curvature | $V \sim \exp[-\lambda\phi]$ negligible curvature |

Table 6.2: This table summarises the asymptotic behaviour of an expanding ($\epsilon = 1$) open FRW universe described by the scaling solutions, when the scalar potential is positive. *Case A* refers to the fluid-scalar field scaling solution, and *Case B* corresponds to the scalar field dominated scaling solution.

We now turn to our general set of results in Table 6.1 and note that in a contracting universe ($\epsilon = -1$), if the scalar potential is negative ($\alpha = -1$) and steep enough ($\lambda^2 > 6$), the scalar field dominated scaling solution can also result in stable attractor behaviour. Such solutions will be unconditionally stable if they exist, and we find the form of the potential together with the time evolution of the scalar field are still given by Eqs. (6.33) and (6.34). In this scenario, $\nu$ is clearly negative, and due to the time reversal of Eq. (6.34) for $\epsilon = -1$, one expects from Table 6.2, an early time power law behaviour of the potential to be followed by an exponential form at late times. This is the generalisation of [144] to general curved space scenarios.

### 6.5.3 Case C: $\lambda \approx 0$

We now return to the special case of $\lambda = 0$, and notice that this corresponds to a constant potential (i.e. de-Sitter space). Solutions for this type of universe are known [164], and it is therefore interesting to see how they would fit into the larger class of solutions we are presenting here. Considering the scaling solutions, we notice that the fixed points described by (6.14) do not exist for the class of constant potentials. However, the attractor solutions (6.15) reduce to $X_c = 0$, and $Y_c = 1$, which describe an exact de-Sitter solution. As mentioned previously, the argument of using $\phi$ as a monotonically varying function of time breaks down when $X_c \propto \dot{\phi} = 0$. For this simplified case, one can clearly solve Eqs. (6.1)-(6.3) directly to find the exact solution. We notice, however, that we should also be able to recover this solution by investigating the behaviour of the scalar potential and the scale factor very close to the fixed point. We do this by considering small values of $\lambda$. From Eq. (6.34), we find the asymptotic dependence of the scalar field to be given by $t \propto \phi$. Substituting this into Eq. (6.33), and using Eq. (6.10) near the fixed point, we find that the scale factor evolves as $a \propto \sinh\left(\frac{\lambda}{2}\nu t\right)$. Thus, recovering the de-Sitter solution as a special case of our larger set of solutions derived here.

We further note from Table 6.1 that when $\lambda \approx 0$, a negative potential scaling solution does not exist in an expanding or a contracting universe; and the positive potential solution, which exists in a contracting universe, is unstable.

## 6.6 Closed FRW universe

Here, we aim to derive the forms of the scalar potentials which would result in stable fixed points (6.14)-(6.15), corresponding to the scaling solutions in a closed FRW universe. In this case, $L(\rho(a))$ is given by

$$L(\rho) = \sqrt{1 - \frac{3}{\rho a^2}}\,. \tag{6.35}$$

In this scenario, the curvature can not dominate the contributions to the total energy density of the universe, otherwise the correction function (6.35) would be imaginary and the right hand side of the Friedmann equation would become negative. The valid range of the correction function is thus given by $0 \leq L \leq 1$. The case of $L = 0$ corresponds to a universe with a constant scale factor, and describes the case where there is no expansion or contraction taking place. The universe may undergo a bounce at this point and change from an expanding (contracting) behaviour to a contracting (expanding) one. We bear this in mind in our following analysis. As in the open case, we first concentrate on the expanding universe ($\epsilon = 1$) with positive potential ($\alpha = 1$) scalar field, and we will comment on the negative potentials and the difference between an expanding and a contracting universe for each set of scaling solutions.

