On a Certain Class of Cyclically Presented Groups

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Abstract

The present thesis is devoted to the study of a class of cyclically presented groups, an important theme in combinatorial group theory.

1. **Introduction**

   We introduce the cyclic presentations which will be the object of our study. We explain why these are important and we state some known results. In view of this we propose a conjecture (Conjecture 1.2.7) to which we give a partial answer in the following chapters. We finally give a definition of irreducibility, p-irreducibility and f-irreducibility for a presentation in the class which we are studying.

2. **Method of proof**

   Here we give a short report on split extensions and (van Kampen) diagrams, which are the two basic ingredients involved in our proofs. We then outline the method of proof, which is a generalization of the method used in [11] and makes use of an analysis of modified diagrams.

3. **The p-irreducible case**

   We start giving a geometric constraint on diagrams and we show, as described in Chapter 2, that a presentation whose diagram respects this constraint gives rise to an infinite group. After studying four particular cases we give conditions on the integer parameters of a presentation in the class considered in order to have a diagram which satisfies the given geometric constraint. Finally, we prove Theorem 2 which partially answers Conjecture 1.2.7 in the p-irreducible case.

4. **The f-irreducible case**

   Given certain constraints the problem is reduced to a particular case. We then study this case as outlined in Chapter 2. We also show that these constraints can be weakened if the number of generators is odd.

5. **Conclusions**

   We show how the results achieved can be used to prove a theorem in a more general setting; we explain what one should prove in order to confirm Conjecture 1.2.7 and why our method fails in these cases.

To my father

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Contents

Chapter 1

Introduction

Combinatorial group theory is essentially the study of groups given by means of presentations. The present thesis aims to study a particular class of *cyclically presented groups* which generalises a classical example of Higman (see [15]) and has been given attention in several recent papers (see, for example, [10, 11] and [14]).

1.1 Cyclic presentations and their relevance

Let $X = \{x_0, \ldots, x_{n-1}\}$ and let σ denote the (unique) automorphism of $F = F(X)$ such that $x_i\sigma = x_{i+1}$ where the subscripts are taken modulo n.

Definition 1.1.1 *With the above notation a* ***cyclic presentation*** *is a presentation $G_n(w) = \langle X | w, w\sigma, \ldots, w\sigma^{n-1}\rangle$ for some cyclically reduced $w \in F$. A group G is said to be* ***cyclically presented*** *or to have a* ***cyclic presentation*** *if there exist $w \in F$ and $n \in \mathbb{N}$ such that $G \cong G_n(w)$.*

There are several motivations for studying cyclic presentations. First of all cyclic presentations are *balanced*, that is they have the same number of generators and defining relators, hence cyclic presentations of the trivial group provide potential counterexamples to the well known Andrews-Curtis conjecture (see, for example, [2, 21]) which is of some interest in topology as well as from a combinatorial group theory point of view. Another connection to topology arises from the following fact. Let M be a closed compact orientable 3-manifold. In 1968 Neuwirth [24] described an algorithm for producing a presentation for $\pi_1(M)$, the fundamental group of M. The output of the algorithm turns out to be a cyclic presentation. These aspects have been studied by many authors (see, for example, [4, 6, 9] and [25]). Dunwoody suggested a connection between cyclic presentations and cyclic branched covers of knots which has been recently confirmed by Cavicchioli, Ruini and Spaggiari [7]. Further investigation of this connection can be found in papers as [3, 13, 22] and [27].

Many of the families of groups studied in the papers above are interesting from a purely combinatorial point of view and are generalizations of classical families of groups which are indeed cyclically presented such as the *Fibonacci groups*, the *fractional Fibonacci groups* and the *Sieradski groups*. These have presentations $F(r,n) = G_n(x_0 \cdots x_{r-1}x_r^{-1})$, $\widetilde{F}_{l,k} = G_n((x_0^{-l}x_1^l)^k x_1 (x_2^{-l}x_1^l)^k)$ and $S(n) = G_n(x_0x_2x_1^{-1})$, respectively.

1.2 The problem and some known results

In [18] Johnson introduced the *polynomial associated* with the cyclic presentation for $G_n(w)$. This is the polynomial $f_w(t) := \sum_{i=0}^{n-1} a_i t^i$ where a_i is the exponent sum of x_i in w. He also showed that the order of the abelianization $G_n(w)^{ab}$ of $G_n(w)$ equals $|\prod_{i=0}^{n-1} f_w(\zeta_i)|$ where the ζ_i are the complex nth roots of unity (here we agree that $|G_n(w)| = \infty$ if this product vanishes) and proved that $G_n(w)^{ab}$ is trivial if and only if the associated polynomial is a unit in the quotient ring $\mathbb{Z}[t]/(t^n - 1)$. It has been conjectured by Dunwoody that $|G_n(w)| = 1 \Longrightarrow f_w(t) = \pm t^i$. This is now known to be false as Edjvet, Hammond and Thomas proved in [10] providing a counterexample for $n = 5$. Nevertheless the conjecture led many authors (see [10, 11, 12] and [14]) to consider the following question.

Problem 1.2.1 *For what n and w is $G_n(w)$ trivial?*

Following Edjvet [10] it suffices to answer the question above for cyclic presentations $G_n(w)$ which are *irreducible*, that is when either $n = 1$ or, for $n \geq 2$, if w involves only $x_{i_1}, \ldots, x_{i_k}$ where $i_j < i_{j+1}$ for $1 \leq j \leq k-1$ then $k \geq 2$ and $h :=\mathrm{hcf}(i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}, n) = 1$.

Notice that if $h > 1$ then $G_n(w)$ decomposes into the free product of h irreducible factors all isomorphic to an irreducible cyclically presented group $G_m(\hat{w})$ where $n = hm$.

Problem 1.2.1 is a very general and difficult question even if we restrict our attention to irreducible presentations. We therefore confine ourselves to study a special class of cyclically presented groups. The following problem has been considered in [11] with the convention that $[a, b] = a^{-1}b^{-1}ab$.

Problem 1.2.2 *For which values of $n, i, j, k; \alpha_r$ is $G = G_n\left(x_i^{\alpha_1}\left[x_j^{\alpha_2}, x_k^{\alpha_3}\right]\right)$ both irreducible and trivial?*

This class of groups generalizes one of the first examples of an irreducible cyclic presentation of the trivial group, namely $G_2\left(x_0\left[x_0^{-1}, x_1\right]\right)$ (see Higman [15]) and has been partially investigated in [10, 11, 12] and [14].

First of all we observe that there is no loss in assuming $\alpha_1 = 1$, for if not the abelianization G^{ab} of G is not trivial being the free product of n copies of the cyclic group of order $|\alpha_1|$; moreover since the presentation is cyclic we can also assume $i = 0$. We can therefore reformulate the given problem as follows.

Problem 1.2.3 *For which values of $n, i, j; \alpha$ and β is $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ both irreducible and trivial?*

In particular Edjvet and Hammond [11] considered Problem 1.2.3 for $(i, j) = (1, 2)$. Their proofs involve an analysis of modified van Kampen diagrams over a split extension of G and some computation for small values of the parameters. Here we want to generalize the geometric construction to the general case and to prove similar statements for different values of i and j. Although the geometric analysis involves only elementary techniques, it turns out to be quite complicated and somewhat involved; in order to simplify this analysis we introduce the so-called *elementary moves* which determine an equivalence relation on the set of presentations of the form $G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ (for a fixed n) which respects isomorphims (that is presentations in the same equivalence class define isomorphic groups). We will then work *modulo* this equivalence relation and study geometrically suitable representatives of classes.

Definition 1.2.4 *Two words $w = x_0[x_i^{\alpha}, x_j^{\beta}]$ and $w' = x_0[x_{i'}^{\alpha'}, x_{j'}^{\beta'}]$ in $X^{\pm}$ are said to be* ***equivalent****, and we write $w \sim w'$, if w' can be obtained from w by a sequence of the following* ***elementary moves****:*

(E1) *changing x_k with x_k^{-1} for each k;*

(E2) *cyclic permutation;*

(E3) *inversion;*

(E4) *permutation induced by an automorphism of indices in* $\mathbb{Z}_n$*.*

Clearly, $\sim$ is an equivalence relation and we have the following.

Lemma 1.2.5 *If* $w \sim w'$ *then* $G_n(w) \cong G_n(w')$*.*

Proof. Each elementary move does not change the isomorphism class.

$\square$

Remark. Notice that elementary moves can only interchange and change the signs of α and β. Since we are working modulo the equivalence relation defined by these moves we will always give symmetric conditions on the parameters α and β. Moreover elementary moves take irreducible cyclic presentations to irreducible cyclic presentations.
The next result is a consequence of a theorem of Pride [26] and is useful in what follows.

Proposition 1.2.6 *Let* $G = G_n(w)$ *be a cyclically presented group. If* $n \geq 4$*,* w *involves the generators* x_0 *and* x_1 *only and if* x_0 *or* x_1 *has zero exponent sum in* w *then* G *is infinite.*

In view of this we can assume in Problem 1.2.3 that if $n \geq 4$ then i and j are distinct and both different from zero. Using elementary moves we can also assume $0 < i < j < n$ and $i \leq \frac{n}{2}$. In fact if $i > j$ then the sequence of elementary moves (E3)-(E2)-(E1) interchanges i and j (so we can assume $i < j$) and if $i < j$ but $i > \frac{n}{2}$ we can apply the automorphism $k \mapsto n - k$

first (E4) and then again (E3)-(E2)-(E1).

We now give some known results concerning Problem 1.2.3.

- $G_n\left(x_0\left[x_0^{-1}, x_1\right]\right)$ is trivial and irreducible for $n = 2$ or 3 (Higman [15]), and it follows that $G_2\left(x_0\left[x_0^{\alpha}, x_1^{\beta}\right]\right)$ is trivial for all α and β; on the other hand $G_3\left(x_0\left[x_0^{-k}, x_1\right]\right)$ is infinite for $k \geq 2$ (Neumann [23]);

- the groups $G_3\left(x_0\left[x_1, x_2\right]\right)$, $G_3\left(x_0\left[x_1^{-1}, x_2^{-1}\right]\right)$, $G_4\left(x_0\left[x_1^{-1}, x_2^{-1}\right]\right)$ and $G_4\left(x_0\left[x_1, x_3\right]\right)$ are both trivial and irreducible (Edjvet, Hammond and Thomas [10]);

- more generally $G_{2k}\left(x_0\left[x_i^{\alpha}, x_{i+k}^{\alpha}\right]\right)$, where $\alpha \in \mathbb{Z} \setminus \{0\}$ and $k \geq 2$, is irreducible and trivial for each i such that $\mathrm{hcf}(i, k) = 1$ (Havas and Robertson [14]);

- for $n \geq 5$ and $(|\alpha|, |\beta|) \neq (1, 1)$ the group $G_n\left(x_0\left[x_1^{\alpha}, x_2^{\beta}\right]\right)$ is infinite (Edjvet and Hammond [11]).

We also point out that using elementary moves we can assume $0 < i < j \leq n - i$. A computation with KBMAG [16] shows that for $5 \leq n \leq 10$, $0 < i < j \leq n - i$ and $0 < |\alpha|, |\beta| \leq 3$ the group $G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ is both irreducible and trivial if and only if it is of the Havas-Robertson type. This prompted the following.

Conjecture 1.2.7 *Let $n \geq 5$, $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$ and $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ be irreducible where $0 < i < j < n$ and $i \leq \dfrac{n}{2}$. Then G is trivial iff $\alpha = \beta$, $n = 2k$ and $j = i + k$.*

Notice that the presentation $G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ is irreducible if and only if $\mathrm{hcf}(n, i, j) = 1$, therefore the conjecture says precisely that for $n \geq 5$ there are no trivial and irreducible groups except from the Havas-Robertson examples. In order to prove partial results towards this conjecture we will distinguish two cases.

Definition 1.2.8 *Let $h_i := hcf(i,n)$ and $h_j := hcf(j,n)$. The presentation $G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ is said to be **p-irreducible** if it is irreducible and $h_i = 1$ or $h_j = 1$; it is said to be **f-irreducible** if it is irreducible and $h_i, h_j > 1$.*

From now on we will assume $n \geq 5$.

In Chapter 3 we will study the p-irreducible case. This study will culminate with a proof of the following.

Theorem 2 *Let $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$. Suppose that $h_i = 1$ or $h_j = 1$ and that $|\alpha| > 1$, $|\beta| > 1$ and $|\alpha| \neq |\beta|$.*

If n is odd and $n \geq 11$ then G is infinite.

Chapter 4 is devoted to the study of the f-irreducible case. The main result is the following theorem.

Theorem 3 *Let $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$. If $h_i > 1$, $h_j > 1$, $|\alpha| > 1$ and $|\beta| > 1$ then G is infinite.*

For n odd we can weaken the hypotheses on α and β and prove the following. **Theorem 4** *Let $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$. If n is odd, $h_i > 1$, $h_j > 1$ and $(|\alpha|, |\beta|) \neq (1,1)$ then G is infinite.*

In the final chapter we give a short report of the results achieved and give conditions under which we can prove $G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ to be infinite using the theorems above. More precisely we look at the hypotheses as constraints on the 5-tuple $((n,i,j,\alpha,\beta))$ and we prove the following.

Theorem 5 *Let $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ be irreducible. If there exists m such that $m \mid n$, $m \nmid j-i$ and the 5-tuple $(m, i', j', \alpha, \beta)$, where $i' \equiv i \bmod m$ and*

$j' \equiv j \bmod m$, *respects one of the conditions in* Theorem 2, Theorem 3 *or* Theorem 4*, then G is infinite.*

We finally discuss which cases are left in order to confirm Conjecture 1.2.7 and explain why our method fails in these cases.

We point out that for $n < 5$ there are examples of presentations defining a non-trivial and finite group (for example $G_3\left(x_0\left[x_1^{-1}, x_2\right]\right)$, which has order 120, has been checked by Martin Edjvet with the computer software **KB-MAG** [16]); nevertheless we believe that for $n \geq 5$ if $G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ is irreducible and non-trivial then it is infinite.

Chapter 2

Method of proof

Our main method of proof involves two ingredients, namely *group extensions* and (*modified*) *van Kampen diagrams*.

2.1 Group extensions.

Definition 2.1.1 *Let E, G, H be groups and 1 denote the trivial group. We say that E is an* ***extension*** *of G by H if there exists a short exact sequence of groups and homomorphisms*

$$1 \longrightarrow G \longrightarrow E \longrightarrow H \longrightarrow 1$$

Then G can be thought as a normal subgroup of E and $E/G \cong H$.

A particularly interesting case of extension is when E is the *semi-direct product* of G and H.

Definition 2.1.2 *Let E, G, H be groups. If $G \lhd E$, $H \leqslant E$, $GH = E$ and $G \cap H = 1$ then we say that H is a* complement *for G in E. Then E is the*

semi-direct product *of G and H and we write $E = G \rtimes H$; we also say that E* splits *over G by H or that E is a* split extension *of G by H.*

It is important to notice that if $E = G \rtimes H$ then E is an extension of G with the notation above and that the correspondence $h \longmapsto \sigma_h \in \mathrm{Aut}(G)$, where $\sigma_h : g \longmapsto h^{-1}gh$, gives a homomorphism $H \longrightarrow \mathrm{Aut(G)}$.
On the other hand given groups G and H and a homomorphism $\alpha : H \longrightarrow \mathrm{Aut(G)}$, then the set $E = H \times G$ equipped with the binary operation $(h_1, g_1)(h_2, g_2) = (h_1h_2, g_1(h_2\alpha)g_2)$ is a group such that $G \lhd E$, $H \leqslant E$, $GH = E$ and $G \cap H = 1$, so it is the semi-direct product (with respect to α) of G and H. Here α says how elements of G conjugate by elements of H in E and we write $E = G \rtimes_\alpha H$. Thus we have an equivalent definition of split extension which will be very useful in our context.
The next result is Corollary 1 on page 140 of [17] and shows how to construct a presentation for $E = G \rtimes_\alpha H$ once we are given presentations for G and H.

Proposition 2.1.3 *Let $G = \langle X|R\rangle$ and $H = \langle Y|S\rangle$ be groups, and let $\alpha : H \longrightarrow Aut(G)$ be a homomorphism such that $x(y\alpha) = w_{y,x}$, a word in $X^{\pm}$, $x \in X$, $y \in Y$. Then the semi-direct product $E = G \rtimes_\alpha H$ has presentation*

$$E = \left\langle X, Y | R, S, \{y^{-1}xyw_{y,x}^{-1}|\ x \in X, y \in Y\}\right\rangle$$

$\square$

We now show why this is particularly important from our point of view. Consider the group $G = G_n\left(x_0\left[x_i^\alpha, x_j^\beta\right]\right)$ and the cyclic group $C_n = \langle t|\ t^n\rangle$. Let $\phi : C_n \longrightarrow \mathrm{Aut}(G)$ be defined by $t \longmapsto \sigma$ where $x_i\sigma = x_{i+1}$ for $i = 0, \ldots, n-1$ (subscripts modulo n) and form the semi-direct product $E = E(n; \alpha, \beta; i, j) := G \rtimes_\phi C_n$. Then $G \lhd E$ and $[G : E] = n$ therefore in order to prove that G is infinite it suffices to show that E is infinite. According to Proposition 2.1.1 we have the following presentation for the extension E:

$$\left\langle x_0, \ldots, x_{n-1}, t \middle| \, t^n, x_k \left[x_{i+k}^{\alpha}, x_{j+k}^{\beta}\right], x_k = t^{-1}x_{k-1}t, (k = 0, \ldots, n-1)\right\rangle$$

where subscripts are taken modulo n.

Now apply Tietze transformations to delete generators using the last $n-1$ relations: we have $x_k = t^{-k}x_0t^k$ for $k = 1, \ldots, n-1$, the $(n+2)$-th relation yields $x_0 = t^{-n}x_0t^n$ and the relations $x_k \left[x_{i+k}^{\alpha}, x_{j+k}^{\beta}\right]$ yield conjugates to $x_0t^{-i}x_0^{-\alpha}t^{i-j}x_0^{-\beta}t^{j-i}x_0^{\alpha}t^{i-j}x_0^{\beta}t^j$. Renaming $x_0 = x$ we have the following:

$$E(n; \alpha, \beta; i, j) = \left\langle x, t \middle| \, t^n, xt^{-i}x^{-\alpha}t^{i-j}x^{-\beta}t^{j-i}x^{\alpha}t^{i-j}x^{\beta}t^j\right\rangle.$$

This presentation will be the main object of our study. The next section provides the geometric preliminaries we need in order to study these extensions.

2.2 Geometric preliminaries

Definition 2.2.1 *Let $\mathcal{P} = \langle X | \, R\rangle$ be a group presentation and w a word in $X^{\pm}$. A* ***van Kampen diagram*** *K over $\mathcal{P}$ for w is a planar and simply connected 2-complex such that the following are satisfied:*

(i) each edge is labelled by an element of X and is given an orientation;

(ii) reading the labels on the boundary of each 2-cell (for some choice of starting point and orientation) yields an element of R (here we agree that if an edge is labelled by x one reads x or x^{-1} if the edge is traversed according or opposite to its orientation, respectively);

(iii) reading the labels on the boundary ∂K of K yields w.

Then w is called the ***boundary word*** *of K.*

We can construct van Kampen diagrams labelling boundaries of 2-cells by elements of R and their inverses R^- and gluing them along edges according to edge labels and orientation. Thus each loop in such a diagram gives a word which is a product of conjugates of words in $R^{\pm}$, hence a word which represents the identity in the group $G = \langle X|\ R\rangle$; in particular the boundary word represents the identity of G. This gives a geometric method for deducing consequences of the relators in $\langle X|\ R\rangle$. Quite remarkably, this statement has a converse, as pointed out in the following result due to van Kampen.

Proposition 2.2.2 *Let $G = \langle X|\ R\rangle$ and w be a word in $X^{\pm}$. Then $w = 1_G$ if and only if w is the boundary word of some van Kampen diagram over $\langle X|\ R\rangle$.*

Let K be a van Kampen diagram (or simply a diagram) over $\langle X|\ R\rangle$ and let Δ and Δ' be distinct regions of K whose boundaries share at least one edge. Let r and r' be the words obtained reading the boundary labels of Δ and Δ' (respectively) starting from one common edge and according to its orientation. If $r = r'$ as words in $F(X)$ then (Δ, Δ') is said to be a *reducible pair.*

A van Kampen diagram over $\langle X|\ R\rangle$ for w is said to be **reduced** if there is no reducible pair in K and is said to be **minimal** if it has minimal number of 2-cells among all other van Kampen diagrams over $\langle X|\ R\rangle$ for w.

If a diagram over $\langle X|\ R\rangle$ is not reduced then there is a reducible pair, say (Δ, Δ'). If we delete common edges of the Δ and Δ' and combine the remaining edges folding and identifying them according to their labels and orientations then we get a new diagram with the same boundary word and fewer regions; it follows that a minimal diagram is reduced.

We will always assume that van Kampen diagrams are minimal.

We now show how we will use this construction to study the extensions $E = E(n;\alpha,\beta;i,j)$. This is a generalization of the construction used in [11] by Edjvet and Hammond to study the extension $E(n;\alpha,\beta;1,2)$.

Since t has order n in the extension E, it follows that we can insert $t^{\pm n}$ in the second relator of the extension in such a way that all the t-exponents are in the interval $\left[-\frac{n}{2},\frac{n}{2}\right]$; since we are assuming $i \leq \frac{n}{2}$ it follows that i is always in the given interval and so we have the following three cases:

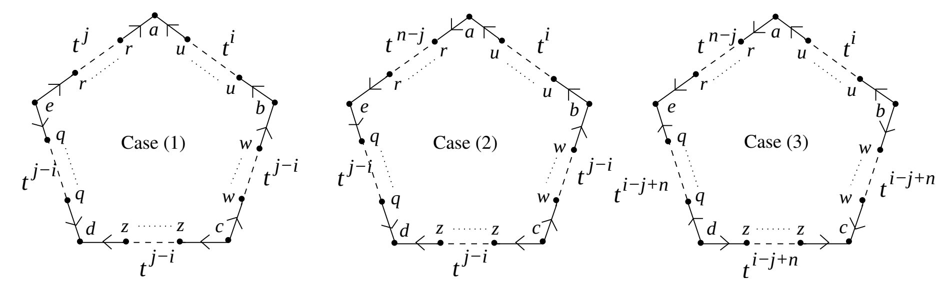
Case (1) $j \leq \frac{n}{2}$, in which case all the t-exponents are already in the given interval and we say they are *reduced modulo* n;

Case (2) $j > \frac{n}{2}$ and $j - i \leq \frac{n}{2}$, in which case j is not reduced modulo n and becomes $j - n$ after reduction;

Case (3) $j - i > \frac{n}{2}$, in which case j, $j - i$ and $i - j$ reduce to $j - n$, $j - i - n$ and $i - j + n$ respectively.

Notice that each reduction changes the sign of the exponent (hence the orientation of the corresponding t-edges).

Suppose we want to prove that the extension is infinite and assume, by way of contradiction, that x is a torsion element in E, say $x^l = 1$ for some minimal $l < \infty$. Then, by Proposition 2.2.1, there exists a van Kampen diagram K over $E(n;\alpha,\beta;i,j)$ for x^l. This diagram is made up by two different types of 2-cells (or *regions*) up to inversion, whose boundary is labelled by the two relators (after reductions of t-exponents) in the relevant presentation. Then the boundary of K is a simple closed curved labelled by x^l. Now collapse to a point each edge in K which is labelled by $x^{\pm 1}$ and retain the labels for the angles created. What we obtain is a so-called **modified van Kampen diagram** over $E(n; \alpha, \beta; i, j)$ for x^l which is a tessellation $\mathscr{D}$ of the 2-sphere whose regions are the regions of K after the collapses. Regions corresponding to the relator t^n are left unchanged by the collapses and are called **s-regions**, while regions corresponding to the other relator are called **m-regions** and are given (up to inversion) by the following figure :



Here we used powers of t to label sequences of edges and the corner labels tell us how many x-edges have been collapsed (and their previous orientation) where they appear. So their values are given by the following table:

a	b	c	d	e	s	u	w	z	q	r
1	$-\alpha$	$-\beta$	α	β	0	0	0	0	0	0

where s is the label of each corner in an s-region. The labels in the inverted regions will be overlined and clearly their values will have opposite sign (for example in an inverted m-region there will be a label $\overline{a}$ whose value is -1). From now on we will omit the orientation of edges in diagrams.

The vertex obtained from collapsing the boundary of K is called the *distinguished vertex* and will be denoted by v_0; the vertices which are not distinguished are said to be *interior* and are denoted by v_r where r is the label of the corresponding corner in the considered region. A region having v_0 as a vertex is called a *boundary region*, otherwise a region is said to be *interior*. The *label* $l(v)$ of a vertex v in $\mathscr{D}$ is the word given by the corner labels at v read anti-clockwise (and so is defined up to cyclic permutation); the *label sum* of $l(v)$ is the sum of the values of the corner labels at v. It follows that the label sum of each interior vertex will be 0 and that the label sum of the distinguished vertex will be l. This is essentially because we are now working over the one-relator product $(\langle x| \rangle * \langle t| t^n \rangle) \,/\, \langle\langle xt^{-i}x^{-\alpha}t^{i-j}x^{-\beta}t^{j-i}x^{\alpha}t^{i-j}x^{\beta}t^{j} \rangle\rangle$ and follows by the minimality of K; since there is no loop made up of interior x-edges in K it follows that the interior x-edges form a forest; the collapse of each connected component of the forest results in the label sum of a single vertex in the modified van Kampen diagram obtained; each x-edge gives a contribution to exactly two values of angle labels in two m-regions created by the collapse and since it is traversed in opposite directions in the two regions before the collapse its contribution to those values are 1 and -1; it follows that if v is an interior vertex then v has zero label sum.

We aim to prove that this construction leads to a contradiction; in order to do that we need some further definitions.

Definition 2.2.3 *The **degree** of a vertex v in $\mathscr{D}$, denoted by $d(v)$, is the number of edges occuring at v.*

Definition 2.2.4 *The **measure** of an angle θ in $\mathscr{D}$ is $m(\theta) := \frac{2\pi}{d}$ where d is the degree of the vertex at which θ occurs.*

Definition 2.2.5 *The **curvature** of a vertex v, denoted by $c(v)$, is equal to $2\pi - \sum_{i\in I} m(\theta_i)$, where I is a set of indices such that the θ_i are the angles occuring at v.*

In view of these definitions all angles at a given vertex are equal and it follows immediately that each vertex in the diagram $\mathscr{D}$ has zero curvature.

Definition 2.2.6 *Let Δ be a region in $\mathscr{D}$ with k edges and whose vertices $v_1, \ldots, v_k$ have degree $d_1, \ldots, d_k$ respectively. Then the interior angles of this region have measures $\dfrac{2\pi}{d_1}, \ldots, \dfrac{2\pi}{d_k}$. We define the **curvature of** Δ, denoted by $c(\Delta)$, as follows:*

$$c(\Delta) := 2\pi - \left[\left(\pi - \frac{2\pi}{d_1} \right) + \cdots + \left(\pi - \frac{2\pi}{d_k} \right) \right].$$

Notice that $c(\Delta) := 2\pi - \sum_{i=1}^{k} \left(\pi - \dfrac{2\pi}{d_i} \right) = 2\pi - k\pi + 2\pi \sum_{i=1}^{k} \dfrac{1}{d_i}$.

Definition 2.2.7 *The **total curvature** T of the tessellation $\mathscr{D}$ is defined as follows:*

$$T := \sum_{v \in \mathscr{D}} c(v) + \sum_{\Delta \in \mathscr{D}} c(\Delta)$$

Since $c(v) = 0 \ \forall v \in \mathscr{D}$ we can write $T = \sum_{\Delta \in \mathscr{D}} c(\Delta)$.

Before describing how we will use this construction to obtain a contradiction we want to clarify the terminology that will be used and introduce the *star graph* associated to $\mathscr{D}$ and the concept of *bridge move* in $\mathscr{D}$ (see [8]). The first is a graph which is very useful in order to find all the possible vertex labels for a given degree and is particularly efficient for small degrees; the latter provides a way to obtain new modified van Kampen diagrams from $\mathscr{D}$ and allows us to exclude some vertex labels or sublabels once we are given certain maximality and/or minimality conditions on $\mathscr{D}$.

Definition 2.2.8 *An **m-segment** is a sequence of t-edges in an m-region whose end corners are labelled by a non-zero power of x and the other corners are labelled by zero-valued labels (in other words it is a sequence of t-edges* *corresponding to the powers of t in the second relator of the extension after reduction modulo n).*

*An **s-segment** is a sequence of edges in an s-region whose vertices have degree 2 except for the end vertices which have degree > 2.*

*A **splitting** in a m-segment of an m-region is a vertex of degree > 2 in the m-segment and which is not an end vertex.*

It follows from the definition that there are five m-segments in each m-region. The m-segment whose end vertices are v_y and v_z will be denoted by yz and we will use $|yz|$ to denote the *length* of yz which is the number of edges in yz. Of course $(y, z) \in \{(a, b), (b, c), (c, d), (d, e), (e, a)\}$ up to inversion.

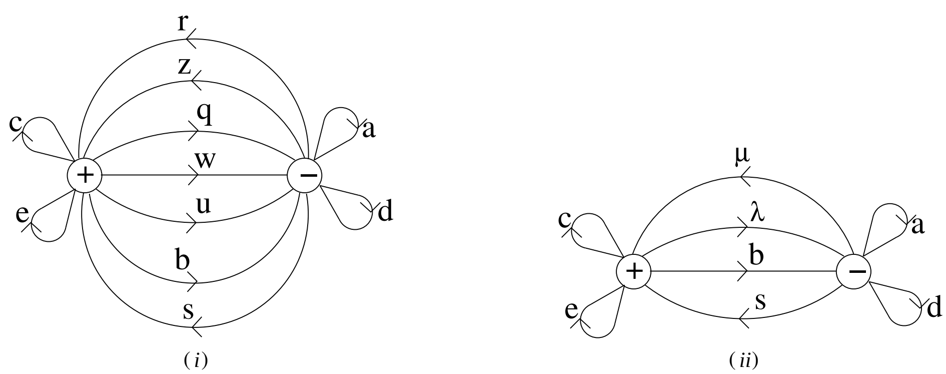
If there is a splitting of degree m in some m-segment we will also say that the segment *splits in degree m*. Moreover we will use the term *segment* to refer to both m-segments and s-segments if no confusion can arise.

Definition 2.2.9 *The associated **star graph** Γ to $\mathscr{D}$ consists of two vertices t, t^{-1} and one oriented edge labelled by y from $t^{-\epsilon_1}$ to t^{ϵ_2} (where $\epsilon_k = \pm 1$ for $k = 1, 2$) if the sequence $t^{\epsilon_1} y t^{\epsilon_2}$ (with $y \in \{a, b, c, d, e, u, w, z, q, r, s\}$) appears in one of the relators (up to cyclic permutation) of $E(n; \alpha, \beta; i, j)$.*

*If in the star graph, for $y \in \{u, w, z, q, r\}$, we substitute $y = \lambda$ or $y = \mu$ when the sequences $t^{-1} y t^{-1}$ or tyt appear respectively, we get the **modified star graph**.*

Definition 2.2.10 *A path in the (modified) star graph is said to be **admissible** if it is closed, cyclically reduced (that is the word spelled out reading the edge labels along the path is cyclically reduced) and has 0 as sum of the edge labels (recall that edges in the star graph correspond to corners in the modified van Kampen diagram $\mathscr{D}$).*

Now the label of an interior vertex in $\mathscr{D}$ corresponds to an admissible path in the (modified) star graph. This gives us a way to construct a list of all possible labels for a vertex of a given degree. Figures 2.1 and 2.2 show the star graph in case (1) and the modified star graphs in cases (1), (2) and (3).



$\lambda \in \{u, w, z^{-1}, q, r^{-1}\}$ and $\mu \in \{u^{-1}, w^{-1}, z, q^{-1}, r\}$

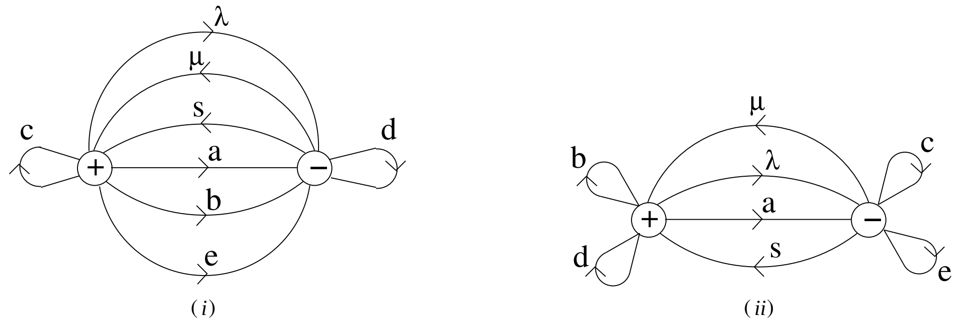
Figure 2.1: Star graph and modified star graph in case (1).

Here we must be careful listing the admissible paths since in the modified star graph the edges labelled by λ and μ cannot be traversed opposite to their orientation.

Remark

As a matter of notation when we need to write multiple labels we will often use curly brackets to list the possible choices (e.g. $a\{\lambda, s\}b$ stays for $a\lambda b$ or asb).

Moreover if there is no difference in using any of the 0-valued labels, or if it is clear from the context to which we are referring, we will sometimes use the generic 0.



$\lambda \in \{u, w, z^{-1}, q, r\}$ $\qquad$ $\lambda \in \{u, w^{-1}, z, q^{-1}, r\}$

$\mu \in \{u^{-1}, w^{-1}, z, q^{-1}, r^{-1}\}$ $\qquad$ $\mu \in \{u^{-1}, w, z^{-1}, q, r^{-1}\}$.

Figure 2.2: Modified star graph in cases (2) and (3).

Definition 2.2.11 *Let Δ be a region in $\mathscr{D}$. The **degree** of Δ, denoted by $d(\Delta)$, is defined to be the number of vertices of Δ of degree ≥ 3.*

We gave this definition because the presence of vertices of degree 2 does not affect the computation of the curvature of a region. Notice that the curvature of a region Δ depends only on the degree of Δ and the degree of its vertices, therefore we will often write only the degree of vertices of degree≥ 3 when computing the curvature. For example if $d(\Delta) = 4$ and Δ has two vertices of degree 3 and two vertices of degree 5 we will write $c(\Delta) = c(3, 3, 5, 5) = \dfrac{2\pi}{15}$.

Lemma 2.2.12 *Let Δ be a region of $\mathscr{D}$. If $d(\Delta) \geq 6$ then $c(\Delta) \leq 0$ with equality if and only if $d(\Delta) = 6$ and Δ has no vertex of degree more than 3.*

Proof

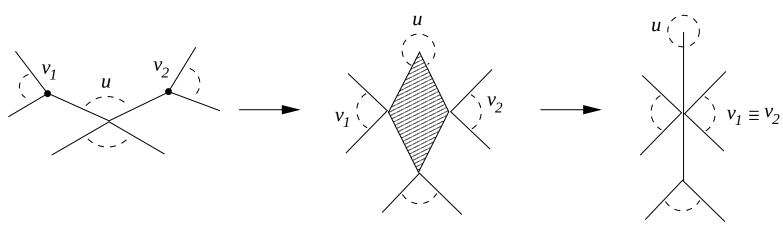
Let $d(\Delta) = k$. We know that $\dfrac{1}{\pi}c(\Delta) = (2 - k) + \sum_{i=1}^{k} \dfrac{2}{d_i}$ where $d_i \geq 3$ for $i = 1, \ldots, k$.

It follows that an upper bound for the sum $\sum_{i=1}^{k} \dfrac{2}{d_i}$ is given by substituting each d_i by 3, therefore $\dfrac{1}{\pi}c(\Delta) \leq (2 - k) + k\dfrac{2}{3}$ and so $c(\Delta) \leq 2\pi - \dfrac{k\pi}{3}$.

Now for $k \geq 7$ we have $c(\Delta) \leq 2\pi - \dfrac{7\pi}{3} < 0$ and for $k = 6$ since $d_i \geq 3$ for every i it follows from the formula that equality holds if and only if $d_i = 3$ for every i.

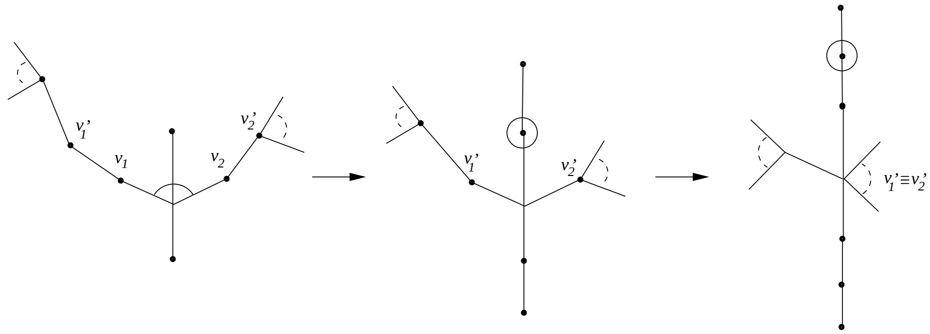
□

Definition 2.2.13 *Let u be a proper subword of a vertex label $l(v)$ (we will often refer to u has a* sublabel of v*). Suppose that u corresponds to an admissible path in the (modified) star graph. If we cut the diagram along the edges occuring at v which are adjacent to the sublabel u and we pull as illustrated in the following figure*



we obtain a new diagram $\mathscr{D}'$ which is still a modified van Kampen diagram over the given extension. Such a move is called ***bridge move at*** v ***relative to the sublabel*** u.

We will use bridge moves to avoid certain proper sublabels once a diagram is given certain conditions of maximality and/or minimality. For example if we assume a diagram to have maximal number of vertices of degree 2 and u is a possible label for a vertex of degree 2 then a bridge move relative to u creates a new vertex of degree 2 and does not kill any such vertex when $d(v_i) > 2$ for $i = 1, 2$. Similarly, if $d(v) = 4$ and u is as above then a bridge move at v relative to u creates two new vertices of degree 2 killing at most one if at least one among v_1 and v_2 is a vertex of degree > 2; if $d(v_1) = d(v_2) = 2$ we can apply a sequence of bridge moves at v and eventually find a bridge move which involves a vertex of degree > 2 as illustrated below.



We will use bridge moves in order to show that certain vertex sublabels force a diagram to be not reduced (see Lemmas 3.1.1 and 3.1.2). More precisely, if there is a vertex sublabel of the form $yu\overline{y}$ with u as in Definition 2.2.13 and $y \in \{a, b, c, d, e\}$ then a bridge move at that vertex relative to u gives a new diagram which is not reduced, hence is not minimal and we can decrease the number of regions.

2.3 Method of proof

Suppose that $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ is finite and let $\mathscr{D}$ be a modified van Kampen diagram for x^l over the extension $E(n; \alpha, \beta; i, j)$. Denote by

- V the number of vertices in $\mathscr{D}$;
- E the number of edges in $\mathscr{D}$;
- F the number of regions (or faces) in $\mathscr{D}$.

Lemma 2.3.1 *The following holds:*

$$T = \sum_{\Delta \in \mathscr{D}} c(\Delta) = 4\pi.$$

Proof. Denote by $e(\Delta)$ the number of edges in Δ, so that we have

$$T = \sum_{\Delta \in \mathscr{D}} (2\pi - \pi e(\Delta)) + \sum_{\Delta \in \mathscr{D}} \left(2\pi \sum_{i=1}^{e(\Delta)} \frac{1}{d_i} \right).$$

Since each edge is shared by two regions it follows that $\sum_{\Delta \in \mathscr{D}} (2\pi - \pi e(\Delta)) = -\pi \sum_{\Delta \in \mathscr{D}} (e(\Delta) - 2) = -\pi \sum_{\Delta \in \mathscr{D}} e(\Delta) + \pi \sum_{\Delta \in \mathscr{D}} 2 = -2\pi E + 2\pi F$.

Now consider the sum $\sum_{\Delta \in \mathscr{D}} \left(2\pi \sum_{i=1}^{e(\Delta)} \frac{1}{d_i} \right)$;

every angle $\frac{2\pi}{d_i}$ appears in the outer sum exactly d_i times, once for each of the regions sharing the corresponding vertex. It follows that each vertex gives a contribution equal to 2π, so we can write

$$\sum_{\Delta \in \mathscr{D}} \left(2\pi \sum_{i=1}^{e(\Delta)} \frac{1}{d_i} \right) = 2\pi V.$$

It follows that $T = -2\pi E + 2\pi F + 2\pi V = 2\pi (V - E + F)$.

It is well known that the Euler characteristic $V - E + F$ of the sphere is 2. It follows that

$$T = \sum_{\Delta \in \mathscr{D}} c(\Delta) = 4\pi.$$

$\square$

In order to obtain the desired contradiction we introduce the **pseudo-curvature** $c^*(\Delta)$ of a region Δ which is given by $c(\Delta)$ plus all the positive curvature Δ receives and minus all the positive curvature transferred from Δ according to a compensation scheme or a distribution process which will be specified case by case. So we will proceed as follows:

1) we will classify the positively curved interior regions and describe a distribution process or a compensation scheme for their curvature;

2) we will then show that each interior region has non-positive pseudo-curvature;

3) we will then distribute the curvature of the boundary regions and show that their pseudo-curvature is strictly less than $\frac{4\pi}{k_0}$ (where k_0 is the degree of the distinguished vertex).

Of course $T = \sum_{\Delta \in \mathscr{D}} c(\Delta) = \sum_{\Delta \in \mathscr{D}} c^*(\Delta)$ and since there are at most k_0 boundary regions it follows from 2) and 3) that $T < 4\pi$, a contradiction to Lemma 2.3.1. This contradiction shows that the considered extension E is infinite, hence so is G.

Chapter 3

The p-irreducible case

3.1 Introduction

In this chapter we want to study the groups $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ where $0 < i < j < n$, $i \leq \dfrac{n}{2}$ and $h_i := \mathrm{hcf}\,(i, n) = 1$ or $h_j := \mathrm{hcf}\,(j, n) = 1$.

In Chapter 2 we have described a geometric construction distinguishing three cases (1), (2) and (3).

From now on $\mathscr{D}$ will denote a modified van Kampen diagram for x^l over the extension $E = E(n; \alpha, \beta; i, j)$ of $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ according to the geometric construction given in Chapter 2. We will only specify in which of the cases (1), (2) or (3) the given presentation falls (often giving constraints on (n, i, j)). Moreover for the remainder of the chapter we will always assume $|\alpha|, |\beta| > 1$ and $|\alpha| \neq |\beta|$.

The idea here is to show first that a presentation falling into cases (1) or (3) for which there is a corresponding modified van Kampen diagram with no s-region of degree ≤ 3, $|\alpha|, |\beta| > 1$ and $|\alpha| \neq |\beta|$, gives rise to an infinite group.

Secondly, we will show that when n is odd we can apply elementary moves in order to find a new presentation $G_n\left(x_0\left[x_{i'}^{\alpha}, x_{j'}^{\beta}\right]\right)$ for G which does satisfy the hypotheses above, except in few cases that will be studied separately in Section 3.3.

We will use the modified star graph in order to find the possible labels for vertices of a given degree in the diagrams.

We start giving some assumptions on diagrams, under which we do not lose generality, and then prove two lemmas which allow us to simplify the geometric study.

Let $G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ be in case (1); we can assume without any loss of generality that each of the following conditions holds:

A1 $\mathscr{D}$ is minimal with respect to the number of regions.

A2 Subject to **A1**, the number of vertices in $\mathscr{D}$ with label ce (up to inversion and cyclic permutation) is maximal.

A3 Subject to **A2**, the number of vertices of $\mathscr{D}$ of degree 2 is maximal.

Let $G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ be in case (3); we can assume without any loss of generality that each of the following conditions holds:

B1 $\mathscr{D}$ is minimal with respect to the number of regions.

B2 Subject to **B1**, the number of vertices in $\mathscr{D}$ with label ce (up to inversion and cyclic permutation) is maximal.

B3 Subject to **B2**, the number of vertices in $\mathscr{D}$ with label bd (up to inversion and cyclic permutation) is maximal.

B4 Subject to **B3**, the number of vertices of $\mathscr{D}$ of degree 2 is maximal.

Lemma 3.1.1 *If $G_n\left(x_0\left[x_i^\alpha, x_j^\beta\right]\right)$ is in case (1), $\mathscr{D}$ satisfies* **A1-A3** *above and $l(v)$ is a vertex label, then the following hold:*

(i) *$l(v)$ is cyclically reduced;*

(ii) *$l(v)$ cannot have as proper sublabel $xw\overline{x}$ or $\overline{x}wx$ where w is a sublabel with zero label sum and $x \in \{a, b, c, d, e\}$;*

(iii) *ce, up to cyclic permutation and inversion, cannot appear as a proper sublabel of $l(v)$;*

(iv) *we cannot have $l(v) = y_1y_2y_3y_4$ where the y_i's are zero-valued labels.*

Proof. Let K be a van Kampen diagram which reduces to $\mathscr{D}$ after the collapses of x-edges. If (i) does not hold then K is not reduced, contradicting its minimality (and so we can reduce the number of regions, a contradiction to **A1**).

If (ii) does not hold we can apply a bridge move relative to w (as explained at the end of Section 2.2) and reduce the number of regions using (i).

If (iii) does not hold we could apply a bridge move at v relative to the sublabel ce creating a new vertex with label ce and not killing any such vertex, hence contradicting **A2**.

If (iv) does not hold we could apply bridge moves (as seen at the end of Section 2.2) at v creating two new vertices of degree 2 and killing at most one vertex of degree 2 which is not labelled by ce, hence contradicting **A3**.

□

Lemma 3.1.2 *If $G_n\left(x_0\left[x_i^\alpha, x_j^\beta\right]\right)$ is in case (3), $\mathscr{D}$ satisfies* **B1-B4** *above and $l(v)$ is a vertex label, then the following hold:*

(i) $l(v)$ *is cyclically reduced;*

(ii) $l(v)$ *cannot have as proper sublabel* $xw\overline{x}$ *or* $\overline{x}wx$ *where* w *is a sublabel with zero label sum and* $x \in \{a, b, c, d, e\}$*;*

(iii) ce*, up to cyclic permutation and inversion, cannot appear as a proper sublabel of* $l(v)$*;*

(iv) we cannot have $l(v) = sxsy$*, up to cyclic permutation and inversion, where* x *and* y *are zero-valued labels;*

(v) bd*, up to cyclic permutation and inversion, cannot appear as a proper sublabel of* $l(v)$*.*

Proof. The proof of (i)-(iv) is analogous to that of the previous lemma. It remains to prove (v). Suppose $\mathscr{D}$ satisfies **B1-B4** and bd is a subword of $l(v)$. A bridge move at v relative to bd creates a new vertex with label bd; moreover it does not change the number of regions and does not kill any vertex labelled by ce (up to cyclic permutation and inversion) since $|de| = n - j + i > 1$. It follows that the new diagram satisfies **B1-B2**, contradicting assumption **B3** on $\mathscr{D}$.

$\square$

The next result is a consequence of Theorem 1.1 in paper [11] by Edjvet and Hammond and will be useful in what follows.

Proposition 3.1.3 *If* $n \geq 5$ *and* $j = 2i$ *then* G *is infinite.*

Proof. Since G is irreducible $i \mapsto 1$ defines an automorphism of $\mathbb{Z}_n$, hence $G \cong G_n\left(x_0\left[x_1^{\alpha}, x_2^{\beta}\right]\right)$ which is infinite as proved in [11].

$\square$

Notice that the constraints $|\alpha|, |\beta| > 1$ and $|\alpha| \neq |\beta|$ are stronger than the hypotheses given in [11], where it is assumed $(|\alpha|, |\beta|) \neq (1, 1)$. In view of the previous proposition we will assume for the rest of the chapter that $j \neq 2i$.

3.2 A geometric constraint.

It is very useful to have a list of possible labels for interior vertices of small degree. Figure 3.1 shows labels in m-regions (up to inversion) and the modified star graphs for $j \leq \dfrac{n}{2}$ and $j - i > \dfrac{n}{2}$, respectively.