### 6.6.1 Case A: Fluid-scalar field scaling solution

In an expanding universe ($\epsilon = 1$), the solution given by (6.14) is an attractor for $\lambda^2 > 3\gamma$, and describes a scenario where the contribution of the scalar field scales with that of the fluid. The scale factor in this case is also given by Eq. (6.27), and after substituting this into Eq. (6.35), the form of the correction function, $L$, is found to be

$$L(\rho) = \sqrt{1 - \frac{3}{A^2}\rho^{\frac{2-3\gamma}{3\gamma}}}. \tag{6.36}$$

Notice there is a maximum level of density, that depends on $A$, beyond which $L$ lies outside its valid range and the situation is unphysical. Following our previous analysis, except for the case $\mu = 0$, where $\mu = (2 - 3\gamma)/3\gamma$, the potential yielding the fixed point solution is given by

$$|V(\phi)|^\mu = Y_c^{2\mu} \left(\frac{A^2}{3}\right) \mathrm{sech}^2 \left(\frac{\lambda}{2}\mu\phi\right) . \tag{6.37}$$

Using $Y_c = \sqrt{|V|/\rho}$, and substituting Eq. (6.37) into Eq. (6.36), the correction function can be written as $L(\phi) = \tanh\left(\frac{\lambda}{2}\mu\phi\right)$. And substituting this form of $L$ back into Eq. (6.25), yields the time dependence of the scalar field as

$$t = \epsilon \frac{\sqrt{3}\lambda}{3\gamma} \left( \frac{3}{A^2} \right)^{\frac{1}{2\mu}} \int^{\phi} d\phi \cosh^{1/\mu} \left( \frac{\lambda\mu}{2} \phi \right) . \qquad (6.38)$$

In order to examine the asymptotic behaviour of the scalar field potential, we follow our previous line of argument and divide up the region of validity of $\mu$ into its positive and negative values ( i.e. $-2/3 < \mu < 0$, and $0 < \mu < \infty$). Once again, making use of the symmetry $\lambda \to -\lambda$ and $\phi \to -\phi$, we restrict our analysis to the potentials lying in the first quadrant, without loss of generality.

For $\mu > 0$ (i.e. $3\gamma < 2$), in an expanding universe at early times, since $a^{-2} > a^{-3\gamma}$, one would expect the curvature to dominate initially. However, as mentioned above, this can not happen in a closed universe, otherwise the correction function (6.36) becomes imaginary and consequently, the right hand side of the Friedmann Eq. (6.1) will be negative. A pragmatic starting point for the expanding evolution is, therefore, where the curvature contribution is only just subdominant to the scalar field and the fluid densities. We set this point to correspond to $t \approx 0$, since going back in time from this point, in the absence of motivations from a fundamental theory, is not physically meaningful. This is when the total energy density approaches its maximum value $\rho \to \rho_{max} \equiv (A^2/3)^{1/\mu}$ and so, the correction function $L(\phi) \to 0$. Eq. (6.1) then suggests that the scale factor does not change as $H \to 0$. Once the universe starts expanding, the contribution of the fluid and the scalar field to the energy density starts to dominate that of the curvature. In the limit where the curvature has become negligible and the universe asymptotes to a flat FRW spacetime, we recover the exponential limit of the potential $V \sim \exp[-\lambda\phi]$.

For $\mu < 0$, as the universe expands, it starts from an asymptotically flat FRW state dominated by the fluid and the scalar field. The potential in this case is that of the exponential form $V \sim \exp[-\lambda\phi]$, and the curvature can be neglected. In time, we reach the maximum level of energy density, where the contribution of the curvature becomes very close to that of the combination of the scalar field and the fluid, and the scale factor seizes to grow as $H \to 0$. After this turning point, the universe begins to collapse, and this provides us with a natural motivation to consider a contracting universe. However, one can see from Table 6.1, that in a contracting universe, late time fluid-scalar field scaling attractor solutions either do not exist at all (for $\alpha = -1$) or are unconditionally unstable when they do exist (for $\alpha = 1$). Once the universe starts to collapse, solutions will asymptote towards the kinetic dominated solutions for $\lambda^2 < 6$. On the other hand, for $\lambda^2 > 6$, solutions will asymptote towards the fluid dominated scaling solutions if the potential is negative ($\alpha = -1$), and will be unstable for positive potentials.

It is worth noting that in the limit of $H \to 0$, as a result of $L \to 0$, the total energy density is a constant close to its maximum value. From Eqs. (6.4) and (6.37), we conclude that in this limit the potential is almost constant $V \approx Y_c^2 \left(\frac{A^2}{3}\right)^{1/\mu}$. One may then naively expect the universe to expand exponentially as it would do in a de-sitter case; however, we note that the correction factor stops this behaviour by keeping the scale factor constant in Eq. (6.1).

The special case of $\mu = 0$ corresponds to where the contribution of the curvature, the fluid, and the scalar field are all scaling at all times during the evolution. We note that this is only valid if the curvature does not dominate over the combination of the scalar field and the fluid, in which case, the correction function of Eq. (6.36) is a constant and the potential has the exponential form.

Similarly to the open universe scenario described earlier, we can see from Table 6.1 that for negative scalar potentials ($\alpha = -1$) this set of scaling solutions do not exist in either an expanding universe or a contracting one.

### 6.6.2 Case B: Scalar field dominated scaling solution

In an expanding universe ($\epsilon = 1$) when the potential of the scalar field is positive ($\alpha = 1$), the fixed points given by (6.14) are attractor solutions for $\lambda^2 < 3\gamma$, and the scale factor evolves according to Eq. (6.31). Therefore, the form of the correction function given by Eq. (6.35) becomes

$$L(\rho) = \sqrt{1 - \frac{3}{B^2}\rho^{\frac{2-\lambda^2}{\lambda^2}}}. \tag{6.39}$$

We remember that in this case, the contribution of the fluid is negligible for these solutions at all times during the evolution of the universe.

Similarly to our previous analysis, we find the corresponding scalar potential, except for the case of $\nu = 0$, to be given by

$$|V(\phi)|^{\nu} = Y_c^{2\nu}\left(\frac{B^2}{3}\right)\mathrm{sech}^2\left(\frac{\lambda}{2}\nu\phi\right), \tag{6.40}$$

and the time dependence of the scalar field is

$$t = \epsilon\,\frac{1}{\sqrt{3}\lambda}\left(\frac{3}{B^2}\right)^{\frac{1}{2\nu}}\int^{\phi} d\phi\cosh^{1/\nu}\left(\frac{\lambda\nu}{2}\phi\right). \tag{6.41}$$

By analogy, the asymptotic behaviour of an expanding universe can be obtained through a similar discussion as the previous case. These results are summarised in Table 6.3.

| | *Case A* | | | *Case B* | | |
|---|---|---|---|---|---|---|
| | $\mu < 0$ | $\mu = 0$ | $\mu > 0$ | $\nu < 0$ | $\nu = 0$ | $\nu > 0$ |
| Early times | $V \sim \exp[-\lambda\phi]$ negligible curvature | $V \sim \exp[-2\phi]$ scaling curvature | $V \approx Y_c^2\left(\frac{A^2}{3}\right)^{1/\mu}$ curvature only just subdominant | $V \sim \exp[-\lambda\phi]$ negligible curvature | $V \sim \exp[-2\phi]$ scaling curvature | $V \approx Y_c^2\left(\frac{B^2}{3}\right)^{1/\nu}$ curvature only just subdominant |
| Late times | $V \approx Y_c^2\left(\frac{A^2}{3}\right)^{1/\mu}$ curvature only just subdominant | $V \sim \exp[-2\phi]$ scaling curvature | $V \sim \exp[-\lambda\phi]$ negligible curvature | $V \approx Y_c^2\left(\frac{B^2}{3}\right)^{1/\nu}$ curvature only just subdominant | $V \sim \exp[-2\phi]$ scaling curvature | $V \sim \exp[-\lambda\phi]$ negligible curvature |

Table 6.3: This table summarises the asymptotic behaviour of an expanding ($\epsilon = 1$) closed FRW universe described by the scaling solutions, when the scalar potential is positive. *Case A* refers to the fluid-scalar field scaling solution, and *Case B* corresponds to the scalar field dominated scaling solution.