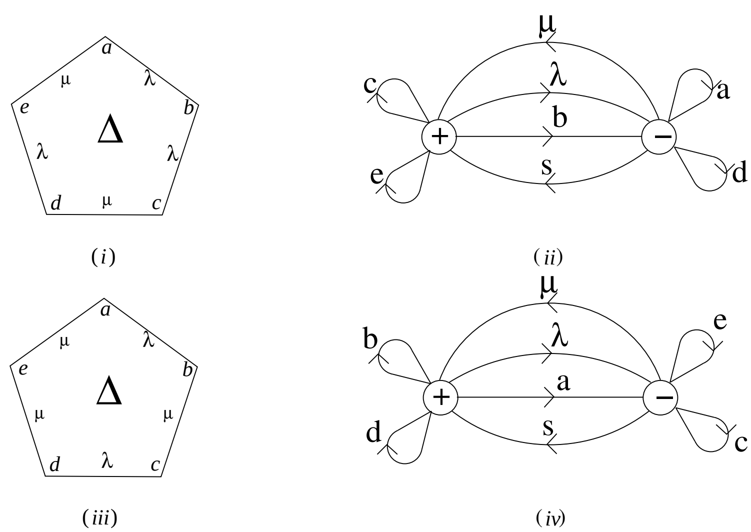


Figure 3.1: labels in an m-region ((i) and (iii)) and modified star graphs ((ii) and (iv)) for $j \leq \dfrac{n}{2}$ and $j - i > \dfrac{n}{2}$, respectively.

Here follows the list of possible labels for vertices of degree 2 and 3 involving a, b, c, d, e or s (up to cyclic permutation and inversion).

For $j \leq \dfrac{n}{2}$:

degree 2 : $ce, s\lambda$ with no consequence on α and β ;

degree 3 : $bds, bd\mu$ with no consequence on α and β ;

$daa \Rightarrow \alpha = -2$;

$\bar{d}aa \Rightarrow \alpha = 2$.

For $j - i > \frac{n}{2}$:

degree 2 : $ce, bd, s\lambda$ with no consequence on α and β ;

degree 3 : none .

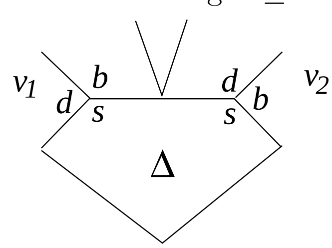
Lemma 3.2.1 *Let Δ be an interior s-region in $\mathscr{D}$. Then the following hold:*

- *if $j \leq \frac{n}{2}$ and $d(\Delta) \geq 5$ then $c(\Delta) < 0$;*
- *if $j - i > \frac{n}{2}$ and $d(\Delta) \geq 4$ then $c(\Delta) \leq 0$.*

Proof.

If $d(\Delta) \geq 6$ the result follows from Lemma 2.2.12; therefore assume $d(\Delta) \leq 5$. First assume $j \leq \frac{n}{2}$ and $d(\Delta) = 5$. There is only one possible label involving s for a vertex of degree 3, namely bds. We now show that two vertices of degree 3 cannot be separated by a sequence of vertices of degree 2.

Suppose v_1 and v_2 are two such vertices. Since db is not a segment label it follows that there must be a vertex of degree$\geq$ 4 between v_1 and v_2.



It follows that Δ cannot have more than two vertices of degree 3, therefore $c(\Delta) \leq c(3,3,4,4,4) = -\dfrac{\pi}{6} < 0$.

Now suppose $j - i > \dfrac{n}{2}$ and let Δ be an s-region of degree ≥ 4; since an interior vertex cannot have degree 3, it follows that $c(\Delta) \leq c(4,4,4,4) = 0$.

$\square$

Lemma 3.2.2 *Let Δ be an interior m-region in $\mathscr{D}$. If $j \leq \dfrac{n}{2}$ then $c(\Delta) \leq 0$.*

Proof.

According to the geometric construction of the modified van Kampen diagram for $j \leq \dfrac{n}{2}$ the lengths of segments in Δ are

$|ab| = |\overline{a}\overline{b}| = i$;

$|ea| = |\overline{ea}| = j$;

$|bc| = |cd| = |de| = |\overline{b}\overline{c}| = |\overline{c}\overline{d}| = |\overline{d}\overline{e}| = j - i$.

First assume $d(v_c) = d(v_e) = 2$. Since $|bc| = j - i < j = |ea|$ the segment ea splits and the splitting has degree≥ 4 since it has sublabel $\{b, \lambda\}\mu$.

If there is a splitting in the segment bc then this splitting must have sublabel $\lambda\mu$ (because $|ea| > |bc|$), hence degree≥ 4, and it follows that $c(\Delta) \leq c(3,3,3,4,4) = 0$. We can therefore assume that the segment bc does not split. Since $|ea| > |bc|$ it follows that v_b has sublabel $b\mu$, therefore $d(v_b) \geq 4$ (see Figure 3.2 (i)).

If there is another splitting $c(\Delta) \leq c(3,3,3,4,4) = 0$ and there is nothing to prove; so assume there is no other splitting in Δ. Since $|cd| = |de|$ it follows that v_d has sublabel ddd so that $d(v_d) \geq 6$ (see Figure 3.2 (ii)).

It follows that $c(\Delta) \leq c(3,4,4,6) = 0$.

Now assume that only one of the vertices v_c and v_e has degree 2.

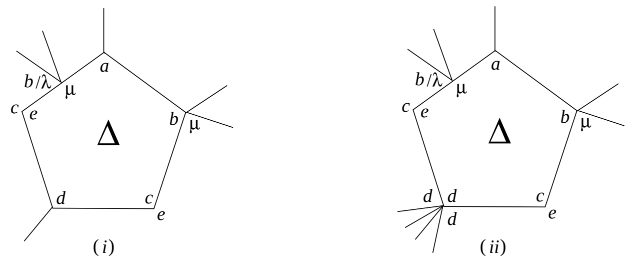


Figure 3.2: interior m-region with $d(v_c) = d(v_e) = 2$.

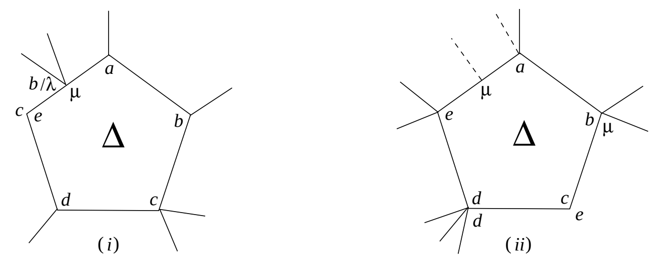


Figure 3.3: interior m-region with either $d(v_e) = 2$ or $d(v_c) = 2$.

If $d(v_e) = 2$ as before the segment ea splits in degree≥ 4; moreover $d(v_c) \geq 4$ and $d(v_a), d(v_b), d(v_d) \geq 3$, therefore $c(\Delta) \leq c(3,3,3,4,4) = 0$ (see Figure 3.3 (i)).

If $d(v_c) = 2$ then $d(v_e) \geq 4$. If one of the segments bc or cd splits then the splitting has degree ≥ 4 because $|ea| > |bc|$ and $|de| = |cd|$, respectively.

So we can assume there is no splitting (otherwise $c(\Delta) \leq 0$) and therefore v_d has sublabel dd and v_b has sublabel $b\mu$; hence $d(v_d) \geq 5$ and $d(v_b) \geq 4$.

If $d(v_a) \geq 4$ then $c(\Delta) \leq c(4,4,4,5) < 0$; if $d(v_a) = 3$ then $l(v_a) \in \{aad, aa\overline{d}\}$ and comparing the segments lengths it turns out that ea must split.

It follows that $c(\Delta) \leq c(3,3,4,4,5) < 0$ (see Figure 3.3 (ii)).

Finally, if $d(v_e), d(v_c) > 2$ then $c(\Delta) \leq c(3,3,3,4,4) = 0$, since v_e and v_c cannot have degree 3.

$\square$

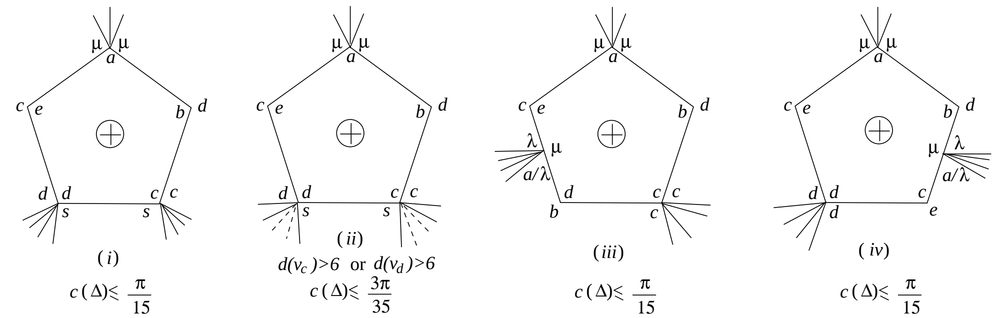


Figure 3.4: positively curved interior m-region for $j - i > \dfrac{n}{2}$.

Lemma 3.2.3 *Let Δ be an interior m-region in $\mathscr{D}$. If $j - i > \dfrac{n}{2}$ and $c(\Delta) > 0$ then Δ is one of the regions in* Figure 3.4*, hence $c(\Delta) \leq \dfrac{3\pi}{35}$.*

Proof. Observe that $d(v_a) \geq 4$; moreover $d(v_i) \neq 3$ for $i \in \{b, c, d, e\}$.
It follows that if three or four of the vertices v_b, v_c, v_d and v_e have degree> 2 then $c(\Delta) \leq c(4, 4, 4, 4) = 0$.
Now suppose that exactly two of the vertices v_b, v_c, v_d and v_e have degree 2. Observe that a splitting cannot have degree 3, therefore we can assume there is no splitting in Δ otherwise $c(\Delta) \leq c(4, 4, 4, 4) = 0$. Moreover since $|ab| < |de|$ and $|ea| < |bc|$, we can assume $d(v_d) \geq 4$ and $d(v_c) \geq 4$ otherwise de splits or bc splits, respectively. It follows that we must have $d(v_b) = d(v_e) = 2$.
Since we are assuming there is no splitting in Δ it follows that v_a has sublabel $\mu a \mu$ which implies $d(v_a) \geq 5$.
Now the segment cd has maximal length, therefore the adjacent region along this segment must be an s-region. To see why that is, one can check that any other label would either imply a splitting or contradict statement (iii) or (v) in Lemma 3.1.2.
It follows that v_c and v_d have sublabels ccs and sdd, respectively.

Using the modified star graph one can easily see the following:

If $d(v_c) = 5$ one the following holds:

$$i) \left\| \begin{array}{l} 2\beta + \alpha = 0 \\ 2\beta - \alpha = 0 \\ 2\beta + \alpha - 1 = 0 \\ 2\beta - \alpha - 1 = 0 \end{array} \right.$$

If $d(v_c) = 6$ one the following holds:

$$ii) \left\| \begin{array}{l} \alpha - 3\beta = 0 \\ \alpha + 3\beta = 0 \\ \alpha - 3\beta + 1 = 0 \\ \alpha + 3\beta - 1 = 0 \end{array} \right.$$

If $d(v_d) = 5$ one the following holds:

$$iii) \left\| \begin{array}{l} 2\alpha + \beta = 0 \\ 2\alpha - \beta = 0 \\ 2\alpha + \beta + 1 = 0 \\ 2\alpha - \beta + 1 = 0 \end{array} \right.$$

If $d(v_d) = 6$ one the following holds:

$$iv) \left\| \begin{array}{l} 3\alpha + \beta = 0 \\ 3\alpha - \beta = 0 \\ 3\alpha + \beta + 1 = 0 \\ 3\alpha - \beta + 1 = 0 \end{array} \right.$$

It follows that if one of the vertices v_c and v_d has degree 5 then the other one has degree≥ 7 hence $c(\Delta) \leq \max\{c(5,5,7), c(5,6,6)\} = \dfrac{3\pi}{35}$ and a positively curved region looks like Figure 3.4 (i) or (ii).

Now suppose that exactly one of the vertices v_b, v_c, v_d and v_e has degree> 2. If $d(v_b) > 2$ or $d(v_e) > 2$ it is easy to see that the segments bc, cd and de split, hence $c(\Delta) \leq c(4,4,4,4,4) < 0$.

So we can assume $d(v_b) = d(v_e) = 2$, $l(v_b) = bd$ and $l(v_e) = ec$.

If $d(v_c) > 2$ and $d(v_d) = 2$ then $l(v_d) = db$ and $d(v_c) \geq 4$.

Since $|ab| < |bc|$ and $|cd| = |de|$ it follows that de splits. We can assume there is no other splitting, otherwise $c(\Delta) \leq c(4,4,4,4) = 0$. It follows that v_a and v_c have sublabels $\mu a \mu$ and ccc respectively, which imply $d(v_a) \geq 5$ and $d(v_c) \geq 6$; moreover the splitting v in de has sublabel $\{a, \lambda\}\mu\lambda$ which implies $d(v) \geq 6$. A positively curved region Δ looks like Figure 3.4 (iii) and $c(\Delta) \leq c(5,6,6) = \dfrac{\pi}{15}$.

If $d(v_c) = 2$ and $d(v_d) > 2$ then $l(v_c) = ce$ and $d(v_d) \geq 4$.

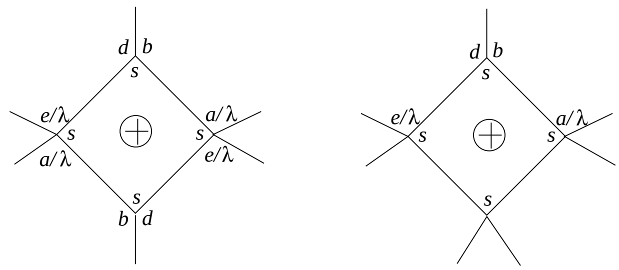
Since $|ea| < |bc|$ and $|cd| = |bc|$ it follows that bc splits. We can assume there is no other splitting, otherwise $c(\Delta) \leq c(4,4,4,4) = 0$. It follows that v_a and v_d have sublabels $\mu a \mu$ and ddd respectively, which imply $d(v_a) \geq 5$ and $d(v_d) \geq 6$; moreover the splitting v in bc has sublabel $\lambda\mu\{a, \lambda\}$ which implies $d(v) \geq 6$. A positively curved region Δ looks like Figure 3.4 (iv) and $c(\Delta) \leq c(5,6,6) = \dfrac{\pi}{15}$.

Finally suppose $d(v_b) = d(v_c) = d(v_d) = d(v_e) = 2$, hence $l(v_b) = l(v_d) = bd$ and $l(v_c) = l(v_e) = ce$.

Since $|ea| < |bc|$, $|ab| < |de|$ and be is not a segment label it follows that the segments bc, de and cd split and so $c(\Delta) \leq c(4,4,4,4) = 0$.

$\square$

Lemma 3.2.4 *Let Δ be an interior s-region of degree 4 in $\mathscr{D}$. If $j \leq \dfrac{n}{2}$ and*



(*i*) $c(\Delta) \leqslant \frac{\pi}{3}$ (*ii*) $c(\Delta) \leqslant \frac{\pi}{6}$

Figure 3.5: positively curved interior s-regions of degree 4 for $j \leq \dfrac{n}{2}$.

$c(\Delta) > 0$ *then* Δ *is one of the regions in* Figure 3.5, *in particular* $c(\Delta) \leq \dfrac{\pi}{3}$.

Proof. If there is no vertex of degree 3 in Δ then $c(\Delta) \leq c(4, 4, 4, 4) = 0$.
Recall that the unique possible label involving s for a vertex of degree 3 is bds and that two such vertices cannot be adjacent.

It follows that a positively curved region is given by Figure 3.5 (i)-(ii) where $c(\Delta) \leq \dfrac{\pi}{3}$ and $c(\Delta) \leq \dfrac{\pi}{6}$ respectively.

$\square$

For the remainder of the section we will assume that diagrams $\mathscr{D}$ do not have s-regions of degree ≤ 3.

We are now able to describe the distribution process of the positive curvature for these diagrams in case (1) and case (3).

Distribution of curvature in case (1).

In this case there is no positively curved interior m-region (Lemma 3.2.2) and the positively curved interior s-regions are given by Figure 3.5 (i)-(ii) (see Lemma 3.2.4).

Let Δ be a positively curved interior s-region; we transfer the curvature $\dfrac{\pi}{6}$

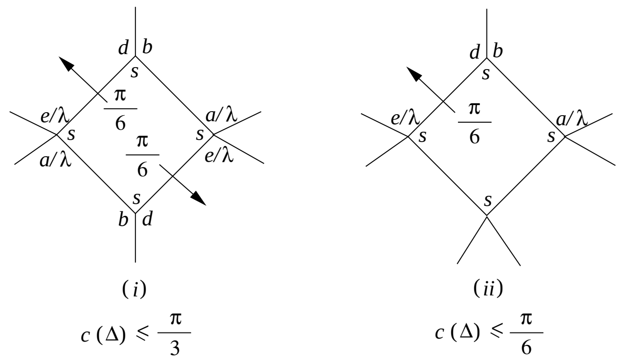


Figure 3.6: distribution scheme for $j \leq \dfrac{n}{2}$.

through each segment labelled by de adjacent to Δ and such that $l(v_d) = bds$. This choice is illustrated in Figure 3.6.

Notice that the m-segments de in Figure 3.6 might or might not split (when the adjacent vertex of degree≥ 4 has sublabel $s\lambda$ or se, respectively).

Distribution of curvature in case (3).

In this case there is no positively curved interior s-region (Lemma 3.2.1) and the positively curved interior m-regions are given by Figure 3.4 (i)-(iv) (see Lemma 3.2.3). Let Δ be a positively curved interior m-region; we transfer the curvature according to Figure 3.7.

Observe that an m-region which receives some positive curvature can not be positively curved in fact if Δ is an m-region receiving curvature then either the segments bc and cd both split or the segments cd and de both do.

Lemma 3.2.5 *If $j \leq \dfrac{n}{2}$ and Δ is an interior region of $\mathscr{D}$ then $c^*(\Delta) \leq 0$.*

Proof. We distributed the curvature in such a way that every interior s-region has non-positive pseudo-curvature; moreover there is no positively

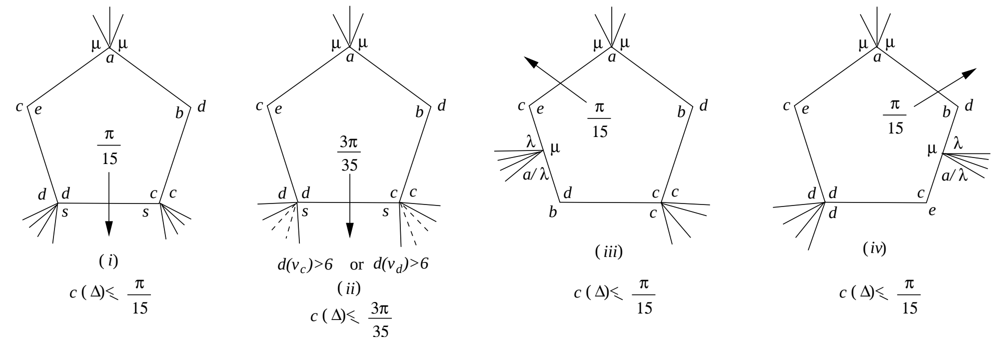


Figure 3.7: distribution scheme for $j - i > \frac{n}{2}$.

curved interior m-region, so we only need to check those m-regions which receive positive curvature from an s-region.

Let Δ be such a region, then it receives the curvature $\frac{\pi}{6}$ through the segment labelled by de, which might split.

Now observe that if de does not split then $d(v_e) \geq 4$; similarly, if cd does not split then $d(v_c) \geq 4$; moreover if ea does not split then $d(v_a) \geq 4$.

Since $d(v_b) \geq 3$, $d(v_d) \geq 3$ and the splittings cannot have degree < 4 it follows that $c^*(\Delta) \leq c(3, 3, 4, 4, 4) + \frac{\pi}{6} = 0$.

$\square$

We denote the distinguished vertex of $\mathscr{D}$ by v_0 and set $d(v_0) = k_0$.

Lemma 3.2.6 *If $j \leq \frac{n}{2}$ and Δ is a boundary region of $\mathscr{D}$ then $c^*(\Delta) < \frac{4\pi}{k_0}$.*

Proof. Let Δ be a boundary s-region.

Then Δ does not receive positive curvature.

Since t has order n in the extension $E(n; \alpha, \beta; 2, 3)$, it follows that each consequence of the relators must have exponent sum of t congruent to 0 modulo n; this implies that the distinguished vertex coincides with exactly one vertex of Δ for if not we would have a loop labelled by t^s, with $s < n$, contradicting the fact that n is the order of t.

Since $d(\Delta) \geq 4$ it follows that $c^*(\Delta) = c(\Delta) \leq c(3,3,3,k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Now let Δ be a boundary m-region. The maximum amount of curvature that Δ can receive is $\dfrac{\pi}{6}$.

Suppose the distinguished vertex v_0 coincides with m vertices of Δ. Notice that $k_0 \geq 2m$.

As seen before the exponent sum of t in each consequence of the relators must be congruent to 0 modulo n. This implies that $m \leq 4$ and we can have $m = 4$ only if v_0 does not coincide with any of the vertices v_a, v_b, v_c, v_d and v_e.

First suppose $m = 4$. Since $d(v_i) \geq 3$ for $i \in \{a, b, d\}$ we have $c^*(\Delta) \leq c(3,3,3,k_0,k_0,k_0,k_0) + \dfrac{\pi}{6} = -3\pi + \dfrac{8\pi}{k_0} + \dfrac{\pi}{6} \leq -3\pi + \pi + \dfrac{\pi}{6} < 0 < \dfrac{4\pi}{k_0}$.

Now suppose $m = 3$. Since $d(\Delta) \geq 4$ and the maximum total amount of curvature Δ can receive is $\dfrac{\pi}{6}$, it follows that $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{6} \leq c(3,k_0,k_0,k_0) + \dfrac{\pi}{6} = -\dfrac{4\pi}{3} + \dfrac{2\pi}{k_0} + \dfrac{4\pi}{k_0} + \dfrac{\pi}{6} \leq -\dfrac{7\pi}{6} + \dfrac{\pi}{3} + \dfrac{4\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Now suppose $m = 2$. We have $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{6} \leq c(3,3,k_0,k_0) + \dfrac{\pi}{6} = -\dfrac{2\pi}{3} + \dfrac{4\pi}{k_0} + \dfrac{\pi}{6} = -\dfrac{\pi}{2} + \dfrac{4\pi}{k_0} < \dfrac{4\pi}{k_0}$.

We can therefore assume that v_0 coincides with a unique vertex of Δ.

Suppose v_0 does not coincide with v_e or with a splitting in the segment ea.

Since $d(v_e) < 4$ implies that ea splits in degree≥ 4, it follows that $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{6} \leq c(3,3,4,k_0) + \dfrac{\pi}{6} = -\dfrac{\pi}{6} + \dfrac{2\pi}{k_0} + \dfrac{\pi}{6} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Finally if v_0 does coincide with v_e or with a splitting in ea, then it follows as in the last part of the proof of Lemma 3.2.2 that there is an interior vertex of degree≥ 4 in Δ, therefore $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{6} \leq c(3,3,4,k_0) + \dfrac{\pi}{6} =$

$$-\frac{\pi}{6}+\frac{2\pi}{k_0}+\frac{\pi}{6}=\frac{2\pi}{k_0}<\frac{4\pi}{k_0}.$$

$\square$

Lemma 3.2.7 *If* $j-i>\frac{n}{2}$ *and* Δ *is an interior region of* $\mathscr{D}$ *then* $c^*(\Delta)\leq 0$.

Proof. We distributed the curvature in such a way that every positively curved interior m-region has non-positive pseudo-curvature.

We only need to check those regions which receive positive curvature according to Figure 3.7. Let Δ be such a region.

Let Δ be an s-region receiving curvature from k positively curved m-regions. Then the incoming curvature is at most $k\frac{3\pi}{35}$ and $c(\Delta)$ is maximised when one of the end vertices of the segment through which the curvature has been transferred has degree 5 and the other one has degree 7. It follows that $c^*(\Delta)\leq -\pi(d(\Delta)-2)+\frac{\pi}{2}(d(\Delta)-k-1)+\frac{k+1}{2}\cdot\frac{2\pi}{5}+\frac{k+1}{2}\cdot\frac{2\pi}{7}+k\frac{3\pi}{35}=-\frac{\pi}{2}d(\Delta)+2\pi-k\frac{5\pi}{70}-\frac{11\pi}{70}<0$ since $d(\Delta)\geq 4$.

If Δ is an m-region receiving curvature in correspondence to the vertex v_c of degree 2, then the segments bc and cd both split in degree 5 and 6, respectively (see Figure 3.7 (*iii*)). Since $d(v_a)\geq 4$ and $d(v_e), d(v_d)<4$ implies that de splits in degree≥ 4, it follows that $c(\Delta)\leq c(4,4,5,6)=-\frac{4\pi}{15}$. The maximum total amount of curvature that Δ can receive is $\frac{\pi}{15}+\frac{\pi}{15}=\frac{2\pi}{15}$, hence $c^*(\Delta)<0$.

It remains to check those m-regions receiving curvature only in correspondence to the vertex v_d of degree 2.

In this case the segments cd and de both split in degree 6 and 5, respectively (see Figure 3.7 (*iv*)); moreover $d(v_a)\geq 4$ and $d(v_b), d(v_c)<4$ implies that the segments bc splits. It follows that $c^*(\Delta)\leq c(\Delta)+\frac{\pi}{15}\leq c(4,4,5,6)+\frac{\pi}{15}<0$.

$\square$

Lemma 3.2.8 *If $j - i > \frac{n}{2}$ and Δ is a boundary region of $\mathscr{D}$ then $c^*(\Delta) < \frac{4\pi}{k_0}$.*

Proof. Let Δ be a boundary s-region. Then Δ receives at most the positive curvature $\frac{3\pi}{35}$ and the distinguished vertex coincides with exactly one vertex of Δ.

Since $d(\Delta) \geq 4$ it follows that $c^*(\Delta) \leq c(\Delta) + \frac{3\pi}{35} \leq c(3,3,3,k_0) + \frac{3\pi}{35} = -\frac{29\pi}{70} + \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Now let Δ be a boundary m-region. The maximum total amount of curvature that Δ can receive is $\frac{\pi}{15} + \frac{\pi}{15} = \frac{2\pi}{15}$ (see Figure 3.7 (*iii*)-(*iv*)).

Suppose the distinguished vertex v_0 coincides with m vertices of Δ. Notice that $k_0 \geq 2m$.

As seen before $m \leq 4$ and we can have $m = 4$ only if v_0 does not coincide with any of the vertices v_a, v_b, v_c, v_d and v_e.

First suppose $m = 4$. Since $d(v_a) \geq 4$ we have $c^*(\Delta) \leq c(4, k_0, k_0, k_0, k_0) + \frac{2\pi}{15} = -3\pi + \frac{\pi}{2} + \frac{8\pi}{k_0} + \frac{2\pi}{15} \leq -3\pi + \frac{\pi}{2} + \pi + \frac{2\pi}{15} < 0 < \frac{4\pi}{k_0}$.

Now suppose $m = 3$. Since $d(\Delta) \geq 4$ it follows that $c^*(\Delta) \leq c(\Delta) + \frac{2\pi}{15} \leq c(4, k_0, k_0, k_0) + \frac{2\pi}{15} = -\frac{3\pi}{2} + \frac{2\pi}{k_0} + \frac{4\pi}{k_0} + \frac{2\pi}{15} \leq -\frac{41\pi}{30} + \frac{\pi}{3} + \frac{4\pi}{k_0} < \frac{4\pi}{k_0}$.

Now suppose $m = 2$. We have $c^*(\Delta) \leq c(\Delta) + \frac{2\pi}{15} \leq c(4, 4, k_0, k_0) + \frac{\pi}{15} = -\pi + \frac{4\pi}{k_0} + \frac{2\pi}{15} = -\frac{13\pi}{15} + \frac{4\pi}{k_0} < \frac{4\pi}{k_0}$.

We can therefore assume that v_0 coincides with a unique vertex of Δ.

Suppose v_0 does not coincide with v_c or with a splitting in the segment bc.

Since $d(v_c) < 4$ implies that bc splits in degree≥ 4, it follows that $c^*(\Delta) \leq c(\Delta) + \frac{2\pi}{15} \leq c(4, 4, 4, k_0) + \frac{2\pi}{15} = -\frac{\pi}{2} + \frac{2\pi}{k_0} + \frac{2\pi}{15} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Finally if v_0 does coincide with v_c or with a splitting in bc, then it is easy to see that there is an interior vertex of degree≥ 4 in Δ, therefore $c^*(\Delta) \leq c(\Delta) + \frac{2\pi}{15} \leq c(4,4,4,k_0) + \frac{2\pi}{15} = -\frac{\pi}{2} + \frac{2\pi}{k_0} + \frac{2\pi}{15} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

□

We can now easily prove the main theorem of this section.

Theorem 1 *Consider the irreducible group presentation* $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ *where* $0 < i < j < n$, $|\alpha|, |\beta| > 1$, $|\alpha| \neq |\beta|$ *and let* $\mathscr{D}$ *be a corresponding modified van Kampen diagram for* x^l *over the extension* $E(n;\alpha,\beta;i,j)$ *satisfying* **A1-A3** *or* **B1-B4***.*

Suppose that there is no s-region of degree≤ 3 *in* $\mathscr{D}$*. If one of the following conditions hold:*

- $j \leq \frac{n}{2}$, *(that is* G *is in case (1));*
- $j - i > \frac{n}{2}$, *(that is* G *is in case (3));*

then the group G *is infinite.*

Proof. If G is in case (1) it follows from Lemma 3.2.5 and Lemma 3.2.6 that the total curvature $T < 4\pi$, a contradiction to Lemma 2.3.1, hence the diagram $\mathscr{D}$ cannot exist and so G is infinite; similarly when G is in case (3) it follows from Lemma 3.2.7 and Lemma 3.2.8 that $T < 4\pi$, again a contradiction to Lemma 2.3.1.

□

We end this section with a corollary and a remark.

Corollary 3.2.9 *Let* $G = G_n\left(x_0\left[x_i^\alpha, x_j^\beta\right]\right)$ *where* $0 < i < j < n$, $|\alpha|, |\beta| > 1$, $|\alpha| \neq |\beta|$.
If $j < \dfrac{n}{3}$ *or* $j - i > \dfrac{2n}{3}$ *then* G *is infinite.*

Proof. Simply observe that in both cases the maximal segment length in an m-region of $\mathscr{D}$ is strictly less than $\dfrac{n}{3}$ hence there is no s-region of degree≤ 3.

□

Remark

Theorem 1 says that if we are in cases (1) or (3) and there exists a modified van Kampen diagram which respects certain geometric constraints then the given presentation defines an infinite group. Observe that the assumption $i \leq \dfrac{n}{2}$ (that we made in the first chapter) is useless in this context since in cases (1) and (3) we always have $i \leq \dfrac{n}{2}$. This is the reason why this assumption does not appear in the statement of the theorem. We also want to clarify why we avoided case (2) in Theorem 1. In Section 3.4 below we will perform elementary moves to transform the given presentation $G = G_n\left(x_0\left[x_i^\alpha, x_j^\beta\right]\right)$ into another one which is either in case (1) or in case (3) and for which any modified van Kampen diagram for x^l over the extension $E(n; \alpha, \beta; i, j)$ (provided it is constructed as outlined in Chapter 2 and satysfies the constraints **A1-A3** or **B1-B4**) respects the geometric constraints of Theorem 1. This will be done ensuring that i and j respect certain conditions under which the maximal length of an m-segment is strictly less than $\dfrac{n}{3}$ so that there is no s-region of degree < 4 and Theorem 1 applies. Observe that in case (2) no assumption on the parameters n, i and j can be given to ensure that there are no s-regions of degree< 4 in the modified van Kampen diagram. In fact the segment lengths in an m-region are $n-j$, i and $j-i$ and they cannot all be strictly less than $\frac{n}{3}$: if $j-i<\frac{n}{3}$ and $i<\frac{n}{3}$ then $j<\frac{2n}{3}$ and so $n-j>\frac{n}{3}$.

3.3 Four particular cases

In this section we will assume that $n \geq 12$ is a multiple of 3, say $n=3n_1$.

Proposition 3.3.1 *Let $n=3n_1$, $n \geq 12$, $|\alpha|,|\beta|>1$ and $|\alpha| \neq |\beta|$. Then the groups $G_n\left(x_0\left[x_1^\alpha, x_{n_1}^\beta\right]\right)$ and $G_n\left(x_0\left[x_1^\alpha, x_{2n_1}^\beta\right]\right)$ are infinite.*

Proof.

Let $G \cong G_n\left(x_0\left[x_1^\alpha, x_{n_1}^\beta\right]\right)$.

Consider the split extension $E(n;\alpha,\beta;1,n_1)$ of G. There is an epimorphism

$$\begin{aligned} E(n;\alpha,\beta;1,n_1) = \quad & \langle x,t | t^n, xt^{-1}x^{-\alpha}t^{1-n_1}x^{-\beta}t^{n_1-1}x^{\alpha}t^{1-n_1}x^{\beta}t^{n_1}\rangle \twoheadrightarrow \\ \twoheadrightarrow \quad & \langle x,t|t^{n_1}, t^{-1}x^{-\alpha}tx^{-\beta}t^{-1}x^{\alpha}tx^{\beta+1}\rangle = E(n_1;\alpha,\beta;1,0) \end{aligned}$$

where the second relator in the last presentation is cyclically reduced since $\beta \neq -1$. This is an extension of $G_{n_1}\left(x_0\left[x_1^\alpha, x_0^\beta\right]\right)$ which is infinite since $n_1 \geq 4$ (see Proposition 1.2.6). It follows that $E(n;\alpha,\beta;1,n_1)$ is infinite, hence so is G. The other case is analogous.

$\square$

Proposition 3.3.2 *Let $n=3n_1$, $n \geq 12$, $|\alpha|,|\beta|>1$ and $|\alpha| \neq |\beta|$. Then the groups $G_n\left(x_0\left[x_2^\alpha, x_{2n_1+2}^\beta\right]\right)$ and $G_n\left(x_0\left[x_1^\alpha, x_{2n_1+1}^\beta\right]\right)$ are infinite.*

According to the method of proof described in Chapter 2 we assume, by way of contradiction, that $x^l=1$ in the corresponding split extension E and we denote by $\mathscr{D}$ a modified van Kampen diagram for x^l over E satifying the assumptions **B1-B4**.

In both cases the given presentation is in case (3), that is $j-i > \dfrac{n}{2}$; moreover the maximal segment length in an m-region is $n-j+i = \dfrac{n}{3}$.

We have already classified the positively curved interior m-regions (Lemma 3.2.3, Figure 3.4) and shown that there is no positively curved interior s-region of degree≥ 4 (Lemma 3.2.1). However the compensation scheme for the positively curved m-regions described in Figure 3.7 no longer works now that there are s-regions of degree 3.

We will proceed as follows:

- first we give a classification of the positively curved interior s-regions of degree 3 and describe a way to compensate their curvature distinguishing several subcases;

- secondly we give a more refined classification of the positively curved m-regions, starting from the result of Lemma 3.2.3;

- we then describe the compensation process for positively curved interior m-regions;

- we finally prove that if Δ is interior then $c^*(\Delta) \leq 0$ and that if Δ is a boundary region then $c^*(\Delta) < \dfrac{4\pi}{k_0}$.

Since each s-region is adjacent to m-regions only and since the maximal segment length in an m-region is exactly $\dfrac{n}{3}$, it follows that the three segments of an s-region of degree 3 are all segments of maximal length in the adjacent m-regions.

This gives restrictions on the possible sublabels of the vertices of such an s-region. More precisely, up to inversion, a possible sublabel is given by

choosing a label in the following brackets: $\{b, c, d, \overline{c}, \overline{d}, \overline{e}\}s\{c, d, e, \overline{b}, \overline{c}, \overline{d}\}$.
An analysis of the modified star graph shows the following easy facts:

- if there is a vertex of degree 4 one of the following holds:

$$i) \quad \alpha \pm \beta \pm 1 = 0;$$

- if there is a vertex of degree 5 one of the following holds:

$$ii) \quad \begin{aligned} &\alpha \pm 2\beta = 0; \\ &2\alpha \pm \beta = 0; \\ &\alpha \pm 2\beta \pm 1 = 0; \\ &2\alpha \pm \beta \pm 1 = 0. \end{aligned}$$

One can check straightforward that any two of the i's and the ii's are mutually exclusive. As a consequence if none of them holds then there is no positively curved interior s-region.
Moreover the occurence of a vertex of degree 4 excludes the presence of vertices of degree 5 in an s-region of degree 3 and conversely.
Therefore we only need to distribute the curvature when either there is a vertex of degree 4 or there is a vertex of degree 5.
One can also check that if Δ is an s-region of degree 3 with a vertex of degree 5 then there is no vertex of degree 6 in Δ.
If there is a vertex of degree 4 then we distribute the curvature according to Figure 3.8.
If there is a vertex of degree 5 then we distribute the curvature according to Figure 3.9.
In both cases each segment is a segment of maximal length in each adjacent m-region.

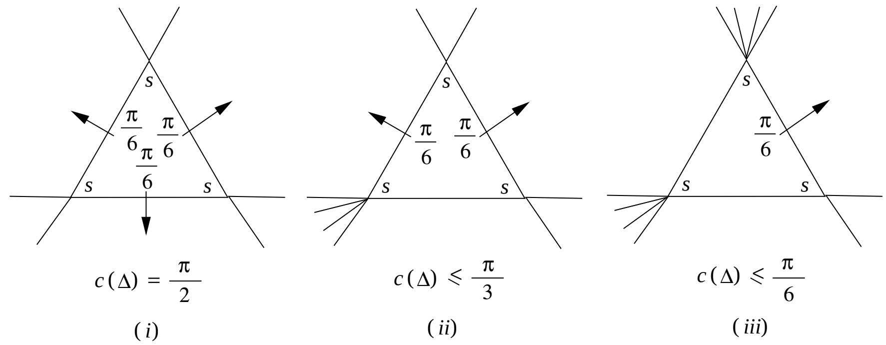


Figure 3.8: distribution for s-regions of degree 3 and at least one vertex of degree 4.

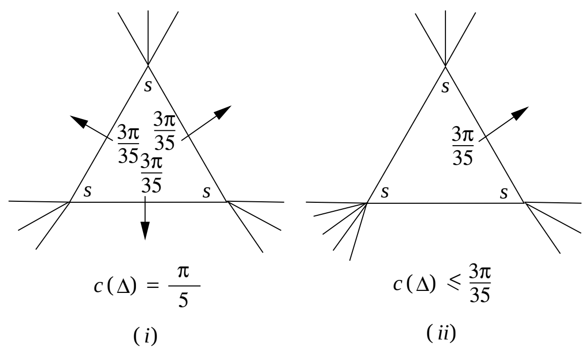


Figure 3.9: distribution for s-regions of degree 3 and at least one vertex of degree 5.

We now go back to the classification of positively curved interior m-regions starting from the results of Lemma 3.2.3.

There we proved that such a region looks like one in Figure 3.4.

We have already mentioned that the distribution process described in Figure 3.7 turns out to be appropriate only if there is no s-region of degree≤ 3.

Consider Figure 3.4 (i)-(iv).

If $|\alpha| > 2$ and $|\beta| > 2$ then $d(v_a) \geq 6$ and Δ can be positively curved only if it looks like Figure 3.4 (ii).

In this case v_c and v_d have sublabels *ccs* and *sdd* respectively; we have already proved that if v_c has degree 5 then v_d has degree≥ 7 and conversely if v_d has degree 5 then v_c has degree≥ 7.

Since Δ can have positive curvature only if there is a vertex of degree< 6 it follows that if Δ is positively curved then $c(\Delta) = c(5,6,7) = \dfrac{2\pi}{105}$ where v_a is the vertex of degree 6 and the vertices of degree 5 and 7 are v_c and v_d (not necessarily in the given order).

An analysis of the star graph shows that if $d(v_c) = 7$ then one the following constraints holds:

$3\alpha - 2\beta = 0;$	$\alpha - 4\beta = 0;$
$3\alpha + 2\beta = 0;$	$\alpha + 4\beta = 0;$
$3\alpha - 2\beta + 1 = 0;$	$\alpha - 4\beta + 1 = 0;$
$3\alpha + 2\beta - 1 = 0;$	$\alpha + 4\beta - 1 = 0;$
$2\alpha - 3\beta + 1 = 0;$	$2\alpha - 3\beta = 0;$
$2\alpha + 3\beta - 1 = 0;$	$2\alpha + 3\beta = 0;$
$\alpha - 2\beta + 1 = 0;$	$\alpha - 2\beta + 2 = 0;$
$\alpha - 2\beta - 1 = 0;$	$\alpha - 2\beta = 0;$
$\alpha + 2\beta + 1 = 0;$	$\alpha + 2\beta = 0;$
$\alpha + 2\beta - 1 = 0;$	$\alpha + 2\beta - 2 = 0.$

Similarly, if $d(v_d) = 7$ then one the following holds:

$$
\begin{array}{||l||l}
4\alpha+\beta+1=0; & 3\alpha+2\beta=0; \\
4\alpha-\beta+1=0; & 3\alpha-2\beta=0; \\
4\alpha+\beta=0; & 2\alpha+3\beta+1=0; \\
4\alpha-\beta=0; & 2\alpha-3\beta+1=0; \\
2\alpha+\beta+1=0; & 2\alpha+\beta+2=0; \\
2\alpha-\beta+1=0; & 2\alpha-\beta+2=0; \\
2\alpha+\beta=0; & 2\alpha+3\beta=0; \\
2\alpha-\beta=0; & 2\alpha-3\beta=0; \\
3\alpha+2\beta+1=0; & 2\alpha+\beta-1=0; \\
3\alpha-2\beta+1=0; & 2\alpha-\beta-1=0.
\end{array}
$$

It follows that we can have $d(v_c) = 5$ and $d(v_d) = 7$ only for $(\alpha, \beta) = (-5, 3)$ and $d(v_c) = 7$ and $d(v_d) = 5$ only for $(\alpha, \beta) = (-3, 5)$.

It follows from Figure 3.8 and 3.9 that Δ cannot receive positive curvature from the s-region adjacent to the segment cd, therefore we distribute the curvature according to Figure 3.10.

Now suppose $|\alpha| = 2$.

Consider Figure 3.4 (ii); since v_c has sublabel ccs it follows that $d(v_c) \geq 7$.

In Figure 3.4 (iii), v_c has sublabel ccc which implies $d(v_c) \geq 8$ hence $c(\Delta) \leq c(5, 6, 8) < 0$.

Similarly for $|\beta| = 2$, in Figure 3.4 (ii) we have $d(v_d) \geq 7$ and in Figure 3.4 (iv) we have $d(v_d) \geq 8$ and $c(\Delta) < 0$.

The distribution processes are described by Figure 3.11 and 3.12, respectively.

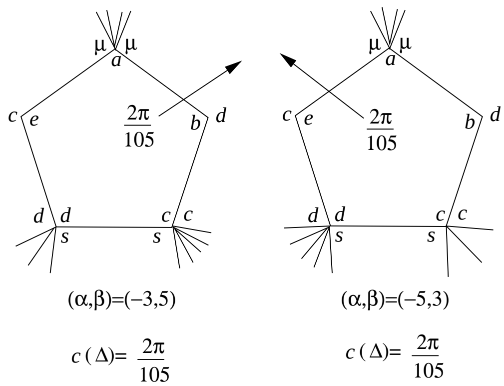


Figure 3.10: distribution for m-regions, $|\alpha| > 2$ and $|\beta| > 2$.

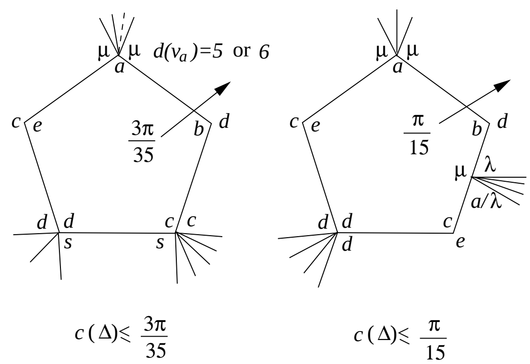


Figure 3.11: distribution for m-regions, $|\alpha| = 2$.

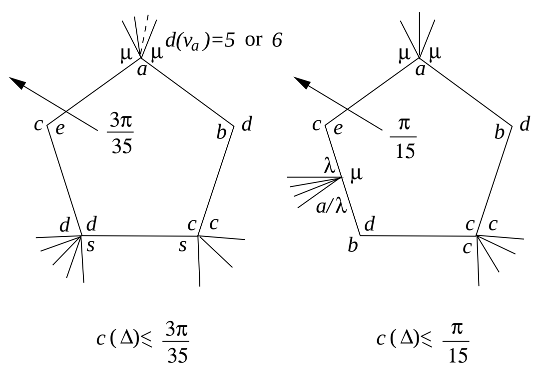


Figure 3.12: distribution for m-regions, $|\beta| = 2$.

We are now able to prove that our choice of distribution of the positive curvature is a good one.

Lemma 3.3.3 *Suppose* $(i, j) \in \{(2, \frac{2n}{3} + 2), (1, \frac{2n}{3} + 1)\}$.
If Δ *is interior then* $c^*(\Delta) \leq 0$.

Proof. Let Δ be an s-region.

We distributed the curvature in such a way that $c^*(\Delta) \leq 0$.

Now let Δ be an m-region which is receiving positive curvature from another m-region. Observe (Figure 3.10, 3.11 and 3.12) that Δ cannot be positively curved, in fact the three segments of maximal length do not split in a positively curved region. Moreover, in each of the subcases $|\alpha| = 2$, $|\beta| = 2$, $(\alpha, \beta) = (-3, 5)$ and $(\alpha, \beta) = (-5, 3)$, Δ cannot receive positive curvature from more than one m-region.

Furthermore, Δ cannot receive other curvature from more than one s-region, in fact this is transferred through maximal length segments bc, cd and de.

The maximum amount of curvature coming from an s-region is $\frac{\pi}{6}$.

If $|\alpha| = 2$ and Δ receives the curvature $\frac{3\pi}{35}$ from an m-region then $d(v_c) \geq 7$, $d(v_a) \geq 4$ and there is a splitting of degree 5 or 6 and sublabel $\mu a \mu$ in the segment de.

If $d(v_e) > 2$ or $d(v_e) = 2$ and there is another splitting in de, then $c^*(\Delta) \leq c(4, 4, 5, 7) + \frac{3\pi}{35} + \frac{\pi}{6} = -\frac{11\pi}{35} + \frac{3\pi}{35} + \frac{\pi}{6} < 0$. If $d(v_e) = 2$ and there is no other splitting in de, then the splitting in de has sublabel $\mu a \mu \lambda$ contradicting the fact that the splitting has degree 5 or 6.

If $|\alpha| = 2$ and Δ receives the curvature $\frac{\pi}{15}$ from an m-region then the segments cd and de both split; the splitting in cd has degree≥ 6 and the splitting in de has exactly degree 5.

But what has been said before implies that the splitting in de can have degree 5 only if there is another splitting in de or $d(v_e) > 2$. It follows that $c^*(\Delta) \leq c(4,4,5,6) + \frac{\pi}{15} + \frac{\pi}{6} = -\frac{4\pi}{15} + \frac{\pi}{15} + \frac{\pi}{6} < 0$.

If $|\beta| = 2$ and Δ receives the curvature $\frac{3\pi}{35}$ from an m-region then $d(v_d) \geq 7$, $d(v_a) \geq 4$ and there is a splitting of degree 5 or 6 and sublabel $\mu a \mu$ in the segment bc.

If $d(v_b) > 2$ or $d(v_b) = 2$ and there is another splitting in bc, then $c^*(\Delta) \leq c(4,4,5,7) + \frac{3\pi}{35} + \frac{\pi}{6} = -\frac{11\pi}{35} + \frac{3\pi}{35} + \frac{\pi}{6} < 0$. If $d(v_b) = 2$ and there is no other splitting in bc, then the splitting in de has sublabel $\lambda\mu a \mu$ contradicting the fact that the splitting has degree 5 or 6.

If $|\beta| = 2$ and Δ receives the curvature $\frac{\pi}{15}$ then the segments bc and cd both split; the splitting in cd has degree$\geq$ 6 and the splitting in bc has exactly degree 5.

But the splitting in bc can have degree 5 only if there is another splitting in bc or $d(v_b) > 2$. It follows that $c^*(\Delta) \leq c(4,4,5,6) + \frac{\pi}{15} + \frac{\pi}{6} = -\frac{4\pi}{15} + \frac{\pi}{15} + \frac{\pi}{6} < 0$.