It is worth noting at this stage that according to the results demonstrated in Table 6.1, an expanding (contracting) universe with a negative (positive) scalar potential does not admit attractor solutions. However, for a contracting universe with a negative potential, if such solutions exist they will be unconditionally stable, and the form of the potential together with the time evolution of the scalar field can be seen to be given by Eqs. (6.40) and (6.41).

### 6.6.3 Case C: $\lambda \approx 0$

As mentioned in the open universe scenario, this case yields an ill-defined solution corresponding to the fixed points described by (6.14). However, $\lambda = 0$ results in a perfectly well-defined scaling solution when substituted in the fixed points given by (6.15). The scaling solution then reduces to $X_c = 0$, and $Y_c = 1$, which is clearly a de-Sitter solution. Once again, the assumption of $\phi$ being a monotonically varying function of time breaks down at $X_c = 0$. Thus, we take the pragmatic approach of investigating the form of the potential and the behaviour of the scale factor near the scaling solution by considering small values of $\lambda$. We then find the potential to be given by $V \propto \cosh^{-2/\nu}\left(\frac{\lambda}{2}\nu\phi\right)$, and the time dependence of the scalar field to be $t \propto \phi$, which from Eq. (6.31), yields the scale factor as $a \propto \cosh\left(\frac{\lambda^2}{2}\nu t\right)$. This is the form of the scale factor evolution in a closed de-sitter universe, and is a special subset of the solutions we have derived here.

In an analogous way to our description for an open scenario, the fixed points (6.15) do not exist in an anti-de-Sitter universe; and even though they exist in a contracting universe sourced by a scalar field with a positive potential, such solutions are unconditionally unstable.

## 6.7 Discussion

In order to understand the asymptotic behaviour of a particular cosmology and determine whether such a background is stable or not, a class of attractor solutions referred to as the scaling solutions play an important role. By now, even though many scaling solutions have been obtained in various cosmologies, they are limited to the case of a flat Friedmann-Robertson-Walker (FRW) universe. It is worth noting that an interesting study has also been done in investigating super-inflation in closed cosmologies [165]. In this chapter, we have analyzed the dynamics of a single scalar field in FRW universes with spatial curvature. We started by generalising the approach developed in [110] to incorporate expanding and contracting universes filled with a perfect fluid and a single scalar field. Due to a growing interest in negative scalar potentials and the possibility of obtaining them from fundamental theories, these are also accommodated in our formalism.

We have identified two sets of scaling solutions. One, where the contribution of the fluid to the total energy density scales with that of the scalar field throughout the evolution, and the other, where the scalar field dominates over the fluid, and the kinetic energy of the field scales with its potential energy. By concentrating on both types of scaling solutions, we obtained the generalised dynamical equations. Once a particular form of the modification function is given and its dependence on the total energy density is known, these equations can be used to derive the form of the scalar potential that leads to the late time attractor solutions. This analysis is explicitly carried out for the cases of an open and a closed universe.

After presenting the general form of the potential, we examined the asymptotic behaviour of the dynamics. In an open universe sourced by a positive scalar potential, we concluded that in regions where the curvature is dominant, the potential can be approximated by a power law function; and where it is negligible, an exponential form is a good approximation. This result is consistent with [161], where the exponential form of the potential was found to be unstable when the curvature term becomes important. We have also shown how the well known de-Sitter solutions can be found as certain limits of our general potential. We then highlighted the case of a collapsing open universe (which could happen in the presence of a negative potential [163]) and concluded that, if the potential is steep enough, the scalar field dominated scaling solution is a late time attractor. This is in line with the known results in the context of an ekpyrotic collapse in the flat FRW universe [80, 83].

The general form of the scalar potential is also presented for a closed universe. Here we are facing an interesting scenario and care needs to be taken since the curvature is forbidden to dominate over the combination of fluid and the scalar field in an expanding universe when the potential is positive. In the limit where the curvature is comparable to this combination, the potential is a constant and so is the scale factor. When the curvature becomes subdominant, the potential asymptotes to an exponential form. In a contracting closed universe we find that a steep negative potential provides the late time attractor in the form of a scalar field dominated solution. Negative exponential potentials have been known to be stable solutions in ekpyrotic models, and this is a special limit of our derived potential.

In this chapter we have concentrated on the scaling solutions, but it is also worth noting that in a contracting universe, if the potential is flat enough ($\lambda^2 < 6$), the kinetic dominated solutions are shown to be late time attractors for both the positive and the negative potential scalar fields. This solution corresponds to the ones obtained in the pre-big bang cosmology [137, 138, 139].

Although the method proposed in [110] had been motivated by generalising a large class of modifications introduced to the Friedmann equation in the context of braneworld cosmologies in flat space, we have demonstrated here that a spatial curvature term can also be written in a similar fashion. We recognise that at the background level it is not possible to distinguish between corrections originating from fundamental theories that can be encoded within the correction function, $L(\rho)$, and information about the spatial curvature which could be represented through a similar modification function. In order to differentiate between the two, one needs to examine signatures such as the behaviour of perturbations in the universe. This analysis is beyond the scope of our work in this chapter. Finally, in this chapter, we have limited our analysis for the canonical scalar fields. We do not envisage the obvious extension to the non-standard scalar fields like non-minimally coupled fields [145], or phantom fields [155] to be complicated. It may be interesting to consider the dynamics of such scalar fields in curved FRW universes.

The method we have presented here has become the standard means of testing the stability of particular potentials or, as we have demonstrated, deriving the forms of the scalar potentials for stability to be maintained throughout the evolution of the universe. An important aspect to note is that when the forms of the potentials are derived in this way, they will only be valid *at* the fixed points of the dynamical system governing the evolutionary behaviour. One may assume that the scalar potential will have a similar form infinitesimally away from these points, but very little can be concluded from this formalism about the shape of the potential further away from the stationary points. We can, therefore, say little about the most general forms of potentials leading to the stable solutions. What one derives using this method is the asymptotic forms once the solutions have been reached. Although we have claimed to have derived the most general forms of potentials in FRW spacetimes here, and we believe this to be the most one can do in this formalism, the caveat above should be kept in mind. We recognise our results being more general than what has been recorded in the literature, but we need to be careful not to exaggerate the power of this analysis. We further note that it is quite possible to find classes of more general potentials which describe a more complete picture of the dynamics through more sophisticated and complex means, but we expect these potentials to asymptote towards what we have derived in this chapter as the fixed points are reached.

# Chapter 7

# Summary and Conclusions

In this thesis we have worked in the semi-classical regime of LQC for the two sets of identified quantum corrections, namely, the inverse volume and the holonomy quantum effects, present at the level of effective equations of motion. We have demonstrated the existence of a super-inflationary phase in both frameworks soon after the universe starts expanding. Considering a scaling solution for a universe sourced by a scalar field, where the energy density is dominated by that of the field, we derived background cosmologies with power-law scale factors. We explicitly showed that only a small number of e-foldings is required during a very slow growth of the universe in order to resolve the horizon problem. This is a significant result, as it had been previously argued and accepted that the inflationary scenarios in the context of LQC can not be fully responsible for resolving the cosmological problems (e.g. in [104, 166]).