If $(\alpha, \beta) = (-3, 5)$ the maximum amount of curvature that Δ can receive from an m-region is $\frac{2\pi}{105}$. We have $d(v_a) \geq 4$, $d(v_c) = 7$ and there is a splitting of degree 6 and sublabel $\mu a \mu$ in the segment de.

As seen before since the splitting has degree 6 it follows that there is another splitting in de or $d(v_e) > 2$. Therefore $c^*(\Delta) \leq c(4,4,6,7) + \frac{2\pi}{105} + \frac{\pi}{6} = -\frac{8\pi}{21} + \frac{2\pi}{105} + \frac{\pi}{6} < 0$.

If $(\alpha, \beta) = (-5, 3)$ the maximum amount of curvature that Δ can receive from an m-region is $\frac{2\pi}{105}$. We have $d(v_a) \geq 4$, $d(v_d) = 7$ and there is a splitting of degree 6 and sublabel $\mu a \mu$ in the segment bc.

Since the splitting has degree 6 it follows that there is another splitting in bc

or $d(v_b) > 2$. Therefore $c^*(\Delta) \leq c(4,4,6,7)+\frac{2\pi}{105}+\frac{\pi}{6} = -\frac{8\pi}{21}+\frac{2\pi}{105}+\frac{\pi}{6} < 0$.

It remains to consider those m-regions which receive positive curvature from s-regions only.

Since we are transferring the positive curvature through the segments of maximal length it follows that Δ cannot receive positive curvature from more than three s-regions.

Recall that in each m-region we have $d(v_a) \geq 4$.

First suppose Δ receives curvature from three s-regions, that is through the segments bc, cd and de.

Then we have $d(v_b) \geq 4$, $d(v_c) \geq 4$, $d(v_d) \geq 4$ and $d(v_e) \geq 4$.

Since the maximum total amount of curvature that Δ can receive is $\frac{\pi}{2}$, it follows that $c^*(\Delta) \leq c(4,4,4,4,4) + \frac{\pi}{2} = 0$.

Now suppose Δ receives curvature from exactly two s-regions.

These can either be bc and cd, bc and de or cd and de.

If the curvature arrives through bc and cd then $d(v_b) \geq 4$, $d(v_c) \geq 4$ and $d(v_d) \geq 4$. The maximum total amount of curvature that Δ can receive is $\frac{\pi}{3}$.

If $d(v_e) > 2$ or there is a splitting in Δ then $c^*(\Delta) \leq c(4,4,4,4,4) + \frac{\pi}{3} < 0$; so assume that $l(v_e) = ec$ and the segments de and ea do not split. This implies that v_d and v_a have sublabels sdd and μa respectively, hence $d(v_d) \geq 5$ and $d(v_a) \geq 5$. Furthermore the vertex v_c has sublabel $\overline{s}cs$, which implies $d(v_c) \geq 5$.

If we are transferring $\frac{3\pi}{35}$ through each segment then we also have $d(v_b) \geq 5$ and $c^*(\Delta) \leq c(5,5,5,5) + \frac{6\pi}{35} < 0$.

If we are transferring $\frac{\pi}{6}$ through each segment then $d(v_c) \geq 6$ and $d(v_d) \geq 6$, therefore $c^*(\Delta) \leq c(4,5,6,6) + \frac{\pi}{3} < 0$.

If the curvature arrives through bc and de then $d(v_b) \geq 4$, $d(v_c) \geq 4$, $d(v_d) \geq 4$ and $d(v_e) \geq 4$, hence $c^*(\Delta) \leq c(4,4,4,4,4) + \dfrac{\pi}{3} < 0$.

If the curvature arrives through cd and de then $d(v_c) \geq 4$, $d(v_d) \geq 4$ and $d(v_e) \geq 4$. The maximum total amount of curvature that Δ can receive is $\dfrac{\pi}{3}$.

If $d(v_b) > 2$ or there is a splitting in Δ then $c^*(\Delta) \leq c(4,4,4,4,4) + \dfrac{\pi}{3} < 0$; so assume that $l(v_b) = bd$ and the segments ab and bc do not split. This implies that v_c and v_a have sublabels ccs and $a\mu$ respectively, hence $d(v_c) \geq 5$ and $d(v_a) \geq 5$. Furthermore the vertex v_d has sublabel $sd\overline{s}$, which implies $d(v_d) \geq 5$.

If we are transferring $\dfrac{3\pi}{35}$ through each segment then we also have $d(v_e) \geq 5$ and $c^*(\Delta) \leq c(5,5,5,5) + \dfrac{6\pi}{35} < 0$.

If we are transferring $\dfrac{\pi}{6}$ through each segment then $d(v_c) \geq 6$ and $d(v_d) \geq 6$, therefore $c^*(\Delta) \leq c(4,5,6,6) + \dfrac{\pi}{3} < 0$.

Finally assume that Δ receives positive curvature from exactly one s-region; this can arrive through either bc, cd or de and can be $\dfrac{\pi}{6}$ or $\dfrac{3\pi}{35}$.

Suppose Δ receives the curvature $\dfrac{\pi}{6}$ through bc only.

If $d(v_e) = d(v_b) = 2$ then the segment de splits, therefore there is another vertex of degree≥ 4. Now we can assume that $d(v_b) = d(v_c) = 4$, for if not v_b or v_c has degree≥ 6 and $c^*(\Delta) \leq c(4,4,4,6) + \dfrac{\pi}{6} = 0$. But $d(v_c) = 4$ implies that v_c has sublabel $\overline{s}c\overline{a}$; since $|\overline{b}\overline{a}| < |cd|$ it follows that cd splits and $c^*(\Delta) \leq c(4,4,4,4,4) + \dfrac{\pi}{6} < 0$.

Suppose Δ receives the curvature $\dfrac{3\pi}{35}$ through bc only.

If $d(v_e) = d(v_b) = 2$ then the segment de splits, therefore there is another vertex of degree≥ 4; moreover we have $d(v_a) \geq 4$, $d(v_b) \geq 5$ and $d(v_c) \geq 5$. It follows that $c^*(\Delta) \leq c(4,4,5,5) + \dfrac{3\pi}{35} < 0$.

Suppose Δ receives the curvature $\frac{\pi}{6}$ through cd only.

If $d(v_c) = 4$ then bc splits and if $d(v_d) = 4$ then de splits. We can therefore assume $d(v_c) = d(v_d) = 4$ for if not then $c^*(\Delta) \leq c(4, 4, 4, 6) + \frac{\pi}{6} = 0$. It follows that bc and de both split and $c^*(\Delta) \leq c(4, 4, 4, 4, 4) + \frac{\pi}{6} < 0$.

Suppose Δ receives the curvature $\frac{3\pi}{35}$ through cd only.

We have $d(v_c) = d(v_d) = 5$, but this implies that either bc or de splits, hence $c^*(\Delta) \leq c(4, 4, 5, 5) + \frac{3\pi}{35} < 0$.

Now suppose Δ receives the curvature $\frac{\pi}{6}$ through de only.

If $d(v_b) = d(v_c) = 2$ then the segment bc splits, therefore there is another vertex of degree≥ 4. Now we can assume that $d(v_d) = d(v_e) = 4$, for if not v_d or v_e has degree≥ 6 and $c^*(\Delta) \leq c(4, 4, 4, 6) + \frac{\pi}{6} = 0$. But $d(v_d) = 4$ implies that v_d has sublabel $\overline{a}d\overline{s}$; since $|\overline{ae}| < |cd|$ it follows that cd splits and $c^*(\Delta) \leq c(4, 4, 4, 4, 4) + \frac{\pi}{6} < 0$.

Finally, suppose Δ receives the curvature $\frac{3\pi}{35}$ through de only.

If $d(v_b) = d(v_c) = 2$ then the segment bc splits, therefore there is another vertex of degree≥ 4; moreover we have $d(v_a) \geq 4$, $d(v_d) \geq 5$ and $d(v_e) \geq 5$. It follows that $c^*(\Delta) \leq c(4, 4, 5, 5) + \frac{3\pi}{35} < 0$.

$\square$

Recall that k_0 denotes the degree of the distinguished vertex v_0.

Lemma 3.3.4 *Suppose* $(i, j) \in \{(2, \frac{2n}{3} + 2), (1, \frac{2n}{3} + 1)\}$.
If Δ *is a boundary region then* $c^*(\Delta) < \frac{4\pi}{k_0}$.

Proof. Let Δ be an s-region.

Recall that the distinguished vertex cannot coincide with more than one vertex in Δ. Moreover an s-region does not receive any positive curvature from adjacent regions. It follows that $c^*(\Delta) = c(\Delta) \leq c(4,4,k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Now let Δ be an m-region.

Suppose that the distinguished vertex coincides with m vertices of Δ.

Notice that $k_0 \geq 2m$.

As seen before the exponent sum of t in each boundary word must be congruent to 0 modulo n. It follows that $m \leq 5$.

If $m = 4$ or 5 then v_0 concides with a splitting in each segment of maximal length and Δ does not receive positive curvature; it follows that $c^*(\Delta) = c(\Delta) \leq c(k_0,k_0,k_0,k_0) = -2\pi + \dfrac{8\pi}{k_0} \leq -\pi < \dfrac{4\pi}{k_0}$.

If $m = 3$ then v_0 must coincide with a splitting in at least two maximal length segments and the maximum amount of curvature that Δ can receive is $\dfrac{\pi}{6}$.

It follows that $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{6} \leq c(k_0,k_0,k_0) + \dfrac{\pi}{6} = -\pi + \dfrac{6\pi}{k_0} + \dfrac{\pi}{6} = -\pi + \dfrac{3\pi}{k_0} + \dfrac{3\pi}{k_0} + \dfrac{\pi}{6} \leq -\pi + \dfrac{\pi}{2} + \dfrac{3\pi}{k_0} + \dfrac{\pi}{6} < \dfrac{3\pi}{k_0} < \dfrac{4\pi}{k_0}$.

If $m = 2$ then v_0 coincides with either a splitting or an end vertex of a segment of maximal length and the maximum amount of curvature that Δ can receive is $\dfrac{\pi}{3}$.

Since $d(\Delta) \geq 3$ it follows that $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{3} \leq c(4,k_0,k_0) + \dfrac{\pi}{3} = -\pi + \dfrac{\pi}{2} + \dfrac{4\pi}{k_0} + \dfrac{\pi}{3} < \dfrac{4\pi}{k_0}$.

If $m = 1$ then the maximum total amount of curvature that Δ can receive is $\dfrac{\pi}{2}$. If $d(\Delta) > 3$ then $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{2} \leq c(4,4,4,k_0) + \dfrac{\pi}{2} = -2\pi + \dfrac{3\pi}{2} + \dfrac{2\pi}{k_0} + \dfrac{\pi}{2} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$. If $d(\Delta) = 3$ then there is an interior vertex of degree≥ 6 and the maximum amount of curvature that Δ can receive is $\dfrac{\pi}{6}$; it follows that $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{6} \leq c(4,6,k_0) + \dfrac{\pi}{6} = -\pi + \dfrac{5\pi}{6} + \dfrac{2\pi}{k_0} + \dfrac{\pi}{6} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

□

We can now prove the proposition stated at the beginning of the section.

Proof of Proposition 3.3.2.

This follows from Lemma 3.3.3 and Lemma 3.3.4 in the same way Theorem 1 follows from Lemmas 3.2.5-3.2.8.

$\square$

3.4 Applying elementary moves

In Section 3.2 we gave a geometric constraint on the modified van Kampen diagrams (no s-region of degree$\leq$ 3) under which the correspondent presentation is the presentation of an infinite group.

Corollary 3.2.9 illustrates some combinatorial hypotheses which ensure that this geometric constraint is satisfied.

In this section we want to show that under reasonable hypotheses on n we can apply a sequence of elementary moves to transform a presentation $G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ into another one for which any modified van Kampen diagram (as constructed in Chapter 2) does not have any s-region of degree$\leq$ 3.

Section 3.3 deals with those cases for which we are unable to perform such a sequence of elementary moves as it will be clear from that which follows.

The key point is given by the following technical lemma.

Lemma 3.4.1 *Let $n \in \mathbb{N}$ be odd, $n \geq 11$ and $j \in \mathbb{Z}_n$, $j \neq 0$ or 1 modulo n, $j \notin \left\{\frac{n}{3}, \frac{n}{3}+1, \frac{2n}{3}, \frac{2n}{3}+1\right\}$ (the so-called **critical values** of j). Then there exists an automorphism of $\mathbb{Z}_n$ sending $(1, j)$ to (i', j') such that one of the following is satisfied:*

- $i', j' < \dfrac{n}{3}$;

- $i' < \frac{n}{3}$, $n - j' + i' < \frac{n}{3}$.

Proof. First assume n is coprime to 3. In this case $1 \mapsto 3$ defines an automorphism of $\mathbb{Z}_n$ and $j \notin \left\{ \frac{n}{3}, \frac{n}{3} + 1, \frac{2n}{3}, \frac{2n}{3} + 1 \right\}$.
If $j < \frac{n}{3}$ there is nothing to prove.
If $\frac{n}{3} < j < \frac{4n}{9}$ apply the automorphism $1 \mapsto 3$.
Then $i' = 3 < \frac{n}{3}$ and $j' = 3j - n < \frac{n}{3}$.
If $\frac{4n}{9} < j < \frac{n}{2}$ apply the automorphism $1 \mapsto 2$.
Then $i' = 2 < \frac{n}{3}$, $j' = 2j$ so that $n - j' + i' < n - \frac{8n}{9} + 2 < \frac{n}{3}$ since $n \geq 11$.
If $\frac{n}{2} < j < \frac{2n}{3}$ apply the automorphism $1 \mapsto 2$.
Then $i' = 2 < \frac{n}{3}$ and $j' = 2j - n < \frac{n}{3}$ since $j < \frac{2n}{3}$.
If $\frac{2n}{3} < j < \frac{7n}{9}$ apply the automorphism $1 \mapsto 3$.
Then $i' = 3 < \frac{n}{3}$ and $j' = 3j - 2n < \frac{n}{3}$ since $j < \frac{7n}{9}$.
Finally if $j > \frac{7n}{9}$ then $n - j + 1 < n - \frac{7n}{9} + 1 < \frac{n}{3}$ since $n \geq 11$.
Now suppose that 3 divides n.
If $j < \frac{n}{3}$ there is nothing to prove.
By assumption $j \notin \left\{ \frac{n}{3}, \frac{n}{3} + 1 \right\}$.
If $\frac{n}{3} + 1 < j < \frac{n}{2}$ apply the automorphism $1 \mapsto 2$.
Then $i' = 2 < \frac{n}{3}$ and $n - j' + i' = n - 2j + 2 < n - \frac{2n}{3} - 2 + 2 = \frac{n}{3}$.
If $\frac{n}{2} < j < \frac{2n}{3}$ apply the automorphism $1 \mapsto 2$.
Then $i' = 2 < \frac{n}{3}$ and $j' = 2j - n < \frac{n}{3}$ since $j < \frac{2n}{3}$.
By assumption $j \notin \left\{ \frac{2n}{3}, \frac{2n}{3} + 1 \right\}$.
Finally if $j > \frac{2n}{3} + 1$ then $n - j + 1 < n - \frac{2n}{3} - 1 + 1 = \frac{n}{3}$.

$\square$

Theorem 2 *Let* $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$. *Suppose that* $h_i = 1$ *or* $h_j = 1$ *and that* $|\alpha| > 1$, $|\beta| > 1$ *and* $|\alpha| \neq |\beta|$.

If n *is odd and* $n \geq 11$ *then* G *is infinite.*

Proof. Since $h_i = 1$ or $h_j = 1$ we can assume, after applying suitable elementary moves if necessary, that $i = 1$. Since the conditions on α and β are symmetric we can assume they are unchanged; it follows that we only need to study $G_n\left(x_0\left[x_1^{\alpha}, x_j^{\beta}\right]\right)$.

Now Lemma 3.4.1 precisely says that there is an elementary move Φ such that $\Phi\left(G_n\left(x_0\left[x_1^{\alpha}, x_j^{\beta}\right]\right)\right)$ satisfies the hypotheses of Theorem 1 (hence G is infinite) except when $3|n$ and $j \in \left\{\frac{n}{3}, \frac{2n}{3}, \frac{n}{3}+1, \frac{2n}{3}+1\right\}$.

If $j \in \left\{\frac{n}{3}, \frac{2n}{3}\right\}$ then G is infinite by Proposition 3.3.1.

If $j = \frac{2n}{3} + 1$ then G is infinite by Proposition 3.3.2.

If $j = \frac{n}{3} + 1$ apply first the elementary move given by the automorphism of $\mathbb{Z}_n$ defined by $1 \mapsto 2$ to see that G is infinite by Proposition 3.3.2.

$\square$

Chapter 4

The f-irreducible case

4.1 Introduction

Consider the group presentation $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ where $0 < i < j < n$ and $i \leq \dfrac{n}{2}$.

Put $h_i := \mathrm{hcf}\,(i, n)$ and $h_j := \mathrm{hcf}\,(j, n)$.

The aim of this chapter is to provide a proof for the following two theorems.

Theorem 3 *Let $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ be irreducible. If $h_i > 1$, $h_j > 1$, $|\alpha| > 1$ and $|\beta| > 1$ then G is infinite.*

Theorem 4 *Let $G = G_n\left(x_0\left[x_i^{\alpha}, x_j^{\beta}\right]\right)$ be irreducible. If n is odd, $h_i > 1$, $h_j > 1$ and $(|\alpha|, |\beta|) \neq (1, 1)$ then G is infinite.*

For the remainder of the chapter we will assume $h_i, h_j > 1$. Before going through the proof of the theorems above as outlined in Chapter 2, we will prove a lemma which allows us to consider only those cases where n is a product of two primes (as long as the conditions on α and β are symmetric with respect to the elementary moves).

Lemma 4.1.1 *In the f-irreducible case it is enough to study the presentations* $G_{pq}\left(x_0\left[x_p^\alpha, x_q^\beta\right]\right)$ *where* p *and* q *are primes such that* $p < q$.

Proof. Let $G = G_n\left(x_0\left[x_i^\alpha, x_j^\beta\right]\right)$ and p and q be primes such that $p \mid h_i$ and $q \mid h_j$. We can assume without any loss that $p < q$. Consider the split extension $E(n;\alpha,\beta;i,j)$; add the relator t^{pq} and reduce the t-exponents modulo pq. Since $p \mid i$ it follows that $i \equiv up \bmod pq$ for some u, similarly $j \equiv vq \bmod pq$ and we have an epimorphism

$$E(n;\alpha,\beta;i,j) = \langle x,t|t^n, xt^{-i}x^{-\alpha}t^{i-j}x^{-\beta}t^{j-i}x^{\alpha}t^{i-j}x^{\beta}t^{j}\rangle \twoheadrightarrow$$

$$\twoheadrightarrow \langle x,t|t^{pq}, xt^{-up}x^{\alpha}t^{vq-up}x^{-\beta}t^{up-vq}x^{\alpha}t^{vq-up}x^{\beta}t^{vq}\rangle = E(pq;\alpha,\beta;up,vq).$$

which is an extension of $G_{pq}\left(x_0\left[x_{up}^\alpha, x_{vq}^\beta\right]\right)$.
Let $\hat{u} := u^{-1} \bmod q$ and $\hat{v} := v^{-1} \bmod p$ (recall that $\mathbb{Z}_q$ and $\mathbb{Z}_p$ are fields). By the Chinese Remainder Theorem there exists λ such that $\lambda \equiv \hat{u} \bmod q$ and $\lambda \equiv \hat{v} \bmod p$. So we have
$q \mid \lambda - \hat{u} \implies q \mid \lambda u - 1 \implies pq \mid \lambda up - p \implies \lambda up \equiv p \bmod pq$; similarly $\lambda vq \equiv q \bmod pq$. It follows that $G_{pq}\left(x_0\left[x_{up}^\alpha, x_{vq}^\beta\right]\right) \cong G_{pq}\left(x_0\left[x_p^\alpha, x_q^\beta\right]\right)$ via the isomorphism induced by the automorphism $1 \mapsto \lambda$ of $\mathbb{Z}_{pq}$. It follows that if $G_{pq}\left(x_0\left[x_p^\alpha, x_q^\beta\right]\right)$ is infinite then G is infinite, hence the claim.

□

4.2 Proof of Theorem 3

This section will be devoted to proving the following.

Proposition 4.2.1 *The group* $G = G_6\left(x_0\left[x_2^\alpha, x_3^\beta\right]\right)$ *where* $|\alpha|, |\beta| \neq 1$ *is infinite.*

Given this we can prove Theorem 3 as follows.

Proof of Theorem 3

Suppose $h_i > 3$. Since G is irreducible, that is $hcf(n, i, j) = 1$, it follows that $hcf(j, h_i) = 1$ and $h_i \nmid j - i$.

Therefore there is an epimorphism

$$E(n; \alpha, \beta; i, j) = \langle x, t | t^n, x t^{-i} x^{-\alpha} t^{i-j} x^{-\beta} t^{j-i} x^{\alpha} t^{i-j} x^{\beta} t^j \rangle \twoheadrightarrow$$
$$\twoheadrightarrow \langle x, t | t^{h_i}, x^{1-\alpha} t^{-j'} x^{-\beta} t^{j'} x^{\alpha} t^{-j'} x^{\beta} t^{j'} \rangle = E(h_i; \alpha, \beta; 0, j').$$

where the second relator is cyclically reduced since $\alpha \neq 1$.

This group is an extension of $G_{h_i}(x_0[x_0^\alpha, x_{j'}^\beta]) \cong G_{h_i}(x_0[x_0^\alpha, x_1^\beta])$ via the isomorphism induced by the automorphism $j' \mapsto 1$ of $\mathbb{Z}_{h_i}$ (since hcf$(h_i, j') = 1$ by irreducibility) which is infinite by S. J. Pride's result ([26]) since $h_i \geq 4$ (see also Proposition 1.2.6). It follows that G is infinite.

Similarly, since $\beta \neq -1$, if $h_j > 3$ then G is infinite.

This leaves the case $1 < h_i, h_j \leq 3$.

Since G is irreducible it follows that $h_i \neq h_j$.

Applying elementary moves if necessary, we can assume that $h_i = 2$ and $h_j = 3$. By Lemma 4.1.1 there is an epimorphism $E(n; \alpha, \beta; i, j) \twoheadrightarrow E(6; \alpha, \beta; 2, 3)$ which is an extension of $G_6(x_0[x_2^\alpha, x_3^\beta])$ and is infinite by Proposition 4.2.1.

It follows that G is infinite.

$\square$

We return now to the proof of Proposition 4.2.1.

In order to prove it, it is enough to show that the element x of the extension

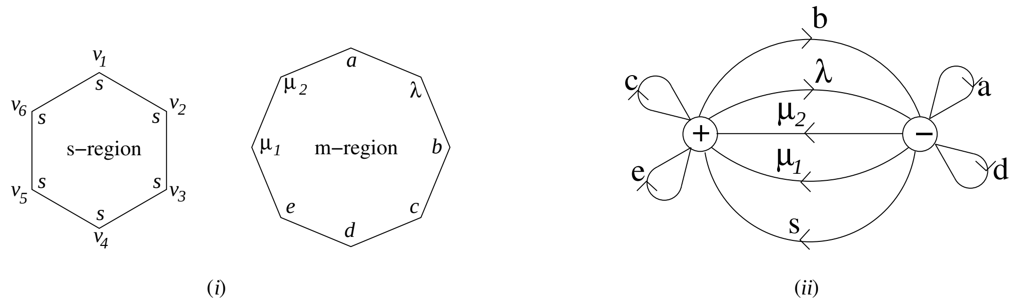


Figure 4.1: regions of $\mathscr{D}$ (i) and star graph (ii)

$E = E(6; \alpha, \beta; 2, 3)$ has infinite order.

Suppose, by way of contradiction that $x^l = 1$ for $l < \infty$. Then there is a modified van Kampen diagram $\mathscr{D}$ for x^l over E.

The regions of $\mathscr{D}$ are given, up to inversion, by Figure 4.1(i), the star graph by Figure 4.1(ii) and the values of labels by the following table:

a	b	c	d	e	s	λ	μ_1	μ_2
1	$-\alpha$	$-\beta$	α	β	0	0	0	0

If $d(v_\lambda) = 2$ we will refer to the pair of edges $(v_a v_\lambda, v_\lambda v_b)$ as a *double edge labelled by* $a\lambda b$; similarly we will use the terms *double edge labelled by* $e\mu_1\mu_2$ or $\mu_1\mu_2 a$ and *triple edge labelled by* $e\mu_1\mu_2 a$ in the obvious sense.

We can assume without any loss of generality that each of the following conditions holds:

- **C1** $\mathscr{D}$ is minimal with respect to the number of regions.

- **C2** Subject to **C1**, the number of vertices in $\mathscr{D}$ with label ce (up to inversion and cyclic permutation) is maximal.

C3 Subject to **C2**, the number of vertices of $\mathscr{D}$ of degree 2 is maximal.

C4 Subject to **C3**, the number of vertices of $\mathscr{D}$ having proper sublabel $s\overline{\mu}_1$ is minimal.

Lemma 4.2.2 *If $\mathscr{D}$ satisfies* **C1-C4** *above and $l(v)$ is a vertex label, then the following hold:*

(i) $l(v)$ is cyclically reduced;

(ii) $l(v)$ cannot have as proper sublabel $xw\overline{x}$ or $\overline{x}wx$ where w is a sublabel with zero label sum and $x \in \{a, b, c, d, e\}$;

(iii) ce, up to cyclic permutation and inversion, cannot appear as a proper sublabel of $l(v)$;

(iv) we cannot have $l(v) = y_1y_2y_3y_4$ where the y_i's are zero-valued labels;

(v) if $l(v)$ has $s\overline{\mu}_1$ a proper sublabel then it has sublabel $s\overline{\mu}_1\overline{b}$ or $s\overline{\mu}_1 s$;

*(vi) if $l(v)$ has $s\overline{\mu}_1 s$ as a proper sublabel then the adjacent vertex v' (*see Figure 4.2*) has label $s\overline{\mu}_1$.*

Proof. The proof of statements (i)-(iv) is analogous to that of Lemma 3.1.1. This leaves (v) and (vi) to be proved. Let v be a vertex in $\mathscr{D}$ with sublabel $s\overline{\mu}_1 x$. It follows from the star graph that $x \in \{a, \overline{a}, d, \overline{d}, \overline{\lambda}, \mu_2, s, \overline{b}\}$.
If $x \neq \overline{b}, s$ applying a bridge move at v relative to $s\overline{\mu}_1$ we obtain a new diagram satisfying **C1-C3** but contradicting **C4**. (See Figure 4.2).
Finally, if $x = s$ and $l(v') \neq s\overline{\mu}_1$ we can contradict assumption **C4** applying the same bridge move.

$\square$

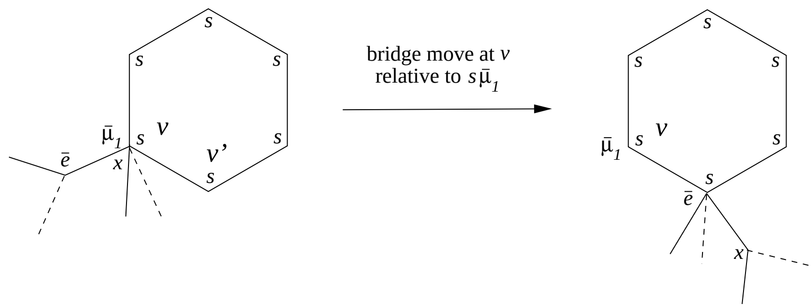


Figure 4.2: Bridge move relative to $s\overline{\mu}_1$

Martin Edjvet has checked by computer that the following holds.

Proposition 4.2.3 *If* $0 < |\alpha|, |\beta| \leq 5$ *then* $G_6(x_0[x_2^{\alpha}, x_3^{\beta}])$ *is an infinite automatic group.*

This has been done case by case with the software **KBMAG** (see [16] for details).

Thus from now on we can assume

C5 it is NOT the case that $0 < |\alpha|, |\beta| \leq 5$.

For the rest of the section we will assume that diagrams satisfy assumptions **C1-C4** and the given presentation satisfies **C5**.

Using the star graph we can make the list of possible labels for interior vertices of degree 2, 3 and 4 (up to cyclic permutation and inversion) and the consequences, if any, on α and β:

degree 2 : $ce, \lambda\mu_1, \lambda\mu_2, \lambda s, \mu_1\overline{\mu}_2, \mu_1\overline{s}, \mu_2\overline{s}$ no consequence on α and β;

degree 3 : $bds, bd\mu_1, bd\mu_2, bd\overline{\lambda}$ no consequence on α and β ;

$$daa \;\Rightarrow\; \alpha = -2 \quad ;$$

$$\overline{d}aa \;\Rightarrow\; \alpha = 2 \quad ;$$

$$b\mu_1\overline{c}, b\mu_2\overline{c}, b\overline{\lambda}\overline{c}, bs\overline{c}, b\mu_1 e, b\mu_2 e, b\overline{\lambda}e, bse \;\Rightarrow\; \alpha = \beta \quad ;$$

$$b\mu_1 c, b\mu_2 c, b\overline{\lambda}c, bsc, b\mu_1\overline{e}, b\mu_2\overline{e}, b\overline{\lambda}\overline{e}, bs\overline{e} \;\Rightarrow\; \alpha = -\beta \quad .$$

degree 4 : $ad\overline{a}\overline{d}$ no consequence on α and β ;

$$aa\{\overline{\lambda}, \mu_1, \mu_2, s\}b \;\Rightarrow\; \alpha = 2 \quad ;$$

$$aa\overline{b}\{\lambda, \overline{\mu}_1, \overline{\mu}_2, \overline{s}\} \;\Rightarrow\; \alpha = -2 \quad ;$$

$$a\{\overline{\lambda}, \mu_1, \mu_2, s\}\{c, \overline{e}\}b \;\Rightarrow\; \alpha + \beta - 1 = 0 \quad ;$$

$$a\{\overline{\lambda}, \mu_1, \mu_2, s\}\{\overline{c}, e\}b \;\Rightarrow\; \alpha - \beta - 1 = 0 \quad ;$$

$$a\overline{b}\{c, \overline{e}\}\{\lambda, \overline{\mu}_1, \overline{\mu}_2, \overline{s}\} \;\Rightarrow\; \alpha - \beta + 1 = 0 \quad ;$$

$$a\overline{b}\{\overline{c}, e\}\{\lambda, \overline{\mu}_1, \overline{\mu}_2, \overline{s}\} \;\Rightarrow\; \alpha + \beta + 1 = 0 \quad ;$$

$$d\{\overline{\lambda}, \mu_1, \mu_2, s\}\{c, \overline{e}\}\{\lambda, \overline{\mu}_1, \overline{\mu}_2, \overline{s}\} \;\Rightarrow\; \alpha = \beta \quad ;$$

$$d\{\overline{\lambda}, \mu_1, \mu_2, s\}\{\overline{c}, e\}\{\lambda, \overline{\mu}_1, \overline{\mu}_2, \overline{s}\} \;\Rightarrow\; \alpha = -\beta \quad ;$$

$$d\overline{b}\{c, \overline{e}\}\{\lambda, \overline{\mu}_1, \overline{\mu}_2, \overline{s}\} \;\Rightarrow\; 2\alpha = \beta \quad ;$$

$$d\overline{b}\{c, \overline{e}\}b \;\Rightarrow\; \alpha = \beta \quad ;$$

$$d\overline{b}\{\overline{c}, e\}\{\lambda, \overline{\mu}_1, \overline{\mu}_2, \overline{s}\} \;\Rightarrow\; 2\alpha = -\beta \quad ;$$

$$d\overline{b}\{\overline{c}, e\}b \;\Rightarrow\; \alpha = -\beta \quad ;$$

$$b\{\overline{\lambda}, \mu_1, \mu_2, s\}\{c, \overline{e}\}\{c, \overline{e}\} \;\Rightarrow\; \alpha = -2\beta \quad ;$$

$$b\{\overline{\lambda}, \mu_1, \mu_2, s\}\{\overline{c}, e\}\{\overline{c}, e\} \;\Rightarrow\; \alpha = 2\beta \quad .$$

Observe that we always have $d(v_a), d(v_b), d(v_d) \geq 3$ and we cannot have $d(v_e) = d(v_{\mu_1}) = 2$, therefore $d(\Delta) \geq 4$ for each interior m-region Δ.

Let k_0 denote the degree of the distinguished vertex of $\mathscr{D}$ and $c^*(\Delta)$ the pseudo-curvature of a region Δ.

As pointed out in the previous chapter, in order to prove that the group presented by $G_6(x_0[x_2^\alpha, x_3^\beta])$ is infinite we only need to show, after describing the distribution process, that $c^*(\Delta) \leq 0$ if Δ is interior and $c^*(\Delta) < \dfrac{4\pi}{k_0}$ if Δ is a boundary region.

Lemma 4.2.4 *Let Δ be an interior m-region of $\mathscr{D}$. Then $c(\Delta) \leq 0$.*

Proof. Distinguish the following cases:

1. $|\alpha| = 2$ (in which case $|\beta| \geq 6$);

2. $|\alpha| > 2$ and $|\beta| \neq |\alpha|$;

3. $|\alpha| > 2$ and $\beta = \alpha$;

4. $|\alpha| > 2$ and $\beta = -\alpha$.

Case 1.

If $d(v_c), d(v_e) > 2$ the assumption **C5** implies $d(v_c), d(v_e) \geq 5$.

It follows that $c(\Delta) \leq c(3, 3, 3, 5, 5) < 0$.

If $d(v_c) = d(v_e) = 2$ then $l(v_c) = l(v_e) = ce$, therefore the vertices v_b, v_d and v_{μ_1} have sublabels $b\mu_1$, ddd and $b\mu_1$ respectively; the constraint $|\alpha| = 2$ implies then $d(v_b), d(v_{\mu_1}) \geq 5$ and $d(v_d) \geq 6$.

It follows that $c(\Delta) \leq c(3, 5, 5, 6) < 0$.

If $d(v_e) = 2$ and $d(v_c) > 2$ we have $d(v_c), d(v_{\mu_1}) \geq 5$ and $d(v_d) \geq 5$.

It follows that $c(\Delta) \leq c(3, 3, 5, 5, 5) < 0$.

Similarly if $d(v_c) = 2$ and $d(v_e) > 2$ we have $d(v_e), d(v_b) \geq 5$ and $d(v_d) \geq 5$.

It follows that $c(\Delta) \leq c(3, 5, 5, 5) < 0$.

Case 2.

We have $d(v_a) \geq 4$, $d(v_b), d(v_d) \geq 3$; moreover the adjacent vertices v_e and v_{μ_1} cannot have both degree ≤ 3.

If $d(v_c) = 2$ then $l(v_c) = ce$ and as before the vertices v_b and v_d have sublabels $b\mu_1$ and dd respectively, which imply $d(v_b) \geq 4$, $d(v_d) \geq 5$.

It follows that $c(\Delta) \leq c(4, 4, 4, 5) < 0$.

If $d(v_c) > 2$ the given assumptions imply $d(v_c) \geq 4$ therefore $c(\Delta) \leq c(3, 3, 4, 4, 4) < 0$.

Case 3.

We have $d(v_a) \geq 4$, $d(v_b), d(v_d) \geq 3$.

If $d(v_e) = 2$ then $l(v_e) = ec$ and so v_d and v_{μ_1} have sublabels dd and $b\mu_1$ respectively; this implies $d(v_d) \geq 5$ and $d(v_{\mu_1}) \geq 3$.

If $d(v_{\mu_1}) > 3$ then $d(v_{\mu_1}) \geq 5$ (see the list of labels for vertices of degree 4 when $\alpha = \beta$) and we have $c(\Delta) \leq c(3, 4, 5, 5) < 0$; so assume $d(v_{\mu_1}) = 3$, in which case $l(v_{\mu_1}) = b\mu_1\{e, \overline{c}\}$ and so v_{μ_2} has sublabel $\{d, \overline{d}\}\mu_2$ and $d(v_{\mu_2}) \geq 3$.

It follows that $c(\Delta) \leq c(3, 3, 3, 4, 5) < 0$.

If $d(v_e) = 3$ then $l(v_e) = eb0$, therefore either $d(v_{\mu_1}) \geq 5$ or $d(v_{\mu_2}) \geq 5$.

It follows that $c(\Delta) \leq c(3, 3, 3, 5, 5) < 0$.

If $d(v_e) \geq 4$ we can assume $d(v_c) = 2$, otherwise $c(\Delta) \leq c(3, 3, 3, 4, 4) = 0$, therefore v_d has sublabel dd and $d(v_d) \geq 5$.

We could have positive curvature only if $d(v_{\mu_1}) = d(v_{\mu_2}) = 2$ in which case Δ must be adjacent to an s-region along the triple edge. It follows that v_a has sublabel $\overline{s}a$ which under the constraint $\alpha = \beta$ implies $d(v_a) \geq 5$.

It follows that $c(\Delta) \leq c(3,4,5,5) < 0$.

Case 4.

We have $d(v_a) \geq 4$, $d(v_b), d(v_d) \geq 3$.

Moreover at least one of the vertices v_e and v_{μ_1} has degree ≥ 3.

If $d(v_c) = 2$ then $l(v_c) = ce$ and so v_d and v_b have sublabels dd and $b\mu_1$ respectively; this implies $d(v_d) \geq 5$ and $d(v_b) \geq 3$.

If $d(v_b) > 3$ the constraint $\alpha = -\beta$ implies $d(v_b) \geq 5$ and we have $c(\Delta) \leq c(3,4,5,5) < 0$; so assume $d(v_b) = 3$, in which case $l(v_b) = b\mu_1\{c, \bar{e}\}$ and so v_λ has sublabel $\lambda\{d, \bar{d}\}$ and $d(v_\lambda) \geq 3$. It follows that $c(\Delta) \leq c(3,3,3,4,4) = 0$.

If $d(v_c) > 2$ then the assumptions imply that at least one of the vertices v_c and v_d has degree ≥ 4, therefore $c(\Delta) \leq c(3,3,3,4,4) = 0$.

$\square$

4.2.1 The case $|\alpha| = 2$.

Assume $|\alpha| = 2$. By Lemma 3.2.2 (which applies also in this particular case) we know that there is no positively curved interior m-region.

In view of **C5** we will exclude, without any specific mention, those labels which have $|\beta| \leq 5$ as a consequence.

Lemma 4.2.5 *Let Δ be an interior s-region of $\mathscr{D}$ which is not adjacent to an m-region along the triple edge labelled by $\overline{a\mu_2}\bar{\mu}_1\bar{e}$; then one of the following holds:*

(a) Δ is NOT adjacent to any double edge labelled by $a\lambda b$ and $c(\Delta) \leq 0$.

(b) Δ is adjacent to three double edges labelled by $a\lambda b$ and $c(\Delta) \leq \dfrac{\pi}{2}$;

(c) Δ is adjacent to exactly two double edges labelled by $a\lambda b$ and $c(\Delta) \leq \dfrac{\pi}{3}$;

(d) Δ is adjacent to exactly one double edge labelled by $a\lambda b$ and $c(\Delta) \leq \dfrac{\pi}{6}$.

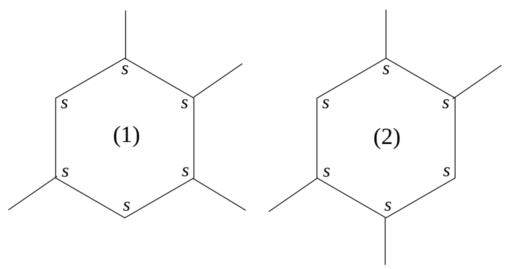
Proof. If Δ is not positively curved there is nothing to prove, therefore we can assume $d(\Delta) \leq 5$ and $c(\Delta) > 0$.

First suppose Δ is not adjacent to any double edge labelled by $a\lambda b$.

If $d(\Delta) = 5$ we can assume without any loss that the unique double edge is $v_1v_2v_3$. By assumptions this double edge can have labels $\overline{a}\overline{\mu}_2\overline{\mu}_1$ or $\overline{\mu}_2\overline{\mu}_1\overline{e}$.

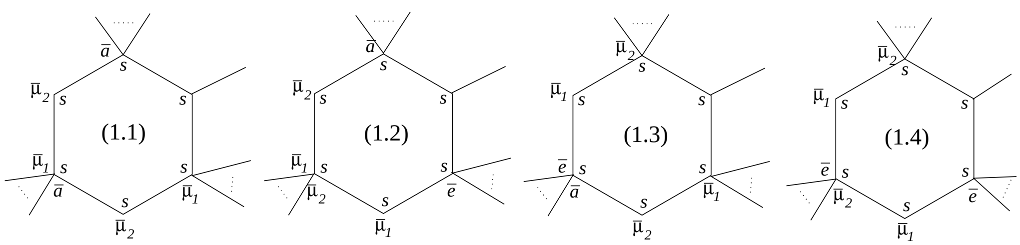
In both cases we have $d(v_1), d(v_3) \geq 4$, therefore $c(\Delta) \leq c(3,3,3,4,4) = 0$.

If $d(\Delta) = 4$, since we are assuming there is no triple edge in Δ, there are two cases:



Case (1)

There are four subcases:



Case (1.1)

The vertices v_1, v_3 and v_5 have sublabels $\overline{a}s$, $s\overline{\mu}_1$ and $\overline{a}s\overline{\mu}_1$ respectively.

It follows that $d(v_1) \geq 4$, $d(v_3), d(v_5) \geq 5$ and so $c(\Delta) \leq c(3,4,5,5) < 0$.

Case (1.2)

The vertices v_1, v_3 and v_5 have sublabels $\overline{a}s$, $s\overline{e}$ and $\overline{\mu}_2s\overline{\mu}_1$ respectively.

It follows that $d(v_1) \geq 4$, $d(v_3), d(v_5) \geq 5$ and so $c(\Delta) \leq c(3,4,5,5) < 0$.

Case (1.3)

The vertices v_1, v_3 and v_5 have sublabels $\overline{\mu}_2 s$, $s\overline{\mu}_1$ and $\overline{a}s\overline{e}$ respectively.

It follows that $d(v_1), d(v_3) \geq 5$, $d(v_5) \geq 6$ and so $c(\Delta) \leq c(3,5,5,6) < 0$.

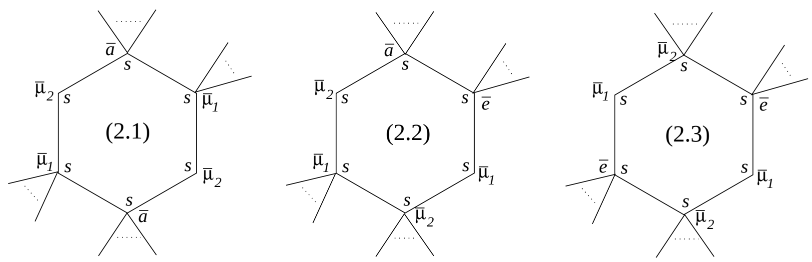
Case (1.4)

The vertices v_1, v_3 and v_5 have sublabels $\overline{\mu}_2 s$, $s\overline{e}$ and $\overline{\mu}_2 s\overline{e}$ respectively.

It follows that $d(v_1), d(v_3) \geq 5$, $d(v_5) \geq 6$ and so $c(\Delta) \leq c(3,5,5,6) < 0$.

Case (2)

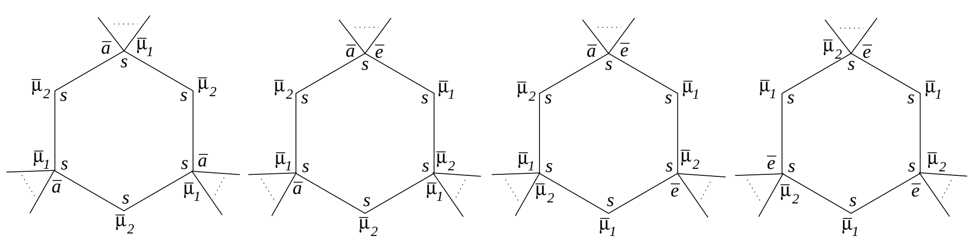
There are three subcases:



In each case we have $d(v_i) \geq 4$ for $i = 1, 2, 4, 5$.

It follows that $c(\Delta) \leq c(4,4,4,4) = 0$.

If $d(\Delta) = 3$ there is only one possible configuration and four subcases:

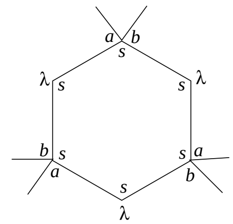


In each case we have $d(v_i) \geq 6$ for $i = 1, 3, 5$.

It follows that $c(\Delta) \leq c(6,6,6) = 0$.

Now suppose Δ is adjacent to three double edges labelled by $a\lambda b$.

There is only one possibility:



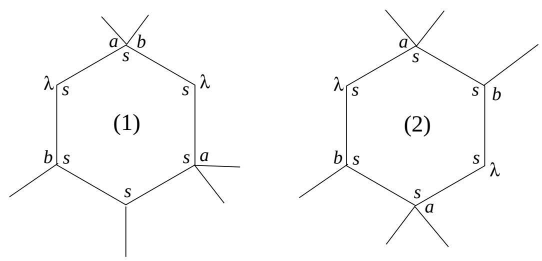
where $d(\Delta) = 3$ and v_1, v_3 and v_5 all have sublabel asb.

This implies $d(v_i) \geq 4$ for $i = 1, 3, 5$ and so $c(\Delta) \leq c(4, 4, 4) = \dfrac{\pi}{2}$.

Now suppose Δ is adjacent to exactly two double edges labelled by $a\lambda b$.

We can have $d(\Delta) = 3$ or $d(\Delta) = 4$.

If $d(\Delta) = 4$ there are two cases:



Case (1)

The vertices v_1, v_3 and v_5 have sublabels asb, as and sb respectively.

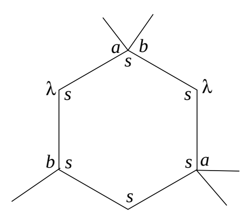
It follows that $d(v_1) \geq 4$ and $d(v_3) \geq 4$, therefore $c(\Delta) \leq c(3, 3, 4, 4) = \dfrac{\pi}{3}$.

Case (2)

The vertices v_1, v_2, v_4 and v_5 have sublabels as, sb, as and sb respectively.

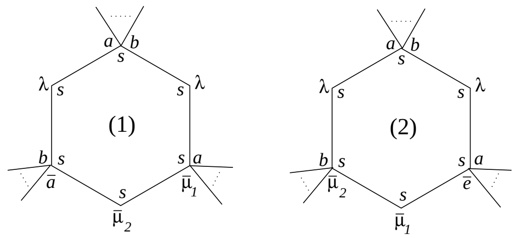
It follows that $d(v_1), d(v_4) \geq 4$, therefore $c(\Delta) \leq c(3, 3, 4, 4) = \dfrac{\pi}{3}$.

If $d(\Delta) = 3$ we can assume, without any loss, that the double edges labelled by $a\lambda b$ are $v_1v_2v_3$ and $v_5v_6v_1$.



This forces $d(v_1), d(v_3) \geq 4$.

Since $d(v_4) = 2$ there are two subcases:



Case (1)

The vertices v_1, v_3 and v_5 have sublabels asb, $as\overline{\mu}_1$ and $\overline{a}sb$ respectively.

It follows that $d(v_1), d(v_5) \geq 4$ and $d(v_3) \geq 5$; moreover either $d(v_1) \geq 5$ or $d(v_5) \geq 5$, therefore $c(\Delta) \leq c(4,5,5) = \dfrac{3\pi}{10} < \dfrac{\pi}{3}$.

Case (2)

The vertices v_1, v_3 and v_5 have sublabels asb, $as\overline{e}$ and $\overline{\mu}_2 sb$ respectively.

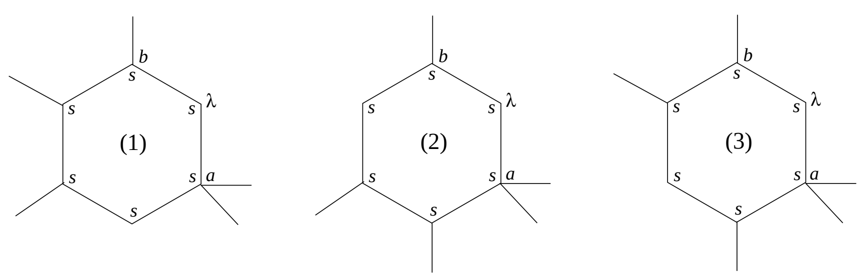
It follows that $d(v_1) \geq 4$, $d(v_3) \geq 6$ and $d(v_5) \geq 5$ and so $c(\Delta) \leq c(4,5,6) = \dfrac{7\pi}{30} < \dfrac{\pi}{3}$.

Finally suppose Δ is adjacent to exactly one double edge labelled by $a\lambda b$.

We can assume without any loss that the double edge is $v_1v_2v_3$, which forces $d(v_3) \geq 4$.

If $d(\Delta) = 5$, since $d(v_3) \geq 4$, we have $c(\Delta) \leq c(3,3,3,3,4) = \dfrac{\pi}{6}$.

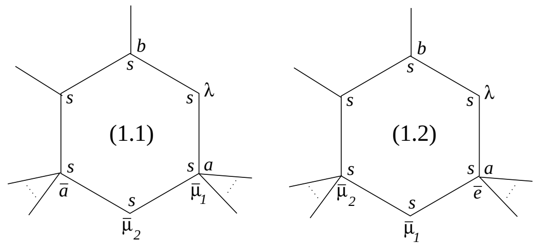
If $d(\Delta) = 4$ there are three cases:



Notice that in each case v_3 has sublabel as which implies $d(v_3) \geq 4$.

Case (1)

There are two subcases:



Case (1.1)

The vertices v_3 and v_5 have sublabels $as\overline{\mu}_1\{\overline{b}, s\}$ and $\overline{a}s$ respectively.

This implies $d(v_3) \geq 6$ and $d(v_5) \geq 4$; therefore $c(\Delta) \leq c(3, 3, 4, 6) = \dfrac{\pi}{6}$.

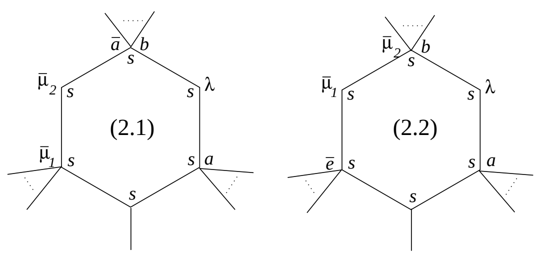
Case (1.2)

The vertices v_3 and v_5 have sublabels $as\overline{e}$ and $\overline{\mu}_2 s$ respectively.

This implies $d(v_3) \geq 6$ and $d(v_5) \geq 5$; therefore $c(\Delta) \leq c(3, 3, 5, 6) < \dfrac{\pi}{6}$.

Case (2)

There are two subcases:



Case (2.1)

The vertices v_1 and v_5 have sublabels $\overline{a}sb$ and $s\overline{\mu}_1$ respectively.

This implies $d(v_1) \geq 4$ and $d(v_5) \geq 5$; therefore $c(\Delta) \leq c(3, 4, 4, 5) < \dfrac{\pi}{6}$.

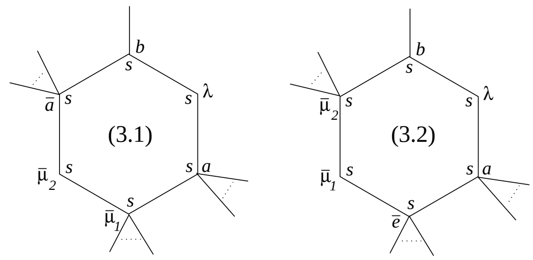
Case (2.2)

The vertices v_1 and v_5 have sublabels $\overline{\mu}_2 sb$ and $s\overline{e}$ respectively.

This implies $d(v_1) \geq 5$ and $d(v_5) \geq 5$; therefore $c(\Delta) \leq c(3, 4, 5, 5) < \dfrac{\pi}{6}$.

Case (3)

There are two subcases:



Case (3.1)

The vertices v_4 and v_6 have sublabels $s\overline{\mu}_1$ and $\overline{a}s$ respectively.

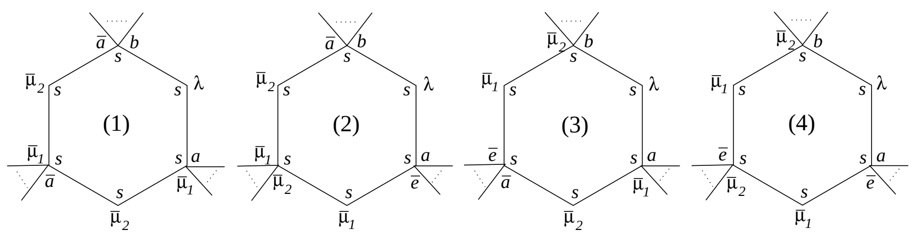
This implies $d(v_4) \geq 5$ and $d(v_6) \geq 4$; therefore $c(\Delta) \leq c(3,4,4,5) < \dfrac{\pi}{6}$.

Case (3.2)

The vertices v_4 and v_6 have sublabels $s\overline{e}$ and $\overline{\mu}_2 s$ respectively.

This implies $d(v_4), d(v_6) \geq 5$; therefore $c(\Delta) \leq c(3,4,5,5) < \dfrac{\pi}{6}$.

If $d(\Delta) = 3$ then there is only one possible configuration and the following four subcases:



Case (1)

The vertices v_1, v_3 and v_5 have sublabels $\overline{a}sb$, $as\overline{\mu}_1\{\overline{b}, s\}$ and $\overline{a}s\overline{\mu}_1\{\overline{b}, s\}$ respectively.

This implies $d(v_1) \geq 4$ and $d(v_3), d(v_5) \geq 6$; therefore $c(\Delta) \leq c(4,6,6) = \dfrac{\pi}{6}$.

Case (2)

The vertices v_1, v_3 and v_5 have sublabels $\overline{a}sb$, $as\overline{e}$ and $\overline{\mu}_2 s\overline{\mu}_1\{s, \overline{b}\}$ respectively.

This implies $d(v_1) \geq 4$, $d(v_3) \geq 6$ and $d(v_5) \geq 6$; therefore $c(\Delta) \leq c(4,6,6) = \dfrac{\pi}{6}$.

Case (3)

The vertices v_1, v_3 and v_5 have sublabels $\overline{\mu}_2 sb$, $as\overline{\mu}_1\{\overline{b}, s\}$ and $\overline{a}s\overline{e}$ respectively.

This implies $d(v_1) \geq 5$, $d(v_3) \geq 6$ and $d(v_5) \geq 6$; therefore $c(\Delta) \leq c(5,6,6) < \frac{\pi}{6}$.

Case (4)

The vertices v_1, v_3 and v_5 have sublabels $\overline{\mu}_2 sb$, $as\overline{e}$ and $\overline{\mu}_2 s\overline{e}$ respectively.

This implies $d(v_1) \geq 5$ and $d(v_3), d(v_5) \geq 6$; therefore $c(\Delta) \leq c(5,6,6) < \frac{\pi}{6}$.

$\square$

Lemma 4.2.6 *Let Δ be an interior s-region of $\mathscr{D}$ which is adjacent to an m-region along the triple edge labelled by $\overline{a\mu_2}\overline{\mu}_1\overline{e}$; then one of the following holds:*

(a) Δ is adjacent to two triple edges and $c(\Delta) \leq \frac{2\pi}{3}$.

(b) Δ is adjacent to exactly one triple edge and $c(\Delta) \leq \frac{\pi}{3}$.

Proof. Firstly suppose Δ is adjacent to two triple edges; then $d(\Delta) = 2$ and there are two vertices, say v_1 and v_4, with sublabel $\overline{a}s\overline{e}$.

This implies $d(v_1), d(v_4) \geq 6$ hence $c(\Delta) \leq c(6,6) = \frac{2\pi}{3}$.

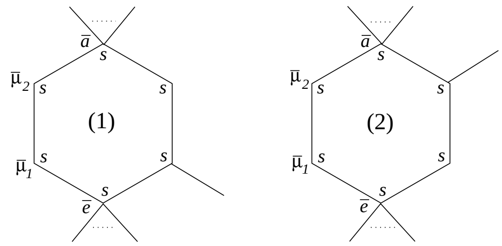
Now suppose Δ is adjacent to exactly one triple edge; we can assume this is $v_4v_5v_6v_1$.

The vertices v_1 and v_4 have sublabels $\overline{a}s$ and $s\overline{e}$ respectively, therefore $d(v_1) \geq 4$ and $d(v_4) \geq 5$.

If $d(\Delta) = 4$ then $c(\Delta) \leq c(3,3,4,5) = \frac{7\pi}{30} < \frac{\pi}{3}$.

We can therefore assume $d(\Delta) = 3$.

There are two different possible configurations:



Case (1)

If the double edge is labelled by $a\lambda b$ the vertex v_3 has sublabel as.

It follows that at least one of the vertices v_1 and v_3 has degree ≥ 5, hence $c(\Delta) \leq c(4,5,5) = \dfrac{3\pi}{10} < \dfrac{\pi}{3}$.

If the double edge is labelled by $\overline{a\mu_2}\overline{\mu}_1$ the vertex v_1 has sublabel $\overline{a}s\overline{\mu}_1$ hence $d(v_1) \geq 5$. It follows that $c(\Delta) \leq c(4,5,5) = \dfrac{3\pi}{10} < \dfrac{\pi}{3}$.

If the double edge is labelled by $\overline{\mu}_2\overline{\mu}_1\overline{e}$ the vertex v_1 has sublabel $\overline{a}s\overline{e}$ hence $d(v_1) \geq 6$, moreover v_3 has sublabel $\overline{\mu}_2 s$ hence $d(v_3) \geq 5$.

It follows that $c(\Delta) \leq c(5,6,6) = \dfrac{\pi}{15} < \dfrac{\pi}{3}$.

Case (2)

If the double edge is labelled by $a\lambda b$ the vertex v_4 has sublabel $as\overline{e}$ which implies $d(v_4) \geq 6$.

If $d(v_2) \geq 4$ we have $c(\Delta) \leq c(4,5,6) = \dfrac{7\pi}{30} < \dfrac{\pi}{3}$.

If $d(v_2) = 3$ then $l(v_2) = bds$ which forces v_1 to have sublabel $\overline{a}se$ and so $d(v_1) \geq 6$. It follows that $c(\Delta) \leq c(3,6,6) = \dfrac{\pi}{3}$.

If the double edge is labelled by $\overline{a\mu_2}\overline{\mu}_1$ the vertex v_2 has sublabel $s\overline{\mu}_1$ hence $d(v_2) \geq 5$. It follows that $c(\Delta) \leq c(4,5,5) = \dfrac{3\pi}{10} < \dfrac{\pi}{3}$.

If the double edge is labelled by $\overline{\mu}_2\overline{\mu}_1\overline{e}$ the vertex v_2 has sublabel $s\overline{e}$ hence $d(v_2) \geq 5$. It follows that $c(\Delta) \leq c(4,5,5) + \dfrac{3\pi}{10} < \dfrac{\pi}{3}$.

$\square$

Distribution process

Recall from Lemma 4.2.4 that there are no positively curved interior m-regions. The **pseudo-curvature** of a region in $\mathscr{D}$, denoted by $c^*(-)$, is obtained redistributing the curvature from every positively curved interior s-region Δ as follows:

- transfer the curvature $\dfrac{\pi}{6}$ through each double edge labelled by $a\lambda b$ adjacent to Δ;

- transfer the curvature $\dfrac{\pi}{3}$ through each triple edge adjacent to Δ when the triple edge is in common with an interior region.

Moreover for the boundary s-regions:

- transfer the curvature $\dfrac{\pi}{6}$ through each adjacent triple edge from each boundary s-region of degree 2 (this affects only the boundary regions).

Lemma 4.2.7 *If Δ is interior then $c^*(\Delta) \leq 0$.*

Proof. In view of Lemma 4.2.5 and Lemma 4.2.6, if Δ is an interior s-region then $c^*(\Delta) \leq 0$.

We only need to prove that $c^*(\Delta) \leq 0$ for every interior m-region Δ.

Distinguish two cases:

A) the region Δ receives positive curvature from exactly one s-region;

B) the region Δ receives positive curvature from exactly two s-regions.

Notice that since the positive curvature is distributed only through particular edges it follows that Δ cannot receive curvature from more than two s-regions.

Case A)

There are two subcases:

$A_1)$ the region Δ receives positive curvature through the double edge $a\lambda b$;

$A_2)$ the region Δ receives positive curvature through the triple edge $e\mu_1\mu_2 a$.

Case $A_1)$

Observe that $d(v_a) \geq 4$; moreover at least one of the vertices v_b and v_c has degree ≥ 5.

If $d(v_e) = 2$ then $l(v_e) = ce$ which implies that v_{μ_1} and v_d have sublabels $b\mu_1$ and dd respectively. This implies $d(v_{\mu_1}), d(v_d) \geq 5$, hence $c^*(\Delta) \leq c(4,5,5,5) + \dfrac{\pi}{6} < 0$.

If $d(v_e) > 2$ then the constraint $|\alpha| = 2$ implies $d(v_e) \geq 5$, moreover $d(v_d) \geq 3$.

If $d(v_c) = 2$ then $l(v_c) = ce$ which implies that v_b and v_d have sublabels $b\mu_1$ and dd respectively. This implies $d(v_b), d(v_d) \geq 5$, hence $c^*(\Delta) \leq c(4,5,5,5) + \dfrac{\pi}{6} < 0$.

If $d(v_c) > 2$ then the constraint $|\alpha| = 2$ implies $d(v_c) \geq 5$. It follows that $c^*(\Delta) \leq c(3,3,4,5,5) + \dfrac{\pi}{6} < 0$.

Case $A_2)$

We have $d(v_1) \geq 4$, $d(v_b), d(v_d) \geq 3$ and $d(v_e) \geq 5$.

If $d(v_c) = 2$ then $l(v_c) = ce$ which implies that v_b and v_d have sublabels $b\mu_1$ and dd respectively. This implies $d(v_b), d(v_d) \geq 5$; moreover either v_d or v_e has degree ≥ 6, hence $c^*(\Delta) \leq c(4,5,5,6) + \dfrac{\pi}{3} < 0$.

If $d(v_c) > 2$ then the constraint $|\alpha| = 2$ implies $d(v_c) \geq 5$.

It follows that $c^*(\Delta) \leq c(3,3,4,5,5) + \dfrac{\pi}{3} < 0$.

Case B)

It follows from the distribution process that v_a has sublabel $\overline{s}as$ which implies $d(v_a) \geq 6$. Moreover $d(v_e) \geq 5$ and $d(v_b), d(v_d) \geq 3$.

If $d(v_c) = 2$ then $l(v_c) = ce$ which implies that v_b and v_d have sublabels $b\mu_1$

and dd respectively. This implies $d(v_b), d(v_d) \geq 5$; moreover either v_d or v_e has degree ≥ 6, hence $c^*(\Delta) \leq c(5,5,6,6) + \frac{\pi}{3} + \frac{\pi}{6} < 0$.

If $d(v_c) > 2$ then the constraint $|\alpha| = 2$ implies $d(v_c) \geq 5$.

It follows that $c^*(\Delta) \leq c(3,3,5,5,6) + \frac{\pi}{3} + \frac{\pi}{6} < 0$.

$\square$

Lemma 4.2.8 *If Δ is a boundary region then $c^*(\Delta) < \frac{4\pi}{k_0}$.*

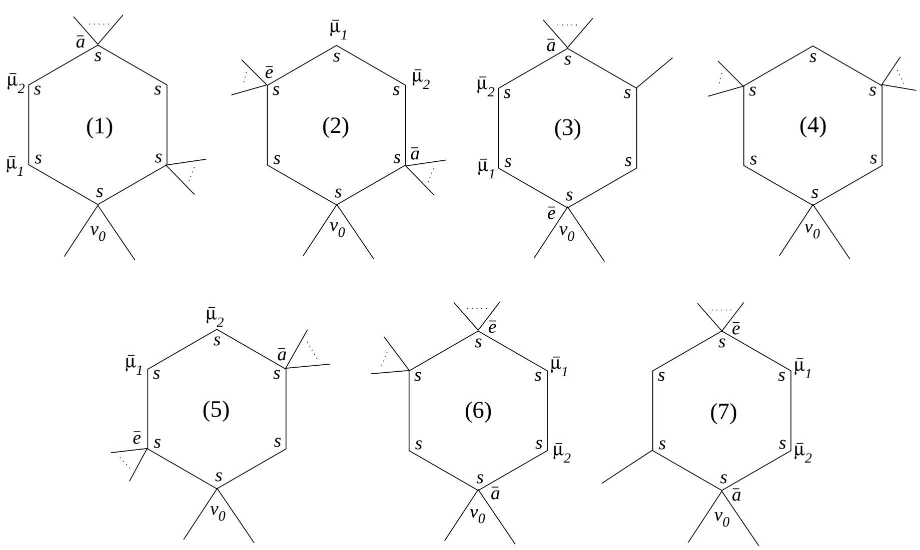
Proof. Let Δ be a boundary s-region.

Then Δ does not receive positive curvature.

Since t has order n in the extension $E(n;\alpha,\beta;2,3)$, it follows that each consequence of the relators must have exponent sum of t congruent to 0 modulo n; this implies that the distinguished vertex coincides with exactly one vertex of Δ.

Observe that if $d(\Delta) \geq 4$ then $c(\Delta) \leq c(3,3,3,k_0) = \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$, so we can assume $d(\Delta) \leq 3$.

If $d(\Delta) = 3$ there are seven subcases:



Observe that in cases (1), (2), (4), (5) and (6) there are two vertices of degree ≥ 4. It follows that $c^*(\Delta) = c(\Delta) \leq c(4, 4, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

In case (3) we can have $d(v_2) = 3$, but only if $l(v_2) = bds$, in which case v_1 has sublabel $\overline{a}se$ hence $d(v_1) \geq 7$.

It follows that $c^*(\Delta) = c(\Delta) \leq c(3, 7, k_0) < \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

In case (7) we have $d(v_1) \geq 6$. It follows that $c^*(\Delta) = c(\Delta) \leq c(3, 6, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

If $d(\Delta) = 2$ then the vertex $v_1 \neq v_0$ has sublabel $\overline{a}s\overline{e}$ which implies $d(v_1) \geq 6$.

According to the distribution process described above we have $c^*(\Delta) \leq c(6, k_0) - \dfrac{\pi}{3} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Now let Δ be a boundary m-region.

Suppose the distinguished vertex v_0 coincides with m vertices of Δ. Notice that $k_0 \geq 2m$.

As seen before the exponent sum of t in each boundary word must be congruent to 0 modulo n. This implies that $m \leq 3$ and we can have $m = 3$ only if $v_0 = v_b = v_d = v_{\mu_1}$.

Suppose $m = 3$. Since $d(\Delta) \geq 4$ and the maximum total amount of curvature Δ can receive is $\dfrac{\pi}{2}$, it follows that $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{2} \leq c(3, k_0, k_0, k_0) + \dfrac{\pi}{2} = -\dfrac{4\pi}{3} + \dfrac{2\pi}{k_0} + \dfrac{4\pi}{k_0} + \dfrac{\pi}{2} \leq -\dfrac{5\pi}{6} + \dfrac{\pi}{3} + \dfrac{4\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Now suppose $m = 2$. We have $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{2} \leq c(3, 3, k_0, k_0) + \dfrac{\pi}{2} = -\dfrac{2\pi}{3} + \dfrac{4\pi}{k_0} + \dfrac{\pi}{2} = -\dfrac{\pi}{6} + \dfrac{4\pi}{k_0} < \dfrac{4\pi}{k_0}$.

We can therefore assume that v_0 coincides with a unique vertex of Δ.

Suppose v_0 concides with v_a, v_λ or v_{μ_2}. We have $d(v_d) \geq 3$ and $d(v_b) \geq 3$.

Moreover the vertices v_e and v_{μ_1} cannot have both degree ≤ 4, and similarly the vertices v_c and v_d cannot both have degree ≤ 4.

Since the maximum total amount of curvature that Δ can receive is $\frac{\pi}{3}$ we have $c^*(\Delta) \leq c(3,5,5,k_0) + \frac{\pi}{3} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Now let v_0 concides with v_b.

We can use the same argument as before with v_a playing the role of v_b.

Suppose v_0 concides with v_c. We have $d(v_a), d(v_b), d(v_d) \geq 3$.

Moreover the vertices v_e and v_{μ_1} cannot have both degree ≤ 4.

Since the maximum total amount of curvature that Δ can receive is $\frac{\pi}{2}$ we have $c^*(\Delta) \leq c(3,3,3,5,k_0) + \frac{\pi}{2} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Now assume v_0 concides with v_d. We have $d(v_a), d(v_b) \geq 3$.

Moreover the vertices v_e and v_{μ_1} cannot have both degree ≤ 4 and the vertices v_c and v_b cannot have both degree ≤ 4.

Since the maximum total amount of curvature that Δ can receive is $\frac{\pi}{2}$ we have $c^*(\Delta) \leq c(3,5,5,k_0) + \frac{\pi}{2} < \frac{2\pi}{k} < \frac{4\pi}{k_0}$.

Let v_0 concides with v_e. We have $d(v_a), d(v_b), d(v_d) \geq 3$.

Since $v_e = v_0$, the maximum total amount of curvature that Δ can receive is $\frac{\pi}{3}$ (in fact the triple edge, if any, is in common with another boundary region).

If $d(v_c) = 2$ then $l(v_c) = ce$ which implies that v_b and v_d have sublabels $b\mu_1$ and dd respectively. This implies $d(v_b), d(v_d) \geq 5$, hence $c^*(\Delta) \leq c(3,5,5,k_0) + \frac{\pi}{3} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

If $d(v_c) > 2$ then the assumption **C5** implies $d(v_c) \geq 5$.

It follows that $c^*(\Delta) \leq c(3,3,3,5,k_0) + \frac{\pi}{3} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Finally suppose v_0 concides with v_{μ_1}. We have $d(v_a), d(v_b), d(v_d) \geq 3$.

Moreover the vertices v_e and v_d cannot have both degree ≤ 4 and the vertices v_c and v_b cannot have both degree ≤ 5.

Since the maximum total amount of curvature that Δ can receive is $\frac{\pi}{6}$ we have $c^*(\Delta) \leq c(3,5,5,k_0) + \frac{\pi}{6} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

$\square$

Now the result follows from what has been said in Chapter 2.

4.2.2 The case $|\alpha| > 2$.

Before describing the distribution process we must produce a list of the positively curved interior regions in $\mathscr{D}$. In view of Lemma 3.1.2 there is no positively curved interior m-region.

We will distinguish three cases:

(a) $|\alpha| \neq |\beta|$;

(b) $\alpha = \beta$;

(c) $\alpha = -\beta$;

Let Δ be an interior s-region of degree 5. We can assume without any loss that the vertex of degree 2 is v_6 (see Figure 4.1 (i)).

Case (a)

The unique possible label for a vertex of degree 3 is bds.

Since we are assuming $d(v_6) = 2$ it follows that $d(v_1) \geq 4$ (simply by checking the possible labels for the adjacent double edge).

If $d(v_5) \geq 4$ then $c(\Delta) \leq c(3,3,3,4,4) = 0$; therefore we can assume $d(v_5) = 3$ and $l(v_5) = bds$. This implies that v_4 has sublabel se which forces $d(v_4) \geq 4$ (see Figure 4.3 (i)). It follows that $c(\Delta) \leq c(3,3,3,4,4) = 0$.

Case (b)

The possible labels involving s for a vertex of degree 3 are:

$$bds\ ,\ bse \text{ and } bs\bar{c}.$$

Since we are assuming $d(v_6) = 2$ it follows that $d(v_1) \geq 4$.

If $d(v_5) \geq 4$ then $c(\Delta) \leq c(3,3,3,4,4) = 0$; therefore we can assume $d(v_5) = 3$.

If $l(v_5) = bse$ or $l(v_5) = bs\bar{c}$ it follows that v_4 has sublabel sc, in which case $d(v_4) \geq 4$ and $c(\Delta) \leq c(3,3,3,4,4) = 0$ (Figure 4.3 (ii)).

We can therefore assume $l(v_5) = bds$. Clearly Δ can be positively curved only if $d(v_4) = d(v_3) = d(v_2) = 3$, but $d(v_4) = 3$ implies $l(v_4) = seb$ and then v_3 has sublabel sc and so $d(v_3) \geq 4$ (Figure 4.3 (iii)).

It follows that $c(\Delta) \leq c(3,3,3,4,4) = 0$.

Case (c)

The possible labels involving s for a vertex of degree 3 are:

$$bds\ ,\ bsc \text{ and } bs\bar{e}.$$

Since we are assuming $d(v_6) = 2$ it follows that $d(v_1) \geq 4$.

If $d(v_5) \geq 4$ then $c(\Delta) \leq c(3,3,3,4,4) = 0$; so we can assume $d(v_5) = 3$.

If $l(v_5) = bsc$ then $d(v_6) > 2$, a contradiction (see Figure 4.3 (iv)).

If $l(v_5) = bds$ then v_4 has sublabel se and this forces $d(v_4) \geq 4$ (Figure 4.3 (v)).

It follows that $c(\Delta) \leq c(3,3,3,4,4) = 0$.

We can therefore assume $l(v_5) = bs\bar{e}$. Clearly Δ can be positively curved only if $d(v_4) = d(v_3) = d(v_2) = 3$ and this forces $l(v_4) = l(v_3) = l(v_2) = bsc$.

A positively curved region would look like Figure 4.3 (vi), in which case

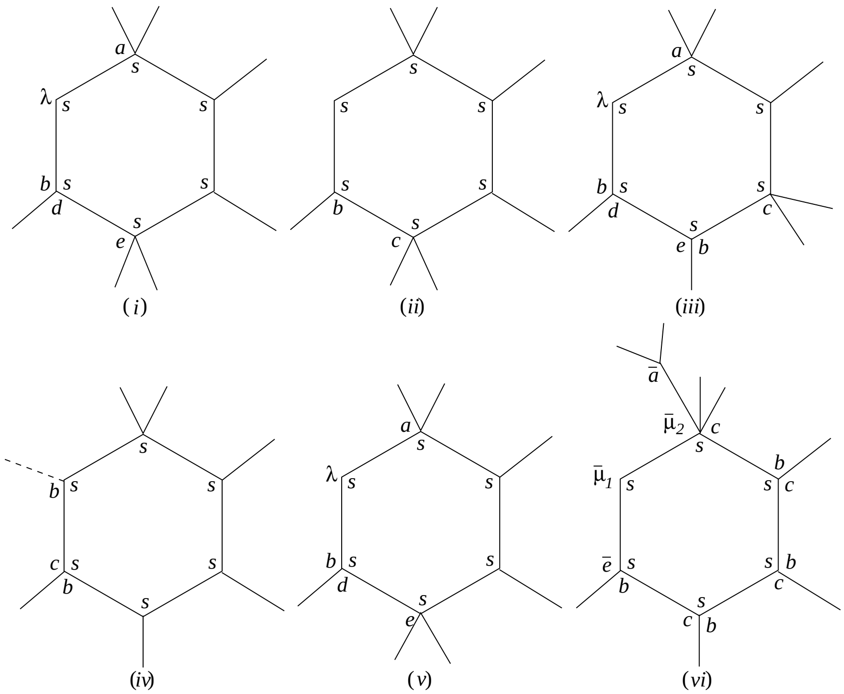


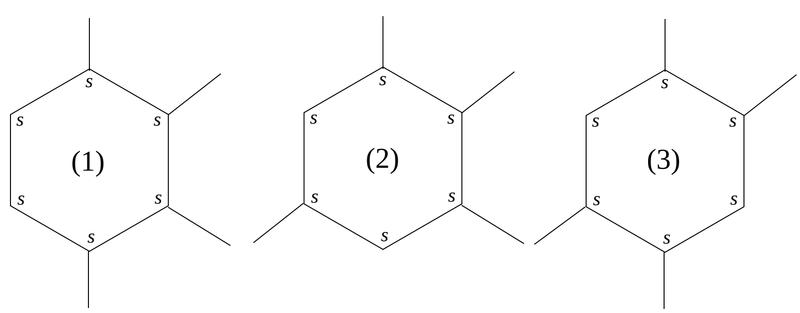
Figure 4.3: interior s-regions of degree 5

applying a bridge move at the vertex v_1 relative to the sublabel $\overline{\mu}_2 s$ we would contradict **C3**. It follows that $c(\Delta) \leq 0$.

We can conclude that there is no positively curved region of degree 5.

Now let Δ be an interior s-region of degree 4 of $\mathscr{D}$.

We can distinguish three different configurations:



Configuration (1)

Case (a)

The vertices v_1 and v_4 have sublabels $\overline{a}s$ and $s\overline{e}$ respectively.

This implies $d(v_1) \geq 4$ and $d(v_4) \geq 4$.

If $d(v_2) = 3$ then $l(v_2) = bds$. This forces v_3 to have sublabel λs, in which case a bridge move at v_3 relative to this sublabel contradicts **C3**.

It follows that $d(v_2) \geq 4$, therefore Δ can be positively curved only if $d(v_3) = 3$.

In this case v_4 has sublabel $\lambda s\overline{e}$ which implies $d(v_4) \geq 6$ (see Figure 4.4 (i)).

It follows that $c(\Delta) \leq c(3, 4, 4, 6) = 0$.

Case (b)

The vertices v_1 and v_4 have sublabels $\overline{a}s$ and $s\overline{e}$ respectively.

This implies $d(v_1) \geq 5$ and $d(v_4) \geq 4$.

In order to have positive curvature at least one of the vertices v_2 and v_3 must have degree 3.

First suppose $d(v_2) = 3$.

If $l(v_2) = bds$ then v_3 has sublabel λs, in which case a bridge move at v_3 relative to this sublabel contradicts **C3** (Figure 4.4 (ii)).

If $l(v_2) = bs\overline{c}$ then v_3 and v_1 have sublabels $\overline{d}s$ and $\overline{a}sc$ respectively, therefore $d(v_3) \geq 4$ and $d(v_1) \geq 6$ (Figure 4.4 (iii)). It follows that $c(\Delta) \leq c(3, 4, 4, 6) = 0$.

If $l(v_2) = bse$ then v_1 has sublabel $\overline{a}sc$, which implies $d(v_1) \geq 6$. Either $d(v_3) \geq 4$, in which case $c(\Delta) \leq 0$, or $d(v_3) = 3$ with $l(v_3) = bds$. In this case v_4 has sublabel $\lambda s\overline{e}$, therefore $d(v_4) \geq 5$.

It follows that Δ can be positively curved only if $d(v_1) = 6, 7$ and $d(v_4) = 5$

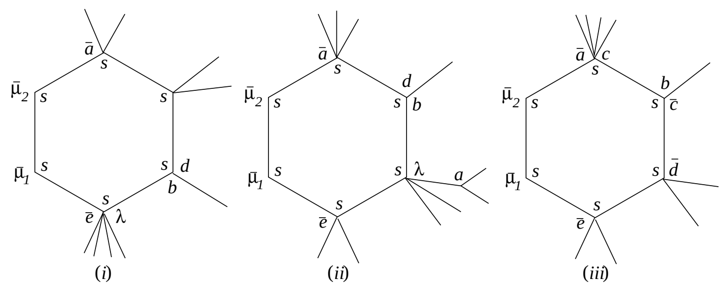


Figure 4.4: interior s-regions of degree 4 (Configuration (1))

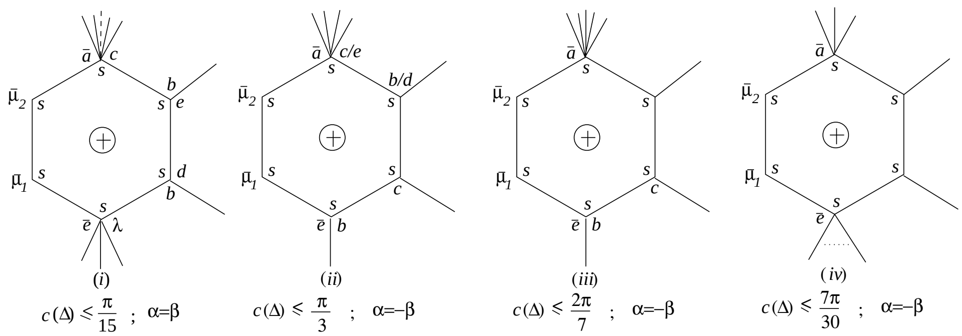


Figure 4.5: positively curved interior s-regions of degree 4 (Configuration (1))

in which case it looks like (Figure 4.5 (i)) where $c(\Delta) \leq c(3,3,5,6) = \dfrac{\pi}{15}$.
Now suppose $d(v_3) = 3$.
We have already found the positive region in the case $l(v_3) = bds$.
If $l(v_3) = bse$ or $l(v_3) = bs\bar{c}$ then v_2 has sublabel sc which implies $d(v_2) \geq 4$. Moreover either $d(v_2) \geq 5$, in which case $c(\Delta) \leq c(3,4,5,5) < 0$, or $d(v_1) \geq 6$ in which case $c(\Delta) \leq c(3,4,4,6) = 0$.

Case (c)

Here we will give different upper bounds for the curvature depending on the degree of the end vertices of the triple edge.

The reason why we do that will be clear when we will describe the distribution process of curvature.

The vertices v_1 and v_4 have sublabels $\bar{a}s$ and $s\bar{e}$ respectively.

This implies $d(v_1) \geq 5$ and $d(v_4) \geq 3$.

First assume $d(v_4) = 3$.

Observe that if $d(v_2) = 3$ then $d(v_1) \geq 6$.

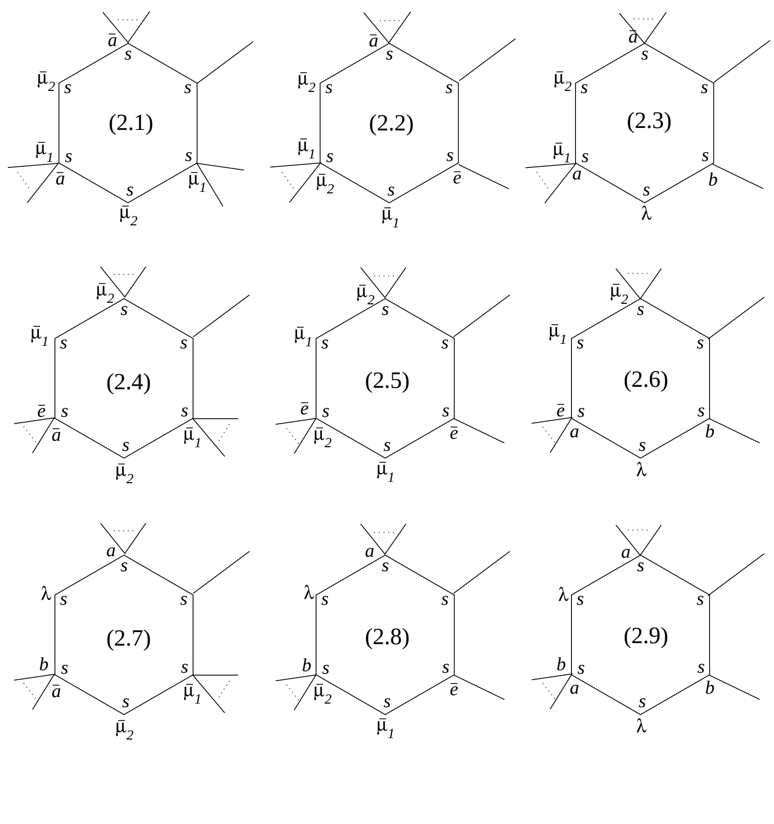
It follows that $c(\Delta) \leq max\{c(3,3,3,6), c(3,3,4,5)\} = \dfrac{\pi}{3}$ (see Figure 4.5 (ii)). Moreover if $d(v_1) \geq 7$ then $c(\Delta) \leq c(3,3,3,7) = \dfrac{2\pi}{7}$ (see Figure 4.5 (iii)).

Now suppose $d(v_4) \geq 4$.

We have $c(\Delta) \leq c(3,3,4,5) = \dfrac{7\pi}{30}$ (see Figure 4.5 (iv)).

Configuration (2)

There are nine subcases:



Configuration (2.1)

The vertices v_1, v_3 and v_5 have subalbels $\overline{a}s$, $s\overline{\mu}_1\{\overline{b}, s\}$ and $\overline{a}s\overline{\mu}_1\overline{b}$ respectively (see Lemma 4.2.2 (v) and (vi)), so $d(v_3) \geq 5$.

Case (a)

Since $|\alpha| > 2$ we have $d(v_1) \geq 4$ and $d(v_5) \geq 6$.

It follows that $c(\Delta) \leq c(3, 4, 5, 6) < 0$.

Case (b)

Since $\alpha = \beta$ we have $d(v_1) \geq 5$ and $d(v_5) \geq 7$.

It follows that $c(\Delta) \leq c(3, 5, 5, 7) < 0$.

Case (c)

Since $\alpha = -\beta$ we have $d(v_1) \geq 5$ and $d(v_5) \geq 7$.

It follows that $c(\Delta) \leq c(3, 5, 5, 7) < 0$.

Configuration (2.2)

The vertices v_1, v_3 and v_5 have subalbels $\overline{a}s$, $s\overline{e}$ and $\overline{\mu}_2 s\overline{\mu}_1\{\overline{b}, s\}$ respectively.

Case (a)

Since $|\alpha| > 2$ and $|\alpha| \neq |\beta|$, we have $d(v_1) \geq 4$, $d(v_5) \geq 6$ and $d(v_3) \geq 4$.

It follows that $c(\Delta) \leq c(3, 4, 4, 6) = 0$.

Case (b)

Since $\alpha = \beta$ we have $d(v_1) \geq 5$, $d(v_5) \geq 5$ and $d(v_3) \geq 4$.

It follows that $c(\Delta) \leq c(3, 4, 5, 5) < 0$.

Case (c)

Since $\alpha = -\beta$ we have $d(v_1) \geq 5$ and $d(v_5) \geq 5$.

It follows that we can have positive curvature only if $d(v_3) = d(v_2) = 3$. This forces $l(v_3) = bs\overline{e}$ and so $l(v_2) = bsc$.

Moreover v_1 has sublabel $\overline{a}sc$, therefore $d(v_1) \geq 6$.

A positively curved region looks like Figure 4.7 (i) where $c(\Delta) \leq c(3,3,5,6) = \frac{\pi}{15}$.

Notice that if v_5 has sublabel $\overline{\mu}_2 s \overline{\mu}_1 s$ then $d(v_5) \geq 6$ and so $c(\Delta) \leq 0$.

Configuration (2.3)

The vertices v_1, v_3 and v_5 have subalbels $\overline{a}s$, sb and $as\overline{\mu}_1\overline{b}$ respectively.

Case (a)

Since $|\alpha| > 2$ and $|\alpha| \neq |\beta|$, we have $d(v_1) \geq 4$ and $d(v_5) \geq 6$.

Moreover at least one of the vertices v_2 and v_3 must have degree ≥ 4. It follows that $c(\Delta) \leq c(3,4,4,6) = 0$.

Case (b)

Since $\alpha = \beta$ we have $d(v_1) \geq 5$ and $d(v_5) \geq 7$.

It follows that we can have positive curvature only if $d(v_3) = d(v_2) = 3$. This is possible only if $l(v_3) = bds$ and $l(v_2) = bse$, in which case v_1 has sublabel $\overline{a}sc$, therefore $d(v_1) \geq 6$ (see Figure 4.6 (i)).

It follows that $c(\Delta) \leq c(3,3,6,7) < 0$.

Case (c)

Since $\alpha = -\beta$ we have $d(v_1) \geq 5$ and $d(v_5) \geq 7$.

It follows that we can have positive curvature only if $d(v_3) = d(v_2) = 3$. This forces $l(v_3) = bds$ and so v_2 has sublabel se and cannot have degree 3. It follows that $c(\Delta) \leq c(3,4,5,7) < 0$.

Configurations (2.4), (2.5) and (2.6)

The vertex v_1 has subalbel $\overline{\mu}_2 s$, therefore $d(v_1) \geq 5$.

Since $|\alpha| \neq 1$, the vertex v in Figure 4.6 (ii) has degree > 2.

Moreover we have $d(v_2) > 2$; it follows that applying a bridge move at v_1 relative to $\overline{\mu}_2 s$ we contradict **C3**.

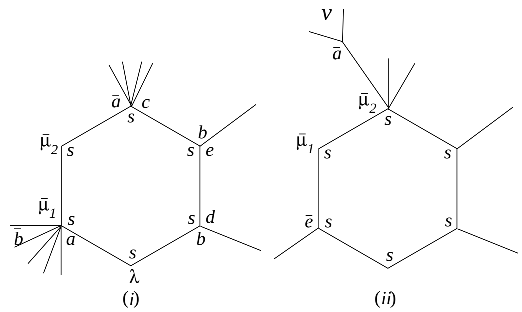


Figure 4.6: interior s-regions of degree 4 (Configuration (2))

Configuration (2.7)

In all cases (a), (b) and (c) we have $d(v_3) \geq 5$.

Case (a)

Since $|\alpha| > 2$ we have $d(v_1) \geq 4$ and $d(v_5) \geq 5$. Moreover $d(v_3) \geq 5$.

It follows that $c(\Delta) \leq c(3,4,5,5) < 0$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 5$ and $d(v_5) \geq 5$. Moreover $d(v_3) \geq 5$.

It follows that $c(\Delta) \leq c(3,5,5,5) < 0$.

Configuration (2.8)

The vertices v_1, v_3 and v_5 have subalbels as, $s\overline{e}$ and $\overline{\mu}_2 sb$ respectively.

Case (a)

Since $|\alpha| > 2$ we have $d(v_1) \geq 4$, $d(v_3) \geq 4$ and $d(v_5) \geq 5$.

We can therefore have positive curvature only if $d(v_2) = 3$, in which case $l(v_2) = bds$ and v_3 has sublabel $\lambda s\overline{e}$. This implies $d(v_3) \geq 6$, hence $c(\Delta) \leq c(3,4,5,6) < 0$.

Case (b)

Since $\alpha = \beta$ we have $d(v_1) \geq 5$, $d(v_3) \geq 4$ and $d(v_5) \geq 5$.

It follows that $c(\Delta) \leq c(3,4,5,5) < 0$.

Case (c)

Since $\alpha = -\beta$ we have $d(v_1) \geq 5$, $d(v_3) \geq 3$ and $d(v_5) \geq 5$.

If $d(v_3) > 3$ then $c(\Delta) \leq c(3, 4, 5, 5) < 0$.

If $d(v_3) = 3$ then $l(v_3) = bs\overline{e}$ and we can have positive curvature only if $d(v_2) = 3$ and $l(v_2) = bsc$, in which case v_1 has sublabel asc hence $d(v_1) \geq 6$.

A positively curved region looks like Figure 4.7 (ii) where $c(\Delta) \leq c(3, 3, 5, 6) = \frac{\pi}{15}$.

Configuration (2.9)

The vertices v_1, v_3 and v_5 have subalbels as, sb and asb respectively.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_1) \geq 4$, $d(v_3) \geq 3$ and $d(v_5) \geq 5$.

If $d(v_2), d(v_3) > 3$ then $c(\Delta) < 0$ therefore we can assume at least one of the vertices v_2 and v_3 has degree 3.

If $d(v_2) = 3$ then v_3 has sublabel λsb which implies $d(v_3) \geq 5$ and so $c(\Delta) \leq c(3, 4, 5, 5) < 0$.

If $d(v_3) = 3$ then v_2 has sublabel se and $d(v_2) \geq 4$.

Moreover at least one of the vertices v_1 and v_2 has degree > 4.

It follows that $c(\Delta) \leq c(3, 4, 5, 5) < 0$.

Case (b)

Since $\alpha = \beta$ we have $d(v_1) \geq 5$ and $d(v_5) \geq 5$.

It follows that Δ can be positively curved only if $d(v_2) = d(v_3) = 3$, which forces $l(v_3) = bds$ and $l(v_2) = bse$.

This also implies that v_1 has sublabel asc and hence $d(v_1) \geq 6$.

A positively curved region looks like Figure 4.7 (iii) where $c(\Delta) \leq c(3, 3, 5, 6) = \frac{\pi}{15}$.

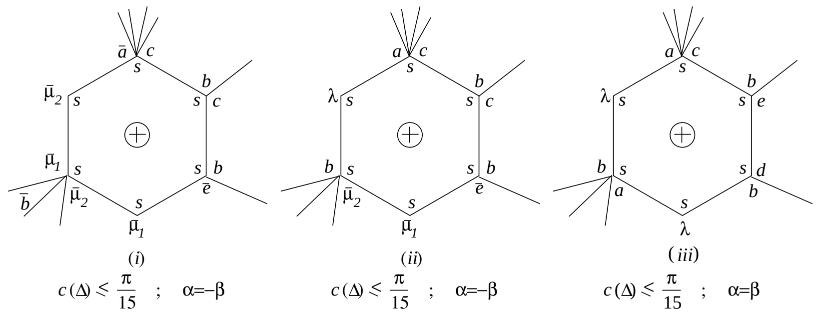


Figure 4.7: positively curved interior s-regions of degree 4 (Configuration (2))

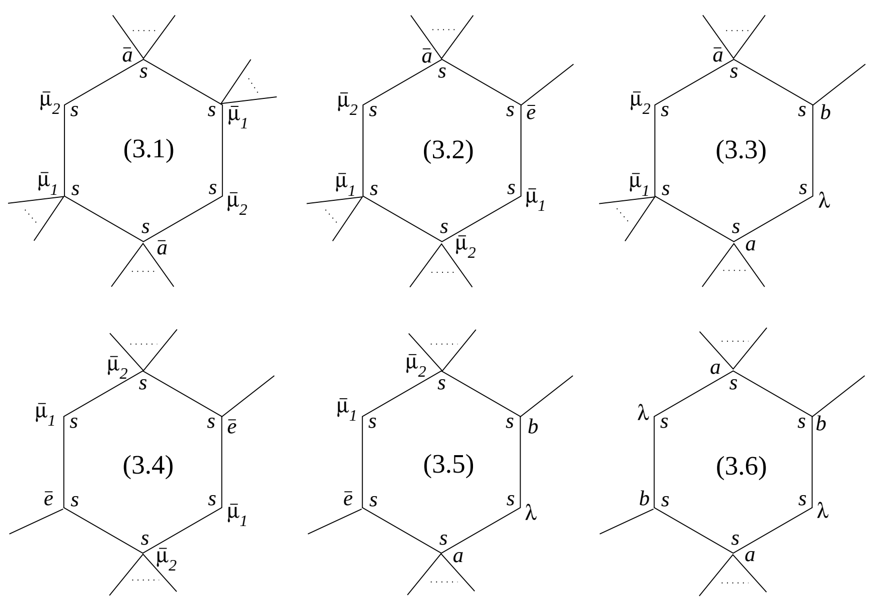
Case (c)

Since $\alpha = -\beta$ we have $d(v_1) \geq 5$ and $d(v_5) \geq 5$.

It follows that Δ can be positively curved only if $d(v_2) = d(v_3) = 3$, but if $d(v_3) = 3$ then $l(v_3) = bds$ and v_2 has sublabel se, which implies $d(v_2) \geq 4$. It follows that $c(\Delta) \leq c(3, 4, 5, 5) < 0$.

Configuration (3)

Due to symmetry we can distinguish the following six configurations:



Configuration (3.1)

We have $d(v_2), d(v_5) \geq 5$.

Cases (a), (b) and (c)

Since $\alpha \neq -1$ we have $d(v_1), d(v_4) \geq 4$.

It follows that $c(\Delta) \leq c(4,4,5,5) < 0$.

Configuration (3.2)

We have $d(v_4), d(v_5) \geq 5$.

Moreover it is easy to see that v_1 and v_2 cannot have degree 3 at the same time. It follows that $c(\Delta) \leq c(3,4,5,5) < 0$.

Configuration (3.3)

We have $d(v_5) \geq 5$.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_1) \geq 4$ and $d(v_4) \geq 4$.

It follows that Δ can be positively curved only if $d(v_2) = 3$, $d(v_1) = d(v_4) = 4$ and $d(v_5) = 5$.

By checking all the possibilities one finds out that $d(v_4) = 4$ and $d(v_5) = 5$ can occur only if $l(v_4) = as\overline{c}b$ and $l(v_5) = \overline{d}s\overline{\mu}_1\overline{b}0$. This implies $\alpha - \beta - 1 = 0$. On the other hand, since $l(v_2) = bds$, the vertex v_1 has sublabel $\overline{a}se$, therefore v_1 can have degree 4 only if $\alpha - \beta + 1 = 0$.

It follows that $c(\Delta) \leq c(3,4,5,5) < 0$.