Having obtained the forms of the potentials that led to this type of evolution, we examined the properties of fluctuations of the scalar field in LQC and presented the forms of the spectral tilt of the power spectrum in terms of the derivatives of these potentials. Both sets of LQC corrections are seen to be able to provide us with an almost scale invariant spectrum of these perturbations and this is in agreement with our most recent observations.

Our work on super-inflationary scenarios in LQC also has a number of possible drawbacks. We have treated the inverse triad and holonomy corrections separately while they should be dealt with together in a realistic set up. It remains to be seen how these corrections relate to one another and how (or if) their regions of validity overlap on small scales. Their relative importance for small values of the scale factor is also unknown. Though the fact that both sets of corrections lead to such similar phenomenology gives us some reassurance that combining them could lead to qualitatively similar results. Furthermore, so far we have not investigated the evolution of scalar metric perturbations. The derivation of the full equations for these is still in progress [119, 167, 70] and although these equations are not required for the calculation of tensor perturbations presented here, they are required to understand how the scale-invariant scalar field perturbation is related to the observed curvature perturbation. We should also mention recent work where it has been shown that the behaviour of the LQC equations with holonomy corrections in the presence of negative exponential scalar field potentials leads to sudden singularities where the Hubble rate is bounded, but the Ricci curvature scalar diverges [168]. Given that for the case of holonomy corrections our super-inflation scenario requires a scalar field potential with a negative exponential part, this serious problem needs to be avoided. In the scenarios we consider, however, the potentials only need to be of the form which gives rise to the power-law behaviour while super-inflation is taking place. After this phase of evolution the potential can change in form. For example, any potential which tended to zero after the field evolved through the region which gives rise to the super-inflation phase would avoid the sudden singularity problem.

We also emphasise that throughout this thesis we have only considered the expanding phase of LQC and have largely ignored any effects feeding through from the collapsing branch. A valid question would be to ask whether a bounce, predicted by LQC, is symmetric or not. There are currently no observational or theoretical bounds on how long a complete phase of a cycle would last in either of the LQC corrections. We have stressed that our scaling solutions are only stable attractors in an expanding universe, and so we limit ourselves to this branch of LQC.

Bearing in mind these assumptions, we turned to the timely question of what spectral index and amplitude for the tensor perturbations would be predicted by LQC. We found that for the scaling solutions we have been considering, the recently calculated tensor contributions to metric perturbations [69] predict a very large blue tilt for both sets of LQC corrections. One thing to note is that for the case of the holonomy corrections, we obtain a mode equation in which a term proportional to the scale factor appears in addition to the conventional $k^2$ and the term of the order of $\tau^{-2}$. Since during the super-inflationary era, the scale factor only grows slightly compared to the length of time it takes for this phase to end, we consider the extra term to be a constant by which the wave number $k$ is shifted. In this way, standard perturbation calculations can be done on very small scales ($k \to \infty$) and the corresponding spectral tilt can be computed, but it becomes clear that for modes close to the shift value the spectral tilt picks up lower order corrections in powers of $k$. This means that our result for the power spectrum is only approximately valid for these modes. A further feature in our formulism is that the value of this shift is negative and this has the effect of setting a minimum limit on $k$ for which our tensor mode perturbation analysis is valid. We have made the assumption that the modes re-entering the horizon today are greater than this value. It is not possible to make predictions for modes below this minimum value as the mode equation will not be physically meaningful. The large blue tilt of the gravitational wave spectral index indicates that the amplitudes of these modes are suppressed on the CMB by many orders of magnitude compared to predictions of the standard inflation.

This may seem like a definite way to distinguish between the inflationary regime of standard cosmology and super-inflation in LQC. However, much of the results we have found for the LQC corrections are also the outcome of the ekpyrotic model and models where the universe is sourced by a phantom field. As mentioned in previous chapters, a triality between these universes had already been identified [112] at the cosmological background level, but no detailed perturbation theory had made a possible link between these models. In this thesis we have carried out the detailed perturbation analysis in LQC using the latest understanding of this theory. By comparing our results with those previously obtained for the ekpyrotic model and a universe sourced by a phantom field we agree that these theories also correspond to one another qualitatively at the level of linear perturbations.