Cases (b) and (c)

In these cases we have $d(v_1), d(v_4) \geq 5$. It follows that $c(\Delta) \leq c(3,5,5,5) < 0$

Configurations (3.4) and (3.5)

Consider Figure 4.8. Since $|\alpha| \neq 1$ the vertex v has degree > 2.

Therefore applying a bridge move at v_1 relative to $\overline{\mu}_2 s$ we contradict **C3**.

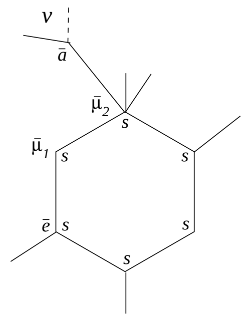


Figure 4.8: interior s-regions of degree 4 (Configurations (3.4) and (3.5))

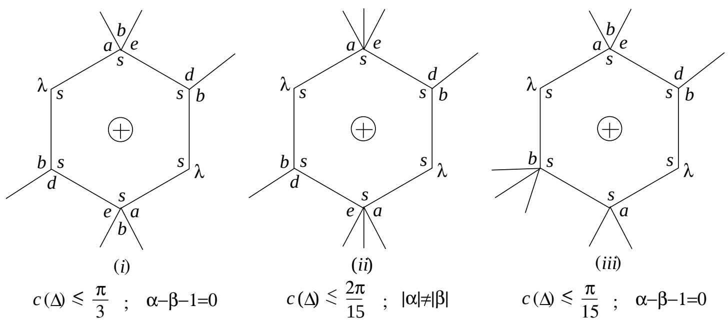


Figure 4.9: positively curved interior s-regions of degree 4 (Configuration (3))

Configuration (3.6)

Case (a)

Since $|\alpha| > 2$ and $|\alpha| \neq |\beta|$ we have $d(v_1) \geq 4$ and $d(v_4) \geq 4$.

It follows that, in order to have positive curvature, at least one of the vertices v_2 and v_5 must have degree 3.

If they both have degree 3 then v_1 and v_4 both have sublabel ase.

The unique possible label in degree 4 is therefore $aseb$ which implies $\alpha - \beta - 1 = 0$.

A positively curved region looks like Figure 4.9 (i) where $c(\Delta) \leq c(3,3,4,4) = \dfrac{\pi}{3}$.

Without this constraint on α and β we have $d(v_1), d(v_4) \geq 5$ and a positively curved region looks like Figure 4.9 (ii), where $c(\Delta) \leq c(3,3,5,5) = \dfrac{2\pi}{15}$.

If only one of v_2 and v_5 has degree 3 the other one must have degree$\geq$ 5; in order to have positive curvature we must have $d(v_1) = d(v_4) = 4$. It follows that a positively curved region looks like Figure 4.9 (iii), where $c(\Delta) \leq c(3,4,4,5) = \dfrac{\pi}{15}$ and $\alpha - \beta - 1 = 0$.

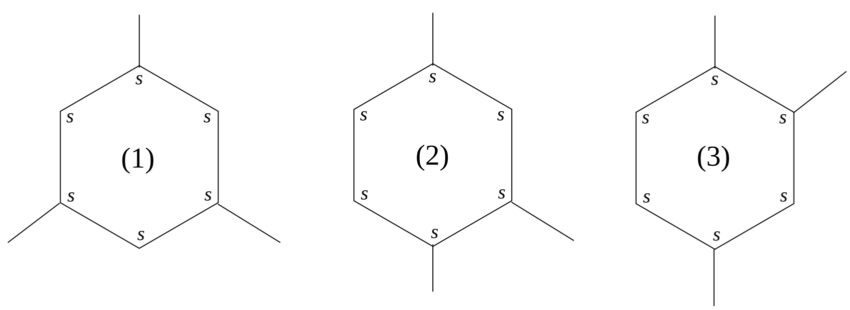
Cases (b) and (c)

In both cases $d(v_1), d(v_4) \geq 5$, moreover the only possible label in degree 3 with sublabel sb is bds.

Since $c(3,4,5,5) < 0$, we can have positive curvature only if $d(v_2) = d(v_5) = 2$ in which case $l(v_2) = l(v_5) = bds$. This forces v_1 and v_4 to have both sublabel ase, hence $d(v_1), d(v_4) \geq 6$ and $c(\Delta) \leq c(3,3,6,6) = 0$.

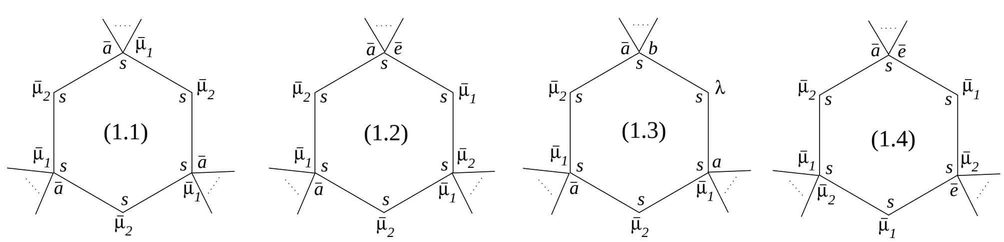
Now let Δ be an interior s-region of degree 3 of $\mathscr{D}$.

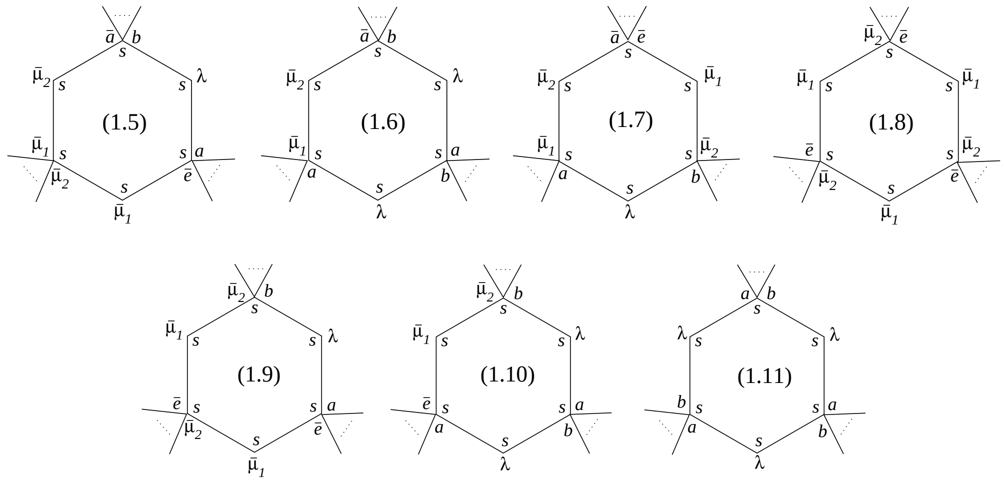
We can distinguish three different configurations:



Configuration (1)

There are eleven subcases:





Configuration (1.1)

Cases (a), (b), (c)

The vertices v_1, v_3 and v_5 all have sublabel $\overline{a}s\overline{\mu}_1\overline{b}$, therefore $d(v_i) \geq 6$ for $i = 1, 3, 5$. It follows that $c(\Delta) \leq c(6, 6, 6) = 0$.

Configuration (1.2)

The vertices v_1, v_3 and v_5 have sublabels $\overline{a}s\overline{e}$, $\overline{\mu}_2 s\overline{\mu}_1\{s, \overline{b}\}$ and $\overline{a}s\overline{\mu}_1\overline{b}$ respectively.

Case (a)

Since $|\alpha| > 2$ and $|\alpha| \neq |\beta|$ we have $d(v_1) \geq 4$, $d(v_3) \geq 6$ and $d(v_5) \geq 6$.

If $d(v_1) = 4$ then $l(v_1) = \overline{a}s\overline{e}b$, therefore $\alpha + \beta + 1 = 0$.

Moreover we have $d(v_3), d(v_5) \geq 7$ and a positively curved region looks like Figure 4.10 (i) where $c(\Delta) \leq c(4, 7, 7) = \dfrac{\pi}{14}$ and $\alpha + \beta + 1 = 0$.

If $d(v_1) \geq 5$ then observe that v_3 and v_5 cannot have both degree 6. A positively curved region looks like Figure 4.10 (ii) where $c(\Delta) \leq c(5, 6, 7) = \dfrac{2\pi}{105}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 6$, $d(v_3) \geq 5$ and $d(v_5) \geq 7$.

It follows that $c(\Delta) \leq c(5, 6, 7) = \dfrac{2\pi}{105}$ and a positively curved region looks

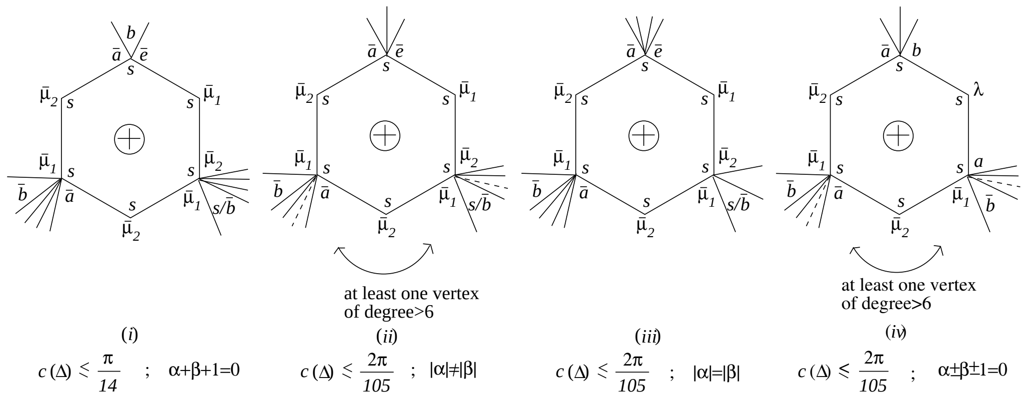


Figure 4.10: positively curved interior s-regions of degree 3 (Configurations (1.2) and (1.3))

like Figure 4.10 (iii).

Configuration (1.3)

The vertices v_1, v_3 and v_5 have sublabels $\overline{a}sb$, $as\overline{\mu}_1\overline{b}$ and $\overline{a}s\overline{\mu}_1\overline{b}$ respectively.

Case (a)

Since $|\alpha| > 2$ it follows that $d(v_1) \geq 5$, $d(v_3) \geq 6$ and $d(v_5) \geq 6$.

If $\alpha \pm 1 \pm \beta \neq 0$ then $d(v_3), d(v_5) \geq 7$ and so $c(\Delta) \leq c(5,7,7) < 0$.

If $\alpha \pm 1 \pm \beta = 0$ then one of the vertices v_3 and v_5 has degree ≥ 7 and a positively curved region looks like Figure 4.10 (iv) where $c(\Delta) \leq c(5,6,7) = \dfrac{2\pi}{105}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ it follows that $d(v_1) \geq 5$, $d(v_3) \geq 7$ and $d(v_5) \geq 7$.

It follows that $c(\Delta) \leq c(5,7,7) < 0$.

Configuration (1.4)

The vertices v_1, v_3 and v_5 have sublabels $\overline{a}s\overline{e}$, $\overline{\mu}_2 s\overline{e}$ and $\overline{\mu}_2 s\overline{\mu}_1$ respectively.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_1) \geq 4$, $d(v_3) \geq 6$ and $d(v_5) \geq 6$.

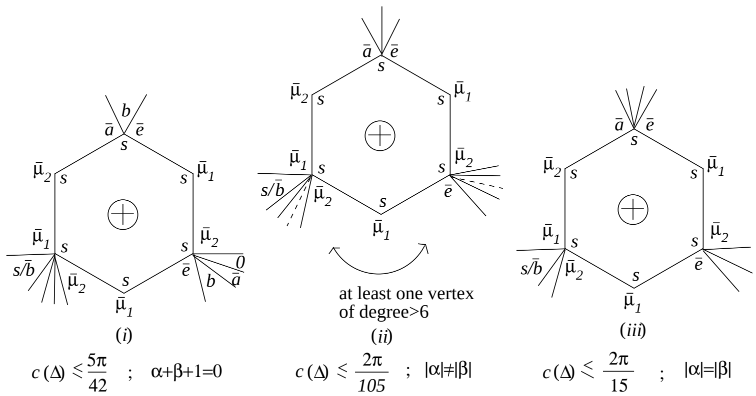


Figure 4.11: positively curved interior s-regions of degree 3 (Configuration (1.4))

If $d(v_1) = 4$ then $l(v_1) = \overline{a}s\overline{e}b$ and $\alpha + \beta + 1 = 0$.

In this case $d(v_5) \geq 7$ and a positively curved region looks like Figure 4.11 (i) where $c(\Delta) \leq c(4, 6, 7) = \dfrac{5\pi}{42}$.

If $d(v_1) = 5$ then either $d(v_3) \geq 7$ or $d(v_5) \geq 7$ and a positively curved region looks like Figure 4.11 (ii) where $c(\Delta) \leq c(5, 6, 7) = \dfrac{2\pi}{105}$.

If $d(v_1) \geq 6$ then $c(\Delta) \leq c(6, 6, 6) = 0$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 6$, $d(v_3) \geq 5$ and $d(v_5) \geq 5$.

A positively curved region looks therefore like Figure 4.11 (iii) where $c(\Delta) \leq c(5, 5, 6) = \dfrac{2\pi}{15}$.

Configuration (1.5)

The vertices v_1, v_3 and v_5 have sublabels $\overline{a}sb$, $as\overline{e}$ and $\overline{\mu}_2 s\overline{\mu}_1\{s, \overline{b}\}$ respectively.

Case (a)

Since $|\alpha| > 2$ and $|\alpha| \neq |\beta|$ we have $d(v_1) \geq 5$, $d(v_3) \geq 4$ and $d(v_5) \geq 6$.

If $d(v_3) = 4$ then $l(v_3) = as\overline{e}b$ and so $\alpha + \beta - 1 = 0$.

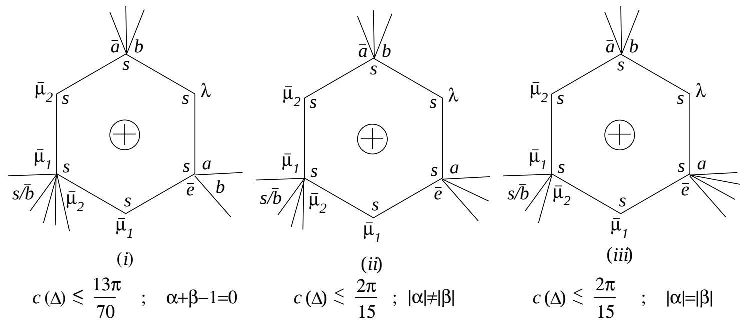


Figure 4.12: positively curved interior s-regions of degree 3 (Configuration (1.5))

In this case $d(v_5) \geq 7$ and a positively curved region looks like Figure 4.12 (i) where $c(\Delta) \leq c(4,5,7) = \dfrac{13\pi}{70}$.

If $d(v_3) \geq 5$ then a positively curved region looks like Figure 4.12 (ii) where $c(\Delta) \leq c(5,5,6) = \dfrac{2\pi}{15}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 5$, $d(v_3) \geq 6$ and $d(v_5) \geq 5$. A positively curved region looks like Figure 4.12 (iii) where $c(\Delta) \leq c(5,5,6) = \dfrac{2\pi}{15}$.

Configuration (1.6)

The vertices v_1, v_3 and v_5 have sublabels $\overline{a}sb$, asb and $as\overline{\mu}_1\overline{b}$ respectively.

Case (a)

Since $|\alpha| > 2$ we have $d(v_1) \geq 5$, $d(v_3) \geq 5$ and $d(v_5) \geq 6$. A positively curved region looks like Figure 4.13 (i) where $c(\Delta) \leq c(5,5,6) = \dfrac{2\pi}{15}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 5$, $d(v_3) \geq 5$ and $d(v_5) \geq 7$. A positively curved region looks like Figure 4.13 (ii) where $c(\Delta) \leq c(5,5,7) = \dfrac{3\pi}{35}$.

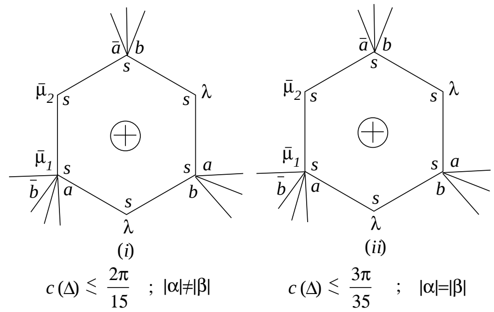


Figure 4.13: positively curved interior s-regions of degree 3 (Configuration (1.6))

Configuration (1.7)

The vertices v_1, v_3 and v_5 have sublabels $\overline{a}s\overline{e}$, $\overline{\mu}_2 sb$ and $as\overline{\mu}_1\overline{b}$ respectively.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_1) \geq 4$, $d(v_3) \geq 5$ and $d(v_5) \geq 6$.

If $d(v_1) = 4$ then $l(v_1) = \overline{a}s\overline{e}b$ and so $\alpha + \beta + 1 = 0$. In this case a positively curved region looks like Figure 4.14 (i) where $c(\Delta) \leq c(4,5,6) = \dfrac{7\pi}{30}$.

If $d(v_1) \geq 5$ then a positively curved region looks like Figure 4.14 (ii) where $c(\Delta) \leq c(5,5,6) = \dfrac{2\pi}{15}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 6$, $d(v_3) \geq 5$ and $d(v_5) \geq 7$. A positively curved region looks like Figure 4.14 (iii) where $c(\Delta) \leq c(5,6,7) = \dfrac{2\pi}{105}$.

Configuration (1.8)

The vertices v_1, v_3 and v_5 all have sublabel $\overline{\mu}_2 s\overline{e}$.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_i) \geq 6$ for $i = 1, 3, 5$.

It follows that $c(\Delta) \leq c(6,6,6) < 0$.

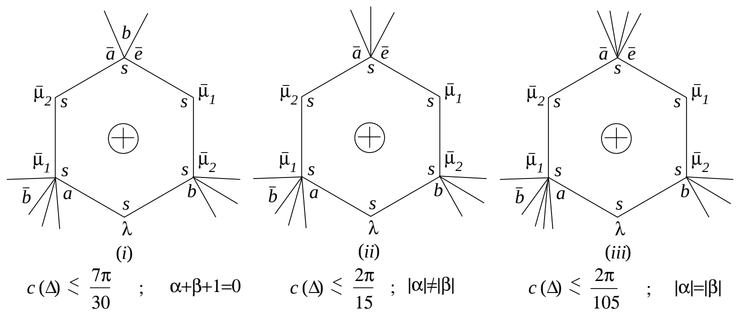


Figure 4.14: positively curved interior s-regions of degree 3 (Configuration (1.7))

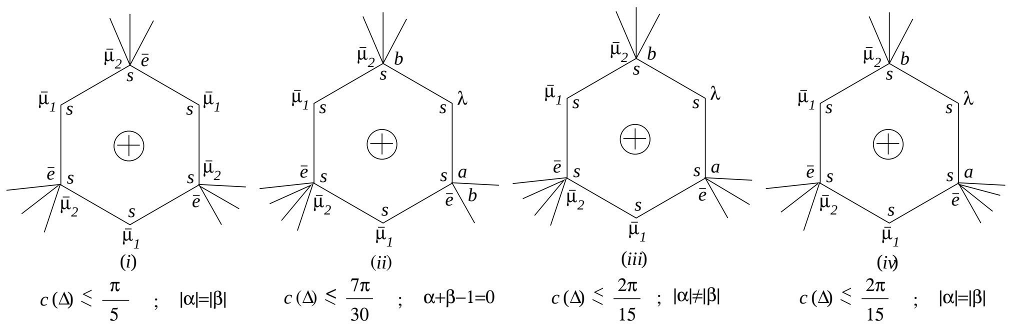


Figure 4.15: positively curved interior s-regions of degree 3 (Configurations (1.8) and (1.9))

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_i) \geq 5$ for $i = 1, 3, 5$. A positively curved region looks like Figure 4.15 (i) where $c(\Delta) \leq c(5, 5, 5) = \dfrac{\pi}{5}$.

Configuration (1.9)

The vertices v_1, v_3 and v_5 have sublabels $\overline{\mu}_2 sb$, $as\overline{e}$ and $\overline{\mu}_2 s\overline{e}$ respectively.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_1) \geq 5$, $d(v_3) \geq 4$ and $d(v_5) \geq 6$.

If $d(v_3) = 4$ then $l(v_3) = as\overline{e}b$ and so $\alpha + \beta - 1 = 0$. In this case positively curved region looks like Figure 4.15 (ii) where $c(\Delta) \leq c(4, 5, 6) = \dfrac{7\pi}{30}$.

If $d(v_3) \geq 5$ then a positively curved region looks like Figure 4.15 (iii) where $c(\Delta) \leq c(5,5,6) = \dfrac{2\pi}{15}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 5$, $d(v_3) \geq 6$ and $d(v_5) \geq 5$. A positively curved region looks like Figure 4.15 (iv) where $c(\Delta) \leq c(5,5,6) = \dfrac{2\pi}{15}$.

Configuration (1.10)

The vertices v_1, v_3 and v_5 have sublabels $\overline{\mu}_2 sb$, asb and $as\overline{e}$ respectively.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_1) \geq 5$, $d(v_3) \geq 5$ and $d(v_5) \geq 4$.

If $d(v_5) = 4$ then $l(v_5) = as\overline{e}b$ and so $\alpha + \beta - 1 = 0$. In this case positively curved region looks like Figure 4.16 (i) where $c(\Delta) \leq c(4,5,5) = \dfrac{3\pi}{10}$.

If $d(v_5) \geq 5$ then a positively curved region looks like Figure 4.16 (ii) where $c(\Delta) \leq c(5,5,5) = \dfrac{\pi}{5}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 5$, $d(v_3) \geq 5$ and $d(v_5) \geq 6$. A positively curved region looks like Figure 4.16 (iii) where $c(\Delta) \leq c(5,5,6) = \dfrac{2\pi}{15}$.

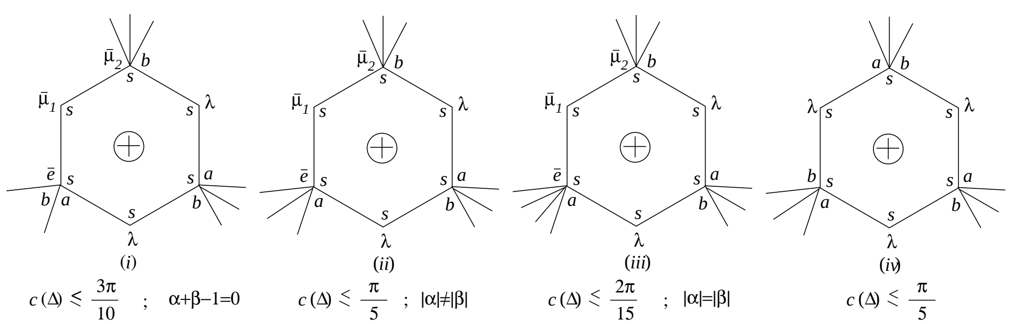


Figure 4.16: positively curved interior s-regions of degree 3 (Configurations (1.10) and (1.11))

Configuration (1.11)

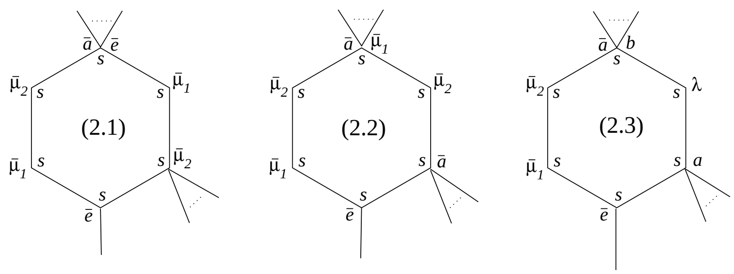
The vertices v_1, v_3 and v_5 all have sublabel asb.

Cases (a), (b) and (c)

In all these cases we have $d(v_i) \geq 5$ for $i = 1, 3, 5$ and a positively curved region looks like Figure 4.16 (iv) where $c(\Delta) \leq c(5,5,5) = \dfrac{\pi}{5}$.

Configuration (2).

We can distinguish the following three subcases:



Notice that in this configuration there is a triple edge adjacent to Δ.

We will give upper bounds for the curvature in cases (b) and (c); this is what we need as will be clear when we will describe the distribution process.

Configuration (2.1)

The vertices v_1, v_3 and v_4 have sublabels $\overline{a}s\overline{e}$, $\overline{\mu}_2 s$ and $s\overline{e}$ respectively.

Figure 4.17 shows that this configuration cannot occur: since $|\alpha| \neq 1$, the

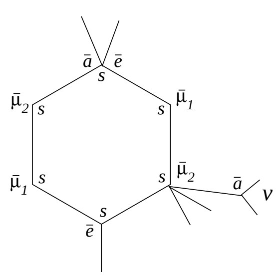


Figure 4.17: Configuration (2.1) cannot occur.

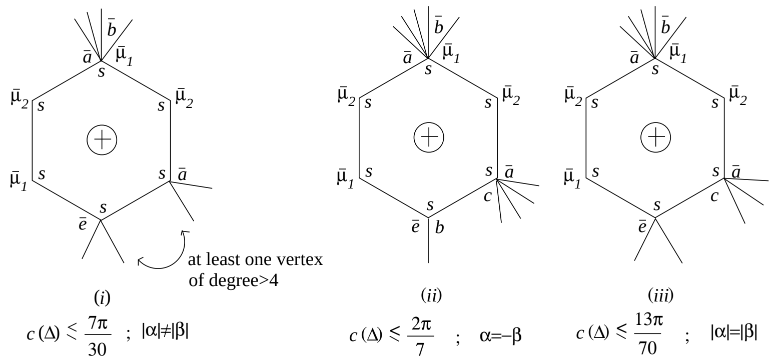


Figure 4.18: positively curved interior s-regions of degree 3 (Configuration (2.2)).

vertex v has degree > 2; applying a bridge move at v_3 relative to $\overline{\mu}_2 s$ we contradict **C3**.

Configuration (2.2)

The vertices v_1, v_3 and v_4 have sublabels $\overline{a}s\overline{\mu}_1\overline{b}$, $\overline{a}s$ and $s\overline{e}$ respectively.

Case (a)

In this case we have $d(v_1) \geq 6$, $d(v_3) \geq 4$ and $d(v_4) \geq 4$.

Moreover at least one of the vertices v_3 and v_4 has degree ≥ 5. A positively curved region looks like Figure 4.18 (i) where $c(\Delta) \leq c(4, 5, 6) = \dfrac{7\pi}{30}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 7$, $d(v_3) \geq 5$ and $d(v_4) \geq 3$.

If $d(v_4) = 3$ then $l(v_4) = bs\overline{e}$ and $\alpha = -\beta$; moreover v_3 has sublabel $\overline{a}sc$ which implies $d(v_c) \geq 6$. It follows that $c(\Delta) \leq c(3, 6, 7) = \dfrac{2\pi}{7}$ (see Figure 4.18 (ii)).

If $d(v_4) \geq 4$ then $c(\Delta) \leq c(4, 5, 7) = \dfrac{13\pi}{70}$.

A region of positive curvature looks like Figure 4.18 (iii).

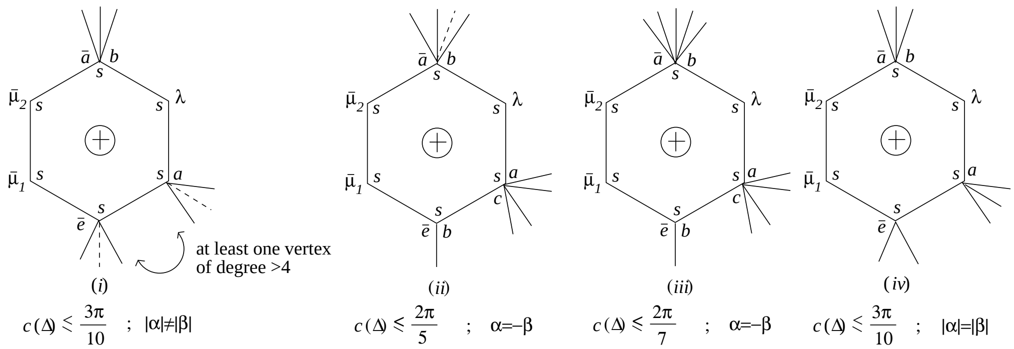


Figure 4.19: positively curved interior s-regions of degree 3 (Configuration (2.3)).

Configuration (2.3)

The vertices v_1, v_3 and v_4 have sublabels $\overline{a}sb$, as and $s\overline{e}$ respectively.

Case (a)

In this case we have $d(v_1) \geq 5$, $d(v_3) \geq 4$ and $d(v_4) \geq 4$.

Moreover the vertices v_3 and v_4 cannot have both degree ≤ 4. A positively curved region looks like Figure 4.19 (i) where $c(\Delta) \leq c(4,5,5) = \dfrac{3\pi}{10}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 5$, $d(v_3) \geq 5$ and $d(v_4) \geq 3$.

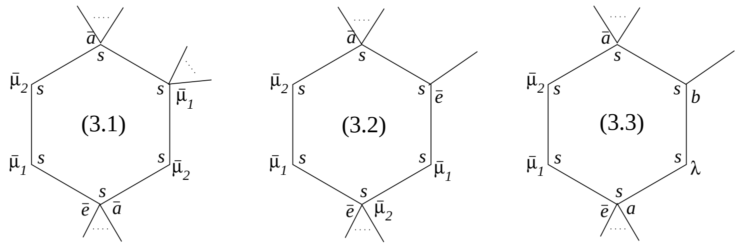
If $d(v_4) = 3$ then $l(v_4) = bs\overline{e}$ and $\alpha = -\beta$; moreover v_3 has sublabel $\overline{a}sc$ which implies $d(v_3) \geq 6$. If $d(v_1) = 5$ or 6 then $c(\Delta) \leq c(3,5,6) = \dfrac{2\pi}{5}$ and a positively curved region looks like Figure 4.19 (ii). If $d(v_1) \geq 7$ then $c(\Delta) \leq c(3,6,7) = \dfrac{2\pi}{7}$ and a region of positive curvature looks like Figure 4.19 (iii).

Finally if $d(v_4) \geq 4$ then $c(\Delta) \leq c(4,5,5) = \dfrac{3\pi}{10}$.

A region of positive curvature looks like Figure 4.19 (iv).

Configuration (3)

We can again distinguish three different subcases as follows:



Configuration (3.1)

The vertices v_1, v_2 and v_4 have sublabels $\overline{a}s$, $s\overline{\mu}_1\{\overline{b}, s\}$ and $\overline{a}s\overline{e}$ respectively.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_1) \geq 4$, $d(v_2) \geq 5$ and $d(v_4) \geq 4$.

If $d(v_2) = 5$ then $l(v_2) \in \{s\overline{\mu_1}\overline{b}0\overline{d}, s\overline{\mu_1}sbd\}$ and so v_1 has sublabel $\overline{a}s\{e, \overline{c}\}$.

It follows that the vertices v_1 and v_4 cannot have both degree ≤ 4. A positively curved region looks like Figure 4.20 (i) where $c(\Delta) \leq c(4, 5, 5) = \dfrac{3\pi}{10}$.

If $d(v_2) \geq 6$ then a positively curved region looks like Figure 4.20 (ii) where $c(\Delta) \leq c(4, 4, 6) = \dfrac{\pi}{3}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 5$, $d(v_2) \geq 5$ and $d(v_4) \geq 6$. It follows that $c(\Delta) \leq c(5, 5, 6) = \dfrac{2\pi}{15}$; a positively curved region looks like Figure 4.20 (iii).

Configuration (3.2)

The vertices v_1, v_2 and v_4 have sublabels $\overline{a}s$, $s\overline{e}$ and $\overline{\mu}_2s\overline{e}$ respectively.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_1) \geq 4$, $d(v_2) \geq 4$ and $d(v_4) \geq 6$.

Moreover the vertices v_1 and v_2 cannot have both degree ≤ 4. A positively curved region looks like Figure 4.21 (i) where $c(\Delta) \leq c(4, 5, 6) = \dfrac{7\pi}{30}$.

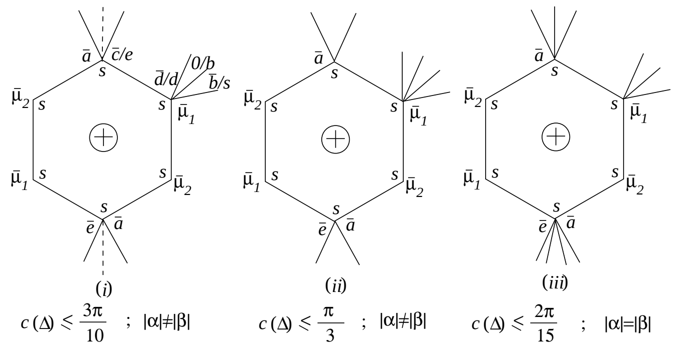


Figure 4.20: positively curved interior s-regions of degree 3 (Configuration (3.1)).

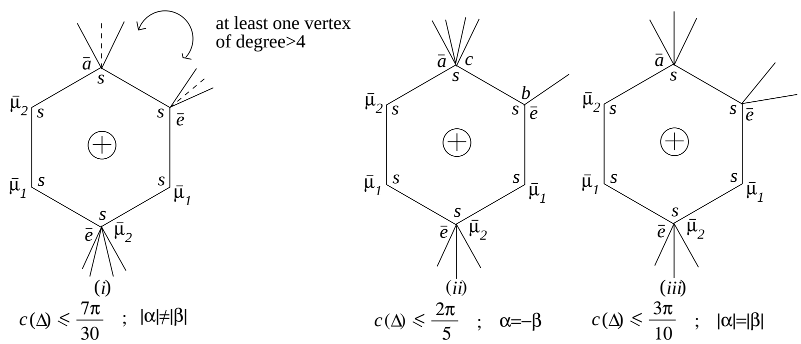


Figure 4.21: positively curved interior s-regions of degree 3 (Configuration (3.2)).

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 5$, $d(v_2) \geq 3$ and $d(v_4) \geq 5$.

If $d(v_2) = 3$ then $l(v_2) = bs\overline{e}$ and $\alpha = -\beta$; it follows that v_1 has sublabel $\overline{a}sc$, hence $d(v_1) \geq 6$. A positively curved region looks like Figure 4.21 (ii) where $c(\Delta) \leq c(3, 5, 6) = \dfrac{2\pi}{5}$.

If $d(v_2) > 3$ then a positively curved region looks like Figure 4.21 (iii) where $c(\Delta) \leq c(4, 5, 5) = \dfrac{3\pi}{10}$.

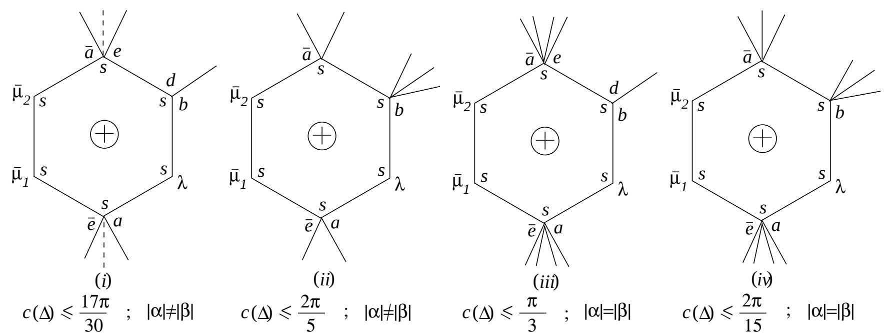


Figure 4.22: positively curved interior s-regions of degree 3 (Configuration (3.3)).

Configuration (3.3)

The vertices v_1, v_2 and v_4 have sublabels $\overline{a}s$, sb and $as\overline{e}$ respectively.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_1) \geq 4$, $d(v_2) \geq 3$ and $d(v_4) \geq 4$.

If $d(v_2) = 3$ then $l(v_3) = bds$ and consequently the vertices v_1 and v_4 cannot have both degree ≤ 4. A positively curved region looks like Figure 4.22 (i) where $c(\Delta) \leq c(3,4,5) = \dfrac{17\pi}{30}$.

If $d(v_2) > 3$ then $d(v_2) \geq 5$ and a positively curved region looks like Figure 4.22 (ii) where $c(\Delta) \leq c(4,4,5) = \dfrac{2\pi}{5}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 5$, $d(v_2) \geq 3$ and $d(v_4) \geq 6$.

If $d(v_2) = 3$ then $l(v_2) = bds$ and so $d(v_1) \geq 6$.

A positively curved region looks like Figure 4.22 (iii) where $c(\Delta) \leq c(3,6,6) = \dfrac{\pi}{3}$.

If $d(v_2) > 3$ then $d(v_2) \geq 5$ and a region of positive curvature looks like Figure 4.22 (iv) where $c(\Delta) \leq c(5,5,6) = \dfrac{2\pi}{15}$.

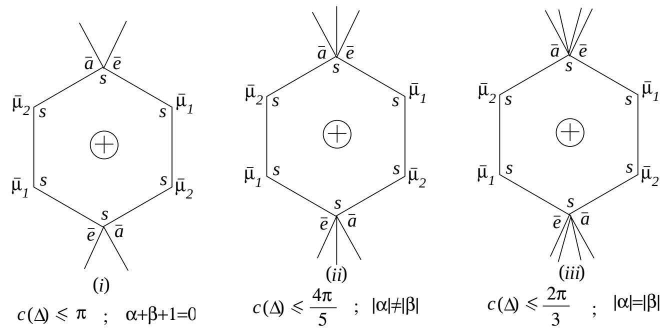


Figure 4.23: positively curved interior s-regions of degree 2.

Finally suppose Δ is an interior s-region of degree 2 of $\mathscr{D}$.

There is one possible configuration, where v_1 and v_4 both have sublabel $\overline{a}s\overline{e}$.

Case (a)

Since $|\beta| \neq |\alpha| > 2$ we have $d(v_1) \geq 4$ and $d(v_4) \geq 4$.

If one (or both) of the vertices v_1 and v_4 has degree 4 then it has label $\overline{a}s\overline{e}b$ which implies $\alpha + \beta + 1 = 0$.

A positively curved region looks like Figure 4.23 (i) where $c(\Delta) \leq c(4,4) = \pi$.

If $\alpha + \beta + 1 \neq 0$ then $d(v_1), d(v_4) \geq 5$ and a positively curved region looks like Figure 4.23 (ii) where $c(\Delta) \leq c(5,5) = \dfrac{4\pi}{5}$.

Cases (b) and (c)

Since $|\alpha| = |\beta|$ we have $d(v_1) \geq 6$ and $d(v_4) \geq 6$. A region of positive curvature looks like Figure 4.23 (iii) where $c(\Delta) \leq c(6,6) = \dfrac{2\pi}{3}$.

This ends the study of positively curved interior s-regions of $\mathscr{D}$.

We can now describe the distribution process for the cases (a), (b) and (c).

Case (a) splits into five subcases:

(a1) $\alpha \pm \beta \pm 1 \neq 0$;

(a2) $\alpha + \beta + 1 = 0$;

(a3) $\alpha - \beta + 1 = 0$;

(a4) $\alpha + \beta - 1 = 0$;

(a5) $\alpha - \beta - 1 = 0$.

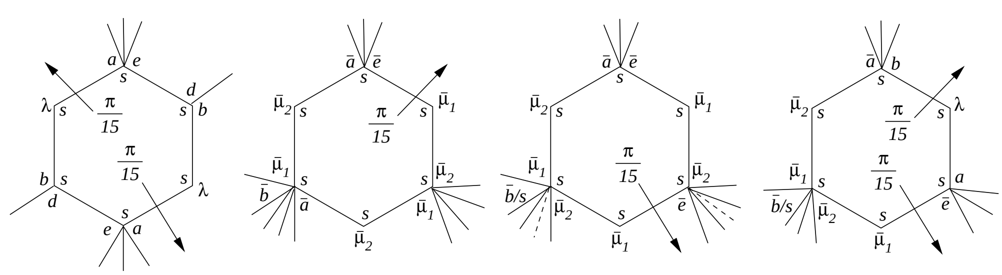
In each diagram we agree to draw the least possible degree for each vertex, unless specified otherwise. For instance if we are transferring a certain amount of curvature through the double edge $a\lambda b$ when $d(v_3) = 3$ (and a different amount of curvature if $d(v_3) \geq 4$), then $d(v_3) = 3$ will be printed somewhere in the figure.

The regions in the compensation scheme are obtained from those classified earlier (which will be indicated) using elementary consequences of the constraints of each case.

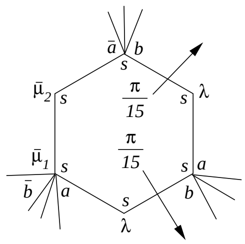
Distribution for case (a1), ($\alpha \pm \beta \pm 1 \neq 0$).

According to the classification we can obtain the positively curved interior regions from Figures 4.9 (ii), 4.10 (ii), 4.11 (ii), 4.12 (ii), 4.13 (i), 4.14 (ii), 4.15 (iii), 4.16 (ii), 4.16 (iv), 4.18 (i), 4.19 (i), 4.20 (i), 4.20 (ii), 4.21 (i), 4.22 (i), 4.22 (ii), 4.23 (ii).

The upper bounds for the curvature are sometimes decreased using the constraint $\alpha \pm \beta \pm 1 \neq 0$ and the positive curvature is distributed according to the scheme which follows.

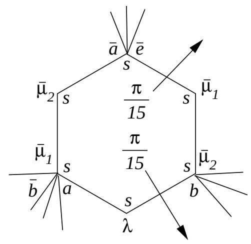


$c(\Delta) \leq \frac{2\pi}{15}$ 4.9 (ii) $c(\Delta) \leq \frac{2\pi}{105}$ 4.10 (ii) $c(\Delta) \leq \frac{2\pi}{105}$ 4.11 (ii) $c(\Delta) \leq \frac{2\pi}{15}$ 4.12 (ii)



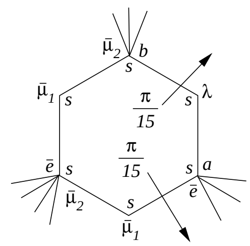
$c(\Delta) \leq \frac{2\pi}{15}$

4.13 (i)



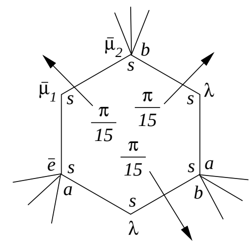
$c(\Delta) \leq \frac{2\pi}{15}$

4.14 (ii)



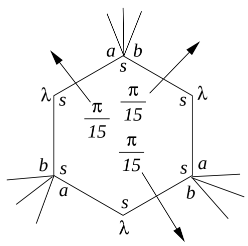
$c(\Delta) \leq \frac{2\pi}{15}$

4.15 (iii)



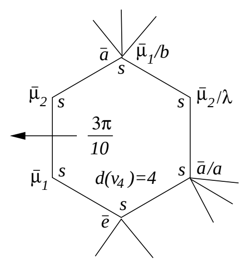
$c(\Delta) \leq \frac{\pi}{5}$

4.16 (ii)



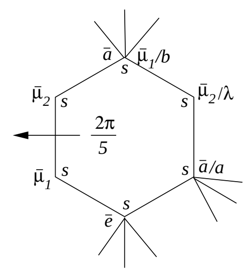
$c(\Delta) \leq \frac{\pi}{5}$

4.16 (iv)



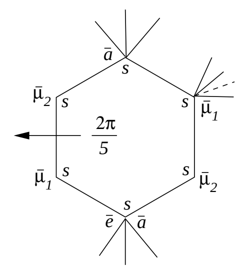
$c(\Delta) \leq \frac{3\pi}{10}$

4.18 (i)−4.19 (i)



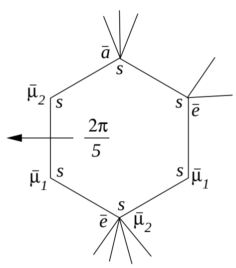
$c(\Delta) \leq \frac{\pi}{5}$

4.18 (i)−4.19 (i)



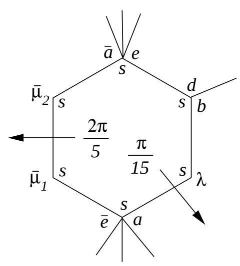
$c(\Delta) \leq \frac{\pi}{5}$

4.20 (i)−(ii)



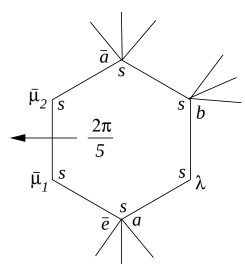
$c(\Delta) \leq \frac{7\pi}{30}$

4.21 (i)



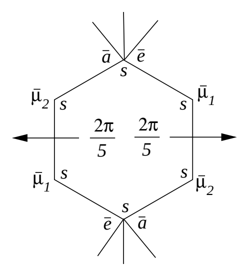
$c(\Delta) \leq \frac{7\pi}{15}$

4.22 (i)



$c(\Delta) \leq \frac{\pi}{5}$

4.22 (ii)



$c(\Delta) \leq \frac{4\pi}{5}$

4.23 (ii)

Lemma 4.2.9 *If Δ is interior then $c^*(\Delta) \leq 0$.*

Proof. We distributed the positive curvature in such a way that every positively curved interior region has non-positive pseudo-curvature.

Let Δ be an interior m-region receiving positive curvature according to the scheme above. Distinguish two cases:

1. the region Δ receives positive curvature from exactly one s-region;

2. the region Δ receives positive curvature from exactly two s-regions.

Notice that since the positive curvature is distributed only through particular edges it follows that Δ cannot receive it from more than two s-regions.

Case 1.)

There are three subcases; the region Δ receives positive curvature through:

1.1) the double edge labelled by $e\mu_1\mu_2$;

1.2) the double edge labelled by $a\lambda b$;

1.3) the triple edge labelled by $e\mu_1\mu_2 a$.

Case 1.1)

We see from the compensation scheme that $d(v_e) \geq 5$ and $d(v_{\mu_2}) \geq 5$.
The maximum amount of positive curvature the region Δ can receive is $\dfrac{\pi}{15}$.
Since $d(v_b), d(v_d) \geq 3$ and $d(v_a) \geq 4$ it follows that $c^*(\Delta) \leq c(3,3,4,5,5) + \dfrac{\pi}{15} < 0$.

Case 1.2)

We have $d(v_a) \geq 5$ and $d(v_b), d(v_d) \geq 3$.
The maximum amount of positive curvature the region Δ can receive is $\dfrac{\pi}{15}$.
If $d(v_e) = 2$ then $l(v_e) = ec$; this implies that v_d and v_{μ_1} have sublabels dd and $b\mu_1$ respectively, which imply $d(v_d) \geq 5$ and $d(v_{\mu_1}) \geq 4$. It is not difficult to see that if $d(v_{\mu_1}) = 4$ then $\alpha = \pm 2\beta$, hence $d(v_d) \geq 6$.
It follows that $c^*(\Delta) \leq \max\{c(3,5,5,5), c(3,4,5,6)\} + \dfrac{\pi}{15} < 0$.
We can therefore assume that $d(v_e) > 2$, which implies $d(v_e) \geq 4$.
If $d(v_c) = 2$ then $l(v_c) = ec$; this implies that v_d and v_b have sublabels dd and $b\mu_1$ respectively, which imply $d(v_d) \geq 5$ and $d(v_b) \geq 4$.
It follows that $c^*(\Delta) \leq c(4,4,5,5) + \dfrac{\pi}{15} < 0$.

We can therefore assume that $d(v_c) > 2$, which implies $d(v_c) \geq 4$.

In this case we have $c^*(\Delta) \leq c(3,3,4,4,5) + \dfrac{\pi}{15} < 0$.

Case 1.3)

We have $d(v_a) \geq 5$, $d(v_b), d(v_d) \geq 3$ and $d(v_e) \geq 4$.

First assume $d(v_e) = 4$. This implies $l(v_e) = e\overline{s}d\overline{b}$ or $l(v_e) = e\overline{s}\overline{b}\{e, \overline{c}\}$, hence $2\alpha + \beta = 0$ or $\alpha + 2\beta = 0$, respectively.

According to the compensation scheme above the maximum amount of curvature the region Δ can receive is $\dfrac{3\pi}{10}$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_d and v_b to have sublabels dd and $b\mu_1$ respectively. This implies $d(v_b) \geq 4$ and $d(v_d) \geq 5$.

Now if $d(v_b) \geq 5$ then $c^*(\Delta) \leq c(4,5,5,5) + \dfrac{3\pi}{10} = 0$.

So assume $d(v_b) = 4$, which implies $l(v_b) = b\mu_1 cc$ or $l(v_b) = b\mu_1\overline{ee}$ (because we have $2\alpha + \beta = 0$ or $\alpha + 2\beta = 0$); in both cases $d(v_\lambda) \geq 3$ and so $c^*(\Delta) \leq c(3,4,4,5,5) + \dfrac{3\pi}{10} = 0$.

If $d(v_c) > 2$ then $d(v_c) \geq 4$. If $d(v_c) \geq 5$ then $c^*(\Delta) \leq c(3,3,4,5,5) + \dfrac{3\pi}{10} < 0$.

If $d(v_c) = 4$ then v_d and v_b cannot both have degree≤ 3; it follows that $c^*(\Delta) \leq c(3,4,4,4,5) + \dfrac{3\pi}{10} < 0$.

Now assume $d(v_e) \geq 5$. The maximum amount of positive curvature the region Δ can receive is $\dfrac{2\pi}{5}$ and we have $d(v_a) \geq 5$ and $d(v_b), d(v_d) \geq 3$.

First assume $d(v_c) = 2$. This implies $l(v_c) = ce$, hence v_d and v_b have sublabels dd and $b\mu_1$ respectively; therefore $d(v_b) \geq 4$ and $d(v_d) \geq 5$.

If $d(v_b) \geq 5$ then $c^*(\Delta) \leq c(5,5,5,5) + \dfrac{2\pi}{5} = 0$.

If $d(v_b) = 4$ then $l(v_b) \in \{b\mu_1 ee, b\mu_1 cc, b\mu_1\overline{ee}, b\mu_1\overline{cc}\}$. If $l(v_b) \neq b\mu_1 ee$ then $d(v_\lambda) \geq 3$ hence $c^*(\Delta) \leq c(3,4,5,5,5) + \dfrac{2\pi}{5} < 0$; if $l(v_b) = b\mu_1 ee$ and $d(v_\lambda) = 2$ then $\alpha = 2\beta$ and v_a has sublabel $\overline{s}a\mu_2$ therefore $d(v_d) \geq 5$ and

$d(v_a) \geq 6$. It follows that $c^*(\Delta) \leq c(4,5,6,6) + \dfrac{2\pi}{5} < 0$.

Now assume $d(v_c) > 2$, which implies $d(v_c) \geq 4$.

If $d(v_c) \geq 5$ then $c^*(\Delta) \leq c(3,3,5,5,5) + \dfrac{2\pi}{5} < 0$.

If $d(v_c) = 4$ then v_d and v_b cannot both have degree< 4; it follows that $c^*(\Delta) \leq c(3,4,4,5,5) + \dfrac{2\pi}{5} < 0$.

Case 2.

There are two subcases; the region Δ receives positive curvature through:

2.1) two double edges;

2.2) one double edge and one triple edge.

Case 2.1)

The maximum amount of curvature that Δ can receive is $\dfrac{\pi}{15} + \dfrac{\pi}{15} = \dfrac{2\pi}{15}$.

We have $d(v_a), d(v_e), d(v_{\mu_2}) \geq 5$ and $d(v_b), d(v_d) \geq 3$.

It follows that $c^*(\Delta) \leq c(3,3,5,5,5) + \dfrac{2\pi}{15} < 0$.

Case 2.2)

The maximum amount of curvature that Δ can receive is $\dfrac{2\pi}{5} + \dfrac{\pi}{15} = \dfrac{7\pi}{15}$.

We have $d(v_b), d(v_d) \geq 3$ and $d(v_e) \geq 4$.

Since v_a has sublabel $\overline{s}as$ it follows that $d(v_a) \geq 7$.

The same argument of case 1.3), with the unique difference on the degree of v_a, shows that $c^*(\Delta) \leq 0$.

$\square$

Lemma 4.2.10 *If Δ is a boundary region then $c^*(\Delta) < \dfrac{4\pi}{k_0}$.*

Proof. In order to prove the lemma we need some extra distribution of curvature which involves boundary regions only. This will not affect the interior regions and so the proof of the previous lemma.

As seen before (see Lemma 4.1.6), we can assume that the distinguished vertex coincides with a unique vertex of each boundary s-region and with no more than three vertices of each boundary m-region.

Distinguish the following cases:

1. Δ is an s-region;

2. Δ is an m-region.

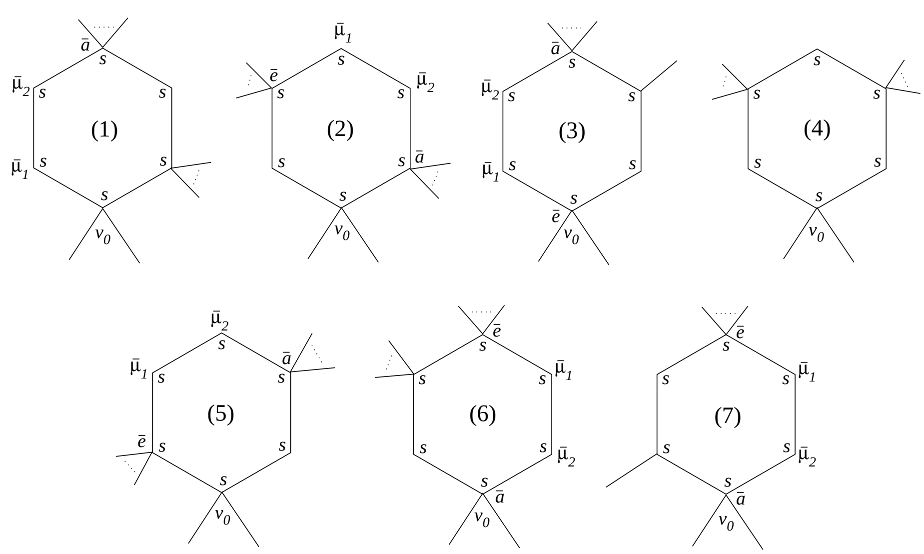
Case 1.

We can assume without any loss that the distinguished vertex coincides with v_4. Since Δ is an s-region it does not receive positive curvature from any interior region.

Now if $d(\Delta) \geq 4$ then $c^*(\Delta) = c(\Delta) \leq c(3,3,3,k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$, so we can assume $d(\Delta) \leq 3$.

Case 1.1)**,** $d(\Delta) = 3$

There are seven subcases:



If there is no interior vertex of degree 3 then $c^*(\Delta) = c(\Delta) \leq c(4,4,k_0) = \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$. Since the unique possible label in degree 3 is bds it follows that in cases (1), (2), (4), (5) and (6) we have two interior vertices of degree≥ 4 and so $c^*(\Delta) < \frac{4\pi}{k_0}$.

In case (3) we can assume $d(v_2) = 3$ and $l(v_2) = bds$, moreover we have $d(v_1) \geq 5$. We choose to transfer the positive curvature $\frac{\pi}{5}$ through the triple edge.

It follows that $c^*(\Delta) = c(\Delta) \leq c(3,5,k_0) - \frac{\pi}{5} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

In case (7) we can assume $d(v_5) = 3$ and $l(v_5) = bds$, moreover we have $d(v_1) \geq 5$. We choose to transfer the positive curvature $\frac{\pi}{5}$ through the triple edge.

It follows that $c^*(\Delta) = c(\Delta) \leq c(3,5,k_0) - \frac{\pi}{5} < \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Case 1.2)**,** $d(\Delta) = 2$

The vertex v_1 has sublabel $\overline{a}s\overline{e}$ which implies $d(v_1) \geq 5$.

We choose to transfer the curvature $\frac{\pi}{5}$ through each triple edge; it follows that $c^*(\Delta) \leq c(5,k_0) - \frac{2\pi}{5} = \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Case 2.

Suppose the distinguished vertex v_0 coincides with m vertices of Δ. Notice that $k_0 \geq 2m$.

Suppose $m = 3$ (hence $v_0 = v_b = v_d = v_{\mu_1}$). In this case v_a is not the distinguished vertex and $d(v_a) \geq 4$. Since $d(\Delta) \geq 4$ and Δ does not receive any positive curvature from interior or from boundary regions (as described above), it follows that $c^*(\Delta) = c(\Delta) \leq c(4,k_0,k_0,k_0) = -2\pi + \frac{\pi}{2} + \frac{6\pi}{k_0} \leq -\frac{\pi}{2} < \frac{4\pi}{k_0}$.

Now suppose $m = 2$. We have $c^*(\Delta) \leq c(\Delta) + \frac{2\pi}{5} + \frac{\pi}{15} = c(\Delta) + \frac{7\pi}{15} \leq$

$c(3,3,k_0,k_0) + \frac{7\pi}{15} = -\frac{2\pi}{3} + \frac{4\pi}{k_0} + \frac{7\pi}{15} = -\frac{\pi}{5} + \frac{4\pi}{k_0} < \frac{4\pi}{k_0}$.

We can therefore assume that v_0 coincides with a unique vertex of Δ.

The maximum amount of curvature that Δ can receive is $\frac{2\pi}{5} + \frac{\pi}{15} = \frac{7\pi}{15}$.

Suppose $d(v_c) = 2$ and v_c is not distinguished. Then $l(v_c) = ce$, moreover v_b and v_d have sublabels $b\mu_1$ and dd respectively. For interior vertices we have $d(v_a) \geq 4$, $d(v_b) \geq 4$, $d(v_d) \geq 5$, moreover v_e and v_{μ_1} cannot both have degree< 4. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one, it follows that $c^*(\Delta) \leq c(4,4,4,k_0) + \frac{7\pi}{15} = \frac{2\pi}{k_0} - \frac{\pi}{30} < \frac{4\pi}{k_0}$.

Suppose $d(v_c) > 2$ and v_c is not distinguished (and so $d(v_c) \geq 4$). For interior vertices we have $d(v_a) \geq 4$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 4. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one, it follows that $c^*(\Delta) \leq c(3,3,4,4,k_0) + \frac{7\pi}{15} = \frac{2\pi}{k_0} - \frac{\pi}{5} < \frac{4\pi}{k_0}$.

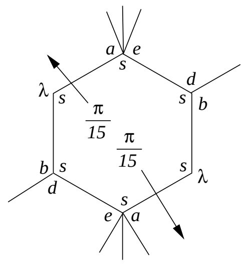
So we can assume $v_0 = v_c$, in which case $d(v_a) \geq 4$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 4. It follows that $c^*(\Delta) \leq c(3,3,4,4,k_0) + \frac{7\pi}{15} = \frac{2\pi}{k_0} - \frac{\pi}{5} < \frac{4\pi}{k_0}$.

□

Distribution for case (a2), ($\alpha + \beta + 1 = 0$).

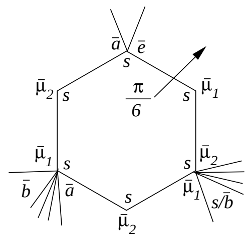
According to the classification we can obtain the positively curved interior regions from Figures 4.9 (*ii*), 4.10 (*i*), 4.10 (*ii*), 4.10 (*iv*), 4.11 (*i*), 4.11 (*ii*), 4.12 (*ii*), 4.13 (*i*), 4.14 (*i*), 4.14 (*ii*), 4.15 (*iii*), 4.16 (*ii*), 4.16 (*iv*), 4.18 (*i*), 4.19 (*i*), 4.20 (*i*), 4.20 (*ii*), 4.21 (*i*), 4.22 (*i*), 4.22 (*ii*), 4.23 (*i*), 4.23 (*ii*).

The upper bounds for the curvature are sometimes decreased using the constraint $\alpha + \beta + 1 = 0$ and the positive curvature is distributed according to the scheme which follows.



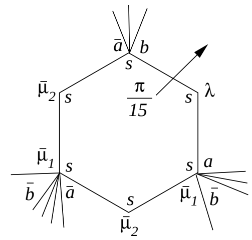
$c(\Delta) \leq \frac{2\pi}{15}$

4.9 (ii)



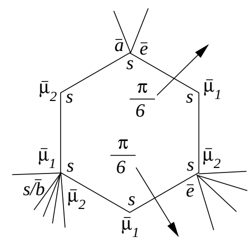
$c(\Delta) \leq \frac{\pi}{14}$

4.10 (i)−(ii)



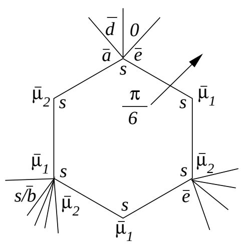
$c(\Delta) \leq \frac{2\pi}{105}$

4.10 (iv)



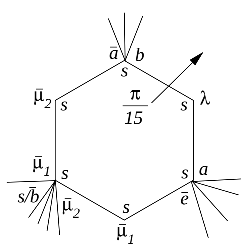
$c(\Delta) \leq \frac{5\pi}{42}$

4.11 (i)



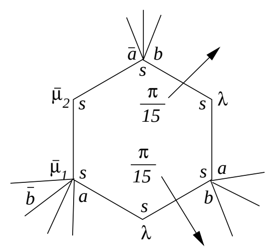
$c(\Delta) \leq \frac{2\pi}{105}$

4.11 (ii)



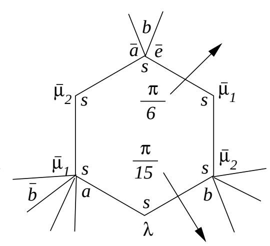
$c(\Delta) \leq \frac{2\pi}{105}$

4.12 (ii)



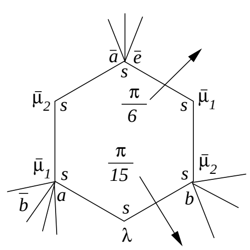
$c(\Delta) \leq \frac{2\pi}{15}$

4.13 (i)



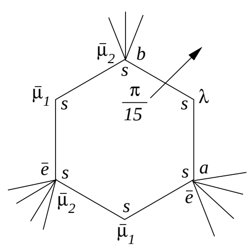
$c(\Delta) \leq \frac{7\pi}{30}$

4.14 (i)



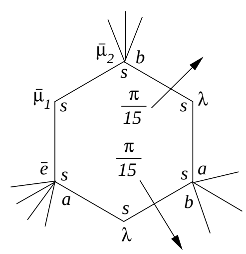
$c(\Delta) \leq \frac{2\pi}{15}$

4.14 (ii)



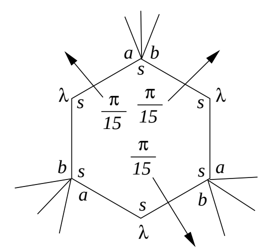
$c(\Delta) \leq \frac{\pi}{15}$

4.15 (iii)



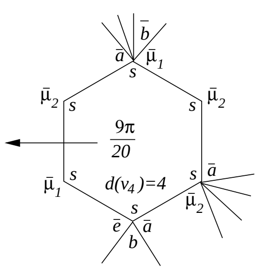
$c(\Delta) \leq \frac{2\pi}{15}$

4.16 (ii)



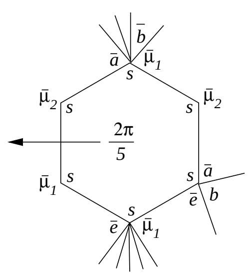
$c(\Delta) \leq \frac{\pi}{5}$

4.16 (iv)



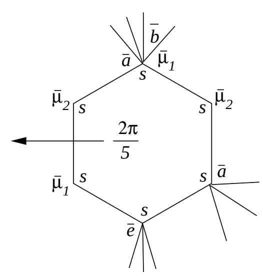
$c(\Delta) \leq \frac{\pi}{6}$

4.18 (i)



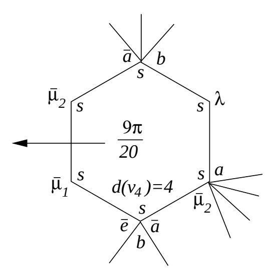
$c(\Delta) \leq \frac{5\pi}{42}$

4.18 (i)



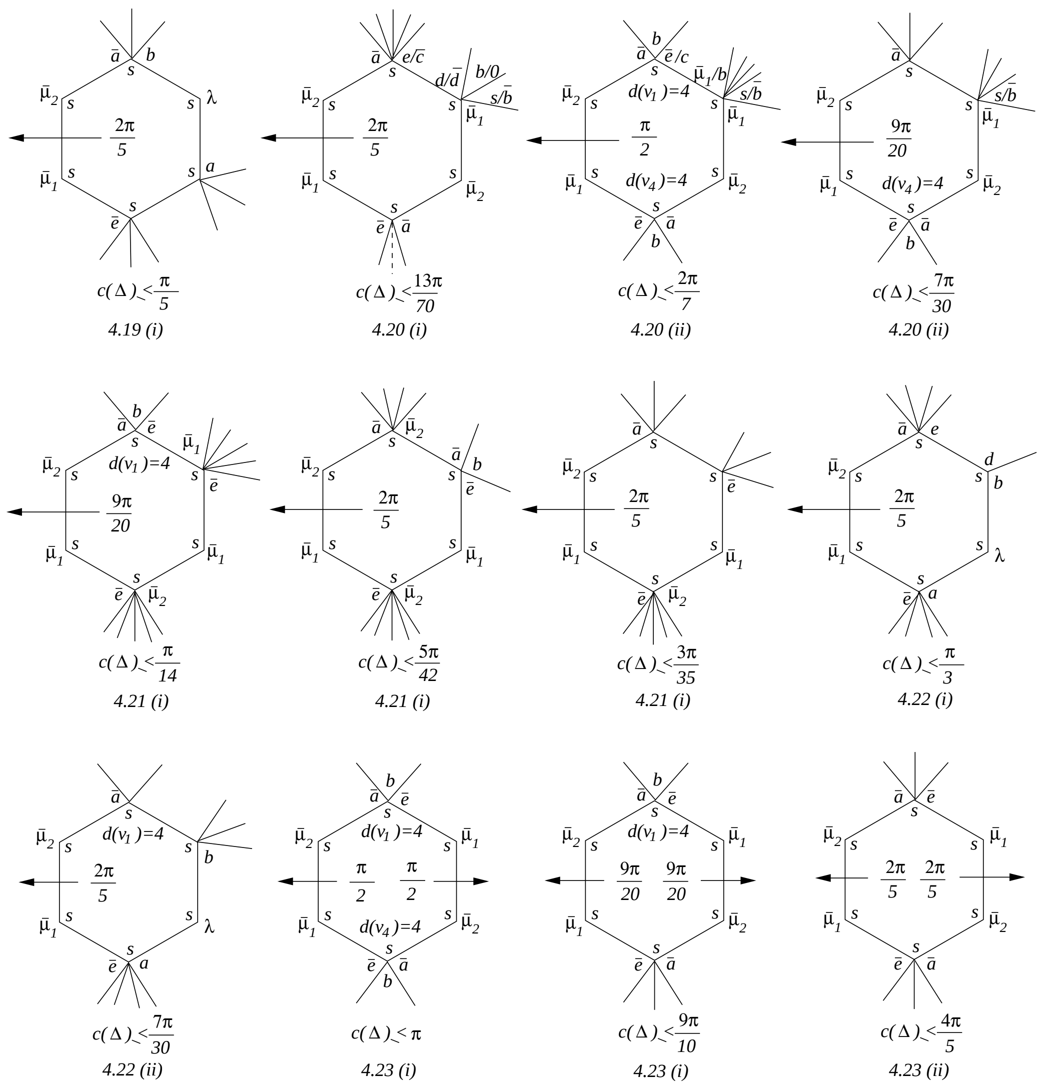
$c(\Delta) \leq \frac{2\pi}{15}$

4.18 (i)



$c(\Delta) \leq \frac{7\pi}{30}$

4.19 (i)



Lemma 4.2.11 *If Δ is interior then $c^*(\Delta) \leq 0$.*

Proof. We distributed the positive curvature in such a way that every positively curved interior region has non-positive pseudo-curvature.

Of course we only need to consider those m-regions which receive some positive curvature according to the scheme above. Distinguish two cases:

1. the region Δ receives positive curvature from exactly one s-region;

2. the region Δ receives positive curvature from exactly two s-regions.

Notice that since the positive curvature is distributed only through particular edges it follows that Δ cannot receive it from more than two s-regions.

Case 1.

There are three subcases; the region Δ receives positive curvature through

1.1) the double edge labelled by $e\mu_1\mu_2$;

1.2) the double edge labelled by $a\lambda b$;

1.3) the triple edge labelled by $e\mu_1\mu_2 a$.

Case 1.1)

We have $d(v_e) \geq 4$ and $d(v_{\mu_2}) \geq 5$. Moreover $d(v_b), d(v_d) \geq 3$ and $d(v_a) \geq 4$. The maximum amount of positive curvature the region Δ can receive is $\frac{\pi}{6}$. It follows that $c^*(\Delta) \leq c(3,3,4,4,5) + \frac{\pi}{6} < 0$.

Case 1.2)

The maximum amount of positive curvature the region Δ can receive is $\frac{\pi}{15}$. Observe that, according to the compensation scheme above, we have $d(v_b) \geq 3$, $d(v_d) \geq 3$ and $d(v_a) \geq 5$. Moreover the vertices v_e and v_{μ_1} cannot both have degree< 4.

If $d(v_c) = 2$ then $l(v_c) = ce$, this forces v_b and v_d to have sublabels $sb\mu_1$ and dd respectively, which imply $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

It follows that $c^*(\Delta) \leq c(4,5,5,6) + \frac{\pi}{15} < 0$.

If $d(v_c) > 2$ then the constraint $\alpha + \beta + 1 = 0$ implies $d(v_c) \geq 4$.

It follows that $c^*(\Delta) \leq c(3,3,4,4,5) + \frac{\pi}{15} < 0$.

Case 1.3)

First assume $d(v_a) = d(v_e) = 4$. This implies $l(v_a) = l(v_e) = a\bar{b}e\bar{s}$, therefore

v_λ has sublabel $\bar{c}\lambda$ and so $d(v_\lambda) \geq 4$. Moreover we have $d(v_b), d(v_d) \geq 3$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{\pi}{2}$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_b and v_d to have sublabels $b\mu_1$ and $dd\overline{\lambda}$ respectively. This implies $d(v_b) \geq 5$ and $d(v_d) \geq 6$. It follows that $c^*(\Delta) \leq c(4,4,4,5,6) + \dfrac{\pi}{2} < 0$.

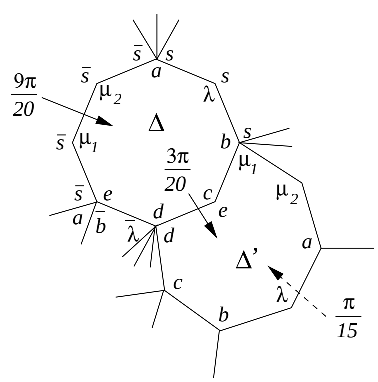
If $d(v_c) > 2$ then the constraint $\alpha + \beta + 1 = 0$ implies $d(v_c) \geq 4$, hence $c^*(\Delta) \leq c(3,3,4,4,4,4) + \dfrac{\pi}{2} < 0$.

Now assume $d(v_e) = 4$ and $d(v_a) \geq 5$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{9\pi}{20}$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_b and v_d to have sublabels $b\mu_1$ and $dd\overline{\lambda}$ respectively. This implies $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

Let Δ' be the m-region adjacent to Δ in the following figure; denote the vertices of Δ' by v'_r for $r = a, \lambda, b, c, d, e, \mu_1, \mu_2$ and redistribute the curvature as follows:



where $d(v'_c) > 2$ when Δ' is interior.

If Δ' is a boundary region there is nothing to prove so assume Δ' is interior.

If $d(v'_c) = 2$ then v_d has sublabel $ddd\overline{\lambda}$ which implies $d(v_d) \geq 8$. We don't have the extra distribution and it follows that $c^*(\Delta) \leq c(4,5,5,8) + \dfrac{9\pi}{20} = 0$.

If $d(v'_c) > 2$ then $c^*(\Delta) \leq c(4,5,5,6) + \dfrac{9\pi}{20} - \dfrac{3\pi}{20} < 0$; moreover we have

$d(v_a') \geq 3$, $d(v_d') \geq 3$, hence $c^*(\Delta') \leq c(3,3,4,5,6) + \dfrac{3\pi}{20} + \dfrac{\pi}{15} < 0$.

Now let $d(v_c) > 2$ so that $d(v_c) \geq 4$. If $d(v_c) = 4$ then $l(v_c) = cb\overline{a}0$, therefore $d(v_b) \geq 5$ and $d(v_d) \geq 4$.

It follows that $c^*(\Delta) \leq c(4,4,4,5,5) + \dfrac{9\pi}{20} < 0$.

If $d(v_c) \geq 5$ observe that if $d(v_b) < 5$ then $d(v_c) \geq 7$, therefore $c^*(\Delta) \leq c(3,3,4,5,7) + \dfrac{9\pi}{20} < 0$.

Now assume $d(v_e), d(v_a) \geq 5$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{2\pi}{5}$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_b and v_d to have sublabels $b\mu_1$ and $dd\overline{\lambda}$ respectively. This implies $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

It follows that $c^*(\Delta) \leq c(5,5,5,6) + \dfrac{2\pi}{5} < 0$.

If $d(v_c) > 2$ then the constraint $\alpha + \beta + 1 = 0$ implies $d(v_c) \geq 4$. Moreover if $d(v_b) < 5$ then $d(v_c) \geq 7$, therefore $c^*(\Delta) \leq c(3,3,5,5,7) + \dfrac{2\pi}{5} < 0$.

Case 2.

There are two subcases; the region Δ receives positive curvature through

2.1) two double edges;

2.2) one double edge and one triple edge.

Case 2.1)

The maximum amount of curvature that Δ can receive is $\dfrac{\pi}{15} + \dfrac{\pi}{6} = \dfrac{7\pi}{30}$.

We have $d(v_a), d(v_{\mu_2}) \geq 5$, $d(v_e) \geq 4$ and $d(v_b), d(v_d) \geq 3$.

It follows that $c^*(\Delta) \leq c(3,3,4,5,5) + \dfrac{7\pi}{30} < 0$.

Case 2.2)

We have $d(v_b), d(v_d) \geq 3$, $d(v_a) \geq 5$ and $d(v_e) \geq 4$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{9\pi}{20} +$

$\dfrac{\pi}{15} = \dfrac{31\pi}{60}$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_b and v_d to have sublabels $b\mu_1$ and dd respectively. This implies $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

Let Δ' be as in case 1.3). The same argument of case 1.3) shows that $c^*(\Delta) \leq 0$ and $c^*(\Delta') \leq 0$.

If $d(v_c) > 2$ then $d(v_c) \geq 4$. If $d(v_c) = 4$ then $l(v_c) = cb\overline{a}0$, therefore $d(v_b) \geq 5$ and $d(v_d) \geq 4$. It follows that $c^*(\Delta) \leq c(4,4,4,5,5) + \dfrac{31\pi}{60} < 0$.

Suppose $d(v_c) \geq 5$.

If $d(v_b) = d(v_d) = 3$ then v_c has sublabel ccc which implies $d(v_c) \geq 10$, therefore $c^*(\Delta) \leq c(3,3,4,5,10) + \dfrac{31\pi}{60} < 0$.

If only one of the vertices v_b and v_d has degree 3, then v_c has sublabel cc which implies $d(v_c) \geq 7$. It follows that $c^*(\Delta) \leq c(3,4,4,5,7) + \dfrac{31\pi}{60} < 0$.

If $d(v_b), d(v_d) > 3$ then $c^*(\Delta) \leq c(4,4,4,5,5) + \dfrac{31\pi}{60} < 0$.

$\square$

Lemma 4.2.12 *If Δ is a boundary region then $c^*(\Delta) < \dfrac{4\pi}{k_0}$.*

Proof. As in the analogous lemma for case (a1) we need some extra distribution of curvature which involves boundary regions only. This will not affect the interior regions and so the proof of the previous lemma.

Moreover we can assume that the distinguished vertex coincides with a unique vertex of each boundary s-region and with no more than three vertices of each boundary m-region. Distinguish the following cases:

1. Δ is an s-region;

2. Δ is an m-region.

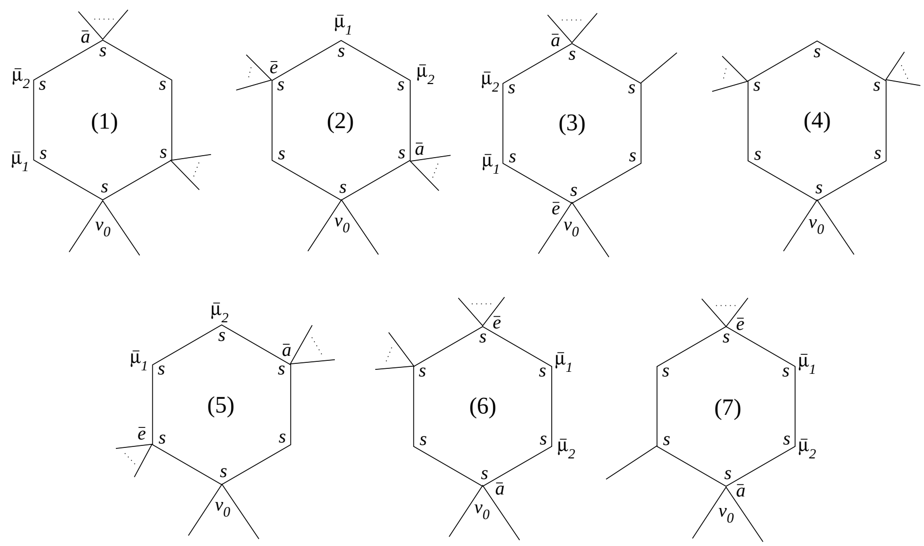
Case 1.

We can assume without any loss that the distinguished vertex coincides with v_4. Since Δ is an s-region it does not receive positive curvature from any interior region.

Now if $d(\Delta) \geq 4$ then $c(\Delta) \leq c(3,3,3,k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$, so we can assume $d(\Delta) \leq 3$.

Case 1.1)**,** $d(\Delta) = 3$

There are seven subcases:



Observe that the only admissible label in degree 3 involving s is bds.

It follows that in all cases, except cases (3) and (7), there are two vertices of degree ≥ 4, hence that $c^*(\Delta) = c(\Delta) \leq c(4,4,k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

We will then consider only cases (3) and (7) and assume that there is an interior vertex of degree 3.

In case (3) we have $d(v_2) = 3$ therefore $l(v_2) = bds$ which forces v_1 to have sublabel $\overline{a}se$. The constraint $\alpha + \beta + 1 = 0$ implies $d(v_1) \geq 6$.

It follows that $c^*(\Delta) = c(\Delta) \leq c(3,6,k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

In case (7) we have $d(v_5) \geq 3$ and so $l(v_5) = bds$. This implies that v_1 has sublabel $as\bar{e}$, hence $d(v_1) \geq 6$.

It follows that $c^*(\Delta) = c(\Delta) \leq c(3, 6, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Case 1.2)**,** $d(\Delta) = 2$

The vertex v_1 has sublabel $\bar{a}s\bar{e}$ which implies $d(v_1) \geq 4$.

We choose to transfer the curvature $\dfrac{\pi}{4}$ through each triple edge; it follows that $c^*(\Delta) \leq c(4, k_0) - \dfrac{\pi}{2} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Notice this further compensation does not affect the pseudo-curvature of any interior region.

Case 2.

Suppose the distinguished vertex v_0 coincides with m vertices of Δ. Notice that $k_0 \geq 2m$.

Suppose $m = 3$, then $v_0 = v_b = v_d = v_{\mu_1}$. Since $d(\Delta) \geq 4$ and Δ does not receive any positive curvature from interior or from boundary regions (as described above), it follows that $c^*(\Delta) = c(\Delta) \leq c(4, k_0, k_0, k_0) = -2\pi + \dfrac{\pi}{2} + \dfrac{6\pi}{k_0} \leq -\dfrac{\pi}{2} < \dfrac{4\pi}{k_0}$.

Now suppose $m = 2$. We have $c^*(\Delta) \leq c(\Delta) + \dfrac{\pi}{2} + \dfrac{\pi}{15} = c(\Delta) + \dfrac{17\pi}{30} \leq c(3, 3, k_0, k_0) + \dfrac{17\pi}{30} = -\dfrac{2\pi}{3} + \dfrac{4\pi}{k_0} + \dfrac{17\pi}{30} = -\dfrac{\pi}{10} + \dfrac{4\pi}{k_0} < \dfrac{4\pi}{k_0}$.

We can therefore assume that v_0 coincides with a unique vertex of Δ.

The maximum amount of curvature that Δ can receive is $\dfrac{\pi}{2} + \dfrac{\pi}{15} = \dfrac{17\pi}{30}$.

Suppose $d(v_c) = 2$ and v_c is not distinguished. Then $l(v_c) = ce$, moreover v_b and v_d have sublabels $b\mu_1$ and dd respectively. For interior vertices we have $d(v_a) \geq 4$, $d(v_b) \geq 5$, $d(v_d) \geq 5$, moreover v_e and v_{μ_1} cannot both have degree< 4. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one, it follows that $c^*(\Delta) \leq c(4, 4, 5, k_0) + \dfrac{17\pi}{30} = \dfrac{2\pi}{k_0} - \dfrac{\pi}{30} < \dfrac{4\pi}{k_0}$.

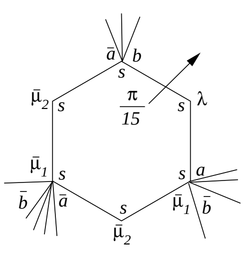
Suppose $d(v_c) > 2$ and v_c is not distinguished (and so $d(v_c) \geq 4$). For interior vertices we have $d(v_a) \geq 4$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 4. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one, it follows that $c^*(\Delta) \leq c(3,3,4,4,k_0) + \frac{17\pi}{30} = \frac{2\pi}{k_0} - \frac{\pi}{10} < \frac{4\pi}{k_0}$.
So we can assume $v_0 = v_c$, in which case $d(v_a) \geq 4$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 4. It follows that $c^*(\Delta) \leq c(3,3,4,4,k_0) + \frac{17\pi}{30} = \frac{2\pi}{k_0} - \frac{\pi}{10} < \frac{4\pi}{k_0}$.

$\square$

Distribution for case (a3), ($\alpha - \beta + 1 = 0$).

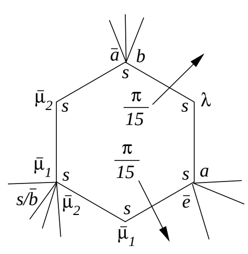
According to the classification we can obtain the positively curved interior regions from Figures 4.9 (ii), 4.10 (ii), 4.10 (iv), 4.11 (ii), 4.12 (ii), 4.13 (i), 4.14 (ii), 4.15 (iii), 4.16 (ii), 4.16 (iv), 4.18 (i), 4.19 (i), 4.20 (i), 4.20 (ii), 4.21 (i), 4.22 (i), 4.22 (ii), 4.23 (ii).

The upper bounds for the curvature are sometimes decreased using the constraint $\alpha - \beta + 1 = 0$ and therefore 4.9 (ii), 4.10 (ii) and 4.11 (ii) are not positively curved. The positive curvature is distributed according to the scheme which follows.



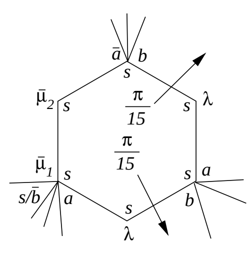
$c(\Delta) \leq \frac{2\pi}{105}$

4.10 (iv)



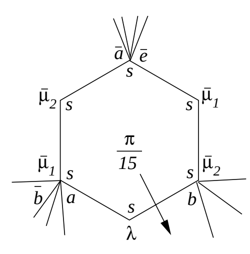
$c(\Delta) \leq \frac{2\pi}{15}$

4.12 (ii)



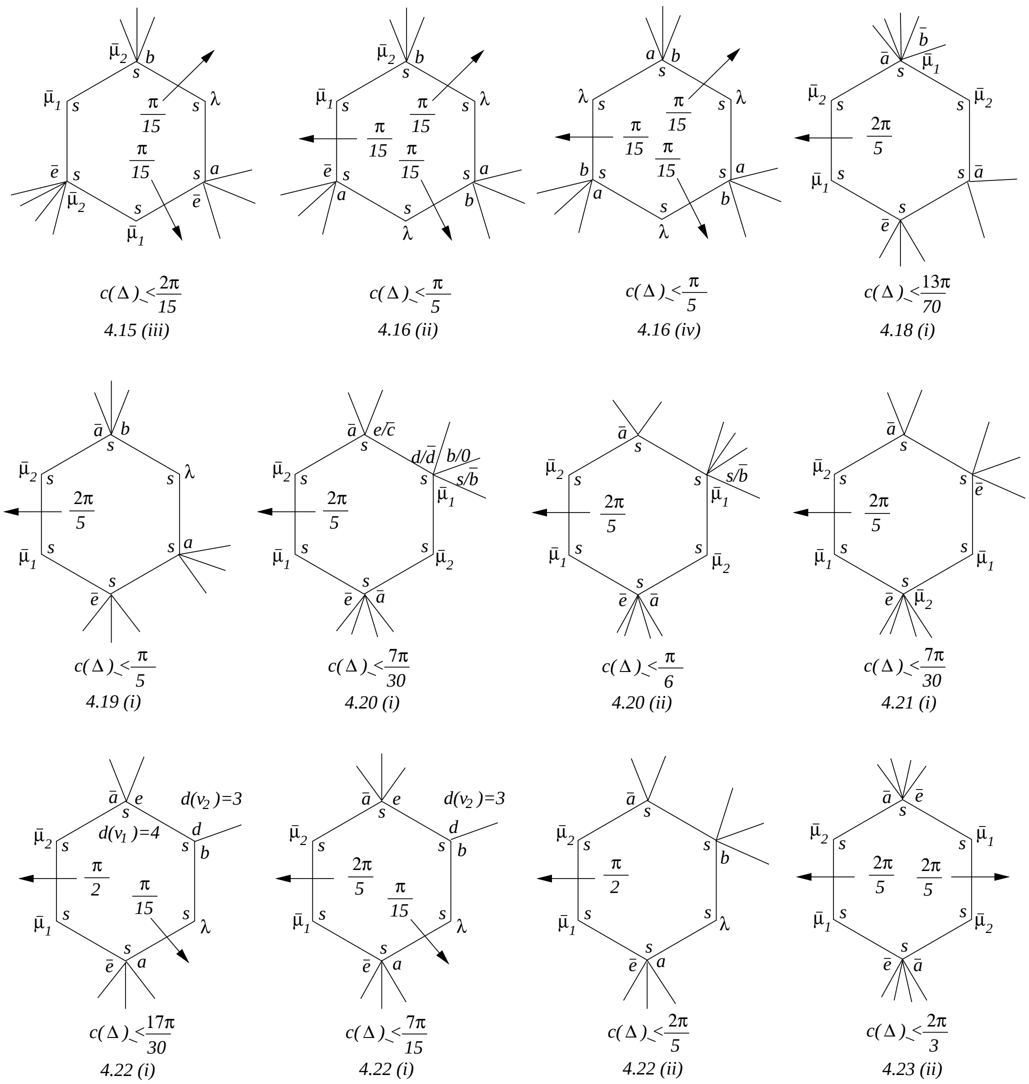
$c(\Delta) \leq \frac{2\pi}{15}$

4.13 (i)



$c(\Delta) \leq \frac{\pi}{15}$

4.14 (ii)



Lemma 4.2.13 *If Δ is interior then $c^*(\Delta) \leq 0$.*

Proof We distributed the positive curvature in such a way that every positively curved interior region has non-positive pseudo-curvature.

Let Δ be an interior m-region receiving positive curvature according to the scheme above. Distinguish two cases:

1. the region Δ receives positive curvature from exactly one s-region;

2. the region Δ receives positive curvature from exactly two s-regions.

Case 1.

There are three subcases; the region Δ receives positive curvature through

1.1) the double edge labelled by $e\mu_1\mu_2$;

1.2) the double edge labelled by $a\lambda b$;

1.3) the triple edge labelled by $e\mu_1\mu_2 a$.

Case 1.1)

We have $d(v_e), d(v_{\mu_2}) \geq 5$. Moreover $d(v_b), d(v_d) \geq 3$ and $d(v_a) \geq 4$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{\pi}{15}$.

It follows that $c^*(\Delta) \leq c(3,3,4,5,5) + \dfrac{\pi}{15} < 0$.

Case 1.2)

The maximum amount of positive curvature the region Δ can receive is $\dfrac{\pi}{15}$.

Observe that, according to the compensation scheme above, we have $d(v_b) \geq 3$, $d(v_d) \geq 3$ and $d(v_a) \geq 5$. Moreover the vertices v_e and v_{μ_1} cannot both have degree< 4.

If $d(v_c) = 2$ then $l(v_c) = ce$, this forces v_b and v_d to have sublabels $sb\mu_1$ and dd respectively, which imply $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

It follows that $c^*(\Delta) \leq c(4,5,5,6) + \dfrac{\pi}{15} < 0$.

If $d(v_c) > 2$ then $d(v_c) \geq 4$. It follows that $c^*(\Delta) \leq c(3,3,4,4,5) + \dfrac{\pi}{15} < 0$.

Case 1.3)

Observe that, according to the compensation scheme above, we have $d(v_e) \geq 5$ and $d(v_a) \geq 4$.

First assume $d(v_a) = 4$.

This implies $l(v_a) = a\overline{b}\{\overline{e}, c\}\overline{s}$, therefore v_λ has sublabel $\overline{c}\lambda$ and so $d(v_\lambda) \geq 5$.

Moreover we have $d(v_b), d(v_d) \geq 3$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{\pi}{2}$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_b and v_d to have sublabels $b\mu_1$ and $dd\overline{\lambda}$ respectively. This implies $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

It follows that $c^*(\Delta) \leq c(4,5,5,5,6) + \dfrac{\pi}{2} < 0$.

If $d(v_c) > 2$ then $d(v_c) \geq 4$. It follows that $c^*(\Delta) \leq c(3,3,4,4,5,5) + \dfrac{\pi}{2} < 0$.

Now assume $d(v_a) \geq 5$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{2\pi}{5}$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_b and v_d to have sublabels $b\mu_1$ and $dd\overline{\lambda}$ respectively. This implies $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

It follows that $c^*(\Delta) \leq c(5,5,5,6) + \dfrac{2\pi}{5} < 0$.

If $d(v_c) > 2$ then $d(v_c) \geq 4$. Moreover the vertices v_b and v_c cannot both have degree< 5. It follows that $c^*(\Delta) \leq c(3,3,5,5,5) + \dfrac{2\pi}{5} < 0$.

Case 2.

There are two subcases; the region Δ receives positive curvature through

2.1) two double edges;

2.2) one double edge and one triple edge.

Case 2.1)

The maximum amount of curvature that Δ can receive is $\dfrac{\pi}{15} + \dfrac{\pi}{15} = \dfrac{2\pi}{15}$.

We have $d(v_a), d(v_{\mu_2}), d(v_e) \geq 5$ and $d(v_b), d(v_d) \geq 3$.

It follows that $c^*(\Delta) \leq c(3,3,5,5,5) + \dfrac{2\pi}{15} < 0$.

Case 2.2)

We have $d(v_b), d(v_d) \geq 3$ and $d(v_a), d(v_e) \geq 5$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{2\pi}{5} + \dfrac{\pi}{15} = \dfrac{7\pi}{15}$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_b and v_d to have sublabels $b\mu_1$ and $dd\overline{\lambda}$ respectively. This implies $d(v_b) \geq 5$ and $d(v_d) \geq 6$.
It follows that $c^*(\Delta) \leq c(5,5,5,6) + \dfrac{7\pi}{15} = 0$.
If $d(v_c) > 2$ then $d(v_c) \geq 4$. Moreover the vertices v_b and v_c cannot both have degree< 5.
It follows that $c^*(\Delta) \leq c(3,4,5,5,5) + \dfrac{7\pi}{15} < 0$.

$\square$

Lemma 4.2.14 *If Δ is a boundary region then $c^*(\Delta) < \dfrac{4\pi}{k_0}$.*

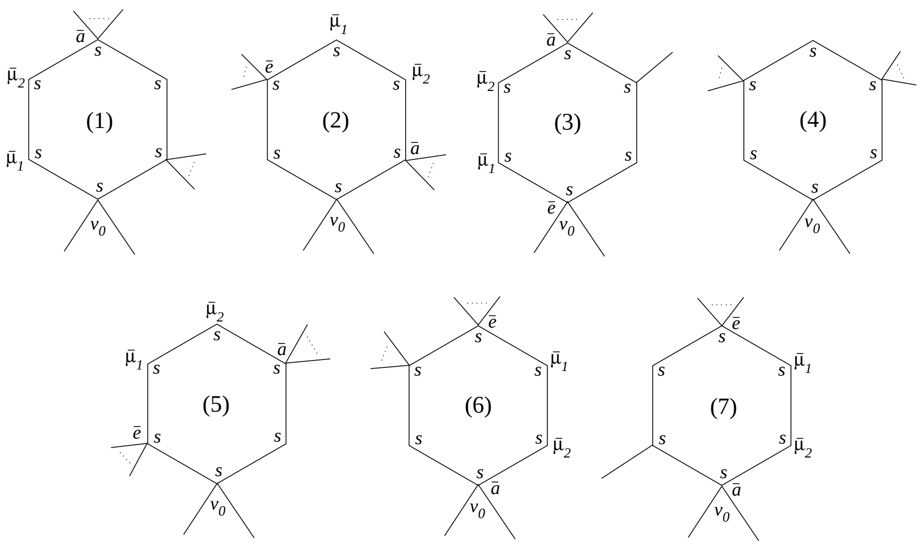
Proof Again, we need some extra distribution of curvature which involves boundary regions only. This will not affect the interior regions and so the proof of the previous lemma.
Moreover we can assume that the distinguished vertex coincides with a unique vertex of each boundary s-region and with no more than three vertices of each boundary m-region.
Distinguish the following cases:

1. Δ is an s-region;

2. Δ is an m-region.

Case 1.

As before we can assume that the distinguished vertex coincides with v_4 and $d(\Delta) \leq 3$.
Case 1.1)**,** $d(\Delta) = 3$
There are seven subcases:



Observe that the only admissible label in degree 3 involving s is bds.

It follows that in all cases, except cases (3) and (7), there are two vertices of degree ≥ 4, hence that $c^*(\Delta) = c(\Delta) \leq c(4, 4, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

We will then consider only cases (3) and (7) and assume that there is a vertex of degree 3.

In case (3) we have $d(v_2) = 3$ therefore $l(v_2) = bds$ which forces v_1 to have sublabel $\overline{a}se$ and so $d(v_1) \geq 4$.

We choose to transfer the curvature $\dfrac{\pi}{6}$ through the triple edge.

It follows that $c^*(\Delta) = c(\Delta) - \dfrac{\pi}{6} \leq c(3, 4, k_0) - \dfrac{\pi}{6} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

In case (7) we have $d(v_5) = 3$ and so $l(v_5) = bds$. This implies that v_1 has sublabel $as\overline{e}$ therefore $d(v_1) \geq 5$.

We choose again to transfer the curvature $\dfrac{\pi}{6}$ through the triple edge.

It follows that $c^*(\Delta) = c(\Delta) - \dfrac{\pi}{6} \leq c(3, 5, k_0) - \dfrac{\pi}{6} < \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Case 1.2**),** $d(\Delta) = 2$

The vertex v_1 has sublabel $\overline{a}s\overline{e}$ which implies $d(v_1) \geq 6$.

We choose to transfer the curvature $\dfrac{\pi}{6}$ through each triple edge; it follows

that $c^*(\Delta) \leq c(6, k_0) - \frac{\pi}{3} = \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.
Notice that this choice does not affect the interior regions.

Case 2.

Suppose the distinguished vertex v_0 coincides with m vertices of Δ. Notice that $k_0 \geq 2m$.