We then turned our attention to a correspondence map between the inverse volume corrected LQC and general braneworld cosmologies at the background level, already mentioned in the literature [20]. This map had been obtained by demanding both sets of cosmologies to result in the same phenomenology. In particular, they calculated the constraint equation for both sets of models to result in the same Hubble parameter. We set out to discover the validity of such a map at the level of linear perturbations. It is reasoned in this thesis that due to the difference in the nature of the Hubble parameter in the slow-roll inflation obtained for braneworld cosmologies and the super-inflation in LQC, it is not possible to extend a map of this kind beyond the background cosmologies discussed in [20]. This is not the first effort in finding a link between braneworld cosmologies and LQC, but it is the only work which has tried to do this for the effective equations of motion in the context of inverse volume corrections of LQC. Other investigations along this line have concentrated purely on the holonomy corrections [111, 135], since it is more natural to compare effective equations that resemble one another.

There has been recent interest in collapsing universe scenarios in both braneworld cosmologies (e.g. the ekpyrotic model) and in LQC. There has also been a lack of a comprehensive dynamical analysis of such a universe in curved space, where quantum effects are likely to be most significant. Motivated by these ideas, we set ourselves the task of exploring the solutions obtained in these circumstances and their stability properties. General forms of scalar potentials are derived for an FRW universe undergoing an expansion/contraction phase. We also allow for the negative sign as well as the more familiar positive sign for these potentials. Certain limits of our results have been obtained in the literature previously, but we believe our study in this field to be the most complete up to date.

An obvious extension to our dynamical analysis is to consider the effective equations of LQC to shed light on possible cosmological background behaviour of a scalar field prior to a bounce. This has not yet been carried out and in most numerical simulations of LQC models evolving through the bounce, either a particular form has been assumed for the potential, or the bounce is assumed to be symmetric. But by careful consideration, we have demonstrated that solutions which are late time attractors in an expanding FRW universe may not be stable attractors in a collapsing phase. It is possible that this may hold in the case of an LQC model and it means the cosmological background may be quite different in these two phases. Although the concept of a cosmological bounce and that of resolving the singularity are results of the full theory and will not be shaken by our findings here, it does mean that the assumptions in the numerical simulations of the bounce should be justified. In particular, if perturbation theory was to be performed on the collapsing phase of LQC, one needs to make sure perturbations are realised about the correct background cosmology.

Another interesting study would be to consider vector mode perturbations in the collapsing branch of LQC. The vector perturbations of the metric have been calculated for LQC [169], but this has not been considered so far in any phenomenological setting. Such fluctuations are of little interest in an expanding universe since they tend to decay in time and leave no observational signatures behind. In a collapsing universe, however, they have been shown to grow [170]. This study has also been done in the context of an ekpyrotic universe [171], but since there are uncertainties about the nature of the bounce in a cyclic model and the need to consider higher dimensions in that regime, no considerable success has been made on predicting any potential observational tests in the field. One possibility for these primordial vector modes to be detected is their ability to give rise to primordial magnetic fields [172], which can in turn be amplified by galactic dynamo mechanisms. A very recent study [173] has concluded that these modes, if produced via standard inflation, will not lead to observed levels of magnetic fields. In search for possible explanations, the collapsing phase of LQC and the unambiguous transition through the bounce may prove to be a worthy candidate.

This thesis is by no means a complete investigation of the phenomenologies arising in the context of LQC, and as such we will not over estimate the power of our results outlined here. Having said that, we do stress that our calculations have been done based on the latest knowledge we currently have about LQC and the full theory. LQC phenomenology is still in its early days of development and there is much room for interesting explorations in the field. We hope to have paved the way and set the framework for future analysis of LQC phenomenology.

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