Suppose $m = 3$, then $v_0 = v_b = v_d = v_{\mu_1}$. Since $d(\Delta) \geq 4$, $d(v_a) \geq 4$ and Δ does not receive any positive curvature from interior or from boundary regions (as described above), it follows that $c^*(\Delta) = c(\Delta) \leq c(4, k_0, k_0, k_0) = -2\pi + \frac{\pi}{2} + \frac{6\pi}{k_0} \leq -\frac{\pi}{2} < \frac{4\pi}{k_0}$.
Now suppose $m = 2$. We have $c^*(\Delta) \leq c(\Delta) + \frac{\pi}{2} + \frac{\pi}{15} \leq c(3, 3, k_0, k_0) + \frac{17\pi}{30} = -\frac{2\pi}{3} + \frac{4\pi}{k_0} + \frac{17\pi}{30} = -\frac{\pi}{10} + \frac{4\pi}{k_0} < \frac{4\pi}{k_0}$.
We can therefore assume that v_0 coincides with a unique vertex of Δ.
The maximum amount of curvature that Δ can receive is $\frac{\pi}{2} + \frac{\pi}{15} = \frac{17\pi}{30}$.
Suppose $d(v_c) = 2$ and v_c is not distinguished. Then $l(v_c) = ce$, moreover v_b and v_d have sublabels $b\mu_1$ and dd respectively. For interior vertices we have $d(v_a) \geq 4$, $d(v_b) \geq 5$, $d(v_d) \geq 5$, moreover v_e and v_{μ_1} cannot both have degree< 4. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one, it follows that $c^*(\Delta) \leq c(4, 4, 5, k_0) + \frac{17\pi}{30} = \frac{2\pi}{k_0} - \frac{\pi}{30} < \frac{4\pi}{k_0}$.
Suppose $d(v_c) > 2$ and v_c is not distinguished (and so $d(v_c) \geq 4$). For interior vertices we have $d(v_a) \geq 4$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 4. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one, it follows that $c^*(\Delta) \leq c(3, 3, 4, 4, k_0) + \frac{17\pi}{30} = \frac{2\pi}{k_0} - \frac{\pi}{10} < \frac{4\pi}{k_0}$.
So we can assume $v_0 = v_c$, in which case $d(v_a) \geq 4$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 4. It follows that $c^*(\Delta) \leq$

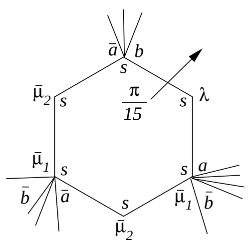
$$c(3,3,4,4,k_0) + \frac{17\pi}{30} = \frac{2\pi}{k_0} - \frac{\pi}{10} < \frac{4\pi}{k_0}.$$

$\square$

Distribution for case (a4), ($\alpha + \beta - 1 = 0$).

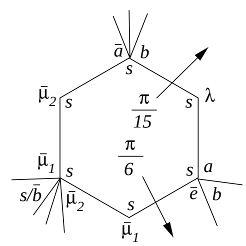
According to the classification we can obtain the positively curved interior regions from Figures 4.9 (ii), 4.10 (ii), 4.10 (iv), 4.11 (ii), 4.12 (i), 4.12 (ii), 4.13 (i), 4.14 (ii), 4.15 (ii), 4.15 (iii), 4.16 (i), 4.16 (ii), 4.16 (iv), 4.18 (i), 4.19 (i), 4.20 (i), 4.20 (ii), 4.21 (i), 4.22 (i), 4.22 (ii), 4.23 (ii).

The upper bounds for the curvature are sometimes decreased using the constraint $\alpha + \beta - 1 = 0$ and therefore 4.9 (ii), 4.10 (ii), 4.11 (ii) and 4.14 (ii) are not positively curved. The positive curvature is distributed according to the scheme which follows.



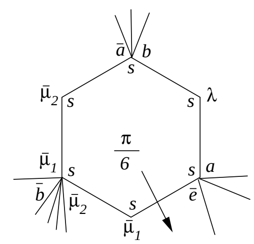
$c(\Delta) \leq \frac{2\pi}{105}$

4.10 (iv)



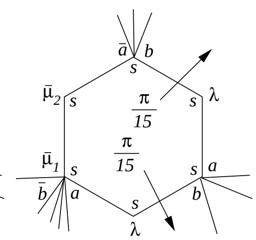
$c(\Delta) \leq \frac{13\pi}{70}$

4.12 (i)



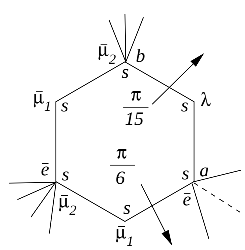
$c(\Delta) \leq \frac{3\pi}{35}$

4.12 (ii)



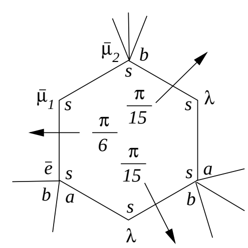
$c(\Delta) \leq \frac{3\pi}{35}$

4.13 (i)



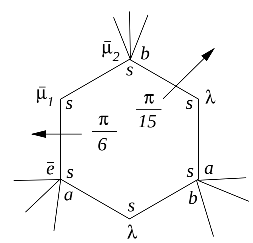
$c(\Delta) \leq \frac{7\pi}{30}$

4.15 (ii)–(iii)



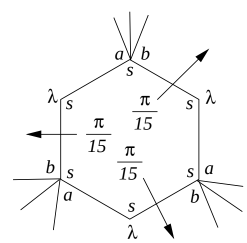
$c(\Delta) \leq \frac{3\pi}{10}$

4.16 (i)



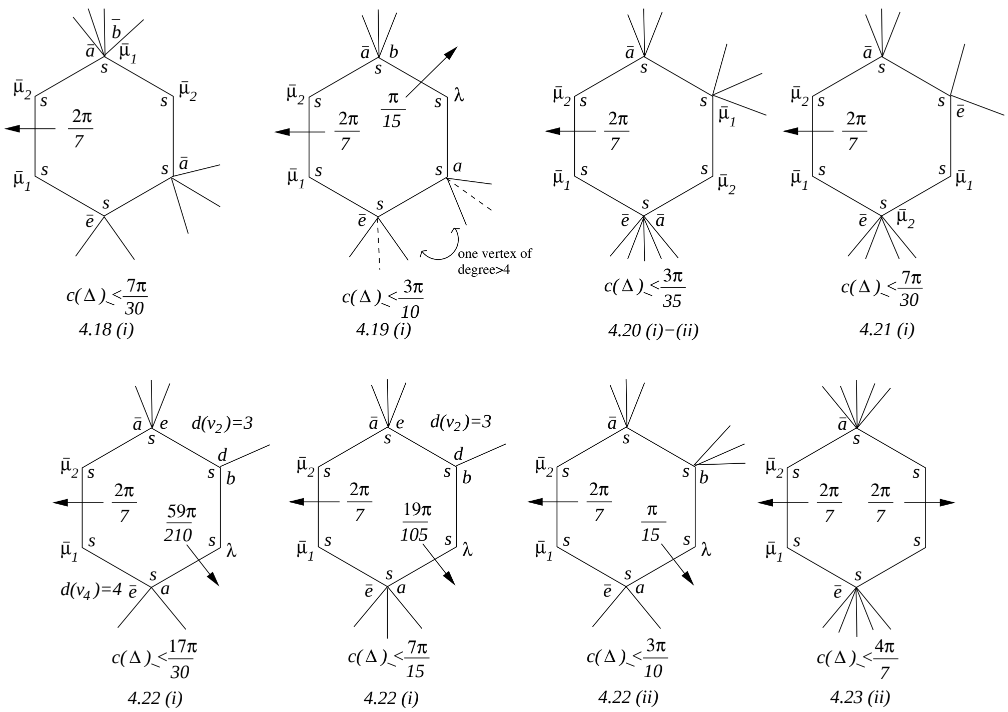
$c(\Delta) \leq \frac{\pi}{5}$

4.16 (ii)



$c(\Delta) \leq \frac{\pi}{5}$

4.16 (iv)



Lemma 4.2.15 *If Δ is interior then $c^*(\Delta) \leq 0$.*

Proof We distributed the positive curvature in such a way that every positively curved interior region has non-positive pseudo-curvature.

Let Δ be an interior m-region receiving positive curvature according to the scheme above. Distinguish two cases:

1. the region Δ receives positive curvature from exactly one s-region;

2. the region Δ receives positive curvature from exactly two s-regions.

Case 1.

There are three subcases; the region Δ receives positive curvature through

1.1) the double edge labelled by $e\mu_1\mu_2$;

1.2) the double edge labelled by $a\lambda b$;

1.3) the triple edge labelled by $e\mu_1\mu_2 a$.

Case 1.1)

We have $d(v_e) \geq 4$ and $d(v_{\mu_2}) \geq 5$. Moreover $d(v_b), d(v_d) \geq 3$ and $d(v_a) \geq 4$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{\pi}{6}$.

It follows that $c^*(\Delta) \leq c(3,3,4,4,5) + \dfrac{\pi}{6} < 0$.

Case 1.2)

Firstly suppose $d(v_b) = 3$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{59\pi}{210}$.

Observe that, according to the compensation scheme above, we have $d(v_b) = 3$, $d(v_d) \geq 3$ and $d(v_a) \geq 4$.

This implies $l(v_b) = bds$, hence v_c has sublabel cc and $d(v_c) \geq 6$.

If $d(v_e) = 2$ then $l(v_e) = ec$, this forces v_{μ_1} and v_d to have sublabels $b\mu_1$ and dd respectively, which imply $d(v_{\mu_1}) \geq 5$ and $d(v_d) \geq 6$.

It follows that $c^*(\Delta) \leq c(3,4,5,6,6) + \dfrac{59\pi}{210} < 0$.

If $d(v_e) > 2$ then $d(v_e) \geq 4$.

If $d(v_a) = 4$ then $l(v_a) = as\{\overline{e}, c\}b$ and so v_{μ_2} has sublabel $\mu_2 c$ and $d(v_{\mu_2}) \geq 4$.

In this case $c^*(\Delta) \leq c(3,3,4,4,4,6) + \dfrac{59\pi}{210} < 0$.

If $d(v_a) \geq 5$ we have $c^*(\Delta) \leq c(3,3,4,5,6) + \dfrac{59\pi}{210} < 0$.

Now assume $d(v_b) > 3$ which indeed implies $d(v_b) \geq 5$ (see the compensation scheme).

The maximum amount of positive curvature the region Δ can receive is $\dfrac{\pi}{15}$.

The vertices v_e and v_{μ_1} cannot both have degree < 4.

If $d(v_c) = 2$ then $l(v_c) = ce$, this forces v_b and v_d to have sublabels $sb\mu_1$ and dd respectively, which imply $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

Since $d(v_a) \geq 4$ it follows that $c^*(\Delta) \leq c(4,4,5,6) + \dfrac{\pi}{15} < 0$.

If $d(v_c) > 2$ then $d(v_c) \geq 4$ and it follows that $c^*(\Delta) \leq c(3,4,4,4,5) + \frac{\pi}{15} < 0$.

Case 1.3)

According to the compensation scheme above we have $d(v_a) \geq 5$ and $d(v_e) \geq 4$; moreover $d(v_b) \geq 3$ and $d(v_d) \geq 3$.

The maximum amount of positive curvature the region Δ can receive is $\frac{2\pi}{7}$.

If $d(v_c) > 2$ then $d(v_c) \geq 4$; moreover if $d(v_d) = 3$ then $d(v_c) \geq 6$.

It follows that $c^*(\Delta) \leq \max\{c(3,3,4,4,6), c(3,4,4,4,5)\} + \frac{2\pi}{7} < 0$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_b and v_d to have sublabels $b\mu_1$ and $dd\overline{\lambda}$ respectively. This implies $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

It follows that $c^*(\Delta) \leq c(4,5,5,6) + \frac{2\pi}{7} < 0$.

Case 2.

There are two subcases; the region Δ receives positive curvature through

2.1) two double edges;

2.2) one double edge and one triple edge.

Case 2.1)

If $d(v_b) = 3$ the maximum amount of curvature Δ can receive is $\frac{59\pi}{210} + \frac{\pi}{6} = \frac{47\pi}{105}$. We have $d(v_a) \geq 4$, $d(v_{\mu_2}) \geq 5$, $d(v_e) \geq 4$, $d(v_c) \geq 6$ and $d(v_d) \geq 3$.

It follows that $c^*(\Delta) \leq c(3,3,4,4,5,6) + \frac{47\pi}{105} < 0$.

If $d(v_b) > 3$ (hence $d(v_b) \geq 5$) then the maximum amount of curvature Δ can receive is $\frac{\pi}{15} + \frac{\pi}{6} = \frac{7\pi}{30}$.

We have $d(v_a) \geq 4$, $d(v_{\mu_2}) \geq 5$, $d(v_e) \geq 4$ and $d(v_d) \geq 3$.

It follows that $c^*(\Delta) \leq c(3,4,4,5,5) + \frac{7\pi}{30} < 0$.

Case 2.2)

Since Δ is receiving curvature through both the double edge labelled by $a\lambda b$

and the triple edge labelled by $e\mu_1\mu_2 a$, the vertex v_a has sublabel $\overline{s}as$, hence $d(v_a) \geq 5$.

Firstly assume $d(v_b) = 3$; the maximum amount of curvature Δ can receive is $\frac{19\pi}{105} + \frac{2\pi}{7} = \frac{7\pi}{15}$.

We have $d(v_a) \geq 5$, $d(v_e) \geq 4$, $d(v_c) \geq 6$ and $d(v_d) \geq 3$.

If $d(v_d) > 3$ then $c^*(\Delta) \leq c(3, 4, 4, 5, 6) + \frac{7\pi}{15} < 0$.

If $d(v_d) = 3$ then v_c has sublabel ccc hence $d(v_c) \geq 8$.

It follows that $c^*(\Delta) \leq c(3, 3, 4, 5, 8) + \frac{7\pi}{15} < 0$.

Now assume $d(v_b) > 3$ which implies $d(v_b) \geq 5$ (v_b has sublabel sb).

We have $d(v_d) \geq 3$, $d(v_e) \geq 4$ and $d(v_a) \geq 5$.

The maximum amount of positive curvature the region Δ can receive is $\frac{2\pi}{7} + \frac{\pi}{15} = \frac{37\pi}{105}$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_b and v_d to have sublabels $sb\mu_1$ and $dd\overline{\lambda}$ respectively. This implies $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

It follows that $c^*(\Delta) \leq c(4, 5, 5, 6) + \frac{37\pi}{105} < 0$.

If $d(v_c) > 2$ then $d(v_c) \geq 4$. It follows that $c^*(\Delta) \leq c(3, 4, 4, 5, 5) + \frac{37\pi}{105} < 0$.

□

Lemma 4.2.16 *If Δ is a boundary region then $c^*(\Delta) < \frac{4\pi}{k_0}$.*

Proof. Again, we need some extra distribution of curvature which involves boundary regions only. This will not affect the interior regions and so the proof of the previous lemma.

Moreover we can assume that the distinguished vertex coincides with a unique vertex of each boundary s-region and with no more than three vertices of each boundary m-region.

Distinguish the following cases:

1. Δ is an s-region;

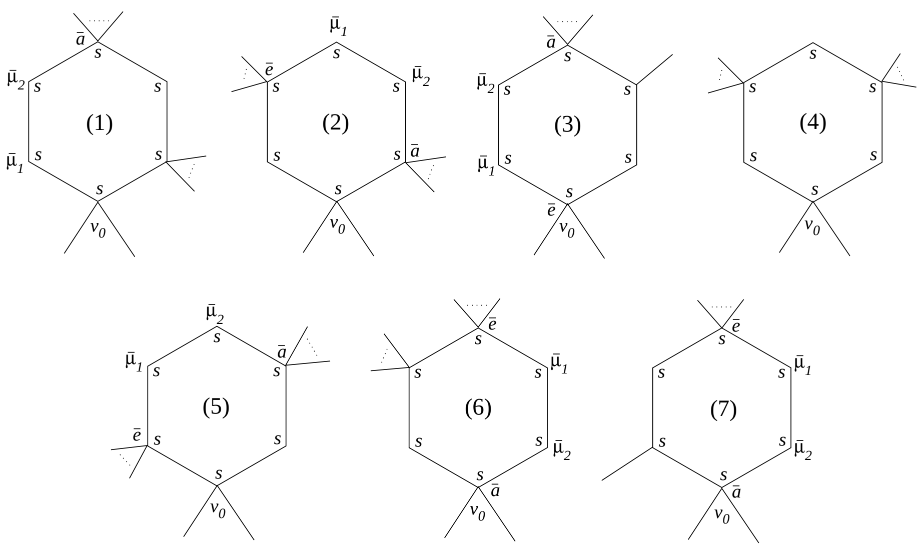
2. Δ is an m-region.

Case 1.

As before we can assume that the distinguished vertex coincides with v_4 and $d(\Delta) \leq 3$.

Case 1.1), $d(\Delta) = 3$

There are seven subcases:



Observe that the only admissible label in degree 3 involving s is bds.

Since de is a single edge, it follows that in all cases, except cases (3) and (7), there are two vertices of degree ≥ 4.

It follows that $c^*(\Delta) = c(\Delta) \leq c(4, 4, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

We will then consider only cases (3) and (7) and assume that there is a vertex of degree 3.

In case (3) we have $d(v_2) = 3$ therefore $l(v_2) = bds$ which forces v_1 to have sublabel $\overline{a}se$. The constraint $\alpha + \beta - 1 = 0$ implies $d(v_1) \geq 5$.

We choose to transfer the curvature $\frac{\pi}{15}$ through the triple edge.

It follows that $c^*(\Delta) = c(\Delta) - \frac{\pi}{15} \leq c(3,5,k_0) - \frac{\pi}{15} = \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

In case (7) we have $d(v_5) = 3$ and so $l(v_5) = bds$. This implies that v_1 has sublabel $as\bar{e}$ therefore $d(v_1) \geq 4$.

We choose to transfer the curvature $\frac{\pi}{6}$ through the triple edge.

It follows that $c^*(\Delta) = c(\Delta) - \frac{\pi}{6} \leq c(3,4,k_0) - \frac{\pi}{6} = \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Case 1.2)**,** $d(\Delta) = 2$

The vertex v_1 has sublabel $\bar{a}s\bar{e}$ which implies $d(v_1) \geq 7$.

We choose to transfer the curvature $\frac{\pi}{7}$ through each triple edge; it follows that $c^*(\Delta) \leq c(7,k_0) - \frac{2\pi}{7} = \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

Notice this choice does not affect the interior regions.

Case 2.

Suppose the distinguished vertex v_0 coincides with m vertices of Δ. Notice that $k_0 \geq 2m$.

Suppose $m = 3$, then $v_0 = v_b = v_d = v_{\mu_1}$. Since $d(\Delta) \geq 4$, $d(v_a) \geq 4$ and Δ does not receive any positive curvature from interior or from boundary regions (as described above), it follows that $c^*(\Delta) = c(\Delta) \leq c(4,k_0,k_0,k_0) = -2\pi + \frac{\pi}{2} + \frac{6\pi}{k_0} \leq -\frac{\pi}{2} < \frac{4\pi}{k_0}$.

Now suppose $m = 2$; then we can have $v_0 = v_b = v_d$ or $v_0 = v_b = v_{\mu_1}$ or $v_0 = v_d = v_{\mu_1}$ or $v_0 = v_\lambda = v_{\mu_2}$ or $v_0 = v_c = v_e$.

According to the distribution described above, the maximum amount of curvature that Δ can receive is $\frac{\pi}{6} + \frac{59\pi}{210} = \frac{47\pi}{210}$ (when $v_0 = v_c = v_e$). We have $c^*(\Delta) \leq c(\Delta) + \frac{47\pi}{210} \leq c(3,3,k_0,k_0) + \frac{47\pi}{210} = -\frac{2\pi}{3} + \frac{4\pi}{k_0} + \frac{47\pi}{210} = -\frac{31\pi}{70} + \frac{4\pi}{k_0} < \frac{4\pi}{k_0}$.

We can therefore assume that v_0 coincides with a unique vertex of Δ.

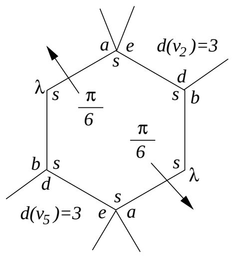
The maximum amount of curvature that Δ can receive is $\frac{2\pi}{7}+\frac{59\pi}{210}=\frac{17\pi}{30}$.
Suppose $d(v_c)=2$ and v_c is not distinguished. Then $l(v_c)=ce$, moreover v_b and v_d have sublabels $b\mu_1$ and dd respectively. For interior vertices we have $d(v_a)\geq 4$, $d(v_b)\geq 5$, $d(v_d)\geq 6$, moreover v_e and v_{μ_1} cannot both have degree<4. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one, it follows that $c^*(\Delta)\leq c(4,4,5,k_0)+\frac{17\pi}{30}=\frac{2\pi}{k_0}-\frac{\pi}{30}<\frac{4\pi}{k_0}$.
Suppose $d(v_c)>2$ and v_c is not distinguished (and so $d(v_c)\geq 4$). For interior vertices we have $d(v_a)\geq 4$, $d(v_b)\geq 3$, $d(v_d)\geq 3$, moreover v_e and v_{μ_1} cannot both have degree<4. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one, it follows that $c^*(\Delta)\leq c(3,3,4,4,k_0)+\frac{17\pi}{30}=\frac{2\pi}{k_0}-\frac{\pi}{10}<\frac{4\pi}{k_0}$.
So we can assume $v_0=v_c$, in which case $d(v_a)\geq 4$, $d(v_b)\geq 3$, $d(v_d)\geq 3$, moreover v_e and v_{μ_1} cannot both have degree<4. It follows that $c^*(\Delta)\leq c(3,3,4,4,k_0)+\frac{17\pi}{30}=\frac{2\pi}{k_0}-\frac{\pi}{10}<\frac{4\pi}{k_0}$.

□

Distribution for case (a5), ($\alpha-\beta-1=0$).

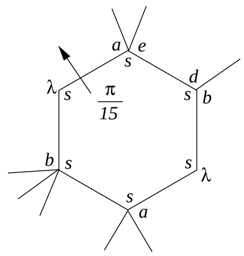
According to the classification we can obtain the positively curved interior regions from Figures 4.9 (*i*), 4.9 (*ii*), 4.9(*iii*), 4.10 (*ii*), 4.10 (*iv*), 4.11 (*ii*), 4.12 (*ii*), 4.13 (*i*), 4.14 (*ii*), 4.15 (*iii*), 4.16 (*ii*), 4.16 (*iv*), 4.18 (*i*), 4.19 (*i*), 4.20 (*i*), 4.20 (*ii*), 4.21 (*i*), 4.22 (*i*), 4.22 (*ii*), 4.23 (*ii*).

The upper bounds for the curvature are sometimes decreased using the constraint $\alpha-\beta-1=0$ and therefore 4.12 (*ii*) is not positively curved. The positive curvature is distributed according to the scheme which follows.



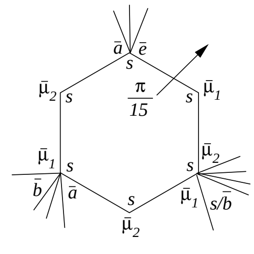
$c(\Delta) \leq \frac{\pi}{3}$

4.9 (i)−(ii)



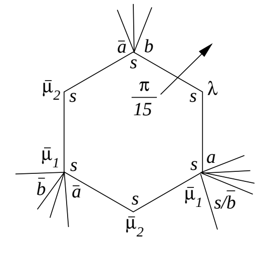
$c(\Delta) \leq \frac{\pi}{15}$

4.9 (iii)



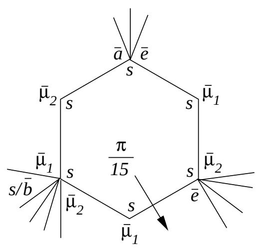
$c(\Delta) \leq \frac{2\pi}{105}$

4.10 (ii)



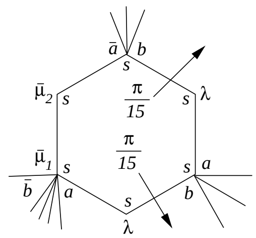
$c(\Delta) \leq \frac{2\pi}{105}$

4.10 (iv)



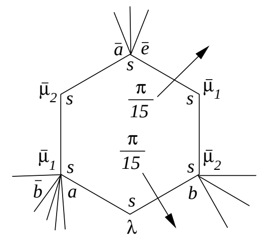
$c(\Delta) \leq \frac{2\pi}{105}$

4.11 (ii)



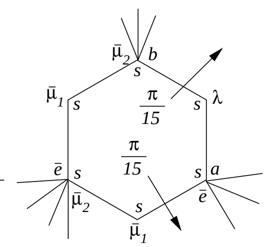
$c(\Delta) \leq \frac{3\pi}{35}$

4.13 (i)



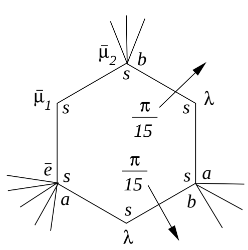
$c(\Delta) \leq \frac{3\pi}{35}$

4.14 (ii)



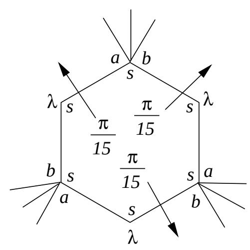
$c(\Delta) \leq \frac{2\pi}{15}$

4.15 (iii)



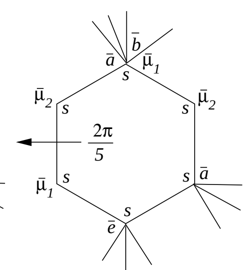
$c(\Delta) \leq \frac{\pi}{5}$

4.16 (ii)



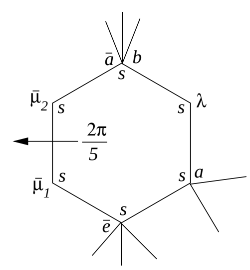
$c(\Delta) \leq \frac{\pi}{5}$

4.16 (iv)



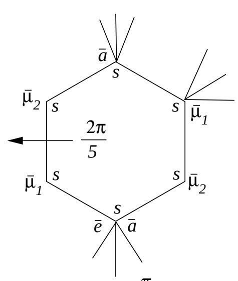
$c(\Delta) \leq \frac{2\pi}{15}$

4.18 (i)



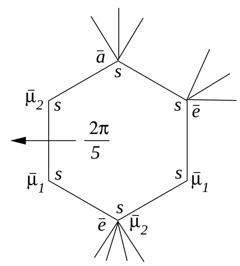
$c(\Delta) \leq \frac{3\pi}{10}$

4.19 (i)



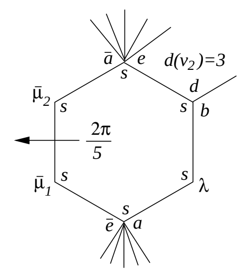
$c(\Delta) \leq \frac{\pi}{5}$

4.20 (i)−(ii)



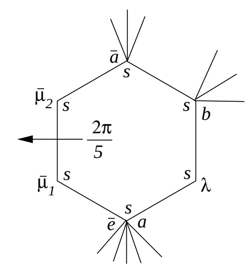
$c(\Delta) \leq \frac{2\pi}{15}$

4.21 (i)



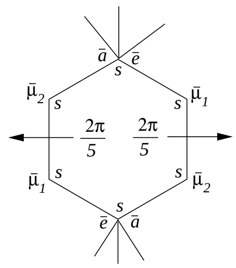
$c(\Delta) \leq \frac{5\pi}{21}$

4.22 (i)



$c(\Delta) \leq \frac{3\pi}{35}$

4.22 (ii)



$c(\Delta) \leq \frac{4\pi}{5}$

4.23 (ii)

Lemma 4.2.17 *If Δ is interior then $c^*(\Delta) \leq 0$.*

Proof We distributed the positive curvature in such a way that every positively curved interior region has non-positive pseudo-curvature.

Let Δ be an interior m-region receiving positive curvature according to the scheme above.

Distinguish two cases:

1. the region Δ receives positive curvature from exactly one s-region;

2. the region Δ receives positive curvature from exactly two s-regions.

Case 1.

There are three subcases; the region Δ receives positive curvature through

1.1) the double edge labelled by $e\mu_1\mu_2$;

1.2) the double edge labelled by $a\lambda b$;

1.3) the triple edge labelled by $e\mu_1\mu_2 a$.

Case 1.1)

We have $d(v_e), d(v_{\mu_2}) \geq 5$. Moreover $d(v_b), d(v_d) \geq 3$ and $d(v_a) \geq 4$.

The maximum amount of positive curvature the region Δ can receive is $\dfrac{\pi}{15}$.

It follows that $c^*(\Delta) \leq c(3,3,4,5,5) + \dfrac{\pi}{15} < 0$.

Case 1.2)

Let $d(v_b) = 3$; the maximum amount of curvature that Δ can receive is $\dfrac{\pi}{6}$.

We have $d(v_a) \geq 4$, moreover the vertices v_e and v_{μ_1} cannot both have degree< 4.

Since we must have $l(v_b) = bds$ it follows that v_c has sublabel cc which implies $d(v_c) \geq 6$. It follows that $c^*(\Delta) \leq c(3,3,4,4,6) + \dfrac{\pi}{6} < 0$.

Now assume $d(v_b) > 3$. The constraint $\alpha - \beta - 1 = 0$ implies $d(v_b) \geq 5$.

Moreover the vertices v_e and v_{μ_1} cannot both have degree< 4; the vertices v_a and v_{μ_2} cannot both have degree< 5 and the vertices v_c and v_d cannot both have degree< 5. It follows that $c^*(\Delta) \leq c(4,5,5,5) + \dfrac{\pi}{15} < 0$.

Case 1.3)

The maximum amount of positive curvature the region Δ can receive is $\dfrac{2\pi}{5}$.

According to the compensation scheme above we have $d(v_e), d(v_a) \geq 5$; moreover $d(v_b), d(v_d) \geq 3$.

First assume $d(v_c) = 2$.

This implies $l(v_c) = ce$, therefore v_b and v_d have sublabels $b\mu_1$ and dd respectively. Since $\alpha - \beta - 1 = 0$ we have $d(v_b) \geq 5$ and $d(v_d) \geq 6$.

It follows that $c^*(\Delta) \leq c(5,5,5,6) + \dfrac{2\pi}{5} < 0$.

Now assume $d(v_c) > 2$, hence $d(v_c) \geq 4$.

Let $d(v_c) = 4$. Then v_c has label $c0\overline{a}\overline{b}$, therefore v_b has sublabel $b\overline{\lambda}$ and $d(v_b) \geq 5$. It follows that $c^*(\Delta) \leq c(3,4,5,5,5) + \dfrac{2\pi}{5} < 0$.

Finally if $d(v_c) \geq 5$ we have $c^*(\Delta) \leq c(3,3,5,5,5) + \dfrac{2\pi}{5} < 0$.

Case 2.

There are two subcases; the region Δ receives positive curvature through

2.1) two double edges;

2.2) one double edge and one triple edge.

Case 2.1)

The maximum amount of curvature that Δ can receive is $\frac{\pi}{15} + \frac{\pi}{6} = \frac{7\pi}{30}$.

We have $d(v_a) \geq 4$, $d(v_{\mu_2}), d(v_e) \geq 5$ and $d(v_b), d(v_d) \geq 3$.

It follows that $c^*(\Delta) \leq c(3, 3, 4, 5, 5) + \frac{7\pi}{30} < 0$.

Case 2.2)

First assume $d(v_b) = 3$.

The maximum amount of positive curvature the region Δ can receive is $\frac{2\pi}{5} + \frac{\pi}{6} = \frac{17\pi}{30}$.

Since $l(v_b) = bds$ the vertex v_c has sublabel cc and therefore $d(v_c) \geq 6$.

We have $d(v_e), d(v_a) \geq 5$ and $d(v_d) \geq 3$.

Moreover if $d(v_d) = 3$ then v_c has sublabel ccc, hence $d(v_c) \geq 8$.

It follows that $c^*(\Delta) \leq \max\{c(3, 3, 5, 5, 8), c(3, 4, 5, 5, 6)\} + \frac{17\pi}{30} < 0$.

Now assume $d(v_b) > 3$ which under the constraint $\alpha - \beta - 1 = 0$ implies $d(v_b) \geq 5$.

The maximum amount of positive curvature the region Δ can receive is $\frac{2\pi}{5} + \frac{\pi}{15} = \frac{7\pi}{15}$.

If $d(v_c) = 2$ then $l(v_c) = ce$ and as in case 1.3) above we have $d(v_b) \geq 5$ and $d(v_d) \geq 6$. It follows that $c^*(\Delta) \leq c(5, 5, 5, 6) + \frac{7\pi}{15} = 0$.

If $d(v_c) > 2$ then the constraint $\alpha - \beta - 1 = 0$ implies $d(v_c) \geq 4$.

It follows that $c^*(\Delta) \leq c(3, 4, 5, 5, 5) + \frac{7\pi}{15} < 0$.

$\square$

Lemma 4.2.18 *If Δ is a boundary region then $c^*(\Delta) < \frac{4\pi}{k_0}$.*

Proof. Again, we need some extra distribution of curvature which involves boundary regions only. This will not affect the interior regions and so the proof of the previous lemma.

Moreover we can assume that the distinguished vertex coincides with a unique vertex of each boundary s-region and with no more than three vertices of each boundary m-region. Distinguish the following cases:

1. Δ is an s-region;

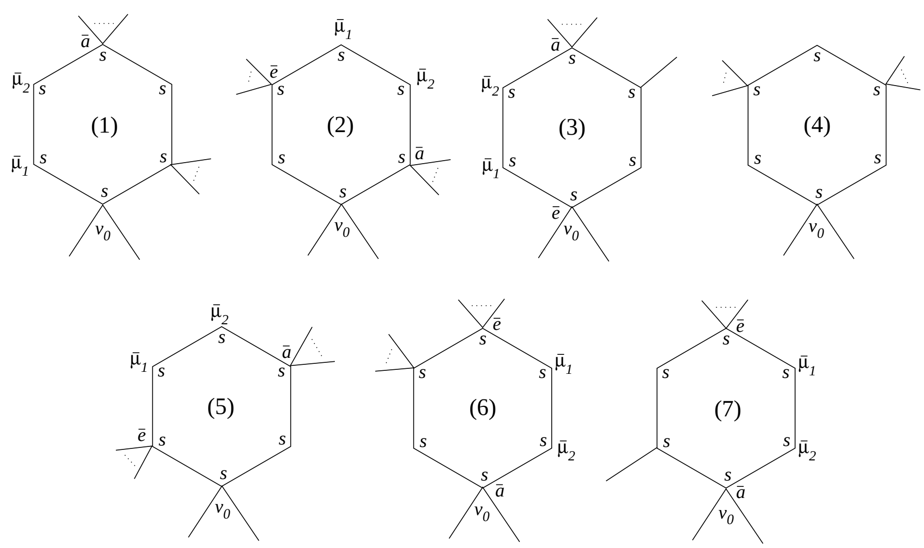
2. Δ is an m-region.

Case 1.

We can assume that the distinguished vertex coincides with v_4 and $d(\Delta) \leq 3$.

Case 1.1)**,** $d(\Delta) = 3$

There are seven subcases:



Observe that the only admissible label in degree 3 involving s is bds.

It follows that in all cases, except cases (3) and (7), there are two interior vertices of degree ≥ 4. It follows that $c^*(\Delta) = c(\Delta) \leq c(4, 4, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

We will then consider only cases (3) and (7) and assume that there is a vertex of degree 3.

In case (3) we have $d(v_2) = 3$ therefore $l(v_2) = bds$ which forces v_1 to have sublabel $\overline{a}se$. The constraint $\alpha - \beta - 1 = 0$ implies $d(v_1) \geq 7$.

It follows that $c^*(\Delta) = c(\Delta) \leq c(3, 7, k_0) = \dfrac{2\pi}{k_0} - \dfrac{\pi}{21} < \dfrac{4\pi}{k_0}$.

In case (7) we have $d(v_5) = 3$ and so $l(v_5) = bds$. This implies that v_1 has sublabel $as\overline{e}$ therefore $d(v_1) \geq 7$.

As above $c^*(\Delta) = c(\Delta) \leq c(3, 7, k_0) = \dfrac{2\pi}{k_0} - \dfrac{\pi}{21} < \dfrac{4\pi}{k_0}$.

Case 1.2)**,** $d(\Delta) = 2$

The vertex v_1 has sublabel $\overline{a}s\overline{e}$ which implies $d(v_1) \geq 5$.

We choose to transfer the curvature $\dfrac{\pi}{5}$ through each triple edge; it follows that $c^*(\Delta) \leq c(5, k_0) - \dfrac{2\pi}{5} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Notice this choice does not affect any interior region.

Case 2.

Suppose the distinguished vertex v_0 coincides with m vertices of Δ. Notice that $k_0 \geq 2m$.

Suppose $m = 3$, then $v_0 = v_b = v_d = v_{\mu_1}$. Since $d(\Delta) \geq 4$, $d(v_a) \geq 4$ and Δ does not receive any positive curvature from interior or from boundary regions (as described above), it follows that $c^*(\Delta) = c(\Delta) \leq c(4, k_0, k_0, k_0) = -2\pi + \dfrac{\pi}{2} + \dfrac{6\pi}{k_0} \leq -\dfrac{\pi}{2} < \dfrac{4\pi}{k_0}$.

Now suppose $m = 2$; recall that the maximum total amount of curvature that Δ can receive is $\dfrac{2\pi}{5} + \dfrac{\pi}{6} = \dfrac{17\pi}{30}$. We have $c^*(\Delta) \leq c(\Delta) + \dfrac{17\pi}{30} \leq c(3, 3, k_0, k_0) + \dfrac{17\pi}{30} = -\dfrac{2\pi}{3} + \dfrac{4\pi}{k_0} + \dfrac{17\pi}{30} = -\dfrac{\pi}{10} + \dfrac{4\pi}{k_0} < \dfrac{4\pi}{k_0}$.

We can therefore assume that v_0 coincides with a unique vertex of Δ.

The maximum amount of curvature that Δ can receive is $\dfrac{2\pi}{5} + \dfrac{\pi}{6} = \dfrac{17\pi}{30}$.

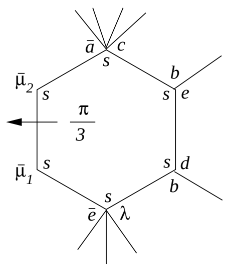
Suppose $d(v_c) = 2$ and v_c is not distinguished. Then $l(v_c) = ce$, moreover v_b and v_d have sublabels $b\mu_1$ and dd respectively. For interior vertices we have $d(v_a) \geq 4$, $d(v_b) \geq 5$, $d(v_d) \geq 6$, moreover v_e and v_{μ_1} cannot both have degree< 4. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one it follows that $c^*(\Delta) \leq c(4,4,5,k_0)+\frac{17\pi}{30} = \frac{2\pi}{k_0}-\frac{\pi}{30} < \frac{4\pi}{k_0}$.
Suppose $d(v_c) > 2$ and v_c is not distinguished (and so $d(v_c) \geq 4$). For interior vertices we have $d(v_a) \geq 4$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 4. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one it follows that $c^*(\Delta) \leq c(3,3,4,4,k_0) + \frac{17\pi}{30} = \frac{2\pi}{k_0} - \frac{\pi}{10} < \frac{4\pi}{k_0}$.
So we can assume $v_0 = v_c$, in which case $d(v_a) \geq 4$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 4. It follows that $c^*(\Delta) \leq c(3,3,4,4,k_0) + \frac{17\pi}{30} = \frac{2\pi}{k_0} - \frac{\pi}{10} < \frac{4\pi}{k_0}$.

$\square$

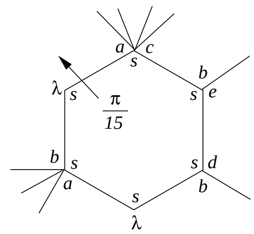
Distribution for case (b), ($\alpha = \beta$).

According to the classification we can obtain the positively curved interior regions from Figures 4.5 (*i*), 4.7 (*iii*), 4.10 (*iii*), 4.11 (*iii*), 4.12 (*iii*), 4.13 (*ii*), 4.14 (*iii*), 4.15 (*i*), 4.15 (*iv*), 4.16 (*iii*), 4.16 (*iv*), 4.18 (*iii*), 4.19 (*iv*), 4.20 (*iii*), 4.21 (*iii*), 4.22 (*iii*), 4.22 (*iv*), 4.23 (*iii*).

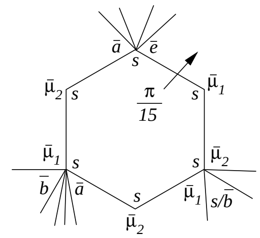
The upper bounds for the curvature are sometimes decreased using the constraint $\alpha = \beta$ and the positive curvature is distributed according to the scheme which follows.



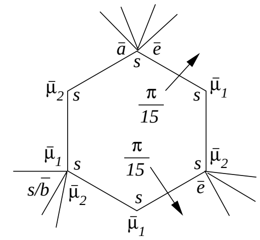
$c(\Delta) \leq \frac{\pi}{15}$
4.5 (i)



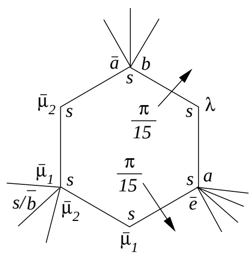
$c(\Delta) \leq \frac{\pi}{15}$
4.7 (iii)



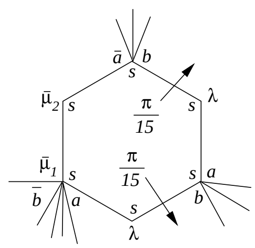
$c(\Delta) \leq \frac{2\pi}{105}$
4.10 (iii)



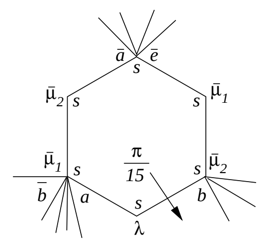
$c(\Delta) \leq \frac{2\pi}{15}$
4.11 (iii)



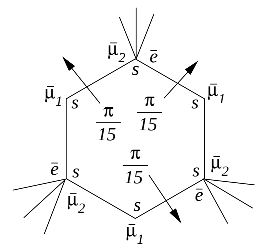
$c(\Delta) \leq \frac{2\pi}{15}$
4.12 (iii)



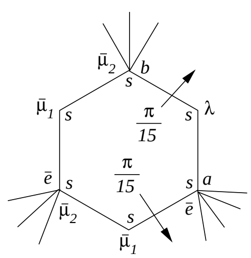
$c(\Delta) \leq \frac{3\pi}{35}$
4.13 (ii)



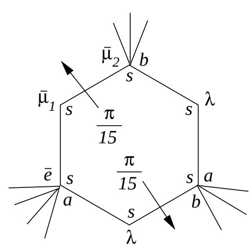
$c(\Delta) \leq \frac{2\pi}{105}$
4.14 (iii)



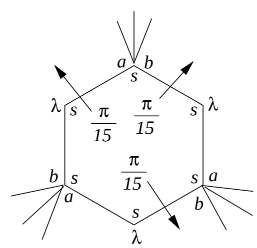
$c(\Delta) \leq \frac{\pi}{5}$
4.15 (i)



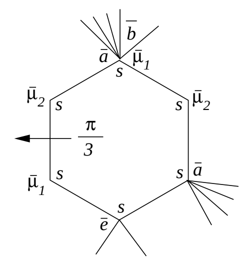
$c(\Delta) \leq \frac{2\pi}{15}$
4.15 (iv)



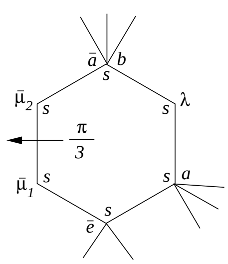
$c(\Delta) \leq \frac{2\pi}{15}$
4.16 (iii)



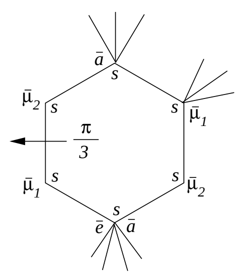
$c(\Delta) \leq \frac{\pi}{5}$
4.16 (iv)



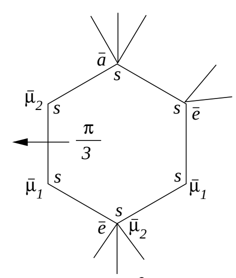
$c(\Delta) \leq \frac{9\pi}{70}$
4.18 (iii)



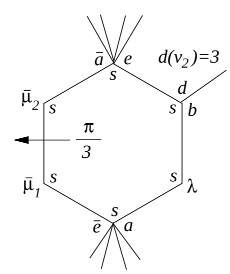
$c(\Delta) \leq \frac{3\pi}{10}$
4.19 (iv)



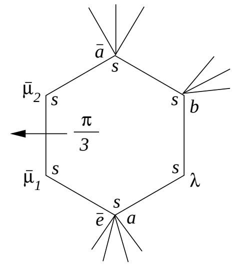
$c(\Delta) \leq \frac{2\pi}{15}$
4.20 (iii)



$c(\Delta) \leq \frac{3\pi}{10}$
4.21 (iii)

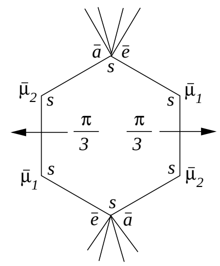


$c(\Delta) \leq \frac{\pi}{3}$
4.22 (iii)



$c(\Delta) \leq \frac{7\pi}{15}$

4.22 (iv)



$c(\Delta) \leq \frac{2\pi}{3}$

4.23 (iii)

Lemma 4.2.19 *If Δ is interior then $c^*(\Delta) \leq 0$.*

Proof We distributed the positive curvature in such a way that every positively curved interior region has non-positive pseudo-curvature.
Let Δ be an interior m-region receiving positive curvature according to the scheme above.
Distinguish two cases:

1. the region Δ receives positive curvature from exactly one s-region;

2. the region Δ receives positive curvature from exactly two s-regions.

Case 1.

There are three subcases; the region Δ receives positive curvature through

1.1) the double edge labelled by $e\mu_1\mu_2$;

1.2) the double edge labelled by $a\lambda b$;

1.3) the triple edge labelled by $e\mu_1\mu_2 a$.

Case 1.1)

The incoming curvature is $\frac{\pi}{15}$. Observe that, according to the distribution scheme, we have $d(v_e), d(v_{\mu_2}) \geq 5$; moreover $d(v_a) \geq 5$ and $d(v_b), d(v_d) \geq 3$.

It follows that $c(\Delta) \leq c(3,3,5,5,5) = -\dfrac{7\pi}{15}$ and so $c^*(\Delta) < 0$.

Case 1.2)

The incoming curvature is $\dfrac{\pi}{15}$. Observe that, according to the distribution scheme, we have $d(v_a), d(v_b) \geq 5$, moreover $d(v_d) \geq 3$.

If $d(v_e) = 2$ then $l(v_e) = ec$, therefore v_d and v_{μ_1} have sublabels dd and $b\mu_1$.

The constraint $\alpha = \beta$ implies $d(v_d) \geq 5$.

It follows that $c(\Delta) \leq c(3,5,5,5) = -\dfrac{2\pi}{15}$ and so $c^*(\Delta) < 0$.

If $d(v_e) > 2$ and $d(v_c) = 2$ then $l(v_c) = ce$ therefore v_d has sublabel dd and so $d(v_d) \geq 5$.

It follows that $c(\Delta) \leq c(3,5,5,5) = -\dfrac{2\pi}{15}$ and so $c^*(\Delta) < 0$.

If $d(v_e), d(v_c) > 2$ then $c(\Delta) \leq c(3,3,3,5,5) = -\dfrac{\pi}{5}$ and so $c^*(\Delta) < 0$.

Case 1.3)

The incoming curvature is $\dfrac{\pi}{3}$. Observe that, according to the distribution scheme, we have $d(v_e) \geq 4$ and $d(v_a) \geq 5$, moreover $d(v_b), d(v_d) \geq 3$.

First suppose $d(v_a) = 5$; since v_a has sublabel $\overline{s}a$ this forces $l(v_a) = \overline{s}a\overline{d}\overline{a}\overline{b}$.

It follows that v_λ has sublabel $\overline{e}\lambda$ and so $d(v_\lambda) \geq 3$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_d to have sublabel dd and so $d(v_d) \geq 5$.

In this case $c(\Delta) \leq c(3,3,4,5,5) = -\dfrac{11\pi}{30}$ and so $c^*(\Delta) < 0$.

If $d(v_c) > 2$ then $c(\Delta) \leq c(3,3,3,3,4,5) = -\dfrac{13\pi}{30}$ and so $c^*(\Delta) < 0$.

Now suppose $d(v_a) \geq 6$.

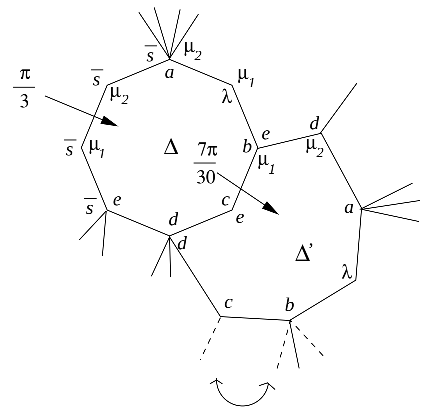
If $d(v_c) = 2$ then $l(v_c) = ce$, which forces v_d and v_b to have sublabels dd and $b\mu_1$ respectively; it follows that $d(v_d) \geq 5$.

If, in addition, $d(v_b) > 3$ then the constraint $\alpha = \beta$ implies $d(v_b) \geq 5$.

It follows that $c(\Delta) \leq c(4,5,5,6) = -\dfrac{11\pi}{30}$ and so $c^*(\Delta) < 0$.

If $d(v_b) = 3$ instead, then the curvature of Δ is not negative enough, in general, to compensate the incoming curvature, and this actually happens when $l(v_b) = b\mu_1 e$ (if $l(v_b) = b\mu_1\bar{c}$ then $d(v_\lambda) \geq 3$ and so $c^*(\Delta) \leq c(3,3,4,5,6) + \frac{\pi}{3} < 0$).

Let Δ' be the m-region adjacent to Δ in the following figure; denote the vertices of Δ' by v'_r for $r = a, \lambda, b, c, d, e, \mu_1, \mu_2$ and redistribute the curvature as follows:



Then we have $c^*(\Delta) \leq 0$.

Since $d(v'_e) = 2$, Δ' cannot receive more than $\frac{\pi}{15}$ from other s-regions.

If Δ' is a boundary region there is nothing to prove, so assume Δ' is interior. We have $d(v'_a) \geq 5$ and $d(v'_{\mu_2}), d(v'_b) \geq 3$; moreover if $d(v'_c) = 2$ then $d(v'_b) \geq 5$, therefore $c(\Delta') \leq \max\{c(3,3,5,5,5), c(3,3,3,3,5,5)\} = -\frac{7\pi}{15}$.

Finally, if $d(v_c) > 2$ then observe that v_c and v_b cannot have degree 3 simultaneously.

It follows that $c(\Delta) \leq c(3,3,4,4,6) = -\frac{\pi}{3}$ and so $c^*(\Delta) \leq 0$.

Case 2.

There are two subcases; the region Δ receives positive curvature through

2.1) two double edges;

2.2) one double edge and one triple edge.

Case 2.1)

The incoming curvature is $\frac{\pi}{15}+\frac{\pi}{15}=\frac{2\pi}{15}$. Observe that, according to the distribution scheme, we have $d(v_a), d(v_{\mu_2}), d(v_e) \geq 5$; moreover $d(v_b), d(v_d) \geq 3$.

It follows that $c(\Delta) \leq c(3,3,5,5,5) = -\frac{7\pi}{15}$ and so $c^*(\Delta) < 0$.

Case 2.2)

The incoming curvature is $\frac{\pi}{15}+\frac{\pi}{3}=\frac{2\pi}{5}$.

Observe that $d(v_e) \geq 4$ and $d(v_b), d(v_d) \geq 3$; moreover since v_a has sublabel $\overline{s}as$ we have $d(v_a) \geq 7$.

If $d(v_c) = 2$ then $l(v_c) = ce$, therefore v_b and v_c have sublabels $sb\mu_1$ and dd respectively; this implies $d(v_b), d(v_d) \geq 5$.

It follows that $c(\Delta) \leq c(4,5,5,7) = -\frac{29\pi}{70}$ and so $c^*(\Delta) \leq \frac{2\pi}{5} - \frac{29\pi}{70} < 0$.

So assume $d(v_c) > 2$. The vertices v_b and v_c cannot both have degree≤ 4.

It follows that $c(\Delta) \leq c(3,3,4,5,7) = -\frac{101\pi}{210}$ and so $c^*(\Delta) \leq \frac{2\pi}{5} - \frac{101\pi}{210} < 0$.

$\square$

Lemma 4.2.20 *If Δ is a boundary region then $c^*(\Delta) < \frac{4\pi}{k_0}$.*

Proof. Again, we need some extra distribution of curvature which involves boundary regions only. This will not affect the interior regions and so the proof of the previous lemma.

Moreover we can assume that the distinguished vertex coincides with a unique vertex of each boundary s-region and with no more than three vertices of each boundary m-region.

Distinguish the following cases:

1. Δ is an s-region;

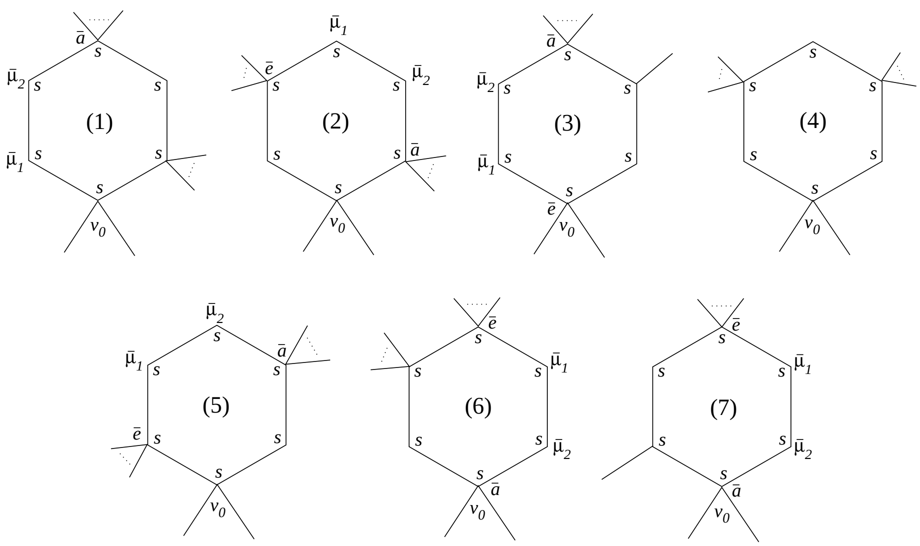
2. Δ is an m-region.

Case 1.

We can assume without any loss that the distinguished vertex coincides with v_4 and that $d(\Delta) \leq 3$.

Case 1.1)**,** $d(\Delta) = 3$

There are seven subcases:



Observe that in all cases except case (3) and case (7), there are two interior vertices of degree ≥ 4. It follows that $c^*(\Delta) = c(\Delta) \leq c(4, 4, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

In case (3) we might have $d(v_2) = 3$, but in this case $l(v_2) = bds$ and so v_1 has sublabel $\overline{a}se$ which implies $d(v_1) \geq 6$.

It follows that $c^*(\Delta) = c(\Delta) \leq c(3, 6, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

In case (7) we might have $d(v_5) = 3$, but in this case $l(v_5) = bds$ and so v_1 has sublabel $as\overline{e}$ which implies $d(v_1) \geq 6$.

It follows that $c^*(\Delta) = c(\Delta) \leq c(3, 6, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Case 1.2)**,** $d(\Delta) = 2$

The vertex v_1 has sublabel $\overline{a}s\overline{e}$ which implies $d(v_1) \geq 6$.

We choose to transfer the curvature $\dfrac{\pi}{6}$ through each triple edge; it follows that $c^*(\Delta) \leq c(6, k_0) - \dfrac{\pi}{3} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Notice that this choice does not change the pseudo-curvature of any interior region.

Case 2.

Suppose the distinguished vertex v_0 coincides with m vertices of Δ. Notice that $k_0 \geq 2m$.

Suppose $m = 3$, then $v_0 = v_b = v_d = v_{\mu_1}$. Since $d(\Delta) \geq 4$, $d(v_a) \geq 5$ and Δ does not receive any positive curvature from interior or from boundary regions (as described above), it follows that $c^*(\Delta) = c(\Delta) \leq c(5, k_0, k_0, k_0) = -2\pi + \dfrac{2\pi}{5} + \dfrac{6\pi}{k_0} \leq -\dfrac{3\pi}{5} < \dfrac{4\pi}{k_0}$.

Now suppose $m = 2$. The maximum amount of curvature that Δ can receive is $\dfrac{\pi}{3} + \dfrac{\pi}{15} = \dfrac{2\pi}{5}$. We have $c^*(\Delta) \leq c(\Delta) + \dfrac{2\pi}{5} \leq c(3, 3, k_0, k_0) + \dfrac{2\pi}{5} = -\dfrac{2\pi}{3} + \dfrac{4\pi}{k_0} + \dfrac{2\pi}{5} = -\dfrac{4\pi}{15} + \dfrac{4\pi}{k_0} < \dfrac{4\pi}{k_0}$.

We can therefore assume that v_0 coincides with a unique vertex of Δ.

The maximum amount of curvature that Δ can receive is $\dfrac{\pi}{3} + \dfrac{\pi}{15} = \dfrac{2\pi}{5}$.

Suppose $d(v_c) = 2$ and v_c is not distinguished. Then $l(v_c) = ce$, moreover v_b and v_d have sublabels $b\mu_1$ and dd respectively. For interior vertices we have $d(v_a) \geq 5$, $d(v_b) \geq 3$, $d(v_d) \geq 5$, moreover either two of the vertices v_e, v_{μ_1} and v_{μ_2} have degree≥ 3 or one of the vertices v_e, v_{μ_1} and v_{μ_2} has degree≥ 4. Since one of the vertices v_a, v_b, v_d, v_e, v_{μ_1} and v_{μ_2} can be the distinguished one it follows that $c^*(\Delta) \leq \max\{c(3, 4, 5, k_0), c(3, 3, 3, 5, k_0)\} + \dfrac{2\pi}{5} = \dfrac{2\pi}{k_0} - \dfrac{\pi}{30} < \dfrac{4\pi}{k_0}$.

Suppose $d(v_c) > 2$ and v_c is not distinguished (and so $d(v_c) \geq 4$). For interior vertices we have $d(v_a) \geq 5$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 3. Since one of the vertices v_a, v_b, v_d, v_e and v_{μ_1} can be the distinguished one it follows that $c^*(\Delta) \leq c(3,3,3,4,k_0) + \frac{2\pi}{5} = \frac{2\pi}{k_0} - \frac{\pi}{10} < \frac{4\pi}{k_0}$.

So we can assume $v_0 = v_c$, in which case $d(v_a) \geq 5$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 3. It follows that $c^*(\Delta) \leq c(3,3,3,5,k_0) + \frac{2\pi}{5} = \frac{2\pi}{k_0} - \frac{\pi}{5} < \frac{4\pi}{k_0}$.

□

Distribution for case (c), ($\alpha = -\beta$).

According to the classification we can obtain the positively curved interior regions from Figures 4.5 (*ii*), 4.5 (*iii*), 4.5 (*iv*), 4.7 (*i*), 4.7 (*ii*), 4.10 (*iii*), 4.11 (*iii*), 4.12 (*iii*), 4.13 (*ii*), 4.14 (*iii*), 4.15 (*i*), 4.15 (*iv*), 4.16 (*iii*), 4.16 (*iv*), 4.18 (*ii*), 4.18 (*iii*), 4.19 (*ii*), 4.19 (*iii*), 4.19 (*iv*), 4.20 (*iii*), 4.21 (*ii*), 4.21 (*iii*), 4.22 (*iii*), 4.22 (*iv*), 4.23 (*iii*).

In this case a compensation scheme would be more complicated than before. In several cases we will distribute different amount of curvature from regions with the same adjacent labels but different degree of some vertices. For this reason it seems to be easier to describe the distribution process in the same fashion as we have done in the case $|\alpha| = 2$. One can easily check that with the following choice of distribution each positively curved interior region from the list above has non-positive pseudo-curvature.

The distribution for the boundary regions will be described directly in Lemma 4.2.22.

Let Δ be a positively curved interior s-region.

- transfer the curvature $\frac{\pi}{15}$ through each double edge labelled by $a\lambda b$

adjacent to Δ and whose end vertices both have degree ≥ 5;

- transfer the curvature $\dfrac{\pi}{15}$ through each double edge labelled by $\overline{\mu}_2\overline{\mu}_1\overline{e}$ adjacent to Δ where $d(v_{\overline{\mu}_2}) \geq 5$ and $d(v_{\overline{e}}) \geq 3$;

- transfer the curvature $\dfrac{\pi}{3}$ through each triple edge adjacent to Δ where $d(v_{\overline{e}}) = 3$ and $d(v_{\overline{a}}) = 5$ or 6;

- transfer the curvature $\dfrac{2\pi}{7}$ through each triple edge adjacent to Δ where $d(v_{\overline{e}}) = 3$ and $d(v_{\overline{a}}) \geq 7$;

- transfer the curvature $\dfrac{3\pi}{10}$ through each triple edge adjacent to Δ where $d(v_{\overline{e}}) = 4$ and $d(v_{\overline{a}}) \geq 5$;

- transfer the curvature $\dfrac{\pi}{3}$ through each triple edge adjacent to Δ where $d(v_{\overline{e}}) \geq 5$ and $d(v_{\overline{a}}) \geq 5$.

Lemma 4.2.21 *If Δ is interior then $c^*(\Delta) \leq 0$.*

Proof. We distributed the positive curvature in such a way that every positively curved interior region has non-positive pseudo-curvature.
Let Δ be an interior m-region receiving positive curvature according to the scheme above.
Distinguish two cases:

1. the region Δ receives positive curvature from exactly one s-region;

2. the region Δ receives positive curvature from exactly two s-regions.

Case 1.
There are three subcases; the region Δ receives positive curvature through

1.1) the double edge labelled by $e\mu_1\mu_2$;

1.2) the double edge labelled by $a\lambda b$;

1.3) the triple edge labelled by $e\mu_1\mu_2 a$.

Case 1.1)

The maximum amount of curvature the region Δ can receive is $\dfrac{\pi}{15}$.

Observe that, according to the distribution process described above, we have $d(v_e) \geq 3$ and $d(v_{\mu_2}) \geq 5$; moreover the constraint $\alpha = -\beta$ implies $d(v_a) \geq 4$ and $d(v_b), d(v_d) \geq 3$. It follows that $c^*(\Delta) \leq c(3,3,3,4,5) + \dfrac{\pi}{15} < 0$.

Case 1.2)

The maximum amount of curvature the region Δ can receive is $\dfrac{\pi}{15}$.

Observe that, according to the distribution process above, we have $d(v_a) \geq 5$ and $d(v_b) \geq 5$; moreover $d(v_d) \geq 3$.

If $d(v_e) = 2$ then $l(v_e) = ec$, therefore v_d and v_{μ_1} have sublabels dd and $b\mu_1$ respectively. This implies $d(v_d) \geq 5$ and $d(v_{\mu_1}) \geq 3$ and it follows that $c^*(\Delta) \leq c(3,5,5,5) + \dfrac{\pi}{15} < 0$.

If $d(v_e) > 2$ and $d(v_c) = 2$ then $l(v_c) = ce$ therefore v_d has sublabel dd and so $d(v_d) \geq 5$. It follows that $c^*(\Delta) \leq c(3,5,5,5) + \dfrac{\pi}{15} < 0$.

If $d(v_e), d(v_c) > 2$ then $c^*(\Delta) \leq c(3,3,3,5,5) + \dfrac{\pi}{15} < 0$.

Case 1.3)

The maximum amount of curvature the region Δ can receive is $\dfrac{\pi}{3}$.

Observe that $d(v_e) \geq 3$ and $d(v_a) \geq 5$, moreover we have $d(v_b), d(v_d) \geq 3$.

First suppose $d(v_a) = 5$ and $d(v_e) = 3$.

In this case, since v_a has sublabel $\overline{s}a$, $l(v_a) = \overline{s}a\overline{d}\overline{a}\overline{b}$ and $l(v_e) = \overline{b}e\overline{s}$.

This implies that v_λ has sublabel $\overline{e}\lambda$ and so $d(v_\lambda) \geq 4$.

If $d(v_c) = 2$ then $l(v_c) = ce$, which implies that v_d has sublabel dd and so

$d(v_d) \geq 5$. It follows that $c(\Delta) \leq c(3,3,4,5,5) = -\dfrac{11\pi}{30}$ and so $c^*(\Delta) < 0$.

If $d(v_c) > 2$ then the vertices v_d and v_c cannot both have degree≤ 4. It follows that $c(\Delta) \leq c(3,3,3,4,5,5) = -\dfrac{7\pi}{10}$ and so $c^*(\Delta) < 0$.

Now suppose $d(v_a) = 6$ and $d(v_e) = 3$.

In this case v_a must have sublabel $\bar{s}ad$ or $\bar{s}a\bar{d}$ therefore v_λ has sublabel $\bar{e}\lambda$ or $c\lambda$ respectively. It follows that $d(v_\lambda) \geq 3$.

The same arguments as above ($d(v_c) = 2$ or $d(v_c) > 2$) show that $c^*(\Delta) < 0$.

Now suppose $d(v_a) \geq 7$ and $d(v_e) = 3$.

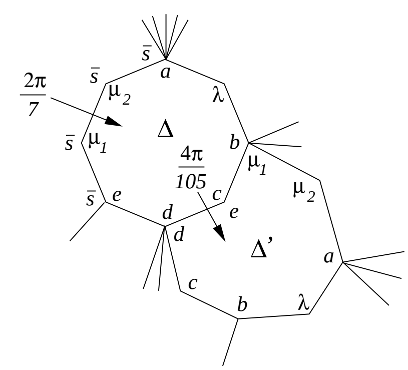
In this case the maximum amount of curvature that Δ can receive is $\dfrac{2\pi}{7}$.

If $d(v_c) = 2$ then $l(v_c) = ce$; it follows that v_b and v_d have sublabels $b\mu_1$ and dd respectively, hence $d(v_d) \geq 5$.

If $d(v_b) = 3$ then $l(v_b) = b\mu_1\{\bar{e}, c\}$, which forces v_λ to have sublabel $\lambda\bar{d}$ or λd and so $d(v_\lambda) \geq 3$. It follows that $c^*(\Delta) \leq c(3,3,3,5,7) + \dfrac{2\pi}{7} < 0$.

If $d(v_b) > 3$ then the constraint $\alpha = -\beta$ implies $d(v_b) \geq 5$ but the curvature of Δ is not negative enough, in general, to compensate the incoming curvature.

Let Δ' be the m-region adjacent to Δ shown in the following figure; denote by v'_r ($r = a, \lambda, b, c, d, e, \mu_1, \mu_2$) its vertices and redistribute the curvature as follows



Then $c^*(\Delta) \leq c(3,5,5,7) + \dfrac{2\pi}{7} - \dfrac{4\pi}{105} \leq 0$ and if Δ' is a boundary region there is nothing else to prove, so assume Δ' is interior.

We have $d(v'_d), d(v'_{\mu_1}), d(v'_a) \geq 5$ and $d(v'_b) \geq 3$.

Notice that since $d(v'_e) = 2$ it follows that Δ' cannot receive more than $\dfrac{\pi}{15}$ of positive curvature from the s-regions, thus we have $c^*(\Delta') \leq c(3,5,5,5) + \dfrac{4\pi}{105} + \dfrac{\pi}{15} < 0$.

Now return to Δ. If $d(v_c) > 2$ then observe that the vertices v_d and v_c cannot both have degree< 5. It follows that $c^*(\Delta) \leq c(3,3,3,5,7) + \dfrac{2\pi}{7} < 0$.

Now assume $d(v_e) = 4$.

According to the compensation process we have $d(v_a) \geq 5$, moreover $d(v_d) \geq 3$ and $d(v_b) \geq 3$.

In this case the maximum amount of curvature that Δ can receive is $\dfrac{3\pi}{10}$.

If $d(v_c) = 2$ then $l(v_c) = ce$ and so v_d has sublabel dd which implies $d(v_d) \geq 5$.

Moreover v_b has sublabel $b\mu_1$ therefore if $d(v_b) < 5$ then $d(v_\lambda) \geq 3$.

It follows that $c^*(\Delta) \leq \max\{c(3,3,4,5,5), c(4,5,5,5)\} + \dfrac{3\pi}{10} \leq 0$.

If $d(v_c) > 2$ then observe that either $d(v_d) \geq 5$ or $d(v_c) \geq 5$.

It follows that $c^*(\Delta) \leq c(3,3,4,5,5) + \dfrac{3\pi}{10} < 0$.

Finally assume $d(v_e) \geq 5$.

The maximum amount of curvature that Δ can receive is $\dfrac{\pi}{3}$.

We have $d(v_a) \geq 5$ and $d(v_d), d(v_b) \geq 3$.

If $d(v_c) = 2$ then $l(v_c) = ce$ and so v_d has sublabel dd which implies $d(v_d) \geq 5$.

Moreover v_b has sublabel $b\mu_1$ therefore if $d(v_b) < 5$ then $d(v_\lambda) \geq 3$.

It follows that $c^*(\Delta) \leq \max\{c(3,3,5,5,5), c(5,5,5,5)\} + \dfrac{\pi}{3} < 0$.

If $d(v_c) > 2$ then observe that the vertices v_d and v_c cannot both have degree< 5. It follows that $c^*(\Delta) \leq c(3,3,5,5,5) + \dfrac{\pi}{3} < 0$.

Case 2.

There are two subcases; the region Δ receives positive curvature through

2.1) two double edges;

2.2) one double edge and one triple edge.

Case 2.1)

The maximum amount of curvature the region Δ can receive is $\dfrac{\pi}{15}+\dfrac{\pi}{15}=\dfrac{2\pi}{15}$. Observe that, according to the compensation process, we have $d(v_a)\geq 5$, $d(v_b)\geq 5$ and $d(v_{\mu_2})\geq 5$; moreover $d(v_d), d(v_e)\geq 3$.

It follows that $c^*(\Delta)\leq c(3,3,5,5,5)+\dfrac{2\pi}{15}<0$.

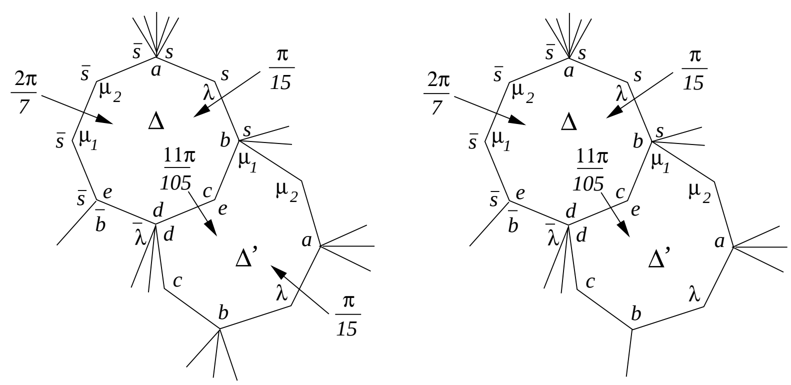
Case 2.2)

Since v_a has sublabel $\bar{s}as$ we have $d(v_a)\geq 7$.

First assume $d(v_e)=3$.

Since $d(v_a)\geq 7$ it follows that the maximum amount of curvature the region Δ can receive is $\dfrac{2\pi}{7}+\dfrac{\pi}{15}=\dfrac{37\pi}{105}$. Observe that, according to the compensation process, we have $d(v_b)\geq 5$; moreover $d(v_d)\geq 3$.

If $d(v_c)=2$ let Δ' be the m-region adjacent to Δ as in case 1.3). Redistribute the curvature as follows:



Then we have $c^*(\Delta) \leq c(3,5,5,7) + \dfrac{37\pi}{105} - \dfrac{11\pi}{105} = 0$ and we can assume Δ' is interior.

If Δ' does receive positive curvature through the double edge $a\lambda b$ then $d(v_b') \geq 5$ therefore $c^*(\Delta') \leq c(5,5,5,5) + \dfrac{11\pi}{105} + \dfrac{\pi}{15} < 0$.

If Δ' does not receive positive curvature through the double edge $a\lambda b$ then $d(v_b') \geq 3$ therefore $c^*(\Delta') \leq c(3,5,5,5) + \dfrac{11\pi}{105} < 0$.

Now return to Δ and assume $d(v_e) = 4$.

The maximum amount of curvature that Δ can receive is $\dfrac{3\pi}{10} + \dfrac{\pi}{15} = \dfrac{11\pi}{30}$.

Observe that, according to the compensation process, we have $d(v_b) \geq 5$; moreover the vertices v_d and v_c cannot both have degree< 5. It follows that $c^*(\Delta) \leq c(4,5,5,7) + \dfrac{11\pi}{30} < 0$.

Finally assume $d(v_e) \geq 5$.

The maximum amount of curvature that Δ can receive is $\dfrac{\pi}{3} + \dfrac{\pi}{15} = \dfrac{2\pi}{5}$.

From the compensation process we know that $d(v_b) \geq 5$; moreover the vertices v_d and v_c cannot both have degree< 5. It follows that $c^*(\Delta) \leq c(5,5,5,7) + \dfrac{2\pi}{5} < 0$.

$\square$

Lemma 4.2.22 *If Δ is a boundary region then $c^*(\Delta) < \dfrac{4\pi}{k_0}$.*

Proof Distinguish the following cases:

1. Δ is an s-region;

2. Δ is an m-region.

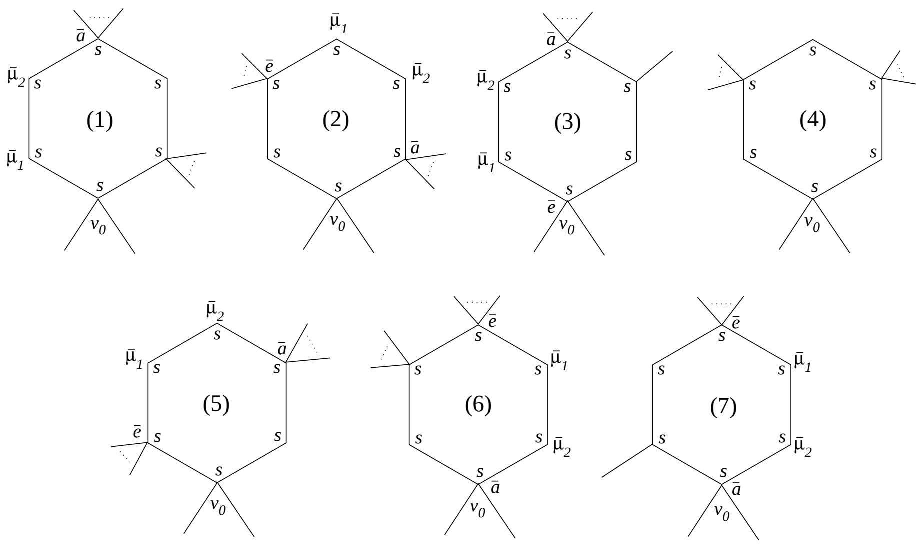
Case 1.

As seen before the distinguished vertex coincides with exactly one vertex of

Δ and we can assume without any loss that $v_0 = v_4$ and that $d(\Delta) \leq 3$.

Case 1.1)**,** $d(\Delta) = 3$

There are seven subcases:



Observe that in cases (1), (2) and (4) there are two interior vertices of degree ≥ 5 (otherwise $d(\Delta) > 3$). It follows that $c^*(\Delta) = c(\Delta) \leq c(5, 5, k_0) = \dfrac{2\pi}{k_0} - \dfrac{\pi}{5} < \dfrac{4\pi}{k_0}$.

In other cases we can assume that there is an interior vertx of degree 3, otherwise $c^*(\Delta) = c(\Delta) \leq c(4, 4, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

In case (3) we can have $d(v_2) = 3$, but only if $l(v_2) = bds$ or $l(v_2) = s\overline{e}b$, in which case v_1 has sublabel $\overline{a}se$ or $\overline{a}sc$ respectively, hence $d(v_1) \geq 6$.

It follows that $c^*(\Delta) = c(\Delta) \leq c(3, 6, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

In case (5) we have $d(v_2) \geq 5$. We choose to transfer a positive curvature of $\dfrac{\pi}{15}$ through the triple edge (which might be in common with an interior region). Notice that this choice respects all the upper bounds given in the distribution process for transferring curvature through a triple edge from an interior region, so the proof of the previuos lemma is not affected.

It follows that $c^*(\Delta) \leq c(3,5,k_0) - \dfrac{\pi}{15} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

In cases (6) and (7) we have $d(v_6) \geq 5$ and $d(v_1) \geq 5$ respectively (otherwise $d(\Delta) > 3$). We choose to transfer a positive curvature of $\dfrac{\pi}{15}$ through the triple edge (which is in common with another boundary region).

It follows that $c^*(\Delta) \leq c(3,5,k_0) - \dfrac{\pi}{15} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Case 1.2)**,** $d(\Delta) = 2$

The vertex v_1 has sublabel $\overline{a}s\overline{e}$ which implies $d(v_1) \geq 6$.

We choose to transfer the curvature $\dfrac{\pi}{6}$ through each triple edge; it follows that $c^*(\Delta) \leq c(6,k_0) - \dfrac{\pi}{6} = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Case 2.

Suppose the distinguished vertex v_0 coincides with m vertices of Δ. Notice that $k_0 \geq 2m$. As seen before we can assume $m \leq 3$. The maximum total amount of curvature that Δ can receive is $\dfrac{\pi}{3} + \dfrac{\pi}{15} = \dfrac{2\pi}{5}$.

Suppose $m = 3$. We have $c^*(\Delta) \leq c(\Delta) + \dfrac{2\pi}{5} \leq c(3,k_0,k_0,k_0) + \dfrac{2\pi}{5} = -\dfrac{4\pi}{3} + \dfrac{2\pi}{k_0} + \dfrac{4\pi}{k_0} + \dfrac{2\pi}{5} \leq -\dfrac{14\pi}{15} + \dfrac{\pi}{3} + \dfrac{4\pi}{k_0} < \dfrac{4\pi}{k_0}$.

Now suppose $m = 2$. We have $c^*(\Delta) \leq c(\Delta) + \dfrac{2\pi}{5} \leq c(3,3,k_0,k_0) + \dfrac{2\pi}{5} = -\dfrac{2\pi}{3} + \dfrac{4\pi}{k_0} + \dfrac{2\pi}{5} = -\dfrac{4\pi}{15} + \dfrac{4\pi}{k_0} < \dfrac{4\pi}{k_0}$.

We can therefore assume that v_0 coincides with a unique vertex of Δ.

Suppose $d(v_c) = 2$ and v_c is not distinguished. Then $l(v_c) = ce$, moreover v_b and v_d have sublabels $b\mu_1$ and dd respectively. For interior vertices we have $d(v_a) \geq 5$, $d(v_b) \geq 3$, $d(v_d) \geq 5$, moreover the vertices v_e and v_{μ_1} cannot have both degree 2 and $d(v_b) < 5$ implies $d(v_\lambda) \geq 3$. Since one of the vertices v_a, v_λ, v_b, v_d, v_e and v_{μ_1} can be the distinguished one it follows that $c^*(\Delta) \leq \max\{c(3,5,5,k_0), c(3,3,3,5,k_0)\} + \dfrac{2\pi}{5} = \dfrac{2\pi}{k_0} - \dfrac{2\pi}{15} < \dfrac{4\pi}{k_0}$.

Suppose $d(v_c) > 2$ and v_c is not distinguished. For interior vertices we have

$d(v_a) \geq 5$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 3 and if $d(v_b) < 5$ then $d(v_\lambda) \geq 3$. Since one of the vertices v_a, v_λ, v_b, v_d, v_e and v_{μ_1} can be the distinguished one it follows that $c^*(\Delta) \leq \max\{c(3,3,3,5,k_0), c(3,3,3,3,3,k_0)\} + \frac{2\pi}{5} = \frac{2\pi}{k_0} - \frac{\pi}{5} < \frac{4\pi}{k_0}$.

So we can assume $v_0 = v_c$, in which case $d(v_a) \geq 5$, $d(v_b) \geq 3$, $d(v_d) \geq 3$, moreover v_e and v_{μ_1} cannot both have degree< 3. It follows that $c^*(\Delta) \leq c(3,3,3,5,k_0) + \frac{2\pi}{5} = \frac{2\pi}{k_0} - \frac{\pi}{5} < \frac{4\pi}{k_0}$.

□

Proof of Proposition 4.2.1

The proof follows from Lemmas 4.2.7-4.2.22 in the same way as Theorem 1 follows from Lemmas 3.2.5-3.2.8.

□

4.3 Proof of Theorem 4

This section will be mainly devoted to proving the following proposition.

Proposition 4.3.1 *The group $G = G_{3q}(x_0[x_3^2, x_q^{-1}])$ where q is a prime $q \geq 5$, is infinite.*

Before going through the proof of this proposition we show how Theorem 4 can be derived from it.

Proof of Theorem 4

We are assuming $h_i, h_j > 1$. By Lemma 4.1.1 it suffices to consider the presentations $G_{pq}\left(x_0\left[x_p^\alpha, x_q^\beta\right]\right)$ where p and q are different primes such that $p|h_i$, $q|h_j$. Moreover, rearranging if necessary, we can assume $p < q$.

Since we are assuming $n = pq$ odd, it follows that $2 < p < q$.

This implies $q \geq 5$. Therefore if $\beta \neq -1$ it follows from Pride's result ([26]) that G is infinite (see proof of Theorem 3).

So we can assume $\beta = -1$ and consequently $|\alpha| \neq 1$.

If $p > 3$ Pride's result implies again that the given group is infinite (since $\alpha \neq 1$). This leaves the cases $G = G_{3q}\left(x_0\left[x_3^{\alpha}, x_q^{-1}\right]\right)$.

To see why we can consider only the case $\alpha = 2$ observe that there is an epimorphism from the extension $E(3q;\alpha,-1;3,q)$ of G to the extension $E(3;\alpha,-1;0,1)$ of $G_3\left(x_0\left[x_0^{\alpha}, x_1^{-1}\right]\right)$ which is infinite by Theorem (4.4) in [23] (To see why that is simply take $p = q = r = \alpha$ and $s = t = u = \alpha + 1$).

$\square$

We now return to the proof of Proposition 4.3.1.

Observe that $q = \dfrac{n}{3}$, therefore the modified star graph is given by Figure 2.1 (ii) and the maximal segment length in an m-region is realized by ea and is exactly q.

Let $\mathscr{D}$ be a modified van Kampen diagram for the given presentation.

It follows that there is no s-region of degree 2 in $\mathscr{D}$, and those of degree 3 which are positively curved are easily classified because of the restrictions on the labels of the adjacent segments.

We will assume $\mathscr{D}$ to satisfy assumptions **C1-C3** and so statements $(i) - (iv)$ in Lemma 4.2.2 hold. Then we have the following list of possible labels for interior vertices of degree 2, 3 and 4 (up to inversion and cyclic permutation).

degree 2 : $ce,\ \lambda\mu,\ \lambda s;$

degree 3 : $bds,\ bd\mu,\ \overline{d}aa;$

degree 4 : $ad\overline{a}\overline{d},\ aa\{\mu,s\}b,\ a\{\mu,s\}\{c,\overline{e}\}b,\ b\{\mu,s\}\{c,\overline{e}\}\{c,\overline{e}\}.$

Lemma 4.3.2 *If Δ is an interior m-region then $c(\Delta) \leq 0$.*

Proof. We have $d(v_a), d(v_b), d(v_d) \geq 3$; moreover v_c and v_e cannot have degree 3.

It follows that if $d(v_c), d(v_e) > 2$ then $c(\Delta) \leq c(3,3,3,4,4) = 0$.

If $d(v_e) = 2$ then $l(v_e) = ec$. Now $|bc| < |ea|$, therefore the segment ea splits and the splitting has sublabel $\{b, \lambda\}\mu$. In both cases the splitting has degree ≥ 4.

In order to have positive curvature we must have $d(v_c) = 2$ and no other splitting in Δ, for if not $c(\Delta) \leq c(3,3,3,4,4) = 0$.

This forces v_d and v_b to have sublabels ddd and $b\mu$ respectively, thus $d(v_d) \geq 6$ and $d(v_b) \geq 4$. It follows that $c(\Delta) \leq c(3,4,5,6) < 0$.

$\square$

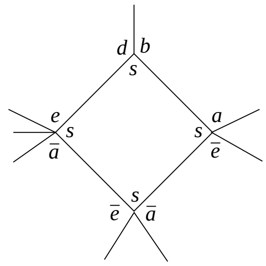
We now proceed to classify the positively curved interior s-regions.

Let Δ be an interior s-region.

Since the unique possible label in degree 3 which involves s is bds and since two such vertices cannot be separated by a sequence of vertices of degree 2, it follows that if $d(\Delta) \geq 5$ then $c(\Delta) < 0$.

Let $d(\Delta) = 4$; if there is no vertex of degree 3 then $c(\Delta) \leq c(4,4,4,4) = 0$, so assume there is a vertex v of degree 3, which must be labelled by bds. It follows that the sum of the length of the two segments of Δ which have v as an end point is less than or equal to $|ab| + |de| = 3 + q - 3 = q$. Since the maximal segment length is q and the total length around an s-region is $3q$, it follows that no vertex of Δ is a splitting in the adjacent m-regions and the remaining segments of Δ must have maximal length, hence are labelled by $\overline{ae}$. Then there is a vertex v' with sublabel $\overline{a}se$ (which implies

$d(v') \geq 5$) and the remaining two vertices have degree$\geq$ 4. It follows that a positively curved interior s-region Δ looks like the following figure where $c(\Delta) \leq c(3,4,4,5) = \dfrac{\pi}{15}$.



Finally let Δ have degree 3.

We have already mentioned that the maximal segment length is $q = \dfrac{n}{3}$ and is realized only by the segment labelled by ea.

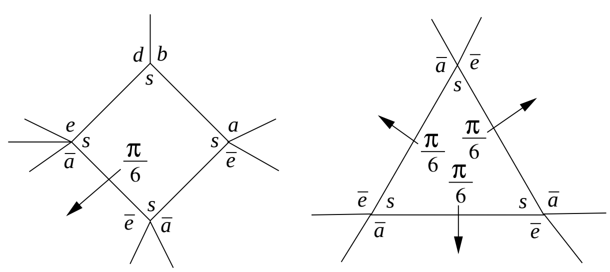
It follows that each of the three vertices of Δ has sublabel $\overline{a}s\overline{e}$ and so degree$\geq$ 4. Therefore $c(\Delta) \leq c(4,4,4) = \dfrac{\pi}{2}$.

We are in the position to describe the distribution process for the positive curvature of the interior regions.

Distribution of curvature for interior regions

Let Δ be a positively curved interior s-region.

We distribute the curvature as follows:



Now Proposition 4.3.1 can be deduced from the following two lemmas as pointed out in Chapter 2.

Lemma 4.3.3 *If Δ is interior then $c^*(\Delta) \leq 0$.*

Proof.

Clearly we only need to consider interior m-regions which receive some positive curvature from adjacent s-regions.

Notice that the positive curvature is transferred only through the segment labelled by ea such that $d(v_a), d(v_e) \geq 4$.

The incoming curvature is $\dfrac{\pi}{6}$ and we have $d(v_b), d(v_d) \geq 3$.

We can assume that $d(v_c) = 2$, hence $l(v_c) = ce$, and that neither bc nor cd splits, otherwise $c^*(\Delta) \leq c(3, 3, 4, 4, 4) + \dfrac{\pi}{6} = 0$.

Since $|bc| < |ea|$ and $|cd| = |de|$ it follows that v_b and v_d have sublabels $b\mu$ and dd respectively.

This implies $d(v_b) \geq 4$ and $d(v_d) \geq 6$, hence $c^*(\Delta) \leq c(4, 4, 4, 6) + \dfrac{\pi}{6} = 0$.

□

Lemma 4.3.4 *If Δ is a boundary region then $c^*(\Delta) < \dfrac{4\pi}{k_0}$.*

Proof. Let Δ be a boundary s-region. Recall that the distinguished vertex v_0 coincides with exactly one vertex of Δ.

If $d(\Delta) \geq 4$ then $c^*(\Delta) = c(\Delta) \leq c(3, 3, 3, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$ and no additional distribution is required.

Similarly, if $d(\Delta) = 3$ then the constraints on the labels in Δ imply $c(\Delta) \leq c(4, 4, k_0) = \dfrac{2\pi}{k_0} < \dfrac{4\pi}{k_0}$.

It remains to check the boundary m-regions.

Observe that $d(\Delta) \geq 4$ since for interior vertices we have $d(v_a), d(v_b), d(v_d) \geq 3$ and if $d(v_e) = 2$ then the segment ea splits.

Since any closed path in $\mathscr{D}$ must have t-exponent divisible by n it follows that no more than two vertices among v_a, v_b, v_d, v_e and a splitting in ea coincide with the distinguished one.

Assume that v_0 coincides with m vertices in Δ; then $k_0 \geq 2m$ and $c(\Delta) \leq c(3, 3, k_0, \ldots, k_0)$ where k_0 appears m times.

Since the maximum amount of curvature that Δ can receive is $\frac{\pi}{6}$, it follows that $c^*(\Delta) \leq c(3, 3, k_0, \ldots, k_0) + \frac{\pi}{6} = -m\pi + \frac{4\pi}{3} + \frac{2m\pi}{k_0} + \frac{\pi}{6} \leq -m\pi + \frac{3\pi}{2} + \frac{2m\pi}{2m} = -m\pi + \frac{5\pi}{2}$.

Therefore $c^*(\Delta) \leq 0$ for $m \geq 3$.

Now suppose that the distinguished vertex coincides with exactly two vertices of Δ. What has been said before implies that $c^*(\Delta) \leq c(3, 3, k_0, k_0) + \frac{\pi}{6} = -\frac{\pi}{2} + \frac{4\pi}{k_0} < \frac{4\pi}{k_0}$.

Finally, assume that v_0 coincides with exactly one vertex of Δ.

For interior vertices we have $d(v_a), d(v_b), d(v_d) \geq 3$, moreover if $d(v_e) < 4$ then there is a splitting of degree ≥ 4 in the segment ea. It follows that $c^*(\Delta) \leq c(3, 3, 4, k_0) + \frac{\pi}{6} = \frac{2\pi}{k_0} < \frac{4\pi}{k_0}$.

$\square$

Proof of Proposition 4.3.1

The proof follows from Lemmas 4.3.3-4.3.4 in the same way as Theorem 1 follows from Lemmas 3.2.5-3.2.8.

$\square$

Chapter 5

Conclusions

In this last short chapter we want to summarize what has been achieved in Chapter 3 and 4 and prove a general statement which gives a partial answer to Conjecture 1.2.7.

5.1 Summary

We first recall the main results obtained in Chapter 3 and 4 for $n \geq 5$.

Theorem 2 *Let* $G = G_n\left(x_0\left[x_i^\alpha, x_j^\beta\right]\right)$. *Suppose that* $h_i = 1$ *or* $h_j = 1$ *and that* $|\alpha| > 1$, $|\beta| > 1$ *and* $|\alpha| \neq |\beta|$.
If n *is odd and* $n \geq 11$ *then* G *is infinite.*

Theorem 3 *Let* $G = G_n\left(x_0\left[x_i^\alpha, x_j^\beta\right]\right)$. *If* $h_i > 1$, $h_j > 1$, $|\alpha| > 1$ *and* $|\beta| > 1$ *then* G *is infinite.*

Theorem 4 *Let* $G = G_n\left(x_0\left[x_i^\alpha, x_j^\beta\right]\right)$. *If* n *is odd,* $h_i > 1$, $h_j > 1$ *and* $(|\alpha|, |\beta|) \neq (1, 1)$ *then* G *is infinite.*

The hypotheses given in the theorems above can be viewed as conditions on the 5-tuple (n, i, j, α, β). With this in mind we can prove the following.

Theorem 5 *Let* $G = G_n\left(x_0\left[x_i^\alpha, x_j^\beta\right]\right)$ *be irreducible. If there exists* m *such that* $m \mid n$, $m \nmid j-i$ *and the 5-tuple* $(m, i', j', \alpha, \beta)$, *where* $i' \equiv i \bmod m$ *and* $j' \equiv j \bmod m$, *respects one of the conditions in* Theorem 2, Theorem 3 *or* Theorem 4, *then* G *is infinite.*

Proof. Notice that since $m \nmid j-i$ there is an epimorphism

$$E(n;\alpha,\beta;i,j) = \langle x,t | t^n, xt^{-i}x^{-\alpha}t^{i-j}x^{-\beta}t^{j-i}x^{\alpha}t^{i-j}x^{\beta}t^{j}\rangle \twoheadrightarrow$$
$$\twoheadrightarrow \langle x,t | t^m, xt^{-i'}x^{-\alpha}t^{i'-j'}x^{-\beta}t^{j'-i'}x^{\alpha}t^{i'-j'}x^{\beta}t^{j'}\rangle = E(m;\alpha,\beta;i',j').$$

which is an extension of $G_m\left(x_0\left[x_{i'}^\alpha, x_{j'}^\beta\right]\right)$ and is infinite by Theorem 2, Theorem 3 or Theorem 4. It follows that G is infinite.

$\square$

5.2 What is left to be proved?

The theorems stated in the previous section do not answer completely Conjecture 1.2.7.

In this section we want to clarify what is left to be proved in order to confirm the conjecture and why our method of proof fails in these cases.

First of all we observe that the Havas-Robertson presentations are all p-irreducible. It follows that in the f-irreducible cases we would need to prove the statement of Theorem 3 also for $|\alpha| = 1$ or $|\beta| = 1$. Now consider the epimorphism we used in the proof of Theorem 3. If $\alpha = 1$ or $\beta = -1$ the co-domain could be the presentation of a finite cyclic group and therefore we cannot conclude that the given group is infinite. Since we are working modulo the equivalence relation determined by the elementary moves and since these moves can interchange α and β and also change their signs, it follows that our conditions on α and β must be symmetric with respect to these moves and the easiest way to ensure that our method works is to assume $|\alpha| > 1$ and $|\beta| > 1$. Here we could have been more precise and weakened these constraints assuming $\alpha \neq 1$ and $\beta \neq -1$. This is due to the fact that we only need to study the presentations under the condition $0 < i < j < n$; if ϕ is a sequence of elementary moves which maps $x_0 \left[x_i^{\alpha}, x_j^{\beta}\right]$ to $x_0 \left[x_{i'}^{\alpha'}, x_{j'}^{\beta'}\right]$ where $0 < i < j < n$ and $0 < i' < j' < n$ then $(\alpha', \beta') \in \{(\alpha, \beta), (-\beta, -\alpha)\}$. In view of these the weaker constraints which ensure that the co-domain of the epimorphism above is not trivially cyclic are $\alpha \neq 1$ and $\beta \neq -1$. In order to prove Theorem 3 with these weaker hypotheses we are left to study $G_6\left(x_0 \left[x_2^{-1}, x_3^{\beta}\right]\right)$ and $G_6\left(x_0\left[x_2^{\alpha}, x_3\right]\right)$. In view of Proposition 4.2.3 we can assume $|\beta| \geq 6$ and $|\alpha| \geq 6$, respectively. These cases will be discussed elsewhere, but we want to remark that they can be proved to be presentations of infinite groups using the same geometric arguments as the present work. We decided not to include these proofs since they are remarkably cumbersome and the hypotheses $|\alpha| > 1$ and $|\beta| > 1$ seem to be reasonably weak in this context. Anyway, provided this is true, we are left to study the cases when $\alpha = 1$ or $\beta = -1$.

Now consider the results achieved in the p-irreducible case.

If n is odd we are left to check what happens for $n = 5, 7$ or 9 and to prove the statement in Theorem 2 without the constraints $|\alpha| > 1$, $|\beta| > 1$ and $|\alpha| \neq |\beta|$. We believe that the method of proof outlined in Chapter 2 can be used to study the cases $n = 5, 7$ and 9. We point out that using elementary moves we can assume $i = 1$ and $j \in \{2, 4\}$, $j \in \{2, 3, 6\}$ or $j \in \{2, 3, 4, 6, 8\}$

when $n = 5, 7$ or 9, respectively; for these cases we know that Conjecture 1.2.7 is true for $0 < |\alpha|, |\beta| \leq 3$; moreover in view of Edjvet and Hammond paper [11] we don't have to study the case $j = 2$.

Furthermore, when $(n, j) \in \{(7, 6), (9, 8)\}$ we know by Corollary 3.2.9 that for $|\alpha|, |\beta| > 1$ and $\alpha \neq \pm\beta$ these groups are infinite; this leaves the cases $|\alpha| = |\beta| \geq 4$, $|\alpha| = 1$ and $|\beta| \geq 4$ or $|\alpha| \geq 4$ and $|\beta| = 1$.

It follows that we are left with five cases for which we have data when $0 < |\alpha|, |\beta| \leq 3$ as pointed out in Chapter 1.

Finally, if $n = 2k$ is even we need to study those cases for which any odd integer dividing k and greater than 3 divides $j - i$ (of course if the remaining cases for n odd are solved). Notice that the Havas-Robertson presentations are of this special case, hence the first case to consider is when $\alpha \neq \beta$.

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